Equivariant Ricci-Flow with Surgery

Jonathan Dinkelbach

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Jonathan Dinkelbach

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Jonathan Dinkelbach
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Erstgutachter:	Prof. Bernhard Leeb, Ph. D.
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Zusammenfassung

In dieser Arbeit untersuchen wir Perelmans Ricci-Fluss mit Chirurgie auf geschlossenen 3–Mannigfaltigkeiten, deren Ausgangsmetrik invariant unter einer vorgegebenen glatten Wirkung einer endlichen Gruppe ist. Eine solche Metrik kann stets durch Mittelung einer beliebigen Riemannschen Metrik erzeugt werden, und wegen der Eindeutigkeit des Ricci-Flusses bleibt dieser bis zum Auftreten von Singularitäten invariant unter der Gruppenwirkung. Die technische Schwierigkeit besteht nun darin, Symmetrien der evolvierenden Metrik zu kontrollieren, wenn sich der Fluss einer Singularität nähert.

Zu diesem Zweck konstruieren wir eine invariante singuläre S^2–Blätterung auf dem Bereich der Mannigfaltigkeit, der von der Chirurgie betroffen ist. Insbesondere ermöglicht es diese, den Chirurgieprozess äquivariant durchzuführen und die Gruppenwirkung auf solchen Komponenten zu analysieren, die bei der Chirurgie komplett entfernt werden. Darüber hinaus lässt sich mit Hilfe der Blätterung beschreiben, wie die Gruppenwirkungen vor und nach der Chirurgie zusammenhängen. Dadurch lassen sich aus dem Langzeitverhalten des Ricci-Flusses und der Gruppenwirkung Rückschlüsse auf die ursprüngliche Wirkung ziehen.

Als Anwendung zeigen wir, dass jede glatte endliche Gruppenwirkung auf einer geschlossenen geometrischen 3–dimensionalen Mannigfaltigkeit mit sphärischer, hyperbolischer oder $S^2 \times \mathbb{R}$–Geometrie verträglich mit der geometrischen Struktur ist, dass also eine invariante vollständige lokalhomogene Riemannsche Metrik existiert. Dies löst eine von William Thurston aufgestellte Frage zu Gruppenwirkungen auf geometrischen 3–Mannigfaltigkeiten, die für die übrigen fünf Geometrien bereits von Meeks und Scott gelöst wurde [MS86].

Abstract

In this thesis we study Perelman's Ricci-flow with surgery on closed 3-manifolds on which the initial metric is invariant under a given smooth finite group action. Such a metric can always be obtained by averaging an arbitrary metric, and due to its uniqueness the Ricci-flow stays invariant until the first singular time. The main technical difficulty is to control the symmetries of the evolving metric when the flow approaches a singular time.

In order to get such a control, we construct an invariant singular S^2–foliation on the part of the manifold which is affected by the surgery. In particular this foliation enables us to perform the surgery process in an equivariant way and to analyze the action on those components which get extinct at the surgery time, since they are completely covered by the foliation. Moreover, it relates the group actions before and after a surgery. Thus, we can conclude properties of the initial group action from the long time behavior of the equivariant flow.

As an application we show that any smooth finite group action on a closed geometric 3-manifold with spherical, hyperbolic or $S^2 \times \mathbb{R}$–geometry is compatible with the geometric structure, i. e. there exists an invariant complete locally homogeneous Riemannian metric. This solves a question of William Thurston for smooth group actions on geometric 3–manifolds, which was proved for the other five geometries by Meeks and Scott [MS86].

Contents

Introduction

A manifold is called *geometric* if it admits a geometric structure in the sense of Thurston, which can be seen as a complete, locally homogeneous Riemannian metric (see Chapter 1.1). A geometric structure contains the essential geometric information of the manifold, and knowing that a particular manifold is geometric greatly helps understanding the manifold.

Now given a smooth action of a finite group G on a geometric manifold M, one might ask whether the geometric structure on M can be choosen such that is compatible with the group action, i. e. the locally homogeneous metric is invariant under the action. If this is possible for any action, this can be interpreted as the geometric structure being *natural* in the sense that it respects any possible (finite) symmetry of the manifold.

In dimension two any compact manifold is geometric, since by the uniformization theorem any closed surface admits a metric of constant positive, flat or negative curvature. Moreover, any smooth finite group action on a two-dimensional closed surface admits a compatible geometric structure, i. e. the constant curvature metric can be chosen such that it is invariant under the group action. This is a consequence of the geometrization and classification of 2–dimensional orbifolds, see [Thu80, Chapter 13]. Alternatively, this also follows from the fact that the Ricci-flow on surfaces converges to a constant curvature metric, see [Ham88] and [Cho91].

Thurston raised the questions whether also 3–dimensional closed geometric manifolds always possess compatible geometric structures [Thu82, Question 6.2], [Thu83, Theorem B][1]:

Question (Thurston). *Let M be a closed geometric 3–manifold and let $\rho\colon G \curvearrowright M$ be a smooth group action of a finite group G. Does there exist a compatible geometric structure on M, i. e. a $\rho(G)$–invariant locally homogeneous metric?*

[1] In this preprint Thurston announced a proof of the statement under certain assumptions. However, this proof has never been published.

In this thesis we give an affirmative answer to the case of spherical, hyperbolic and $S^2 \times \mathbb{R}$–geometry, using techniques of Hamilton and Perelman, namely an equivariant version of the Ricci-flow with surgery. Together with the results of Meeks and Scott [MS86] this yields a complete solution of the question.

In the following we give an overview over the historical developments towards the solution of the question and describe different previously known partial results. We then end this introduction by sketching our approach and giving an outline of the structure of this thesis.

The *Smith-conjecture* was the first case where this question was investigated. The original motivation however was different: Smith has studied periodic orientation preserving homeomorphisms of S^3 of order $p \geq 3$, i. e. orientation preserving group actions of the cyclic group $\mathbb{Z}_p \curvearrowright S^3$. Computing the Čech-homology of the fixed-point set L of such a homeomorphism, he found out that L is either empty or a simply closed curve $S^1 \subset S^3$ ([Smi38, Theorem 4.15] and [Smi39, Theorem 4]). He then conjectured that this S^1 is topologically trivially embedded, i. e. it is a trivial knot. Assuming the fix-point set is non-empty, this conjecture can be seen to be equivalent to the question whether the $\mathbb{Z}_p$–action is conjugate to an orthogonal action by rotations, see [Moi62] for smooth or piecewise linear case and [Moi79] for the topological category.

For the case of an orientation reversing $\mathbb{Z}_2$–action, the fixed point set is either a two-sphere or consists of two points [Smi39]. In the later case, the action is conjugate to the orthogonal involution $(x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, -x_3, -x_4)$, as was proved by Livesay [Liv63], correcting a mistake in an earlier proof by Hirsch and Smale [HS59]. If the fixed point set is a tamely embedded two-sphere, then it is easy to see that this sphere can be isotoped to an equator-sphere and the action is conjugate to the involution at this sphere, the map $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)$. If however the fix-point 2–sphere is wildly embedded, it is impossible to find a conjugation: Let $\iota \colon S^2 \to M$ be a wild embedding, i. e. there is a point $x \in S^2$ such that there exists no homeomorphism from a neighborhood U of $\iota(x)$ to $\mathbb{R}^3$ mapping $\iota(S^2) \cap U$ to $\mathbb{R}^2 \times \{0\}$. It is obvious that wildness of the fixed-point set is preserved by conjugation, but fixed-point sets of orthogonal maps clearly cannot be wild.

In fact, there are counter-examples to the Smith-conjecture in the topological category: Bing gave such an example of an involution with fixed point set Σ an Alexander horned sphere [Bin52]. He showed that when gluing together two non-simply-connected solid horned spheres along their boundaries, the result is homeomorphic to S^3. Interchanging the two components gives the involution. Now if there were a self-homeomorphism of S^3 conjugating that involution to an orthogonal one, this homeomorphism would map Σ to an equator-sphere and each complement of Σ to a hemisphere homeomorphic to a 3–ball, which gives a contradiction.

Similarly, Montgomery and Zippin [MZ54] have modified Bing's construction in order to obtain examples of orientation preserving periodic homeomorphisms, where the fixed point set is a wildly embedded S^1. Also these group actions cannot be conjugated to orthogonal actions.

This illustrates the necessity of restricting to smooth actions. For those actions the Smith-conjecture could be verified in the late 1970s by contributions from different mathematicians such as W. Thurston, W. Meeks, S.-T. Yau, H. Bass and C. Gordon, combining methods and results from hyperbolic geometry, minimal-surface theory and algebra. For a collection of relevant papers for the proof of the Smith-conjecture and related work we refer to [MB84].

We call a smooth group action of a finite group on a closed geometric 3–manifold *standard* if there exists a compatible geometric structure, i. e. an invariant locally homogeneous metric, compare Chapter 1.2. So Thurston's question asks whether all such actions are standard and the Smith conjecture is equivalent to cyclic non-free orientation preserving actions on S^3 being standard, see Corollary 1.13.

Starting with the Smith conjecture, for a variety of different cases the question of Thurston has been verified. In the following we state some of the known results.

Meeks and Scott show that if M is a Seifert fibered space and the action $G \curvearrowright M$ preserves the Seifert fibration up to homotopy, then there exists a G–invariant Seifert fibration homotopic to the original one [MS86, Theorem 2.2]. They use this result to solve the question for geometries different from S^3, $\mathbb{H}^3$ and $S^2 \times \mathbb{R}$:

Theorem ([MS86, Theorem 2.1]). *Let M be a closed geometric 3–manifold, such that the model geometry is one of $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL}(2,\mathbb{R})$, Nil, $\mathbb{R}^3$ or Sol. Then any smooth finite group action on M is standard.* □

Meeks and Yau combine techniques of minimal surfaces with ideas of Papakyriakopoulos' proof of the Dehn Lemma to obtain an equivariant version of the Dehn Lemma [MY81]. They apply this to show that certain actions on $S^2 \times \mathbb{R}$ (viewed as $\mathbb{R}^3 - \{0\}$) are standard:

Theorem ([MY84, Theorem 4]). *If $\rho\colon G \curvearrowright \mathbb{R}^3$ is a smooth orientation preserving action of a compact group G such that $\rho(G)$ is not isomorphic to the orientation preserving icosahedral group A_5, then the action is conjugate to an orthogonal action.* □

Meeks and Yau actually show that *any* finite group that acts orientation preserving on $\mathbb{R}^3$ is isomorphic to a subgroup of $SO(3)$. However, to realize this subgroup embedding by a conjugation, they need solvability of the group, which leads to the exclusion of A_5. Also note that it is not obvious how to apply this Theorem to finite actions on $S^2 \times S^1$, since the lift of such an action needs not to be finite.

Finding an invariant geometric structure is equivalent to showing that the quotient orbifold is geometric, see Chapter 1.2. For orientable, irreducible 3–orbifolds with non-trivial ramification locus this holds by the Orbifold Theorem of Boileau, Leeb and Porti [BLP05, Corollary 1.2]:

Theorem (Orbifold Theorem, [BLP05]). *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible 3–orbifold with non-empty ramification locus. If $\mathcal{O}$ is topologically atoroidal, then $\mathcal{O}$ is geometric.* □

Thus, if G acts *non-freely and orientation preserving* on a geometric manifold M, and if the quotient orbifold $\mathcal{O} = M/G$ is irreducible, then $\mathcal{O}$ satisfies the assumption of the orbifold theorem and so the action is standard.

In fact, for a finite orientation preserving group action $\rho\colon G \curvearrowright S^3$ Boileau, Leeb and Porti show (using their orbifold theorem) that the quotient orbifold *is irreducible* and thus solve the case of non-free orientation preserving spherical group actions:

Corollary ([BLP05, Corollary 1.1]). *Any orientation preserving, smooth, non-free finite group action on S^3 is smoothly conjugate to an orthogonal action.* □

Finally, for *free* actions on S^3 the question is also known as "spherical space-form conjecture" and follows as a consequence of Perelman's Geometrization Theorem for closed 3–manifolds [Per03a, Per03b].

Theorem ([Per03a, Theorem 8.2(a)]). *Any orientation preserving, smooth, free finite group action on S^3 is smoothly conjugate to an orthogonal action.* □

Outline of the paper In this thesis, we present a unified approach for spherical, $S^2 \times \mathbb{R}$– and hyperbolic manifolds. Given a closed 3–manifold M with a smooth finite group action, we equip M with an invariant initial metric and study Perelman's Ricci-flow with surgery. Until the first singular time, any symmetry of the initial metric will be preserved by the flow due to its uniqueness. Thus, the group action will stay isometric under the flow.

The basic idea now is the following: Assume that no singularity occurs and the metric converges (up to rescaling) to a locally homogeneous limit metric—as in the case of positive Ricci-curvature [Ham82]. Then the limit metric is still invariant, and therefore the action is standard.

By the results of Perelman, one can also deal with singularities occurring during the flow. However, his Ricci-flow with (r, δ)–cutoff is a-priori not equivariant. There are three main issues which need to be resolved:

- First, one has to control the symmetries of the evolving metric when the flow approaches a singular time and to show that the surgery procedure can be done equivariantly.
- Second, at a surgery time there might be components on which scalar curvature gets uniformly large, even though the metric might not converge to a geometric one (the curvature operator gets only almost non-negative). Those components are diffeomorphic to spherical space forms, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ or $S^2 \times S^1$ and they are thrown away, so one has to ensure that the action is standard when restricted to them.
- Finally, having obtained that the action is standard on all limits, one needs to get back to the original manifold and action. Therefore, one has to relate the actions before and after a surgery.

In Chapter 1 we recall terminology and notations, and give some preliminary considerations. We recall a result of Grove and Karcher on close group actions (Chapter 1.4) and generalize a result of Munkres to equivariant diffeomorphism of the 2–sphere (Chapter 1.5). In Chapter 1.6 we define how to do a connected sum construction in a way that is compatible with the group action. We observe that similar to the fact that connected sum with S^3 is a trivial move, also the equivariant connected sum with a standard action on a union of 3–spheres is trivial (if the 3–spheres are attached along trees).

As our approach is based on Perelman's Ricci-flow with surgery, we summarize the main arguments and steps in the construction of a Ricci-flow with surgery in Chapter 2. We do not intend to give a self-contained proof thereof, but rather discuss those constructions that are necessary for the equivariant version. In particular, we focus on a precise description of the neck–cap decomposition of highly curved regions and on the surgery process.

In order to deal with the issues mentioned above, we construct an invariant singular S^2–foliation on the part of the manifold which is affected by the surgery. This singular foliation is a smooth foliation except for a finite number of points, and the smooth leaves are diffeomorphic to S^2 except for a finite number of $\mathbb{R}P^2$–leaves. The construction is done in Chapter 3, first on the neck-like region in Chapter 3.2 and then extended to the non-neck-like caps in Chapter 3.4. The surgery is applied at neck-like regions, and there our foliation is close to the standard cylindrical foliation by totally geodesic round 2–spheres. Thus, the foliation can be used to construct equivariant surgery necks and define an equivariant surgery, which is done in Chapter 4. Components which get extinct at the surgery time are completely covered by the singular foliation. Since the induced action on the one-dimensional leaf-space as well as on the leaves is easy to understand, it can be shown that the action on such a foliated component is standard, see Proposition 3.2.

Finally, the effect of equivariant surgery on the group action is studied in Chapter 4.2. To relate the group actions before and after a surgery, one follows the arguments of Perelman that describe the effect of surgery on the topology and keeps track of the action. The gluing-in of 3–balls and extending the action by spherical suspension on the balls corresponds to an equivariant connected sum construction as described in Chapter 1.6.

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1 Topological and geometrical preliminaries

This chapter summarizes general facts and terminology that are useful for the later discussion. In Sections 1.1–1.3 we recall definitions of geometric structures, standard actions and closeness of maps to isometries. We then explain a result of Grove and Karcher on how to explicitly conjugate actions which are close enough by a center of mass construction (Chapter 1.4).

We adapt a well-known fact concerning diffeomorphisms of the two-sphere to the equivariant setting in Chapter 1.5 using standard methods and finally give a definition of equivariant connected sum in Chapter 1.6. Here we also show that equivariant connected sum of a manifold with a standard action on a union of 3–spheres is a trivial operation if the spheres are attached along trees. This is an essential observation for understanding the effect of equivariant surgery on the group action.

Let us fix the following notations: We denote the Riemann curvature operator by R, sectional curvature by K, the Ricci-tensor by Ric and scalar curvature by S. For a group action we write $\rho\colon G \curvearrowright M$ which means that ρ is a homomorphism from G to $\mathrm{Diff}(M)$. Besides being smooth there are no implicit assumptions on the group actions, so they may be non-orientation-preserving and non-free.

1.1 Geometric structures

Following the approach to geometry of Felix Klein, William Thurston thought of geometric properties and structures of manifolds as being encoded in the group of transformations of the underlying space that are allowed for coordinate changes. For 3–manifolds, he focused on a special and—as it turned out—very suitable such structure, namely the one induced by isometries of homogeneous spaces, which leads to the characterization of eight 3–dimensional "geometries". Most of these concepts are described in detail in [Thu97] and [Thu80]. An additional source for a precise description of the 3–dimensional geometries is the survey article by Scott [Sco83].

Definition 1.1 ((X,Γ)–structure). Let X be a topological manifold and Γ a group acting on X. An (X,Γ)*–structure* for a manifold M is a maximal (X,Γ)–compatible collection of charts $\phi_i\colon U_i \to X$ covering M. Two charts ϕ_i, ϕ_j are (X,Γ)*–compatible*, if on each component V of $U_i \cap U_j$ the coordinate change $\phi_j \circ \phi_i^{-1}|_{\phi_i^{-1}(V)}$ is the restriction of some element $g \in \Gamma$.

Example 1.2. An $(\mathbb{R}^n, \mathcal{C}^k(\mathbb{R}^n))$–structure on M is a differentiable structure of class k.

Definition 1.3 (model geometry, geometric manifold). A *model geometry* is a smooth, simply connected manifold X with a Lie-group Γ acting transitively on X such that

1. Γ has compact point stabilizer,
2. Γ is maximal in the sense that it is not contained in any larger group of diffeomorphisms of X with compact point stabilizer,
3. there exists at least one compact manifold with an (X,Γ)–structure.

If (X,Γ) is a model geometry, then an (X,Γ)–structure is called *geometric structure* and a manifold with a geometric structure is called *geometric manifold.*

Remark 1.4. It follows from condition 1 and the fact that Γ acts transitively that X carries a homogeneous Γ–invariant Riemannian metric g.

Therefore, a geometric manifold carries a locally homogeneous metric. Vice versa, a locally homogeneous metric on a complete manifold M defines compatible local charts and hence is equivalent to a geometric structure.

Recall that a geometric structure on a manifold M defines a local isometry $\mathrm{dev}\colon \widetilde{M} \to X$, called the *developing map*, by gluing the local charts together along paths, using the fact that coordinate changes are in $\Gamma = \mathrm{Isom}(X)$. The compatibility of charts gives uniqueness of this map up to isometries of X (by choosing an arbitrary base point and a frame for the chart around the base point in M). The developing map induces an homomorphism $\mathrm{hol}\colon \pi_1(M) \to \mathrm{Isom}(X)$, called the *holonomy representation.*

If M is a complete geometric manifold, then dev is a covering map and therefore (X is simply connected) an isomorphism. Thus, complete geometric manifolds are always quotients X/G, where $G = \mathrm{hol}(\pi_1(M))$ is the group of deck transformations.

Similar to the two-dimensional situation where there are only three model geometries (the hyperbolic plane, flat space and the round 2–sphere with Γ the isometry group in each case), the 3–dimensional model geometries can be classified as follows, see [Thu97, Theorem 3.8.4]:

Theorem 1.5 (model geometries). *In dimension 3 there are eight model geometries, namely $X = S^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\widetilde{SL}(2,\mathbb{R})$ and Sol, with Γ the Lie-group of isometries on these spaces with their canonical metric.* □

Definition 1.6 (geometric orbifold). Let (X, Γ) be a model geometry. A metric space which is locally isometric to a quotient of X by a finite subgroup $\Gamma_x < \Gamma$ is called (X, Γ)*–orbifold* or *geometric orbifold*.

Remark 1.7. Note that this definition includes the possibility of $\mathcal{O}$ being a geometric manifolds, namely if all Γ_x can be chosen trivial. On the other hand, the underlying space does in general not even topologically need to be manifold, for instance the link of a point does not need to be homeomorphic to an $(n-1)$–sphere. However, in dimension 2 the only finite group actions on $X = S^2, \mathbb{R}^2$ or $\mathbb{H}^2$ are (conjugate to) rotations and reflections, such that any 2–orbifold (as topological space) is a 2–manifold (with boundary).

1.2 Standard actions

We consider smooth actions $\rho\colon G \curvearrowright M$ of a finite group G on a smooth manifold M. In case that M carries a geometric structure, we define what it means for the action to be compatible with this structure:

Definition 1.8 (standard action). Let (X, Γ) be a model geometry and M an (X, Γ)–manifold. We say, the action $\rho\colon G \curvearrowright M$ *is standard*, if there exists a $\rho(G)$–invariant complete locally homogeneous metric on M.

Remark 1.9. $\rho\colon G \curvearrowright M$ is standard if and only if the quotient space $M/\rho(G)$ admits a metric such that it becomes a complete geometric orbifold.

Proof. This is clear since a given complete geometric orbifold metric on $M/\rho(G)$ lifts to a $\rho(G)$–invariant complete locally homogeneous metric on M. On the other hand, a $\rho(G)$–invariant complete locally homogeneous metric on M descends to a complete metric on $M/\rho(G)$, which then is locally isometric to a quotient of X by a finite group. $\square$

Note that although the *type* of a closed geometric manifold is uniquely determined, the geometric *structure* itself needs not to be unique. So if M is a geometric manifold and the action $\rho\colon G \curvearrowright M$ is standard, then the geometric structure induced by the $\rho(G)$–invariant locally homogeneous metric on M might be different from an originally given geometric structure. In other words, the question whether a group action on a geometric manifold is standard does not depend on the choice of a particular geometric structure.

Example 1.10. This can be nicely illustrated by different flat structures on a torus. Consider a flat 2–torus with hectagonal fundamental domain and the dihedral group D_6 acting on it by isometries. If we regard the same group action, but equip the torus with a flat metric with rectangular fundamental domain, the group action can not be made isometric only by conjugation—one needs to change the geometric structure.

If however the geometric structure is unique as in the case of hyperbolic manifolds (by the Mostow rigidity theorem [Mos68]) or spherical manifolds (see Proposition 1.12 below), then any two locally homogeneous metrics on M are isometric. Therefore we get:

Proposition 1.11. *If (M, g) is geometric such that the geometric structure is unique, and $\rho\colon G \curvearrowright M$ is standard, then ρ is smoothly conjugate to an isometric action $\tilde{\rho}\colon G \curvearrowright (M, g)$.*

Proof. The conjugation diffeomorphism is given by the isometry between M with its $\rho(G)$–invariant locally homogeneous metric and M with the given locally homogeneous metric g. $\square$

Proposition 1.12. *Spherical structures on 3–manifolds are unique, i. e. if M_1 and M_2 are diffeomorphic compact spherical 3–manifolds, then they are isometric.*

Proof. A spherical structure on M_i induces via the holonomy representation an isomorphism between $\pi_1(M_i)$ and a finite subgroup $G_i <$

$SO(4)$ acting freely on S^3. These subgroups are classified by Seifert and Threlfall [TS31, TS33], using the homomorphism $SO(4) \to SO(3) \times SO(3)$ with kernel $\{\mathrm{id}, -\mathrm{id}\}$ (compare [Sco83, Theorem 4.10, Theorem 4.11], [Thu97, Theorem 4.4.14]).

From this isometry classification follows that if G_i is not cyclic, then any other freely acting subgroup of $SO(4)$ that is isomorphic to G_i is actually conjugate to G_i in $O(4)$. Hence, if $\pi_1(M_1) \cong \pi_1(M_2)$ is not cyclic, then G_1 and G_2 are conjugate and therefore M_1 and M_2 are isometric.

On the other hand, if the groups G_i are cyclic of order p, then M_i is isometric to a lens space $L(p, q_i)$, i. e. a quotient of $S^3 \subset \mathbb{C}^2$ by the group of rotations generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q_i/p} z_2)$. It now follows from the topological classification of lens spaces by Brody that $L(p, q_1)$ and $L(p, q_2)$ are diffeomorphic if and only if $q_1 \equiv \pm q_2^{\pm 1}$ modulo p ([Bro60, §4 Example I], see also [Hat00, Theorem 2.5]). If this is the case, it easy to see that $L(p, q_1)$ and $L(p, q_2)$ are isometric (a change of signs corresponds to complex conjugation in the z_2–plane respectively interchanging z_1 and z_2). □

Corollary 1.13. *Let $\rho\colon G \curvearrowright (M, g)$ be a smooth standard action of a finite group on an round spherical space-form. Then ρ is conjugate to an isometric action on (M, g).* □

On open manifolds geometric structures are much less rigid. For instance, an open ball carries any geometric structure (if we do not require completeness), and it can carry two different complete geometric structures ($\mathbb{R}^3$ and $\mathbb{H}^3$). However, this ambiguity plays no role for the definition of a finite group action on B^3 being standard, since B^3 carries an invariant complete hyperbolic metric if and only if it carries an invariant complete Euclidean one (note that the action must have a fixed point and is determined by its differential in that point).

The only open manifolds that shall be considered are B^3, $S^2 \times (0, 1)$ and $\mathbb{R}P^3 - \bar{B}^3$. On them, we can explicitly describe which actions are standard:

Remark 1.14 (standard action on B^3, $S^2 \times (0, 1)$, $\mathbb{R}P^3 - \bar{B}^3$). An action on the open ball B^3, on $S^2 \times (0, 1)$ or on $\mathbb{R}P^3 - \bar{B}^3$ is standard, if and only if it is smoothly conjugate to an isometric action on a Euclidean unit ball, on the round cylinder respectively on its orientable $\mathbb{Z}_2$–quotient. (The last case is conjugate to an isometric action on round $\mathbb{R}P^3$ minus a round ball of radius $< \pi/2$.)

1.3 Comparing Riemannian metrics

Definition 1.15. Let M be a $\mathcal{C}^k$–manifold and let $U \subseteq M$ be an open subset. We define the *$\mathcal{C}^k(U)$–norm* on $\mathrm{Sym}^2 T^*U$ with respect to a fixed reference metric g_0 as

$$\|\omega\|^2_{\mathcal{C}^k(U)} = \sup_{x \in U} \left(|\omega(x)|^2_{g_0} + \sum_{l=1}^{k} \left| \nabla^l_{g_0} \omega(x) \right|^2_{g_0} \right) ,$$

where ∇_{g_0} is the Levi-Cevita connection with respect to the metric g_0.

This defines a distance between Riemannian metrics g_1, g_2 on U by

$$d_{\mathcal{C}^k(U)}(g_1, g_2) = \|g_1 - g_2\|_{\mathcal{C}^k(U)} .$$

We call the induced topology on the space of Riemannian metrics on U *$\mathcal{C}^k(U)$–topology.* Note that this topology is independent of g_0. For convenience, we define the *ϵ–neighborhood* $\mathcal{U}(g, \epsilon, k, U)$ of a metric g on U as the set of metrics g' on U with $d_{\mathcal{C}^k(U)}(g, g') \leq \epsilon$ *where we take g a the reference metric.*

We observe that a $\mathcal{C}^2$–small variation of the metric changes the initial directions of unique minimizing geodesics only by a small angle. This will later help us to analyze (long) minimizing geodesics in almost cylindrical regions, see Chapter 3.2.

Proposition 1.16. *Let (M, g) be a (not necessarily complete) Riemannian manifold, $B(2R) \subseteq M$ a metric ball with compact closure and $\gamma \subset B = B(R)$ a unique minimizing geodesic with endpoints p and q in $B(R)$. Given ν and $k \geq 2$, there is an ϵ–neighborhood $\mathcal{U}(g, \epsilon, k, B)$ of $g|_B$ such that for any $h \in \mathcal{U}(g, \epsilon, k, B)$ a minimizing geodesic γ_h (with respect to h) from p to q satisfies*

$$\angle_g(\dot\gamma(0), \dot\gamma_h(0)) < \nu .$$

Proof. First note that since q is not in the cut locus of p, there is a neighborhood V of $v = \exp^{-1}(q) \in T_p(M)$ such that $\gamma_w \colon t \mapsto \exp(tw)$, $t \in [0, 1]$ is a minimizing geodesic for all $w \in V$. Choose V such that $\angle(w_1, w_2) \leq \nu$ for any $w_1, w_2 \in V$ and that $\exp(V)$ has positive distance from the cut locus. Now since geodesics are solutions of the second order differential equation $\nabla_{\dot\gamma}\dot\gamma = 0$, a variation of the metric in $\mathcal{C}^k(B)$–topology for $k \geq 2$ results in a continuous variation of $\exp(V)$. Thus, for ϵ small enough and $h \in \mathcal{U}(g, \epsilon, k, B)$ we have $q \in \exp_h(V)$, say $q =$

$\exp_h(w_h)$, and $\gamma_h\colon\ t \mapsto \exp(tw_h)$, $t \in [0,1]$ is an h–geodesic from p to q with $\angle_g(\dot\gamma(0), \dot\gamma_h(0)) = \angle_g(v, w_h) < \nu$.

Since lengths of curves depend continuously on h, for ϵ sufficiently small V still has positive distance to the tangential h–cut locus. Hence γ_h is the unique minimizing h–geodesic from p to q. □

Definition 1.17 (ϵ–isometry, ϵ–homothety). Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds, and let $\phi\colon\ M_1 \hookrightarrow M_2$ be a diffeomorphism on its image. We call ϕ an *ϵ–isometry*, if $\phi^* g_2$ is ϵ–close to g_1 in $\mathcal{C}^{[\frac{1}{\epsilon}]}$–topology, i. e. if $\phi^* g_2 \in \mathcal{U}(g_1, \epsilon, [\frac{1}{\epsilon}], M_1)$ (compare Definition 1.15).

We say ϕ is an *ϵ–homothety*, if it is an ϵ–isometry from (M_1, g_1) to $(M_2, \lambda^{-2} g_2)$ for some $\lambda > 0$, which we call the *scale* of ϕ.

We say that a smooth action $\rho\colon\ G \curvearrowright (M,g)$ on a Riemannian manifold is *ϵ–isometric*, if $\rho(\gamma)\colon\ (M,g) \to (M,g)$ is an ϵ–isometry for all $\gamma \in G$.

Note that all these definitions are not scale-invariant (ϵ depends on the scale). As long as we are comparing compact manifolds, this can be resolved by normalizing e. g. the maximal scalar curvature. However, when approximating pointed (and possibly non-compact) manifolds *up to scale*, we need a notion that relates distance and closeness to the curvature scale.

Definition 1.18 (relative distance). Let $x \in (M,g)$ be a point with $S(x) > 0$. We define the *distance from x relative to its curvature scale* as $\tilde d(x,\cdot) = S(x)^{\frac{1}{2}} d(x,\cdot)$. Accordingly, we define the *relative r–ball*

$$\tilde B(x,r) := B(x, S(x)^{-\frac{1}{2}} r) = \{\tilde d(x,\cdot) < r\}$$

the *relative r–sphere*

$$\tilde S(x,r) := S(x, S(x)^{-\frac{1}{2}} r) = \{\tilde d(x,\cdot) = r\}$$

and the *relative radius* $\widetilde{\mathrm{rad}}(x,X) := \sup_{y \in X}\{\tilde d(x,y)\}$.

To formulate a pointed version of closeness up to scale, we compare manifolds on relative $\frac{1}{\epsilon}$–balls:

Definition 1.19 (ϵ–approximation). We say that the pointed Riemannian manifold (M_1, x_1, g_1) *ϵ–approximates* (M_2, x_2, g_2) if there is a diffeomorphism $\phi\colon\ \tilde B(x_1, \frac{1}{\epsilon}) \hookrightarrow M_2$ with $\phi(x_1) = x_2$, such that after normalizing curvature at x by replacing g_1 with $S(x_1) g_1$, ϕ is an

ϵ–homothety. Equivalently we also say that (M_1, x_1, g_1) is *ϵ–close* to (M_2, x_2, g_2).

Definition 1.20 (pointed smooth convergence). We say that a sequence (N_i, x_i, g_i) of pointed Riemannian manifolds *converges smoothly* to a Riemannian manifold $(N_\infty, x_\infty, g_\infty)$ if there exists a sequence of ϵ_i–approximations

$$\phi_i\colon\ \tilde{B}(x_\infty, \tfrac{1}{\epsilon_i}) \to (N_i, x_i)$$

with $\epsilon_i \to 0$.

1.4 Conjugation of close actions

In order to show that certain actions are standard, it is useful to observe that being standard is an open condition in $\mathcal{C}^1$–topology. So if one manages to conjugate an action ρ sufficiently close to a standard action, then ρ itself is already standard.

The fact that $\mathcal{C}^1$–close actions are conjugate was first proved by Palais [Pal61]. Later, Grove and Karcher gave a differential geometric proof [GK73] using a center of mass construction, which we shall recall here.

Definition 1.21 (close actions). Given a finite group G and a Riemannian manifold (M, g), we say that two smooth actions $\rho_1, \rho_2\colon G \curvearrowright M$ are *ϵ–close in $\mathcal{C}^k$–topology*, if for each $\gamma \in G$ the diffeomorphisms $\rho_1(\gamma), \rho_2(\gamma)$ are ϵ–close in $\mathcal{C}^k$–topology. If ρ_1 acts by isometries, this is equivalent to $\rho_1(\gamma^{-1}) \circ \rho_2(\gamma)$ being ϵ–close to the identity.

If two actions $\rho_1, \rho_2\colon G \curvearrowright (M, g)$ are $\mathcal{C}^0$–close, then for any point $p \in M$ the image of the map

$$f_p\colon G \to M, \qquad f_p(\gamma) = \rho_1(\gamma)^{-1} \circ \rho_2(\gamma)(p)$$

is contained in an ϵ–ball around p. For such *almost constant* maps $f\colon G \to M$ (i. e. satisfying $f(G) \subset B(\epsilon)$ and ϵ sufficiently small) Grove and Karcher [GK73] define a *center of mass* $\mathscr{C}(f)$ with the property that for any isometry $A\colon M \to M$ holds $\mathscr{C}(A \circ f) = A \circ \mathscr{C}(f)$ and for R_γ the right multiplication with γ holds $\mathscr{C}(f \circ R_\gamma) = \mathscr{C}(f)$.

Moreover, if the actions are sufficiently $\mathcal{C}^1$–close, then the *center of mass map*

$$c\colon M \to M, \qquad p \mapsto \mathscr{C}(f_p)$$

is a diffeomorphism within prescribed $\mathcal{C}^1$–distance to the identity [GK73, Proposition 3.7 and (3.16)]. (Note that $\mathcal{C}^1$–closeness to the identity is not explicitly stated in [GK73], but it follows from their calculation (3.16) by first making k_1, k_2 sufficiently small by bounding the $\mathcal{C}^0$–distance, and then ensuring (3.15) by bounding the $\mathcal{C}^1$–distance).

It is direct from the definition that the map c is (ρ_1, ρ_2)–equivariant:

$$\begin{aligned} c \circ \rho_2(\gamma)(p) &= \mathscr{C}(f_{\rho_2(\gamma)p}) \stackrel{(*)}{=} \mathscr{C}(\rho_1(\gamma) \circ f_p \circ R_\gamma) \\ &= \rho_1(\gamma) \circ \mathscr{C}(f_p) = \rho_1(\gamma) \circ c(p) \end{aligned}$$

where the second equality $(*)$ uses that $f_{\rho_2(\gamma)p}$ is the map

$$\begin{aligned} \beta \mapsto \rho_1(\beta)^{-1}\rho_2(\beta)\rho_2(\gamma)p &= \rho_1(\beta)^{-1}\rho_2(\beta\gamma)p \\ &= \rho_1(\gamma)\rho_1(\beta\gamma)^{-1}\rho_2(\beta\gamma)p = \rho_1(\gamma)f_p(R_\gamma\beta). \end{aligned}$$

Therefore, the two actions ρ_1, ρ_2 are smoothly conjugate, and we summarize:

Theorem 1.22 (conjugating close actions). *Let $\rho_1, \rho_2 \colon G \curvearrowright M$ be two smooth actions on a connected compact Riemannian manifold (M, g). If ρ_1 acts by isometries and ρ_2 is sufficiently $\mathcal{C}^1$–close to ρ_1, then the two actions are smoothly conjugate by a diffeomorphism $c \colon M \to M$ within prescribed $\mathcal{C}^1$–distance to the identity.*

The result of [GK73] is actually more general: They study actions not only of finite groups but of compact Lie groups, and furthermore they give explicit bounds on the required $\mathcal{C}^1$–closeness of the actions, depending only on curvature bounds of the metric g. In this sense they generalize the result of Palais, which does not give an explicit conjugation map. While we do not need the bounds of [GK73] on closeness, the fact that the conjugation map is $\mathcal{C}^1$–close to the identity is important for our application, see Chapter 3.4.

1.5 Equivariant diffeomorphisms of the 2–sphere

The diffeomorphisms group of S^2 allows a lot of freedom in deforming a given diffeomorphism. For instance, any diffeomorphism of S^2 is isotopic to an isometry, as was shown by Munkres [Mun60a] (compare also [Thu97, Theorem 3.10.11]). In fact, Smale has strengthened this result by proving that $O(3)$ is a strong deformation retract of $\mathrm{Diff}(S^2)$, i. e. all isotopies can be done simultaneously [Sma59]. Note that the analogue result holds also in dimension 3. This was conjectured by Smale and proved by Hatcher [Hat83].

For our setting we need the following equivariant version for diffeomorphisms of the 2–sphere: If the given diffeomorphism is equivariant with respect to an orthogonal action, then the isotopy can also be made through equivariant diffeomorphisms.

Proposition 1.23. *Let $\rho\colon H \curvearrowright S^2(1)$ be an orthogonal action by a finite group on the 2–dimensional unit sphere. Then every ρ–equivariant diffeomorphism $S^2 \to S^2$ is ρ–equivariantly isotopic to an isometric one.*

In fact, we will show an equivalent formulation in terms of quotient orbifolds:

Proposition 1.24. *Any diffeomorphism $\mathcal{O} \to \mathcal{O}$ of a spherical two-orbifold is isotopic to an isometry.*

For the case of $\mathcal{O} = S^2$ (with no singularities) the proposition is shown by Munkres [Mun60a]. For general surfaces isotopy classes of diffeomorphisms were investigated by Epstein [Eps66], working in the category of piecewise linear manifolds (which in dimension 2 is equivalent to the smooth category). In particular, Epstein showed that the proposition holds for $\mathcal{O} = \mathbb{R}P^2$ [Eps66, Theorem 5.5].

Precisely the same methods work also in the orbifold setting, one only needs to check that the constructed isotopies can be made compatible with the orbifold structure, i. e. they lift locally to equivariant isotopies of the orbifold charts.

Proof (of Proposition 1.24). Since $\mathcal{O}$ is two-dimensional, an isolated singularity must be a *cone point* (i. e. the group action on the orbifold chart is generated by a rotation of angle $\frac{2\pi}{p}$), and non-isolated singularities are *reflector boundaries* (the action on the orbifold chart is by a reflection) or *corner reflectors* (the action is by a dihedral group of reflections and rotations).

Also note that spherical 2–orbifolds are always good, i. e. quotients of S^2. Since the proposition is true for non-singular orbifolds (S^2 and $\mathbb{R}P^2$), we restrict to orbifolds which have singularities. The spherical 2–orbifolds with singularities can be classified by the following list:

- S^2 with two or three cone points,

- $\mathbb{R}P^2$ with one cone points, or
- D^2 with reflector boundary, with at most 3 corner reflector points and possibly one interior cone point. (It can only occur in case of at most 2 corner reflectors and must exist if there is precisely 1 corner reflector.)

Denote the isolated singularities and the corner reflector points by x_i. A diffeomorphism of $\mathcal{O}$ is a homeomorphism that lifts locally to a diffeomorphism of the orbifold charts. Therefore, it permutes the x_i of the same type and maps the reflector boundary onto itself. There clearly exists an isometry ψ of $\mathcal{O}$ that does the same permutation of the x_i and therefore $\phi' = \psi^{-1} \circ \phi$ fixes all x_i. If $\mathcal{O}$ is orientable, we may further choose ψ to be orientation preserving if and only if ϕ is. Hence ϕ' is orientation preserving in this case. If $\mathcal{O}$ is not orientable then $\mathcal{O} = \mathbb{R}P^2(x_1)$ and we can choose ψ such that ϕ' is locally orientation preserving near the cone point x_1. As a consequence, ϕ' preserves isolated cone points and orientations near the cone points, and ϕ' preserves reflector boundary segments and is an orientation preserving reparameterization on them.

In order to prove the claim, it suffices to show that ϕ' is isotopic to the identity, since this implies that ϕ is isotopic to ψ. As a first step, we want to locally isotope ϕ' to the identity near the singularities.

Lemma 1.25. *Let $D \subset \mathbb{R}^2$ be an open round disk around the origin and let $\rho\colon G \curvearrowright D$ be an orthogonal action of a finite group G (so $\rho(G)$ is a finite cyclic group of rotations, a two-elementary reflection group or a dihedral group of rotations and reflections). Let $\phi\colon D \to D$ be a $\rho(H)$–equivariant orientation preserving diffeomorphism with $\phi(0) = 0$. Then ϕ is $\rho(H)$–equivariantly isotopic to a $\rho(H)$–equivariant diffeomorphism which is equal to $\pm\,\mathrm{id}$ near 0 if $\rho(H)$ contains reflections, and equal to id if $\rho(H)$ contains only rotations. The isotopy is compactly supported, i. e. it fixes ϕ on a neighborhood of ∂D.*

Proof. In the non-equivariant setting this was proved by [Mun60b, Lemma 8.1]. Provided the given diffeomorphism is equivariant, then the construction in the proof can also be made equivariant. We will follow the proof presented in [Thu97, 3.10.12], which uses the same ideas but is clearer in its exposition:

We first isotope ϕ such that it becomes linear near 0. Let $A = d\phi_0$ be the differential at 0 (which automatically is $\rho(H)$–equivariant). We then use a rotational symmetric smooth test function $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ ($\theta \equiv 1$ on $B(\frac{1}{2})$ and $\theta \equiv 0$ outside the unit-disk $B(1)$), and interpolate for $0 \leq t \leq 1$

$$\phi_t(x) := \phi(x) + t\theta_\lambda(x) \cdot (A - \phi)(x)\,, \quad \theta_\lambda(x) := \theta\big(\tfrac{x}{\lambda}\big)$$

Then $\|d\phi_t - d\phi\| = t\,\|d(\theta_\lambda(A - \phi))\| \leq C\lambda t$, where C only depends on θ, so for $\lambda > 0$ sufficiently small this gives an isotopy, which clearly is $\rho(G)$–equivariant. Note that the isotopy fixes ϕ outside of $B(\lambda)$, and $\phi_1 \equiv A$ on $B(\frac{\lambda}{2})$.

It remains to find a $\rho(G)$–equivariant isotopy of A to $\pm\,\mathrm{id}$ with support on $B(\frac{\lambda}{2})$. The property of being a $\rho(H)$–equivariant linear map implies for A the following: If $\rho(H)$ contains a reflection γ, then A commutes with γ and therefore must preserve its ± 1–eigenspaces. Hence A has two orthogonal eigenvectors with eigenvalues $\mu_1, \mu_2 \in \mathbb{R} - \{0\}$.

If $\rho(G)$ contains a reflection and a rotation of order ≥ 3, then it follows from the above that $\mu_1 = \mu_2$ and therefore $A = \mu_1\,\mathrm{id}$ (if the rotation is by $\frac{\pi}{2}$, then $\mu_1 = \mu_2$ is clear; otherwise $\rho(G)$ contains two reflections with non-orthogonal axis). It is obvious that ϕ can be $\rho(G)$–equivariantly isotoped near 0 to $\pm\,\mathrm{id}$.

If $\rho(G)$ contains only the reflection γ as non-trivial element, then ϕ can equivariantly be isotoped to $\pm\,\mathrm{id}$ by stretching in the directions of the eigenvectors. The same is true if $\rho(G)$ contains in addition $-\,\mathrm{id}$ (so it is generated by two orthogonal reflections).

Finally, consider the case that $\rho(G)$ is generated by a rotation β: If $\beta = -\,\mathrm{id}$ then A can be any orientation preserving linear map. Here one can first make ϕ orthonormal and then rotate on a small ball around 0. For both steps the $\pm\,\mathrm{id}$–equivariance is no obstruction.

If $\beta \neq -\,\mathrm{id}$, then A must be a homothety, i. e. the product of a dilation and a rotation. Thus, ϕ can first be equivariantly isotoped such that it is just a rotation on a ball around 0, and then this rotation can be isotoped to id on a smaller ball. □

We apply Lemma 1.25 to neighborhoods of all isolated cone points and of all corner reflector points. So we can assume that ϕ' is the identity near those points. Since ϕ' is a reparameterization of the boundary edges and the identity near the ends of each edge, we can furthermore apply an isotopy such that ϕ' fixes a neighborhood of the boundary pointwise. Thus, we may assume that ϕ' is the identity near all singular points.

In the case of $\mathbb{R}P^2(x_1)$ regard a non-contractible curve γ through the cone point x_1. Then $\phi'(\gamma)$ can be isotoped back to γ (fixing a neighborhood of the cone point), see [Eps66, Theorem 3.3]. So we may assume that ϕ' is the identity in a neighborhood of γ and cut along γ to obtain a 2–disk without singularities. On this disk, ϕ' fixes a neighborhood of the boundary and so ϕ' can be isotoped to the identity using the classical result [Mun60a, Theorem 1.3], [Sma59], see also [Thu97, end of proof of Theorem 3.10.11]. This result also applies to the case $\mathcal{O} = D^2$ with reflector boundary and no cone-points.

Proposition 1.24 now follows from the next Lemma. □

Lemma 1.26. *(i) Let D be a disk with 1 cone point, and let $\phi\colon D^2 \to D^2$ be an orientation preserving diffeomorphism which is the identity in a neighborhood of the boundary and of the cone point. Then ϕ can be isotoped to the identity map on D^2, where the isotopy fixes a neighborhood of the boundary. (It may however rotate around the cone-point.)*

(ii) Let $\mathcal{O}$ be S^2 with 2 or 3 cone-points, and let $\phi\colon \mathcal{O} \to \mathcal{O}$ be an orientation preserving diffeomorphism which is the identity in a neighborhood of the cone points. Then ϕ is isotopic to the identity map on $\mathcal{O}$. (The isotopy may rotate around the cone-points.)

Proof. (i) Denote the cone-point by x_1 and choose an arc γ from the cone point to the boundary ∂D. We may assume that $\phi(\gamma)$ is transversal to γ except near the boundary and near x_1, where the two arcs coincide. If there are transversal interior intersection points of γ and $\phi(\gamma)$ let y_1 be the first one along $\phi(\gamma)$. Let α be the sub-arc of $\phi(\gamma)$ from x_1 to y_1, and let β be the sub-arc of γ from x_1 to y_1. Because of the choice of y_1, α and β are disjoint (except for x_1 and y_1) and thus bound a disk D (there may however be intersections of $\phi(\gamma) - \alpha$ with β). By pushing the disk D through γ we can remove the intersection point y_1 (and maybe other intersection points on β) and create no new intersection points. After repeating this finitely many times, we reach that γ and $\phi(\gamma)$ have no interior intersection point and hence bound a disk, along which we can isotope $\phi(\gamma)$ to γ. We may further isotope ϕ such that it fixes a neighborhood of γ pointwise, which reduces the assertion to the case of a disk without singularities.

(ii) We proceed similar as in case (i): Denote the cone points by x_1, x_2 and (possibly) x_3 and choose a smooth arc γ connecting x_1 and x_2, such that γ avoids x_3 if it exists. We may assume that $\phi(\gamma)$ is transversal to γ. If there are interior transversal intersection points, let y_1 and y_2 be the first and the last such point along $\phi(\gamma)$ ($y_1 = y_2$ is possible). Denote by α_i the sub-arc of $\phi(\gamma)$ from x_i to y_i, and by β_i the corresponding sub-arc of γ. As before, α_1 and β_1 are disjoint, and so are α_2 and β_2. So for $i = 1$

and 2, the union of α_i and β_i cuts $\mathcal{O}$ into two disks. Denote by D_i the (unique) one of them, which does not contain a piece of γ in its interior. We claim that the interior of the disks D_1 and D_2 is disjoint: Note that the arcs α_1 and α_2 cannot intersect. So if there were a common interior point, β_1 must transversally intersect D_2 or vice versa, contradicting the assumption that the interior D_i is disjoint from γ.

We conclude that either D_1 or D_2 does not contain x_3, so we can push that disk through γ by an isotopy of ϕ and reduce the number of intersection points (obviously we can do the same if there is no third cone point x_3). After finitely many steps we achieve γ and $\phi(\gamma)$ have no interior intersection point. Thus we can isotope ϕ such that $\gamma = \phi(\gamma)$ and ϕ fixes a neighborhood of γ pointwise (ϕ preserves the orientation of γ). Cutting along γ reduces now to case (i) if x_3 exists, or the case without singularities. □

Proposition 1.27. *Let $\rho_1, \rho_2\colon G \curvearrowright \bar{B}^3$ be two isometric actions of a finite group G on the closed unit ball. Then any (ρ_1, ρ_2)–equivariant diffeomorphism $\alpha\colon \partial\bar{B}^3 \to \partial\bar{B}^3$ can be extended to a (ρ_1, ρ_2)–equivariant diffeomorphism $\hat{\alpha}\colon \bar{B}^3 \to \bar{B}^3$.*

Proof. We regard $\bar{B}^3 - B^3(\frac{1}{2})$ as product $S^2 \times [\frac{1}{2}, 1]$ and use Proposition 1.23 in order to equivariantly isotope α along the interval factor. So α extends to a (ρ_1, ρ_2)–equivariant diffeomorphism $\hat{\alpha}'\colon \bar{B}^3 - B^3(\frac{1}{2}) \to \bar{B}^3 - B^3(\frac{1}{2})$ which is isometric on the inner boundary sphere. Therefore $\hat{\alpha}'|_{\partial B^3(\frac{1}{2})}$ is the restriction of an orthogonal and (ρ_1, ρ_2)–equivariant map, which yields the extension to the rest of $\bar{B}^3$. □

1.6 Equivariant connected sum

Given a smooth action $\rho\colon G \curvearrowright M$ on a compact manifold and a $\rho(G)$–invariant smoothly embedded 2–sphere $S \subset M$, one can cut M along S and obtain a smooth action on a compact manifold with boundary $\check{\rho}\colon G \curvearrowright \check{M}$, the boundary of $\check{M}$ consisting of two S^2–components S_1 and S_2. The restricted actions $\check{\rho}|_{S_i}$ are conjugate and also conjugate to (the restriction to the boundary of) an orthogonal action on the round unit ball $\check{\rho}_B\colon G \curvearrowright B^3$. We therefore find $(\check{\rho}_B, \check{\rho})$–equivariant diffeomorphism $\phi_i\colon \partial B^3 \to S_i$, by which we glue in copies of B^3 to each boundary, and obtain a smooth action ρ' on the resulting compact manifold M'. The smooth conjugacy class of ρ' does not depend on the special choice of ϕ_i

by Proposition 1.27. Also note that the infinitesimal actions $d\rho'_{x_i}\colon G \curvearrowright T_{x_i}M'$ are conjugate, where x_i denotes the points in M' corresponding to the center of B^3.

This construction can easily be generalized to a finite $\rho(G)$–invariant family $\{S_j\}$ of disjoint 2–spheres in M, by replacing G with the stabilizer of each S_j and choosing ϕ_{ji} equivariantly for each orbit of S_j. We call the resulting action $\rho'\colon G \curvearrowright M'$ an *equivariant connected sum decomposition* of $\rho\colon G \curvearrowright M$. We now define a construction which reverses the equivariant connected sum decomposition:

Definition 1.28 (equivariant connected sum). Let $\rho\colon G \curvearrowright M$ be a smooth finite group action on a compact manifold. Let $\mathcal{P}$ be a $\rho(G)$–invariant finite family of pairs of points $(x_i, y_i) \in M \times M$ such that all x_i, y_i are pairwise disjoint. Denote by G_i the stabilizer of x_i (or of y_i, which is the same). Suppose that $d\rho_{x_i}\colon G_i \curvearrowright T_{x_i}M$ and $d\rho_{y_i}\colon G_i \curvearrowright T_{y_i}M$ are conjugate by a linear maps $\alpha_{x_i}\colon T_{x_i} \to T_{y_i}$ respectively $\alpha_{y_i} = \alpha_{x_i}^{-1}$ for all i, and that the family α of all these maps is $(d\rho,\rho)$-equivariant, i. e. for each pair (x, y) and each $g \in G$ holds $d\rho(g)_x \circ \alpha_y = \alpha_{gx} \circ d\rho(g)_x$.

Then we define the *equivariant connected sum along $\mathcal{P}$* as follows: For an arbitrary smooth $\rho(G)$–invariant Riemannian metric on M cut out small r–balls around all points x_i and y_i (r chosen so small that $2r$–balls are disjoint and $2r < \operatorname{inj} M$). For each pair (x_i, y_i) consider the map

$$\begin{aligned}\beta_{x_i}\colon\ B(x_i, 2r) - \bar{B}(x_i, \tfrac{1}{2}r) &\to B(y_i, 2r) - \bar{B}(y_i, \tfrac{1}{2}r)\\ z &\mapsto \exp_{y_i} \circ\, \iota_r \circ \alpha_{x_i} \circ \exp_{x_i}^{-1}(z)\,,\end{aligned}$$

and $\beta_{y_i} = \beta_{x_i}^{-1}$, where ι_r is the involution on the sphere of radius r. β_{x_i} is $\rho(G_i)$–equivariant, and the family β of all these maps is $\rho(G)$–equivariant. Now cut out the balls $\bar{B}(x_i, \frac{r}{2})$ and $\bar{B}(y_i, \frac{r}{2})$ and identify what remains from $B(x_i, 2r)$ and $B(y_i, 2r)$ using the map β_{x_i}. One obtains a smooth manifold $M_{\mathcal{P}}$ and a smooth action $\rho_{\mathcal{P}}\colon G \curvearrowright M_{\mathcal{P}}$. The smooth conjugacy type of $\rho_{\mathcal{P}}$ does only depend on $\mathcal{P}$ and on the family α. Since in all applications the particular choice of α is either canonical or irrelevant, we suppress the dependence on α in our notation.

Proposition 1.29. *Assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint $\rho(G)$–invariant families as in Definition 1.28. Then holds $(\rho_{\mathcal{P}_1})_{\mathcal{P}_2} = \rho_{(\mathcal{P}_1 \cup \mathcal{P}_2)}$.*

Proof. Supposed r is chosen small enough that none of the $2r$–balls around x_i, y_i intersect, the gluing procedure in Definition 1.28 can be carried out for each $\rho(G)$–orbit of pairs separately. Since the families $\mathcal{P}_i$ are disjoint and $\rho(G)$–invariant, this implies the claim. □

Definition 1.30. We define the following *graph* $\Gamma_{\mathcal{P}}$ *associated to the equivariant connected sum* $\mathcal{P}$: For each component of M take a vertex, and for each pair $(x, y) \in \mathcal{P}$ take an edge connecting the vertices corresponding to the components that contain x and y. Then the group action on M induces a group action on $\Gamma_{\mathcal{P}}$.

Proposition 1.31. *Assume that the associated graph* $\Gamma_{\mathcal{P}}$ *of an equivariant connected sum* $\mathcal{P}$ *is a tree. Suppose furthermore that at most one component of* M *is different from* S^3 *and the action* $\rho\colon G \curvearrowright M$ *is standard on the union of all components of* M *diffeomorphic to* S^3*. Then either*

(i) $M_{\mathcal{P}} \cong S^3$ *and* $\rho_{\mathcal{P}}$ *is standard or*

(ii) $M_{\mathcal{P}} \ncong S^3$ *and there exists a unique component* M_0 *of* M *diffeomorphic to* $M_{\mathcal{P}}$*.* M_0 *is* $\rho(G)$*–invariant and* $\rho_{\mathcal{P}}$ *is smoothly conjugate to* $\rho|_{M_0}$*.*

Proof. If there is a component not diffeomorphic to S^3, denote it by M_0; otherwise put $M_0 = \emptyset$. First consider case (ii) where $M_{\mathcal{P}} \ncong S^3$. Then $M_0 \neq \emptyset$ since the connected sum of only S^3–components along a tree is an S^3.

Let $M_1 \subset M$ be the union of all S^3–components of M that correspond to ends of the tree $\Gamma_{\mathcal{P}}$ (i. e. vertices with at most one edge). Let $\mathcal{P}_1 \subseteq \mathcal{P}$ be the corresponding family of pairs, i. e. the pairs $(x_i, y_i) \in \mathcal{P}$ with $x_i \in M_1$, and let $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$. It is clear that both families are $\rho(G)$–invariant.

If $\mathcal{P}_1 = \emptyset$, then there can be no S^3–components, so $M = M_0$. Because there are no loops it follows that $\mathcal{P} = \emptyset$ and the assertion is trivial.

If $\mathcal{P}_1 \neq \emptyset$, then we shall show that $\rho_{\mathcal{P}_1}$ is conjugate to $\rho|_{M_2}$, where $M_2 = M - M_1$. Let S^3_i be the components of M_1 and let $\mathcal{P}_1 = \{(x_i, y_i)\}$ with $x_i \in S^3_i$, $y_i \in M_2$. Then x_i is a fixed point for $\rho(G_i)$. A standard action on S^3 which has a fixed point must be the suspension of the action on the equator, so the restriction of $\rho(G_i)$ to $S^3_i - \bar{B}(x_i, \frac{r}{2})$ is conjugate to the restriction on $B(y_i, \frac{r}{2})$.

This gives $\rho_{\mathcal{P}_1} \cong \rho|_{M_2}$. Since $M_0 \subseteq M_2 \subset M$ and M_2 has fewer S^3–components than M after finitely many iterations of this process we reach at the situation $M = M_0$. This proves the Proposition in case (ii).

In case (i) we can apply the same reduction of the graph as long as it is not a point or an interval. If it is a point, then again the assertion is trivial. If it is an interval, then $\mathcal{P} = \{(x_1, y_1)\}$ with $x_1 \in S^3_1$ and $y_1 \in S^3_2$. $G_1 = G$ or $[G_1 : G] = 2$ if there are group elements that interchange S^3_1 and S^3_2 (and thus interchange x_1 and y_1). Precisely the same argument as before gives that the restriction of $\rho(G_1)$ to $S^3_1 - \bar{B}(x_1, \frac{r}{2})$ is conjugate to the restriction on $B(y_1, \frac{r}{2})$. Therefore $\rho_{\mathcal{P}}(G_1) \cong \rho(G_1)|_{S^3_1} \cong \rho(G_1)|_{S^3_2}$ is the suspension of an action on S^2 and hence standard. Elements in $G - G_1$ interchange the two 3–balls $S^3_1 - B(x_1, r)$ and $S^3_2 - B(y_1, r)$ (and the antipodal points of x_1 and y_1), so the complete action $\rho_{\mathcal{P}}(G)$ is standard. □

Remark 1.32. Consider the special case that $M_{\mathcal{P}}$ is irreducible, i. e. every embedded 2–sphere bounds a 3–ball. Then every embedded 2–sphere is separating and therefore $\Gamma_{\mathcal{P}}$ contains no loops. Since $M_{\mathcal{P}}$ is connected, also $\Gamma_{\mathcal{P}}$ is connected and hence is a tree.

Also note that if $M_{\mathcal{P}}$ is irreducible, at most one component of M is *not* diffeomorphic to S^3. Thus if in addition the action restricted to all S^3–components is standard, the hypothesis for Proposition 1.31 is fulfilled.

2 Perelman's Ricci-flow with surgery

The study of Ricci-flow on Riemannian manifolds was developed by Richard Hamilton, as an attempt to find distinguished metrics on the manifolds. Indeed, he could show that for 3–manifolds with an initial metric of positive Ricci curvature, the Ricci-flow converges up to rescaling to a metric of constant positive sectional curvature [Ham82]. For general Riemannian 3–manifolds he proved under the extra hypothesis that the solution is non-collapsing and non-singular that the Ricci-flow converges to a metric of constant sectional curvature, where the convergence is up to rescaling and diffeomorphisms (in case of negative sectional curvature the limit might also be non-compact and the (pointed) convergence depends on the choice of base-points) [Ham99].

This property of homogenizing Riemannian metrics makes the Ricci-flow interesting for the purpose of geometrization. However, for general initial metrics the flow can develop singularities. This major obstacle was removed by Perelman [Per02], who succeeded in excluding certain "bad" (locally collapsing) types of singularities and describing the possible singularities by local models, so-called κ–solutions. With this control on the singularities he managed to prove existence of a Ricci-flow with surgery for all times which has a similar long-time behavior as Hamilton's non-singular solutions [Per03a]. The analysis of the long-time behavior proves the geometrization conjecture of Thurston.

In the following chapter we will present some well-known facts concerning the Ricci-flow with surgery. Most of this is based on Perelman's papers [Per02, Per03a] and their detailed elaborations by Kleiner and Lott [KL07], Morgan and Tian [MT07] and Bamler [Bam07].

We will especially focus on those properties of Ricci-flow (and Ricci-flow with surgery) which are important for our applications, namely the geometry of the Ricci-flow close to a singular time and the surgery process. In particular we are interested in a precise description of the neck–cap decomposition of the region which is affected by the surgery (see Chapter 2.2 for this discussion in the local models and Chapter 2.3 for its application to the original manifold). The chapter ends with a description of the surgery process by (r, δ)–cutoff in Chapter 2.4.

Definition 2.1 (Ricci-flow). Given a Riemannian manifold (M, g_0), a *Ricci-flow on M with initial metric* g_0 is a solution $(M, g(t))_{t\in[a,b)}$ of the partial differential equation

$$\tfrac{\partial}{\partial t} g(t) = -2Ric(g(t))\,, \qquad g(a) = g_0\,.$$

The Ricci-operator is an almost elliptic operator, where the non-ellipticity only stems from the invariance of Ric under diffeomorphisms of M. This was observed by DeTurk and used to give a short proof of unique short-time existence of the Ricci-flow [DeT83].

If $(M, g(t))_{t\in[a,b)}$ is a Ricci-flow, then also $(M, \lambda^2 g(\frac{1}{\lambda^2}t))_{t\in[\lambda^2 a, \lambda^2 b)}$ is a Ricci-flow (note that Ric is scale-invariant!):

$$\tfrac{\partial}{\partial t}\lambda^2 g(\tfrac{1}{\lambda^2}t) = -2Ric(g(\tfrac{1}{\lambda^2}t)) = -2Ric(\lambda^2 g(\tfrac{1}{\lambda^2}t))$$

Therefore, a rescaling by λ in space and λ^2 in time is called *parabolic rescaling* with factor λ.

Definition 2.2 (parabolic ball). A *parabolic ball* $B(x_0, t_0, r, \tau)$ in a Ricci-flow $(M, g(\cdot))$ is a space-time product $B(x_0, t_0, r) \times [t_0, t_0 + \tau]$ (respectively $[t_0 + \tau, t_0]$ if $\tau < 0$), where $B(x_0, t_0, r)$ is the r–ball around x_0 in the t_0–time-slice $(M, g(t_0))$.

One often considers parabolic balls of the form $B(x_0, t_0, \lambda, -\lambda^2)$, since they correspond under parabolic rescaling (and possibly time-shifting) to (backward) parabolic "unit-balls" $B(x_0, t_0, 1, -1)$.

2.1 κ– and standard solutions

We start with describing the local singularity-models (κ–solutions) and models for the post-surgery behavior in a surgery region (standard solutions). The main issues that shall be recalled are the classification of three-dimensional orientable κ–solutions and the compactness of the space of κ– and standard solutions.

An essential property of the Ricci-flow is that under certain conditions it is non-collapsing in the sense that when normalizing curvature to 1 at a point, the volume of balls (and thus injectivity radius) is bounded, as was proved by Perelman using the analysis of the $\mathcal{L}$–length and reduced volume comparison [Per02, Theorem 8.2]. This makes it possible to extract limit flows using the compactness result for Ricci-flows of Hamilton [Ham95a].

κ–non-collapsedness

We recall the definition of κ–non-collapsing compare [KL07, Definition 26.1], [MT07, Definition 9.1]:

Definition 2.3 (κ–non-collapsed). Let $(M^n, g(t))$, $t \in [a, b)$ be a Ricci-flow such that $(M^n, g(t))$ is a complete n–dimensional manifold for all t. We say the flow is *κ–non-collapsed on scale $< \rho$*, if for any $(x_0, t_0) \in M^n \times [a, b)$ and any $r < \rho$ with $a \le t_0 - r^2$ holds: Either $S(x, t) > r^{-2}$ for some (x, t) in the parabolic ball $B(x_0, t_0, r, -r^2)$ or $\mathrm{vol}(B(x_0, r)) \ge \kappa r^n$.

Equivalently one can formulate this as follows: The Ricci-flow is κ–non-collapsed if after parabolic rescaling of the flow with factor r^{-1}, the volume of $B(x_0, t_0, 1)$ is greater then κ whenever the flow is defined on the parabolic ball $B(x_0, t_0, 1, -1)$ and satisfies $S(x, t) \le 1$ there.

Note that a Ricci-flow that is κ–non-collapsed on scale ρ also is κ'–non-collapsed on scale ρ' for any $0 < \kappa' \le \kappa$ and $0 < \rho' \le \rho$.

Example 2.4. An important example of a κ–non-collapsed solution is the product of a shrinking round sphere with Euclidean space, $M = S^k \times \mathbb{R}^{n-k}$, $1 \le k \le n$ defined on some time interval $[a, b)$. This is κ–non-collapsed on all scales r for a suitable $\kappa = \kappa(n, k)$: If in the t_0–time slice the sphere has radius $< \sqrt{k(k-1)}r$, then $S(x_0, t_0) > r^{-2}$ and the statement is trivial. Otherwise, the lower bound on the radius guaranties a certain volume.

Standard solutions

Standard solutions are designed as local models for the flow after removing a singularity by surgery.

Definition 2.5. A *standard initial metric* is a Riemannian metric g_{stand} on $\mathbb{R}^3$ with the following properties:

- g_{stand} is complete and has non-negative sectional curvature
- g_{stand} is rotational symmetric around the origin, called the *tip* of the standard metric,
- g_{stand} is isometric to $S^2(\sqrt{2}) \times \mathbb{R}^+$ outside a compact ball around the origin,

- g_{stand} is isometric to a subset of the round sphere $S^3(\sqrt{2})$ near the origin.

Definition 2.6 (standard solution). A *standard solution* is a Ricci-flow on $\mathbb{R}^3$ defined on a time interval $[0, T)$ with the properties that $g(0)$ is a standard initial metric, $|Rm|$ is bounded on each time-slice and T is maximal with this property, i. e. the solution cannot be extended such that curvature stays bounded on each time-slice.

It is a result of Shi [Shi89], that also for non-compact, complete initial metrics with bounded curvature the solution for the Ricci-flow exists for some short time period and one has curvature control for a short time. Therefore, for any standard initial metric, there exists a standard solution.

The following facts about standard solutions are claimed in [Per03a, Section 2] and proved in detail in [KL07, Sections 60–66], [MT07, Chapter 12], [Bam07, Section 7.3].

Proposition 2.7. *Every standard solution is defined on* $[0, 1)$ *and for* $t \to 1$ *scalar curvature gets uniformly large. That is, there is a constant* c *such that* $S(x,t) > \frac{c}{1-t}$ *on any standard solution. Moreover, any standard solution is* κ*–non-collapsed on scales below 1.* □

It can be showed that for a given standard initial metric the solution of the Ricci-flow equation is unique. More generally, Chen and Zhu show that there is unique short time existence for complete non-compact n–dimensional manifolds if the initial metric has bounded curvature operator [CZ05]. However, this result is not required since instead it suffices to have compactness of the space of standard solutions with a fixed standard initial metric. This can be obtained without uniqueness, see [KL07, Lemma 64.1]:

Proposition 2.8 (compactness of pointed standard solutions). *The space of pointed standard solutions* $(\mathbb{R}^3, 0, g(\cdot))$ *with base point at their tip and with a fixed standard initial metric is compact with respect to pointed smooth convergence of flows.* □

Remark 2.9. In the following we will always fix a standard initial metric g_{stand} and assume that every standard solution has this metric as initial metric. The specific choice of g_{stand} does not play a role. However, the constants derived to control the geometry of standard solutions may depend on g_{stand} since the space of standard initial metrics is *not* compact.

κ–solutions: Definition and examples

The fact that a Ricci-flow is κ–non-collapsed (on scales depending on the initial metric and on time, see [Per02, Theorem 8.1]) makes it possible "zoom into" a singularity and to extract a limit flow. The greater the curvature gets, the longer is the life-time of the rescaled solution and thus such a limit flow is defined on $(-\infty, 0]$ and κ–non-collapsed on all scales. Moreover, if the Φ–pinching holds (see Definition 2.37), the limit has non-negative curvature. This motivates the definition of κ–solutions as local singularity-models, compare [KL07, Definition 38.1], [MT07, Definition 9.2].

Definition 2.10 (κ–solution). A Ricci-flow $(M, g(t))$, defined for $t \in (-\infty, 0]$, is called an *ancient solution* if for any $t \in (-\infty, 0]$, the time-slice $(M, g(t))$ is a complete, non-flat manifold with non-negative Riemann curvature R with $|R(x,t)| < C(t)$ for some number $C(t) > 0$ which may depend on t. A *κ–solution* is an ancient solution which in addition is κ–non-collapsed on all scales (see Definition 2.3).

Remark 2.11. It follows that on a κ–solution scalar curvature is positive everywhere: If it were zero at (x,t), then also $R(x,t) = 0$. Due the strong maximum principle [Ham86, Lemma 8.2] the null-space of curvature tensor field R is invariant under parallel translation and in time, so $R \equiv 0$ holds everywhere, but this is excluded.

Example 2.12. $M = S^k \times \mathbb{R}^{n-k}$ as in Example 2.4, defined on $(-\infty, 0]$, is a κ–solution for $\kappa \leq \kappa(n,k)$ if $k \geq 2$ (otherwise it is flat). It is easy to see that the Ricci-flow can be defined on $(-\infty, 0]$: The inverse Ricci-flow $\frac{\partial}{\partial t} g(t) = 2Ric(g(t))$ is just the product of an expanding sphere with $\mathbb{R}^k$.

This gives the most import 3–dimensional examples for κ–solutions: the shrinking round cylinder $S^2 \times \mathbb{R}$ ($k = 2$) and the shrinking round sphere S^3 ($k = 3$).

Example 2.13. If one takes a finite volume quotient of the $\mathbb{R}^{n-k}$–factor in Example 2.12, i.e. $M = S^k \times F^{n-k}$ where F^{n-k} is a flat compact $(n-k)$–manifold, then the Ricci-flow is still defined on $(-\infty, 0]$. However, it is not κ–non-collapsed on *any* scale for *any* κ: For $t \to -\infty$ the sphere is expanding, so it can be rescaled with smaller and smaller r while keeping $S \leq 1$. This makes the F^{n-k}–factor arbitrarily small and therefore the volume of the rescaled ball $B(x,1)$ gets below any κ.

In dimension 3 this yields that the ancient Ricci-flow on $S^2 \times S^1$ with the round shrinking metric on S^2 is not a κ–solution. More generally:

Proposition 2.14. *The only quotient of the round shrinking cylinder $S^2 \times \mathbb{R}$ which is a κ–solution is $S^2 \times_{\mathbb{Z}_2} \mathbb{R}$, i. e. the quotient under the map $(x,t) \mapsto (-x,-t)$.*

Proof. Let $\rho\colon\ G \curvearrowright S^2 \times \mathbb{R}$ be an isometric action, such that the quotient $S^2 \times \mathbb{R}/\rho(G)$ (with the evolving round quotient metric) is a κ–solution.

By Example 2.13, $S^2 \times \mathbb{R}/\rho(G)$ must have infinite volume. Thus, the induced action on $\mathbb{R}$ can only be the trivial action or the $\mathbb{Z}_2$–action given by $x \mapsto -x$. If it is trivial, then ρ fixes every S^2, but there is no non-trivial orientation preserving isometry of S^2; so ρ is trivial. If it is the $\mathbb{Z}_2$–action, then it leaves $S^2 \times \{0\}$ invariant, so it must be the antipodal map on the S^2–factor. On the other hand, it is easy to verify that $S^2 \times_{\mathbb{Z}_2} \mathbb{R}$ is κ–non-collapsed, since balls contain at least half the volume of the corresponding balls in the universal cover $S^2 \times \mathbb{R}$, which is κ–non-collapsed. □

Classification of 3–dimensional κ–solutions

We give an overview about how to classify 3–dimensional orientable κ–solutions. An essential step for this classification is a dimension-reduction argument, for which one needs the following non-trivial result about 2–dimensional κ–solutions.

Proposition 2.15. *The only orientable two-dimensional κ–solution is the round shrinking two-sphere.*

Proof. See [KL07, Corollary 40.1 and Section 43] or [Bam07, Section 5.3]. □

The three-dimensional orientable κ–solutions are topologically classified as follows:

Proposition 2.16. *Let $(N, h(t))$ be an orientable three-dimensional κ–solution.*

If $(N, h(t))$ has not strictly positive curvature for some time t, then N is isometric to a round shrinking cylinder $S^2 \times \mathbb{R}$ or to its smooth orientable $\mathbb{Z}_2$–quotient $S^2 \times_{\mathbb{Z}_2} \mathbb{R}$.

If $(N, h(t))$ has stricly positive curvature for all t, then N is diffeomorphic to a spherical space form S^3/Γ if N is compact, or diffeomorphic to $\mathbb{R}^3$ if N is non-compact.

Proof. In case of not strictly positive curvature, by Hamilton's strong maximum principle N must locally split off an $\mathbb{R}$–factor [Ham86, Lemma 8.2 and 9]. So the universal cover splits off a line and $\tilde{N} = \mathbb{R} \times N^2$ as a metric product by the Splitting Theorem of Cheeger and Gromoll [CG72], where the splitting is invariant under the flow, since the null-space of R is one-dimensional and perserved in time due to the maximum principle. After reparameterization of time this induces a κ'–solution on N^2. Thus, N^2 is a round shrinking 2–sphere by Proposition 2.15. N cannot be a finite volume quotient of $\tilde{N}$, see Example 2.13. So $S^2 \times \mathbb{R}$ and $S^2 \times_{\mathbb{Z}_2} \mathbb{R}$ remain as the only possibilities.

If N is compact (and thus has strictly positive curvature), then the conclusion follows from [Ham82]. For non-compact N with positive curvature it is a consequence of the Soul Theorem [CG72]: The soul is then a point and its tangent space diffeomorphic to N. □

We shall later derive a more precise characterization of compact 3–dimensional κ–solutions, namely that if the solution is not diffeomorphic to S^3 or $\mathbb{R}P^3$, then it must be *round* (see Proposition 2.29).

Compactness of the space of κ– and standard solutions

From now on we restrict to 3–dimensional orientable κ–solutions. So for abbreviation the term "κ–solution" shall always mean "3–dimensional orientable κ–solution", if not explicitly stated otherwise. There is the following compactness result, compare [Per02, Theorem 11.7], [KL07, Theorem 46.1].

Theorem 2.17 (compactness of κ–solutions). *For given κ, the space of (orientable 3–dimensional) κ–solutions is compact modulo scaling. That is, for any sequence (N_i, x_i, h_i) with $S(x_i) = 1$ there exists a subsequence that is smoothly converging to a limit flow, which again is a κ–solution.* □

Combining the compactness results for $\kappa-$ and standard solutions, one obtains a compactness result for the space of time-slices of $\kappa-$ and standard solutions.

Theorem 2.18 (compactness of $\kappa-$ and standard solutions). *The space of pointed time-slices of $\kappa-$ and standard solutions is compact modulo scaling with respect to the pointed smooth convergence.*

Proof. For pointed standard solutions with base-point the tip, the corresponding compactness property was already stated in Proposition 2.8.

Consider now sequences of pointed standard solutions $(N_i, x_i, g(\cdot))$, and of times t_i. If (for a subsequence) t_i stays bounded away from 1 and $d(x_i, 0)$ stays bounded with respect to $g(t_i)$, then t_i subconverge to a time $t_\infty < 1$ and under the convergence $(N_i, 0, g(\cdot)) \to (N_\infty, 0, g(\cdot))$ also x_i subconverge to a point $x_\infty \in N_\infty$. Therefore, (N_i, x_i, g_{t_i}) subconverges to the t_∞–time-slice of a standard-solution.

If $x_i \to \infty$, then up to rescaling the sequence subconverges to a round cylinder, see [KL07, Lemma 61.1]. Finally, if $t_i \to 1$, then S gets uniformly large on N_i and (using that standard solutions are κ–non-collapsed on scales ≤ 1) one can apply [KL07, Theorem 52.7] (compare [Per02, Theorem 12.1]) to conclude that $(N_i, x_i, g(t_i)$ is ϵ_i–close to a subset of a κ–solution, with $\epsilon_i \to 0$. □

Proposition 2.19 (universal κ_0). *There exists $\kappa_0 > 0$ such that any κ–solution is a either a shrinking spherical space-form or a κ_0–solution.*

Proof. See [KL07, Proposition 50.1], [MT07, Proposition 9.58]. □

2.2 Geometry of $\kappa-$ and standard solutions

The compactness theorem yields bounds on the variation of curvature on κ–solutions, as well as on their asymptotic geometry.

Proposition 2.20 (bounded curvature at bounded distance). *For fixed κ there are positive functions $\alpha_1, \alpha_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for any time-slice of a κ–solution $(N, h(t))$ and any points $x, y \in N$ holds*

$$\alpha_1\big(\tilde{d}(x,y)\big) \leq \frac{S(y)}{S(x)} \leq \alpha_2\big(\tilde{d}(x,y)\big)\,.$$

Moreover, there is a positive function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{s\to\infty}\beta(s) = \infty$ *such that* $\tilde{d}(y,x) \geq \beta(\tilde{d}(x,y))$.

Proof. If there were no such functions α_1 and α_2, then there would exist a positive number s and there would be sequences of time-slices of κ–solutions $(N_i, h_i(t_i))$ and of points $x_i, y_i \in N_i$ such that $\tilde{d}(x_i, y_i) = s$ and the quotient $\frac{S(y_i)}{S(x_i)}$ tending towards 0 or ∞. We normalize scalar curvature at x_i to 1 and thus obtain a time-slice of a limit flow, for which scalar curvature is zero or unbounded at distance s from x_∞. This is a contradiction.

For the second claim we use that in bounded distance to y the ratio $S(x)/S(y)$ is bounded above, so for any r on the relative r–ball $\tilde{B}(y,r)$ holds

$$\tilde{d}(x,y) = \tilde{d}(y,x)\left(\frac{S(x)}{S(y)}\right)^{\frac{1}{2}} \leq r\alpha_2(r)^{\frac{1}{2}}\,.$$

In other words, $\tilde{d}(x,y) \to \infty$ forces also $\tilde{d}(y,x) \to \infty$. □

Lemma 2.21. *For fixed κ there is a function $\hat{r}\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds for any time-slice of a κ–solution: Suppose the relative r–balls $\tilde{B}(x,r)$ and $\tilde{B}(y,r)$ intersect. Then $\tilde{B}(y,r) \subseteq \tilde{B}(x,\hat{r}(r))$.*

Proof. Let $z \in \tilde{B}(x,r) \cap \tilde{B}(y,r)$. Then by Proposition 2.20 the ratios of $S(x)$, $S(z)$ and $S(y)$ are bounded: We have $\frac{S(x)}{S(z)} \leq \alpha_1(r)^{-1}$ and $\frac{S(z)}{S(y)} \leq \alpha_2(r)$, and therefore $\frac{S(x)}{S(y)} \leq \frac{\alpha_2(r)}{\alpha_1(r)} =: C(r)$. So any point $z' \in \tilde{B}(y,r)$ satisfies

$$\begin{aligned}\tilde{d}(x,z') &\leq S(x)^{\frac{1}{2}}\big(d(x,z) + d(z,y) + d(y,z')\big)\\ &\leq \tilde{d}(x,z) + \frac{S^{\frac{1}{2}}(x)}{S^{\frac{1}{2}}(y)}\big(\tilde{d}(y,z) + \tilde{d}(y,z')\big) \leq r + 2C(r)^{\frac{1}{2}}r\,.\end{aligned}$$

Hence $\hat{r}(r) := r + 2C(r)^{\frac{1}{2}}r$ gives the desired function. □

We will later on often make use of the following dichotomy: κ–solutions with an upper radius bound have a uniform lower sectional curvature bound. On the other hand, κ–solutions with large radii have a good control on their geometry in the sense that they are almost cylindrical except for at most two regions with bounded radius. These two essential observations shall be derived in the rest of this section.

Proposition 2.22. *There is a constant $c_1' = c_1'(D'', \kappa) > 0$ such that the following holds: Let N be a compact κ–solution, $x \in N$ with $\widetilde{\text{rad}}(x, N) < D''$, then the sectional curvature on N is bounded below by $c_1' S(x)$.*

Proof. Assume by contradiction that there are sequences of compact κ–solutions N_i, of points $x_i \in N_i$ with $\widetilde{\text{rad}}(x_i, N_i) < D''$ such that for each i there is point y_i where the sectional curvature K on a two-plane $P_i \subset T_{y_i} N_i$ satisfies $K_{y_i}(P_i) < \nu_i S(x_i)$ with $\nu_i \to 0$ for $i \to \infty$.

We rescale the N_i such that $S(x_i) = 1$ and $K_{y_i}(P_i) < \nu_i$. By compactness of the space of κ–solutions there is a subsequence for which (N_i, x_i) converges to a κ–solution (N_∞, x_∞), which satisfies $S(x_\infty) = 1$ and $\widetilde{\text{rad}}(x_\infty, N_\infty) \leq D''$. By the definition of convergence there are points $y_{\infty,i} \in N_\infty$ and two-planes $P_{\infty,i} \in T_{y_{\infty,i}} N_\infty$ with $K_{y_{\infty,i}}(P_{\infty,i}) < \nu_i'$ with $\nu_i' \to 0$. Since N_∞ is compact, due to Arzelà-Ascoli a subsequence of the $y_{\infty,i}$ converges to a point $y_\infty \in N_\infty$ and for a further subsequence also the planes converge to a two-plane $P_\infty \subset T_{y_\infty} N_\infty$. Thus $K_{y_\infty}(P_\infty) = 0$, which contradicts the positive curvature of compact κ–solutions, see Proposition 2.16. □

Note that although this section deals with κ–solutions, the following definitions are for general Riemannian manifolds.

Definition 2.23 (ϵ–neck). Let (M, g) be a Riemannian manifold and $x \in M$. An ϵ–homothety

$$\eta\colon\ S^2(\sqrt{2}) \times (-\tfrac{1}{\epsilon}, \tfrac{1}{\epsilon}) \hookrightarrow M$$

with $x \in \eta(S^2(\sqrt{2}) \times \{0\})$ is called an *ϵ–neck around x*. x is called *center* of the ϵ–neck and $\eta(S^2(\sqrt{2}) \times \{0\})$ the *central leaf* of the neck. Sometimes we will also refer to the image of η as an ϵ–neck.

We define the *neck-like region* M_ϵ^{neck} as the set of points in M that are center of an ϵ–neck.

Note that an ϵ–neck $\eta\colon\ S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \hookrightarrow M$ gives an ϵ–approximation of (M, x, g) by a round cylinder $S^2(\sqrt{2}) \times \mathbb{R}$. However, there is a slight technical difference, since an ϵ–approximation maps only an $\frac{1}{\epsilon}$–*ball* in $S^2(\sqrt{2}) \times \mathbb{R}$ to M and not the (slightly larger) cylinder $S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$.

Definition 2.24 (strong ϵ–neck). Let $(M, g(\cdot))$ be a Ricci-flow solution defined for $[0, T)$ and $(x, t) \in M \times [0, T)$. A *strong ϵ–neck* around

(x,t) is an ϵ–neck $\eta\colon\ S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \hookrightarrow M$ around x, such that after parabolic rescaling of $(M, g(\cdot))$ with factor $S(x,t)^{\frac{1}{2}}$, η is an ϵ–isometry between the parabolic regions $(S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})) \times [-1, 0]$ and $\mathrm{im}(\eta) \times [t-1, t]$, where the first region is in a round evolving cylinder with final time-slice of scalar curvature 1.

Definition 2.25 ((ϵ, d)–cap). Let (M, g) be a Riemannian manifold and $x \in M$. We call an open subset $C \subset M$ with $x \in C$ an *(ϵ, d)–cap around x*, if the following holds:

- the scalar curvature on C is strictly positive,
- C is diffeomorphic to B^3 or $\mathbb{R}P^3 - B^3$,
- $\widetilde{\mathrm{rad}}(x, C) < d$ and
- $\tilde{B}(x, d) - C \subset M_\epsilon^{neck}$, in particular $\partial C \subset M_\epsilon^{neck}$.

Remark 2.26. The idea behind the definition of an (ϵ, d)–cap is to assemble the characteristic properties of the part of a κ–solution which is not neck-like, see the following Proposition 2.27. These properties will carry over to regions of the Ricci-flow with high scalar curvature, which are approximated by κ–solutions. Therefore, there is some flexibility which properties to put into the definition and which to deduce separately from the approximating κ–solutions. For instance, one can omit the upper radius bound for the cap and show a-posteriori that it suffices to consider canonical neighborhoods of bounded radius as done in [KL07, Lemma 59.7 and Definition 69.1].

Also note that in our definition the end of a cap is contained in M_ϵ^{neck}, but we do not require that the boundary neck is *part of* the cap.

Proposition 2.27 (neck–cap decomposition I). *There exists $\epsilon^{(0)} = \epsilon^{(0)}(\kappa) > 0$ such that for all $0 < \epsilon < \epsilon^{(0)}$ there are constants $0 < d'(\epsilon, \kappa) < D'(\epsilon, \kappa)$ such that the following holds:*

If N is a κ– or standard solution, $x \in N - N_\epsilon^{neck}$ with $\widetilde{\mathrm{rad}}(x, N) > D'$, then x is center of an (ϵ, d')–cap C.

Furthermore, if for any $r > d'$ holds $D' > \hat{r}(r)$ ($\hat{r}$ from Lemma 2.21) then $\tilde{B}(x, r) - \tilde{B}(x, d') \subseteq N_\epsilon^{neck}$. In particular, if $\hat{C}$ is another (ϵ, d')–cap around a point $\hat{x}' \notin N_\epsilon^{neck}$, then either $C - N_\epsilon^{neck} = \hat{C} - N_\epsilon^{neck}$ or $C \cap \hat{C} = \emptyset$. In the later case N is a compact κ–solution.

The proof of Proposition 2.27 makes use of the following description of κ–solutions in terms of their neck-like and non-neck-like parts, which is proved in [KL07] as a corollary of the compactness result. We state this Lemma with adapted notation (and a slightly stronger formulation of case C, which actually is proved in [KL07]):

Lemma 2.28 ([KL07, Corollary 48.1]). *For all κ there exists an $\epsilon^{(0)}(\kappa) > 0$ such that for all $0 < \epsilon < \epsilon^{(0)}$ there exists an $\alpha = \alpha(\epsilon, \kappa)$ such that for any time-slice (N, h) of a κ–solution precisely one of the following holds:*

A. (N, h) is a round infinite cylinder, so every point is center of an ϵ–neck for any $\epsilon > 0$.

B. N is non-compact, $N \neq N_\epsilon^{neck}$ and for any points $x, y \in N - N_\epsilon^{neck}$ holds $\tilde{d}(x, y) < \alpha$. So $N = \tilde{B}(x, \alpha) \cup N_\epsilon^{neck}$ for any $x \in N - N_\epsilon^{neck}$.

C. N is compact and there is a pair of points $x, y \in N - N_\epsilon^{neck}$ with $\tilde{d}(x, y) > \alpha$. For any such pair x, y holds

$$N = \tilde{B}(x, \alpha) \cup N_\epsilon^{neck} \cup \tilde{B}(y, \alpha)\,,$$

and there is a minimizing geodesic $\overline{xy}$ such that every $z \in N_\epsilon^{neck}$ satisfies $\tilde{d}(z, \overline{xy}) < \alpha$.

D. N is compact and there exists a point $x \in N - N_\epsilon^{neck}$ such that $\widetilde{\mathrm{rad}}(x, M) < \alpha$, i. e. $N = \tilde{B}(x, \alpha)$. □

Lemma 2.28 leads to the following description of compact κ–solutions (see also [Bam07, Theorem 5.4.12] for an alternative proof):

Proposition 2.29. *If a κ–solution $(N, g(\cdot))$ is compact, then either it is a round shrinking space form (and thus $\widetilde{\mathrm{rad}}(x, N) \leq \sqrt{6}\pi$) or N is diffeomorphic to S^3 or $\mathbb{R}P^3$.*

Proof. Recall that any κ–solution has an associated *asymptotic soliton*, which shows up as a limit flow of the restricted solutions $(N, g(\cdot))_{(-\infty, t_i]}$ (shifted back to final time 0) for any sequence $t_i \to -\infty$ (compare [KL07, Section 39]).

If the asymptotic soliton is compact, it follows from [Ham82] that it must be a round shrinking space form. So let us assume that it is non-compact.

Then there must be a sequence of times $t_i \to -\infty$ such that each t_i–time-slice is of type C. Let x_i respectively y_i be points in the non-neck-like balls. Since the asymptotic soliton is non-compact, $\tilde{d}(x_i, y_i)$

and $\tilde{d}(y_i, x_i)$ tend towards infinity and a pointed limit flow with base-points x_i or y_i is non-compact as well. Such a limit must either be round $S^2 \times_{\mathbb{Z}_2} \mathbb{R}$ (note that the base-points are non-neck-like) or diffeomorphic to $\mathbb{R}^3$ (by the soul theorem). So for i large enough $\tilde{B}(x_i, \alpha)$ and $\tilde{B}(y_i, \alpha)$ are disjoint and each diffeomorphic to $\mathbb{R}P^3 - \bar{B}^3$ or B^3 (use $\partial B(x_i, \alpha) \cong S^2$ to apply Alexander's Theorem [Hat00, Theorem 1.1]), and the complement of the two balls is diffeomorphic to $S^2 \times [0, 1]$. It is impossible that both ends are $\mathbb{R}P^3 - \bar{B}^3$'s by the topological classification of κ–solutions in Proposition 2.16. □

The corresponding result of Lemma 2.28 for standard solutions says that for them only case B occurs. Note that we only regard standard solutions with a fixed standard initial metric g_{stand} (α may depend on the choice of g_{stand}).

Proposition 2.30 (standard solutions mostly neck-like). *There exists a positive number $\alpha = \alpha(\epsilon, g_{stand})$ such that for any time-slice (N, h) of a standard solution holds $N = \tilde{B}(0, \alpha) \cup N_\epsilon^{neck}$.*

Proof. For a fixed standard solution this follows from the asymptotically cylindrical geometry. That is, by [KL07, Lemma 61.1] the sequence of pointed flows obtained by moving the base-point towards infinity, converges uniformly to the round cylindrical flow. This implies that for base-points outside some compact ball around the origin all time-slices are ϵ–close to the round cylinder. Now compactness of the space of all standard solution yields a radius of this ball independent on the solution. □

Proof of Proposition 2.27. Let $x \in N - N_\epsilon^{neck}$ and $\widetilde{\text{rad}}(x, N) > D'$. This excludes case A of Lemma 2.28, and if $D' > \beta(\alpha)$ with β from Proposition 2.20, then it also excludes case D. Note that if (N, h) is time-slice of a standard solution, then by Proposition 2.30 it always satisfies the conclusion of case B by Proposition 2.30. So we are in case B if N is non-compact and in case C if N is compact, and in either case the non-neck-like part is contained in the relative α–ball around x (and possibly another relative α–ball around a point y). In the non-compact case B it is immediate that $\tilde{B}(x, \alpha)$ is an (ϵ, d')–cap. (Note that $\partial \tilde{B}(x, \alpha) \subset N_\epsilon^{neck}$ is diffeomorphic to S^2 and so $\tilde{B}(x, \alpha)$ is either diffeomorphic to $\mathbb{R}P^3 - B^3$ or B^3 by Alexander's Theorem [Hat00, Theorem 1.1].)

If in case C the relative r–balls $\tilde{B}(x, r)$ and $\tilde{B}(y, r)$ intersect for some $r \geq \alpha$, then by Lemma 2.21 we have $\tilde{B}(y, r) \subset \tilde{B}(x, \hat{r}(r))$. Since any $z' \in$

$N - (\tilde{B}(x,r) \cup \tilde{B}(y,r))$ is center of an ϵ–neck and satisfies $\tilde{d}(z', \overline{xy}) < \alpha$, it follows that all of N is contained in $\tilde{B}(x, \hat{r}(r))$ and thus $\widetilde{\mathrm{rad}}(x,N) \leq \hat{r}$. From this we conclude, that assuming $D' > \hat{r}(r)$ forces the relative r–balls around x and y to be disjoint. In particular, the region $\tilde{B}(x,r) - \tilde{B}(x,\alpha)$ is neck-like.

Due to the lower radius bound D', N cannot be a round spherical space-form, so by Proposition 2.29 $N \cong S^3$ or $\mathbb{R}P^3$. Again it follows from Alexander's Theorem that the ball $\tilde{B}(x,\alpha)$ is diffeomorphic to $\mathbb{R}P^3 - B^3$ or B^3. This shows the existence of an (ϵ, α)–cap around x. We therefore put $d'(\epsilon) = \alpha$.

Now if there is another (ϵ, d')–cap $\hat{C}$ and if some non-neck-like points of $\hat{C}$ are not contained in C, then in particular they are not contained in $\tilde{B}(x,\alpha)$, so we are in the case of a compact κ–solution again. The above argument then yields that the two caps are disjoint. □

Remark 2.31 (constants independend of κ). Since all statements made in this section are trivial in the case that the κ–solution is a shrinking round spherical space-form, the existence of a universal κ_0 (see Proposition 2.19) yields that all constants or functions $\epsilon^{(0)}$, α_1, α_2, $\hat{r}$, c_1', D', d', α can actually be chosen independent of κ, and in the following we shall always assume that this is the case.

2.3 Regions approximated by local models

κ– and standard solutions are local models for parts of the Ricci-flow on a manifold shortly before a singular time and shortly after carrying out the surgery. Thus, properties of κ– and standard solutions carry over to the part of the manifold that is approximated. This shall be examined in the current section.

Throughout the surgery process, we use ϵ and ϵ_1 as global parameter which take fixed, sufficiently small values. The parameter ϵ is used to control the quality of the necks, i. e. the closeness to round cylinders. On the other hand, $\epsilon_1 \ll \epsilon$ controls the quality of approximations by local models, i. e. by κ– or standard solution. At several steps of the argument we need to improve this quality and so derive upper bounds for ϵ and ϵ_1. In order to keep track of their dependencies, we shall denote them by $\epsilon^{(0)}$, $\epsilon^{(1)}$, etc. respectively $\epsilon_1^{(1)}$, $\epsilon_1^{(2)}$ etc.

Note that in [Per03a], [KL07] and [MT07] only one parameter ϵ is used (which corresponds to our ϵ_1). However, we find it more transparent to distinguish between the two different types of quality.

In order to distinguish between the Riemannian manifold M and an approximating κ– or standard solution N, we make the convention to decorate objects in N with a dash $'$.

Definition 2.32 ($A_0(\epsilon_1)$). Let (M, g) be a connected, closed, orientable 3–dimensional Riemannian manifold. For $\epsilon_1 \ll \epsilon$ we define the ϵ_1*–model-like part* $A_0 = A_0(\epsilon_1)$ as the subset of those points $x \in M$, for which (M, x, g) is ϵ_1–approximated (in the sense of Definition 1.19) by a time-slice of a κ– or standard solution.

Recall the definitions of ϵ–necks and (ϵ, d)–caps from Chapter 2.2. Now the neck–cap decomposition of Proposition 2.27 translates as follows to A_0:

Proposition 2.33 (neck–cap decomposition II). *For* $0 < \epsilon < \epsilon^{(0)}$ *there are constants* $0 < d(\epsilon) < D(\epsilon)$*,* $D(\epsilon) > 10$*, and* $0 < \epsilon_1^{(1)}(\epsilon) \ll \epsilon$*, such that for any* $\epsilon_1 \leq \epsilon_1^{(1)}$ *the following holds:*

If $x \in A_0(\epsilon_1) - M_\epsilon^{neck}$ *with* $\widetilde{\mathrm{rad}}(x, M) > D$*, then* x *is center of an* (ϵ, d)*–cap* C*.*

Furthermore, if $\hat{C}$ *is another* (ϵ, d)*–cap around a point* $\hat{x} \in A_0 - M_\epsilon^{neck}$ *with* $\widetilde{\mathrm{rad}}(\hat{x}, M) > D$*, then either* $C - M_\epsilon^{neck} = \hat{C} - M_\epsilon^{neck}$ *or* $C \cap \hat{C} = \emptyset$*.*

Proof. Let (N, x', h) be the κ– or standard solution which ϵ_1–approximates (M, x, g). For ϵ_1 sufficiently small (with respect to ϵ), x' cannot be center of an $\frac{\epsilon}{2}$–neck. Suppose $D > 2D'(\frac{\epsilon}{2})$, then $\widetilde{\mathrm{rad}}(x', N) > D'$ and hence by Proposition 2.27 x' is center of an $(\frac{\epsilon}{2}, d'(\frac{\epsilon}{2}))$–cap C'. We put $d(\epsilon) := 2d'(\frac{\epsilon}{2})$.

By definition, each $y' \in \partial C'$ is contained in an $\frac{\epsilon}{2}$–neck, and this neck (composed with ϕ) gives an ϵ–neck around $\phi(y')$. So $\phi(\partial C') \subset M_\epsilon^{neck}$ and $\phi(C')$ is an (ϵ, d)–cap centered at x.

For the second claim assume that $z \in C \cap \hat{C}$. Because both caps are approximated by a κ– or standard solution, the ratios $S(x)/S(z)$ and $S(z)/S(\hat{x})$ are bounded, and so $\widetilde{\mathrm{rad}}(x, \hat{C}) < c$ for some constant $c = c(d, \epsilon) = c(\epsilon)$. Now if $\epsilon_1 < \frac{1}{2c}$, then $\hat{x}$ and the ϵ–cap $\hat{C}$ lie in the ϵ_1–approximated region $\phi(N)$.

For any non-neck-like point $y \in \hat{C} - M_\epsilon^{neck}$, $y' := \phi^{-1}(y)$ cannot be center of an $\frac{\epsilon}{2}$–neck in N. For D large enough we have $\widetilde{\mathrm{rad}}(x', N) > \hat{r}(c)$, so Proposition 2.27 yields that all non-neck-like points in $\tilde{B}(x', c)$ must lie in $\tilde{B}(x', d') \subset C'$. We get $y \in C - M_\epsilon^{neck}$, and by exchanging the roles of $\hat{C}$ and C we obtain $C - M_\epsilon^{neck} = \hat{C} - M_\epsilon^{neck}$. □

In the following we shall always assume that $\epsilon < \epsilon^{(0)}$ and $\epsilon_1 \leq \epsilon_1^{(1)}(\epsilon)$.

Definition 2.34 (ϵ–cap, $A_1(\epsilon, \epsilon_1)$). We denote the set of points $x \in A_0$ satisfying $\widetilde{\mathrm{rad}}(x, M) > D(\epsilon)$ by $A_1 = A_1(\epsilon, \epsilon_1) \subseteq A_0(\epsilon_1)$. For $x \in A_1 - M_\epsilon^{neck}$, we will refer to an $(\epsilon, d(\epsilon))$–cap C simply as an *ϵ–cap*.

Remark 2.35. If $x \in A_1$, then the bound on $\widetilde{\mathrm{rad}}(x, M)$ excludes the possibility that (M, x, g) is approximated by a round spherical space form. Therefore, for all $x \in A_1$, (M, x, g) is approximated by a κ_0– or standard solution.

2.4 Ricci-flow with (r, δ)–cutoff

Definition 2.36. A *Ricci-flow with surgery* is a collection of finite or infinite sequences of increasing times t_k, of Ricci-flows $(M_k, g(\cdot))$ defined on $[t_k, t_{k+1})$ and of smooth embeddings $\phi_k \colon X_k \to M_{k+1}$ where $X_k \subset M_k$, with the following properties:

- t_k is a singular time for the Ricci-flow $(M_{k-1}, g(\cdot))$, i. e. the curvature explodes and the flow cannot be extended.
- The set of singular times $\{t_k\}$ is discrete.
- X_k is a compact 3–manifold with boundary, contained in the region $\Omega_k = \{x \in M_{k-1} \mid \sup_{t<t_k} \|R(x,t)\| < \infty\}$, on which the limit metric $g^-(t_k) = \lim_{t \nearrow t_k} g(t)$ exists.
- ϕ_k is isometric with respect to the limit metric on X_k and the metric $g(t_k)$ on M_k.

It is allowed that $M_k = \emptyset$ for some k, so one can assume a Ricci-flow with surgery always to be defined on $[0, \infty)$. Furthermore, it is useful to associate a Ricci-flow with surgery with a space-time $\mathcal{M}_{t\in[0,\infty)}$, such that $\mathcal{M}_t = (M, g(t))$ if t is not a singular time. For the singular times, there is a backward time-slice $\mathcal{M}_{t_k}^- = (\Omega_k, g^-(t_k))$ and a forward time-slice $\mathcal{M}_{t_k}^+ = (M_k, g(t_k))$. The backward and forward time-slices shall be identified on X_k respectively $\phi_k(X_k)$ via the isometry ϕ_k.

Note that Ω_k may be empty or consist of infinitely many components, and the metric on Ω_k in general is incomplete.

A-priori assumptions

There are two properties that are important for the control of the behavior of the Ricci-flow shortly before a singularity and which shall be explained in the following: the Φ–pinching and the canonical neighborhood assumptions. If both properties are valid, one says that the *a-priori assumptions* hold.

The meaning of Φ–pinching is that whenever the scalar curvature gets large, the positive eigenvalues of R grow much faster then the negative ones. This is quantified as follows:

Definition 2.37 (Φ–pinching). Let $\Phi\colon [1,\infty) \to (0,\infty)$ be a decreasing function with $\lim_{S\to\infty} \Phi(S) = 0$. We say, a Ricci-flow with surgery $\mathcal{M}$ satisfies the *Φ–pinching condition*, if $R \geq -\Phi(S)S$ on every point $(x,t) \in \mathcal{M}$ with $S(x,t) \geq 1$.

Before the first surgery, Φ–pinching is guaranteed by the Hamilton–Ivey curvature pinching [Ham95b, Ive93], see also [Ham99, Theorem 4.1] for a time-improving Φ–pinching version.

Definition 2.38 (canonical neighborhoods). Let $\mathcal{M}_{t\in[a,b)}$ be a solution of the Ricci-flow and $r\colon [a,b) \to (0,\infty)$ a non-increasing function. We say $\mathcal{M}$ satisfies the *$(r, \epsilon_1, \epsilon)$–canonical neighborhood assumptions* if all $x \in \mathcal{M}_t$ with $S(x,t) > r(t)^{-2}$ are contained in $A_0(\epsilon_1)$, i. e. $(M, x, g(t))$ is ϵ_1–approximated by the time slice of a κ– or standard solution.

Moreover, if $\tilde{B}(x, \frac{1}{\epsilon_1})$ is not a closed manifold and x is not center of an ϵ–cap, then x is center of a *strong* ϵ–neck.

Remark 2.39. The existence of *strong* necks (as assumed in the canonical neighborhood assumptions) is not important for our application, for which approximation of time-slices are sufficient. However, strong necks are needed to prove the existence of Ricci-flow with (r,δ)–cutoff. For this purpose, the advantage of strong necks is that when taking a limit one gets not only a limit manifold, but a limit Ricci-flow and can apply the strong maximum principle to this flow.

Before the first surgery, the canonical neighborhood assumptions are guaranteed by [Per02, Theorem 12.1] (see also [KL07, Theorem 52.7]) for those points (x,t), for which the flow is defined at least for the time-period $[t - S(x)^{-1}, t]$ (i. e. on the scale of x, the flow runs already for at least time 1).

In order to obtain universal bounds on curvature and life-time and thus a universal function r for the canonical neighborhood assumptions, one considers flows with normalized initial conditions.

Definition 2.40. We say a closed, orientable Riemannian 3–manifold (M, g_0) has *normalized initial conditions* if the following holds:

1. $|R(x)| \leq 1$ for all $x \in M$
2. (M, g_0) is $\frac{\omega}{2}$–non-collapsed on scales less then 1, where ω is the volume of the unit-ball in Euclidean 3-space.

Note that due to the first condition the second one is equivalent to the condition that any 1–ball in M has at least half the volume of a Euclidean unit ball. Given any Riemannian metric g_0' on a compact orientable 3–manifold, one can always find a scale λ such that $g_0 := \lambda g_0'$ has normalized initial condition.

Ricci-flow with (r, δ)–cutoff

We next describe a very special way of doing surgery, the so-called Ricci-flow with (r, δ)–cutoff. This process makes use of the a-priori assumptions. They are guaranteed before the first singular time, and the surgery will be done very carefully in order to preserve the assumptions also after the surgery.

The a-priori assumptions imply some control on the behavior of Ω_k in the case that it carries an incomplete metric. First note that towards the ends of Ω_k the Riemann curvature cannot stay bounded, and the Φ–pinching then implies that the scalar curvature gets arbitrarily large. Therefore, for $\rho > 0$ the sets

$$\Omega_{k,\rho} := \{x \in \Omega_k \mid S(x, t_k) \leq \rho^{-2}\}$$

are compact subsets of Ω_k.

For ρ taken small enough (will be quantified later on), scalar curvature is large on the complement of $\Omega_{k,\rho}$ and thus the canonical neighborhood assumptions can be applied on $\Omega_k - \Omega_{\rho,k}$. It follows that components of $\Omega_k - \Omega_{\rho,k}$ which have boundary in $\partial\Omega_{k,\rho}$ and have unbounded curvature can only be approximated by cylindrical regions of higher and higher curvature (i. e. smaller and smaller cross-sections). Therefore, these components are completely contained in Ω_ϵ^{neck} and each of them is diffeomorphic to $S^2 \times (0, \infty)$, with curvature unbounded towards ∞. One calls a component with these properties an *ϵ–horn*.

It is possible to show that when moving towards the end of an ϵ–horn, the regions are not only ϵ–approximated by cylinders but even δ–approximated for arbitrarily small δ. More precisely, there is a universal constant $h = h(\delta, \epsilon, \epsilon_1, r, \Phi)$ such that if x is a point in an ϵ–horn in a singularity-limit Ω_k and $S(x, t_k) \geq h^{-2}$, then x is center of an δ–neck [Per03a, Lemma 4.3], [KL07, Lemma 71.1].

This control on the ends of Ω_k makes it possible to perform surgery in the following way, compare [KL07, Definition 73.1]:

Definition 2.41. Let $r, \delta\colon [0, T) \to (0, \infty)$ be non-increasing functions. A *Ricci-flow with (r, δ)–cutoff* is a Ricci-flow with surgery $\mathcal{M}$, which satisfies the Φ–pinching condition and where in addition $\mathcal{M}^+_{t_k}$ is obtained from $\mathcal{M}^-_{t_k}$ as follows:

1. Throw away components of Ω_k that do not intersect $\Omega_{k,\rho}$, where $\rho = \delta(t_k) r(t_k)$.
2. Each end of an (remaining) incomplete component of Ω_k is an ϵ–horn. For each such ϵ–horn $\mathcal{H}_i$ find a strong $\delta(t_k)$–neck around a point x_i satisfying $S(x_i, T) = h$, where h only depends on $\delta, r, \epsilon, \epsilon_1$ and Φ (see [KL07, Lemma 71.1] for the existence of such δ–necks). Cut along their central spheres and throw away the unbounded side of the horn. The remaining part of Ω_k gives X_k.
3. Do surgery at the necks according to Lemma 2.42 in order to obtain the closed forward manifold $\mathcal{M}^+_{t_k}$.

Note that complete components of Ω_k which intersect Ω_ρ are part of X_k, so they are not affected by the surgery.

Lemma 2.42 (gluing). *There exists $\delta_0 = \delta_0(\Phi, g_{stand}) > 0$ (where g_{stand} is a standard initial metric) such that for any $\delta \leq \delta_0$ the following holds: Let (M, g) be a Riemannian 3–manifold that satisfies the Φ–pinching condition. Let $\eta\colon S^2(\sqrt{2}) \times (-\frac{1}{\delta}, \frac{1}{\delta}) \to M$ be a δ–neck. Then there exists a metric h on $S^2 \times (-\frac{1}{\delta}, 20) \cup \{pt\}$ (where the point is added to the positive end), such that*

- *on $S^2 \times (-\frac{1}{\delta}, 0]$ holds $h = \eta^* g$*
- *on $S^2 \times [10, 20)$ holds $h = g_{stand}$, where g_{stand} is a fixed standard initial metric, and we identify $S^2(\sqrt{2}) \times [10, 20)$ with the pointed*

3–ball $B(0,10) - \{0\} \subset \mathbb{R}^3$ via the map

$$(x,t) \mapsto \frac{x}{\sqrt{2}}(20-t)\,.$$

- *h satisfies the Φ–pinching condition.*

Proof. For a detailed proof see [MT07, Theorem 13.2], [KL07, Lemma 72.20]. The idea is to define h as interpolation

$$h = \beta_1 S(x)^{-1} g_{stand} + \beta_2 e^{2f(t)} \eta^* g\,,$$

where β_1, β_2 is a partition of unity subordinate to the cover $\{S^2 \times (5,20), S^2 \times (-\frac{1}{\delta}, 10)\}$, and $f\colon (-\frac{1}{\delta}, 10) \to (-\infty, 0]$ is a suitable function satisfying $f \equiv 0$ on $(-\frac{1}{\delta}, 0]$. The technical difficulty is to choose f in such a way that the Φ–pinching is preserved. Indeed, one can find f such that the smallest eigenvalue of R_h is greater than the one of R_g, see [MT07, Corollaries 13.11 and 13.12], [KL07, Lemma 72.1]. □

Remark 2.43. The Φ–pinching is not affected by the surgery process, i. e. if $\mathcal{M}^-_{t_k}$ satisfies the Φ–pinching, then so does $\mathcal{M}^+_{t_k}$. Therefore, the Φ–pinching condition can be replaced with a time-improving Φ–pinching which exists on the non-singular flow periods by [Ham99, Theorem 4.1], see also [KL07, Appendix B]. This will only be relevant for the discussion of long-time behavior of the Ricci-flow with (r,δ)–cutoff.

Theorem 2.44 (existence of Ricci-flow with (r,δ)–cutoff). *There exists $\epsilon^{(0)} > 0$ such that for any fixed $\epsilon < \epsilon^{(0)}$ and $\epsilon_1 \le \epsilon_1^{(1)}(\epsilon)$, there exist non-increasing functions $r, \bar{\delta}, \kappa\colon [0,\infty) \to (0,\infty)$ such that for any $\delta\colon [0,\infty) \to (0,\infty)$ with $\delta(\cdot) \le \bar{\delta}(\cdot)$ the following holds: Let (M, g_0) be a Riemannian manifold with normalized initial condition. Then there exists a Ricci-flow with (r,δ)–cutoff for all times, it is $\kappa(t)$–non-collapsed on scales below ϵ and satisfies the Φ–pinching and the (r,ϵ_1,ϵ)–canonical neighborhood assumptions.*

Proof. See [Per03a, Proposition 5.1], [KL07, Proposition 77.2] or [MT07, Theorem 15.9]. □

3 Invariant singular S^2–foliations

The aim of this chapter is to construct an invariant singular S^2–foliation on the part A_1 of a manifold M which is well-approximated by local models, as defined in Chapter 2.3. At the surgery process, this is the region of M where scalar curvature gets large and which is affected by the surgery, compare Chapter 2.4.

The purpose of the invariant singular foliation on A_1 is threefold: First, it allows us to find equivariant surgery necks and slightly modify the surgery process such that it becomes equivariant (see Chapter 4.1). Second, we use it to show that the action is standard on all components which get extinct or are thrown away at a surgery time (see Chapter 3.1 and Corollary 3.20). Finally, it is essential for relating the actions before and after the surgery as an equivariant connected sum construction (compare Chapter 4.2).

The construction of the foliation goes in two steps: It is relatively obvious that the neck-like part carries an invariant foliation by almost round almost totally geodesic two-spheres, since each neck-approximation has such a foliation and one only needs to interpolate between them. This is carried out in Chapter 3.2, and applied in Chapter 3.3 to find equivariant necks. Using the neck–cap decomposition from Chapter 2.3, it then remains to extend the foliation on an invariant family of disjoint ϵ–caps (see Chapter 3.4).

3.1 Singular S^2–foliations

Definition 3.1. Let M be a smooth orientable 3–manifold. A partition $\mathcal{F}$ of M in disjoint smooth submanifolds, called leaves, is called *singular S^2–foliation*, if

- finitely many leaves are points $x_1, \ldots, x_k$ and near each x_i the partition is diffeomorphic to the one by distance spheres to the origin in Euclidean space,
- $\mathcal{F} - \{x_1, \ldots, x_k\}$ is a smooth foliation (in the usual sense, compare e. g. [Law74]) with leaves diffeomorphic to S^2 except finitely many leaves diffeomorphic to $\mathbb{R}P^2$.

We call the S^2–leaves *regular leaves*, and the $\mathbb{R}P^2$–leaves and points *singular leaves* (despite the fact that the $\mathbb{R}P^2$–leaves of course are smooth leaves in the usual definition of smooth foliations).

If one removes the singular leaves, one gets a smooth S^2–foliation of an open 3–manifold $M_{regular}$. The foliation must have local product structure and hence is an S^2–fibration over a one-dimensional manifold, which we call *leaf space*. So a component $M^{(i)}_{regular}$ of $M_{regular}$ either is closed (the 1–manifold is S^1) and is diffeomorphic to $S^2 \times S^1$, or the 1–manifold is an interval and the component is diffeomorphic to $S^2 \times (0, 1)$.

One gets the original component back from $M^{(i)}_{regular} \cong S^2 \times (0, 1)$ by gluing in the singular leaves, i. e. by first adding boundary leaves $S^2 \times \{0, 1\}$ and then identifying antipodal points on a boundary leaf of $S^2 \times [0, 1]$ or by identifying the whole leaf with a point.

It follows that each singular leaf has a neighborhood with a standard foliation, and furthermore each closed component of M is diffeomorphic to either $S^2 \times S^1$, S^3 (both ends to points), $\mathbb{R}P^3$ (one end to a point, the other to $\mathbb{R}P^2$) or $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ (both ends to $\mathbb{R}P^2$). Similarly, an open component of M is diffeomorphic to $S^2 \times (0, 1)$, B^3 or $\mathbb{R}P^3 - \bar{B}^3$.

Proposition 3.2. *Let M be a connected (open or closed) 3–manifold which has a singular S^2–foliation $\mathcal{F}$. Let $\rho\colon H \curvearrowright M$ be a smooth finite group action that preserves the foliation, i. e. maps leaves to leaves. Then ρ is standard.*

Proof. First note that if x is a singular point, then we can find an invariant spherical metric on a saturated neighborhood N_x of the H–orbit of x (consisting of at most two points), such that leaves around x are distance spheres in the metric.

Similarly, if $F \cong \mathbb{R}P^2$ is a singular leaf, choose a $\rho(H)$–invariant spherical metric on $\rho(H)F$ and extend it $\rho(H)$–invariantly to an $(S^2 \times \mathbb{R})$–metric on a saturated neighborhood N_F of $\rho(H)F$, such that leaves near F are metrically double covers of F.

Denote the union of these regions by N. It now suffices to construct an $\rho(H)$–invariant $(S^2 \times \mathbb{R})$–metric on $M - N$, which agrees on the boundary spheres with the (2–dimensional) spherical metric of ∂N. (In case that $M \cong S^3$ or $\mathbb{R}P^3$ this metric can easily be made spherical by warping along the $\mathbb{R}$–factor).

The leaf space Λ of $M - N$ is either S^1 or an interval I. Since ρ preserves the foliation, there is an induced action ρ_Λ on Λ. Denote the kernel of this action by H_0. Choose a connected fundamental domain Λ_0 for the ρ_Λ–action, and let M_0 be the corresponding subset of $M - N$. Let Σ be the boundary spheres of M_0 which are not contained in ∂N and choose a $\rho(H)$–invariant spherical metric on $\rho(H)\Sigma$.

In the special case of $\Lambda = S^1$ and ρ_Λ the trivial action ($H_0 = H$) take an arbitrary leaf F, equip it with a ρ–invariant spherical metric and let $M_0 \cong S^2 \times I$ be M cut open along F (with F added on both ends of the interval).

Now ∂M_0 is equipped with a spherical metric, and it remains to extend this metric $\rho(H_0)$–invariantly to the interior of M_0. Since Λ_0 is a fundamental domain for ρ_Λ, such a metric on M_0 lifts to the desired $\rho(H)$–invariant metric on $M - N$.

In order to construct a $\rho(H_0)$–invariant $(S^2 \times \mathbb{R})$–metric on M_0, note that the foliation has a $\rho(H_0)$–invariant transversal line field. This induces a $\rho(H_0)$–equivariant diffeomorphism ϕ between the two boundary-spheres $\Sigma_0 = S^2 \times \{0\}$ and $\Sigma_1 = S^2 \times \{1\}$. We take a fixed $\rho(H_0)$–equivariant isometry to identify Σ_0 and Σ_1 and apply Proposition 1.23 to conclude that the diffeomorphism $\phi\colon \Sigma_0 \to \Sigma_1$ is equivariantly isotopic to an isometric one. Using this isotopy, we equivariantly modify the line field such that the induced diffeomorphism gets an isometry. Now take the product metric on M_0 with respect to the trivialization of the modified line field, i. e. leaves are totally geodesic and orthogonal to the line-field. ☐

3.2 Invariant foliation of the neck-like region

Let M be a (not necessarily complete) Riemannian manifold, and let $\rho\colon G \curvearrowright M$ be an isometric finite group action. Recall that M_ϵ^{neck} is the set of points in M, which are centers of ϵ–necks. Since the action is by isometries, M_ϵ^{neck} is $\rho(G)$–invariant. In this section we construct a $\rho(G)$–invariant S^2–foliation of M_ϵ^{neck}.

Definition 3.3. We call a (unit) tangent vector $v \in T_pM$ at a point $p \in M_\epsilon^{neck}$ a *distant direction*, if there exists a minimizing geodesic γ of length $S^{-\frac{1}{2}}(p)\frac{1}{3\epsilon}$ starting from $\gamma(0) = p$ with $\dot\gamma(0) = v$.

Definition 3.4. A smoothly embedded surface $\Sigma \subseteq M$ is called *ν–horizontal*, if angles between Σ and all distant directions are $> \frac{\pi}{2} - \nu$.

Each neck $\eta\colon S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \to M$ defines a *height function* $h_\eta = \pi_{(-\frac{1}{\epsilon},\frac{1}{\epsilon})} \circ \eta^{-1}$ on its image, where $\pi_{(-\frac{1}{\epsilon},\frac{1}{\epsilon})}$ is the projection on the interval factor. We call the level sets $h_\eta^{-1}(t) = \eta(S^2 \times \{t\})$ *leaves* of the neck η.

For an ϵ–neck $\eta\colon S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \to M$ define the *inner half* $V_\eta = \eta(S^2 \times (-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}))$, the *inner third* $W_\eta = \eta(S^2 \times (-\frac{1}{3\epsilon}, \frac{1}{3\epsilon}))$, and the *inner quarter* $U_\eta = \eta(S^2 \times (-\frac{1}{4\epsilon}, \frac{1}{4\epsilon}))$.

To simplify notation, we use the convention that a 0–*neck* is an infinite round cylinder.

Proposition 3.5 (distant directions in ϵ–necks). *There is a monotonically increasing function $\theta_1\colon [0, \frac{1}{100}) \to [0, \infty)$ with $\lim_{\epsilon\to 0} \theta_1(\epsilon) = 0$ such that the following holds:*

Let $\eta\colon S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \to M$ be an ϵ–neck with $\epsilon < \frac{1}{100}$, then for any $p \in V_\eta$ there are distant directions v_1, v_2 with $\angle_p(v_1, v_2) \geq \pi - \theta_1(\epsilon)$ and for any two distant directions w_1, w_2 holds either $\angle_p(w_1, w_2) \geq \pi - \theta_1(\epsilon)$ or $\angle_p(w_1, w_2) \leq \theta_1(\epsilon)$.

Proof. To simplify notation assume that $S(p) = 1$, so η is an ϵ–isometry. We denote $\eta^{-1}(p)$ by $(x, t) \in S^2(\sqrt{2}) \times (-\frac{1}{2\epsilon}, \frac{1}{2\epsilon})$.

Let $q_1 = \eta((x, t - \frac{1}{2\epsilon}))$, $q_2 = \eta((x, t + \frac{1}{2\epsilon}))$. Then $\left|d(q_i, p) - \frac{1}{2\epsilon}\right| \leq 1$ and $\left|d(q_1, q_2) - \frac{1}{\epsilon}\right| \leq 1$. Therefore minimizing geodesics γ_i $(i = 1, 2)$ from p to q_i have length $> \frac{1}{3\epsilon}$ and by triangle comparison $\angle_p(\dot{\gamma_1}(0), \dot{\gamma_2}(0)) \to \pi$ for $\epsilon \to 0$ (note that curvature on $\mathrm{im}(\eta)$ is almost non-negative). We put $v_i := \frac{\dot{\gamma}_i(0)}{\|\dot{\gamma}_i(0)\|}$ and choose a monotonically increasing function θ_1 with $\lim_{\epsilon\to 0} \theta_1(\epsilon) = 0$ such that $\angle_p(\dot{\gamma_1}(0), \dot{\gamma_2}(0)) \geq \pi - \frac{\theta_1(\epsilon)}{3}$.

On the other hand, assume that w is any distant directions at p. Let γ be a geodesics starting from p with $\dot{\gamma}(0) = w$ and let $q = \gamma(\frac{1}{3\epsilon})$. Then $\eta^{-1}(q)$ has distance at most 1 from $S^2 \times \{t - \frac{1}{3\epsilon}\}$ or to $S^2 \times \{t + \frac{1}{3\epsilon}\}$. From the two distant direction constructed above let v_i be the one for which γ_i does *not* get close to q and put $\bar{q} := \gamma_i(\frac{1}{3\epsilon})$. Then holds

$$d(\bar{q}, q) > \tfrac{2}{3\epsilon} - 2 \quad \text{and} \quad d(p, q) = d(p, \bar{q}) = \tfrac{1}{3\epsilon}$$

so as before for $\epsilon \to 0$ we have $\angle_p(\dot{\gamma_i}(0), \dot{\gamma}(0)) \to \pi$, and we may modify θ_1 such that $\angle_p(\dot{\gamma_i}(0), \dot{\gamma}(0)) \geq \pi - \frac{\theta_1(\epsilon)}{3}$. This shows that $\angle_p(w, -v_i) \leq \frac{\theta_1(\epsilon)}{3}$, so any distant direction has angle $\leq \frac{\theta_1(\epsilon)}{3}$ with either $-v_1$ or $-v_2$. This proves the proposition. □

Proposition 3.6 (leaves almost horizontal). *There is a monotonically increasing function $\theta_2\colon [0, \frac{1}{100}) \to [0, \infty)$ with $\lim_{\epsilon\to 0} \theta_2(\epsilon) = 0$ such* *that the following holds: Let η be an ϵ–neck and Σ a leaf in the inner half V_η, i. e. $\Sigma = h_\eta^{-1}(t)$ with $t \in (-\frac{1}{2\epsilon}, \frac{1}{2\epsilon})$. Then Σ is $\theta_2(\epsilon)$–horizontal.*

Proof. Let γ be a minimizing geodesic of length $\frac{\epsilon}{3}$ starting from p in any distant direction. Then $\eta^{-1}(\gamma)$ is a minimizing $\eta^* g$–geodesic in $S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$, and it is uniquely minimizing between $x := \eta^{-1}(p)$ and $y := \eta^{-1}(\gamma(\frac{1}{6\epsilon}))$. By Proposition 1.16 the angle at x between $\eta^{-1}(\gamma)$ and the minimizing g_{cyl}–geodesic γ_{cyl} from x to y goes to 0 for $\epsilon \to 0$. But since γ_{cyl} is minimizing of length $\geq \frac{1}{8\epsilon}$ (with respect to g_{cyl}), it must be almost orthogonal on $S^2 \times \{t\} = \eta^{-1}(\Sigma)$, with $\angle_{g_{cyl}}(\eta^{-1}(\Sigma), \dot\gamma_{cyl}(0)) \to 0$ for $\epsilon \to 0$. The claim follows now from $\mathcal{C}^1$–closeness of g and g_{cyl}. □

Corollary 3.7. *Let $\eta_1, \eta_2\colon S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \to M$ be ϵ–necks around x_i that intersect. Then*

1. *if the inner half intersect, then at $x \in V_{\eta_1} \cap V_{\eta_2}$ holds*

$$\angle(dh_{\eta_1}, dh_{\eta_2}) < 2\theta_2(\epsilon) \quad \text{or} \quad \angle(dh_{\eta_1}, dh_{\eta_2}) > \pi - 2\theta_2(\epsilon)\,.$$

2. *if the inner thirds intersect and $\theta_2(\epsilon) < \frac{1}{10}$, then any leaf $\Sigma = h_{\eta_1}^{-1}(t)$ of η_1 is isotopic to any leaf of η_2.*

Proof. Let Σ_1, Σ_2 be the leaves through a point $x \in V_{\eta_1} \cap V_{\eta_2}$, and choose a distant directions $v \in T_x M$. By Proposition 3.6, v is almost orthogonal on both Σ_1 and Σ_2, thus the angle between Σ_1 and Σ_2 is less $2\theta_2(\epsilon)$. This implies the first statement.

For the second statement, first isotope Σ to a leaf $\Sigma' = h_{\eta_1}^{-1}(t')$ intersecting $W_{\eta_1} \cap W_{\eta_2}$. Since the diameter of Σ' is close to $\sqrt{2}\pi S(x_1)^{-\frac{1}{2}}$ by Proposition 3.6, Σ' lies in the inner halves $V_1 \cap V_2$. Thus at every point $x \in \Sigma'$, the angle between Σ' and the η_2–leaf through x is $\leq 2\theta_2(\epsilon)$. This implies that $\eta_2^{-1}(\Sigma')$ hits every $\{a\} \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ exactly once, and therefore Σ' can be isotoped to a leaf of η_2. □

We now construct a global $\rho(G)$–invariant smooth S^2–foliation on the union U_ϵ of all inner quarters of ϵ–necks,

$$W_\epsilon := \bigcup_{\eta \text{ is } \epsilon\text{–neck}} W_\eta \supseteq U_\epsilon := \bigcup_{\eta \text{ is } \epsilon\text{–neck}} U_\eta \supseteq \overline{M_\epsilon^{neck}}.$$

Lemma 3.8. *For given K there exists a monotonically increasing function $\theta\colon [0, \frac{1}{100}) \to [0, \infty)$ with $\lim_{s\to 0} \theta(s) = 0$ such that the following holds:*

Let $\rho\colon G \curvearrowright (M,g)$ be an isometric finite group action with $|G| \le K$ on a complete connected Riemannian manifold (M,g) and let $\epsilon \le \frac{1}{100}$. Then there exists a $\rho(G)$–invariant smooth S^2–foliation $\mathcal{F}_\epsilon$ on an open set F_ϵ, $U_\epsilon \subseteq F_\epsilon \subseteq M$ with the following properties:

If x is center of an $\tilde\epsilon$–neck for $\tilde\epsilon \le \epsilon$ and Σ is a leaf of $\mathcal{F}_\epsilon$ through x, then Σ is $\theta(\tilde\epsilon)$–horizontal. Furthermore, the foliation $\mathcal{F}_\epsilon$ on $\tilde B(x, \frac{1}{\theta(\tilde\epsilon)}) \cap U_\epsilon$ is $\theta(\tilde\epsilon)$–close in $\mathcal{C}^{\frac{1}{\theta(\tilde\epsilon)}}$–topology to the metric product foliation of $S^2(\sqrt{2}) \times \mathbb{R}$.

Proof. Of course, the $\tilde\epsilon$–necks $\eta\colon S^2 \times (-\frac{1}{\tilde\epsilon}, \frac{1}{\tilde\epsilon}) \to M$ give local S^2–foliations, with almost horizontal leaves and the foliation $\tilde\epsilon$–close to the cylindrical standard foliation. The problem is that those approximations need not to be ρ–invariant and different local foliations have to be glued together. In order to get a global $\rho(G)$–invariant foliation, the main idea is here to average the height functions (respectively its gradient) of a $\rho(G)$–invariant family of $\tilde\epsilon$–necks.

For each $x \in M_\epsilon^{neck}$ choose $\tilde\epsilon(x) \in [0,\epsilon]$ and an $\tilde\epsilon(x)$–neck η_x such that $\tilde\epsilon(x)$ is almost as small as possible, i. e. any $\tilde\epsilon'$–neck around x satisfies $\tilde\epsilon' > \frac{2}{3}\tilde\epsilon(x)$. (If $\tilde\epsilon(x) = 0$ for some point, then (M,g) is isometric to a round cylinder and the claim is trivial. So we may exclude this case.) Note that for any y in the inner halves V_{η_x} holds

$$3\tilde\epsilon(x) \ge \tilde\epsilon(y) \ge \tfrac{1}{3}\tilde\epsilon(x)\,, \tag{3.2.1}$$

since η_x is also a $2\tilde\epsilon$–neck around y, and a $\frac{1}{3}\tilde\epsilon(x)$–neck around y would be an $\frac{2}{3}\tilde\epsilon(x)$–neck around x, which cannot exist.

Now from this family of necks choose a sub-family

$$\Phi = \{\eta_x\colon\ S^2 \times (-\tfrac{1}{\tilde\epsilon(x)}, \tfrac{1}{\tilde\epsilon(x)}) \to M\}$$

of $\tilde\epsilon(x)$–necks such that their inner halves $\{V_{\eta_x} \mid \eta_x \in \Phi\}$ form a locally finite covering of W_ϵ and choose a subordinate partition of unity $\{\beta_\eta \mid \eta \in \Phi\}$. We may assume that the covering and the partition of unity is $\rho(G)$–invariant, for if Φ is a given collection, than $\{\rho(g)\eta \mid \eta \in \Phi,\ g \in G\}$ is a $\rho(G)$–invariant collection of ϵ–necks, and $\{\frac{1}{|G|}\beta_\eta \circ \rho(g)^{-1} \mid \eta \in \Phi,\ g \in G\}$ is $\rho(G)$–invariant subordinate partition of unity. Since the regions W_{η_x} are almost cylindrical, it is possible to bound the multiplicity of a (non-invariant) covering by 3 and thus the multiplicity of the invariant covering by $3\,|G|$.

By Proposition 3.6 all leaves in V_{η_x} are $\theta_2(\tilde\epsilon(x))$–horizontal, so $\ker dh_{\eta_x}$ has angle $\ge \frac{\pi}{2} - \theta_2(\tilde\epsilon(x))$ with all distant directions in V_{η_x}.

For any $y \in W_\epsilon$ we can choose signs $\epsilon_{y,\eta} \in \{\pm 1\}$ such that by Corollary 3.7 and (3.2.1) holds for z near y

$$\angle_z(\epsilon_{y,\eta} dh_\eta(z), \epsilon_{y,\eta'} dh_{\eta'}(z)) < 2\theta_2(3\tilde{\epsilon}(y)) \tag{3.2.2}$$

for all η, η' with $y \in V_\eta \cap V_{\eta'}$. Using these neck-orientations, we can interpolate to define a local one-form near y

$$\alpha(z) := \sum_{\eta \in \Phi} \beta_\eta(z) \epsilon_{y,\eta} dh_\eta(z) \in T_z^* M$$

α is locally well-defined up to sign. Thus, we get a well-defined global one-form $\tilde{\alpha}$ as a section in $T^*M/\pm 1$ on $W_\epsilon \supseteq U_\epsilon \supseteq M_\epsilon^{neck}$. Note that $\tilde{\alpha}$ is $\rho(G)$–invariant since Φ and β are so.

Since small balls in the space of directions are convex, interpolation improves the horizontality, and so we conclude from Proposition 3.6 that $\ker \alpha$ is $\theta_2(3\tilde{\epsilon}(y))$–horizontal.

α has a local primitive f near y, namely

$$f(z) = \sum_{\eta \in \Phi} \beta_\eta(z) \epsilon_{y,\eta} h_\eta(z).$$

So the (globally on W_ϵ defined) plane field $\ker \tilde{\alpha}$ is integrable and defines a foliation $\mathcal{F}_\epsilon$ by integral surfaces which are level sets of f.

Note that on the overlap $V_{\eta_x} \cap V_{\eta_{x'}}$ the height functions are θ'–close in $\mathcal{C}^{[\frac{1}{\theta'}]}$–topology, with $\theta' \to 0$ for $\max\{\tilde{\epsilon}(x), \tilde{\epsilon}(x')\} \to 0$. Thus, if η is any $\tilde{\epsilon}$–neck around y, the pulled back height functions and also the interpolated function f is close to the cylindrical height function, and so $\mathcal{F}_\epsilon$ is close to the standard cylindrical foliation (in particular the leaves of $\mathcal{F}_\epsilon$ are spheres close to the $\tilde{\epsilon}$–neck-leaves). Using (3.2.1) and the bound on the multiplicity of the covering, one finds a function $\theta(\tilde{\epsilon})$ as in the statement of the Lemma measuring the closeness of $\mathcal{F}_\epsilon$ and the cylindrical foliation.

Since $\tilde{\alpha}$ is defined on W_ϵ and for each $x \in U_\epsilon$ holds $\tilde{B}(x, \frac{1}{20\epsilon}) \subset W_\epsilon$, we can take F_ϵ to be the union of all $\ker \tilde{\alpha}$–leaves through U_ϵ. □

We get an immediate consequence if M is completely neck-like:

Corollary 3.9. *Suppose M is a connected, orientable, closed 3–manifold and $M = M_\epsilon^{neck}$ with $\epsilon \leq \frac{1}{100}$. Let $\rho\colon G \curvearrowright M$ be an isometric finite group action. Then $M \cong S^2 \times S^1$ and the action ρ is standard.*

Proof. Lemma 3.8 yields that M has a $\rho(G)$–invariant (smooth) S^2–foliation. Hence $M \cong S^2 \times S^1$, and the action is standard by Proposition 3.2. □

We finally note that invariant, almost horizontal spheres are isotopic to leaves of $\mathcal{F}_\epsilon$:

Lemma 3.10. *Let $\epsilon \leq \frac{1}{100}$ be such that $\theta(\epsilon) < \frac{1}{10}$ and $\Sigma \subset U_\epsilon$ an embedded 2–sphere that is $\frac{1}{10}$–horizontal and $\rho(H)$–invariant for a subgroup $H \leq G$. Then Σ can be $\rho(H)$–equivariantly isotoped to a leaf of the foliation $\mathcal{F}_\epsilon$ from Lemma 3.8.*

Proof. This is an equivariant version of the second claim in Corollary 3.7, and the proof is analogous: Since Σ is $\frac{1}{10}$–horizontal and leaves of $\mathcal{F}_\epsilon$ are $\theta(\epsilon)$–horizontal (with $\theta < \frac{1}{10}$), Σ intersects all leaves of $\mathcal{F}_\epsilon$ transversally. It follows that Σ hits each integral curve of $\tilde{\alpha}$ exactly once. Since $\tilde{\alpha}$ is invariant, the isotopy can now be done $\rho(H)$–equivariantly along these integral curves. □

3.3 Equivariant approximations by local models

Definition 3.11 (equivariant ϵ–neck). Given a subgroup $H \leq G$, an ϵ–neck $\eta\colon S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \hookrightarrow M$ is called $\rho(H)$*–equivariant*, if the image of η is $\rho(H)$–invariant and the pulled back action $\eta^*\rho\colon H \curvearrowright S^2(\sqrt{2}) \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ is isometric.

In the following we shall fix a group G and some constant

$$\epsilon^{(1)}(|G|) < \min\{\tfrac{1}{100}, \epsilon^{(0)}\}$$

such that for $\epsilon \leq \epsilon^{(1)}(|G|)$ holds $\theta(\epsilon, |G|) < \frac{1}{100}$ with θ from Lemma 3.8. This gives a "minimal quality" for the foliation $\mathcal{F}_\epsilon$, and guaranties that the non-foliated, well-approximated region $A_1 - U_\epsilon$ consists of disjoint ϵ–caps. In the following, we shall always assume that $\epsilon < \epsilon^{(1)}(|G|)$.

We now use the invariant foliation $\mathcal{F}_\epsilon$ in order to construct *equivariant* necks, where the quality of the necks depends on the (local) quality of the foliation:

Lemma 3.12 (finding equivariant necks). *For any $\delta > 0$ there exists $\tilde{\delta} = \tilde{\delta}(\delta, |G|) > 0$ such that the following holds:*

Suppose that $\rho\colon G \curvearrowright M$ is an isometric action, $x \in M_\epsilon^{neck}$ and Σ is the leaf through x of the $\rho(G)$–invariant foliation $\mathcal{F}_\epsilon$. Let $H = Stab_G(\Sigma)$. If x is the center of a $\tilde{\delta}$–neck, then x is the center of an H–equivariant δ–neck.

Proof. By Lemma 3.8 the $\rho(H)$–invariant metric $g|_\Sigma$ on Σ is almost round, so for $\tilde{\delta}$ sufficiently small we find an isometric action $\hat{\rho}_0\colon H \curvearrowright S^2(\sqrt{2})$ and a $(\hat{\rho}_0, \rho|_H)$–equivariant δ–approximation $\hat{\eta}_0\colon S^2 \to \Sigma$. We then extend $\hat{\eta}_0$ along integral lines of $\mathcal{F}_\epsilon$ to a diffeomorphism $\hat{\eta}\colon S^2(\sqrt{2})\times(-\frac{1}{\tilde{\delta}}, \frac{1}{\tilde{\delta}}) \to M$ such that $\hat{\eta}|_{S^2(\sqrt{2})\times\{0\}} = \hat{\eta}_0$. Since the integral lines are $\rho(H)$–invariant, the pulled back action is isometric. Now for $\tilde{\delta}$ sufficiently small, the restriction of $\hat{\eta}$ to $S^2(\sqrt{2}) \times (-\frac{1}{\delta}, \frac{1}{\delta})$ gets arbitrarily close to an homothety. □

As soon as one can guaranty equivariant necks, the surgery process itself can be made equivariant without the need of any further modification:

Proposition 3.13 (equivariant gluing). *Let $\rho\colon G \curvearrowright M$ be an isometric action and $\eta\colon S^2(\sqrt{2}) \times (-\frac{1}{\delta}, \frac{1}{\delta}) \to M$ be an H–equivariant δ–neck, where $H \leq G$ such that $\eta^*\rho(H)$ acts trivial on the interval factor and assume $\delta \leq \delta_0$ from Lemma 2.42. Then surgery at η can be done equivariantly: That is, if (M', g') is the manifold obtained by surgery along η, then there exists an isometric action $\rho'\colon H \curvearrowright M'$ that agrees with $\rho|_H$ on the part that is not affected by surgery, and the restriction of $\rho'(H)$ on the surgery cap is conjugate to a spherical suspension of $\eta^*\rho|_{S^2\times\{0\}}(H)$.*

Proof. The metric h constructed in the proof of Lemma 2.42 on $S^2 \times (-\frac{1}{d}, 20)$ preserves any symmetry of $\eta^* g$ (with which it agrees on $(-\frac{1}{\delta}, 0]$), because the standard initial metric g_{stand} is rotational symmetric and the interpolation function f only depends on the height variable (the $\mathbb{R}$–coordinate of the neck). □

Away from the foliated neck-like region, it is less obvious how one can find equivariant approximations:

If $x \in A_1$ then there is an ϵ–approximation $\phi\colon \tilde{B}(\frac{1}{\epsilon}, x') \to (M, x)$, where x' lies in a κ_0– or standard solution N. The pulled back action $\phi^*\rho$ is then only defined on a subset of N and is only ϵ–isometric (assuming that we have normalized $S(x') = 1$). We say an approximation is equivariant, if the pulled back action extends to an *isometric* action *on all of N.*

Regarding this problem for κ_0– or standard solution, it is a consequence of the compactness of the space of model solutions that partially defined almost-isometries can be replaced by globally defined isometries, supposed the region where the action is defined is large enough and the orbit of the base point x' is contained in a not too large ball:

Lemma 3.14 (finding equivariant approximations). *For $a, \nu > 0$ and a finite group G exists $\tilde{\epsilon}_1(a, \nu, G) > 0$ such that:*

Let (N, x') be a time slice of a κ_0– or renormalized standard solution, normalized so that $S(x') = 1$, and let $\rho\colon G \curvearrowright V$ be an ϵ_1–isometric action on an open subset V, $\tilde{B}(x', \frac{1}{2\epsilon_1}) \subseteq V \subseteq N$ with $0 < \epsilon_1 \leq \tilde{\epsilon}_1$. Suppose that A is a $\rho(G)$–invariant open subset, $x' \in A \subset V$, with $\widetilde{\mathrm{rad}}(x', A) < a$.

Then there exists a time slice $(\hat{N}, \hat{x}')$ of a κ_0– or renormalized standard solution, a ν–approximation $\hat{\phi}\colon (\hat{N}, \hat{x}') \to (N, x')$, an isometric action $\hat{\rho}\colon G \curvearrowright \hat{N}$ and a $(\rho, \hat{\rho})$–equivariant smooth embedding $\iota\colon A \hookrightarrow \hat{N}$ which is ν–close to $\hat{\phi}^{-1}$ in $\mathcal{C}^1$–topology.

Proof. Assume the statement is false. This means, we can find sequences of positive numbers $\epsilon_{1,i} \to 0$, of time-slices of κ_0– or renormalized standard solutions (N_i, x'_i, h_i) with $S(x'_i) = 1$, of $\epsilon_{1,i}$–isometric actions $\rho_i\colon G \curvearrowright V_i$ on open subsets satisfying $B(x'_i, \frac{1}{2\epsilon_{1,i}}) \subseteq V_i \subseteq N_i$, and of $\rho_i(G)$–invariant open subsets A_i with $x'_i \in A_i \subset V_i$ and $\widetilde{\mathrm{rad}}(x_{1,i}, A_i) < a$, such that for all i the conclusion of the Lemma is not satisfied.

Since the argument is by contradiction, we may pass to any subsequence, and hence we can assume that the (N_i, x'_i, h_i) converge to a time slice of a κ_0– or renormalized standard solution $(N_\infty, x'_\infty, h_\infty)$ (using the compactness property of the space of κ_0– and standard solutions). Hence for i sufficiently large, (N_i, x'_i, h_i) is ν–close to $(N_\infty, x'_\infty, h_\infty)$.

We claim that also the actions converge: By passing to a subsequence, we can assume that (N_i, x'_i, h_i) is $\frac{1}{i}$–close to $(N_\infty, x'_\infty, h_\infty)$ and that $B(x'_i, i+1) \subseteq V_i$. This implies that there is an $\frac{1}{i}$–isometric map $\psi_i\colon B(x'_\infty, i) \to N_i$ with $\psi_i(x'_\infty) = x'_i$ and $B(x'_i, i-1) \subseteq \mathrm{im}(\psi_i) \subseteq V_i$. Because the $\rho_i(G)$–orbit of x'_i is contained in A_i (and $\widetilde{\mathrm{rad}}(x'_i, A_i) < a$), the orbit of $B(x'_i, i-a-2)$ is contained in $B(x'_i, i-1)$. We conclude that there is a $\rho_i(G)$–invariant subset U_i with

$$B(x'_i, i-a-2) \subseteq U_i \subseteq \mathrm{im}(\psi_i)\,.$$

We now consider the pull-back action $\psi_i^*\rho_i$ on $\psi_i^{-1}(U_i)$. Since ρ_i is $\epsilon_{1,i}$–isometric and ψ is an $\frac{1}{i}$–isometry, the action $\psi_i^*\rho_i$ is $\hat{\epsilon}_i$–isometric with $\lim_{i\to\infty}\hat{\epsilon}_i = 0$. Furthermore,

$$\psi^*\rho_i(G)x'_\infty \subset B(x'_\infty, (1+\epsilon_{1,i})a)$$

and the sets $\psi_i^{-1}(U_i)$ exhaust N_∞. Therefore, a subsequence of the actions $\psi_i^*\rho_i$ converges to an isometric limit action $\rho_\infty\colon G \curvearrowright N$ with $\rho_\infty(G)x'_\infty \subset B(x'_\infty, a)$.

In order to construct an (ρ_i, ρ_∞)–equivariant embedding of A_i, note that the two actions ρ_∞ and $\psi_i^*\rho_i$ get arbitrarily close on $\psi^{-1}(U_i)$ for i large enough. Since $B(x'_\infty, 2a) \subset \psi_i^{-1}(U_i)$, for each $\gamma \in G$ the restriction

$$\rho_\infty^{-1}(\gamma) \circ \psi^*\rho_i(\gamma)\colon\ B(x'_\infty, 2a) \to N_\infty$$

converges to the identity in $\mathcal{C}^k$–topology for any k.

It thus follows from Chapter 1.4 that there are smooth conjugation maps $c_i\colon B(x'_\infty, 2a) \to N_\infty$ such that $\rho_\infty \circ c_i = c_i \circ \psi_i^*\rho_i$ on $B(x'_\infty, 2a)$. Since $\psi_i^{-1}(A_i) \subset B(x'_\infty, 2a)$, it follows that $\rho_\infty(h)\circ(c_i\circ\psi_i^{-1}) = (c_i\circ\psi_i^{-1})\circ\rho_i(h)$ on A_i for all $h \in G$. For i large enough, c_i gets arbitrarily $\mathcal{C}^1$–close to the identity map. Hence $\iota_i = c_i \circ \psi_i^{-1}\colon\ A_i \hookrightarrow N_\infty$ is a (ρ_i, ρ_∞)–equivariant embedding and is ν–close to ψ^{-1} for i large enough. This contradicts the assumption that for all i the conclusion of the Lemma is not satisfied. □

3.4 Extending the foliation to the caps

In Chapter 3.2 we have constructed a $\rho(G)$–invariant S^2–foliation $\mathcal{F}_\epsilon$ on a region containing all inner quarters of ϵ–necks. In this section we want to modify and extend this foliation to a $\rho(G)$–invariant singular S^2–foliation on a region containing A_1. This will later allow us to apply Proposition 3.2 in order to control the action on regions affected by the surgery process.

In the following we will only consider ϵ–caps around points in $A_1 - M_\epsilon^{neck}$. We say two such ϵ–caps C_1, C_2 are equivalent, if $C_1 - M_\epsilon^{neck} = C_2 - M_\epsilon^{neck}$. By Proposition 2.33 follows that inequivalent ϵ–caps are disjoint.

Proposition 3.15. *In a closed manifold (M, g) there can be only finitely many equivalence classes of ϵ–caps around points in $A_1 - M_\epsilon^{neck}$.*

Proof. If $x_1, x_2 \in A_1 - M_\epsilon^{neck}$ are inequivalent and if r is such that $\hat{r}(r) < D$, then by Proposition 2.27 $\tilde{d}(x_1, x_2) > r$. Since M is compact,

S has an upper bound c and thus $d(x_1, x_2) > \frac{r}{\sqrt{c}}$, which implies the claim. □

Clearly for each $\gamma \in G$, the image $\rho(\gamma)(C)$ is again an ϵ–cap around $\rho(\gamma)(x) \in A_1 - M_\epsilon^{neck}$.

Definition 3.16. The stabilizer of an ϵ–cap C is the subgroup $G_C \leq G$ such that for all $\gamma \in G_C$ the cap $\rho(\gamma)C$ is equivalent to C. In other words, $G_C = stab_G(C - M_\epsilon^{neck})$.

It follows that for $\gamma \notin G_C$ we have $\rho(\gamma)C \cap C = \emptyset$. A given cap C can be made G_C–invariant as a subset of M by adding all leaves of $\mathcal{F}_\epsilon$ to C that intersect C.

The strategy for extending the foliation is now as follows: Given a $\rho(G_C)$–invariant ϵ–cap C, we first apply Lemma 3.14 to find an approximation of the cap on which the pulled back action is by real isometries and not only by ϵ–isometries.

The second step is then to find a singular S^2–foliation *in the approximating κ_0– or standard solution.* One could do so by observing that isometric actions on κ_0– or standard solution are standard, which follows from Hamilton's result [Ham82] for compact κ–solutions and an equivariant version of the soul theorem for non-compact κ_0– or standard solutions. However, the problem of this approach is that one needs to ensure that the singular foliation which results from the standard action is equivariantly isotopic to the pushed forward foliation from M_ϵ^{neck}. This amounts to showing that an isometric action on the round S^3 is standard when restricted to any invariant smooth 3–ball. Indeed this is true but the proof is rather laborious, see [DL08, Section 2.4]. Instead, we present an alternative argument based on the following idea: The long neck-region in the local model can be used to perform equivariant surgery in such a way that the action on the neck-leaves is encoded in the tangential action at the tip of the glued-in 3–ball. This will be explained in detail in the proof of Proposition 3.18.

Proposition 3.17. *Given $\nu > 0$ and $\epsilon < \epsilon^{(1)}(|G|)$ there exists $\epsilon_1^{(2)} = \epsilon_1^{(2)}(\nu, \epsilon, G) \leq \epsilon_1^{(1)}(\epsilon)$ such that for $\epsilon_1 < \epsilon_1^{(2)}$ the following holds:*

Let C be a G_C–invariant ϵ–cap centered at $x \in A_1 - M_\epsilon^{neck}$, then there exists a κ_0– or renormalized standard solution $(\hat{N}, \hat{x}', \hat{h})$, an isometric action $\hat{\rho}\colon G_C \curvearrowright \hat{N}$, a ν–approximation $\hat{\phi}\colon (\hat{N}, \hat{x}', \hat{h}) \to (M, x, g)$ and a $(\rho, \hat{\rho})$–equivariant embedding $\hat{\iota}\colon C \hookrightarrow \hat{N}$, which is ν–close in $\mathcal{C}^1$–topology to $\hat{\phi}^{-1}$.

Proof. Let (N, x', h) be a κ_0– or renormalized standard solution which ϵ_1–approximates (M, x, g) via the map $\phi\colon (N, x', h) \to (M, x, g)$. Since the $\rho(G_C)$–orbit of x stays in $C \subseteq \tilde{B}(x, d)$, the $\rho(G_C)$–invariant set

$$V := \rho(G_C)\tilde{B}(x, \tfrac{2}{3\epsilon_1})$$

satisfies

$$\tilde{B}(x, \tfrac{2}{3\epsilon_1}) \subseteq V \subseteq \tilde{B}(x, \tfrac{2}{3\epsilon_1} + d) \subseteq \tilde{B}(x, \tfrac{1}{\epsilon_1}).$$

We denote the pre-image of V by $V' = \phi^{-1}(V)$. Then V' satisfies $\tilde{B}(x', \frac{1}{2\epsilon_1}) \subseteq V' \subseteq N$ and on V' the action $\rho' = \phi^*\rho\colon G_C \curvearrowright V'$ is defined and is ϵ_1–isometric. Let $A' := \phi^{-1}(C) \subset V'$, then A' is $\rho'(G_C)$–invariant and satisfies $\widetilde{\mathrm{rad}}(x', A') < 2d$.

Thus, all requirements for Lemma 3.14 are fulfilled, and for $\epsilon_1 \leq \tilde{\epsilon}_1(2d, \frac{\nu}{2}, G_C)$ the Lemma yields a κ_0– or renormalized standard solution $(\hat{N}, \hat{x}', \hat{h})$ with an isometric action $\hat{\rho}\colon G_C \curvearrowright \hat{N}$ and a $\frac{\nu}{2}$–approximation $\psi\colon (\hat{N}, \hat{x}') \to (N, x')$. The composition $\hat{\phi} := \phi \circ \psi$ gives the required ν–approximation. Furthermore, the $(\rho', \hat{\rho})$–equivariant embedding $\iota\colon A' \hookrightarrow \hat{N}$ can be pre-composed with ϕ^{-1} to give a $(\rho, \hat{\rho})$–equivariant embedding $\hat{\iota} := \iota \circ \phi^{-1}\colon C \hookrightarrow \hat{N}$. ι is $\frac{\nu}{2}$–close to ψ^{-1} in $\mathcal{C}^1$–topology, so $\hat{\iota}$ is ν–close to $\hat{\phi}^{-1}$. Therefore,

$$\epsilon_1^{(2)}(\nu, \epsilon, G) := \min\bigl\{\tilde{\epsilon}_1\bigl(2d(\epsilon), \tfrac{\nu}{2}, H\bigr) \mid H \leq G\bigr\}$$

gives the desired constant. □

Proposition 3.18. *There is a constant* $0 < \epsilon^{(2)}(|G|) \leq \epsilon^{(1)}(|G|)$ *such that for* $\epsilon < \epsilon^{(2)}$ *holds: Let* (N, h) *be a* κ_0*– or standard solution and let* $\rho\colon H \curvearrowright N$ *be an isometric action by a finite subgroup* $H \leq G$*. Assume that* C' *is a* $\rho(H)$*–invariant* ϵ*–cap around* $x' \in N$*. Then the* $\rho(H)$*–invariant* S^2*–foliation* $\mathcal{F}_\epsilon^N$ *on* $N_\epsilon^{neck} - C'$ *can be extended to a* $\rho(H)$*–invariant singular* S^2*–foliation on* $N_\epsilon^{neck} \cup C'$*.*

Proof. If (N, h) is a quotient of the round cylinder, then we may assume that $\mathcal{F}_\epsilon^N$ is that standard foliation by round cross-sections (if not, we could equivariantly isotoped it to that foliation using Lemma 3.10). The standard foliation obviously extends to the cap. We therefore only have to consider the case that (N, h) has strictly positive sectional curvature.

Let $z \in \partial C'$, so z is center of an ϵ–neck. By Lemma 3.12, for

$$\epsilon < \min\bigl\{\tilde{\delta}(\delta_0, |G|), \epsilon^{(1)}\bigr\} =: \epsilon^{(2)}$$

holds that z is center of an $\rho(H)$–equivariant δ_0–neck

$$\eta\colon\ S^2(\sqrt{2}) \times (-\tfrac{1}{\delta_0}, \tfrac{1}{\delta_0}) \to N,$$

where δ_0 is the constant from Lemma 2.42. $\rho(H)$ preserves the cap and therefore also the orientation of the neck, so it preserves each leaf. We assume that the negative half $\mathcal{N}^- := \eta(S^2 \times (-\frac{1}{\delta_0}, 0])$ of the neck lies in the direction towards the cap, and the positive half $\mathcal{N}^+ := \eta(S^2 \times [0, \frac{1}{\delta_0}))$ lies in the direction away from the cap.

We can now apply Proposition 3.13 in order to do surgery on the equivariant δ_0–neck η. We denote by C_0' the part of C' up to the central leaf $\eta(S^2 \times \{0\})$ of the neck η and replace the positive half $\mathcal{N}^+$ by a 3–ball. The result of this equivariant surgery is a smooth compact manifold $\tilde{C} = C_0' \cup B^3$ diffeomorphic to S^3 or $\mathbb{R}P^3$, and an isometric action $\tilde{\rho}\colon\ H \curvearrowright \tilde{C}$, which agrees with the action ρ on C_0', and which on B^3 is the suspension of the action $\rho|_{\partial C_0' = \partial B^3}$ (It has a fixed point in the tip p of B^3).

$\mathcal{N}^- \cup B^3$ carries a $\tilde{\rho}(H)$–invariant singular S^2–foliation $\mathcal{F}_0$ by the leaves $S^2 \times \{t\}$ for $t \in (-\frac{1}{\delta}, 20)$, and with p as a singular point. Obviously, this is an extension of the foliation $\mathcal{F}_\eta|_{\mathcal{N}^-}$.

Note that the metric on $\tilde{C}$ has strictly positive sectional curvature. This is because the original metric is strictly positive and hence eigenvalues of the curvature are increased by the surgery process, compare Lemma 2.42.

It then follows from [Ham82] that the Ricci-flow on $\tilde{C}$ converges to a spherical metric. Let $\mathcal{F}_\infty$ be the (singular) foliation by distances spheres to p with respect to the spherical limit metric. Because p is a fix-point of $\tilde{\rho}(H)$, this foliation is $\tilde{\rho}(H)$–invariant. Note that the singular leaves are the point p and either its antipodal point (if $\tilde{C} \cong S^3$) or an $\mathbb{R}P^2$ (if $\tilde{C} \cong \mathbb{R}P^3$).

On small balls around p both foliations, $\mathcal{F}_0$ and $\mathcal{F}_\infty$, are equivariantly isotopic. Thus, $\mathcal{F}_\infty$ can also be equivariantly isotoped such that it agrees with $\mathcal{F}_0$ on $\mathcal{N}^- \cup B^3$. We obtain a $\rho(H)$–invariant singular foliation on C_0' that agrees with $\mathcal{F}_\eta$ on $\mathcal{N}^-$.

The claim now follows from Lemma 3.10: Since leaves of $\mathcal{F}_\eta$ are clearly $\frac{1}{10}$–horizontal, they can be equivariantly isotoped to leaves of $\mathcal{F}_\epsilon^N$. □

We now put the pieces together to obtain an extension of the foliation to the ϵ–caps:

Theorem 3.19 (Existence of invariant singular S^2–foliation). *Let $\epsilon < \epsilon^{(2)}(|G|)$ and $\epsilon_1 < \epsilon_1^{(2)}(\theta, \epsilon, G)$ for some fixed $\theta < \frac{1}{100}$. Assume that M is connected and $A_0 \neq \emptyset$. Then either $A_1 \neq A_0$ and M is ϵ_1–approximated by a compact κ–solution or $A_1 = A_0$ and there is a $\rho(G)$–invariant singular foliation on A_0 that agrees with $\mathcal{F}$ away from ϵ–caps.*

Proof. In the case of $A_1 \neq A_0$, there is a point $x \in A_0$ with $\widetilde{\mathrm{rad}}(x, M) < D(\epsilon)$. Since $D < \frac{1}{\epsilon_1}$, the approximation model must be a compact manifold (hence a compact κ–solution) and ϵ_1–approximates M.

It remains the case where $A_1 = A_0$. If x is a point in $A_1 - M_\epsilon^{neck}$, then by Proposition 2.33 x is center of an ϵ–cap C_x. By Proposition 3.15, $A_1 - M_\epsilon^{neck}$ is covered by a finite collection of such ϵ–caps which are pairwise disjoint. By adding leaves of $\mathcal{F}_\epsilon$, we may assume that each C_x contains the ball $\tilde{B}(x, \frac{2}{3}d)$ and the collection is $\rho(G)$–invariant (see the remark before Proposition 3.17).

To extend the foliation $\mathcal{F}_\epsilon$ equivariantly on the caps C_x, it suffices to show that this is possible for a single cap: This will define a $\rho(G)$–invariant foliation on its orbit $\rho(G)C_x$, and we can repeat the procedure for any remaining cap.

So let C_x be any of the ϵ–caps in the collection. Proposition 3.17 yields a θ–approximation $\hat{\phi}\colon (\hat{N}, \hat{x}', \hat{h}) \to (M, x, g)$ by a κ_0– or renormalized standard solution $\hat{N}$, an isometric action $\hat{\rho}\colon G_{C_x} \curvearrowright \hat{N}$ and a $(\rho, \hat{\rho})$–equivariant embedding $\hat{\imath}\colon C_x \hookrightarrow \hat{N}$.

As in the proof of Proposition 2.33, x' can not be center of an $\frac{\epsilon}{2}$–neck, so it is center of an $\frac{\epsilon}{2}$–cap C' with

$$\widetilde{\mathrm{rad}}(x', C') \leq d'(\tfrac{\epsilon}{2}) < \tfrac{1}{2} d(\epsilon)$$

(Of course, C' is also an ϵ–cap). Lemma 3.8 yields a foliation on $\hat{N}_\epsilon^{neck}$, which we denote by $\mathcal{F}_\epsilon^{\hat{N}}$.

Since the action $\hat{\rho}\colon G_C \curvearrowright \hat{N}$ is isometric, we can apply Proposition 3.18 in order to extend the foliation $\mathcal{F}_\epsilon^{\hat{N}}$ to a singular foliation $\hat{\mathcal{F}}$ on $\hat{N}_\epsilon^{neck} \cup C'$ that agrees with $\mathcal{F}_\epsilon^{\hat{N}}$ outside of C'. In particular, the two foliation agree on $\tilde{B}(x', 2d) - \tilde{B}(x', \frac{1}{2}d)$.

Note that $\Sigma := \hat{\imath}(\partial C_x)$ is contained in this region. Moreover, since $\hat{\imath}$ is $\mathcal{C}^1$–close to the θ–approximation $\hat{\phi}^{-1}$, Σ is almost horizontal, and by Lemma 3.10 we can equivariantly isotope Σ to a leaf of $\hat{\mathcal{F}}$. Equivalently, we can also modify $\hat{\mathcal{F}}$ by an equivariant isotopy to $\hat{\mathcal{F}}'$, such that $\hat{\imath}(\partial C_x)$ now is a leaf of $\hat{\mathcal{F}}'$.

Using $\hat{\iota}$, we pull back the singular S^2–foliation $\hat{\mathcal{F}}'$ from $\hat{\iota}(C_x)$ to C_x. Because $\hat{\iota}$ is $(\rho, \hat{\rho})$–equivariant, the resulting singular foliation $\hat{\iota}^*\hat{\mathcal{F}}'$ is $\rho(G_c)$–invariant. This gives the desired extension of $\mathcal{F}_\epsilon$. □

The following corollary will be applied later to components which go extinct at a singular time:

Corollary 3.20. *Assume that $M = A_0$, and $\rho\colon G \curvearrowright M$ is a smooth action of a finite group. Then M is a union of components diffeomorphic to spherical space forms, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ or $S^2 \times S^1$ and the action ρ is standard.*

Proof. It suffices to consider each connected component separately, therefore we assume that M is connected.

In the case of $A_1 \neq A_0$, the approximating compact κ–solution has uniformly positive sectional curvature $K > c_1' \cdot S(x)$ (see Proposition 2.22), and therefore M has positive sectional curvature. It now follows from [Ham82] that the metric converges up to rescaling to a spherical metric and thus the action is standard.

In the case of $A_1 = A_0 = M$, Theorem 3.19 provides an invariant singular S^2–foliation on all of M. Hence the conclusion is a direct application of Proposition 3.2. □

4 Equivariant Ricci-flow with cutoff

4.1 Existence of equivariant Ricci-flow with cutoff

Definition 4.1. An *equivariant Ricci-flow with surgery* is a Ricci-flow with surgery $\mathcal{M} = \{(M_k, g_k(t))\}$ together with actions $\rho_k \colon G \curvearrowright M_k$ such that

1. ρ_k is isometric with respect to $g_k(t)$ for all $t \in [t_k, t_{k+1})$
2. The region $X_k = \mathcal{M}^-_{t_k} \cap \mathcal{M}^+_{t_k}$ which not affected by surgery is invariant under $\rho_{k-1}(G)$ and $\rho_k(G)$ and the restrictions of the two actions to X_k coincide.

Analogously, an *equivariant Ricci-flow with* (r, δ)*–cutoff* is an equivariant Ricci-flow with surgery for which in addition $\mathcal{M}$ is a Ricci-flow with (r, δ)–cutoff.

It is relatively easy to deduce the existence of an equivariant Ricci-flow with (r, δ)–cutoff from the existence of a Ricci-flow with (r, δ)–cutoff: All one has to do is to make the surgery equivariant. The following is the equivariant version of Theorem 2.44:

Proposition 4.2. *For a fixed finite group G and for constants $\epsilon < \epsilon^{(0)}$, $\epsilon_1 < \epsilon_1^{(2)}(\frac{1}{100}, \epsilon, G)$ there exist non-increasing functions $r, \bar{\delta}, \kappa \colon [0, \infty) \to (0, \infty)$ such that for any $\delta \colon [0, \infty) \to (0, \infty)$ with $\delta(\cdot) \leq \bar{\delta}(\cdot)$ the following holds: Let (M, g_0) be a Riemannian manifold with normalized initial condition, and $\rho_0 \colon G \curvearrowright (M, g_0)$ is an isometric action the group G. Then there exists an equivariant Ricci-flow with (r, δ)–cutoff for all times, it is $\kappa(t)$–non-collapsed on scales below ϵ and satisfies the $(r, \epsilon_1, \epsilon)$–canonical neighborhood assumptions.*

Proof. We denote the corresponding functions from Theorem 2.44 that guaranty the existence of a non-equivariant Ricci-flow with cutoff by r, κ and $\bar{\delta}'$. We claim that we can find $\bar{\delta} < \bar{\delta}'$ such that r, $\bar{\delta}$ and κ are satisfying the claim of the proposition.

Note that by uniqueness of the Ricci-flow solution, any symmetry of the initial metric g_0 is preserved until the first singular time, and the same holds later on for any symmetry of $g(t_k)$ in the time interval $[t_k, t_{k+1})$. Thus it suffices to make the surgery process equivariant, that is to choose X_k ρ_k–invariant and to extend the action on X_k to an isometric action $\rho_{k+1}\colon G \curvearrowright \mathcal{M}^+_{t_{k+1}}$.

This can be achieved as follows: Choose $\bar{\delta}(t)$ such that $\tilde{\delta}(\bar{\delta}(t)) \leq \bar{\delta}'(t)$ with $\tilde{\delta}$ from Lemma 3.12. This implies that every center of an $\bar{\delta}(t)$–neck is center of an equivariant $\bar{\delta}'$–neck.

When doing the (r, δ)–cutoff as in Definition 2.41 with $\delta \leq \bar{\delta}$, glue in the cap (in step 2 of the construction) using a $\rho_{k-1}(H)$–equivariant $\tilde{\delta}$–neck with $\tilde{\delta} \leq \bar{\delta}'$, where H is the stabilizer of the corresponding ϵ–horn $\mathcal{H}_i$ and apply Proposition 3.13. Of course, this can be done equivariantly for each $\rho_{k-1}(G)$–orbit of $\mathcal{H}_i$.

Therefore, $\rho_{k-1}(G)|_{X_k}$ can be isometrically extended by suspensions on the glued in 3–balls. This gives the desired isometric action ρ_k as an extension of $\rho_{k-1}|_{X_k}$. □

Proposition 4.3. *Let $\mathcal{M}$ be an equivariant Ricci-flow with (r, δ)–cutoff, and let t_k be a singular time. Then for t sufficiently close to t_k the part $M_{k-1} - \Omega_\rho$ is contained in A_0.*

Proof. It follows from the bounds on ∇S and $\frac{\partial}{\partial t} S$ and from the Φ–pinching that for t sufficiently close to t_k, on $M_{k-1} - \Omega_\rho$ still holds $S(x,t) > \frac{1}{2}\rho^{-2}$. Note that the surgery parameter $\rho = \delta(t_k) r(t_k)$ is chosen so small, that

$$S(x,t) > \tfrac{1}{2}\rho^{-2} = \tfrac{1}{2}\delta^{-2} r^{-2}(t_k) > r^{-2}(t_k) \geq r^{-2}(t)\,.$$

Thus by the $(r, \epsilon_1, \epsilon)$–canonical neighborhood assumptions (see Definition 2.38) we have $M_{k-1} - \Omega_\rho \subseteq A_0$ with respect to the metric $g(t)$. □

4.2 Effect of surgery on the group action

Theorem 4.4. *For each $k \geq 1$, the action $\rho_{k-1}\colon G \curvearrowright M_{k-1}$ is obtained from $\rho_k\colon G \curvearrowright M_k$ as follows:*

1. *First, let $\tilde{\rho}_k$ be a disjoint union of $\rho_k\colon G \curvearrowright M_k$ with a standard action of G on a finite union $\dot{\bigcup}_i C_i$ of components $C_i \cong \mathbb{R}P^3$, such that the stabilizer of each C_i has a fix point $y_i \in C_i$.*

2. *Then form an equivariant connected sum of $\tilde{\rho}_k$ along $\mathcal{P}$ (compare Chapter 1.6). For each $\mathbb{R}P^3$–component added in step (1), there*

is exactly one pair (x_i, y_i) in $\mathcal{P}$ with $x_i \in M_k$. The other tuples in $\mathcal{P}$ (if there are) have both points in M_k.

3. *Finally, ρ_{k-1} is the disjoint union of $(\tilde{\rho}_k)_{\mathcal{P}}$ with a standard action of G on a finite union $\dot{\bigcup}_i C'_i$ of components C'_i diffeomorphic to spherical space forms, to $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ or to $S^1 \times S^2$.*

(Note that any of the three steps might be trivial, i. e. the finite unions of standard actions might be empty, or the connected sum might consist only of the pairs (x_i, y_i).)

Proof. The proof follows the same line as in the non-equivariant version [KL07, Lemma 67.13 and 73.4], [MT07, Proposition 15.2]. Note that our notion of equivariant connected sum allows loops, i. e. connected sum construction with base-points in the same component. For this reason, our statement of the theorem does not need a connected sum construction with additional $S^2 \times S^1$–components.

In the proof, we will keep track of the action while following the *forward* surgery process. Thus going backwards reverses the order of the steps.

Recall from Chapter 2.4 that $X = \mathcal{M}^-_{t_k} \cap \mathcal{M}^+_{t_k}$ is the subset common to the backward and forward time slice at the surgery time t_k (so X is the part which is not affected by the surgery). ∂X consists of finitely many 2–spheres, which are central spheres of the surgery necks. M_k is obtained from M_{k-1} by replacing $M_{k-1} - X$ by a union of 3–balls, such that a ball is glued onto each boundary sphere of X. The families of surgery necks and boundary spheres are equivariant, and the action ρ_k is obtained from ρ_{k-1} by suspension of the action $\rho_{k-1}|_{\partial X}$ onto the balls.

By Proposition 4.3 we have $M_{k-1} - X \subseteq M_{k-1} - \Omega_\rho \subseteq A_0$. Thus, closed components of $M_{k-1} - X$ are diffeomorphic to spherical space forms, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ or $S^2 \times S^1$ and ρ is standard on them by Corollary 3.20. These components are thrown away in the surgery process, so in the reversed construction they are added. This gives step (3).

On the other hand, let Y be a component of $M_{k-1} - X$ which is not closed. We claim that $Y \subseteq A_1$. Otherwise, there is a point $y \in Y$ with $\widetilde{\mathrm{rad}}(y, M_{k-1}) < D(\epsilon)$ and $(M_{k-1}, y, g(t))$ is ϵ_1–approximated by a compact κ–solution (N, y') with $\widetilde{\mathrm{rad}}(y', N) \leq D$. On N, scalar curvature is pinched by the constant $2c'_1(D, \kappa)$, see Proposition 2.22. However, for t sufficiently close to t_k, on Y the pinching is arbitrarily bad, since (part of) Y approximates an ϵ–horn in $\mathcal{M}^-_{t_k}$ (and scalar curvature is unbounded towards the tip of a horn).

So indeed $Y \subseteq A_1$, and by Theorem 3.19 there is a $\rho(G_Y)$–invariant singular S^2–foliation on Y which has ∂Y as boundary spheres (Note that a collar-neighborhood of ∂Y is neck-like with respect to both $g(t)$ and $g(t_k)$, and that the two corresponding foliations near ∂Y are equivariantly isotopic.)

We now see that the interior of Y is a connected open 3–manifold with $\rho(G_Y)|_Y$–invariant singular S^2–foliation. Therefore, it is diffeomorphic to either $S^2 \times (0,1)$, B^3 or $\mathbb{R}P^3 - B^3$ and the restricted action is standard by Proposition 3.2.

We first consider the $S^2 \times (0,1)$–components: Replacing them by two 3–balls, on which the action is given as suspension of the boundary action, is clearly the same as doing an equivariant connected sum decomposition. The inverse is done by forming an equivariant connected sum along pairs in $\mathcal{P}$ with both points in M_k in step (2).

Finally, the B^3– and $(\mathbb{R}P^3 - B^3)$–components are replaced. For the B^3–components this is a trivial move. Replacing an $(\mathbb{R}P^3 - B^3)$–component by a 3–ball is the same as first doing an equivariant connected sum decomposition and then throwing away the obtained $\mathbb{R}P^3$–component. This gives the first part of step (2) and step (1). □

4.3 Group actions with finite extinction time

If all components get extinct after finite time, then one can reconstruct the original action with the help of Theorem 4.4 by equivariant connected sum construction of standard actions.

Corollary 4.5 (of Theorem 4.4). *Let (M, g) be a closed orientable 3–manifold, and $\rho\colon G \curvearrowright M$ be an isometric finite group action. Suppose furthermore that the Ricci-flow with (r, δ)–cutoff gets extinct in finite time. Then ρ is an equivariant connected sum of standard actions on components diffeomorphic to spherical spaceforms, $S^2 \times S^1$ and $\mathbb{R}P^3 \sharp \mathbb{R}P^3$.*

Proof. Consider the equivariant Ricci-flow with (r, δ)–cutoff on M. By the assumption after a finite time T and finitely many surgeries, we end up with the trivial flow on an empty manifold $M_{k_T} = \emptyset$ (Note that the *equivariant* Ricci-flow with (r, δ)–cutoff is just a special choice of cut-off parameters and surgery-necks in the ordinary Ricci-flow with (r, δ)–cutoff, but the property of finite extinction time does not depend on such particular choices).

To obtain the original manifold and action, we apply Theorem 4.4 finitely many times. Each time, standard actions on space forms, $S^2 \times S^1$ and $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ may be added and an equivariant connected sum is build. From the transitivity of the equivariant connect sum construction (Proposition 1.29) follows, that this is the same as forming an equivariant connected sum in one step. □

It is known that the Ricci-flow with (r, δ)–cutoff gets extinct for manifolds which do not contain aspherical factors in their prime decomposition (or equivalently, for manifolds with fundamental group a free product of finite groups and infinite cyclic groups), see [Per03b, Theorem 1.1], [CM05, Corollary 1.2], [MT07, Theorem 18.1].

In the irreducible case this yields the following conclusion (which has nothing to do with the group action and follows already from [Per03b] and [Per03a]):

Proposition 4.6. *Let (M, g) be a closed, orientable, irreducible 3–manifold. Then the Ricci-flow with (r, δ)–cutoff on M has finite extinction time if and only if $\pi_1(M)$ is finite.*

Proof. If $\pi_1(M)$ is finite, then finite extinction follows from [Per03b, Theorem 1.1.]. On the other hand, if M has finite extinction time, then by Corollary 4.5 M is a connected sum of finitely many S^3–components and possibly a spherical spaceform (note that $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ and $S^2 \times S^1$ are not irreducible). So M is a spherical spaceform. □

5 Actions on geometric manifolds

Except for Chapter 4.3, there were no requirements on the initial manifold so far besides being closed and orientable. In this final chapter, we derive some more specific results in the case that the manifold allows an S^3–, $S^2 \times \mathbb{R}$– or $\mathbb{H}^3$–geometry. We use that for those geometries the Ricci-flow with surgery either goes extinct in finite time (in the first two cases) or that the long-time behaviour is well understood (in the last case). In either case, the analysis of the Ricci-flow with surgery leads to a standard limit action, from which we then deduce (by Theorem 4.4 and Corollary 4.5) that the original action is already standard.

5.1 Actions on spherical manifolds

Theorem 5.1. *Let M be a closed, connected, orientable 3–manifold with finite fundamental group, and let $\rho\colon G \curvearrowright M$ be a smooth finite group action. Then ρ is a standard action on a spherical space form.*

Proof. Since the Ricci-flow with (r, δ)–cutoff on M gets extinct in finite time (see Chapter 4.3), we can apply Corollary 4.5. However, since M has finite fundamental group, the only components that may occur in the equivariant connected sum are standard actions on S^3 and at most one component with a standard action on a higher spherical space form. Moreover, there can be no loops in the graph associated to the equivariant connected sum, so this graph is a tree. Now Proposition 1.31 yields the assertion. □

Remark 5.2. Note that the assumption of orientability in Theorem 5.1 can be dropped: If there were a non-orientable closed connected 3–manifold M with finite π_1, consider its orientable double cover $\hat{M}$ with the deck action $\hat{\rho}\colon \mathbb{Z}_2 \curvearrowright \hat{M}$. Now Theorem 5.1 implies that $\hat{M}$ is a spherical space form and $\hat{\rho}$ is standard, so also M is a spherical space-form. However, any finite free quotient of an odd-dimensional sphere is orientable e. g. due to the Lefschetz fixed point Theorem, which gives a contradiction.

Due to the uniqueness of spherical structures (Proposition 1.11 and Proposition 1.12) Theorem 5.1 yields directly:

Corollary 5.3. *Let (M, g) be a spherical 3–manifold and $\rho\colon G \curvearrowright M$ a smooth finite group action. Then the action ρ is smoothly conjugate to an isometric action on (M, g).* □

5.2 Actions on $S^2 \times \mathbb{R}$–manifolds

Remark 5.4. The geometry of $S^2 \times \mathbb{R}$ is special in the way that there exist only four closed manifolds which admit this geometry, namely $S^2 \times S^1$, $\mathbb{R}P^3 \sharp\, \mathbb{R}P^3$, $S^2 \tilde{\times} S^1$ and $\mathbb{R}P^2 \times S^1$. The last two of them are non-orientable and all are $\mathbb{Z}_2$–quotients of $S^2 \times S^1$.

Thus a smooth finite group action on a closed $S^2 \times \mathbb{R}$–manifold lifts to a smooth finite group action on $S^2 \times S^1$, and to show that the action is standard on the quotients it suffices to prove this for $S^2 \times S^1$ (compare the end of the proof of Theorem 5.8 for non-orientable hyperbolic manifolds). We conclude that the following Theorem yields an affirmative answer to Thurston's question in the case of $S^2 \times \mathbb{R}$–geometry.

Theorem 5.5. *Let $M = S^2 \times S^1$, and $\rho\colon G \curvearrowright M$ a smooth finite group action. Then the action ρ is standard.*

Proof. By Corollary 4.5 the action ρ on $S^2 \times S^1$ is an equivariant connected sum of standard actions on components diffeomorphic to spherical spaceforms, $S^2 \times S^1$ and $\mathbb{R}P^3 \sharp\, \mathbb{R}P^3$. So there is a $\rho(G)$–invariant family of disjoint embedded spheres S_i^2 in M, such that the equivariant connected sum *decomposition* along these spheres yields standard actions on the mentioned components, i. e. $\rho = \hat{\rho}_{\mathcal{P}}$ where $\hat{\rho}\colon G \curvearrowright \hat{M}$ is a standard action. In fact, since $S^2 \times S^1$ is prime, $\hat{M}$ consists only of S^3's and at most one $S^2 \times S^1$–component (this component exists if and only if all S_i^2 are separating).

Let $\mathcal{P}_1 \subseteq \mathcal{P}$ (respectively $\mathcal{P}_2 \subseteq \mathcal{P}$) be those pairs for which the corresponding sphere S_i^2 is non-separating (respectively separating). Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, each $\mathcal{P}_i$ is ρ–invariant and we have $\rho = \hat{\rho}_{\mathcal{P}} = (\hat{\rho}_{\mathcal{P}_1})_{\mathcal{P}_2}$ by Proposition 1.29.

Since the connected sum along $\mathcal{P}_2$ is topologically trivial, $\hat{M}_{\mathcal{P}_1}$ contains a unique $(S^2 \times S^1)$–component M_0. We first show that $\hat{\rho}_{\mathcal{P}_1}|_{M_0}$ is standard. If $\mathcal{P}_1 = \emptyset$ there is nothing to prove, so assume there are non-separating 2–spheres among the S_i^2. Then the path-component $\Gamma_0 \subseteq \Gamma_{\mathcal{P}_1}$

corresponding to M_0 is a circle (possibly a loop), and the vertices of Γ_0 correspond to 3–spheres S_i^3, $i \in \mathbb{Z}_q$ for some $q \geq 1$. We can number the pairs of $\mathcal{P}_1 = \{(x_i, y_i)\}$ such that $x_i \in S_i^3$ and $y_i \in S_{i+1}^3$. Observe that for the stabilizers of the pairs holds $G_i = G_{i+1}$ (otherwise the graph would branch and could not be a circle), so all G_i are equal. This allows us to choose a spherical $\hat{\rho}|_{M_0}$–invariant metric on the union of the S_i^3 such that x_{i+1} and y_i are antipodal points (recall that $\hat{\rho}$ is standard). Now the $\hat{\rho}|_{M_0}$–invariant singular S^2–foliation by distance spheres to $\{x_i\}$ gives an $\hat{\rho}_{\mathcal{P}_2}|_{M_0}$–invariant smooth S^2–foliation on $M_0 = S^2 \times S^1$ (the equivariant connected sum construction glues distant spheres of x_i onto distance spheres of y_i). This shows that $\hat{\rho}_{\mathcal{P}_1}|_{M_0}$ is standard.

For the equivariant connected sum along $\mathcal{P}_2$ the associated graph is a tree. A unique vertex v_0 of that tree corresponds the component $M_0 \cong S^2 \times S^1$ and all the other components are 3–spheres. So Proposition 1.31 (ii) yields that $\rho \cong \hat{\rho}_{\mathcal{P}_1}|_{M_0}$, which we have seen to be standard above. □

As a direct consequence of Theorem 5.5 we obtain a generalization of [MS86, Theorem 8.1], where F is assumed not to be S^2 or $\mathbb{R}P^2$:

Theorem 5.6. *If F is a compact surface and if $\rho\colon G \curvearrowright F \times I$ is a finite smooth group action which preserves $F \times \partial I$, then ρ is conjugate to an action which preserves the product structure.*

Proof. The proof is exactly the same as the one of [MS86, Theorem 8.1], in view of which we only have to consider the case $F \cong S^2$ or $\mathbb{R}P^2$: We double $F \times I$, i. e. we regard $F \times I$ as $\mathbb{Z}_2$–quotient of $F \times S^1$ by a reflection in the S^1–factor, and consider the lifted action $\hat{\rho}\colon \hat{G} = G \times \mathbb{Z}_2 \curvearrowright F \times S^1$. This action is standard by Theorem 5.5 (respectively Remark 5.4), so the $\hat{\rho}$–invariant $(S^2 \times \mathbb{R})$–metric on $F \times S^1$ induces an invariant product structure on $F \times I$. □

5.3 Actions on hyperbolic manifolds

If M is a manifold which admits a hyperbolic metric, then by Proposition 4.6 the Ricci-flow with surgery does not get extinct in finite time for any initial metric on M. One therefore has to study the long-time behavior of the flow. This is very similar to the behavior of the Ricci-flow in case that there are no singularities, which was analyzed by Hamilton [Ham99].

The following Proposition describes the fact that in the long-time picture the (almost) hyperbolic pieces cover more and more of the manifold. The idea why this is similar in the case of Ricci-flow with surgery is that surgeries only occur in regions with positive scalar curvature, so its effect on the formation of the hyperbolic pieces can be controlled.

Proposition 5.7 ([KL07, 90.1]). *There exist a number $T_0 < \infty$, a non-increasing function $\alpha\colon [T_0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} \alpha(t) = 0$, a (possibly empty) collection $\mathcal{H} = \{(H_1, x_1), \dots, (H_k, x_k)\}$ of complete, connected, pointed, finite-volume hyperbolic 3–manifolds and a family of smooth maps*

$$f_i(t)\colon\ B_{t,i} = B\big(x_i, \tfrac{1}{\alpha(t)}\big) \to \mathcal{M}_t,$$

defined for $t \in [T_0, \infty)$, such that

1. *$f_i(t)$ is an $\alpha(t)$–approximation for all i (with scale t^{-1})*

2. *$f_i(t)$ defines a smooth family of maps which changes slowly with time, i. e. $\big|\dot{f}_i(p,t)\big| < \alpha(t)t^{-\frac{1}{2}}$ for all $p \in B_{t,i}$ and*

3. *the $\alpha(t)$–thick part $M^+(\alpha(t), t)$ (as defined in [KL07, Definition 89.8]) is contained in the image $\bigcup_i f_i(B_{t,i})$ for $t \geq T_0$.* □

In case of a closed manifold M which admits a hyperbolic metric, the long-time picture is even much easier: The collection $\mathcal{H}$ cannot be empty, because this would imply that the thick part gets empty for large t and $\mathcal{M}_t$ would be a graph manifold, which is not possible. On the other hand, M does not contain any incompressible tori, so also $\mathcal{M}_t \cong M_0 \cup \bigcup S_i^3$ does not. Therefore, the hyperbolic pieces H_i cannot have cusps and hence must be closed. It follows that the approximation f_i covers all of H_i for large t, so there is precisely *one* hyperbolic piece $\{H_1\} = \mathcal{H}$. Furthermore, for large t the ball $B(x_1, \frac{1}{\alpha(t)})$ covers H_1 and there is a $T_1 \geq T_0$ such that the approximation f_1 is *onto* a component $(M_0, g(t))$ for $t \geq T_1$. On this component, no further surgery can occur because of negative scalar curvature. After finite time T_2, all other components are extinct, so only a non-singular flow on M_0 remains.

Theorem 5.8. *Let M_0 be a closed, connected hyperbolic 3–manifold and let $\rho_0\colon G \curvearrowright M_0$ be a smooth finite group action. Then the action ρ_0 is standard.*

Proof. First consider the case that M_0 is orientable. We start with a $\rho(G)$–invariant Riemannian metric g_0 on M_0 and consider the equivariant Ricci-flow with surgery for M_0 with initial metric $g(0) = g_0$.

By the observation preceding this theorem, there exists a time T_2, a hyperbolic manifold H and a smooth family of maps $f(t)\colon H \to \mathcal{M}_t$ for $t \geq T_2$ such that $f(t)$ is an $\alpha(t)$–approximation onto $\mathcal{M}_t$, and there are no more surgeries for $t > T_2$. In other words, $\mathcal{M}_t$ converges smoothly to a hyperbolic manifold.

Assume that there where k surgery times before T_2, so M_k is the underlying manifold and the action is $\rho_k\colon G \curvearrowright M_k$ is isometric with respect to $g(t)$ for all $t \geq T_2$. Consider the pulled back action $\hat{\rho}_t := f(t)^*\rho_k\colon G \curvearrowright H_1$ for $t \geq T_2$. It varies smoothly with t, so for each $\gamma \in G$, $\hat{\rho}_t(\gamma)$ stays in a fixed homotopy class for all $t \geq T_2$. In this class there exists a unique isometry by the Mostow rigidity theorem [Mos68], which we denote by $\bar{\rho}(\gamma)$. This defines a isometric action $\bar{\rho}\colon G \curvearrowright H_1$ which is homotopic to $\hat{\rho}_t$ for all $t \geq T_2$.

We claim that $\bar{\rho}$ is also conjugate to $\hat{\rho}_t$ for large t. To deduce this, note that by Arzelà-Ascoli for any γ and any sequence of times $\tau_i \nearrow \infty$, $\tau_i \geq T_2$, the sequence of diffeomorphisms $\hat{\rho}_{\tau_i}(\gamma)$ subconverges. The limit must be an isometry since $\tau_i \to \infty$ and therefore the limit must coincide with $\bar{\rho}(\gamma)$. Because this holds for *any* sequence of times we conclude that $\hat{\rho}_t$ smoothly converges to $\bar{\rho}$. So for t sufficiently large, say $t \geq T_3$, $\hat{\rho}_t$ and $\bar{\rho}$ are conjugate by Theorem 1.22.

Putting the conjugations together, this yields for $t \geq T_3$ that $\bar{\rho} \cong \hat{\rho}_t \cong \rho_k$, where the second conjugation is given by $f(t)$.

Furthermore, ρ_0 is an equivariant connect sum of ρ_k and standard actions on 3–spheres. By Proposition 1.31 (and Remark 1.32), the standard actions are trivial summands for the equivariant connected sum and so $\rho_k\colon G \curvearrowright M_k$ is conjugate to the original action $\rho_0\colon G \curvearrowright M_0$. Hence the original action is conjugate to an isometric action on a hyperbolic manifold and thus ρ_0 is standard.

For the case of M_0 not being orientable, consider an orientable double covering $\check{M}_0 \to M_0$, such that M_0 is the quotient of $\check{M}$ by a smooth involution ι. The action ρ_0 lifts to an action $\check{\rho}_0\colon \check{G} \curvearrowright \check{M}$, where $\check{G}$ is an index two extension of G. If ι_G is the non-trivial element in the kernel of the natural projection $\check{G} \twoheadrightarrow G$, then $\check{\rho}_0(\iota_G) = \iota$.

The above proof of the orientable case yields a $\check{\rho}_0$–invariant hyperbolic metric $\check{g}$ on $\check{M}$. Since $\iota = \check{\rho}_0(\iota_G)$ is an isometry with respect to $\check{g}$, the metric descends to a ρ_0–invariant hyperbolic metric on M_0. □

Again as in the spherical case, uniqueness of hyperbolic structures (as a consequence of Mostow rigidity) gives:

Corollary 5.9. *Let* (M, g) *be a closed hyperbolic 3–manifold and let* $\rho\colon G \curvearrowright M$ *be a smooth finite group action. Then* ρ *is conjugate to an isometric action on* (M, g)*.* □

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