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Motives of projective homogeneous varieties

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ZUSAMMENFASSUNG

Das Hauptthema dieser Arbeit sind lineare algebraische Gruppen G, projektive G-homogene Varietäten (getwistete Flaggenvarietäten) und deren Chow Motive mit $\mathbb{Z}$-Koeffizienten. Wir untersuchen das Zerlegungsverhalten der Objekten dieser Kategorie und interessieren uns insbesondere für die Frage unter welchen Umständen zwei Objekte dieser Kategorie isomorph sind. Von besonderer Bedeutung sind für uns die Zerlegungen der Ausnahmevarietäten, wobei die verallgemeinerten Rost Motive auftreten. Mittels dieser Zerlegungen untersuchen wir u.a. die Chow Gruppen dieser Varietäten.

Eine der wichtigsten Hilfsmittel in den Beweisen von unseren Resultate sind Hasse Diagramme. Diese Diagramme erlauben es das Rechnen in den Chow Ringen von projektiven homogenen Varietäten zu visualisieren und erwiesen sich dadurch als ein effizientes Instrument in der Theorie der Chow Motive. Eine wichtige Rolle spielen auch die Zerlegungseigenschaften der einfachen algebraischen Gruppen, deren Tits Algebren, sowie Rost's Nilpotenzsatz.

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Die wichtigsten und die interessantesten Ergebnisse sind die Folgenden:

Theorem. *Das Krull-Schmidt Theorem gilt nicht in der Kategorie der Chow Motive* $\mathcal{M}(\mathrm{PGL}_1(A),\mathbb{Z})$*, wobei* A *eine zentral einfache Algebra vom Grad* 5 *ist.*

Theorem. *Seien* X *und* Y *zwei nicht isomorphe getwistete Flaggenvarietäten von der Dimension kleiner oder gleich* 5 *vom inneren Typ über einem Körper* k *der Charakteristik ungleich* 2*, dessen Chow Motive isomorph sind.*

1. *Angenommen es ist* $X_s := X \times_k k_s \simeq Y_s := Y \times_k k_s$*. Dann gilt: Entweder*
   (a) $X \simeq \mathrm{SB}(A)$ *und* $Y \simeq \mathrm{SB}(A^{\mathrm{op}})$ *sind Severi-Brauer Varietäten, die einer zentral einfachen Algebra* A *und ihrer Opposite-Algebra* A^{op} *entsprechen, wobei* $\deg(A) = 3, 4, 5, 6$ *und* $\exp(A) > 2$*, oder*
   (b) $X \simeq \mathrm{SB}_{2,3}(A)$ *und* $Y \simeq \mathrm{SB}_{2,3}(A^{\mathrm{op}})$*, wobei die zentral einfache Algebra* A *den Grad* 4 *und den Exponenten* 4 *hat.*
2. *Angenommen es ist* $X_s \not\simeq Y_s$*. Dann gilt: Entweder*
   (a) $X \simeq \mathbb{P}^n$ *und* $Y \simeq Q^n$ *für ungerade* $1 < n \leq 5$*, oder*

(b) $X \simeq \mathrm{SB}_{1,3}(A)$ *und* $Y \simeq \mathrm{SB}_{2,3}(A')$, *wobei* $\deg(A) = 4$ *und* $A \simeq A'$ *oder* A'^{op}, *oder*

(c) $X \simeq {}_\xi(G/P_1)$ *und* $Y \simeq {}_\xi(G/P_2)$ *sind die getwisteten Formen von den Varietäten* G/P_i, $i = 1, 2$, *wobei* G *eine Ausnahmegruppe vom Typ* G_2 *ist und die* P_i *eine ihrer maximalen parabolischen Untergruppen sind, oder*

(d) $X \simeq \mathrm{G}_2/P_2$ *und* $Y \simeq \mathbb{P}^5$.

Theorem. *Sei* k *ein Körper der Charakteristik ungleich* 2 *und* 3. *Sei* X *eine projektive* G*-homogene Varietät über* k, *wobei* G *eine anisotrope Gruppe vom Typ* F_4 *ist, die aus der 1. Tits-Konstruktion hervorgeht. Angenommen, dass über einem separablen Abschluß* X *zu* G_s/P *isomorph ist, wobei* P *die maximale parabolische Untergruppe ist, die den ersten (letzten) drei Ecken des Dynkin Diagramms* F_4 *entspricht. Dann gilt die folgende Zerlegung des Chow Motivs von* X *mit* $\mathbb{Z}$*-Koeffizienten*

$$\mathcal{M}(X) \cong \oplus_{i=0}^{7} R(i),$$

wobei das Motiv $R = (X, p)$ *ein verallgemeinertes Rost Motiv mit* $\mathbb{Z}$*-Koeffizienten ist, d.h. dass es sich über einem separablen Abschluß* k_s *von* k *als die direkte Summe von Lefschetz Motiven* $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ *darstellen lässt.*

Theorem. *Angennomen wir sind in der Situation des letzten Theorems. Seien* X_1 *und* X_2 *zwei projektive homogene Varietäten, die den ersten drei bzw. den letzten drei Ecken des Dynkin Diagramms entsprechen. Dann sind die Motive von* X_1 *und* X_2 *isomorph.*

Theorem. *Sei* G *eine anisotrope Gruppe vom Typ* F_4, *die mit Hilfe der 1. Tits-Konstruktion entstand. Sei* X *eine projektive homogene Varietät, die über einem algebraischen Abschluß zu* G_s/P_4 *isomorph ist, wobei* P_4 *die parabolische Untegruppe von* G_s *ist, die den ersten drei Ecken des Dynkin Diagramms* F_4 *entspricht. Dann hat die Gruppe* $\mathrm{CH}^*(X)$ *Torsion in der Kodimension* 13 *(Dimension* 2*).*

Theorem. *Seien* A *eine zentral einfache Algebra vom Grad* 3 *über einem Körper* k, $c \in k^*$, *und* $D = D(A, c)$ *eine Varietät, die durch Galois Abstieg von der Varietät*

$$\{\alpha \oplus \beta \in (A \oplus A)_s \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \mathrm{Nrd}(\alpha) = c\,\mathrm{Nrd}(\beta)\}/\mathrm{GL}_1(A_s),$$

entstanden ist, wobei $\mathrm{GL}_1(A_s)$ *auf* $A_s \oplus A_s$ *durch die Linksmultiplikation wirkt. Dann gilt*

$$\mathcal{M}(D) \simeq R \oplus (\oplus_{i=1}^{5} R'(i)),$$

wobei R *ein Motiv ist, das über einem algebraischen Abschluß zu* $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ *isomorph ist und* $R' \simeq \mathcal{M}(\mathrm{SB}(A))$.

SUMMARY

The main topic of our investigations are linear algebraic groups G, projective G-homogeneous varieties (twisted flag varieties), and their Chow motives with $\mathbb{Z}$-coefficients. We investigate decompositions and isomorphism criteria in this category. Of particular impotance are for us the motivic decompositions of exceptional varieties, where the generalized Rost motives appear. Using these decompositions we investigate the Chow groups of these varieties.

One of the main ingredients of the proof of our results is the usage of Hasse diagrams. These diagrams allow visualizing of the calculations in the Chow rings of projective homogeneous varieties and turn out to be a very efficient tool in the theory of Chow motives. Further important ingredients of the proofs are the splitting properties of simple algebraic groups, their Tits algebras, and the Rost nilpotence theorem.

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The most important and interesting results are the following ones:

Theorem. *The Krull-Schmidt theorem fails in the category of Chow motives* $\mathcal{M}(\mathrm{PGL}_1(A),\mathbb{Z})$, *where A is a central simple division algebra of degree* 5.

Theorem. *Let X and Y be non-isomorphic twisted flag varieties of dimension less than or equal to* 5 *of inner type over a field k of characteristic not* 2, *which have isomorphic Chow motives.*

1. *If* $X_s := X \times_k k_s \simeq Y_s := Y \times_k k_s$, *then either*
   - (a) $X \simeq \mathrm{SB}(A)$ *and* $Y \simeq \mathrm{SB}(A^{\mathrm{op}})$ *are Severi-Brauer varieties corresponding to a central simple algebra A and its opposite* A^{op}, *where* $\deg(A) = 3, 4, 5, 6$ *and* $\exp(A) > 2$, *or*
   - (b) $X \simeq \mathrm{SB}_{2,3}(A)$ *and* $Y \simeq \mathrm{SB}_{2,3}(A^{\mathrm{op}})$, *where the central simple algebra A has degree* 4 *and exponent* 4.
2. *If* $X_s \not\simeq Y_s$, *then either*
   - (a) $X \simeq \mathbb{P}^n$ *and* $Y \simeq Q^n$ *for odd* $1 < n \leq 5$, *or*
   - (b) $X \simeq \mathrm{SB}_{1,3}(A)$ *and* $Y \simeq \mathrm{SB}_{2,3}(A')$, *where* $\deg(A) = 4$ *and* $A \simeq A', A'^{\mathrm{op}}$, *or*
   - (c) $X \simeq {}_{\xi}(G/P_1)$ *and* $Y \simeq {}_{\xi}(G/P_2)$ *are the twisted forms of the variety* G/P_i, $i = 1, 2$, *where G is an exceptional group of type* G_2 *and* P_i *is one of its maximal parabolic subgroups, or*

(d) $X \simeq \mathrm{G}_2/P_2$ *and* $Y \simeq \mathbb{P}^5$.

Theorem. *Let k be a field of characteristic different from 2 and 3. Let X be a projective G-homogeneous variety over k, where G is an anisotropic group of type F_4 obtained by the first Tits process, such that over a separable closure it becomes isomorphic to G_s/P, where P is the maximal parabolic subgroup corresponding to the first (last) three vertices of the respective Dynkin diagram. Then the (integral) Chow motive of X decomposes as*

$$\mathcal{M}(X) \cong \oplus_{i=0}^{7} R(i),$$

where the motive $R = (X, p)$ is the (integral) generalized Rost motive, i.e., over a separable closure k_s of k it splits as the direct sum of Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

Theorem. *Under the hypotheses of the previous theorem let X_1 and X_2 be two projective homogeneous varieties corresponding to the maximal parabolic subgroups generated by the last (first) three vertices of the Dynkin diagram respectively. Then the motives of X_1 and X_2 are isomorphic.*

Theorem. *Let G be an anisotropic group of type F_4 of the 1st Tits process. Consider the projective homogeneous variety X such that over a separable closure it becomes isomorphic to G_s/P_4, where P_4 is the standard parabolic subgroup of G_s, corresponding to the first three vertices of the Dynkin diagram F_4 (we follow the enumeration of Bourbaki). Then the group $\mathrm{CH}^*(X)$ has torsion in codimension 13 (dimension 2).*

Theorem. *Let A denote a central simple algebra of degree 3 over a field k, $c \in k^*$, and $D = D(A, c)$ denote a variety obtained by Galois descent from the variety*

$$\{\alpha \oplus \beta \in (A \oplus A)_s \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \mathrm{Nrd}(\alpha) = c\,\mathrm{Nrd}(\beta)\}/\mathrm{GL}_1(A_s),$$

where $\mathrm{GL}_1(A_s)$ acts on $A_s \oplus A_s$ by the left multiplication. Then

$$\mathcal{M}(D) \simeq R \oplus (\oplus_{i=1}^{5} R'(i)),$$

where R is a motive such that over a separably closed field it becomes isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ and $R' \simeq \mathcal{M}(\mathrm{SB}(A))$.

Contents

1 Introduction

The present thesis is devoted to study of projective homogeneous varieties and their Chow motives.

One of the motivations for this problem is the recent progress achieved in proving celebrated conjectures relating Galois cohomology and Milnor K-theory. Namely, Milnor's conjecture was proven and proofs of the Bloch-Kato conjecture were proposed by Voevodsky, Rost, and Suslin in the series of papers ([SV96], [Ro98], [SV99], [FSV00], [Vo01], [Vo03]). One of the main ingredients of those proofs are norm varieties and their motivic decompositions. Observe that the general construction of norm varieties provided by Rost is very implicit as well as their motivic decompositions. An attempt to describe explicitely some norm varieties and their motivic decompositions is made in the thesis (see Chapters 5, 7 and 9).

The text is organized as follows. Chapter 2 describes background information on motivic categories and Chow rings. In Chapter 3 we introduce the notion of a Hasse diagram and explain its connection with Chow rings. We also translate the result of [CGM] about a decomposition of the motive of an isotropic projective homogeneous variety into this framework. Chapter 4 deals with motivic isomorphisms in the category of projective homogeneous varieties in the completely split case. The result of this Chapter was obtained in the seminar "Motivic decompositions of projective homogeneous varieties" taken place in Bielefeld University, 2004. I use it in Chapter 6, so I decided to put it into my thesis.

The main purpose of Chapter 5 is to express the Chow motive of a twisted flag variety in terms of motives of "minimal" flags, i.e., those G-homogeneous varieties that correspond to maximal parabolic subgroups of G. As a by-product, a counter-example to the uniqueness of a direct sum decomposition in the category of Chow motives with integral coefficients is provided. The results of this Chapter are joint work with B. Calmès, V. Petrov, and K. Zainoulline.

Chapter 6 can be viewed as a further application of the methods and results obtained by N. Karpenko [Ka00]. Namely, we give a complete classification of motivic isomorphisms of projective homogeneous varieties of inner type of dimension up to 5. In this Chapter the results of the previous one play a crucial role. The results of this Chapter is a joint work with K. Zainoulline.

In Chapter 7 we provide a shortened and explicit construction of a generalized Rost motive for a norm variety that corresponds to a symbol $(3, 3)$. By the next result, we provide the first known "purely exceptional" example of two non-isomorphic varieties with isomorphic motives. The results of this Chapter is a joint work with S. Nikolenko and K. Zainoulline.

In Chapter 8 we investigate the torsion part of the Chow group of some F_4-variety. We find a torsion element using motivic decomposition of the previous Chapter.

The last Chapter is devoted to certain twisted forms of a smooth hyperplane section of $\mathrm{Gr}(3, 6)$. These varieties have a lot of interesting geometrical properties. We provide a complete decomposition of the Chow motives of these varieties.

The present thesis is a very natural continuation of the celebrated papers of O. Izhboldin, N. Karpenko, A. Merkurjev, M. Rost, A. Vishik and others (see references).

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2 Category of Chow motives

2.1 Preliminaries

In the present section, we introduce the category of Chow motives following [Ma68]. We formulate and prove the Grassmann Bundle Theorem. At the end we recall the notion of a functor of points following [Ka01, section 8] and provide some examples.

2.1 (Chow motives)**.** Let k be a field and $\mathcal{V}ar_k$ be the category of smooth projective varieties over k. We define the category $\mathcal{C}or_k$ of *correspondences* over k. Its objects are smooth projective varieties over k. As morphisms, called correspondences, we set $\mathrm{Mor}(X,Y) := \coprod_{l=1}^{n} \mathrm{CH}_{d_l}(X_l \times Y)$, where $X_1, \dots, X_n$ are the irreducible components of X of dimensions $d_1, \dots, d_n$. For any two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$

$$\beta \circ \alpha = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)),$$

where pr_{ij} denotes the projection on the i-th and j-th factors of $X \times Y \times Z$ respectively and ${\mathrm{pr}_{ij}}_*$, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups. Observe that the composition $\circ$ induces the ring structure on the abelian group $\mathrm{End}_{\mathcal{M}}(X)$. The unit element of this ring is the class of the diagonal Δ_X.

The pseudo-abelian completion of $\mathcal{C}or_k$ is called the category of *Chow motives* and is denoted by $\mathcal{M}_k$. The objects of $\mathcal{M}_k$ are pairs (X,p), where X is a smooth projective variety and $p \in \mathrm{Mor}(X,X)$ is a projector, that is, $p \circ p = p$. The motive (X, Δ_X) will be denoted by $\mathcal{M}(X)$.

2.2. By construction, $\mathcal{M}_k$ is a tensor additive category, where the tensor product is given by the usual product $(X,p) \otimes (Y,q) = (X \times Y, p \times q)$. For any cycle α we denote by α^t the corresponding transposed cycle. Moreover, the Chow functor $\mathrm{CH}\colon \mathcal{V}ar_k \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_k$, i.e., one has the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{V}ar_k & \xrightarrow{\mathrm{CH}} & \mathbb{Z}\text{-}\mathcal{A}b \\ & {\scriptstyle\Gamma}\searrow \quad \nearrow{\scriptstyle r} & \\ & \mathcal{M}_k & \end{array}$$

where $\Gamma\colon f \mapsto \Gamma_f$ is the graph and $r\colon (X,p) \mapsto \mathrm{Im}(p_\star)$ is the realization.

2.3. Consider the morphism $(e, \mathrm{id})\colon \{pt\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. The image by means of the induced push-forward $(e, \mathrm{id})_*(1)$ doesn't depend on the choice of a point $e\colon \{pt\} \to \mathbb{P}^1$ and defines a projector in $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1. The motive $\mathbb{Z}(1) := L := (\mathbb{P}^1, p_1)$ is called the Lefschetz motive. For a motive M and a nonnegative integer i we denote by $M(i) = M \otimes L^{\otimes i}$ its twist.

We will extensively use the following fact that easily follows from Manin's Identity Principle [Ma68, p. 450] and the Grassmann Bundle Theorem for Chow groups [Ful].

2.4 Proposition (Grassmann Bundle Theorem)**.** *Let X be a variety over k and $\mathcal{E}$ be a vector bundle over X of rank n. Then the motive of the Grassmann bundle* $\mathrm{Gr}(d, \mathcal{E})$ *over X is isomorphic to*

$$\mathcal{M}(\mathrm{Gr}(d, \mathcal{E})) \simeq \bigoplus_{\lambda} \mathcal{M}(X)(d(n-d) - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $n - d \geq \lambda_1 \geq \ldots \geq \lambda_d \geq 0$.

Proof. We follow the notation of [Ma68]. Denote $\mathrm{Gr}(d, \mathcal{E})$ by Y and the canonical projection of Y to X by ψ. It is known (see [Ful, Prop. 14.6.5]) that $\mathrm{CH}^*(Y)$ as $\mathrm{CH}^*(X)$-module (via ψ^*) has a basis consisting of elements $\Delta_\lambda \in \mathrm{CH}^{|\lambda|}(Y)$ parameterized by partitions λ. For any partition λ denote by λ^{op} the partition defined by $\lambda_i^{\mathrm{op}} = n - d - \lambda_{d+1-i}$. Then for every two partitions λ and μ with $|\lambda| + |\mu| \leq d(n-d)$ and any element $a \in \mathrm{CH}^*(X)$ the following duality formula holds (see [Ful, Prop. 14.6.3]):

$$\psi_*(\Delta_\lambda \Delta_\mu \psi^*(a)) = \begin{cases} a, \ \mu = \lambda^{\mathrm{op}}, \\ 0, \ \text{otherwise}. \end{cases}$$

For every partition λ, we define an element f_λ of $\mathrm{End}(\mathcal{M}(Y))$ inductively. For the unique partition $\lambda_{\max}$ with $|\lambda_{\max}| = d(n-d)$ set $f_{\lambda_{\max}} = c(\psi) \circ c(\psi)^t$. Now, by the decreasing induction on $|\lambda|$, set

$$f_\lambda = f_{\lambda_{\max}} \circ c_{\Delta_{\lambda^{\mathrm{op}}}} \circ (\Delta_Y - \sum_{|\mu|>|\lambda|} c_{\Delta_\mu} \circ f_\mu).$$

Finally, set $p_\lambda = c_{\Delta_\lambda} \circ f_\lambda$.

Now, the duality formula implies that

$$(p_\lambda)_e(\sum_\mu \Delta_\mu \psi^*(a_\mu)) = \Delta_\lambda \psi^*(a_\lambda).$$

Therefore, by Manin's identity principle, the p_λ form a complete orthogonal system of projectors. The identification of their images with twisted motives of X can be done as in [Ma68, §7] and we omit it. □

2.5 (Functors of Points)**.** In sections 5.3, 5.5 and 5.6 we use the functorial language, that is consider k-schemes as functors from the category of k-algebras to the category of sets. Fix a scheme X. By an X-algebra we mean a pair (R, x), where R is a k-algebra and x is an element of $X(R)$. X-algebras form a category with obvious morphisms. The morphisms $\varphi\colon Y \to X$ can be considered as the functors from the category of X-algebras to the category of sets, by sending a pair (R, x) to its preimage in $Y(R)$.

2.6. Let X be a variety over k. To any vector bundle $\mathcal{F}$ over X we can associate the Grassmann bundle $Y = \mathrm{Gr}(d, \mathcal{F})$. Fix an X-algebra (R, x). The value of the functor corresponding to $\mathrm{Gr}(d, \mathcal{F})$ at (R, x) is the set of direct summands of rank d of the projective R-module $\mathcal{F}_x \otimes_k R$ (cf. [Ka01, section 9]), where $\mathcal{F}_x = \mathcal{F}(R, x)$.

Next, we need to recall some properties of rational cycles on projective homogeneous varieties.

2.7. Let G be a split linear algebraic group over k. Let X be a projective G-homogeneous variety, i.e., $X = G/P$, where P is a parabolic subgroup of G. The abelian group structure of $\mathrm{CH}(X)$, as well as its ring structure, is well-known. Namely, X has a cellular filtration and the generators of the Chow groups of the bases of this filtration correspond to the free additive generators of $\mathrm{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \simeq \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two homogeneous cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{CH}(Y \times X)$ is given by (cf. [Bo03, Lem. 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta),$$

where $\deg\colon \mathrm{CH}(Y) \to \mathrm{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

2.8. From now on we assume that all varieties under consideration are irreducible. Moreover, in view of the duality given by the transposition of cycles, we freely switch between covariant and contravariant notation for the Chow motives.

Let X be a projective variety of dimension n over a field k. Let k_s be a separable closure of the field k. Consider the scalar extension $X_s = X \times_k k_s$. We say a cycle $J \in \mathrm{CH}(X_s)$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X_s)$. For instance, there is an obvious rational cycle Δ_{X_s} on $\mathrm{CH}^n(X_s \times X_s)$ that is given by the diagonal class. Clearly, all linear combinations, intersections and correspondence products of rational cycles are rational.

2.9. We will use the following fact (see [CGM, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let X be a twisted flag variety and p_s be a non-trivial rational projector in $\mathrm{CH}^n(X_s \times X_s)$, i.e., $p_s \circ p_s = p_s$. Then there exists a non-trivial projector p on $\mathrm{CH}^n(X \times X)$ such that $p \times_k k_s = p_s$. Hence, the existence of a non-trivial rational projector p_s on $\mathrm{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, \Delta_X - p).$$

2.10. Observe that

$$\mathrm{Mor}((X,p)(m), (Y,q)(l)) = q \circ \mathrm{CH}_{\dim X + m - l}(X, Y) \circ p.$$

An isomorphism between twisted motives $(X,p)(m)$ and $(Y,q)(l)$ is given by correspondences $j_1 \in q \circ \mathrm{CH}_{\dim X+m-l}(X \times Y) \circ p$ and $j_2 \in p \circ \mathrm{CH}_{\dim Y + l - m}(Y \times X) \circ q$ such that $j_1 \circ j_2 = q$ and $j_2 \circ j_1 = p$. If X and Y are twisted flag varieties then by the Rost nilpotence theorem (see [CM06, Theorem 8.2] and [CGM, Corollary 8.4]) it suffices to give a rational j_1 and some j_2 satisfying these conditions over a separable closure (note that j_2 will automatically be rational).

2.2 Rational cycles on projective homogeneous varieties

Several techniques are available to produce rational cycles. We shall use the following:

(i) Consider a variety Y and a morphism $X \to Y$ such that $X_s = Y_s \times_Y X$, where $Y_s = Y \times_k k_s$. Then any rational cycle on $\mathrm{CH}(Y_s)$ gives rise to a rational cycle on $\mathrm{CH}(X_s)$ by the induced pull-back $\mathrm{CH}(Y_s) \to \mathrm{CH}(X_s)$.

(ii) Consider a variety Y and a projective morphism $Y \to X$ such that $Y_s = X_s \times_X Y$. Then any rational cycle on $\mathrm{CH}(Y_s)$ gives rise to a rational cycle on $\mathrm{CH}(X_s)$ by the induced push-forward $\mathrm{CH}(Y_s) \to \mathrm{CH}(X_s)$.

(iii) Let X and Y be projective homogeneous varieties over k that split completely over the function fields $k(Y)$ and $k(X)$ respectively. Consider the following diagram

$$\begin{array}{ccc} \mathrm{CH}^i(X \times Y) & \xrightarrow{g} & \mathrm{CH}^i(X_s \times Y_s) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_s} \\ \mathrm{CH}^i(X_{k(Y)}) & \xrightarrow{=} & \mathrm{CH}^i(X_{k_s(Y_s)}) \end{array}$$

where the vertical arrows are surjective by [IK00, §5]. Now take any cycle $\alpha \in \mathrm{CH}^i(X_s \times Y_s)$, $i \le \dim X$. Let $\beta = g(f^{-1}(f_s(\alpha)))$. Then $f_s(\beta) = f_s(\alpha)$ and β is rational. Hence, $\beta = \alpha + J$, where $J \in \mathrm{Ker} f_s$, and we conclude that $\alpha + J \in \mathrm{CH}^i(X_s \times Y_s)$ is rational.

3 Hasse diagrams

3.1 Notation

Let k be a field, G a Chevalley group over k, i.e., a split algebraic (semi)simple group defined over k.

Consider an irreducible linear representation $\pi\colon G \to \mathrm{GL}(V)$ of G on a k-vector space V. Let T be a split maximal torus of G defined over k. An element $\lambda \in T^* = \mathrm{Hom}(T, k^*)$ is called a character. If

$$V^\lambda = \{v \in V \mid \forall\, t \in T\ \pi(t)v = \lambda(t)v\} \neq 0,$$

then it is called a weight space and the corresponding character λ is called a weight.

By definition the dimension of V^λ is the multiplicity of λ. As $\overline{\Lambda}(\pi)$ we denote the set of all weights of π. We also use the following notation: $\Lambda(\pi)$ the set of all weights of π with multiplicities, $\overline{\Lambda}^*(\pi)$ the set of all nonzero weights, and $\Lambda^*(\pi)$ the set of all nonzero weights with multiplicities.

If $\pi = \mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$ is the adjoint representation of G, then $\overline{\Lambda}^*(\mathrm{Ad}) =: \Phi$ is the root system of G. We choose a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ ($l = \mathrm{rk}\,\Phi = \dim T$) of Φ.

There exists a highest weight ω such that any other weight of π has the form $\omega - \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Z}^{\geq 0}$ (in additive notation).

We construct a labelled graph which is called a weight diagram as follows. Its vertices are the elements of $\Lambda(\pi)$. Two vertices λ and μ are connected by an edge going from μ to λ with a label i iff $\lambda - \mu = \alpha_i$.

The following fact is well known.

3.1 (Chevalley)**.** The representation π is uniquely determined by the set of its weights or by its highest weight.

By $x_\alpha(\xi)$ we denote the elementary root unipotents of G, or the elementary Steinberg generators ($\alpha \in \Phi$, $\xi \in k$). Consider

$$w_\alpha(\varepsilon) := x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon) \in G, \quad \varepsilon \in k^*.$$

The group $\widetilde{W} := \langle w_\alpha(1), \alpha \in \Phi(\text{or } \Pi)\rangle$ is the extended Weyl group.

The ordinary Weyl group W acts naturally on the set of weights of the representation π. We say that π is a microweight representation, if all its weights lie in one orbit under this action. There exists *a weight basis* $\{v^\lambda\}_{\lambda\in\Lambda(\pi)}$ of V. It has several nice properties. E.g., for all $\lambda \in \Lambda^*(\pi)$ and for all $w \in \widetilde{W}$ there exists $\nu \in \Lambda^*(\pi)$ such that $wv^\lambda = v^\nu$ or $wv^\lambda = -v^\nu$. The vectors $v^\lambda \in V^\lambda$ are called weight vectors. For a precise definition and further properties of weight vectors see [Va90, ch. 1].

3.2 Hasse diagrams and Chow rings

3.2. To each projective homogeneous variety X we may associate an oriented labeled graph $\mathcal{H}$ called Hasse diagram. It is known that the ring structure of $\mathrm{CH}(X)$ is determined by $\mathcal{H}$. In the present section we recall several facts concerning relations between Hasse diagrams and Chow rings. For a precise reference on this account see [De74], [Hi82a], and [Ko91].

3.3. Let G be a split simple algebraic group defined over a field k. We fix a split maximal torus T in G and a Borel subgroup B of G containing T and defined over k. Denote by Φ the root system of G, by $\Pi = \{\alpha_1, \dots, \alpha_{\mathrm{rk}\, G}\}$ the set of simple roots of Φ corresponding to B, by W the Weyl group, and by $S = \{s_1 = s_{\alpha_1}, \dots, s_{\mathrm{rk}\, G} = s_{\alpha_{\mathrm{rk}\, G}}\}$ the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be a (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta,\, \theta \in \Theta\rangle$. Denote

$$W^\Theta = \{w \in W \mid \forall\, \alpha_i \in \Theta \quad l(ws_i) = l(w) + 1\},$$

where l is the length function. The pairing

$$W^\Theta \times W_\Theta \to W \qquad (w, v) \mapsto wv$$

is a bijection and $l(wv) = l(w) + l(v)$. It is easy to see that W^Θ consists of all representatives in the left cosets W/W_Θ which have minimal length. Sometimes it is also convenient to consider the set of all representatives of maximal length. We shall denote this set as ${}^\Theta W$. Observe that there is a bijection $W^\Theta \to {}^\Theta W$ given by $v \mapsto vw_\theta$, where w_θ is the longest element of W_Θ. The longest element of W^Θ corresponds to the longest element w_0 of the Weyl group.

3.4. To a subset Θ of the finite set Π we associate an oriented labelled graph, which we call a Hasse diagram and denote by $\mathcal{H}_W(\Theta)$. This graph is constructed as follows. The vertices of this graph are the elements of W^Θ. There is an edge from a vertex w to a vertex w' with a label i if and only if $l(w) < l(w')$ and $w' = s_i w$. The example of such a graph is provided in 7.8. Observe that the diagram $\mathcal{H}_W(\emptyset)$ coincides with the Cayley graph associated to the pair (W, S).

3.5 Lemma. *The assignment $\mathcal{H}_W \colon \Theta \mapsto \mathcal{H}_W(\Theta)$ is a contravariant functor from the category of subsets of the finite set Π (with embeddings as morphisms) to the category of oriented graphs.*

Proof. It is enough to embed the diagram $\mathcal{H}_W(\Theta)$ to the diagram $\mathcal{H}_W(\emptyset)$. We do this as follows. We identify the vertices of $\mathcal{H}_W(\Theta)$ with the subset of vertices of $\mathcal{H}_W(\emptyset)$ by means of the bijection $W^\Theta \to {}^\Theta W$. Then the edge from w to w' of ${}^\Theta W \subset W$ has a label i if and only if $l(w) < l(w')$ and $w' = s_i w$ (as elements of W). Clearly, the obtained graph will coincide with $\mathcal{H}_W(\Theta)$. $\square$

3.6. Now consider the Chow ring of a projective homogeneous variety G/P_Θ. It is well known that $\mathrm{CH}(G/P_\Theta)$ is a free abelian group with a basis given by the varieties $[X_w]$ that correspond to the vertices w of the Hasse diagram $\mathcal{H}_W(\Theta)$. The degree of the basis element $[X_w]$ corresponds to the minimal number of edges needed to connect the respective vertex w with w^θ (which is the longest one). The multiplicative structure of $\mathrm{CH}(G/P_\Theta)$ depends only on the root system of G and the diagram $\mathcal{H}_W(\Theta)$.

3.7 Lemma. *The contravariant functor* $\mathrm{CH}\colon \Theta \mapsto \mathrm{CH}(G/P_\Theta)$ *factors through the category of Hasse diagrams* $\mathcal{H}_W$*, i.e., the pull-back (ring inclusion)*

$$\mathrm{CH}(G/P_{\Theta'}) \hookrightarrow \mathrm{CH}(G/P_\Theta)$$

arising from the embedding $\Theta \subset \Theta'$ *is induced by the embedding of the respective Hasse diagrams* $\mathcal{H}_W(\Theta') \subset \mathcal{H}_W(\Theta)$.

3.8 Corollary. *Let* B *be a Borel subgroup of* G *and* P *its (standard) parabolic subgroup. Then* $\mathrm{CH}(G/P)$ *is a subring of* $\mathrm{CH}(G/B)$. *The generators of* $\mathrm{CH}(G/P)$ *are* $[X_w]$*, where* $w \in {}^\Theta W \subset W$. *The cycle* $[X_w]$ *in* $\mathrm{CH}(G/P)$ *has codimension* $l(w_0) - l(w)$.

Proof. Apply the lemma to the case $B = P_\emptyset$ and $P = P_{\Theta'}$. □

Hence, in order to compute $\mathrm{CH}(G/P)$ it is enough to compute $\mathrm{CH}(X)$, where $X = G/B$ is the variety of complete flags. The following results provide tools to perform such computations.

3.9 (Poincaré duality)**.** In order to multiply two basis elements h and g of $\mathrm{CH}(G/P)$ such that $\deg h + \deg g = \dim G/P$ we use the following formula (see [Ko91, 1.4]):

$$[X_w] \cdot [X_{w'}] = \delta_{w, w_0 w' w_\theta} \cdot [pt].$$

3.10 (Pieri's formula)**.** In order to multiply two basis elements of $\mathrm{CH}(X)$ one of which is of codimension 1 we use the following formula (see [De74, Cor. 2 of 4.4]):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta \in \Phi^+,\, l(w s_\beta) = l(w) - 1} \langle \beta^\vee, \overline{\omega}_\alpha \rangle [X_{w s_\beta}],$$

where the sum runs through the set of positive roots $\beta \in \Phi^+$, s_α denotes the simple reflection corresponding to α and $\bar{\omega}_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0 s_\alpha}]$ is the element of codimension 1.

3.11 (Giambelli's formula)**.** Let $P = P(\Phi)$ be the weight space. We denote as $\bar{\omega}_1, \dots \bar{\omega}_l$ the basis of P consisting of fundamental weights. The symmetric algebra $S^*(P)$ is isomorphic to $\mathbb{Z}[\bar{\omega}_1, \dots \bar{\omega}_l]$. The Weyl group W acts on P, hence, on $S^*(P)$. Namely, for a simple root α_i,

$$w_{\alpha_i}(\bar{\omega}_j) = \begin{cases} \bar{\omega}_i - \alpha_i, & i = j, \\ \bar{\omega}_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c\colon S^*(P) \to \mathrm{CH}^*(G/B)$ as follows. For a homogeneous $u \in \mathbb{Z}[\bar{\omega}_1, \dots, \bar{\omega}_l]$

$$c(u) = \sum_{w \in W,\, l(w) = \deg(u)} \Delta_w(u)[X_{w_0 w}],$$

where for $w = w_{\alpha_1} \dots w_{\alpha_k}$ we denote by Δ_w the composition of derivations $\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_k}$ and the derivation $\Delta_{\alpha_i}\colon S^*(P) \to S^{*-1}(P)$ is defined by $\Delta_{\alpha_i}(u) = \frac{u - w_{\alpha_i}(u)}{\alpha_i}$. Then (see [Hi82a, ch. IV, 2.4])

$$[X_w] = c(\Delta_{w^{-1}}(\frac{d}{|W|})),$$

where d is the product of all positive roots in $S^*(P)$. In other words, the element $\Delta_{w^{-1}}(\frac{d}{|W|}) \in c^{-1}([X_w])$.

Hence, in order to multiply two basis elements $h, g \in \mathrm{CH}(X)$ take their preimages under the map c and multiply them in $S^*(P) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\bar{\omega}_1, \dots \bar{\omega}_l]$. Then apply c to their product.

3.3 Hasse diagrams

In this section we collect some information concerning Hasse diagrams in general. In particular, we explain the connection between Hasse diagrams and the Chernousov-Gille-Merkurjev method of a motivic decomposition of an isotropic projective homogeneous variety.

We start with some general remarks.

3.12. Every Hasse diagram can be considered as a thin building. Hasse diagrams describe the weak Bruhat order of Schubert cells. The weight diagram for a microweight representation of a split simple algebraic group coincides with the respective Hasse diagram (see [PSV, 1.1 and 2.2]).

3.13. For microweight representations we can extract a lot of information about the representation itself from the Hasse diagram purely combinatorially.

Let us give some examples. Let G be a Chevalley group and π its microweight representation. The branching rules can be described as follows. If we restrict π to a subsystem subgroup of G we just need do erase/add some edges of the respective Hasse diagram (weight diagram) (see [PSV, 3.7]).

The tensor product of two representations of the same group and its decomposition into irreducible representations can be described as some combinatorial operations on the Hasse graph. For details see [Kash], [PSV, 3.6].

As an application of the rules described there we can "fold" the diagram corresponding to the representation $(\mathrm{E}_6, \bar{\omega}_1)$ to the representation $(\mathrm{F}_4, \bar{\omega}_4)$. First one should identify the labels i, j on the edges of $(\mathrm{E}_6, \bar{\omega}_1)$ by the relation

$$i \sim j \Leftrightarrow \text{the images of } i \text{ and } j \text{ under the folding } \mathrm{E}_6 \to \mathrm{F}_4 \text{ coincide.}$$

After that one should apply the rules of [PSV, 3.6] to the middle squares of the resulting diagram. After this procedure one obtains the weight diagram $(\mathrm{F}_4, \bar{\omega}_4)$.

3.14. The equations on the orbit of the highest weight vector are hidden in the Hasse diagram (see [PSV, 3.8] and [Va97, 3.3]).

The action of the elementary transvections, the action of the Weyl group can be seen from the picture ([PSV, 3.4] and [Va97, 2.3]).

Centers and axis of the elementary transvections can also be obtained from the Hasse diagram. Consider the case of the Freudenthal transvections (see [Va90], [Va97]). Take an elementary transvection $x_\alpha(\xi)$ of the simply-connected Chevalley group of type E_6 in the representation with the highest weight $\bar{\omega}_1$. Consider the decomposition of $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_m}$ into a sum of simple roots and find in the Hasse diagram, corresponding to this representation, all paths with labels $i_1, \ldots, i_m$ (in any order). There are precisely six such paths (the residue of a transvection of E_6 equals six).

Let $v^{\lambda_1}, \ldots, v^{\lambda_6}$ (resp. $v^{\mu_1}, \ldots, v^{\mu_6}$) denote weight vectors which are initial (resp. terminal) points of these paths. Take any vectors $x \in \langle v^{\lambda_1}, \ldots, v^{\lambda_6} \rangle$ and $y \in \langle v^{\mu_1}, \ldots, v^{\mu_6} \rangle$. The Freudenthal transvection, constructed by x and y, will lie in the same root subgroup as $x_\alpha(\xi)$. Moreover, any center and axis of any transvection from this root subgroup can be obtained in this way. Note that in general position such vectors x and y give a non-trivial element of the root subgroup.

Recall (see [Va97, § 4.1] and [SV68]) that a Freudenthal transvection is given by the following formula:

$$T_{uv}(\xi)x = x + (v,x)\xi u - \xi v \times (u \times x),$$

where $u \in V$, $v \in V^*$, $\xi \in k$, $vu = 0$, $u \times u = 0$, and $v \times v = 0$. The vectors u and v are called the center and the axis of the transvection T_{uv}.

3.15. In fact, we can find multilinear invariants for some representations. E.g., the cubic form of E_6 is a particular case of the following construction.

Let G be a Chevalley group and $\pi\colon G \to \mathrm{GL}(V)$ its microweight representation.

Fix $m \geq 1$ and consider the set

$$M = \{v^{\lambda_1}, \dots, v^{\lambda_m}\}$$

of (distinct) weight vectors ($\lambda_i \in \overline{\Lambda}^*(\pi)$) such that all differences

$$\lambda_i - \lambda_j \notin \Phi.$$

The group $\widetilde{W}$ acts naturally on the monomials

$$ax_{\nu_1} \dots x_{\nu_m} \in k[x_\lambda, \lambda \in \Lambda(\pi)],$$

where $a \in k$, $\nu_i \in \Lambda(\pi)$. Namely,

$$w(ax_{\nu_1} \dots x_{\nu_m}) := a(-1)^p x_{w(\nu_1)} \dots x_{w(\nu_m)},$$

where $p = \#\{1 \leq i \leq m \mid \text{for some } \lambda \in \Lambda(\pi)\, wv^{\nu_i} = -v^\lambda (\lambda \text{ depends on } i)\}$, and $\widetilde{W}$ acts on the weights naturally (there is a natural epimorphism $\widetilde{W} \twoheadrightarrow W$, where W is the ordinary Weyl group).

Consider the orbit of the monomial $x_{\lambda_1} \dots x_{\lambda_m}$ under the action of $\widetilde{W}$:

$$q := \sum_{w \in \widetilde{W}} w(x_{\lambda_1} \dots x_{\lambda_m}),$$

$$f := \text{the (full) polarization of } q.$$

It is obvious that q and f are $\widetilde{W}$-invariant.

Taking the representation with the highest weight $\bar{\omega}_1$ of the simply-connected Chevalley group of type E_6 one obtains a trilinear form being preserved by E_6. This form is the norm form of the split simple exceptional 27-dimensional Jordan algebra.

Using this construction we also get the pfaffian invariant of the representation $(A_n, \bar{\omega}_2)$ and some quadratic invariants, e.g., the bilinear form for $(D_n, \bar{\omega}_1)$. In general, one should not expect that the algorithm above will give a G-invariant form. In general, one should take sums by several orbits of the extended Weyl group.

3.16. The double cosets $W_P\backslash W/W_Q$ are obtained by the branching rules (see [PSV, 3.3]).

The generating function f (see 4.2 below) has the following property: the i-th coefficient of $f(G, P)$ equals the number of vertices in the respective Hasse diagram at the distance i from the leftmost one.

This coefficient is also equal to $\operatorname{rk} \mathrm{CH}^i(G/P)$. Hence we can study these ranks using the combinatorial structure of the Hasse diagrams. E.g., it follows from the results of Stanley (see [Hi82a]) that

$$\operatorname{rk} \mathrm{CH}^i(G/P) \leq \operatorname{rk} \mathrm{CH}^{i+1}(G/P),$$

where $i \leq [\dim(G/P)/2]$. From this interpretation it is also obvious that $\operatorname{rk} \mathrm{CH}^i(G/P) = \operatorname{rk} \mathrm{CH}^{n-i}(G/P)$ for any P, where $n = \dim(G/P)$ (Poincaré duality).

There is also a combinatorial interpretation of Pieri's formula for the multiplication in the Chow rings of projective homogeneous varieties (see [Hi82b, cor. 3.3]).

3.17. There is a very nice interpretation of the Chernousov-Gille-Merkurjev method of a motivic decomposition of an isotropic projective homogeneous variety (see [CGM]). The crucial point here are the branching rules for computing the double cosets of the Weyl group.

We shall illustrate this method in the case when G is a simple adjoint algebraic group and P is a maximal parabolic subgroup of G defined over the base field (G/P has a rational point). Apart from this, we assume that the $*$-action of the Galois group on the set of the double cosets $W_P\backslash W/W_P$ is trivial (see [CGM]). The case of an arbitrary parabolic subgroup is similar.

Suppose $P = P_i$ is the maximal parabolic subgroup corresponding to the i-th root $\alpha_i \in \Pi$. Using the branching rules we construct the double cosets $W_P\backslash W/W_P$ cutting the Hasse diagram (G, P) along the edges with a label i (we draw the Hasse diagram assuming that G is split). The diagram splits into several parts which we denote as H_j.

Consider the semisimple part G_P of the Levi subgroup of P. This is a group of type $\Pi \setminus \{\alpha_i\}$ (we delete the i-th simple root from the Dynkin diagram of G).

Fix a component H_j. Consider the set of labels $\{i_1, \dots i_l\}$ on its edges, whose initial vertex is the rightmost one. Recall that the Hasse diagram is an oriented graph and the rightmost vertex of H_j is the vertex, which is a terminal one for no edges.

For each root i_s ($1 \leq s \leq l$) there exists a unique simple component $(G_P)_{i_s}$ of G_P which Dynkin diagram contains the root with number i_s (the enumeration of roots of the simple components of G_P inherits the enumeration of roots of G_P, which inherits the enumeration of roots of G).

Consider the variety $Z_j = (G_P)_{i_1}/Q_{i_1} \times \ldots \times (G_P)_{i_l}/Q_{i_l}$, where Q_{i_s} is the maximal parabolic subgroup of $(G_P)_{i_s}$, corresponding to the root i_s. Now the Hasse sub-diagram H_j coincides with the Hasse diagram (G_P, Q_j), where the parabolic subgroup Q_j corresponds to the product of parabolic subgroups Q_{i_s}. In turn, this last Hasse diagram is the product of the Hasse diagrams $(G_P)_{i_s}/Q_{i_s}$, $1 \leq s \leq l$.

In notation of [CGM] the varieties Z_j coincide with the varieties Z_{D_j} over a separably closed field. They are precisely the building blocks of the motive of G/P. It is important to notice that the Hasse diagrams just give information how the varieties Z_{D_j} look like after the base change to a separably closed field.

4 Motivic isomorphisms in the split case

4.1 Notation

Let G be a split simple algebraic group defined over a field k. We fix a split maximal torus T in G and a Borel subgroup B containing T and defined over k. Denote by Φ the root system of G, by Π the set of simple roots of Φ corresponding to B, by W the Weyl group, and by S the corresponding set of fundamental reflections.

For a smooth projective variety X we denote as $\mathcal{M}(X)$ its motive in the category of Chow motives (see [Ma68] and [Ka01] for the detailed description of the category of Chow motives).

We would like to get a complete solution to the following problem:

4.1 Problem. *When the motives of two projective split homogeneous varieties are isomorphic?*

4.2 Main theorem

Let $P = P_\Theta$ be a (standard) parabolic subgroup, where $\Theta \subset \Pi$, $P = BW_\Theta B$, $W_\Theta = \langle s_\theta, \theta \in \Theta \rangle$.

Denote

$$W^\Theta = \{w \in W \mid \forall\, \alpha_i \in \Theta \quad l(ws_i) = l(w) + 1\},$$

where l is the length function. The pairing

$$W^\Theta \times W_\Theta \to W$$

$$(w, v) \mapsto wv$$

is a bijection and $l(wv) = l(w) + l(v)$ (see [Ko91]).

It is easy to see that W^Θ consists of all representatives in the cosets W/W_Θ which have minimal length.

We know

$$\mathcal{M}(G/P) \simeq \oplus_{w \in\ W^\Theta} \mathrm{L}^{\otimes l(w)},$$

where L is the Lefschetz motive (see [Ko91]).

Consider the following function

$$(G, P) \mapsto f(G, P) = \sum_{w \in W^\Theta} x^{l(w)} \in \mathbb{Z}[x].$$

It is obvious that $\mathcal{M}(G/P) \simeq \mathcal{M}(G'/P')$ if and only if $f(G, P) = f(G', P')$.

For a subgroup $V \le W$ denote

$$r(V) = \sum_{w \in V} x^{l(w)} \in \mathbb{Z}.$$

It is clear that $f(G, P)r(W_\Theta) = r(W)$, i.e., $f(G, P) = \frac{r(W)}{r(W_\Theta)} = \frac{r(W)}{r(W_1)\ldots r(W_k)}$, where $W_\Theta = W_1 \times \ldots \times W_k$ and W_i are the Weyl groups of the irreducible parts of W_Θ.

We need the following well known theorem of Solomon (see [Ca72]).

4.2 Proposition.
$$r(W)=\prod_{i=1}^{l}\frac{x^{d_i(W)}-1}{x-1},$$
where $d_i(W)$ are the degrees of the basic polynomial invariants of W (see [Ca72] and [PV94]).

G	$d_i(G)$
A_l	$2,3,\dots,l+1$
$\mathrm{B}_l,\mathrm{C}_l$	$2,4,\dots,2l$
D_l	$2,4,\dots,2l-2,l$
G_2	$2,6$
F_4	$2,6,8,12$
E_6	$2,5,6,8,9,12$
E_7	$2,6,8,10,12,14,18$
E_8	$2,8,12,14,18,20,24,30$

In order to answer the question when the motives of two varieties G/P and G'/P' are isomorphic, we need to do some computations. $\mathcal{M}(G/P)\simeq\mathcal{M}(G'/P')$ if and only if
$$\frac{r(W)}{r(W_1)\dots r(W_k)}=\frac{r(W')}{r(W'_1)\dots r(W'_{k'})},$$
where $W'=W(G')$ and W_i, W'_j are the irreducible parts as above.

It is easy to see that
$$\prod_{a\in A}\frac{x^a-1}{x-1}=\prod_{b\in B}\frac{x^b-1}{x-1}\Leftrightarrow A=B,$$
where A and B are some multisets of indices (i.e., the same factors in the formula above can occur several times). For a subgroup $V\leq W$ consider the following function
$$v(V)=\sum_{i=1}^{l}v_{d_i(V)},$$
where v_i are independent variables of some vector space (e.g., $\mathbb{Q}^{\mathbb{Z}}$).

We have $\mathcal{M}(G/P)\simeq\mathcal{M}(G'/P')$ if and only if
$$v(W)-v(W_1)-\dots-v(W_k)=v(W')-v(W'_1)-\dots-v(W'_{k'}),$$

i.e., $\Phi - \Phi_1 - \ldots - \Phi_k = \Phi' - \Phi'_1 - \ldots - \Phi'_{k'}$, where Φ_i (resp. Φ'_j) are the root subsystems of $\Phi = \Phi(G)$ (resp. $\Phi' = \Phi(G')$) corresponding to W_i (resp. W'_j) (we can see this subsystems on the Dynkin diagram of G (resp. G'): just delete the vertices corresponding to P (resp. P')).

4.3 Theorem. *$\mathcal{M}(G/P) \simeq \mathcal{M}(G'/P')$ in the category of Chow motives if and only if (at least) one of the following conditions holds*

1. *$G/P \simeq G'/P'$ (e.g., $\mathrm{C}_l - \mathrm{C}_{l-1} = \mathrm{A}_{2l-1} - \mathrm{A}_{2l-2}$ and $\mathrm{D}_{l+1} - \mathrm{A}_l = \mathrm{B}_l - \mathrm{A}_{l-1}$).*
2. *$W(G) = W(G')$, $W(P) \simeq W(P')$ (as $W(P)$ we denote the Weyl group of the semisimple part of P).*
3. *$\mathrm{B}_l - \mathrm{B}_{l-1} = \mathrm{A}_{2l-1} - \mathrm{A}_{2l-2}$ (the quadric Q_{2l-1} and the projective space $\mathbb{P}^{2l-1}$).*
4. *$\mathrm{D}_{l+1} - \mathrm{A}_l = \mathrm{C}_l - \mathrm{A}_{l-1}$.*
5. *$\mathrm{B}_l - \mathrm{B}_k - \mathrm{A}_{2k-2} - \ldots - \mathrm{A}_m - \sum \mathrm{A} = \mathrm{B}_l - \mathrm{B}_m - \mathrm{A}_{2k-1} - \ldots - \mathrm{A}_{2m+1} - \sum \mathrm{A}$.*
6. *$\mathrm{D}_{l+1} - \mathrm{D}_{k+1} - \mathrm{A}_{2k-2} - \ldots - \mathrm{A}_m - \mathrm{A}_{k-1} - A_m - \sum \mathrm{A} =$
$\mathrm{D}_{l+1} - \mathrm{D}_{m+1} - \mathrm{A}_{2k-1} - \ldots - \mathrm{A}_{2m+1} - \mathrm{A}_k - \mathrm{A}_{m-1} - \sum \mathrm{A}$.*
7. *$\mathrm{G}_2 - \mathrm{A}_1 = \mathrm{A}_5 - \mathrm{A}_4 = \mathrm{B}_3 - \mathrm{B}_2$.*

Everywhere in the list above B *can be substituted by* C *and vice versa.*

Proof. We should analyze all cases when the above identity on v can be fulfilled. We must start with Φ and Φ' s.t. the degree of $v(W(\Phi))$ is equal to the degree of $v(W(\Phi'))$. After that, we look at the next term (we order the terms by their degrees) of the polynomial $v(W(\Phi))$ or $v(W(\Phi'))$ which doesn't cancel. We should subtract some polynomial v corresponding to a root subsystem of Φ or Φ' in order to kill this term. Going further case by case we obtain the full list of the theorem. □

4.4 Remark. By a result of Demazure if $G/P \simeq G'/P'$ then $G \simeq G'$ apart from the following cases: $\mathrm{G}_2/P_1 \simeq \mathrm{B}_3/P_1$, $\mathrm{C}_n/P_1 \simeq \mathrm{A}_{2n-1}/P_1 \simeq \mathrm{A}_{2n-1}/P_{2n-1}$, $\mathrm{B}_n/P_n \simeq \mathrm{D}_{n+1}/P_n \simeq \mathrm{D}_{n+1}/P_{n+1}$.

5 Chow motives of twisted flag varieties

Let G be an adjoint simple algebraic group of inner type over a field k. Let X be a twisted flag variety, i.e., a projective G-homogeneous variety over k. The main purpose of this chapter is to express the Chow motive of X in terms of the motives of "minimal" flags, i.e., those G-homogeneous varieties that correspond to maximal parabolic subgroups of G.

Observe that the motive of an *isotropic* G-homogeneous variety can be decomposed in terms of the motives of simpler G-homogeneous varieties using the techniques developed by Chernousov, Gille, Merkurjev [CGM] and Karpenko [Ka01]. For G-varieties, when G is *isotropic*, one obtains a similar decomposition following the arguments of Brosnan [Br05]. In the case of G-varieties, where G is *anisotropic*, no general decomposition methods are known except several particular cases of quadrics (see for example Rost [Ro98]) and Severi-Brauer varieties (see Karpenko [Ka95]).

In the present chapter we provide methods that allow to decompose the motives of some *anisotropic* twisted flag G-varieties, where the root system of G is of types A_n, B_n, C_n, G_2 and F_4, i.e., has a Dynkin diagram which does not branch.

As an application, we provide another counter-example to the uniqueness of a direct sum decomposition in the category of Chow motives with integral coefficients (see 5.6). Observe that such a counter-example was already constructed by Chernousov and Merkurjev (see [CM06, Example 9.4]) and is given by a G-homogeneous variety, where G is a product of two simple groups. Our example is given by a G-variety, where G is a *simple* group.

The chapter is organized as follows. In section 5.2 we state the main results. In the other sections we give proofs of the results for varieties of type A_n (section 5.3), of types B_n and C_n (section 5.5), and exceptional varieties of types G_2 and F_4 (section 5.6). Section 5.4 is devoted to the motivic decomposition of generalized Severi-Brauer varieties.

5.1 Notation and Conventions

By G we denote an adjoint simple algebraic group of inner type over k and by n its rank. G' stands for a split group of the same type as G. All varieties that appear in the chapter are projective G-homogeneous varieties over k. They can be considered as twisted forms of the varieties G'/P, where P is a parabolic subgroup of G'. The Chow motive of a variety X is denoted by

$\mathcal{M}(X)$. By A we denote a central simple algebra over k of index $\operatorname{ind}(A)$ and by $\operatorname{SB}(A)$ the corresponding Severi-Brauer variety. I is always a right ideal of A and $\operatorname{rdim} I$ stands for its reduced dimension. V is a vector space over k.

By $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we denote a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0$ with $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_l$. Integers d_1, d_2, …, d_k always satisfy the condition $1 \leq d_1 < d_2 < \ldots < d_k \leq n$ and are the dimensions of some flag. For each $i = 0, \ldots, k$ we define δ_i to be the difference $d_{i+1} - d_i$ (assuming here $d_0 = 0$ and $d_{k+1} = n+1$).

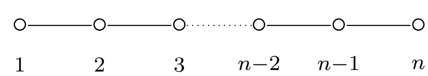
5.2 Statements of Results

We follow [MPW96, Appendix] and [CG06] for the description of projective G-homogeneous varieties that appear below. According to the type of the group G, we obtain the following results.

A_n: In this case $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n+1$, $n > 0$, and the set of points of a projective G-homogeneous variety X can be identified with the set of flags of (right) ideals

$$X(d_1, \ldots, d_k) = \{I_1 \subset I_2 \subset \ldots \subset I_k \subset A\}$$

of fixed reduced dimensions $1 \leq d_1 < d_2 < \ldots < d_k \leq n$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



The following result reduces the computation of the motive of X to the motives of "smaller" flags.

5.1 Theorem. *Suppose that* $\gcd(\operatorname{ind}(A), d_1, \ldots, \hat{d}_m, \ldots, d_k) = 1$*, then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ *such that* $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 5.21. □

As a consequence, for the variety of complete flags we obtain

5.2 Corollary. *The motive of the variety $X = X(1,\dots,n)$ of complete flags is isomorphic to*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where the a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^n \frac{z^k-1}{z-1}$.

Proof. Apply Theorem 5.1 recursively to the sequence of varieties $X(1,\dots,n)$, $X(1,\dots,n-1)$, …, $X(1,2)$ and $X(1) = \mathrm{SB}(A)$. □

Another interesting example is the "incidence" variety $X(1,n)$:

5.3 Corollary. *The motive of $X(1,n)$ is isomorphic to*

$$\mathcal{M}(X(1,n)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}(\mathrm{SB}(A))(i).$$

In order to complete the picture we need to know how to decompose the motive of a "minimal" flag, i.e., a generalized Severi-Brauer variety.

Note that for some rings of coefficients (fields, discrete valuation rings) one easily obtains the desired decomposition using the Krull-Schmidt Theorem (the uniqueness of a direct sum decomposition). More precisely, consider the subcategory $\mathcal{M}(G,R)$ of the category of motives with coefficients in a ring R that is the pseudo-abelian completion of the category of motives of projective G-homogeneous varieties (see [CM06, section 8]). Then we have the following

5.4 Proposition. *Let $X(d) = \mathrm{SB}_d(A)$, $1 < d < n$, be a generalized Severi-Brauer variety for a central simple algebra A of degree $n+1$ such that* $\gcd(\mathrm{ind}(A), d) = 1$. *Let R be a ring such that the Krull-Schmidt Theorem holds in the category $\mathcal{M}(G,R)$. Then the motive of $\mathrm{SB}_d(A)$ with coefficients in R is isomorphic to*

$$\mathcal{M}(\mathrm{SB}_d(A)) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where the integers a_i are the coefficients of the polynomial $\frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)}$ at terms z^i and the set of indices $\mathcal{I} = \{i \mid a_i \neq 0\}$.

Proof. See 5.23. □

It turns out that the motives of some generalized Severi-Brauer varieties with integral coefficients can still be decomposed but in a slightly unexpected way.

5.5 Theorem. *Let* $\mathrm{SB}_2(A)$ *be a generalized Severi-Brauer variety for a division algebra of degree* 5*. Then there is an isomorphism*

$$\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}(B)) \oplus \mathcal{M}(\mathrm{SB}(B))(2),$$

where B *is a division algebra Brauer-equivalent to the tensor square* $A^{\otimes 2}$*.*

Proof. See 5.30. □

As an immediate consequence of Theorems 5.1 and 5.5 we obtain

5.6 Corollary. *The Krull-Schmidt Theorem fails in the category of motives* $\mathcal{M}(\mathrm{PGL}_1(A), \mathbb{Z})$*, where* A *is a division algebra of degree* 5*.*

Proof. Apply Theorem 5.1 recursively to the sequences of varieties $X(1,2), X(1)$ and $X(1,2), X(2)$, where $X(1,2)$ is the twisted flag G-variety for $G = \mathrm{PGL}_1(A)$. We obtain two decompositions of the motive of $X(1,2)$

$$\bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(A))(i) \simeq \mathcal{M}(X(1,2)) \simeq \mathcal{M}(\mathrm{SB}_2(A)) \oplus \mathcal{M}(\mathrm{SB}_2(A))(1).$$

Applying now Theorem 5.5 to the components of the second decomposition, we obtain two different decompositions of the motive $\mathcal{M}(X(1,2))$ into indecomposable objects

$$\bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(A))(i) \simeq \mathcal{M}(X(1,2)) \simeq \bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(B))(i).$$

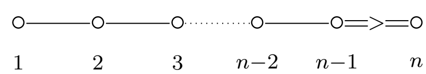
By [Ka95, Theorem. 2.2.1] and [Ka00, Criterion 7.1] the motives $\mathcal{M}(\mathrm{SB}(A))$ and $\mathcal{M}(\mathrm{SB}(B))$ are indecomposable and non-isomorphic. This finishes the proof of the corollary. □

5.7 Remark. Observe that the counter-example provided by Chernousov and Merkurjev (see [CM06, Ex.9.4]) is the product of two Severi-Brauer varieties $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$ which is a G-homogeneous variety for the *semi-simple* group $G = \mathrm{PGL}_1(A) \times \mathrm{PGL}_1(B)$, where A and B are two division algebras satisfying some conditions. The example that we provide, i.e., the flag $X(1,2)$, is a G-homogeneous variety for the *simple* group $G = \mathrm{PGL}_1(A)$.

B$_n$**:** We assume that the characteristic of the base field k is different from 2. It is known that $G = \mathrm{O}^+(V, q)$, where (V, q) is a regular quadratic space of dimension $2n+1$, $n > 0$, and projective G-homogeneous varieties can be described as flags of totally q-isotropic subspaces

$$X(d_1, \ldots, d_k) = \{V_1 \subset \ldots \subset V_k \subset V\}.$$

of fixed dimensions $1 \leq d_1 < \ldots < d_k \leq n$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



The following result shows that some motives of flag varieties can be decomposed into a direct sum of twisted motives of "smaller" flags.

5.8 Theorem. *Suppose that $m < k$, then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 5.35. □

In particular, for the variety of complete flags we obtain a formula similar to the one of Corollary 5.2.

5.9 Corollary. *The motive of the variety of complete flags $X = X(1, 2, \ldots, n)$ is isomorphic to*

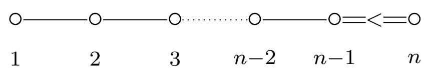
$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(X(n))(i)^{\oplus a_i},$$

where the a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^{n} \frac{z^k - 1}{z-1}$, and $X(n)$ is the twisted form of the maximal orthogonal Grassmannian.

C$_n$**:** We assume that the characteristic of the base field k is different from 2. In this case $G = \mathrm{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$, $n \geq 2$, with an involution σ of symplectic type on A, and a projective G-homogeneous variety can be described as the set of flags of (right) ideals

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid I_i \subseteq I_i^{\perp}\}$$

of fixed reduced dimensions $1 \leq d_1 < \dots < d_k \leq n$, where $I^{\perp} = \{x \in A \mid \sigma(x)I = 0\}$ is the right ideal of reduced dimension $2n - \mathrm{rdim}\, I$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



Again, the motives of some flag varieties can be decomposed into a direct sum of twisted motives of "smaller" flags.

5.10 Theorem. *Suppose that d_i is odd for some $i < k$ and $d_k - d_{k-1} = 1$. Then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{i=0}^{2n-2d_{k-1}-1} \mathcal{M}(X(d_1, \dots, d_{k-1}))(i).$$

In particular, for the variety of complete flags we obtain

5.11 Corollary. *The motive of the variety of complete flags $X = X(1, 2, \dots, n)$ is isomorphic to*

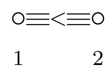
$$\mathcal{M}(X(1, \dots, n)) \simeq \bigoplus_{i=0}^{n(n-1)} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where a_i are the coefficients of the polynomial $\psi_n(z) = \prod_{k=1}^{n-1} \frac{z^{2k}-1}{z-1}$.

G$_2$**:** We suppose that the characteristic of k is not 2. It is known that $G = \mathrm{Aut}(C)$, where C is a Cayley algebra over k. By an i*-space*, where $i = 1, 2$, we mean an i-dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$. The only flag variety corresponding to a non-maximal parabolic is the variety of complete flags $X(1, 2)$ which is described as follows

$$X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C\}.$$

We enumerate the simple roots on the Dynkin diagram as follows:



In this case we obtain

5.12 Theorem. *The motive of the variety of complete flags $X = X(1,2)$ is isomorphic to*

$$\mathcal{M}(X) \simeq \mathcal{M}(X(2)) \oplus \mathcal{M}(X(2))(1).$$

Proof. See 5.45 □

Observe that by the result of Bonnet [Bo03] the motives of $X(1)$ and $X(2)$ are isomorphic (here $X(1)$ is a 5-dimensional quadric).

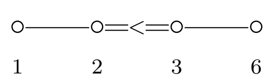
F_4: We suppose that the characteristic of k is not 2 and 3. It is known that $G = \mathrm{Aut}(J)$, where J is an exceptional simple Jordan algebra of dimension 27 over k. Set $\mathcal{I} = \{1,2,3,6\}$. By an *$i$-space*, $i \in \mathcal{I}$, we mean an i-dimensional subspace V of J such that every $u, v \in V$ satisfy the following condition:

$$\mathrm{tr}(u) = 0,\ u \times v = 0,\ \text{and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

A projective G-homogeneous variety can be described as the set of flags of subspaces

$$X(d_1, \ldots, d_k) = \{V_1 \subset \ldots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } J\}.$$

where the integers $d_1 < \ldots < d_k$ are taken from the set $\mathcal{I}$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



In this case we obtain

5.13 Theorem. *Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$, then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 5.50. □

5.3 Groups of type A_n

The goal of the present section is to prove Theorem 5.1 and Proposition 5.4. We use the notation of section 5.2.

5.14. Let G be an adjoint group of inner type A_n defined over a field k. It is well known that $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n+1$ and points of projective G-homogeneous varieties are flags of (right) ideals of A

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid \operatorname{rdim} I_i = d_i\}.$$

For convenience we set $d_0 = 0$, $d_{k+1} = n+1$, $I_0 = 0$, $I_{k+1} = A$.

5.15. The value of the functor of points corresponding to the variety $X(d_1, \dots, d_k)$ at a k-algebra R (see 2.5) equals the set of all flags $I_1 \subset \dots \subset I_k$ of right ideals of $A_R = A \otimes_k R$ having the following properties (see [IK00, section 4])

- the injection of A_R-modules $I_i \hookrightarrow A_R$ splits;
- $\operatorname{rdim} I_i = d_i$.

5.16. On the scheme $X = X(d_1, \dots, d_k)$ there are "tautological" vector bundles $\mathcal{J}_i$, $i = 0, \dots, k+1$, of ranks $(n+1)d_i$. The value of $\mathcal{J}_i$ on an X-algebra (R, x), where $x = (I_1, \dots, I_k)$, is the ideal I_i considered as a projective R-module. The bundle $\mathcal{J}_i$ also has a structure of a right A_X-module, where A_X is the constant sheaf of algebras on X determined by A.

For every $m \in \{1, \dots, k\}$ there exists an obvious morphism

$$X(d_1, \dots, d_k) \to X(d_1, \dots, \hat{d}_m, \dots, d_k)$$
$$(I_1, \dots, I_k) \mapsto (I_1, \dots, \hat{I}_m, \dots, I_k)$$

that turns $X(d_1, \dots, d_k)$ into an $X(d_1, \dots, \hat{d}_m, \dots, d_k)$-scheme.

5.17 Lemma. *Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X. Assume there exists a vector bundle $\mathcal{E}$ over X such that $A_X \simeq \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle*

$$\mathcal{F} = \mathcal{J}_{m+1}\mathcal{E}/\mathcal{J}_{m-1}\mathcal{E} = \mathcal{J}_{m+1}/\mathcal{J}_{m-1} \otimes_{A_X} \mathcal{E}$$

of rank $d_{m+1} - d_{m-1}$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{F})$ over X.

Proof. We use essentially the same method as in [IK00, Proposition 4.3].

Fix an X-algebra (R, x) where $x = (I_1, \dots, \hat{I}_m, \dots, I_k)$. The fiber of Y over x, i.e., the value at (R, x), can be identified with the set of all ideals I_m satisfying the conditions 5.15 such that $I_{m-1} \subset I_m \subset I_{m+1}$. The fiber of Z over x is the set of all R-submodules N of $\mathcal{F}_y = \mathcal{F}(R, y)$ such that the injection $N \hookrightarrow \mathcal{F}_y$ splits and $\operatorname{rk}_R N = d_m - d_{m-1}$.

We define a natural bijection between the fibers of Y and Z over x as follows.

Consider the following mutually inverse bijections between the set of all right ideals of reduced dimension r in A_R (satisfying 5.15) and the set of all direct summands of rank r of the R-module $\mathcal{E}_x$

$$\begin{aligned} \Phi \colon I &\mapsto I\mathcal{E}_x \\ \Psi \colon N &\mapsto \operatorname{Hom}_R(\mathcal{E}_x, N) \subset \operatorname{End}_R(\mathcal{E}_x) \simeq A_R \end{aligned}$$

Observe that these bijections preserve the respective inclusions of ideals and modules. So the ideals of reduced dimension d_m between I_{m-1} and I_m correspond to the submodules of rank d_m between $I_{m-1}\mathcal{E}_x$ and $I_{m+1}\mathcal{E}_x$, and, therefore, to the submodules of rank $d_{m+1} - d_{m-1}$ in $I_{m+1}\mathcal{E}_x / I_{m-1}\mathcal{E}_x = \mathcal{F}_x$. This gives the desired natural bijection on the fibers. □

5.18 Lemma. *Suppose that* $\gcd(\operatorname{ind}(A), d_1, \dots, d_k) = 1$. *Then there exists a vector bundle* $\mathcal{E}$ *over* $X = X(d_1, \dots, d_k)$ *of rank* $n+1$ *such that* $A_X \simeq \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$.

Proof. We have to prove that the class $[A_X]$ in $\operatorname{Br}(X)$ is trivial. Since X is a regular Noetherian scheme the canonical map

$$\operatorname{Br}(X) \to \operatorname{Br}(K)$$

where $K = k(X)$ is the function field of X, is injective by [Gr, 1.10] and [AG60, Theorem 7.2]. So it is enough to prove that $A \otimes_k K$ splits. But the generic point of X defines a flag of ideals of $A \otimes_k K$ of reduced dimensions $d_1, \dots, d_k$. Since the index $\operatorname{ind}(A \otimes_k K)$ divides $d_1, \dots, d_k$ and $\operatorname{ind} A$, by the assumption of the lemma it must be equal to 1. So $A \otimes_k K$ is split and this finishes the proof of the lemma. □

5.19 Remark. In the case $d_1 = 1$ one can take $\mathcal{E} = \mathcal{J}_1^\vee$.

5.20 Remark. It can be shown using the Index Reduction Formula (see [MPW96]) that the condition on the gcd is necessary and sufficient for the central simple algebra $A_{k(X)}$ to be split.

We are now ready to finish the proof of Theorem 5.1.

5.21 (Proof of Theorem 5.1)**.** By Lemma 5.18 there exists a vector bundle $\mathcal{E}$ over $X = X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$ of rank $n+1$ such that $A_X \simeq \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$. By Lemma 5.17 we conclude that $Y = X(d_1, \ldots, d_k)$ is a Grassmann bundle over X. Now by Proposition 2.4 we obtain the isomorphism of Theorem 5.1.

5.22 Remark. Note that the assumption of Theorem 5.1 on the reduced dimensions $d_1, \ldots, d_k$ is essential. Indeed, suppose the Theorem holds for any twisted flag variety. Consider the flag $X = X(1, d)$ with $\gcd(\mathrm{ind}(A), d) > 1$. Then we have an isomorphism of motives

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{d-1} \mathcal{M}(\mathrm{SB}_d(A))(i)$$

which appears after applying Theorem to the flags $X(1, d)$ and $X(d)$. Consider the group $\mathrm{CH}_0(X) = \mathrm{Mor}_{\mathcal{M}_k}(\mathcal{M}(pt), \mathcal{M}(X))$. The isomorphism above induces the isomorphism of groups

$$\begin{aligned}\mathrm{Coker}(\mathrm{CH}_0(X) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(X_{k_s})) &\cong \mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}_d(A)) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(\mathrm{Gr}(d, n+1))) \\ &\cong \mathbb{Z}/(\tfrac{\mathrm{ind}(A)}{\gcd(\mathrm{ind}(A), d)})\mathbb{Z},\end{aligned}$$

where res is the pull-back induced by the scalar extension k_s/k (here k_s denotes a separable closure of k) and the last isomorphism follows by [Bl91, Theorem 3]. On the other hand, applying Theorem 5.1 to the flags $X(1, d)$ and $X(1)$ we obtain an isomorphism

$$\mathcal{M}(X) \simeq \bigoplus_{\lambda} \mathcal{M}(\mathrm{SB}(A))((n+1-d)(d-1) - |\lambda|)$$

which induces the isomorphism of groups

$$\begin{aligned}\mathrm{Coker}(\mathrm{CH}_0(X) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(X_{k_s})) &\cong \mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}(A)) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(\mathbb{P}^n)) \\ &\cong \mathbb{Z}/\operatorname{ind}(A)\mathbb{Z},\end{aligned}$$

that leads to a contradiction.

We now prove Proposition 5.4.

5.23 (Proof of Proposition 5.4)**.** Let $G = \mathrm{PGL}_1(A)$ and let $\mathcal{M}(G,R)$ be the tensor category of Chow motives of G-homogeneous varieties with coefficients in a ring R for which the Krull-Schmidt theorem holds. It is the case, e.g., when R is a field or, more general, a discrete valuation ring (see [CM06, Theorem 9.6]).

Consider the G-homogeneous variety $X(1,d)$, $1 < d < n$. Applying Theorem 5.1 to the sequences of flags $X(1,d)$, $X(d)$ and $X(1,d)$, $X(1)$. We obtain two isomorphisms in $\mathcal{M}(G,R)$

$$\bigoplus_{i=0}^{d-1} \mathcal{M}(\mathrm{SB}_d(A))(i) \simeq \mathcal{M}(X) \simeq \bigoplus_{\lambda} \mathcal{M}(\mathrm{SB}(A))((n+1-d)(d-1)-|\lambda|), \quad (*)$$

where the sum on the right hand side is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{d-1})$ such that $n+1-d \geq \lambda_1 \geq \ldots \geq \lambda_{d-1} \geq 0$. Since the Krull-Schmidt Theorem holds in $\mathcal{M}(G,R)$, the motive $\mathrm{SB}(A)$ has a unique decomposition into the direct sum of indecomposable objects H_i, $i \in \mathcal{I}$, and their twists

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \bigoplus_{i \in \mathcal{I}} (\oplus_{j \in \mathcal{J}_i} H_i(j)).$$

Consider the subcategory $\mathcal{M}(G,R)_{\mathcal{I}}$ additively generated by the motives H_i, $i \in \mathcal{I}$, and their twists. The abelian group of isomorphism classes of objects of this category can be equipped with a structure of a free module over the polynomial ring $R[z]$. Namely, multiplication by z is given by the twist. Clearly, the classes $[H_i]$, $i \in \mathcal{I}$, form the basis of this $R[z]$-module.

By (*) we have $\mathcal{M}(\mathrm{SB}_d(A)) \in \mathcal{M}(G,R)_{\mathcal{I}}$ and the isomorphisms (*) can be rewritten as

$$\begin{aligned} \frac{z^d-1}{z-1}[\mathrm{SB}_d(A)] &= \frac{\varphi_n(z)}{\varphi_{d-1}(z)\varphi_{n+1-d}(z)}[\mathrm{SB}(A)] \\ &= \frac{z^d-1}{z-1}\frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)}[\mathrm{SB}(A)] \end{aligned}$$

where $\varphi_n(z) = \prod_{k=2}^{n} \frac{z^k-1}{z-1}$. This immediately implies the equality

$$[\mathrm{SB}_d(A)] = \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)}[\mathrm{SB}(A)],$$

i.e., the isomorphism in $\mathcal{M}(G,R)_{\mathcal{I}}$ between $\mathcal{M}(\mathrm{SB}_d(A))$ and the respective sum of twists of $\mathcal{M}(\mathrm{SB}(A))$. This finishes the proof of the proposition.

5.4 Motivic decomposition of $\mathrm{SB}_2(A)$

This section is devoted to the proof of Theorem 5.5.

We now recall some properties of Grassmann varieties and describe their Chow rings.

5.24. Consider the Grassmann variety $\mathrm{Gr}(d, n+1)$, $1 \leq d \leq n$, of d-planes in the $(n+1)$-dimensional affine space. It has dimension $d(n+1-d)$. A twisted form of it is a generalized Severi-Brauer variety $\mathrm{SB}_d(A)$, where A is a central simple algebra of degree $n+1$. For any two integers d and d', $1 \leq d, d' \leq n$, there is the product diagram

$$\begin{array}{ccc} \mathrm{Gr}(d, n+1) \times \mathrm{Gr}(d', n+1) & \xrightarrow{\mathrm{Seg}} & \mathrm{Gr}(dd', (n+1)^2) \\ \downarrow & & \downarrow \\ \mathrm{SB}_d(A) \times \mathrm{SB}_{d'}(A^{\mathrm{op}}) & \xrightarrow{\mathrm{Seg}} & \mathrm{SB}_{dd'}(A \otimes_k A^{\mathrm{op}}) \end{array} \qquad (1)$$

where the horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension k_s/k (here k_s is a separable closure of k).

5.25. The diagram (1) induces the commutative diagram of rings

$$\begin{array}{ccc} \mathrm{CH}(\mathrm{Gr}(d, n+1) \times \mathrm{Gr}(d', n+1)) & \xleftarrow{\mathrm{Seg}^*} & \mathrm{CH}(\mathrm{Gr}(dd', (n+1)^2)) \\ \uparrow{\scriptstyle \mathrm{res}} & & \simeq \uparrow{\scriptstyle \mathrm{res}} \\ \mathrm{CH}(\mathrm{SB}_d(A) \times \mathrm{SB}_{d'}(A^{\mathrm{op}})) & \xleftarrow{\mathrm{Seg}^*} & \mathrm{CH}(\mathrm{SB}_{dd'}(A \otimes_k A^{\mathrm{op}})) \end{array} \qquad (2)$$

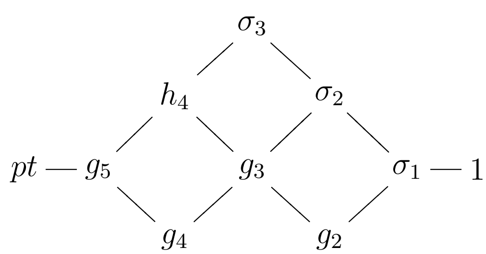
Observe that the right vertical arrow is an isomorphism since $A \otimes_k A^{\mathrm{op}}$ splits.

Consider a vector bundle E over $\mathrm{Gr}(dd', (n+1)^2)$. It is easy to see that the pull-back of the total Chern class $\mathrm{Seg}^*(c(E))$ is a rational cycle on $\mathrm{CH}(\mathrm{Gr}(d, n+1) \times \mathrm{Gr}(d', n+1)) = \mathrm{CH}(\mathrm{Gr}(d, n+1)) \otimes \mathrm{CH}(\mathrm{Gr}(d', n+1))$. In particular, if $E = \tau_{dd'}$ is the tautological bundle of $\mathrm{Gr}(dd', (n+1)^2)$ we obtain the following

5.26 Lemma. *The total Chern class $c(\mathrm{pr}_1^*\tau_d \otimes \mathrm{pr}_2^*\tau_{d'})$ of the tensor product of the pull-backs (induced by the projection maps) of the tautological bundles τ_d and $\tau_{d'}$ of $\mathrm{Gr}(d, n+1)$ and $\mathrm{Gr}(d', n+1)$ respectively is rational.*

From now on we restrict ourselves to the case $n = 4$, $d = 2$ and $d' = 1$, i.e., to the Grassmannian $\mathrm{Gr}(2,5)$ and the projective space $\mathbb{P}^4 = \mathrm{Gr}(1,5)$.

5.27. We describe the generators and relations of the Chow ring $\mathrm{CH}(\mathrm{Gr}(2,5))$ following [Ful, section 14.7]. Set $\sigma_m = c_m(Q)$, $m = 1,2,3$, where $Q = \mathcal{O}^5/\tau_2$ is the universal quotient bundle of rank 3 over $\mathrm{Gr}(2,5)$. It is known that the elements σ_m generate the Chow ring $\mathrm{CH}(\mathrm{Gr}(2,5))$. More precisely, as an abelian group this ring is generated by the Schubert cycles $\Delta_\lambda(\sigma)$ that are parameterized by all partitions $\lambda = (\lambda_1, \lambda_2)$ such that $3 \geq \lambda_1 \geq \lambda_2 \geq 0$. In particular, $\sigma_m = \Delta_{(m,0)}$, $m = 1,2,3$. For other generators we set the following notation $g_2 = \Delta_{(1,1)}$, $g_3 = \Delta_{(2,1)}$, $h_4 = \Delta_{(3,1)}$, $g_4 = \Delta_{(2,2)}$, $g_5 = \Delta_{(3,2)}$, $pt = \Delta_{(3,3)}$. These generators corresponds to the vertices of the Hasse diagram of $\mathrm{Gr}(2,5)$



The multiplication rules can be determined using Pieri's formulae

$$\Delta_\lambda \cdot \sigma_m = \sum_\mu \Delta_\mu,$$

where the sum is taken over all partitions $\mu = (\mu_1, \mu_2)$ such that $3 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq 0$.

5.28. Consider the tautological bundle τ_2 of the Grassmannian $\mathrm{Gr}(2,5)$. Its total Chern class is

$$c(\tau_2) = c(Q)^{-1} = \frac{1}{1 + \sigma_1 + \sigma_2 + \sigma_3} = 1 - \sigma_1 - \sigma_2 + \sigma_1^2 + \dots$$

where the rest consists of the summands of degree greater than 2. Hence, we obtain $c_1(\tau_2) = -\sigma_1$ and $c_2(\tau_2) = -\sigma_2 + \sigma_1^2 = g_2$.

5.29. The Chow ring of the projective space $\mathbb{P}^4$ can be identified with the factor ring $\mathbb{Z}[H]/(H^5)$, where $H = c_1(\mathcal{O}(1))$ is the class of a hyperplane section. Thus, the first Chern class of the tautological bundle of $\mathbb{P}^4$ equals to $c_1(\tau_1) = c_1(\mathcal{O}(-1)) = -H$.

We are now ready to prove Theorem 5.5.

5.30 (Proof of Theorem 5.5)**.** By Lemma 5.26 we obtain the following rational cycles in $\mathrm{CH}^*(\mathrm{Gr}(2,5)\times\mathbb{P}^4)$

$$\begin{aligned}
r &= c_1(\mathrm{pr}_1^*(\tau_2)\otimes\mathrm{pr}_2^*(\tau_1)) = c_1(\mathrm{pr}_1^*(\tau_2) + 2c_1(\mathrm{pr}_2^*(\tau_1)) = -\sigma_1\times 1 - 2(1\times H),\\
\rho &= c_2(\mathrm{pr}_1^*(\tau_2)\otimes\mathrm{pr}_2^*(\tau_1)) = c_2(\mathrm{pr}_1^*(\tau_2)) + c_1(\mathrm{pr}_1^*(\tau_2))c_1(\mathrm{pr}_2^*(\tau_1)) + c_1(\mathrm{pr}_2^*(\tau_1))^2\\
&= g_2\times 1 + \sigma_1\times H + 1\times H^2
\end{aligned}$$

For two cycles x and y we shall write $x =_5 y$ if there exists a cycle z such that $x-y=5z$. Note that $=_5$ is an equivalence relation that preserves rationality of cycles. Then the following cycles are rational

$$\begin{aligned}
\rho^2 &=_5 1\times H^4 + 2\sigma_1\times H^3 + (\sigma_2+3g_2)\times H^2 + 2g_3\times H + g_4\times 1,\\
\rho^3 &=_5 (3\sigma_2+g_2)\times H^4 + (\sigma_3+3g_3)\times H^3 + (g_4+3h_4)\times H^2 + 3g_5\times H + pt\times 1.
\end{aligned}$$

and a direct computation shows that $\rho^3\circ(\rho^2)^t =_5 \Delta_{\mathbb{P}^4}$ is the class of the diagonal in $\mathrm{CH}^4(\mathbb{P}^4\times\mathbb{P}^4)$.

Consider the composition

$$\begin{aligned}
(\rho^2)^t\circ\rho^3 =_5 &(3\sigma_2+g_2)\times g_4 + (2\sigma_3+g_3)\times g_3\\
&+ (g_4+3h_4)\times(\sigma_2-2g_2) + g_5\times\sigma_1 + pt\times 1.
\end{aligned}$$

Note that the right-hand side is a rational projector (over $\mathbb{Z}$) and, therefore, by the Rost Nilpotence Theorem (see [CGM, Corollary 8.3]) has the form $p\times_k k_s$ where p is a projector in $\mathrm{End}(\mathcal{M}(\mathrm{SB}_2(A)))$. The latter determines an object $(\mathrm{SB}_2(A),p)$ in the category of motives (actually in $\mathcal{M}(G,\mathbb{Z})$) which we denoted by $\mathcal{H}$.

Set $q=\Delta_{\mathrm{SB}_2(A)}-p$. We then show that

$$(\mathcal{M}(\mathrm{SB}_2(A)),q)\simeq(\mathcal{M}(\mathrm{SB}_2(A)),p^t)\simeq\mathcal{H}(2),$$

which gives the claimed decomposition $\mathcal{M}(\mathrm{SB}_2(A))\simeq\mathcal{H}\oplus\mathcal{H}(2)$.

Observe that an isomorphism $(\mathrm{SB}_2(A),q)\simeq(\mathrm{SB}_2(A),p^t)$ is given by the two mutually inverse motivic isomorphisms $p_s^t\circ q_s$ and $q_s\circ p_s^t$ over k_s which are rational. An isomorphism $\mathcal{H}(2)\simeq(\mathcal{M}(\mathrm{SB}_2(A)),p^t)$ is given by the following two cycles

$$\begin{aligned}
j_1 &= (3\sigma_2+g_2)\times pt - (2\sigma_3+g_3)\times g_5\\
&\quad + (g_4+3h_4)\times(g_4+3h_4) - g_5\times(2\sigma_3+g_3) + pt\times(3\sigma_2+g_2),\\
j_2 &= 1\times g_4 - \sigma_1\times g_3 + (\sigma_2-2g_2)\times(\sigma_2-2g_2) - g_3\times\sigma_1 + g_4\times 1.
\end{aligned}$$

Note that j_1 is rational, since $j_1 =_5 (1\times(3\sigma_2+g_2))p$, and $1\times(3\sigma_2+g_2) =_5 3(\rho+r^2)^t\circ\rho^2$ is rational.

Since A is a division algebra of degree 5, there is a division algebra B of degree 5 Brauer-equivalent to the tensor square $A^{\otimes 2}$. We claim that $\mathcal{H}\simeq\mathcal{M}(\mathrm{SB}(B))$. By the exact sequence (see [Ka00, Remark 7.17])

$$\mathrm{CH}^1(\mathrm{SB}(A^{\mathrm{op}})\times\mathrm{SB}(B)) \xrightarrow{\mathrm{res}_{F_s/F}} \mathrm{CH}^1(\mathbb{P}^4\times\mathbb{P}^4) \xrightarrow[1\times H\mapsto[B]]{H\times 1\mapsto[A^{\mathrm{op}}]} \mathrm{Br}(F) \qquad (3)$$

the following cycle in $\mathrm{CH}^1(\mathbb{P}^4\times\mathbb{P}^4)$ is rational

$$u = 2H\times 1 + 1\times H.$$

Therefore the cycles

$$\alpha = pt\times 1 + g_5\times H - (g_4+3h_4)\times H^2 - (g_3+2\sigma_3)\times H^3 + (3\sigma_2+g_2)\times H^4 \equiv u^4\circ\rho^3,$$
$$\beta = 1\times g_4 - H\times g_3 - H^2\times(\sigma_2-2g_2) + H^3\times\sigma_1 + H^4\times 1 \equiv (\rho^2)^t\circ(u^4)^t$$

are rational. A direct computation shows that $\alpha\circ\beta=\Delta_{\mathbb{P}^4}$ and $\beta\circ\alpha=p_s$. Therefore, the by Rost nilpotence theorem $\mathcal{H}\simeq\mathcal{M}(\mathrm{SB}(B))$. This finishes the proof of the theorem.

5.5 Groups of types B_n and C_n

The goal of the present section is to prove Theorems 5.8 and 5.10.

5.31. Let G be an adjoint group of type B_n. From now on we suppose that the characteristic of k is not 2. It is known that $G=\mathrm{O}^+(V,q)$, where (V,q) is a regular quadratic space of dimension $2n+1$ and projective G-homogeneous varieties can be described as flags of q-totally isotropic subspaces

$$X(d_1,\ldots,d_k)=\{V_1\subset\ldots\subset V_k\subset V \mid \dim V_i=d_i\}.$$

5.32. The value of the functor corresponding to the variety $X(d_1,\ldots,d_k)$ at a k-algebra R equals the set of all flags $V_1\subset\ldots\subset V_k$, where V_i is a q_R-totally isotropic direct summand of V_R of rank d_i.

For convenience we set $d_0=0$, $V_0=0$.

5.33. On the scheme $X = X(d_1, \dots, d_k)$ there are "tautological" vector bundles $\mathcal{V}_i$ of ranks d_i. The value of $\mathcal{V}_i$ on an X-algebra (R, x) is V_i, where $x = (V_1, \dots, V_k)$. For every m there exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\to X(d_1, \dots, \hat{d}_m, \dots, d_k) \\ (V_1, \dots, V_k) &\mapsto (V_1, \dots, \hat{V}_m, \dots, V_k) \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$-scheme.

5.34 Lemma. *Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k))$ by X. Suppose that $m < k$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X.*

Proof. Fix an X-algebra (R, x), where $x = (V_1, \dots, \hat{V}_m, \dots, V_k)$. We define a natural bijection between the fibers over the point x of Y and Z as follows. The fiber of Y over x can be identified with the set of all direct summands V_m of V_R of rank d_m such that $V_{m-1} \subset V_m \subset V_{m+1}$ (note that V_m is automatically q_R-isotropic since V_{m+1} is so). This fiber is clearly isomorphic to the fiber of Z over x which is the set of all direct summands of $(\mathcal{V}_{m+1}/\mathcal{V}_{m-1})_x = V_{m+1}/V_{m-1}$ of rank d_m. $\square$

5.35 (Proof of Theorem 5.8)**.** Applying Lemma 5.34 to the varieties $Y = X(d_1, \dots, d_k)$ and $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$. We obtain that Y is a Grassmann bundle over X. To finish the proof we apply Proposition 2.4.

5.36. Let G be an adjoint group of type C_n over k. It is known that $G = \mathrm{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$ with an involution σ of symplectic type on A, and projective G-homogeneous varieties can be described as flags of (right) ideals of A

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid I_i \subseteq I_i^{\perp},\ \mathrm{rdim}\, I_i = d_i\}.$$

Here $I^{\perp} = \{x \in A \mid \sigma(x) I = 0\}$ is a right ideal of reduced dimension $2n - \mathrm{rdim}\, I$.

5.37. The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a k-algebra R equals to the set of all flags $I_1 \subset \dots \subset I_k$ of right ideals of $A_R = A \otimes_k R$ having the following properties

- the injection of A_R-modules $I_i \hookrightarrow A_R$ splits;

- $I_i \subseteq I_i^\perp$;
- $\operatorname{rdim} I_i = d_i$.

For convenience we set $I_0 = 0$.

5.38. On the scheme $X = X(d_1, \dots, d_k)$ there are "tautological" vector bundles $\mathcal{J}_i$ of ranks $2nd_i$ and their "orthogonal complements" $\mathcal{J}_i^\perp$ of rank $2n(2n - d_i)$. The value of $\mathcal{J}_i$ (resp. $\mathcal{J}_i^\perp$) on an X-algebra (R, x), where $x = (I_1, \dots, I_k)$, is I_i (resp. $I_i^\perp$) considered as a projective R-module. The bundles $\mathcal{J}_i$ and $\mathcal{J}_i^\perp$ also have structures of right A_X-modules, where A_X is a constant sheaf of algebras on X determined by A. There exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\to X(d_1, \dots, d_{k-1}) \\ (I_1, \dots, I_k) &\mapsto (I_1, \dots, I_{k-1}), \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, d_{k-1})$-scheme.

5.39 Lemma. *Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, d_{k-1})$ by X. Suppose that $d_k = d_{k-1} + 1$ and there exists a vector bundle $\mathcal{E}$ over X such that $A_X \simeq \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle*

$$\mathcal{F} = \mathcal{J}_{k-1}^\perp \mathcal{E} / \mathcal{J}_{k-1} \mathcal{E} = \mathcal{J}_{k-1}^\perp / \mathcal{J}_{k-1} \otimes_{A_X} \mathcal{E}$$

of rank $2(n - d_{k-1})$. Then Y as a scheme over X can be identified with the projective bundle $Z = \mathbb{P}(\mathcal{F}) = \operatorname{Gr}(1, \mathcal{F})$ over X.

Proof. Fix an X-algebra (R, x), where $x = (I_1, \dots, I_{k-1})$. We define a natural bijection between the fibers over the point x of Y and Z. The fiber of Y can be identified with the set of all ideals I_k containing I_{k-1} and satisfying the conditions 5.37. The fiber of Z is the set of all direct summands of $\mathcal{F}_x = \mathcal{F}(R, x)$ of rank 1.

The involution σ induces an isomorphism $h \colon \mathcal{E}_x \otimes \mathcal{L} \to \mathcal{E}_x^*$ for some invertible R-module $\mathcal{L}$ (see [Knus, Lemma III.8.2.2]) such that

$$\begin{aligned} \sigma(f) \otimes 1 &= h^{-1} f^* h \text{ for all } f \in A \\ h^* \mathrm{can} \otimes 1 &= -h \end{aligned}$$

where $\mathrm{can} \colon \mathcal{E}_x \to \mathcal{E}_x^{**}$ is the canonical isomorphism.

Let U_1 and U_2 be direct summands of $\mathcal{E}_x$. We write $U_2 \subseteq U_1^\perp$ if $h(u \otimes l)(v) = 0$ for all $u \in U_1$, $v \in U_2$, $l \in \mathcal{L}$. We call a direct summand U of $\mathcal{E}_x$ *totally isotropic* if $U \subseteq U^\perp$. Note that any direct summand of rank 1 is totally isotropic (it can be proved easily using localization).

Define Φ and Ψ as in the proof of Theorem 5.17. Direct computations show that $I_1 \subseteq I_2^\perp$ if and only if $\Phi(I_1) \subseteq \Phi(I_2)^\perp$.

So the fiber of Y over x is naturally isomorphic to the set of all totally isotropic direct summands U_k of $\mathcal{E}_x$ of rank d_k containing $U_{k-1} = \Phi(I_k)$. One can represent U_k as the direct sum $U_{k-1} \oplus U$ where U is a direct summand of rank 1 (since $d_k = d_{k-1} + 1$). This U is totally isotropic and, therefore, U_k is totally isotropic if and only if $U_k \subseteq U_{k-1}^\perp$. Hence the set of all U_{k-1} is naturally isomorphic to the set of all direct summands of $\Phi(I_{k-1}^\perp)$ of rank d_k containing $\Phi(I_{k-1})$. The latter can be identified with $\mathbb{P}(\mathcal{F}_x)$. This finishes the proof of the lemma. □

We are now ready to prove Theorem 5.10.

5.40 (Proof of Theorem 5.10)**.** Consider the flag varieties $Y = X(d_1, \ldots, d_k)$ and $X(d_1, \ldots, d_{k-1})$. Since $\operatorname{ind} A = 2^r$ for some r and there is an odd d_i, we have $\gcd(\operatorname{ind}(A), d_1, \ldots, d_{k-1}) = 1$. By Lemma 5.18 there exists a bundle $\mathcal{E}$ over X such that $A_X = \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$. Applying Lemma 5.39 to the varieties X, Y and the bundle $\mathcal{E}$, we obtain that Y is a projective bundle over X. Now we use Proposition 2.4 to finish the proof.

5.6 Groups of types G_2 and F_4

This section is devoted to the proofs of Theorems 5.12 and 5.13.

5.41. Let G be a group of type G_2. We suppose that characteristic of k is not 2. It is known that $G = \operatorname{Aut}(C)$ where C is a Cayley algebra over k. By *i-space* where $i = 1, 2$ we mean an i-dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$.

The only flag variety corresponding to a non-maximal parabolic is the complete flag variety $X(1,2)$ which is described as follows (see [Bo03]):

$$X(1,2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C\}.$$

Similarly one can describe the homogeneous flag variety corresponding to the maximal parabolic

$$X(2) = \{V \mid V \text{ is a 2-subspace of } C\}.$$

5.42. Let R be a k-algebra. By an i-submodule in $C_R = C \otimes_k R$ we mean a direct summand V_i of C_R of rank i such that $uv = 0$ for every two elements $u, v \in V_i$. The value of the functor corresponding to the variety $X(1,2)$ (respectively $X(2)$) at a k-algebra R equals the set of all flags $V_1 \subset V_2$ (respectively submodules V_2) where V_i is an i-submodule of C_R.

5.43. On the scheme $X = X(2)$ there is a "tautological" vector bundle $\mathcal{V}$ of rank 2. The value of $\mathcal{V}$ on an X-algebra (R, x) is V, where $x = V$.

There exists an obvious morphism

$$X(1,2) \to X(2)$$
$$(V_1, V_2) \mapsto V_2$$

which makes $X(1,2)$ into an $X(2)$-scheme.

5.44 Lemma. *$X(1,2)$ as a scheme over $X(2)$ can be identified with the projective bundle $\mathbb{P}(\mathcal{V}) = \mathrm{Gr}(1, \mathcal{V})$ over $X(2)$.*

Proof. The proof goes as in B_n-case (note that if V_2 is a 2-submodule then each of its direct summands of rank 1 is a 1-submodule). □

5.45 (Proof of Theorem 5.12)**.** Apply Lemma 5.44 and Proposition 2.4.

5.46. Let G be a group of type F_4. We suppose that characteristic of k is not $2, 3$. It is known that $G = \mathrm{Aut}(J)$ where J is an exceptional simple Jordan algebra of dimension 27 over k. Set $I = \{1, 2, 3, 6\}$. By *i-space* where $i \in I$ we mean an i-dimensional subspace V_i of J such that every $u, v \in V_i$ satisfy the following condition:

$$\mathrm{tr}(u) = 0,\ u \times v = 0,\ \text{and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

It is known that projective G-homogeneous varieties are parameterized by sequences of numbers $d_1 < \ldots < d_k$ from I and can be described as follows:

$$X(d_1, \ldots, d_k) = \{V_1 \subset \ldots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } A\}.$$

5.47. Let R be a k-algebra. By an i-submodule in $J_R = J \otimes_k R$ we mean a direct summand V_i of J_R of rank i such that every two elements $u, v \in V_i$ satisfy the conditions above. The value of the functor corresponding to the variety $X(d_1, \ldots, d_k)$ at a k-algebra R equals the set of all flags $V_1 \subset \ldots \subset V_k$ where V_i is a d_i-submodule of J_R.

For convenience we set $d_0 = 0$, $V_0 = 0$.

5.48. On the scheme $X = X(d_1, \dots, d_k)$ there are "tautological" vector bundles $\mathcal{V}_i$ of rank d_i. The value of $\mathcal{V}_i$ on an X-algebra (R, x) is V_i, where $x = (V_1, \dots, V_k)$.

There exists an obvious morphism

$$X(d_1, \dots, d_k) \to X(d_1, \dots, \hat{d}_m, \dots, d_k)$$
$$(V_1, \dots, V_k) \mapsto (V_1, \dots, \hat{V}_m, \dots, V_k)$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$-scheme.

5.49 Lemma. *Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X. Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X.*

Proof. The proof goes as in B_n-case (note that under our restrictions if V_{m+1} is a d_{m+1}-submodule then each of its direct summands of rank d_m is a d_m-submodule). □

5.50 (Proof of Theorem 5.13)**.** Applying Lemma 5.49 to the varieties $Y = X(d_1, \dots, d_k)$ and $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$, we obtain that Y is a Grassmann bundle over X. To finish the proof it remains to apply Proposition 2.4.

6 Classification of motives of projective homogeneous varieties up to dimension 5

6.1 Introduction

The present chapter can be viewed as a further application of the methods and results obtained by N. Karpenko [Ka00].

Let k be a field of characteristic not 2 and k_s denote its separable closure. For a variety X over k we denote by X_s the base change $X \times_k k_s$. Recall (see [MPW96, section 1]) that X is a twisted flag variety of inner type over k if X is a twisted form ${}_\xi(G/P)$ of the variety G/P, where G is a split simple adjoint algebraic group and the 1-cocycle $\xi \in Z^1(k, G(k_s))$.

The present chapter is devoted to the following

6.1 Problem. *Describe all pairs (X, Y) of non-isomorphic twisted flag varieties X and Y of inner type over k, which have isomorphic Chow motives.*

This problem can be subdivided into two subproblems:

(i) Describe all such pairs (X,Y) with $X_s \simeq Y_s$;

(ii) Describe all such pairs (X,Y) with $X_s \not\simeq Y_s$.

Let us briefly recall what is known so far. The complete solution of the problem (i) is known for quadrics and Severi-Brauer varieties due to Izhboldin, Karpenko, Merkurjev, Rost, Vishik and others (see [I98], [Ka95], [Ka00], [Ro98], [Vi03]). Concerning (ii), an example (of dimension 5) was provided by Bonnet in [Bo03]. It deals with twisted flag varieties of type G_2.

In the present chapter we provide a complete solution of the mentioned above problem for projective homogeneous varieties of dimension less than 6. Namely, we prove the following (using the notation of 6.6)

6.2 Theorem. *Let X and Y be non-isomorphic twisted flag varieties of dimension ≤ 5 of inner type over k, which have isomorphic Chow motives.*

1. *If $X_s \simeq Y_s$, then either*
   - *(a) $X = \mathrm{SB}(A)$ and $Y = \mathrm{SB}(A^{\mathrm{op}})$ are Severi-Brauer varieties corresponding to a central simple algebra A and its opposite A^{op} respectively, where $\deg(A) = 3, 4, 5, 6$ and $\exp(A) > 2$, or*
   - *(b) $X = \mathrm{SB}_{2,3}(A)$ and $Y = \mathrm{SB}_{2,3}(A^{\mathrm{op}})$, where the central simple algebra A has degree 4 and exponent 4.*
2. *If $X_s \not\simeq Y_s$, then either*
   - *(a) $X = \mathbb{P}^n$ and $Y = Q^n$ for odd $1 < n \leq 5$, or*
   - *(b) $X = \mathrm{SB}_{1,3}(A)$ and $Y = \mathrm{SB}_{2,3}(A')$, where $\deg(A) = \deg(A') = 4$ and $A \simeq A', A'^{\mathrm{op}}$, or*
   - *(c) $X = {}_{\xi}(G/P_1)$ and $Y = {}_{\xi}(G/P_2)$ are the twisted forms of the variety G/P_i, $i = 1, 2$, where G is the split exceptional group of type G_2 and P_i is one of its maximal parabolic subgroups, or*
   - *(d) $X = \mathrm{G}_2/P_2$ and $Y = \mathbb{P}^5$.*

6.3 Remark. Observe that the case $X = {}_{\xi}(\mathrm{G}_2/P_1)$ and $Y = {}_{\xi}(\mathrm{G}_2/P_2)$ of the theorem is the example of Bonnet mentioned above and, hence, is the minimal one in the sense of dimension.

6.4 Remark. The case $X = \mathrm{SB}_{1,3}(A)$ and $Y = \mathrm{SB}_{2,3}(A')$ with $A \simeq A', A'^{\mathrm{op}}$ provides another minimal example of two non-isomorphic varieties, which have isomorphic Chow motives.

Apart from Theorem 6.2, for every prime $p > 3$, we provide new examples of twisted flag varieties of type A_{p-1} that satisfy conditions (i) and (ii). Namely, we prove the following

6.5 Theorem. *Let $X = \mathrm{SB}_{d_1,\dots,d_k}(A)$ and $Y = \mathrm{SB}_{e_1,\dots,e_k}(A')$ be twisted flag varieties of inner type A_n, $n \geq 2$, over k, where the central simple algebras A and A' have exponents 1, 2, 3, 4, or 6. Assume that*

(i) $\mathcal{M}(X_s) \simeq \mathcal{M}(Y_s)$;

(ii) $d_1 = 1$ *or* $d_k = n$

(iii) $e_1 = 1$ *or* $e_k = n$.

Then $\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow A \simeq A'$ *or* A'^{op}.

The chapter is organized as follows. In section 6.2 we consider the case of a split group and state several facts which will be extensively used in the proofs. Section 6.3 is devoted to the case by case proof of Theorem 6.2. In section 6.4 we prove Theorem 6.5 and provide several results that we need for the proof of Theorem 6.2.

6.2 Preliminaries

In the chapter we use the following notation.

6.6. Let G be a split simple algebraic group defined over a field k. We fix a split maximal torus T of G and a Borel subgroup B of G containing T and defined over k. Denote by Φ the root system of G, by $\Pi = \{\alpha_1, \dots, \alpha_{\mathrm{rk}\, G}\}$ the set of simple roots of Φ corresponding to B, by W the Weyl group, and by $S = \{s_1, \dots, s_{\mathrm{rk}\, G}\}$ the corresponding set of fundamental reflections. Let P_Θ be the standard parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P_\Theta = BW_\Theta B$, where $W_\Theta = \langle s_i,\, \alpha_i \in \Theta\rangle$. Denote by P_i the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$. By Φ/P_Θ we denote the flag variety G/P_Θ. The root enumeration follows Bourbaki.

The notation $\mathrm{SB}_{n_1,\dots,n_r}(A)$, $1 \leq n_1 < \ldots < n_r \leq n$, is used for the twisted form of the variety A_n/P_Θ, where $\Theta = \Pi \setminus \{\alpha_{n_1}, \dots, \alpha_{n_r}\}$ and A is a central simple algebra of degree $n+1$ corresponding to the twisting. Observe that $\mathrm{SB}_{n_1,\dots,n_r}(A) = X(A; n_1,\dots,n_r)$ in the notation of [MPW96, Appendix] and $\mathrm{SB}(A) = \mathrm{SB}_1(A)$ is the usual Severi-Brauer variety defined by A. By $\mathrm{ind}(A)$ we denote the index of A and by $\exp(A)$ its exponent. The split projective quadric of dimension n is denoted by Q^n.

6.7. According to [Ko91] the Chow motive of the flag variety $X = G/P_\Theta$, when G is a split group, is isomorphic to

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{\dim X} \mathbb{Z}(i)^{\oplus a_i(X)},$$

where the positive integers (ranks) $a_i(X)$ are the coefficients of the generating polynomial $p_X(z) = \sum_{i=0}^{\dim X} a_i(X) z^i$. The latter is defined by the following explicit formula:

$$p_X(z) = \Big(\prod_{i=1}^{\mathrm{rk}\, G} \frac{z^{d_i(W_\Theta)}-1}{z-1}\Big) / \Big(\prod_{j=1}^{m}\prod_i \frac{z^{d_i(W_j)}-1}{z-1}\Big).$$

Here $W_1 \times \ldots \times W_m$ is the decomposition of W_Θ into a product of the Weyl groups corresponding to the irreducible root systems and $d_i(W_j)$ are the degrees of the respective fundamental polynomial invariants (see [Ca72, 9.4 A]).

The coefficients $a_i(X)$ can be also computed as follows (see [Ko91]):

$$a_i(X) = \#\{w \in W \mid \forall\, \alpha_i \in \Theta\ l(ws_i) = l(w)+1, l(w) = i\}.$$

The dimension of a projective homogeneous variety X can be computed by the following formula:

$$\dim X = \Big(\sum_i d_i(G) - \mathrm{rk}\, G\Big) - \sum_j \Big(\sum_i d_i(P_j) - \mathrm{rk}\, P_j\Big) = |\Phi^+| - |\Phi_P^+|.$$

The following observation follows from the above isomorphism.

6.8. *The motives of flag varieties X and Y of dimension n over a separably closed field are isomorphic iff the corresponding sequences of ranks $(a_0(X),\dots,a_n(X))$ and $(a_0(Y),\dots,a_n(Y))$ are equal.*

We shall need the following two facts:

6.9. (See [Ka00, Criterion 7.1]) Let A, A' be central simple algebras over k and $\mathrm{SB}(A)$, $\mathrm{SB}(A')$ be the respective Severi-Brauer varieties. Then

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \mathcal{M}(\mathrm{SB}(A')) \Leftrightarrow A \simeq A', A'^{\mathrm{op}}.$$

6.10. (See [I98, Cor. 2.9 and Prop. 3.1]) Let q, q' be regular quadratic forms of rank n and X_q, $X_{q'}$ be the respective projective quadrics. If n is odd or $n < 7$, then

$$\mathcal{M}(X_q) \simeq \mathcal{M}(X_{q'}) \Leftrightarrow X_q \simeq X_{q'}.$$

Finally, we shall need the following observation:

6.11. (See [Ka00, Proof of Lemma 2.3]) Let X and Y be smooth projective varieties over k with isomorphic Chow motives. Then there is an isomorphism of abelian groups

$$\mathrm{Coker}(\mathrm{CH}_0(X) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(X_s)) \simeq \mathrm{Coker}(\mathrm{CH}_0(Y) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(Y_s)).$$

6.3 Small dimensions

In this section we classify all pairs (X, Y) of non-isomorphic twisted flag varieties of inner type over k of dimension ≤ 5 with isomorphic Chow motives and, hence, prove Theorem 6.2.

Dimension 1. Twisted flag varieties of dimension 1 are the twisted forms of the projective line $\mathbb{P}^1$. The twisted forms of $\mathbb{P}^1$ are Severi-Brauer varieties $\mathrm{SB}(H)$, where H is a quaternion algebra. By 6.9

$$\mathcal{M}(\mathrm{SB}(H)) \simeq \mathcal{M}(\mathrm{SB}(H')) \Leftrightarrow H \simeq H', H'^{\mathrm{op}}$$

Since $H \simeq H^{\mathrm{op}}$, we conclude that the motives are isomorphic iff the varieties are isomorphic.

Dimension 2. All twisted flag varieties of dimension 2 are the twisted forms of the projective space $\mathbb{P}^2$ or the split quadric surface $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Observe that Q^2 is a projective homogeneous variety for a group of type D_2 which is not simple, but semisimple. Nevertheless, we shall consider this case too.

The motives of $\mathbb{P}^2$ and Q^2 are not isomorphic, since the respective sequences of ranks $(1, 1, 1)$ and $(1, 2, 1)$ are different.

The twisted forms of Q^2 of inner type over k are 2-dimensional quadrics (see [Inv, Cor. (15.12)]). By 6.10 the motives of two quadrics of dimension 2 are isomorphic iff the quadrics are isomorphic.

The twisted forms of $\mathbb{P}^2$ are Severi-Brauer varieties $\mathrm{SB}(A)$, where A is a central simple algebra of degree 3. Again by 6.9 we have

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \mathcal{M}(\mathrm{SB}(A')) \Leftrightarrow A \simeq A', A'^{\mathrm{op}}.$$

Since the varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A^{\mathrm{op}})$ are isomorphic iff A is split, we conclude that all pairs of non-isomorphic varieties with isomorphic motives are of the kind $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 3.

Dimension 3. Computing the generating functions (see 6.7) we conclude that there are only three projective homogeneous varieties of dimension 3 over k_s. Namely, the projective space $\mathbb{P}^3$, the quadric Q^3 and the variety of complete flags A_2/B (B denotes a Borel subgroup). The respective sequences of ranks look as follows:

$$\begin{array}{rcl} \mathbb{P}^3 \simeq \mathrm{A}_3/P_1 & : & (1,1,1,1) \\ Q^3 \simeq \mathrm{B}_2/P_1 & : & (1,1,1,1) \\ \mathrm{A}_2/B & : & (1,2,2,1) \end{array}$$

In particular, we see that the motives of $\mathbb{P}^3$ and Q^3 are isomorphic but the motives of Q^3 and A_2/B are not.

By 6.9 all non-isomorphic twisted forms of $\mathbb{P}^3$, which have isomorphic motives, form pairs $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 4 and exponent 4. Observe that all non-isomorphic twisted forms of Q^3 are quadrics as well and by 6.10 the motive of a quadric determines this quadric uniquely. Therefore it remains to describe all possible motivic isomorphisms between the twisted forms ${}_\xi\mathbb{P}^3$ and ${}_\zeta Q^3$ and the twisted forms ${}_\xi(\mathrm{A}_2/B)$ and ${}_\zeta(\mathrm{A}_2/B)$ of the variety of complete flags A_2/B.

According to Corollary 6.18 there are no non-isomorphic twisted forms of A_2/B with isomorphic Chow motives. And the next lemma shows that there are no such (non-trivial) twisted forms of $\mathbb{P}^3$ and Q^3.

6.12 Lemma. *Let ξ, ζ be 1-cocycles. Then $\mathcal{M}({}_\xi\mathbb{P}^3) \simeq \mathcal{M}({}_\zeta Q^3)$ iff ξ and ζ are trivial.*

Proof. This is a particular case of a more general result (see Lemma 6.16) proven using the Index Reduction Formula. Here we give an elementary proof. It uses only well-known facts about quadrics and Severi-Brauer varieties.

Observe that any twisted form of $\mathbb{P}^3$ is a Severi-Brauer variety $\mathrm{SB}(A)$ for some central simple algebra A of degree 4 and any twisted form of Q^3 is a non-singular quadric of dimension 3.

As in 6.11 for a variety X consider the abelian group $\mathrm{Coker}(\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_s))$. If $X = \mathrm{SB}(A)$ is a Severi-Brauer variety of a central simple algebra A, then this cokernel is equal to $\mathbb{Z}/\operatorname{ind}(A)\mathbb{Z}$ (see [Ka00]), where $\operatorname{ind}(A)$ is the index of A. In particular, this cokernel is trivial iff A is split. If X is a quadric then this cokernel is trivial iff X is isotropic. In the case X is an anisotropic quadric this cokernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In our case we have two varieties $X = \mathrm{SB}(A)$ and $Y = {}_{\zeta}Q^3$ with isomorphic motives. Hence, by 6.11 the respective cokernels must be isomorphic.

Hence, if the quadric Y is isotropic, then the algebra A is split. The latter implies that the motive $\mathcal{M}(\mathrm{SB}(A))$ splits into a direct sum of the Lefschetz motives and so is $\mathcal{M}(Y)$, i.e., Y is split as well by 6.10.

Assume q is anisotropic, then there exists a quadratic field extension l/k such that the Witt index of $Y_l = Y \times_k l$ is one (see [Vi03, §7.2]). Since the motives of X and Y are still isomorphic over l, we conclude that A is split over l. Then Y_l is split as well. This leads to a contradiction. □

6.13 Remark. Observe that the pair of twisted forms $({}_{\xi}(\mathrm{B}_2/P_1), {}_{\xi}(B_2/P_2))$ can be viewed as a low-dimensional analog of the pair $({}_{\xi}(\mathrm{G}_2/P_1), {}_{\xi}(\mathrm{G}_2/P_2))$ considered by Bonnet. The lemma says that contrary to the G_2-case the motives of ${}_{\xi}(\mathrm{B}_2/P_1)$ and ${}_{\xi}(\mathrm{B}_2/P_2)$ are not isomorphic (if ξ is non-trivial).

Dimension 4**.** There are three non-isomorphic projective homogeneous varieties of dimension 4 over k_s. Namely, the projective space $\mathbb{P}^4$, the 4-dimensional quadric $Q^4 \simeq \mathrm{Gr}(2,4)$ and the variety of complete flags B_2/B. The respective sequences of ranks in these cases are all different and look as follows:

$$\begin{array}{rcl} \mathbb{P}^4 \simeq \mathrm{A}_4/P_1 & : & (1,1,1,1,1) \\ Q^4 \simeq \mathrm{A}_3/P_2 & : & (1,1,2,1,1) \\ \mathrm{B}_2/B & : & (1,2,2,2,1) \end{array}$$

Hence, the motives of $\mathbb{P}^4$, Q^4 and B_2/B are non-isomorphic to each other.

By 6.9 all non-isomorphic twisted forms of $\mathbb{P}^4$ with isomorphic motives form pairs $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 5. By

Corollary 6.20 there are no non-isomorphic twisted forms of B_2/B with isomorphic Chow motives. Therefore the only case left is the case of inner twisted forms of Q^4.

The inner forms of Q^4 are the generalized Severi-Brauer varieties $\mathrm{SB}_2(A)$, where A is a central simple algebra of degree 4. The next lemma shows that there are no non-isomorphic forms of $\mathrm{SB}_2(A)$, which have isomorphic motives.

6.14 Lemma. *Let A, A' be central simple algebras of degree 4. Then*

$$\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}_2(A')) \Leftrightarrow \mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A')$$

Proof. Let $\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}_2(A'))$. It suffices to prove that for all field extensions l/k one has $\mathrm{ind}(A_l) = \mathrm{ind}(A'_l)$. Indeed, by [Ka00, Lemma 7.13] $\langle A\rangle = \langle A'\rangle$ in $\mathrm{Br}(k)$, hence, $A \simeq A'$ or A'^{op}. But $\mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A^{\mathrm{op}})$ for any central simple algebra A of degree 4.

Assume that there exists a field extension l/k such that $\mathrm{ind}(A_l) \neq \mathrm{ind}(A'_l)$. Depending on the indices of A and A' we distinguish the following cases:

Case 1. $\mathrm{ind}(A) = 4$ and $\mathrm{ind}(A') = 1$ or 2.

In this case $\mathrm{SB}_2(A')$ has a rational point. By [Inv, Case $\mathrm{A}_3 = \mathrm{D}_3$], the variety $\mathrm{SB}_2(A')$ is an isotropic quadric, hence, the group

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}_2(A')) \to \mathrm{CH}_0(\mathrm{SB}_2(A'_{k_s}))$$

is trivial. By 6.11 the cokernel

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}_2(A)) \to \mathrm{CH}_0(\mathrm{SB}_2(A_{k_s}))$$

must be trivial as well. If $\exp(A) = 2$, then A is a biquaternion algebra and by [Inv, Cor. (15.33)] $\mathrm{SB}_2(A)$ is an anisotropic quadric. Then the cokernel above must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction. If $\exp(A) = 4$, then by [Inv, Cor. (15.33)] $A \simeq C^{\pm}(B, \sigma, f)$, where $(B, \sigma, f) \in {}^1\mathrm{D}_3$ and B is a central simple algebra of degree 6 and index 2. Therefore the cokernel above must be again isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction.

Case 2. $\mathrm{ind}(A) = 2$ and $\mathrm{ind}(A') = 1$.

In this case A' is split, hence, the corresponding variety is a split quadric. On the other hand, $\mathrm{SB}_2(A) \simeq X_q$, where q is some 6-dimensional quadratic form and X_q is the corresponding projective quadric. Using 6.10, we conclude that $\mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A')$, a contradiction. □

Dimension 5. There are five non-isomorphic projective homogeneous varieties over k_s of dimension 5. Namely, the projective space $\mathbb{P}^5$, the quadric Q^5, the exceptional Fano variety G_2/P_2, the flag varieties $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$. The respective sequences of ranks look as follows:

$$\begin{array}{ccc}
\mathbb{P}^5 \simeq \mathrm{A}_5/P_1 & : & (1,1,1,1,1,1) \\
Q^5 \simeq \mathrm{B}_3/P_1 & : & (1,1,1,1,1,1) \\
\mathrm{G}_2/P_2 & : & (1,1,1,1,1,1) \\
\mathrm{A}_3/P_{\{\alpha_1\}} \simeq \mathrm{A}_3/P_{\{\alpha_3\}} & : & (1,2,3,3,2,1) \\
\mathrm{A}_3/P_{\{\alpha_2\}} & : & (1,2,3,3,2,1)
\end{array}$$

Therefore, the motives of $\mathbb{P}^5$, Q^5 and G_2/P_2 are isomorphic and the motives of $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$ are isomorphic.

As mentioned before, the twisted forms of $\mathbb{P}^5$ and Q^5 were completely classified up to motivic isomorphisms by Karpenko and Izhboldin (see 6.9 and 6.10). Namely, all such non-isomorphic forms are of the kind $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a central simple algebra of degree 6 with $\exp(A) > 2$. Moreover, by Lemma 6.16 there there is only one pair $({}_\xi\mathbb{P}^5, {}_\zeta Q^5)$ of twisted forms with isomorphic motives.

By the result of Bonnet [Bo03] the motive of a twisted form ${}_\xi(\mathrm{G}_2/P_2)$ is isomorphic to the motive of ${}_\xi(\mathrm{G}_2/P_1)$, which is a 5-dimensional quadric.

By Corollary 6.19 the motives of the twisted forms of $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$ are isomorphic iff the respective central simple algebras of degree 4 are isomorphic or opposite. This provides the last example (see Theorem 6.2) of a pair of non-isomorphic varieties of dimension 5 with isomorphic motives.

6.4 Arbitrary dimensions

In the present section we prove several classification results. We start with the following

6.15 Lemma. *Let X and Y be twisted flag varieties of inner type over k with isomorphic Chow motives. Assume X is not of type* E_8 *and splits over its function field $k(X)$, i.e., the group corresponding to X splits over $k(X)$. Then X splits over the function field of Y.*

Proof. Since the motives are isomorphic, there is an isomorphism of cokernels (see 6.11) and, hence, an isomorphism of cokernels over $k(Y)$

$$\mathrm{Coker}(\mathrm{CH}_0(X_{k(Y)}) \to \mathrm{CH}_0(X_{k(Y)_s})) \simeq \mathrm{Coker}(\mathrm{CH}_0(Y_{k(Y)}) \to \mathrm{CH}_0(Y_{k(Y)_s}))$$

Since $Y_{k(Y)}$ is isotropic, the right cokernel is trivial and so is the left one. The fact that the map $\mathrm{res}\colon \mathrm{CH}_0(X_{k(Y)}) \to \mathrm{CH}_0(X_{k(Y)_s})$ is surjective and the group $\mathrm{CH}_0(X_{k(Y)_s})$ is a free abelian group of rank one generated by the class of a rational point $[pt]$ implies that the preimage $\mathrm{res}^{-1}([pt])$ is a 0-cycle of degree 1 in $\mathrm{CH}_0(X_{k(Y)})$. Then, by [To04, Q. 0.2] $X_{k(Y)}$ is isotropic. Since X splits over its function field, $X_{k(Y)}$ splits as well. □

6.16 Lemma. *Let γ, δ be 1-cocycles and $X = {}_\gamma\mathbb{P}^n$, $Y = {}_\delta Q^n$ be the respective twisted forms for $n > 1$ odd. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow \gamma \text{ and } \delta \text{ are trivial.}$$

Proof. Observe that X is a Severi-Brauer variety corresponding to a central simple algebra A and Y is an n-dimensional quadric.

Assume that $\mathcal{M}(X) \simeq \mathcal{M}(Y)$ and γ is not trivial. By Lemma 6.15 applied to X and Y, the algebra $A_{k(Y)}$ splits, i.e., $\mathrm{ind}(A_{k(Y)}) = 1$. On the other hand by the Index Reduction Formula (see [MPW96]) we obtain

$$\mathrm{ind}(A_{k(Y)}) = \min\{\mathrm{ind}(A), 2^{(n-1)/2}\,\mathrm{ind}(A \otimes_k C_0(q))\} > 1,$$

where $C_0(q)$ is the even part of the Clifford algebra of the quadratic form corresponding to Y. This leads to a contradiction. □

Note that the same proof works for twisted forms of types B_n and C_n. Namely,

6.17 Proposition. *Let γ, δ be 1-cocycles and $X = {}_\gamma(\mathrm{C}_n/P_l)$, $Y = {}_\delta(\mathrm{B}_n/P_l)$ be the respective twisted forms for an odd $1 \leq l < n$. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow \gamma \text{ and } \delta \text{ are trivial.}$$

The rest of this section is devoted to the twisted forms of flag varieties. In particular, we obtain the description of motivic isomorphisms for twisted forms of the flag varieties A_2/B, B_2/B and $\mathrm{A}_3/P_{\{\alpha_i\}}$, $i = 1, 2, 3$. We start with the proof of Theorem 6.5.

Proof of Theorem 6.5. W.l.o.g. we may assume that $d_1 = e_1 = 1$. Assume $\mathcal{M}(X) \simeq \mathcal{M}(Y)$. Since X and Y are twisted forms of flags "containing" the subspace of a minimal dimension, the motives of X and Y can be decomposed into a direct sum of twisted motives of Severi-Brauer varieties (see 5.1). Namely,

$$\mathcal{M}(X) \simeq \bigoplus_i \mathcal{M}(\mathrm{SB}(A))(i), \quad \mathcal{M}(Y) \simeq \bigoplus_j \mathcal{M}(\mathrm{SB}(A'))(j). \qquad (*)$$

This together with 6.11 implies the isomorphism of abelian groups

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}(A)) \to \mathrm{CH}_0(\mathbb{P}^n)) \simeq \mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}(A')) \to \mathrm{CH}_0(\mathbb{P}^n))$$

and, hence, the isomorphism $\mathbb{Z}/\operatorname{ind}(A)\mathbb{Z} \simeq \mathbb{Z}/\operatorname{ind}(A')\mathbb{Z}$, i.e., $\operatorname{ind}(A) = \operatorname{ind}(A')$. Since the motivic isomorphism is preserved under the base extensions, we obtain that $\operatorname{ind}(A_l) = \operatorname{ind}(A'_l)$ for any finite field extension l/k. The latter is equivalent to the condition $\langle A\rangle = \langle A'\rangle$ in $\mathrm{Br}(k)$. By conditions of the theorem this is equivalent to $A \simeq A'$ or A'^{op}.

In the opposite direction, let $A \simeq A'$ or A'^{op}. By conditions (i)-(iii) one has two motivic decompositions (*) with the same sets of indices $\{i\}$ and $\{j\}$. Now according to 6.9 the motives of $\mathrm{SB}(A)$ and $\mathrm{SB}(A')$ are isomorphic and, hence, so are $\mathcal{M}(X)$ and $\mathcal{M}(Y)$. □

The following obvious consequences of Theorem 6.5 are used in the proof of Theorem 6.2.

6.18 Corollary. *Let X and Y be twisted forms of the variety of complete flags A_n/B. Let A and A' denote the central simple algebras corresponding to X and Y respectively. Assume that* $\exp(A)$, $\exp(A')$ *equal* 1, 2, 3, 4 *or* 6. *Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow X \simeq Y.$$

6.19 Corollary. *Let $X = \mathrm{SB}_{1,3}(A)$ and $Y = \mathrm{SB}_{2,3}(A')$. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow A \simeq A' \text{ or } A'^{\mathrm{op}}.$$

6.20 Corollary. *Let X and Y be twisted forms of the variety of complete flags B_2/B. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow X \simeq Y.$$

Proof. The proof repeats the proof of 6.5 observing that the motivic decompositions (*) are provided in 5.9. □

6.5 Incidence varieties

In this section we give elementary proofs of some particular cases of the classification theorem above.

Let us fix some notation. Let X, Y be any smooth projective varieties. For any cycle J on X we denote as $\overline{J}$ its image in $\mathrm{CH}(X_s)$ under the restriction map $\mathrm{res}_{k_s/k}$. If $\varphi\colon X \to Y$ is a morphism, then $\overline{\varphi}$ denotes its scalar extension $\varphi \times_k k_s$.

A cycle J on X is called rational if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X_s)$.

First we consider the variety G/B, where G is the split group of type A_2. The ring $\mathrm{CH}((G/B)_s)$ has generators

$$[X_1], \quad [X_{s_1s_2s_1}] = 1, \quad [X_{s_1s_2}] := h, \quad [X_{s_2s_1}] := g, \quad [X_{s_1}] = g^2, \quad [X_{s_2}] = h^2.$$

We calculated the multiplicative structure using Pieri's formula.

Consider the smooth projective morphism $\varphi\colon {}_\gamma(G/B) \to {}_\gamma(G/P_1)$ (γ is a 1-cocycle). The corresponding push-forward morphism $\overline{\varphi}_*$ acts on the generators as follows:

$$1 \mapsto 0, \quad h \mapsto 0, \quad g \mapsto 1, \quad g^2 \mapsto h, \quad h^2 \mapsto 0, \quad [X_1] \mapsto h^2.$$

6.21 Proposition. *Let X, X' be inner twisted forms of G/B, where G is the split group of type A_2 and B its Borel subgroup. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(X') \Leftrightarrow X \simeq X'.$$

Proof. Denote as A, A' the central simple algebras of degree 3 corresponding to $X = {}_\gamma(G/B)$ and $X' = {}_{\gamma'}(G/B)$. Let J be a motivic isomorphism between X and X'. $\overline{J}$ has the following form:

$$\begin{aligned}\overline{J} = a_1 1 \times [X_1] + a_2[X_1] \times 1 + a_3 h \times h^2 + a_4 h \times g^2 \\ + a_5 g \times h^2 + a_6 g \times g^2 + a_7 h^2 \times h + a_8 h^2 \times g \\ + a_9 g^2 \times h + a_{10} g^2 \times g \in \mathrm{CH}^3((G/B)_s \times (G/B)_s),\end{aligned}$$

where $a_i \in \mathbb{Z}$. Since all object under consideration split completely over a suitable field extension of degree 3, we may work modulo 3 w.l.o.g. Since $\overline{J}$ is an isomorphism, it has a summand $\pm g^2 \times \ldots$

We claim that there exists a maximal standard parabolic subgroup P' in G such that the push-forward map

$$(\overline{\varphi}_* \times \overline{\varphi}'_*)(\overline{J}) = \pm h \times 1 + a 1 \times h' \in \mathrm{CH}^1((G/P_1)_s \times (G/P')_s)$$

for some $a \in \{0, \pm 1\}$ ($\varphi' \colon {}_{\gamma'}(G/B) \to {}_{\gamma'}(G/P')$).

Since push-forwards preserve rationality of cycles, the cycle $h \times 1 \pm a 1 \times h'$ is rational. If $a = \pm 1$, then we apply the theorem of Karpenko which says that the cycle $h \times 1 \pm 1 \times h'$ is rational if and only if $A \simeq A', A'^{\mathrm{op}}$ (see [Ka00]).

If $a = 0$, then h is rational because of the structure of the Picard group of ${}_{\gamma}(G/P_1)$ (see [MT95]). Hence A is split. Therefore A' is split.

It remains to notice that the varieties X and X' are isomorphic (see [Inv, Prop. (1.19)]), if $A \simeq A'^{\mathrm{op}}$. □

Case B_2/B**.** Let G be the split group of type B_2. The ring $\mathrm{CH}(G/B)$ has generators

$$[X_1], \quad [X_{s_2 s_1 s_2 s_1}] = 1, \quad [X_{s_1 s_2 s_1}] := h, \quad [X_{s_2 s_1 s_2}] := g,$$
$$[X_{s_1 s_2}] = g^{(2)}, \quad [X_{s_2 s_1}] = h^2, \quad [X_{s_2}] = g^{(3)} = g g^{(2)}, \quad [X_{s_1}] = h^3.$$

We calculated the multiplicative structure using Pieri's formula.

Consider the smooth projective morphism $\varphi \colon {}_{\gamma}(G/B) \to {}_{\gamma}(G/P_2)$. The variety ${}_{\gamma}(G/P_2)$ is a twisted form of $\mathbb{P}^3$. The corresponding push-forward morphism $\overline{\varphi}_*$ acts on the generators as follows:

$$1 \mapsto 0, \quad h \mapsto 0, \quad g \mapsto 1, \quad g^{(2)} \mapsto h,$$
$$h^2 \mapsto 0, \quad [X_1] \mapsto h^2, \quad h^3 \mapsto 0, \quad g^{(3)} \mapsto h^2.$$

6.22 Proposition. *Let X and X' be twisted forms of G/B, where G is the split group of type B_2 and B its Borel subgroup. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(X') \Leftrightarrow X \simeq X'.$$

Proof. We have $X = {}_{\gamma}(G/B)$ and $X' = {}_{\gamma'}(G/B)$. Denote as A, A' the central simple algebras of degree 3 corresponding to ${}_{\gamma}(G/P_2)$ and ${}_{\gamma'}(G/P_2)$. Let $J \in \mathrm{CH}^4(X \times X')$ be a motivic isomorphism between X and X'.

We proceed similar to the case A_2/B. W.l.o.g. we may work modulo 4. Apply the push-forward map $\varphi_* \times \varphi'_*$ to J, where φ' is a projection ${}_{\gamma'}(G/B) \to$

${}_{\gamma'}(G/P_1)$ or ${}_{\gamma'}(G/B) \to {}_{\gamma'}(G/P_2)$. The image of J lies in $\mathrm{CH}^2({}_\gamma(G/P_2) \times {}_{\gamma'}(G/P'))$ for some P' (P' is either P_1 or P_2).

Apply now the pull-back homomorphism

$$\mathrm{CH}^2({}_\gamma(G/P_2) \times {}_{\gamma'}(G/P')) \to \mathrm{CH}^2({}_\gamma(G/P_2) \times {}_{\gamma'}(G/B)).$$

Again using some push-forward φ'' we get a (rational) cycle r in $\mathrm{CH}^1({}_\gamma(G/P_2)\times {}_{\gamma'}(G/P''))$ (P'' is either P_1 or P_2). Since J is an isomorphism, we can always choose P', P'' in such a way that $\overline{r} = \pm h \times 1 + a1 \times d'$, where d' is either h or g in $\mathrm{CH}^1((G/B)_s)$, $a \in \mathbb{Z}/4$.

If d' is g (the generator for a quadric) or $a = 0$, then d' is rational. Therefore $h \times 1$ is rational. Hence A is split. If d' corresponds to h and $a = \pm 1$, then applying Karpenko's arguments we get $A \simeq A', A'^{\mathrm{op}}$. If $a = \pm 2$, then we apply the same arguments to A and A' interchanged. We get that the cycle $h \times 1 + a'1 \times d$ is rational. So it remains to consider only the case, when $d = h$ and $a' = \pm 2$. In this case $\operatorname{ind} A' = 2$. Therefore $h \times 1$ is rational. Hence A is split.

Since $A = C_0(q) \simeq A^{\mathrm{op}}$, where q is the 5-dimensional quadratic form corresponding to ${}_\gamma G$, we are done. □

Using similar arguments it can be shown that

$$\mathcal{M}(X) \simeq \mathcal{M}(X') \Leftrightarrow X \simeq X',$$

if X, X' are twisted forms of G/B, where G is a split group of type G_2 and B its Borel subgroup. We leave this as an exercise.

7 The case of dimension 15

The main motivation for this chapter was the result of N. Karpenko where he gave a shortened construction of a Rost motive for a norm quadric [Ka98]. In the present chapter we provide a shortened and explicit construction of a generalized Rost motive for a norm variety that corresponds to a symbol $(3, 3)$. The latter is given by the Rost-Serre invariant g_3 for an Albert algebra. Namely, we prove the following

7.1 Theorem. *Let k be a field of characteristic different from 2 and 3. Let X be a projective G-homogeneous variety over k, where G is an anisotropic group of type F_4 obtained by the first Tits process, such that over a separable* *closure it becomes isomorphic to G_s/P, where P is the maximal parabolic subgroup corresponding to the first (last) three vertices of the respective Dynkin diagram. Then the (integral) Chow motive of X decomposes as*

$$\mathcal{M}(X) \cong \oplus_{i=0}^{7} R(i),$$

where the motive $R = (X, p)$ is the (integral) generalized Rost motive, i.e., over a separable closure k_s of k it splits as the direct sum of Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

By the next result, we provide the first known "purely exceptional" example of two non-isomorphic varieties with isomorphic motives. Recall that a similar result for groups of type G_2 obtained in [Bo03] provides a motivic isomorphism between a quadric and a Fano variety.

7.2 Theorem. *Under the hypotheses of theorem 7.1 let X_1 and X_2 be two projective homogeneous varieties corresponding to the maximal parabolic subgroups generated by the first (last) three vertices of the Dynkin diagram respectively. Then the motives of X_1 and X_2 are isomorphic.*

7.3. Our proof uses well-known facts concerning linear algebraic groups and projective homogeneous varieties, a computer program that computes the Chow ring $\mathrm{CH}(G/P)$ for a split group G, the Rost Nilpotence Theorem and several procedures that allow to produce rational cycles on $\mathrm{CH}(G/P \times G/P)$. Moreover, the proof works not only for projective homogeneous varieties of type F_4. Applying the similar arguments to Pfister quadrics and their maximal neighbours one obtains the well-known decompositions into Rost motives [Ro98]. For exceptional groups of type G_2 one immediately obtains the motivic decomposition of the variety G_2/P_2 together with the motivic isomorphism found by J.-P. Bonnet [Bo03].

7.4. The chapter is organized as follows. In Section 7.1 we apply the formulae introduced in Section 3.2 to projective homogeneous varieties X_1 and X_2 of type F_4. In Section 7.2 we prove Theorem 7.1. Section 7.3 is devoted to the proof of Theorem 7.2.

7.1 Projective homogeneous varieties of type F_4

7.5. From now on, we assume that the characteristics of the base field k is not equal to 2 or 3. In the present section we remind several well-known facts concerning Albert algebras, groups of type F_4 and respective projective homogeneous varieties (see [PR94], [Inv], [Ga97]). At the end we provide partial computations of the Chow rings of these varieties.

We start with the following observation concerning the Picard group of a projective homogeneous variety of type F_4

7.6 Lemma. *Let X be a projective homogeneous variety such that over a separable closure it becomes isomorphic to G/P, where G is a split group of type F_4 and P its maximal parabolic subgroup. Let $X' = X \times_k k_s$ be its scalar extension to a separable closure k_s. Then the Picard group $\mathrm{Pic}(X')$ is a free abelian group of rank 1 with a rational generator.*

Proof. Since P is maximal, $\mathrm{Pic}(X')$ is a free abelian group of rank 1. We use the following exact sequence (see [Ar82] and [MT95, 2.3]):

$$0 \longrightarrow \mathrm{Pic}\, X \longrightarrow (\mathrm{Pic}\, X')^{\Gamma} \xrightarrow{\alpha_X} \mathrm{Br}(k),$$

where $\Gamma = \mathrm{Gal}(k_s/k)$ is the absolute Galois group and $\mathrm{Br}(k)$ the Brauer group of k. The map α_X is explicitly described in [MT95] in terms of Tits classes. Since groups of type F_4 are adjoint and simply-connected, their Tits classes are trivial and so is α_X. Since Γ acts trivially on $\mathrm{Pic}(X')$ and the image of α_X is trivial, we have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X')$. □

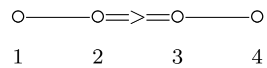
7.7. It is well known that the classification of algebraic groups of type F_4 is equivalent to the classification of Albert algebras (those are 27-dimensional exceptional simple Jordan algebras). All Albert algebras can be obtained from one of the two Tits constructions.

An Albert algebra A obtained by the first Tits construction is produced from a central simple algebra of degree 3. By using the Rost-Serre invariant g_3 (if the input central simple algebra is split, then $g_3 = 0$) one can show that for the respective group $G = \mathrm{Aut}(A)$ only two Tits diagrams ([Ti66, Table II]) are allowed, namely the completely split case and the anisotropic case. This means that

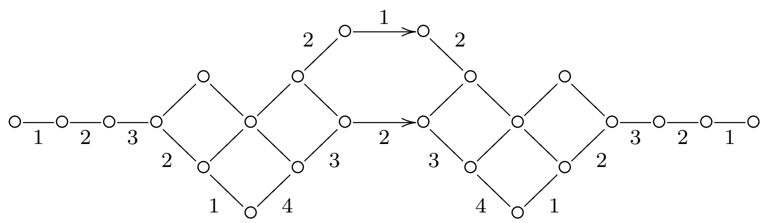
(i) anisotropic G splits completely by a cubic field extension;

(ii) for each i the variety X_i of maximal parabolic subgroups of G of type i splits completely over the function field $k(X_j)$, $j = 1, 2, 3, 4$.

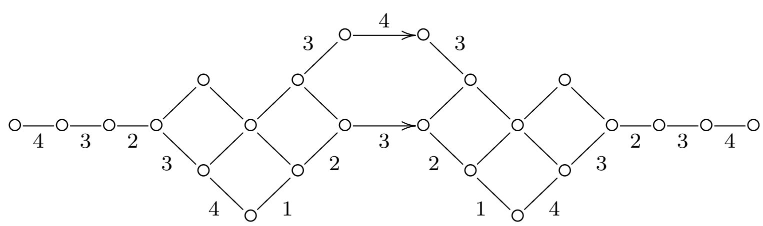
7.8. From this point on we consider a split group G of type F_4. Let $X_1 = G/P_1$ and $X_2 = G/P_4$ be projective homogeneous varieties, corresponding to maximal parabolic subgroups P_1 and P_4 generated by the last $\{2, 3, 4\}$ and the first $\{1, 2, 3\}$ three vertices of the Dynkin diagram



Varieties X_1 and X_2 are not isomorphic and have dimension 15. We provide the Hasse diagrams (graphs) for X_1:



and X_2:



We draw the diagrams in such a way that the labels on the opposite sides of a parallelogram are equal and in that case we omit all labels but one.

Recall that (see 3.6) the vertices of this graph correspond to the basis elements of the Chow group $\mathrm{CH}(X_2)$. The rightmost vertex is the unit class $1 = [X_{w^\theta}]$ and the leftmost one is the class of a 0-cycle of degree 1.

7.9. We denote the basis elements of the respective Chow groups as follows

$$\mathrm{CH}^i(X_1) = \begin{cases} \langle h_1^i \rangle, & i = 0 \ldots 3, 12 \ldots 15, \\ \langle h_1^i, g_1^i \rangle, & i = 4 \ldots 11. \end{cases}$$

$$\mathrm{CH}^i(X_2) = \begin{cases} \langle h_2^i \rangle, & i = 0 \ldots 3, 12 \ldots 15, \\ \langle h_2^i, g_2^i \rangle, & i = 4 \ldots 11. \end{cases}$$

The generators h_i correspond to the upper vertices of the respective Hasse diagrams, and g_i to the lower ones (if the corresponding rank is 2).

7.10. Applying 3.9 we immediately obtain the following partial multiplication table

$$h_k^s g_k^{15-s} = 0, \qquad h_k^s h_k^{15-s} = g_k^s g_k^{15-s} = h_k^{15},$$

where $k = 1, 2$, for all s.

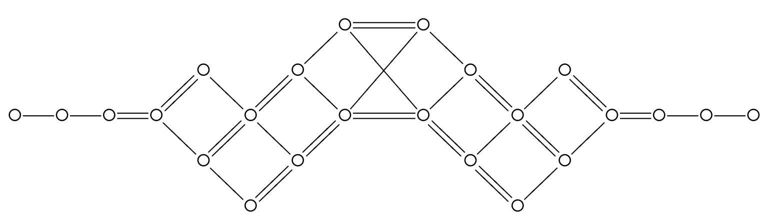
7.11. By Pieri's formula 3.10 we obtain the following partial multiplication tables for $\mathrm{CH}(X_1)$:

$h_1^1 h_1^1 = h_1^2$, $\quad h_1^1 h_1^2 = 2h_1^3$, $\quad h_1^1 h_1^3 = 2h_1^4 + g_1^4$, $\quad h_1^1 h_1^4 = h_1^5$,
$h_1^1 g_1^4 = 2h_1^5 + g_1^5$, $\quad h_1^1 h_1^5 = 2h_1^6 + g_1^6$, $\quad h_1^1 g_1^5 = 2g_1^6$, $\quad h_1^1 h_1^6 = h_1^7 + g_1^7$,
$h_1^1 g_1^6 = 2g_1^7$, $\quad h_1^1 h_1^7 = 2h_1^8 + g_1^8$, $\quad h_1^1 g_1^7 = h_1^8 + 2g_1^8$, $\quad h_1^1 h_1^8 = h_1^9$,
$h_1^1 g_1^8 = h_1^9 + 2g_1^9$, $\quad h_1^1 h_1^9 = 2h_1^{10}$, $\quad h_1^1 g_1^9 = h_1^{10} + 2g_1^{10}$, $\quad h_1^1 h_1^{10} = h_1^{11} + 2g_1^{11}$,
$h_1^1 g_1^{10} = g_1^{11}$, $\quad h_1^1 h_1^{11} = 2h_1^{12}$, $\quad h_1^1 g_1^{11} = h_1^{12}$, $\quad h_1^1 h_1^{12} = 2h_1^{13}$,
$h_1^1 h_1^{13} = h_1^{14}$, $\quad h_1^1 h_1^{14} = h_1^{15}$.

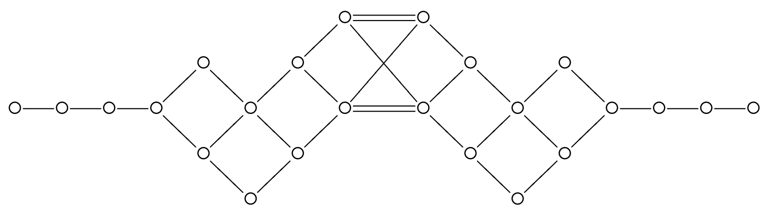
for $\mathrm{CH}(X_2)$:

$h_2^1 h_2^1 = h_2^2$, $\quad h_2^1 h_2^2 = h_2^3$, $\quad h_2^1 h_2^3 = h_2^4 + g_2^4$, $\quad h_2^1 h_2^4 = h_2^5$,
$h_2^1 g_2^4 = h_2^5 + g_2^5$, $\quad h_2^1 h_2^5 = h_2^6 + g_2^6$, $\quad h_2^1 g_2^5 = g_2^6$, $\quad h_2^1 h_2^6 = h_2^7 + g_2^7$,
$h_2^1 g_2^6 = g_2^7$, $\quad h_2^1 h_2^7 = 2h_2^8 + g_2^8$, $\quad h_2^1 g_2^7 = h_2^8 + 2g_2^8$, $\quad h_2^1 h_2^8 = h_2^9$,
$h_2^1 g_2^8 = h_2^9 + g_2^9$, $\quad h_2^1 h_2^9 = h_2^{10}$, $\quad h_2^1 g_2^9 = h_2^{10} + g_2^{10}$, $\quad h_2^1 h_2^{10} = h_2^{11} + g_2^{11}$,
$h_2^1 g_2^{10} = g_2^{11}$, $\quad h_2^1 h_2^{11} = h_2^{12}$, $\quad h_2^1 g_2^{11} = h_2^{12}$, $\quad h_2^1 h_2^{12} = h_2^{13}$,
$h_2^1 h_2^{13} = h_2^{14}$, $\quad h_2^1 h_2^{14} = h_2^{15}$.

7.12. Observe that the multiplication tables 7.11 can be visualized by means of the Hasse diagrams. Namely, for the variety X_1 consider the following graph which is obtained from the respective Hasse diagram by adding a few more edges and erasing all the labels:



and for X_2:



The multiplication rules can be restored from this graph as follows: for a vertex u (that corresponds to a basis element of the Chow group) we set

$$h_i^1 u = \sum_{u \to v} v,$$

where the sum runs through all the edges going from u one step to the right (cf. [Hi82b, Cor. 3.3]), $i = 1, 2$.

7.13. Applying Giambelli's formula 3.11 we obtain the following products which will be essentially used in the next section (see Appendix)

$$g_1^4 g_1^4 = 6h_1^8 + 8g_1^8, \qquad g_2^4 g_2^4 = 3h_2^8 + 4g_2^8.$$

7.2 Construction of rational projectors

The goal of the present section is to prove Theorem 7.1.

7.14. According to 2.9 in order to decompose the motive $\mathcal{M}(X)$ it is enough to construct rational projectors on $\mathrm{CH}^{15}(X' \times X')$, where $X' = X \times_k k_s$. Observe that our anisotropic group G of type F_4 is obtained by the first Tits construction, so by 7.7(i) there is a splitting field extension of degree 3. Therefore, for any basis element h of $\mathrm{CH}(X')$ the cycle $3h$ is rational (it follows immediately by transfer arguments). Hence, in order to construct rational cycles in $\mathrm{CH}(X' \times X')$ it is enough to work modulo 3. We shall write $x =_3 y$ iff $x - y = 3z$ for some cycle z. Please note that all results hold for Chow groups with integral coefficients.

7.15. Recall that the cycles h_1^1 and h_2^1 are rational by Lemma 7.6. Hence, their powers $(h_1^1)^i$ and $(h_2^1)^i$, $i = 2 \dots 7$, are rational as well. Using multiplication tables 7.11 or their graph interpretation 7.12 we immediately obtain the following rational cycles: in codimensions 2 through 7 for $\mathrm{CH}(X_1')$:

$$h_1^2,\ h_1^3,\ h_1^4 - g_1^4,\ h_1^5 + g_1^5,\ h_1^6,\ h_1^7 + g_1^7,$$

and for $\mathrm{CH}(X_2')$:

$$h_2^2,\ h_2^3,\ h_2^4+g_2^4,\ g_2^5-h_2^5,\ h_2^6,\ h_2^7+g_2^7.$$

7.16. Apply the arguments of 2.2(iii) to $\mathrm{CH}^4(X_2'\times X_1')$ (this can be done because all the properties for X_1' and X_2' hold by 7.7(ii)). There exists a rational cycle $\alpha_1\in\mathrm{CH}^4(X_2'\times X_1')$ such that $f'(\alpha_1)=g_2^4\times 1$. This cycle must have the following form:

$$\alpha_1=g_2^4\times 1+a_1h_2^3\times h_1^1+a_2h_2^2\times h_1^2+a_3h_2^1\times h_1^3+a_4 1\times h_1^4+a'1\times g_1^4,$$

where $a_i,a'\in\{-1,0,1\}$. We may reduce α_1 by adding cycles that are known to be rational (by 7.15) to

$$\alpha_1=(g_2^4\times 1)+a'(1\times g_1^4).$$

Repeating the same procedure for a rational cycle $\alpha_2\in\mathrm{CH}^4(X_2'\times X_1')$ such that $f'(\alpha_2)=1\times g_1^4$ and reducing it we obtain the rational cycle

$$\alpha_2=b(g_2^4\times 1)+(1\times g_1^4),$$

where $b\in\{-1,0,1\}$. Hence, there is a rational cycle of the form

$$r=g_2^4\times 1-a(1\times g_1^4),$$

where $a\in\{-1,1\}$.

7.17. To obtain a rational projector p_1 we proceed as follows. First, we obtain the following rational cycles in $\mathrm{CH}(X_2'\times X_1')$ modulo 3.

$$r^2=(g_2^4\times 1-a\cdot 1\times g_1^4)^2=_3 g_2^8\times 1+a(g_2^4\times g_1^4)-1\times g_1^8,$$

$$r_1=(1\times(h_1^7+g_1^7))r^2=_3 g_2^8\times(h_1^7+g_1^7)-1\times h_1^{15}-a(g_2^4\times(g_1^{11}+h_1^{11})),$$

$$r_2=((h_2^7+g_2^7)\times 1)r^2=_3 -(h_2^7+g_2^7)\times g_1^8+h_2^{15}\times 1+a((g_2^{11}-h_2^{11})\times g_1^4),$$

$$r_3=(h_2^1\times h_1^6)r^2=_3 -h_2^1\times h_1^{14}+(g_2^9+h_2^9)\times h_1^6+a((h_2^5+g_2^5)\times(g_1^{10}-h_1^{10})),$$

$$r_4=(h_2^6\times h_1^1)r^2=_3 h_2^6\times(g_1^9-h_1^9)+h_2^{14}\times h_1^1+a((h_2^{10}+g_2^{10})\times(h_1^5-g_1^5)),$$

$$r_5=(h_2^2\times(h_1^5+g_1^5))r^2=_3 h_2^2\times h_1^{13}+(g_2^{10}-h_2^{10})\times(h_1^5+g_1^5)+a((h_2^6-g_2^6)\times g_1^9),$$

$$r_6=((g_2^5-h_2^5)\times h_1^2))r^2=_3 -h_2^{13}\times h_1^2-(g_2^5-h_2^5)\times(h_1^{10}+g_1^{10})+a(g_2^9\times(h_1^6+g_1^6)),$$

$$r_7=(h_2^3\times(h_1^4-g_1^4))r^2=_3 h_2^3\times h_1^{12}-h_2^{11}\times(h_1^4-g_1^4)+a(h_2^7\times(g_1^8-h_1^8)),$$

$$r_8=((h_2^4+g_2^4)\times h_1^3)r^2=_3 -h_2^{12}\times h_1^3+(h_2^4+g_2^4)\times h_1^{11}+a((h_2^8-g_2^8)\times h_1^7).$$

7.18. To obtain motivic decompositions, we need to find rational projectors in $\mathrm{CH}(X_1'\times X_1')$ and $\mathrm{CH}(X_2'\times X_2')$. By compositioning (still modulo 3), we obtain the following rational cycles:

$$-(r_2\circ r_1^t) =_3 h_1^{15}\times 1 + (g_1^{11}+h_1^{11})\times g_1^4 + (g_1^7+h_1^7)\times g_1^8 =: p_0,$$
$$-(r_4\circ r_3^t) =_3 h_1^{14}\times h_1^1 + (h_1^{10}-g_1^{10})\times (g_1^5+2h_1^5) + h_1^6\times (h_1^9-g_1^9) =: p_1,$$
$$-(r_6\circ r_5^t) =_3 h_1^{13}\times h_1^2 - (h_1^5+g_1^5)\times (h_1^{10}-2g_1^{10}) + g_1^9\times (h_1^6+g_1^6) =: p_2,$$
$$-(r_8\circ r_7^t) =_3 h_1^{12}\times h_1^3 + (h_1^8-g_1^8)\times h_1^7 + (h_1^4-g_1^4)\times h_1^{11} \in \mathrm{CH}^{15}(X_1'\times X_1') =: p_3;$$
$$-(r_1^t\circ r_2) =_3 h_2^{15}\times 1 + (g_2^{11}-h_2^{11})\times g_2^4 + (g_2^7+h_2^7)\times g_2^8 =: q_0,$$
$$-(r_4^t\circ r_3) =_3 h_2^1\times h_2^{14} + (g_2^9+h_2^9)\times h_2^6 - (h_2^5+g_2^5)\times (-2h_2^{10}+g_2^{10}) =: q_1,$$
$$-(r_6^t\circ r_5) =_3 h_2^2\times h_2^{13} - (g_2^{10}-h_2^{10})\times (-2g_2^5-h_2^5) + (g_2^6-h_2^6)\times g_2^9 =: q_2,$$
$$-(r_7^t\circ r_8) =_3 h_2^{12}\times h_2^3 + (h_2^4+g_2^4)\times h_2^{11} + (h_2^8-g_2^8)\times h_2^7 \in \mathrm{CH}^{15}(X_2'\times X_2') =: q_3.$$

It remains to note that

$$p_i\circ p_i = p_i \quad \text{in} \quad \mathrm{CH}^{15}(X_1'\times X_1'),$$
$$q_i\circ q_i = q_i \quad \text{in} \quad \mathrm{CH}^{15}(X_2'\times X_2'),$$

where the equalities hold with integral coefficients (not just modulo 3).

Also note that

$$p_0+p_1+p_2+p_3+p_0^t+p_1^t+p_2^t+p_3^t = \Delta_{X_1'},$$
$$q_0+q_1+q_2+q_3+q_0^t+q_1^t+q_2^t+q_3^t = \Delta_{X_2'},$$

where $\Delta_{X_i'}$ are the diagonal cycles, and all these projectors are orthogonal to each other. Hence, by 2.9 we obtain the decomposition of the motive of X_1 and X_2:

$$\mathcal{M}(X_1) = (X_1,p_0)\oplus(X_1,p_1)\oplus(X_1,p_2)\oplus(X_1,p_3)\oplus$$
$$(X_1,p_0^t)\oplus(X_1,p_1^t)\oplus(X_1,p_2^t)\oplus(X_1,p_3^t),$$

$$\mathcal{M}(X_2) = (X_2,q_0)\oplus(X_2,q_1)\oplus(X_2,q_2)\oplus(X_2,q_3)\oplus$$
$$(X_2,q_0^t)\oplus(X_2,q_1^t)\oplus(X_2,q_1^t)\oplus(X_2,q_3^t),$$

It is also easy to see that over the separable closure k_s the motives (X_1,p_0) and (X_2,q_0) split as a direct sums of the Lefschetz motives $\mathbb{Z}\oplus\mathbb{Z}(4)\oplus\mathbb{Z}(8)$. Indeed, the images of the realizations of p_0 and q_0 are free abelian groups $\langle 1, g_1^4, g_1^8\rangle$ and $\langle 1, g_2^4, g_2^8\rangle$ respectively.

7.3 Motivic isomorphism between $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$

The goal of this section is to prove Theorem 7.2.

7.19. We continue to use the notation from the previous section. Consider the following cycle in $\mathrm{CH}^{15}(X_2' \times X_1')$

$$\begin{aligned} J = {} & g_2^8 \times g_1^7 - 1 \times h_1^{15} - ag_2^4 \times g_1^{11} - ah_2^{12} \times h_1^3 + ah_2^4 \times h_1^{11} + h_2^8 \times h_1^7 \\ & - g_2^7 \times g_1^8 + h_2^{15} \times 1 + ag_2^{11} \times g_1^4 + ah_2^3 \times h_1^{12} - ah_2^{11} \times h_1^4 - h_2^7 \times h_1^8 \\ & - h_2^1 \times h_1^{14} + h_2^9 \times h_1^6 + ah_2^5 \times h_1^{10} - ag_2^5 \times g_1^{10} + ah_2^{13} \times h_1^2 - g_2^9 \times g_1^6 \\ & - h_2^6 \times h_1^9 + h_2^{14} \times h_1^1 - ah_2^2 \times h_1^{13} + g_2^6 \times g_1^9 + ag_2^{10} \times g_1^5 - ah_2^{10} \times h_1^5. \end{aligned}$$

Observe that

$$J \circ (-J^t) = \Delta_{X_1'}, \quad (-J^t) \circ J = \Delta_{X_2'}.$$

In other words, the correspondence J provides a motivic isomorphism between X_2' and X_1' with the inverse $(-J)^t$. On the other hand,

$$J =_3 (r_1 + ar_8) + (r_2 + ar_7) + (r_3 - ar_6) + (r_4 - ar_5)$$

and, hence, is rational. Theorem 7.2 is proved.

8 Torsion part of $\mathrm{CH}^*(\mathrm{F}_4/P_4)$

In [Vo03] Voevodsky constructed a direct summand of the Chow motive of a norm variety corresponding to a symbol (n, p). This direct summand is called a generalized Rost motive. It has been conjectured that for norm varieties in the sense of Rost the torsion elements of Chow groups in the realization of the generalized Rost motive are concentrated in dimensions $p^i - 1$, $i > 0$. This was proved by Karpenko and Merkurjev in the case of norm quadrics.

The goal of the present Chapter is to provide an evidence of this conjecture for the F_4-varieties considered above and to prove the following theorem.

8.1 Theorem. *Let G be an anisotropic group of type F_4 of the 1st Tits process. Consider the projective homogeneous variety X such that over a separable closure it becomes isomorphic to G_s/P_4, where P_4 is the standard parabolic subgroup of G_s, corresponding to the first three vertices of the Dynkin diagram F_4 (we follow the enumeration of Bourbaki). Then the group $\mathrm{CH}^*(X)$ has torsion in codimension 13 (dimension 2).*

According to the conjecture this element is the only torsion element in the realization of the generalized Rost motive q_0 constructed in the previous Chapter.

The proof of the theorem is similar to that of Karpenko-Merkurjev's theorem on the structure of the torsion part in the Chow group of a norm quadric ([KM02, cor. 4.9]). Our main tool will be the Steenrod operations modulo 3.

In notation of Chapter 7 there exists a projector $\rho \in \mathrm{CH}^{15}(X)$ such that

$$\rho_s = 1 \times h^{(15)} + g^{(4)} \times (g^{(11)} - h^{(11)}) + g^{(8)} \times (g^{(7)} + h^{(7)}) \in \mathrm{CH}^{15}(X_s).$$

In Chapter 7 this projector was denoted as q_0. Note that we perform all computation modulo 3.

8.2 Lemma.

$$c(T_{X_s}) = 1 - h + h^2 - h^3 + h^4 - h^5 + h^6 - h^7,$$

$$c(T_{X_s})^{-1} = 1 + h.$$

Proof. These formulae immediately follow from the following one:

$$c(T_{X_s}) = c(\prod_{i=1}^{\dim(T_{X_s})} (1 - h_i)),$$

where h_i are the weights of the tangent bundle T_{X_s}, and the map

$$c\colon \mathbb{Z}[\bar{\omega}_1, \ldots \bar{\omega}_n] \to \mathrm{CH}^*(X_s) \quad (n = \operatorname{rk} G)$$

was described above. □

8.3 Corollary.

$$c(T_X)^{-1} = 1 + h + e_2 + d,$$

for some torsion element $e_2 \in \mathrm{CH}^2(X)$ and some element $d \in \mathrm{CH}^{\geq 3}(X)$.

8.4 Lemma. *Let S be the total Steenrod operation modulo 3 and h be a hyperplane section of $X \hookrightarrow \mathbb{P}^{25}$ (note that $\mathrm{Pic}(X_s)$ is rational). Then*

$$S(\rho_s) = 1 \times h^{(15)} + S(g^{(4)}) \times (g^{(11)} - h^{(11)}) + S(g^{(8)}) \times (g^{(7)} + h^{(7)}),$$

$$\rho_{s\star}(h^i) = 0, \text{ if } i \neq 7.$$

Proof. It suffices to calculate $S(g^{(11)} - h^{(11)})$ and $S(g^{(7)} + h^{(7)})$.

$$S(g^{(7)} + h^{(7)}) = S(-h^7) = -(S(h))^7 = -(h + h^3)^7 = -h^7 = g^{(7)} + h^{(7)}.$$

Calculating degrees by the Pieri formula, it is easy to see that the Schubert varieties corresponding to the cycles $g^{(11)}$ and $h^{(11)}$ are isomorphic to $\mathbb{P}^4$. By the Riemann-Roch theorem

$$S(g) = c(T_{X_s}) f^g_*(S_{\mathbb{P}^4}(\mathbb{P}^4) c(T_{\mathbb{P}^4})^{-1}),$$

where $g = g^{(11)}$ or $h^{(11)}$ and $f^g \colon g \hookrightarrow X$.

Therefore $S(g) = c(T_{X_s}) f^g_*((1 + H)^{-5}) = c(T_{X_s}) f^g_*(1 + H + H^3 + H^4)$, where H is a hyperplane section of $\mathbb{P}^4$. Therefore $S(g) = c(T_{X_s})(g + h^{(12)} + h^{(14)} + h^{(15)})$ and $S(g^{(11)} - h^{(11)}) = (g^{(11)} - h^{(11)}) c(T_{X_s}) = g^{(11)} - h^{(11)}$ by 8.2. □

8.5 Lemma.

$$S(\rho_\star(\alpha)) = S_{X \times X}(\rho)_\star(S_X(\alpha) c(T_X)^{-1}), \quad \alpha \in \mathrm{CH}(X).$$

This lemma is nothing else as Lemma 3.1 in [KM02].

8.6 Corollary.

$$S(\rho_\star(h^i)) = S(\rho)_\star(h^i(1 + h^2)^i c(T_X)^{-1}).$$

The following lemma is obvious:

8.7 Lemma. *For all* $\alpha \in \mathrm{CH}_i(X_s)$ *and* $i \geq 8$*,* $\deg(h^i \alpha)$ *is divisible by* 3*.*

8.8 Lemma. *If* $\mathrm{CH}_0(X)$ *has no torsion, then*

$$S^k(\rho)_\star(h^i) = 0$$

for $k = \frac{15-i}{2}$*,* $k > 0$*,* $i \geq 8$*.*

Proof. In the sequel we shall use this lemma only for $i = 13$. By the assumption the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is injective. This map is the multiplication by 3, since X is a variety of the 1st Tits process (see [PR94, Cor. on page 205]).

Therefore it suffices to prove that $\deg(S^k(\rho)_\star(h^i))$ is divisible by 9. Now we proceed similar to [KM02, Cor. 4.5]. We have

$$\begin{aligned}\deg(S^k(\rho)_\star(h^i)) &= \deg(\mathrm{pr}_{2*}(S^k(\rho)\mathrm{pr}_1^*(h^i)))\\ &= \deg(\mathrm{pr}_{1*}(S^k(\rho)\mathrm{pr}_1^*(h^i))) = \deg(h^i\mathrm{pr}_{1*}(S^k(\rho))).\end{aligned}$$

Since the degree does not change under the scalar extensions, it suffices to calculate it over k_s. By the definition

$$\mathrm{pr}_{1*}(a\times b) = \begin{cases} a, & \text{if } \dim a = \dim a\times b, \text{ i.e., } \deg b = 15,\\ 0, & \text{otherwise.}\end{cases}$$

By Lemma 8.4 for $k > 0$ $S^k(\rho_s)$ has no summands $a\times b$, where $\deg b = 15$. Therefore $\mathrm{pr}_{1*}(S^k(\rho))$ is divisible by 3. Now we are done because of Lemma 8.7. □

8.9 Lemma. *If* $\mathrm{CH}_0(X)$ *has no torsion, then*

$$S^1(\rho_\star(h^{13})) = \rho_\star(h^{15}).$$

Proof. By Corollary 8.6

$$S(\rho_\star(h^i)) = S(\rho)_\star(h^i(1+h^2)^i c(T_X)^{-1}).$$

Therefore $S^1(\rho_\star(h^{13})) =$ 15-codimensional component of $S(\rho)_\star(h^{13}(1+h^2)^{13}c(T_X)^{-1})$. We are done because of Lemma 8.8 and Corollary 8.3, since $h^{13}e_2 = 0$. □

8.10 Lemma. *If* $\mathrm{CH}_0(X)$ *has no torsion, then*

$$\rho_\star(h^{15}) = h^{15}.$$

Proof. By the assumption the degree map $\deg\colon \mathrm{CH}_0(X)\to\mathbb{Z}$ is injective. Therefore it suffices to prove that ρ_s acts on $\mathrm{CH}_0(X_s)$ identically. But this is obvious. □

8.11 Lemma. *Under the assumptions of Theorem 8.1 if* $\mathrm{CH}_0(X)$ *has no torsion, then* h^{15} *is nontrivial.*

Proof. The statement follows from the fact that X is anisotropic of the 1st Tits construction and $\deg h^{15} = 78$. □

Now we are able to prove Theorem 8.1. Assume that $\mathrm{CH}^{15}(X) = \mathrm{CH}_0(X)$ has no torsion. By Lemma 8.4 $\rho_{s\star}(h^{13}) = 0$. The previous lemmas imply that $\rho_\star(h^{13}) \neq 0$. Hence $\rho_\star(h^{13})$ is a nontrivial torsion element in $\mathrm{CH}^{13}(X)$. It remains to notice that indeed $\mathrm{CH}^{15}(X) = \mathrm{CH}_0(X)$ has no torsion (this was announced by M. Rost and proved in [PSZ05]).

9 Motivic decomposition of a compactification of a Merkurjev-Suslin variety

9.1 Introduction

In the present chapter we study certain twisted forms of a smooth hyperplane section of $\mathrm{Gr}(3,6)$. These twisted forms are smooth $\mathrm{SL}_1(A)$-equivariant compactifications of a Merkurjev-Suslin variety corresponding to a central simple algebra A of degree 3. On the other hand, these twisted forms are norm varieties corresponding to symbols in $\mathrm{K}_3^M/3$ given by the Serre-Rost invariant g_3. In the present paper we provide a complete decomposition of the Chow motives of these varieties.

The history of this question goes back to Rost and Voevodsky. Namely, Rost obtained the celebrated decomposition of a norm quadric (see [Ro98]) and later Voevodsky found some direct summand, called a generalized Rost motive, in the Chow motive of any norm variety (see [Vo03]). Note that the F_4-varieties from chapter 7 can be considered as a mod-3 analog of a Pfister quadric (more precisely, of a maximal Pfister neighbour). In turn, our variety can be considered as a mod-3 analog of a norm quadric.

The main ingredients of our proofs are results of Białynicki-Birula [BB73], the Lefschetz hyperplane theorem, and the Segre embedding.

9.2 Decomposition

9.1. We use Galois descent language, i.e., identify a (quasi-projective) variety X over a field k with the variety $X_s = X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k_s$ over a separable closure k_s equipped with an action of the absolute Galois group $\Gamma = \mathrm{Gal}(k_s/k)$. The set of k-rational points of X is precisely the set of k_s-rational points of X_s stable under the action of Γ.

The generating function for a variety X is, by definition, the polynomial $\sum a_i t^i \in \mathbb{Z}[t]$ with $a_i = \mathrm{rk}\,\mathrm{CH}^i(X)$.

The structure of the Chow ring of a Grassmann variety is of particular interest for us. We do a lot of computations using formulae from Schubert calculus (see [Ful] 14.7).

From now on we assume the characteristic of the base field k is 0.

It is well-known (see [GH, Ch. 1, § 5, p. 193]) that the Grassmann variety $\mathrm{Gr}(l,n)$ can be represented as the variety of $l \times n$ matrices of rank l modulo an obvious action of the group GL_l. Having this in mind we give the following definition.

9.2 Definition. Let A be a central simple algebra of degree 3 over a field k, $c \in k^*$. Fix an isomorphism $A_s \simeq \mathrm{M}_3(k_s)$. Consider the variety $D = D(A, c)$ obtained by Galois descent from the variety

$$\{\alpha \oplus \beta \in (A \oplus A)_s \simeq \mathrm{M}_{3,6}(k_s) \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \mathrm{Nrd}(\alpha) = c\,\mathrm{Nrd}(\beta)\}/\mathrm{GL}_1(A_s),$$

where $\mathrm{GL}_1(A_s)$ acts on $A_s \oplus A_s$ by the left multiplication.

This variety was first considered by M. Rost.

Consider the Plücker embedding of $\mathrm{Gr}(3,6)$ into a projective space (see [GH, Ch. 1, § 5, p. 209]). It is obvious that under this embedding for all c the variety $D(\mathrm{M}_3(k), c)$ is a hyperplane section of $\mathrm{Gr}(3,6)$.

9.3 Lemma. *The variety D is smooth.*

Proof. (M. Florence) We can assume k is separably closed. Consider first the variety

$$V = \{\alpha \oplus \beta \in \mathrm{M}_3(k) \oplus \mathrm{M}_3(k) = \mathrm{M}_{3,6}(k) \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \det(\alpha) = c\det(\beta)\}.$$

An easy computation of differentials shows that V is smooth. The variety V is a GL_3-torsor over D and, since GL_3 is smooth, this torsor is locally trivial for étale topology. Therefore to prove its smoothness we can assume that this torsor is split.

Since $D \times_k \mathrm{GL}_3$ is smooth, $D \times_k \mathrm{M}_3$ is also smooth. Therefore it suffices to prove that if $D \times_k \mathbb{A}^1$ is smooth, then D is smooth. But this is true for any variety. Indeed, for any point x on D we have $T_{(x,0)}(D \times_k \mathbb{A}^1) = T_x D \oplus T_0 \mathbb{A}^1 = T_x D \oplus k$ and $\dim T_x D = \dim T_{(x,0)}(D \times_k \mathbb{A}^1) - 1 = \dim(D \times_k \mathbb{A}^1) - 1 = \dim D$. $\square$

9.4 Remark. One can associate to the variety D a Serre-Rost invariant $g_3(D) = (A) \cup (c) \in \mathrm{H}^3(k, \mathbb{Z}/3)$ (see [Inv, § 40]). This invariant is trivial if and only if D is isotropic.

It is easy to see that $D^0 := \mathrm{MS}(A, c) := \{a \in A \mid \mathrm{Nrd}(a) = c\}$ is an open orbit under the natural right $\mathrm{SL}_1(A)$- or $\mathrm{SL}_1(A) \times \mathrm{SL}_1(A)$-action on D. Namely, the open orbit consists of all $\alpha \oplus \beta$ with $\mathrm{rk}(\alpha) = 3$. D^0 is called a Merkurjev-Suslin variety. In other words, the variety $D(A, c)$ is a smooth $\mathrm{SL}_1(A)$-equivariant compactification of the Merkurjev-Suslin variety $\mathrm{MS}(A, c)$.

Denote as $\imath\colon D \to \mathrm{SB}_3(\mathrm{M}_2(A))$ the corresponding closed embedding.

9.5 Lemma. *For the variety D_s the following properties hold.*

1. There exists a $\mathbb{G}_m$-action on D_s with 18 fixed points. In particular, D_s is a cellular variety.

2. The generating function for $\mathrm{CH}(D_s)$ is equal to $g = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$.

3. Picard group $\mathrm{Pic}(D_s)$ is rational.

Proof. 1. We can assume $c = 1$. The right action of $\mathbb{G}_m$ on D_s is induced by the following action:

$$\begin{aligned}(\mathrm{M}_3(k_s) \oplus \mathrm{M}_3(k_s)) \times \mathbb{G}_m(k_s) &\to \mathrm{M}_3(k_s) \oplus \mathrm{M}_3(k_s)\\ (\alpha \oplus \beta, \lambda) &\mapsto \alpha\,\mathrm{diag}(\lambda, \lambda^5, \lambda^6) \oplus \beta\,\mathrm{diag}(\lambda^2, \lambda^3, \lambda^7)\end{aligned}$$

Note that this action is compatible with the left action of $\mathrm{GL}_3(k_s)$.

The 18 fixed points of D are the $\binom{6}{3} = 20$ 3-dimensional standard subspaces of $\mathrm{Gr}(3, 6)$ minus 2 subspaces, generated by the first and by the last 3 basis vectors.

2. By the Lefschetz hyperplane theorem (see [GH]) the pull-back $\imath_s^*$ is an isomorphism in codimensions $i < \frac{\dim(\mathrm{Gr}(3,6))-1}{2}$. Therefore $\mathrm{rk}\,\mathrm{CH}^i(D_s) = \mathrm{rk}\,\mathrm{CH}^i(\mathrm{Gr}(3, 6))$ for such i's. Since Poincaré duality holds, we have $\mathrm{rk}\,\mathrm{CH}_i(D_s) = \mathrm{rk}\,\mathrm{CH}_i(\mathrm{Gr}(3, 6))$ for $i < \frac{\dim(\mathrm{Gr}(3,6))-1}{2} = 4$.

It remains to determine only the rank in the middle codimension. To do this observe that $\mathrm{rk}\,\mathrm{CH}^*(D_s) = 18$ (see [BB73]). Therefore $\mathrm{rk}\,\mathrm{CH}^4(D_s) = 2\,\mathrm{rk}\,\mathrm{CH}^4(\mathrm{Gr}(3, 6)) - 2 = 4$.

3. Consider the following commutative diagram:

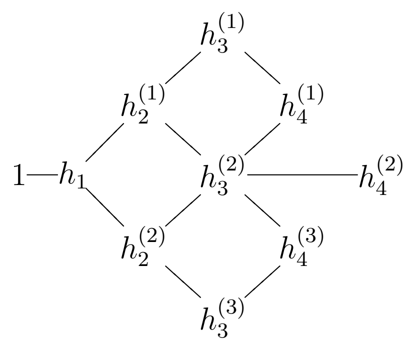
$$\begin{array}{ccc} \mathrm{Pic}(\mathrm{SB}_3(\mathrm{M}_2(A))) & \xrightarrow{\imath^*} & \mathrm{Pic}(D) \\ \downarrow & & \downarrow \mathrm{res}^* \\ \mathrm{Pic}(\mathrm{Gr}(3,6)) & \xrightarrow{\imath_s^*} & \mathrm{Pic}(D_s) \end{array} \qquad (4)$$

where the vertical arrows are the morphisms of scalar extension. By the Lefschetz hyperplane theorem the map $\imath_s^*$ restricted to $\mathrm{Pic}(\mathrm{Gr}(3, 6))$ is an isomorphism. Since $\mathrm{Pic}(\mathrm{SB}_3(\mathrm{M}_2(A)))$ is rational (see [MT95] and Lemma 7.6), i.e., the left vertical arrow is an isomorphism, the restriction map res^* is surjective. On the other hand, it follows from the Hochschild-Serre spectral sequence (see [Ar82, § 2]) that $\mathrm{Pic}(D)$ can be identified with a subgroup of $\mathbb{Z}$. We are done. □

9.6 Remark. It immediately follows from this Lemma that the variety D is not a twisted flag variety. Indeed, the generating functions of all twisted flag varieties over a separabely closed field are well-known and all of them are different from the generating function of D_s.

9.7. We must determine partially the multiplicative structure of $\mathrm{CH}(D_s)$. By the Lefschetz hyperplane theorem the generators in codimensions 0, 1, 2, and 3 are pull-backs of the canonical generators $\Delta_{(0,0,0)}$, $\Delta_{(1,0,0)}$, $\Delta_{(1,1,0)}$, $\Delta_{(2,0,0)}$, $\Delta_{(1,1,1)}$, $\Delta_{(2,1,0)}$, $\Delta_{(3,0,0)}$ of $\mathrm{Gr}(3,6)$ (see [Ful, 14.7]). We denote these pull-backs as 1, h_1, $h_2^{(1)}$, $h_2^{(2)}$, $h_3^{(1)}$, $h_3^{(2)}$, and $h_3^{(3)}$ respectively. In codimension 4 the pull-back is injective and the pull-backs $h_4^{(1)} := \imath_s^*(\Delta_{(2,1,1)})$, $h_4^{(2)} := \imath_s^*(\Delta_{(2,2,0)})$, $h_4^{(3)} := \imath_s^*(\Delta_{(3,1,0)})$, where $\imath$ is as above, form a subbasis of $\mathrm{CH}^4(D_s)$.

Consider the following diagram:



Since pull-backs are ring homomorphisms, it immediately follows that

$$h_1 \cdot u = \sum_{u \to v} v,$$

where u is a vertex in the diagram, which corresponds to a generator of codimension less than 4, and the sum runs through all the edges going from u one step to the right.

Next we compute some products in the middle codimension.

Since $\Delta_{(3,1,0)}\Delta_{(2,1,1)} = \Delta^2_{(2,2,0)} = 0$ and $\Delta^2_{(2,1,1)} = \Delta^2_{(3,1,0)} = \Delta_{(2,2,0)}\Delta_{(2,1,1)} = \Delta_{(2,2,0)}\Delta_{(3,1,0)} = \Delta_{(3,3,2)}$ (see [Ful, 14.7]), we have $h_4^{(1)}h_4^{(3)} = (h_4^{(2)})^2 = 0$ and $(h_4^{(1)})^2 = (h_4^{(3)})^2 = h_4^{(2)}h_4^{(3)} = h_4^{(1)}h_4^{(2)} = \imath_s^*(\Delta_{(3,3,2)}) = pt$, where pt denotes the class of a rational point on D_s.

The next theorem shows that the Chow motive of D with $\mathbb{Z}/3$-coefficients is decomposable. Note that for any cycle h in $\mathrm{CH}(D_s)$ or in $\mathrm{CH}(D_s \times D_s)$ the cycle $3h$ is rational.

9.8 Theorem. *Let A denote a central simple algebra of degree 3 over a field k, $c \in k^*$, and $D = D(A, c)$. Then*

$$\mathcal{M}(D) \simeq R \oplus (\oplus_{i=1}^{5} R'(i)),$$

where R is a motive such that over a separably closed field it becomes isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ and $R' \simeq \mathcal{M}(\mathrm{SB}(A))$.

Proof. Consider the following commutative diagram (see 5.24):

$$\begin{array}{ccccc} D_s \times \mathbb{P}^2 & \xrightarrow{\imath_s \times \mathrm{id}_s} & \mathrm{Gr}(3,6) \times \mathbb{P}^2 & \xrightarrow{\mathrm{Seg}_s} & \mathrm{Gr}(3,18) \\ \downarrow & & \downarrow & & \downarrow \\ D \times \mathrm{SB}(A^{\mathrm{op}}) & \xrightarrow{\imath \times \mathrm{id}} & \mathrm{SB}_3(\mathrm{M}_2(A)) \times \mathrm{SB}(A^{\mathrm{op}}) & \xrightarrow{\mathrm{Seg}} & \mathrm{SB}_3(\mathrm{M}_2(A) \otimes_k A^{\mathrm{op}}) \end{array} \tag{5}$$

where the right horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension k_s/k.

This diagram induces the commutative diagram of rings

$$\begin{array}{ccccc} \mathrm{Ch}^*(D_s \times \mathbb{P}^2) & \xleftarrow[(\imath_s \times \mathrm{id}_s)^*]{} & \mathrm{Ch}^*(\mathrm{Gr}(3,6) \times \mathbb{P}^2) & \xleftarrow[\mathrm{Seg}_s^*]{} & \mathrm{Ch}^*(\mathrm{Gr}(3,18)) \\ \uparrow & & \uparrow & & \simeq \uparrow \\ \mathrm{Ch}^*(D \times \mathrm{SB}(A^{\mathrm{op}})) & \xleftarrow[(\imath \times \mathrm{id})^*]{} & \mathrm{Ch}^*(\mathrm{SB}_3(\mathrm{M}_2(A)) \times \mathrm{SB}(A^{\mathrm{op}})) & \xleftarrow[\mathrm{Seg}^*]{} & \mathrm{Ch}^*(\mathrm{SB}_3(\mathrm{M}_2(A) \otimes_k A^{\mathrm{op}})) \end{array} \tag{6}$$

Observe that the right vertical arrow is an isomorphism since $\mathrm{M}_2(A) \otimes A^{\mathrm{op}}$ splits.

Let τ_3 and τ_1 be tautological vector bundles on $\mathrm{Gr}(3,6)$ and $\mathbb{P}^2$ respectively and let e denote the Euler class (the top Chern class). By Lemma 5.26 the cycle $(\imath_s \times \mathrm{id}_s)^*(e(\mathrm{pr}_1^*\tau_3 \otimes \mathrm{pr}_2^*\tau_1)) \in \mathrm{Ch}(D_s \times \mathbb{P}^2)$ is rational. A straightforward computation (cf. 5.29 and 5.30) shows that $r := -(\imath_s \times \mathrm{id}_s)^*(e(\mathrm{pr}_1^*\tau_3 \otimes \mathrm{pr}_2^*\tau_1)) = h_3^{(1)} \times 1 + h_2^{(1)} \times H + h_1 \times H^2 \in \mathrm{Ch}^3(D_s \times \mathbb{P}^2)$, where H is the class of a smooth hyperplane section of $\mathbb{P}^2$.

Define the following rational cycles $\rho_i = r(h_1^i \times 1) \in \mathrm{Ch}^{3+i}(D_s \times \mathbb{P}^2)$ for $i = 1, \ldots, 4$, $\rho_0 = r + h_1^3 \times 1 \in \mathrm{Ch}^3(D_s \times \mathbb{P}^2)$ and $\rho_1' = r(h_1 \times 1) + h_1^4 \times 1$. A straightforward computation using the multiplication rules in 9.7 shows that $(-\rho_1') \circ \rho_3^t$ as well as $(-\rho_{4-i}) \circ \rho_i^t \in \mathrm{Ch}^2(\mathbb{P}^2 \times \mathbb{P}^2)$ is the diagonal $\Delta_{\mathbb{P}^2}$. Moreover, the opposite compositions $(-\rho_0)^t \circ \rho_4$, $(-\rho_1)^t \circ \rho_3$, $(-\rho_2)^t \circ \rho_2$, $(-\rho_3)^t \circ \rho_1'$, and $(-\rho_4)^t \circ \rho_0$ give rational pairwise orthogonal idempotents in $\mathrm{Ch}_8(D_s \times D_s)$.

To finish the proof of the theorem it remains by 2.10 to lift all these rational cycles ρ_i, ρ_j^t to $\mathrm{Ch}(D\times \mathrm{SB}(A^{\mathrm{op}}))$ and to $\mathrm{Ch}(\mathrm{SB}(A^{\mathrm{op}})\times D)$ respectively in such a way that the corresponding compositions of their preimages give the diagonal $\Delta_{\mathrm{SB}(A^{\mathrm{op}})}$.

Fix an $i = 0, \ldots, 4$. Consider first any preimage $\alpha \in \mathrm{Ch}(D \times \mathrm{SB}(A^{\mathrm{op}}))$ of $-\rho_{4-i}$ and any preimage $\beta \in \mathrm{Ch}(\mathrm{SB}(A^{\mathrm{op}}) \times D)$ of ρ_i^t. The image of the composition $\alpha\circ\beta$ under the restriction map is the diagonal $\Delta_{\mathbb{P}^2}$. Therefore by the Rost Nilpotence theorem for Severi-Brauer varieties $\alpha \circ \beta = \Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n$, where n is a nilpotent element in $\mathrm{End}(\mathcal{M}(\mathrm{SB}(A^{\mathrm{op}})))$. Since n is nilpotent $\alpha \circ \beta$ is invertible and $((\Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n)^{-1} \circ \alpha) \circ \beta = \Delta_{\mathrm{SB}(A^{\mathrm{op}})}$. In other words, we can take $(\Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n)^{-1} \circ \alpha$ as a preimage of $-\rho_{4-i}$ and β as a preimage of ρ_i^t.

Denote as R the remaining direct summand of the motive of D. Comparing the left and the right hand sides of the decomposition over k_s it is easy to see that $R_s \simeq \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$. □

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Appendix I

The multiplication table for X_1:

$$
\begin{array}{llll}
h^4h^4 = h^8 + g^8, & h^4h^5 = 2h^9 + 2g^9, & h^4h^6 = 2h^{10} + g^{10}, & h^4h^7 = h^{11} + 2g^{11}, \\
h^4h^8 = h^{12}, & h^4h^9 = 2h^{13}, & h^4h^{10} = h^{14}, & h^4g^4 = 2h^8 + 3g^8, \\
h^4g^5 = h^9 + 2g^9, & h^4g^6 = 2h^{10} + 2g^{10}, & h^4g^7 = h^{11} + 3g^{11}, & h^4g^8 = 2h^{12}, \\
h^4g^9 = h^{13}, & h^4g^{10} = 0, & g^4g^4 = 6h^8 + 8g^8, & g^4g^5 = 4h^9 + 4g^9, \\
g^4g^6 = 6h^{10} + 4g^{10}, & g^4g^7 = 3h^{11} + 8g^{11}, & g^4g^8 = 6h^{12}, & g^4g^9 = 4h^{13}, \\
g^4g^{10} = h^{14}, & g^4h^5 = 5h^9 + 6g^9, & g^4h^6 = 5h^{10} + 4g^{10}, & g^4h^7 = 2h^{11} + 6g^{11}, \\
g^4h^8 = 2h^{12}, & g^4h^9 = 4h^{13}, & g^4h^{10} = 2h^{14}, & h^5h^5 = 6h^{10} + 4g^{10}, \\
h^5h^6 = 2h^{11} + 5g^{11}, & h^5h^7 = 4h^{12}, & h^5h^8 = 2h^{13}, & h^5h^9 = 2h^{14}, \\
h^5g^5 = 4h^{10} + 4g^{10}, & h^5g^6 = 2h^{11} + 6g^{11}, & h^5g^7 = 5h^{12}, & h^5g^8 = 4h^{13}, \\
h^5g^9 = h^{14}, & g^5g^5 = 4h^{10}, & g^5g^6 = 2h^{11} + 4g^{11}, & g^5g^7 = 4h^{12}, \\
g^5g^8 = 4h^{13}, & g^5g^9 = 2h^{14}, & g^5h^6 = h^{11} + 4g^{11}, & g^5h^7 = 2h^{12}, \\
g^5h^8 = 0, & g^5h^9 = 0, & h^6h^6 = 3h^{12}, & h^6h^7 = 3h^{13}, \\
h^6h^8 = h^{14}, & h^6g^6 = 3h^{12}, & h^6g^7 = 3h^{13}, & h^6g^8 = h^{14}, \\
g^6g^6 = 4h^{12}, & g^6g^7 = 4h^{13}, & g^6g^8 = 2h^{14}, & g^6h^7 = 2h^{13}, \\
g^6h^8 = 0, & h^7h^7 = 2h^{14}, & h^7g^7 = h^{14}, & g^7g^7 = 2h^{14}.
\end{array}
$$

The multiplication table for X_2:

$$
\begin{array}{llll}
h^4h^4 = 2h^8 + 2g^8, & h^4h^5 = 4h^9 + 2g^9, & h^4h^6 = 4h^{10} + g^{10}, & h^4h^7 = 2h^{11} + 2g^{11}, \\
h^4h^8 = h^{12}, & h^4h^9 = h^{13}, & h^4h^{10} = h^{14}, & h^4g^4 = 2h^8 + 3g^8, \\
h^4g^5 = h^9 + g^9, & h^4g^6 = 2h^{10} + g^{10}, & h^4g^7 = 2h^{11} + 3g^{11}, & h^4g^8 = 2h^{12}, \\
h^4g^9 = h^{13}, & h^4g^{10} = 0, & g^4g^4 = 3h^8 + 4g^8, & g^4g^5 = 2h^9 + g^9, \\
g^4g^6 = 3h^{10} + g^{10}, & g^4g^7 = 3h^{11} + 4g^{11}, & g^4g^8 = 3h^{12}, & g^4g^9 = 2h^{13}, \\
g^4g^{10} = h^{14}, & g^4h^5 = 5h^9 + 3g^9, & g^4h^6 = 5h^{10} + 2g^{10}, & g^4h^7 = 2h^{11} + 3g^{11}, \\
g^4h^8 = h^{12}, & g^4h^9 = h^{13}, & g^4h^{10} = h^{14}, & h^5h^5 = 6h^{10} + 2g^{10}, \\
h^5h^6 = 4h^{11} + 5g^{11}, & h^5h^7 = 4h^{12}, & h^5h^8 = h^{13}, & h^5h^9 = h^{14}, \\
h^5g^5 = 2h^{10} + g^{10}, & h^5g^6 = 2h^{11} + 3g^{11}, & h^5g^7 = 5h^{12}, & h^5g^8 = 2h^{13}, \\
h^5g^9 = h^{14}, & g^5g^5 = h^{10}, & g^5g^6 = h^{11} + g^{11}, & g^5g^7 = 2h^{12}, \\
g^5g^8 = h^{13}, & g^5g^9 = h^{14}, & g^5h^6 = h^{11} + 2g^{11}, & g^5h^7 = h^{12}, \\
g^5h^8 = 0, & g^5h^9 = 0, & h^6h^6 = 6h^{12}, & h^6h^7 = 3h^{13}, \\
h^6h^8 = h^{14}, & h^6g^6 = 3h^{12}, & h^6g^7 = 3h^{13}, & h^6g^8 = h^{14}, \\
g^6g^6 = 2h^{12}, & g^6g^7 = 2h^{13}, & g^6g^8 = h^{14}, & g^6h^7 = h^{13}, \\
g^6h^8 = 0, & h^7h^7 = 2h^{14}, & h^7g^7 = h^{14}, & g^7g^7 = 2h^{14}.
\end{array}
$$

Appendix II

In this appendix we describe how we obtained the necessary multiplication tables. Our root enumeration follows Bourbaki ([Bou]). We fix an orthonormal base $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{R}^4$. F_4 has the following simple roots:

$$
\begin{array}{ll}
\alpha_1 = e_3 - e_2, & \alpha_2 = e_2 - e_1, \\
\alpha_3 = e_1, & \alpha_4 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 + \frac{1}{2}e_4.
\end{array}
$$

The set of fundamental weights:

$$
\begin{array}{ll}
\bar{\omega}_1 = e_3 + e_4, & \bar{\omega}_2 = e_2 + e_3 + 2e_4, \\
\bar{\omega}_3 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{3}{2}e_4, & \bar{\omega}_4 = e_4.
\end{array}
$$

For the expressions of other positive roots by the base roots we refer to [Bou]. We list the expressions of these roots in the basis of the fundamental weights:

$$
\begin{array}{rlrl}
-\bar{\omega}_3+2\bar{\omega}_4, & -\bar{\omega}_2+2\bar{\omega}_3-\bar{\omega}_4, & -\bar{\omega}_1+2\bar{\omega}_2-2\bar{\omega}_3, & 2\bar{\omega}_1-\bar{\omega}_2,\\
-\bar{\omega}_2+\bar{\omega}_3+\bar{\omega}_4, & -\bar{\omega}_1+\bar{\omega}_2-\bar{\omega}_4, & -\bar{\omega}_1+2\bar{\omega}_3-2\bar{\omega}_4, & \bar{\omega}_1+\bar{\omega}_2-2\bar{\omega}_3,\\
-\bar{\omega}_1+\bar{\omega}_2-\bar{\omega}_3+\bar{\omega}_4, & \bar{\omega}_1-\bar{\omega}_4, & -\bar{\omega}_1+2\bar{\omega}_4, & \bar{\omega}_1-\bar{\omega}_2+2\bar{\omega}_3-2\bar{\omega}_4,\\
-\bar{\omega}_1+\bar{\omega}_3, & \bar{\omega}_1-\bar{\omega}_3+\bar{\omega}_4, & \bar{\omega}_1-\bar{\omega}_2+2\bar{\omega}_4, & \bar{\omega}_2-2\bar{\omega}_4,\\
\bar{\omega}_1-\bar{\omega}_2+\bar{\omega}_3, & \bar{\omega}_2-2\bar{\omega}_3+2\bar{\omega}_4, & \bar{\omega}_2-\bar{\omega}_3, & -\bar{\omega}_2+2\bar{\omega}_3,\\
\bar{\omega}_3-\bar{\omega}_4, & -\bar{\omega}_1+\bar{\omega}_2, & \bar{\omega}_4, & \bar{\omega}_1.
\end{array}
$$

Using the Giambelli formula, we obtain the preimages of g_i^4 in $S^*(P)\otimes_{\mathbb{Z}}\mathbb{Q}$. Here is the list:

$$
\begin{aligned}
g_1^4 = c(&\frac{11}{6}\bar{\omega}_1^2\bar{\omega}_4^2+\frac{3}{4}\bar{\omega}_1^2\bar{\omega}_2^2-\frac{4}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3^2+\frac{11}{6}\bar{\omega}_1^2\bar{\omega}_3^2-\frac{2}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4+\frac{11}{12}\bar{\omega}_1^4+\\
&\frac{1}{6}\bar{\omega}_2^4-\frac{4}{3}\bar{\omega}_2\bar{\omega}_3^2\bar{\omega}_4+\frac{4}{3}\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4^2+\frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3\bar{\omega}_4+\frac{2}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2-\frac{11}{6}\bar{\omega}_1^2\bar{\omega}_3\bar{\omega}_4+\\
&2\bar{\omega}_1\bar{\omega}_3^2\bar{\omega}_4-2\bar{\omega}_1\bar{\omega}_3\bar{\omega}_4^2-\frac{7}{12}\bar{\omega}_1^3\bar{\omega}_2-\frac{11}{6}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_3+\frac{4}{3}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_3+\frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3^2-\\
&\frac{2}{3}\bar{\omega}_2^3\bar{\omega}_3-\frac{1}{3}\bar{\omega}_1\bar{\omega}_2^3-\frac{2}{3}\bar{\omega}_2^2\bar{\omega}_4^2),
\end{aligned}
$$

$$
\begin{aligned}
g_2^4 = c(&\frac{11}{6}\bar{\omega}_4^4-\frac{7}{6}\bar{\omega}_3\bar{\omega}_4^3+\frac{11}{12}\bar{\omega}_1^2\bar{\omega}_4^2+\frac{3}{2}\bar{\omega}_3^2\bar{\omega}_4^2-\frac{11}{6}\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4^2+\frac{11}{12}\bar{\omega}_2^2\bar{\omega}_4^2-\\
&\frac{11}{12}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2-\frac{2}{3}\bar{\omega}_3^3\bar{\omega}_4-\frac{1}{2}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_4+\frac{1}{3}\bar{\omega}_1^2\bar{\omega}_3\bar{\omega}_4+\frac{4}{3}\bar{\omega}_2\bar{\omega}_3^2\bar{\omega}_4+\frac{1}{2}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_4-\\
&\frac{2}{3}\bar{\omega}_2^2\bar{\omega}_3\bar{\omega}_4-\frac{1}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4+\frac{1}{3}\bar{\omega}_3^4-\frac{1}{3}\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_3+\frac{1}{3}\bar{\omega}_1\bar{\omega}_2\bar{\omega}_3^2-\frac{1}{3}\bar{\omega}_1^2\bar{\omega}_3^2+\\
&\frac{1}{3}\bar{\omega}_1^2\bar{\omega}_2\bar{\omega}_3+\frac{1}{3}\bar{\omega}_2^2\bar{\omega}_3^2-\frac{2}{3}\bar{\omega}_2\bar{\omega}_3^3).
\end{aligned}
$$

Multiplying the correspondent polynomials and taking the c function, we find the products.

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