Closed 3-Forms and Random Worldlines

Dissertation

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1 Introduction and Overview

We are considering a random worldline. This should be thought of as the path of a particle in the space-time manifold (which might be 4-dimensional Minkowski space, but might as well be a more general d-dimensional manifold), this path being dependent on some randomness. That is, we have a submanifold-valued random variable. A narrower case we are particularly interested in is that of a *deterministic* random worldline, i.e. a random worldline L with the property that if you know one point of it, you know what the entire worldline is; in other words, it is only the "initial position" of the particle that is chosen at random, and once this initial position is chosen, the future motion is fixed in a deterministic way.

An example of a deterministic random worldline is provided by Bohmian mechanics, a physical theory about a moving point particle depending on the wave function of quantum mechanics; in this theory, all the randomness comes from the unknown initial position of the particle. In chapter 4, I will illustrate my results in terms of Bohmian mechanics, and apply them to a version of Bohmian mechanics adapted to the one-particle Klein–Gordon (resp. Dirac) equation. Another application lies in the context of the N-particle dynamics of the "hypersurface Bohm–Dirac model".

We discuss the connection between random worldlines and closed differential 3-forms on the 4-dimensional space-time manifold. (All my results hold as well for other dimensions; so whenever I say 3, I mean a number ≥ 1; and whenever I say 4, I mean 3+1.) Differential 3-forms, or 3-forms, are completely antisymmetric tensor fields of rank 3; in usual index notation, a 3-form β reads

$$\beta_{\lambda\mu\nu} = -\beta_{\lambda\nu\mu} = -\beta_{\mu\lambda\nu}\,.$$

On the other hand, a 3-form can be understood as what topologists call a 3-cochain, that is a mapping that assigns a real number to every oriented 3-dimensional hypersurface (and, indeed, to formal linear combinations thereof), this number usually being written as $\int_H \beta$, the integral of the 3-form β over the hypersurface H. A 3-form is said to be closed if its integral over H vanishes whenever H is the boundary of some 4-volume, or, equivalently, if its exterior (skew) derivative vanishes.

It turns out that closed 3-forms are closely related to random worldlines. For every random worldline (that means, worldline-valued random variable, not worldline chosen at random), there is a naturally associated closed 3-cochain containing information about the probability density; given appropriate smoothness properties of the random worldline, the closed 3-cochain corresponds to a closed 3-form.

We explain that closed 3-forms are the natural way of covariantly expressing probability density. And covariant, in this case, means not only Lorentz-invariant (in case of space-time being Minkowski space) but actually diffeomorphism-invariant (or general-relativistic, one might say).

We now describe what the connection between 3-forms and random worldlines is. To begin with, the immediate question to ask about a random worldline L and a 3-dimensional hypersurface H is: what is the probability of L crossing H? This question is, of course, highly relevant for detection probabilities (where H is a piece of a $t =$ const. surface) and for scattering cross sections (where H is time-axis $\times$ a sphere around the scattering center), for example. The number of crossings through H is an integer-valued random variable, and one may ask for its expectation value. However, we are going to ask a slightly different question. If we give orientations to the worldlines, to the hypersurface and to space-time, we may define which crossings are *positively oriented* and which are *negatively oriented*. The number of signed crossings, defined essentially as the number of positive crossings minus the number of negative crossings, is again an integer-valued random variable which we call $N(H)$, and we will see its expectation, as a function of the hypersurface H, is a 3-cochain, essentially because it adds for disjoint unions of hypersurfaces, and it changes sign when reverting the orientation of H. This is the connection between random worldlines and 3-forms we are talking of.

The reason why the expected number of signed crossings $\mathbb{E}N$ is more relevant for our purposes than the expected total number of crossings or the probability of crossing, is just the fact that (sufficient smoothness properties assumed, and finiteness of the expectation) it can be read as a 3-form, i.e. as a tensor field! In contrast, the probability of crossing H as a function of H is a very abstract mapping on a very abstract space (the set of all suitable H's), and so is the expected total number of crossings. A tensor field is a comparably simple and familiar object; in many cases the relevant 3-form can be specified by some explicit formula.

Our central enterprise is to discuss the converse question: given a closed 3-form β, is there a random worldline L such that β provides the expected number of signed crossings $\mathbb{E}N$ with respect to L? Is L unique? Let's start with uniqueness: there is no reason to expect uniqueness—but if a *deterministic* random worldline exists, it is unique (for the precise statement, see p. 19). In section 3, we describe the construction of this deterministic random worldline $\Gamma(\beta)$ from the 3-form β. In general, the construction can only be carried out on a "set of well-behaved points" in space-time, the wandering set. Our construction provides a 1-dimensional foliation (or congruence) of the wandering set, and a measure on the worldlines; this measure is not necessarily normalized, so only a suitably normalized 3-form β will define a $\Gamma(\beta)$. The following diagram roughly summarizes the correspondences:

$$\begin{array}{c}\{\text{random worldlines}\}\\ \downarrow \mathbb{E}N\\ \{\text{normalized closed 3-forms}\}\\ \downarrow \Gamma\\ \{\text{deterministic random worldlines}\}\end{array}$$

That $\mathbb{E}N(L)$ is indeed a smooth 3-form is, of course, a nontrivial smoothness property L may have or not. With this restriction, however, $\mathbb{E}N$ defines a many-to-one mapping. On the set of deterministic random worldlines, $\mathbb{E}N$ and Γ are essentially inverse mappings, i.e. $\Gamma(\mathbb{E}N_L) = L$ for every deterministic random worldline L, while $\mathbb{E}N\big(\Gamma(\beta)\big) = \beta$ holds only on the wandering set.

We already remarked above that the mapping $\mathbb{E}N$ that assigns closed 3-forms to random worldlines, does not depend on the Lorentzian metric on space-time, it is not related to any metric. The same applies to the other mapping: Γ can be constructed on any orientable manifold.

Since the reasoning of chapters 2 and 3 relies in no way on properties of the numbers 3 and 4, but applies just as well to closed $d-1$-forms on d-dimensional manifolds, the phase flow of Newtonian mechanics is an example of a deterministic random worldline, with space-time replaced by phase space $\times$ time axis. Furthermore, for the hypersurface Bohm–Dirac model, a relativistic dynamics for N particles, the differential form method can be fruitfully applied to establish rigorously the equivariance statement, even for a curved space-time, which is a novel result.

There is a relation between $d-1$-forms, d-forms, and vector fields on a d-dimensional manifold. A d-form α and a vector field j together define a $d-1$-form by plugging the vector into the first slot of the d-form, $\beta(X_1, \ldots, X_{d-1}) = \alpha(j, X_1, \ldots, X_{d-1})$. If a nowhere-vanishing d-form (which is often called a *volume form*) is given then the translation from a vector field to a $d-1$-form is one-to-one and onto, so vector fields and $d-1$-forms amount to the same thing. This is sometimes called the *duality* between $d-1$-forms and vector fields; but keep in mind this depends on the volume form. A volume form is automatically selected by the metric (together with the orientation) in Minkowski space, or more generally in any Lorentzian or Riemannian manifold; as well, a volume form is selected by the structure of a Galilei space (see p. 12). In all these cases, closedness of the $d-1$-form corresponds to j being divergence-free. The orientation alone does not select a volume form, but rather a class of volume forms that differ by a positive function.

Most of our examples and applications exploit the duality by defining a divergence-free current vector field and translating it into a closed $d-1$-form, which again defines a deterministic random worldline. The possible outcomes are worldlines tangent to the current vector. Nevertheless, I wish to stress that the object naturally associated with the random worldline is the $d-1$-form rather than the current vector field which comes into play only through the duality induced by a volume form. To speak of probabilities, one needs a form. This is one of the main lessons from our construction of random worldlines on arbitrary manifolds; another notable lesson—seeming more obvious than it is—is that the integral of the form over a hypersurface H does not give the *probability* of crossing H but rather the *expected number of signed crossings*.

There is a divergence-free vector field j^μ associated with a solution ψ of the single-particle Klein–Gordon equation. When we apply Γ to the closed 3-form that corresponds to j^μ, we end up with a deterministic random worldline. This is remarkable in so far as some authors [4, 7] have claimed that since the temporal component j^0 is sometimes negative, there cannot be any random worldline associated with ψ. This is discussed in more detail in section 4.4.

2 Random Worldlines Crossing Hypersurfaces

2.1 Definitions

This section is more technical in character. We give definitions for the terms *worldline, random worldline, deterministic random worldline, hypersurface*, and *signed crossing.*

2.1.1 Worldlines

Space-time is an oriented (connected C^∞) 4-manifold. We define a *worldline* to be an (oriented) equivalence class of curves, where curves are understood to be injective C^∞ mappings $\gamma : \mathbb{R} \to$ (space-time) with the property $\mathrm{d}\gamma/\mathrm{d}t \neq 0$ everywhere, and equivalence of two curves γ, γ' means each is a reparametrisation of the other, i.e. there is a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{d}\varphi/\mathrm{d}t > 0$ everywhere and $\gamma' = \gamma \circ \varphi$ resp. $\gamma = \gamma' \circ \varphi^{-1}$.

Equivalently, we may say that a worldline is (up to orientation) a connected 1-dimensional submanifold (in the broad sense of "submanifold": an *immersed* rather than *imbedded* manifold) that is diffeomorphic to the real line. Those connected 1-dimensional submanifolds that are not worldlines are diffeomorphic to the circle. If ℓ is a worldline, we will write $p \in \ell$ to indicate that the point p lies on the worldline ℓ and often speak of ℓ as if it was a set, although ℓ does not merely denote a set of points (in some cases, like the one of the figure "8", there are two different submanifolds on the same set of points). We will sometimes say the wordline ℓ' is a part of the worldline ℓ when we mean there is a parametrization γ of ℓ and a parametrization γ' of ℓ' such that $\gamma' = \gamma \circ \varphi$ where φ is a diffeomorphism $\mathbb{R} \to (a, b)$ with $-\infty \leq a < b \leq \infty$ and $\mathrm{d}\varphi/\mathrm{d}t > 0$.

We do not require worldlines to be timelike, as we do not presume any metric on space-time.

We define a *random worldline* to be a worldline-valued random variable. For this to be well-defined, we need to know which sets of worldlines count as measurable, i.e. we need to define a σ-algebra $\mathcal{W}$ on the set of worldlines.

$\mathcal{W}$ should contain the subsets of the form {all worldlines intersecting A} where A is a Borel-measurable subset of space-time. But the σ-algebra generated by these sets is not enough, since it does not distinguish orientation; $\mathcal{W}$ should also contain the subsets of the form {all worldlines having a representing curve γ such that $\omega(\mathrm{d}\gamma/\mathrm{d}t) > 0$} where ω is a 1-form on space-time. We define $\mathcal{W}$ to be the σ-algebra generated by these two families of sets. Note that sets of the form {all worldlines lying completely within A} ("cylinder sets") are measurable for Borel sets A, since they are complements of generating sets.

We usually denote a random worldline as L or L'. Be warned that there is a possible source of confusion, as always with random variables, in that the same symbol L (and the same name "random wordline") is used for both (a) the random variable and (b) its outcome; to stress the difference, (a) is not random at all, but a fixed mapping from a probability space to the set of worldlines whereas (b) is a worldline chosen by Tyche in an unforeseeable way.

We will speak a lot about deterministic random worldlines, meaning by that that merely the initial position is random, but the motion is not; in other words that it is enough to know one point on L to know the entire path. As a first step towards a definition, we say a random worldline L has the *determinism property* if it is possible to associate with every $p \in$ (space-time) a worldline ℓ_p such that the event

$$\{\forall p \in (\text{space-time}) : \text{ if } p \in L, \text{ then } L = \ell_p\} \tag{1}$$

has probability 1. The mapping $p \mapsto \ell_p$ has the property that up to null events, if $q \in \ell_p$ then $\ell_q = \ell_p$; more precisely, if S is the set

$$S = \{p \in (\text{space-time}) | \forall q \in \ell_p : \ \ell_q = \ell_p\} \tag{2}$$

then the event $\{L \subseteq S\}$ has probability 1 (since the complement of this event is that there is a $p \in L$ and a $q \in \ell_p$ such that $\ell_q \neq \ell_p$, which implies that either $p \in L, L \neq \ell_p$ or $q \in L, L \neq \ell_q$, which by assumption is a null event). L determines the mapping $p \mapsto \ell_p$ uniquely up to null events, i.e. if there is another mapping $p \mapsto \ell'_p$ such that the event (1) has still probability 1 when ℓ is replaced by ℓ', then almost certainly, L is a subset of $\{p \in (\text{space-time}) | \ell_p = \ell'_p\}$ (since the complementary event is that there is a $p \in L$ such that $\ell_p \neq \ell'_p$, which implies that either $L \neq \ell_p$ or $L \neq \ell'_p$, which by assumption is the union of null events). It is clear from the definition (2) of S that the curves ℓ_p form a decomposition of S into disjoint lines.

Although every possible outcome of a random worldline with the determinism property is a smooth worldline, such a random variable may be nonsmooth in several respects: The field of tangents to the worldlines ℓ_p may be nonsmooth, as illustrated in fig. 1. The orientation may change discontinuously. And even if the family of worldlines is completely well-behaved, the probability measure may still be too wild: as an example, think of $\mathbb{R}^4$ as space-time and the foliation into the parallels to the x^0 axis as the ℓ_p; then it is still a nontrivial property that

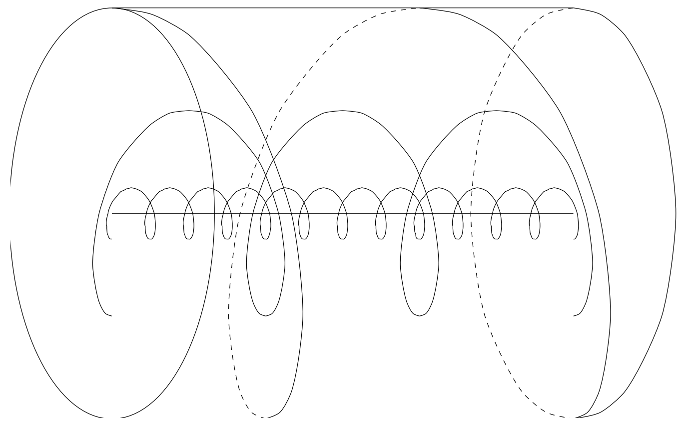

Figure 1: An example for a decomposition of a manifold (here, $\mathbb{R}^3$) into lines such that each line is smooth, but together they do not form a foliation of the manifold. One of the lines is the z axis, and the other lines are helices around the z axis with winding frequencies approaching infinity near the z axis. More precisely, the curves—except the z axis itself—are defined by $x = R\cos(z/R + \alpha)$, $y = R\sin(z/R + \alpha)$ where $R > 0$ and $0 \leq \alpha < 2\pi$ parametrize the family.

the probability measure on the worldlines (as parametrized by $\mathbb{R}^3$) has a smooth density.

We define a *smooth deterministic random worldline* to be a random worldline with the determinism property and the following additional ones:

(i) There is an open set $S_0 \subseteq S$ containing either all or nothing of a given ℓ_p and containing almost-all of them (i.e., the event $\{L \subseteq S_0\}$ has probability 1).

(ii) The field of tangents to the worldlines we decompose S_0 into is smooth. This is equivalent to saying the curves ℓ_p form a foliation[1] of S_0.

(iii) On S_0, the orientation of ℓ_p varies continuously with p, i.e., the positively oriented direction of the tangent to ℓ_p forms a smooth field of directions.

(iv) Every $p \in S_0$ has a neighborhood U such that for every $q \in U$, $\ell_q \cap U$ is connected. (In particular, no ℓ_q recurs to q arbitrarily close.)

(v) L has smooth probability densities in the following sense: for every coordinate chart[2] (U, x) in S_0 which maps the ℓ_p worldlines to lines parallel to

[1] A *foliation* is a decomposition of a manifold of dimension n into disjoint submanifolds of dimension $m < n$ (called the *leaves*) in a way locally diffeomorphic to the way $\mathbb{R}^n$ can be decomposed into parallel m-dimensional planes. A 1-dimensional foliation, where the leaves are curves, is sometimes called a *congruence*.

[2] A coordinate chart consists of an open subset U of the manifold and a mapping $x : U \to \mathbb{R}^d$ that is a diffeomorphism onto its range.

the x^0 axis (with the same orientation), and which has the property that for every $q \in U$, $\ell_q \cap U$ is connected, the induced measure on the projection to $\mathbb{R}^3$ of $x(U)$ has smooth density.

The role of axiom (iv) is that without it, axiom (v) would be vacuous, and axiom (v) must suppose that for every $q \in U$, $\ell_q \cap U$ is connected, because otherwise different points of $\mathbb{R}^3$ would represent the same worldline ℓ_q, and the probability measure on the worldlines would not define a measure on $\mathbb{R}^3$, so we could not speak of the probability density.

We will usually drop the word "smooth" and assume smoothness implicitly when speaking of a deterministic random worldline.

2.1.2 Hypersurfaces

A *hypersurface* is a 3-dimensional, oriented, imbedded submanifold. Imbedded means its topology and differential structure is inherited from space-time through being a subset, or, equivalently, every point on the hypersurface has arbitrarily small neighbourhoods in the hypersurface that are intersections of certain neighbourhoods in space-time with the hypersurface; in other words, every point on the hypersurface has a neighbourhood in space-time through which the hypersurface "cuts only once". An example of a non-imbedded submanifold is a geodesic on the torus $S^1 \times S^1$ having irrational slope, i.e., the image of a straight line (having irrational slope) in $\mathbb{R}^2$ when taken modulo the lattice $\mathbb{Z}^2$; this submanifold is dense in the torus, so it intersects every neighbourhood "infinitely often". Another example is shown in fig. 2.

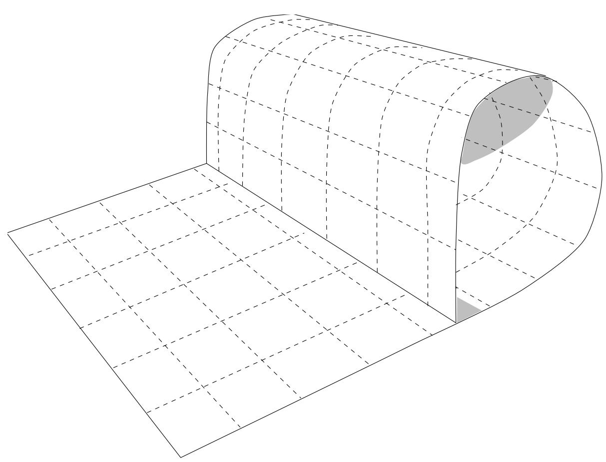

Figure 2: An example of a non-imbedded submanifold of $\mathbb{R}^3$ is a sheet of paper twisted in such a way that "its boundary touches the surface."

"Oriented" means a hypersurface H is equipped with an *orientation* defining for every basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of every tangent space T_pH whether it is *positively* or *negatively oriented*. Of course the orientation is required to vary continuously with p, so a given connected submanifold can at most have two different orientations. Note that not all submanifolds allow for an orientation, e.g. the Möbius band does not. Relative to the orientation of space-time, the orientation of a hypersurface defines which side is "the up side" and which is "the bottom side": if $\boldsymbol{e}_0$ is a tangent vector at p, but not tangent to H, and if $(\boldsymbol{e}_0, \ldots, \boldsymbol{e}_3)$ is a positively oriented basis of T_p(space-time) while $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_3)$ is a positively oriented basis of T_pH, then the side to which $\boldsymbol{e}_0$ is pointing is the *up side*.

We will sometimes say "a piece of hypersurface" rather than "a hypersurface" to remind you that we do not require a hypersurface to cut space-time into two parts.

2.1.3 Crossings

We now define the *sign* of a worldline's crossing through a hypersurface. In short, a crossing is positive if the worldline comes from the bottom side and leaves on the up side.

Before we give the definition, we consider a simple case first: by an *isolated intersection* of a worldline ℓ through a hypersurface H we mean a point $p \in \ell \cap H$ such that for every parametrisation $\gamma : \mathbb{R} \to$ (space-time) of ℓ, there is a neighbourhood U around the parameter value s_0 corresponding to p such that if $s \in U$ and $\gamma(s) \in H$ then $s = s_0$. Examples of *non*-isolated intersections are: a worldline that moves along a hypersurface for an entire parameter interval of positive length, or a worldline that intersects a hypersurface at parameter times 0 and $1/n$ for every positive integer n.

Proposition. *Four types of isolated intersections are possible:*

(i) The worldline comes from the bottom side, crosses through the hypersurface, and leaves on the up side. (This we call a positive crossing.*)*

(ii) The worldline comes from the bottom side, touches the hypersurface and leaves on the bottom side again. (This we do not count as a crossing.)

(iii) The worldline comes from the up side, crosses through the hypersurface, and leaves on the bottom side. (This we call a negative crossing.*)*

(iv) The worldline comes from the up side, touches the hypersurface, and leaves on the up side again. (This we do not count as a crossing.)

Proof. Since H is imbedded, there is a coordinate chart on a neighbourhood V of the isolated intersection p that "straightens" $H \cap V$ to a hyperplane in the coordinate space $\mathbb{R}^4$. Choose a parametrization γ of ℓ, and let $s_0 := \gamma^{-1}(p)$. There is a neighbourhood U' of s_0 such that $\gamma(U') \subseteq V$, and a neighbourhood $U \subseteq U'$ such that $U \cap \gamma^{-1}(H) = \{s_0\}$. Since the coordinate image of $H \cap V$ is a hyperplane, a curve cannot get on the other side without intersecting H. As a consequence, "which-side" is constant before and after s_0. □

Note that an isolated crossing is not necessarily transverse, but may be tangent. That is, the tangent to the worldline in the point of intersection may be tangent to the hypersurface (like the graph of the $x \mapsto x^3$ function when crossing the x axis).

If a given worldline possesses only isolated intersections with a given hypersurface H, it is clear what the *number of positive crossings* $N^+(\ell, H)$ and the *number of negative crossings* $N^-(\ell, H)$ mean. The *number of signed crossings* is the number of positive crossings minus the number of negative crossings, $N(\ell, H) = N^+(\ell, H) - N^-(\ell, H)$. If both N^+ and N^- are infinite, we define N to be $+\infty$, to avoid that N is not defined in some cases.

Whether an isolated crossing is positive or negative depends on the orientation of space-time, on the orientation of the hypersurface, and on the orientation of the worldline. If any of these orientations is reversed, the crossing changes sign. The question whether an isolated intersection is a crossing at all is not affected by reversing an orientation.

Before we extend our definition to non-isolated intersections, let us consider an example. Say the smooth curve γ in $\mathbb{R}^4$ intersects the $x^1 = 0$ hypersurface at times $-1/n$ and 0, with $x^1(\gamma(t)) > 0$ for $t > 0$ and $-\frac{1}{2n-1} < t < -\frac{1}{2n}$, whereas $x^1(\gamma(t)) < 0$ for $t < -1$ and $-\frac{1}{2n} < t < -\frac{1}{2n+1}$. Then γ has infinitely many positive and infinitely many negative crossings through the hypersurface; nevertheless it passes from $x^1 < 0$ to $x^1 > 0$ in total, and that is why we want to set the number of signed crossings to 1. The number of signed crossings is sometimes finite even if the number of positive and negative crossings are not.

For any pair $s_1 < s_2$ of parameter values, we define the *number of signed crossings between s_1 and s_2* like this:

(i) It is 0 if in the interval $[s_1, s_2]$, γ did not intersect H.

(ii) If there is a coordinate chart (U, x) mapping $H \cap U$ onto $\{x^0 = 0\} \cap x(U)$ preserving the orientation of space-time and of the hypersurface, such that $\gamma(t) \in U$ for $s_1 \leq t \leq s_2$, then

$N(s_1, s_2)$ =	$x^0(\gamma(s_2)) < 0$	$x^0(\gamma(s_2)) = 0$	$x^0(\gamma(s_2)) > 0$
$x^0(\gamma(s_1)) < 0$	0	1/2	1
$x^0(\gamma(s_1)) = 0$	−1/2	0	1/2
$x^0(\gamma(s_1)) > 0$	−1	−1/2	0

(This means landing on the hypersurface counts as half a crossing; this is of course undone when starting towards the side the worldline came from.)

(iii) If there is a coordinate chart (U, x) mapping $H \cap U$ to an open subset of the $\{x^0 = 0\}$ hyperplane, such that $\gamma(t) \in U$ and $x^0(\gamma(t)) = 0$ for $s_1 \le t \le s_2$, then $N(s_1, s_2) = 0$. (This means leaving the hypersurface through its boundary does not buy a crossing.)

(iv) Furthermore, for any $s_1 < s_2 < s_3$, $N(s_1, s_3) = N(s_1, s_2) + N(s_2, s_3)$.

The (total) number of signed crossings is the limit of $N(s_1, s_2)$ for $s_1 \to -\infty$ and $s_2 \to \infty$.

One easily checks that the local coordinate definitions are consistent with the additivity and with each other, that $N(s_1, s_2)$ does not depend on the choice of coordinates, that $N(s_1, s_2)$ takes on half-integer values, that this definition is consistent with the earlier definition for isolated intersections, and that the limit does not always exist (not even when allowing $\pm\infty$ as a limit, which we do). E.g., it does not exist for the graph of the sine function, and not either for a curve that crosses a hypersurface positively at every positive integer time, and negatively at every negative integer time. In any case, infinitely many positive and negative crossings are involved when the limit does not exist. When the limit does not exist, we set $N = +\infty$ to avoid ill-defined expressions.

The number of signed crossings depends on the orientations of space-time, the hypersurface, and the worldline in just the same way as the sign of an isolated crossing: it changes sign when any of the orientations is reversed.

When H is the disjoint union of two hypersurfaces, $H = H_1 \cup H_2$, then the crossing numbers add: $N^+(\ell, H) = N^+(\ell, H_1) + N^+(\ell, H_2)$, $N^-(\ell, H) = N^-(\ell, H_1) + N^-(\ell, H_2)$, $N(\ell, H) = N(\ell, H_1) + N(\ell, H_2)$. Here we calculate as if $+\infty - \infty = +\infty$, as we will generally. In case H_1 and H_2 are not disjoint, additivity still holds if we count the crossings with a suitable multiplicity.

As a consequence, $N(\ell, \cdot)$ has the formal properties of a 3-cochain (over the ring of integers), i.e., it accepts a (3-dimensional, oriented) hypersurface (and a formal linear combination thereof) as an argument and is $\mathbb{Z}$-linear—except at the infinities. Unfortunately, the infinities cannot be avoided since there is always some hypersurface that contains ℓ, so there is a nonisolated intersection, and then by definition N is infinite.

2.1.4 3-Forms Integrated Over Hypersurfaces

There is a point we must clarify about the integration of a form over a submanifold (as, e.g., a hypersurface). The integral over a d-form needs a d-dimensional oriented "domain of integration". The definition [10, p. 141-3] of such an integral supposes that the domain is a so-called d-chain, that is a formal linear combination of so-called singular[3] d-simplices, which are smooth mappings from (a neighborhood of) the standard simplex in $\mathbb{R}^d$ into the manifold. This strategy ensures that (a) the domain of integration is compact, so the d-form is bounded and the integral is

[3] The name "singular" comes from the fact that the mapping need not have full rank.

finite, and that (b) the boundary of the domain of integration is piecewise smooth again.

To extend this definition to submanifolds as the domain of integration, one may simply decompose the submanifold into countably many simplices σ_k and add the integrals over σ_k. The only problem that arises is that the sum may not converge. Four cases are possible:

(a) The sum converges for every triangulation, and the value is independent of the choice of triangulation. In this case, there is no ambiguity about the value of the integral over the submanifold.

(b) The sum diverges to ∞ for every triangulation. In this case, we set the value of the integral to ∞.

(c) The same with $-\infty$.

(d) For some triangulation, the positive summands and the negative ones are both infinite. Now the integral is really ill-defined. To ensure it always has *some* value, we set the integral to $+\infty$ also in this case.

2.1.5 Minkowski and Galilei Space

An *affine Minkowski* (resp. *Galilei*) *space* is a set on which the additive group of a Minkowski (resp. Galilei) vector space acts freely and transitively. A *Minkowski vector space* is a four-dimensional real vector space endowed with a symmetric bilinear form η which in a suitable basis has the matrix form $\mathrm{diag}(1,-1,-1,-1)$. A *Galilei vector space* is a four dimensional real vector space endowed with a nonzero linear form ω (the *absolute time*) and a positive definite symmetric bilinear form on $\ker\omega$. The affine Minkowski (resp. Galilei) space may equivalently be characterized as a manifold diffeomorphic to $\mathbb{R}^4$ endowed with a flat connection (on the tangent bundle) and the structure of a Minkowski (resp. Galilei) vector space on every tangent space, such that the Minkowski (resp. Galilei) structure is parallel ("covariantly constant") w.r.t. the connection. We say a basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of a Minkowski (resp. Galilei) vector space is *inertial*, if the matrix representing the metric is $\eta = \mathrm{diag}(1,-1,-1,-1)$ (resp. if $\omega(\boldsymbol{e}_0) = 1$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ form an orthonormal basis of $\ker\omega$). It is clear there exist global coordinate charts of a Minkowskian (or Galileian) space-time that are inertial in every point; such coordinates we call *inertial coordinates*. Note that in inertial coordinates, like in every coordinate chart, the equation $x^0 = \text{const.}$ defines a hypersurface.

2.1.6 Volume Forms and Current Vector Fields

There is a relation between $d-1$-forms, d-forms, and vector fields on a d-dimensional manifold. A d-form α and a vector field j together define a $d-1$-form by plugging the vector into the first slot of the d-form, $\beta(X_1, \ldots, X_{d-1}) = \alpha(j, X_1, \ldots, X_{d-1})$. In usual index notation, this translation reads for $d = 4$:

$$\beta_{\lambda\mu\nu} = j^\kappa \alpha_{\kappa\lambda\mu\nu} , \tag{3}$$

At point p, in other words, α defines a linear mapping from the tangent space T_p to the space $\Lambda^{d-1}(T_p)$, called the $d-1$-th exterior power of T_p and being the value space of $d-1$-forms at p. If $\alpha_p \neq 0$ then this mapping is injective and, since both spaces have the same dimension d, also surjective and hence an isomorphism.

So if a nowhere-vanishing d-form (which is often called a *volume form*) is given then a vector field and a $d-1$-form amount to the same thing. This is sometimes called the *duality* between $d-1$-forms and vector fields; but keep in mind this depends on the volume form. A volume form is automatically selected by the metric (together with the orientation) in Minkowski space since Lorentz transformations have determinant ± 1: choose a (correctly oriented) inertial frame $\boldsymbol{e}_0 \ldots \boldsymbol{e}_3$, and let α be the 4-form having components

$$\alpha_{\kappa\lambda\mu\nu} = \varepsilon_{\kappa\lambda\mu\nu} ,$$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is the Levi-Cività symbol (the unit skew symbol, i.e., if $\kappa\lambda\mu\nu$ is a permutation of 0123, ε is the sign ± 1 of the permutation, and 0 otherwise). In other correctly orientated frames, α will have the same components because the transformation matrix has determinant 1. Equivalently, we may say

$$\alpha = \hat{\boldsymbol{e}}_0 \wedge \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3$$

where $\hat{\boldsymbol{e}}_0 \ldots \hat{\boldsymbol{e}}_3$ is the dual basis. In such a frame, we can spell out the inverse formula to (3):

$$j^\kappa = -\frac{1}{3!} \varepsilon^{\kappa\lambda\mu\nu} \beta_{\lambda\mu\nu} .$$

Since Galilei transformations, when orientation-preserving, have determinant 1, too, every (oriented) Galilei space has a selected volume form as well. Similarly, a volume form is selected on any Lorentzian or Riemannian manifold by the metric and the orientation. (Note that the role of the metric in the duality is solely to select a volume form and nothing else; as an illustration, duality also works in Galilei space where no metric is available.)

With a given volume form, (3) can be inverted, and the so distilled j is called the *current vector field*. Many examples and applications use this channel in the other direction: they define a current vector field first, and then exploit the volume form to translate it into a 3-form, which again serves to define a deterministic random worldline. In all these cases, the random worldline is only obtained by mediation of the 3-form, and the 3-form is only reached by the duality mechanism. It is fair to say that the natural tensor field representing the random worldline is the 3-form rather than the current vector field, since a vector field will not define a random worldline in the absence of a selected volume form.

In all our example cases (Galilei space, Lorentzian and Riemannian manifolds) a covariant derivative ∇_μ is available with respect to which the volume form α is parallel, i.e., $\nabla\alpha = 0$. Note that $\nabla\alpha$ distinct from $\mathrm{d}\alpha$, which identically vanishes by definition. The relation between d and ∇ is $\mathrm{d}\alpha = \nabla \wedge \alpha$, or, in components: $(\mathrm{d}\alpha)_{\lambda_1 \ldots \lambda_{d+1}} = \nabla_{[\lambda_1} \alpha_{\lambda_2 \ldots \lambda_{d+1}]}$ where the square brackets stand for anti-symmetrization. The covariant derivative allows us to speak of derivatives of j, and to ask into what condition the closedness of β translates. The general answer is

$$0 = \nabla_{[\xi} \beta_{\lambda\mu\nu]} = \Big(\nabla_{[\xi|} \alpha_{\kappa|\lambda\mu\nu]} \Big) j^\kappa + \Big(\nabla_{[\xi|} j^\kappa \Big) \alpha_{\kappa|\lambda\mu\nu]} \,,$$

and in case $\nabla\alpha = 0$ this reduces to $\nabla_\mu j^\mu = 0$; in words, closed 3-forms correspond to divergence-free current vector fields.

2.2 3-Forms From Random Worldlines

2.2.1 The Expected Number of Signed Crossings

Given a random worldline L and a hypersurface H, the number of signed crossings $N(L, H)$ is a $\mathbb{Z} \cup \{\infty, -\infty\}$-valued random variable, and we may ask for its expectation value.

There is a subtlety about expectation values of random variables that may assume infinite values. Let X be a $\mathbb{Z} \cup \{\infty, -\infty\}$-valued random variable; the expectation of the positive part of X is always well defined,

$$\mathbb{E}(X \cdot \mathbf{1}_{\{X>0\}}) = \begin{cases} \infty & \text{if } \mathrm{Prob}\{X = \infty\} > 0 \\ \sum\limits_{n=1}^{\infty} n \mathrm{Prob}\{X = n\} & \text{otherwise,} \end{cases}$$

where the infinite sum may well be infinite. Similarly, the expectation of the negative part of X is well-defined; the problem arises when both partial expectation values are infinite. In that case, we define the expectation $\mathbb{E}X$ to be $+\infty$, to make sure it is always defined.

$\mathbb{E}N(L, \cdot)$ is a $\mathbb{R} \cup \{\infty, -\infty\}$-valued functional on the set of hypersurfaces. Due to the linearity of expectations, it inherits the additivity of $N(\ell, \cdot)$, i.e., $\mathbb{E}N(L, \cdot)$ adds on disjoint unions of hypersurfaces. Moreover, it changes sign when the orientation of the hypersurface is reversed. It can thus be extended to a linear functional on the formal linear combinations of hypersurfaces with real coefficients (where $-H$ is identified with the reversed orientation of H, and $H_1 + H_2$ with $H_1 \cup H_2$ provided $H_1 \cap H_2 = \emptyset$). This means $\mathbb{E}N(L, \cdot)$ is a real 3-cochain (except at the infinities, where linearity breaks down).

In other words, $\mathbb{E}N(L, \cdot)$ is a "formal" 3-form if we disregard differentiability questions.

2.2.2 Closedness

Our reason to restrict attention to closed 3-forms is that under suitable conditions, the formal 3-form $\mathbb{E}N(L,\cdot)$ is closed in the sense that it vanishes on the boundary of a compact 4-volume. The rough idea behind this is that what flows in has to flow out again some time, so the net flux across the surface should be zero. We elaborate a bit on what conditions this would be.

A mapping is called *proper* if the pre-image of every compact set is compact. For a smooth curve $\gamma : \mathbb{R} \to$ (space-time) this means it sooner or later leaves ultimately every compact set, in both time directions. Since properness is preserved under reparametrization, we may speak of proper wordlines. The notion of properness is interesting in the context of the question whether the worldlines start or end somewhere, and from the physical point of view, it is a very reasonable property for a worldline to have.

Consider a compact 4-volume R in space-time with smooth surface ∂R. For every (nonrandom) worldline ℓ that is proper, the number of signed crossings $N(\ell, \partial R)$ is zero: since ℓ ultimately leaves R in both time directions, there was a time s_1 before the first encounter with R, and $N(s_1, t)$ is 1 when $\gamma(t)$ is inside R and 0 when outside because there is no way to get inside (resp. out) without crossing the boundary.

Now if a random worldline L has the property that it is proper with probability one, then $N(L, \partial R) = 0$ with probability one, thus $\mathbb{E}N(L, \partial R) = 0$.

2.2.3 Smoothness

There is no general condition ensuring the formal 3-form $\mathbb{E}N(L,\cdot)$ is a smooth 3-cochain, i.e., that there is a 3-form β such that $\mathbb{E}N(L,\cdot) = \int_\cdot \beta$. It is simply a property a random worldline L may have or not. More can be said about deterministic random worldlines:

Proposition. *Let L be a (smooth) deterministic random worldline. Then there is a unique smooth 3-form β on S_0 such that*

$$\mathbb{E}N(L, H) = \int_H \beta \,. \tag{4}$$

whenever either side is finite. Moreover, β is closed.

Proof. Choose $p \in S_0$; by the rectification theorem (see p. 21), there is a coordinate system (U, x) on a neighborhood around p that maps the worldlines ℓ_q to lines parallel to the x^0 axis (and with the same orientation); by axiom (iv) of a smooth deterministic random worldline, we may assume $\ell_q \cap U$ is connected for every $q \in U$. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ denote the canonical projection; every point $u \in \pi \circ x(U)$ represents a worldline $\pi^{-1}(u)$. By axiom (v), the induced measure on the projection $\pi \circ x(U)$ has smooth density $\rho(x^1, x^2, x^3)$. Define $\beta|U := \rho(x^1, x^2, x^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

Certainly, β is smooth and closed:

$$\mathrm{d}\beta = \mathrm{d}\rho \wedge \mathrm{d}x^1 \dots \mathrm{d}x^3 = \sum_{a=1}^{3} \frac{\partial \rho}{\partial x^a} \, \mathrm{d}x^a \wedge \mathrm{d}x^1 \dots \mathrm{d}x^3 = 0 \, .$$

Once we show that for hypersurfaces in U, (4) is true, we are finished, because

(a) This ensures uniqueness of β on U, because for $\beta' \neq \beta$, one can find an arbitrarily small piece of hypersurface H such that $\int_H \beta' \neq \int_H \beta$.

(b) As a consequence of uniqueness, β_p does not depend on the choice of (U, x), so the pieces $\beta|U$ match up to give a 3-form β on S_0.

(c) Eq. (4) is fulfilled for arbitrary hypersurfaces since they can be decomposed into smaller pieces lying within a U. Note that ignoring or double-counting the boundaries between pieces does no harm: since they form a null set in H, they do not change $\int_H \beta$, and since the image of a null set under a smooth mapping is a null set [6, p. 73], their images in $\mathbb{R}^3$ under $\pi \circ x$ are null sets in $\mathbb{R}^3$ and thus do not influence the expectation value.

The rest of this proof will show that (4) holds for hypersurfaces H in U. $x(H)$ is a hypersurface in $x(U) \subseteq \mathbb{R}^4$. The expectation value may be expressed as

$$\mathbb{E}N(L, H) = \int_{\pi \circ x(U)} N\big(\pi^{-1}(u), x(H)\big) \, \rho(u) \, \mathrm{d}^3 u \, . \tag{5}$$

We now remove from H all points where $\beta|H$ vanishes, that is we set $H' = \{p \in H | (\beta|H)_p \neq 0\}$ and observe $\int_H \beta = \int_{H'} \beta$. We claim that all the points we removed did not contribute to the left hand side of (4) either:

(i) Points with $\rho(\pi(p)) = 0$ can be left out of the domain of integration in (5) without changing the expectation value.

(ii) All other points with $(\beta|H)_p = 0$ must have the property that the tangent plane to H at p is vertical in coordinate space, i.e. it must contain a parallel to the x^0 axis. But the projection of these points forms a null set in $\mathbb{R}^3$, according to the lemma below; so we may leave the projection out of the domain of integration in (5) without damage.

The connected components H'_i of the remaining H' have the property that $\beta|H'_i$ does not change sign: since $\Lambda^3(T_p H)$ is 1-dimensional, the orientation of H defines a *positive half-space* and a *negative half-space* of $\Lambda^3(T_p H)$; let σ_i be $+1$ if $\beta|H'_i$ is positive, and -1 otherwise. If σ_i is positive (resp. negative), each point of H'_i is a positive (resp. a negative) transverse crossing of some worldline. The projection π when restricted to H'_i has full rank, so it is a local diffeomorphism. Divide H'_i into (countably many) pieces H'_{ij} such that on each H'_{ij}, π is a diffeomorphism onto its image, which allows us to parametrize H'_{ij} from its projection, and we find

$$\int_{H'_{ij}} \beta = \sigma_i \int_{\pi(H'_{ij})} \rho(u)\, \mathrm{d}^3 u\,,$$

and summing over i and j we add the signs of all transverse crossings and arrive at the right hand side of (5), which completes the proof. □

Lemma. *Given a ($d-1$-dimensional) hypersurface H in $\mathbb{R}^d$ ($d \geq 2$), the set $S = \pi\{p \in H | \partial/\partial x^0 \in T_p H\}$, that is the projection in $\mathbb{R}^{d-1}$ of all $p \in H$ such that the tangent plane at p is vertical, has Lebesgue measure zero.*

Proof. Consider $d = 2$ first: choose a curve γ parametrizing 1-dimensional H. Remove all the points having horizontal tangent; this does not interfere with the set of points having vertical tangent. Since the parameter values s with horizontal tangent are the zeroes of $x^0(\dot{\gamma}(s))$, this set is closed, and the remaining set of parameter values is open, and thus is the disjoint union of countably many open intervals U_i. To show that $\lambda(S) = 0$, it suffices to show that $\lambda(S_i) = 0$ where $S_i = S \cap \pi\big(\gamma(U_i)\big)$. On U_i, the hypersurface may be smoothly parametrized by the x^0 coordinate; we assume γ is such a parametrization, $\gamma(t) = \big(t, f(t)\big)$. Then $S_i = f\big(\{t \in U_i | \dot{f}(t) = 0\}\big)$, and the statement follows from the fact that U_i can be covered by countably many compact intervals, and the lemma of Sard:

Sard's Lemma. *[9, p. 247] Let U be an open interval, let $f : U \to \mathbb{R}$ be C^1 and Lipschitz, and let $C = \{t \in U | \dot{f} = 0\}$ be the set of critical points. Then $f(C)$ has Lebesgue measure zero.*

Now consider $d > 2$. We only need to show that in a sufficiently small neighborhood U of any $p \in H$ where the tangent plane $T_p H$ is vertical, the set of points with a vertical tangent plane projects to a null set in $\mathbb{R}^{d-1}$. Choose U such that, as q varies in U, the plane $T_q H$ varies not too much in the set of $d-1$-dimensional subspaces of $\mathbb{R}^d$; this includes that $T_q H$ does not become horizontal, and where it is vertical, its projection $\pi(T_q H)$ does not vary much in the set of $d-2$-dimensional subspaces of $\mathbb{R}^{d-1}$. Then there is a direction $\boldsymbol{e}$ in $\mathbb{R}^{d-1}$ which is transverse to all the $T_q H$, and we consider the family of parallel 2-planes in $\mathbb{R}^d$ containing the direction $\boldsymbol{e}$ and the vertical. Using these 2-planes, we reduce everything to the 2-dimensional case.

The intersection of H with such a 2-plane P is an imbedded submanifold of dimension 1 because H and P are transverse (i.e., $T_q H$ and $T_q P$ span the $\mathbb{R}^d$ in every $q \in H \cap P$), and the transverse intersection of imbedded submanifolds is again an imbedded submanifold, while the codimensions add [10, p. 31].

$H \cap P$ has vertical tangent at every point in $U \cap H \cap P$ where H has a vertical tangent plane. Thus, by the statement of the lemma for dimension 2, the projected vertical tangents over any line $g \subseteq \mathbb{R}^{d-1}$ with direction $\boldsymbol{e}$ form a null subset of g. But by Fubini's theorem (applied to an indicator function), a set S is a null set in $\mathbb{R}^{d-1}$ if $S \cap g$ is a null set in 1 dimension for every parallel g to some fixed axis. □

3 Constructing a Deterministic Random Worldline From a 3-Form

3.1 Construction of the Worldlines

3.1.1 Directions From a 3-Form

A 3-form has the property that it selects a direction in every tangent space where it does not vanish. This fact is simple yet crucial for our entire enterprise. Say β is a 3-form, and β_p is β taken at point p.

Proposition. *Assume $\beta_p \in \Lambda^3(T_p)$ does not vanish. Then β_p has a 1-dimensional kernel (= subspace of T_p containing such vectors X that for all $Y, Z \in T_p$: $\beta(X, Y, Z) = 0$).*

Proof. Let $\Lambda^k(E)$ denote the space of skew-symmetric k-linear forms on a given vector space E of dimension d. Choose an element $\alpha \neq 0$ from $\Lambda^d(E)$. α induces a linear mapping $E \to \Lambda^{d-1}(E)$ by $Y \mapsto i_Y\alpha$ where $i_Y\alpha$ denotes the skew-symmetric $d-1$-linear form $(X_1, \ldots, X_{d-1}) \mapsto \alpha(Y, X_1, \ldots, X_{d-1})$. This linear mapping $E \to \Lambda^{d-1}(E)$ is injective, and as $\Lambda^{d-1}(E)$ has the same dimension d as E, it is an isomorphism. Thus every given $d-1$-form β_p can be written as $\alpha(Y, -, \cdots, -)$. If $\beta_p \neq 0$, then $Y \neq 0$. Obviously, the equation $i_X\beta_p = \beta_p(X, -, \cdots, -) = 0$ holds if and only if $\alpha(Y, X, -, \cdots, -) = 0$ which is equivalent to $X = \lambda Y$ for some real λ. □

In case a volume form is given, this is precisely the mechanism of duality between vector fields and $d-1$-forms described on p. 12. But if no particular α_p is selected, every nonzero element of $\Lambda^d(E)$ is good enough. Since $\Lambda^d(E)$ has dimension 1, two possible choices for α_p can only differ by a nonzero factor. That is why β_p defines a 1-dimensional subspace (its kernel), but not a particular vector.

The nonzero elements of $\Lambda^d(E)$ (which has dimension 1) form two classes within which the elements differ only by *positive* factors. The given orientation on space-time (and thus on $E = T_p$) selects one of these classes. This again selects one of the two half-spaces of $\ker \beta_p$, or, equivalently, an orientation of $\ker \beta_p$.

So a 3-form β induces a smooth field $\tau = \ker \beta$ of tangent lines (1-dimensional subspaces of the tangent spaces) on the open subset $D = \{p | \beta_p \neq 0\}$, and even a smooth field of directions τ^+ on D, where τ_p^+ is the positive half-line of τ_p. The relevance of this direction field for a deterministic random worldline is made clear by the following observation: if L is a deterministic random worldline and X_p a vector tangent to ℓ_p, then $i_{X_p}\beta_p = 0$ by construction, that is, $X_p \in \ker \beta_p$.

Corollary. *Let L, L' be deterministic random worldlines. If their associated 3-forms coincide, $\beta = \beta'$, which presupposes $S_0 = S_0'$, then L and L' are identically distributed.*

Proof. Since $\ker \beta_p$ is the tangent to ℓ_p, on the open set $M := S_0 \cap \{p | \beta_p \neq 0\}$ the tangent field to the foliation formed by the ℓ_p must agree with that of ℓ_p', so the two foliations must agree on M. Thus L and L' share the same possible worldlines; furthermore, rectifying coordinates (U, x) also rectify the ℓ_p', and in terms of x we have $\rho \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \beta = \rho' \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, so $\rho = \rho'$. The probability for L to lie in $S_0 \setminus M$ is zero. □

Corollary. *Let L, L' be deterministic random worldlines, and β, β' their associated 3-forms. Let $U \subseteq S_0 \cap S_0'$ be open, and $\ell_p \subseteq U$ for every $p \in U$. If $\beta | U = \beta' | U$ then ℓ_p is part of ℓ_p' for all $p \in U$ up to null events, and the mapping $\ell_p \mapsto \ell_p'$ preserves the measure.*

Proof. Set $M := U \cap \{p | \beta_p \neq 0\}$. Note $L \in U \setminus M$ is a null event. On M, the kernels of β_p' and β_p coincide and thus the tangents to the foliations, implying ℓ_p is a part of ℓ_p' for every $p \in M$. In rectifying coordinates the density functions of L and L' coincide, implying the probability for L to lie in a certain set of worldlines equals the probability for L' to lie in the corresponding set of worldlines under the mapping $\ell_p \mapsto \ell_p'$. □

These two corollaries answer a uniqueness question, the existence counterpart of which is the topic of this chapter. But before we study the construction of a random worldline from β, we draw another useful conclusion:

Proposition. *Given a deterministic random worldline (with associated 3-form β) and a coordinate chart (U, x), then the probability density for crossing the $x^0 = \text{const.}$ hypersurface at p is $|\beta_{123}(p)| = |\beta(\partial_1, \partial_2, \partial_3)(p)|$. More precisely, if $\beta_{123}(p) \neq 0$ then there is a 3-neighborhood H of p (on the $x^0 = \text{const.}$ hypersurface) that each worldline intersects at most once, and as a transverse crossing, and with constant sign on H; and for Borel subsets $A \subseteq H$, the probability of hitting A is $\int_A |\beta_{123}(x)| \, \mathrm{d}^3x$.*

Proof. If $\beta_{123}(p) \neq 0$ then $\beta_p \neq 0$, and the kernel of β_p is not tangent to $\{x^0 = \text{const.}\}$. Choose a 3-neighborhood $H' \subseteq \{x^0 = \text{const.}\} \subseteq U$ of p with $\beta_{123} \neq 0$ on H'. There is a neighborhood U' of p such that every worldline runs only once through U' (axiom (iv) of a deterministic random worldline). Define $H = H' \cap U'$. Then a worldline cannot travers H twice since the sign of β is constant. From the fact that every worldline can intersect H at most once and with a fixed sign $\sigma = \pm 1$, it follows that the expected number of signed crossings equals σ times the probability of crossing, for every polyhedral subset of H; thus, the probability density is $|\beta_{123}|$. □

In case a volume form α is selected, and in case the coordinates satisfy $\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3 = \alpha$, we observe $|\beta_{123}(p)| = |j^0(p)|$ for the current j.

We now take β to be any given closed 3-form. The maximal integral curves of τ are what we will and have to use as the worldlines of the deterministic random worldline, and the orientations of these worldlines are fixed by the condition that $\dot\gamma_p \in \tau_p^+$ for positive parametrization γ.

Note that the worldlines the (yet to be constructed) deterministic random worldline consists of are tangent to the kernel of β. Relative to a given (positively oriented) volume form, β corresponds to a current vector field j, and we observe $j_p \in \tau_p^+$ (this is immediate from the definition $\beta = \alpha(j, \ldots)$ of j). As a consequence, all the worldlines are tangent to the current.

We will often choose a τ^+-compatible vector field, i.e. a smooth vector field X such that $X_p = 0 \Leftrightarrow \beta_p = 0$ and $X_p \in \tau_p^+$ wherever $\beta_p \neq 0$. In case a volume form is selected, the current does the job; but for our purposes it is enough to take *some* τ^+-compatible vector field, as produced by *some* positively oriented nowhere-vanishing 4-form (which exists due to orientability of space-time). We also sometimes say a vector field X on space-time is τ-compatible if $X_p = 0 \Leftrightarrow \beta_p = 0$ and $X_p \in \tau_p$ wherever $\beta_p \neq 0$. The integral curves of τ, of course, are the integral curves of X, except that the integral curves of τ do not have a defined parametrization. Different choices of X differ on $D = \{p|\beta_p \neq 0\}$ by a (smooth) nowhere-vanishing scalar function, $X'|D = fX|D, f \neq 0$ (it may happen that f does not possess a smooth continuation outside D).

3.1.2 Facts About Integral Curves

We note the main facts about integral curves:

Existence and uniqueness theorem. *[10, p. 37] Let X be a C^∞ vector field on a (differentiable) manifold M. For each $p \in M$ there exist $a(p)$ and $b(p)$ in $\mathbb{R} \cup \{\infty, -\infty\}$, and a smooth curve*

$$\gamma_p : (a(p), b(p)) \to M$$

such that

(i) $0 \in (a(p), b(p))$ *and* $\gamma_p(0) = p$.

(ii) γ_p *is an integral curve of* X, *i.e.,* $\dot\gamma_p(t) = X_{\gamma_p(t)}$ *for all* t.

(iii) If $\gamma' : (c, d) \to M$ *is a smooth curve satisfying conditions (i) and (ii), then* $(c, d) \subseteq (a(p), b(p))$ *and* $\gamma' = \gamma|(c, d)$.

(iv) *For each $p \in M$, there exists an open neighborhood U of p and an $\varepsilon > 0$ such that the flow map*

$$(p,t) \overset{\phi}{\longmapsto} \gamma_p(t)$$

is defined and is C^∞ from $U \times (-\varepsilon, \varepsilon)$ into M.

(v) *For each $t \in \mathbb{R}$, $\mathcal{D}_t := \{p \in M | t \in (a(p), b(p))\}$ is open.*

(vi) *$\phi^t : p \mapsto \phi(t,p)$ is a diffeomorphism $\mathcal{D}_t \to \mathcal{D}_{-t}$ with inverse ϕ^{-t}.*

(vii) *Let t and t' be real numbers. Then the domain of $\phi^t \circ \phi^{t'}$ is contained in $\mathcal{D}_{t+t'}$ and is equal to $\mathcal{D}_{t+t'}$ in case t and t' have the same sign. On the domain of the left hand side, $\phi^t \circ \phi^{t'} = \phi^{t+t'}$.*

Rectification theorem. *[10, p. 41] Let X be a smooth vector field on the manifold M, and $p \in M$ such that $X_p \neq 0$. Then there exists a coordinate system (U,x) with coordinate functions $x_1, \ldots, x_d$ on a neighborhood of p such that $X|U = \frac{\partial}{\partial x_1}|U$.*

Proposition. *The set $\mathcal{D} = \{(p,t) \in M \times \mathbb{R} | t \in (a(p), b(p))\}$ is open, and the flow map $\phi : \mathcal{D} \to M$ is C^∞.*

Proof. We show that every pair $(p,t) \in \mathcal{D}$ has an open neighborhood $V(p,t) \subseteq M \times \mathbb{R}$ such that ϕ exists and is smooth on $V(p,t)$. It is sufficient to fix p and show that the set of such t that a $V(p,t)$ exists is nonempty, open and closed in $(a(p), b(p))$, and thus equal to the whole interval. Nonemptyness follows from statement (iv) of the existence theorem, while openness is trivial.

For closedness, consider a sequence $t_n \to T$ converging within $(a(p), b(p))$, that is, $a(p) < T < b(p)$; without loss of generality, we may assume $T > 0$. We distinguish two cases: (a) $X_{\phi^T(p)} = 0$, and (b) $X_{\phi^T(p)} \neq 0$.

(a) By uniqueness of the integral curve, it must be constant, $\phi^t(p) = p \; \forall t$, so $X_p = 0$. According to statement (iv) of the existence theorem, there is a neighborhood U of p and an $\varepsilon > 0$ such that ϕ is C^∞ on $U \times (-\varepsilon, \varepsilon)$. Choose a coordinate chart (U', x) such that $U' \subseteq U$. By means of the coordinates, we can assign distances to pairs of points, and norms to the velocities $X_{p'}$.

We want to have a neighborhood $\widetilde{U}$ of p such that the velocities in $\widetilde{U}$ are bounded so close to 0 that the integral curves starting in $\widetilde{U}$ cannot leave U' (or cease to exist) before time $T + \delta$ with $\delta > 0$. Then choose an integer $n > (T+\delta)/\varepsilon$ and observe that $\phi^t = (\phi^{t/n})^n = \phi^{t/n} \circ \cdots \circ \phi^{t/n}$ is smooth since $t/n < \varepsilon$ for $t < T + \delta$.

Since X is differentiable, there is a constant $K > 0$ such that

$$\|x_* X\| \leq K \|x - x(p)\|$$

(that is to say, velocity is less than K times distance from p) on a neighborhood $U'' \subseteq U'$ of p. U'' contains the ball (in terms of x-coordinates) of radius (say) r around p; as $\widetilde{U}$, choose the ball around p of radius $re^{-K(T+\delta)}$. A point in $\widetilde{U}$ cannot leave U'' within time $T + \delta$.

(b) Choose rectifying coordinates (U, x) around $\phi^T(p)$. Sufficiently far in the sequence t_n, the difference $\delta := |t_n - T|$ is so small that $\{\phi^t(p)|T - 2\delta < t < T + 2\delta\} \subseteq U$ and $T - 2\delta > 0$. By definition of T, ϕ exists and is smooth on a neighborhood V' of (p, t_n) (keep in mind $t_n = T \pm \delta$) while on U the flow (for small times) is simply (the pull-back through x of) the translation on $\mathbb{R}^4$ in the x^0-direction. Thus if we choose a neighborhood U' of $\phi^{t_n}(p)$ such that $U' \subseteq U$ and $\phi^{\mp\delta}(U') \subseteq U$, then $\phi^{\mp\delta}$ is smooth on U'. By continuity of ϕ on V' there is a neighborhood V'' of (p, t_n) with $\phi(V'') \subseteq U'$. Now by statement (vii) of the existence theorem, for $(p', t) \in V''$ we have $\phi^{t\mp\delta}(p') = \phi^{\mp\delta}\phi^t(p')$, so $(p', t \mp \delta) \in \mathcal{D}$, and ϕ is smooth on $\{(p', t \mp \delta)|(p', t) \in V''\}$ which is open and contains (p, T), as desired.

□

Since two τ^+-compatible vector fields X, X' differ on $D = \{p|\beta_p \neq 0\}$ by a positive function, the integral curves γ, γ' through a point p differ only by an orientation-preserving reparametrization. Besides, an integral curve cannot intersect itself (due to the uniqueness of part of the existence theorem above). Thus, we have: *The X-integral curve through p defines a worldline if and only if it is not closed; in this case, the worldline does only depend on τ^+ but not on the choice of X.*

Every such worldline lies within the open set D. In the following, we will speak of τ^+-integral worldlines; sometimes also of τ-integral (or τ^+-integral) curves, meaning equivalence classes of curves integral to *some* τ-compatible (or τ^+-compatible) vector field.

3.1.3 The Wandering Set

An integral curve of τ^+ might feature two undesirable properties. First, it might be closed (and fail to define a worldline in the sense used here). Then, it might recur to a each neighborhood of a point it previously traversed infinitely many times; why this is bad for our purposes we discuss with a simple case:

Example. Consider the geodesics on the torus[4] $\mathbb{R}^2/\mathbb{Z}^2$ of a fixed irrational slope; such curves are sometimes called solenoids; they are everywhere dense. They arise indeed from a closed $d-1$-form in just the manner described earlier, namely $\beta = \mathrm{d}x^1 + \lambda \mathrm{d}x^2$ where λ is an irrational constant. Now the number of signed

[4] Of course, similar cases can be found in dimension 4, but the relevant phenomenon is already present in dimension $d = 2$.

crossings through a piece of hypersurface, however small, is infinite, so it is not possible to understand integrals of the $d-1$-form as the expected number of signed crossings. The form does not locally define probabilities either.

We thus restrict our attention to the well-behaved integral curves of τ^+:

Definition. Let ϕ^t denote the flow map of X. We say a point p *wanders* w.r.t. X if there is an open neighborhood U of p and a number $T > 0$ such that for all $t > T$, $\phi^t\{q \in U|\phi^t(q) \text{ exists}\} \cap U = \emptyset$. The set of all wandering points w.r.t. any τ-compatible vector field X we call the *wandering set* of τ.

Proposition. *The wandering set is open, and with every p it contains the entire τ-integral curve through p.*

Proof. Openness is immediate from the definition, since U is a subset of the wandering set. Now we show that if p wanders w.r.t. X then so does $p' = \phi^{t'}(p)$ (if $t' > 0$); consider $T' = T$ and $U' = \phi^{t'}\{q \in U|\phi^{t'}(q) \text{ exists}\}$, which is open (because U and $\mathcal{D}_t$ are open, and $\phi^{t'}$ is a diffeomorphism); for $t > T$,

$$\phi^t\{q' \in U'|\phi^t(q') \text{ exists}\} \cap U' = \{\phi^t(q')|q' \in U' \text{ and } \phi^t(q') \text{ exists}\} \cap U' =$$

$$= \{\phi^{t+t'}(q)|q \in U \text{ and } \phi^{t+t'}(q) \text{ exists}\} \cap U' =$$

$$= \phi^{t'}\{\phi^t(q)|q \in U \text{ and } \phi^{t+t'}(q) \text{ exists}\} \cap \phi^{t'}\{r \in U|\phi^{t'}(r) \text{ exists}\} =$$

$$= \phi^{t'}\left\{r \in \phi^t\{q \in U|\phi^t(q) \text{ exists}\} \cap U \middle| \phi^{t'}(r) \text{ exists}\right\} = \emptyset$$

since $\phi^t\{q \in U|\phi^t(q) \text{ exists}\} \cap U$ is empty. In the case $t' < 0$, p' is still contained in the wandering set, since (according to what we have just shown) it is wandering w.r.t. $-X$. □

Moreover, the wandering set does not contain any closed integral curves, since if there is a $t > 0$ such that $\phi^t(p) = p$, the disjointness condition cannot be fulfilled.

We also note that the integral curves form a foliation of $\{\beta \neq 0\}$, and of the wandering set; this follows from the rectification theorem. Let Ξ denote the set of τ^+-integral worldlines in the wandering set; in the following we assume it is not empty.

3.2 Construction of the Measure

3.2.1 A Simple Case First

Assume there is a hypersurface H such that every wordline $\in \Xi$ intersects H precisely once, and in a positive crossing. Without loss of generality, we may assume that H lies within the wandering set. Then there is a one-to-one relation between H and Ξ, so H may be taken to parametrize Ξ. The restriction of β to H defines a measure on the Borel subsets of H in an obvious manner: roughly speaking, the measure of a set $A \subseteq H$ equals $\int_A \beta$; for a rigorous treatment, see the general discussion below. This provides a measure on the subsets of Ξ. This measure is normalized (and thus a probability measure) if and only if

$$\int_H \beta = 1\,. \tag{6}$$

Let us assume this is the case. Then we have arrived at a deterministic random worldline. The question remains whether two different hypersurfaces, both satisfying the requirements we asked, give rise to the same random worldline. The answer is yes, and this follows from considerations we have to go through anyway when we are not given such a nice hypersurface H.

If (6) is not the case, then β cannot be the expected number of signed crossings of any deterministic random worldline (and perhaps, not of any random worldline at all). However, it may still be the expected number of signed crossings of a random finite set of worldlines. This amounts to a broadening of the notion of worldline, namely dropping the axiom that a worldline be connected. As a simple example, suppose β has norm 2; then $\frac{1}{2}\beta$ satisfies (6) and defines a deterministic random worldline; now take two identically distributed copies of this random variable; the expectation of the *sum* of the two numbers of signed crossings (which may be understood as the number of signed crossing for one "non-connected worldline") then agrees with the integral of the given β. In case the norm $\int_H \beta$ of β is noninteger but finite, we may first pick a random integer number $K \geq 0$ of worldlines distributed in a way that $\mathbb{E}K$ equals the norm of β, and then consider K independent identically distributed copies of the deterministic random worldline associated with the normalization of β, $(\int_H \beta)^{-1}\beta$.

A similar procedure can be applied if the norm of β is infinite: then use a covering of H such that on each covering set, the integral of β is finite, and a partition of unity subordinate to this covering, get a countable (locally finite) family of (automatically closed) 3-forms on H adding up to β, turn these into a countable family of independent deterministic random worldlines such that β gives the expected sum of the numbers of signed crossings.

In general, however, a hypersurface H as just considered does not exist. There exist only pieces of hypersurface cross-secting batches of worldlines, parametrizing patches of Ξ. On each patch, the measure will be defined in essentially the same way as above, and then the complete measure must be pieced together.

3.2.2 Outline

We now turn to the construction of the measure $\mathbb{M}$ on the space of worldlines. The definition goes roughly like this:

Given a coordinate chart (U, x) rectifying τ (so $U \subseteq \{\beta \neq 0\}$) such that $x(U) = (-1, 1) \times B^3_1$ where $B^3_1 \subseteq \mathbb{R}^3$ is the open 3-ball of radius 1, and with the property that every integral curve of τ intersects $H := x^{-1}(\{0\} \times B^3_1)$ at most once. Note that H is a (piece of) hypersurface. The 3-form β induces a measure on H and thus on the set of τ-integral worldlines intersecting H, and this measure coincides by definition with the restriction of $\mathbb{M}$ to the subsets of the set of τ^+-integral worldlines intersecting H. One has to show, of course, that these measures can be pieced together to give $\mathbb{M}$, i.e., that the constructed local measures are compatible on overlaps of patches—here the key ingredient is Stokes's theorem. It is understood that $\mathbb{M}$ of the complement of Ξ is 0.

3.2.3 Local Construction of the Measure

Proposition. *For every p in the wandering set, there is an open neighborhood U and a coordinate system $x : U \to \mathbb{R}^4$ such that*

1. *$U \subseteq$ wandering set*
2. *$x(U) = (-1, 1) \times B_1^3$ where $B_1^3 \subseteq \mathbb{R}^3$ is the open 3-ball of radius 1*
3. *τ^+ is mapped to the positive x^0-direction in $\mathbb{R}^4$*
4. *every integral curve of τ intersects $H := x^{-1}(\{0\} \times B_1^3)$ at most once.*
5. *$0 < \int_H \beta < \infty$.*

Such a coordinate system we will call a standard *coordinate system.*

Proof. Since p lies in the wandering set, there is a τ^+-compatible vector field X and an X-wandering open neighborhood U' of p. Now by the rectification theorem, there is an open neighborhood $\widetilde{U}$ of p (without loss of generality, $\widetilde{U} \subseteq U'$) and a coordinate chart $\widetilde{x}$ such that $X|\widetilde{U} = \partial/\partial\widetilde{x}^0$. x will simply be a suitable magnification of $\widetilde{x}$, restricted on a smaller neighborhood. As soon as we have an open neighborhood U'' of p such that every τ-integral curve intersects $\widetilde{x}^{-1}(\{0\} \times B_1^3) \cap U''$ *at most once*, we take U to be a suitable subset of $\widetilde{U} \cap U''$, and we are finished.

Now assume such an U'' did not exist, i.e., there are points u_n (for all integers $n > n_0$) in the $\{x^0 = 0\}$ plane of coordinate space at distance less than $1/n$ from the origin such that the τ-integral curves through the points $p_n = \widetilde{x}^{-1}(u_n)$ they represent sooner or later meet the hyperplane $\widetilde{H} := \widetilde{x}^{-1}\{x^0 = 0\}$ again at points p'_n whose coordinate representatives $u'_n = \widetilde{x}(p'_n)$ are again at distance less than $1/n$ from the origin; say, when read as X-integral curves, they meet $\widetilde{H}$ again at parameter times t_n, i.e., $\phi^{t_n}(p_n) = p'_n$. Without loss of generality, $\widetilde{U}$ may be chosen such that its image in coordinate space is a cylinder $(-r, r) \times B_r^3$; then the recurrence to $\widetilde{H}$ takes a minimum time $2r$ (since in coordinates, a point moves in x^0-direction at speed 1 just after leaving and just before recurring to $\{x^0 = 0\}$). By the wandering property, all t_n must be less than $T < \infty$, since after time T, all $\phi^t(p_n)$, if existent, must be outside U'. So we have an infinite sequence t_n between $2r$ and T, thus t_n must have an accumulation point $\widetilde{t}$ with $2r \leq \widetilde{t} \leq T$. Since $p_n \to p$ and $p'_n \to p$ as $n \to \infty$, and since ϕ is continuous, $\phi^{\widetilde{t}}(p)$ must be defined and equal to p, thus the integral curve through p is closed, in contradiction to the wandering of p.

Finiteness of the integral of β over H can be achieved by making the neighborhood smaller; the integral is nonnegative anyway (thanks to the correct orientation of our coordinates), and nonzero because β is nonzero where X is nonzero. □

Now comes the core of the construction of the measure $\mathbb{M}$. We pick a standard coordinate chart (U, x); let Ξ_H denote the set of τ^+-integral worldlines that intersect $H = x^{-1}(\{0\} \times B_1^3)$; these worldlines intersect H precisely once, that means we have a bijection between H and Ξ_H. In the following paragraph, we construct a measure $\mathbb{M}[U, x]$ on the ($\mathcal{W}$-measurable) subsets of Ξ_H; in the end this will agree with the restriction of $\mathbb{M}$, i.e., for every ($\mathcal{W}$-measurable) subset B of Ξ_H, $\mathbb{M}(B) = \mathbb{M}[U, x](B)$.

x induces a parametrization of the set Ξ_H with B_1^3 as the parametrizing domain, in the following way: Let $\widetilde{x} : B_1^3 \to$ (set of worldlines) map $u \in B_1^3$ to the unique τ^+-integral worldline intersecting H in the point $x^{-1}(0, u)$. $x_*\beta$ is a 3-form on $(-1, 1) \times B_1^3$, and by restriction a 3-form on B_1^3, which is a function f times the canonical 3-form $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. The density function f is real-valued, smooth, and positive (by construction of standard coordinates). f times the Lebesgue measure is a measure on the Borel σ-algebra of B_1^3, and this measure agrees with the 3-form $x_*\beta$ in the following sense: whenever A is a polyhedral or open subset of B_1^3 and g is a smooth function on B_1^3,

$$\int_A g(u) f(u) \lambda(\mathrm{d}u) = \int_A g(u) x_*\beta \,,$$

where the left hand side is a Lebesgue integral and the right hand side is an integral of a 3-form. Now $\widetilde{x}$ carries this measure over to the set Ξ_H, i.e., for a subset B of Ξ_H, define

$$\mathbb{M}[U, x](B) = \int_{\widetilde{x}^{-1}(B)} f(u) \lambda(\mathrm{d}u) \,.$$

Note that $\mathbb{M}[u, x]$ is a finite measure thanks to property 5 of standard coordinate charts.

3.2.4 Measurability

One point we have to show is that $\widetilde{x}$ carries Borel sets $A \subseteq B_1^3$ to $\mathcal{W}$-measurable sets of worldlines.

Proposition. *Given an open subset U of space-time and a nowhere-vanishing vector field X on U having no closed integral curves. Then the set Ξ' of X-integral worldlines lies in $\mathcal{W}$.*

Proof. First note that the integral curves of X indeed define worldlines (by forgetting the parametrization except for the orientation), since they cannot self-intersect and are not closed by hypothesis.

We define a topology on the set of worldlines lying in U, generated by the sets of the form {all worldlines lying within U'} where U' is an open subset of U. This topology—seen as a family of subsets of space-time—clearly forms a subset of $\mathcal{W}$. Now define Ξ'' to be the set containing precisely all the elements of Ξ' and their inverse (i.e., with orientation changed). First we show that Ξ'' is a closed set w.r.t. this topology. This implies $\Xi'' \in \mathcal{W}$. Then we show there is a 1-form ω on U such that for every X-integral curve γ, $\omega(\dot{\gamma}) > 0$ everywhere. This property singles out Ξ' from Ξ'' again, and it is a $\mathcal{W}$-measurable property; thus $\Xi' \in \mathcal{W}$.

To see that Ξ'' is closed, we establish its complement is open. If it were not, there would be a worldline ℓ that is not X-integral (nor its inverse) such that every neighborhood of ℓ in U contains a complete worldline of Ξ''. But this is only possible if ℓ contains an X-integral worldline ℓ' as a proper subset, which again is only possible if ℓ' has an endpoint p inside U, which again is only possible if $X_p = 0$, which is excluded by assumption.

To construct a 1-form as indicated above, choose a nondegenerate Riemann metric on U (for the existence of which see the lemma below), and use this metric to turn X into a 1-form $\omega_\mu = g_{\mu\nu} X^\nu$. Since $X \neq 0$ everywhere and g is nondegenerate, $\omega(X) > 0$, as desired. □

Lemma. *Every manifold M admits a (nondegenerate, smooth) Riemannian metric $g_{\mu\nu}$.*

Proof. Choose a countable family of coordinate charts covering M (such a family exists by definition of a manifold) and a partition of unity[5] on M subordinate to this family of charts. Use the charts to pull back the Euclidean metric from coordinate space $\mathbb{R}^d$ to the domain of the chart, and combine them by means of the partition of unity. □

As a consequence, the set Ξ of τ^+-integral worldlines in the wandering set is $\mathcal{W}$-measurable.

Since $x^{-1}(A) \subseteq H$ is a Borel subset of space-time, the set of worldlines intersecting $x^{-1}(A)$ is $\mathcal{W}$-measurable, thus also the intersection with Ξ is, which is the set of τ^+-integral worldlines intersecting $x^{-1}(A)$. This is precisely the image of A under $\widetilde{x}$, which we wanted to show to be $\mathcal{W}$-measurable.

3.2.5 Compatibility on the Overlap of Patches

We have defined measures $\mathbb{M}[U, x]$ on "patches" Ξ_H (the set of τ^+-integral worldlines intersecting H) covering Ξ (the set of all τ^+-integral worldlines in the wan-

[5] A *partition of unity* is a collection $\{f_i\}$ of smooth functions such that for every p, $f_i(p) \geq 0$, $\sum_i f_i(p) = 1$, and there are only finitely many is with $f_i(p) \neq 0$. It is called *subordinate* to the cover $\{U_\alpha\}$ if for each i there is an α such that the support of f_i is contained in U_α. The existence of a countable partition of unity subordinate to a given open cover is a known fact [10, p. 10].

dering set). Before we put these measures together to form a measure $\mathbb{M}$ on Ξ in the next section, we have to check that on the overlap of patches, the "local" measures agree. This means, given two standard coordinate charts $(U, x), (U', x')$, we consider a ($\mathcal{W}$-measurable) subset B of $\Xi_H \cap \Xi_{H'}$ and establish that $\mathbb{M}[U, x](B) = \mathbb{M}[U', x'](B)$.

Proposition. *Given two standard coordinate charts (U, x) and (U', x'), let $C \subseteq H$ (resp. $C' \subseteq H'$) be the set of points p such that the worldline through p also intersects H' (resp. H). Then C is open in H, i.e., $x(C)$ is open in $\mathbb{R}^3$.*

Proof. Choose a τ^+-compatible vector field X. Say $p \in C$ and $\phi^t(p) = p' \in H'$, then due to continuity of ϕ an entire neighborhood of p in H will be mapped by ϕ^t to U', and thanks to the rectifying coordinates, a suitable correction of t will move the image to H'. This shows that C is open. □

Proposition. *With respect to a τ^+-compatible vector field X, the unique time parameter $\widetilde{t}(q)$ when the integral curve starting in $q \in C$ will hit H' is a smooth function of q, $\widetilde{t} : C \to \mathbb{R}$.*

Proof. It is clear that $\widetilde{t}(q)$ exists and is unique for every $q \in C$. Since $\phi^{\widetilde{t}(q)}(q) \in U'$, $x'(\phi^{\widetilde{t}(q)}(q))$ is defined, and we observe its 0-component is identically zero. That is, $\widetilde{t}$ solves the equation

$$f(q, \widetilde{t}) = x^{0\prime}\Big(\phi^{\widetilde{t}}(q)\Big) = 0\,,$$

and by the implicit function theorem [10, p. 31], it is smooth. □

Proposition. *Some points $p \in C$ will have the property that the worldline through p meets C' at a* later *time (call this subset C_+), others at* earlier *times (call this subset C_-). The definition of C_+ and C_- does not depend on X but only on τ^+. C_+ and C_- are open in H, and $C \setminus (C_+ \cup C_-) = H \cap H'$.*

Proof. Since the worldline through p itself (oriented as it is) determines whether or not p belongs to C_+, X was not involved in the definition; nevertheless, C_+ can be characterized as the set where $\widetilde{t} > 0$ (which implies C_+ is open), and the function $\widetilde{t}$ is indeed defined only w.r.t. some vector field X. Obviously, $C \setminus (C_+ \cup C_-) = H \cap H'$. □

On $H \cap H'$ the constructed measures coincide trivially.

Proposition. *There is $\varepsilon > 0$ and a diffeomorphism $\Phi : C_+ \times (-\varepsilon, 1 + \varepsilon) \to$ (space-time) such that*

(i) $\Phi(p, 0) = p$ *for* $p \in C_+$

(ii) $\Phi(C_+, 1) \subseteq H'$

(iii) for fixed $p \in C_+$, $\Phi(p,t)$ is a correctly oriented parametrization of a τ-integral curve.

Proof. Let X again be a τ^+-compatible vector field, and choose $\varepsilon > 0$. Let $\widetilde{t}_1(p)$ be the unique (negative) time parameter when the worldline through p crosses the $x^0 = -1/2$ hypersurface; let $\widetilde{t}_2(p)$ be the unique time parameter ($> \widetilde{t}(p)$) when the worldline through p crosses the $x^{0\prime} = 1/2$ hypersurface; just as with $\widetilde{t}$, the functions $\widetilde{t}_1$ and $\widetilde{t}_2$ are smooth. Let $f : (0,\varepsilon) \times (0,\infty) \times (0,\infty) \to (0,\infty)$ a smooth function such that (1) $f(t,u,v)$ is increasing in t, (2) at $t \searrow 0$, $f(t,u,v)$ approaches 0, (3) at $t \nearrow \varepsilon$, $f(t,u,v)$ approaches u, (4) at $t \searrow 0$, all derivatives approach zero except $\partial f/\partial t \to v$. Set

$$\Phi(p,t) := \begin{cases} \phi^{t\widetilde{t}(p)}(p) & \text{if } 0 \le t \le 1\,, \\ \phi^{\widetilde{t}(p)+f(t-1,\widetilde{t}_1(p)-\widetilde{t}(p),\widetilde{t}(p))}(p) & \text{if } 1 < t\,, \\ \phi^{-f(-t,-\widetilde{t}_2(p),\widetilde{t}(p))}(p) & \text{if } t < 0\,. \end{cases}$$

Note that Φ is smooth. Next, Φ is injective, since if $p, p' \in C$ and $\phi^t(p) = \phi^{t'}(p')$ then $p = p'$ (because every worldline crosses H at most once) and $\phi^{t-t'}(p) = p$ which implies $t = t'$. Finally, we show that the tangent map $\mathrm{d}\Phi$ has full rank, which then implies Φ is a diffeomorphism:

Consider $0 \le t \le 1$. The first column of the 4×4 matrix $\mathrm{d}\Phi$ is $\partial\Phi/\partial t = \widetilde{t}(p) X_{\Phi(p,t)}$, and the remaining 3 columns, so to speak, are $\partial\Phi/\partial p = t X_{\Phi(p,t)} \otimes \mathrm{d}\widetilde{t} + (\partial\phi^s/\partial p)(s = t\widetilde{t}(p))$. According to statement (vi) of the existence theorem, ϕ^s (for fixed s) is a diffeomorphism. $X_{\Phi(p,t)}$ is linearly independent of the the columns of $(\partial\phi^s/\partial p)(s = t\widetilde{t}(p))$, so the full rank of $\mathrm{d}\Phi$ follows from the following simple fact of linear algebra: if vectors $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_3$ are linearly independent and if $\alpha_0 \ne 0$, then $\alpha_0\boldsymbol{e}_0, \boldsymbol{e}_1 + \alpha_1\boldsymbol{e}_0, \boldsymbol{e}_2 + \alpha_2\boldsymbol{e}_0, \boldsymbol{e}_3 + \alpha_3\boldsymbol{e}_0$ are linearly independent.

For $t < 0$ or $t > 1$, a similar reasoning applies. □

Remark. On the basis of this diffeomorphism, one can find a diffeomorphism $C \to C'$. As a consequence, a differentiable structure is defined on the set Ξ of τ^+-integral worldlines in the wandering set, which would turn Ξ into a manifold if the Hausdorff axiom did not fail. As an example that the Hausdorff property can fail, consider space-time to be $\mathbb{R}^4 \setminus \{0\}$ and τ^+ the positive x^0-direction; then the wandering set is all of space-time, and the positive and the negative part of the x^0-axis form two distinct worldlines having no disjoint neighborhoods.

C_+ can be treated as a subset of B_1^3. For every polyhedron $P \subseteq C_+$, we now show that the measure of $P' = \Phi(P,1)$ as defined on the basis of (U', x') equals the measure of P as defined on the basis of (U, x). This is based on Stokes's theorem:

Stokes's integral theorem. *[10, p. 144] Let c be a k-chain[6] in a (d-dimensional)*

[6] "Chain" means a formal linear combination, with real coefficients, of finitely many oriented singular simplices, i.e., images of standard simplices through a smooth mapping defined on some neighborhood of the standard simplex.

differentiable manifold M, and let β be a smooth $k-1$-form defined on a neighborhood of the image of c. Then

$$\int_{\partial c} \beta = \int_c \mathrm{d}\beta \,.$$

The cylinder $P \times [0,1]$ is, of course, a chain, and we take $\Phi(P \times [0,1])$ as the c in Stokes's formula. In our case, β is closed, so the right hand side vanishes. The boundary of c consists of three parts: the bottom $-P$, the lid P', and the mantle $\Phi(\partial P \times [0,1])$. Since the mantle is everywhere tangent to τ, the integral of β over the mantle vanishes. Thus we have

$$\int_{P'} \beta - \int_P \beta = 0 \,,$$

which establishes that the measures induced by (U, x) and (U', x') give the same value on the set of worldlines intersecting P.

Now the polyhedra in C_+ generate the Borel σ-algebra of C_+ (indeed, already the axiparallel cuboids do), and if the two measures agree on a $\cap$-stable generator, then they agree on the entire Borel σ-algebra of C_+, as the following theorem guarantees:

Measure uniqueness theorem. *[2, p. 33-4] Suppose that P_1 and P_2 are probability measures on $\sigma(G)$, the σ-algebra generated by G, where G is closed under the formation of finite intersections. If P_1 and P_2 agree on G, then they agree on $\sigma(G)$.*

3.2.6 Composing the Local Measures

Now define $\mathbb{M}(B)$ for a ($\mathcal{W}$-measurable) subset B of Ξ like this: partition B into countably many disjoint parts $B = \bigcup_{k=1}^{\infty} B_k$ such that each B_k lies within some Ξ_{H_k} associated with a standard coordinate chart (U_k, x_k), and set

$$\mathbb{M}(B) = \sum_{k=1}^{\infty} \mathbb{M}[U_k, x_k](B_k) \,.$$

First we have to show that such a partition exists. This follows from

Lemma. *There are countably many standard coordinate neighborhoods that cover the wandering set.*

Proof. By the definition of a manifold, there is a countable dense set in space-time and hence in the wandering set, say $\{p_k | k \in \mathbb{N}\}$. We will use standard coordinate neighborhoods of these p_k. To ensure that the entire wandering set is covered, one has to take care to choose sufficiently large standard neighborhoods.

To be able to control this, choose a Riemann metric on space-time (see p. 27); it serves to define what the ball of radius r around a point is. Every standard neighborhood of p_k contains some ball around p_k; the set of all radii r such that there exists a standard neighborhood of p_k containing the r-ball around p_k has the property that with any r it contains every positive $r' < r$. Let $\widetilde{R}_k$ be the sup of this set, and define $R_k = \widetilde{R}_k/2$ in case $\widetilde{R}_k$ is finite, and $R_k = 1$ in case $\widetilde{R}_k = \infty$. Then there exists a standard neighborhood of p_k containing the R_k-ball. This we use for our countable covering.

Now let p be any point in the wandering set, and (U, x) a standard coordinate neighborhood. U contains some r-ball around p; put $\varepsilon = \min(r/4, 1)$; pick k such that p_k lies in the ε-ball around p; as a consequence, the 3ε-ball around p_k lies in the 4ε-ball around p and thus in U. Therefore there exists a standard neighborhood of p_k containing a 3ε-ball around p_k, and thus R_k is at least ε, implying the chosen standard neighborhood around p_k contains p. □

Next we have to show that the value of $\mathbb{M}(B)$ does not depend on the choice of partition $B = \bigcup_k B_k = \bigcup_m B'_m$. We observe

$$\sum_{k=1}^{\infty} \mathbb{M}[U_k, x_k](B_k) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{M}[U_k, x_k](B_k \cap B'_m) =$$

$$= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{M}[U'_m, x'_m](B_k \cap B'_m) = \sum_{m=1}^{\infty} \mathbb{M}[U'_m, x'_m](B'_m)$$

because $\mathbb{M}[U, x]$ is a measure and therefore σ-additive, and because on $\Xi_H \cap \Xi_{H'}$, the measures $\mathbb{M}[U, x]$ and $\mathbb{M}[U', x']$ agree.

Finally, we have to check that $\mathbb{M}$ is a measure. It will be helpful to fix a partition of the entire wandering set $\Xi = \bigcup_k B_k$ such that $B_k \subseteq \Xi_H$ for some standard coordinate chart. For a countable family A_n of disjoint sets,

$$\mathbb{M}\Big(\bigcup_n A_n\Big) = \mathbb{M}\Big(\bigcup_{k,n} (A_n \cap B_k)\Big) = \sum_{k=1}^{\infty} \mathbb{M}[U_k, x_k]\Big(\bigcup_n (A_n \cap B_k)\Big) =$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathbb{M}[U_k, x_k](A_n \cap B_k) = \sum_{n=1}^{\infty} \mathbb{M}(A_n)\,,$$

and certainly, $\mathbb{M}(\emptyset) = 0$.

To sum up, starting from a closed 3-form β we have constructed a measure $\mathbb{M}$ on the set Ξ of τ^+-integral worldlines in the wandering set of τ. It may as well be understood as a measure on the set of worldlines by setting $\mathbb{M}(B) = \mathbb{M}(B \cap \Xi)$. In general, $\mathbb{M}$ is not a probability (i.e. normalized) measure, but by construction it is σ-finite, since every $\mathbb{M}[U, x]$ is a finite measure.

We will focus on the case that $\mathbb{M}$ is indeed a probability measure; any other case can be dealt with in the way indicated in section 3.2.1.

Then $\mathbb{M}$ defines a deterministic random worldline $\Gamma(\beta)$: take the wandering set as S_0, and the integral worldlines of τ^+ as ℓ_p; the standard coordinate neighborhoods have the property that the intersections with the worldlines ℓ_p are connected.

Proposition. *For every hypersurface H in the wandering set of τ, $\int_H \beta = \mathbb{E}N(\Gamma(\beta), H)$, whenever either side is finite.*

Proof. This follows from the proposition of p. 15, and the fact that the 3-form associated with $\Gamma(\beta)$ is β. □

3.3 The Global Existence Question

We recall that a mapping is called proper if the pre-image of every compact set is compact (see p. 15), and that for a smooth curve $\gamma : \mathbb{R} \to U \subseteq$ (space-time) this means it sooner or later leaves ultimately every compact subset of U, in both time directions. Note that this depends on U: a curve may well be proper in U which is not proper in space-time; this is because there may be a compact subset of space-time (but not contained in U) whose pre-image fails to be compact.

Proposition. *Every τ^+-integral worldline in the wandering set is proper as a mapping into the wandering set.*

Proof. Any compact subset K of the wandering set can be covered by finitely many standard coordinate charts as defined on p. 25, which in the course of this proof we call *radar screens*. Consider a parametrization γ of a τ^+-integral worldline, possibly inversely oriented. Let us see whether γ ultimately leaves K. If it doesn't intersect K at positive times, there is nothing to prove. If it does, it appears on some radar screen. Let I be the set of radar screens the curve will appear on at positive times; since there are but finitely many radar screens, I is finite, too. By construction of standard coordinates, the curve cannot travers the same radar screen twice. So it may either leave once and for all the screens listed in I, in which case it also left K, or it stays an infinite amount of time on some last radar screen, or several ones. Since γ must go on as long as a continuation along the τ-direction (which on the screen is mapped to the x^0-direction) is possible, it can only slow down close to the boundary of each radar screen on which it is still visible, and approach, as $t \to \infty$, a point on the boundary. But this point is outside K, so for sufficiently large t, the curve has left K. □

The case we would regard as regular behavior of worldlines is that they come from $t = -\infty$ and go to $t = +\infty$. Our general perspective treating space-time as a bare manifold lacks any notion of timelikeness and time-orientation on space-time, so in this context the only question is whether the worldlines start and end "on the boundary" of space-time. This is not always the case. A worldline in the

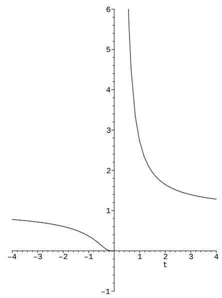

Figure 3: The graph of the exp$(1/t)$ function, which is used in the example 3-form. When approaching 0 from the left, the function and all its derivatives tend to zero.

wandering set may well approach the nonwandering set. In particular, a worldline may run into a zero of β.

Example. Here is an example showing that indeed infinitely many worldlines may run into the same zero of β: we take space-time to be $\mathbb{R}^4$, and we express β by its dual vector field j where duality is mediated by the 4-form $\varepsilon = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3$ (see p. 12). Instead of $x^0, \ldots, x^3$ we write t, x, y, z, and set $j^\kappa = \rho v^\kappa$ with

$$v = \Big(1,\ -\frac{3x}{t|t|},\ \frac{y}{t|t|},\ \frac{z}{t|t|}\Big)$$

$$\text{and}\ \ \rho = \frac{\exp\Big(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\Big)}{(2\pi)^{3/2}\sigma_x(t)\sigma_y(t)\sigma_z(t)}\ ,$$

where $\sigma_x(t) = e^{-3/|t|}$, $\sigma_y(t) = e^{1/|t|}$, and $\sigma_z(t) = e^{1/|t|}$. The so defined j is smooth and divergence-free, so β is smooth and closed. For a plot of the $e^{1/t}$ function, see fig. 3.

The integral curves are easy to compute: for $t > 0$, they solve the decoupled linear ODE $\dot{x} = -3x/t^2$, $\dot{y} = y/t^2$, $\dot{z} = z/t^2$, which implies $x(t) = \xi e^{3/t}$, $y(t) = \eta e^{-1/t}$, $z(t) = \zeta e^{-1/t}$. For $t < 0$, they solve $\dot{x} = 3x/t^2$, $\dot{y} = -y/t^2$, $\dot{z} = -z/t^2$, which implies $x(t) = \xi e^{-3/t}$, $y(t) = \eta e^{1/t}$, $z(t) = \zeta e^{1/t}$ (for a plot, see fig. 4). The family of integral curves is symmetric against reflection at the $\{t = 0\}$ hyperplane. All the integral curves with $\xi = 0$ run into the zero at the origin; together they fill the 3-dimensional $\{x = 0\}$ hyperplane. These are the only integral curves running into a zero.

Nevertheless, such irregular behavior seems not to be the rule but the exception.

Conjecture. *Given a closed $d-1$-form β on an oriented d-dimensional manifold, let τ^+ be the field of oriented kernels of β on $D = \{p|\beta_p \neq 0\}$, and let Ξ denote*

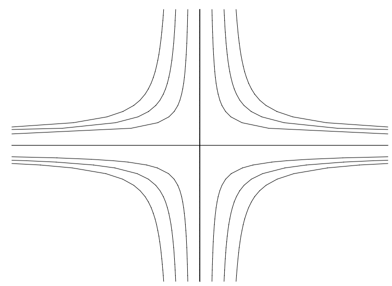

Figure 4: The integral curves of the example 3-form in the xt plane.

the set of correctly oriented integral curves (modulo parametrization) of τ^+ *in the wandering set.*

Then the curves in Ξ *are proper as mappings into space-time, except for a set of measure zero (w.r.t. the measure defined in the previous section).*

Some consequences of this conjecture: Almost-surely, a worldline in Ξ does not run into a zero of β, nor intersect every neighborhood of a zero; there cannot be any endpoints $\lim_{t\to\infty}\gamma(t)$ or accumulation points (limits for certain sequences $t_n \to \infty$). In particular, the worldline does not approach (in the sense of intersecting every neighborhood) any compact part of the nonwandering set. All this is understood, of course, as almost-sure statements. Another consequence: If space-time is compact, the wandering set is empty (since no worldline in a compact space-time can be proper).

The conjecture generalizes a theorem of Münch-Berndl [1, p. 38] stating that almost-surely, the worldlines of nonrelativistic Bohmian mechanics do not run into a zero of the 3-form (see also section 4.1).

As an overview, we may classify the different "quality classes" of τ-integral curves as follows: the really bad ones are the ones in the nonwandering set, including the single-point curves (zeroes of β), closed integral curves, and integral curves recurring infinitely many times to every neighborhood of some point they have traversed before. The really good ones, on the other end, are the proper curves (proper as mappings into space-time) in the wandering set, which behave as worldlines reasonably should. Inbetween, there are nonproper worldlines in the wandering set; but these, the conjecture says, are negligibly few.

4 Examples and Applications

4.1 Nonrelativistic Bohmian Mechanics

Bohmian mechanics, a nonrelativistic (more precisely, Galilei-invariant) theory of point particle motion, serves as an illustration of our result. Bohmian mechanics of a single particle states that the motion through Euclidean 3-space satisfies the first-order equation

$$\dot{Q} = \frac{\hbar}{m} \Im \frac{\nabla \psi(Q,t)}{\psi(Q,t)} \tag{7}$$

where ψ solves the Schrödinger equation, and the initial position at time $t = t_0$ is chosen at random with distribution $|\psi(t_0)|^2$ times Lebesgue measure. It has been shown [1] that (7) possesses global solutions for $|\psi(t_0)|^2$-almost every initial position, and that the distribution of $Q(t)$ has density $|\psi(t)|^2$.

These global solutions can be understood as a deterministic random worldline L in a Galilei space-time[7] in the following way: let L arise in the usual way from a closed 3-form β which again arises in the usual way (3) from a divergence-free current vector field j^μ, which again is defined as

$$j^0 = |\psi|^2 \,, \quad j^a = \frac{\hbar}{m} \Im(\psi^* \partial_a \psi) \text{ for } a = 1,2,3\,.$$

It is the wave function that defines the 3-form.

To see that L is the usual Bohmian trajectory, check the following: since $j^0 > 0$ whenever $j^\mu \neq 0$, the time coordinate x^0 is a Lyapunov (i.e., ever-increasing) function, which implies that $\{\psi \neq 0\} = \{j^\mu \neq 0\} = \{\beta \neq 0\}$ is the wandering set (proof: around p with $\psi(p) \neq 0$, choose rectifying coordinates on a neighborhood so small that j^0 is bounded away from 0, and then choose a sufficiently small time interval). Since the worldlines are tangent to j^μ, the velocity of the particle is $\dot{Q}^a = j^a/j^0$ for $a = 1,2,3$, thus (7) is satisfied. Due to the Lyapunov character of x^0, every $\{x^0 = \text{const.}\}$ hyperplane can intersect any τ^+-integral curve at most once, and this hyperplane (when cut into sufficiently small parts) may serve as the hypersurface in a standard coordinate system defining the probabilities. These probabilities correspond to the density $\beta_{123} = j^0 = |\psi|^2$. The trajectories which do not intersect this hyperplane form a null set, as shown in [1], hence the measure constructed in earlier sections is normalized and thus defines a deterministic random worldline which coincides with the Bohmian trajectory.

The properness conjecture of p. 33 implies for Bohmian mechanics that if the wave function is C^∞ then for $|\psi|^2$-almost-every initial position, the Bohmian trajectory either exists globally in time, or flees to spatial infinity in finite time. This statement is known to be true [1, p. 38] under an assumption on the wave function that may be called finiteness of the kinetic energy in the time average. Münch-Berndl indeed showed more than this, namely that the trajectory almost-surely

[7] Indeed, (7) is Galilei-invariant, provided one assumes the usual transformation behavior of wave functions under Galilei boosts.

does not flee to spatial infinity. Our approach in terms of manifolds, of course, is blind towards the distinction between spatial and temporal infinity.

4.2 Equivariance

Equivariance of Bohmian mechanics is the statement that if $Q(0)$ is chosen $|\psi(0)|^2$-distributed, then $Q(t)$ as determined by the equation of motion is $|\psi(t)|^2$-distributed, that is, the probability density remains the same functional of the wave function. This generalizes in the following sense: if a 3-form is given as some function f of a (possibly spinor-valued) wave function,

$$\beta_p = f(\psi(p), \mathrm{d}\psi(p))\,,$$

and if β defines a deterministic random worldline (i.e., if β is closed and the measure $\mathbb{M}$ arising from β is normalized), then everywhere in the wandering set the probability of crossing a "suitable" piece of hypersurface H is $|\int_H \beta|$, so the functional dependence of ψ is the same everywhere. "Suitable" means transverse to the foliation of the wandering set into worldlines, and sufficiently small such that each worldline intersects H at most once.

Hence, whenever a (normalized) closed 3-form is defined from a wave function—as it is in Bohmian mechanics and in the cases below—then the dynamics is equivariant in the wandering set.

This implies that on a hypersurface of constant x^0 coordinate, the density of the crossing probability is $|\beta_{123}|$, but keep in mind it does not mean that the probability of crossing this hypersurface is 1.

4.3 A Bohm-Type Mechanics for the Klein–Gordon Equation

We now present how a Bohm-type mechanics can be constructed for a point particle guided by a Klein–Gordon wave function.

In a system of units in which $\hbar = c = 1$, the Klein–Gordon equation for a single particle of mass m and charge e is:

$$\eta^{\mu\nu}(\mathrm{i}\partial_\mu - eA_\mu)(\mathrm{i}\partial_\nu - eA_\nu)\psi = m^2\psi\,, \tag{8}$$

where ψ is a complex scalar function on Minkowski space, A_μ is the external electromagnetic potential, and $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric of Minkowski space. For the vector field j^μ, usually called the Klein–Gordon current and defined as

$$\begin{aligned} j^\mu &= \frac{1}{m}\eta^{\mu\nu}\Re\Big(\psi^*(\mathrm{i}\partial_\nu - eA_\nu)\psi\Big) \\ &= -\frac{1}{m}\eta^{\mu\nu}\Im\Big(\psi^*(\partial_\nu + \mathrm{i}eA_\nu)\psi\Big)\,, \end{aligned} \tag{9}$$

with $\Re$ denoting the real part and $\Im$ the imaginary part, the Klein–Gordon equation (8) implies

$$\partial_\mu j^\mu = 0\,.$$

This is just what is needed for a deterministic random worldline, except for normalization. The integral worldlines satisfy $\mathrm{d}Q^a/\mathrm{d}x^0 = j^a/j^0$ wherever $j^0 \neq 0$.

This random worldline is only defined on the wandering set, and I have nothing general to say about what the wandering set looks like for the Klein–Gordon particle. It may occur that the wandering set is empty (e.g., when ψ is real and $A_\mu = 0$), in which case there is no associated deterministic random worldline. It also fails to exist in case the measure on Ξ is nonzero but not normalized; but this can be cured through changing ψ by a factor (in case the measure is finite), or in the way indicated in section 3.2.1. Note also that the wandering set in general depends on the one-form A_μ on all of space-time, so whether or not a point p lies in the wandering set might depend on A_μ in the future of p.

4.4 The Objections of Bohm and Hiley, and Holland

The analogy between Bohmian mechanics and the guidance law $\dot{Q}^a = j^a/j^0$ with the Klein–Gordon current (9) breaks down at the point that in Bohmian mechanics, $j^0 = |\psi|^2$ is the probability density whereas the Klein–Gordon j^0 is sometimes negative and thus cannot be a probability density. Holland [7, p. 500] points out that even restricting ψ to the positive energy subspace of Hilbert space does not cure this problem. Secondly, j^μ is not always timelike, so according to the guidance law, the particle sometimes moves faster than light, or even goes back in time. This feature, Holland writes [7, p. 500], "is clearly unacceptable". Regarding both of these problems, Holland comments [7, p. 501]: "We conclude that the Klein–Gordon equation does not have a consistent single-particle interpretation and the naive transcription of the trajectory interpretation of nonrelativistic Schrödinger quantum mechanics into this context does not work." On the basis of the same two problems, Bohm and Hiley [4, p. 233-4] arrive at the same judgment: that a particle interpretation of the Klein–Gordon equation "cannot be carried out consistently". They suggest that in this context, one should replace the particle ontology by field beables.

Such conclusions are too hasty, however, since our treatment makes clear that these problems are not serious enough to keep the Klein–Gordon current from defining a random worldline. Let us first clear up the mystery of the negative j^0: in the wandering set (the outside is irrelevant anyway), the probability of crossing an infinitesimal piece of the $\{x^0 = \text{const.}\}$ hyperplane is $|j^0|$ times the 3-volume of the piece, since j^0 times 3-volume gives the expected number of signed crossings through the piece—and thus, since the piece is infinitesimal, the sign of crossing times the crossing probability. The sign of j^0 reflects the orientation of the worldlines passing through the piece, as compared to the orientation of the x^0 axis, and does neither refer to negative probabilities nor to the opposite charge, as suggested by some authors (e.g. [8, p. 7]) who call j^μ the *charge current*, as opposed to the probability current.

Now concerning the second problem, the spacelike trajectories: there is no problem at all, as long as a worldline is defined, and it is. Be this worldline superluminal, or even go back and forth in time, it is nonetheless a worldline. I wish to stress that as a consequence of our mathematical treatment there is no room for causal paradoxes (in this 1-particle system).

On the contrary, worldlines oriented backwards in time have often been linked to antiparticles [3, section 9.1], and certainly a $\bigcap$-shaped worldline resembles the annihilation of a particle with its antiparticle, just as a $\bigcup$-shaped worldline resembles pair creation. These phenomena seem to downright suggest the existence of worldlines turning around in time. The 1-particle theory provides, of course, a rather limited frame for such a discussion. Nevertheless it is worth noting that the complex conjugate wave function ψ^* solves (8) again, but with the opposite charge $-e$, and induces the negative current $j^\mu[\psi^*] = -j^\mu[\psi]$, the negative 3-form and thus the same worldlines but with the opposite orientation; this means, particle and antiparticle change roles. That is to say, it is a symmetry of this Bohm-type 1-particle Klein–Gordon theory to stay invariant under exchange of particles and antiparticles.

4.5 The Bohm–Dirac Model

The Dirac equation reads

$$\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi = m\psi$$

and implies $\partial_\mu j^\mu = 0$ where $j^\mu = \overline{\psi}\gamma^\mu\psi$ and $\overline{\psi} = \psi^\dagger\gamma^0$. The wave function is $\mathbb{C}^4$-valued, and γ^μ are the 4×4 Dirac matrices. Again, j^μ induces a deterministic random worldline, known as the Bohm–Dirac model [4, p. 272].

Since $j^0 > 0$ whenever $j^\mu \neq 0$, the time coordinate of any Lorentz frame serves as a Lyapunov function, showing that $\{\psi \neq 0\}$ is the wandering set. j^μ, if nonzero, is always timelike or null, thus a worldline cannot flee to spatial infinity, and the properness conjecture of p. 33 implies that the solutions to the ODE $\dot{Q}^a = j^a/j^0$ exist globally in time for $\psi^\dagger\psi$-almost every initial value.

4.6 Many Particle Dynamics

Everything I said about random worldlines and differential forms applies, of course, not only to dimension 4, but to manifolds of any dimension. In this section, we replace space-time by some "configuration space-time" manifold $\mathcal{C}$ for N particles. The results of chapters 2 and 3 prove rigorously the equivariance of the hypersurface Bohm–Dirac model first derived in a less rigorous fashion in [5, p. 2733-5].

We assume space-time to be a Lorentzian manifold, i.e., we allow for a curved space-time, which makes our discussion slightly more general than that in [5] focussing on Minkowski space. It is one of the advantages of the differential form method developed here that it easily generalizes to a curved space-time.

It is reasonable furthermore to suppose of the space-time metric that there exist no closed timelike-or-null curves, and this hypothesis will exert an pleasant influence on how big the wandering set is.

In the rest of this section, we will construct a deterministic random worldline in $\mathcal{C}$ via a closed $3N$-form, based on a many-particle Dirac wave function. This random worldline is equivalent to that discussed by [5]; what is novel about my discussion is that it employs closed forms, features full mathematical rigor, and is valid in curved space-times.

4.6.1 Definition of the Form

The configurations we are interested in are the simultaneous configurations w.r.t. the time-foliation, i.e., N-tuples of space-time points that lie on the same leaf. The foliation can be written (we assume) as the level sets of a function F on space-time, where $\mathrm{d}F$ must be a nowhere-vanishing timelike 1-form to make the leaves smooth and spacelike. Let n denote the (unique) future-pointing timelike 1-form of unit length whose (3-dimensional) kernel is tangent to the leaves of the foliation; that is, $n_\mu = (\partial^\nu F \partial_\nu F)^{-1/2}\, \partial_\mu F$. The set $\mathcal{C}$ of simultaneous configurations can be written as

$$\mathcal{C} = \left\{ (p_1, \ldots, p_N) \in (\text{space-time})^N \middle| F(p_1) = \ldots = F(p_N) \right\},$$

and, by the implicit function theorem [10, p. 31], is an imbedded submanifold of dimension $3N + 1$.

Let TM denote the tangent bundle of the manifold M. If E and E' are two vector bundles over M, then $E \otimes E'$ denotes the vector bundle with fibers $(E \otimes E')_p = E_p \otimes E'_p$. Accordingly, $T^{\otimes N}(\text{space-time})^N$ is a bundle of $(4N)^N$-dimensional vector spaces over a $4N$-dimensional manifold. Regarding $T_p(\text{space-time})^N$ as $\bigoplus_i T_{p_i}(\text{space-time})$, we may form $E_p := \bigotimes_i T_{p_i}(\text{space-time})$, which is a 4^N-dimensional subspace of $T_p^{\otimes N}(\text{space-time})^N$, so $E = \bigcup_p E_p$ is a subbundle of $T^{\otimes N}(\text{space-time})^N$. Consider a section J through this bundle E; at $p \in (\text{space-time})^N$ it takes values in $\bigotimes_i T_{p_i}(\text{space-time})$. In index notation, it may be written $J^{\mu_1 \cdots \mu_N}$ where μ_i refers to $T_{p_i}(\text{space-time})$.

Such a section is given by

$$J^{\mu_1 \cdots \mu_N} = \overline{\psi}\, \gamma_{p_1}^{\mu_1} \otimes \cdots \otimes \gamma_{p_N}^{\mu_N}\, \psi\,,$$

where ψ is a smooth section through the bundle $\bigotimes_i D_{p_i}$ over $(\text{space-time})^N$, D_{p_i} being the space of Dirac spinors at space-time point p_i, and $\gamma_{p_i} : \overline{D}_{p_i} \otimes D_{p_i} \to T_{p_i}$ being the gamma tensor at space-time point p_i, satisfying $\gamma_{p_i}^\mu \gamma_{p_i}^\nu + \gamma_{p_i}^\nu \gamma_{p_i}^\mu = 2g^{\mu\nu}\mathbf{1}$. If ψ solves the multi-time Dirac equation,

$$\mathrm{i}\gamma_{p_i}^{\mu_i} \nabla_{i,\mu_i} \psi = m\psi\,,$$

(where ∇ is the covariant derivative including, when acting on spinors, the vector potential) then the section J satisfies

$$\nabla_{i,\mu_i} J^{\mu_1\dots\mu_N} = 0\,, \tag{10}$$

since by definition, the covariant derivative of γ vanishes. We define a $3N$-form β on (space-time)^N by

$$\beta_{\Delta_1\dots\Delta_{3N}} = (-1)^{N(N+1)/2}\, \varepsilon_{\Delta_1\dots\Delta_{4N}}\, J^{\Delta_{3N+1}\dots\Delta_{4N}}\,,$$

where the Δ indices run through the $4N$ dimensions of (space-time)^N, the value space $\bigotimes_i T_{p_i}$ of J is seen as a subspace of $(T_p(\text{space-time})^N)^{\otimes N}$, and $\varepsilon_{\Delta_1\dots\Delta_{4N}}$ is the $4N$-form on (space-time)^N arising from the metric-induced 4-form $\varepsilon_{\mu_1\dots\mu_4}$ in the sense $\varepsilon^{(4N)} = \varepsilon(p_1)\wedge\ldots\wedge\varepsilon(p_N)$.

4.6.2 The Associated Random Path

Proposition. *$\beta|\mathcal{C}$ is closed.*

Proof. Choose a coordinate system on space-time that has the F function as its x^0-coordinate; in fact, what we need is only a collection of N local coordinate systems having F as their x^0 functions, being defined on some open neighborhood of the space-time points $p_1\dots p_N$, with $p = (p_1,\dots,p_N)\in\mathcal{C}$. Note that the $4N$ functions $x^{1,0}\dots x^{N,3}$ form a coordinate system on a neighborhood of p in (space-time)^N; the corresponding $4N$ canonical vector fields will be denoted $\partial_{10}\dots\partial_{N3}$. On $\mathcal{C}$ the x^{i0} functions coincide, so call them x^0; it follows that $\mathrm{d}x^{i0}|\mathcal{C} = \mathrm{d}x^0$. The x^0 function together with the $3N$ functions $x^{11}\dots x^{N3}$ form a coordinate system of $\mathcal{C}$ (in a neighborhood of p); the corresponding $3N+1$ canonical vector fields are $\partial_0 = \sum_{i=1}^N \partial_{i0}|\mathcal{C}$ and $\partial_{11}|\mathcal{C},\dots,\partial_{N3}|\mathcal{C}$.

In order to establish that the restriction of β to $\mathcal{C}$ is closed, we only have to calculate $\mathrm{d}\beta(\partial_0\wedge\partial_{11}\wedge\ldots\wedge\partial_{N3}) = 0$. Since the covariant derivative of ε is zero, we have

$$\mathrm{d}\beta = \nabla_{[\Delta'}\,\beta_{\Delta_1\dots\Delta_{3N}]} = (-1)^{N+N(N+1)/2}\varepsilon_{[\Delta_1\dots\Delta_{3N}|\Delta_{3N+1}\dots\Delta_{4N}}\nabla_{|\Delta']}\,J^{\Delta_{3N+1}\dots\Delta_{4N}}\,,$$

and thus $\mathrm{d}\beta(\partial_0\wedge\partial_{11}\wedge\ldots\wedge\partial_{N3}) = \sum\limits_{i=1}^N \mathrm{d}\beta(\partial_{i0}\wedge\partial_{11}\wedge\ldots\wedge\partial_{N3}) =$

$$= (-1)^{N+N(N+1)/2}\sum_{i=1}^N \varepsilon_{[11\dots N3|\Delta_{3N+1}\dots\Delta_{4N}}\nabla_{|i0]}\,J^{\Delta_{3N+1}\dots\Delta_{4N}} =$$

[because $J^{\Delta_{3N+1}\dots\Delta_{4N}}$ is nonzero only if $\Delta_{3N+1} = (1,\mu_1),\dots,\Delta_{4N} = (N,\mu_N)$]

$$= \frac{(-1)^{N+N(N+1)/2}}{3N+1}\,\varepsilon_{11\dots N3,10\dots N0}\sum_{i=1}^N \nabla_{i0}\,J^{10\dots N0} +$$

$$+\frac{(-1)^{N+N(N+1)/2+1}}{3N+1}\sum_{i=1}^{N}\sum_{f=1}^{3N}\varepsilon_{11\ldots\overset{f}{i0}\ldots N3,10\ldots\overset{i}{f}\ldots N0}\nabla_f\, J^{10\ldots\overset{i}{f}\ldots N0}=$$

[the last term is zero unless $f=(i,a)$ with $a\in\{1,2,3\}$]

$$=\frac{(-1)^N}{3N+1}\sum_{i=1}^{N}\nabla_{i0}\, J^{10\ldots N0}+$$

$$+\frac{(-1)^{N+N(N+1)/2+1}}{3N+1}\sum_{i=1}^{N}\sum_{a=1}^{3}\varepsilon_{11\ldots\overset{ia}{i0}\ldots N3,10\ldots\overset{i}{ia}\ldots N0}\nabla_{ia}\, J^{10\ldots\overset{i}{ia}\ldots N0}=$$

$$=\frac{(-1)^N}{3N+1}\sum_{i=1}^{N}\nabla_{i0}\, J^{10\ldots N0}+$$

$$+\frac{(-1)^{N+N(N+1)/2+1}}{3N+1}\sum_{i=1}^{N}\sum_{a=1}^{3}(-1)^{N(N+1)/2-1}\nabla_{ia}\, J^{10\ldots\overset{i}{ia}\ldots N0}=$$

$$=\frac{(-1)^N}{3N+1}\sum_{i=1}^{N}\nabla_{i\mu}\, J^{10\ldots\overset{i}{i\mu}\ldots N0}=0$$

□

I wish to stress that while this proof may appear sophisticated, it is not. It is essentially a straightforward calculation, not much more than bookkeeping about which terms vanish and how many signs one collects when permuting the indices.

Now that we have checked that the key requirement on the form, closedness, is satisfied, the machinery we have developed does the work and defines a foliation of the wandering set and a measure on the curves. In case the measure is normalized, we have arrived at a deterministic random "worldline" in $\mathcal{C}$.

How big is the wandering set? Provided that space-time has no closed timelike-or-null curves, the wandering set is $\{p\in\mathcal{C}|\psi(p)\neq 0\}=\{p\in\mathcal{C}|\beta_p\neq 0\}$, since all the 4-velocities are timelike-or-null, and the parameter F of the time-foliation may serve as a Lyapunov function. This is the largest set possible for a wandering set.

The properness conjecture of p. 33 implies the almost-certain global existence of the solution curve in $\mathcal{C}$ tangent to $\ker\beta$. It also implies that the norm of the measure equals the L^2 norm of the wave function, which is usually set to 1.

4.6.3 Equivalent Formulations

We can be more explicit about the probability densities and the velocities. In terms of the coordinates used in the proof above, the probability density of crossing the hypersurface $x^0=\text{const.}=F(p_i)$ at configuration p is

$$\beta(\partial_{11}\wedge\ldots\wedge\partial_{N3})=(-1)^{N(N+1)/2}\,\varepsilon_{11\ldots N3,10\ldots N0}\,J^{10\ldots N0}\,.$$

The number $\varepsilon_{11\dots N3,10\dots N0}$ can be computed as $(-1)^{N(N+1)/2}\prod_{i=1}^{N}\sqrt{-g(p_i)}$, g being the determinant of the matrix representing the metric in the given coordinates. If, moreover, $\partial_{i0}\dots\partial_{i3}$ form an orthonormal basis of T_{p_i} for each i, then the probability density simply equals $J^{10\dots N0} = \overline{\psi}\,\gamma^0\otimes\dots\otimes\gamma^0\,\psi = \psi^\dagger\psi = \sum_\sigma|\psi_\sigma|^2$ where the index σ refers to the basis in $D_{p_1}\otimes\dots\otimes D_{p_N}$ formed by tensor products of the basis vectors in D_{p_i} corresponding to $\partial_{i0}\dots\partial_{i3}$.

To compute explicit coordinate expressions for the velocities, note that the kernel of β (in $\mathcal{C}$) is the ray through

$$J^{0\dots0}\partial_0 + \sum_{i=1}^{N}\sum_{a=1}^{3} J^{0\dots\overset{i}{a}\dots0}\partial_{ia}\,.$$

This means the velocity of particle number i is

$$\dot{Q}_i^a = \frac{J^{0\dots\overset{i}{a}\dots0}}{J^{0\dots0}}\,.$$

These are precisely the probabilities and velocities of the hypersurface Bohm–Dirac model, which in [5] are expressed in a slightly different way, as we explain in the following.

In [5], the probabilities and velocities are expressed in terms of N smooth mappings $j_i:(\text{space-time})^N\to T(\text{space-time})$ such that p_i is the base point $\pi(j_i(p))$ of $j_i(p)$, which are defined from J by

$$j_i^{\mu_i} = J^{\mu_1\dots\mu_N}n_{\mu_1}(p_1)\cdots\widehat{n_{\mu_i}(p_i)}\cdots n_{\mu_N}(p_N)$$

and which satisfy (a) $\nabla_{i\mu}j_i^\mu = 0$ and (b) $j_i^\mu n_\mu(p_i)$ is independent of i. As the authors argue, (a) and (b) are sufficient for defining an equivariant hypersurface dynamics on Minkowski space. Indeed on $\mathcal{C}$, j_i can be recovered from β and vice versa (note $\mathrm{d}x^{i0} = n(p_i)$):

$$j_i^\mu n_\mu(p_i) = (-1)^{N(N+1)/2}\,\frac{\beta(\partial_{11}\wedge\dots\wedge\partial_{N3})}{\varepsilon(\partial_{11}\wedge\dots\wedge\partial_{N3}\wedge\partial_{10}\wedge\dots\wedge\partial_{N0})}\,,$$

$$j_i^a = (-1)^{i+a+1+N(N+1)/2}\frac{\beta(\partial_0\wedge\partial_{11}\wedge\dots\widehat{\partial_{ia}}\dots\wedge\partial_{N3})}{\varepsilon(\partial_{11}\wedge\dots\wedge\partial_{N3}\wedge\partial_{10}\wedge\dots\wedge\partial_{N0})}\,,$$

$$\beta_{\Delta_1\dots\Delta_{3N}}|\mathcal{C} = (-1)^{N(N+1)/2}\varepsilon_{\Delta_1\dots\Delta_{4N}}\Big(j_i^\mu n_\mu(p_i)\,\partial_{10}^{\Delta_{3N+1}}\wedge\dots\wedge\partial_{N0}^{\Delta_{4N}} + $$
$$+\sum_{i=1}^{N}\sum_{a=1}^{3} j_i^a\,\partial_{10}^{\Delta_{3N+1}}\wedge\dots\partial_{ia}^{\Delta_{3N+i}}\dots\wedge\partial_{N0}^{\Delta_{4N}}\Big)\,.$$

And indeed, (a) and (b) are sufficient for $\beta|\mathcal{C}$ to be closed, also in curved space-time: (we use a special space-time coordinate system making the connection coefficients vanish at $p_1,\dots,p_N$)

$$\mathrm{d}\beta(\partial_0\wedge\partial_{11}\wedge\dots\wedge\partial_{N3}) = \frac{1}{3N+1}\,\nabla_0\beta_{11\dots N3} +$$

$$+\frac{1}{3N+1}\sum_{i=1}^{N}\sum_{a=1}^{3}(-1)^{i+a-1}\nabla_{ia}\beta_{0,11\ldots\widehat{ia}\ldots N3} =$$

$$= \frac{(-1)^{N(N+1)/2}}{3N+1}\varepsilon_{11\ldots N0}\Big(\nabla_0(\underbrace{j_i^\mu n_\mu(p_i)}_{=j_i^0}) + \sum_i\sum_a \nabla_{ia}j_i^a\Big) =$$

$$= \ldots \Big(\sum_i \nabla_{i0}j_i^0 + \sum_i\sum_a \nabla_{ia}j_i^a\Big) = 0\,.$$

4.7 Newtonian Mechanics

Newtonian mechanics is the theory about the motion of N point particles in 3-dimensional Euclidean space stating the equation of motion

$$\ddot{x}_{i,a} = \sum_{\substack{j=1\\ j\neq i}}^{N}\Big(Gm_j - \frac{q_iq_j}{4\pi\varepsilon_0 m_i}\Big)\frac{x_{j,a}-x_{i,a}}{|\boldsymbol{x}_j-\boldsymbol{x}_i|^3} \tag{11}$$

in a Cartesian (orthonormal) coordinate system for $i = 1,\ldots,N$ and $a = 1,2,3$. G, m_j, q_j, and ε_0 are constants. The existence and uniqueness theorem (p. 20) for ODEs implies there exists a unique local solution for every set of $6N$ initial conditions $x_{ia}(0) = \widetilde{x}_{ia}, \dot{x}_{ia} = \widetilde{v}_{ia}$ provided $\widetilde{\boldsymbol{x}}_i \neq \widetilde{\boldsymbol{x}}_j$ for $i \neq j$. Thus, we define the *phase space* as

$$\Phi = \Big\{(\boldsymbol{x}_1,\boldsymbol{v}_1,\ldots,\boldsymbol{x}_N,\boldsymbol{v}_N)\in\mathbb{R}^{6N}\Big|\boldsymbol{x}_i\neq\boldsymbol{x}_j \text{ for } i\neq j\Big\}$$

and consider $\Phi\times$ time axis as the manifold on which we will define a (non-normalized) deterministic random "worldline" using the current

$$j^0 = 1\,,\quad (j^{i1},\ldots,j^{i6}) = (v^{i1},v^{i2},v^{i3},a^{i1},a^{i2},a^{i3})\,,$$

where the six components per particle are numbered in the order $(x^{i1}, x^{i2}, x^{i3}, v^{i1}, v^{i2}, v^{i3})$, the 0-component refers to the time axis, and a^{ia} stands for the right hand side in (11). The current is divergence-free in the sense

$$\partial_0\, j^0 + \sum_{i=1}^{N}\sum_{A=1}^{6}\partial_{iA}\, j^{iA} = 0\,,$$

because it vanishes termwise. With the help of the $6N+1$-dimensional Levi-Cività symbol, we may define a closed $6N$-form

$$\beta_{\Delta_1\ldots\Delta_{6N}} = \varepsilon_{\Delta_0,\Delta_1\ldots\Delta_{6N}}\, j^{\Delta_0}\,.$$

The associated congruence are the maximal integral curves of j, or solutions to (11), and the associated measure can be understood as Lebesgue's measure on Φ for any fixed time t.

Since time is a Lyapunov function ($j^0 > 0$), and β vanishes nowhere, $\Phi \times$ time axis is the wandering set, and the properness conjecture holds (as proved on p. 32). This does *not* imply (almost-certain) global existence of the solution curves, however, which would only follow from being proper as a mapping into $\mathbb{R}^{6N} \times$ time axis, rather than into $\Phi \times$ time axis.

In terms of β, Liouville's theorem states that

$$\int_{P \times \{t\}} \beta = \int_{P' \times \{t'\}} \beta$$

provided P and P' are two polyhedra in Φ such that $P \times \{t\}$ and $R' \times \{t'\}$ intersect precisely the same trajectories. In particular, the assumption implies that none of the trajectories starting in P at time t (or in P' at t') run into a singularity in $\mathbb{R}^{6N} \setminus \Phi$ within the time span between t and t'. Liouville's theorem follows from Stokes's theorem by much the same reasoning as applied in section 3.2.5.

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