Transversality Results and Computations in Symplectic Field Theory

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Abstract

Although the definition of symplectic field theory suggests that one has to count holomorphic curves in cylindrical manifolds $\mathbb{R}\times V$ equipped with a cylindrical almost complex structure J, it is already well-known from Gromov-Witten theory that, due to the presence of multiply-covered curves, we in general cannot achieve transversality for all moduli spaces even for generic choices of J.

In this thesis we treat the transversality problem of symplectic field theory in two important cases. In the first part of this thesis we are concerned with the rational symplectic field theory of Hamiltonian mapping tori, which is also called the Floer case. For this observe that in the general geometric setup for symplectic field theory, the contact manifolds can be replaced by mapping tori M_ϕ of symplectic manifolds (M,ω_M) with symplectomorphisms ϕ. While the cylindrical contact homology of M_ϕ is given by the Floer homologies of powers of ϕ, the other algebraic invariants of symplectic field theory for M_ϕ provide natural generalizations of symplectic Floer homology. For symplectically aspherical M and Hamiltonian ϕ we study the moduli spaces of rational curves and prove a transversality result, which does not need the polyfold theory by Hofer, Wysocki and Zehnder and allows us to compute the full contact homology of $M_\phi \cong S^1\times M$.

The second part of this thesis is devoted to the branched covers of trivial cylinders over closed Reeb orbits, which are the trivial examples of punctured holomorphic curves studied in rational symplectic field theory. Since all moduli spaces of trivial curves with virtual dimension one cannot be regular, we use obstruction bundles in order to find compact perturbations making the Cauchy-Riemann operator transversal to the zero section and show that the algebraic count of elements in the resulting regular moduli spaces is zero. Once the analytical foundations of symplectic field theory are established, our result implies that the differential in rational symplectic field theory and contact homology is strictly decreasing with respect to the natural action filtration. After introducing additional marked points and differential forms on the target manifold we finally use our result to compute the E^2-page of the corresponding spectral sequence for filtered complexes.

Zusammenfassung

Obwohl es die Definition der symplektischen Feldtheorie nahelegt, dass holomorphe Kurven in zylindrischen Mannigfaltigkeiten $\mathbb{R} \times V$ gezählt werden, die mit einer zylindrischen fast-komplexen Struktur J versehen sind, ist es bereits von der Gromov-Witten-Theorie wohlbekannt, dass man wegen des Vorhandenseins von mehrfach überlagerten Kurven auch für generische Wahlen von J keine Transversalität für alle Modulräume erreichen kann.

In dieser Arbeit behandeln wir das Transversalitätsproblem der symplektischen Feldtheorie in zwei wichtigen Fällen. Im ersten Teil dieser Arbeit beschäftigen wir uns mit der rationalen symplektischen Feldtheorie von Hamiltonischen Abbildungstori, was auch als der Floer-Fall bezeichnet wird. Dafür beobachtet man, dass im verallgemeinerten geometrischen Formalismus der symplektischen Feldtheorie die Kontaktmannigfaltigkeiten durch Abbildungstori M_ϕ von symplektischen Mannigfaltigkeiten (M, ω_M) mit Symplektomorphismen ϕ ersetzt werden können. Während die zylindrische Kontakthomologie von M_ϕ durch die Floer-Homologien der Potenzen von ϕ gegeben ist, bieten die anderen algebraischen Invarianten der symplektischen Feldtheorie von M_ϕ natürliche Verallgemeinerungen der symplektischen Floer-Homologie. Wir untersuchen die Modulräume rationaler Kurven für symplektisch-asphärisches M und Hamiltonisches ϕ und beweisen ein Transversalitätsresultat, welches nicht auf die Polyfold-Theorie von Hofer, Wysocki und Zehnder zurückgreift und uns die Berechnung der vollen Kontakthomologie von $M_\phi \cong S^1 \times M$ erlaubt.

Der zweite Teil dieser Arbeit ist den verzweigten Überlagerungen von trivialen Zylindern über geschlossenen Reeb-Orbiten gewidmet, welche die trivialen Beispielen für holomorphen Kurven sind, die in der rationalen symplektischen Feldtheorie untersucht werden. Da alle Modulräume mit virtueller Dimension eins nicht regulär sein können, benutzen wir Obstruktionsbündel, um kompakte Störungen zu finden, welche den Cauchy-Riemann-Operator transversal zum Nullschnitt machen und zeigen, dass das algebraische Zählen der Elemente in dem sich ergebenen regulären Modulraum Null ergibt. Wenn die analytischen Grundlagen der symplektischen Feldtheorie einmal bewiesen sind, wird unser Resultat zeigen, dass das Differential in der rationalen symplektischen Feldtheorie wie auch der Kontakthomologie strikt absenkend ist bezüglich der natürlichen Aktionsfiltration. Nach dem Einführen zusätzlicher markierter Punkte und Differentialformen auf der Zielmannig-

faltigkeit benutzen wir zu guter Letzt unser Resultat, um die E^2-Seite der zugehörigen Spektralsequenz für filtrierte Komplexe zu berechnen.

Chapter 0

Introduction

0.1 Symplectic field theory

Symplectic field theory (SFT) is a very large project, initiated by Y. Eliashberg, A. Givental and H. Hofer in their paper [EGH], designed to describe in a unified way the theory of pseudoholomorphic curves in symplectic and contact topology. Besides providing a unified view on well-known theories like symplectic Floer homology and Gromov-Witten theory, it shows how to assign algebraic invariants to closed manifolds with a stable Hamiltonian structure.

Following [BEHWZ] and [CM2] a Hamiltonian structure on a closed $(2m-1)$-dimensional manifold V is a closed two-form ω on V, which is maximally nondegenerate in the sense that $\ker\omega = \{v \in TV : \omega(v,\cdot) = 0\}$ is a one-dimensional distribution. Note that here we (and [CM2]) differ slightly from [EKP]. The Hamiltonian structure is required to be stable in the sense that there exists a one-form λ on V such that $\ker\omega \subset \ker d\lambda$ and $\lambda(v) \neq 0$ for all $v \in \ker\omega - \{0\}$. Any stable Hamiltonian structure (ω,λ) defines a symplectic hyperplane distribution $(\xi = \ker\lambda, \omega_\xi)$, where ω_ξ is the restriction of ω, and a vector field R on V by requiring $R \in \ker\omega$ and $\lambda(R) = 1$, which is called the Reeb vector field of the stable Hamiltonian structure. Examples for closed manifolds V with a stable Hamiltonian structure (ω,λ) are contact manifolds, symplectic mapping tori and principal circle bundles over symplectic manifolds ([BEHWZ],[CM2]):

First observe that when λ is a contact form on V, it is easy to check that $(\omega := d\lambda, \lambda)$ is a stable Hamiltonian structure and the symplectic hyperplane distribution agrees with the contact structure. For the other two cases, let (M,ω_M) be a symplectic manifold. Then every principal circle bundle $S^1 \to V \to M$ and every symplectic mapping torus $M \to V \to S^1$, i.e., $V = M_\phi = \mathbb{R}\times M/\{(t,p) \sim (t+1,\phi(p))\}$ for $\phi \in \mathrm{Symp}(M,\omega)$ also carries a stable Hamiltonian structure. For the circle bundle the Hamiltonian structure is given by the pullback $\pi^*\omega$ under the bundle projection and we can choose as one-form λ any S^1-connection form. On the other hand, the stable Hamiltonian structure on the mapping torus $V = M_\phi$ is given by lifting the symplectic form to $\omega \in \Omega^2(M_\phi)$ via the natural flat connection $TV = TS^1 \oplus TM$ and setting $\lambda = dt$ for the natural S^1-coordinate t on M_ϕ. While in the mapping torus case ξ is always integrable, in the circle bundle case the hyperplane distribution ξ may be integrable or non-integrable, even contact.

Symplectic field theory assigns algebraic invariants to closed manifolds V with a stable Hamiltonian structure. The invariants are defined by counting $\underline{J}$-holomorphic curves in $\mathbb{R} \times V$ with finite energy, where the underlying closed Riemann surfaces are explicitly allowed to have punctures, i.e., single points are removed. The almost complex structure $\underline{J}$ on the cylindrical manifold $\mathbb{R} \times V$ is required to be cylindrical in the sense that it is $\mathbb{R}$-independent, links the two natural vector fields on $\mathbb{R} \times V$, namely the Reeb vector field R and the $\mathbb{R}$-direction ∂_s, by $\underline{J}\partial_s = R$, and turns the symplectic hyperplane distribution on V into a complex subbundle of TV, $\xi = TV \cap \underline{J}TV$. It follows that a cylindrical almost complex structure $\underline{J}$ on $\mathbb{R} \times V$ is determined by its restriction $\underline{J}_\xi$ to $\xi \subset TV$, which is required to be ω_ξ-compatible in the sense that $\omega_\xi(\cdot, \underline{J}_\xi \cdot)$ defines a metric on ξ. Note that in [CM2] such almost complex structures $\underline{J}$ are called compatible with the stable Hamiltonian structure and that the set of these almost complex structures is non-empty and contractible. On the other hand, following [BEHWZ], the energy $E(u)$ of a punctured J-holomorphic curve $u = (a, f) : \dot{S} \to \mathbb{R} \times V$ is given by the sum of the λ- and the ω-energy of u,

$$E_\lambda(u) = \sup_A \int_{\dot{S}} \alpha(a)\; da \wedge f^*\lambda, \;\; E_\omega(u) = \int_{\dot{S}} f^*\omega,$$

where A denotes the set of all smooth functions $\alpha : \mathbb{R} \to \mathbb{R}_0^+$ with compact support and L^1-norm equal to one. It follows that $E_\lambda(u), E_\omega(u)$ are nonnegative and, following proposition 5.8 in [BEHWZ], that all punctured J-holomorphic curves with $E(u) < \infty$ are asymptotically cylindrical over a periodic orbit of the Reeb vector field R in the neighborhood of each puncture as long as all periodic orbits are nondegenerate in the sense of [BEHWZ], i.e., one is not an eigenvalue of the linearized return map restricted to the symplectic hyperplane distribution.

While the punctured curves in symplectic field theory may have arbitrary genus and arbitrary numbers of positive and negative punctures, it is shown in [EGH] that there exist algebraic invariants counting only special types of curves: While in rational symplectic field theory one counts punctured curves with genus zero, contact homology is defined by further restricting to punctured spheres with only one positive puncture. Further restricting to spheres with both just one negative and one positive puncture, i.e., cylinders, the resulting algebraic invariant is called cylindrical contact homology. Note however that contact homology and cylindrical contact homology are not always defined. In order to prove the well-definedness of (cylindrical) contact homology it however suffices to show that there are no punctured holomorphic curves where all punctures are negative (or all punctures are positive). While the existence of holomorphic curves without positive punctures can be excluded for all contact manifolds using the maximum principle, which shows that contact homology is well-defined for all contact manifolds, it can be seen from homological reasons that for mapping tori M_ϕ there cannot exist holomorphic curves in $\mathbb{R} \times M_\phi$ carrying just one type of punctures, which shows that in this case both contact homology and cylindrical contact homology are defined.

0.2 Main theorems

The first part of this thesis essentially agrees with the paper [F1], i.e., we are concerned with the moduli spaces of pseudoholomorphic curves studied in rational symplectic field theory for Hamiltonian mapping tori, where the symplectomorphism ϕ is Hamiltonian, i.e., the time-one map of the symplectic flow of a Hamiltonian $H : S^1 \times M \to \mathbb{R}$. In this case the Hamiltonian flow ϕ^H provides us with a natural diffeomorphism $M_\phi \cong S^1 \times M$, so that we can replace M_ϕ by $S^1 \times M$ equipped with the pullback stable Hamiltonian structure (ω^H, λ^H) on $S^1 \times M$ given by $\omega^H = \omega + dH \wedge dt$, $\lambda^H = dt$ with symplectic bundle $\xi^H = TM$ and Reeb vector field $R^H = \partial_t + X^H_t$, where X^H_t is the symplectic gradient of $H_t = H(t, \cdot)$. In [EKP] this is also called the Floer case. Furthermore $(\mathbb{R} \times M_\phi, \underline{J})$ can be identified with $(\mathbb{R} \times S^1 \times M, \underline{J}^H)$ equipped with the pullback cylindrical almost complex structure, which is nonstandard in the sense that the splitting $T(\mathbb{R} \times S^1 \times M) = \mathbb{R}^2 \oplus TM$ is not $\underline{J}^H$-complex.

Observe that the closed orbits of the Reeb vector field R^H on $S^1 \times M$ have integer periods, where the set of closed orbits of period $T \in \mathbb{N}$ is naturally identified with the T-periodic orbits of X^H on M. It follows that the chain complex $(\mathfrak{A}, \partial)$ for contact homology naturally splits, $\mathfrak{A} = \bigoplus_{T \in \mathbb{N}} \mathfrak{A}^T$, where $\mathfrak{A}^T$ is generated by all monomials $q_{(x_1,T_1)} \dots q_{(x_n,T_n)}$, with T_i-periodic orbits (x_i, T_i) and $T_1 + \dots + T_n = T$, and it is easily seen from homological reasons that this splitting is respected by the differential ∂. Furthermore, given two different Hamiltonian functions $H_1, H_2 : S^1 \times M \to \mathbb{R}$ the corresponding chain map $\Phi : (\mathfrak{A}_1, \partial_1) \to (\mathfrak{A}_2, \partial_2)$, defined as in [EGH] by counting holomorphic curves in $\mathbb{R} \times S^1 \times M$ equipped with a non-cylindrical almost complex structure $\underline{J}^{\tilde{H}}$, which itself can be defined using a homotopy $\tilde{H} : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ from H_1 to H_2, also respects the splittings $\mathfrak{A}_1 = \bigoplus_{T \in \mathbb{N}} \mathfrak{A}_1^T$, $\mathfrak{A}_2 = \bigoplus_{T \in \mathbb{N}} \mathfrak{A}_2^T$.

For our computation of the contact homology we choose Hamiltonian functions $H : S^1 \times M \to \mathbb{R}$, which are S^1-independent and so small in the C^2-norm such that in particular all closed orbits of the Reeb vector field for any given period $T \in \mathbb{N}$ are critical points of $H : M \to \mathbb{R}$. Furthermore we assume that $H : M \to \mathbb{R}$ is Morse, which in turn implies that all periodic orbits are nondegenerate in the sense of [BEHWZ], i.e., one is not an eigenvalue of the linearized return map restricted to the symplectic hyperplane distribution. We achieve this by rescaling any given Morse function on M, where the scaling factor however has to depend on the period $T \in \mathbb{N}$, which in turn implies that we have to compute the contact homology using an infinite sequence of different Hamiltonian functions. Making use of the splitting of the chain complex for contact homology into chain complexes for different periods $T \in \mathbb{N}$ and the fact that the chain map Φ introduced above should lead to an isomorphism on the level of homology once the analytical program for defining symplectic field theory is completed, we can formulate our result using a direct limit as follows:

Let $T_N \in \mathbb{N}$ be a sequence of (maximal) periods with $T_N \leq T_{N+1}$ and $\lim_{N\to\infty} T_N = \infty$ and let $H_N : S^1 \times M \to \mathbb{R}$, $N \in \mathbb{N}$ be a sequence of Hamiltonians with corresponding chain complexes $(\mathfrak{A}_N, \partial_N)$, $N \in \mathbb{N}$. Assume that for every $N \in \mathbb{N}$ we have defined a chain map $\Phi_N : (\mathfrak{A}_N, \partial_N) \to (\mathfrak{A}_{N+1}, \partial_{N+1})$ using a homotopy $\tilde{H}_N : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ interpolating between H_N and H_{N+1}, which by the above arguments restricts to a map from $\mathfrak{A}_N^T$ to $\mathfrak{A}_{N+1}^T$ for every $T \in \mathbb{N}$. Setting

$$HC_*^{\leq T_N}(S^1 \times M, \underline{J}^{H_N}) = H_*(\mathfrak{A}_N^{\leq T_N}, \partial_N) = \bigoplus_{T \leq T_N} H_*(\mathfrak{A}_N^T, \partial_N)$$

we obtain a directed system $(C_N, \Phi_{N,M})$ with $C_N = HC_*^{\leq T_N}(S^1 \times M, \underline{J}^{H_N})$ and $\Phi_{N,M} = \Phi_N \circ \Phi_{N+1} \circ ... \circ \Phi_{M-1} \circ \Phi_M$ for $N \leq M$.

Main Theorem A: *Let (M, ω) be a closed symplectic manifold, which is symplectically aspherical, $\langle c_1(TM), \pi_2(M)\rangle = 0 = \langle [\omega], \pi_2(M)\rangle$. Then for every S^1-independent Hamiltonian $H : M \to \mathbb{R}$, which is sufficiently small in the C^2-norm and Morse, there is an isomorphism*

$$\lim_{N\to\infty} HC_*^{\leq 2^N}(S^1 \times M, \underline{J}^{H/2^N}) \;\;\cong\;\; \mathfrak{S}\big(\bigoplus_{\mathbb{N}} H_{*-2}(M, \mathbb{Q})\big) \otimes \mathbb{Q}[H_2(M)],$$

where $\mathfrak{S}$ is the graded symmetric algebra functor.

In order to understand the relevance of this result note that our result implies, once the analytical foundations for symplectic field theory are established and hence the rational symplectic field theory for $(S^1 \times M, \omega^H, \lambda^H)$ is defined for all choices of Hamiltonians $H : S^1 \times M \to \mathbb{R}$, that the contact homology of $(S^1 \times M, \omega^H, \lambda^H)$ with symplectically aspherical M is isomorphic as a graded algebra to the tensor product of the coefficient ring with the graded symmetric algebra generated by countably infinitively many copies of the singular homology of M with rational coefficients (with degree shift) for *any* chosen $H : S^1 \times M \to \mathbb{R}$. Indeed, assuming that the analytical program for defining symplectic field theory is carried out and, in particular, proves that $\Phi_N : H_*(\mathfrak{A}_N^T, \partial_N) \to H_*(\mathfrak{A}_{N+1}^T, \partial_{N+1})$ is an isomorphism for every $N \in \mathbb{N}$ and $T \in \mathbb{N}$, it follows that the direct limit $\lim_{N\to\infty} C_N = \lim_{N\to\infty} HC_*^{\leq T_N}(S^1 \times M, \underline{J}^{H_N})$ is isomorphic to $HC_*(S^1 \times M, \underline{J}^H)$ for any chosen $H : S^1 \times M \to \mathbb{R}$.

The second part of this thesis is made up of the results in [F2], where we studied the trivial examples of punctured holomorphic curves in rational symplectic field theory, where we again assume that the stable Hamiltonian structure is generic in the sense that all periodic orbits are nondegenerate in the sense of [BEHWZ], i.e., one is not an eigenvalue of the linearized return map restricted to the symplectic hyperplane distribution. While the contribution of the trivial curves in cylindrical contact homology, namely trivial cylinders staying over one orbit, is still immediately clear, observe that the trivial examples of punctured holomorphic curves studied in general symplectic field theory are not only these trivial cylinders but also their branched covers. We show that these branched covers are in fact the reason why transversality for generic J in general fails in symplectic field theory and whose contribution to the theory is therefore hard to determine. Indeed it is easy to show that in every case where these trivial curves would contribute to the algebraic invariants by index reasons, transversality for the Cauchy-Riemann operator can never be satisfied, so that one has to perturb the Cauchy-Riemann operator appropriately and count elements in the resulting regular moduli spaces. Here it is important that the perturbation chosen for different moduli spaces are compatible with compactness and gluing in symplectic field theory. In order to obtain these compact perturbations we study sections in the cokernel bundle over the compactified moduli space, i.e., we generalize the technique of computing Euler numbers of obstruction bundles for determining the contribution of nonregular moduli spaces from Gromov-Witten theory to the case of moduli spaces with codimension one boundary, as appearing in the study of pseudo-holomorphic curves with punctures and/or boundary in (Lagrangian) Floer homology, (relative) symplectic field theory, the work by Fukaya-Oh-Ohta-Ono and Cornea-Lalonde's cluster homology. With this we can show:

Main Theorem B: *We can choose compact perturbations of the Cauchy-Riemann operator, which make all moduli spaces of trivial curves regular in a way compatible with compactness and gluing, such that the algebraic counts of elements in all resulting zero-dimensional regular moduli spaces (modulo $\mathbb{R}$-shift) are zero.*

For the significance of this result for symplectic field theory we claim that, once the analytical foundations of symplectic field theory are established, our result proves that the differential in contact homology and rational symplectic field theory is strictly decreasing with respect to the natural action filtration. In particular, the statement of the theorem should be true for any choice of coherent compact perturbations chosen to make the moduli spaces of symplectic field theory regular. We introduce the rational symplectic field theory of a single closed Reeb orbit and use our result to compute the underlying generating function. Including the even more general picture outlined in [EGH] needed to view Gromov-Witten theory as a part of symplectic field theory, we further prove what we get when we additionally introduce a string of closed differential forms $\Theta = (\theta_1, ..., \theta_N) \in (\Omega^*(V))^N$. Here we prove by simple means (but using our main result) that the generating function only sees the homology class represented by the underlying closed Reeb orbit. It follows that the generating function is in general no longer equal to zero when a string of differential forms is chosen, which implies that the differential in rational symplectic field theory and contact homology is no longer strictly decreasing with respect to the action filtration. However, we follow [FOOO] in employing the spectral sequence for filtered complexes to prove the following important consequence of our main theorem B, which we however only prove for contact manifolds and symplectic mapping tori:

Corollary: *Consider a contact manifold or a symplectic mapping torus. Then there exists a spectral sequence* (E^r, d^r) *computing the contact homology,* $E^\infty = H_*(\mathfrak{A}, \partial)$*, where the* E^2*-page is given by the graded commutative algebra* $\mathfrak{A}_0$ *which, in contrast to* $\mathfrak{A}$*, is now only freely generated by the formal variables* q_γ *with* $\int_\gamma \theta_i = 0$ *for all* $i = 1, ..., N$*.*

Note that this in turn provides us with an easy method to show the vanishing of contact homology:

Corollary: *Assume that the string of closed differential forms is chosen in such a way that it indeed generates the cohomology of the target manifold (and that none of the corresponding formal variables is set to zero). Then the contact homology vanishes if there are no null-homologous Reeb orbits, like in the case of symplectic mapping tori and unit cotangent bundle of tori.*

Chapter 1

Rational SFT in the Floer case

1.0 Summary

In this first chapter we are concerned with the moduli spaces of pseudoholomorphic curves studied in rational symplectic field theory for Hamiltonian mapping tori, where the symplectomorphism ϕ is Hamiltonian, i.e., the time-one map of the flow of a Hamiltonian $H: S^1 \times M \to \mathbb{R}$. More precisely, we prove main theorem A from the introduction:

Main Theorem A: *Let (M,ω) be a closed symplectic manifold, which is symplectically aspherical, $\langle c_1(TM), \pi_2(M)\rangle = 0 = \langle [\omega], \pi_2(M)\rangle$. Then for every S^1-independent Hamiltonian $H: M \to \mathbb{R}$, which is sufficiently small in the C^2-norm and Morse, there is an isomorphism*

$$\lim_{N\to\infty} HC_*^{\leq 2^N}(S^1 \times M, \underline{J}^{H/2^N}) \;\cong\; \mathfrak{S}\big(\bigoplus_{\mathbb{N}} H_{*-2}(M,\mathbb{Q})\big) \otimes \mathbb{Q}[H_2(M)],$$

where $\mathfrak{S}$ is the graded symmetric algebra functor.

As we outlined above, note that our result implies that, once the analytical foundations for symplectic field theory are established, the contact homology for $(S^1 \times M, \omega^H, \lambda^H)$ with symplectically aspherical M is isomorphic as a graded algebra to the tensor product of the coefficient ring with the graded symmetric algebra generated by countably infinitively many copies of the singular homology of M with rational coefficients.

For the proof we show that for S^1-independent C^2-small Hamiltonians and a given maximal period for the periodic orbits we can naturally enlarge the class of cylindrical almost complex structures $\underline{J}^H$ on $\mathbb{R} \times S^1 \times M$, so that we achieve transversality for all moduli spaces and additionally have an S^1-symmetry on all moduli spaces of curves, where the underlying punctured spheres are stable. Since non-constant holomorphic spheres and holomorphic planes do not exist, it follows for every chosen maximal period T that the subcomplex of the contact homology, which is generated by orbits of period $\leq T$, can be computed by only counting holomorphic cylinders, that is, Floer trajectories for a Hamiltonian symplectomorphism on M.

The cylindrical almost complex structure $\underline{J}^H$ on $\mathbb{R} \times S^1 \times M$ is specified by the choice of an S^1-family of almost complex structures J_t on M and an S^1-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$. In order to get an S^1-symmetry on moduli spaces of curves with more than three punctures, we restrict ourselves to almost complex structures J_t and Hamiltonians H_t, which are independent of $t \in S^1$. We achieve transversality for all moduli spaces by considering domain-dependent Hamiltonian perturbations. This means that, for defining the Cauchy-Riemann operator for curves, we allow the Hamiltonian to depend explicitly on points on the punctured sphere underlying the curve whenever the punctured sphere is stable, i.e., there are no nontrivial automorphisms. Here we follow the ideas in [CM1] in order to define domain-dependent almost complex structures, which vary smoothly with the positions of the punctures. In [CM1] the authors use this method to achieve transversality for moduli spaces in Gromov-Witten theory. Besides that we make the Hamiltonian and not the almost complex structure on M domain-dependent in order to achieve transversality also for the trivial curves, i.e., branched covers of trivial cylinders (see the second chapter), observe that in contrast to the Gromov-Witten case we now have to make coherent choices for the different moduli spaces simultaneously, i.e., the different Hamiltonian perturbations must be compatible with gluing of curves in rational symplectic field theory. We use the absence of holomorphic disks to present an easy algorithm for defining these coherent choices and finally show that the resulting class of perturbations is indeed large enough to achieve transversality for all moduli spaces of curves with three or more punctures.

For the cylindrical moduli spaces the Hamiltonian perturbation is domain-independent, and it is known from Floer theory that in general we must allow H to depend explicitly on $t \in S^1$ to achieve nondegeneracy of the periodic orbits and transversality for the moduli spaces of Floer trajectories. However, the gluing compatibility requires that also the Hamiltonian perturbation for the cylindrical moduli spaces is independent of $t \in S^1$. The important observation is now that we can indeed solve this problem by considering Hamiltonians H, which are so small in the C^2-norm that all orbits up to given maximal period T are critical points of H and all cylinders between these orbits correspond to gradient flow lines between the underlying critical points. Choosing H and J additionally so that the resulting pair of H and the metric $\omega(\cdot, J\cdot)$ on M is Morse-Smale, it follows that all periodic orbits up to the maximal period are nondegenerate and we achieve transversality for all corresponding cylindrical moduli spaces.

We emphasize that it is in fact the gluing-compatibility of the perturbations for the moduli spaces, which forces us to use S^1-independent Hamiltonian perturbations for cylindrical moduli spaces, although we are actually looking for an S^1-symmetry on the moduli spaces of curves with three or more punctures. Note that in order to achieve transversality for moduli spaces of cylinders one could alternatively introduce asymptotic markers at the punctures in order to fix S^1-coordinates on the cylinders. However, since the asymptotic markers are required to be mapped to marked points on the periodic orbits, the S^1-symmetry on moduli spaces of stable curves gets destroyed.

To any monomial in the chain algebra underlying contact homology one can assign a total period given by the sum of the periods of the occuring orbits. For mapping tori it follows from homological reasons that the differential respects this splitting of the algebra into subspaces of elements with the same total period. Since our statements only hold up to a maximal period for the asymptotic orbits, we cannot use the given coherent Hamiltonian perturbation to compute the full contact homology, but we must rescale the Hamiltonian for the cylindrical moduli spaces, which clearly affects the Hamiltonian perturbations for all punctured spheres. To this end we construct chain maps between the differential algebras for the different coherent Hamiltonian perturbations which are defined by counting holomorphic curves in an almost complex manifold with cylindrical ends. We prove by the same methods as above that we only have to count trivial gradient flow lines, which shows that all chain maps are just the identity when the total period is small enough.

This first chapter is organized as follows:

While we prove in 1.1.1 all the fundamental results about pseudoholomorphic curves in Hamiltonian mapping tori, we show in subsection 1.1.2 how we get an S^1-symmetry on all moduli spaces of domain-stable curves, but still have nondegeneracy for the closed orbits and transversality for all moduli spaces. We collect all the important results about the moduli spaces in theorem 1.1.6. Recall that we achieve the latter by combining the relation between Morse homology and symplectic Floer homology with the introduction of domain-dependent cylindrical almost complex structures. After recalling the definition of the Deligne-Mumford space of stable punctured spheres in 1.2.1, we define the underlying domain-dependent Hamiltonian perturbations in 1.2.2 and prove in 1.2.3 that the construction is compatible with the SFT compactness theorem. After describing in detail the neccessary Banach manifold setup for our Fredholm problems in 1.3.1, we prove in 1.3.2 the fundamental transversality result for the Cauchy-Riemann operator. Since all our results only hold up to a maximal period for the asymptotic orbits, i.e., we have to rescale our Hamiltonian perturbation during our computation of contact homology, we generalize all our previous results to homotopies of Hamiltonian perturbations in 1.4.1 and 1.4.2. After describing the chain complex underlying contact homology in 1.5.1, we prove the main theorem A using our previous results about moduli spaces of holomorphic curves in $\mathbb{R} \times S^1 \times M$.

1.1 Moduli spaces

1.1.1 Holomorphic curves in $\mathbb{R} \times S^1 \times M$

Let (M, ω) be a closed symplectic manifold and let ϕ be a symplectomorphism on it. As already explained in the introduction, the corresponding mapping torus $M_\phi = \mathbb{R} \times M/\{(t,p) \sim (t+1, \phi(p))\}$ carries a natural stable Hamiltonian structure (ω, λ) given by lifting the symplectic form ω to a two-form on M_ϕ via the flat connection $TM_\phi = TS^1 \oplus TM$ and setting $\lambda = dt$. It follows that the corresponding symplectic vector bundle $\xi = \ker \lambda$ is given by TM and the Reeb vector field R agrees with the S^1-direction ∂_t on M_ϕ. In this paper we restrict ourselves to the case where (M, ω) is symplectically aspherical,

$$\langle [\omega], \pi_2(M) \rangle = 0 = \langle c_1(TM), \pi_2(M) \rangle$$

and ϕ is Hamiltonian, i.e., the time-one map of the flow of a Hamiltonian $H : S^1 \times M \to \mathbb{R}$. In this case observe that the Hamiltonian flow ϕ^H provides us with the natural diffeomorphism

$$\Phi : S^1 \times M \xrightarrow{\cong} M_\phi,\ (t,p) \mapsto (t, \phi^H(t,p)),$$

so that we can replace M_ϕ by $S^1 \times M$ equipped with the pullback stable Hamiltonian structure.

Proposition 1.1.1: *The pullback stable Hamiltonian structure (ω^H, λ^H) on $S^1 \times M$ is given by*

$$\omega^H \;=\; \omega + dH \wedge dt, \quad \lambda^H \;=\; dt$$

with symplectic bundle ξ^H and Reeb vector field R^H given by

$$\xi^H \;=\; TM, \quad R^H \;=\; \partial_t + X^H_t,$$

where X^H_t is the symplectic gradient of $H_t = H(t, \cdot)$.

Proof: Using

$$d\Phi = (\mathbf{1}, X^H_t \otimes dt + d\Phi^H_t) : TS^1 \oplus TM \to TS^1 \oplus TM$$

we compute for $v_1 = (v_{11}, v_{12}), v_2 = (v_{21}, v_{22}) \in TS^1 \oplus TM$,

$$\begin{aligned}
&\omega^H(v_1, v_2) = \omega(d\Phi(v_1), d\Phi(v_2)) \\
&= \omega((X^H_t \otimes dt)(v_{11}) + d\Phi^H_t(v_{12}), (X^H_t \otimes dt)(v_{21}) + d\Phi^H_t(v_{22})) \\
&= \omega(X^H_t, X^H_t) dt(v_{11}) dt(v_{21}) + \omega(d\Phi^H_t(v_{12}), d\Phi^H_t(v_{22})) \\
&+ \omega(X^H_t, d\Phi^H_t(v_{22})) dt(v_{11}) + \omega(d\Phi^H_t(v_{12}), X^H_t) dt(v_{21}) \\
&= \omega(v_{12}, v_{22}) + \omega(d\Phi^H_t(v_{12}), X^H_t) dt(v_{21}) - \omega(d\Phi^H_t(v_{22}), X^H_t) dt(v_{11}) \\
&= \omega(v_1, v_2) + (dH \wedge dt)(v_1, v_2)
\end{aligned}$$

and $\lambda^H = \lambda \circ d\Phi = dt$. On the other hand, it directly follows that $\xi^H = TM$, while $R^H = \partial_t - X_t^H$ spans the kernel of ω^H,

$$\begin{aligned}\omega^H(\cdot, R^H) &= \omega(\cdot, \partial_t - X_t^H) + dH \cdot dt(\partial_t + X_t^H) - dH(\partial_t + X_t^H) \cdot dt \\ &= -\omega(\cdot, X_t^H) + dH = 0\end{aligned}$$

with $\lambda^H(R^H) = dt(\partial_t - X_t^H) = 1$. $\square$

As in the introduction we consider an almost complex structure $\underline{J}$ on the cylindrical manifold $\mathbb{R} \times S^1 \times M$, which is required to be cylindrical in the sense that it is $\mathbb{R}$-independent, links the Reeb vector field R^H and the $\mathbb{R}$-direction ∂_s, by $\underline{J}\partial_s = R^H = \partial_t + X_t^H$ and turns the symplectic hyperplane distribution $\xi^H = TM$ into a complex subbundle of $T(S^1 \times M)$. It follows that $\underline{J}$ on $\mathbb{R} \times S^1 \times M$ is determined by its restriction to $\xi^H = TM$, which is required to be ω_{ξ^H}-compatible, so that $\underline{J}$ is determined by the S^1-dependent Hamiltonian H_t and an S^1-family of ω-compatible almost complex structures J_t on the symplectic manifold (M, ω).

Let us recall the definition of moduli spaces of holomorphic curves studied in rational SFT in the general setup. Let (V, ω, λ) be a closed manifold with stable Hamiltonian structure with symplectic hyperplane distribution ξ and Reeb vector field R and let $\underline{J}$ be a compatible cylindrical almost complex structure on $\mathbb{R} \times V$. Let P^+, P^- be two ordered sets of closed orbits γ of the Reeb vector field R on V, i.e., $\gamma : \mathbb{R} \to V$, $\gamma(t + T) = \gamma(t)$, $\dot\gamma = R$, where $T > 0$ denotes the period of γ. Then the (parametrized) moduli space $\mathcal{M}^0(V; P^+, P^-, \underline{J})$ consists of tuples $(F, (z_k^\pm))$, where $\{z_1^\pm, ..., z_{n^\pm}^\pm\}$ are two disjoint ordered sets of points on $\mathbb{CP}^1$, which are called positive and negative punctures, respectively. The map $F : \dot S \to \mathbb{R} \times V$ starting from the punctured Riemann surface $\dot S = \mathbb{CP}^1 - \{(z_k^\pm)\}$ is required to satisfy the Cauchy-Riemann equation

$$\bar\partial_{\underline{J}} F = dF + \underline{J}(F) \cdot dF \cdot i = 0$$

with the complex structure i on $\mathbb{CP}^1$. Assuming we have chosen cylindrical coordinates $\psi_k^\pm : \mathbb{R}^\pm \times S^1 \to \dot S$ around each puncture $z_k^\pm$ in the sense that $\psi_k^\pm(\pm\infty, t) = z_k^\pm$, the map F is additionally required to show for all $k = 1, ..., n^\pm$ the asymptotic behaviour

$$\lim_{s \to \pm\infty} (F \circ \psi_k^\pm)(s, t + t_0) = (\pm\infty, \gamma_k^\pm(T_k^\pm t))$$

with some $t_0 \in S^1$ and the orbits $\gamma_k^\pm \in P^\pm$, where $T_k^\pm > 0$ denotes period of $\gamma_k^\pm$. Observe that the group $\mathrm{Aut}(\mathbb{CP}^1)$ of Moebius transformations acts on elements in $\mathcal{M}^0(V; P^+, P^-, \underline{J})$ in an obvious way,

$$\varphi.(F, (z_k^\pm)) = (F \circ \varphi^{-1}, \varphi(z_k^\pm)), \quad \varphi \in \mathrm{Aut}(\mathbb{CP}^1),$$

and we obtain the moduli space $\mathcal{M}(V; P^+, P^-, \underline{J})$ studied in symplectic field theory by quotiening out this action.

It remains to identify the occuring objects in our special case. First, it follows that all closed orbits γ of the vector field $R^H = \partial_t - X^H_t$ on $S^1 \times M$ are of the form

$$\gamma(t) = (t + t_0, x(t)),$$

and therefore have natural numbers $T \in \mathbb{N}$, i.e., the winding number around the S^1-factor, as periods. Since we study closed Reeb orbits up to reparametrization, we can set $t_0 = 0$, so that γ can be identified with $x : \mathbb{R}/T\mathbb{Z} \to M$, which is a T-periodic orbit of the Hamiltonian vector field,

$$\dot{x}(t) = X^H_t(x(t)).$$

Hence we will in the following write $\gamma = (x, T)$, where $T \in \mathbb{N}$ is the period and x is a T-periodic orbit of the Hamiltonian H. We denote the set of T-periodic orbits of the Reeb vector field R^H on $S^1 \times M$ by $P(H, T)$.

For the moduli spaces of curves observe that in $\mathbb{R} \times S^1 \times M$ we can naturally write the holomorphic map F as a product,

$$F = (h, u) : \dot{S} \to (\mathbb{R} \times S^1) \times M\ .$$

Proposition 1.1.2: *$F : \dot{S} \to \mathbb{R} \times S^1 \times M$ is $\underline{J}$-holomorphic precisely when $h = (h_1, h_2) : \dot{S} \to \mathbb{R} \times S^1$ is holomorphic and $u : \dot{S} \to M$ satisfies the h-dependent perturbed Cauchy-Riemann equation of Floer type,*

$$\begin{aligned}\bar{\partial}_{J,H,h} u &= \Lambda^{0,1}(du + X^H(h_2, u) \otimes dh_2) \\ &= du + X^H(h_2, u) \otimes dh_2 + J(h_2, u) \cdot (du + X^H(h_2, u) \otimes dh_2) \cdot i.\end{aligned}$$

Proof: Observing that $\underline{J}(t, p) : T(\mathbb{R} \times S^1) \oplus TM \to T(\mathbb{R} \times S^1) \oplus TM$ is given by

$$\underline{J}(t, p) = \begin{pmatrix} i & 0 \\ \Delta(t, p) & J_t(p) \end{pmatrix}$$

with $\Delta(t, p) = -X^H_t(p) \otimes ds + J_t(p) X^H_t(p) \otimes dt$ we compute

$$\begin{aligned} & (dh, du) + \underline{J}(h, u) \cdot (dh, du) \cdot i \\ = \; & (dh + i \cdot dh \cdot i, \\ & du + (J(h_2, u) \cdot du - X^H(h_2, u) \otimes dh_1 + J(h_2, u) X^H(h_2, u) \otimes dh_2) \cdot i) \\ = \; & (\bar{\partial} h, du - X^H(h_2, u) \otimes dh_1 \cdot i + J(h_2, u) \cdot (du + X^H(h_2, u) \otimes dh_2) \cdot i).\end{aligned}$$

Finally observe that $dh_1 \cdot i = -dh_2$ if $\bar{\partial} h = 0$. $\square$

Recalling that our orbit sets are given by $P^\pm = \{(x^\pm_1, T^\pm_1), ..., (x^\pm_{n^\pm}, T^\pm_{n^\pm})\}$, we use the rigidity of holomorphic maps to prove the following statement about the map component $h : \dot{S} \to \mathbb{R} \times S^1$. Let $T^\pm = T^\pm_1 + ... + T^\pm_{n^\pm}$ denote the total period above and below, respectively.

Lemma 1.1.3: *The map $h = (h_1, h_2)$ exists if and only if $T^+ = T^-$ and is unique up a shift $(s_0, t_0) \in \mathbb{R} \times S^1$,*

$$h(z) = h^0(z) + (s_0, t_0)$$

for some fixed map $h^0 = (h_1^0, h_2^0)$. In particular, every holomorphic cylinder has a positive and a negative puncture, there are no holomorphic planes and all holomorphic spheres are constant.

Proof: The asymptotic behavior of the map F near the punctures implies that

$$h \circ \psi_k(s, t + t_0) \overset{s \to \pm\infty}{\longrightarrow} (\pm\infty, T_k t)$$

with some $t_0 \in S^1$. Identifying $\mathbb{R} \times S^1 \cong \mathbb{CP}^1 - \{0, \infty\}$, it follows that h extends to a meromorphic function h on $\mathbb{CP}^1$ with $z_1^+, ..., z_{n^+}^+$ poles of order $T_1^+, ..., T_{n^+}^+$ and $z_1^-, ..., z_{n^-}^-$ zeros of order $T_1^-, ..., T_{n^-}^-$. Since the zeroth Picard group of $\mathbb{CP}^1$ is trivial, i.e., every divisor of degree zero is a principal divisor, we get that such meromorphic functions exist precisely when $T^+ = T^-$. On the other hand it follows from Liouville's theorem that they are uniquely determined up to a nonzero multiplicative factor, i.e., $h = a \cdot h^0$ with $a \in \mathbb{C}^* \cong \mathbb{R} \times S^1$ for some fixed $h_0 : \mathbb{CP}^1 \to \mathbb{CP}^1$. For every $\underline{J}^H$-holomorphic sphere (h, u) observe that h is constant, $h = (s_0, t_0)$, and therefore u is a J_{t_0}-holomorphic sphere in M, which must be constant by $\langle [\omega], \pi_2(M) \rangle = 0$. □

Note that the lemma also holds when ϕ is no longer Hamiltonian when we define $h = \pi \circ F$ using the holomorphic bundle projection $\pi : \mathbb{R} \times M_\phi \to \mathbb{R} \times S^1$.

It follows that we only have to study punctured $\underline{J}^H$-holomorphic curves $(h, u) : \dot{S} \to \mathbb{R} \times S^1 \times M$, $\dot{S} = \mathbb{CP}^1 - \{(z_k^\pm)\}$ with two or more punctures, where it remains to understand the map u. Note that by proposition 1.1.2 the perturbed Cauchy-Riemann equation for u depends on the S^1-component $h_2 = h_2^0 + t_0$ of the map h. Starting with the case of two punctures, we make precise the well-known connection between symplectic Floer homology and symplectic field theory for Hamiltonian mapping tori:

Proposition 1.1.4: *The $\underline{J}^H$-holomorphic cylinders connecting the R^H-orbits (x^+, T) and (x^-, T) in $\mathbb{R} \times S^1 \times M$ correspond to the Floer connecting orbits in M between the one-periodic orbits $x^+(T\cdot)$ and $x^-(T\cdot)$ of the Hamiltonian $H_T(t, \cdot) = T \cdot H(Tt, \cdot)$ and the family $J_T(t, \cdot) = J(Tt, \cdot)$ of ω-compatible almost complex structures.*

Proof: When $n = 2$, i.e., $\underline{z} = (z^-, z^+)$, we find an automorphism $\varphi \in \mathrm{Aut}(\mathbb{CP}^1)$ with $\varphi(z^-) = 0$, $\varphi(z^+) = \infty$. Since in the moduli space two elements are considered equal when they agree up to an automorphism of the domain, we can assume that $\underline{z} = (0, \infty)$. It follows from lemma 1.1.3. that $h : \mathbb{CP}^1 - \{0, \infty\} \cong \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is of the form

$$h(s, t) = (Ts + s_0, Tt + t_0)$$

with $T = T^+ + T^-$. We can assume that h is given by $h(s,t) = (Ts, Tt)$ after composing with the automorphism $\varphi(s,t) = (s - s_0/T, t - t_0/T)$ of $\mathbb{R} \times S^1$. Now the claim follows from the fact that the Cauchy-Riemann equation for $u : \mathbb{R} \times S^1 \to M$ reads as

$$\bar{\partial}_{J,H} u \cdot \partial_s = \partial_s u + J(Tt, u) \cdot (\partial_t u + T \cdot X^H(Tt, u)) = 0,$$

with $T \cdot X^H = X^{T \cdot H}$. □

1.1.2 S^1-symmetry, nondegeneracy and transversality

For understanding the curves with more than two punctures, observe that in these cases the underlying punctured Riemann spheres $\dot{S}$ are stable, so that every automorphism φ of $\dot{S}$ is the identity. While this implies that different maps $h = h^0 + (s_0, t_0)$ give different elements in the moduli space, the main problem is that the solutions for u moreover depend on the S^1-component $h_2 = h_2^0 + t_0$ of the chosen map h, that is, the S^1-parameter t_0.

Instead of studying how the solution spaces for u vary with $t_0 \in S^1$, it is natural to restrict to special situations when the solution spaces are t_0-independent. Moreover, when this can be arranged so that all asymptotic orbits are nondegenerate and we can achieve transversality for the moduli spaces, we can use the resulting S^1-symmetry on the moduli spaces to show that they do not contribute to the algebraic invariants in rational symplectic field theory.

It is easily seen that the Cauchy-Riemann equation is independent of $t_0 \in S^1$ when both the family of almost complex structures $J(t, \cdot)$ and the Hamiltonian $H(t, \cdot)$ are independent of $t \in S^1$. Hence for the following we will always assume that

$$J(t, \cdot) \equiv J, \quad H(t, \cdot) \equiv H.$$

and it remains to address the problem of nondegeneracy and transversality.

It is well-known from symplectic Floer homology that we can achieve that all one-periodic orbits $(x, 1) \in P(S^1 \times M, H)$ are nondegenerate by choosing H to be a time-independent Morse function $H : M \to \mathbb{R}$ with a sufficiently small C^2-norm, so that, in particular, only the one-periodic orbits of H are the critical points of H. While this sounds promising to solve the first of our two problems, note that in contrast to symplectic Floer homology we do not only study curves which are asymptotically cylindrical to one-periodic orbits $(x, 1)$ but allow periodic orbits (x, T) of arbitrary period $T \in \mathbb{N}$. Now the problem is that the T-periodic orbits of H are in natural correspondence with one-periodic orbits of the Hamiltonian $T \cdot H$, while $T \cdot H$ need no longer be C^2-small enough. In order to solve this problem, we fix a maximal period $T = 2^N$ and replace the original Hamiltonian H by $H/2^N$, so that all orbits up to the maximal period 2^N are nondegerate, in particular, critical points of $H/2^N$, i.e., of H.

So it remains the problem of transversality. Although the definition of the algebraic invariants of symplectic field theory suggests that all we have to do is counting true $\underline{J}^H$-holomorphic curves in $\mathbb{R} \times S^1 \times M$, it is implicit in the definition of all pseudoholomorphic curve theories that before counting the geometric data has to be perturbed in such a way that the Cauchy-Riemann operator becomes transversal to the zero section in a suitable Banach space bundle over a suitable Banach manifold of maps. It is the main problem of symplectic field theory, as well as Gromov-Witten theory and symplectic Floer homology for general symplectic manifolds, that transversality for all moduli spaces cannot be achieved even for generic choices for $\underline{J}^H$. While in Gromov-Witten theory and symplectic Floer theory this problem can be solved by restricting to special geometric situations like semi-positive symplectic manifolds, this does not work in symplectic field theory. In fact the problem already occurs for the trivial curves, i.e., trivial examples of curves in symplectic field theory, see the second chapter. In order to solve these problems virtual moduli cycle techniques were invented, furthermore they were the starting point for the polyfold theory by Hofer et al.

In order to solve the transversality problem in our S^1-symmetric special case, we combine the approach in [CM1] for achieving transversality in Gromov-Witten theory with the well-known connection between symplectic Floer homology and Morse homology in [SZ]:

It is well-known, see e.g. [Sch], that transversality in Floer homology and Gromov-Witten theory can be achieved by allowing the almost complex structure on the symplectic manifold (M, ω) to depend on points on the punctured Riemann surface underlying the holomorphic curves, i.e., introducing domain-dependent almost complex structures. In this paper we fix the S^1-independent almost complex structure J and introduce domain-dependent Hamiltonian perturbations H, which however are still S^1-independent. Here we let H rather than J depend on the underlying punctured spheres, so that we achieve transversality also for the trivial curves, i.e., the branched covers of trivial cylinders. Note that in order to make the latter transversal, it is clearly neccessary to make the stable Hamiltonian structure on $S^1 \times M$ domain-dependent.

In order to make the choices for the domain-dependent Hamiltonian perturbations H compatible with gluing of curves in symplectic field theory, the perturbations must vary smoothly with the position of the punctures $\underline{z} = (z_1^{\pm}, ..., z_{n^{\pm}}^{\pm})$,

$$H = H_{\underline{z}} : \mathbb{CP}^1 - \{z_1^{\pm}, ..., z_{n^{\pm}}^{\pm}\} \times M \to \mathbb{R}\,.$$

In order to guarantee that finite energy solutions are still asymptotically cylindrical over periodic orbits of the original domain-independent Hamiltonian H, we require that $H_{\underline{z}}$ agrees with H over the cylindrical neighborhoods of the punctures. Furthermore, in order to asure that the automorphism group of $\mathbb{CP}^1$ still acts on the moduli space, they must satisfy

$$H_{\varphi(\underline{z})} = \varphi_* H_{\underline{z}} = H_{\underline{z}} \circ \varphi^{-1}.$$

When the number of punctures is greater or equal than three, i.e., the punctured Riemann sphere is stable, it follows that $H_{\underline{z}}$ should depend only on the class $[\underline{z}] \in \mathcal{M}_{0,n}$ in the moduli space of n-punctured Riemann spheres. For the construction of such domain-dependent structures we follow the ideas in [CM1]. Further we show that the resulting class of domain-dependent cylindrical almost complex structures $\underline{J}^H$ on $\mathbb{R} \times S^1 \times M$ is still large enough to achieve transversality for all moduli spaces of curves with three or more punctures.

For curves with two or less punctures, the compatibility with the action of $\mathrm{Aut}(\mathbb{CP}^1)$ implies that $H_{\underline{z}}$ must be *in*dependent of points on the domain, i.e., just a function on M. For this observe that for given two punctures $\underline{z} = (z^-, z^+)$ and $z, w \in \mathbb{CP}^1 - \{z^-, z^+\}$ we always find $\varphi \in \mathrm{Aut}(\mathbb{CP}^1)$ with $\varphi(\underline{z}) = \underline{z}$, $\varphi(z) = w$, so that

$$H_{\underline{z}}(w) = H_{\varphi(\underline{z})}(w) = (\varphi_* H_{\underline{z}})(w) = H_{\underline{z}}(\varphi^{-1}(w)) = H_{\underline{z}}(z).$$

On the other hand it is known from symplectic Floer homology that for fixed almost complex structure J it is important to let the Hamiltonian explicitly be S^1-dependent to have transversality for generic choices, which seems to destroy our hopes for computing the symplectic field theory of $\mathbb{R} \times S^1 \times M$ with S^1-independent H and J. To overcome this problem, we remind ourselves that we already assume H to be so small such that all one-period orbits are nondegenerate, in particular, critical points of H. Furthermore by proposition 1.1.4 we know that the $\underline{J}^H$-holomorphic cylinders naturally correspond to Floer connecting orbits. The trick is now to use the following connection between Floer homology and Morse homology:

If we choose H possibly smaller in the C^2-norm, e.g. by rescaling, we can achieve that all Floer trajectories u are indeed Morse trajectories, i.e., gradient flow lines $u(s,t) \equiv u(s)$ of H between the critical points x^- and x^+ with respect to the metric $\omega(\cdot, J\cdot)$ on M. When the pair $(H, \omega(\cdot, J\cdot))$ is Morse-Smale, the linearization F_u of the gradient flow operator is surjective, and it is shown in [SZ] that this indeed suffices to show that the linearization D_u of the Cauchy-Riemann operator is surjective as well. More precisely, we use the following lemma, which is proven in [SZ]:

Lemma 1.1.5: *Let (H,J) be a pair of a Hamiltonian H and an almost complex structure J on a closed symplectic manifold with $\langle [\omega], \pi_2(M)\rangle = 0$ so that $(H, \omega(\cdot, J\cdot))$ is Morse-Smale. Then the following holds:*

- *If $\tau > 0$ is sufficiently small, all finite energy solutions $u : \mathbb{R} \times S^1 \to M$ of $\bar{\partial}_{J,\tau H} u = \partial_s u + J(u)(\partial_t u + X^{\tau H}(u)) = 0$ are independent of $t \in S^1$.*

- *In this case, the linearization D_u^τ of $\bar\partial_{J,\tau H}$ is onto at any solution $u:\mathbb{R}\times S^1\to M$.*

Recall that we fixed a maximal period $T=2^N$ and let $P(H/2^N,\leq 2^N)$ denote the set of periodic orbits of the Reeb vector field $R^{H/2^N}$ for the Hamiltonian $H/2^N$ with period less or equal than 2^N. We collect our results about moduli spaces of holomorphic curves in $\mathbb{R}\times S^1\times M$ in the following

Theorem 1.1.6: *Let (M,ω) be a closed symplectic manifold, which is symplectically aspherical, equipped with a ω-compatible almost complex structure J and $H:M\to\mathbb{R}$ so that lemma 1.1.5 is satisfied with $\tau=1$. Further assume that for any ordered set of punctures $\underline{z}=(z_1^\pm,...,z_{n^\pm}^\pm)$ containing three or more points we have constructed a domain-dependent Hamiltonian perturbation $H_{\underline{z}}:(\mathbb{CP}^1-\{\underline{z}\})\times M\to\mathbb{R}$ of H with the properties outlined above. Then, depending on the number of punctures n we have the following result about the moduli spaces of $\underline{J}^H$-holomorphic curves in $\mathbb{R}\times S^1\times M$:*

- $n=0$*: All holomorphic spheres are constant.*
- $n=1$*: Holomorphic planes do not exist.*
- $n=2$*: For $T\leq 2^N$ the automorphism group $\mathrm{Aut}(\mathbb{CP}^1)$ acts on the moduli space of parametrized curves $\mathcal{M}^0(S^1\times M,(x^+,T),(x^-,T),\underline{J}^{H/2^N})$ of holomorphic cylinders with constant finite isotropy group $\mathbb{Z}/T\mathbb{Z}$ and the quotient can be naturally identified with the space of gradient flow lines of H with respect to the metric $\omega(\cdot,J\cdot)$ on M between the critical points x^+ and x^-.*
- $n\geq 3$*: For $P^+,P^-\subset P(H/2^N,\leq 2^N)$ the action of $\mathrm{Aut}(\mathbb{CP}^1)$ on the parametrized moduli space is free and the moduli space is given by the product*

$$\mathbb{R}\times S^1\quad\times\quad\{(u,\underline{z}):u:\mathbb{CP}^1-\{\underline{z}\}\to M:(*1),(*2)\}/\,\mathrm{Aut}(\mathbb{CP}^1)$$

  with

$$(*1):\quad du+X_{\underline{z}}^{H/2^N}(z,u)\otimes dh_2^0+J(u)\cdot(du+X_{\underline{z}}^{H/2^N}(z,u)\otimes dh_2^0)\cdot i=0\,,$$
$$(*2):\quad u\circ\psi_k^\pm(s,t)\overset{s\to\pm\infty}{\longrightarrow}x_k^\pm.$$

  In particular, there remains a free S^1-action on the moduli space after quotiening out the $\mathbb{R}$-translation.

Proof: Observe that all statements rely on proposition 1.1.2 and lemma 1.1.3. For $n=2$ we additionally use proposition 1.1.4 and lemma 1.1.5 and remark that the critical points and gradient flow lines of $H/2^N$ are naturally identified with those of H. For the statement about the isotropy groups observe that for $h(s,t)=(Ts,Tt)$ and $u(s,t)=u(s)$ we have

$$(h,u)=(h\circ\varphi,u\circ\varphi)\Leftrightarrow\varphi(s,t)=(s,t+\frac{k}{T}),\ k\in\mathbb{Z}/T\mathbb{Z}\,.$$

For the case $n \geq 3$ observe that the action of $\mathrm{Aut}(\mathbb{CP}^1)$ is already free on the underlying set of punctures and that the moduli space of parametrized curves is given by the product

$$\mathbb{R} \times S^1 \times \{(u, \underline{z}) : u : \mathbb{CP}^1 - \{\underline{z}\} \to M : (*1), (*2)\}.$$

$\square$

1.2 Domain-dependent Hamiltonians

Based on the ideas in [CM1] for achieving transversality in Gromov-Witten theory, we describe in this section a method to define domain-dependent Hamiltonian perturbations. In the following we drop the superscript for the punctures, $\underline{z} = (z_k)$, since for the assignment of Hamiltonians we do not distinguish between positive and negative punctures.

1.2.1 Deligne-Mumford space

We start with the following definition.

Definition 1.2.1: *A n-labelled tree is a triple (T, E, Λ), where (T, E) is a tree with the set of vertices T and the edge relation $E \subset T \times T$. The set $\Lambda = (\Lambda_\alpha)$ is a decomposition of the index set $I = \{1, ..., n\} = \bigcup \Lambda_\alpha$. We write $\alpha E \beta$ if $(\alpha, \beta) \in E$.*

A tree is called *stable* if for each $\alpha \in T$ we have $n_\alpha = \sharp\Lambda_\alpha + \sharp\{\beta : \alpha E \beta\} \geq 3$. For $n \geq 3$ a n-labelled tree can be stabilized in a canonical way. First delete vertices α with $n_\alpha < 3$ to obtain $\mathrm{st}(T) \subset T$ and modify E in the obvious way. We get a surjective tree homomorphism $\mathrm{st} : T \to \mathrm{st}(T)$, which by definition collapses some subtrees of T to vertices of $\mathrm{st}(T)$. If $\alpha E \beta$ with $\alpha \notin \mathrm{st}(T)$ but $\beta \in \mathrm{st}(T)$, the new subset Λ_β in the decomposition of the index set is given by the union $\Lambda_\beta \cup \Lambda_\alpha$. Note that $\Lambda_\alpha \neq \emptyset$ only if $\sharp\{\beta : \alpha E \beta\} = 1$.

Definition 1.2.2: *A nodal curve of genus zero modelled over $T = (T, E, \Lambda)$ is a tuple $\underline{z} = ((z_{\alpha\beta})_{\alpha E \beta}, (z_k))$ of special points $z_{\alpha\beta}, z_k \in \mathbb{CP}^1$ such that for each $\alpha \in T$ the special points in $Z_\alpha = \{z_{\alpha\beta} : \alpha E \beta\} \cup \{z_k : k \in \Lambda_\alpha\}$ are pairwise distinct.*

To any nodal curve $\underline{z}$ we can naturally associate a nodal Riemann surface $\Sigma_{\underline{z}} = \coprod_{\alpha \in T} S_\alpha / \{z_{\alpha\beta} \sim z_{\beta\alpha}\}$ with punctures (z_k), obtained by gluing a collection of Riemann spheres $S_\alpha \cong \mathbb{CP}^1$ at the points $z_{\alpha\beta} \in \mathbb{CP}^1$.

A nodal curve $\underline{z}$ is called *stable* if the underlying tree is stable, i.e., every sphere S_α carries at least three special points. Stabilization of trees immediately leads to a canonical stabilization $\underline{z} \to \mathrm{st}(\underline{z})$ of the corresponding nodal curve given as follows:

If $\alpha \in T$ is removed, we have $\sharp\{\beta \in \mathrm{st}(T) : \alpha E \beta\} \in \{1,2\}$. If there is precisely one $\beta \in \mathrm{st}(T)$ with $\alpha E \beta$, let $z_{\beta\alpha} =: z_{k'} \in \Lambda_\beta$. If there exist stable $\beta_1, \beta_2 \in T$ with $\alpha E \beta_1$, $\alpha E \beta_2$, we set $z_{\beta_1\alpha} =: z_{\beta_1\beta_2} \in \mathrm{st}(\underline{z})$ and $z_{\beta_2\alpha} =: z_{\beta_2\beta_1} \in \mathrm{st}(\underline{z})$. Observe that we get a natural map $st : \Sigma_{\underline{z}} \to \Sigma_{\mathrm{st}(\underline{z})}$ by projecting all points on $\alpha \notin \mathrm{st}(T)$ to $z_{k'}$ or $z_{\beta_1\beta_2} \sim z_{\beta_2\beta_1} \in \Sigma_{\mathrm{st}(\underline{z})}$, respectively.

Denote by $\widetilde{\mathcal{M}}_T \subset (\mathbb{CP}^1)^E \times (\mathbb{CP}^1)^n$ the space of all nodal curves (of genus zero) modelled over the tree $T = (T, E, \Lambda)$. An isomorphism between nodal curves $\underline{z}, \underline{z}'$ modelled over the same tree is a tuple $\phi = (\phi_\alpha)_{\alpha \in T}$ with $\phi_\alpha \in \mathrm{Aut}(\mathbb{CP}^1)$ so that $\phi(\underline{z}) = \underline{z}'$, i.e., $z'_{\alpha\beta} = \phi_\alpha(z_{\alpha\beta})$ and $z'_k = \phi_\alpha(z_k)$ if $k \in \Lambda_\alpha$. Observe that ϕ induces a biholomorphism $\phi : \Sigma_{\underline{z}} \to \Sigma_{\underline{z}'}$. Let G_T denote the group of biholomorphisms. For stable T the action of G_T on $\widetilde{\mathcal{M}}_T$ is free and the quotient $\mathcal{M}_T = \widetilde{\mathcal{M}}_T / G_T$ is a (finite-dimensional) complex manifold.

Definition 1.2.3: *For $n \geq 3$ denote by $\mathcal{M}_{0,n}$ the moduli space of stable genus zero curves modelled over the n-labelled tree with one vertex, i.e, the moduli space of Riemann spheres with n marked points. Taking the union of all moduli spaces of stable nodal curves modelled over n-labelled trees, we obtain the Deligne-Mumford space*

$$\overline{\mathcal{M}}_{0,n} = \coprod_T \mathcal{M}_T,$$

which, equipped with the Gromov topology, provides the compactification of the moduli space $\mathcal{M}_{0,n}$ of punctured Riemann spheres.

By a result of Knudsen (see [CM1], theorem 2.1) the Deligne-Mumford space $\overline{\mathcal{M}}_{0,n}$ carries the structure of a compact complex manifold of complex dimension $n-3$. For each stable n-labelled tree T the space $\mathcal{M}_T \subset \overline{\mathcal{M}}_{0,n}$ is a complex submanifold, where any $\mathcal{M}_T \neq \mathcal{M}_{0,n}$ is of complex codimension at least one in $\overline{\mathcal{M}}_{0,n}$.

It is a crucial observation that we have a canonical projection $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ by forgetting the $(k+1)$.st marked point and stabilizing. The map π is holomorphic and the fibre $\pi^{-1}([\underline{z}])$ is naturally biholomorphic to $\Sigma_{\underline{z}}$. Moreover, for $[\underline{z}] \in \overline{\mathcal{M}}_{0,n}$, every component $S_\alpha \subset \Sigma_{\underline{z}}$ is an embedded holomorphic sphere in $\overline{\mathcal{M}}_{0,n+1}$. Note that $\mathcal{M}_{0,n+1} \overset{\subset}{\neq} \pi^{-1}(\mathcal{M}_{0,n})$ as $\pi^{-1}([\underline{z}]) \cap \mathcal{M}_{0,n+1} = \mathbb{CP}^1 - \{(z_k)\}$ for $[\underline{z}] \in \mathcal{M}_{0,n}$.

1.2.2 Definition of coherent Hamiltonian perturbations

With this we are now ready to describe the algorithm how to find domain-dependent Hamiltonians $H_{\underline{z}}$ on M:

For $n = 2$ let $H^{(2)} : M \to \mathbb{R}$ be the domain-*in*dependent Hamiltonian from theorem 1.1.6, i.e., such that with the fixed almost complex structure J on M lemma 1.1.5 is satisfied with $\tau = 1$.

For $n \geq 3$ we choose smooth maps $H^{(n)} : \overline{\mathcal{M}}_{0,n+1} \to C^\infty(M)$. For $[\underline{z}] \in \overline{\mathcal{M}}_{0,n}$ we then define $H_{\underline{z}}$ to be the restriction of $H^{(n)}$ to the fibre $\pi^{-1}([\underline{z}]) \cong \Sigma_{\underline{z}}$. In particular, for $\underline{z} \in \mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$ we get from $\Sigma_{\underline{z}} \cong \mathbb{CP}^1$ a map

$$H_{\underline{z}} = H^{(n)}|_{\pi^{-1}([\underline{z}])} : \mathbb{CP}^1 \to C^\infty(M)\,,$$

where the biholomorphism $\Sigma_{\underline{z}} \cong \mathbb{CP}^1$ is fixed by requiring that (z_1, z_2, z_3) are mapped to $(0, 1, \infty)$. Further let $d_{\underline{z}} = \inf\{d(z_k, z_l) : 1 \leq k < l \leq n\}$ denote the minimal distance between two marked points with respect to the Fubini-Study metric on $\mathbb{CP}^1$, let $D_{\underline{z}}(z)$ be the ball of radius $d_{\underline{z}}/2$ around $z \in \mathbb{CP}^1$ and set $N_{\underline{z}} = D_{\underline{z}}(z_1) \cup ... \cup D_{\underline{z}}(z_n)$. Then we choose $H^{(n)}$ so that $H_{\underline{z}}$ agrees with $H^{(2)}$ on $N_{\underline{z}}$.

The gluing compatibility is ensured by specifying $H^{(n)}$ on the boundary $\partial\,\mathcal{M}_{0,n+1} = \overline{\mathcal{M}}_{0,n+1} - \mathcal{M}_{0,n+1}$, which consists of the fibres $\pi^{-1}([\underline{z}]) = \Sigma_{\underline{z}}$ over $[\underline{z}] \in \partial\,\mathcal{M}_{0,n} = \overline{\mathcal{M}}_{0,n} - \mathcal{M}_{0,n}$ and the points $z_1, ..., z_n \in \mathbb{CP}^1 = \Sigma_{\underline{z}}$ in the fibres over $[\underline{z}] \in \mathcal{M}_{0,n}$:

Note that we have already set $H_{\underline{z}}(z_k) = H^{(2)}$. For $[\underline{z}] \in \partial\,\mathcal{M}_{0,n} = \overline{\mathcal{M}}_{0,n} - \mathcal{M}_{0,n}$ we have $H_{\underline{z}} = H^{(n)}|_{\pi^{-1}([\underline{z}])} : \Sigma_{\underline{z}} \to C^\infty(M)$ with $\Sigma_{\underline{z}} = \coprod S_\alpha/\sim$ and $\sharp T \geq 2$. As before let $Z_\alpha = \{z_1^\alpha, ..., z_{n_\alpha}^\alpha\}$ denote the set of special points on S_α. Then we want that

$$H_{\underline{z}}|_{S_\alpha} = H_{\underline{z}^\alpha}$$

for $\underline{z}^\alpha = (z_k^\alpha)$.

Since $n_\alpha = \sharp Z_\alpha < n$, this requirement implies that a choice for the map $H^{(n)} : \overline{\mathcal{M}}_{0,n+1} \to C^\infty(M)$ also fixes the maps $H^{(n')} : \overline{\mathcal{M}}_{0,n'+1} \to C^\infty(M)$ for $n' < n$.

If $H^{(k)} : \overline{\mathcal{M}}_{0,k+1} \to C^\infty(M)$, $k = 2, ..., n-1$ are compatible in the above sense we call them coherent. We show how to find $H^{(n)} : \overline{\mathcal{M}}_{0,n+1} \to C^\infty(M)$ so that $H^{(2)}, ..., H^{(n)}$ are coherent:

Let $[\underline{z}] \in \partial\,\mathcal{M}_{0,n}$ with $\Sigma_{\underline{z}} = \coprod S_\alpha/\sim$. Under the assumption that $H_{\underline{z}^\alpha}$ was chosen to agree with $H^{(2)}$ on the neighborhood $N_{\underline{z}^\alpha}$ of the special points it follows that all $H_{\underline{z}^\alpha}$ fit together to a smooth assignment $H_{\underline{z}} : \Sigma_{\underline{z}} \to C^\infty(M)$. Let $T = (T, E, \Lambda)$ be the tree underlying $\underline{z}$. Then it follows by the same arguments that the maps $H^{(n_\alpha)}$ fit together to a smooth map $H^T : \pi^{-1}(\overline{\mathcal{M}}_T) \to C^\infty(M)$. Now let $\tau : T \to T'$ be a surjective tree homomorphism with $\sharp T' \geq 2$. Then $\overline{\mathcal{M}}_T \subset \overline{\mathcal{M}}_{T'}$ and it follows from the compatibility of $H^{(2)}, ..., H^{(n-1)}$ that H^T and $H^{T'}$ agree on $\pi^{-1}(\overline{\mathcal{M}}_T)$. Hence we get a unique assigment on $\partial\,\mathcal{M}_{0,n+1} = \pi^{-1}(\coprod\{\mathcal{M}_T : \sharp T \geq 2\})$.

After having specified the map $H^{(n)} : \overline{\mathcal{M}}_{0,n+1} \to C^\infty(M)$ on the boundary $\partial\,\mathcal{M}_{0,n+1}$, we choose $H^{(n)}$ in the interior $\mathcal{M}_{0,n+1}$ so that $H^{(n)}$ is smooth (on the compactification $\overline{\mathcal{M}}_{0,n+1}$) and $H^{(n)}$ agrees with $H^{(2)}$ on $N_{\underline{z}} \subset \pi^{-1}([\underline{z}])$ for all $[\underline{z}] \in \mathcal{M}_{0,n}$

Assuming we have determined $H^{(n)}$ for $n \geq 2$, we organize all maps into a map

$$H : \coprod_n \mathcal{M}_{0,n+1} \to C^\infty(M).$$

Note that for $n = 2$ the space $\mathcal{M}_{0,n+1}$ just consists of a single point. A map H as above, i.e., for which all restrictions $H^{(n)} : \mathcal{M}_{0,n+1} \to C^\infty(M)$, $n \in \mathbb{N}$ are coherent, is again called coherent.

Together with the almost complex structure J recall that this defines a domain-dependent cylindrical almost complex structure $\underline{J}^H$ on $\mathbb{R} \times S^1 \times M$,

$$\underline{J}^H : \coprod_n \mathcal{M}_{0,n+1} \to \mathcal{J}_{\text{cyl}}(\mathbb{R} \times S^1 \times M).$$

With this generalized notion of cylindrical almost complex structure we call, according to theorem 1.1.6, a map $F = (h,u) : \mathbb{CP}^1 - \{\underline{z}\} \to \mathbb{R} \times S^1 \times M$ $\underline{J}^H$-holomorphic when it satisfies the domain-dependent Cauchy-Riemann equation

$$\bar{\partial}_{\underline{J}}(h,u) = d(h,u) + \underline{J}^H_{\underline{z}}(z,h,u) \cdot d(h,u) \cdot i = 0,$$

which by proposition 1.1.2 is equivalent to the set of equations $\bar{\partial} h = 0$ and

$$\bar{\partial}_{J,H} = du + X^H_{\underline{z}}(z,u) \otimes dh^0_2 + J(u) \cdot (du + X^H_{\underline{z}}(z,u) \otimes dh^0_2) \cdot i = 0$$

with $X^H_{\underline{z}}(z,\cdot)$ denoting the symplectic gradient of $H_{\underline{z}}(z,\cdot) : M \to \mathbb{R}$.

Since $H_{\underline{z}}(z,\cdot)$ agrees with the Hamiltonian $H^{(2)} : M \to \mathbb{R}$ near the punctures, it follows that any finite-energy solution of the modified perturbed Cauchy-Riemann equation again converges to a periodic orbit of the Hamiltonian flow of $H^{(2)}$ as long as all possible asymptotic orbits are nondegenerate. Observe that it follows from the definition of $H_{\underline{z}}$ that the group of Moebius transformations still acts on the resulting moduli space of parametrized curves. We show in the section on transversality that for any given almost complex structure J on M we can find Hamiltonian perturbations $H : \coprod_n \mathcal{M}_{0,n+1} \to C^\infty(M)$, so that all moduli spaces $\mathcal{M}^0(S^1 \times M; P^+, P^-; \underline{J}^{H/2^N})$ are cut out transversally simultaneously for all maximal periods 2^N, $N \in \mathbb{N}$.

1.2.3 Compatibility with SFT compactness

It remains to show that the notion of coherent cylindrical almost complex structures $\underline{J}^H$ is actually compatible with Gromov convergence of $\underline{J}^H$-holomorphic curves in $\mathbb{R} \times S^1 \times M$:

Definition 1.2.4: *A $\underline{J}^H$-holomorphic level ℓ map $(h,u,\underline{z})$ consists of the following data:*

- *A nodal curve $\underline{z} = \coprod S_\alpha/\sim \in \overline{\mathcal{M}}_{0,n}$ and a labeling $\sigma : T \to \{1,...,\ell\}$, called levels, such that two components $\alpha,\beta \in T$ with $\alpha E \beta$ have levels differing by at most one.*
- *$\underline{J}^H$-holomorphic maps $F_\alpha : S_\alpha \to \mathbb{R} \times S^1 \times M$ (satisfying $d(h_\alpha,u_\alpha) + \underline{J}^H_{\underline{z}^\alpha}(z,h_\alpha,u_\alpha)\cdot d(h_\alpha,u_\alpha)\cdot i = 0$) with the following behaviour at the nodes:*
  If $\sigma(\alpha) = \sigma(\beta)+1$ then $z_{\alpha\beta}$ is a negative puncture for (h_α,u_α) and $z_{\beta\alpha}$ a positive puncture for (h_β,u_β) and they are asymptotically cylindrical over the same periodic orbit; else, if $\sigma(\alpha)=\sigma(\beta)$, then $(h_\alpha,u_\alpha)(z_{\alpha\beta}) = (h_\beta,u_\beta)(z_{\beta\alpha})$.

With this we can give the definition of Gromov convergence of $\underline{J}^H$-holomorphic maps.

Definition 1.2.5: *A sequence of stable $\underline{J}^H$-holomorphic maps $(h^\nu,u^\nu,\underline{z}^\nu)$ converges to a level ℓ holomorphic map $(h,u,\underline{z})$ if for any $\alpha \in T$ (T is the tree underlying $\underline{z}$) there exists a sequence of Moebius transformations $\phi^\nu_\alpha \in \mathrm{Aut}(\mathbb{CP}^1)$ so that:*

- *for $(h,u) = (h_1,h_2,u) = (h_{1,\alpha},h_{2,\alpha},u_\alpha)_{\alpha\in T}$ there exist sequences s^ν_i, $i=1,...,\ell$ with*

$$h^\nu_1 \circ \phi^\nu_\alpha + s^\nu_{\sigma(\alpha)} \overset{\nu\to\infty}{\longrightarrow} h_{1,\alpha},\ \ (h^\nu_2,u^\nu)\circ\phi^\nu_\alpha \overset{\nu\to\infty}{\longrightarrow} (h_{2,\alpha},u_\alpha)$$

  for all $\alpha \in T$ in $C^\infty_{\mathrm{loc}}(\dot{S})$,
- *for all $k=1,...,n$ we have $(\phi^\nu_\alpha)^{-1}(z^\nu_k) \to z_k$ if $k \in \Lambda_\alpha$ ($z_k \in S_\alpha$),*
- *and $(\phi^\nu_\alpha)^{-1}\circ\phi^\nu_\beta \to z_{\alpha\beta}$ for all $\alpha E\beta$.*

Note that a level ℓ holomorphic map $(h,u,\underline{z})$ is called stable if for any $l \in \{1,...,\ell\}$ there exists $\alpha \in T$ with $\sigma(\alpha) = l$ and (h_α,u_α) is not a trivial cylinder and, furthermore, if (h_α,u_α) is constant then the number of special points $n_\alpha = \sharp Z_\alpha \geq 3$. Although any holomorphic map $(h^\nu,u^\nu,\underline{z}^\nu) \in \mathcal{M}^0(S^1\times M;P^+,P^-;\underline{J}^H)$ with $n = \sharp P^+ + \sharp P^- \geq 3$ is stable, the nodal curve $\underline{z}$ underlying the limit level ℓ holomorphic map $(h,u,\underline{z})$ need not be stable. However, we can use the absence of holomorphic planes and (non-constant) holomorphic spheres in $\mathbb{R}\times S^1\times M$ to prove the following lemma about the boundary of $\mathcal{M}(S^1\times M;P^+,P^-;\underline{J}^H)/\mathbb{R}$:

Lemma 1.2.6: *Assume that the sequence $(h^\nu,u^\nu,\underline{z}^\nu) \in \mathcal{M}(S^1\times M;P^+,P^-;\underline{J}^H)$ Gromov converges to the level ℓ holomorphic map $(h,u,\underline{z})$. For the number of special points n_α on the component $S_\alpha \subset \Sigma_{\underline{z}}$ it holds*

- *$n_\alpha \leq n = \sharp P^+ + \sharp P^-$ for any $\alpha \in T$,*

- *if $n_\alpha = n$ for some $\alpha \in T$ then all other components are cylinders, i.e., carry precisely two special points.*

Proof: We prove this statement by iteratively letting circles on $\mathbb{CP}^1$ collapse to obtain the nodal surface $\Sigma_{\underline{z}}$:
For increasing the maximal number of special points on spherical components on a nodal surface we must collapse a special circle with all special points on one hemisphere. Even after collapsing further circles to nodes there always remains one component with just one special point (a node). Since by $\langle[\omega], \pi_2(M)\rangle = 0$ there are no holomorphic planes and bubbles (except 'ghost bubbles' which we drop) this cannot happen, which shows the first part of the statement. For the second part observe that collapsing circles with more than one special point on each hemisphere leads to two new spherical components which carry strictly less special points than the original one. □

For chosen $H : \coprod_n \mathcal{M}_{0,n+1} \to C^\infty(M)$ recall that for stable nodal curves $\underline{z}$ we defined $H_{\underline{z}} = H|_{\pi^{-1}([\underline{z}])} : \Sigma_{\underline{z}} \to C^\infty(M)$. For general nodal curves $\underline{z}$ we can use the stabilization $\underline{z} \to \mathrm{st}(\underline{z})$ and the induced map $\mathrm{st} : \Sigma_{\underline{z}} \to \Sigma_{\mathrm{st}(\underline{z})}$ to define

$$H_{\underline{z}}(z) := H_{\mathrm{st}(\underline{z})}(\mathrm{st}(z)), \quad z \in \Sigma_{\underline{z}}$$

(compare [CM1], section 4) with corresponding cylindrical almost complex structure $\underline{J}^H_{\underline{z}}(z) := \underline{J}^H_{\mathrm{st}(\underline{z})}(\mathrm{st}(z)) \in \mathcal{J}_{\mathrm{cyl}}(S^1 \times M)$.

Proposition 1.2.7: *A $\underline{J}^H$-holomorphic level ℓ map $(h, u, \underline{z})$ is $\underline{J}^H_{\underline{z}}$-holomorphic.*

Proof: If $\underline{z}$ is stable this follows directly from the construction of $\underline{J}^H$ as the restriction of $\underline{J}^H_{\underline{z}}$ to a component $S_\alpha \subset \Sigma_{\underline{z}}$ agrees with $\underline{J}^H_{\underline{z}^\alpha}$ when $\underline{z}^\alpha = (z_1^\alpha, ..., z_{n_\alpha}^\alpha)$ denotes the ordered set of special points on S_α. If $\underline{z}$ is not stable the proposition relies on the following two observations:
Since there are no spherical components with just one special point all special points on stable components of $\Sigma_{\underline{z}}$ are preserved under stabilization, i.e., a node connecting a stable component with an unstable one is not removed but becomes a marked point on $\Sigma_{\mathrm{st}(\underline{z})}$.
On the other hand points on a cylindrical component (a tree of cylinders) are mapped under stabilization to the node connecting it to a stable component (which then is a marked point for the nodal surface $\Sigma_{\mathrm{st}(\underline{z})}$). Since $\underline{J}^H_{\mathrm{st}(\underline{z})}$ near special points agrees with complex structure $\underline{J}^{H,(2)}$ chosen for cylinder we have $\underline{J}^H_{\underline{z}}(z) = \underline{J}^H_{\mathrm{st}(\underline{z})}(\mathrm{st}(z)) = \underline{J}^{H,(2)}$ for any $z \in \Sigma_{\underline{z}}$ lying on a cylindrical component. □

In order to show the gluing compatibility we prove the following proposition.

Proposition 1.2.8: *Let $(h^\nu, u^\nu, \underline{z}^\nu)$ be a sequence of $\underline{J}^H_{\underline{z}^\nu}$-holomorphic maps converging to the level ℓ map $(h, u, \underline{z})$. Then $(h, u, \underline{z})$ is $\underline{J}^H_{\underline{z}}$-holomorphic.*

Proof: Recall from the definition of Gromov convergence that for any $\alpha \in T$ (the tree underlying $\underline{z}$) there exists a sequence $\phi_\alpha^\nu \in \mathrm{Aut}(\mathbb{CP}^1)$ and for any $i \in \{1,...,\ell\}$ sequences $s_i^\nu \in \mathbb{R}$ such that $h_1^\nu \circ \phi_\alpha^\nu + s_{\sigma(\alpha)}^\nu \to h_{1,\alpha}$ and $(h_2^\nu, u^\nu) \circ \phi_\alpha^\nu \to (h_{1,\alpha}, u_\alpha)$. Hence it remains to show that

$$\underline{J}_{\underline{z}^\nu}^H \circ \phi_\alpha^\nu \to \underline{J}_{\underline{z}}^H$$

in $C^\infty(S_\alpha, \mathcal{J}_{\mathrm{cyl}}(S^1 \times M))$ as $\nu \to \infty$ for all $\alpha \in T$:

Since the projection from the compactified moduli space to the Deligne-Mumford space $\overline{\mathcal{M}}_{0,n}$ is smooth (see theorem 5.6.6 in [MDSa]), it follows from $(h^\nu, u^\nu, \underline{z}^\nu) \to (h, u, \underline{z})$ that $\underline{z}^\nu = \mathrm{st}(\underline{z}^\nu) \to \mathrm{st}(\underline{z})$ in $\overline{\mathcal{M}}_{0,n}$.
For $\alpha \in \mathrm{st}(T)$ and $z \in S_\alpha$ we have $\mathrm{st}(z) = z$ and it follows that

$$(\underline{z}^\nu, \phi_\alpha^\nu(z)) \to (\mathrm{st}(\underline{z}), z) \in \overline{\mathcal{M}}_{0,n+1}\,.$$

Since $\underline{J}^{H,(n)} : \overline{\mathcal{M}}_{0,n+1} \to \mathcal{J}_{\mathrm{cyl}}(S^1 \times M)$ is continuous, we have

$$\underline{J}_{\underline{z}^\nu}^H(\phi_\alpha^\nu(z)) \to \underline{J}_{\mathrm{st}(\underline{z})}^H(z) = \underline{J}_{\underline{z}}^H(z)$$

in $\mathcal{J}_{\mathrm{cyl}}(S^1 \times M)$ for all $z \in S_\alpha$. The uniform convergence in all derivatives follows by the same argument using the smoothness of $\underline{J}^{H,(n)}$.
On the other hand, if $\alpha \notin \mathrm{st}(T)$ and $z \in S_\alpha$, then $\mathrm{st}(z) = z_{\beta\alpha} \in \mathrm{st}(\underline{z})$ if $\alpha E \beta$. In $\overline{\mathcal{M}}_{0,n+1}$ we have that

$$(\underline{z}^\nu, \phi_\alpha^\nu(z)) \to (\underline{z}, z_{\beta\alpha})$$

since $(\phi_\beta^\nu)^{-1}(\phi_\alpha^\nu(z)) \to z_{\beta\alpha} \in S_\beta$ and therefore

$$\underline{J}_{\underline{z}^\nu}^H(\phi_\alpha^\nu(z)) \to \underline{J}_{\mathrm{st}(\underline{z})}^H(\mathrm{st}(z)) = \underline{J}_{\underline{z}}^H(z)\,.\ \square$$

1.3 Transversality

We follow [BM] for the description of the analytic setup of the underlying Fredholm problem. More precisely, we take from [BM] the definition of the Banach space bundle over the Banach manifold of maps, which contains the Cauchy-Riemann operator studied above as a smooth section.

1.3.1 Banach space bundle and Cauchy-Riemann operator

For a chosen coherent Hamiltonian perturbation $H : \coprod_n \mathcal{M}_{0,n+1} \to C^\infty(M)$ and fixed $N \in \mathbb{N}$, we choose ordered sets of periodic orbits

$$P^\pm = \{(x_1^\pm, T_1^\pm), ..., (x_{n^\pm}^\pm, T_{n^\pm}^\pm)\} \subset P(H^{(2)}/2^N, \leq 2^N).$$

Instead of considering $\mathbb{CP}^1 \cong S^2$ with its unique conformal structure, we fix punctures $z_1^{\pm,0}, ..., z_n^{\pm,0} \in S^2$ and let the complex structure on $\dot{S} = S^2 - \{z_1^{\pm,0}, ..., z_n^{\pm,0}\}$ vary. Following the constructions in [BM] we see that the appropriate Banach manifold $\mathcal{B}^{p,d}(\mathbb{R}\times S^1 \times M;(x_k^\pm,T_k^\pm))$ for studying the underlying Fredholm problem is given by the product

$$\mathcal{B}^{p,d}(\mathbb{R}\times S^1\times M,(x_k^\pm,T_k^\pm)) = H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})\times\mathcal{B}^p(M;(x_k^\pm))\times\mathcal{M}_{0,n}$$

with $d>0$ and $p>2$, whose factors are defined as follows:

The Banach manifold $\mathcal{B}^p(M;(x_k^\pm))$ consists of maps $u\in H^{1,p}_{\text{loc}}(\dot{S},M)$, which converge to the critical points $x_k^\pm\in\text{Crit}(H^{(2)})$ as $z\in\dot{S}$ approaches the puncture $z_k^{\pm,0}$. More precisely, if we fix linear maps $\Theta_k^\pm:\mathbb{R}^{2m}\to T_{x_k^\pm}M$, the curves satisfy

$$u\circ\psi_k^\pm(s,t)=\exp_{x_k^\pm}(\Theta_k^\pm\cdot v_k^\pm(s,t))$$

for some $v_k^\pm\in H^{1,p}(\mathbb{R}^\pm\times S^1,\mathbb{R}^{2m})$, where exp denotes the exponential map for the metric $\omega(\cdot,J\cdot)$ on M.

The space $H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})$ consists of maps $h\in H^{1,p}_{\text{loc}}(\dot{S},\mathbb{C})$, for which there exist $(s_0^{\pm,k},t_0^{\pm,k})\in\mathbb{R}^2\cong\mathbb{C}$, so that $h_k^\pm=h\circ\psi_k^\pm$ differs from the constant $(s_0^{\pm,k},t_0^{\pm,k})$ by a function, which is not only in $H^{1,p}(\mathbb{R}^\pm\times S^1,\mathbb{C})$, but still in this space after multiplication with the asymptotic weight $(s,t)\mapsto e^{\pm d\cdot s}$,

$$\begin{aligned}&\mathbb{R}^\pm\times S^1\to\mathbb{R}^2,\ (s,t)\mapsto(h_k^\pm(s,t)-(s_0^{\pm,k},t_0^{\pm,k}))\cdot e^{\pm d\cdot s}\\&\in H^{1,p}(\mathbb{R}^\pm\times S^1,\mathbb{C}).\end{aligned}$$

Loosely spoken, $H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})$ consists of maps differing asymptotically from a constant one by a function, which converges exponentially fast to zero.

Finally $\mathcal{M}_{0,n}$ denotes, as before, the moduli space of complex structures on the punctured sphere $\dot{S}$, which clearly is naturally identified with its originally defined version, the moduli space of Riemann spheres with n punctures.
Here we represent $\mathcal{M}_{0,n}$ explicitly by finite-dimensional families of (almost) complex structures on $\dot{S}$, so that $T_j\,\mathcal{M}_{0,n}$ becomes a finite-dimensional subspace of

$$\{y\in\text{End}(T\dot{S}):yj+jy=0\}.$$

Note that in [BM] the authors work with Teichmueller spaces, since the corresponding moduli spaces of complex structures, obtained by quotienting out the mapping class group, become orbifolds for non-zero genus.

Given $\bar{h}\in H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})$ observe that the corresponding map $h:\dot{S}\to\mathbb{R}\times S^1$ is given by $h=h^0+\bar{h}$, where h^0 denotes an arbitrary fixed holomorphic map $h^0:\dot{S}\to\mathbb{R}\times S^1\cong\mathbb{CP}^1-\{0,\infty\}$, so that $z_k^{\pm,0}$ is a pole/zero of order $T_k^\pm$. Note that we do not use asymptotic exponential weights (depending on $d\in\mathbb{R}^+$) for the Banach manifold $\mathcal{B}^p(M;(x_k^\pm))$, since we are dealing with nondegenerate asymptotics.

Let $H^{1,p}(u^*TM)$ consist of sections $\xi \in H^{1,p}_{\mathrm{loc}}(u^*TM)$, such that

$$\xi \circ \psi_k^\pm(s,t) = (d\exp_{x_k^\pm})(\Theta_k^\pm \cdot v_k^\pm(s,t)) \cdot \Theta_k^\pm \xi_k^{\pm,0}(s,t)$$

with $\xi_k^{\pm,0} \in H^{1,p}(\mathbb{R}^\pm \times S^1, \mathbb{R}^{2m})$ for $k = 1,...,n$. Note that here we take the differential of $\exp_{x_k^\pm} : T_{x_k^\pm}M \to M$ at $\Theta_k^\pm \cdot v_k^\pm(s,t) \in T_{x_k^\pm}M$, which maps the tangent space to M at $x_k^\pm$ to the tangent space to M at

$$\exp_{x_k^\pm}(\Theta_k^\pm \cdot v_k^\pm(s,t)) = u \circ \psi_k^\pm(s,t).$$

Then the tangent space to $\mathcal{B}^{p,d}(\mathbb{R} \times S^1 \times M; (x_k^\pm, T_k^\pm))$ at $(\bar{h}, u, j)$ is given by

$$T_{(\bar{h},u,j)}\,\mathcal{B}^{p,d}(\mathbb{R} \times S^1 \times M; (x_k^\pm, T_k^\pm)) = H^{1,p,d}_{\mathrm{const}}(\dot{S}, \mathbb{C}) \oplus H^{1,p}(u^*TM) \oplus T_j\,\mathcal{M}_{0,n}\,.$$

Consider the bundle $T^*\dot{S} \otimes_{j,J} u^*TM$, whose sections are (j,J)-antiholomorphic one-forms α on $\dot{S}$ with values in the pullback bundle u^*TM,

$$\alpha - J(u) \cdot \alpha \cdot j = 0.$$

The space $L^p(T^*\dot{S} \otimes_{j,J} u^*TM)$ is defined similarly as $H^{1,p}(u^*TM)$: it consists of sections $\alpha \in L^p_{\mathrm{loc}}$, which asymptotically satisfy

$$(\psi_k^\pm)^*\alpha(s,t) \cdot \partial_s = (d\exp_{x_k^\pm})(\Theta_k^\pm \cdot v_k^\pm(s,t)) \cdot \Theta_k^\pm \alpha_k^{\pm,0}(s,t)$$

with $\alpha_k^{\pm,0} \in L^p(\mathbb{R}^\pm \times S^1, \mathbb{R}^{2m})$.

Over $\mathcal{B}^{p,d} = \mathcal{B}^{p,d}(\mathbb{R} \times S^1 \times M; (x_k^\pm, T_k^\pm))$ consider the Banach space bundle $\mathcal{E}^{p,d} \to \mathcal{B}^{p,d}$ with fibre

$$\mathcal{E}^{p,d}_{\bar{h},u,j} = L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J} u^*TM).$$

Recall that we have fixed a coherent Hamiltonian perturbation $H : \coprod \mathcal{M}_{0,n+1} \to C^\infty(M)$. Our convention at the beginning of this section, i.e., fixing the punctures on S^2 but letting the almost complex structure $j : T\dot{S} \to T\dot{S}$ vary, now leads to a dependency $H(j,z) = H^{(n)}(j,z)$ on the complex structure j on $\dot{S}$ and points $z \in \dot{S}$. For the following exposition let us assume $N = 0$ in order to keep the notation simple.

The Cauchy-Riemann operator

$$\bar{\partial}_{\underline{J}^H}(h,u,j) = \bar{\partial}_{j,\underline{J}^H}(h,u) \;=\; d(h,u) + \underline{J}^H(j,z,h,u) \cdot d(h,u) \cdot j$$

is a smooth section in $\mathcal{E}^{p,d} \to \mathcal{B}^{p,d}$ and naturally splits,

$$\bar{\partial}_{j,\underline{J}^H}(h,u) = (\bar{\partial}h, \bar{\partial}_{J,H}u) \in L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J} u^*TM).$$

Here $\bar{\partial} = \bar{\partial}_{j,i}$ is the standard Cauchy-Riemann operator for maps $h : (\dot{S}, j) \to \mathbb{R} \times S^1$ and $\bar{\partial}_{J,H}$ is the perturbed Cauchy-Riemann operator given by

$$\bar{\partial}_{J,H}(u) = du + X^H(j,z,u) \otimes dh_2^0 + J(u) \cdot (du + X^H(j,z,u) \otimes dh_2^0) \cdot j,$$

where again $X^H(j,z,\cdot)$ denotes the symplectic gradient of the Hamiltonian $H(j,z,\cdot): M \to \mathbb{R}$

It follows that the linearization $D_{\bar{h},u,j}$ of $\bar{\partial}_{\underline{J}^H}$ at a solution $(\bar{h},u,j)$ splits,

$$D_{\bar{h},u,j} = D_{\bar{h},u} \oplus D_j,$$

with $D_j : T_j \mathcal{M}_{0,n} \to \mathcal{E}^{p,d}_{\bar{h},u,j}$ and

$$\begin{aligned} D_{\bar{h},u} = \mathrm{diag}(\bar{\partial}, D_u) : \quad & H^{1,p,d}_{\mathrm{const}}(\dot{S},\mathbb{C}) \oplus H^{1,p}(u^*TM) \\ & \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J} u^*TM), \end{aligned}$$

where

$$\begin{aligned} D_u : \ & H^{1,p}(u^*TM) \to L^p(T^*\dot{S} \otimes_{j,J} u^*TM), \\ D_u\xi \ & = \nabla\xi + J(u)\cdot\nabla\xi\cdot j + \nabla_\xi J(u)\cdot du\cdot j \\ & +\nabla_\xi X^H(j,z,u) \otimes dh^0_2 + \nabla_\xi \nabla H(j,z,u) \otimes dh^0_1 \end{aligned}$$

is the linearization of the perturbed Cauchy-Riemann operator $\bar{\partial}_{J,H}$.

1.3.2 Universal moduli space

Let $\mathcal{H}^\ell_n(M; H^{(2)},...,H^{(n-1)})$ denote the Banach manifold consisting of C^ℓ-maps $H^{(n)} : \mathcal{M}_{0,n+1} \to C^\ell(M)$, which extend as C^ℓ-maps to $\overline{\mathcal{M}}_{0,n+1}$ as induced by $H^{(k)}$, $k=2,...,n-1$ and $H^{(n)}(j,\cdot) = H^{(2)}$ on a neighborhood $N_0 \subset \dot{S}$ of the punctures.

Note that it is essential to work in the C^ℓ-category since the corresponding space of C^∞-structures just inherits the structure of a Frechet manifold and we later cannot apply the Sard-Smale theorem.

The tangent space to $\mathcal{H}^\ell = \mathcal{H}^\ell_n(M; H^{(2)},...,H^{(n-1)})$ at $H = H^{(n)}$ is given by

$$T_H \mathcal{H}^\ell_n(M; H^{(2)},...,H^{(n-1)}) \ = \ \mathcal{H}^\ell_n(M; 0,...,0).$$

The universal Cauchy-Riemann operator $\bar{\partial}_J(\bar{h},u,j,H) := \bar{\partial}_{\underline{J}^H}(h,u,j)$ extends to a smooth section in the Banach space bundle $\hat{\mathcal{E}}^{p,d} \to \mathcal{B}^{p,d} \times \mathcal{H}^\ell$ with fibre

$$\hat{\mathcal{E}}^{p,d}_{\bar{h},u,j,H} = \mathcal{E}^{p,d}_{\bar{h},u,j} = L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J} u^*TM).$$

Letting $\underline{J}^{H,(2)},...,\underline{J}^{H,(n-1)} : \mathcal{M}_{0,n} \to \mathcal{J}^\ell_{\mathrm{cyl}}(\mathbb{R}\times S^1 \times M)$ denote the domain-dependent cylindrical almost complex structures on $\mathbb{R}\times S^1 \times M$ induced by J and $H^{(2)},...,H^{(n-1)} : \mathcal{M}_{0,n} \to C^\ell(M)$, we define the universal moduli space $\mathcal{M}(S^1 \times M; P^+,P^-; \underline{J}^{H,(2)},...,\underline{J}^{H,(n-1)})$ as the zero set of the universal Cauchy-Riemann operator,

$$\begin{aligned} & \mathcal{M}(S^1 \times M; P^+,P^-; (\underline{J}^{H,(k)})^{n-1}_{k=2}) = \\ & \{(\bar{h},u,j,H) \in \mathcal{B}^{p,d} \times \mathcal{H}^\ell : \bar{\partial}_J(\bar{h},u,j,H) = 0\}. \end{aligned}$$

Theorem 1.3.1: *For $n \geq 3$ let $H^{(2)},...,H^{(n-1)}$ be fixed. Then for any chosen (P^+,P^-) with $\sharp P^+ + \sharp P^- = n$, the universal moduli space $\mathcal{M}(S^1 \times M; P^+, P^-; (\underline{J}^{H,(k)})_{k=2}^{n-1})$ is transversally cut out by the universal Cauchy-Riemann operator $\bar{\partial}_J : \mathcal{B}^{p,d} \times \mathcal{H}^\ell \to \hat{\mathcal{E}}^{p,d}$ for $d > 0$ sufficiently small. In particular, it carries the structure of a C^∞-Banach manifold.*

The proof relies on the following two lemmata:

Lemma 1.3.2: *The operator $\bar{\partial} : H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C})$ is onto.*

Proof: Fix a splitting
$$H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) = H^{1,p,d}(\dot{S},\mathbb{C}) \oplus \Gamma^n$$
where $\Gamma^n \subset C^\infty(\dot{S},\mathbb{C})$ is a $2n$-dimensional space of functions storing the constant shifts (see [BM]). Given a function $\varphi_d : \dot{S} \to \mathbb{R}$ with $(\varphi_d \circ \psi^\pm_k)(s,t) = e^{\pm d \cdot s}$, multiplication with φ_d defines isomorphisms
$$\begin{aligned} H^{1,p,d}(\dot{S},\mathbb{C}) &\xrightarrow{\cong} H^{1,p}(\dot{S},\mathbb{C}), \\ L^{p,d}(T^*\dot{S} \otimes_{i,i} \mathbb{C}) &\xrightarrow{\cong} L^{p}(T^*\dot{S} \otimes_{i,i} \mathbb{C}), \end{aligned}$$
under which $\bar{\partial}$ corresponds to a perturbed Cauchy-Riemann operator
$$\bar{\partial}_d = \bar{\partial} + S_d : H^{1,p}(\dot{S},\mathbb{C}) \to L^p(T^*\dot{S} \otimes_{i,i} \mathbb{C}).$$
With the asymptotic behaviour of φ_d one computes
$$S^{\pm,k}_d(t) = (S_d \circ \psi^\pm_k)(\pm\infty, t) = \operatorname{diag}(\mp d, \mp d)$$
so that the Conley-Zehnder index for the corresponding paths $\Psi^{\pm,k} : \mathbb{R} \to \mathrm{Sp}(2m)$ of symplectic matrices is ∓ 1 for $d > 0$ sufficiently small. Hence the index of $\bar{\partial} : H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \to L^{p,d}(T^*\dot{S} \otimes_{i,i} \mathbb{C})$ is given by
$$\operatorname{ind} \bar{\partial} = \dim \Gamma^n + \operatorname{ind} \bar{\partial}_d = 2n - n + 1 \cdot (2-n) = 2,$$
where the first summand is the dimension of Γ^n and the second is the sum of the Conley-Zehnder indices. On the other hand, it follows from Liouville's theorem that the kernel of $\bar{\partial}$ consists of the constant functions on $\dot{S}$, so that $\dim \operatorname{coker} \bar{\partial} = 0$. $\square$

Lemma 1.3.3: *For $n \geq 3$ the linearization $D_{u,H}$ of $\bar{\partial}_J(u,H) = \bar{\partial}_{J,H}(u)$ is surjective at any $(\bar{h},u,j,H) \in \mathcal{M}(S^1 \times M; P^+,P^-; (\underline{J}^{H,(k)})_{k=2}^{n-1})$.*

Proof: The operator $D_{u,H}$ is the sum of the linearization D_u of the perturbed Cauchy-Riemann operator $\bar{\partial}_{J,H}$ and the linearization of $\bar{\partial}_J$ in the $\mathcal{H}^\ell$-direction,
$$\begin{aligned} D_H :\ & T_H \mathcal{H}^\ell \to L^p(T^*\dot{S} \otimes_{j,J} u^*TM), \\ & D_H G = X^G(j,z,u) \otimes dh^0_2 + J(u) X^G(j,z,u) \otimes dh^0_1 \,. \end{aligned}$$

We show that $D_{u,H}$ is surjective using well-known arguments:

Since D_u is Fredholm, the range of $D_{u,H}$ in $L^p(T^*\dot{S}\otimes_{j,J}u^*TM)$ is closed, and it suffices to prove that the annihilator of the range of $D_{u,H}$ is trivial.

We identify the dual space of $L^p(T^*\dot{S}\otimes_{j,J}u^*TM)$ with $L^q(T^*\dot{S}\otimes_{j,J}u^*TM)$, $1/p+1/q=1$ using the L^2-inner product on sections in $T^*\dot{S}\otimes_{j,J}u^*TM$, which is defined using the standard hyperbolic metric on $(\dot{S},j)$ and the metric $\omega(\cdot,J\cdot)$ on M.

Let $\eta\in\hat{\mathcal{E}}^{p,d}_{\tilde{h},u,j,H}=L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J}u^*TM)$ so that $\langle D_{u,H}\cdot(\xi,G),\eta\rangle=0$ for all $\xi\in H^{1,p}(u^*TM)$ and $G\in T_H\,\mathcal{H}^\ell$. Then surjectivity of $D_{u,H}$ is equivalent to showing $\eta\equiv 0$:

From $\langle D_u\xi,\eta\rangle=0$ for all $\xi\in H^{1,p}(u^*TM)$, we get that η is a weak solution of the perturbed Cauchy-Riemann equation $D_u^*\eta=0$, where D_u^* is the adjoint of D_u. By elliptic regularity, it follows that η is smooth and hence a strong solution. By unique continuation, which is an immediate consequence of the Carleman similarity principle, it follows that $\eta\equiv 0$ whenever η vanishes identically on an open subset of $\dot{S}$.

On the other hand we have

$$\begin{aligned}0\;=\;\langle D_HG,\eta\rangle\;&=\;\int_{\dot{S}}\langle J(u)X^G(j,z,u)\otimes dh_1^0+X^G(j,z,u)\otimes dh_2^0,\eta(z)\rangle\,dz\\&=\;\int_{\dot{S}}\langle\nabla G(j,z,u)\otimes dh_1^0-J(u)\nabla G(j,z,u)\otimes dh_2^0,\eta(z)\rangle\,dz\end{aligned}$$

for all $G\in T_H\,\mathcal{H}^\ell$. When $z\in\dot{S}$ is not a branch point of the map $h^0:\dot{S}\to\mathbb{R}\times S^1$, observe that we can write $\eta(z)=\eta_1(z)\otimes dh_1^0+\eta_2(z)\otimes dh_2^0$ with $\eta_2(z)+J(u)\eta_1(z)=0$, since η is (j,J)-antiholomorphic. It follows that

$$\begin{aligned}&\langle\nabla G(j,z,u)\otimes dh_1^0-J(u)\nabla G(j,z,u)\otimes dh_2^0,\eta(z)\rangle\\&=\langle\nabla G(j,z,u)\otimes dh_1^0-J(u)\nabla G(j,z,u)\otimes dh_2^0,\\&\quad\eta_1(z)\otimes dh_1^0-J(u)\eta_1(z)\otimes dh_2^0\rangle\\&=\langle\nabla G(j,z,u),\eta_1(z)\rangle\cdot\|dh_1^0\|^2\;+\;\langle J(u)\nabla G(j,z,u),J(u)\eta_1(z)\rangle\cdot\|dh_2^0\|^2\\&=\|dh_1^0\|^2\;\cdot\langle\nabla G(j,z,u),\eta_1(z)\rangle\;=\;\|dh_1^0\|^2\;\cdot dG(j,z,u)\cdot\eta_1(z),\end{aligned}$$

where $dG(j,z,\cdot)$ denotes the differential of $G(j,z,\cdot):M\to\mathbb{R}$.

With this we prove that η vanishes identically on the complement of the set of branch points of h^0, which by unique continuation implies $\eta=0$:

Assume to the contrary that $\eta(z_0) \neq 0$ for some $z_0 \in \dot{S}$, which is not a branch point, so that by (j,J)-antiholomorphicity $\eta_1(z_0) \neq 0$. We obviously can find $G_0 \in C^\infty(M)$ such that

$$dG_0(u(z_0)) \cdot \eta_1(z_0) > 0.$$

Setting $j_0 := j$, let $\varphi \in C^\infty(\overline{\mathcal{M}}_{0,n+1}, [0,1])$ be a smooth cut-off function around $(j_0, z_0) \in \mathcal{M}_{0,n+1}$ with $\varphi(j_0, z_0) = 1$ and $\varphi(j,z) = 0$ for $(j,z) \notin U(j_0, z_0)$. Here the neighborhood $(j_0, z_0) \in U_1(j_0) \times U_2(z_0) = U(j_0, z_0) \subset \overline{\mathcal{M}}_{0,n+1}$ is chosen so small that

$$U(j_0, z_0) \cap (\overline{\mathcal{M}}_{0,n+1} - \mathcal{M}_{0,n+1}) = \emptyset,\; U_2(z_0) \cap N_0 = \emptyset,$$

and $dG_0(z, u(z)) \cdot \eta_1(z) \geq 0$ for all $z \in U_2(z_0)$.

With this define $G : \overline{\mathcal{M}}_{0,n+1} \times M \to \mathbb{R}$ by $G(j,z,p) := \varphi(j,z) \cdot G_0(p)$. But this leads to the desired contradiction since we found $G \in T_H H^\ell = \mathcal{H}^\ell_n(M; 0, ..., 0)$ with

$$\langle D_H \cdot G, \eta \rangle = \int_{U_2(z_0)} \frac{1}{2} \|dh^0(z)\|^2 \cdot dG(j,z,u) \cdot \eta_1(z)\, dz > 0. \;\square$$

Proof of theorem 1.3.1: For $n \geq 3$ we must show that the linearization $D_{\bar{h},u,j,H}$ of the universal Cauchy-Riemann operator $\bar{\partial}_J$ is surjective at any $(\bar{h}, u, j, H) \in \mathcal{M}(S^1 \times M; P^+, P^-; (\underline{J}^{H,(k)})_{k=2}^{n-1})$. Using the splitting $D_{\bar{h},u,j,H} = D_{\bar{h},u,H} + D_j$ we show that the first summand

$$\begin{aligned} D_{\bar{h},u,H} : \quad & H^{1,p,d}_{\mathrm{const}}(\dot{S}, \mathbb{C}) \oplus T_u \mathcal{B}^p(M; P^+, P^-) \oplus T_H \mathcal{H}^\ell \\ & \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J} u^*TM) \end{aligned}$$

is onto. However, since

$$D_{\bar{h},u,H} = \mathrm{diag}(\bar{\partial}, D_{u,H}),$$

this follows directly from the surjectivity of $\bar{\partial}$ and $D_{u,H} = D_u + D_H$. $\square$

The importance of the above theorem is that, combined with lemma 1.1.5, we obtain transversality for all moduli spaces of holomorphic curves in $\mathbb{R} \times S^1 \times M$ asymptotically cylindrical over periodic orbits up to the given maximal period 2^N. Moreover we can achieve that this holds for all maximal periods simultaneously.

Corollary 1.3.4: *For $n = 2$ and $T \leq 2^N$ the moduli spaces $\mathcal{M}(S^1 \times M; (x^+, T), (x^-, T); \underline{J}^{H/2^N})$ are transversally cut out by the Cauchy-Riemann operator for all $N \in \mathbb{N}$. For $n \geq 3$ we can choose $H^{(n)} \in \mathcal{H}^\ell$, simultaneously for all $N \in \mathbb{N}$, so that the moduli spaces $\mathcal{M}(S^1 \times M; P^+, P^-; \underline{J}^{H/2^N})$ are transversally cut out by the resulting Cauchy-Riemann operator for all $P^+, P^- \subset P(H^{(2)}/2^N, \leq 2^N)$ with $\#P^+ + \#P^- = n$.*

Proof: For $n = 2$ the linear operator

$$D_{\bar{h},u} = \mathrm{diag}(\bar{\partial}, D_u)$$

is surjective since D_u is onto by lemma 1.1.5. Indeed, recall that we have chosen the pair $(H^{(2)}, J)$ to be regular in the sense that $(H^{(2)}, \omega(\cdot, J\cdot))$ is Morse-Smale, which implies that all pairs $(H^{(2)}/2^N, J)$ for any $N \in \mathbb{N}$ are again regular, since the stable and unstable manifolds are the same.

For $n \geq 3$ and $N = 0$ the Sard-Smale theorem applied to the map

$$\mathcal{M}(S^1 \times M; P^+, P^-; (\underline{J}^{H,(k)})_{k=2}^{n-1}) \to \mathcal{H}^\ell_n(M; (H^{(k)})_{k=2}^{n-1}),\ (\bar{h}, u, j, H) \mapsto H$$

tells us that the set of Hamiltonian perturbations $\mathcal{H}^\ell_{\mathrm{reg}}(P^+, P^-) = \mathcal{H}^\ell_{\mathrm{reg}}(P^+, P^-, 0)$, for which the moduli space $\mathcal{M}(S^1 \times M; P^+, P^-; \underline{J}^H)$ is cut out transversally by the Cauchy-Riemann operator $\bar{\partial}_{\underline{J}^H}$, is of the second Baire category in $\mathcal{H}^\ell = \mathcal{H}^\ell_n(M; (H^{(k)})_{k=2}^{n-1})$. Since there exist just a countable number of tuples (P^+, P^-) with $\sharp P^+ + \sharp P^- = n$, it follows that $\mathcal{H}^\ell_{\mathrm{reg}} = \mathcal{H}^\ell_{\mathrm{reg}}(0) = \bigcap\{\mathcal{H}^\ell_{\mathrm{reg}}(P^+, P^-, 0) : \sharp P^+ + \sharp P^- = n\}$ is still of the second category.

Replacing $H^{(2)}, ..., H^{(n-1)}$ in the above argumentation by $H^{(2)}/2^N, ..., H^{(n-1)}/2^N$ for each $N \in \mathbb{N}$, we obtain sets of regular structures $\mathcal{H}^\ell_{\mathrm{reg}}(N)$, for which the moduli spaces $\mathcal{M}(S^1 \times M; P^+, P^-; \underline{J}^{H/2^N})$ are cut out transversally for all $P^+, P^- \subset P(H^{(2)}/2^N, \leq 2^N)$. However, it follows that $\mathcal{H}^\ell_{\mathrm{reg}} = \bigcap\{\mathcal{H}^\ell_{\mathrm{reg}}(N) : N \in \mathbb{N}\}$ is still of the second category in $\mathcal{H}^\ell$. $\square$

1.4 Cobordism

Since our statements only hold up to a maximal period for the asymptotic orbits, we cannot use the same coherent Hamiltonian perturbation to compute the full contact homology. As seen above we must rescale the Hamiltonian for the cylindrical moduli spaces, which clearly affects the Hamiltonian perturbations for all punctured spheres. For showing that the graded vector space isomorphism we obtain is actually an isomorphism of graded algebras, we construct chain maps between the differential algebras for the different coherent Hamiltonian perturbations, which are defined by counting holomorphic curves in an almost complex manifold with cylindrical ends.

1.4.1 Moduli spaces

For a given Hamiltonian $H : M \to \mathbb{R}$ let $\tilde{H} : \mathbb{R} \times M \to \mathbb{R}$ be a smooth homotopy with $\tilde{H}(s, \cdot) = H/2$ for $s \leq -1$ and $\tilde{H}(s, \cdot) = H$ for $s \geq +1$. Besides that $\tilde{H}$ defines a homotopy of stable Hamiltonian structures $(\omega^{\tilde{H}},\lambda^{\tilde{H}})$ with corresponding (constant) symplectic hyperplane bundles $\xi^{\tilde{H}}=TM$ and $\mathbb{R}$-dependent Reeb vector fields $R^{\tilde{H}}(s,t,p)=\partial_t+X^{\tilde{H}}(s,t,p)$, it equips $\mathbb{R}\times S^1\times M$ with the structure of a symplectic manifold with stable cylindrical ends

$$((-\infty,-1]\times S^1\times M,\omega^{H/2},\lambda^{H/2}) \text{ and } ([+1,+\infty)\times S^1\times M,\omega^H,\lambda^H),$$

where the symplectic structure on the compact, non-cylindrical part $(-1,+1)\times S^1\times M$ is given by

$$\underline{\omega}^{\tilde{H}}=\omega^{\tilde{H}}+ds\wedge dt$$

with $\omega^{\tilde{H}}=\omega+d\tilde{H}\wedge dt$.

Together with the fixed ω-compatible almost complex structure J on M, the homotopy $\tilde{H}$ further equips $\mathbb{R}\times S^1\times M$ with an almost complex structure $\underline{J}^{\tilde{H}}$ by requiring that it turns $\xi^{\tilde{H}}=TM$ into a complex subbundle with complex structure J and

$$\underline{J}^{\tilde{H}}\cdot\partial_s=R^{\tilde{H}}(s,\cdot)=\partial_t+X^{\tilde{H}}(s,\cdot).$$

It follows that $(\mathbb{R}\times S^1\times M,\underline{J}^{\tilde{H}})$ is an almost complex manifold with cylindrical ends $((-\infty,-1]\times S^1\times M,\underline{J}^{H/2})$ and $([+1,+\infty)\times S^1\times M,\underline{J}^H)$. Note that $\underline{J}^{\tilde{H}}$ is indeed $\underline{\omega}^{\tilde{H}}$-compatible.

For our applications we clearly have to replace the Hamiltonian $H:M\to\mathbb{R}$ by the domain-dependent Hamiltonian perturbation $H:\coprod_n\mathcal{M}_{0,n+1}\times M\to\mathbb{R}$ from before. It follows that the Hamiltonian homotopy $\tilde{H}$ has to depend explicitly on points on the underlying stable punctured spheres, i.e., for the following we consider coherent Hamiltonian homotopies

$$\tilde{H}:\coprod_n\mathcal{M}_{0,n+1}\times\mathbb{R}\times M\to\mathbb{R},$$

with corresponding domain-dependent almost complex structures

$$\underline{J}^{\tilde{H}}:\coprod_n\mathcal{M}_{0,n+1}\to\mathcal{J}(S^1\times M).$$

While it is again clear that the moduli spaces of $\underline{J}^{\tilde{H}}$-holomorphic curves with more than two punctures come with an S^1-symmetry, it remains to verify nondegeneracy for the asymptotic orbits and transversality for the curves. Note for the first that we again have to consider rescaled versions $\tilde{H}_N:\coprod_n\mathcal{M}_{0,n+1}\times\mathbb{R}\times M\to\mathbb{R}$ with $\tilde{H}_N(s)=\tilde{H}(s/2^N)/2^N$. Since $\tilde{H}_N(s)=H/2^{N+1}$ for $s\leq -2^N$ and $\tilde{H}_N(s)=H/2^N$ for $s\geq +2^N$, it is clear that the nondegeneracy holds for all asymptotic orbits of period less or equal to 2^N.

While we show below that we can again achieve transversality for all $\underline{J}^{\tilde{H}}$-holomorphic curves with more than three punctures making use of the domain-dependency of the almost complex structure, it remains to guarantee transversality for $\underline{J}^{\tilde{H}}$-holomorphic cylinders. Note that in analogy to proposition 1.1.6 it follows that all $\underline{J}^{\tilde{H}}_N$-holomorphic cylinders connecting orbits (x^+,T) and (x^-,T) with $T \leq 2^N$ are in natural correspondence to cylinders in M connecting the critical points x^+,x^-, which satisfy the $\mathbb{R}$-dependent perturbed Cauchy-Riemann equation

$$\bar{\partial}_{J,H} u \cdot \partial_s = \partial_s u + J(u) \cdot (\partial_t u + T \cdot X^{\tilde{H}}(Ts,u)) = 0.$$

While in general transversality generically only holds for t-dependent Hamiltonian homotopies $\tilde{H}$, we can now make use of the following natural generalization of lemma 1.1.5:

Lemma 1.4.1: *Let (H,J) be a pair of a Hamiltonian H and an almost complex structure J on a closed symplectic manifold with $\langle [\omega], \pi_2(M)\rangle = 0$ so that $(H, \omega(\cdot, J\cdot))$ is Morse-Smale. Choose $\varphi \in C^\infty(\mathbb{R},\mathbb{R}^+)$ with $\varphi(s) = 1/2$ for $s \leq -1$ and $\varphi(s) = 1$ for $s \geq 1$, and let $\tilde{H} : \mathbb{R} \times M \to \mathbb{R}$, $\tilde{H}(s,p) = \varphi(s) \cdot H(p)$. Then the following holds:*

- *The linearization $\tilde{F}_u$ of $\nabla_{J,\tilde{H}} u = \partial_s u + J(u) X^{\tilde{H}}(s,u)$ is surjective at all solutions.*
- *If $\tau > 0$ is sufficiently small, all finite energy solutions $u : \mathbb{R} \times S^1 \to M$ of $\bar{\partial}_{J,\tilde{H}^\tau} u = \partial_s u + J(u)(\partial_t u + X^{H^\tau}(s,u)) = 0$ with $\tilde{H}^\tau(s,\cdot) = \tau \tilde{H}(\tau s, \cdot)$ are independent of $t \in S^1$.*
- *In this case, the linearization $\tilde{D}_u = \tilde{D}^\tau_u$ of $\bar{\partial}_{J,\tilde{H}^\tau}$ is onto at any solution $u : \mathbb{R} \times S^1 \to M$.*

Proof: The proof is a simple generalization of the arguments given in [SZ] and we just show the first statement. Let $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}^+$ with $\partial_s \tilde{\varphi} = \varphi$. Then $\tilde{u}(s) = u(\tilde{\varphi}(s))$ satisfies $\nabla_{J,\tilde{H}} \tilde{u} = 0$ whenever $u : \mathbb{R} \to M$ is a solution of $\nabla_{J,H} u = 0$, since

$$\partial_s \tilde{u} + \nabla \tilde{H}(s,\tilde{u}) = \partial_s \tilde{\varphi}(s) \cdot \partial_s u + \varphi(s) \cdot \nabla H(u)\,.$$

For $\tilde{\eta} \in L^p(\tilde{u}^* TM)$ we find $\eta \in L^p(u^*TM)$ so that $\tilde{\eta}(s) = \eta(\tilde{\varphi}(s))$. Assuming that $\langle F_{\tilde{u}} \tilde{\xi}, \tilde{\eta}\rangle = 0$ for all $\tilde{\xi} \in H^{1,p}(\tilde{u}^*TM)$, it follows that $\langle F_u \xi, \eta \rangle = 0$ for all $\xi \in H^{1,p}(u^*TM)$ by identifying $\tilde{\xi}(s) = \xi(\tilde{\varphi}(s))$, where $\tilde{F}_{\tilde{u}}$, F_u denote the linearizations of $\nabla_{J,\tilde{H}}$, $\nabla_{J,H}$ at $\tilde{u}, u$, respectively. The regularity of (H,J) provides us with the surjectivity of F_u at any solution $u : \mathbb{R} \to M$, so that η and therefore $\tilde{\eta}$ must vanish. $\Box$

With the fixed Hamiltonian $H^{(2)} : M \to \mathbb{R}$ for the cylinders we choose the Hamiltonian homotopy for the cylinders $\tilde{H}^{(2)} : \mathbb{R} \times M \to \mathbb{R}$ to be

$$\tilde{H}^{(2)}(s,p) = \varphi(s) \cdot H^{(2)}(p),$$

so that $\tilde{H}^{(2)}(s,\cdot) = H^{(2)}/2$ for $s \leq -1$ and $\tilde{H}^{(2)}(s,\cdot) = H^{(2)}$. After possibly rescaling $H^{(2)}$, we can and will assume that both lemma 1.1.5 and lemma 1.4.1 hold with $\tau = 1$ for the fixed J and the chosen $H^{(2)}$, $\tilde{H}^{(2)}$, respectively.

Before we prove transversality in the next subsection, let us state the following analogue of theorem 1.1.6. Denote by $\underline{J}_N^{\tilde{H}}$ the domain-dependent almost complex structure on $\mathbb{R} \times S^1 \times M$ induced by $\tilde{H}_N$.

Theorem 1.4.2: *Depending on the number of punctures n we have the following result about the moduli spaces of $\underline{J}_N^{\tilde{H}}$-holomorphic curves in $\mathbb{R} \times S^1 \times M$:*

- $n = 0$*: All holomorphic spheres are constant.*
- $n = 1$*: Holomorphic planes do not exist.*
- $n = 2$*: For $T \leq 2^N$ the automorphism group $\mathrm{Aut}(\mathbb{CP}^1)$ acts on the moduli space of parametrized curves $\mathcal{M}^0(S^1 \times M, (x^+, T), (x^-, T), \underline{J}_N^{\tilde{H}})$ of holomorphic cylinders with constant finite isotropy group $\mathbb{Z}_T$ and the quotient can be naturally identified with the space of gradient flow lines of $H^{(2)}$ with respect to the metric $\omega(\cdot, J\cdot)$ on M between the critical points x^+ and x^- of $H^{(2)}$. In particular, we have*

  $$\sharp\, \mathcal{M}(\mathbb{R} \times S^1 \times M; (x^+, T), (x^-, T); \underline{J}_N^{\tilde{H}}) = \delta_{x^-, x^+}$$

  since the zero-dimensional components are empty for $x^+ \neq x^-$ and just contain the constant path for $x^+ = x^-$.

- $n \geq 3$*: For $P^+ \subset P(H^{(2)}/2^N, \leq 2^N)$ and $P^- \subset P(H^{(2)}/2^{N+1}, \leq 2^N)$ the action of $\mathrm{Aut}(\mathbb{CP}^1)$ on the moduli space of parametrized curves is free and the moduli space is given by the product*

  $$S^1 \times \{(s_0, u, \underline{z}) : s_0 \in \mathbb{R}, u : \mathbb{CP}^1 - \{\underline{z}\} \to M : (*1), (*2)\} / \mathrm{Aut}(\mathbb{CP}^1)$$

  with

  $$\begin{aligned} (*1) : \quad & du + X_{\underline{z}}^{\tilde{H}_N}(z, h_1^0 + s_0, u) \otimes dh_2^0 \\ & + J(u) \cdot (du + X_{\underline{z}}^{\tilde{H}_N}(z, h_1^0 + s_0, u) \otimes dh_2^0) \cdot i = 0\,, \\ (*2) : \quad & u \circ \psi_k^{\pm}(s, t) \stackrel{s \to \pm\infty}{\longrightarrow} x_k^{\pm}. \end{aligned}$$

  In particular, it remains a free S^1-action on the moduli space.

Proof: The proof is completely analogous to the one of theorem 1.1.6. Note that it follows by lemma 1.1.3 that $h : \mathbb{CP}^1 - \{\underline{z}\} \to \mathbb{R} \times S^1$ can be identified with $(s_0, t_0) \in \mathbb{R} \times S^1$ and that the map u now satisfies an s_0-dependent perturbed Cauchy-Riemann equation. For $n=2$ observe that by lemma 1.3.1 we can identify $\mathcal{M}(S^1\times M;(x^+,T),(x^-,T);\underline{J}_N^{\tilde{H}})$ with the space of all $u:\mathbb{R}\to M$ satisfying $\nabla_{J,\tilde{H}^{(2)}}u=0$, $u(s,t)\to x^{\pm}$, which following the proof of lemma 1.3.1 can be identified with the space of $\tilde{u}(s)=u(\tilde{\varphi}(s))$ satisfying $\nabla_{J,H^{(2)}}u=0$. □

1.4.2 Transversality

For the remaining part of this section we discuss transversality, where we again restrict ourselves to the case $N=0$:

Since $\bar{\partial}_{\underline{J}^{\tilde{H}}}(h,u)=(\bar{\partial}h,\bar{\partial}_{J,\tilde{H},s_0}u)$ with

$$\begin{aligned}\bar{\partial}_{J,\tilde{H},s_0}u &= du+X^{\tilde{H}}(j,z,h_1^0+s_0,u)\otimes dh_2^0\\ &+ J(u)\cdot(du+X^{\tilde{H}}(j,z,h_1^0+s_0,u)\otimes dh_2^0)\cdot i,\end{aligned}$$

where $X^{\tilde{H}}(j,z,s,u)$ denotes the symplectic gradient of $\tilde{H}(j,z,s,\cdot):M\to\mathbb{R}$, it follows that the linearization $D_{h,u}$ of $\bar{\partial}_{\underline{J}^{\tilde{H}}}$ is again of diagonal form.

It follows that for $n=2$ we get transversality from lemma 1.3.2 and lemma 1.4.1 by the special choice of $\tilde{H}^{(2)}$.

For $n\geq 3$ let us describe the setup for the underlying universal Fredholm problem:

As before the Cauchy-Riemann operator extends to a C^∞-section in a Banach space bundle $\tilde{\mathcal{E}}^{p,d}\to\mathcal{B}^{p,d}\times\tilde{\mathcal{H}}^{\ell}$. Here $\mathcal{B}^{p,d}=\mathcal{B}^{p,d}(\mathbb{R}\times S^1\times M;P^+,P^-)$ denotes the manifold of maps from section 5, which is given by the product

$$\mathcal{B}^{p,d}(\mathbb{R}\times S^1\times M;(x_k^{\pm},T_k^{\pm}))=H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})\times\mathcal{B}^p(M;(x_k^{\pm}))\times\mathcal{M}_{0,n}\,,$$

while the set of coherent Hamiltonian perturbations $\mathcal{H}_n^{\ell}(M;(H^{(k)})_{k=2}^{n-1})$ is now replaced by the set of coherent Hamiltonian homotopies

$$\tilde{\mathcal{H}}^{\ell}\;=\;\tilde{\mathcal{H}}_n^{\ell}(M;H;(\tilde{H}^{(k)})_{k=2}^{n-1})$$

for fixed coherent Hamiltonian $H:\coprod_n\mathcal{M}_{n+1}\times M\to\mathbb{R}$ and $\tilde{H}^{(2)},...,\tilde{H}^{(n-1)}$:
Any $\tilde{H}^{(n)}\in\tilde{\mathcal{H}}^{\ell}$ is a C^{ℓ}-map

$$\tilde{H}^{(n)}:\mathcal{M}_{0,n+1}\times\mathbb{R}\times M\to\mathbb{R},$$

which extends to a C^{ℓ}-map on $\overline{\mathcal{M}}_{0,n+1}\times\mathbb{R}\times M$, so that

- on $\big((\overline{\mathcal{M}}_{0,n+1}-\mathcal{M}_{0,n+1})\cup(\mathcal{M}_{0,n}\times N_0)\big)\times\mathbb{R}\times M$ it is given by $\tilde{H}^{(2)},...,\tilde{H}^{(n-1)}$,
- $\tilde{H}^{(n)}=H^{(n)}/2$ on $\mathcal{M}_{0,n+1}\times(-\infty,-2^N)\times M$,
- and $\tilde{H}^{(n)}=H^{(n)}$ on $\mathcal{M}_{0,n+1}\times(+2^N,+\infty)\times M$,

where $N_0\subset\dot{S}$ again denotes the fixed neighborhood of the punctures. It follows that the tangent space at $\tilde{H}=\tilde{H}^{(n)}\in\tilde{\mathcal{H}}^\ell$ is given by

$$T_{\tilde{H}}\tilde{\mathcal{H}}^\ell_n \;=\; \tilde{\mathcal{H}}^\ell_n(M;0;(0)_{k=2}^{n-1}).$$

Since the linearization of $\bar{\partial}_{\underline{J}^{\tilde{H}}}$ at $(\bar{h},u,j,\tilde{H})\in\mathcal{B}^{p,d}\times\tilde{\mathcal{H}}^\ell$ is again of diagonal form,

$$\begin{aligned}&D_{\bar{h},u,j,\tilde{H}} \;=\; D_j+\mathrm{diag}(\bar{\partial},D_{u,\tilde{H}}):\\ &T_j\,\mathcal{M}_{0,n}\oplus H^{1,p,d}_{\mathrm{const}}(\dot{S},\mathbb{R}^2)\oplus H^{1,p}(u^*TM)\oplus T_{\tilde{H}}\tilde{\mathcal{H}}^\ell\\ &\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{R}^2)\oplus L^p(T^*\dot{S}\otimes_{j,J}u^*TM)\end{aligned}$$

it remains by lemma 1.3.2 to prove surjectivity of $D_{u,\tilde{H}}$, which is the linearization of the perturbed Cauchy-Riemann operator $\bar{\partial}_{J,s_0}(u,\tilde{H})=\bar{\partial}_{J,\tilde{H},s_0}(u)$. Since the proof is in the central arguments completely similar to lemma 1.3.3, we just sketch the main points:

Assume for some $\eta\in L^p(T^*\dot{S}\otimes_{j,J}u^*TM)$ that $\langle D_{u,\tilde{H}}(\xi,\tilde{G}),\eta\rangle=0$ for all $(\xi,\tilde{G})\in H^{1,p}(u^*TM)\oplus T_{\tilde{H}}\tilde{\mathcal{H}}^\ell$. From $\langle\eta,D_u\xi\rangle=0$ for all ξ we already know that it suffices to show that η vanishes on an open and dense subset.

Now observe that it follows from the same arguments used to prove lemma 1.3.3 that

$$0\;=\;\langle D_{\tilde{H}}\tilde{G},\eta\rangle\;=\;\int_{\dot{S}-B}\|dh^0_1(z)\|^2\cdot d\tilde{G}(j,z,h^1_0(z)+s_0,u(z))\cdot\eta_1(z)\,dz$$

for all $\tilde{G}\in T_{\tilde{H}}\tilde{\mathcal{H}}^\ell$, where B is the set of branch points of the map $h^0:\dot{S}\to\mathbb{R}\times S^1$, we again write $\eta(z)=\eta_1(z)\otimes dh^0_1+\eta_2(z)\otimes dh^0_2$ with $\eta_2(z)+J(u)\eta_1(z)=0$ for $z\in\dot{S}-B$ and where $d\tilde{G}(j,z,h^1_0(z)+s_0,\cdot)$ denotes the differential of $\tilde{G}(j,z,h^1_0(z)+s_0,\cdot):M\to\mathbb{R}$. But with this we can prove as before that η vanishes identically on the open and dense subset $\dot{S}-B$:

Assume to the contrary that $\eta(z_0)\neq0$, i.e., $\eta_1(z_0)\neq0$ for some $z_0\in\dot{S}-B$. As in the proof of lemma 1.3.3 we find $G_0\in C^\infty(M)$ so that

$$dG_0(u(z_0))\cdot\eta_1(z_0)>0.$$

Setting $j_0:=j$, observe that we can organize all fixed maps $h_0:\dot{S}\to\mathbb{R}\times S^1$ for different j on $\dot{S}$ into a map $h_0:\mathcal{M}_{0,n+1}\to\mathbb{R}\times S^1$. Let

$\tilde{\varphi} \in C^\infty(\overline{\mathcal{M}}_{0,n+1} \times \mathbb{R}, [0,1])$ be a smooth cut-off function around $(j_0, z_0, h_0^1(j_0, z_0) + s_0) \in \mathcal{M}_{0,n+1} \times \mathbb{R}$ with $\varphi(j_0, z_0, h_0^1(j_0, z_0) + s_0) = 1$ and $\varphi(j, z, h_0^1(j, z) + s) = 0$ for $(j, z, s) \notin U(j_0, z_0, s_0)$. Here the neighborhood $U(j_0, z_0, s_0) \subset \overline{\mathcal{M}}_{0,n+1} \times \mathbb{R}$ is chosen so small that

$$\begin{aligned} U(j_0, z_0, s_0) \cap \Big(\big((\overline{\mathcal{M}}_{0,n+1} - \mathcal{M}_{0,n+1}) \cup (\mathcal{M}_{0,n+1} \times N_0) \big) \times \mathbb{R} \Big) &= \emptyset, \\ U(j_0, z_0, s_0) \cap \Big(\overline{\mathcal{M}}_{0,n+1} \times \big((-\infty, -1) \cup (+1, +\infty) \big) \Big) &= \emptyset, \end{aligned}$$

and $dG_0(z, u(z)) \cdot \eta_1(z) \geq 0$ for all $(z, j, h_0^1(j, z) + s) \in U(j_0, z_0, s_0)$.

Defining $\tilde{G} : \overline{\mathcal{M}}_{0,n+1} \times \mathbb{R} \times M \to \mathbb{R}$ by $\tilde{G}(j, z, s, p) := \varphi(j, z, s) \cdot G_0(p)$, this leads to the desired contradiction since we found $\tilde{G} \in T_{\tilde{H}} \tilde{\mathcal{H}}^\ell = \tilde{\mathcal{H}}_n^\ell(M; 0; 0, ..., 0)$ with

$$\langle D_{\tilde{H}} \cdot \tilde{G}, \eta \rangle = \int_{\dot{S}-B} \|dh_1^0(z)\|^2 \cdot d\tilde{G}(j_0, z, h_0^1(j_0, z) + s_0, u(z)) \cdot \eta_1(z)\, dz > 0.$$

So we have shown that the corresponding universal moduli space $\mathcal{M}(\mathbb{R} \times S^1 \times M; P^+, P^-; \underline{J}^H; (\underline{J}^{\tilde{H},(k)})_{k=2}^{n-1})$ is again transversally cut out by the Cauchy-Riemann operator $\bar{\partial}_J$. Further it follows by the same arguments as in section 4 that we can choose a (smooth) coherent Hamiltonian homotopy $\tilde{H} : \coprod_n \mathcal{M}_{0,n+1} \times \mathbb{R} \to C^\infty(M)$ such that for all $N \in \mathbb{N}$ and P^+, P^- the moduli spaces $\mathcal{M}(\mathbb{R} \times S^1 \times M; P^+, P^-; \underline{J}_N^{\tilde{H}})$ are transversally cut out by the Cauchy-Riemann operator.

1.5 Contact homology

1.5.1 Chain complex

The contact homology of $S^1 \times M$ equipped with the stable Hamiltonian structure (ω^H, λ^H) is defined as the homology of a differential graded algebra $(\mathfrak{A}, \partial)$, which is generated by closed orbits of the Reeb vector field R^H and whose differential counts $\underline{J}^H$-holomorphic curves with one positive puncture. As in [EGH] we start with assigning to any $(x, T) \in P(H)$, which is *good* in the sense of [BM], a graded variable $q_{(x,T)}$ with

$$\deg q_{(x,T)} = \dim M/2 - 2 + \mu_{CZ}(x, T).$$

Here μ_{CZ} denotes the Conley-Zehnder index for (x, T), which is defined as in [EGH] after fixing a basis for $H_1(S^1 \times M)$ and choosing a spanning surface between the orbit (x, T) and suitable linear combinations of these basis elements. Note that in the corresponding definition in [EGH] one adds $n-3$, where n denotes the complex dimension of $\mathbb{R} \times S^1 \times M$. Further we assume, as in [EGH], that $H_1(S^1 \times M)$ and hence $H_1(M)$ is torsion-free, where we use that the torsion-freedom of $H_*(S^1)$ also yields the Kuenneth formula for $H_*(S^1 \times M)$. Let

$$\mathbb{Q}[H_2(S^1 \times M)] = \{\sum q(A) e^A : A \in H_2(S^1 \times M), q(A) \in \mathbb{Q}\}$$

be the group algebra generated by $H_2(S^1 \times M) \cong H_2(M) \oplus (H_1(S^1) \otimes H_1(M))$. Since $c_1(TM)$ clearly vanishes on $H_1(S^1) \otimes H_1(M)$ we can and will work with the reduced group ring $\mathbb{Q}[H_2(M)]$. With this let $\mathfrak{A}_*$ be the graded commutative algebra of polynomials in the good periodic orbits

$$f = \sum_{\underline{q}} f(\underline{q})\, q_{(x_1,T_1)}^{j_1} \cdots q_{(x_n,T_n)}^{j_n}\,,$$

where

$$\underline{q} = (\overbrace{q_{(x_1,T_1)}, ..., q_{(x_1,T_1)}}^{j_1-\text{times}}, \overbrace{q_{(x_2,T_2)}, ..., q_{(x_2,T_2)}}^{j_2-\text{times}}, ...)$$

with coefficients $f(\underline{q})$ in $\mathbb{Q}[H_2(M)]$.

Let C_* be the vector space over $\mathbb{Q}$ freely generated by the graded variables $q_{(x,T)}$, which naturally splits, $C_* = \bigoplus_T C_*^T$ with C_*^T generated by the good orbits in $P(H,T)$. Since C_* is graded, we can define a graded symmetric algebra $\mathfrak{S}(C_*)$: Denoting by $\mathfrak{T}(C_*)$ the tensor algebra over C_*, the symmetric algebra is defined as quotient, $\mathfrak{S}(C_*) = \mathfrak{T}(C_*)/\mathfrak{I}$, where $\mathfrak{I}$ is the ideal freely generated by elements

$$a \otimes b + (-1)^{\deg a + \deg b + 1} b \otimes a \in \mathfrak{T}(C_*)$$

for pairs a, b of homogeneous elements in $\mathfrak{T}(C_*)$. Let $\mathfrak{S} : \mathfrak{T}(C_*) \to \mathfrak{S}(C_*)$ denote the projection. One easily sees that $\mathfrak{S}(C_*)$ is the graded commutative algebra freely generated by the basis elements of C_* with rational coefficients, so that $\mathfrak{A}_*$ agrees with the tensor product of the graded symmetric algebra over C_* with the group algebra $\mathbb{Q}[H_2(M)]$,

$$\mathfrak{A}_* = \mathfrak{S}(C_*) \otimes \mathbb{Q}[H_2(M)].$$

For the following we assume that all occuring periodic orbits are good.

Note that to any holomorphic curve in $\mathcal{M}(S^1 \times M; P^+, P^-; \underline{J}^H)$ we assign as in [EGH] a homology class $A \in H_2(S^1 \times M)$ after fixing a basis for $H_1(S^1 \times M)$ and choosing spanning surfaces between the asymptotic orbits in $P^+, P^- \subset P(H)$ and suitable linear combinations of these basis elements. For fixed $(x_0, T_0) \in P(H)$ we follow [EGH] and denote by $\mathrm{h}_{(x_0,T_0)} \in \mathfrak{A}$ the generating function, which counts the algebraic number of holomorphic curves with $P^+ = \{(x_0, T_0)\}$ but arbitary orbit set $P^- = \{(x_1^-, T_1^-), ..., (x_n^-, T_n^-)\}$,

$$\mathrm{h}_{(x_0,T_0)} = \sum_{P^-,A} \sharp\, \mathcal{M}_A(S^1 \times M; P^+, P^-; \underline{J}^H)/\mathbb{R}\;\; q_{(x_1^-,T_1^-)} \cdots q_{(x_n^-,T_n^-)}\; e^A,$$

where $\mathcal{M}_A(S^1 \times M; P^+, P^-; \underline{J}^H)$ denotes the one-dimensional component of the moduli space, whose curves represent the class $A \in H_2(M) \cong H_2(S^1 \times M)/(H_1(S^1) \otimes H_1(M))$. Note that in comparison to [EGH] we have not introduced asymptotic markers at the punctures, so we do not have to quotient by the number of their possible positions. Then the differential $\partial : \mathfrak{A} \to \mathfrak{A}$ is defined by (see [EGH],p.621)

$$\partial f = \sum_{(x_0,T_0) \in P(H)} \mathrm{h}_{(x_0,T_0)} \frac{\partial f}{\partial q_{(x_0,T_0)}}.$$

Setting $d_k = \deg(q_{(x_k,T_k)})$, we get for the monomial $f = q^{j_1}_{(x_1,T_1)} \dots q^{j_n}_{(x_n,T_n)}$ that

$$\begin{aligned}
&\partial\big(q^{j_1}_{(x_1,T_1)} \cdots q^{j_n}_{(x_n,T_n)}\big) \\
= \; & \sum_{k=1}^{n} \mathrm{h}_{(x_k,T_k)} \, \frac{\partial}{\partial q_{(x_k,T_k)}} \big(q^{j_1}_{(x_1,T_1)} \cdots q^{j_n}_{(x_n,T_n)}\big) \\
= \; & \sum_{k} \sum_{l=1}^{j_k} (-1)^{j_1 d_1 + \dots + j_{k-1} d_{k-1} + (l-1) d_k} q^{j_1}_{(x_1,T_1)} \cdots q^{j_{k-1}}_{(x_{k-1},T_{k-1})} \\
& \cdot \, q^{l-1}_{(x_k,T_k)} \, \cdot \, \Big(\mathrm{h}_{(x_k,T_k)} \cdot \frac{\partial}{\partial q_{(x_k,T_k)}} q_{(x_k,T_k)}\Big) \, \cdot \, q^{j_k - l}_{(x_k,T_k)} q^{j_{k+1}}_{(x_{k+1},T_{k+1})} \cdots q^{j_n}_{(x_n,T_n)} \\
= \; & \sum_{k} \sum_{l=1}^{j_k} (-1)^{j_1 d_1 + \dots + j_{k-1} d_{k-1} + (l-1) d_k} q^{j_1}_{(x_1,T_1)} \cdots q^{j_{k-1}}_{(x_{k-1},T_{k-1})} \, \cdot \, q^{l-1}_{(x_k,T_k)} \\
& \partial q_{(x_k,T_k)} \, \cdot \, q^{j_k - l}_{(x_k,T_k)} q^{j_{k+1}}_{(x_{k+1},T_{k+1})} \cdots q^{j_n}_{(x_n,T_n)}
\end{aligned}$$

with

$$\begin{aligned}
\partial q_{(x_k,T_k)} \; &= \; \mathrm{h}_{(x_k,T_k)} \cdot \frac{\partial}{\partial q_{(x_k,T_k)}} q_{(x_k,T_k)} \\
&= \; \sum_{P^-,A} \sharp\, \mathcal{M}_A(S^1 \times M; P^+, P^-; \underline{J}^H)/\,\mathbb{R} \; \cdot q_{(x_1^-,T_1^-)} \dots q_{(x_n^-,T_n^-)} \; e^A,
\end{aligned}$$

i.e., ∂ satisfies a graded Leibniz rule. Note that for commuting the variables we made use of

$$\deg(\mathrm{h}_{(x_0,T_0)} \cdot \partial/\partial q_{(x_k,T_k)}) = 1,$$

which follows from

$$\deg(\partial/\partial q_{(x_k,T_k)}) = \deg(q_{(x_k,T_k)}), \;\; \deg \mathrm{h}_{(x_k,T_k)} = \deg(q_{(x_k,T_k)}) - 1.$$

For $(T_1, ..., T_n) \in \mathbb{N}^n$ let $\mathfrak{A}^{(T_1,...,T_n)}$ denote the subspace of $\mathfrak{A}$ spanned by monomials $q_{(x_1,T_1)} \cdots q_{(x_n,T_n)}$,

$$\mathfrak{A}^{(T_1,...,T_n)} = \mathfrak{S}^{(T_1,...,T_n)}(C_*) \otimes \mathbb{Q}[H_2(M)] := \mathfrak{S}(\mathfrak{T}^{(T_1,...,T_n)}(C_*)) \otimes \mathbb{Q}[H_2(M)],$$

where

$$\mathfrak{T}^{(T_1,...,T_n)}(C_*) = C_*^{T_1} \otimes ... \otimes C_*^{T_n}.$$

Note in particular that $\mathfrak{A}^{(T_1,...,T_n)}$ does not depend on the ordering of the $T_1, ..., T_n$. Since $\sharp\, \mathcal{M}(S^1 \times M; P^+, P^-; \underline{J}^H)/\,\mathbb{R} = 0$ for $T_1^- + ... + T_n^- \neq T_k$ by lemma 1.1.3, it follows from the above calculations that the differential ∂ respects the splitting

$$\mathfrak{A} = \bigoplus_{T \in \mathbb{N}} \mathfrak{A}^T,$$

where $\mathfrak{A}^T = \bigoplus_{T_1 + ... + T_n = T} \mathfrak{A}^{(T_1,...,T_n)}$.

1.5.2 Proof of main theorem A

In what follows we use our results about holomorphic curves in $\mathbb{R}\times S^1\times M$ to prove main theorem A. At first we compute $H_*(\mathfrak{A}^{\leq 2^N},\partial)=\bigoplus_{T\leq 2^N}H_*(\mathfrak{A}^T,\partial)$ using our results about moduli spaces of holomorphic curves in $\mathbb{R}\times S^1\times M$ in theorem 1.1.6 together with the transversality results:

With the fixed almost complex structure J on M let $H:\coprod\mathcal{M}_{0,n+1}\to C^\infty(M)$ be a coherent Hamiltonian perturbation as before, in particular, $H^{(2)}$ satisfies lemma 1.1.5 with $\tau=1$. Following corollary 1.3.4 we further assume that H is chosen such that transversality holds for all moduli spaces $\mathcal{M}(S^1\times M;P^+,P^-;\underline{J}^{H/2^N})$, $P^\pm\subset P(H^{(2)}/2^N,\leq 2^N)$, simultaneously for all $N\in\mathbb{N}$. Together with theorem 1.1.6 it then follows that for defining the algebraic invariants we only have to count gradient flow lines of the function $H^{(2)}$ on M with respect to the metric $g_J=\omega(\cdot,J\cdot)$ on M.

For $N\in\mathbb{N}$ let $(\mathfrak{A}_N,\partial_N)$ denote the differential algebra for the domain-dependent Hamiltonian $H/2^N:\coprod\mathcal{M}_{0,n+1}\to C^\infty(M)$ and the fixed almost complex structure J on M. For the computation of the contact homology subcomplex we use special choices for the basis elements in $H_1(S^1\times M)$ and the spanning surfaces as follows: Choose a basis for $H_1(S^1\times M)=H_1(S^1)\oplus H_1(M)$ containing the canonical basis element $[S^1]$ of $H_1(S^1)$, which is represented by the circle $(x^*,1):S^1\to S^1\times M$, $t\mapsto(t,x^*)$ for some point $x^*\in M$. For any periodic orbit $(x,T)\in P(H^{(2)}/2^N,\leq 2^N)$ we have $[(x,T)]=T[S^1]\in H_1(S^1\times M)$, since x is a constant orbit in M, and we naturally specify a spanning surface $S_{(x,T)}$ between (x,T) and the T-fold cover of $(x^*,1)$ by choosing a path $\gamma_x:[0,1]\to M$ from x^* to x and setting $S_{(x,T)}:S^1\times[0,1]\to S^1\times M$, $S_{(x,T)}(t,r)=(Tt,\gamma_x(r))$.

Lemma 1.5.1 *Let $HM_*=HM_*(M,-H^{(2)},g_J;\mathbb{Q})$ denote the Morse homology for the Morse function $-H^{(2)}$ and the metric $g_J=\omega(\cdot,J\cdot)$ on M with rational coefficients. Then we have*

$$H_*(\mathfrak{A}_N^{\leq 2^N},\partial_N)=\mathfrak{S}^{\leq 2^N}(\bigoplus_{\mathbb{N}}HM_{*-2})\otimes\mathbb{Q}[H_2(M)]\,,$$

where

$$\mathfrak{S}^{\leq 2^N}(\bigoplus_{\mathbb{N}}HM_{*-2})=\bigoplus_{T_1+\ldots+T_n\leq 2^N}\mathfrak{S}^{(T_1,\ldots,T_n)}(\bigoplus_{\mathbb{N}}CM_{*-2}).$$

Proof: For the grading of the q-variables we have

$$\deg q_{(x,T)}=\dim M/2-2+\mu_{CZ}(x,T)=\mathrm{ind}_{-H}(x)-2,$$

when we choose a canonical trivialization of TM over $(x^*,1)$ and extend it over the spanning surfaces to a canonical trivialization over (x,T), i.e., the map $\Theta:S^1\times\mathbb{R}^{2m}\to x^*TM=S^1\times T_xM$ is independent of S^1. It follows that C_*^T agrees with the chain group CM_{*-2} for the Morse homology for $T\leq 2^N$ and therefore

$$\mathfrak{A}_N^{\leq 2^N}=\mathfrak{S}^{\leq 2^N}(\bigoplus_{\mathbb{N}}CM_{*-2})\otimes\mathbb{Q}[H_2(M)]\,.$$

Here it is important to observe that any $(x,T) \in P(H^{(2)}/2^N, \leq 2^N)$ is indeed good in the sense of [BM]: note that it follows from $\mu_{CZ}(x,T) = \mathrm{ind}_{-H}(x) - \dim M/2$ that $\mu_{CZ}(x,T)$ has the same parity for all $T \leq 2^N$.

It follows from theorem 1.1.6 that the generating function for $(x_0,T_0) \in P(H^{(2)}/2^N, \leq 2^N)$ is given by

$$\mathrm{h}^N_{(x_0,T_0)} = \sum_{x,A} \sharp\, \mathcal{M}_A((x_0,T),(x,T))/\,\mathbb{R}\;\; q_{(x,T_0)} e^A.$$

where all curves in $\mathcal{M}((x_0,T),(x,T))/\,\mathbb{R}$ are gradient flow lines. Further it follows from the above choice of spanning surfaces that they all represent the trivial class $A \in H_2(M) = H_2(S^1 \times M)/(H_1(S^1) \otimes H_1(M))$: Indeed, letting u denote the gradient flow line between x_0 and x it follows that u represents the class $A = T[S^1] \otimes [\gamma_{x_0} \sharp u \sharp - \gamma_x] \in H_1(S^1) \otimes H_1(M)$. Hence we in fact have

$$\mathrm{h}^N_{(x_0,T_0)} = \sum_x \sharp(x_0,x)\;\, q_{(x,T_0)} = \partial^{\mathrm{Morse}} q_{(x_0,T_0)}$$

with $\sharp(x,x_0)$ denoting the algebraic number of gradient flow lines of $-H^{(2)}$ from x_0 to $x \in \mathrm{Crit}(H^{(2)})$. It follows that the differential ∂_N on $\mathfrak{A}^{\leq 2^N}_N$ is given by

$$\begin{aligned} & \partial_N\big(q^{j_1}_{(x_1,T_1)} \cdots q^{j_n}_{(x_n,T_n)}\big) \\ = \;\; & \sum_k \sum_{l=1}^{j_k} (-1)^{j_1 d_1 + \ldots + j_{k-1} d_{k-1} + (l-1) d_k} q^{j_1}_{(x_1,T_1)} \cdots q^{j_{k-1}}_{(x_{k-1},T_{k-1})} \\ & \cdot\, q^{l-1}_{(x_k,T_k)} \,\cdot\, \partial^{\mathrm{Morse}} q_{(x_k,T_k)} \,\cdot\, q^{j_k - l}_{(x_k,T_k)} q^{j_{k+1}}_{(x_{k+1},T_{k+1})} \cdots q^{j_n}_{(x_n,T_n)}, \end{aligned}$$

in particular, ∂_N respects the natural splitting

$$\mathfrak{A}^{\leq 2^N}_N = \bigoplus_{T_1+\ldots+T_n \leq 2^N} \mathfrak{A}^{(T_1,\ldots,T_n)}_N = \bigoplus_{T_1+\ldots+T_n \leq 2^N} \mathfrak{S}^{(T_1,\ldots,T_n)}\big(\bigoplus_{\mathbb{N}} CM_{*-2}\big) \otimes \mathbb{Q}[H_2(M)].$$

Using the graded Leibniz rule, the Morse boundary operator $\partial^{\mathrm{Morse}}$ on CM_{*-2} extends to a differential $\partial^{\mathrm{Morse}}_{\otimes n}$ on the tensor product

$$\mathfrak{T}^{(T_1,\ldots,T_n)}\big(\bigoplus_{\mathbb{N}} CM_{*-2}\big) = CM^{\otimes n}_{*-2}.$$

With the projection

$$\mathfrak{S} : \mathfrak{T}^{(T_1,\ldots,T_n)}\big(\bigoplus_{\mathbb{N}} CM_{*-2}\big) \to \mathfrak{S}^{(T_1,\ldots,T_n)}\big(\bigoplus_{\mathbb{N}} CM_{*-2}\big)$$

it directly follows from the definition of $\partial^{\mathrm{Morse}}_{\otimes n}$ and the above computation for ∂ that

$$\partial \circ \mathfrak{S} \;=\; \mathfrak{S} \circ \partial^{\mathrm{Morse}}_{\otimes n}\,.$$

With the theorem of Künneth we get

$$\begin{aligned}
H_*(\mathfrak{A}_N^{(T_1,\dots,T_n)},\partial) &= H_*(\mathfrak{S}^{(T_1,\dots,T_n)}(\bigoplus_{\mathbb{N}} CM_{*-2})\otimes\mathbb{Q}[H_2(M)],\partial)\\
&= \mathfrak{S}\left(H_*(\mathfrak{T}^{(T_1,\dots,T_n)}(\bigoplus_{\mathbb{N}} CM_{*-2}),\partial^{\mathrm{Morse}}_{\otimes n})\right)\otimes\mathbb{Q}[H_2(M)]\\
&= \mathfrak{S}\left(\mathfrak{T}^{(T_1,\dots,T_n)}(H_*(\bigoplus_{\mathbb{N}} CM_{*-2},\partial^{\mathrm{Morse}}))\right)\otimes\mathbb{Q}[H_2(M)]\\
&= \mathfrak{S}\big(\mathfrak{T}^{(T_1,\dots,T_n)}(\bigoplus_{\mathbb{N}} HM_{*-2})\big)\otimes\mathbb{Q}[H_2(M)]\\
&= \mathfrak{S}^{(T_1,\dots,T_n)}\big(\bigoplus_{\mathbb{N}} HM_{*-2}\big)\otimes\mathbb{Q}[H_2(M)]
\end{aligned}$$

and the claim follows. □

With this we can now complete the proof of main theorem by using theorem 1.4.2 and the transversality result of section four:

To this end choose a coherent Hamiltonian homotopy $\tilde{H}:\coprod_n \mathcal{M}_{0,n+1}\times\mathbb{R}\to C^\infty(M)$ as in section four, i.e., with $\tilde{H}(j,z,s,p) = H(j,z,p)/2$ for small s and $\tilde{H}(j,z,s,p) = H(j,z,p)$ for large s such that for all $N\in\mathbb{N}$ and P^+,P^- the moduli spaces $\mathcal{M}(\mathbb{R}\times S^1\times M;P^+,P^-;\underline{J}_N^{\tilde{H}})$ are transversally cut out. Let $\underline{J}_N^{\tilde{H}}$ denotes the coherent non-cylindrical almost complex structure on $\mathbb{R}\times S^1\times M$ induced by J and $\tilde{H}/2^N$.

Let $\Psi_N:(\mathfrak{A}_N,\partial_N)\to(\mathfrak{A}_{N+1},\partial_{N+1})$ be the chain homotopy, defined as in [EGH], by counting holomorphic curves with one positive puncture and an arbitrary number of negative punctures in the resulting almost complex manifold $(\mathbb{R}\times S^1\times M,\underline{J}_N^{\tilde{H}})$ with cylindrical ends. Then it follows from theorem 1.4.2 that the restriction $\Psi_N^T:(\mathfrak{A}_N^T,\partial_N)\to(\mathfrak{A}_{N+1}^T,\partial_{N+1})$ is the identity for $T\leq 2^N$, since again all curves with three or more punctures come in S^1-families and all zero-dimensional cylindrical moduli spaces just consist of trivial gradient flow lines.

Chapter 2

Trivial curves in rational SFT

2.0 Introduction

2.0.1 Summary

The second part of this thesis is concerned with the trivial examples of punctured holomorphic curves studied in rational symplectic field theory. Recall from the introduction that for all our expositions we assume that *the stable Hamiltonian structure is generic in the sense that all periodic orbits are nondegenerate in the sense of [BEHWZ].*

As we already mentioned at the beginning it is in general impossible to achieve transversality for all moduli spaces in symplectic field theory even for generic choices of J due to the presence of multiply covered curves. On the other the trivial examples of holomorphic curves studied in rational symplectic field theory are not only the trivial cylinders over closed Reeb orbits but also their multiple covers. Indeed we will show in this chapter that the branched covers of trivial cylinders are in fact the reason why transversality for generic J in general fails in symplectic field theory and whose contribution to the theory is not immediately clear. Indeed, as we already quoted at the beginning, it is easy to show that in every case where these trivial curves would contribute to the algebraic invariants by index reasons, transversality for the Cauchy-Riemann operator can never be satisfied, so that one has to perturb the Cauchy-Riemann operator appropriately and count elements in the resulting regular moduli spaces. Here it is important that the perturbation chosen for different moduli spaces are compatible with compactness and gluing in symplectic field theory. In order to obtain these compact perturbations we study sections in the cokernel bundle over the compactified moduli space, i.e., we generalize the obstruction bundle technique for determining the contribution of multiple covers to the algebraic invariants from Gromov-Witten theory to symplectic field theory. With this we prove the second main theorem of this thesis:

Main Theorem B: *We can choose compact perturbations of the Cauchy-Riemann operator, which make all moduli spaces of trivial curves regular in a way compatible* *with compactness and gluing, such that the algebraic counts of elements in all resulting zero-dimensional regular moduli spaces (modulo* $\mathbb{R}$*-shift) are zero.*

For the proof we show that for every moduli space of trivial curves the cokernels of the linearizations of the Cauchy-Riemann operator indeed fit together to give a global vector bundle over the corresponding compactified moduli space, and prove that there exists an Euler number for coherent (that is, gluing-compatible) sections in the cokernel bundle which is zero. While in Gromov-Witten theory the existence of the Euler number is immediately clear since all moduli spaces are pseudo-cycles, i.e., homologically have no boundary, but their computation is hard in general, the opposite is true here: Since the algebraic invariants of symplectic field theory rely on the codimension one boundary phenomena of the moduli spaces of punctured curves, i.e., the regular moduli spaces define relative rather than absolute virtual moduli cycles, Euler numbers for Fredholm problems in general do not exist since the count of zeroes in general depends on the compact perturbations chosen for the moduli spaces in the boundary. In this paper we make use of the fact that the moduli spaces in the boundary again consist of branched covers of trivial cylinders and prove the existence of the Euler number by induction on the number of punctures. For the induction step we do not only use that there exist Euler numbers for the moduli spaces in the boundary, but it is further important that all these Euler numbers are in fact trivial. The vanishing of the Euler number in turn can be deduced from the different parities of the actual and the virtual dimensions of the moduli spaces following the idea for the vanishing of the Euler characteristic for odd-dimensional manifolds. From some invariance argument we deduce that, once the analytical foundations of symplectic field theory are established, the result about sections in the cokernel bundles suffices to prove that the algebraic number of elements in the regular moduli spaces, obtained by adding general compact perturbations to the Cauchy-Riemann operator are still zero even when the abstract perturbations no longer result from sections in the cokernel bundles. Despite the analytical work in order to show that the cokernels fit together to give a nice vector bundle and showing that studying sections in it gives the right result, the strategy of our proof indeed only relies on the difference of the parity of the Fredholm index, i.e., the virtual dimension of the moduli space, and the actual dimension of the moduli space, including the moduli spaces in the boundary. Hence it should be applicable to a wide range of other multiple cover problems in pseudoholomorphic curve theories.

Remark: Note that in order to prove $d^2 = 0$ in embedded contact homology and periodic Floer homology the authors of [HT1] and [HT2] also study sections in obstruction bundles over moduli spaces of branched covers of trivial cylinders. Beside the fact that their papers became available shortly before this project was finished, we emphasize that there is an essential difference between their project and ours: While we view the branched covers of orbit cylinders as trivial examples of curves counted in the differential of rational symplectic field theory and therefore count trivial curves of Fredholm index *one*, M. Hutchings and C. Taubes developed a generalized gluing theory for symplectic field theory in dimension four where trivial curves of Fredholm index *zero* are inserted between the curves to be glued.

After describing the geometric setup underlying symplectic field theory, we focus on the basic facts about trivial curves in symplectic field theory. Since we have to deal with nonregular moduli spaces we introduce coherent abstract perturbations. We then rigorously describe the moduli spaces $\mathcal{M}$ and $\mathcal{M}^0$ of trivial curves, obtained by quotiening out or not quotiening out the $\mathbb{R}$-action, and their compactifications. We show that $\mathcal{M}$ and $\mathcal{M}^0$ are given as products involving the moduli space of punctured spheres and use the conservation of energy to describe their compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ which are again made up of moduli spaces of trivial curves. Introducing the notion of a tree with (based) level structure $(T,\mathcal{L})$, $(T,\mathcal{L},\ell_0)$ we show that $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ carry natural stratifications and prove that $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ are smooth manifolds with corners. While for this we explicitly describe the compactifications using Fenchel-Nielsen coordinates on the moduli space of punctured spheres, we emphasize that the compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ are different from the one obtained using the Deligne-Mumford-Knudsen compactification of the moduli space of punctured spheres, in particular, $\overline{\mathcal{M}}$ (and $\overline{\mathcal{M}^0}$) have codimension one boundary strata. We then introduce the cokernel bundles $\overline{\mathrm{Coker}}\bar{\partial}_J$ and $\overline{\mathrm{Coker}_0}\bar{\partial}_J$ over the compactified moduli spaces $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$. After describing the neccessary Banach space bundle setup, we study the linearization of the Cauchy-Riemann operator $\bar{\partial}_J$ and prove that, by energy reasons, the kernel of the linearization of $\bar{\partial}_J$ agrees with the tangent space to the moduli space. This proves in particular that the cokernel of the linearization of $\bar{\partial}_J$ has the same dimension at every point in $\mathcal{M}$ (and $\mathcal{M}^0$) which is sufficient to prove that $\overline{\mathrm{Coker}}\bar{\partial}_J$ (and $\overline{\mathrm{Coker}_0}\bar{\partial}_J$) naturally carry the structure of a smooth vector bundle over the strata of $\overline{\mathcal{M}}$ (and $\overline{\mathcal{M}^0}$). In order to show that these bundles over the strata fit together to a smooth vector bundle over the manifold with corners $\overline{\mathcal{M}}$ (and $\overline{\mathcal{M}^0}$) we prove a linear gluing result for cokernel bundles. While we show that the construction of coherent orientations in [BM] together with the complex orientations of the strata of $\overline{\mathcal{M}}$ (and $\overline{\mathcal{M}^0}$) equips the cokernel bundle with an orientation over each stratum, it follows from the results in [BM] that these orientations in general do not fit together to give an orientation of the whole cokernel bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$ ($\overline{\mathrm{Coker}_0}\bar{\partial}_J$) but differ by a fixed sign due to reordering of the punctures. Equipped with the neccessary analytical results about $\overline{\mathrm{Coker}}\bar{\partial}_J$ (and $\overline{\mathrm{Coker}_0}\bar{\partial}_J$) we finally prove the main theorem. After showing that sections in the cokernel bundle indeed provide us with the desired compact perturbations for the Cauchy-Riemann operator, we discuss the gluing compatibility for sections in the cokernel bundle and define the notion of coherent sections in $\overline{\mathrm{Coker}}\bar{\partial}_J$. We finally prove by induction that there exists an Euler number for coherent sections in $\overline{\mathrm{Coker}}\bar{\partial}_J$ and show that it is zero. For this we study sections in the cokernel bundle $\overline{\mathrm{Coker}_0}\bar{\partial}_J$ over $\overline{\mathcal{M}^0}$. We again emphasize that the induction step does not only need the existence result of Euler numbers for the moduli spaces in the boundary but also that these numbers are indeed zero. After this we discuss the implications of our result on rational symplectic field theory once the analytical foundations are proven. After explaining why the conclusion of the main result should continue to hold for all choices of coherent compact perturbations, we introduce the natural action filtration on symplectic field theory. Finally we introduce the rational symplectic field theory of a single closed Reeb orbit and use our result to compute the underlying generating function. Including the even more general picture outlined in [EGH] needed to view Gromov-Witten theory as a part of symplectic field theory, we further prove what we get when we additionally introduce a string of closed differential forms. Here we prove by simple means (but using our main result) that the generating function only sees the homology class represented by the underlying closed Reeb orbit. It follows that the generating function is in general no longer equal to zero when a string of differential forms is chosen, which implies that the differential in rational symplectic field theory and contact homology is no longer strictly decreasing with respect to the action filtration. However, we follow [FOOO] in employing the spectral sequence for filtered complexes, where we use our result to show that after passing from the E^1-page to the E^2-page we only have to consider those formal variables, where the homology class of the underlying closed orbit is annihilated by all chosen differential forms.

This chapter is organized as follows: After two introductory subsections on trivial curves and coherent compact perturbations, section one is concerned with the nonregular moduli spaces of unperturbed branched covers of trivial cylinders. While section two is devoted to establishing the existence and the properties of the cokernel bundle, we prove the main theorem in section three. In section four we finally discuss the implications of our result on rational symplectic field theory once the analytical foundations of symplectic field theory are proven.

2.0.2 Trivial curves in symplectic field theory

Beside the constant curves with no punctures, which do not contribute to the differential by algebraic reasons, note that for each closed orbit γ of the vector field R we have the trivial cylinder $\mathbb{R}\times\gamma$ as trivial example of a J-holomorphic curve in $\mathbb{R}\times V$, where the J-holomorphicity follows from $J\partial_s = R = \dot{\gamma}$. While these trivial cylinders correspond to the trivial connecting orbits in Floer homology and by the same arguments turn out to be irrelevant for the algebraic invariants, it is important to observe that in contact homology and (rational) symplectic field theory we get from a single trivial cylinder infinitely many other trivial examples of punctured J-holomorphic curves with two or more punctures by considering branched and unbranched covers of the given trivial cylinder. While the unbranched covers are again trivial cylinders over a multiple of the underlying Reeb orbit, it follows (see proposition 2.1.1) that the branched covers are in one-to-one correspondence with meromorphic functions on the underlying closed Riemann surface by removing zeroes and poles and identifying $\mathbb{CP}^1 - \{0,\infty\} \cong \mathbb{R}\times S^1 \cong (\mathbb{R}\times\gamma, J)$. While these curves are clearly trivial in the above sense, it is important to observe that they are also trivial from another viewpoint:

Like the constant curves and cylinders staying over one orbit are the only holomorphic curves in Gromov-Witten theory and symplectic Floer homology with trivial energy, the branched and unbranched covers of trivial cylinders are the only punctured holomorphic curves with vanishing ω-energy. Indeed, if $u = (a,f) : \dot{S} \to \mathbb{R} \times V$ has $E_\omega(u) = 0$ it follows, see lemma 5.4 in [BEHWZ], that $df \in \ker\omega = \mathbb{R} \times R$, so that the image of the V-component f is a closed Reeb orbit. On the other hand, assuming as in [EGH] that the first homology group of V is torsion-free, observe that after choosing a basis for $H_1(V)$ and choosing for each simple orbit γ a spanning surface f_γ in V realizing a cobordism between γ and a suitable linear combination of these basis elements as in [EGH], we can define an action

$$S(\gamma) = \int f_\gamma^* \omega,$$

for every simple closed Reeb orbit γ. On the other hand, note that for a multiply covered orbit γ^m we can use the formal multiple f_γ^m of the spanning surface f_γ to realize a cobordism between γ^m and a linear combination of basis elements, so that $S(\gamma^m) = m \cdot S(\gamma)$. Then $E_\omega(u)$ can be expressed as the difference of the actions of the closed orbits $\gamma_1^\pm, ..., \gamma_{n^\pm}^\pm$ corresponding to positive, respectively negative punctures of u and the ω-area of the homology class $A \in H_2(V)$ which we can assign to u using the spanning surfaces for the simple orbits underlying $\gamma_1^\pm, ..., \gamma_{n^\pm}^\pm$,

$$E_\omega(u) = \sum_{k=1}^{n^+} S(\gamma_k^+) - \sum_{\ell=1}^{n^-} S(\gamma_\ell^-) + \omega(A).$$

In particular, it follows that the moduli spaces $\mathcal{M}_{g,0}(\gamma^{m_1^+}, ..., \gamma^{m_{n^+}^+}; \gamma^{m_1^-}, ..., \gamma^{m_{n^-}^-})$ of J-holomorphic curves of genus g in $\mathbb{R} \times V$ which are asymptotically cylindrical over the multiple covers $\gamma^{m_1^+}, ..., \gamma^{m_{n^+}^+}$ of γ at the positive, over $\gamma^{m_1^-}, ..., \gamma^{m_{n^-}^-}$ at the negative punctures and represent the homology class $A = 0 \in H_2(V)$, entirely consist of multiple covers of the trivial cylinder over γ. For this observe that $m_1^+ + ... + m_{n^+}^+ = m_1^- + ... + m_{n^-}^-$ since else the moduli space is empty by homological reasons, so that

$$\sum_{k=1}^{n^+} S(\gamma^{m_k^+}) - \sum_{\ell=1}^{n^-} S(\gamma^{m_\ell^-}) = (\sum_{k=1}^{n^+} m_k^+ - \sum_{\ell=1}^{n^-} m_\ell^-) \cdot S(\gamma) = 0.$$

For the rest of this paper we restrict ourselves to the case of rational curves, i.e., with genus $g = 0$. Note that the moduli space $\mathcal{M}_{0,0}(\gamma^{m_1^+}, ..., \gamma^{m_{n^+}^+}; \gamma^{m_1^-}, ..., \gamma^{m_{n^-}^-})$ contributes to the differential in rational symplectic field theory only when its virtual dimension given by the Fredholm index of the linearization of the Cauchy-Riemann operator $\bar\partial_J$,

$$\operatorname{ind} \bar\partial_J = \sum_{k=1}^{n^+} \mu_{CZ}(\gamma^{m_k^+}) - \sum_{\ell=1}^{n^-} \mu_{CZ}(\gamma^{m_\ell^-}) + (m-3) \cdot (2-n),$$

is equal to one, where $n = n^+ + n^-$ is the number of punctures and $\dim V = 2m - 1$. For this observe that under the assumption that the Cauchy-Riemann operator meets the zero section transversally in a suitable Banach space bundle over a Banach manifold of maps this implies that the moduli space is one-dimensional, i.e., discrete after quotiening out the natural $\mathbb{R}$-action. While for trivial cylinders the Fredholm index is always zero, there indeed exist examples of branched covers with Fredholm index one. For example it is easy to check that the moduli spaces $\mathcal{M}_{0,0}(\gamma^2;\gamma,\gamma)$ and $\mathcal{M}_{0,0}(\gamma,\gamma;\gamma^2)$ of pairs of pants mapping to the trivial cylinder over an arbitrary hyperbolic orbit γ in a three-manifold have virtual dimension equal to one and therefore, in contrast to the underlying trivial cylinder, possibly contribute to the algebraic invariants of rational symplectic field theory. On the other hand we prove in proposition 2.1.1 that when the number of punctures $n=n^++n^-$ is greater or equal to three the moduli space is given by

$$\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-}) = \mathbb{R}\times S^1\times\mathcal{M}_{0,n^++n^-}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-},$$

where $\mathcal{M}_{0,n^++n^-}$ is the moduli space of stable n-punctured spheres, which is a complex manifold of complex dimension $n-3$. In particular, the moduli space is a complex manifold of *complex* dimension greater or equal to one so that, when the Fredholm index is assumed to be one, the actual dimension of the moduli space must be strictly larger than its virtual dimension expected by the Fredholm index. Note that this in turn implies that the moduli cannot be transversally cut out by the Cauchy-Riemann operator, in other words: *Even for generic choices of J, each moduli space of trivial curves with Fredholm index one must be nonregular in the sense that the the Cauchy-Riemann operator does not meet the zero section transversally.*

In order to see why the Fredholm index can be smaller than the actual dimension, observe that the index is sensitive to the underlying periodic orbit γ and the dimension of V, while the actual dimension is not. On the other hand the nontrivial behaviour of the Conley-Zehnder index under replacing an orbit by some multiple cover makes it hard to exclude trivial curves with Fredholm index one. Restricting to contact homology for simplicity, note that the best way to get a hand on the possible range of the Fredholm index of trivial curves for the general case, i.e., without further assumptions on the underlying Reeb orbit γ, is to combine the formula for the virtual dimension of the moduli space $\mathcal{M}_{0,0}(\gamma^{n-1};\gamma,...,\gamma)$,

$$\operatorname{ind}\bar{\partial}_J = \mu_{CZ}(\gamma^{n-1}) - (n-1)\cdot\mu_{CZ}(\gamma) + (m-3)\cdot(2-n)$$

with the estimate for the Conley-Zehnder index of multiply covered orbits in [L],

$$\begin{aligned}&(n-1)(\mu_{CZ}(\gamma)-(m-1))+(m-1)\ \le\ \mu_{CZ}(\gamma^{n-1})\\ &\le\ (n-1)(\mu_{CZ}(\gamma)+(m-1))-(m-1)\end{aligned}$$

to obtain

$$(2-n)(2m-4)\ \le\ \operatorname{ind}\bar{\partial}_J\ \le\ 2n-4.$$

While the right hand side agrees with the actual dimension of the moduli space $\mathcal{M}_{0,0}(\gamma^{n-1};\gamma,...,\gamma)$ and is strictly greater than one, the left hand side is nonpositive for

$m \geq 2$, i.e., $\dim V \geq 3$. Hence we cannot exclude branched covers of trivial cylinders with Fredholm index one for any number of punctures greater or equal to three as well as any dimension of V greater or equal to three (without imposing further assumptions on the underlying Reeb orbit).

2.0.3 Coherent compact perturbations

Since the actual dimension of the moduli spaces does not agree with the virtual dimension expected by the Fredholm index, we already deduced that the Cauchy-Riemann operator $\bar{\partial}_J$ cannot be transversal to the zero section in a suitable Banach space bundle over a Banach manifold of maps. The general way to remedy this is to add compact perturbations to the Cauchy-Riemann operator $\bar{\partial}_J$ so that it becomes transversal. Since the linearization of the perturbed Cauchy-Riemann operator then differs from the linearization of the original one only by a compact operator, it is still a Fredholm operator with the same index, which now by the implicit function theorem agrees with the local dimension of the zero set of the underlying nonlinear perturbed operator. In order to obtain a compactness result for this new zero set one also has to add compact perturbations to the Cauchy-Riemann operator over the moduli spaces forming the boundary. In particular, the compact perturbations chosen for any moduli space must be compatible with the compact perturbations chosen for the moduli spaces forming its boundary. The algebraic invariants are then defined by replacing the original compactified moduli space by the compactified zero set of the perturbed Cauchy-Riemann operator. Note that this can be achieved by either thinking about the specialities of the problem and then using special perturbations as in the first chapter or by building a general framework allowing for arbitrary compact perturbations. The observation that one is only interested in the zero set of the perturbed Cauchy-Riemann operator led to the (relative) virtual moduli cycle techniques in symplectic Floer homology and Gromov-Witten theory for general symplectic manifolds, see [LiuT], [LT], [FO], [MD], where the construction of the relative virtual moduli cycles in symplectic field theory is sketched in [B]. On the other hand, the wish to obtain the (relative) virtual moduli cycle directly as the zero set of the perturbed Cauchy-Riemann operator, viewed as a section in some kind of infinite-dimensional bundle over an infinite-dimensional space of maps, led to the invention of polyfolds by Hofer, Wysocki and Zehnder, see [HWZ] and the references therein.

While the virtual moduli cycles techniques as well as the polyfold theory provide us with the correct setup to handle the problem of transversality in symplectic field theory, it seems that one has to give up any hope to finally compute the desired algebraic invariants. However it is is a folk's theorem in Gromov-Witten theory, see e.g. [MD], [MDSa], that in some good cases the situation can be drastically simplified:

Although the Cauchy-Riemann operator $\bar{\partial}_J$ is not transversal to the zero section, it might happen that its zero set is still a manifold and that the virtual moduli cycle can be represented by the zero set of a generic section in a finite-dimensional obstruction bundle over the compactification of the nonregular moduli space. In particular, the zero set agrees with (the compactification of) the regular moduli space obtained by adding to the Cauchy-Riemann operator a suitably extension of the given obstruction bundle section. The standard example of such an obstruction bundle is the cokernel bundle, where one has to show that the cokernels of the linearization of $\bar{\partial}_J$ at every zero always have the right dimension so that, in particular, they fit together to give a finite-dimensional vector bundle. Note however that the dimension of the cokernel usually jumps, so that the cokernels in general only fit together to local obstruction bundles, which leads to the definition of Kuranishi structures in [FO].

Using the characterization of trivial curves as curves with trivial ω-energy we can prove that we indeed have a global obstruction bundle over the compactification of every moduli space of trivial curves. While in Gromov-Witten theory the count of elements in the moduli space, more general, the cobordism class of the moduli space, is independent of the chosen abstract perturbation of the Cauchy-Riemann operator, this no longer holds for the moduli spaces in symplectic field theory. This follows from the fact that the moduli spaces in symplectic field theory typically have codimension one boundary strata, while in Gromov-Witten theory the regular moduli spaces form pseudo-cycles in the sense that the boundary strata have codimension at least two, i.e., from the homological point of view have no boundary. So while in Gromov-Witten theory the moduli spaces can be studied separately, the interplay between the different moduli spaces is the reason why the algebraic invariants of symplectic field theory are defined as differential algebras, which can be shown to be independent of extra choices like the cylindrical almost complex structure and the compact perturbation. In our case this problem is expressed by the fact that we have to study sections in vector bundles over moduli spaces with codimension one boundary, so that the count of zeroes in general depends on the choice of sections in the boundary, i.e., the chosen perturbations of the Cauchy-Riemann operator used to define the regular moduli spaces in the boundary. However we outline below that in our case we indeed have a well-defined count of zeroes so that, as in the Gromov-Witten case, we can (iteratively) define Euler numbers for our Fredholm problems.

2.1 Moduli space of trivial curves

2.1.1 Branched covers of trivial cylinders

Choosing closed orbits $\gamma_{1,\pm}^{m_1^\pm},...,\gamma_{n^\pm,\pm}^{m_{n^\pm}^\pm}$ of the vector field R on V, where γ^m denotes the m.th iterate of the simple orbit γ, and a homology class $A\in H_2(V)$, the moduli space $\mathcal{M}_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ of punctured J-holomorphic curves in $\mathbb{R}\times V$ of genus zero is defined as follows (see [EGH]):

Fix positive and negative punctures $z_1^\pm,...,z_{n^\pm}^\pm \in S^2$ and pairwise disjoint embeddings of half-cylinders $\psi_k^\pm : \mathbb{R}^\pm \times S^1 \hookrightarrow \dot{S}$ with $\lim_{r\to\pm\infty} \psi_k^\pm(r,\cdot) = z_k^\pm$, where $\dot{S} = S^2 - \{z_1^\pm,...,z_{n^\pm}^\pm\}$. Then the moduli space $\mathcal{M}^0_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ of parametrized curves consists of tuples $u = (u,j,\mu^\pm)$, where j denotes a complex structure on the punctured sphere $\dot{S}$ which agrees with the standard complex structure on the cylindrical coordinate neighborhoods of the punctures, $\mu^\pm = (\mu_1^\pm,...,\mu_{n^\pm}^\pm)$, $\mu_k^\pm \in (T_{z_k^\pm}S^2 - \{0\})/\mathbb{R}_+ \cong S^1$ is a collection of directions at the punctures $z_1^\pm,...,z_{n^\pm}^\pm$, called asymptotic markers, and $u : (\dot{S},j) \to (\mathbb{R}\times V, J)$ is a (j,J)-holomorphic map which is asymptotically cylindrical over the closed orbit $\gamma_{k,\pm}^{m_k^\pm}$ at the puncture $z_k^\pm$,

$$(u\circ\psi_k^\pm)(s,t+\mu_k^\pm) \to \gamma(m_k^\pm T^{\gamma_{\pm,k}}t),\quad k=1,...,n^\pm.$$

Here T^γ denotes the period of the simple orbit γ and it follows from the chosen S^1-shift in the asymptotic condition that the asymptotic marker $\mu_k^\pm \in S^1$ is mapped to the point $z^{\gamma_{\pm,k}} = \gamma_{\pm,k}(0)$ on the underlying simple orbit. Note that when the asymptotic condition is fulfilled with the asymptotic marker $\mu_k^\pm$, then it also holds for the asymptotic markers $\mu_k^\pm + \ell/m_k^\pm$, $\ell = 1,...,m_k^\pm - 1$. Representing a basis of $H_1(V)$, which is assumed to be torsion-free as in [EGH], by circles in V and choosing for each simple orbit γ a spanning surface in V between γ and a suitable linear combination of these circles as in 0.3, one can assign an absolute homology class in $H_2(V)$ to each map u. With this we require that the map u represents the given homology class $A \in H_2(V)$.

Note that when $n^+ + n^- \leq 3$ we have a unique complex structure i on $\dot{S}$ and we obtain the moduli space $\mathcal{M}_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ as quotient of $\mathcal{M}^0_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ under the obvious action of the automorphism group $\mathrm{Aut}(\dot{S},i)$. On the other hand, when $n^+ + n^- \geq 3$ the automorphism group of $(\dot{S},j)$ is trivial, so that the desired moduli space $\mathcal{M}_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ agrees with the moduli space $\mathcal{M}^0_{A,0}(\gamma_{1,+}^{m_1^+},...,\gamma_{n^+,+}^{m_{n^+}^+};\gamma_{1,-}^{m_1^-},...,\gamma_{n^-,-}^{m_{n^-}^-})$ of parametrized curves from before.

When all chosen simple orbits agree, $\gamma_{\pm,k} = \gamma$, $k = 1,...,n^\pm$, and $A = 0 \in H_2(V)$, we already outlined in 0.2 that all curves have trivial ω-energy $E_\omega(u) = 0$, and therefore have V-image contained in a trajectory of the Reeb vector field. When there is at least one puncture it follows that the moduli space $\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})$ entirely consists of branched covers of the trivial cylinder over a single closed orbit γ. For every (simple) closed orbit γ of the vector field R, the trivial cylinder $\mathbb{R}\times\gamma$ represents a curve in the above sense with $u_0 : (\mathbb{R}\times S^1, i) \to (\mathbb{R}\times V, J)$, $(s,t) \mapsto (T^\gamma s, \gamma(T^\gamma t))$, which is holomorphic by $J\partial_s = \dot{\gamma} = R$. It follows that every curve u in $\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})$ is of the form $u = h\circ u_0$ with the branched covering map

$$h : (\dot{S},j) \to \mathbb{R}\times S^1$$

between the punctured Riemann spheres $(\dot{S},j)$ and $\mathbb{R}\times S^1\cong\mathbb{C}^*=\mathbb{CP}^1-\{0,\infty\}$.

It directly follows from the asymptotic conditions for the curve u in $\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})$ that h extends to a holomorphic map from $(S^2,j)\cong\mathbb{CP}^1$ to $\mathbb{CP}^1$. More precisely, it represents a meromorphic function h on (S^2,j), where the positive punctures $z_1^+,...,z_{n^+}^+$ are poles of order $m_1^+,...,m_{n^+}^+$, the negative punctures $z_1^-,...,z_{n^-}^-$ are zeroes of order $m_1^-,...,m_{n^-}^-$. For the rest of the paper we make the convention to identify u directly with the branched covering h. Furthermore we make the convention that, unless otherwise mentioned, all considered branched covers are connected and have no nodes. Choosing the standard complex structure i on S^2, $(S^2,i)=\mathbb{CP}^1$, and letting the positions of $z_1^\pm,...,z_{n^\pm}^\pm\in\mathbb{CP}^1$ vary, it follows that the moduli space $\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})$ agrees with the moduli space of meromorphic functions on $\mathbb{CP}^1$ with the given number of poles and zeroes with multiplicities $m_1^\pm,...,m_{n^\pm}^\pm$, where we just must take care of the possible different choices for the asymptotic markers.

For the following expositions we assume that $m_1^+ + ... + m_{n^+}^+ = m_1^- + ... + m_{n^-}^-$ since else the moduli space is obviously empty by homological reasons. In particular, there are no holomorphic planes ($n=n^++n^-=1$). For $n=2$ the moduli space $\mathcal{M}_{0,0}(\gamma^m;\gamma^m)/\mathbb{R}$ consists precisely of m^2 elements, namely the unique trivial cylinder over the iterated orbit γ^m together with the m^2 possible choices for the asymptotic marker above and below. Note that here the actual and the virtual dimension given by the Fredholm index agree to be zero, so that they are not interesting from the viewpoint of symplectic field theory. Hence it suffices to restrict our considerations to the stable case $n\geq 3$.

Proposition 2.1.1: *For $n=n^++n^-\geq 3$ the moduli space of trivial curves (connected, without nodes) with fixed multiplicities $m_1^\pm,...,m_{n^\pm}^\pm$ is given by*

$$\mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})/\mathbb{R}\cong S^1\times\mathcal{M}_{0,n^++n^-}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-},$$

where $\mathcal{M}_{0,n}$ denotes the moduli space of n-punctured spheres and $m^\pm=m_1^\pm\cdot...\cdot m_{n^\pm}^\pm$.

Proof: For the proof we fix the natural complex structure $j=i$ on S^2, $(S^2,i)=\mathbb{CP}^1$, and let instead the positions of the punctures $z_1^\pm,...,z_{n^\pm}^\pm$ vary. Since the zeroth Picard group $\mathrm{Pic}^0(\mathbb{CP}^1)$ is trivial, i.e., all degree zero divisors on $\mathbb{CP}^1$ are in fact principal divisors, it follows that a meromorphic function exists for any choice of zeroes and poles with multiplicities, as long as the number of poles with multiplicities agrees with the number of zeroes with multiplicities. More explicitly, an example of h is

$$h^0(z)=\frac{\prod_{k=1}^{n^-}(z-z_k^-)^{m_k^-}}{\prod_{k=1}^{n^+}(z-z_k^+)^{m_k^+}}$$

and it follows from Liouville's theorem that such a map is uniquely determined up to a nonzero multiplikative factor, i.e., $h=a\cdot h_0$ with $a\in\mathbb{C}^*$. Since for $n\geq 3$ the automorphism group $\mathrm{Aut}(\mathbb{CP}^1)$ already acts freely on the ordered set of punctures $(z^\pm_1,...,z^\pm_{n^\pm})$, it follows that the moduli space agrees with the product $\mathbb{C}^* \times \mathcal{M}_{0,n^++n^-}$ with $\mathbb{C}^* \cong \mathbb{R}\times S^1$. On the other hand there are $m^\pm_k$ possible directions for the asymptotic marker $\mu^\pm_k$ at each puncture $z^\pm_k$, $k=1,...,n^\pm$, for each (h,j) as outlined in the definition of the moduli spaces, so that $\mu^\pm_k \in \mathbb{Z}_{m^\pm_k}$, i.e., $\mu^\pm = (\mu^\pm_1,...,\mu^\pm_{n^\pm}) \in \mathbb{Z}_{m^\pm_1}\times...\times\mathbb{Z}_{m^\pm_{n^\pm}} \cong \mathbb{Z}_{m^\pm}$. $\square$

In what follows we fix the multiplicities $m^\pm_1,...,m^\pm_{n^\pm}$ and abbreviate the corresponding moduli space of trivial curves by

$$\mathcal{M} = \mathcal{M}_{0,0}(\gamma^{m^+_1},...,\gamma^{m^+_{n^+}};\gamma^{m^-_1},...,\gamma^{m^-_{n^-}})/\,\mathbb{R}\,.$$

Note that here we view the target $\mathbb{R}\times S^1$ as a cylindrical cobordism in the sense of [BEHWZ], so that we quotient out the corresponding $\mathbb{R}$-symmetry on the moduli space. Later, for the proof of the main theorem, we further have to consider the corresponding moduli space of holomorphic curves in $\mathbb{R}\times S^1$ without quotiening out the $\mathbb{R}$-translations,

$$\mathcal{M}^0 = \mathcal{M}_{0,0}(\gamma^{m^+_1},...,\gamma^{m^+_{n^+}};\gamma^{m^-_1},...,\gamma^{m^-_{n^-}}),$$

i.e., we view the holomorphic curves as sitting in a noncylindrical cobordism by just ignoring the natural $\mathbb{R}$-action.

2.1.2 Compactification

While introducing abstract perturbations we must asure that these are compatible with the curve splitting phenomena described in the compactness theorem of symplectic field theory. Hence we must also include the compactification of the moduli space of trivial curves into our considerations which is, of course, not too bad. Recall that by [BEHWZ] the compactification of a moduli space of curves in a cylindrical or noncylindrical cobordism consists of holomorphic curves in cobordisms together with a level structure. Calling a level (non-)cylindrical whenever the corresponding cobordism is (non-)cylindrical, observe that when we start with curves in a cylindrical cobordism the resulting levels are all cylindrical. On the other hand, when we start with curves in a noncylindrical cobordism, there is precisely one noncylindrical level, while all other levels are cylindrical. Furthermore we call a connected component of a holomorphic curve (non-)cylindrical when it is (not) a cylinder. This leads to the following compactness statement:

Proposition 2.1.2: *The boundary of the compactified moduli space $\overline{\mathcal{M}}$ consists of level holomorphic curves in the sense of [BEHWZ], which are connected or disconnected nodal branched covers of the same orbit cylinder, such that the punctured spheres underlying all noncylindrical components are stable and on each level there is at least one noncylindrical component. The same holds true for the compactification $\overline{\mathcal{M}^0}$, except that the last part of the statement need not be satisfied for the noncylindrical level. For $\overline{\mathcal{M}}$ it follows that all connected components carry strictly less than n punctures, whereas for $\overline{\mathcal{M}^0}$*

this is true only up to the case of a two level curve where all curves on the noncylindrical level are cylinders.

Proof: Choosing a sequence of holomorphic curves in $\mathcal{M}$, it follows from the compactness theorem in [BEHWZ] that a suitable subsequence converges to a level holomorphic map in the sense of [BEHWZ]. It follows from lemma 5.4 in [BEHWZ] together with the preservation of the ω-energy that the connected components in each level of the limiting level curve are again, after resolving the nodes, multiple covers of the corresponding orbit cylinder. Since there are no multiple covers with one puncture and every curve with no punctures is constant it follows that every component of the limit level holomorphic map has at least two punctures, i.e., that every noncylindrical component has positive Euler characteristic. Furthermore there always must be a noncylindrical component on each cylindrical level, since otherwise the $\mathbb{R}$-action is trivial. The remaining statements on the number of punctures follow from the additivity of the Euler characteristic. $\square$

Definition 2.1.3: *A (n^+,n^-)-labelled tree with level structure is a tuple $(T,\mathcal{L}) = (T,E,\Lambda^+,\Lambda^-,\mathcal{L})$, where (T,E) is a tree with the set of vertices T and the edge relation $E \subset T \times T$, the sets $\Lambda^\pm = (\Lambda^\pm_\alpha)_{\alpha \in T}$ are decompositions of $\{1,...,n^\pm\}$, i.e.,*

$$\bigcup_{\alpha\in T} \Lambda^\pm_\alpha = \{1,...,n^\pm\},\ \Lambda^\pm_\alpha \cap \Lambda^\pm_\beta = \emptyset \,\text{when}\, \alpha \neq \beta,$$

and $\mathcal{L}: T \to \{1,...,L\}$ is surjective map, which is called a level structure. Furthermore, a tuple $(T,\mathcal{L},\ell_0) = (T,E,\Lambda^+,\Lambda^-,\mathcal{L},\ell_0)$ with $\ell_0 \in \{1,...,L\}$ is called a (n^+,n^-)-labelled tree with based level structure.

Observe that every level branched cover in $\overline{\mathcal{M}}$ represents a (n^+,n^-)-labelled tree with level structure, where the tree structure (T,E) represents the underlying nodal curve, i.e., bubble tree, and the elements $k \in \{1,...,n^\pm\}$ represent positive or negative punctures. On the other hand, a level branched cover in the boundary of $\mathcal{M}^0$ represents a tree with based level structure $(T,\mathcal{L},\ell_0)$ with ℓ_0 denoting the noncylindrical level. It follows that $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ carry natural stratifications

$$\overline{\mathcal{M}} = \bigcup_{T,\mathcal{L}} \mathcal{M}_{T,\mathcal{L}},\ \ \overline{\mathcal{M}^0} = \bigcup_{T,\mathcal{L},\ell_0} \mathcal{M}^0_{T,\mathcal{L},\ell_0}$$

where $\mathcal{M}_{T,\mathcal{L}}$ and $\mathcal{M}^0_{T,\mathcal{L},\ell_0}$ can be described as follows:

First we can assign to every labelled tree with level structure $(T,\mathcal{L}) = (T,E,\Lambda^\pm,\mathcal{L})$ a nodal surface with positive and negative punctures by assigning to each $\alpha \in T$ a sphere $S_\alpha = S^2$, to any edge $(\alpha,\beta) \in E$ a marked point $z_{\alpha\beta} \in S_\alpha$ and to any $k \in \Lambda^\pm_\alpha$, $\alpha \in T$ a positive, respectively negative puncture $z^\pm_k \in S_\alpha$. Since to each positive, respectively negative puncture we assign a fixed multiple $\gamma^{m^\pm_k}$ of the underlying simple orbit γ, we can naturally assign a multiplicity with sign $m_{\alpha\beta} \in \mathbb{Z}$ to each edge in E by requiring for each $\alpha \in T$ that

$$\sum_{\beta:\alpha E\beta} m_{\alpha\beta} + \sum_{k\in\Lambda_\alpha^+} m_k^+ - \sum_{k\in\Lambda_\alpha^-} m_k^- = 0.$$

Note that each edge (α, β) with $m_{\alpha\beta} \neq 0$ corresponds to a positive or negative puncture for the components α and β and $m_{\alpha\beta} = -m_{\beta\alpha}$ denotes the period with sign. In particular, when $m_{\alpha\beta} > 0$ then $\mathcal{L}(\alpha) > \mathcal{L}(\beta)$, whereas by similar arguments the edges with trivial multiplicity $m_{\alpha\beta} = 0$ corresponds to nodes between components α and β in the same level, $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$. With this we define sets of positive, respectively negative punctures on S_α by

$$\begin{aligned} Z_\alpha^+ &= \{z_k^+ : k \in \Lambda_\alpha^+\} \cup \{z_{\alpha\beta} : \mathcal{L}(\beta) > \mathcal{L}(\alpha)\} \\ &= \{z_{\alpha,k}^+ : k = 1, ..., n_\alpha^+\}, \\ Z_\alpha^- &= \{z_k^- : k \in \Lambda_\alpha^-\} \cup \{z_{\alpha\beta} : \mathcal{L}(\beta) < \mathcal{L}(\alpha)\} \\ &= \{z_{\alpha,k}^- : k = 1, ..., n_\alpha^-\} \end{aligned}$$

and denote the corresponding punctured sphere by $\dot{S}_\alpha = S_\alpha - \{z_{\alpha,1}^\pm, ..., z_{\alpha,n_\alpha^\pm}^\pm\}$, while $Z_\alpha^0 = \{z_{\alpha\beta} : \mathcal{L}(\alpha) = \mathcal{L}(\beta)\}$ is the set of nodes connecting S_α with S_β of the same level. Note that by the above definitions we assign a positive multiplicity $m_{\alpha,k}^\pm$ to any point $z_{\alpha,k}^\pm$ in $Z_\alpha^\pm$. Finally note that we did not fix the complex structure on any of the punctured spheres S_α.

We want to describe the moduli space $\mathcal{M}_{T,\mathcal{L}}$ using the corresponding moduli spaces of nodal curves on the different levels. For this observe that to any labelled tree with level structure $(T, E, \Lambda^\pm, \mathcal{L})$ we can assign a tuple of labelled trees $T_\ell = (T_\ell, E_\ell, \Lambda_\ell^\pm)$, $\ell \in \{1, ..., L\}$, where $T_\ell = \{\alpha \in T : \mathcal{L}(\alpha) = \ell\}$, $E_\ell = E \cap (T_\ell \times T_\ell)$ and $\Lambda_\ell^\pm = (\Lambda_{\ell,\alpha}^\pm)_{\alpha\in T_\ell}$ with $\Lambda_{\ell,\alpha}^\pm = \Lambda_\alpha^\pm \cup \{\beta \in T_{\ell\pm1} : \alpha E\beta\}$.

For every $T_\ell = (T_\ell, E_\ell, \Lambda_\ell^\pm)$, $\ell \in \{1, ..., L\}$ we now introduce the moduli space $\mathcal{M}_{T_\ell}$ as follows: Every element in $\mathcal{M}_{T_\ell}$ is a tuple $(h_\ell, j_\ell, \mu_\ell^\pm) = (h_\alpha, j_\alpha, \mu_\alpha^\pm)_{\alpha\in T_\ell}$, where j_α is a complex structure on $\dot{S}_\alpha$ and $h_\alpha : (\dot{S}_\alpha, j_\alpha) \to \mathbb{R} \times S^1$ extends to a meromorphic function on $(S_\alpha = S^2, j_\alpha)$ with poles, respectively zeroes $z_{\alpha,1}^\pm, ..., z_{\alpha,n_\alpha^\pm}^\pm$ of multiplicities $m_{\alpha,1}^\pm, ..., m_{\alpha,n_\alpha^\pm}^\pm$, such that $h_\alpha(z_{\alpha\beta}) = h_\beta(z_{\beta\alpha})$ if $z_{\alpha\beta} \in Z_\alpha^0$, i.e., $z_{\beta\alpha} \in Z_\beta^0$. Further $\mu_\alpha^\pm = (\mu_{\alpha,1}^\pm, ..., \mu_{\alpha,n_\alpha^\pm}^\pm)$ denotes the collection of asymptotic markers $\mu_{\alpha,k}^\pm \in \mathbb{Z}_{m_{\alpha,k}^\pm}$.

Note that in general the trees T_ℓ are not connected. Denoting the connected components by $T_{\ell,1}, ..., T_{\ell,N_\ell}$, the moduli space $\mathcal{M}_{T_\ell}$ can be written as direct product

$$\mathcal{M}_{T_\ell} = \mathcal{M}_{T_{\ell,1}} \times ... \times \mathcal{M}_{T_{\ell,N_\ell}} \times \mathbb{R}^{N_\ell - 1}$$

of moduli spaces $\mathcal{M}_{T_{\ell,k}}$, $k = 1, ..., N_\ell$ of connected nodal branched covers, where the $\mathbb{R}$-factors encode the relative $\mathbb{R}$-position of the connected components of the curves in $\mathcal{M}_{T_\ell}$.

With the moduli spaces $\mathcal{M}_{T_1},...,\mathcal{M}_{T_L}$ we can finally describe the moduli spaces $\mathcal{M}_{T,\mathcal{L}}$ and $\mathcal{M}^0_{T,\mathcal{L},\ell_0}$:

While the definitions of complex structures and holomorphic maps is straightforward, we explicitly want that two tuples $(h_\ell, j_\ell, \mu_\ell)_{\ell=1,...,L}$ represent the same element in $\mathcal{M}_{T,\mathcal{L}}$ if the asymptotic markers at pairs of positive and negative punctures, which correspond to edges between components in neighboring levels, describe the same decorations. Note that this convention is implicit in the proof of the master equation of (rational) symplectic field theory, which is derived by studying the codimension boundary strata of moduli spaces. Indeed we will show below that this convention guarantees that the compactified moduli space $\overline{\mathcal{M}}$ (and $\overline{\mathcal{M}^0}$) carries the structure of a manifold with boundary. Going back to the goal of describing $\mathcal{M}_{T,\mathcal{L}}$ explicitly, we assign to any tuple $(h_\ell, j_\ell, \mu^\pm_\ell)_{\ell=1,...,L} \in \mathcal{M}_{T_1}\times ... \times \mathcal{M}_{T_L}$ a tuple $(h,j,\mu^\pm,\theta)\in\mathcal{M}_{T,\mathcal{L}}$, where $(h,j)=(h_\ell,j_\ell)_{\ell=1,...,L}=(h_\alpha,j_\alpha)_{\alpha\in T}$. For the asymptotic markers $\mu^\pm$ and decorations θ we recall that

$$\mu^\pm_\ell = (\mu^\pm_\alpha)_{\alpha\in T_\ell}, \qquad \begin{aligned} \mu^+_\alpha &= ((\mu^+_k)_{k\in\Lambda^+_\alpha}, (\mu_{\alpha\beta})_{\mathcal{L}(\beta)>\mathcal{L}(\alpha)}), \\ \mu^-_\alpha &= ((\mu^-_k)_{k\in\Lambda^-_\alpha}, (\mu_{\alpha\beta})_{\mathcal{L}(\beta)<\mathcal{L}(\alpha)}). \end{aligned}$$

From this we get asymptotic markers $\mu^\pm = (\mu^\pm_k)_{k=1,...,n^\pm}$ and decorations $\theta=(\theta_{\alpha\beta})_{\mathcal{L}(\alpha)>\mathcal{L}(\beta)}$ by setting

$$\theta_{\alpha\beta} = [(\mu_{\alpha\beta},\mu_{\beta\alpha})] \in \frac{\mathbb{Z}_{|m_{\alpha\beta}|}\times\mathbb{Z}_{|m_{\alpha\beta}|}}{\Delta_{\alpha\beta}},$$

where $\Delta_{\alpha\beta}=\Delta_{\beta\alpha}$ denotes the diagonal in $\mathbb{Z}_{|m_{\alpha\beta}|}\times\mathbb{Z}_{|m_{\beta\alpha}|}$. For this recall that $m_{\alpha\beta}=-m_{\beta\alpha}$ and observe that two pairs of asymptotic markers $(\mu_{\alpha\beta},\mu_{\beta\alpha})$ and $(\mu'_{\alpha\beta},\mu'_{\beta\alpha})$ represent the same decoration if there exists some $\mu_0\in\mathbb{Z}_{|m_{\alpha\beta}|}$ with $(\mu'_{\alpha,\beta},\mu'_{\beta,\alpha})=(\mu_{\alpha\beta}+\mu_0,\mu_{\beta\alpha}+\mu_0)$. With this it follows that the moduli space $\mathcal{M}_{T,\mathcal{L}}$ is given by

$$\mathcal{M}_{T,\mathcal{L}} = \frac{\mathcal{M}_{T_1}\times ... \times \mathcal{M}_{T_L}}{\Delta}.$$

with $\Delta = \prod_{\mathcal{L}(\alpha)>\mathcal{L}(\beta)}\Delta_{\alpha\beta}$. On the other hand, it follows from the same arguments that $\mathcal{M}^0_{T,\mathcal{L},\ell_0}$ is given by

$$\mathcal{M}^0_{T,\mathcal{L},\ell_0} = \frac{\mathcal{M}_{T_1}\times ... \times \mathcal{M}^0_{T_{\ell_0}}\times ... \times \mathcal{M}_{T_L}}{\Delta},$$

Here $\mathcal{M}^0_{T_{\ell_0}}$ is the moduli space of trivial curves on the noncylindrical level, so that $\mathcal{M}^0_{T_{\ell_0}} = \mathbb{R}\times\mathcal{M}_{T_{\ell_0}}$ whenever T_{ℓ_0} represents a curve with at least one noncylindrical component, and just consists of a point if all components are trivial cylinders.

Observe that each $\mathcal{M}_{T,\mathcal{L}}$ is a smooth manifold of codimension

$$\dim\mathcal{M} - \dim\mathcal{M}_{T,\mathcal{L}} = L-1+2N,$$

where L is the number of levels and $N=\frac{1}{2}\sharp\{\alpha E\beta : \mathcal{L}(\alpha)=\mathcal{L}(\beta)\}$ denotes the number of nodes between components in the same level. For this observe that creating a new level we indeed only loose one dimension corresponding to the $\mathbb{R}$-coordinate on the new level which is quotiented out. It follows that the compactified moduli space $\overline{\mathcal{M}}$ is a stratified space with natural stratification

$$\mathcal{M}=\overline{\mathcal{M}}^0\subset\overline{\mathcal{M}}^1\subset\overline{\mathcal{M}}^2\subset ...\subset\overline{\mathcal{M}}^k\subset ...\subset\overline{\mathcal{M}}^\infty=\overline{\mathcal{M}},$$

where

$$\overline{\mathcal{M}}^k=\bigcup_{(T,\mathcal{L}):L-1+2N\leq k}\mathcal{M}_{T,\mathcal{L}}\,.$$

contains the components of the compactified moduli space of codimension at most k. In the same way we have

$$\mathcal{M}^0=\overline{\mathcal{M}^0}^0\subset\overline{\mathcal{M}^0}^1\subset\overline{\mathcal{M}^0}^2\subset ...\subset\overline{\mathcal{M}^0}^k\subset ...\subset\overline{\mathcal{M}^0}^\infty=\overline{\mathcal{M}^0},$$

where

$$\overline{\mathcal{M}^0}^k=\bigcup_{(T,\mathcal{L},\ell_0):L-1+2N\leq k}\mathcal{M}^0_{T,\mathcal{L},\ell_0}\,.$$

Observe that $\overline{\mathcal{M}}^1$, defined as disjoint union of the moduli space with the codimension one boundary components, consists of curves with two level and no nodes. More precisely, the connected components of the codimension one boundary are given by fibre products

$$\mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2=\frac{\mathcal{M}_1\times\mathcal{M}_2}{\Delta},$$

where $\mathcal{M}_1=\mathcal{M}_{T_1}$, $\mathcal{M}_2=\mathcal{M}_{T_2}$ denote moduli spaces of possibly disconnected branched covers without nodes. Note that here T_1,T_2 are trees with trivial edge relation and $\mathbb{Z}_{m_{1,2}}=\prod_{\mathcal{L}(\alpha)=2,\mathcal{L}(\beta)=1}\mathbb{Z}_{|m_{\alpha\beta}|}$ acts on $\mathcal{M}_1$ and $\mathcal{M}_2$ in the obvious way. On the other hand, observe that the connected components of the codimension one boundary of $\mathcal{M}^0$ are given either given by products of the form

$$\mathcal{M}^0_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2,\;\;\mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}^0_2$$

with $\mathcal{M}^0_1=\mathbb{R}\times\mathcal{M}_1$ and $\mathcal{M}^0_2=\mathbb{R}\times\mathcal{M}_2$ or

$$\{\text{point}\}\times\mathcal{M},\;\;\mathcal{M}\times\{\text{point}\}$$

corresponding to $\mathcal{M}^0_1=\{\text{point}\}$, $\mathcal{M}^0_2=\{\text{point}\}$, respectively, i.e., where on the noncylindrical level we just find trivial cylinders.

We close this section with an important technical lemma about the compactified moduli spaces $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$.

Proposition 2.1.4: *The compactified moduli spaces $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ naturally carry the structure of a manifold with corners.*

Proof: We prove the statement only for the compactification of $\mathcal{M}$, since the statement about the compactification of $\mathcal{M}^0$ follows the same arguments. Essentially it follows from an explicit description of the moduli space $\mathcal{M}$ and its compactification in terms of Fenchel-Nielsen coordinates:

Recall from the definition of the moduli spaces that we fixed n^+ positive and n^- negative punctures $z_1^{\pm},...,z_{n^{\pm}}^{\pm} \in S^2$ and fixed cylindrical coordinates

$$\psi_k^{\pm} : \mathbb{R}_0^{\pm} \times S^1 \hookrightarrow \dot{S}$$

around each puncture $z_k^{\pm}$, $k \in \{1,...,n^{\pm}\}$ on the punctured sphere $\dot{S} = S^2 - \{z_1^{\pm},...,z_{n^{\pm}}^{\pm}\}$. Beside the mentioned embeddings of half-cylinders we now embed $n-3$ finite cylinders $\psi_k : [-1,+1] \times S^1 \hookrightarrow \dot{S}$, $k \in \{1,...,n-3\}$ such that their images are pairwise disjoint, disjoint from the cylindrical coordinate neighborhoods of the punctures and such that the circles $\psi_k(\{0\} \times S^1) \subset \dot{S}$, $k =\in \{1,...,n-3\}$ define a pair of pants decomposition of $\dot{S}$. Observe that this naturally defines a (n^+,n^-)-labelled tree (T^0,E^0,Λ^0), where T^0 is the set of pair-of-pants components,

$$\dot{S} = \bigcup_{\alpha \in T^0} Y_\alpha$$

with the obvious edge relation

$$(\alpha,\beta) \in E^0 \quad \Leftrightarrow \quad Y_\alpha \cap Y_\beta \neq \emptyset,$$

and the decompositions $\Lambda^{0,\pm} = (\Lambda_\alpha^{0,\pm})_{\alpha \in T^0}$ of the sets $\{1,...,n^{\pm}\}$ given by

$$k \in \Lambda_\alpha^{0,\pm} \subset \{1,...,n^{\pm}\} \quad \Leftrightarrow \quad z_k^{\pm} \in Y_\alpha.$$

We fix a complex structure j_0 on $\dot{S}$ such that it agrees with the natural complex structures on the embedded cylinders. Let $\bar{E}^0 = E^0/\{(\alpha,\beta) \sim (\beta,\alpha)\}$ be the set of undirected edges and for every $\tau \in \bar{E}^0$ let $\psi_\tau : [-1,+1] \times S^1 \hookrightarrow \dot{S}$ denote the corresponding embedding of the finite cylinder. For every $(r_\tau,t_\tau) \in (\mathbb{R}_0^+ \times S^1)^{\bar{E}^0}$ let $\dot{S}_{(r_\tau,t_\tau)}$ denote the punctured Riemann sphere obtained from $\dot{S}$ by replacing for each $\tau \in \bar{E}^0$ the embedded cylinders $\psi_\tau([-1,0] \times S^1)$ by $[-r_\tau,0] \times S^1$, $\psi_\tau([0,+1] \times S^1)$ by $[0,+r_\tau] \times S^1$, and gluing $[-r_\tau,0] \times S^1$ and $[0,+r_\tau] \times S^1$ with a twist $t_\tau \in S^1$. Note that for any $(r_\tau,t_\tau) \in (\mathbb{R}_0^+ \times S^1)^{\bar{E}^0}$ the punctured Riemann sphere $\dot{S}_{(r_\tau,t_\tau)}$ represents an element in $\mathcal{M}_{0,n}$ and we assume without loss of generality that the complex structure j_0 on the noncylindrical part of $\dot{S}$ is chosen such that the map from $(\mathbb{R}_0^+ \times S^1)^{\bar{E}^0}$ to $\mathcal{M}_{0,n}$ is indeed a coordinate chart for $\mathcal{M}_{0,n}$.

Assuming that we have covered $\mathcal{M}_{0,n}$ by coordinate charts of the above form, we are now ready to describe the compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ by compactifying each coordinate neighborhood in the following nonstandard way. First observe (compare [BEHWZ]) that when we compactify each coordinate neighborhood by viewing it as a subset of $(\mathbb{R} \times S^1)^{\bar{E}^0}$ with compactification $(\overline{\mathbb{R}} \times S^1)^{\bar{E}^0}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, then we obtain the Deligne-Mumford compactification $\overline{\mathcal{M}}^{\$}_{0,n}$ with decorations at each node. On the other hand, note that when we use the compactification $(\mathbb{CP}^1)^{\bar{E}^0}$ of $(\mathbb{R}\times S^1)^{\bar{E}^0}$ by identifying $\mathbb{R}\times S^1\cong\mathbb{C}^*$, then we obtain the usual Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ without decorations. In order to obtain $\overline{\mathcal{M}} = S^1\times\widetilde{\mathcal{M}}_{0,n}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ we need yet another compactification $\widetilde{\mathcal{M}}_{0,n}$ of $\mathcal{M}_{0,n}$. Besides that we want decorations only at those nodes which correspond to a pair of a positive and a negative puncture, we must keep track of the relative $\mathbb{R}$-shift of the different components when they are mapped to the trivial cylinder.

To this end, recall that each $k\in\Lambda^{0,\pm}_{\alpha}$ represents a positive, respectively negative puncture to which we assign a fixed multiple $\gamma^{m^{\pm}_k}$ of the underlying simple orbit γ. Hence we can again naturally assign a multiplicity with sign $m_{\alpha\beta}\in\mathbb{Z}$ to each directed edge in E^0 by requiring for each $\alpha\in T^0$ that

$$\sum_{\beta:\alpha E^0\beta} m_{\alpha\beta} + \sum_{k\in\Lambda^{0,+}_{\alpha}} m^+_k - \sum_{k\in\Lambda^{-,0}_{\alpha}} m^-_k = 0.$$

Note that $m_{\beta\alpha} = -m_{\alpha\beta}$. Now we identify the coordinate subset of $\mathcal{M}_{0,n}$ not with $(\mathbb{R}^+_0\times S^1)^{\bar{E}^0}$, but view it as a linear subspace of $(\mathbb{R}^+_0\times S^1)^{\bar{E}^0}\times\mathbb{R}^{T^0\times T^0}$ by setting for $(\alpha,\beta)\in T^0\times T^0$

$$s_{\alpha\beta} = \sum_{i=1}^{k} m_{(\gamma_{i-1},\gamma_i)} r_{[\gamma_{i-1},\gamma_i]},$$

where $\alpha=\gamma_0,...,\gamma_k=\beta$ is the enumeration of vertices on the unique directed path in (T^0,E^0) from α to β.

Distinguishing further the undirected edges in $\bar{E}^0$ by whether their multiplicity is zero or not, $\bar{E}^0 = \bar{E}^0_0\cup\bar{E}^0_{\pm}$, we now obtain $\widetilde{\mathcal{M}}_{0,n}$ by viewing it as a subset of $(\mathbb{R}\times S^1)^{\bar{E}^0_0}\times(\mathbb{R}\times S^1)^{\bar{E}^0_{\pm}}\times\mathbb{R}^{T^0\times T^0}$ with compactification given by $(\mathbb{CP}^1)^{\bar{E}^0_0}\times(\overline{\mathbb{R}}\times S^1)^{\bar{E}^0_{\pm}}\times\overline{\mathbb{R}}^{T^0\times T^0}$. It directly follows from the construction of $\widetilde{\mathcal{M}}_{0,n}$ that $\widetilde{\mathcal{M}}_{0,n}$ carries the structure of a manifold with corners. Further the boundary of $\mathcal{M}_{0,n}$ in $\widetilde{\mathcal{M}}_{0,n}$ consists of tuples $((r_\tau,t_\tau),(s_{\alpha\beta}))$ with $r_\tau=\infty$ for some edge $\tau\in\bar{E}^0$. While the coordinates (r_τ,t_τ) describe a nodal curve with decorations at nodes corresponding to edges in $\bar{E}^0_{\pm}$, we show that the coordinates $(s_{\alpha\beta})$ describes a level structure with relative $\mathbb{R}$-shifts. More precisely, recalling that $\mathcal{M}\cong S^1\times\mathcal{M}_{0,n}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$, we show in the following that there is a natural identification of $S^1\times\widetilde{\mathcal{M}}_{0,n}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ with the compactified moduli space $\overline{\mathcal{M}}$ of trivial curves. To this end we assign to any tuple $(t_0,((r_\tau,t_\tau),(s_{\alpha\beta})),\mu^{\pm})$ a level branched covering $(h,j,\mu^{\pm},\theta)$ as follows:

First observe that the underlying nodal curve is described by the coordinates $(r_\tau,t_\tau)\in(\mathbb{CP}^1)^{\bar{E}^0_0}\times(\overline{\mathbb{R}}\times S^1)^{\bar{E}^0_{\pm}}$, where $\alpha,\beta\in T^0$ belong to the same connected component when $r_\tau<\infty$ for each edge on the unique path from α to β. Note that the latter defines an equivalence relation $\approx$ on T^0, such that the quotient $T=T^0/\approx$ with induced edge relation $E \subset T \times T$ is the tree representing the nodal curve. Distinguishing the undirected edges in $\bar{E}$ by whether they have a nonzero multiplicity or not, $\bar{E} = \bar{E}_0 \cup \bar{E}_\pm$, note that the undirected edges in $\bar{E}_0$ now correspond to nodes connecting components in the same level, while the edges in $\bar{E}_\pm$ correspond to pairs of components living on neighboring levels connected by a positive, respectively negative puncture. Since each branched cover of the trivial cylinder is determined up to $\mathbb{R} \times S^1$-shift by the underlying punctured sphere in $\mathcal{M}_{0,n}$, it follows that the level branched cover in $\overline{\mathcal{M}}$ is already known up to the S^1-shifts, decorations in $\mathbb{Z}_{|m_{\alpha\beta}|} \times \mathbb{Z}_{|m_{\alpha\beta}|} / \Delta \cong \mathbb{Z}_{|m_{\alpha\beta}|}$ at the punctures between levels and the level structure with the relative $\mathbb{R}$-shifts.

First, in order to see how the coordinates $s_{\alpha\beta} \in \bar{\mathbb{R}}$, $\alpha, \beta \in T^0$ fix the level structure and the relative $\mathbb{R}$-shifts, let $((r^n_\tau, t^n_\tau), (s^n_{\alpha\beta})) \in (\mathbb{R}^+_0 \times S^1)^{E^0} \times \mathbb{R}^{T^0 \times T^0}$ be a sequence converging to $((r_\tau, t_\tau), (s_{\alpha\beta}))$, where without loss of generality $t^n_\tau = t_\tau$. Let $\dot{S}_n = \dot{S}_{(r^n_\tau, t^n_\tau)}$ be the corresponding sequence of punctured spheres converging to the punctured nodal surface $\dot{S}$ with connected components $\dot{S}_{[\alpha]}$, $[\alpha] \in T = T^0/\approx$, and let $h^n = (h^n_1, h^n_2) : \dot{S}_n \to \mathbb{R} \times S^1$ be a corresponding sequence of branched covering maps converging to the level branched cover $h = (h^{[\alpha]})_{[\alpha] \in T} : \dot{S} \to \mathbb{R} \times S^1$. In order to see the relation between $(s^n_{\alpha\beta})_{\alpha,\beta}$ and the level structure and relative $\mathbb{R}$-shifts of the limit level curve h, fix points z_α, z_β on the pair of pants components corresponding to two chosen $\alpha, \beta \in T^0$. For each $(\gamma, \delta) \in E^0$ on the unique path from α to β, set $h^n_{\gamma\delta} = \int_0^1 h^n_1 \circ \psi^n_{\gamma\delta}(r^n_{\gamma\delta}, t)dt$, with the embedding $\psi^n_{\gamma\delta} : [-r^n_{\gamma\delta}, +r^n_{\gamma\delta}] \times S^1 \to \dot{S}_n$ of the finite cylinder at the edge $(\gamma, \delta) \in E^0$, where $r^n_{\gamma\delta} = r^n_{[\gamma,\delta]}$ and $\psi^n_{\delta\gamma} : [-r^n_{\delta\gamma}, r^n_{\delta\gamma}] \times S^1 \to \dot{S}_n$, $\psi^n_{\delta\gamma}(s,t) = \psi^n_{\gamma\delta}(-s,-t)$. Observe that we have

$$\begin{aligned} h^n_{\gamma\delta} - h^n_{\delta\gamma} &= \int_0^1 \int_{-r^n_{\gamma\delta}}^{+r^n_{\gamma\delta}} \partial_s (h^n_1 \circ \psi^n_{\gamma\delta})(s,t)\, ds\, dt \\ &= \int_{-r^n_{\gamma\delta}}^{+r^n_{\gamma\delta}} \int_0^1 \partial_t (h^n_2 \circ \psi^n_{\gamma\delta})(s,t)\, dt\, ds \\ &= \int_{-r^n_{\gamma\delta}}^{+r^n_{\gamma\delta}} ((h^n_2 \circ \psi^n_{\gamma\delta})(s,1) - (h^n_2 \circ \psi^n_{\gamma\delta})(s,0))\, ds \\ &= 2 \cdot m_{\gamma\delta} \cdot r^n_{\gamma\delta}. \end{aligned}$$

Now let $\alpha = \gamma_0, \gamma_1, ..., \gamma_k = \beta$ be the enumeration of vertices in T^0 on the unique path from α to β and set $h^n_{i,j} = h^n_{\gamma\delta}$ for $\gamma = \gamma_i$, $\delta = \gamma_j$. Then we have

$$\begin{aligned} h^n_1(z_\alpha) - h^n_1(z_\beta) &= h^n_1(z_\alpha) - h^n_{0,1} + \sum_{i=0}^{k-1} \big(h^n_{i,i+1} - h^n_{i+1,i}\big) \\ &\quad + \sum_{i=1}^{k-1} \big(h^n_{i,i-1} - h^n_{i,i+1}\big) + h^n_{k,k-1} - h^n_1(z_\beta). \end{aligned}$$

With $m_{i,j} = m_{\gamma\delta}$, $r^n_{i,j} = r^n_{\gamma\delta}$ for $\gamma = \gamma_i$, $\delta = \gamma_j$ we have

$$\sum_{i=0}^{k-1} \big(h^n_{i,i+1} - h^n_{i+1,i}\big) = \sum_{i=0}^{k-1} 2m_{i,i+1} r^n_{i,i+1} = 2s^n_{\alpha\beta},$$

so that

$$\begin{aligned}
&(h^n_1(z_\alpha) - h^n_1(z_\beta)) - 2s^n_{\alpha\beta} \\
&= \big(h^n_1(z_\alpha) - h^n_{0,1}\big) + \sum_{i=1}^{k-1} \big(h^n_{i,i-1} - h^n_{i,i+1}\big) + \big(h^n_{k,k-1} - h^n_1(z_\beta)\big) \\
&\stackrel{n\to\infty}{\longrightarrow} \big(h_{[\alpha],1}(z_\alpha) - h_{[\alpha],0,1}\big) + \sum_{i=1}^{k-1} \big(h_{[\gamma_i],i,i-1} - h_{[\gamma_i],i,i+1}\big) \\
&+ \big(h_{[\beta],k,k-1} - h_{[\beta],1}(z_\beta)\big).
\end{aligned}$$

Note that the last expression depends only on the underlying nodal curve and is independent of the $\mathbb{R} \times S^1$-shifts. But this shows how the coordinates $s_{\alpha\beta} \in \mathbb{R}$ describe the level structure and the relative $\mathbb{R}$-shifts, in particular, two connected components belong to the same level precisely when $-\infty < s_{\alpha\beta} < +\infty$ for each $\alpha, \beta \in T^0$ representing the connected components in $T = T^0/\approx$.

In order to fix the S^1-shifts and decorations in $\mathbb{Z}_{|m_{\alpha\beta}|}$ at punctures between levels, observe that the coordinates $t_\tau \in S^1$ with $\tau \in \bar{E}^0_\pm$ determine decorations t_τ at the nodes $\tau \in \bar{E}_\pm$ corresponding to pairs of punctures connecting components on neighboring levels. Together with the S^1-coordinate t_0 they fix the S^1-shifts on each connected component of the level branched covering map as follows:

First for $\alpha \in T$ with $1 \in \Lambda^+_\alpha$ we fix h_α by requiring that h_α maps the asymptotic marker at z^+_1 to $t_0 \in S^1$. On the other hand, if h_α is fixed for some $\alpha \in T$, we can fix the S^1-shift for maps h_β with $\alpha E \beta$ as follows: On the one hand, when $m_{\alpha\beta} = 0$, i.e., when α and β represent curves in the same level connected by a node $z_{\alpha\beta} \sim z_{\beta\alpha}$, the condition $h_\alpha(z_{\alpha\beta}) = h_\beta(z_{\beta\alpha})$ immediately fixes the S^1-shift for h_β. Now consider the case when $m_{\alpha\beta} \neq 0$, i.e., $z_{\alpha\beta}$ and $z_{\beta\alpha}$ are positive or negative punctures. After choosing an asymptotic marker at $z_{\alpha\beta}$, which is mapped to $0 \in S^1$ under h_α, we can use the decoration $t_{[\alpha,\beta]} \in S^1$, $[\alpha,\beta] \in \bar{E}_\pm$ to get an asymptotic marker at $z_{\beta\alpha}$, and choose $h_\beta : (\dot{S}_\beta, j_\beta) \to \mathbb{R} \times S^1$ so that it maps the asymptotic marker at $z_{\beta\alpha}$ to $0 \in S^1$. Since $h_\alpha : (\dot{S}_\alpha, j_\alpha) \to \mathbb{R} \times S^1 \cong \mathbb{R} \times \gamma$ is asymptotically cylindrical over the multiple $\gamma^{|m_{\alpha\beta}|}$, it follows that there are $|m_{\alpha\beta}|$ different possible choices for the asymptotic marker at $z_{\alpha\beta}$. Using the decoration $t_{\alpha\beta}$ this leads to $|m_{\beta\alpha}| = |m_{\alpha\beta}|$ different possible choices for the asymptotic marker at $z_{\beta\alpha}$, which however all lead to the same map $h_\beta : (\dot{S}_\beta, j_\beta) \to \mathbb{R} \times S^1$. Note that in this way we do not only get the holomorphic maps $h_\alpha : (\dot{S}_\alpha, j_\alpha) \to \mathbb{R} \times S^1$ (up to the common $\mathbb{R}$-shift in each level), but also the decorations $\theta_{\alpha\beta} \in \mathbb{Z}_{|m_{\alpha\beta}|}$, i.e., we see that each element $(t_0,((r_\tau,t_\tau),(s_{\alpha\beta})),\mu^\pm)\in S^1\times\widetilde{\mathcal{M}}_{0,n}\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ uniquely defines an element $(h,j,\mu,\theta)\in\overline{\mathcal{M}}$.

For the reverse, assume we are given an element $(h,j,\mu,\theta)\in\overline{\mathcal{M}}$, i.e., we are given maps h_α and h_β for two components α,β connected by an edge in $(T,\mathcal{L})$, where we must only consider the case where α and β live on different levels. Here we simultaneously have $|m_{\alpha\beta}|$ different possible choices for the asymptotic marker at $z_{\alpha\beta}$ and $|m_{\alpha\beta}|$ different possible choices for the asymptotic marker at $z_{\alpha\beta}$, which lead to $|m_{\alpha\beta}|$ different possible choices for the decoration $t_{[\alpha,\beta]}\in S^1$, which is then fixed using $\theta_{\alpha\beta}\in\mathbb{Z}_{|m_{\alpha\beta}|}$. □

2.2 Obstruction bundle and Fredholm theory

For determining the contribution of the moduli spaces of branched covers of trivial cylinders to the differential in rational symplectic field theory and contact homology, we show in section 2.3.1 that it suffices to study sections in a natural candidate for an obstruction bundle over the compactified moduli space of branched covers, the so-called cokernel bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$ of the Cauchy-Riemann operator $\bar{\partial}_J$. Hence we follow the standard approach in Gromov-Witten theory of using obstruction bundles in order to deal with moduli spaces which are not regular in the sense that they are not transversally cut out by the Cauchy-Riemann operator.

2.2.1 Cokernel bundle

Denoting by $D_{h,j}$ the linearization of the Cauchy-Riemann operator $\bar{\partial}_J$ at $(h,j,\mu^\pm)\in\mathcal{M}$, which we discuss in detail in the upcoming subsection, the fibre at $(h,j,\mu^\pm)$ of the bundle $\mathrm{Coker}\,\bar{\partial}_J$ over $\mathcal{M}$ as well as the bundle $\mathrm{Coker}_0\,\bar{\partial}_J$ over $\mathcal{M}^0$ is given by the cokernel of $D_{h,j}$

$$(\mathrm{Coker}\,\bar{\partial}_J)_{(h,j,\mu^\pm)}=(\mathrm{Coker}_0\,\bar{\partial}_J)_{(h,j,\mu^\pm)}=\mathrm{coker}\,D_{h,j}.$$

For the extensions $\overline{\mathrm{Coker}}\bar{\partial}_J$, $\overline{\mathrm{Coker}_0}\bar{\partial}_J$ over the compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^0}$ we require that the fibre over (h,j,μ,θ) in the stratum $\mathcal{M}_{T,\mathcal{L}}$ or $\mathcal{M}^0_{T,\mathcal{L},\ell_0}$ is given by

$$\begin{aligned}&(\overline{\mathrm{Coker}}\bar{\partial}_J)_{(h,j,\mu,\theta)}=(\overline{\mathrm{Coker}_0}\bar{\partial}_J)_{(h,j,\mu,\theta)}\\&=\bigoplus_{\ell=1}^{L}\{(\eta_\alpha)_{\alpha\in T_\ell}:\eta_\alpha\in\mathrm{coker}\,D_{h_\alpha,j_\alpha},\ \eta_\alpha(z_{\alpha\beta})=\eta_\beta(z_{\beta\alpha})\}.\end{aligned}$$

Since the fibre does not depend on the position of the asymptotic markers $\mu^\pm\in\mathbb{Z}_{m^\pm}$, it follows that $\overline{\mathrm{Coker}}\bar{\partial}_J$ ($\overline{\mathrm{Coker}_0}\bar{\partial}_J$) naturally lives over the quotient $\overline{\mathcal{M}}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ ($\overline{\mathcal{M}^0}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$) rather than $\overline{\mathcal{M}}$ ($\overline{\mathcal{M}^0}$) and we will view it this way. However it will later become important to consider it as a bundle over $\overline{\mathcal{M}}$ ($\overline{\mathcal{M}^0}$) when we talk about orientations.

Denoting by $\mathrm{Coker}^{T,\mathcal{L}}\,\bar\partial_J$, $\mathrm{Coker}_0^{T,\mathcal{L}}\,\bar\partial_J$ the restrictions of $\overline{\mathrm{Coker}}\bar\partial_J$, $\overline{\mathrm{Coker}_0}\bar\partial_J$ to $\mathcal{M}_{T,\mathcal{L}}$, $\mathcal{M}^0_{T,\mathcal{L}}$, observe that they are given as direct sums

$$\begin{aligned}\mathrm{Coker}^{T,\mathcal{L}}\,\bar\partial_J &= \pi^*_{1,1}\,\mathrm{Coker}^{T_{1,1}}\,\bar\partial_J \oplus ... \oplus \pi^*_{L,N_L}\,\mathrm{Coker}^{T_{L,N_L}}\,\bar\partial_J,\\ \mathrm{Coker}_0^{T,\mathcal{L}}\,\bar\partial_J &= \pi^*_{1,1}\,\mathrm{Coker}^{T_{1,1}}\,\bar\partial_J \oplus ... \oplus \pi^*_{\ell_0,N_{\ell_0}}\,\mathrm{Coker}_0^{T_{\ell_0,N_{\ell_0}}}\,\bar\partial_J \oplus\\ &\quad ... \oplus \pi^*_{L,N_L}\,\mathrm{Coker}^{T_{L,N_L}}\,\bar\partial_J,\end{aligned}$$

with the projections

$$\pi_{\ell,k} : \mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}) = \frac{\mathcal{M}_{T_1}\times ... \times \mathcal{M}_{T_L}}{\Delta\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}} \to \mathcal{M}_{T_{\ell,k}}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}}),$$

and similar for $\mathcal{M}^0_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, where $m^\pm_{\ell,k} = \prod_{\alpha\in T_{\ell,k}} m^\pm_\alpha$, $m^\pm_\alpha = m^\pm_{\alpha,1}\cdot ... \cdot m^\pm_{\alpha,n^\pm_\alpha}$ and $\mathrm{Coker}^{T_{\ell,k}}$ ($\mathrm{Coker}_0^{T_{\ell,k}}$) denotes the cokernel bundle over $\mathcal{M}_{T_{\ell,k}}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}})$ ($\mathcal{M}^0_{T_{\ell,k}}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}})$) for $\ell = 1,...,L$, $k = 1,...,N_\ell$. Note that there exists no natural map from $\mathcal{M}_{T,\mathcal{L}}$ ($\mathcal{M}^0_{T,\mathcal{L}}$) to $\mathcal{M}_{T_1},...,\mathcal{M}_{T_L}$ and hence to $\mathcal{M}_{T_{1,1}},...,\mathcal{M}_{T_{L,N_L}}$, since we quotient out the diagonal Δ, i.e., identify pairs of asymptotic markers if they represent the same decoration.

Recall from subsection 2.1.2 that when $\mathcal{M}_{T,\mathcal{L}}$ belongs to the codimension one boundary of $\mathcal{M}$ it is of the form $\mathcal{M}_{T,\mathcal{L}} = \mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are moduli spaces of possibly disconnected trivial curves without nodes. Note that the compactification of the fibre product $\overline{\mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2}\subset\partial\overline{\mathcal{M}}$ can directly be identified with the fibre product of the compactifications,

$$\overline{\mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2} = \overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{1,2}}}\overline{\mathcal{M}}_2.$$

For this observe that the partitioning of the levels of a limiting curve in $\overline{\mathcal{M}_1\times_{\mathbb{Z}_{m_{1,2}}}\mathcal{M}_2}$ into levels belonging to the compactification $\overline{\mathcal{M}}_1$ or $\overline{\mathcal{M}}_2$, respectively, follows from the conservation of the total Euler characteristic under degeneration of punctured Riemann surfaces. Denoting by $\overline{\mathrm{Coker}}^1\bar\partial_J$ and $\overline{\mathrm{Coker}}^2\bar\partial_J$ the extensions of the cokernel bundles over $\mathcal{M}_1$, $\mathcal{M}_2$ to the corresponding compactified moduli spaces, it directly follows from the form of $\overline{\mathrm{Coker}}\bar\partial_J$ over the strata $\mathcal{M}_{T,\mathcal{L}}$ that

$$\overline{\mathrm{Coker}}\bar\partial_J|_{\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{1,2}}}\overline{\mathcal{M}}_2} = \pi_1^*\overline{\mathrm{Coker}}^1\bar\partial_J\oplus\pi_2^*\overline{\mathrm{Coker}}^2\bar\partial_J,$$

with the projections $\pi_{1,2} : \overline{\mathcal{M}}/\,\mathbb{Z}_{m^\pm}\to\overline{\mathcal{M}}_{1,2}/\,\mathbb{Z}_{m^\pm_{1,2}}$. For the cokernel bundle $\overline{\mathrm{Coker}_0}\bar\partial_J$ it follows in the same way that

$$\begin{aligned}\overline{\mathrm{Coker}_0}\bar\partial_J|_{\overline{\mathcal{M}^0_1}\times_{\mathbb{Z}_{m_{1,2}}}\overline{\mathcal{M}}_2} &= \pi_1^*\overline{\mathrm{Coker}_0}^1\bar\partial_J\oplus\pi_2^*\overline{\mathrm{Coker}}^2\bar\partial_J,\\ \overline{\mathrm{Coker}_0}\bar\partial_J|_{\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{1,2}}}\overline{\mathcal{M}^0_2}} &= \pi_1^*\overline{\mathrm{Coker}}^1\bar\partial_J\oplus\pi_2^*\overline{\mathrm{Coker}_0}^2\bar\partial_J\end{aligned}$$

and

$$\overline{\mathrm{Coker}_0}\bar{\partial}_J|_{\{\mathrm{point}\}\times\overline{\mathcal{M}}} = \overline{\mathrm{Coker}_0}\bar{\partial}_J|_{\overline{\mathcal{M}}\times\{\mathrm{point}\}} = \overline{\mathrm{Coker}}\bar{\partial}_J.$$

In order to show that $\overline{\mathrm{Coker}}\bar{\partial}_J$ indeed serves as an obstruction bundle, we show in the upcoming subsection 2.2.2 that on every stratum $\mathcal{M}_{T,\mathcal{L}} \subset \overline{\mathcal{M}}$ we have

$$\ker D_{h,j} = T_{h,j}\,\mathcal{M}_{T,\mathcal{L}}$$

at every $(h, j, \mu^{\pm}, \theta) \in \mathcal{M}_{T,\mathcal{L}}$, see subsection 2.3.1 below, which then automatically implies that $\mathrm{Coker}^{T,\mathcal{L}}$ is indeed a smooth vector bundle over $\mathcal{M}_{T,\mathcal{L}}$. In order to show that these vector bundle naturally fit together to a smooth vector bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$ over the manifold with corners $\overline{\mathcal{M}}$, we prove in subsection 2.2.3 a linear gluing theorem relating the cokernel bundles over different strata of $\overline{\mathcal{M}}$.

2.2.2 Linearized operator

For all this we first need to understand the linearization $D_{h,j}$ of the Cauchy-Riemann operator $\bar{\partial}_J$ at $(h, j) \in \mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$. In what follows we formulate our statements only for the cokernel bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$, since the statements for $\overline{\mathrm{Coker}_0}\bar{\partial}_J$ then follow immediately. For the Banach manifold setup we follow [BM] and the expositions from the first chapter.

Recall from the definition of the moduli spaces that we fixed n^+ positive and n^- negative punctures $z^{\pm}_1, ..., z^{\pm}_{n^{\pm}} \in S^2$ and fixed cylindrical coordinates

$$\psi^{\pm}_k : \mathbb{R}^{\pm}_0 \times S^1 \hookrightarrow \dot{S}$$

around each puncture $z^{\pm}_k$, $k \in \{1, ..., n^{\pm}\}$ on the punctured sphere $\dot{S} = S^2 - \{z^{\pm}_1, ..., z^{\pm}_{n^{\pm}}\}$. Let the space $H^{1,p,d}_{\mathrm{const}}(\dot{S}, \mathbb{C})$ consist of all maps from $\dot{S}$ to $\mathbb{C}$ differing asymptotically from a constant one by a function, which is still in $H^{1,p}$ after multiplication with an asymptotic weight. To be precise, any $v \in H^{1,p,d}_{\mathrm{const}}(\dot{S}, \mathbb{C})$ is in $H^{1,p}_{\mathrm{loc}}$ and for any puncture $z^{\pm}_k$ there exist $(s^{\pm,k}_0, t^{\pm,k}_0) \in \mathbb{R}^2 \cong \mathbb{C}$, so that the function

$$\mathbb{R}^{\pm}\times S^1 \to \mathbb{C},\ (s,t) \mapsto [(v \circ \psi^{\pm}_k)(s,t) - (s^{\pm,k}_0, t^{\pm,k}_0)] \cdot e^{\pm d\cdot s}$$

is in $H^{1,p}$. Let further $L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C})$ denote the space of (j, i)-antiholomorphic one-forms on $\dot{S}$ with values in $\mathbb{C}$, which are still in L^p after multiplication with the asymptotic weight $e^{\pm d\cdot s}$.

With $h^*\xi$ denoting the pullback of the subbundle $\xi \subset TV$ under the branched covering map $h : (\dot{S}, j) \to (\mathbb{R}\times S^1, i) \cong (\mathbb{R}\times\gamma, J)$, we introduce the spaces $H^{1,p}(h^*\xi)$ of sections and $L^p(T^*\dot{S}\otimes_{j,J_\xi} h^*\xi)$ of (j,J_ξ)-antiholomorphic one-forms on $\dot{S}$ with values in $h^*\xi$, where the $H^{1,p}$- and L^p-topologies are defined with respect to any trivialization of ξ along the fixed Reeb orbit γ.

Following [Sch] and [BM] there exists a Banach space bundle $\mathcal{E}^{p,d}$ over a Banach manifold of maps $\mathcal{B}^{p,d}$ in which the Cauchy-Riemann operator $\bar{\partial}_J$ extends to a smooth section. In our special case it follows that the fibre is given by

$$\mathcal{E}^{p,d}_{h,j} = L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J_\xi} h^*\xi),$$

while the tangent space to the Banach manifold of maps $\mathcal{B}^{p,d} = \mathcal{B}^{p,d}(\gamma^{m_1^+},...,\gamma^{m_{n^+}^+};\gamma^{m_1^-},...,\gamma^{m_{n^-}^-})$ at $(h,j)\in\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ is given by

$$T_{h,j}\,\mathcal{B}^{p,d}(V;(\gamma^{m_i^\pm})) = H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})\oplus H^{1,p}(h^*\xi)\oplus T_j\,\mathcal{M}_{0,n}\,.$$

Note that we use the complex splitting of the tangent bundle $T(\mathbb{R}\times V)=\mathbb{C}\oplus\xi$ in order to write tangent spaces and fibres as direct sums.

In order to give an explicit formula for the linearization $D_{h,j}$ of $\bar{\partial}_J$ we choose a complex connection on (ξ,J_ξ) which we extend to a connection ∇ on $T(\mathbb{R}\times V)=\mathbb{C}\oplus\xi$, $\mathbb{C}=\mathbb{R}\cdot\partial_s\oplus\mathbb{R}\cdot R$ by requiring $\mathbb{R}$-invariance and $\nabla\partial_s=\nabla R=0$, where ∂_s is the $\mathbb{R}$-direction and R the Reeb vector field of the stable Hamiltonian structure. For this connection it follows that the linearization $D_{h,j}$ of $\bar{\partial}_J$ at branched covers of orbit cylinders (h,j) is of a special form.

Proposition 2.2.1: *With respect to the complex connection ∇ on $T(\mathbb{R}\times V)$ from above, the linearization $D_{h,j}$ of $\bar{\partial}_J$ at $(h,j)\in\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ is given by*

$$\begin{aligned} D_{h,j}:\ & H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})\oplus H^{1,p}(h^*\xi)\oplus T_j\,\mathcal{M}_{0,n}\\ &\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J_\xi} h^*\xi),\\ & D_{h,j}\cdot(v_1,v_2,y) = (\bar{\partial}v_1 + D_j y, D^\xi_h v_2),\end{aligned}$$

where $\bar{\partial}: H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C})\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})$ is the standard Cauchy-Riemann operator,

$$\begin{aligned} &D^\xi_h: H^{1,p}(h^*\xi)\to L^p(T^*\dot{S}\otimes_{j,J_\xi} h^*\xi),\\ &D^\xi_h v_2 = \nabla v_2 + J_\xi\cdot\nabla v_2\cdot j + \nabla_{dh}v_2 + J_\xi\nabla_{i\,dh}v_2\end{aligned}$$

describes the linearization of $\bar{\partial}_J$ in the direction of $\xi\subset TV$ and

$$D_j: T_j\,\mathcal{M}_{0,n}\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C}),\ D_j y = i\cdot dh\cdot y.$$

describes the variation of $\bar{\partial}_J$ with $j\in\mathcal{M}_{0,n}$.

Proof: Since ∇ is a complex connection, it is well-known, see e.g. [Sch], that the linearization $D_h : H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \oplus H^{1,p}(h^*\xi) \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J_\xi} h^*\xi)$ of $\bar{\partial}_J$ fixing the complex structure $j \in \mathcal{M}_{0,n}$ is given by

$$D_h \cdot v \;=\; \nabla v + J \cdot \nabla v \cdot j + \operatorname{Tor}(dh, v) + J \operatorname{Tor}(J \cdot dh, v),$$

where $\operatorname{Tor}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. First it follows from the special form of ∇ that

$$\nabla v + J \cdot \nabla v \cdot j = (\bar{\partial} v_1, \nabla v_2 + J_\xi \cdot \nabla v_2 \cdot j).$$

for $v = (v_1, v_2) \in H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \oplus H^{1,p}(h^*\xi)$. On the other hand,

$$\begin{aligned}
&\operatorname{Tor}(dh, v) + J \operatorname{Tor}(J \cdot dh, v) = \\
&\nabla_{dh} v + J \cdot \nabla_{J\,dh} v - \nabla_v dh - J \cdot \nabla_v (J \cdot dh) - ([dh, v] + J[J\,dh, v]) = \\
&\nabla_{dh} v + J \cdot \nabla_{J\,dh} v + J \cdot (L_v(J\,dh) - J \cdot L_v dh) = \\
&\nabla_{dh} v + J \cdot \nabla_{J\,dh} v + J \cdot L_v J \cdot dh = \nabla_{dh} v + J \cdot \nabla_{J\,dh} v.
\end{aligned}$$

From $\nabla \partial_s = \nabla R = 0$ it follows that $\operatorname{Tor}(dh, v_1) + J \operatorname{Tor}(J \cdot dh, v_1) = 0$, while for $v_2 \in \xi$ we have $\nabla_{dh} v_2 + J \cdot \nabla_{J\,dh} v_2 \in \xi$, so that $D_h \cdot (v_1, v_2) = (\bar{\partial} v_1, D^\xi_h v_2)$ with D^ξ_h as in the lemma. Finally note that for the linearization of $\bar{\partial}_J$ in the direction of $\mathcal{M}_{0,n}$ there is obviously no variation in the ξ-direction,

$$D_j y = (i \cdot dh \cdot y, 0). \;\square$$

Based on this result, the following lemmata describe kernel and cokernel of $D_{h,j}$.

Proposition 2.2.2: *The standard Cauchy-Riemann operator $\bar{\partial} : H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C})$ is onto, so that*

$$\operatorname{coker} D_{h,j} = \operatorname{coker} D^\xi_h.$$

Proof: The first part of the statement is the content of lemma 1.3.2 from the first chapter, while the the second part of the statement follows from the upper-triangle-form of $D_{h,j}$. $\square$

Proposition 2.2.3: *The operator D^ξ_h has a trivial kernel, so that*

$$\ker D_{h,j} = T_{h,j}(\mathbb{R} \times \mathcal{M}).$$

Proof: Note that here the second part of the statement follows from the first one using proposition 2.2.2 as follows: First it is clear that we have the inclusion $T_{h,j}(\mathbb{R} \times \mathcal{M}) \subset \ker D_{h,j}$, since $\mathbb{R} \times \mathcal{M} = \bar{\partial}_J^{-1}(0)$. On the other hand, using the first part of the statement we know that the kernel of $D_{h,j}$ consists of all pairs $(\bar{h}, y) \in H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \oplus T_j \mathcal{M}_{0,n}$ satisfying $\bar{\partial}\bar{h} + D_j y = 0$. Since $\bar{\partial} : H^{1,p,d}_{\text{const}}(\dot{S},\mathbb{C}) \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C})$ is surjective, it follows that $\ker D_{h,j}$ projects surjectively onto $T_j \mathcal{M}_{0,n}$, where the fibre can be identified with $\ker \bar{\partial} = \mathbb{C}$. In particular, we have that the dimension of $\ker D_{h,j}$ agrees with the dimension of $T_{h,j}(\mathbb{R}\times S^1\times\mathcal{M}_{0,n}) = T_{h,j}(\mathbb{R}\times\mathcal{M})$, so that the inclusion must indeed be an equality.

The statement about the kernel of D_h^ξ is the linearized version of lemma 5.4 in [BEHWZ]. For chosen $h=(h_1,h_2):(\dot{S},j)\to\mathbb{R}\times S^1\cong(\mathbb{R}\times\gamma,J)$ and $v_2\in\ker D_h^\xi\subset H^{1,p}(h^*\xi)$ we can use the exponential map of some Riemannian metric on $\mathbb{R}\times V$ to get for $r>0$ sufficiently small a family of curves

$$\exp_h rv_2 = (h_1,\exp_{h_2} rv_2):(\dot{S},j)\to\mathbb{R}\times V.$$

Note that their ω-energies $E_\omega(\exp_h rv_2)=\int_{\dot{S}}(\exp_{h_2} rv_2)^*\omega$ by homological reasons agree with the ω-energy of h and hence vanish,

$$E_\omega(\exp_h rv_2) = E_\omega(h) = 0,$$

since all curves in the family are asymptotically cylindrical over the same closed Reeb orbits near the punctures. Choosing an atlas $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$ for the complex manifold $\dot{S}$ with local holomorphic coordinates (s_α,t_α) on $U_\alpha\subset\dot{S}$, together with a subordinate partition of unity $(\psi_\alpha)_{\alpha\in A}$, observe that the above integral can be rewritten as

$$\begin{aligned}
&\int_{\dot{S}}(\exp_{h_2} rv_2)^*\omega\\
&=\sum_\alpha\int_{U_\alpha}\psi_\alpha\cdot\omega(\partial_{s_\alpha}\exp_{h_2} rv_2,\partial_{t_\alpha}\exp_{h_2} rv_2)ds_\alpha\wedge dt_\alpha\\
&=\sum_\alpha\int_{U_\alpha}\psi_\alpha\cdot\omega_\xi(\pi_\xi\partial_{s_\alpha}\exp_{h_2} rv_2,\pi_\xi\partial_{t_\alpha}\exp_{h_2} rv_2)ds_\alpha\wedge dt_\alpha,
\end{aligned}$$

where π_ξ denotes the projection $TV=\mathbb{C}\oplus\xi\to\xi$ and the second equality follows from $R\in\ker\omega$. With the metric $\langle\cdot,\cdot\rangle_\xi=\omega_\xi(\cdot,J_\xi\cdot)$ on ξ we get that the latter is equal to

$$\begin{aligned}
&\sum_\alpha\int_{U_\alpha}\psi_\alpha\cdot\langle\pi_\xi\partial_{s_\alpha}\exp_{h_2} rv_2,-\pi_\xi\,J\,\partial_{t_\alpha}\exp_{h_2} rv_2\rangle_\xi\; ds_\alpha\wedge dt_\alpha=\\
&\sum_\alpha\int_{U_\alpha}\psi_\alpha\cdot\langle\pi_\xi\partial_{s_\alpha}\exp_{h_2} rv_2,\\
&\pi_\xi\partial_{s_\alpha}\exp_{h_2} rv_2-\pi_\xi\bar\partial_J\exp_{h_2} rv_2\cdot\partial_{s_\alpha}\rangle_\xi\; ds_\alpha\wedge dt_\alpha.
\end{aligned}$$

For $r=0$ observe that we have $\pi_\xi\bar\partial_J\exp_{h_2} rv_2=\pi_\xi\bar\partial_J h_2=0$ and $\pi_\xi\partial_{s_\alpha}\exp_{h_2} rv_2=\pi_\xi\partial_{s_\alpha}h_2=0$, where the latter uses that $h=(h_1,h_2)$ is a branched cover of a trivial cylinder, i.e., h_2 is contained in a trajectory of the Reeb vector field. Letting $\Phi_{h_2}rv_2$ denote parallel transport on ξ starting from h_2 in the direction v_2 with respect to the complex connection ∇ from before, where we additionally assume that it preserves ω_ξ, i.e., the metric $\langle\cdot,\cdot\rangle_\xi$, the Leibniz rule implies

$$\begin{aligned}
&\frac{d^2}{dr^2}|_{r=0}\big\langle \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2, \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2 - \pi_\xi \bar{\partial}_J \exp_{h_2} rv_2 \cdot \partial_{s_\alpha}\big\rangle_\xi = \\
&\frac{d^2}{dr^2}|_{r=0}\big\langle (\Phi_{h_2} rv_2)^{-1} \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2, \\
&(\Phi_{h_2} rv_2)^{-1} \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2 - (\Phi_{h_2} rv_2)^{-1} \pi_\xi \bar{\partial}_J \exp_{h_2} rv_2 \cdot \partial_{s_\alpha}\big\rangle_\xi = \\
&\Big\langle \frac{d}{dr}|_{r=0} (\Phi_{h_2} rv_2)^{-1} \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2, \\
&\frac{d}{dr}|_{r=0} (\Phi_{h_2} rv_2)^{-1} \pi_\xi \partial_{s_\alpha} \exp_{h_2} rv_2 \\
&-\frac{d}{dr}|_{r=0} (\Phi_{h_2} rv_2)^{-1} \pi_\xi \bar{\partial}_J \exp_{h_2} rv_2 \cdot \partial_{s_\alpha}\Big\rangle_\xi = \\
&\langle \nabla_{s_\alpha} v_2, \nabla_{s_\alpha} v_2 - D_h^\xi v_2 \cdot \partial_{s_\alpha}\rangle_\xi . = |\nabla_{s_\alpha} v_2|_\xi^2
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= \frac{d^2}{dr^2} E_\omega(\exp_h rv_2) \\
&= \sum_\alpha \int_{U_\alpha} \psi_\alpha \cdot |\nabla_{s_\alpha} v_2|_\xi^2 \, ds_\alpha \wedge dt_\alpha,
\end{aligned}$$

so that $\nabla_{s_\alpha} v_2 = 0$. Since by the same arguments $\nabla_{t_\alpha} v_2 = 0$ we indeed have $\nabla v_2 = 0$ on $\dot{S}$, which by $v_2 \in H^{1,p}(h^*\xi)$ implies $v_2 = 0$ as desired. □

Since the kernel of the linearized operator agrees with the tangent space to the moduli space of trivial curves, the dimension of the kernel of the linearization of $\bar{\partial}_J$ is constant on the moduli space. Together with the constancy of the Fredholm index it proves that the cokernel bundle is of constant rank over $\mathcal{M}/(\mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-}) = S^1 \times \mathcal{M}_{0,n}$. By the same arguments it follows that the cokernel bundle over the moduli space $\mathcal{M}_{T,\mathcal{L}}$ is of constant rank for any tree with level structure $(T, \mathcal{L})$. As in [MDSa] this rank constancy proves that $\mathrm{Coker}^{T,\mathcal{L}}\,\bar{\partial}_J$ is indeed a smooth vector bundle over the smooth manifold $\mathcal{M}_{T,\mathcal{L}}$:

Corollary 2.2.4: $\mathrm{Coker}^{T,\mathcal{L}}\,\bar{\partial}_J$ *naturally carries the structure of a smooth vector bundle over* $\mathcal{M}_{T,\mathcal{L}}$.

2.2.3 Linear gluing

This subsection is concerned with the following extension of the above result:

Proposition 2.2.5: *Using a linear gluing construction (relating the cokernel bundle over the moduli space with the cokernel bundles over the boundary strata) we can equip* *the cokernel bundle* $\overline{\mathrm{Coker}\bar{\partial}_J}$ *over the compactified moduli space* $\overline{\mathcal{M}}$ *with the structure of a smooth vector bundle over a smooth manifold with corners.*

Recall that we have shown in proposition 2.1.4 that the compactified moduli space $\overline{\mathcal{M}}$ carries the structure of a smooth manifold with corners. For the proof it suffices to establish linear gluing theorems for the cokernel bundle under gluing of the underlying moduli spaces of branched covers. For the gluing theorems we must distinguish the case of gluing of curves on different levels, i.e., gluing at punctures, and gluing of curves in the same level, which corresponds to gluing at a node.

Gluing of moduli spaces:

In order to describe gluing of the cokernel bundles, we must start with gluing of the underlying moduli spaces of branched covers. Although these moduli spaces are nonregular and we hence cannot apply the usual gluing theorems, the gluing can explicitly described as follows:

Starting with the case of gluing at a puncture and using the notation introduced in 2.1.2, let $(T,\mathcal{L}) = (T,E,\Lambda^{\pm},\mathcal{L})$ denote the tree with level structure given by $T = \{1,2\}$, $1E2$ and $\mathcal{L}(1) = 1$, $\mathcal{L}(2) = 2$. Note that the moduli space $\mathcal{M}_{T,\mathcal{L}}$ is given by the fibre product $\mathcal{M}_1 \times_{\mathbb{Z}_{m_{12}}} \mathcal{M}_2$ where $\mathcal{M}_1$, $\mathcal{M}_2$ denote moduli spaces of connected branched covers without nodes. Let $(h,j,\theta,\mu^{\pm}) \in \mathcal{M}_{T,\mathcal{L}}$ with $h = (h_1,h_2)$, $j = (j_1,j_2)$, $\theta = \theta_{12} \in \mathbb{Z}_{m_{1,2}}$ and $\mu^{\pm} = (\mu_1^{\pm},\mu_2^{\pm})$ with $\mu_1^{\pm} = (\mu_k^{\pm})_{k\in\Lambda_1^{\pm}}$, $\mu_2^{\pm} = (\mu_k^{\pm})_{k\in\Lambda_2^{\pm}}$. Then the underlying punctured spheres are $\dot{S}_1 = S^2 - (Z_1^+ \cup Z_1^-)$, $\dot{S}_2 = S^2 - (Z_2^+ \cup Z_2^-)$, where the connecting pair of punctures is (z_{12},z_{21}) with $z_{12} \in Z_1^-$ and $z_{21} \in Z_2^+$. We define the family of glued curves

$$(h^r,j^r,\mu^{\pm}) \;=\; \sharp_r(h,j,\theta,\mu^{\pm}) \;=\; (h_1,j_1,\mu_1)\sharp_{r,\theta}(h_2,j_2,\mu_2)$$

as follows, where $r = r_{12} \in \mathbb{R}^+$ denotes the gluing parameter:

When $\psi_{12} : \mathbb{R}^- \times S^1 \to \dot{S}_1$, $\psi_{21} : \mathbb{R}^+ \times S^1 \to \dot{S}_2$ denote the fixed cylindrical coordinates around $z_{12} \in \dot{S}_1$, $z_{21} \in \dot{S}_2$, let $\dot{S}_1^r$, $\dot{S}_2^r$ denote the punctured surfaces with boundary given by cutting out the half-cylinders $(-\infty,-r) \times S^1$, $(+r,+\infty) \times S^1$, respectively,

$$\dot{S}_1^r = \dot{S}_1 - \psi_{12}((-\infty,-r)\times S^1),\;\; \dot{S}_2^r = \dot{S}_2 - \psi_{21}((+r,+\infty)\times S^1).$$

We introduce the punctured surface $\dot{S}^r$ underlying $(h^r,j^r,\mu^{\pm})$ by gluing $\dot{S}_1^r$ and $\dot{S}_2^r$ along the boundary with the twist given by the maps h_1 and h_2 and the decoration θ_{12},

$$\dot{S}^r \;=\; \dot{S}_1^r\sharp_{\theta_{12}}\dot{S}_2^r \;=\; \dot{S}_1^r \coprod \dot{S}_2^r/\{\psi_{12}(-r,t) \sim \psi_{21}(+r,t+\theta_{12})\}.$$

Note that here the decoration θ_{12} is viewed as an element in S^1 rather than in $\mathbb{Z}_{m_{12}}$. For this recall that the maps h_1, h_2 determine $m_{1,2}$ different asymptotic markers at $z_{12} \in \dot{S}_1$ and $z_{21} \in \dot{S}_2$, which determine S^1-coordinates in the cylindrical coordinates ψ_{12} and ψ_{21}. Hence there are m_{12} possible ways to glue $\dot{S}_1^r$ and $\dot{S}_2^r$ so that these S^1-coordinates match, and the element in $\mathbb{Z}_{m_{12}}$ singles out the unique gluing twist. Note that $\dot{S}^r$ is again diffeomorphic to a punctured sphere and the complex structures j_1 on $\dot{S}_1$ and j_2 on $\dot{S}_2$ determine a complex structure j^r on $\dot{S}^r$ since both agree with the standard complex structure on the embedded half-cylinders determined by ψ_{12} and ψ_{21}. On the other hand, the branched covering map $h^r : (\dot{S}^r, j^r) \to \mathbb{R} \times S^1$ is unique up to $\mathbb{R}$-shift by the requirement that the asymptotic markers of h^r match with those of the maps h_1 on $\dot{S}_1^r$ and h_2 on $\dot{S}_2^r$ and exists by the choice of the gluing twist $\theta_{12} \in S^1$, since it is chosen so that the S^1-shifts for h_1 and h_2 agree. Hence we found a natural gluing map for gluing at punctures

$$\sharp : (\mathcal{M}_1 \times_{\mathbb{Z}_{m_{12}}} \mathcal{M}_2) \times (0,+\infty) \hookrightarrow \mathcal{M}, \;\; ((h,j,\theta,\mu^{\pm}),r) \mapsto (h^r, j^r, \mu^{\pm}).$$

On the other hand, for the case of gluing at a node we want a gluing map from $\mathcal{M}_T$ to the moduli space $\mathcal{M}$ for a tree $T = \{1,2\}$, $1E2$ with trivial level structure $\mathcal{L}(1) = \mathcal{L}(2) = 1$, i.e.,

$$\mathcal{M}_T = \{(h_1, j_1, \mu_1) \in \mathcal{M}_1, (h_2, j_2, \mu_2) \in \mathcal{M}_2 : \; h_1(z_{12}) = h_2(z_{21})\}.$$

Here everything follows the expositions from above, except that now the maps h_1 and h_2 in $h = (h_1, h_2)$, $j = (j_1, j_2)$, $(h, j, \mu^{\pm}) \in \mathcal{M}_T$ satisfy $h_1(z_{12}) = h_2(z_{21})$ and cannot be used to fix the gluing twist $\theta_{12} \in S^1$. Hence we now have two gluing parameters $r = r_{12}$ and $\theta = \theta_{12}$ and the gluing procedure is given the map

$$\sharp : \mathcal{M}_T \times (0,+\infty) \times S^1 \to \mathcal{M}, \;\; ((h,j,\mu^{\pm}),r,\theta) \mapsto (h^{r,\theta}, j^{r,\theta}, \mu^{\pm}).$$

Linear gluing of the cokernel bundle:

We now start with the gluing of the cokernel bundles. It follows from proposition 2.2.2 in the last subsection that the fibres of the cokernel bundle over $(h, j) \in \mathcal{M}$ are given by

$$(\operatorname{Coker} \bar{\partial}_J)_{(h,j)} \; = \; \operatorname{coker} D_{h,j} \; = \; \operatorname{coker} D_h^{\xi} \; = \; \ker (D_h^{\xi})^*,$$

where

$$(D_h^{\xi})^* : H^{1,q}(T^*\dot{S} \otimes_{j,J_\xi} h^*\xi) \to L^q(h^*\xi), \;\; 1/p + 1/q = 1$$

denotes the formal adjoint of the linearization $D_h^{\xi} : H^{1,p}(h^*\xi) \to L^p(T^*\dot{S} \otimes_{j,J_\xi} h^*\xi)$ of $\bar{\partial}_J$ in the direction of the hyperplane distribution $\xi \subset TV$. Since by elliptic regularity all occuring kernels and hence cokernels are independent of the choice of $p \geq 2$, see [Sch], we set in the following $p = q = 2$. Note that since $\ker D_h^{\xi} = \{0\}$ by proposition 2.2.3, the operators $(D_h^{\xi})^*$ are surjective.

In the case of gluing at punctures we want to define a gluing map

$$\sharp : \operatorname{Coker}^{T,\mathcal{L}} \bar{\partial}_J \times (0,+\infty) \to \operatorname{Coker} \bar{\partial}_J$$

where $T = 1, 2,\ 1E2$, $\mathcal{L}(1) = 1$, $\mathcal{L}(2) = 2$, while for gluing at nodes we are looking for a map

$$\sharp : \operatorname{Coker}^T \bar{\partial}_J \times (0, +\infty) \times S^1 \to \operatorname{Coker} \bar{\partial}_J$$

where $T = (T, E)$ is given as before but with the trivial level structure $\mathcal{L}(1) = 1 = \mathcal{L}(2)$. Both gluing maps are constructed in such a way that they are bundle maps over the corresponding gluing maps for the underlying moduli spaces of branched covers. Following the expositions in [Sch] about linear gluing we start with the definition of pregluing operations:

Linear pregluing at punctures. Starting again with the case of gluing at punctures, recall that the cokernel bundle over $\mathcal{M}_{T,\mathcal{L}} = \mathcal{M}_1 \times_{\mathbb{Z}_{m_{12}}} \mathcal{M}_2$ is given as direct sum,

$$\operatorname{Coker}^{T,\mathcal{L}} \bar{\partial}_J \;=\; \pi_1^* \operatorname{Coker}^1 \bar{\partial}_J \oplus \pi_2^* \operatorname{Coker}^2 \bar{\partial}_J,$$

where $\operatorname{Coker}^1, \operatorname{Coker}^2 \bar{\partial}_J$ denote the cokernel bundles over $\mathcal{M}_1, \mathcal{M}_2$ and π_1, π_2 the projections from $\mathcal{M}^{T,\mathcal{L}} / (\mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-})$ to $\mathcal{M}_1 / (\mathbb{Z}_{m_1^+} \times \mathbb{Z}_{m_1^-})$, $\mathcal{M}_2 / (\mathbb{Z}_{m_2^+} \times \mathbb{Z}_{m_2^-})$, respectively. Let $(h, j, \theta, \mu^{\pm}) \in \mathcal{M}_{T,\mathcal{L}} = \mathcal{M}_1 \times_{\mathbb{Z}_{m_{12}}} \mathcal{M}_2$ with $h = (h_1, h_2)$, $j = (j_1, j_2)$. For

$$\eta = (\eta_1, \eta_2) \in (\operatorname{Coker}^{T,\mathcal{L}} \bar{\partial}_J)_{(h,j)} = (\operatorname{Coker}^1 \bar{\partial}_J)_{(h_1,j_1)} \oplus (\operatorname{Coker}^2 \bar{\partial}_J)_{(h_2,j_2)}$$

with

$$\eta_1 \in (\operatorname{Coker}^1 \bar{\partial}_J)_{(h_1,j_1)} = \ker(D^{\xi}_{h_1})^* \subset H^{1,2}(T^*\dot{S}_1 \otimes_{j_1,J_\xi} h_1^*\xi),$$
$$\eta_2 \in (\operatorname{Coker}^2 \bar{\partial}_J)_{(h_2,j_2)} = \ker(D^{\xi}_{h_2})^* \subset H^{1,2}(T^*\dot{S}_2 \otimes_{j_2,J_\xi} h_2^*\xi)$$

we define a preglued section

$$\eta^r_0 = \sharp^0_r \eta \;=\; \eta_1 \sharp^0_r \eta_2 \in H^{1,2}(T^*\dot{S}^r \otimes_{j^r,J_\xi} (h^r)^*\xi)$$

in the bundle of j^r, J_ξ-antiholomorphic one-forms over the glued surface $(\dot{S}^r, j^r)$ with values in the pull-back bundle $(h^r)^*\xi$. Note that the integration measure for defining the $H^{1,2}$-norm agrees on the connecting cylindrical neck $\psi_{21}((0, +r] \times S^1) \sharp_{\theta_{12}} \psi_{12}([-r, 0) \times S^1)$ with the standard measure $ds \wedge dt$ on the cylinder.

For $r > 0$ let $\beta^r : [0, +r] \to [0, 1]$ be a smooth cut-off function such that $\beta^r(s) = 1$ for $0 \leq s \leq r/4$ and $\beta^r(s) = 0$ for $3r/4 \leq s \leq r$ with $|\partial_s \beta^r| \leq 4/r$. Let

$$\beta^r_1, \beta^r_2 : \dot{S}^r \to [0, 1]$$

be the two cut-off functions which are constant equal to zero on $\dot{S}^r_2$, $\dot{S}^r_1$, constant equal to one on $\dot{S}^r_1 - \psi_{12}([-r, 0] \times S^1)$, $\dot{S}^r_2 - \psi_{21}([0, +r] \times S^1)$ and are on $\psi_{12}([-r, 0) \times S^1) \subset \dot{S}^r_1$, $\psi_{21}((0, +r] \times S^1) \subset \dot{S}^r_2$ given by

$$\beta^r_1(\psi_{12}(s, t)) = \beta^r(-s), \;\; \beta^r_2(\psi_{21}(s, t)) = \beta^r(+s),$$

respectively. With this we define the preglued section $\eta_1\sharp^0_r\eta_2$ on $\dot{S}^r=\dot{S}^r_1\sharp\dot{S}^r_2$ by

$$\eta^r_0=\eta_1\sharp^0_r\eta_2=\beta^r_1\eta_1+\beta^r_2\eta_2.$$

It follows that η^r_0 agrees with η_1, η_2 over $\dot{S}^r_1-\psi_{12}([-r,0]\times S^1)$, $\dot{S}^r_2-\psi_{21}([0,+r]\times S^1)$, respectively, while over the connecting neck we have

$$(\eta^r_0\circ\psi_{12})(s,t)=\beta^r(-s)\cdot(\eta_1\circ\psi_{12})(s,t),$$
$$(\eta^r_0\circ\psi_{21})(s,t)=\beta^r(+s)\cdot(\eta_2\circ\psi_{21})(s,t).$$

Observe that by $\beta^r(s)=0$ for $3r/4\le s\le r$ this indeed yields a well-defined section in $H^{1,2}(T^*\dot{S}^r\otimes_{j^r,J_\xi}(h^r)^*\xi)$.

Linear pregluing at nodes. In the case of gluing at a node, recall that the cokernel bundle $\operatorname{Coker}^T\bar\partial_J$ over $\mathcal{M}_T=\{(h_1,j_1,\mu_1)\in\mathcal{M}_1,(h_2,j_2,\mu_2)\in\mathcal{M}_2: h_1(z_{12})=h_2(z_{21})\}$ has fibre

$$\begin{aligned}(\operatorname{Coker}^T\bar\partial_J)_{(h,j)} \;&=\; \{(\eta_1,\eta_2)\in(\operatorname{Coker}^1\bar\partial_J)_{(h_1,j_1)}\oplus(\operatorname{Coker}^2\bar\partial_J)_{(h_2,j_2)}:\\ &\qquad \eta_1(z_{12})=\eta_2(z_{21})\},\end{aligned}$$

where again

$$\eta_1\in(\operatorname{Coker}^1\bar\partial_J)_{(h_1,j_1)}=\ker(D^\xi_{h_1})^*\subset H^{1,2}(T^*\dot{S}_1\otimes_{j_1,J}h_1^*\xi),$$
$$\eta_2\in(\operatorname{Coker}^2\bar\partial_J)_{(h_2,j_2)}=\ker(D^\xi_{h_2})^*\subset H^{1,2}(T^*\dot{S}_2\otimes_{j_2,J}h_2^*\xi).$$

Note that since z_{12} and z_{21} are now points on the punctured surfaces $\dot{S}_1$, $\dot{S}_2$, the measure on $\dot{S}_1$, $\dot{S}_2$ underlying the $H^{1,2}$-norm now does not agree with the cylindrical measure on $\psi_{12}([-r,0]\times S^1)$, $\psi_{21}([0,+r]\times S^1)$ but with the standard measure as a subset of $S^2\cong\dot{S}_1,\dot{S}_2$.

For $\eta=(\eta_1,\eta_2)\in(\operatorname{Coker}^T\bar\partial_J)_{(h,j)}$ we define the preglued section

$$\eta^{r,\theta}_0=\sharp^0_{r,\theta}\eta=\eta_1\sharp^0_{r,\theta}\eta_2\in H^{1,2}(T^*\dot{S}^{r,\theta}_0\otimes_{j^{r,\theta},J_\xi}(h^{r,\theta})^*\xi),$$

where the subscript at the glued punctured surface $\dot{S}^{r,\theta}_0$ should indicate that for gluing at nodes we do not use the standard cylindrical measure on the connecting cylindrical neck $\psi_{21}((0,+r]\times S^1)\sharp_{\theta_{12}}\psi_{12}([-r,0)\times S^1)$, but again take the measure as subset of the standard sphere $S^2\cong\dot{S}^{r,\theta}$:

As above, we require $\eta^{r,\theta}_0$ to agree with η_1, η_2 over $\dot{S}^r_1-\psi_{12}([-r,0]\times S^1)$, $\dot{S}^r_2-\psi_{21}([0,+r]\times S^1)$, respectively, while over the connecting neck we use the cutoff function β^r to set

$$(\eta^{r,\theta}_0\circ\psi_{12})(s,t)=\beta^r(-s)\cdot(\eta_1\circ\psi_{12})(s,t)+(1-\beta^r(-s))\cdot\eta_1(z_{12}),$$
$$(\eta^{r,\theta}_0\circ\psi_{21})(s,t)=\beta^r(+s)\cdot(\eta_2\circ\psi_{21})(s,t)+(1-\beta^r(+s))\cdot\eta_2(z_{21}).$$

Observe that this gives a well-defined section $H^{1,2}(T^*\dot{S}_0^{r,\theta}\otimes_{j^{r,\theta},J_\xi}(h^{r,\theta})^*\xi)$ since $\beta^r(+r)=0$ and $\eta_1(z_{12})=\eta_2(z_{21})$.

The gluing lemma. For $r\in(0,+\infty)$, $\theta\in S^1$ let

$$\begin{aligned}
\sharp_r^0(\mathrm{Coker}^{T,\mathcal{L}}\bar\partial_J)_{(h,j)} &= \ker(D^\xi_{h_1})^*\sharp^0_{r,\theta}\ker(D^\xi_{h_2})^*\\
&= \{\sharp^0_r(\eta_1,\eta_2):\eta_i\in\ker(D^\xi_{h_i})^*, i=1,2\}\\
&\subset H^{1,2}(T^*\dot{S}^r\otimes_{j^r,J_\xi}(h^r)^*\xi),\\
\sharp^0_{r,\theta}(\mathrm{Coker}^T\bar\partial_J)_{(h,j)} &= \{\sharp^0_{r,\theta}(\eta_1,\eta_2):\eta_i\in\ker(D^\xi_{h_i})^*, i=1,2,\eta_1(z_{12})=\eta_2(z_{21})\}\\
&\subset H^{1,2}(T^*\dot{S}_0^{r,\theta}\otimes_{j^{r,\theta},J_\xi}(h^{r,\theta})^*\xi)
\end{aligned}$$

denote the subspaces of preglued sections. With the orthogonal projections

$$\begin{aligned}
\pi_r: H^{1,2}(T^*\dot{S}^r\otimes_{j^r,J_\xi}(h^r)^*\xi) &\to \mathrm{coker}\,D_{h^r,j^r}=\ker(D^\xi_{h^r})^*\\
\pi_{r,\theta}:, H^{1,2}(T^*\dot{S}_0^{r,\theta}\otimes_{j^{r,\theta},J_\xi}(h^{r,\theta})^*\xi) &\to \mathrm{coker}\,D_{h^{r,\theta},j^{r,\theta}}=\ker(D^\xi_{h^{r,\theta}})^*
\end{aligned}$$

we can state and prove the gluing lemma:

Lemma 2.2.6: *The projections from the spaces of preglued sections on the fibres of the cokernel bundles over the underlying glued branched covers,*

$$\begin{aligned}
\pi_r:&\quad \sharp^0_r(\mathrm{Coker}^{T,\mathcal{L}}\bar\partial_J)_{(h,j)}\to(\mathrm{Coker}\,\bar\partial_J)_{(h^r,j^r)},\ \ (h^r,j^r)=\sharp_r(h,j,\theta)\\
\pi_{r,\theta}:&\quad \sharp^0_{r,\theta}(\mathrm{Coker}^T\bar\partial_J)_{(h,j)}\to(\mathrm{Coker}\,\bar\partial_J)_{(h^{r,\theta},j^{r,\theta})},\ \ (h^{r,\theta},j^{r,\theta})=\sharp_{r,\theta}(h,j)
\end{aligned}$$

are isomorphisms for all $r>0$ sufficiently large, and additionally for all gluing twists $\theta\in S^1$ in the case of gluing at nodes.

Proof: For the proof we follow the proof of proposition 3.2.9 in [Sch]. However we emphasize that we cannot directly apply the linear gluing lemma in [Sch], since the linear operator $D^\xi_{h^r}$ over the glued surface does not agree with the glued operator $D^\xi_{h_1,j_1}\sharp_{r,\theta}D^\xi_{h_2,j_2}$ studied in [Sch]. We outline the proof for the case of gluing at punctures, and claim that the arguments for gluing at nodes are similar:

Observe that it suffices to find for every $r>0$ sufficiently large a constant $c>0$ such that $\|(D^\xi_{h^r,j^r})^*\eta\|_2\geq c\|\eta\|_{1,2}$ for all $\eta\in(\sharp^0_r\,\mathrm{Coker}^{T,\mathcal{L}}\bar\partial_J)^\perp_{(h,j)}=(\ker(D^\xi_{h_1})^*\sharp^0_{r,\theta}\ker(D^\xi_{h_2})^*)^\perp$. Indeed, it then follows that

$$\ker(D^\xi_{h^r})^*\cap(\ker(D^\xi_{h_1})^*\sharp^0_{r,\theta}\ker(D^\xi_{h_2})^*)^\perp=\{0\},$$

which proves that the orthogonal projection is surjective. On the other hand, since $\dim\ker D^\xi_{h^r,j^r}=\dim\ker D^\xi_{h_1,j_1}=\dim\ker D^\xi_{h_2,j_2}=0$ by proposition 2.2.3 and the index of $D^\xi_{h^r,j^r}$ equals the sum of the indices of $D^\xi_{h_1,j_1}$ and $D^\xi_{h_2,j_2}$, it follows that

$$\dim\ker(D^\xi_{h^r})^* = \dim\ker(D^\xi_{h_1})^*+\dim\ker(D^\xi_{h_2})^*.$$

Since the latter agrees with the dimension of the space $\ker(D^{\xi}_{h_1})^* \sharp^0_{r,\theta} \ker(D^{\xi}_{h_2})^*$ of preglued sections, the surjectivity of the orthogonal projection directly implies that it is an isomorphism.

Assume to the contrary that there exists a sequence

$$\eta_n \in (\ker(D^{\xi}_{h_1})^* \sharp^0_{r_n,\theta} \ker(D^{\xi}_{h_2})^*)^{\perp},\;\; r_n \to \infty$$

with $\|\eta_n\|_{1,2} = 1$ but $\|(D^{\xi}_{h^{r_n}})^*\eta_n\|_2 \to 0$ as $n \to \infty$. Now observe that

$$\begin{aligned}\|(D^{\xi}_{h^{r_n}})^*(\beta_1^{r_n}\eta_n)\|_2 &\leq \|(D^{\xi}_{h^{r_n}})^*\eta_n\|_2 + c_1\|d\beta_1^{r_n}\cdot\eta_n\|_2\\ &\leq \|(D^{\xi}_{h^{r_n}})^*\eta_n\|_2 + c_1\|d\beta_1^{r_n}\|_\infty\cdot\|\eta_n\|_2\end{aligned}$$

for some $c_1 > 0$ with $\|d\beta_1^{r_n}\|_\infty \leq 4/r_n$ and $\|\eta_n\|_2 \leq \|\eta_n\|_{1,2} = 1$, so that $\|(D^{\xi}_{h^{r_n}})^*(\beta_1^{r_n}\eta_n)\|_2 \to 0$ for $n \to \infty$. But since $(h^{r_n}, j^{r_n}) \to (h_1, j_1)$ on $\dot{S}_1^{r_n} = \dot{S}_1 - \psi_{12}((-\infty,-r_n)\times S^1)$, this directly implies that

$$\|(D^{\xi}_{h_1})^*(\beta_1^{r_n}\eta_n)\|_2 \to 0$$

in the $L^2(\dot{S}_1)$-sense and we can use the semi-Fredholm property of $(D^{\xi}_{h_1})^*$ and the boundedness of (η_n) to deduce that, possibly after passing to a suitable subsequence,

$$\beta_1^{r_n}\eta_n \stackrel{H^{1,2}}{\to} \eta_1,\;\; \eta_1 \in \ker(D^{\xi}_{h_1})^*.$$

Using the same arguments we deduce $\beta_2^{r_n}\eta_n \to \eta_2 \in \ker(D^{\xi}_{h_2})^*$. We use this to prove the desired contradiction by computing

$$\begin{aligned}1 = \lim_{n\to\infty}\|\eta_n\|_{1,2} &= \lim_{n\to\infty}\langle(\beta_1^{r_n})^2\eta_n + (\beta_2^{r_n})^2\eta_n, \eta_n\rangle_{1,2}\\ &+ \lim_{n\to\infty}\langle(1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2)\cdot\eta_n,\eta_n\rangle_{1,2}\\ &= \lim_{n\to\infty}\langle\beta_1^{r_n}\eta_1 + \beta_2^{r_n}\eta_2, \eta_n\rangle_{1,2}\\ &= \lim_{n\to\infty}\langle\eta_1\sharp^0_{r_n,\theta}\eta_2, \eta_n\rangle_{1,2} = 0,\end{aligned}$$

since $\eta_n \in (\ker(D^{\xi}_{h_1})^* \sharp^0_{r_n,\theta} \ker(D^{\xi}_{h_2})^*)^{\perp}$, where it only remains to prove that

$$\lim_{n\to\infty}\langle(1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2)\cdot\eta_n,\eta_n\rangle_{1,2} = 0.$$

For this we use that $1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2$ has only support in the middle part

$$\psi_{21}([+r_n/4,+r_n]\times S^1)\sharp_{\theta_{12}}\psi_{12}([-r_n,-r_n/4]\times S^1) \cong [-3r_n/4,+3r_n/4]\times S^1$$

of the cylindrical neck to prove that the $H^{1,2}$-norm of $(1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2)\eta_n$ tends to zero as $n \to \infty$:

Choosing a unitary trivialization of the symplectic hyperplane bundle ξ over the simple orbit γ, the restriction of the the differential operator $(D^{\xi}_{h^r})^*$ to $[-3r_n/4,+3r_n/4]\times S^1\subset \dot{S}^r$ is of the form

$$\begin{aligned} D_n=\partial_s+J_0\partial_t+S_n: \quad & H^{1,2}([-3r_n/4,+3r_n/4]\times S^1,\mathbb{R}^{2m-2}) \\ \to \quad & L^2([-3r_n/4,+3r_n/4]\times S^1,\mathbb{R}^{2m-2}) \end{aligned}$$

with $S_n(s,t)\in\mathbb{R}^{(2m-2)\times(2m-2)}$, which we extend to an operator on the full cylinder $\mathbb{R}\times S^1$ by setting $S_n(+s,t)=S_n(+3r_n/4,t)$, $S_n(-s,t)=S_n(-3r_n/4,t)$ for $s>3r_n/4$. In order to study the operator D_n let $h_n=h^{r_n}|_{[-3r_n/4,+3r_n/4]\times S^1}:[-3r_n/4,+3r_n/4]\times S^1\to\mathbb{R}\times S^1$ and $x_n=h_n(0,\cdot):S^1\to\mathbb{R}\times S^1$. Since for $n\to\infty$ the length of the cylindrical neck goes to infinity, it follows that h_n converges on each compact subinterval uniformly with all derivatives to the $\mathbb{R}$-independent function $x_\infty=\lim_{n\to\infty}x_n:S^1\to\mathbb{R}\times S^1$ of the form $x_\infty(t)=(s_0,m_{12}t+t_0)$. From this it follows that $S_n(s,t)\to S_\infty(t)$, i.e., D_n is converging in the operator norm to a translation invariant operator D_∞.

Finishing the proof observe that from

$$\begin{aligned} &\|D_n(\eta_n-(\beta_1^{r_n})^2\eta_n-(\beta_2^{r_n})^2\eta_n)\|_2 \\ &\le \|D_n\eta_n\|_2+c_2(\|d\beta_1^{r_n}\|_\infty\|\beta_1^{r_n}\eta_n\|_2+\|d\beta_2^{r_n}\|_\infty\|\beta_2^{r_n}\eta_n\|_2) \\ &\le \|D_n\eta_n\|_2+c_2(\|d\beta_1^{r_n}\|_\infty+\|d\beta_2^{r_n}\|_\infty)\|\eta_n\|_2 \end{aligned}$$

and $\|d\beta_1^{r_n}\|_\infty,\|d\beta_2^{r_n}\|_\infty\to 0$, $\|\eta_n\|_2=1$, $\|D_n\eta_n\|_2\to 0$ it follows that

$$\|D_\infty(\eta_n-(\beta_1^{r_n})^2\eta_n-(\beta_2^{r_n})^2\eta_n)\|_2\to 0,\quad n\to\infty.$$

But now we can use the fact that the operator $D_\infty: H^{1,2}(\mathbb{R}\times S^1,\mathbb{R}^{2m-2})\to L^2(\mathbb{R}\times S^1,\mathbb{R}^{2m-2})$ is an isomorphism ([Sch]) and hence

$$\|(1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2)\eta_n\|_{1,2}\le c_3\cdot\|D_\infty(\eta_n-(\beta_1^{r_n})^2\eta_n-(\beta_2^{r_n})^2\eta_n)\|_2$$

for some $c_3>0$ to deduce that $\|(1-(\beta_1^{r_n})^2-(\beta_2^{r_n})^2)\eta_n\|_{1,2}\to 0$ as n goes to infinity. □

2.2.4 Orientations

In this section we show how the techniques by [BM] and [HWZ] for defining coherent orientations of the moduli spaces in symplectic field theory define an orientation of the cokernel bundle $\operatorname{Coker}\bar{\partial}_J$ over the non-compactified moduli space $\mathcal{M}$ and discuss the extension over the boundary strata. Although we have seen in the last section that the cokernel bundle $\operatorname{Coker}\bar{\partial}_J$ naturally lives over the quotient $\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, obtained by forgetting the asymptotic markers $\mu^\pm\in\mathbb{Z}_{m^\pm}$, we show in this section that in general we can orient $\operatorname{Coker}\bar{\partial}_J$ only over the full moduli space $\mathcal{M}$. For this we start with recalling the main points of the constructions of coherent orientations in [BM]:

Let $\dot{S} = S^2 - \{z^{\pm}_{1,0}, ..., z^{\pm}_{n^{\pm},0}\}$ denote as in 2.2.2 the punctured sphere underlying the moduli space $\mathcal{M}$. For regular paths of symplectic matrices

$$A^{\pm}_1, ..., A^{\pm}_{n^{\pm}} : [0,1] \to \mathrm{Sp}(2m-2), \;\; \det(A^{\pm}_k(1) - A^{\pm}_k(0)) \neq 0$$

with $A^{\pm}_k(0) = \mathbf{1}$, $\dot{A}^{\pm}_k(0)A^{\pm}_k(0)^{-1} = \dot{A}^{\pm}_k(1)A^{\pm}_k(1)^{-1}$ and where $2m-2$ is the rank of ξ, let $\mathcal{O}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ denote the set of Cauchy-Riemann operators

$$\begin{aligned} D: \quad & H^{1,p,d}_{\text{const}}(\dot{S}, \mathbb{C}) \oplus H^{1,p}(\dot{S}, \mathbb{R}^{2m-2}) \\ & \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,J_0} \mathbb{R}^{2m-2}) \\ & D \cdot v = dv + J_0 \cdot dv \cdot j + S \cdot v \end{aligned}$$

where $S : \dot{S} \to \mathbb{R}^{2m \times 2m}$ is a family of symmetric matrices such that the limit matrices are of the form

$$(S \circ \psi^{\pm}_k)(s, t + \mu^{\pm}_k) \stackrel{s \to \pm\infty}{\longrightarrow} (S \circ \psi^{\pm}_k)(\pm\infty, t + \mu^{\pm}_k) = \begin{pmatrix} 0 & 0 \\ 0 & S^{\pm}_k(t) \end{pmatrix},$$

and where $S^{\pm}_1, ..., S^{\pm}_{n^{\pm}} : S^1 \to \mathbb{R}^{(2m-2)\times(2m-2)}$ are related to $A^{\pm}_1, ..., A^{\pm}_{n^{\pm}} : [0,1] \to \mathrm{Sp}(2m-2)$ via

$$S^{\pm}_k(t) = -J_0 \cdot \dot{A}^{\pm}_k(t) \cdot A^{\pm}_k(t)^{-1}$$

for all $k = 1, ..., n^{\pm}$.

Since every operator $D \in \mathcal{O}(\dot{S}, (A^{\pm}_k)^{n^{\pm}}_{k=1}) = \bigcup_{j,\mu} \mathcal{O}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ is a Fredholm operator, we have the determinant line bundle $\mathrm{Det}(\dot{S}, (A^{\pm}_k)^{n^{\pm}}_{k=1}))$ over $\mathcal{O}$ with fibre

$$\mathrm{Det}(\dot{S}, (A^{\pm}_k)^{n^{\pm}}_{k=1}))_D = \mathrm{Det}(D) = \Lambda^{\max} \ker D \otimes \Lambda^{\max} \operatorname{coker} D.$$

Since the space of Fredholm operators $\mathcal{O}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ is contractible, it follows that the restriction $\mathrm{Det}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ of $\mathrm{Det}(\dot{S}, (A^{\pm}_k)^{n^{\pm}}_{k=1}))$ to $\mathcal{O}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ is trivial. On the other hand, it is shown in proposition 11 in [BM] that the determinant line bundle remains trivial when we allow the complex structure j on the punctured sphere $\dot{S}$ to vary.

In [BM] the authors describe a method to orient how the resulting bundles $\mathrm{Det}((\dot{S}, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ over $\mathcal{O}(\dot{S}, \mu^{\pm}, (A^{\pm}_k)^{n^{\pm}}_{k=1}) = \bigcup_j \mathcal{O}((\dot{S}, j, \mu^{\pm}), (A^{\pm}_k)^{n^{\pm}}_{k=1})$ for any number of punctures, directions $\mu^{\pm}$ and regular paths $A^{\pm}_1, ..., A^{\pm}_{n^{\pm}}$ of symplectic matrices. The construction is based on arbitrarily fixing orientations for determinant bundles over the space $\mathcal{O}((\mathbb{C}^*, 0), A)$ of Cauchy-Riemann operators on the holomorphic plane, constructing a gluing map for determinant bundles under gluing of Riemann surfaces and finally observing that we have a natural orientation of $\mathrm{Det}(S^2)$ induced by the complex orientation of the determinant line over the standard Cauchy-Riemann operator on $(S^2, i) = \mathbb{CP}^1$. Note that at this point the specification of the asymptotic markers $\mu = (\mu^+, \mu^-)$, $\mu^{\pm} = (\mu^{\pm}_k)^{n^{\pm}}_{k=1}$

becomes important, as they describe how to glue the holomorphic planes to the punctured sphere $\dot{S}$ to obtain the closed sphere S^2. However it directly follows from the construction that the orientations on $\mathrm{Det}((\dot{S},\mu^{\pm}),(A_k^{\pm})_{k=1}^{n^{\pm}})$ for different asymptotic markers μ fit together to give an orientation of the whole determinant bundle $\mathrm{Det}(\dot{S},(A_k^{\pm})_{k=1}^{n^{\pm}}))$.

Observe that the linearization of $\bar{\partial}_J$ at some $(h,j,\mu^{\pm})\in\mathcal{M}$,

$$\begin{aligned} D_{h,j}:\ & H^{1,p,d}_{\mathrm{const}}(\dot{S},\mathbb{C})\oplus H^{1,p}(h^*\xi)\oplus T_j\,\mathcal{M}_{0,n} \\ & \to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J_\xi}h^*\xi), \end{aligned}$$

can be written as sum $D_{h,j}=D_h+D_j$ with

$$\begin{aligned} D_j:\ & T_j\,\mathcal{M}_{0,n}\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J_\xi}h^*\xi) \\ D_h:\ & H^{1,p,d}_{\mathrm{const}}(\dot{S},\mathbb{C})\oplus H^{1,p}(h^*\xi)\to L^{p,d}(T^*\dot{S}\otimes_{j,i}\mathbb{C})\oplus L^p(T^*\dot{S}\otimes_{j,J_\xi}h^*\xi) \end{aligned}$$

where D_h is a Cauchy-Riemann operator. Using a unitary trivialization of the hyperplane bundle ξ over the closed simple orbit γ, we get a unitary trivialization of $h^*\xi$ and a natural map

$$\mathrm{op}:\mathcal{M}\to\mathcal{O}(\dot{S},(A_k^{\pm})_{k=1}^{n^{\pm}}),\ \ (h,j,\mu^{\pm})\mapsto D_h,$$

where the regular paths of symplectic matrices $A_1^{\pm},...,A_{n^{\pm}}^{\pm}$ are determined by the restriction to ξ of the linearized Reeb flow along γ. Using the map op we can pull-back the determinant bundle $\mathrm{Det}=\mathrm{Det}(\dot{S},(A_k^{\pm})_{k=1}^{n^{\pm}})$ to obtain the line bundle $\mathrm{op}^*\,\mathrm{Det}$ over $\mathcal{M}$. On the other hand, following the arguments in [BM], we deduce from the fact that $D_{h,j}=D_j\oplus D_h$ is homotopic to the stabilization $0\oplus D_h$ with the complex vector space $T_j\,\mathcal{M}_{0,n}$ that the determinant spaces of the linearization $D_{h,j}$ and the Cauchy-Riemann operator D_h are canonically isomorphic, so that the pull-back of the determinant bundle over the space of Cauchy-Riemann operators is isomorphic to the determinant bundle of the fully linearized operator

$$\mathrm{op}^*\,\mathrm{Det}\cong\Lambda^{\max}\,\mathrm{Ker}\,\bar{\partial}_J\otimes\Lambda^{\max}\,\mathrm{Coker}\,\bar{\partial}_J$$

with fibre $\Lambda^{\max}\ker D_{h,j}\otimes\Lambda^{\max}\operatorname{coker}D_{h,j}$ over $(h,j,\mu^{\pm})\in\mathcal{M}$.

Since $\mathrm{Ker}\,\bar{\partial}_J$ and $\mathrm{Coker}\,\bar{\partial}_J$ are bundles over $\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, it follows that the action of $\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ lifts in an obvious way to an action on the vector bundle $\Lambda^{\max}\,\mathrm{Ker}\,\bar{\partial}_J\otimes\Lambda^{\max}\,\mathrm{Coker}\,\bar{\partial}_J$ which is trivial on the fibres. On the other hand, the fibres over $(h,j,\mu^{\pm}),(h,j,\mu'^{\pm})\in\mathcal{M}$ do not neccessarily carry the same orientation. Indeed it is shown in theorem 3 in [BM] that this action is orientation-preserving if γ is good, else, the action is orientation-preserving or -reversing if $\mu'-\mu\in\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ is even or odd, respectively. In this case the even iterates γ^{2k} of the simple orbit γ are called bad.

Proposition 2.2.7: *For every tree with level structure $(T,\mathcal{L})$ with trees $T_1,...,T_L$, the choice of coherent orientations in [BM] equip the cokernel bundles* $\mathrm{Coker}^{T_1}\,\bar{\partial}_J$*,*

..., $\mathrm{Coker}^{T_L}\,\bar{\partial}_J$ over $\mathcal{M}_{T_1}$, ..., $\mathcal{M}_{T_L}$ with orientations, which descend to an orientation of the cokernel bundle $\mathrm{Coker}^{T,\mathcal{L}}\,\bar{\partial}_J = \pi_1^*\,\mathrm{Coker}^{T_1}\,\bar{\partial}_J \oplus ... \oplus \pi_L^*\,\mathrm{Coker}^{T_L}\,\bar{\partial}_J$ over $\mathcal{M}_{T,\mathcal{L}} = \mathcal{M}_{T_1} \times ... \times \mathcal{M}_{T_L}/\Delta$. The orientations of the cokernel bundles over the strata $\mathcal{M}_{T,\mathcal{L}} \subset \overline{\mathcal{M}}$ in general do not *fit together to an orientation of the cokernel bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$ over the compactified moduli space $\overline{\mathcal{M}}$, but differ by a fixed sign due to reordering the punctures.*

We remark that the fact that the orientations of the cokernel bundles over the different strata differ by a fixed sign is not completely trivial, since the strata are in general not connected due to the possible choices for the asymptotic markers. Furthermore it directly follows from theorem 3 in [BM] that the cokernel bundle $\mathrm{Coker}\,\bar{\partial}_J$ is orientable over the quotient $\mathcal{M}/(\mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-})$ only when all asymptotic orbits $\gamma^{m_1^\pm}, ..., \gamma^{m_{n^\pm}^\pm}$ are good.

Proof: In the way described above the choice of coherent orientations in symplectic field theory following [BM] provides us with an orientation of the determinant bundles $\Lambda^{\max}\,\mathrm{Ker}\,\bar{\partial}_J \otimes \Lambda^{\max}\,\mathrm{Coker}\,\bar{\partial}_J$ of the Cauchy-Riemann operator $\bar{\partial}_J$ over the moduli space of branched covers $\mathcal{M}$. But since by lemma 2.2.3 $\mathrm{Ker}\,\bar{\partial}_J$ agrees with the tangent space to $\mathbb{R} \times \mathcal{M}$ and $\mathbb{R} \times \mathcal{M} = \mathbb{R} \times S^1 \times \mathcal{M}_{0,n} \times \mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-}$ is a complex manifold, we always have a natural orientation of $\mathrm{Ker}\,\bar{\partial}_J$, which directly fixes an orientation on the cokernel bundle $\mathrm{Coker}\,\bar{\partial}_J$ over $\mathcal{M}$ by requiring that the orientations on $\mathrm{Ker}\,\bar{\partial}_J$ and $\mathrm{Coker}\,\bar{\partial}_J$ determine the orientation of the determinant bundle $\Lambda^{\max}\,\mathrm{Ker}\,\bar{\partial}_J \otimes \Lambda^{\max}\,\mathrm{Coker}\,\bar{\partial}_J$.

In order to see that the same arguments can be used to orient the cokernel bundles $\mathrm{Coker}^{T_\ell}\,\bar{\partial}_J$ over the moduli spaces $\mathcal{M}_{T_\ell}$ of nodal curves for $\ell = 1, ..., L$, observe that the constructions in [BM] immediately generalize to nodal curves in such a way that the orientation of the determinant bundle for the nodal surface fits with the orientation for the determinant bundle over the glued surface. Indeed this follows, using the gluing argument for the determinant line bundles, simply from the fact that also on closed surface with nodes we have a standard Cauchy-Riemann operator providing us with a natural orientation of the determinant line bundle over the space of Fredholm operators on a closed nodal surface, which clearly fits with the natural orientation of the determinant bundle over the space of Fredholm operators over the glued surface. In order to see that the orientations of $\mathrm{Coker}^{T_1}\,\bar{\partial}_J$, ..., $\mathrm{Coker}^{T_L}\,\bar{\partial}_J$ determine an orientation of the cokernel bundle over the stratum $\mathcal{M}_{T,\mathcal{L}} = \mathcal{M}_{T_1} \times ... \times \mathcal{M}_{T_L}/\Delta$, we must show that the lift of the action of Δ on $\mathcal{M}_{T_1} \times ... \times \mathcal{M}_{T_L}$ to the cokernel bundle $\mathrm{Coker}^{T,\mathcal{L}}\,\bar{\partial}_J = \pi_1^*\,\mathrm{Coker}^{T_1}\,\bar{\partial}_J \oplus ... \oplus \pi_L^*\,\mathrm{Coker}^{T_L}\,\bar{\partial}_J$ is orientation-preserving:

For this recall that $\Delta = \prod_{\mathcal{L}(\alpha)>\mathcal{L}(\beta)} \Delta_{\alpha\beta}$, where $\Delta_{\alpha\beta}$ is the diagonal in $\mathbb{Z}_{|m_{\alpha\beta}|} \times \mathbb{Z}_{|m_{\beta\alpha}|}$ so that $\Delta_{\alpha\beta}$ acts on $\mathcal{M}_{T_k} \times \mathcal{M}_{T_\ell}$ for $k = \mathcal{L}(\alpha)$, $\ell = \mathcal{L}(\beta)$. Now it follows from theorem 3 in [BM] that the $\mathbb{Z}_{|m_{\alpha\beta}|}$-actions on the cokernel bundles $\mathrm{Coker}^{T_k}\,\bar{\partial}_J$ and $\mathrm{Coker}^{T_\ell}\,\bar{\partial}_J$ are orientation-preserving if $\gamma^{|m_{\alpha\beta}|}$ is good, and simultaneuously orientation-preserving or -reversing for even or odd elements in $\mathbb{Z}_{|m_{\alpha\beta}|}$ if $\gamma^{|m_{\alpha\beta}|}$ is bad. Hence the action on the direct sum $\pi_k^* \operatorname{Coker}^{T_k} \bar{\partial}_J \oplus \pi_\ell^* \operatorname{Coker}^{T_\ell} \bar{\partial}_J$ is orientation-preserving in all cases.

The statement about the behaviour of the orientations on the cokernel bundles under gluing directly follows from theorem 1 in [BM] which states that the gluing diffeomorphisms preserve the orientations up to a sign due to reordering of the punctures. This is however an immediate consequence of the behaviour of the orientation of moduli spaces under reordering the punctures. □

2.3 Perturbation theory and Euler numbers

2.3.1 Perturbed Cauchy-Riemann operator

As outlined in the section about the linearized operator, the Cauchy-Riemann operator $\bar{\partial}_J$ can be viewed as a smooth section in a Banach space bundle $\mathcal{E}^{p,d}$ over a Banach manifold of maps $\mathcal{B}^{p,d}$. Since for the contribution to the differential in contact homology and rational symplectic field theory we are interested in moduli spaces of branched covers $\mathcal{M}$ of virtual dimension zero while the actual dimension is always strictly greater than zero, it follows that in the cases of interest the Cauchy-Riemann operator $\bar{\partial}_J$ does not meet the zero section transversally. In other words, the image bundle $\operatorname{Im}\bar{\partial}_J$ of $\bar{\partial}_J$ over $\mathcal{M}$ with fibre $(\operatorname{Im}\bar{\partial}_J)_{h,j} = \operatorname{im} D_{h,j}$ is a true closed subbundle of the Banach space bundle $\mathcal{E}^{p,d}$ over the moduli space of branched covers $\mathcal{M} = \bar{\partial}_J^{-1}(0) \subset \mathcal{B}^{p,d}$, where the closedness of $\operatorname{im} D_{h,j}$ in $\mathcal{E}^{p,d}_{h,j}$ follows from the semi-Fredholm property of $D_{h,j} : T_{h,j}\mathcal{B}^{p,d} \to \mathcal{E}^{p,d}_{h,j}$. In particular, observe that we have a natural splitting

$$\mathcal{E}^{p,d}|_{\bar{\partial}_J^{-1}(0)} = \operatorname{Im}\bar{\partial}_J \oplus \operatorname{Coker}\bar{\partial}_J$$

with the cokernel bundle $\operatorname{Coker}\bar{\partial}_J$ introduced in section two.

For determining the contribution of $\mathcal{M}$ to the differential in contact homology and rational symplectic field theory it follows that the Cauchy-Riemann operator $\bar{\partial}_J$ has to be perturbed slightly to a transversal section in the Banach space bundle $\mathcal{E}^{p,d} \to \mathcal{B}^{p,d}$ in the sense that it meets the zero section transversally. This means that we have to add a compact perturbation ν to the Cauchy-Riemann operator to make it transversal and count elements in the regular moduli space $\mathcal{M}^\nu$,

$$\mathcal{M}^\nu = (\bar{\partial}_J^\nu)^{-1}(0) \subset \mathcal{B}^{p,d}, \ \ \bar{\partial}_J^\nu = \bar{\partial}_J + \nu.$$

We first prove the folk's theorem that it indeed suffices to study smooth sections in the cokernel bundle $\operatorname{Coker}\bar{\partial}_J \subset \mathcal{E}^{p,d}|_{\mathcal{M}}$ over the moduli space $\mathcal{M}$, i.e., the zero set of the Cauchy-Riemann operator $\bar{\partial}_J$; for a different proof in the context of Gromov-Witten theory, see proposition 7.2.3 in [MDSa]. For this we extend a section in $\operatorname{Coker}\bar{\partial}_J$ over $\mathcal{M}$ to a smooth section in the Banach space bundle $\mathcal{E}^{p,d}$ over the whole Banach manifold $\mathcal{B}^{p,d}$

as follows:

Choosing a unitary trivialization of (ξ, ω_ξ, J_ξ) along the Reeb orbit γ, note that it can be extended to a unitary trivialization of (ξ, ω_ξ, J_ξ) over a sufficiently small neighborhood N of γ using parallel transport along geodesics with respect to a unitary connection ∇. Further identifying N with a neighborhood of the zero section in $\gamma^*\xi \cong S^1 \times \mathbb{C}^{m-1}$ we assume that $N \cong S^1 \times B_\epsilon(0)$ with $B_\epsilon(0) = \{z \in \mathbb{C}^{m-1} : |z| < \epsilon\}$.

Now observe that for a section ν in the cokernel bundle $\operatorname{Coker}\bar{\partial}_J$ over $\mathcal{M}$ we have $\nu(h,j) \in L^p(T^*\dot{S} \otimes_{j,i} \mathbb{C}^{m-1})$ for every $(h,j) \in \mathcal{M}/(\mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-})$, which for every tuple (h,j,z), $z \in \dot{S}$ defines an element $\nu(h,j,z) \in T^*_z\dot{S} \otimes_{j,i} \mathbb{C}^{m-1}$. Identifying for fixed complex structure j the branched covering map h with the direction $t \in S^1$ of the asymptotic marker, i.e., $(h,j) \equiv (t,j) \in S^1 \times \mathcal{M}_{0,n} \cong \mathcal{M}/(\mathbb{Z}_{m^+} \times \mathbb{Z}_{m^-})$, note that this defines for every (j,z) a smooth map $\nu_0(j,z) : S^1 \to T^*_z\dot{S} \otimes_{j,i} \mathbb{C}^{m-1}$. With the choice of a smooth cut off function $\varphi_\epsilon : [0,\epsilon] \to [0,1]$ with $\varphi_\epsilon(0) = 1$ and $\varphi_\epsilon(\epsilon) = 0$ we can extend $\nu_0(j,z)$ to a map starting from $N \cong S^1 \times B_\epsilon(0)$ by setting $\nu_0(j,z)(t,v) := \varphi_\epsilon(|v|) \cdot \nu_0(j,z)(t)$ for $(t,v) \in S^1 \times B_\epsilon(0)$.

Let $\mathcal{U} = \mathcal{U}^{p,d}$ denote the small neighborhood of $\mathcal{M}$ in $\mathcal{B}^{p,d}$ of all maps $u : \dot{S} \to \mathbb{R} \times V$ having image contained in the neighborhood N of γ in V. Writing $u = (h,v) : \dot{S} \to (\mathbb{R} \times S^1) \times B_\epsilon(0) \subset \mathbb{R} \times V$ we can define an extension of ν from $\mathcal{M}$ to $\mathcal{U}^{p,d}$ by setting $\nu(u,j)(z) := \nu_0(j,z)(t(u), v(z))$ with $t(u) \in S^1$ denoting the direction of the asymptotic marker defined by the map u. Note that this indeed defines an extension and that $\nu(u,j) \in L^p(T^*\dot{S} \otimes_{j,i} \mathbb{C}^{m-1})$. In particular, ν defines a section in trivial bundle $\mathcal{E}^{p,d}|_{\mathcal{U}^{p,d}}$ with fibre $L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,i} \mathbb{C}^{m-1})$, which in turn after extending by zero defines a section in the Banach space bundle $\mathcal{E}^{p,d}$ over the whole Banach manifold $\mathcal{B}^{p,d}$. Then the following holds:

Proposition 2.3.1: *Let ν be a section in the cokernel bundle* $\operatorname{Coker}\bar{\partial}_J \subset \mathcal{E}^{p,d}|_{\mathcal{M}}$ *over the moduli space $\mathcal{M} = \bar{\partial}_J^{-1}(0) \subset \mathcal{B}^{p,d}$, which is extended to a section in $\mathcal{E}^{p,d}$ as described above. Then it holds:*

- *The moduli space $\mathcal{M}^\nu$ agrees with the zero set of ν in $\mathcal{M}$,*

$$\mathcal{M}^\nu = \{(h,j) \in \mathcal{M} : \nu(h,j) = 0\}.$$

- *If ν is a transversal section in* $\operatorname{Coker}\bar{\partial}_J$*, then $\bar{\partial}_J^\nu$ is a transversal section in $\mathcal{E}^{p,d}$, i.e., $\mathcal{M}^\nu$ is regular.*

- *The linearization of ν at every zero is a compact operator, so that the linearizations of $\bar{\partial}_J$ and $\bar{\partial}_J^\nu$ belong to the same class of Fredholm operators.*

Proof: First we find no zeroes of $\bar{\partial}_J^\nu$ outside of the neighborhood $\mathcal{U}$ of $\mathcal{M}$ since there

$\bar{\partial}_J^\nu = \bar{\partial}_J$. For every $(u,j) \in \mathcal{U}$ with $u = (h,v) : \dot{S} \to (\mathbb{R} \times S^1) \times B_\epsilon(0)$ let π_1 denote the projection onto the first factor in $\mathcal{E}^{p,d}_{u,j} = L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C}) \oplus L^p(T^*\dot{S} \otimes_{j,i} \mathbb{C}^{m-1})$. Then we have by construction that $\pi_1 \circ \nu(u,j) = 0$ while $\pi_1 \circ \bar{\partial}_J(u) = \bar{\partial} h$ with the standard Cauchy-Riemann operator $\bar{\partial} : H^{1,p,d}_{\mathrm{const}}(\dot{S},\mathbb{C}) \to L^{p,d}(T^*\dot{S} \otimes_{j,i} \mathbb{C})$. For $(u,j) \in \mathcal{U} - \mathcal{M}$ it follows that $\pi_1 \circ \bar{\partial}_J^\nu(u) = \bar{\partial} h \neq 0$, so that we find no zeroes of $\bar{\partial}_J^\nu$ in $\mathcal{U} - \mathcal{M}$. Finally, on $\mathcal{M}$ we have $\bar{\partial}_J^\nu = \nu$.

With respect to the splittings $T_{h,j}\,\mathcal{B}^{p,d} = (\ker D_{h,j})^\perp \oplus \ker D_{h,j}$ and $\mathcal{E}^{p,d}_{h,j} = \operatorname{im} D_{h,j} \oplus \operatorname{coker} D_{h,j}$ at $(h,j) \in \mathcal{M}$, observe that the linearization $D_{h,j}\bar{\partial}_J^\nu : T_{h,j}\,\mathcal{B}^{p,d} \to \mathcal{E}^{p,d}_{h,j}$ at a zero $\nu(h,j) = 0$, $(h,j) \in \mathcal{M}$, is of upper triangle form

$$D_{h,j}\bar{\partial}_J^\nu = \begin{pmatrix} D_{h,j} & 0 \\ D^0_{h,j}\nu & D^1_{h,j}\nu \end{pmatrix},$$

where D^0, D^1 denotes differentiation in the direction of $(\ker D_{h,j})^\perp$ and $\ker D_{h,j} = T_{h,j}\,\mathcal{M}$, respectively. Since ν is a transversal section in $\operatorname{Coker} \bar{\partial}_J$ over $\mathcal{M}$ precisely when $D^1_{h,j}\nu : \ker D_{h,j} \to \operatorname{coker} D_{h,j}$ is surjective at every $\nu(h,j) = 0$, the second statement follows from the fact that $D_{h,j} : (\ker D_{h,j})^\perp \to \operatorname{im} D_{h,j}$ is an isomorphism.

For the last statement it suffices to see that the linearization $D_{h,j}\nu : T_{h,j}B^{p,d} \to \operatorname{coker} D_{h,j}$ is an operator with finite-dimensional image. $\square$

Since the cokernel bundle $\operatorname{Coker} \bar{\partial}_J$ as well as its base space $\mathcal{M}$ are oriented, it follows that the regular moduli space $\mathcal{M}^\nu$ carries an orientation, which by the construction of orientations for $\operatorname{Coker} \bar{\partial}_J$ agrees with the orientation of moduli spaces in symplectic field theory constructed in [BM]. The contribution of branched covers of orbit cylinders to the differential in rational symplectic field theory is given by the algebraic count of elements in $\mathcal{M}^\nu$, which however might explicitly depend on the chosen perturbation ν.

2.3.2 Gluing compatibility

In order to have transversality for all moduli spaces of connected branched covers without nodes we choose transversal sections $\nu = \nu_{\vec{m}}$ in the cokernel bundles over the moduli spaces

$$\mathcal{M} = \mathcal{M}_{\vec{m}} = \mathcal{M}_{0,0}(\gamma^{m_1^+},...,\gamma^{m^+_{n^+}};\gamma^{m_1^-},...,\gamma^{m^-_{n^-}})/\,\mathbb{R}$$

for all tuples $\vec{m} = (\vec{m}^+,\vec{m}^-)$, $\vec{m}^\pm = (m_1^\pm,...,m^\pm_{n^\pm})$ with

$$|\vec{m}| := m_1^+ + ... + m^+_{n^+} = m_1^- + ... + m^-_{n^-},$$

i.e., for which $\mathcal{M}_{\vec{m}} \neq \emptyset$.

To be precise we choose transversal sections ν in the cokernel bundles over the quotient $\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, where we forget the position of the asymptotic markers at the positive, respectively negative punctures. In this way we ensure that the chosen abstract perturbation and hence the contribution of curves to the differential does not depend on the choice of asymptotic markers, which is implicit in the algebraic setup of symplectic field theory. At this point recall that although the cokernel bundle $\operatorname{Coker}\bar{\partial}_J$ naturally lives over the quotient $\mathcal{M}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, it is in general only orientable over the complete moduli space $\mathcal{M}$, since the orientation of a fibre in general depends on the choice of the asymptotic markers at the punctures.

In order to have the compactness and gluing results for the resulting regular moduli spaces which are implicit in the definition of algebraic invariants in symplectic field theory we consider only sets of cokernel sections $(\nu_{\vec{m}})_{\vec{m}}$, which are compatible with compactness and gluing in symplectic field theory in the following sense:

Let (h^q,j^q), $q\in\mathbb{N}$ be a sequence of curves in the regular moduli space $\mathcal{M}^\nu=\mathcal{M}^{\nu_{\vec{m}}}_{\vec{m}}$, which converges for $q\to\infty$ to a level branched covering $(h,j)\in\mathcal{M}_{T,\mathcal{L}}\subset\overline{\mathcal{M}}$ with

$$\mathcal{M}_{T,\mathcal{L}}=\mathcal{M}_{T_1}\times...\times\mathcal{M}_{T_L}/\Delta$$

and $\mathcal{M}_{T_\ell}=\mathcal{M}_{T_{\ell,1}}\times...\times\mathcal{M}_{T_{\ell,N_\ell}}\times\mathbb{R}^{N_\ell-1}$. Then all components $(h_{\ell,k},j_{\ell,k})\in\mathcal{M}_{T_{\ell,k}}$, $\ell=1,...,L$, $k=1,...,N_\ell$ again satisfy a perturbed Cauchy-Riemann equation. When the moduli space $\mathcal{M}_{T,\mathcal{L}}$ is made up of curves with no nodes, i.e., for which the trees $T_{1,1},...,T_{L,N_L}$ are trivial, the moduli spaces $\mathcal{M}_{\ell,k}=\mathcal{M}_{T_{\ell,k}}$ are again moduli spaces of connected branched covers without nodes,

$$\mathcal{M}_{\ell,k}=\mathcal{M}_{\vec{m}_{\ell,k}}$$

for new tuples $\vec{m}_{\ell,k}=(\vec{m}^+_{\ell,k},\vec{m}^-_{\ell,k})$, $\vec{m}^\pm_{\ell,k}=(m^\pm_{\ell,k,1},...,m^\pm_{\ell,k,n^\pm_{\ell,k}})$. Assuming that the abstract perturbations $\nu_{\ell,k}=\nu_{\vec{m}_{\ell,k}}$ for the moduli spaces $\mathcal{M}_{\ell,k}$ are already chosen, compatibility with gluing in symplectic field theory now means that the abstract perturbation $\nu=\nu_{\vec{m}}$ is chosen in such a way that $(h_{\ell,k},j_{\ell,k})$ satisfies the Cauchy-Riemann equation with perturbation $\nu_{k,\ell}$, i.e., is an element in the regular moduli space $\mathcal{M}^{\nu_{\ell,k}}_{\ell,k}\subset\mathcal{M}_{\ell,k}$. Observe that for every level $\ell=1,...,L$ we have

$$|\vec{m}_{\ell,1}|+...+|\vec{m}_{\ell,N_\ell}|=|\vec{m}|,$$

while for the number of punctures $\sharp\vec{m}=n=n^++n^-$ and $\sharp\vec{m}_{\ell,k}=n_{\ell,k}=n^+_{\ell,k}+n^-_{\ell,k}$ we have

$$\sharp\vec{m}_{\ell,k}<\sharp\vec{m}.$$

It follows that the choice of the abstract perturbation $\nu_{\vec{m}}$ depends only on abstract perturbations $\nu_{\vec{m}'}$ with $\sharp\vec{m}'<\sharp\vec{m}$ and $|\vec{m}'|\leq|\vec{m}|$.

The correct setup for constructing perturbations $\nu=\nu_{\vec{m}}$ with the desired properties is to study smooth transversal sections $\bar{\nu}$ in the cokernel bundle $\overline{\operatorname{Coker}\bar{\partial}_J}$ over the compactification $\overline{\mathcal{M}}$ of the moduli space $\mathcal{M}=\mathcal{M}_{\vec{m}}$. More precisely, we study smooth transversal sections in the cokernel bundle over the quotient $\overline{\mathcal{M}}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, i.e., we again forget the positions of the asymptotic markers. Then the abstract perturbation $\nu=\nu_{\vec{m}}$ for the moduli space $\mathcal{M}$ is given by the restriction $\nu=\bar{\nu}|_{\mathcal{M}}$ of the section to the interior, while the abstract perturbations $\nu_{\vec{m}'}$ for the moduli spaces for tuples $\vec{m}'$ with $\sharp\vec{m}'<\sharp\vec{m}$ and $|\vec{m}'|\leq|\vec{m}|$ determine $\bar{\nu}$ on the boundary $\partial\overline{\mathcal{M}}=\overline{\mathcal{M}}-\mathcal{M}$ as follows:

Let $(T,\mathcal{L})$ be a tree with level structure which represents curves with no nodes, i.e., for which all trees $T_{\ell,k}$, $\ell=1,...,L$, $k=1,...,N_\ell$ are trivial, and denote again $\mathcal{M}_{\ell,k}=\mathcal{M}_{T_{\ell,k}}$ the corresponding moduli spaces of branched covers. Let us further assume that $\mathcal{M}_{T,\mathcal{L}}$ is indeed a boundary stratum, i.e., does not agree with the top stratum $\mathcal{M}$. Denoting by $\mathrm{Coker}^{\ell,k}\,\bar{\partial}_J$ the cokernel bundle over $\mathcal{M}_{\ell,k}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}})$ with the sets $\mathbb{Z}_{m^\pm_{\ell,k}}=\mathbb{Z}_{m^\pm_{\ell,k,1}}\times...\times\mathbb{Z}_{m^\pm_{\ell,k,n^\pm_{\ell,k}}}$ of asymptotic markers at the positive, respectively negative punctures, recall that the cokernel bundle over $\mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ is given as sum of pullback bundles

$$\mathrm{Coker}^{T,\mathcal{L}}\,\bar{\partial}_J=\pi^*_{1,1}\,\mathrm{Coker}^{1,1}\,\bar{\partial}_J\oplus...\oplus\pi^*_{L,N_L}\,\mathrm{Coker}^{L,N_L}\,\bar{\partial}_J$$

under the projections

$$\pi_{\ell,k}:\mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})\to\mathcal{M}_{\ell,k}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}}).$$

For the section $\bar{\nu}$ in the cokernel bundle over $\overline{\mathcal{M}}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ we now require that the restriction $\nu_{T,\mathcal{L}}$ to $\mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ is given by

$$\nu_{T,\mathcal{L}}(h,j)=(\nu_{1,1}(h_{1,1},j_{1,1}),...,\nu_{L,N_L}(h_{L,N_L},j_{L,N_L}))$$

for $(h,j)\in\mathcal{M}_{T,\mathcal{L}}$ with $h=(h_{1,1},...,h_{L,N_L})$, $j=(j_{1,1},...,j_{L,N_L})$. In other words, $\bar{\nu}$ is over $\mathcal{M}_{T,\mathcal{L}}$ given as sum of pullback sections

$$\bar{\nu}|_{\mathcal{M}_{T,\mathcal{L}}}=\pi^*_{1,1}\nu_{1,1}\oplus...\oplus\pi^*_{L,N_L}\nu_{L,N_L}$$

with $\nu_{1,1}$, ..., ν_{L,N_L} chosen before. Note that this makes sense, since all sections $\nu_{\ell,k}$ indeed live in the cokernel bundle $\mathrm{Coker}^{\ell,k}\,\bar{\partial}_J$ over the quotient $\mathcal{M}_{\ell,k}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}})$.

We define $\overline{\mathcal{M}}^{\bar{\nu}}\subset\overline{\mathcal{M}}$ by pulling back the section $\bar{\nu}$ from the cokernel bundle over $\overline{\mathcal{M}}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ to the cokernel bundle over $\overline{\mathcal{M}}$ and setting

$$\overline{\mathcal{M}}^{\bar{\nu}}=\bar{\nu}^{-1}(0).$$

Recall that we have seen in proposition 2.2.5 that the cokernel bundle $\overline{\mathrm{Coker}\bar{\partial}_J}$ can be equipped with the structure of a smooth vector bundle over the compactified moduli space $\overline{\mathcal{M}}$, which by proposition 2.1.4 is a smooth manifold with corners. Since $\bar{\nu}$ is assumed to be smooth and transversal to the zero section, it follows from a version of the implicit function theorem that $\overline{\mathcal{M}}^{\bar{\nu}}$ is a smooth submanifold with corners of $\overline{\mathcal{M}}$, which is furthermore neat in the sense that

$$\partial\overline{\mathcal{M}}^{\bar{\nu}}=\overline{\mathcal{M}}^{\bar{\nu}}\cap\partial\overline{\mathcal{M}}.$$

More precisely it follows that $\overline{\mathcal{M}}^{\bar{\nu}}$ is again a stratified space with strata

$$\mathcal{M}^{\bar{\nu}}_{T,\mathcal{L}} = \overline{\mathcal{M}}^{\bar{\nu}} \cap \mathcal{M}_{T,\mathcal{L}}\,.$$

Since $\bar{\nu}$ is independent of the directions of the asymptotic markers at the punctures, it follows that $\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}$ still acts on the regular moduli space $\overline{\mathcal{M}}^{\bar{\nu}}$. Furthermore the conditions on the section $\bar{\nu}$ imply that for $\mathcal{M}^{\bar{\nu}}_{T,\mathcal{L}}$ with trivial trees $T_{1,1},...,T_{L,N_L}$ we have

$$\mathcal{M}^{\bar{\nu}}_{T,\mathcal{L}} = \prod_{\ell=1}^{L} \mathbb{R}^{N_\ell-1}\times\mathcal{M}^{\nu_{1,1}}_{1,1}\times...\times\mathcal{M}^{\nu_{L,N_L}}_{L,N_L}\,/\Delta.$$

This motivates the following definition:

Definition 2.3.2: *A section $\bar{\nu}$ in the cokernel bundle $\overline{\mathrm{Coker}}\bar{\partial}_J$ over the compactified moduli space $\overline{\mathcal{M}}$ is called* coherent *if it is the pullback of a section in the cokernel bundle over the quotient $\overline{\mathcal{M}}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, which over each boundary stratum $\mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$ for trees with level structure $(T,\mathcal{L})$ with trivial trees $T_{1,1},...,T_{L,N_L}$ is given as sum*

$$\bar{\nu}|_{\mathcal{M}_{T,\mathcal{L}}} = \pi^*_{1,1}\nu_{1,1}\oplus...\oplus\pi^*_{L,N_L}\nu_{L,N_L}$$

of pullbacks of sections in the cokernel bundles $\mathrm{Coker}^{T_{1,1}}\,\bar{\partial}_J$,...,$\mathrm{Coker}^{T_{L,N_L}}\,\bar{\partial}_J$ *under the projections*

$$\pi_{\ell,k}:\mathcal{M}_{T,\mathcal{L}}\,/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})\to\mathcal{M}_{T_{\ell,k}}\,/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}}).$$

We emphasize that our notion of coherency is weaker than the usual definition: While we just require that the abstract perturbations are of a special form over each boundary stratum, one usually additionally requires that for every moduli space one chooses a unique abstract perturbation in the sense that if a moduli space appears in two different boundary strata the two perturbations agree. However it follows from our proof of theorem 2.3.3 that our weaker assumption indeed suffices to prove our desired result.

Let $\mathcal{M}_1\times_{\mathbb{Z}_{m_{12}}}\mathcal{M}_2\subset\partial\overline{\mathcal{M}}$ be an arbitrary codimension one boundary stratum. Recall from subsection 2.2.1 that the restriction of $\overline{\mathrm{Coker}}\bar{\partial}_J$ to $\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2=\overline{\mathcal{M}_1\times_{\mathbb{Z}_{m_{12}}}\mathcal{M}_2}\subset\overline{\mathcal{M}}$ is given by the sum of pullback bundles

$$\overline{\mathrm{Coker}}\bar{\partial}_J|_{\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{1,2}}}\overline{\mathcal{M}}_2} \;=\; \pi_1^*\overline{\mathrm{Coker}}^1\bar{\partial}_J\oplus\pi_2^*\overline{\mathrm{Coker}}^2\bar{\partial}_J.$$

It directly follows from the definition that any coherent (and transversal) section $\bar{\nu}$ in $\overline{\mathrm{Coker}}\bar{\partial}_J$ is given over $\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2$ by

$$\bar{\nu}|_{\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2}=\pi_1^*\bar{\nu}_1\oplus\pi_2^*\bar{\nu}_2$$

with coherent (and transversal) sections in $\overline{\mathrm{Coker}}^1\bar{\partial}_J$, $\overline{\mathrm{Coker}}^2\bar{\partial}_J$, respectively. Furthermore

$$\overline{\mathcal{M}_1\times_{\mathbb{Z}_{m_{12}}}\mathcal{M}_2}^{\bar{\nu}}=\overline{\mathcal{M}}_1^{\bar{\nu}_1}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2^{\bar{\nu}_2}.$$

2.3.3 Euler numbers for Fredholm problems

We have seen that the perturbation chosen for a moduli space explicitly depends on the perturbations chosen for the moduli spaces forming the boundary of its compactification. However, in this last section we prove that for any coherent and transversal section $\bar{\nu}$ in $\overline{\text{Coker}}\bar{\partial}_J$ in the sense of definition 2.3.2 the algebraic count of elements in the regular compactified moduli space $\overline{\mathcal{M}}^{\bar{\nu}}$ is zero, independent of all choices. Together with the discussion in 0.3 we have then shown that branched covers over trivial orbit cylinders do not contribute to the differential of rational symplectic field theory, i.e., we have proven the main theorem.

Theorem 2.3.3: *For the cokernel bundle $\overline{\text{Coker}}\bar{\partial}_J$ over the compactification $\overline{\mathcal{M}}$ of every moduli space of branched covers over a trivial cylinder with* $\dim_{\text{virt}}\mathcal{M} = \dim\mathcal{M} - \text{rank}\,\text{Coker}\,\bar{\partial}_J = 0$ *the following holds:*

- *For every pair $\bar{\nu}^0$, $\bar{\nu}^1$ of coherent and transversal sections in $\overline{\text{Coker}}\bar{\partial}_J$ the algebraic count of zeroes of $\bar{\nu}^0$ and $\bar{\nu}^1$ are finite and agree, so that we can define an Euler number $\chi(\overline{\text{Coker}}\bar{\partial}_J)$ for coherent sections in $\overline{\text{Coker}}\bar{\partial}_J$ by*
$$\chi(\overline{\text{Coker}}\bar{\partial}_J) := \sharp(\bar{\nu}^0)^{-1}(0) = \sharp(\bar{\nu}^1)^{-1}(0).$$
- *This Euler number is $\chi(\overline{\text{Coker}}\bar{\partial}_J) = 0$.*

Proof: We prove this statement for all moduli spaces of trivial curves by induction on the number of punctures $n \geq 3$.

Let $\bar{\nu}$ be a coherent and transversal section in $\overline{\text{Coker}}\bar{\partial}_J$. In order to see that the zeroes of $\bar{\nu}$ can be counted to give a finite number, observe that it follows from $\dim\mathcal{M} - \text{rank}\,\overline{\text{Coker}}\bar{\partial}_J = 0$ and the implicit function theorem that $\bar{\nu}^{-1}(0)$ is a neat zero-dimensional submanifold of $\overline{\mathcal{M}}$, i.e., a discrete set of points in $\mathcal{M} \subset \overline{\mathcal{M}}$, which is compact as a closed subset of a compact set.

Now let $\bar{\nu}^0$ and $\bar{\nu}^1$ be two coherent and transversal sections in $\overline{\text{Coker}}\bar{\partial}_J$. In order to see that the numbers of zeroes $\sharp(\bar{\nu}^0)^{-1}(0)$ and $\sharp(\bar{\nu}^1)^{-1}(0)$ indeed agree, let $\bar{\nu}^{01}$ be a section in the cokernel bundle $\overline{\text{Coker}_0}\bar{\partial}_J$ over $\overline{\mathcal{M}^0}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})$, which is coherent and compatible with $\bar{\nu}^0$ and $\bar{\nu}^1$ in the sense that over each stratum

$$\begin{aligned}&\mathcal{M}^0_{T,\mathcal{L},\ell_0}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})\\ &= \frac{\mathcal{M}_{T_1}\times ...\times\mathcal{M}_{T_{\ell_0-1}}\times\mathcal{M}^0_{T_{\ell_0}}\times\mathcal{M}_{T_{\ell_0+1}}\times ...\times\mathcal{M}_{T_L}}{\Delta\times\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-}}\end{aligned}$$

the restriction $\nu^{01}_{T,\mathcal{L},\ell_0} = \bar{\nu}^{01}|_{\mathcal{M}^0_{T,\mathcal{L},\ell_0}}$ is of the form

$$\begin{aligned}\nu^{01}_{T,\mathcal{L},\ell_0} &= \pi_1^*\nu^0_{T_1}\oplus ...\oplus\pi^*_{\ell_0-1}\nu^0_{T_{\ell_0-1}}\oplus\pi^*_{\ell_0}\nu^{01}_{T_{\ell_0}}\\ &\quad\oplus\pi^*_{\ell_0+1}\nu^1_{T_{\ell_0+1}}\oplus ...\oplus\pi^*_L\nu^1_{T_L},\end{aligned}$$

with the projections

$$\pi_{\ell,k}:\mathcal{M}^0_{T,\mathcal{L},\ell_0}/(\mathbb{Z}_{m^+}\times\mathbb{Z}_{m^-})\to\mathcal{M}^{(0)}_{T_{\ell,k}}/(\mathbb{Z}_{m^+_{\ell,k}}\times\mathbb{Z}_{m^-_{\ell,k}}),$$

where $\nu^0_{T_1},...,\nu^0_{T_{\ell_0-1}}$ and $\nu^1_{T_{\ell_0+1}},...,\nu^1_{T_L}$ are given by $\bar{\nu}^0$ and $\bar{\nu}^1$, respectively. Note that this implies that

$$\begin{aligned}\bar{\nu}^{01}|_{\overline{\mathcal{M}^0_1}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2} &= \pi_1^*\bar{\nu}^{01}_1\oplus\pi_2^*\bar{\nu}^1_2,\\ \bar{\nu}^{01}|_{\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}^0_2}} &= \pi_1^*\bar{\nu}^0_1\oplus\pi_2^*\bar{\nu}^{01}_2,\\ \bar{\nu}^{01}|_{\{\text{point}\}\times\overline{\mathcal{M}}}=\bar{\nu}^1 \quad\text{and}&\quad \bar{\nu}^{01}|_{\overline{\mathcal{M}}\times\{\text{point}\}}=\bar{\nu}^0\end{aligned}$$

and that we can always find a section $\bar{\nu}^{01}$ in $\overline{\text{Coker}_0}\bar{\partial}_J$ with the above properties by iteratively extending as in 2.3.2 the sections from the boundary of $\mathcal{M}^0$ to the interior of the moduli space. In particular, observe that by proposition 2.1.2 the numbers of punctures of the curves in $\mathcal{M}^0_1$ and $\mathcal{M}^0_2$ in the codimension one boundary of $\mathcal{M}^0$ are strictly smaller than the number of punctures of the curves in $\mathcal{M}^0$.

Note that by the propositions 2.1.4 and 2.2.5 the cokernel bundle $\overline{\text{Coker}_0}\bar{\partial}_J$ over $\overline{\mathcal{M}^0}$ can also be equipped with the structure of a smooth vector bundle over a manifold with corners. With this we further again assume that $\bar{\nu}^{01}$ is a smooth and transversal section in $\overline{\text{Coker}_0}\bar{\partial}_J$, which in turn implies that for each stratum $\mathcal{M}^0_{T,\mathcal{L},\ell_0}$ the underlying sections $\nu^0_{T_1}$, ..., $\nu^0_{T_{\ell_0-1}}$, $\nu^{01}_{T_{\ell_0}}$, $\nu^1_{T_{\ell_0+1}}$, ..., $\nu^1_{T_{\ell_0}}$ of the cokernel bundles $\text{Coker}^{T_1}\bar{\partial}_J$, ..., $\text{Coker}^{T_{\ell_0}}_0\bar{\partial}_J$, ..., $\text{Coker}^{T_L}\bar{\partial}_J$ are again smooth and transversal. Now it follows from

$$\dim\mathcal{M}^0-\operatorname{rank}\overline{\text{Coker}_0}\bar{\partial}_J=1+\dim\mathcal{M}-\operatorname{rank}\overline{\text{Coker}}\bar{\partial}_J=1$$

that the resulting regular moduli space

$$\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}=(\bar{\nu}^{01})^{-1}(0)\subset\overline{\mathcal{M}^0}$$

is a neat one-dimensional submanifold of $\overline{\mathcal{M}^0}$. In other words, we have that $\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}$ is a one-dimensional manifold with boundary given by

$$\partial\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}=\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}\cap\partial\overline{\mathcal{M}^0}.$$

In order to determine the boundary of $\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}$ observe that after setting

$$\begin{aligned}\big(\overline{\mathcal{M}^0_1}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2\big)^{\bar{\nu}^{01}} &:= \overline{\mathcal{M}^0}^{\bar{\nu}^{01}}\cap\big(\overline{\mathcal{M}^0_1}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}}_2\big),\\ \big(\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}^0_2}\big)^{\bar{\nu}^{01}} &:= \overline{\mathcal{M}^0}^{\bar{\nu}^{01}}\cap\big(\overline{\mathcal{M}}_1\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}^0_2}\big),\\ \big(\{\text{point}\}\times\overline{\mathcal{M}}\big)^{\bar{\nu}^{01}} &:= \overline{\mathcal{M}^0}^{\bar{\nu}^{01}}\cap\big(\{\text{point}\}\times\overline{\mathcal{M}}\big),\\ \text{and}\quad\big(\overline{\mathcal{M}}\times\{\text{point}\}\big)^{\bar{\nu}^{01}} &:= \overline{\mathcal{M}^0}^{\bar{\nu}^{01}}\cap\big(\overline{\mathcal{M}}\times\{\text{point}\}\big),\end{aligned}$$

the boundary conditions for $\bar{\nu}^{01}$ yield

$$\begin{aligned}
\big(\overline{\mathcal{M}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2}\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2}^{\bar{\nu}_2^1},\\
\big(\overline{\mathcal{M}_1}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}_1}^{\bar{\nu}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}},\\
\big(\{\text{point}\}\times\overline{\mathcal{M}}\big)^{\bar{\nu}^{01}} &= \{\text{point}\}\times\overline{\mathcal{M}}^{\bar{\nu}^1},\\
\text{and}\quad \big(\overline{\mathcal{M}}\times\{\text{point}\}\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}}^{\bar{\nu}^0}\times\{\text{point}\}.
\end{aligned}$$

All together it follows that the boundary of $\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}$ is given by

$$\begin{aligned}
\partial\overline{\mathcal{M}^0}^{\bar{\nu}^{01}} &= (\overline{\mathcal{M}}^{\bar{\nu}^0}\times\{\text{point}\})\,\cup\,(\{\text{point}\}\times\overline{\mathcal{M}}^{\bar{\nu}^1})\\
&\cup\bigcup_{2<n_1,n_2<n}\Big((\overline{\mathcal{M}_1}^{\bar{\nu}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}})\,\cup\,(\overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2}^{\bar{\nu}_2^1})\Big),
\end{aligned}$$

where we take the union over all those codimension one boundary strata of $\overline{\mathcal{M}^0}$ where the number of punctures n_1,n_2 for $\overline{\mathcal{M}_1^0}$, $\overline{\mathcal{M}_2}$ (and $\overline{\mathcal{M}_1}$, $\overline{\mathcal{M}_2^0}$) is strictly between two and the number of punctures n for $\overline{\mathcal{M}_0}$, i.e., $\overline{\mathcal{M}_1^0}$, $\overline{\mathcal{M}_2^0}\neq\{\text{point}\}$.

Now since $\partial\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}$ is the boundary of a one-dimensional manifold and taking into account the orientation of the codimension one boundary of the base space $\overline{\mathcal{M}^0}$ it follows that

$$\begin{aligned}
0 &= \#(\overline{\mathcal{M}}^{\bar{\nu}^0}\times\{\text{point}\})\;-\;\#(\{\text{point}\}\times\overline{\mathcal{M}}^{\bar{\nu}^1})\\
&+\sum_{2<n_1,n_2<n}\Big(\#(\overline{\mathcal{M}_1}^{\bar{\nu}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}})\;-\;\#(\overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2}^{\bar{\nu}_2^1})\Big).
\end{aligned}$$

Note that here $\#$ refers to the orientation as boundary of $(\mathcal{M}^0)^{\bar{\nu}^{01}}$, which itself is induced by the orientation of the cokernel bundle $\mathrm{Coker}_0\,\bar{\partial}_J$ over $\mathcal{M}^0$. In order to show that

$$\#(\bar{\nu}^0)^{-1}(0)=\#\overline{\mathcal{M}}^{\bar{\nu}^0}=\#\overline{\mathcal{M}}^{\bar{\nu}^1}=\#(\bar{\nu}^1)^{-1}(0),$$

i.e., to prove the existence of the Euler number $\chi(\overline{\mathrm{Coker}}\bar{\partial}_J)$, it hence suffices to show that

$$\begin{aligned}
&\#(\overline{\mathcal{M}_1}^{\bar{\nu}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}})=0\\
\text{and}\quad&\#(\overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2}^{\bar{\nu}_2^1})=0
\end{aligned}$$

for every other boundary stratum:

For this observe that in order to have

$$\#(\overline{\mathcal{M}_1}^{\bar{\nu}_1^0}\times_{\mathbb{Z}_{m_{12}}}\overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}})\neq 0,$$

we in particular must have

$$\overline{\mathcal{M}}_1^{\bar{\nu}_1^0} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}} \neq \emptyset,$$

which is equivalent to

$$\overline{\mathcal{M}}_1^{\bar{\nu}_1^0} \neq \emptyset \qquad \text{and} \qquad \overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}} \neq \emptyset.$$

Now since both $\bar{\nu}_1^0$ and $\bar{\nu}_2^{01}$ are transversal, i.e., have zero as a regular value, it follows that

$$\begin{array}{rcl} \dim \mathcal{M}_1 - \operatorname{rank} \overline{\operatorname{Coker}}^1 \bar{\partial}_J = \dim \mathcal{M}_1^{\nu_1^0} & \geq & 0 \\ \text{and} \qquad \dim \mathcal{M}_2^0 - \operatorname{rank} \overline{\operatorname{Coker}_0}^2 \bar{\partial}_J = \dim (\mathcal{M}_2^0)^{\nu_2^{01}} & \geq & 0. \end{array}$$

On the other hand, since

$$\begin{array}{rcl} 1 & = & \dim \mathcal{M}^0 - \operatorname{rank} \overline{\operatorname{Coker}_0} \bar{\partial}_J \\ & = & 1 + \dim \mathcal{M}_1 + \dim \mathcal{M}_2^0 - \operatorname{rank} \overline{\operatorname{Coker}}^1 \bar{\partial}_J - \operatorname{rank} \overline{\operatorname{Coker}_0}^2 \bar{\partial}_J \end{array}$$

it follows that we indeed must have equality, i.e.,

$$\begin{array}{rcl} \dim \mathcal{M}_1 - \operatorname{rank} \overline{\operatorname{Coker}}^1 \bar{\partial}_J = \dim \mathcal{M}_1^{\nu_1^0} & = & 0 \\ \text{and} \qquad \dim \mathcal{M}_2^0 - \operatorname{rank} \overline{\operatorname{Coker}_0}^2 \bar{\partial}_J = \dim (\mathcal{M}_2^0)^{\nu_2^{01}} & = & 0. \end{array}$$

In other words, we can immediately forget about all boundary components $\overline{\mathcal{M}}_1^{\bar{\nu}_1^0} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}}$ where the virtual dimension of $\mathcal{M}_1$ (and of $\mathcal{M}_2^0$) is not equal to zero, i.e., the underlying Fredholm index of $\bar{\partial}_J$ is not equal to one, where a corresponding statement clearly also holds for the boundary strata $\overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}}_2^{\bar{\nu}_2^1}$. In particular, observe that this would directly prove the existence of the desired Euler number $\chi(\overline{\operatorname{Coker}} \bar{\partial}_J)$ if we were able to show that none of the moduli spaces $\mathcal{M}_1$ or $\mathcal{M}_2$ appearing in the codimension one boundary has virtual dimension zero. While this is typically the case when the compactification is not "too large" , note that here there is no way to exclude the latter from happening. However, at this point, we can now make use of the induction hypothesis as follows:

Since the number of punctures for the moduli space $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$ is strictly smaller than the number of punctures for the original moduli space $\overline{\mathcal{M}}$, it follows that we do not only have Euler numbers $\chi(\overline{\operatorname{Coker}}^1 \bar{\partial}_J)$ and $\chi(\overline{\operatorname{Coker}}^2 \bar{\partial}_J)$ for coherent and transversal sections in the cokernel bundles $\overline{\operatorname{Coker}}^1 \bar{\partial}_J$ and $\overline{\operatorname{Coker}}^2 \bar{\partial}_J$, but by assumption further know that they are zero. In other words, we already know that

$$\#_1 \overline{\mathcal{M}}_1^{\bar{\nu}_1^0} \; = \; \chi(\overline{\operatorname{Coker}}^1 \bar{\partial}_J) \; = \; 0, \quad \#_2 \overline{\mathcal{M}}_2^{\bar{\nu}_2^1} \; = \; \chi(\overline{\operatorname{Coker}}^2 \bar{\partial}_J) \; = \; 0,$$

where $\#_1$, $\#_2$ refers to the algebraic count with respect to the orientation on the cokernel bundle $\operatorname{Coker}^1 \bar{\partial}_J$, $\operatorname{Coker}^2 \bar{\partial}_J$ over $\mathcal{M}_1$, $\mathcal{M}_2$, respectively. Denoting by $\#_1$, $\#_2$ further the algebraic count with respect to the induced orientation on $\mathrm{Coker}^1_0\,\bar\partial_J$, $\mathrm{Coker}^2_0\,\bar\partial_J$ it follows that

$$\begin{aligned}
\#_{12}(\overline{\mathcal{M}_1}^{\bar\nu_1^0} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2^0}^{\bar\nu_2^{01}}) &= \frac{1}{m_{12}} \cdot \#_1\overline{\mathcal{M}_1}^{\bar\nu_1^0} \cdot \#_2\overline{\mathcal{M}_2^0}^{\bar\nu_2^{01}} = 0,\\
\#_{12}(\overline{\mathcal{M}_1^0}^{\bar\nu_1^{01}} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2}^{\bar\nu_2^1}) &= \frac{1}{m_{12}} \cdot \#_1\overline{\mathcal{M}_1^0}^{\bar\nu_1^{01}} \cdot \#_2\overline{\mathcal{M}_2}^{\bar\nu_2^1} = 0,
\end{aligned}$$

where $\#_{12}$ refers to the induced orientations on $\pi_1^*\,\mathrm{Coker}^1\,\bar\partial_J \oplus \pi_2^*\,\mathrm{Coker}^2_0\,\bar\partial_J$. But since the algebraic counts $\#$ and $\#_{12}$ differ only by sign by proposition 2.2.7, it follows that

$$\#(\overline{\mathcal{M}_1}^{\bar\nu_1^0} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2^0}^{\bar\nu_2^{01}}) = 0, \quad \#(\overline{\mathcal{M}_1^0}^{\bar\nu_1^{01}} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2}^{\bar\nu_2^1}) = 0,$$

which proves the first part of the theorem.

It remains to prove $\chi(\overline{\mathrm{Coker}}\bar\partial_J) = 0$:

But for this we must only observe that the rank of $\overline{\mathrm{Coker}}\bar\partial_J$ is always odd, since it agrees with the dimension of $\mathcal{M}$, which itself is the product of a one-dimensional manifold with a complex manifold. Indeed, we have

$$\mathrm{rank}\,\overline{\mathrm{Coker}}\bar\partial_J = \dim\mathcal{M} = \dim(S^1 \times \mathcal{M}_{0,n}) = 2(n-3) + 1 \equiv 1 \mod 2.$$

Following the idea of proving the vanishing of the Euler characteristic for odd-dimensional closed manifolds, observe that for any coherent and transversal section $\bar\nu$ in $\overline{\mathrm{Coker}}\bar\partial_J$ the section $-\bar\nu$ has the same property and we have

$$\chi(\overline{\mathrm{Coker}}\bar\partial_J) = \sharp(-\bar\nu)^{-1}(0) = -\sharp\bar\nu^{-1}(0) = -\chi(\overline{\mathrm{Coker}}\bar\partial_J),$$

implying $\chi(\overline{\mathrm{Coker}}\bar\partial_J) = 0$. □

2.4 Consequences

2.4.1 Action filtration on rational symplectic field theory

In this section we want to discuss the implications of our main theorem on rational symplectic field theory. While we have seen that the problem of achieving regularity for moduli spaces already appears in the case of trivial curves, which we however have settled above using obstruction bundles, note that our method does not allow us to solve the problem for the other moduli spaces studied in rational symplectic field theory. Beside the fact that we cannot assume the nonregular moduli spaces to be manifolds in general, we further cannot assume that the cokernels of the linearizations of the Cauchy-Riemann operator fit together to give a vector bundle of the right rank over the nonregular moduli space. For this recall that we have proven the latter by a linearized energy argument in proposition 2.2.3 which is not available in the general case. In order to settle the transversality problem in symplectic field theory H. Hofer, K. Wysocki and E. Zehnder invented the theory of polyfolds, which however at the moment of writing this paper is still on its way of being completed. While our result about trivial curves in rational symplectic field theory is itself independent of the methods used to achieve regularity in the general case, let us outline how our result can be embedded in the general story:

While the most natural way consists in using our obstruction bundle perturbations for the moduli spaces of trivial curves and extending them via the polyfold theory to abstract perturbations for all other moduli spaces, we claim that the statement of the main theorem is true independent of the method used to define the coherent compact perturbations. In particular it should hold for the abstract perturbations constructed using the polyfold theory of [HWZ] as well as the domain-dependent Hamiltonian perturbations used in the first chapter. Since the analytical foundations of symplectic field theory are not yet established, we cannot make the above statement rigorous in full detail. However, let us point out the important consequences of our result to symplectic field theory of which we are confident that they can be shown once the analytical tools from polyfold theory are available. Despite the fact that we can not make them rigorous by the aforementioned reasons, we decided to state them as propositions with proofs as it is common in recent papers on symplectic field theory, see e.g. [B] and [EGH].

Proposition 2.4.1: *For all choices of coherent compact perturbations ν which make the perturbed Cauchy-Riemann operator $\bar{\partial}^{\nu}_J = \bar{\partial}_J + \nu$ transversal to the zero section in an appropriate Banach space bundle (or polyfold) setup, the algebraic count of elements in the resulting regular moduli space $\overline{\mathcal{M}}^{\nu} = (\bar{\partial}^{\nu}_J)^{-1}(0)$ is zero. It follows that branched covers over orbit cylinders do not contribute to the algebraic invariants of rational symplectic field theory.*

Proof: Here we proceed as in the proof of theorem 2.3.3 and prove the statement by induction on the number of punctures. For every moduli space of trivial curves $\mathcal{M}$ assume we are given an arbitrary coherent perturbations $\bar{\nu}^0$ and $\bar{\nu}^1$, constructed e.g. using the polyfold theory of [HWZ], which, after being added to $\bar{\partial}_J$, make all strata of the compactification $\overline{\mathcal{M}}$ regular. Using polyfold theory we can construct a compact perturbation $\bar{\nu}^{01}$ of $\overline{\mathcal{M}}^0$ so that, in the notation from before, the codimension one boundary strata of the resulting regular moduli space $\overline{\mathcal{M}^0}^{\bar{\nu}^{01}}$ are again given by

$$\begin{aligned}
\big(\overline{\mathcal{M}}^0_1 \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}}_2\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}^0_1}^{\bar{\nu}^{01}_1} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2}^{\bar{\nu}^1_2}, \\
\big(\overline{\mathcal{M}}_1 \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}^0_2}\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}_1}^{\bar{\nu}^0_1} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}^0_2}^{\bar{\nu}^{01}_2}, \\
\big(\{\text{point}\} \times \overline{\mathcal{M}}\big)^{\bar{\nu}^{01}} &= \{\text{point}\} \times \overline{\mathcal{M}}^{\bar{\nu}^1}, \\
\text{and} \quad \big(\overline{\mathcal{M}} \times \{\text{point}\}\big)^{\bar{\nu}^{01}} &= \overline{\mathcal{M}}^{\bar{\nu}^0} \times \{\text{point}\}.
\end{aligned}$$

In particular we again have

$$\begin{aligned}\#\overline{\mathcal{M}}^{\bar{\nu}^0} &- \#\overline{\mathcal{M}}^{\bar{\nu}^1} \\ &= \sum_{2<n_1,n_2<n} \Big(\#(\overline{\mathcal{M}_1^0}^{\bar{\nu}_1^{01}} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2}^{\bar{\nu}_2^1}) \;-\; \#(\overline{\mathcal{M}_1}^{\bar{\nu}_1^0} \times_{\mathbb{Z}_{m_{12}}} \overline{\mathcal{M}_2^0}^{\bar{\nu}_2^{01}})\Big).\end{aligned}$$

Using the induction hypothesis it follows as before that the right hand side of the equation is equal to zero, so that

$$\#\overline{\mathcal{M}}^{\bar{\nu}^0} \;=\; \#\overline{\mathcal{M}}^{\bar{\nu}^1},$$

i.e., the number of elements in the regular moduli space is independent of *any* choice of coherent compact perturbations. Assuming in particular that $\bar{\nu}^0$ is a coherent compact perturbation resulting from a section in the cokernel bundle $\overline{\mathrm{Coker}\bar{\partial}_J}$ as studied before, it follows that this number is zero. □

Like in Gromov-Witten theory and symplectic Floer homology the trivial curves in symplectic field theory can be characterized by the fact that they carry no energy in a certain sense, which, as in Floer homology, can be expressed as difference of actions assigned to the asymptotic periodic orbits. More precisely, we can introduce a natural action filtration on rational symplectic field theory as follows:

The action

$$S(\gamma) = \int f_\gamma^* \omega,$$

which we defined in 0.2 using the spanning surface f_γ for every closed Reeb orbit γ, naturally defines an action filtration $\mathcal{F}$ on the chain algebras $\mathfrak{A}$ and $\mathfrak{P}$ underlying contact homology and rational symplectic field theory. For this observe that over the group ring over $H_2(V)$ $\mathfrak{A}$ and $\mathfrak{P}$ are generated by the formal variables q_γ (and p_γ) assigned to every good orbit γ in the sense of [BM], so that for every monomial we can define

$$\mathcal{F}(q_{\gamma_1^-}...q_{\gamma_{n^-}^-} p_{\gamma_1^+}...p_{\gamma_{n^+}^+} e^A) := \sum_{k=1}^{n^-} S(\gamma_k^-) - \sum_{\ell=1}^{n^+} S(\gamma_\ell^+) + \omega(A).$$

Note that in the contact case, i.e., where the one-form λ of the Hamiltonian structure on V is contact and $\omega = d\lambda$, we have $\omega(A) = 0$ and the action for the periodic orbits γ, i.e., the closed Reeb orbits, is given by integrating the one-form λ along γ.

Corollary 2.4.2: *Like in cylindrical contact homology the differential in contact homology and rational symplectic field theory is strictly decreasing with respect to the action filtration.*

Proof: Since the differential $d = d^{\mathbf{h}} = \{\mathbf{h}, \cdot\} : \mathfrak{P} \to \mathfrak{P}$ in rational symplectic field theory, given by the generating function $\mathbf{h} \in \mathfrak{P}$ counting holomorphic curves in $\mathbb{R} \times V$, satisfies a graded Leibniz rule, it is strictly decreasing with respect to $\mathcal{F}$ precisely when for every orbit γ,

$$\begin{aligned} &\langle dp_\gamma, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle \neq 0 \quad \text{implies} \quad \mathcal{F}(p_\gamma) > \mathcal{F}(p^{\Gamma^+} q^{\Gamma^-} e^A) \\ \text{and} \quad &\langle dq_\gamma, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle \neq 0 \quad \text{implies} \quad \mathcal{F}(q_\gamma) > \mathcal{F}(p^{\Gamma^+} q^{\Gamma^-} e^A), \end{aligned}$$

where $\langle dp_\gamma, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle$ and $\langle dq_\gamma, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle$ denote the coefficients of

$$p^{\Gamma^+} q^{\Gamma^-} e^A = p_{\gamma_1^+} ... p_{\gamma_{n^+}^+} q_{\gamma_1^-} ... q_{\gamma_{n^-}^-} e^A$$

in the series expansion of dp_γ and dq_γ, respectively. On the other hand it follows from the definition of d that

$$\begin{aligned} \langle dp_\gamma, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle &= \langle\{\mathbf{h}, p_\gamma\}, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle \\ &= \kappa_\gamma \, \langle \frac{\partial \mathbf{h}}{\partial q_\gamma}, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle \\ &= \pm\kappa_\gamma \, \langle \mathbf{h}, p^{\Gamma^+} (q^{\Gamma^-} q_\gamma) e^A\rangle \end{aligned}$$

with the Hamiltonian $\mathbf{h} \in \mathfrak{P}$ of rational symplectic field theory, and similar for dq_γ, so that the requirement on d is equivalent to requiring that

$$\langle \mathbf{h}, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle \neq 0 \quad \text{implies} \quad \mathcal{F}(p^{\Gamma^+} q^{\Gamma^-} e^A) > 0.$$

Note that here we use $\mathcal{F}(q_\gamma) = -\mathcal{F}(p_\gamma)$. In order to see how this follows from the above proposition, recall that $\langle \mathbf{h}, p^{\Gamma^+} q^{\Gamma^-} e^A\rangle$ is given by the algebraic count of elements in the moduli space described by the monomial $p^{\Gamma^+} q^{\Gamma^-} e^A$, which consists of the curves which are asymptotically cylindrical over the orbits $\gamma_1^\pm, ..., \gamma_{n^\pm}^\pm$ at the positive, respectively negative punctures and represent the homology class $A \in H_2(V)$. On the other hand recall from 0.2 that the ω-energy of a holomorphic curve u in the moduli space can be expressed in terms of the actions of the closed orbits $\gamma_1^\pm, ..., \gamma_{n^\pm}^\pm$ and the integral of ω over the homology class $A \in H_2(V)$ by

$$E_\omega(u) = \sum_{k=1}^{n^+} S(\gamma_k^+) - \sum_{\ell=1}^{n^-} S(\gamma_\ell^-) + \omega(A),$$

i.e., $E_\omega(u) = \mathcal{F}(q_{\gamma_1^-} ... q_{\gamma_{n^-}^-} p_{\gamma_1^+} ... p_{\gamma_{n^+}^+} e^A)$. But since the algebraic count of curves in moduli spaces of curves with $E_\omega(u) = 0$ is zero by proposition 2.4.1, we get the desired result. $\square$

Recall that this statement is trivial in the case of cylindrical contact homology and symplectic Floer homology since the only trivial curves in these cases are trivial cylinders.

2.4.2 Marked points, differential forms and the spectral sequence for filtered complexes

Since trivial curves are characterized by the fact that they have trivial ω-energy and this quantity is preserved under taking boundaries and gluing of moduli spaces, it follows that every algebraic invariant of rational symplectic field theory has a natural analog defined by counting only those trivial curves. More precisely, observe that the generating function $\mathbf{h} \in \mathfrak{P}$ counting holomorphic curves in $(\mathbb{R} \times V, J)$ can be written as a sum $\mathbf{h} = \mathbf{h}_0 + \mathbf{h}_{>0}$ where $\mathbf{h}_0 \in \mathfrak{P}$ is the generating function for the curves with trivial ω-energy and $\mathbf{h}_{>0}$ the one for the curves with strictly positive ω-energy, which in turn immediately implies that also the differential $d = d^{\mathbf{h}} : \mathfrak{P} \to \mathfrak{P}$ is given as a sum $d = d_0 + d_{>0}$ with $d_0 = d^{\mathbf{h}_0}$, $d_{>0} = d^{\mathbf{h}_{>0}}$.

In the same way as we use the study of the boundaries of one-dimensional moduli spaces (after quotiening out the $\mathbb{R}$-action) to deduce the fundamental identity $\{\mathbf{h}, \mathbf{h}\} = 0$ implying $d^2 = 0$, it follows from the aforementioned fact that the ω-energy is preserved under taking boundaries and gluing of moduli spaces that we already have $\{\mathbf{h}_0, \mathbf{h}_0\} = 0$ and therefore $d_0^2 = 0$. Even further it is clear that we already have $\{\mathbf{h}_{0,\gamma}, \mathbf{h}_{0,\gamma}\} = 0$ where $\mathbf{h}_{0,\gamma}$ is the generating function counting all trivial curves over the closed Reeb orbit $\gamma \in P(V)$ so that $\mathbf{h}_0 = \sum_{\gamma \in P(V)} \mathbf{h}_{0,\gamma}$.

Denoting by $\mathfrak{P}_\gamma$ the graded Poisson subalgebra of $\mathfrak{P}$ generated only by the variables p_{γ^k}, q_{γ^k} assigned to multiple covers of the chosen Reeb orbit γ, observe that we have $\mathbf{h}_{0,\gamma} \in \mathfrak{P}_\gamma$ so that $d_{0,\gamma} = \{\mathbf{h}_{0,\gamma}, \cdot\}$ defines a differential on $\mathfrak{P}_\gamma$. We call its homology $H_*(\mathfrak{P}_\gamma, d_{0,\gamma})$ the *rational symplectic field theory of* γ.

While it follows from our main theorem that $h_{0,\gamma} = 0$ and therefore $H_*(\mathfrak{P}_\gamma, d_{0,\gamma}) = \mathfrak{P}_\gamma$ when no differential forms are chosen, let us spend the remaining time studying what can be said about the general case described in [EGH] when a string of closed differential forms is introduced:

To this end, let $\Theta = (\theta_1, ..., \theta_N) \in (\Omega^*(V))^N$ be a tuple of closed differential forms. Abbreviating $\gamma^{\vec{m}^\pm} = (\gamma^{m_1^\pm}, ..., \gamma^{m_{n^\pm}^\pm})$, note that on every moduli space $\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})$ of trivial curves with additional r marked points $\underline{w} = (w_1, ..., w_r) \in \dot{S}^r$ we have r evaluation maps

$$\mathrm{ev}_i : \mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R} \to V, \ \ i = 1, ..., r$$

given by mapping the tuple $(h, j, \mu, \underline{w}) \in \mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R}$ to $h(w_i) \in V$, which extend to the compactified moduli space $\overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R}}$. Since we still cannot expect the moduli space $\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})$ to be transversally cut out by the Cauchy-Riemann operator, we must proceed as before and choose coherent sections $\bar{\nu}$ in the cokernel bundles $\mathrm{Coker}\bar{\partial}_J$ over the compactified moduli spaces $\overline{\mathcal{M}} = \overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R}}$ to obtain the regular moduli spaces

$$\overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R}}^{\bar{\nu}} = \bar{\nu}^{-1}(0) \subset \overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+}, \gamma^{\vec{m}^-})/\mathbb{R}}.$$

Assigning to each chosen differential form $\theta_i \in \Omega^*(V)$ a graded formal variable t_i with $\deg t_i = \deg\theta_i - 2$ and abbreviating $p_m = p_{\gamma^m}$ and $q_m = q_{\gamma^m}$ we let $\mathfrak{P}_\gamma$ be the graded Poisson algebra of formal power series in the variables p_m with coefficients which are polynomials in the q_m's and formal power series of the t_i's. Following [EGH] we define the generating function $h_{0,\gamma} \in \mathfrak{P}_\gamma$ by

$$\mathbf{h}_{0,\gamma} = \sum_{\vec{m}^\pm,\vec{i}} \frac{1}{n^+!n^-!r!} \int_{\overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})/\mathbb{R}}^{\vec{\nu}}} \mathrm{ev}_1^*\,\theta_{i_1} \wedge ... \wedge \mathrm{ev}_r^*\,\theta_{i_r} \quad p_{\vec{m}^+} q_{\vec{m}^-} t_{\vec{i}}.$$

Theorem 2.4.3: *For chosen string of closed differential forms $\Theta = (\theta_1, ..., \theta_N) \in (\Omega^*(V))^N$ the generating function $\boldsymbol{h}_{0,\gamma} \in \mathfrak{P}_\gamma$ is given by*

$$\mathbf{h}_{0,\gamma} = \sum_{i:\deg\theta_i=1} \sum_{m\in\mathbb{N}} m \int_\gamma \theta_i \;\cdot\; p_m q_m t_i.$$

Proof: Since the positions of the marked points are not fixed, it follows that the dimension of the regular moduli space $\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})^\nu$ is given by $2r$ plus the dimension of the underlying regular moduli space $\mathcal{M}_{0,0}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})^\nu$ with no additional marked points. In the case of true branched covers, i.e., $\mathcal{M}_{0,0}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-}) \neq \mathcal{M}_{0,0}(\gamma^m,\gamma^m)$ it follows that $\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})^\nu$ has dimension greater or equal to $2r+1$. In other words, the top stratum of $\overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})/\mathbb{R}}^{\vec{\nu}}$ has dimension greater or equal to $2r$, which in turn must agree with the degree of the differential form $\mathrm{ev}_1^*\,\theta_{i_1} \wedge ... \wedge \mathrm{ev}_r^*\,\theta_{i_r}$ in order to get a nonzero contribution to $\mathbf{h}_{0,\gamma}$. In particular, at least one differential form θ_{i_k}, $k \in \{1,...,r\}$ must have degree greater or equal to two. On the other hand, observing that the image of the evaluation map ev_k from $\overline{\mathcal{M}_{0,0,r}(\gamma^{\vec{m}^+},\gamma^{\vec{m}^-})/\mathbb{R}}$ to V is clearly contained in the closed Reeb orbit γ and that the pullback of a form on V under the inclusion map $\gamma \hookrightarrow V$ is nonzero only for forms of degree zero or one, it follows that $\mathrm{ev}_k^*\,\theta_{i_k} = 0$. So, while we have shown in this paper that moduli spaces of true branched covers without additional marked points do not contribute to the generating function $\mathbf{h}_{0,\gamma}$, it follows from the last observation that this remains true when we introduce additional marked points and differential forms by simple topological reasons. Finally, observe that for moduli spaces of trivial cylinders the top stratum of $\overline{\mathcal{M}_{0,0,r}(\gamma^m,\gamma^m)/\mathbb{R}}$ has dimension $2r-1$, so that here we might get nonzero contributions from moduli spaces with one additional marked point if the corresponding differential form has degree one. Since the moduli spaces of trivial cylinders are automatically regular, it is easily seen that this contribution is given by integrating the one-form along the closed Reeb orbit γ. □

Observe that the generating function is in general no longer equal to zero when a string of differential forms is chosen, which implies that the differential in rational symplectic field theory and contact homology is no longer strictly decreasing with respect to the action filtration, where we have set $\mathcal{F}(t_i) = 0$ for each formal variable t_i. However, in order to show how theorem 2.4.3 can be used to compute SFT invariants, we follow [FOOO] in employing the spectral sequence for filtered complexes, where for simplicity we restrict our attention only to the computation of the contact homology for contact manifolds and symplectic mapping tori. Recall from the introduction that in both cases the contact homology is indeed well-defined.

Corollary 2.4.4: *Let V be a contact manifold or a symplectic mapping torus. Then there exists a spectral sequence (E^r,d^r) computing the contact homology, $E^\infty = H_*(\mathfrak{A},\partial)$, where the E^2-page is given by the graded commutative algebra $\mathfrak{A}_0$ which, in contrast to $\mathfrak{A}$, is now only freely generated by the formal variables q_γ with $\int_\gamma \theta_i = 0$ for all $i=1,...,N$.*

Proof: First observe that it follows from the theorem of Arzela-Ascoli that for any given maximal period $T>0$ the set of closed Reeb orbits of period $\leq T$ is compact. Together with the assumption that the contact one-form λ is chosen generically in the sense that every closed orbit is nondegenerate and hence isolated, it follows that the number of closed orbits with period less or equal T is finite for every $T>0$, so that, in particular, the action spectrum $\{\int_\gamma \lambda : \gamma \in P(V)\}$ is a discrete subset of $\mathbb{R}^+$. Note that this automatically implies that the set of action values $\mathcal{F}(q^\Gamma) \in \mathbb{R}^+$, $\Gamma \subset P(V)$ is discrete, and hence can be identified with the discrete set $\{a_1,a_2,...\} \subset \mathbb{R}^+$ with $a_k \leq a_{k+1}$.

Using this we equip the chain complex $(\mathfrak{A},\partial)$ underlying contact homology with a filtration $(\mathcal{F}^k\,\mathfrak{A})_{k\in\mathbb{N}}$ by requiring that $\mathcal{F}^k\,\mathfrak{A}$ is spanned by monomials q^Γ with $\mathcal{F}(q^\Gamma) \leq a_k$. Note that it follows from the fact all curves have nonnegative contact area that the differential is indeed respecting the filtration, $\partial : \mathcal{F}^k\,\mathfrak{A} \to \mathcal{F}^k\,\mathfrak{A}$. Now we can use as in [FOOO] the spectral sequence (E^r,d^r) for filtered complexes to compute the homology of $(\mathfrak{A},\partial)$. In order to see how the theorem implies the corollary it suffices to observe that the differential $d^1 : E^1_{k,\ell} \to E^1_{k,\ell-1}$ agrees with the part ∂_0 of the differential $\partial : \mathfrak{A} \to \mathfrak{A}$, which is counting only curves with zero contact area, i.e., trivial curves. Hence $E^2 = H_*(\mathfrak{A},\partial_0)$ and it is easily deduced from the fact that ∂_0 satisfies the Leibniz rule that the latter agrees with $\mathfrak{A}_0$ as defined above.

On the other hand, for symplectic mapping tori one can use as in the first chapter the splitting of the chain complex with respect to the total period and again use the compactness of the set of closed orbits of bounded period to get discreteness of the action spectrum. □

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Erklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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