Quantum Aspects of Black Holes

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Abstract

In this thesis we study two quantum aspects of black holes, their entropy and the Hawking effect. First, we present a model for the statistical interpretation of black hole entropy and show that this entropy emerges as a result of missing information about the exact state of the matter from which the black hole was formed. We demonstrate that this idea can be applied to black holes made from both ultra-relativistic and nonrelativistic particles.

In the second part we focus our attention on several features of black hole evaporation. We discuss the dependence of the Hawking radiation on the vacuum definition of different observers. It becomes evident that in certain cases the choice of observer has an influence on the particle spectrum. In particular, we study the meaning of the Kruskal vacuum on the horizon. After that we determine the Hawking flux for nonstationary black holes. We find approximate coordinates which are regular on the time dependent horizon and calculate the particle density measured by an observer at infinity.

Finally, we derive the response of a particle detector in curved background. In our approach we use the Unruh detector to quantify the spectrum of radiation seen by general observers in Minkowski, Schwarzschild and Vaidya space-times. We find that an arbitrarily accelerated detector in flat space-time registers a particle flux with a temperature proportional to a time dependent acceleration parameter. A detector moving in Schwarzschild space-time will register a predominantly thermal spectrum with the exact temperature depending on the observer's trajectory. If the detector is located at constant distance from the black hole it measures a shifted temperature which diverges on the horizon. On the other hand, a detector in free fall towards the black hole does not register a thermal particle flux when it crosses the horizon. In this framework corrections to the temperature measured by a detector moving in Vaidya space-time are obtained as well. We argue that our result also clarifies the role of horizons in black hole radiation.

Contents

Notation

We use the conventions of [31, 33].
The signature of the space-time metric is $\{+, -, -, -\}$,
Greek indices μ, ν, ... range from 0 to 3,
whereas Latin indices i, j, ... range from 0 to 1 and denote time and space components in two dimensions,
and repeated indices are summed.
If not mentioned otherwise we use natural units, $\hbar = c = k_B = G = 1$.

Introduction

The existence and simple geometrical properties of black holes are predicted by the theory of general relativity. Astrophysical observations have confirmed their existence with almost certainty. It is believed that supermassive black holes exist in the centres of most galaxies, including our own [14, 45]. Astrophysical black holes can be formed during gravitational collapse. If the mass of a collapsing star is large enough no inner structure survives and the star becomes a black hole. The essential feature of black holes is the existence of the so-called horizon that defines a region from which no signals, not even light, can escape. According to the no-hair theorem, stationary black holes – the asymptotic final state of the collapse – are uniquely described by only three parameters: mass, electric charge and angular momentum. This suggests an analogy to gases which can be described macroscopically by few parameters, such as temperature, pressure, volume and entropy.

In 1972 Bekenstein showed that black holes possess entropy proportional to the horizon area and deduced that they should therefore emit radiation [4, 5]. Soon afterwards Hawking confirmed this conjecture. In his famous work of 1974 he calculated the particle flux from black holes in the framework of quantum field theory in curved backgrounds [20, 21]. In effect a black hole is not completely black but emits radiation with a low but nonzero temperature. An intuitive picture of black hole radiation involves virtual particle-antiparticle creation in the vicinity of the horizon due to quantum fluctuations. It may happen that two particles with opposite momenta are created, one particle inside the horizon and the other particle on the outside. The first virtual particle always falls into the black hole. If the momentum of the particle outside is directed away from the black hole, it has a nonzero probability of moving away from the horizon and becoming a real radiated particle. The mass of the black hole will decrease in this process since the energy of the particle falling into the black hole is formally negative. The result of careful calculations is that the black hole emits a flux of thermally distributed particles with a temperature inversely proportional to its mass. Actually, the spectrum of the emitted particles contains an additional grey factor since particles with low energies are backscattered by a potential barrier. Although the Hawking effect is negligible for astrophysical black holes, it becomes significant for very small, primordial black holes which might have been formed in the very early stage of our universe's evolution when it was still extremely dense and hot.

The derivation of Hawking radiation confirmed the existence of black hole entropy and led to the formulation of black hole thermodynamics [2, 6]. Nevertheless, even today there remain open questions such as the information paradox, the microscopical origin of entropy and the final state of evaporation [15]. A complete understanding of those problems is only possible within a consistent quantum theory of gravity. In recent years promising progress in this direction has been made within loop quantum gravity and string theory, see for example [38, 50, 53] and references therein. However, at least for very large black holes, quantum effects can be studied within semi-classical theory as well.

In the following we investigate two aspects of quantum black holes, the statistical origin of entropy and the properties of the Hawking radiation for general observers and for nonstationary black holes. To motivate our considerations we briefly discuss previous work in these areas.

In usual thermodynamics, the entropy definition based on information theory assumes a direct relation between the lack of information about a physical system and its entropy. By analogy, one tries to find a similar understanding of why $S_{BH} = \frac{1}{4}A$ represents the entropy of a black hole by identifying its quantum dynamical degrees of freedom [5, 66]. Zurek and Thorne suggested that the entropy of a black hole can be interpreted as "the logarithm of the number of quantum-mechanical distinct ways that the hole could have been made" [66]. Another interesting approach is to attempt quantization of the black hole mass [7, 32]. However, neither approach gives a complete solution to the problem [15].

The computation of Hawking radiation appears to be reasonably robust. Schützhold showed in [46] that the particle flux at late times is insensitive to the presence of particles in the initial state. One might also anticipate that the spectrum depends on physics beyond the Planck scale, since outgoing modes which contribute to the particle spectrum originate from modes with extremely large wave numbers [10, 55]. In a recent article Unruh *et al.* presented examples where changes in the dispersion relation in the Planckian regime are visible in the particle spectrum [56]. On the other hand, properties of Hawking radiation will depend on the motion of the observer and the choice of vacuum state.

The Hawking flux for an eternal black hole is usually derived using the maximally extended Kruskal manifold. Then the initial vacuum state is specified by imposing boundary conditions on the past event horizon, where the Kruskal coordinate U is a Killing vector, see also [9, 12, 16, 17, 47]. Some work has been done concerning the dependence of the particle spectrum on the choice of other possible coordinates, in essence other possible vacua. Shankaranarayanan *et al.* studied the derivation of the Hawking effect in Lemaitre coordinates which describe a freely falling observer and cover only half of the Kruskal manifold. Their calculations were performed using the Euclidean formulation of field theory. They find that in this case the change of coordinates does not affect the particle spectrum [48].

The Hawking effect is certainly sensitive to changes in the black hole mass. When a black hole emits radiation it loses energy and therefore its mass decreases. In this process the temperature increases and the black hole will emit more radiation. A model for black holes with linear mass decrease has been studied in [23, 60]. However, the real situation is more complicated since the mass decrease of a black hole due to evaporation is inversely proportional to the square of its mass. A number of researchers studied the particle flux for general black hole mass using the method of analytic continuation [27, 63, 64]. However, the results are limited since they are only valid in the vicinity of the horizon [49].

From the physical perspective, perhaps the most appealing derivation of the Hawking effect is to determine the particle flux registered by a detector moving in the space-time. Often the "Unruh thermometer", introduced in [54], is used as a model for this particle detector. The detector is coupled to a quantum field with the interaction being turned on and off in some proper detector time. Given the motion of the detector in the space-time, the transition probability to its various energy eigenstates can be calculated using standard time dependent perturbation theory. The transition probability for a uniformly accelerated detector in flat space-time which registers a thermal flux of particles in the Minkowski vacuum has been studied extensively [17, 54]. In a recent article [28] Lin *et al.* showed that including the backreaction of the detector on the field does not change the particle spectrum significantly and therefore can be neglected. It has also been argued that as long as the detector is turned on and off smoothly the spectrum changes only by transients [51]. As noted by Schlicht [44], however, it is still an open question if the existence of a horizon is crucial for the emergence of a particle flux for an arbitrary observer.

In this thesis we present a new statistical explanation for black hole entropy and study some properties of the Hawking radiation in detail. Using the Unruh detector the particle flux measured by different observers in Minkowski and Schwarzschild space-time is determined. In this framework we also derive the Hawking effect for an observer in a time dependent black hole space-time.

The outline of our thesis is as follows. In chapter 1 we discuss stationary black hole solutions in general relativity and review the development of black hole thermodynamics. After that we calculate the particle density emitted by eternal black holes as well as black holes formed by gravitational collapse.

We present a new statistical explanation for the origin of black hole entropy in chapter 2 and show that our model can be used for black holes made from relativistic as well as nonrelativistic particles.

In chapter 3 some properties of the Hawking radiation are examined in detail. To make the problem mathematically tractable we work in the $1 + 1$ dimensional theory. Of special interest is the sensitivity of the observed particle spectrum on the choice of vacuum, and we discuss the influence of the presence of particles in the initial state. Lastly, we compute the particle flux measured by different observers and clarify the meaning of the Kruskal coordinates.

We calculate the Hawking flux for a black hole with time dependent mass in chapter 4. As a model for such nonstationary black holes we use the Vaidya solution. We find coordinates which are regular on the time dependent horizon and examine their relation to an observer very far away from the black hole. After that we determine the Hawking spectrum for particular examples.

The Unruh detector is introduced in chapter 5 to quantify the spectrum of radiation seen by general observers. First we study detectors moving in Minkowski space-time. After that, the particle spectrum registered at finite distance from the black hole is derived. We consider both static and freely falling detectors and show that in both cases the Hawking temperature is multiplied by a factor depending on the detector's radial distance from the black hole. Finally, we compute the particle flux registered by a detector in Vaidya space-time, and compare the results to chapter 4. This sheds light on the role of horizons for black hole radiation.

In chapter 6 we review the general results and discuss still open problems.

Chapter 1

Black holes and Hawking radiation

Classical general relativity describes black holes as massive objects with such a strong gravitational field that even light cannot escape their surface. On the other hand, quantum theory predicts that black holes do emit particles. The particle flux has a thermal distribution, and a temperature inversely proportional to the mass can be assigned to the black hole. Accordingly, black holes have nonzero entropy and, analogous to usual thermodynamical systems, laws of black hole thermodynamics can be formulated. On the classical level this is a purely formal analogy, whereas a physical interpretation is possible within quantum theory.

In the following we will first give a brief introduction to black holes in general relativity. Then we present the laws of black hole thermodynamics and discuss the analogy to classical thermodynamics. Finally we review the derivation of Hawking radiation for eternal black holes as well as black holes formed by gravitational collapse.

1.1 Black holes in general relativity

1.1.1 Schwarzschild metric

A spherically symmetric nonrotating, uncharged black hole is described by the well-known Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit two-sphere. The Schwarzschild radius $r_s = 2M$ is also called the black hole horizon since it causally separates the exterior region from the interior. The Schwarzschild space-time is the unique spherically symmetric solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. The components of the metric tensor do not depend on time t, and the generator of this time symmetry transformation is the Killing vector ∂_t. At large distances, $r \to \infty$, the metric reduces to the usual Minkowski metric. Hence, r is the radial space- and t the time-coordinate for an observer located at infinity.

The metric (1.1) has two singular points. The singularity at $r = r_s$ represents a breakdown of these particular coordinates, which means it is a coordinate singularity similar to that at the origin $r = 0$ of polar coordinates in flat space. In the next section we will show that there exist coordinates which are regular on the horizon. The other singularity occurs at the origin $r = 0$ and is a real curvature singularity which can be seen from the divergence of the invariant $I = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 12\frac{r_s^2}{r^6}$. Inside the horizon, for $r < r_s$, the coordinate t is space-like and r is time-like. Therefore the coordinates (t, r) may be used with the normal interpretation of time and space only in the exteriour region $r > r_s$.

1.1.2 Kruskal coordinates

Since the Schwarzschild metric (1.1) only describes the space-time outside the black hole horizon, it is useful to find coordinates which do not show a singularity at that point and can be extended beyond. The most convenient way to study the behavior near $r = r_s$ is to choose coordinates along ingoing and outgoing radial null geodesics. For this purpose we introduce the tortoise coordinate

$$r^\star = r - r_s + r_s \ln\left(\frac{r}{r_s} - 1\right), \quad dr^\star = \frac{dr}{(1 - \frac{r_s}{r})}\,, \tag{1.2}$$

which results in the alternative from of the metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right)\left(dt^2 - dr^{\star 2}\right) - r^2(r^\star)d\Omega^2. \tag{1.3}$$

In these coordinates the horizon is mapped to infinity, that means $r = r_s$ corresponds to $r^\star = -\infty$. This is why $r^\star$ is called tortoise coordinate: An object approaching the horizon has to cross an infinite distance in $r^\star$. If we take a set of photons at a fixed t and assign to each of them a number v which remains constant during the motion of the photon, this v can be chosen as a new coordinate. The light cone coordinate for ingoing photons, i.e. photons moving radially towards the black hole centre, is

$$v = t + r^\star(r). \tag{1.4}$$

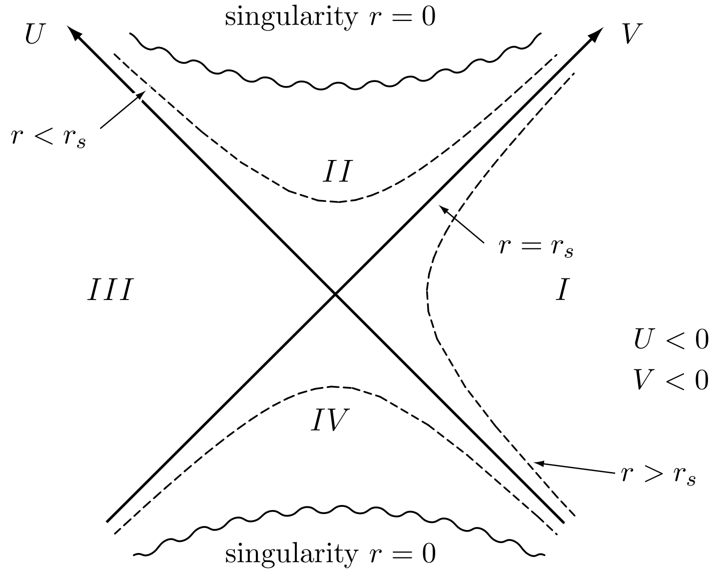

Figure 1.1: Spacetime diagram of the extended Kruskal manifold.

The distance to the black hole decreases with time, so v is called the advanced time. Since no observer can move together with a photon, the new frame strictly speaking does not fulfill the requirement of a reference frame. Nevertheless, the system of test photons proves to be convenient. The outgoing light cone coordinate u is called the retarded time and is defined by

$$u = t - r^{\star}(r). \tag{1.5}$$

Now we can rewrite the Schwarzschild metric (1.1) using either the ingoing coordinate v,

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2 \,, \tag{1.6}$$

or using the outgoing coordinate u,

$$ds^2 = \left(1 - \frac{r_s}{r}\right) du^2 + 2dudr - r^2 d\Omega^2. \tag{1.7}$$

These coordinates (u, v) are called Eddington-Finkelstein coordinates and are defined only outside the horizon. However, since they are regular on the horizon they can be analytically continued to the origin $r = 0$.

The two line elements (1.6) and (1.7) describe different physical scenarios. Examining the light cone equation $ds^2 = 0$, for (1.6) we find the two equations

$\frac{dv}{dr} = 0$ and $\frac{dv}{dr} = \frac{2}{1-\frac{r_s}{r}}$. The solution describes a black hole, since everything inside $r = r_s$ falls towards the black hole centre $r = 0$. Only massless particles can move on the horizon. Since the Einstein equations are invariant under time reflection, general relativity allows the existence of both black holes and white holes. In distinction to black holes, white holes need special initial conditions and are unstable, which is why in nature there can only exist black holes [15]. The second metric (1.7) describes a white hole. The light rays are given through $u = const$ and $\frac{du}{dr} = -\frac{2}{1-\frac{r_s}{r}}$.

Our next step is to introduce the Kruskal coordinates

$$U = -\frac{1}{\kappa} e^{-\kappa u} , \tag{1.8}$$

$$V = \frac{1}{\kappa} e^{\kappa v} , \tag{1.9}$$

where $\kappa = (4M)^{-1} = (2r_s)^{-1}$ is the black hole surface gravity. In these coordinates the black hole metric becomes

$$ds^2 = f(U,V)dUdV - r^2 d\Omega^2, \tag{1.10}$$

where

$$f(U,V) = \frac{r_s}{r} e^{1-\frac{r}{r_s}}. \tag{1.11}$$

The radial coordinate r is a function of U and V:

$$\frac{r}{r_s} - 1 = W\left(-\frac{UV}{4r_s^2}\right), \tag{1.12}$$

where W is the Lambert function which satisfies $W(x)e^{W(x)} = x$. We get

$$f(U,V) = \frac{W(-\frac{UV}{4r_s^2})}{1 + W(-\frac{UV}{4r_s^2})} \left(-\frac{4r_s^2}{UV}\right). \tag{1.13}$$

The Kruskal coordinates given by (1.8), (1.9) are defined only in the exteriour region $r > r_s$, which corresponds to the range of $U \in (-\infty, 0]$ and $V \in [0, \infty)$, but can be analytically continued to $U > 0$ and $V < 0$ since metric (1.10) is regular on the horizon. Then the singularity at $r = 0$ corresponds to $UV = 1$, the horizon $r = r_s$ to either $U = 0$ or $V = 0$. The resulting Kruskal diagram of the Schwarzschild black hole is shown in Fig. 1.1. The Kruskal coordinates are the maximal analytical extension of the Schwarzschild coordinate, meaning that every geodesic either extends to all values of its affine parameter or encounters a singularity.

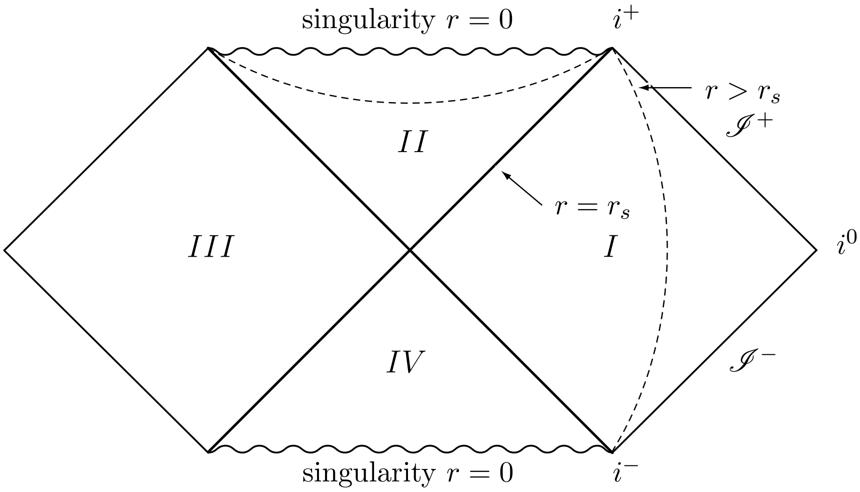

Figure 1.2: Conformal diagram of an eternal black hole.

The Kruskal space-time can be separated into four different regions: a black hole (region II), a white hole (region IV), and two regions outside the horizons (I and III). The null surface $U = 0$ which separates the exteriour region I and the black hole interiour II is called event horizon and is denoted by H_+. The null surface $V = 0$ on the other hand which separates the exteriour region and white hole region is called past event horizon H_-. One can see from Fig. 1.1 that the singularity $r = 0$ is space-like and therefore labels not a certain point, but a certain time. As mentioned earlier the Schwarzschild coordinates change their role inside the horizon and the metric inside the horizon is no longer static.

Another very convenient way to examine the causal structure of a space-time is via its conformal diagram. Using conformal transformations, regions at infinity are mapped to a finite boundary. Light rays are always mapped to lines at 45 degrees. Since the diagram is two-dimensional, every point on the diagram represents a two-dimensional sphere. For a pedagogical introduction to conformal diagrams see [31, 52]. The conformal diagram of an eternal Schwarzschild black hole is shown in Fig. 1.2.

1.1.3 Gravitational collapse

In this section we analyze the formation of a black hole as a result of contraction of a spherical mass to a size less than r_s. For simplicity we consider a spherically symmetric collapse. In this case the space-time outside the collapsing body can be described by the Schwarzschild metric, whereas the

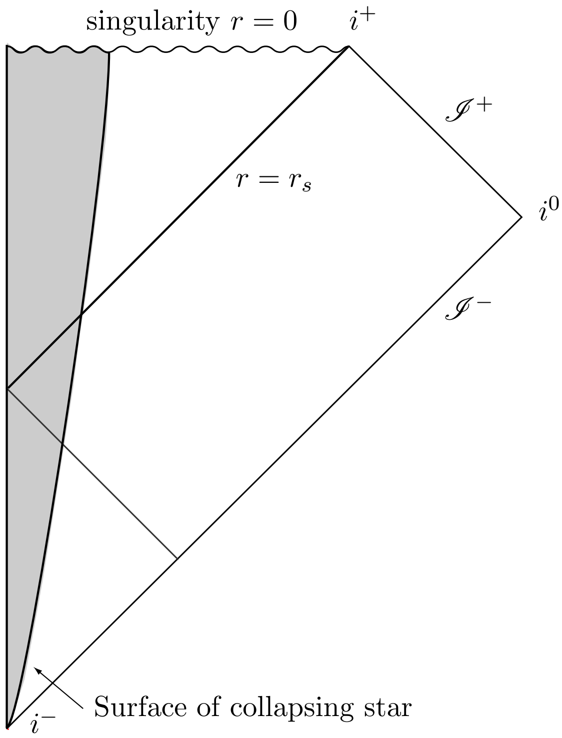

Figure 1.3: Conformal diagram of a black hole formed by gravitational collapse. The vertical line $r = 0$ represents the centre of the collapsing body.

metric inside the collapsing body is of course different and depends on the properties of the collapse. It can be seen from Fig. 1.2 that only regions I and II of the conformal diagram of the eternal black hole are relevant for the collapsing model. In Fig. 1.3 the conformal diagram of a black hole formed by a gravitational collapse is shown. The vertical line $r = 0$ is not singular but represents the centre of the collapsing body.

Often the well known Vaidya metric is used to describe the space-time in case of a collapsing body [57, 58]. In terms of ingoing coordinates (r, v) it looks as follows,

$$ds^2 = \left(1 - \frac{2M(v)}{r}\right) dv^2 - 2drdv - r^2 d\Omega^2 \, . \tag{1.14}$$

The metric (1.14) is a solution to the Einstein equations with an energy momentum tensor

$$T_{vv} = \frac{\epsilon(v)}{4\pi r^2} \, , \tag{1.15}$$

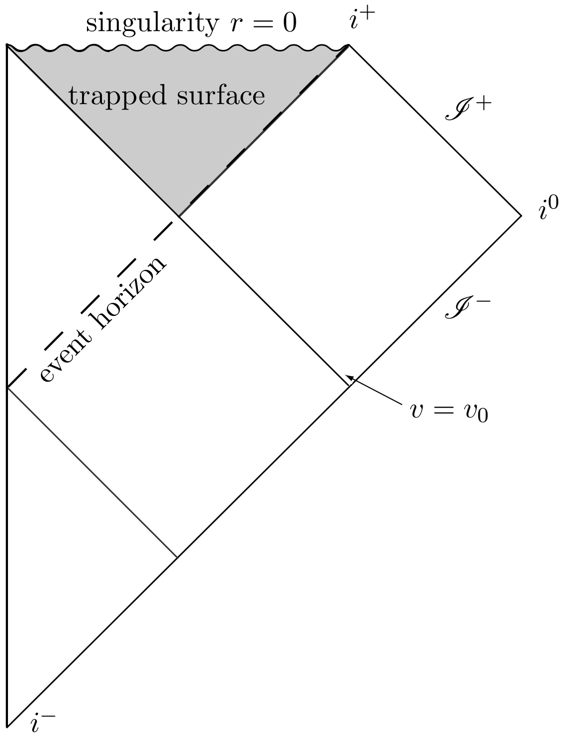

Figure 1.4: Conformal diagram of a black hole formed by spherically symmetric collapse of a null shock wave.

all other components of $T_{\mu\nu}$ being zero. The relation between total mass $M(v)$ inside v and energy density ϵ is then given by

$$M(v) = \int_{-\infty}^{v} dx \epsilon(x) \, . \tag{1.16}$$

As a simple example we consider the gravitational collapse of a null shock wave. In this case the energy momentum tensor is nonzero only along $v = v_0$,

$$T_{vv} = \frac{M_0 \delta(v - v_0)}{4\pi r^2} \, , \tag{1.17}$$

and the total mass is

$$M(v) = M_0 \theta(v - v_0) \, . \tag{1.18}$$

In this case the region above $v = v_0$ is described by the Schwarzschild metric, whereas for v_0 the space-time is flat and is described by the Minkowski metric. The conformal diagram for a black hole formed by a spherically symmetric null shock wave is shown in Fig. 1.4. A characteristic feature of the gravitational collapse is the occurrence of a future event horizon that prevents the singularity being seen from outside. Of course, there is no past event horizon for a black hole formed by gravitational collapse.

1.1.4 Charged and rotating black holes

In addition to the Schwarzschild solution there exist other stationary black hole solutions which have charge and/or angular momentum. In general relativity stationary black holes, the asymptotic final state after gravitational collapse, are uniquely described by the three parameters mass M, electric charge q, and angular momentum J. All other degrees of freedom are radiated away during the collapse. Wheeler expressed this property of stationary black holes in the following way: "Black holes have no hair" [39].

In the presence of non-Abelian gauge fields, the no-hair theorem no longer necessarily holds, but the corresponding black hole solutions are usually unstable, see [15] and references therein.

The spherically symmetric black hole solution with electric charge q can be formally generated from the Schwarzschild metric (1.1) by substituting $M \to M - \frac{q^2}{2r}$. The resulting metric is called Reissner-Nordstrøm metric and is given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (1.19)$$

It is the unique spherically symmetric and asymptotically flat solution of the Einstein-Maxwell equations. For $|q| < M$, the metric has coordinate singularities at

$$r_\pm = M \pm \sqrt{M^2 - q^2}. \quad (1.20)$$

The point r_+ denotes the position of the event horizon, which can be seen taking the limit $q \to 0$, whereas r_- characterizes the Cauchy horizon. (For the meaning of different types of horizons see 1.1.5.) The conformal diagram of the Reissner-Nordstrøm space-time is shown in Fig. 1.5. The singularity at $r = 0$ is now time-like and therefore is a naked singularity in some regions, e.g. region III. The special case $|q| = M$ is referred to as extremal black hole. The two horizons now coincide. For $|q| > M$ there is no event horizon, instead of a black hole there is now a naked singularity.

Whereas it is most unlikely that we will ever observe astrophysical black holes with a considerable net charge, most black holes are expected to be rotating. The solution for a rotating stationary black hole, the so-called Kerr

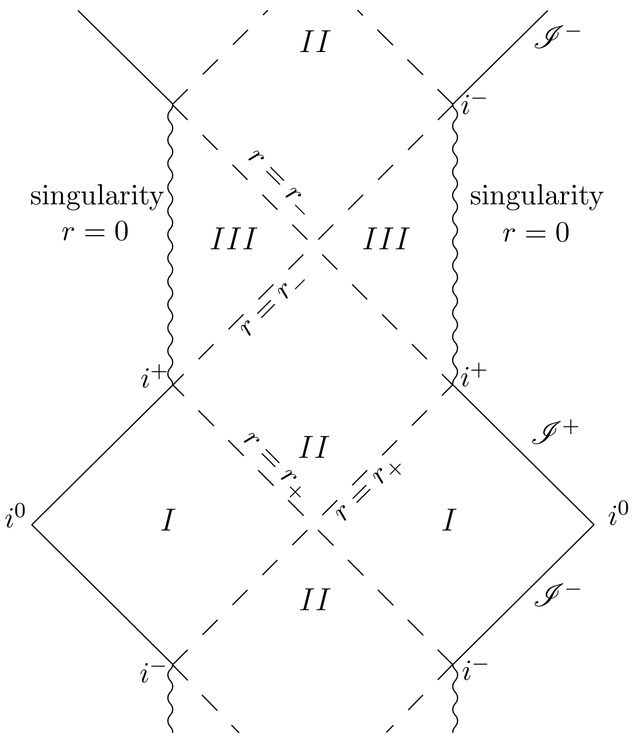

Figure 1.5: Conformal diagram of a Reissner-Nordstrøm black hole with $|q| < M$.

solution, is no longer spherically symmetric and static, but it is axisymmetric and stationary. The solution is characterized by the mass M and angular momentum J. There exist two Killing vectors ∂_t and ∂_ϕ representing the symmetries of the space-time. The Kerr solution, in the frequently used Boyer-Lindquist coordinates, is given by

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2, \tag{1.21}$$

where

$$a = \frac{J}{M}, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad \text{and} \quad \Delta = r^2 - 2Mr + a^2. \tag{1.22}$$

For the sake of completeness we also note the most general solution for a stationary black hole which is the Kerr-Newman solution characterized by mass M, angular momentum J and electric charge q. It can be obtained from the Kerr solution (1.21) substituting $M - \frac{q^2}{2r}$ for M.

Formally, laws of thermodynamics can be established for these stationary black hole solutions. The three parameters M, J and q characterizing black holes are identified with thermodynamic variables. This will be discussed in section 1.2.

1.1.5 Event horizon, apparent horizon, trapped surfaces

In this section we discuss the previously used notions of event horizon, apparent horizon, Cauchy horizon and trapped surfaces. We will not provide rigorous mathematical definitions, but explain their physical meaning in spherically symmetric space-times.

The **future event horizon** defines a region from behind which it is impossible to escape to $\mathscr{I}^+$ without exceeding the speed of light. The **past event horizon** is the time reverse of this, that means it is impossible to get behind starting from $\mathscr{I}^-$. The maximally extended Kruskal manifold contains both past and future event horizon at $r = r_s$, as can be seen in Fig. 1.2, whereas a black hole formed by gravitational collapse only possesses a future event horizon, see Fig. 1.3.

The interiour of a black hole generally contains a region of trapped surfaces. To illustrate this notion consider a two-sphere in flat Minkowski space. There are two families of null geodesics which emanate from the two-sphere, those that are outgoing and those that are ingoing. The former diverge, while the latter converge. A **trapped surface** is one for which both families of null geodesics are everywhere converging, due to gravitational forces. It is easy to check that two-spheres of constant r behind the future event horizon in the Schwarzschild space-time are trapped. Outgoing light rays at $r = r_s$ of course generate the horizon itself, whose area is constant in Schwarzschild space-time. Thus this two-sphere is marginally trapped.

An **apparent horizon** is the outer boundary of a region of trapped surfaces. In the Schwarzschild space-time the apparent horizon is located at $r = r_s$. Thus event horizon and apparent horizon coincide for the Schwarzschild black hole. This sometimes leads to confusion about the two surfaces which in general are quite different. An event horizon is a global concept, and the structure of the entire space-time must be known to define it. The location of an apparent horizon can be determined from the initial data on a space-time slice.

To illustrate the difference let us have a look at a black hole with increasing mass. At time t_0 the black hole would have a fixed apparent horizon. Throwing matter into the black hole at some time $t > t_0$ will have no effect on the area or location of the apparent horizon at t_0 (though it will at later times), but the infalling matter changes the event horizon at the earlier time

t_0 to move out to larger distances. Of course, the position of the event horizon can only be determined if the mass function for the whole space-time is known. If we assume that there is no additional mass change at later times then at t_0 the apparent horizon lies inside the event horizon.

The above example is a typical situation in classical general relativity where the apparent horizon is usually a null or space-like surface which lies inside or coincides with the event horizon (assuming cosmic censorship). We will later consider evaporating black holes. In this case the apparent horizon moves outside the event horizon, since the black hole mass decreases. This will be discussed in detail in chapter 4.

The location of the apparent and event horizon for a black hole formed by a null shock wave are shown in Fig. 1.4. The solid line is the apparent horizon, which bounds the shaded region of trapped surfaces. The dashed line is the event horizon, which coincides with the apparent horizon after the collapse is completed.

The **Cauchy horizon** is a light-like surface which is the boundary of the domain of validity of the Cauchy problem. That means it is impossible to use the laws of physics to predict the structure of the region beyond the Cauchy horizon. Thus it signals the onset of unpredictability. An example is the surface $r = r_-$ in the Reissner-Nordstrøm space-time in Fig. 1.5.

1.2 Black hole thermodynamics

Even before Hawking's discovery of black hole evaporation it has been known that black holes require a thermodynamical description involving a nonzero intrinsic entropy.

The situation is as follows. Imagine that a black hole swallows a hot body possessing a certain amount of entropy. Then an observer outside the black hole finds that the total entropy in the part of the world accessible to him has decreased. This would contradict the second law of thermodynamics, however, which states that the entropy of a closed system can never decrease in time. We can avoid the decrease of entropy if we assign the entropy of the investigated body to the interiour region of the black hole. This attempt is quite unsatisfactory since by no means could any outside observer measure this entropy. If we are not inclined to give up the law of nondecreasing entropy we have to conclude that the black hole itself possesses a certain amount of entropy and that a body which falls into it not only transfers its mass, angular momentum and charge to the black hole, but also its entropy S. As a result the entropy of the black hole increases at least by S.

The first hint that black holes are endowed with thermodynamic properties was the area theorem of black holes [19], according to which the total horizon area A in black hole spaces has to increase in all reasonable physical processes. The area theorem is a result only using classical general relativity together with certain reasonable assumptions about the behavior of matter. The area law seems to exhibit a close analogy to the second law of thermodynamics. This was discussed by Bekenstein in 1972 [4, 5] who therefore postulated a connection between black hole area and entropy. In 1974 Hawking discovered that black holes emit particles with thermal spectrum [20, 21]. Thus a temperature can be assigned to black holes and the connection between black hole area and entropy was confirmed. In this framework laws of black hole thermodynamics have been established [2], see also [15].

The first law of black hole thermodynamics is just the conservation of energy. An arbitrary black hole, similar to thermodynamical systems, reaches an equilibrium state after the relaxation processes are completed. As mentioned earlier it is then uniquely described by the three parameters mass M, angular momentum J, and electric charge q. The area of the black hole in terms of these parameters is

$$A = 4\pi\Big(2M^2 - q^2 + 2M\sqrt{M^2 - q^2 - J^2/M^2}\Big). \tag{1.23}$$

If dM is an infinitesimal change of the black hole mass, then

$$dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \phi dq\ , \tag{1.24}$$

where κ is the black hole surface gravity

$$\kappa = 4\pi\frac{\sqrt{M^2 - q^2 - J^2/M^2}}{A}\ , \tag{1.25}$$

$\Omega = 4\pi J/MA$ is the angular velocity and $\phi = 4\pi q r_+/A$ the electrostatic potential. The second and third term in (1.24) describe the change in the rotational and electric energy. Since M can be identified with the energy of the black hole, (1.24) is the analogue to the first law of thermodynamics

$$dE = TdS - pdV + \mu dN. \tag{1.26}$$

In classical thermodynamics temperature is the conjugate variable to the entropy. Comparing (1.24) and (1.26) it becomes obvious that in black hole thermodynamics the quantity analogous to the temperature is the surface gravity κ. At this point this analogy is purely formal. However, considering quantum field theory in black hole space-time not only supports this analogy but provides the correct coefficients (in SI units),

$$T = \frac{\hbar\kappa}{2\pi c k_B}\ . \tag{1.27}$$

This formula will be derived in the next section.

The second law of black hole thermodynamics is a direct consequence of Hawking's area theorem. It can be formulated as follows: In any classical process the area of the black hole A, and hence its entropy S_{BH}, does not decrease,

$$dA \geqslant 0, \quad dS_{BH} \geqslant 0. \tag{1.28}$$

Both for usual and black hole thermodynamics the second law signals the irreversibility inherent in a system as a whole, and thus provides an arrow of time.

Lastly, Bardeen, Carter and Hawking [2] also formulated the analogue of the third law for black holes. It states that it is impossible by any procedure, no matter how idealized, to reduce the black hole surface gravity κ to zero by a finite sequence of operations. This corresponds to the formulation of the third law of thermodynamics due to Nernst. It must be emphasized that Planck's formulation of the third law of thermodynamics, stating that the entropy of a system vanishes at zero absolute temperature, does not hold for black holes. Extremal black holes have temperature zero, nevertheless they possess nonzero area A and entropy.

1.3 Black hole evaporation

It has been shown in the last section that entropy and temperature can be assigned to black holes. Using the formula $T_{BH} dS_{BH} = \frac{c^2 \kappa}{8\pi G} dA$ (now using SI units), on dimensional grounds we have

$$T_{BH} \propto \frac{\hbar \kappa}{c k_B}, \quad S_{BH} \propto \frac{c^3 k_B A}{G \hbar}. \tag{1.29}$$

The exact coefficients in (1.29) cannot be determined within the classical theory, since there this analogy is only formal. Studying quantum fields in a classical black hole background provides the exact coefficients. For this it is necessary to introduce the framework of quantum theory in curved space-time.

As mentioned earlier the first rigorous calculation of the rate of particle creation by a black hole was presented by Hawking in 1974 [20]. An intuitive picture of Hawking radiation involves the creation of virtual particle-antiparticle pairs in the vicinity of the black hole horizon. It may happen that two particles with opposite momenta are created on the horizon, one particle inside the horizon and the other particle outside the horizon. The first virtual particle always falls into the black hole. If the momentum of the second particle is directed away from the black hole, it has a nonzero probability for moving away from the horizon and becoming a real radiated particle. In the following we will calculate the density of particles emitted by a static black hole, as registered by observers far away from the horizon.

In quantum theory, particles are excitations of quantum fields, so we consider a scalar field ϕ in the presence of a single nonrotating black hole with mass M which is described by the Schwarzschild metric (1.1). As an example we consider a $1+1$-dimensional black hole with coordinates (t, r). It will be shown later that, apart from an irrelevant transmission coefficient, this reduction of dimensionality does no injustice to the physics. The two-dimensional line element is

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 = g_{kl} dx^k dx^l. \tag{1.30}$$

The action for a minimally coupled scalar field is

$$S[\phi] = \frac{1}{2} \int d^2x \sqrt{-g} g^{kl} \frac{\partial \phi}{\partial x^k} \frac{\partial \phi}{\partial x^l} \, , \tag{1.31}$$

where $g \equiv \det(g) = -1$. It is more convenient to work with the metric in conformally flat form. Using the tortoise coordinate (1.2), the conformally flat metric is

$$ds^2 = \left(1 - \frac{r_s}{r}\right) \left(dt^2 - dr^{\star 2}\right) . \tag{1.32}$$

The action for the scalar field in the tortoise coordinate becomes

$$S[\phi] = \frac{1}{2} \int dt dr^\star \left[(\partial_t \phi)^2 - (\partial_{r^\star} \phi)^2 \right] . \tag{1.33}$$

From this we can now derive the classical equation of motion

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial r^{\star 2}} = 0 \, , \tag{1.34}$$

which has the general solution

$$\phi(t, r^\star) = P(t - r^\star) + Q(t + r^\star) \, . \tag{1.35}$$

Here $P(t - r^\star)$ and $Q(t + r^\star)$ are arbitrary smooth functions.

1.3.1 Quantization in Schwarzschild space-time

We shall now quantize the scalar field ϕ in two different reference frames and compare the vacuum states. Suitable choices are the Schwarzschild reference frame with tortoise coordinate $r^\star$ and the Kruskal frame. At spatial infinity, $r \to \infty$, the Schwarzschild metric tends to the Minkowski metric, so the time coordinate t coincides with proper time in this limit. Therefore the tortoise coordinates are suitable coordinates for observers very far away from the black hole. On the horizon we found that the Kruskal coordinates U and V are regular.

The familiar mode expansion in Kruskal coordinates, with proper time $T = \frac{1}{2}(U+V)$ and spatial coordinate $X = \frac{1}{2}(V-U)$, is

$$\hat{\phi}(T,X) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2|k|}} \left(e^{-i|k|T+ikX} \hat{a}_k^- + e^{i|k|T-ikX} \hat{a}_k^+ \right) . \qquad (1.36)$$

Here the normalization factor which is $(2\pi)^{\frac{3}{2}}$ in the four-dimensional theory is replaced by $(2\pi)^{\frac{1}{2}}$. The creation and annihilation operators $\hat{a}_k^\pm$ defined in (1.36) satisfy the usual commutation relations, see appendix (A.26), and describe particles moving with momentum k either in the positive ($k > 0$) or negative ($k < 0$) direction. The vacuum state in the Kruskal reference frame is the zero eigenvector of all annihilation operators $\hat{a}_k^-$,

$$\hat{a}_k^- |0_K\rangle = 0 \quad \text{for all } k \, . \qquad (1.37)$$

The mode expansion in the tortoise coordinate frame $(t, r^\star)$ is formally very similar to (1.36),

$$\hat{\phi}(t,r^\star) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2|k|}} \left(e^{-i|k|t+ikr^\star} \hat{b}_k^- + e^{i|k|t-ikr^\star} \hat{b}_k^+ \right) . \qquad (1.38)$$

Then the corresponding vacuum state is defined as

$$\hat{b}_k^- |0_S\rangle = 0 \quad \text{for all } k \, , \qquad (1.39)$$

where the index S stands for Schwarzschild. The mode expansion (1.36) can be rewritten in light cone coordinates U and V. In these coordinates the field is decomposed into out- and ingoing waves

$$\hat{\phi}(U,V) = \int_0^{\infty} \frac{d\omega}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega U} \hat{a}_\omega^- + e^{i\omega U} \hat{a}_\omega^+ + e^{-i\omega V} \hat{a}_{-\omega}^- + e^{i\omega V} \hat{a}_{-\omega}^+ \right), \qquad (1.40)$$

where we have used $\omega = |k|$. The light cone mode expansion in the Schwarzschild reference frame has the same form, using Eddington-Finkelstein coordinates $u = t - r^\star$ and $v = t + r^\star$. We use $\Omega = |k|$ for the integration variable to distinguish the mode expansion in the Schwarzschild reference frame from that in Kruskal coordinates,

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\Omega}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega u} \hat{b}^-_\Omega + e^{i\Omega u} \hat{b}^+_\Omega + e^{-i\Omega v} \hat{b}^-_{-\Omega} + e^{i\Omega v} \hat{b}^+_{-\Omega} \right). \quad (1.41)$$

1.3.2 Hawking effect

The vacua $|0_S\rangle$ and $|0_K\rangle$, annihilated by $\hat{b}^-_\Omega$ and $\hat{a}^-_\omega$, respectively, are different. Therefore $\hat{b}^-_\Omega|0_K\rangle \neq 0$ and vice versa, which means the state $|0_K\rangle$ contains particles with frequency $\pm\Omega$. Or in other words, particles are created due to the gravitational field.

The operators $\hat{b}^\pm_\Omega$ can be expressed in terms of the operators $\hat{a}^\pm_\omega$ by a so-called Bogolyubov transformation. This is, see also appendix (A.30),

$$\hat{b}^-_\Omega = \int_0^\infty d\omega \left(\alpha_{\omega\Omega} \hat{a}^-_\omega + \beta_{\omega\Omega} \hat{a}^+_\omega \right), \quad (1.42)$$

where the Bogolyubov coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ are determined as

$$\left.\begin{array}{c} \alpha_{\omega\Omega} \\ \beta_{\omega\Omega} \end{array}\right\} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u \mp i\omega U}. \quad (1.43)$$

For general properties of the Bogolyubov transformation see appendix A. We keep in mind that $U = U(u)$. To determine the number of particles which are created in the Schwarzschild background we use the transformation (1.8) between Kruskal coordinate U and Eddington-Finkelstein coordinate u explicitly,

$$\left.\begin{array}{c} \alpha_{\omega\Omega} \\ \beta_{\omega\Omega} \end{array}\right\} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u \pm i\frac{\omega}{\kappa} e^{-\kappa u}}. \quad (1.44)$$

The average density of the created particles as measured by an observer at infinity can be determined as follows. The b-particle operator is $\hat{N}_\Omega = \hat{b}^+_\Omega \hat{b}^-_\Omega$, so the average number of particles in the Kruskal vacuum $|0_K\rangle$ is equal to its expectation value

$$\langle \hat{N}_\Omega \rangle = \langle 0_K | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_K \rangle = \int_0^\infty d\omega |\beta_{\omega\Omega}|^2 . \quad (1.45)$$

Evaluating the integral (1.44), the coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ can be expressed in terms of the Γ-function. The relation between them is $|\alpha_{\omega\Omega}|^2 = e^{2\pi\Omega/\kappa} |\beta_{\omega\Omega}|^2$.

Finally, using the normalization condition of the Bogolyubov coefficients (A.34), the total number of emitted particles in the mode Ω is

$$\langle \hat{N}_\Omega \rangle = \delta(0) \frac{1}{e^{2\pi\Omega/\kappa} - 1} \,. \tag{1.46}$$

Since $\langle \hat{N}_\Omega \rangle$ represents the total number of particles in the entire space we expect it to be divergent. To extract the mean density of particles from $\langle \hat{N}_\Omega \rangle$ the divergent volume factor $\delta(0)$ has to be separated. Hence, the mean density of particles in the mode Ω is

$$n_\Omega = \frac{1}{e^{2\pi\Omega/\kappa} - 1} \,. \tag{1.47}$$

The above result has been derived for positive modes $\Omega > 0$. The particle density for negative frequencies is obtained by replacing Ω by $|\Omega|$ in (1.47). The spectrum of emitted particles is thermal, corresponding to the spectrum of black body radiation with the temperature

$$T_H = \frac{\kappa}{2\pi} \tag{1.48}$$

in natural units, which in SI units becomes

$$T_H = \frac{\hbar\kappa}{2\pi c k_B} \,. \tag{1.49}$$

The surface gravity of the Schwarzschild black hole in SI units is $\kappa = \frac{c^4}{4GM}$, so the Hawking temperature

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \tag{1.50}$$

is inversely proportional to the black hole mass M. This leads to an unusual thermal property of black holes. When a black hole emits particles and in this way loses energy, its mass decreases. In this process the black hole temperature increases since it is inversely proportional to the mass. The black hole gets hotter and will emit more radiation which means that the black hole's heat capacity is negative.

In the final state of black hole evaporation the semiclassical treatment is no longer justified, since quantum gravitational effects can no longer be neglected. Whether the black hole will disappear with an explosion, or whether there might be a remnant of Planckian size ($10^{-5}g$), still remains an open question.

1.3.3 Gravitational collapse

In the previous section we studied the thermal radiation emitted by an eternal black hole. In a realistic scenario a black hole is formed by gravitational collapse. In the following we will calculate the particle flux emitted by such black holes. We will show that the particle spectrum is identical to that of an eternal black hole.

We consider the simple model of a collapsing null shock wave which was introduced in section 1.1.3. Let us recall that the space-time inside the shell is flat, whereas outside the shell it is described by the Schwarzschild metric. In distinction to the eternal black hole one considers light rays which emerge at infinity $\mathscr{I}^-$, cross the collapsing shell, and at some later time escape to infinity $\mathscr{I}^+$. Such a light ray γ is shown in Fig. 1.6, where γ_{H_+} depicts the last light ray escaping to infinity. It follows from the no-hair theorem that during the collapse of an arbitrary matter shell all inhomogeneities are radiated away. Hence the Hawking spectrum at late times u cannot depend on the details of the collapse. It is sufficient to consider light rays γ which reach $\mathscr{I}^+$ at late time u and use the geometric optics approximation to trace them back to $\mathscr{I}^-$.

At $\mathscr{I}^+$ the light ray is described by the parameter u which is constant along the outgoing light ray. Tracing it back to $\mathscr{I}^-$ it is reflected at the origin $r = 0$ and becomes an ingoing light ray described by $v = const$. Thus the task is to find the transformation $v(u)$ connecting these two coordinates.

The line element of the space-time is given by the Vaidya metric (1.14) with the mass function (1.18). Without loss of generality we assume $v_0 = 0$. Along the outgoing light ray the light cone coordinate u is constant. Outside the shell the space-time is described by the Schwarzschild metric and we can again use the tortoise coordinate $r^\star$. In terms of advanced and retarded time the tortoise coordinate is $2r^\star(u, v) = v - u$. Inside the shell space-time is flat, so the ingoing light ray is given by $v = t + r = const$ and the outgoing ray by $u = t - r = const$. The coordinates inside and outside are matched along $v = 0$. The radial coordinate inside the shell is

$$r = \frac{1}{2}v - r(u, v = 0) \,. \tag{1.51}$$

The light ray γ reaches $\mathscr{I}^+$ at some late time u, which means it crosses the shell at a radius near r_s. In this limit the relation between tortoise coordinate $r^\star$ and radial coordinate r is $r^\star \simeq r_s \ln(r/r_s - 1)$. Then the radial coordinate at $v = 0$ is

$$r(u, 0) \simeq r_s(1 + e^{-\kappa u}) \,. \tag{1.52}$$

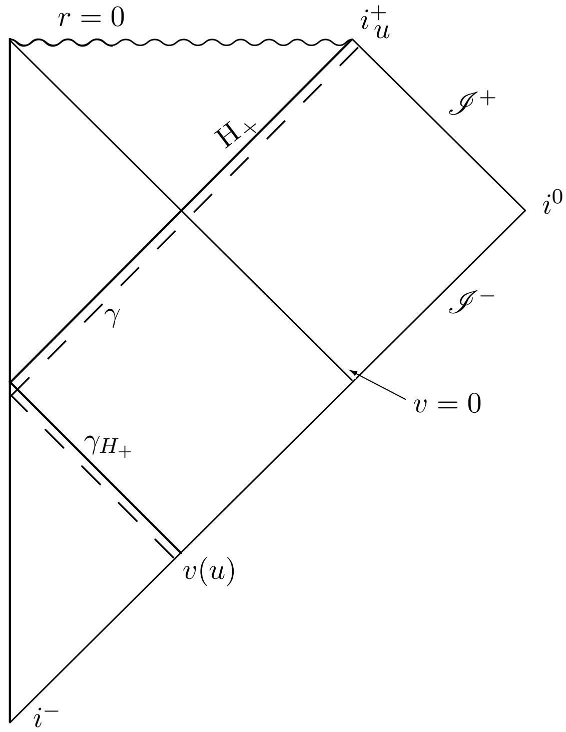

Figure 1.6: The light ray γ emerges at infinity $\mathscr{I}^-$, is reflected at the origin $r=0$ and at late time u escapes to infinity. γ_{H_+} is the last light ray escaping to infinity.

Substituted into (1.51) this yields the desired coordinate relation

$$v(u) \simeq -\frac{1}{\kappa} e^{-\kappa u} \,, \tag{1.53}$$

which connects the time v when the light ray γ emerges from $\mathscr{I}^-$ with the time u when it reaches $\mathscr{I}^+$.

Now we define suitable vacuum states for the ingoing wave at $\mathscr{I}^-$ and the outgoing state at $\mathscr{I}^+$. On $\mathscr{I}^+$, we can expand the field operator in Schwarzschild coordinates in the same way as we did in the case of the eternal black hole, see (1.38) with the corresponding vacuum state.

Inside the shell space-time is flat, the light cone coordinates are u and v. Hence, on $\mathscr{I}^-$ the field can be resolved into positive and negative frequencies as well. Calculating the Bogolyubov coefficients we find

$$\left.\begin{array}{c}\alpha_{\omega\Omega}\\ \beta_{\omega\Omega}\end{array}\right\} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i\Omega u \mp i\omega v(u)} \,. \tag{1.54}$$

The transformation $v(u)$ for late times u is identical to the Kruskal coordinate $U = -\frac{1}{\kappa}e^{-\kappa u}$. Hence, the Bogolyubov coefficients are the same in both cases and the particle spectrum is thermal with a temperature $T_H = \frac{\kappa}{2\pi}$.

An observer at infinity, who examines the collapse using a detector, will measure almost no radiation during the time when the radius of the shell is still very large. With decreasing size the particle flux increases exponentially and asymptotically approaches the radiation of the eternal black hole.

The above calculations have shown that at late times u the particle spectrum emitted by a black hole created by gravitational collapse is identical to the particle spectrum of an eternal black hole. This result is not surprising since according to the no-hair theorem an uncharged nonrotating black hole is asymptotically described solely by its mass. Thus the particle spectrum of a black hole should not depend on the details of its formation.

1.3.4 Black hole wave equation

The above calculations have been done in the two-dimensional theory. In four dimensions the derivation of Hawking radiation is essentially the same. The resulting particle spectrum is thermal, but it will be suppressed due to the existence of a potential barrier of the black hole in the four-dimensional theory.

Consider again a scalar field in the Schwarzschild background. Since the background space-time is spherically symmetric the Klein-Gordon equation $(\Box + m^2)\phi = 0$ may be separated into spherical harmonics,

$$\phi = \frac{f(r,t)}{r} Y_{lm} e^{-i\omega t} \,. \tag{1.55}$$

Using the tortoise coordinate $r^\star$ the resulting radial wave equation is

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r^{\star 2}} - V_l(r) f = 0 \,, \tag{1.56}$$

with the potential

$$V_l(r) = \left(1 - \frac{r_s}{r}\right)\left[\frac{r_s}{r^3} + \frac{l(l+1)}{r^2} + m^2\right] \,, \tag{1.57}$$

where m is the mass of the scalar field. Near the black hole horizon, as $r^\star \to -\infty$, the potential falls off exponentially, $V \sim e^{r^\star / r_s}$. For $r^\star \to \infty$ the potential behaves as $V \sim m^2(1 - r_s/r^\star)$ in the massive case and $V \sim l(l+1)/r^2$ in the massless case. The detailed form of the potential is irrelevant in the geometric optics approximation. The incoming waves, however, will partially scatter off the gravitational field on the potential (1.57) which results in a superposition of incoming and outgoing waves. The backscattering is a function of Ω, and the spectrum is not precisely thermal. The luminosity of the black hole is given by [61]

$$L_H = \frac{1}{2\pi} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} d\Omega \frac{\Gamma_{\Omega l}}{e^{8\pi M \Omega} - 1} , \tag{1.58}$$

where $\Gamma_{\Omega l}$ is the so-called grey factor of the black hole. A possible dependency on the angular momentum and spin of the particles resides in $\Gamma_{\Omega l}$.

1.3.5 Black hole life time

During evaporation the mass of the black hole decreases. It is still unknown whether a black hole can lose its whole mass or if there is a black hole remnant of Planckian size as the final stage of evaporation. This difference does not play any role for the estimate of black hole life time. If M_0 is the initial mass of the black hole, then after evaporation its mass M in any case is much smaller, $M \ll M_0$. We estimate the life time of a black hole using the assumption that the decrease of the black hole mass can be identified with the energy radiated to infinity using the Stefan-Boltzmann law

$$\frac{dM}{dt} \sim -A T_H^4 \sim -M^2 \cdot M^{-4} = -\frac{1}{M^2} . \tag{1.59}$$

Integration of this equation yields

$$t(M) \sim \left(M_0^3 - M^3 \right) \simeq M_0^3 . \tag{1.60}$$

The life time of the black hole can now be estimated as [15]

$$t_H \sim \left(\frac{M_0}{m_p} \right) t_p \sim 10^{65} \left(\frac{M_0}{M_\odot} \right) \text{years.} \tag{1.61}$$

If in the early universe primordial black holes with $M_0 \sim 10^{14}$ g have been created, they should evaporate at the present age of the universe.

During the process of evaporation different kinds of particles can be emitted depending on the black hole mass. For black holes with a mass of 10^{17} g the temperature is less than 10^9 K and only massless particles can be emitted, photons, gravitons, and neutrinos. Black holes with smaller mass will also emit electrons and positrons, and if the mass is below 10^{14} g, heavier particles such as protrons and neutrons can be emitted. In this case the black hole temperature is more than 10^{13} K. A detailed investigation of this topic has been performed by Page in [34, 35, 36].

1.3.6 Charged and rotating black holes

Let us now consider the particle spectrum for rotating and charged black holes. It turns out that the presence of charge and/or angular momentum changes the black hole temperature as well as the particle spectrum.

First consider a charged black hole with mass M and charge q, described by the Reissner-Nordstrøm metric (1.19). The event horizon is located at $r_+ = M + \sqrt{M^2 - q^2}$, and the black holes surface area is given by $A = 4\pi r_+^2$. Substituting the expression for the surface gravity κ we find the radiation temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{1 - 16\pi^2 q^2/A^2}{8\pi M} \,. \tag{1.62}$$

Thus the presence of a charge q reduces the Hawking temperature of a black hole. For an extremal black hole with $q^2 = M^2$ the temperature is zero, whereas the area is not. According to the second law of black hole thermodynamics, see section 1.2, the entropy of such a black hole is $S = \frac{1}{4}A \neq 0$. This implies that the formulation of the third law, which states that the entropy becomes zero for the limit where the temperature tends to zero, cannot be valid in the case of extremal black holes.

For rotating black holes with angular velocity Σ the situation is more complicated. A black hole with mass M and angular momentum J is described by the Kerr metric (1.21). In this case the scalar field potential has additional terms, and the solution of the scalar field equation is more complicated. However, at large distances the difference to nonrotating black holes can be mainly described by replacing the frequency Ω by $\Omega - m\Sigma$, where m is the azimuthal quantum number of the spheroidal harmonics. Hence, the particle spectrum on $\mathscr{I}^+$ becomes

$$n_\Omega = \frac{1}{e^{2\pi(\Omega - m\Sigma)/\kappa} - 1} \,. \tag{1.63}$$

The rotation of the black hole is reflected as a chemical potential in the particle spectrum. Since the spectrum depends on m the radiation is asymmetric. The black hole tends to emit those particles which result in a decrease of the black hole's angular momentum.

In the case of $\Omega < m\Sigma$ the particle flux becomes negative. The black hole loses energy to an ingoing wave with lower frequency Ω which then becomes outgoing with a larger amplitude. This phenomenon of rotating black holes is called super-radiance.

Chapter 2

Origin of black hole entropy

Hawking's discovery of black hole evaporation confirmed the formal analogy between usual thermodynamics and black hole thermodynamics. It verified Bekenstein's conjecture that black holes are endowed with entropy and proved the validity of the second law which seemed to be violated in processes where matter is swallowed by a black hole.

Nowadays, more than thirty years after the discovery of black hole entropy, its statistical origin is still a rather hot topic in the literature, see [15] and references therein. We believe that the proper understanding of this quantity can be extremely useful and could highlight important aspects of black hole physics. In the following we present some considerations concerning the origin of black hole entropy. We show that this entropy can be related to missing information about the exact state of the matter from which the black hole was formed.

The entropy of ordinary matter can be understood by counting different possibilities of preparing the system in a final state with given macroscopic parameters from microscopically different states

$$S^I = -\sum_\alpha p_\alpha \ln p_\alpha \, , \tag{2.1}$$

where p_α are the probabilities of different initial states. This definition directly relates the entropy of a given system to the information lost in the process of its formation. In the informational approach to black hole entropy one tries to find a similar understanding of why $S_{BH} = \frac{1}{4}A$ represents the entropy of a black hole by identifying its quantum dynamical degrees of freedom. In the following we discuss two different approaches to this topic.

2.1 Concerning black hole entropy

In 1985 Zurek and Thorne proposed that the entropy of a black hole can be interpreted as "the logarithm of the number of quantum-mechanical distinct ways that the hole could have been made" [66]. In this model one estimates the time it takes to form a black hole of mass M to be of the order of the black hole life time $t_H \sim M^3$, see (1.61). The vast majorities of ways to form a black hole involve building it up by accretion of one quantum after the other where the energy ϵ of the quanta should be as small as possible $\epsilon \sim M^{-1}$. Then the total number of quanta which are required to form a black hole of total mass M is $N_q \sim M/\epsilon \sim M^2$. The effective cross-section of the black hole is proportional to M^2, so the total volume in phase space which the particles occupy before the injection is $\sim M^2 t_H \epsilon^3$. Hence, $N \sim M^2 \sim N_q$ is the total number of single-particle states in which the N_q quanta can be injected into the black hole. With that the number of ways to build the black hole is the number of ways to distribute the N_q quanta, for simplicity bosons, among the N states

$$\mathcal{N} \sim \frac{(N-1+N_q)!}{(N-1)!N_q!} \,. \tag{2.2}$$

The logarithm of this expression can be identified with the black hole entropy

$$S_{BH} \sim \ln \mathcal{N} \sim N_q \sim M^2. \tag{2.3}$$

This result is in approximate agreement with the formula for black hole entropy $S_{BH} = 4\pi M^2$. In this approach the expression for the black hole entropy does not depend on the number of possible different particle species. One might expect that the number of possibilities to create the black hole increases if more than one particle species are involved, but at the same time the black hole life time decreases and N stays constant. Thus Zurek and Thorne [66] showed that the entropy (2.3) is related in a simple way to the amount of information lost by stretching the horizon.

In another interesting approach to explain the origin of black hole entropy Bekenstein and Mukhanov consider quantization of the black hole mass [7]. They assume that the black hole area should be quantized in integers as $A = \alpha \hbar n$, where $n \in \mathbb{N}$. If the black hole has mass M, the system is in the state $n \sim M^{\frac{1}{2}}$. The absorption and emission of one quantum is described by the transition of the energy state from n to $n \pm 1$. In general the degeneracy of an energy level labeled by n is $g(n)$ and is identified with the number of different ways to reach the level n by injecting quanta into the black hole. It is easy to see that the number of possible ways to reach the n-th level from

$n = 0$ is 2^{n-1}. Since the logarithm of the degeneracy of states, $\ln g(n)$, can be identified with the entropy the parameter α can be determined as $\alpha = 4 \ln 2$.

The radiation emitted by a black hole with quantized mass will be concentrated in lines at integer multiples of the fundamental frequency

$$\bar{\omega} = \frac{\ln 2}{8\pi M} \, . \tag{2.4}$$

No quanta are expected with frequencies below this fundamental frequency. Even if some broadening occurs or if the spectrum is blurred by multiquanta emission per jump, the spectrum will be radically different to the originally derived Hawking spectrum. Thus if one day primordial black holes can be produced in the laboratory and quantum aspects of black holes can be investigated this will exclude at least one model.

There are of course more attempts to explain black hole entropy which we did not mention here, e.g. relating the entropy to the properties of the vacuum in strong gravitational fields, for references see [15].

In the following we present some considerations concerning the origin of black hole entropy. We show that the entropy of a black hole can be interpreted as lost information of the matter forming the black hole. To do so we study the physically most relevant case of the evaporating black hole and avoid discussion of the eternal black hole in which case the considerations presented below fail (at least when applied naively).

2.2 Entropy of a nonequilibrium gas

In view of the following sections we recall the calculation of entropy for nonequilibrium systems. According to (2.1) the informational entropy is related to missing information about the exact state of the system, where p_α characterizes the probability to find the system in the states α. It takes its maximum value when all states are equally probable, that is when $p_\alpha = 1/\Gamma$. Hence, the entropy is equal to

$$S = \ln \Gamma, \tag{2.5}$$

where Γ is the total number of possible microstates which the system can occupy. Let us characterize an ideal gas of N Bose particles located within a box of volume V by its energy spectrum. In this case the number of particles with energy in the interval between ϵ and $\epsilon + \Delta\epsilon$ is equal to ΔN_ϵ, and the information about the exact microscopic configuration of the system is missing. In fact, even if the energy spectrum is completely specified every particle can still have an arbitrary, nonspecified direction of propagation, and can be located in any place inside the box. If we require that the unspecified directions of the particle propagation are restricted by the solid angle ΔO then the number of possible indistinguishable microstates for every particle with energy between ϵ and $\epsilon + \Delta\epsilon$ is equal to

$$\Delta g_\epsilon = \frac{g_p}{(2\pi)^3} \int d^3x d^3p \simeq \frac{g_p}{(2\pi)^3} V \Delta O \sqrt{\epsilon^2 - m^2}\epsilon\Delta\epsilon \,, \tag{2.6}$$

where g_p is the helicity of the particle species and m the particle mass. The total number of possible configurations for ΔN_ϵ Bose particles is equal to the number of ways to distribute those particles among Δg_ϵ states,

$$\Delta G_\epsilon = \frac{(\Delta g_\epsilon - 1 + \Delta N_\epsilon)!}{(\Delta g_\epsilon - 1)!(\Delta N_\epsilon)!} \,. \tag{2.7}$$

The total number of states is given by

$$\Gamma = \prod_\epsilon \Delta G_\epsilon \,. \tag{2.8}$$

Since for a given energy spectrum $\{\Delta N_\epsilon\}$ all these states are equally probable the entropy is

$$S = \ln \Gamma = \sum_\epsilon \ln \Delta G_\epsilon \,. \tag{2.9}$$

Assuming that $\Delta N_\epsilon, \Delta g_\epsilon \gg 1$ and using Stirling's formula to approximate the factorials in (2.7) we finally get

$$S = \sum_\epsilon \Delta g_\epsilon \left[(n_\epsilon - 1) \ln(1 - n_\epsilon) - n_\epsilon \ln(n_\epsilon) \right] , \tag{2.10}$$

where $n_\epsilon = \Delta N_\epsilon / \Delta g_\epsilon$ is the occupation number. The entropy for a gas of fermions in nonequilibrium state can be calculated along the same lines, see [31]. The only difference is that for fermions the Pauli exclusion principle forbids two fermions from simultaneously occupying the same microstates. The general result for the entropy for a gas of bosons or fermions, respectively, is

$$S = \sum_\epsilon \Delta g_\epsilon \left[(n_\epsilon \mp 1) \ln(1 \mp n_\epsilon) - n_\epsilon \ln(n_\epsilon) \right] , \tag{2.11}$$

where $-$ is valid for bosons and $+$ for fermions. If the total number of particles $N = \sum \Delta g_\epsilon n_\epsilon$ and the energy $E = \sum \epsilon \Delta g_\epsilon n_\epsilon$ are conserved, the entropy (2.11) takes its maximum value for thermally distributed particles

$$n_\epsilon = \frac{1}{e^{(\epsilon - \mu)/T} \mp 1} \,, \tag{2.12}$$

where μ denotes the chemical potential of the particles.

If one considers an ideal gas of photons, which have zero mass and thus zero chemical potential, the energy and entropy can easily be calculated. The number of possible indistinguishable microstates for every photon with energy between ω and $\Delta\omega$ in one-particle phase space is equal to

$$\Delta g_\omega = 2 \int \frac{d^3x d^3k}{(2\pi)^3} \simeq \frac{1}{4\pi^3} V\omega^2 \Delta O \Delta\omega \ , \tag{2.13}$$

where the factor two accounts for the different possible polarizations of the photons. The total energy of the photon gas can be determined by

$$E = \frac{V\Delta O}{4\pi^3} \int d\omega \omega^3 n_\omega \ , \tag{2.14}$$

where n_ω is the occupation number of the photons. According to the above considerations the photons have maximal entropy if they have Planckian spectrum $n_\omega = \frac{1}{e^{\omega/T}-1}$. In this case their energy can be expressed in terms of volume V, solid angle ΔO and temperature T, see [26], which yields

$$E = \frac{\pi}{60} T^4 V \Delta O \ . \tag{2.15}$$

The entropy of the photon gas is equal to

$$S = \frac{4}{3}\frac{E}{T} \ . \tag{2.16}$$

This entropy characterizes the missing information about the exact microscopic state of a nonequilibrium gas of photons which can propagate only within the solid angle ΔO and have Planckian spectrum.

In the next section we will also need expressions for the entropy and energy for a gas of nonrelativistic particles. In this case the temperature is much smaller than the rest mass $\frac{m}{T} \gg 1$, and in addition $\frac{m-\mu}{T} \gg 1$. Hence, the spin-statistics does not play an essential role and the occupation number (2.12) becomes $n_m = e^{-(m-\mu)/T}$. For nonrelativistic particles with nonvanishing chemical potential we can neglect the antiparticle density as it is suppressed by $e^{-\frac{2\mu}{T}}$.

In the leading order the integrals for the energy density $\epsilon = E/V$, particle density $n = N/V$, and pressure p can be evaluated up to corrections of

$\mathcal{O}\left(1/(\frac{m}{T})^2\right)$, see [31],

$$\begin{aligned} \epsilon &= \frac{gT(Tm)^{3/2}\Delta O}{2^{7/2}\pi^{5/2}} e^{\frac{\mu-m}{T}} \left(\frac{m}{T} + \frac{27}{8} + \frac{45}{8}\frac{T}{m}\right), &(2.17)\\ n &= \frac{g(Tm)^{3/2}\Delta O}{2^{7/2}\pi^{5/2}} e^{\frac{\mu-m}{T}} \left(1 + \frac{15}{8}\frac{T}{m}\right), &(2.18)\\ p &= \frac{gT(Tm)^{3/2}\Delta O}{2^{7/2}\pi^{5/2}} e^{\frac{\mu-m}{T}} \left(1 + \frac{15}{8}\frac{T}{m}\right). &(2.19) \end{aligned}$$

In terms of the particle density n the above expressions can be written as

$$\begin{aligned} \epsilon &\cong \left(\frac{m}{T} + \frac{3}{2}\right) nT\,, &(2.20)\\ p &\cong nT\,. &(2.21) \end{aligned}$$

Then the entropy density $s = \frac{\epsilon+p-\mu n}{T}$ takes the form

$$s \cong \left(\frac{m-\mu}{T} + \frac{5}{2}\right) n\,. \tag{2.22}$$

Equipped with various formulae for the thermodynamic quantities we now turn to the question of the origin of black hole entropy.

2.3 Statistical interpretation of black hole entropy

2.3.1 Remarks on the entropy of Hawking radiation

Hawking radiation emitted in empty space is far from equilibrium. Nevertheless it possesses entropy and has Planckian spectrum. Hawking radiation has temperature

$$T_H = \frac{1}{8\pi M}\,. \tag{2.23}$$

If we neglect the influence of the potential barrier of the black hole, the entropy of emitted radiation is given by (2.16). When the black hole emits energy ΔM it carries the entropy

$$\Delta S_H = \frac{4}{3}\frac{\Delta M}{T_H}\,. \tag{2.24}$$

At the same time the entropy of the black hole decreases by the amount

$$\Delta S_{BH} = -\frac{\Delta M}{T_H} \,. \tag{2.25}$$

Hence, while the black hole evaporates the total entropy of the system consisting of Hawking radiation and black hole increases by

$$\Delta S = \Delta S_H + \Delta S_{BH} = \frac{1}{3}\frac{\Delta M}{T_H} \,. \tag{2.26}$$

As it is clear from (2.16) the entropy of the emitted radiation does not change as long as the radiation propagates in the space since its total energy and temperature remain the same. At first glance this seems to contradict the fact that the total volume occupied by the Hawking quanta, emitted for instance within the time interval Δt, increases as $V \simeq 4\pi t^2 \Delta t$ for big t. However, the indeterminacy of the concrete photon propagation direction characterized by the solid angle

$$\Delta O \simeq 4\pi \frac{\sigma_{BH}}{4\pi t^2} \,, \tag{2.27}$$

decreases with time, and $V\Delta O$ in (2.15) remains constant which is in complete agreement with the fact that the temperature does not change. Here $\sigma_{BH} \sim M^2$ is the classical effective cross-section of the black hole.

In this consideration we neglect the grey factor in the Hawking spectrum. However, if one takes it into account, the main conclusion that the total entropy increases remains unchanged.

2.3.2 Black hole entropy

The idea of a statistical interpretation of black hole entropy is rather simple and can roughly be formulated as follows. Take the emitted Hawking radiation and reverse it forming a new black hole. Then the entropy of this black hole should be about the entropy of the Hawking radiation. For simplicity we consider photon radiation in the following, but the same arguments also hold for ultrarelativistic bosons and fermions. In the next section our considerations will be generalized to the case of nonrelativistic particles. In the calculations we neglect the influence of the potential barrier of the black hole on the particle spectrum.

Actually, if we want to build a black hole from photons with total energy M we first put them into a box of volume $V \sim D^3$, where D is the typical size of the box and then collect these photons inside the volume M^3. How this might be done realistically is not important for the general question of the origin of the entropy. Could we then identify the maximal possible entropy of these photons with the entropy of the formed black hole? As mentioned above, the entropy of the photons has its maximal value when the photons have thermal spectrum with temperature

$$T \sim M^{1/4} D^{-3/4} (\Delta O)^{-1/4} \,. \tag{2.28}$$

Then the entropy is

$$S \sim M/T \sim M^{3/4} D^{3/4} (\Delta O)^{1/4} \,. \tag{2.29}$$

If one takes the box size D to be of the order of the black hole size M and $\Delta O \sim \mathcal{O}(1)$, then the entropy S is approximately $M^{3/2}$ which is too small to explain the black hole entropy $S_{BH} \sim M^2$. On the other hand, if the box is very large, $D \to \infty$, then S tends to infinity, too. The size of the box will be restricted by the life time of the black hole t_H, otherwise the black hole which was already formed would radiate away before we have put all matter inside. Therefore we take $D \sim t_H$, which for a black hole of mass M is $t_H \sim M^3$. With this value of D the entropy is $S \sim M^3 (\Delta O)^{\frac{1}{4}}$. Moreover, in order to form the black hole the propagation of the photons will not be completely arbitrary. Most of the photons are located near the border of the box. If the black hole is formed in the centre of the box the photons have to be directed in such a way to arrive in a small region of size M in the centre of the box. This restricts the solid angle ΔO for most of them by

$$\Delta O \sim \frac{M^2}{D^2} \sim \frac{1}{M^4} \,. \tag{2.30}$$

Putting together the above considerations we get the correct estimate for the entropy of the black hole,

$$S \sim M^2. \tag{2.31}$$

The system of mirrors which one could use to redirect the photons complicates the consideration but leaves the conclusion unchanged. Thus this approach opens the way to understand the statistical origin of black hole entropy as a result of lost information about the exact microscopical initial state of the matter from which the black hole was formed.

This interpretation of black hole entropy does not depend on the number of fields. In the case of N massless fields the total energy increases by a factor N,

$$M \sim N T^4 D^3 \Delta O. \tag{2.32}$$

On the other hand, the rate of evaporation of the black hole is proportional to the number of fields. Thus the life time of the black hole and the size of the box, respectively, should be N times smaller. Taking into account that $\Delta O \sim \frac{M^2}{D^2}$ we see that the number of fields N is cancelled in (2.32). Therefore temperature and entropy, $S \sim M/T$, should not depend on the number of fields.

Now we address the question about the exact numerical coefficient in the formula for the entropy. As already mentioned in the above approach we have just reversed the Hawking flux and explained the black hole entropy via the entropy of the Hawking radiation itself. Therefore the result might be 4/3 times bigger than it should be. However, the real situation is more complicated. To get a more precise picture of what is going on let us find how much entropy matter can contribute to the black hole. Let us consider an already existing black hole of mass M and the amount of matter ΔE. We then address the question how much entropy this matter can "add to the black hole", taking into account that during the process of absorption the black hole continues to evaporate losing mass and respectively entropy. During the time Δt when the matter ΔE is absorbed, the black hole also emits Hawking radiation with luminosity L_H, so the mass of the black hole only increases by ΔM, given by

$$\Delta M = \Delta E - L_H \Delta t \, . \tag{2.33}$$

The photons emitted by the black hole fill a shell of width Δt, the radius of which grows with time. The energy of the Hawking radiation emitted during Δt is easily determined. The photons have temperature T_H and the total amount of energy they carry can be calculated as

$$L_H \Delta t = \frac{\pi}{60} T_H^4 V \Delta O = \frac{\pi^2}{15} \sigma_{BH} T_H^4 \Delta t \, , \tag{2.34}$$

where we took into account $V = 4\pi t^2 \Delta t$ and used the formula (2.27) for the solid angle. Since we assume that the photons which are responsible for the mass increase will add maximal entropy to the black hole we find

$$\Delta M = \frac{\pi^2}{15} \sigma_{BH} T_\star^4 \Delta t \, , \tag{2.35}$$

where $T_\star$ is the temperature of the photons. Given ΔE we get from (2.35)

$$\Delta t = \left(\frac{\pi^2}{15} \sigma_{BH} \right)^{-1} \frac{\Delta E}{T_\star^4 + T_H^4} \, . \tag{2.36}$$

Accordingly, the amount of entropy which matter adds to the black hole is

$$\Delta S = \frac{4}{3}\frac{\Delta M}{T_\star} = \frac{4}{3}\frac{T_\star^3}{T_\star^4 + T_H^4}\Delta E\,. \qquad (2.37)$$

Keeping ΔE fixed, we maximize ΔS with respect to the temperature $T_\star$ and find that it takes its maximal value at

$$T_\star = 3^{1/4} T_H\,. \qquad (2.38)$$

Therefore the maximal amount of entropy matter can contribute when the mass of the black hole increases by ΔM is

$$\Delta S = \frac{4}{3^{5/4}}\frac{\Delta M}{T_H} \simeq 1.0131\frac{\Delta M}{T_H}\,. \qquad (2.39)$$

We compare this result to the change of the black hole entropy (2.25). There is only a mismatch of one percent, which is probably due to the neglected influence of the black hole potential barrier in the calculation. The exact calculations in this case are rather complicated. However, one can easily see that including the grey factor decreases the coefficient in (2.39), that is, at least acts in the correct direction.

2.3.3 Nonrelativistic particles

Now we apply the considerations concerning black hole entropy to nonrelativistic particles. We show that our statistical explanation for the origin of black hole entropy can also be applied in this case. During evaporation the black hole also emits massive particles, but the probability is exponentially suppressed by $e^{-m/T}$, i.e. by the mass m of the emitted particles. Hence this process becomes important only if the temperature of the black hole is much bigger than the electron rest mass, $T_H \gg m_e$. If we artificially separate the emission of massless and massive particles and assume that the black hole evaporates only by emitting nonrelativistic particles with mass m, the life time of the black hole will certainly be very big,

$$t_H \sim e^{8\pi M m}. \qquad (2.40)$$

The following considerations are very similar to the case of photons discussed before, so we will just review the basic steps. Let us assume an ensemble of nonrelativistic particles of mass m. To build a black hole of mass M from those particles first we place the total number of particles, $N = M/m$, inside a box with the volume $V = D^3$ and then collect them inside M^3. The entropy of the particles reaches its maximal value for thermally distributed particles, so the particle density is given by (2.18). The total number of particles N can be expressed in terms of the particle density n and the occupied volume

$$N = n \cdot D^3 = g\Delta O D^3 (mT)^{3/2} e^{\frac{\mu - m}{T}}, \tag{2.41}$$

where g is the number of possible polarizations, T is the particle temperature and μ their chemical potential. Since the size of the box is restricted by the black hole life time t_H we assume

$$D = t_H v\,, \tag{2.42}$$

where $v := \overline{|\vec{v}|} = \sqrt{\frac{8T}{\pi m}}$ is the particles' mean thermal velocity. If the black hole is formed in the center of the box the particles' direction of propagation cannot be completely arbitrary. As for the photons the solid angle for most of them will be restricted to $\Delta O \sim \frac{M^2}{D^2}$, which yields

$$N = g(mT)^{3/2} M^2 e^{8\pi M m} e^{\frac{\mu - m}{T}}. \tag{2.43}$$

Solving this equation for $\frac{m-\mu}{T}$ implies

$$\frac{m - \mu}{T} \simeq 8\pi M m + \mathcal{O}\left(\ln\left(m/T\right)\right). \tag{2.44}$$

The logarithmic corrections can be neglected, so the entropy that the non-relativistic particles can add to the black hole is

$$S \sim \frac{m - \mu}{T} N \sim Mm \cdot \frac{M}{m} \sim M^2. \tag{2.45}$$

Thus black hole entropy can be interpreted as a result from lost information about the exact microscopical initial state of the matter from which the black hole was formed, even if we consider a black hole built from nonrelativistic particles, instead of photon radiation considered before.

Our next step is to determine the exact coefficient for the formula of black hole entropy. Consider an existing black hole of mass M and a given amount of matter ΔE, which will be absorbed by the black hole. During the process of absorption the black hole will continue to evaporate. When the matter ΔE is absorbed by the black hole during the time interval Δt, during this time the black hole will emit the energy $L_H \Delta t$, where L_H is the Hawking flux due to the emission of nonrelativistic particles, see also (2.33).

The emitted particles with mass m carry the energy

$$L_H \Delta t = 2n(m, T_H) \cdot mV = \frac{gm(mT_H)^{3/2} V_{T_H} \Delta O_{T_H}}{2^{5/2}\pi^{5/2}} e^{-\frac{m}{T_H}}. \tag{2.46}$$

Here the factor two arises since for a nonrotating black hole the Hawking particles have zero chemical potential and both particles and antiparticles contribute to the energy in the same amount. We write $V_T \Delta O_T$ with subscript T to indicate that these quantities also depend on the temperature. The particles which are responsible for the black hole mass increase ΔM in general will have nonzero chemical potential μ, their temperature will be denoted by $T_\star$. Since $\mu \neq 0$ the number of antiparticles is exponentially suppressed by the factor $e^{-\frac{2\mu}{T_\star}}$. The particles can add maximal entropy to the black hole if they have thermal distribution, and we write

$$\Delta M = n(m, T_\star) m \cdot V = \frac{gm(T_\star m)^{3/2} V_{T_\star} \Delta O_{T_\star}}{2^{7/2} \pi^{5/2}} e^{\frac{\mu - m}{T_\star}} . \tag{2.47}$$

Now we have to consider $V_T \Delta O_T$ in detail. For big t the volume occupied by the Hawking particles changes as

$$V_T = 4\pi r^2 \Delta r = 4\pi v^3 t^2 \Delta t \,, \tag{2.48}$$

where the particles move with constant mean velocity v. Strictly speaking the particles move with the mean radial velocity $\overline{v_r}$, but since we only consider particles very far away from the black hole, which means large t, the indeterminacy of the concrete direction of the particles, characterized by the solid angle

$$\Delta O_T = 4\pi \frac{\sigma_{BH}}{4\pi r^2} \,, \tag{2.49}$$

is very small. Hence, $\overline{v_r}$ can be replaced by the mean thermal velocity of particles

$$\overline{v_r} \simeq v = \sqrt{\frac{8T}{\pi m}} \,. \tag{2.50}$$

The quantity $V \Delta O = 4\pi \sigma_{BH} v \Delta t$ does not depend on time t since the mean velocity v is a function of temperature and particle mass only. Now (2.33) can be solved for

$$\Delta t = \left(\frac{gm^2 \sigma_{BH}}{\pi^2} \right)^{-1} \frac{\Delta E}{T_\star^2 e^{\frac{\mu - m}{T_\star}} + 2 T_H^2 e^{-\frac{m}{T_H}}} \,, \tag{2.51}$$

where T_H is the temperature of the black hole and $T_\star$ the temperature assigned to the particles. These contribute to the black hole mass change by

$$\Delta M = \Delta E \frac{T_\star^2 e^{\frac{\mu - m}{T_\star}}}{T_\star^2 e^{\frac{\mu - m}{T_\star}} + 2 T_H^2 e^{-\frac{m}{T_H}}} \,. \tag{2.52}$$

The increase of the entropy due to nonrelativistic particles is

$$\Delta S \simeq \frac{\Delta M}{T_\star}\left(1 - \frac{\mu}{m} + \mathcal{O}\left((m/T_\star)\right)^{-1}\right) . \tag{2.53}$$

Substituting (2.52) yields

$$\Delta S \simeq \frac{\Delta M}{T_\star}\Big(1 - \frac{\mu}{m}\Big) = \Delta E \frac{(1-\frac{\mu}{m})T_\star e^{\frac{\mu-m}{T_\star}}}{T_\star^2 e^{\frac{\mu-m}{T_\star}} + 2T_H^2 e^{-\frac{m}{T_H}}} , \tag{2.54}$$

which can be maximized with respect to the chemical potential μ. It takes its maximal value for

$$\frac{m-\mu}{T_\star} \cong \frac{m}{T_H} - \ln\left(2\frac{m}{T_H}\right) , \tag{2.55}$$

up to higher order logarithmic corrections. The maximal amount of entropy matter can contribute to the black hole is

$$\Delta S \simeq \frac{\Delta M}{T_H} . \tag{2.56}$$

One might wonder why this result is in such good agreement with the expected change of entropy, even though we neglected the influence of the grey factor in our calculation. This can be explained easily, having a brief look at the scattering problem for massive scalar fields. As shown in appendix B.1.2 for nonrelativistic particles, $mM \gg 1$, the four-dimensional potential barrier coincides with the two-dimensional one and the reflection of particles in this case is exponentially suppressed. Thus the potential barrier of the black hole should be of no relevance for nonrelativistic particles.

Note that in principle ΔS can also be maximized with respect to the temperature $T_\star$ instead of the chemical potential, which is why we can also apply the above arguments for particles with $\mu = 0$. The only difference is that in the case of vanishing chemical potential the antiparticle density is equal to the particle density. We have

$$\Delta M = 2n(m, T_\star) \cdot mV . \tag{2.57}$$

The resulting temperature of the particles $T_\star$ will be equal to the Hawking temperature T_H. Nevertheless the black hole mass increases by

$$\Delta M = \frac{1}{2}\Delta E . \tag{2.58}$$

At first sight this result seems to contradict the well-known fact that there is no mass change for a black hole in equilibrium with a thermal bath, $T = T_H$.

However, we should not forget the meaning of $T_\star$, which is the temperature assigned to the particles contributing to the effective mass change ΔM of the black hole. The energy thrown in was taken arbitrarily and did not necessarily possess thermal spectrum. In the case of $\mu = 0$, if one afterwards assigns a temperature to ΔE, it is bigger than the Hawking temperature T_H.

Chapter 3

Stability of Hawking radiation

The Hawking radiation measured by an observer at rest at infinity has thermal distribution with a temperature inversely proportional to the black hole mass. This is valid for eternal black holes as well as for the more realistic case when a black hole is formed during gravitational collapse [20]. In the latter case, due to the formation of a horizon, the initial state might not be the vacuum. The question arises whether the presence of particles in the initial state is visible in the Hawking flux at infinity. Hawking argued that any finite number of particles do not change the radiation spectrum as they suffer an infinite red shift and thus "never" reach the observer at infinity. This was confirmed by [46], see also [8] and references therein.

The Hawking effect might still depend on other parameters. For example, the particle spectrum might be sensitive to physics beyond the Planck scale, as discussed in [10, 55]. The contributing outgoing modes originate from modes with extremely large wave numbers (due to the exponential red shift near the horizon). Thus changes in the dispersion relation in the Planckian regime might be visible in the particle spectrum. It has been shown in [56] that under certain conditions the Hawking effect is insensitive to changes in the dispersion relation. However, the authors also present examples where there will be strong deviations from Hawking's result.

In the following we address the question whether the vacuum definition of different observers has any influence on the Hawking spectrum. In the maximally extended Kruskal space-time the initial vacuum state is usually defined by the boundary conditions on the past event horizon H_-, where the outgoing modes can be defined using the Killing vector ∂_U. The corresponding vacuum is called the Unruh vacuum [54] and is a good approximation to the more realistic case of graviational collapse since the surface H_- resembles the last outgoing light ray γ in the collapsing model, see [9, 12, 16, 17, 47], as well as Fig. 1.4. However, the maximally extended Kruskal manifold is a quite unphysical scenario, since it contains two eternal black and two white holes. As shown in chapter 1.3, in the derivation of the Hawking effect we only consider the exterior region of the black hole, corresponding to region I in Fig. 1.2. The initial vacuum is defined using proper time coordinates in the vicinity, but outside the event horizon H_+. Then the Kruskal coordinates are just one possible choice of coordinates which are regular on the horizon, and we can transformation to other coordinates. Besides, since the Kruskal coordinates define a freely falling frame only at one particular moment on the horizon, they are not special from this point of view either. The vacuum states for the outgoing modes for each new coordinate frame is then defined by means of the corresponding Killing vector $\partial_{\tilde{U}}$. In general the resulting vacuum states will not be equivalent since the Kruskal vacuum is invariant only under Lorentz transformations.

In the following we show how to compute the particle spectrum for quite arbitrary coordinate transformations. Since the Bogolyubov coefficients often cannot be determined explicitly we will compute the particle density in a different way. First we study the influence of Lorentz transformations on the Hawking spectrum. After that we discuss the presence of particles in the initial state. Finally we consider different observers on the horizon and calculate the corresponding particle flux measured at infinity. We will show that this choice of initial vacuum does not change the Hawking spectrum.

3.1 Hawking radiation

In section 1.3 we have determined the particle flux from the black hole using the explicit form of the Bogolyubov coefficients (1.44). The total number of emitted particles in the mode Ω has been calculated from

$$\langle \hat{N}_\Omega \rangle = \langle 0_K | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_K \rangle = \int_0^\infty d\omega |\beta_{\omega\Omega}|^2 \, . \tag{3.1}$$

Afterwards we have been able to separate the particle density n_Ω from this diverging quantity,

$$\langle \hat{N}_\Omega \rangle = n_\Omega \delta(0) \, . \tag{3.2}$$

It will prove convenient for us to write (3.1) in a different way,

$$\langle \hat{N}_\Omega \rangle = \lim_{\Omega' \to \Omega} N(\Omega, \Omega') \, , \tag{3.3}$$

where we introduce

$$N(\Omega, \Omega') = \int_0^\infty \frac{d\omega}{\omega} \sqrt{\Omega\Omega'} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u} e^{+i\omega U(u)} \int_{-\infty}^\infty \frac{du'}{2\pi} e^{-i\Omega' u'} e^{-i\omega U(u')} . \tag{3.4}$$

We may as well compute the regularized energy momentum tensor using the method of point splitting. In two dimensions the contribution to the energy momentum tensor is

$$T_{uu,reg} = \frac{1}{2\pi} \lim_{u' \to u} \int_0^\infty d\Omega \int_0^\infty d\Omega' \sqrt{\Omega\Omega'} \Re\Big(e^{-i\Omega u} e^{i\Omega u'} N(\Omega, \Omega')\Big). \tag{3.5}$$

The above expression can be verified using the completeness relation of the Bogolyubov coefficients (A.35). Other possible terms in the energy momentum tensor are proportional to $\delta(\Omega + \Omega')$ and do not contribute.

The computation of the Bogolyubov coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ depends crucially on the coordinate transformation $U = U(u)$. In the case of the transformation between the Schwarzschild and the Kruskal coordinates (1.8) we have already determined the particle spectrum in section 1.3.2. For arbitrary coordinate transformations $U = U(u)$ it is impossible to calculate the Bogolyubov coefficients explicitly. Therefore we calculate the particle flux in a different way.

We assume that the order of integration in (3.4) can be reversed. Then we need to solve the integral

$$K(x) = \int_0^\infty \frac{d\omega}{\omega} e^{i\omega x} , \tag{3.6}$$

which has an infrared (IR) and ultraviolet (UV) divergence, the latter when $x = 0$. We use a smoothing function $e^{-\alpha\omega} \tanh(\beta\omega)$, which introduces a cutoff in the IR region at β^{-1} and an UV cutoff at α^{-1}. After the calculation we let $\alpha \to 0$ and $\beta \to \infty$. According to appendix B.2.1, that yields

$$K(x) = \ln\beta + \frac{i\pi}{2}\mathrm{sgn}(x) - \ln|x| . \tag{3.7}$$

This result can be used to determine (3.4). We get

$$\begin{aligned} N(\Omega, \Omega') \quad &= \quad \sqrt{\Omega\Omega'} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u} \int_{-\infty}^\infty \frac{du'}{2\pi} e^{i\Omega' u'} \\ &\quad \times \left[\ln\beta + \frac{i\pi}{2}\mathrm{sgn}(U - U') - \ln|U - U'|\right] , \end{aligned} \tag{3.8}$$

which is valid for general coordinate transformations $U(u)$. The first two terms of (3.8) do not depend on the special relation between the coordinates U and u, but for the computation of the third part we need the explicit transformation $U(u)$. The first term in (3.8) can be easily evaluated

$$N_1 = \ln(\beta)\sqrt{\Omega\Omega'}\delta(\Omega)\delta(\Omega') , \tag{3.9}$$

which gives us a zero mode contribution which is zero for finite β and can be ignored for $\beta \to \infty$. The second term simplifies since $\mathrm{sgn}(U-U') = \mathrm{sgn}(u-u')$. Introducing the new coordinates

$$\begin{aligned} x &= u - u' \,, \\ y &= u + u' \,, \end{aligned} \tag{3.10}$$

the integral takes the form

$$\begin{aligned} N_2 &= \frac{i\pi}{4}\sqrt{\Omega\Omega'}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{\frac{1}{2}i(\Omega-\Omega')y}\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{\frac{1}{2}i(\Omega+\Omega')x}\mathrm{sgn}(x) \\ &= -\pi\Omega\delta(\Omega-\Omega')\int_0^{\infty}\frac{dx}{2\pi}\sin\Omega x \\ &= -\frac{1}{2}\delta(\Omega-\Omega')\,. \end{aligned} \tag{3.11}$$

We note that this contribution is negative and interpret it as accounting for the zero point occupancies of the various modes of the field.

To determine the last term in (3.8) we need to know the function $U(u)$. As a first example we use the usual Kruskal coordinates on the horizon, see (1.8). Introducing the new coordinate $x = -U$, we have

$$N_3 = -\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^{\infty}\frac{dx}{2\pi\kappa}\int_0^{\infty}\frac{dx'}{2\pi\kappa}x^{-1-i\Omega/\kappa}x'^{-1+i\Omega'/\kappa}\ln|x-x'|. \tag{3.12}$$

It turns out to be more convenient to introduce yet other coordinates of integration,

$$\begin{aligned} x' &= \frac{s}{t}\,, \\ x &= st\,. \end{aligned} \tag{3.13}$$

We find that the contribution to the total number of particles is

$$\begin{aligned} N_3 &= -2\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^{\infty}\frac{ds}{2\pi\kappa}\int_0^{\infty}\frac{dt}{2\pi\kappa} \\ &\quad \times s^{-1-i(\Omega-\Omega')/\kappa}t^{-1-i(\Omega+\Omega')/\kappa}\ln|1/t-t| \\ &= \frac{1}{2}\coth\left(\frac{\pi\Omega}{a}\right)\delta(\Omega-\Omega')\,. \end{aligned} \tag{3.14}$$

Putting together the contributions N_2 and N_3 we have

$$N(\Omega,\Omega') = \frac{1}{2}\left[\coth\left(\frac{\pi\Omega}{\kappa}\right)-1\right]\delta(\Omega-\Omega') = \frac{1}{e^{2\pi\Omega/\kappa}-1}\delta(\Omega-\Omega')\,. \tag{3.15}$$

From that we can easily extract total number of particles in the limit $\Omega \to \Omega'$. This yields

$$\langle \hat{N}_\Omega \rangle = \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{1}{e^{2\pi\Omega/\kappa} - 1} , \tag{3.16}$$

where the integral over u corresponds to the diverging δ-function. The above expression clarifies the infinite nature of the total number of particles as emerging as a result of an integral over an infinite time interval. The particle density itself is a time independent quantity,

$$n_\Omega = \frac{1}{e^{2\pi\Omega/\kappa} - 1} , \tag{3.17}$$

which gives us the expected result, identical to (1.47). Thus the particle flux from the black hole has the expected thermal spectrum with the temperature

$$T = \frac{\kappa}{2\pi} . \tag{3.18}$$

3.2 Defining vacuum and choice of observer

As mentioned earlier, the initial vacuum in the maximally extended Kruskal space-time is defined by the boundary conditions on the past event horizon, where ∂_U is a Killing vector. If we now work in the Schwarzschild space-time and choose different coordinates by a suitable transformation $U \to \tilde{U}(U)$, the initial vacuum state with respect to the new observer can be defined along the surface where $\partial_{\tilde{U}}$ is a Killing vector. One special type of coordinate transformations are Lorentz transformations. The observers move with constant velocity with respect to each other, but still they agree on their definition of the vacuum state.

In the following we study the particle numbers for different scenarios in detail and discuss possible effects on the Hawking spectrum.

3.2.1 Moving observers

We examine the influence of Lorentz transformations between observers on the Hawking spectrum. First we assume that an observer at spatial infinity, who examines the particle emission of a black hole, moves with constant velocity with respect to the Schwarzschild observer, who is at rest at infinity. As his distance from the black hole is infinite, he does not feel any gravitational acceleration. We obtain the coordinate system of the moving observer using the Lorentz transformation between the Schwarzschild coordinates and the new coordinate frame. If the observer's proper velocity is given by $\mathbf{v}$, then the Lorentz transformation between the two frames is

$$\begin{aligned} t' &= \frac{t - \mathbf{v}r}{\sqrt{1-\mathbf{v}^2}} \, , \\ r' &= \frac{r - \mathbf{v}t}{\sqrt{1-\mathbf{v}^2}} \, . \end{aligned} \tag{3.19}$$

Alternatively, we can rewrite the transformation (3.19) in terms of light cone coordinates

$$\begin{aligned} u' &= \frac{1+\mathbf{v}}{\sqrt{1-\mathbf{v}^2}} u \;=\; \alpha u \, , \\ v' &= \frac{1-\mathbf{v}}{\sqrt{1-\mathbf{v}^2}} v \;=\; \frac{v}{\alpha} \, , \end{aligned} \tag{3.20}$$

where α denotes the Doppler factor. If the observer moves towards the black hole, then $\mathbf{v}$ is positive, otherwise $\mathbf{v}$ is negative. The particle density can be calculated as shown in section 3.1. It follows that the moving observer measures a thermal spectrum with the Doppler shifted temperature

$$T'_H = \frac{\kappa}{2\pi\alpha} \, . \tag{3.21}$$

In terms of $\mathbf{v} = \tanh(\beta)$ we determine the Doppler factor and get

$$\alpha = \sinh(\beta) + \cosh(\beta) = e^{\beta}. \tag{3.22}$$

The metric in light cone coordinates remains unchanged under the above coordinate transformation

$$ds^2 = du dv = du \alpha \frac{dv}{\alpha} = du' dv'. \tag{3.23}$$

Besides, we note that if the Doppler-factor is equal to the surface curvature $\alpha = 2r_s$ the temperature measured by the moving observer is equal to the Planckian temperature.

Now we consider an observer on the horizon, moving with constant velocity $\tilde{\mathbf{v}}$ with respect to the Kruskal observer. According to (3.20), the appropriate Lorentz transformation is

$$\begin{aligned} U' &= \frac{1+\tilde{\mathbf{v}}}{\sqrt{1-\tilde{\mathbf{v}}^2}} U = \tilde{\alpha} U \, , \\ V' &= \frac{1-\tilde{\mathbf{v}}}{\sqrt{1-\tilde{\mathbf{v}}^2}} V = \frac{V}{\tilde{\alpha}} \, . \end{aligned} \tag{3.24}$$

When we calculate the particle flux measured by an observer at rest at infinity, the particle density and the Hawking temperature remain unchanged.

We have seen that linear coordinate transformations $\tilde{U}(U)$ will not change the particle spectrum measured at infinity. On the other hand, an observer at infinity moving with constant velocity will measure a Doppler shifted temperature. At first sight this seems to contradict the fact that this observer has the same vacuum state as an observer at rest at infinity. However, the Doppler shifted temperature can easily be explained if we bear in mind that the invariant quantity in the thermal spectrum is $\frac{\omega}{T}$. Thus the shift in the particle frequencies compensates the Doppler shifted temperature.

3.2.2 Kruskal particles on the horizon

In the previous calculations we studied the particle spectrum for an eternal black hole where the regular state on the horizon is taken to be the initial vacuum. However, the initial state on the horizon might be occupied if the black hole was formed by a gravitational collapse. In the following we show that a finite number of particles on the horizon does not change the particle spectrum measured at infinity.

For simplicity we consider the one-particle state defined as

$$|1_{\omega_1}\rangle = \sqrt{\omega_1}\hat{a}^{\dagger}_{\omega_1}|0\rangle \; . \tag{3.25}$$

The results obtained for this case can be easily generalized to n particles. The factor proportional to the particle energy ensures Lorentz invariance of the state. Note that the vacuum state is normalized as

$$\langle 0|0\rangle = 1 \; , \tag{3.26}$$

whereas the one-particle state is normalized according to

$$\langle 1_\omega|1_{\omega'}\rangle = \omega\delta(\omega-\omega') \; . \tag{3.27}$$

We calculate the average particle number in the one-particle state

$$\begin{aligned} \langle\hat{N}_\Omega\rangle &= \frac{\langle 1_{\omega_1}|\hat{b}^+_\Omega\hat{b}^-_\Omega|1_{\omega_1}\rangle}{\langle 1_{\omega_1}|1_{\omega_1}\rangle} \\ &= \int_0^\infty\!\!\int_0^\infty \frac{d\omega d\omega'}{\langle 1_{\omega_1}|1_{\omega_1}\rangle}\left(\alpha^\star_{\omega\Omega}\alpha_{\omega'\Omega}\langle 1_{\omega_1}|\hat{a}^+_\omega\hat{a}^-_{\omega'}|1_{\omega_1}\rangle + \beta^\star_{\omega\Omega}\beta_{\omega'\Omega}\langle 1_{\omega_1}|\hat{a}^-_\omega\hat{a}^+_{\omega'}|1_{\omega_1}\rangle\right) \\ &= \int_0^\infty d\omega|\beta_{\omega\Omega}|^2 + \frac{\omega_1}{\langle 1_{\omega_1}|1_{\omega_1}\rangle}\left(|\alpha_{\omega_1\Omega}|^2+|\beta_{\omega_1\Omega}|^2\right) \\ &= \langle\hat{N}_\Omega\rangle_0 + \delta N_\Omega \; , \end{aligned} \tag{3.28}$$

where $\langle \hat{N}_\Omega \rangle_0$ is the total number of particles per unit energy, evaluated from Kruskal vacuum, and δN_Ω denotes the change of the particle number. According to (1.46) these are

$$\langle \hat{N}_\Omega \rangle_0 = \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{1}{e^{2\pi\Omega/\kappa} - 1} \tag{3.29}$$

and

$$\delta N_\Omega = \frac{1}{2\pi a \langle 1_{\omega_1} | 1_{\omega_1} \rangle} \coth(\pi\Omega/a) \,. \tag{3.30}$$

As expected, the total number of particles $\langle \hat{N}_\Omega \rangle_0$ is divergent, see also section 3.1. The change in the particle number is a finite quantity and does not contribute to the particle density. If we separate the divergent volume factor $L \sim \delta(0)$ from (3.30), we immediately see that the change in the particle density is suppressed by the factor L^{-2}. Thus we conclude that the particle density measured by an observer at infinity remains unchanged.

3.2.3 Considering different freely falling observers

We have seen that neither Lorentz transformations between observers on the horizon nor the presence of a finite number of particles in the horizon state change the Hawking spectrum at infinity. Now we address the question whether the general choice of vacuum on the horizon affects the Hawking flux at infinity. Corresponding to each coordinate transformation $U \to \tilde{U}$ the vacuum can be defined along the new Killing surface $\partial_{\tilde{U}}$. One possibility is to consider freely falling observers which cross the horizon at some moment $V = V_0$. It can be easily seen that the Kruskal observer is freely falling along $U = V$ and crosses the horizon at $V = 0$. The nonvanishing Christoffel symbols of the Kruskal metric (1.10) are

$$\Gamma^U_{UU} = \frac{f_{,U}}{f} \quad \text{and} \quad \Gamma^V_{VV} = \frac{f_{,V}}{f} \,. \tag{3.31}$$

On the horizon, when $U = 0$ and $V \neq 0$, the Γ^V_{VV} component vanishes but Γ^U_{UU} tends to a constant

$$\Gamma^U_{UU} = \frac{1}{2} \frac{r + r_s}{r^2} e^{1 - \frac{r_s}{r}} e^{\frac{v}{2r_s}} \to \frac{1}{2r_s} e^{\frac{v(r_s)}{r_s}} = \frac{1}{2r_s^2} V \,. \tag{3.32}$$

Hence Γ^U_{UU} vanishes only at $U = V = 0$.

We can always find a freely falling observer who crosses the horizon at some $V = V_0$ by a transformation, such that a quadratic term in the original

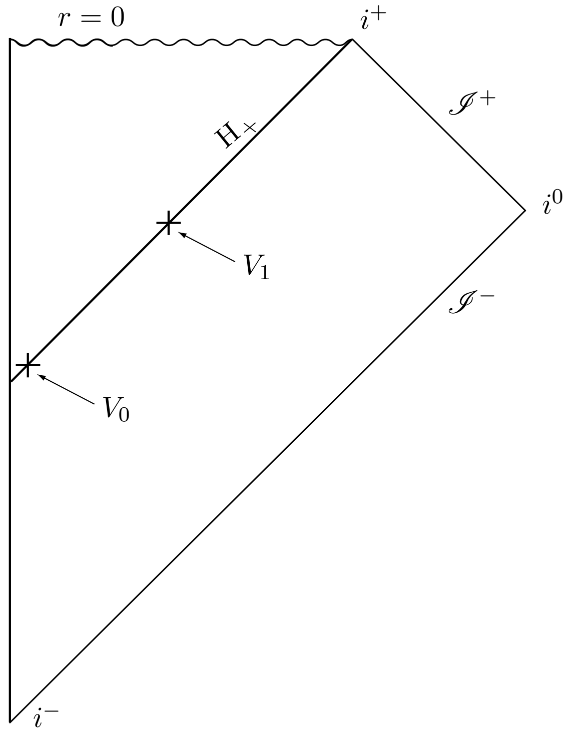

Figure 3.1: Possible choices of initial vacuum in the vicinity of the horizon, for example at some moment $V_0 = 2r_s^2\Gamma^U_{UU}|_{V=V_0}$.

coordinate U cancels the value of the Christoffel symbol on the horizon, that is

$$\tilde{U} = U + \frac{1}{2}\alpha U^2 + \dots \,, \tag{3.33}$$

where $\alpha = V_0/2r_s^2$ is the value of the Christoffel symbol Γ^U_{UU} at V_0. The dots represent a whole class of inertial observers, which differ by arbitrary higher order terms in the transformation. For the definition of a new observer we also have to change the coordinate V correspondingly, but since these modes are not important for the computation of the total number of particles we skip this part here. One can easily check that in the new coordinates the metric still has Minkowski form on the horizon and all Christoffel symbols vanish at $U = 0$ and $V = 2\alpha r_s^2$.

The coordinate transformation $\tilde{U}(U)$ has to be unambiguous. Therefore it is not sufficient to use the quadratic transformation, as it maps different values of U to the same $\tilde{U}$. Possible coordinate transformations are for examples

$$\tilde{U} = \begin{cases} -\frac{1}{\alpha}\ln(1-\alpha U) \\ \frac{1}{\alpha}(e^{\alpha U}-1) \\ \frac{U}{1-\frac{1}{2}\alpha U} \end{cases} , \tag{3.34}$$

which map $U \in [0,-\infty)$ to $\tilde{U} \in \{[0,-\infty),[0,-1/\alpha),[0,-2/\alpha)\}$, respectively. The inverse transformations are given by

$$U = \begin{cases} -\frac{1}{\alpha}(e^{-\alpha\tilde{U}}-1) \\ \frac{1}{\alpha}\ln(1+\alpha\tilde{U}) \\ \frac{\tilde{U}}{1+\frac{1}{2}\alpha\tilde{U}} \end{cases} . \tag{3.35}$$

Since for general coordinate transformations it is impossible to determine the Bogolyubov coefficients analytically we apply our method introduced in section 3.1. As an example we consider the transformation

$$\tilde{U} = \frac{U}{1-\frac{1}{2}\alpha U} = \frac{-\kappa^{-1}e^{-\kappa u}}{1+\frac{1}{2}\alpha\kappa^{-1}e^{-\kappa u}} , \tag{3.36}$$

where we have used the defining equation for the Kruskal coordinate (1.8). To determine the total number of particles we solve the integrals in (3.8). As shown above only the third term N_3 depends on the particular coordinate transformation. We have

$$N_3 = -\sqrt{\Omega\Omega'}\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{i\Omega u}\int_{-\infty}^{\infty}\frac{du'}{2\pi}e^{i\Omega' u'}\ln|U-U'| . \tag{3.37}$$

Again we introduce the coordinates $x=\kappa^{-1}e^{-\kappa u}$, $x'=\kappa^{-1}e^{-\kappa u'}$ and find

$$\ln|\tilde{U}-\tilde{U}'| = \ln|x'-x| - \ln\left(1+\frac{1}{2}\alpha x\right) - \ln\left(1+\frac{1}{2}\alpha x'\right) . \tag{3.38}$$

The first term in (3.38) represents the usual contribution to the total number of particles, but the other two terms are corrections to the result found above. However, since they are proportional to $\sqrt{\Omega'}\delta\Omega'$ and $\sqrt{\Omega}\delta\Omega$ these terms are infinitely red shifted, their contribution to the energy momentum tensor vanishes. Thus we conclude that there is no change in the thermal spectrum measured by an observer at infinity.

After demonstrating that the particle flux is not influenced by the particular transformation (3.36) we turn to the question if any other well defined

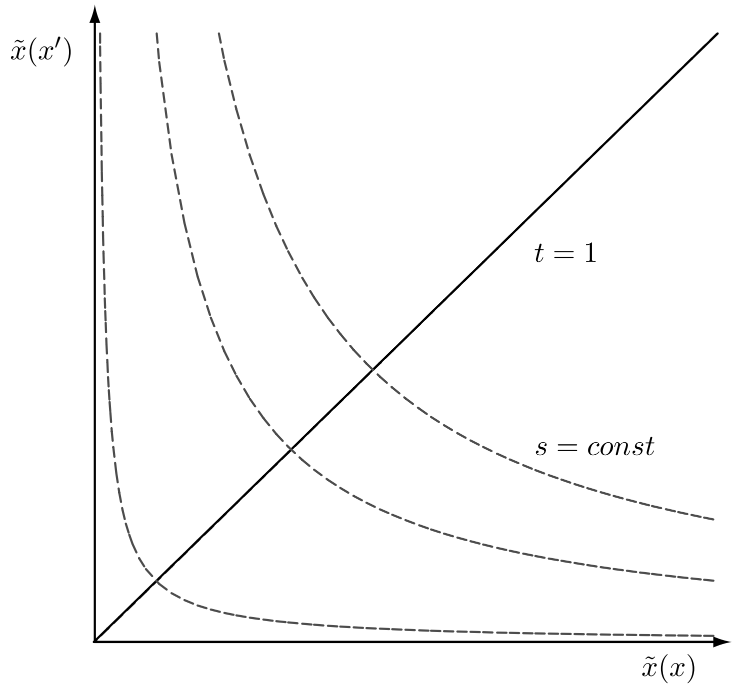

Figure 3.2: The range of integration $x \in [0, \infty)$, $x' \in [0, \infty)$ is covered by the coordinates t and s. The line $t = 1$ corresponds to the divergent $\ln|\tilde{x}' - \tilde{x}|$.

coordinate transformation $\tilde{U}(U)$ might induce a change in the Hawking spectrum at infinity. To do so we try to extend our particular example to a general statement.

Since only the last term in (3.8) depends on the particular form of the transformation we focus on its general form,

$$N_3 = -\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^\infty \frac{dx}{2\pi\kappa}\int_0^\infty \frac{dx'}{2\pi\kappa} x^{-1-i\Omega/\kappa} x'^{-1+i\Omega'/\kappa} \ln|\tilde{x}' - \tilde{x}|, \quad (3.39)$$

where $\tilde{x}' = \tilde{x}(x')$. We can always separate the term contributing to the usual particle spectrum in this expression, which yields

$$N_3 = -\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^\infty \frac{dx}{2\pi\kappa}\int_0^\infty \frac{dx'}{2\pi\kappa} x^{-1-i\Omega/\kappa} x'^{-1+i\Omega'/\kappa} \ln|x' - x| + \delta N_3,$$

where

$$\delta N_3 = -\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^\infty \frac{dx}{2\pi\kappa}\int_0^\infty \frac{dx'}{2\pi\kappa} x^{-1-i\Omega/\kappa} x'^{-1+i\Omega'/\kappa} \ln\left|\frac{\tilde{x}' - \tilde{x}}{x' - x}\right|. \quad (3.40)$$

Again we transform to coordinates s and t as defined in (3.13), writing the general expression for N_3 as

$$\begin{aligned} N_3 &= -2\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^\infty \frac{ds}{2\pi\kappa}\int_0^\infty \frac{dt}{2\pi\kappa} \\ &\quad \times s^{-1-i(\Omega-\Omega')/\kappa} t^{-1-i(\Omega+\Omega')/\kappa} \ln\left|\tilde{x}(\frac{s}{t}) - \tilde{x}(st)\right| . \end{aligned} \tag{3.41}$$

Lines of constant s and t are shown in Fig. 3.2. The line $t = 0$ corresponds to $x = 0$ and $t = 1$ to $\tilde{x}' - \tilde{x} = x' - x = 0$. For arbitrary coordinate transformations it seems to be impossible to calculate the integral (3.41). However, it can be solved using an approximation. Our claim is that changes to the integral (3.14) mainly originate in the region where $t = 1$. Therefore we examine the function $\tilde{f} = \frac{1}{st}\ln|\tilde{x}(\frac{s}{t}) - \tilde{x}(st)|$. As shown in Fig. 3.3 the diverging contributions to the original integral evaluated along $s = const = 1$, $\int_0^\infty \frac{1}{t}\ln|\frac{1}{t} - t|dt$, result from the divergence at $t = 0$ and $t = \infty$, whereas the contribution from the divergence at $t = 1$ is finite. Proper regularization of the integral smoothes out the divergences at $t = 0$ and $t = \infty$ and the finite value of the integral can be calculated. If we now perform the transformation $x \to \tilde{x}(x)$ the regularization of the integral is still valid as long as $\tilde{x}$ is a polynomial of x, $\tilde{x} \sim x^n$. Thus we conclude that the change to the original N_3 mainly originates from the region around $\tilde{x}' - \tilde{x} = 0$. Near $x = x'$ we can apply

$$\lim_{x'\to x} \frac{\tilde{x}(x') - \tilde{x}(x)}{x' - x} = \frac{\partial\tilde{x}(x)}{\partial x} , \tag{3.42}$$

and approximate δN_3 by

$$\delta N_3 = -\kappa^{-i(\Omega-\Omega')/\kappa}\sqrt{\Omega\Omega'}\int_0^\infty \frac{dx}{2\pi\kappa}\int_0^\infty \frac{dx'}{2\pi\kappa} x^{-1-i\Omega/\kappa} x'^{-1+i\Omega'/\kappa} \ln\left|\frac{\partial\tilde{x}(x)}{\partial x}\right| . \tag{3.43}$$

It is now easy to see that the integral is proportional to $\sqrt{\Omega'}\delta(\Omega')$. Therefore this term is infinitely red shifted and its contribution to the particle density vanishes. Therefore the particle spectrum as measured by an observer at infinity remains unchanged with the usual Hawking temperature

$$T_H = \frac{\kappa}{2\pi} . \tag{3.44}$$

The above approximation can be checked with the transformation (3.36) which we have studied earlier. In that case the approximation (3.43) is even exact since we can symmetrize the derivative

$$\ln\left|\frac{\partial\tilde{x}(x)}{\partial x}\right| = \frac{1}{2}\ln\left|\frac{\partial\tilde{x}(x)}{\partial x}\right| + \frac{1}{2}\ln\left|\frac{\partial\tilde{x}(x')}{\partial x'}\right| = \frac{1}{1+\frac{1}{2}\alpha x} + \frac{1}{1+\frac{1}{2}\alpha x'} . \tag{3.45}$$

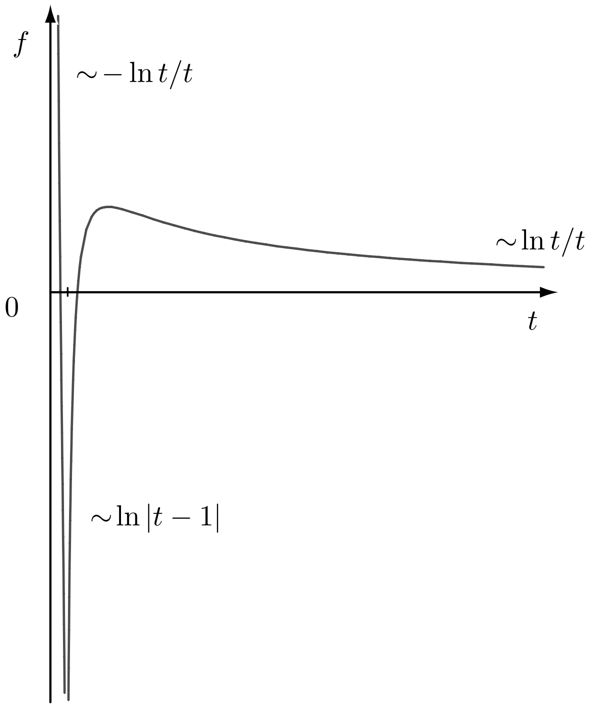

Figure 3.3: The diverging pieces in the integral over $f(t) = \frac{1}{t} \ln |1/t - t|$ originate from the regions around $t = 0$ and $t = \infty$, whereas the contribution around $t = 1$ is finite.

This example supports the validity of our approximation. Thus we conclude that the particle spectrum remains unchanged by the coordinate transformation as long as $\tilde{x}$ is a polynomial of x, $\tilde{x} \sim x^n$. Possible transformations in comparsion to the exponential $e^{\alpha x}$ are shown in Fig. 3.4.

The Hawking flux measured by an observer at infinity is not affected by general transformations of the coordinates on the horizon. Since the vacuum is only invariant under Lorentz transformations, the vacuum states defined by different observers on the horizon in general do not agree. As mentioned before, for each transformation $U \to \bar{U}$ the initial vacuum state is defined using the corresponding Killing vector $\partial_{\bar{U}}$. It can be shown that $\partial_{\bar{U}}$ is a Killing vector along the line which starts at the point on the horizon where the coordinates are inertial and end at $\mathscr{I}^+$. This is quite different from the Kruskal coordinates, for which ∂_U is a Killing vector along the past event horizon H_-. This difference does not affect our argument since we define the vacuum in the vicinity of the horizon H_+. The physical interpretation of our result is as follows: Outside the horizon the Kruskal coordinates are inertial

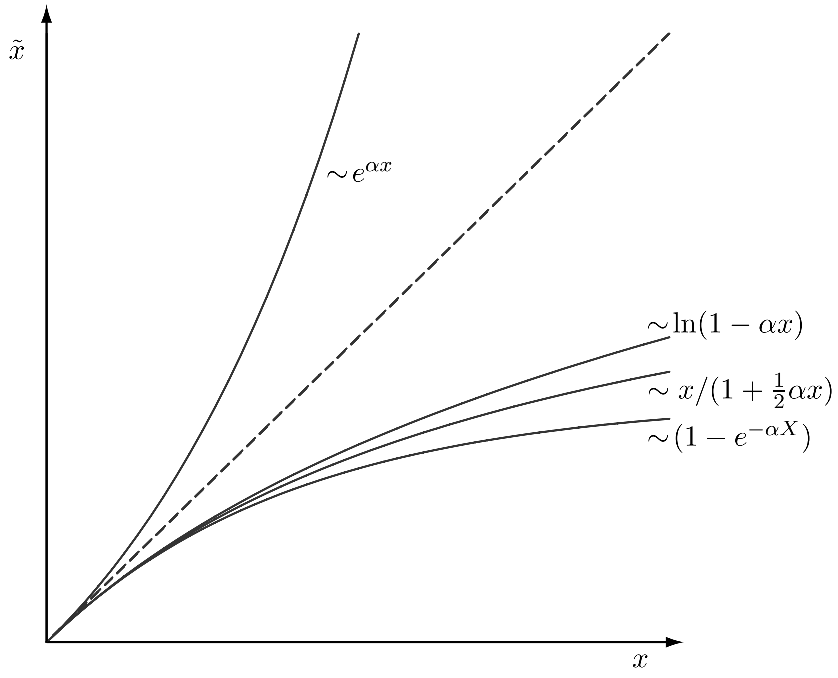

Figure 3.4: Possible coordinate transformations $X \to \tilde{x}$ are shown for $\alpha = \frac{1}{2}$. They correspond to transformations $\tilde{U}(U)$ through $x = -U$ and $\tilde{x} = -\tilde{U}$.

only at $U = V = 0$. To define the proper vacuum state in the vicinity of the horizon, in principle we have to introduce a new freely falling system for every moment V_0. Nevertheless the Hawking spectrum measured at infinity is not sensitive to this subtlety. The above considerations have been done in the two-dimensional theory. The calculations in four dimensions will be more complicated, but we expect the same result.

Chapter 4

Nonstationary black holes

Naturally, the mass of a black hole increases by the absorption of matter. On the other hand, we expect that due to Hawking radiation the mass decreases. Since this time dependence will of course influence the emitted radiation itself it is physically interesting to study the spectrum of radiation of nonstationary black holes.

For a nonrotating uncharged black hole with constant mass M the Schwarzschild radius $r_s = 2M$ marks the location of both apparent and event horizon. If the mass of the black hole changes, event horizon and apparent horizon no longer coincide. As an example we draw the conformal diagram of a black hole with linearly decreasing mass in Fig. 4.1. We assume that after a gravitational collapse the mass of the black hole is M_0. After that, there is a period of linear decrease until the black hole has completely vanished. For $0 < v < v_0$, the apparent horizon is constant, $r = 2M_0$, whereas between $v_0 < v < v_1$ it decreases linearly. After the black hole has evaporated completely space-time is flat, hence the space-time does not possess an event horizon. This example shows that since the event horizon is a global quantity one needs to know the development of the mass in the whole space-time before it can be determined.

However, we expect the particle spectrum at infinity to depend on the local change of the black hole mass. For this reason the event horizon cannot be meaningful for the derivation of Hawking radiation. The importance of other horizons is often suggested, see for example Ashtekar's dynamical horizon or Hayward's work on isolated horizons, [1] and references therein. Recently Visser *et al.* introduced the notion of an evolving horizon which essentially coincides with the apparent horizon [59].

In the following we derive the Hawking flux for a black hole with time dependent mass. As a model for an evaporating black hole we consider the Vaidya solution. We will focus our attention on light rays which stay near the apparent horizon for a very long time. It turns out that they can be used to define the line separating the light rays which escape from the black hole from those falling in. We find an approximate solution for this time dependent horizon and introduce coordinates which are regular on it. After that we consider particular examples for the mass change.

There has been earlier work concerning the issue of Hawking flux from black holes with changing mass. For some particular types of mass functions the space-time possesses an extra conformal Killing vector. In this case it becomes easier to determine the particle flux, e.g. for linear mass decrease [60, 23]. However, the situation is more complicated if one considers general mass change, in particular including the physically relevant mass change due to Hawking radiation. There have been some articles considering general mass [27, 49, 63, 64] which focus on the behaviour of the wave function in the vicinity of the event horizon and use the method of analytic continuation suggested for the Schwarzschild black hole by Damour and Ruffini [11]. Analytic continuation may work well in the case of black holes with constant mass, but in the nonstationary case we doubt the validity of such a procedure. Furthermore, as noted in [27], the resulting temperature is meaningful only in the vicinity of the event horizon. Besides, it remains unclear in which way the black hole parameters are connected to the time of observation. Therefore it is desirable to gain a better understanding of the Hawking effect and the meaning of horizons of nonstationary black holes.

4.1 Vaidya space-time

For the sake of completeness we start with the general metric for a spherically symmetric gravitational field

$$ds^2 = e^{2\psi}\left(1 - \frac{2m}{r}\right) dv^2 - 2e^{\psi} dr dv - r^2 d\Omega^2, \tag{4.1}$$

where ψ and m are both functions of v and r, see also [62]. If $\psi = 0$ and $m = const > 0$, then (4.1) is the Schwarzschild metric in (r, v) coordinates, see (1.6). In the case of $\psi = 0$ and $m = M(v)$, (4.1) is the Vaidya metric which can be used as a model to describe nonstationary black holes with increasing or decreasing mass. It has already been introduced in section 1.1.3 to model the mass increasing in the case of black hole formation. The corresponding line element for an arbitrary mass function is given by

$$ds^2 = \left(1 - \frac{2M(v)}{r}\right) dv^2 - 2dr dv - r^2 d\Omega^2. \tag{4.2}$$

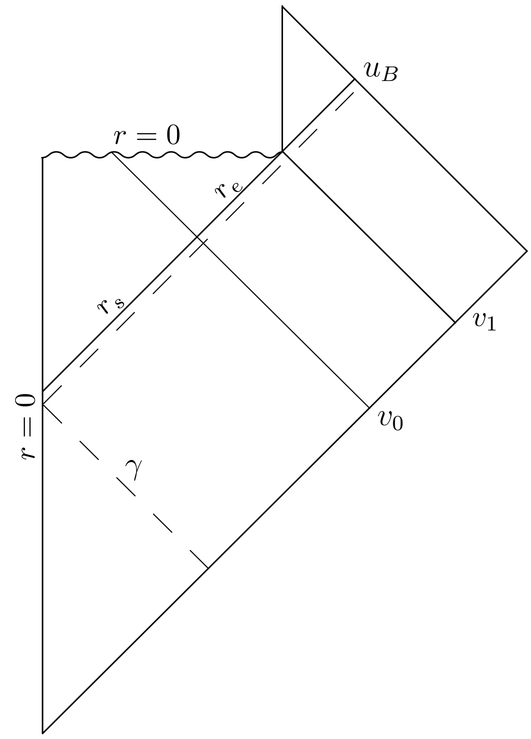

Figure 4.1: Conformal diagram of a black hole with linearly decreasing mass in the region $v \in [v_0, v_1]$. The light ray γ reaches infinity at u_B.

In the following we consider decreasing mass $\frac{dM}{dv} < 0$, the physically most relevant progression given by (1.59). The Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ for (4.2) are

$$\partial_v M = -4\pi r^2 T^r_{\ v} \,, \tag{4.3}$$

with all other components equal to zero. For the general metric (4.1) we have in addition $\partial_r m = 4\pi r^2 T^v_{\ v}$ and $\partial_r \psi = 4\pi r^2 T_{rr}$. The geodesic equation for radially outgoing light rays in the Vaidya space-time is

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right) \,. \tag{4.4}$$

This equation can be solved exactly only for special cases such as mass linear or exponential in v [24]. Here we want to study a general mass change and thus have to solve the geodesic equation approximately.

4.1.1 Horizons

First we study important horizons in the Vaidya space-time. The location of the apparent horizon r_a can be determined from $\frac{dr}{dv} = 0$, thus we have

$$r_a(v) = 2M(v). \tag{4.5}$$

Note that the apparent horizon r_a is not a solution to the geodesic equation (4.4) for non-constant mass.

The event horizon is defined by the outmost locus traced by outgoing photons that can never reach arbitrarily large distances. To determine the location of the event horizon one needs to know the development of the black hole mass at all times. A more practical approximate condition for this horizon is to look for outgoing light rays which stay near the apparent horizon for an infinitely long time. Strictly speaking, this is not an event horizon in the sense of section 1.1.5. Nevertheless it proves useful for the calculation of the particle flux.

Expanding the geodesic equation near the apparent horizon, we find that there is only one light ray $r_e(v)$, which does not deviate exponentially from the apparent horizon. We introduce a small parameter ϵ by assuming that the mass and therefore the apparent horizon is slowly changing in v and write $r_a = r_a(\epsilon v)$. To first order in ϵ our ansatz is

$$r(v) = r_a(\epsilon v)\left(1 + \epsilon z(v)\right). \tag{4.6}$$

Using $r'_a = \frac{dr_a}{d(\epsilon v)}$ the linearized geodesic equation reads

$$\frac{1}{2} z(v) = r'_a + r_a(\epsilon v)\frac{dz}{dv}\ . \tag{4.7}$$

The homogeneous solution can be obtained straightforwardly

$$z_h(v) = z_0 e^{\int_v \frac{dv'}{r_a(v')}}\ , \tag{4.8}$$

where z_0 is a constant of integration. The particular solution to zeroth order in ϵ is

$$z_p(v) = 2r'_a\ . \tag{4.9}$$

From this follows the existence of one light ray, $z_0 = 0$, which does not deviate exponentially quickly from the apparent horizon. Returning ϵ to unity, the location of our time dependent horizon to first order is

$$r_e^{(1)}(v) = r_a(v)\left(1 + 2\frac{dr_a}{dv}\right)\ . \tag{4.10}$$

This result is in agreement with the approximation obtained by York in [62], who studied the surface of "unaccelerated" photons with $\frac{d^2r}{dv^2} = 0$.

Let us briefly explain the relation between the energy emission of a black hole and the corresponding change of its horizon area. During the process of evaporation the black hole mass decreases according to

$$\frac{dM}{dv} = -\frac{c}{M^2} , \tag{4.11}$$

where c is some constant, see (1.59). The approximate solution to the geodesic equation near the apparent horizon r_a is given by (4.10). Using (4.11) yields

$$r_e^{(1)} = 2M - \frac{8c}{M} . \tag{4.12}$$

From the above result we see immediately that the difference between the two horizons is very small as long as the black hole mass is larger than the Planck mass,

$$r_e^{(1)} - r_a = -\frac{8\pi c}{M} . \tag{4.13}$$

Furthermore, assuming that the emission of one quantum causes a change in the horizon as (4.13) the black hole area is changed by one Planckian area,

$$\Delta A = 4\pi \left(r_a^2 - r_e^2\right) \sim A_{pl}. \tag{4.14}$$

It seems astonishing that this result is independent from the black hole mass.

4.1.2 Double-null coordinates

To determine the particle flux from a nonstationary black hole we first have to solve the wave equation $\Box\phi = 0$. The following calculations will again be performed in the two-dimensional model. For the case of constant black hole mass we have already shown that apart from an irrelevant transmission coefficient this reduction of dimensionality does not affect the physics. Therefore we expect that the physically relevant information for the Vaidya space-time can be extracted from the two-dimensional model as well. It is useful to write the metric (4.2) in a conformally flat form. For this purpose we introduce a new coordinate defined by

$$y = r - r_e(v) . \tag{4.15}$$

In the case of an arbitrarily changing black hole mass the location of the horizon r_e can only be determined approximately. In the following we use the approximate solution from above. Using the new coordinate frame the Vaidya metric can be written as

$$ds^2 = \frac{y}{y + r_e} r_e dv \left[dv \frac{r_a}{r_e^2} - 2d\ln(y) - 2\frac{dy}{r_e} \right] . \tag{4.16}$$

It turns out to be useful to introduce the rescaled coordinate

$$d\tilde{v} = \frac{r_a(v)}{r_e(v)^2} dv , \tag{4.17}$$

which gives $\tilde{v} = \frac{v}{2M_0}$ in the limit of constant mass, corresponding to a Lorentz transformation. For $y < \frac{r_e r_a}{r_a - r_e}$ we can replace $\frac{dy}{r_e}$ in (4.16) by the total differential $d(\frac{y}{r_e})$ and define the outgoing light cone coordinate

$$\tilde{u} = \tilde{v} - 2\ln(y) - 2\frac{y}{r_e} . \tag{4.18}$$

Using the coordinate $\tilde{u}$ the metric reads

$$ds^2 = \frac{y}{y + r_e} r_e dv d\tilde{u} . \tag{4.19}$$

Now we check the range of validity of our light cone coordinates. Since the emission of one quantum changes the black hole area only by one Planckian length (4.14), the difference between the apparent horizon r_a and r_e is much smaller than one as long as the black hole mass is larger than the Planck mass, $|r_a - r_e| \sim 1/M \ll 1$. In this case the above approximation can be used for $y < M^3$ and therefore is still valid far away from the horizon. The metric (4.19) is not regular on the horizon, but we can define coordinates analogous to the Kruskal coordinates in the following way

$$d\tilde{U} = e^{-\frac{1}{2}\tilde{u}} d\tilde{u} , \tag{4.20}$$

$$d\tilde{V} = e^{\frac{1}{2}\tilde{v}} dv . \tag{4.21}$$

Then the resulting form of the metric looks very familiar,

$$ds^2 = \frac{y}{y + r_e} r_e dv d\tilde{u} = \frac{r_e}{r} e^{1 - \frac{r}{r_e}} d\tilde{V} d\tilde{U}. \tag{4.22}$$

Thus we found coordinates which are regular on the time dependent horizon $r_e(v)$. These coordinates will be used to define the vacuum state on the horizon. Far away from the horizon we have to use a different approximation for the retarded time coordinate. This can be done in the following way: For $y \gg r_e$ the logarithmic term in (4.16) can be neglected and the metric becomes

$$ds^2 = \frac{y}{y + r_e} dv du \ , \tag{4.23}$$

where the retarded time coordinate u is given by

$$u = v - 2y \ . \tag{4.24}$$

The coordinates (4.23) reduce to the Minkowski metric at spatial infinity. Hence they can be used to define the vacuum state of an observer at rest at infinity.

4.1.3 Hawking effect

We introduced two appropriate coordinate frames in the previous section. The coordinates $(\tilde{U}, \tilde{V})$ can be used to define a regular coordinate frame on the horizon, the coordinates (u, v) at spatial infinity. Since the two coordinate frames are valid in different regions of the space-time we have to match them in the region

$$r_e < y < \frac{r_e r_a}{r_a - r_e} \ , \tag{4.25}$$

where both $\tilde{u} = const$ and $u = const$ are good approximations for the solution to the geodesic equation. The coordinates will be matched along an arbitrary line $v = v_g$ in this region. Accordingly, for $v < v_g$ the light rays are described by constant $\tilde{u}$ and for $v > v_g$ they obey $u = const$. The relation between $\tilde{u}$ and u can be written down immediately,

$$\tilde{u} = \frac{u}{2M(v_g)} - 2 \ln \left(\frac{M(v_g) y}{m_0} \right), \tag{4.26}$$

where we keep in mind that $y = y(u, v_g)$. The light cone coordinates in the Vaidya space-time which have been introduced above are valid for an arbitrary function $M(v)$. For simplicity we consider a rate of change according to (1.59) in the following, since this is the expected mass change due to Hawking radiation. The corresponding mass of the black hole is

$$M(v) = m_0 (1 - 3cv)^{\frac{1}{3}} \ . \tag{4.27}$$

We will show later that these results can also be applied to other mass functions.

To determine the time dependent Hawking flux we consider a light ray that reaches an observer at infinity at very late time u_B. This light ray stays near the horizon r_e for a very long time, so we linearize the geodesic equation around $r(v) = r_e(v) + \epsilon(v)$ and obtain the following solution

$$r(v) = r_e(v) + \epsilon_i \exp\left(\int_{v_i}^{v} dv' \frac{r_a(v')}{2r_e(v')^2}\right) . \tag{4.28}$$

We expect that this light ray brings information to an observer at infinity about the black hole mass at the time when its distance from the horizon was of order ϵ. Using the change of the mass according to (1.59) we have

$$r(v) = r_e(v) + \epsilon_i \frac{M(v_i)}{M(v)} e^{\frac{1}{8c}\left(M(v_i)^2 - M(v)^2\right)} . \tag{4.29}$$

Tracing back the light ray $\tilde{u}$ to the moment v_i when its distance from the horizon was very small, $r(v_i) = r_e(v_i) + \epsilon$, it turns out that during this time the mass of the black hole does not change significantly. This is due to the exponential deviation of the light ray from the horizon. The relative mass change can be estimated

$$\frac{\Delta M}{M(v_i)} \simeq \frac{1}{M(v_i)^2}\left[\ln\left(\frac{M(v_g)y}{M(v_i)\epsilon}\right) + \frac{y}{r_e}\right] . \tag{4.30}$$

Thus we write

$$\tilde{u} = \frac{u}{2M(v_i)}\left[1 + \frac{1}{M(v_i)^2}\ln\left(\frac{M(v_g)y}{M(v_i)\epsilon}\right) + \frac{y}{r_e M(v_i)^2}\right] - 2\ln\left(\frac{M(v_g)y}{m_0}\right) . \tag{4.31}$$

If we consider an observer $r_B = r(u_B, v_B)$ at very large distance, we can connect u_B to the initial moment v_i, when the light ray was just a distance ϵ from the horizon,

$$u_B = v_i\left(1 + \frac{\Delta M}{M(v_i)}\right) + u_\star , \tag{4.32}$$

where $u_\star = 4M(v_g)\ln\left(\frac{M(v_g)y}{M(v_i)}\right)$, and we define

$$\mathcal{M}(u_B) \simeq M(u_B - u_\star) \simeq M(v_i) . \tag{4.33}$$

Thus we have been able to find the relation between the mass of the black hole at the moment v_i, when the light ray was still very close to the horizon, and the time u_B, when the observer measures the radiation. Now we can determine the coordinate transformation $\tilde{U}(u)$. This yields

$$\tilde{U} = e^{-\frac{1}{2}\tilde{u}(u)} \simeq e^{-\frac{u}{4\mathcal{M}(u_B)}} . \tag{4.34}$$

The Bogolyubov coefficients are calculated as

$$\left.\begin{array}{l}\alpha_{\omega\Omega}\\ \beta_{\omega\Omega}\end{array}\right\} = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} du e^{i\Omega u} e^{\mp i\omega\tilde{U}(u)} \ . \tag{4.35}$$

We can read off the temperature measured by an observer r_B which is equal to

$$T = \frac{1}{8\pi\mathcal{M}(u_B)}, \tag{4.36}$$

and depends on the retarded time u_B. We conclude that this time dependence of the black hole mass is reflected in a time dependent Hawking flux at infinity. Using the geometric optics approximation it becomes obvious that an observer at infinity receives information about the state of the black hole when the distance of the light ray from the apparent horizon was very small. We also expect nonthermal corrections in the Hawking spectrum, but these corrections probably cannot be recovered by the above method.

4.2 Examples

In the following we calculate the particle flux for two particular choices of black hole mass. In some cases our method to determine Hawking radiation can be used to determine the main effects of the mass change quite straightforwardly.

Our first example is a black hole with fluctuating horizon. This model was also examined by Barrabes *et al.* in [3]. This scenario is very interesting physically since metric fluctuations are expected due to quantum corrections. To describe these fluctuations quantum mechanically and to determine their influence on the Hawking radiation in principle requires a full theory of quantum gravity. However, it might be possible to obtain the main effects via semiclassical theory. The fluctuations of the black hole geometry are approximated by periodic changes in its mass

$$M(v) = M_0\left[1 + \mu_0 \sin(\omega v)\right]\theta(v) \ , \tag{4.37}$$

where μ_0 is the dimensionless amplitude of the fluctuations. The step function $\theta(v)$ shows that the black hole results from gravitational collapse of a massive null shell propagating along $v = 0$. Following [3] we assume $\mu_0 = \alpha(\frac{m_{pl}}{M})$, where α is a number of order unity. In particular this means $\mu_0 \ll 1$ for a black hole mass $M \gg m_{pl}$. In this case the rescaled coordinate

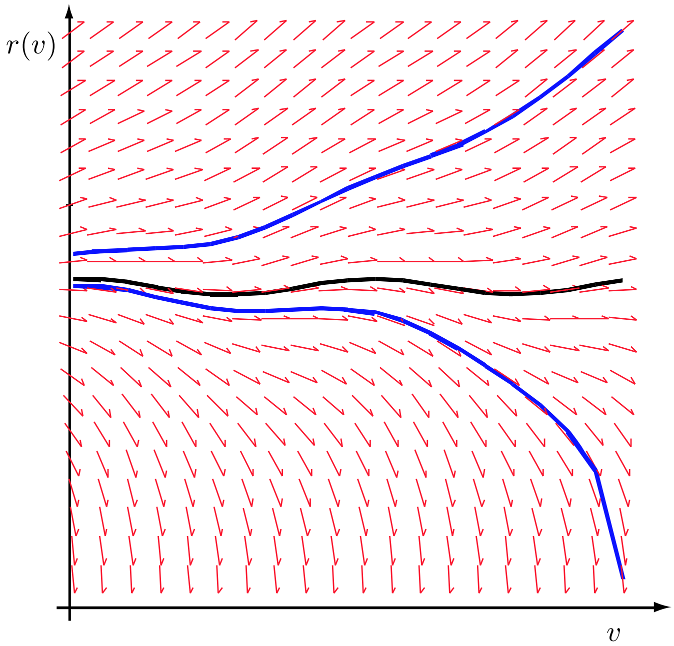

Figure 4.2: Numerical plot of (r, v)-diagram of Vaidya space-time with oscillating mass.

(4.17) can be integrated explicitly. The black hole mass changes periodically with $v_T = \frac{2\pi}{\omega}$. This property can be used to define the average surface gravity

$$\bar{\kappa} \simeq \overline{\left(\frac{r_a}{2r_e^2}\right)} = \frac{1}{v_T}\int_0^{v_T} dv \frac{r_a(v)}{2r_e(v)^2} \ . \tag{4.38}$$

Since the average surface gravity (4.38) is constant the light cone coordinate u can be defined as

$$\tilde{u} = 2\bar{\kappa}u \ . \tag{4.39}$$

Correspondingly, the Kruskal-like coordinate is $\tilde{U} \sim e^{-\bar{\kappa}u}$. The thermal corrections to the temperature can now be read off the transformation immediately,

$$T_H = \frac{\bar{\kappa}}{2\pi} \ . \tag{4.40}$$

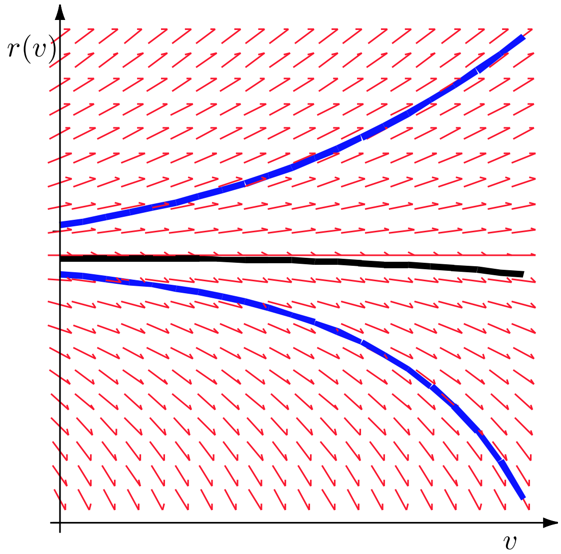

Figure 4.3: Numerical plot of (r, v)-diagram of Vaidya space-time with linearly decreasing mass.

This agrees with the result derived in [3], where the particle flux was obtained by including corrections to the light rays and the temperature was proportional to some rescaled surface gravity, that coincided with the averaged surface gravity only by chance. In our derivation it becomes immediately obvious that the Hawking temperature is indeed proportional to the average surface gravity. In Fig. 4.2 we show a numerical plot of the (r, v)-diagram for oscillating mass showing in- and outgoing light rays. For oscillating mass the approximation (4.10) is not a good approximation, but r_e can be determined as a perturbation around $\bar{r}_a = 2M_0$, see [3]. Both apparent and event horizon oscillate around the same mean value $\bar{r}_a = \bar{r}_e = 2M_0$. However, their amplitudes are different depending on μ_0 and ω, and there is a phase shift between them. We conclude that our derivation of time dependent Hawking temperature is especially useful in cases where we can exploit the characteristic features of the metric. In the above case this is the periodicity of the black hole mass.

Our next example is the linear mass function

$$M(v) = M_0 \left(1 - \frac{\alpha}{M_0} v \right), \tag{4.41}$$

where we restrict v to positive values and $\frac{\alpha}{M_0} v < 1$. The conformal diagram of a Vaidya black hole with linearly decrasing mass is shown in Fig. 4.1. The geodesic equation (4.4) can be solved analytically with the result

$$r_e = \frac{r_a}{8\alpha}\left(\sqrt{1+16\alpha}-1\right), \tag{4.42}$$

which coincides with the Cauchy horizon of the space-time satisfying the relation $\frac{d^2r}{dv^2}=0$. Since the derivative of the mass function with respect to v is $\dot{M}=-\alpha$, the distance between r_a and r_e is constant. For small α we obtain

$$r_e \simeq r_a(1-4\alpha)\,. \tag{4.43}$$

We determine the average surface gravity defined over the interval $\Delta v = v_f - v_i$,

$$\bar{\kappa} = \frac{1}{\Delta v}\int_{v_i}^{v_f} dv \frac{r_a(v)}{2r_e(v)^2} = \frac{1}{4\alpha\Delta v\,(1-4\alpha)^2}\ln\left(\frac{M(v_i)}{M(v_f)}\right). \tag{4.44}$$

For a typical interval Δv this can be approximated by

$$\bar{\kappa} \simeq \frac{1}{4\alpha\Delta v\,(1-4\alpha)^2}\frac{\alpha\Delta v}{M(v_i)} \simeq \frac{1}{4M(v_i)}\,(1+8\alpha)\,. \tag{4.45}$$

If we consider a light ray which at $v=v_i$ has distance ϵ from r_e we define the mass dependence for an observer at infinity by $\mathcal{M}(u_B)=M(u_B-u_\star)\simeq M(v_i)$. Hence, we can compute the corrections to the Hawking temperature

$$T_H = \frac{\bar{\kappa}}{2\pi} = \frac{1}{8\pi\mathcal{M}(u_B)}\,(1+8\alpha)\,. \tag{4.46}$$

It turns out that we found an additional correction in temperature in formula (4.46) which we have not been able to determine in (4.36).

The previous examples illustrate the derivation of the Hawking flux for black holes with changing mass. In some cases thermal corrections can be determined quite easily, for example for linearly decreasing mass. The temperature of the particle flux has been extracted from the total number of particles emitted in the space-time. For nonstationary black holes the question arises which horizon is the important one. Here we have chosen an approximate solution to the geodesic equation, which at every moment of time describes a light ray which does not deviate exponentially fast from the apparent horizon. In the next chapter we will turn back to this question, examining the response of a detector moving in a time dependent background. In this way further corrections to the temperature can be determined as well.

Chapter 5

Particle detectors

What do our previous considerations imply for the detection of particles in curved space-time? Necessarily, one has to specify all details of the process used to detect the presence of quanta. In particular the state of motion of the detector has an affect on the outcome. For example a freely falling detector will not register the same spectrum as a noninertial accelerated detector. This is so even in flat space-time: A detector moving with constant proper acceleration registers a thermal flux of particles in the Minkowski vacuum with a temperature proportional to its acceleration. This is the well known Unruh effect [54].

In the previous chapters we studied the particle flux from black holes by calculating the total number of emitted particles $\langle \hat{N}_\Omega \rangle$, which is integrated over all times. However, as shown in the following example, the physical interpretation of this approach is problematic since it is difficult to account for a time dependent scenario. In contrast, a particle detector can be switched on for a measurement over a finite period of time and consequently reveals information about the spectrum of radiation at this time.

In the following we use a model particle detector to quantify the particle spectrum measured by observers moving in a curved background. The finite nature of the physical measurement will be taken into account by a suitable window function in the detector-field interaction. Since formally the derivation of the Hawking radiation is equivalent to the Unruh effect we start with some considerations in Minkowski space-time.

5.1 Accelerated observers in flat space-time

Particle creation in Minkowski space-time was first studied by Unruh for the case of a uniformly accelerated observer [54]. He found that this so-called

Rindler observer detects a thermal flux of particles in the Minkowski vacuum with a temperature proportional to his acceleration. Formally the derivation of this effect is identical to the derivation of Hawking radiation since the transformation between Minkowski coordinates (u_M, v_M) and the observer's rest frame $(\tilde{u}, \tilde{v})$ can be written as

$$\begin{aligned} u_M &= -\frac{1}{a_0} e^{-a_0 \tilde{u}} , \\ v_M &= \frac{1}{a_0} e^{a_0 \tilde{v}} , \end{aligned} \tag{5.1}$$

where a_0 is the proper acceleration of the Rindler observer. This transformation is identical to (1.9) with the identification

$$\begin{aligned} (U, V) &\hat{=} (u_M, v_M), \\ (u, v) &\hat{=} (\tilde{u}, \tilde{v}). \end{aligned} \tag{5.2}$$

Thus the Kruskal observer in the black hole space-time corresponds to an inertial observer in Minkowski space-time and Eddington-Finkelstein correspond to Rindler coordinates. Correspondingly, $\frac{1}{4M}$ is substituted by the acceleration a_0.

We consider an observer moving with general acceleration. The Bogolyubov approach of transformations to scalar particle creation in flat space-time has been considered for some classes of accelerated frames in $1+1$ dimensions [40, 41, 42, 44, 65] and in four dimensions [43]. Dolby [13] extended the considerations to fermionic particle production for some special examples of observers in $1+1$ dimensions.

Calculating the controlling equations for an arbitrarily accelerated observer is straightforward. We start with the two-dimensional line element in Minkowski space-time

$$ds^2 = dt^2 - dr^2 = \eta_{ik} dx^i dx^k, \tag{5.3}$$

with Latin indices $i, k = 0, 1$. If τ is the proper time parametrizing the observer's trajectory $x^k(\tau) = (t(\tau), r(\tau))$, then the observer's two-velocity is given by

$$u^k(\tau) = \frac{dx^k}{d\tau} = \left(\dot{t}(\tau), \dot{r}(\tau)\right) , \tag{5.4}$$

which is normalized according to

$$u^k(\tau) u_k(\tau) = \eta_{ik} \dot{x}^i \dot{x}^k = 1 . \tag{5.5}$$

The two-acceleration is defined as $a^k = \frac{d^2x^k}{d\tau^2} = \dot{u}^k$ and is orthogonal to the velocity

$$u^k(\tau)a_k(\tau) = 0\ , \tag{5.6}$$

which can be shown by taking the derivative of (5.5). In an instantly comoving inertial frame the observer is at rest and thus $\dot{r}(\tau) = 0$. Then the observer's velocity is $u^k(\tau) = (1, 0)$ and the acceleration $a^k(\tau) = (0, a(\tau))$. The absolute value of the acceleration is given by

$$\eta_{ik}a^i(\tau)a^k(\tau) = -a(\tau)^2\ . \tag{5.7}$$

This relation is valid in the comoving frame and since it is completely covariant it is valid in an arbitrary inertial frame. In the case of the Rindler observer the proper acceleration is constant in time, $a(\tau) = a_0 = const.$ To understand the physical meaning of constant acceleration consider a spaceship with infinite energy supply and a propulsion engine that exerts a constant force. Then the acceleration of the spaceship in its comoving frame is constant. In light cone coordinates, $u_M = t - r$, $v_M = t + r$, the two-dimensional line element in Minkowski space-time is given by

$$ds^2 = du_M dv_M\ . \tag{5.8}$$

In the following we skip the index M for simplicity. The two-velocity of the observer in light cone coordinates is

$$u^k(\tau) = \frac{dx^k}{d\tau} = (\dot{u}(\tau), \dot{v}(\tau))\,. \tag{5.9}$$

The observer's trajectory can be determined from the following equations,

$$\dot{u}\dot{v} = 1\ , \tag{5.10}$$

$$\ddot{u}\ddot{v} = -a(\tau)^2\ . \tag{5.11}$$

Differentiating (5.10) with respect to τ we get $\ddot{v} = -\frac{\ddot{u}}{\dot{u}^2}$. Substituting this result into (5.11) yields

$$\frac{\ddot{u}}{\dot{u}} = -\frac{\ddot{v}}{\dot{v}} = -a(\tau)\ . \tag{5.12}$$

This equation can be integrated and using (5.10) we obtain

$$\dot{u} = c_1 e^{-\int a(t)dt} \equiv A(\tau)\ , \tag{5.13}$$

$$\dot{v} = c_1^{-1} e^{\int a(t)dt} \equiv A(\tau)^{-1}\ , \tag{5.14}$$

where c_1 is a constant of integration. The appropriate comoving frame (ξ^0, ξ^1), where the accelerated observer is at rest, can be defined choosing $\xi^1 = const = 0$, and the time coordinate ξ^0 to be the proper time τ along the observer's world line. Then the world line $\xi^k(\tau)$ of the observer is given by

$$\xi^0(\tau) = \tau \ , \quad \xi^1(\tau) = 0 \ . \tag{5.15}$$

Assuming that the coordinate basis vectors ξ^k are orthogonal, the metric in this frame is

$$ds^2 = G^2(\xi^0, \xi^1) \left[(d\xi^0)^2 - (d\xi^1)^2 \right] , \tag{5.16}$$

where $G(\xi^0, \xi^1)$ is the conformal factor of the metric which still has to be determined. Now we can introduce light cone coordinates for the comoving frame $\tilde{u} = \xi^0 - \xi^1$ and $\tilde{v} = \xi^0 + \xi^1$. In these coordinates the observer's trajectory is

$$\tilde{v}(\tau) = \tilde{u}(\tau) = \tau \ . \tag{5.17}$$

The coordinates (u, v) and $(\tilde{u}, \tilde{v})$ describe Minkowski space-time in two different frames,

$$ds^2 = dudv = G^2(\tilde{u}, \tilde{v}) d\tilde{u} d\tilde{v} \ , \tag{5.18}$$

where $G^2(\tilde{v} = \tau, \tilde{u} = \tau) = 1$. The observer's trajectory in terms of the two coordinate systems is

$$\frac{du(\tau)}{d\tau} = \frac{du(\tilde{u})}{d\tilde{u}} \frac{d\tilde{u}(\tau)}{d\tau} \ . \tag{5.19}$$

In addition, we know that

$$\frac{du(\tau)}{d\tau} = A(\tau) \ , \quad \frac{d\tilde{u}(\tau)}{d\tau} = 1 \ . \tag{5.20}$$

Therefore the exact form of the transformation can be determined. This yields

$$\frac{du}{d\tilde{u}} = A(\tilde{u}) \ , \tag{5.21}$$

$$\frac{dv}{d\tilde{v}} = A(\tilde{v})^{-1} \ . \tag{5.22}$$

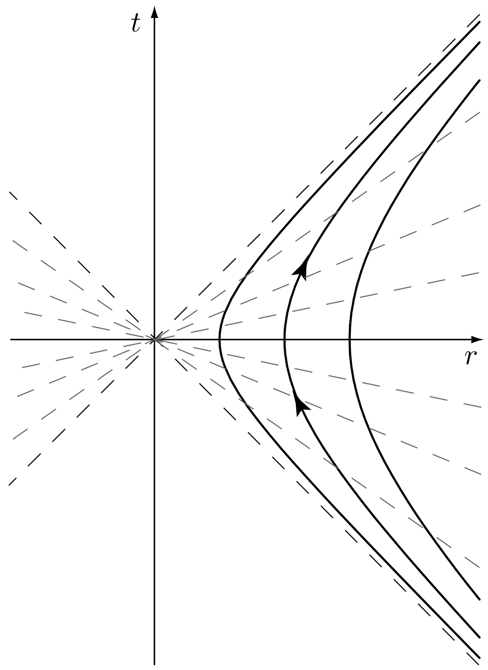

Figure 5.1: World lines of uniformly accelerated observers. The solid hyperbolae are lines of constant proper distance ξ. Lines of constant τ are dashed. The outer dashed lines show the light cone which corresponds to $\xi = -a_0^{-1}$.

Now we consider the Rindler observer, where $a(\tau) = a_0 = const$, and $A(\tau) = c_1 e^{-a_0 \tau}$. In this case (5.22) can be integrated with the result

$$u = -\frac{c_1}{a_0} e^{-a_0 \tilde{u}} + B \,, \tag{5.23}$$

$$v = \frac{1}{a_0 c_1} e^{a_0 \tilde{v}} + C \,. \tag{5.24}$$

With a suitable choice of the integration constants c_1 and B, C, which correspond to Lorentz transformation and shifting the origin of the coordinate system, we arrive at equation (5.1). In Fig. 5.1 world lines of uniformly accelerated observers are drawn. The solid hyperbolae are lines of constant proper distance. Lines of constant τ are dashed. For large $|t|$ the trajectories approach the light cone. Using the arguments of section 1.3.2 we can determine the total number of particles,

$$\langle \hat{N}_\Omega \rangle = \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{e^{2\pi\Omega/a_0} - 1} \,. \tag{5.25}$$

Since the acceleration of the observer a_0 is constant we can read off the particle density immediately,

$$n_\Omega = \frac{1}{e^{2\pi\Omega/a_0} - 1} \ . \tag{5.26}$$

Therefore the uniformly accelerated observer in Minkowki space-time measures a thermal particle flux with temperature proportional to its acceleration a_0.

As an example for nonuniformly accelerated motion we consider an observer in Minkowski space-time who at $\tau = -\infty$ is at rest. His acceleration $a(\tau)$ constantly increases for $-\infty < \tau < \infty$ until finally it tends to a constant. Such an observer can be described by

$$A(\tilde{v})^{-1} = 1 + e^{a_f \tilde{v}} \ , \tag{5.27}$$

where a_f is the final value of the acceleration at $\tau = \infty$. The proper acceleration $a(\tau)$ of the observer can be determined from

$$\ddot{u}\ddot{v} = -a(\tau)^2 \ , \tag{5.28}$$

which yields

$$a(\tau) = \frac{a_f}{1 + e^{-a_f \tau}} \ . \tag{5.29}$$

A possible trajectory is shown in Fig. 5.2. For $\tau \to -\infty$ the observer coincides with the Minkowski observer, for $\tau \to \infty$ he approaches the light cone. The transformation between Minkowski coordinates and the observer's comoving frame is

$$u = -\frac{1}{a_f} \ln(1 + e^{-a_f \tilde{u}}) \ , \tag{5.30}$$

$$v = \tilde{v} + \frac{1}{a_f} e^{a_f \tilde{v}} \ . \tag{5.31}$$

The computation of the Bogolyubov coefficients (1.44) in this case yields

$$\left.\begin{array}{c} \alpha_{\omega\Omega} \\ \beta_{\omega\Omega} \end{array}\right\} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} e^{i\Omega\tilde{u}} \left(1 + e^{a_f \tilde{u}}\right)^{\pm i\omega/a_f}$$

$$= \sqrt{\frac{\Omega}{\omega}} \frac{1}{2\pi a_f} \frac{\Gamma(-\frac{i\Omega}{a_f})\Gamma(\frac{i(\Omega\mp\omega)}{a_f})}{\Gamma(\mp\frac{i\omega}{a_f})} \ . \tag{5.32}$$

Accordingly, we can naively determine the total number of particles (1.45)

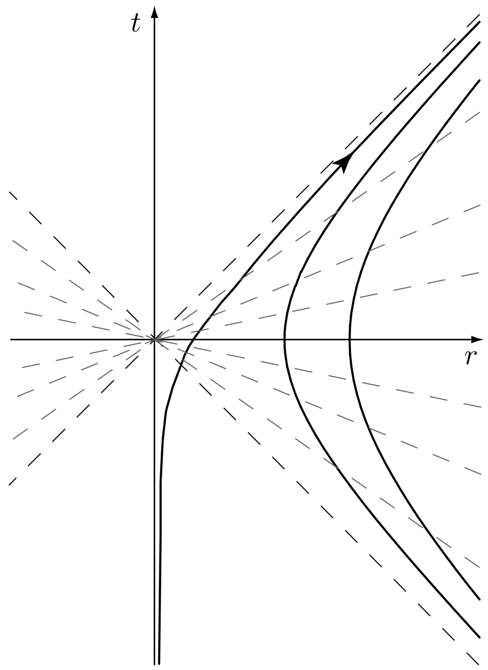

Figure 5.2: World line of a nonuniformly accelerated observer. The observer starts at rest at $\tau = -\infty$. He is accelerated until $a(\tau) \to a_f$ and for $\tau \to \infty$ he approaches the light cone.

as

$$\langle \hat{N}_\Omega \rangle = \int_0^\infty d\omega \frac{1}{4\pi a_f(\Omega + \omega)} \frac{\sinh(\frac{\pi\omega}{a_f})}{\sinh(\frac{\pi\Omega}{a_f}) \sinh(\frac{\pi(\Omega+\omega)}{a_f})} \ . \tag{5.33}$$

It is now impossible, however, to extract the proper time-dependent particle density from the above result. At early times we expect that the observer does not measure any particles or at least very few, whereas at late times the spectrum should resemble a thermal spectrum with temperature $T = \frac{a_f}{2\pi}$. This example reveals the limited utility of computing $\langle \hat{N}_\Omega \rangle$. Because of the formal analogy between accelerated observers and black hole radiation we expect that we also have to be careful in the case of Vaidya space-time.

5.2 Unruh detector

The Unruh detector [8, 54] can be implemented in a number of ways. Here we take the coupling between the detector and the field to be of linear monopole type. The detector itself is an idealized point-like detector defined solely through its energy levels E_m. The states evolve as

$$|m,\tau\rangle = e^{iH_0\tau}|m\rangle = e^{iE_m\tau}|m\rangle \ , \tag{5.34}$$

where H_0 is the unperturbed Hamiltonian of the detector and τ denotes the proper time measured along its trajectory $x(\tau)$. The detector-field interaction is described by an interaction Lagrangian of the form

$$\hat{U}_I = -i\epsilon \int_{-\infty}^{\infty} d\tau W(\tau)\hat{Q}(\tau)\hat{\phi}(x) \ , \tag{5.35}$$

where $\hat{Q}$ is the detector's monople moment and ϵ is a small coupling parameter. Usually, the response of the detector is studied for its entire history, from the infinite past to the infinite future in the detectors proper time, but in any realistic situation the detector is only switched on for a finite period of time. The window function $W(\tau)$ models the finite duration of the measurement and it allows for the time dependence of the detector response.

During a measurement the detector will undergo a transition from its ground state E_i to an excited state E_f due to its interaction with the scalar field. If we suppose the field $\hat{\phi}$ is initially in its vacuum state $|0\rangle$ then the transition amplitude in first order perturbation theory is

$$\mathcal{A}(\Sigma) = \langle\Sigma|\langle f|\hat{U}_I|i\rangle|0\rangle \ , \tag{5.36}$$

where $|\Sigma\rangle$ is the final state of the field. Since $|\Sigma\rangle$ is immaterial for the detector the probability of transition is

$$\begin{aligned}\mathcal{P} &= \int d\Sigma \mathcal{A}(\Sigma)^* \mathcal{A}(\Sigma) \\ &= \epsilon^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' W(\tau)W(\tau')Q_{fi}e^{i\Delta E(\tau-\tau')}\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle \ ,\end{aligned} \tag{5.37}$$

where $Q_{fi} = |\langle f|\hat{U}_I|i\rangle|^2$.

We will now compute the transition probability (5.37) explicitly. The detector's trajectory is parametrized in terms of in- and outgoing light cone coordinates $(u(\tau), v(\tau))$. Using the plane wave expansion for the scalar field

$$\hat{\phi}(u,v) = \int_0^{\infty} \frac{d\Omega}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega u}\hat{b}_{\Omega}^- + e^{i\Omega u}\hat{b}_{\Omega}^+ + e^{-i\Omega v}\hat{b}_{-\Omega}^- + e^{i\Omega v}\hat{b}_{-\Omega}^+ \right) \tag{5.38}$$

and the result for the integral (3.7) we obtain

$$\begin{aligned}\mathcal{P} &\propto \int_{-\infty}^{\infty} \frac{d\tau}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\tau' e^{i\Delta E(\tau-\tau')} e^{-\frac{(\tau-\tau_0)^2}{2\sigma^2}} e^{-\frac{(\tau'-\tau_0)^2}{2\sigma^2}} \\ &\times \left[-\ln|u-u'| + \frac{i\pi}{2}\mathrm{sgn}(u-u') - \ln|v-v'| + \frac{i\pi}{2}\mathrm{sgn}(v-v') \right] ,\end{aligned} \tag{5.39}$$

where $u' = u(\tau')$ and $v' = v(\tau')$. We use the Gaussian window function

$$W(\tau) = e^{-\frac{(\tau-\tau_0)^2}{2\sigma^2}} , \tag{5.40}$$

where τ_0 denotes the time of the measurement and σ is the proper time interval of the measurement. Other possible choices for the window function are a δ-function or a step-function. However, as noted in [22], instantaneously switching on and off the detector leads to divergences in the detector response, whereas smooth window functions are expected to change the registered spectrum only by transients. This has been shown explicitly in the case of an uniformly accelerated detector in [51].

Without a suitable finite window function one would expect to measure an infinite number of particles but, as it stands, the transition probability (5.39) has an undesirable dependence on the interaction time σ. Besides, it typically exceeds unity after a finite time. It is sufficient, however, to consider the transition rate of the detector which is given by

$$\mathcal{T} \equiv \lim_{\sigma\to\infty} \frac{\mathcal{P}}{\sigma} . \tag{5.41}$$

This quantity is independent of the details of the interaction and from this we can determine the spectrum of radiation measured by the observer.

Now, given the motion of an observer in the space-time, we are able to calculate its transition rate, assuming that the integrals (5.39) can be computed at least approximately. To be physically sensible, we require a restriction on the detector motion

$$\frac{a}{\dot{a}} \gg \sigma \gg (\Delta E)^{-1} , \tag{5.42}$$

where $a = a(\tau)$ is a parameter describing the observer's acceleration in proper time. In flat space-time this quantity coincides with the proper acceleration. This means, that the interaction time must be less than the time scale of any changes in the system we hope to measure, and we can only detect radiation with energy greater than the inverse of the interaction time.

5.2.1 Rindler observer

The simplest application of the Unruh detector is the Rindler observer in flat space-time. In this case the trajectory of the observer is given by (5.1). In this section we let $\tau_0 = 0$ without loss of generality and write a instead of a_0. The transition probability (5.37) can be calculated explicitly. The first term in (5.39) yields

$$\mathcal{P}_1 \;\propto\; -\int_{-\infty}^{\infty} \frac{dx}{4\sqrt{\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{x^2}{4\sigma^2}} e^{-\frac{y^2}{4\sigma^2}} e^{i\Delta E x} \ln|1 - e^{-ax}| , \tag{5.43}$$

with the change of integral variables $x = \tau - \tau'$ and $y = \tau + \tau'$. The y-integral can be performed easily and gives an overall multiplicative factor of σ. Then we find

$$\mathcal{T}_1 \;\; \propto \;\; \frac{\pi}{2} \frac{\coth(\Delta E \pi / a)}{\Delta E} \; . \tag{5.44}$$

Since $\text{sgn}(u - u') = \text{sgn}(\tau - \tau')$, the contribution of the second term of (5.39) to the transition rate is

$$\mathcal{T}_2 \;\; \propto \;\; -\frac{\pi}{2\Delta E} \; . \tag{5.45}$$

The v-terms in (5.39) result in exactly the same contributions, therefore in- and outgoing particles contribute to the particle density in the same amount. The corresponding transition rate is

$$\mathcal{T} \propto 2 \left[\frac{\pi}{2} \frac{\coth(\Delta E \pi / a)}{\Delta E} - \frac{\pi}{2\Delta E} \right] = \frac{2\pi}{\Delta E} \frac{1}{e^{2\pi \Delta E / a} - 1} \; , \tag{5.46}$$

which means that the particle detector behaves as if it is immersed in a thermal bath of radiation with temperature $T = \frac{a}{2\pi}$.

5.2.2 Nonuniformly accelerated observer

The trajectory of an observer in Minkowski space-time whose acceleration changes with time can by derived from (5.13) and (5.14). To solve the integrals in (5.39) we first note that the argument of the logarithm in the first term in (5.39) can always be written as

$$u(\tau) - u(\tau') = \int_{\tau'}^{\tau} ds \; \dot{u}(s) \; , \tag{5.47}$$

where $\dot{u} = \frac{du}{d\tau}$ is given by (5.13). We assume that the detector is used for a measurement at the moment τ_0. Then we can expand $\dot{u}$ around τ_0,

$$\dot{u} \simeq \dot{u}_0 e^{-\frac{\ddot{u}_0}{\dot{u}_0}(\tau - \tau_0)} e^{-\frac{1}{2}(\frac{\ddot{u}_0}{\dot{u}_0})^{\cdot}(\tau - \tau_0)^2} . \tag{5.48}$$

This approximation is valid as long as the observer's acceleration is in the adiabatic regime,

$$\frac{\dot{a}}{a^2} \ll 1 \; . \tag{5.49}$$

It can be easily seen that the adiabatic condition coincides with the restriction (5.42), since we expect the temperature to be $T \sim a$ and therefore the difference between the detector states $\Delta E \sim a$. In this case it is enough to use the first order term in (5.48). This yields

$$u(\tau) - u(\tau') = \int_{\tau'}^{\tau} ds\, \dot{u}(s) \simeq \frac{\dot{u}_0^2}{\ddot{u}_0} \left[e^{-\frac{\ddot{u}_0}{\dot{u}_0}(\tau'-\tau_0)} - e^{-\frac{\ddot{u}_0}{\dot{u}_0}(\tau-\tau_0)} \right] . \tag{5.50}$$

Analogous to the calculation of the Rindler observer we can now determine the temperature. Since the time τ_0 is arbitrary we have

$$T \simeq -\frac{1}{2\pi} \frac{\ddot{u}}{\dot{u}} . \tag{5.51}$$

To check the validity of this approximation we apply it to the observer who is initially at rest and then is accelerated and finally approaches constant acceleration. In this case the result is

$$T \simeq \frac{a(\tau)}{2\pi} = \frac{1}{2\pi} \frac{a_f}{1 + e^{-a_f \tau}} . \tag{5.52}$$

Initially the observer will measure no radiation whereas during his late-time acceleration he detects particles. Finally the particle flux converges to that seen by a Rindler observer with constant acceleration a_f. Of course, we can only trust the result for large τ, when the motion of the accelerator satisfies the adiabatic condition (5.49).

We have shown that for time dependent acceleration a model particle detector is able to extract the time dependence of the particle flux. Further corrections to the temperature (5.51) can be determined as well. This question will be addressed in [37].

5.3 Hawking effect

Now we determine the transition rates for particle detectors in black hole space-time. In this section we consider a Schwarzschild black hole with constant mass M. In curved space-time the equations of motion of the detector are in general more complicated than in flat space-time. Furthermore, one has to impose proper boundary conditions for the scalar field. We assume that there are no ingoing modes, i.e. nothing falling into the black hole $\hat{\phi}(u) \sim e^{-i\Omega u}$. Ingoing v-modes originate from the past event horizon of an eternal black hole. This corresponds to choosing the Unruh vacuum state $|U\rangle$, see [54].

5.3.1 Static detector

We consider a detector that is located at constant distance from the black hole, also called a static detector. Of course, to stay at constant finite distance from the black hole we assume the detector is fitted with a suitable rocket. We recall that the background metric is given by

$$ds^2 = \bar{f}(u,v)dudv = f(U,V)dUdV \ , \tag{5.53}$$

where (u,v) and (U,V) are the Eddington-Finkelstein coordinates and the Kruskal coordinates, respectively, $\bar{f} = 1-\frac{2M}{r}$, and $f(U,V)$ is given by (1.11). From the transformation between the coordinates (1.8) and (1.9) we define

$$f(U,V) = \bar{f}(u,v)\frac{\partial u}{\partial U}\frac{\partial v}{\partial V} \equiv \bar{f}(u,v)B(U)A(V) = \bar{f}(u,v)e^{\left(\frac{u}{4M}-\frac{v}{4M}\right)} \ . \tag{5.54}$$

The acceleration of an observer at rest at $r = R$ can be determined from

$$a^k = \frac{\mathcal{D}u^k}{\mathcal{D}\tau} = \frac{du^k}{d\tau} + \Gamma^k_{ij}u^i u^j, \tag{5.55}$$

where u^k denotes the observer's proper velocity. With respect to the Schwarzschild background the observer feels the acceleration

$$a = \sqrt{a_k a^k} = \frac{M}{R^2}\frac{1}{\sqrt{1-\frac{2M}{R}}} \ . \tag{5.56}$$

This implies that if the observer is at rest at infinity he does not feel any acceleration, whereas on the horizon $R = 2M$ he feels an infinite force.

The detector's world line $(U(\tau), V(\tau))$ is parametrized by its proper time τ. The equations defining the observer's trajectory are

$$\dot{r} = r_{,U}\dot{U} + r_V\dot{V} = 0 \ , \tag{5.57}$$

$$f(U,V)\dot{U}\dot{V} = 1 \ , \tag{5.58}$$

where

$$\begin{aligned} r_{,U} &= B(U)r_{,u} = -\frac{1}{2}\bar{f}(u,v)B(U) \ , \\ r_{,V} &= A(V)r_{,u} = \frac{1}{2}\bar{f}(u,v)A(V) \ . \end{aligned} \tag{5.59}$$

Here and in the following $,v$ denotes partial derivative with respect to v and so on. Both $r_{,u}$ and $r_{,v}$ are obtained from the light ray equation in Eddington-Finkelstein coordinates. We have

$$r_{,v} = -r_{,u} = \frac{1}{2}\left(1-\frac{2M}{R}\right) \equiv \frac{1}{2}C_0^2 \ . \tag{5.60}$$

Using (5.57) and (5.58) we obtain

$$\dot{V} = \frac{1}{A(V)C_0}\,, \quad \text{and} \quad \dot{U} = \frac{1}{B(U)C_0}\,. \tag{5.61}$$

This implies the following relation between the proper time τ of the observer and the light cone coordinates u and v, respectively,

$$\frac{\partial v}{\partial \tau} = \frac{1}{C_0}\,, \quad \frac{\partial u}{\partial \tau} = \frac{1}{C_0}\,. \tag{5.62}$$

Then the observer's trajectory is given by

$$\begin{aligned} V(\tau) &= 4Me^{\frac{\tau}{4MC_0}}\,, \\ U(\tau) &= -4Me^{-\frac{\tau}{4MC_0}}\,. \end{aligned} \tag{5.63}$$

At infinity, $R \to \infty$, we find $C_0 = 1$ and $\tau = u = v$ up to an irrelevant constant. Using the transformation between the coordinates we obtain

$$B(\tau) = \frac{1}{A(\tau)}\,. \tag{5.64}$$

This result is in agreement with (5.58) even though $f(U,V)$ tends to zero for $R \to \infty$. Expanding $f((U(u),V(v)))$ for large arguments we find

$$f \to e^{\frac{(u-v)}{4M}}\,, \tag{5.65}$$

which exactly cancels the other contribution in (5.58).

The transition rate can be calculated analogously to (5.46). The only difference is that now only the outgoing modes contribute to the spectrum of radiation. We find

$$\mathcal{T} \propto \frac{\pi}{\Delta E}\frac{1}{e^{\frac{2\pi\Delta E}{4MC_0}} - 1}\,. \tag{5.66}$$

Hence, the temperature measured by a static detector includes a multiplicative factor that depends on its distance from the black hole

$$T_H(R) = \frac{1}{8\pi M\sqrt{1 - \frac{2M}{R}}}\,. \tag{5.67}$$

If the observer is at rest at infinity, $R \to \infty$, the detected temperature (5.67) coincides with the usual Hawking temperature. On the horizon $R = 2M$ the detector would measure an infinite temperature which corresponds to the fact that the proper acceleration of the static observer diverges on the horizon.

5.3.2 Freely falling observer

If we parametrize the world line of a massive test particle in this space-time by its proper time τ, then the equations of motion of the particle falling radially into the black hole can be derived from the Lagrangian

$$\mathcal{L} = f(U,V)\dot{U}\dot{V}, \tag{5.68}$$

where the dots denote the derivative with respect to proper time τ, see appendix B.3. In the Schwarzschild space-time where we have $f(U,V) = f(U \cdot V)$ the equations of motion of a freely falling observer become

$$\begin{aligned} 1 &= f(UV)\dot{U}\dot{V}, && (5.69)\\ \frac{V}{\dot{V}} &= \frac{U}{\dot{U}} + k\,, && (5.70)\\ \dot{r}^2 &= 2M\left(\frac{1}{r} - \frac{1}{R}\right), && (5.71) \end{aligned}$$

where k is a constant of integration, $r = r(UV)$, and R denotes the point where the observer is instantaneously at rest. If the observer is at rest at infinity (5.71) implies

$$\dot{r} = -\sqrt{\frac{2M}{r}}\,, \tag{5.72}$$

and k can be determined as $k = 8M$. If the freely falling detector starts at $r = R < \infty$ the constant of integration is

$$k = 8M\sqrt{1 - \frac{2M}{R}}\,. \tag{5.73}$$

From the above equations we compute

$$\begin{aligned} \frac{\dot{V}}{V} &= \frac{1}{4M} \cdot \frac{\sqrt{1 - \frac{2M}{r}} - \sqrt{\frac{2M}{r} - \frac{2M}{R}}}{\left(1 - \frac{2M}{r}\right)}\,, && (5.74)\\ \frac{\dot{U}}{U} &= -\frac{1}{4M} \cdot \frac{\sqrt{1 - \frac{2M}{r}} + \sqrt{\frac{2M}{r} - \frac{2M}{R}}}{(1 - \frac{2M}{r})}\,. && (5.75) \end{aligned}$$

By analogy to (5.12) we define

$$-\frac{\dot{f}}{f} = \frac{\ddot{U}}{\dot{U}} + \frac{\ddot{V}}{\dot{V}} \equiv a_U + a_V\,. \tag{5.76}$$

After that we interpret $T = -\frac{a_U}{2\pi}$ as the temperature measured by the freely falling detector, since there are no V-modes. Differentiating (5.75) with respect to τ we extract the relevant information:

$$-a_U = \left(1+\frac{2M}{r}\right)\left(\sqrt{1-\frac{2M}{R}}+\sqrt{\frac{2M}{r}-\frac{2M}{R}}\right), \quad (5.77)$$

$$a_V = \left(1+\frac{2M}{r}\right)\left(\sqrt{1-\frac{2M}{R}}-\sqrt{\frac{2M}{r}-\frac{2M}{R}}\right). \quad (5.78)$$

We conclude that the freely falling detector measures the approximate temperature

$$T(r,R) = \frac{1}{8\pi M}\left(1+\frac{2M}{r}\right)\left(\sqrt{1-\frac{2M}{R}}+\sqrt{\frac{2M}{r}-\frac{2M}{R}}\right), \quad (5.79)$$

and when it is instantaneously at rest measures the temperature

$$T(R) = \frac{1}{8\pi M}\left(1+\frac{2M}{R}\right)\sqrt{1-\frac{2M}{R}}\,. \quad (5.80)$$

Thus we find that a freely falling observer at rest on the horizon detects no thermal flux of particles. In [29, 30] Massar *et al.* calculated the energy momentum tensor in coordinates that are inertial at one point R. Their resulting temperature $T = T_H\sqrt{1-\frac{2M}{R}}$ also vanishes on the horizon[1]. By construction, in this case the observer is inertial at $R = 2M$ and his vacuum definition agrees with the Kruskal vacuum. It is not clear where the discrepancy between the results arises though it may be explained by a simple Doppler shift.

We find from (5.80) that a freely falling detector at rest at infinity measures the temperature

$$T(r) = \frac{1}{8\pi M}\left(1+\frac{2M}{r}\right)\left(1+\sqrt{\frac{2M}{r}}\right). \quad (5.81)$$

This result gives the usual Hawking temperature at infinity, but near the horizon implies a nonvanishing temperature. However, in this limit, adiabaticity

[1] Their choice of coordinates does not lead to Minkowski coordinates at the point R where the observer is at rest. They find the temperature $T = T_H(1-\frac{2M}{R})$ but proper rescaling of the coordinates gives $T = T_H\sqrt{1-\frac{2M}{R}}$.

is violated and we cannot trust our approximation. Another interesting approach is the derivation of Hawking radiation in Lemaitre coordinates by the complex path approach [48]. The authors showed that in this case they obtain Hawking's result with no changes to the thermal spectrum. The difference of this result to (5.81) is surprising since the Lemaitre coordinates are natural for a freely falling observer. Either the complex path approach is no proper non-adiabatic treatment of the problem or, more likely, it is not calculating what it claims to. The authors simply assert that there is no connection between their method and that of the Unruh detector.

5.4 Evaporating black hole

The detector response can also be calculated for black holes with time dependent mass. We determine the spectrum of radiation registered by a detector in the Vaidya space-time. It turns out that using the detector approach corrections to the temperature derived in chapter 4 can be determined.

5.4.1 Static observer

Converting the Vaidya solution (4.2) to double-null coordinates is straightforward, but it is not unique. We start with the form

$$ds^2 = \bar{F}\,(u,v)\,dudv - r(u,v)^2 d\Omega^2, \tag{5.82}$$

where

$$r_{,v} = \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right). \tag{5.83}$$

The conformal metric function $\bar{F}\,(u,v)$ as well as the outgoing light cone coordinate u still have to be determined from the physical properties of the space-time, the controlling equation for $\bar{F}$ being

$$\bar{F} = -2r_{,u} \tag{5.84}$$

which in addition has to satisfy the Einstein equation

$$\frac{\bar{F}_{,v}}{\bar{F}} = \frac{m}{r^2}\,. \tag{5.85}$$

With the requirement $\bar{F} \to 1$ for $r \to \infty$, the coordinates (5.82) are inertial at infinity and we call this the Schwarzschild-like solution. We now introduce other light cone coordinates U and V and connect them to the original ones via $U(u)$ and $V(v)$. The metric (5.82) becomes

$$ds^2 = F\,(U,V)\,dUdV - R(U,V)^2 d\Omega^2, \tag{5.86}$$

where $F\,(U,V)$ can be expressed in terms of the function $\bar{F}$. It is clear that $R = r$ and the transformation between the conformal factors is

$$F = \bar{F}\frac{\partial u}{\partial U}\frac{\partial v}{\partial V} \equiv \bar{F}B(U)A(V)\;. \tag{5.87}$$

The geodesic equation (5.83) for outgoing light rays in the new coordinates becomes

$$r_{,V} = \frac{1}{2}A(V)\left(1 - \frac{2M(v)}{r}\right)\;, \tag{5.88}$$

the conformal factor satisfies

$$F = -2A(V)r_{,U}\;, \tag{5.89}$$

and using the Einstein equation (5.85) becomes

$$\frac{F_{,V}}{F} = \frac{A'(V)}{A(V)} + \frac{A(V)M(v)}{r^2}\;, \tag{5.90}$$

where $A'(V) = \frac{dA(V)}{dV}$. Since the information about the registered particle flux is given by the U-modes we need to determine $B(U)$ which in principle results from

$$B(U) = \frac{F}{\bar{F}A(V)}\;, \tag{5.91}$$

once we have chosen the normalization of F. In fact, as we will see, $B(U)$ can be used to ensure the regularity of F on some surface.

We now turn to the considerations of the transformation $A(V)$. The nonvanishing Christoffel symbols of the metric (5.86) are

$$\Gamma^U_{UU} = \frac{F_{,U}}{F}\;, \quad \Gamma^V_{VV} = \frac{F_{,V}}{F}\;. \tag{5.92}$$

Using (5.90) it follows that the choice of $A(V)$ dictates the properties of Γ^V_{VV}, in particular where it vanishes. If we choose $A = 1$, then Γ^V_{VV} vanishes at infinity. This corresponds to an observer at rest at infinity and was therefore called Schwarzschild-like. For the Schwarzschild black hole, where $M(v) =$

M_0, the choice of vacuum was justified by two conditions on the coordinates: the vanishing of Γ^V_{VV} and regularity on the horizon $r = 2M_0$. In general we wish that the Christoffel symbols, at least Γ^V_{VV} vanish along some curve $r = r_C(v)$. This curve is defined in terms of Schwarzschild time v. From this we determine

$$A = e^{-\int dv \frac{M(v)}{r_C(v)^2}} \ . \tag{5.93}$$

After this we may write

$$V = \int dv e^{\int_v dv' \frac{M(v')}{r_C(v')^2}} . \tag{5.94}$$

For the Schwarzschild black hole with mass M_0 this yields $A = e^{-\frac{v}{4M_0}}$, which implies

$$V = 4M_0 e^{\frac{v}{4M_0}} \ . \tag{5.95}$$

Thus we call the solution (5.93) the Kruskal-like solution.

Analogous to the equations describing a static detector in stationary black hole space-time (5.57) and (5.58), we have

$$\dot{r} = r_{,U}\dot{U} + r_{,V}\dot{V} \ = \ 0 \ , \tag{5.96}$$

$$F(U,V)\dot{U}\dot{V} \ = \ 1 \ . \tag{5.97}$$

We are interested in the temperature $T \sim \frac{\ddot{U}}{\dot{U}}$ measured by a detector at $r \to \infty$. Using the above equations we obtain

$$\dot{V} = \frac{1}{A(V)\sqrt{1 - \frac{2M(v)}{r}}} \ , \tag{5.98}$$

and then find

$$\frac{\ddot{V}}{\dot{V}} = \frac{1}{\sqrt{1 - \frac{2M(v)}{r}}} \left(\frac{M(v)}{r_C(v)^2} + \frac{M'(v)}{r - 2M(v)} \right) \overset{r\to\infty}{\longrightarrow} \frac{M(v)}{r_C^2(v)} \ . \tag{5.99}$$

Using the fact that $\ddot{r} = 0$, we first calculate $\ddot{U}$ and finally

$$\frac{\ddot{U}}{\dot{U}} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \left[\frac{M}{r} + \frac{M'}{r - 2M} + \frac{AF_U}{F^2}\left(1 - \frac{2M}{r}\right) \right] \overset{r\to\infty}{\longrightarrow} \frac{AF_{,U}}{F^2} \ , \tag{5.100}$$

where for the sake of simplicity we omit the dependence on v in the formula. The relation between the coordinates and the proper time of the detector at infinity follows from

$$\dot{v} = \frac{1}{\sqrt{1 - \frac{2M(v)}{r}}} \quad \Rightarrow \quad \tau = \int dv \sqrt{1 - \frac{2M(v)}{r}} \, . \tag{5.101}$$

As expected, at infinity the v coordinate coincides with proper time τ up to a constant. The same result holds for u, where we have used the fact that $\bar{F} \to 1$ at infinity. For the physical measurement this constant is of importance. Since the space-time is asymptotically flat $\bar{F} \to 1$ we obtain that $r = \frac{1}{2}(v - u) + k$, where k is some constant. The outgoing light cone coordinate u can then be related to v via

$$u = v + 2(k - r). \tag{5.102}$$

Furthermore, by analogy to (5.64) we expect that

$$B\left(U(\tau)\right) = \frac{1}{A(V(\tau))} \, . \tag{5.103}$$

It is then straightforward to show that in this limit we also have

$$\frac{AF_{,U}}{F^2}\Big|_{u=\tau} = \frac{B_{,u}}{B}\Big|_{u=\tau} = -\frac{A_{,v}}{A}\Big|_{v=\tau} \, . \tag{5.104}$$

Let us now choose $r_C = r_e$, the time dependent horizon we introduced in chapter 4. It can be shown that this choice of r_e also insures that F is regular on the horizon and

$$F(U, V)\big|_{r=r_e} = const. \tag{5.105}$$

Another possibility for r_C is the apparent horizon r_a, as suggested by Visser *et al.* [59]. However, since F will not be constant along this line this spoils a proper vacuum definition. Then the thermal particles registered by the detector at τ have the temperature

$$T = \frac{1}{2\pi} \frac{M(\tau)}{r_e^2(\tau)} = \frac{1}{2\pi} \frac{M(v - v_*)}{r_e^2(v - v_*)} \, , \tag{5.106}$$

where $v_* = 2(r - k)$. The measurement of the detector at τ therefore provides information about the black hole parameters at the retarded time in $v - v_*$. This is a physically sensible result which becomes obvious in the light of our results in chapter 4. In the geometric optics approximation light rays arriving at infinity carry information about the initial moment $v_i = v - v_*$ when their distance to the horizon was very small.

Using the Unruh detector we have been able to determine thermal corrections to the temperature of the Hawking flux of the Vaidya solution compared to (4.36). The corrections depend on r_e which is the surface separating in- and outgoing light rays. Our result also clarifies the role of horizons in black hole evaporation and implies that r_e is the important horizon for nonstationary black holes. The fact that F is constant on this surface further confirms this claim. In our calculation we did not need the explicit form of the conformal factors F and $\bar{F}$. However, to study the response of a freely falling detector in Vaidya space-time at least numerical solutions will be needed.

Chapter 6

Discussion

The examination of quantum effects in black hole physics provides important hints on the way towards a unified theory of quantum gravity. We showed that the microscopic origin of black hole entropy can be understood as a result of missing information about the exact state of the matter from which the black hole was formed. When matter falls into a black hole, the overall change of its mass is the result of two competing effects. The black hole gains mass due to matter absorption, but in the meantime loses mass due to evaporation. We showed that the maximal amount of entropy matter can contribute in this process can be identified with the change of the black hole entropy, and that this applies to black holes made from ultra-relativistic particles as well as nonrelativistic particles.

Secondly, we examined the influence of the observer's motion and the choice of vacuum on the spectrum of Hawking radiation. Concerning observers at infinity, we found that the Hawking effect is indeed sensitive to their motion. The most obvious example is the Doppler shifted spectrum measured by an observer moving with constant velocity with respect to an observer at rest. On the other hand, the particle flux for an observer at infinity is not affected by a general transformation of the coordinates defining the initial vacuum state. To be precise, the Kruskal coordinates which are regular on the horizon but define an inertial frame at only one particular moment, can be replaced by other coordinates which are inertial on the horizon at any later moment.

Next, we determined the Hawking flux for black holes with time dependent mass, our model for the evaporating black hole being the Vaidya solution. We found that an observer at infinity in this space-time measures a time dependent thermal radiation with a temperature inversely proportional to the black hole mass at retarded time. As expected, the mass decrease leads to an increasing temperature and therefore enhances evaporation. Previous work on the particle flux in the Vaidya space-time with arbitrarily changing mass [63, 64, 49] was performed using analytic continuation and the results were only valid in the vicinity of the horizon; whereas we were able to consider observers very far away from the black hole. Unlike the other results our derivation allows a physical interpretation: In the geometric optics approximation the light rays at infinity carry information about the black hole mass at the time when they were very close to the apparent horizon.

Finally, we derived the response of a model particle detector, the Unruh thermometer, moving in curved background. In this way we were able to determine the particle flux for different observers in Minkowski, Schwarzschild and Vaidya space-time. We found that an arbitrarily accelerated detector in flat space-time registers a flux of radiation with a temperature proportional to its time dependent acceleration parameter. A detector moving in Schwarzschild space-time registers a predominantly thermal spectrum with the observed black hole temperature depending on the observer's trajectory. For example, a detector in free fall towards the black hole registers a particles flux depending on its radial distance from the black hole which vanishes when it crosses the horizon. This result is in agreement with that of [30] but not with [48]. In the latter case, the particle flux is computed within the complex path approach and even though coordinates describing a freely falling observer are used the thermal spectrum obtained corresponds to that of an observer at rest at infinity. This and the fact that the Unruh detector is physically motivated suggests that the complex path approach may not always be reliable.

Most importantly, using the Unruh detector we were able to determine thermal corrections to the Hawking spectrum for an observer in the Vaidya space-time. We found that the temperature does not only depend on the black hole mass, but picks up a correction depending on r_e, which is the horizon separating in- and outgoing light rays. Thus we showed that the important horizon for black hole evaporation is r_e and not the apparent horizon. From the physical perspective our result is quite appealing, since it connects the time of measurement to the black hole parameters in a natural way.

In addition to our results, it would be interesting to determine the particle flux registered by a freely falling detector in the Vaidya space-time as well. By analogy to a freely falling detector in the Schwarzschild space-time one expects that the thermal particle flux will vanish when the detector crosses the horizon r_e. Perhaps the spectrum measured when the detector crosses the apparent horizon gives further information about its meaning. It may be that numerical calculations will be needed to solve the controlling equations in this case.

Appendix A

Quantum fields in curved space-time

Here we collect some basic properties of quantum field theory (QFT) in curved backgrounds. For a review on this topic see also [8, 25, 33, 61].

A.1 QFT in flat space-time

In usual field theory, physical fields such as scalar or electromagnetic fields satisfy a wave equation in flat Minkowski space. For example a free real scalar field $\phi(x)$ with mass m satisfies the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0 \, , \tag{A.1}$$

where $\Box = \partial_\mu \partial_\nu \eta^{\mu\nu} = \partial_t^2 - \nabla^2$.

The field ϕ can be decomposed into positive and negative frequencies with respect to the coordinates, corresponding to the Killing vector ∂_t according to

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x} - i\omega_{\mathbf{k}} t} + a^\star_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x} + i\omega_{\mathbf{k}} t} \right) , \tag{A.2}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. In quantum theory the classical field ϕ is replaced by the field operator $\hat{\phi}$ which can be expanded in terms of annihilation and creation operators, $\hat{a}_k$ and $\hat{a}_k^\dagger$, resp. analogous to (A.2). The field operator $\hat{\phi}(x)$ satisfies the commutation relation

$$\left[\hat{\phi}(\mathbf{x}), \hat{\Pi}(\mathbf{x}') \right] = i\delta^3(\mathbf{x} - \mathbf{x}') \, , \tag{A.3}$$

where $\Pi(\mathbf{x}) = \dot{\hat{\phi}}(\mathbf{x})$ is the canonical momentum operator. Starting with these relations a standard Fock space can be constructed. The vacuum state $|0\rangle$ is the unique eigenvector of all annihilation operators $\hat{a}_\mathbf{k}$ with eigenvalue 0,

$$\hat{a}_\mathbf{k}|0\rangle = 0 \quad \text{for all } \mathbf{k} \ , \tag{A.4}$$

and excited states are obtained by the application of the creation operators $\hat{a}^\dagger_\mathbf{k}$ on the vacuum.

A.2 QFT in curved backgrounds

Now we consider quantum field theory on a globally hyperbolic space-time with metric $g_{\mu\nu}$ and $g = \det g_{\mu\nu}$. The Klein-Gordon equation for a massive scalar field now reads

$$(\Box + m^2)\phi(x) = 0 \ , \tag{A.5}$$

where

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\partial_\nu\right) . \tag{A.6}$$

Consider two solutions f_1 and f_2 of (A.5). These solutions are generalizations of the plane wave solutions,

$$f_k = \frac{1}{(2\pi)^{3/2}} \frac{e^{i\mathbf{kx} - i\omega_\mathbf{k} t}}{\sqrt{2\omega_\mathbf{k}}} \ . \tag{A.7}$$

The corresponding canonical momentum is defined as $\Pi = n^\mu \nabla_\mu f$, where n^μ is the normal vector with respect to a space-like hypersurface Σ. The inner product of the solutions is

$$(f_1, f_2) = i \int_\Sigma (f_1^* \Pi_2 - \Pi_1 f_2) d^3x = (f_2, f_1)^*, \tag{A.8}$$

which can be shown to be is independent of the choice of the hypersurface Σ, but is not positive definite. A complete set of solutions $\{f_k, f_k^*\}$ can be normalized according to

$$(f_k, f_k') = \delta(k, k') \quad \text{and} \quad (f_k, f_k'^*) = 0 \ . \tag{A.9}$$

Since $\{f_k, f_k^*\}$ is a complete set of solutions the field ϕ can be expanded as

$$\phi(x) = \int d\mu(k) \, (f_k a_k + f_k^* a_k^*) \, , \tag{A.10}$$

where $d\mu(k)$ stands for the integral measure.

A.2.1 Quantization

In flat space-time the classical field ϕ is replaced by the operator $\hat{\phi}$ which can be expanded in terms of the annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$. Excitations of the vacuum state, like $\hat{a}_k^\dagger|0\rangle$, are interpreted as particles. In curved space-time the interpretation is more complicated. Here the vacuum state can be defined as the eigenvector for all $\hat{a}_k$ with eigenvalue 0,

$$\hat{a}_k|0_k\rangle = 0 \quad \text{for all } k \,. \tag{A.11}$$

We consider a solution to the equation of motion (A.10), written in terms of field operators,

$$\hat{\phi}(x) = \int d\mu(k) \left(f_k \hat{a}_k + f_k^* \hat{a}_k^\dagger \right) . \tag{A.12}$$

In general relativity there are no distinguished coordinate frames, so the notion of positive and negative frequencies is no longer unambiguous and the definition of vacuum depends on the chosen set of solutions. This is a consequence of the general covariance of general relativity. We take this into account by labeling the vacuum state as $|0_k\rangle$. The annihilation and creation operators satisfy the commutation relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta(k - k') \,. \tag{A.13}$$

One can expand the field into a different complete set of solutions $\{g_K, g_K^*\}$ which , analogous to (A.12), yields

$$\hat{\phi}(x) = \int d\mu(K) \left(g_K \hat{b}_K + g_K^* \hat{b}_K^\dagger \right) . \tag{A.14}$$

The corresponding vacuum state is

$$\hat{b}_K|0_K\rangle = 0 \quad \text{for all } K. \tag{A.15}$$

The two different sets $\{f_k, f_k^*\}$ and $\{g_K, g_K^*\}$ can be connected by a transformation, called a Bogolyubov transformation. The relation between f_k and g_K is

$$g_K = \int d\mu(k) \left(\alpha_{K,k} f_k + \beta_{K,k} f_k^\star \right) , \tag{A.16}$$

where $\alpha_{K,k}$ and $\beta_{K,k}$ are the Bogolyubov coefficients, given by

$$\alpha_{K,k} = (f_k, g_K) \,, \quad \text{and} \quad \beta_{K,k} = -(f_k^\dagger, g_K) \,. \tag{A.17}$$

The Bogolyubov coefficients satisfy a normalization and completeness relation. In matrix notation (skipping the indices K and k for the sake of readability) we have

$$\begin{aligned} \alpha\alpha^\dagger - \beta\beta^\dagger &= 1\,, \\ \beta\alpha^T - \alpha\beta^T &= 0\,. \end{aligned} \tag{A.18}$$

It is possible also to connect the creation and annihilation operators of one frame $\hat{a}_k$ to those of the other frame $\hat{b}_K$ by

$$(\hat{a},\hat{a}^\dagger) = (\hat{b},\hat{b}^\dagger)\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} \alpha^* & -\beta^* \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}. \tag{A.19}$$

Since the vacua $|0_k\rangle$ and $|0_K\rangle$ corresponding to the operators $\hat{a}_k$ and $\hat{b}_K$ are different, one vacuum contains particles with respect to the other, and vice versa.

The particle number operator of particles of type k in a given state is $\hat{\hat{N}}_k = \hat{a}_k^\dagger \hat{a}_k$. Its expectation value with respect to the vacuum $|0_k\rangle$ is of course zero. On the other hand, if we compute the expectation value of the particle number operator $\hat{N}_K = \hat{b}_K^\dagger \hat{b}_K$, which gives us the number of "$\hat{b}_K$-particles", in the vacuum $|0_k\rangle$, the result does not vanish in general. The expectation value of $\hat{N}_K$ is

$$\langle \hat{N}_K \rangle = \langle 0_k | \hat{b}_K^\dagger \hat{b}_K | 0_k \rangle = \int d\mu(k) |\beta_{K,k}|^2\,. \tag{A.20}$$

The $\hat{a}_k$-vacuum $|0_k\rangle$ contains particles with frequency K. For quantum field theory in a curved background the vacuum definition and thus the whole particle concept is ambiguous if the Bogolyubov coefficients $\beta_{K,k}$ are nonzero.

A.2.2 QFT in two dimensions

In the following the above considerations will be applied to a scalar field in $1+1$ dimensions. A detailed consideration of this topic can be found in [33]. The action for a massless scalar field is

$$S = \int d^2x \sqrt{-g}\left(\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi\right), \tag{A.21}$$

where a, b take values 0 and 1. The corresponding equation of motion $(g^{ab}\phi_{,b}\sqrt{-g})_{,a}$ is conformally invariant, i.e. the form of (A.21) remains unchanged if metric in one coordinate frame differs only by an overall factor from the metric in another coordinate frame.

In two dimensions every metric can be written as Minkowski metric with a conformal factor, $ds^2 = F(t,x)(dt^2 - dx^2)$, so that the equation of motion can always be written as

$$\partial_t^2 \phi - \partial_x^2 \phi = 0 \, . \tag{A.22}$$

This equation has the solution

$$\phi(t,x) = A(t-x) + B(t+x) \tag{A.23}$$

with arbitrary smooth functions A and B. We can also rewrite the solution with mode expansion in k,

$$\phi(t,x) = \int d\mu(k) \, (f_k a_k + f_k^* a_k^*) \, , \tag{A.24}$$

where the mode functions f_k are functions of $x-t$. Using suitable coordinates (t_1, x_1) we can substitute ϕ by the operator $\hat{\phi}$ and express the solution in terms of ingoing and outgoing waves. In light cone coordinates $u_1 = t_1 - x_1$ and $v_1 = t_1 + x_1$ we have

$$\hat{\phi}(u_1, v_1) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \Big(e^{-i\omega u_1} \hat{a}_\omega + e^{i\omega u_1} \hat{a}_\omega^\dagger + e^{-i\omega v_1} \hat{a}_{-\omega} + e^{i\omega v_1} \hat{a}_{-\omega}^\dagger \Big) \tag{A.25}$$

where $\omega = |k|$ and $(2\pi)^{\frac{1}{2}}$ replaces the four-dimensional factor $(2\pi)^{3/2}$, u_1 is the advanced time coordinate, and v_1 is the retarded time coordinate. The two terms in (A.25) depending on u_1 describe outgoing waves, whereas the two other an ingoing ones. The operators $\hat{a}_\omega$ satisfy the usual commutation relation

$$[\hat{a}_\omega^-, \hat{a}_{\omega'}^+] = \delta(\omega - \omega') \, , \tag{A.26}$$

where $\hat{a}_\omega^- = \hat{a}_\omega$ and $\hat{a}_\omega^+ = \hat{a}_\omega^\dagger$. The vacuum state is defined as

$$\hat{a}_\omega |0_1\rangle = 0 \quad \text{for all } \omega \, . \tag{A.27}$$

In different coordinates (t_2, x_2) the field can be expanded in a very similar way. By analogy, using $u_2 = t_2 - x_2$ and $v_2 = t_2 + x_2$ we have

$$\hat{\phi}(u_2, v_2) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} \Big(e^{-i\Omega u_2} \hat{b}_\Omega + e^{i\Omega u_2} \hat{b}_\Omega^\dagger + e^{-i\Omega v_2} \hat{b}_{-\Omega} + e^{i\Omega v_2} \hat{b}_{-\Omega}^\dagger \Big) \tag{A.28}$$

In regions where the coordinates overlap we can expand the field $\hat{\phi}$ in both frames

$$\begin{aligned} \hat{\phi} &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \Big(e^{-i\omega u_1} \hat{a}_\omega + e^{i\omega u_1} \hat{a}_\omega^\dagger + e^{-i\omega v_1} \hat{a}_{-\omega} + e^{i\omega v_1} \hat{a}_{-\omega}^\dagger \Big) \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} \Big(e^{-i\Omega u_2} \hat{b}_\Omega + e^{i\Omega u_2} \hat{b}_\Omega^\dagger + e^{-i\Omega v_2} \hat{b}_{-\Omega} + e^{i\Omega v_2} \hat{b}_{-\Omega}^\dagger \Big) . \end{aligned} \tag{A.29}$$

A.2.3 Bogolyubov transformation

The operators $\hat{b}^{\pm}_{\Omega}$ can be expressed in terms of $\hat{a}^{\pm}_{\omega}$. We have

$$\hat{b}^{-}_{\Omega} = \int_{0}^{\infty} d\omega (\alpha_{\omega\Omega}\hat{a}^{-}_{\omega} + \beta_{\omega\Omega}\hat{a}^{+}_{\omega})\,, \tag{A.30}$$

with Bogolyubov coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$. The relation for the operator $\hat{b}^{+}_{\Omega}$ is the Hermitian conjugate of (A.30). Note that for simplicity a different convention for the Bogolyubov coefficients is used, that is $\alpha \to \alpha^*$ and $\beta \to -\beta^*$ compared to subsection A.2.1. Substituting (A.30) into (A.29) we find

$$\frac{1}{\sqrt{\omega}} e^{-i\omega u_1} = \int_{0}^{\infty} \frac{d\Omega'}{\sqrt{\Omega'}} \left(\alpha_{\omega\Omega'} e^{-i\Omega' u_2} - \beta_{\omega\Omega'} e^{i\Omega' u_2}\right). \tag{A.31}$$

By carrying out a Fourier transformation of the above equation with respect to u_2, the Bogolyubov coefficients can be calculated explicitly. This yields

$$\left.\begin{matrix}\alpha_{\omega\Omega} \\ \beta_{\omega\Omega}\end{matrix}\right\} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \frac{du_2}{2\pi} e^{i\Omega u_2 \mp i\omega u_1}\,, \tag{A.32}$$

where u_1 can be expressed in terms of u_2. Let us briefly consider the properties of the general Bogolyubov transformation

$$\hat{b}^{-}_{\Omega} = \int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega}\hat{a}^{-}_{\omega} + \beta_{\omega\Omega}\hat{a}^{+}_{\omega}\right)\,, \tag{A.33}$$

The relation is equal to (A.30) except for the range of integration which now covers $-\infty$ to ∞. This is justified since in (A.30) all Bogolyubov coefficients relating momenta of opposite sign vanish, i.e. $\alpha_{-\omega,\Omega} = 0$, $\beta_{-\omega,\Omega} = 0$. The normalization and completeness relation of the Bogolyubov coefficients read, see [33],

$$\int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega}\alpha^*_{\omega\Omega'} - \beta_{\omega\Omega}\beta^*_{\omega\Omega'}\right) = \delta(\Omega - \Omega')\,, \tag{A.34}$$

$$\int_{-\infty}^{\infty} d\omega \left(\alpha_{\omega\Omega}\beta_{\omega\Omega'} - \alpha_{\omega\Omega'}\beta_{\omega\Omega}\right) = 0\,, \tag{A.35}$$

analogous to the general form in (A.18). From this we can formally define the inverse Bogolyubov transformation

$$\hat{a}^{-}_{\omega} = \int_{-\infty}^{\infty} d\Omega \left(\alpha^*_{\omega\Omega}\hat{b}^{-}_{\Omega} - \beta_{\omega\Omega}\hat{b}^{+}_{\Omega}\right). \tag{A.36}$$

This relation can be easily verified by substituting in (A.30).

The b-particle operator is $\hat{N}_\Omega = \hat{b}^+_\Omega \hat{b}^-_\Omega$ and the average number of particles in the vacuum $|0_1\rangle$ is equal to its expectation value

$$\langle \hat{N}_\Omega \rangle = \langle 0_1 | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_1 \rangle = \int_0^\infty d\omega |\beta_{\omega\Omega}|^2. \tag{A.37}$$

Appendix B

Calculations

B.1 Potential barrier for Schwarzschild black hole

B.1.1 Black hole wave equation

The Klein-Gordon equation for a scalar field with mass m in Schwarzschild background (1.1) reads

$$\Box\Phi = \frac{1}{\sqrt{g}}\partial_\mu\left(\frac{1}{\sqrt{g}}g^{\mu\nu}\partial_\nu\Phi\right) = m^2\Phi\ . \tag{B.1}$$

Separating the angular variables as well as the time dependence $e^{-i\omega t}$ as usual, we find the wave equation

$$\left[\frac{d^2}{dr_\star^2} + \omega^2 - V(r)\right]\phi(r) = 0\ , \tag{B.2}$$

with the potential

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right]\ , \tag{B.3}$$

where $dr_\star = dr/(1-2M/r)$ is the well-known tortoise coordinate. If we introduce the dimensionless variable $x = r/(2M)$, equation (B.2) becomes

$$\phi'' + \frac{1}{x(x-1)}\phi' + \left[\frac{4M^2\omega^2x^2}{(x-1)^2} - \frac{4M^2m^2x}{(x-1)} - \frac{l(l+1)}{x(x-1)} - \frac{1}{x^2(x-1)}\right]\phi = 0 \tag{B.4}$$

where the prime denotes the derivative with respect to x.

B.1.2 Potential barrier for nonrelativistic particles

If we want to determine the Gray factor of the Hawking radiation of nonrelativistic particles, we have to calculate the absorption probability of particles of mass m in the black hole background. We compare the above potential (B.3) to the two-dimensional case, ${}^2\Box\Phi - m^2\Phi = 0$, which yields the wave equation

$$\left[\frac{d^2}{dr_\star^2} + \omega^2 - V_{2D}(r)\right]\phi(r) = 0\,, \tag{B.5}$$

with the potential

$$V_{2D}(r) = \left(1 - \frac{2M}{r}\right)m^2. \tag{B.6}$$

It is obvious that in the limit of large particle mass $mM \gg 1$ the four-dimensional problem reduces to the 2-dimensional one, $V(r) \to V_{2D}(r)$. We know that in the nonrelativistic limit the temperature T of a particle is much smaller than its mass, $\frac{m}{T} \gg 1$, and is inversely proportional to the black hole mass M. Hence, it is justified to use the two-dimensional approximation. On the other hand, in distinction to the massless case, even in two dimensions there is a potential well for nonrelativistic particles. The ingoing wave will be scattered at the potential barrier. The corresponding boundary conditions are

$$\phi = \begin{cases} \sqrt{\frac{k}{\omega}}T(\omega)e^{-i\omega r_\star} & ,\ r_\star \to -\infty \\ \sqrt{\frac{\omega}{k}}\left[e^{-ikr_\star}(2kr_\star)^{-iMm^2/k} + R(\omega)e^{ikr_\star}(2kr_\star)^{iMm^2/k}\right] & ,\ r_\star \to +\infty\,, \end{cases} \tag{B.7}$$

where $k = \sqrt{\omega^2 - m^2} \gg \omega$, as we are studying the nonrelativistic limit. Using the above normalization we ensure that $1 = |R|^2 + |T|^2$. We substitute $\phi = \sqrt{\frac{x}{x-1}}\xi$, and get the approximate equation,

$$\frac{d^2\xi}{dx^2} + \left[\frac{4M^2\omega^2x^2}{(x-1)^2} - \frac{4M^2m^2x}{(x-1)}\right]\xi(x) = 0\,. \tag{B.8}$$

Equivalently, in terms of k that is

$$\frac{d^2\xi}{dx^2}\frac{(x-1)^2}{4M^2} + (k^2x^2 + m^2x)\xi(x) = 0\,. \tag{B.9}$$

Using ξ the boundary conditions are

$$\xi = \begin{cases} \sqrt{\frac{k}{\omega}}T(\omega)(x-1)^{\frac{1}{2}-2iM\omega} & ,\ r_\star \to -\infty \\ \sqrt{\frac{\omega}{k}}\Big\{e^{-2iMk(x-1)}(x-1)^{-2iMk}\left[4Mk(x-1)\right]^{-\frac{iMm^2}{k}} & \\ +R(\omega)e^{2iMk(x-1)}(x-1)^{2iMk}\left[4Mk(x-1)\right]^{\frac{iMm^2}{k}}\Big\} & ,\ r_\star \to +\infty\,. \end{cases} \tag{B.10}$$

The above equation can be solved in terms of generalized hypergeometric functions of first order, so-called Kummer functions $\mathcal{M}$. Using $\omega > m$ and $M\omega \gg 1$ we find

$$\begin{aligned}\xi(x) &= C_1(x-1)^{\frac{1+a}{2}} e^{-\frac{1}{2}b(x-1)} \mathcal{M}\left(\frac{1+a}{2}+\tilde{k}, 1+a, b(x-1)\right) \\ &+ C_2(x-1)^{\frac{1-a}{2}} e^{-\frac{1}{2}b(x-1)} \mathcal{M}\left(\frac{1-a}{2}+\tilde{k}, 1-a, b(x-1)\right)\end{aligned} \quad (B.11)$$

where $a = 4i\omega M$, $b = 4iMk$ and $\tilde{k} = i\frac{M}{k}(2k^2+m^2)$. From the boundary conditions on the horizon we conclude that $C_1 = 0$ and $C_2 = 1\sqrt{\frac{\omega}{k}}$. The solution also has to satisfy the boundary conditions at infinity, where we can use the asymptotics of the Kummer function for large z, see [18],

$$\frac{\mathcal{M}(a_1,a_2,z)}{\Gamma(a_2)} \to \frac{e^{i\pi a_1} z^{-a_1}}{\Gamma(a_2-a_1)} + \frac{e^z z^{a_1-a_2}}{\Gamma(a_1)} . \quad (B.12)$$

We then have

$$\xi \to C_2 b^{\frac{a-1}{2}} \Gamma(1-a) \left\{ \frac{e^{i\pi(\frac{1-a}{2}+\tilde{k})} \left[b(x-1)\right]^{-\tilde{k}} e^{-\frac{1}{2}b(x-1)}}{\Gamma(\frac{a-1}{2}-\tilde{k})} + \frac{\left[b(x-1)\right]^{-\tilde{k}} e^{\frac{1}{2}b(x-1)}}{\Gamma(\frac{a-1}{2}+\tilde{k})} \right\}.$$

Then the transmission and reflection coefficient, resp., are

$$\begin{aligned}|T|^2 &= \frac{1-e^{-8\pi M\omega}}{1+e^{-4\pi M(\omega+k)} e^{-\frac{2\pi M m^2}{k}}} , \\ |R|^2 &= e^{-4\pi M\omega} \frac{\cosh(2\pi M(\omega-k) - \frac{\pi M m^2}{k})}{\cosh(2\pi M(\omega+k) + \frac{\pi M m^2}{k})} ,\end{aligned} \quad (B.13)$$

which satisfy the relation $|T|^2 = 1 - |R|^2$. The physical interpretation of the above boundary condition is that no wave should leave the black hole, which corresponds to the classical scattering problem. In the situation of particle creation we have of course an outgoing wave, which is reflected at the black hole barrier. But as we will show this yields the same result for the coefficients. Considering the general boundary condition

$$\phi = \begin{cases} \sqrt{\frac{k}{\omega}} \left(Ae^{i\omega r_\star} + Be^{-i\omega r_\star}\right) & , \; r_\star \to -\infty \\ \sqrt{\frac{\omega}{k}} \left[Ce^{ikr_\star}(2kr_\star)^{iMm^2/k} + De^{-ikr_\star}(2kr_\star)^{-iMm^2/k}\right] & , \; r_\star \to +\infty , \end{cases} \quad (B.14)$$

we determine the constraint on the coefficients from the constant Wronskian

$$|A|^2 - |B|^2 = |C|^2 - |D|^2. \quad (B.15)$$

The classical scattering corresponds to $A = 0, B = T, C = R$, and $D = 1$, whereas for particle creation we have $A = 1, B = R, C = T$ and $D = 0$. We also check limiting cases for small and large k,

$$\begin{aligned} k &\rightarrow 0 \\ |T|^2 &\rightarrow (1 - e^{-8\pi M\omega}) \\ |R|^2 &\rightarrow e^{-4\pi M\omega} e^{-4\pi M\omega} = e^{-8\pi M\omega} , \end{aligned} \tag{B.16}$$

$$\begin{aligned} k &\rightarrow \infty \\ |T|^2 &\rightarrow (1 - 2e^{-8\pi M\omega}) \\ |R|^2 &\rightarrow e^{-4\pi M\omega}(e^{-4\pi M\omega} + e^{-4\pi M\omega}) = 2e^{-8\pi M\omega} . \end{aligned} \tag{B.17}$$

B.2 Stability of Hawking radiation

B.2.1 Explicit calculation of the integral K(x)

Here we compute the integral

$$K(x) = \int_0^\infty \frac{d\omega}{\omega} e^{i\omega x} . \tag{B.18}$$

As noted in chapter 3, we first calculate the renormalized integral

$$K(x) = \int_0^\infty \frac{d\omega}{\omega} e^{-(\alpha - ix)\omega} \tanh(\beta\omega) , \tag{B.19}$$

and then let $\alpha \rightarrow 0$ and $\beta \rightarrow \infty$. For simplicity we substitute $\alpha - ix \rightarrow \alpha$ in the calculation and later convince ourselves that the result is also valid for complex α. We find

$$\frac{\partial K(\alpha, \beta)}{\partial \alpha} = -J(\alpha, \beta) , \tag{B.20}$$

where

$$J = \int_0^\infty d\omega e^{-\alpha\omega} \tanh(\beta\omega) . \tag{B.21}$$

The calculation of this integral is quite straight forward,

$$\begin{aligned} J &= \int_0^\infty d\omega e^{-\alpha\omega} \frac{1 - e^{-2\beta\omega}}{1 + e^{-2\beta\omega}} \\ &= \sum_{m=0}^\infty (-1)^m \int_0^\infty d\omega e^{-\alpha\omega} (1 - e^{-2\beta\omega}) e^{-2\beta\omega m} \\ &= \sum_{m=0}^\infty \frac{(-1)^m}{2\beta} \left(\frac{1}{\frac{\alpha}{2\beta} + m} - \frac{1}{\frac{\alpha}{2\beta} + 1 + m} \right) , \end{aligned} \tag{B.22}$$

and implies, see [18] for the function $\Psi(z)$,

$$\frac{\partial K(\alpha,\beta)}{\partial\alpha} = -\frac{1}{\alpha} + \Psi\left(\frac{\alpha}{4\beta}+1\right) - \Psi\left(\frac{\alpha}{4\beta}+\frac{1}{2}\right) . \tag{B.23}$$

Now the equation (B.23) can easily be integrated. Using the proper boundary condition $K \to 0$ for $\alpha \to \infty$ we get

$$K(x) = -\ln\frac{\alpha}{4\beta} + 2\ln\frac{\Gamma(\frac{\alpha}{4\beta}+1)}{\Gamma(\frac{\alpha}{4\beta}+\frac{1}{2})} \tag{B.24}$$

Replacing $\alpha \to \alpha - ix$ we determine the limit $\alpha \to 0$,

$$\begin{aligned} K(x) &= -\ln\frac{\alpha-ix}{4\beta} + 2\ln\frac{\Gamma(\frac{\alpha-ix}{4\beta}+1)}{\Gamma(\frac{\alpha-ix}{4\beta}+\frac{1}{2})} \\ &\to \ln\beta - \ln(\alpha-ix) \\ &\to \ln\beta - \ln|x| + \frac{i\pi}{2}\mathrm{sgn}(x). \end{aligned} \tag{B.25}$$

B.3 Freely falling observers

B.3.1 Euler-Lagrange equations

The trajectory of a massive test particle in an arbitrary background can be determined using the Euler-Lagrange equations. If the geodesic world line of the observer is parametrized by its proper time τ, then the equations of motion can be derived from the Lagrangian

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{B.26}$$

where the dots denote the derivative with respect to proper time τ. The corresponding line element of the background space-time is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{B.27}$$

The particle's four-velocity $\dot{x}^\mu$ is normalized as usual,

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1. \tag{B.28}$$

The Euler-Lagrange equations read

$$\partial_\tau\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = \frac{\partial\mathcal{L}}{\partial x^\mu} , \tag{B.29}$$

which are equivalent to

$$\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0. \tag{B.30}$$

In the following we focus on spherically symmetric background space-times and consider the general spherically symmetric time dependent line element

$$ds^2 = g(r,t)dt^2 - g(r,t)^{-1}dr^2 - r^2 d\Omega^2. \tag{B.31}$$

The Lagrangian for a massive test particle falling radially towards the centre of the black hole can be derived easily

$$\mathcal{L} = g(r,t)\dot{t}^2 - g(r,t)^{-1}\dot{r}^2. \tag{B.32}$$

Using the Euler-Lagrange equations (B.29) we get

$$\partial_\tau \left[2g(r,t)\dot{t}\right] = \partial_t f\dot{t}^2 - \partial_t \left(f^{-1}\right)\dot{r}^2\,, \tag{B.33}$$
$$\partial_\tau \left[2g(r,t)^{-1}\dot{r}\right] = -\partial_r f\dot{t}^2 + \partial_r \left(f^{-1}\right)\dot{r}^2\,. \tag{B.34}$$

In addition, we use the normalization condition of the particle's four-velocity

$$g(r,t)\dot{t}^2 - g(r,t)^{-1}\dot{r}^2 = 1\,. \tag{B.35}$$

First we study an observer falling into a static black hole. In this case we have $g = g(r)$. The first equation of (B.34) can be integrated which yields

$$\dot{t} = \frac{c_1}{2f(r)}\,, \tag{B.36}$$

where c_1 is a constant of integration. Hence we get

$$\dot{r} = -\sqrt{\frac{c_1^2}{4} - f(r)}\,, \tag{B.37}$$

where we have chosen the minus sign since we consider an infalling observer. The particle's rest energy is denoted by E. Using the identification $E = \frac{1}{2}c_1$ the observer's trajectory is

$$\frac{dr}{dt} = -\frac{f(r)}{E}\sqrt{E^2 - f(r)}. \tag{B.38}$$

If the observer is at rest at infinity then its rest energy is $E = 1$, and at infinity t coincides with the proper time coordinate τ. For the Schwarzschild black hole we have $f(r) = 1 - \frac{2M}{r}$. With respect to the observer's proper time we now have

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}. \tag{B.39}$$

The integration of this equation is quite straight forward. As a result we determine the proper time $\Delta\tau$ which the observer takes to fall from r_1 to r.

$$\Delta\tau = \tau - \tau_0 = \frac{2}{3} 2M \left[\left(\frac{r_1}{2M} \right)^{3/2} - \left(\frac{r}{2M} \right)^{3/2} \right] . \tag{B.40}$$

This shows that even though for an outside observer it takes an infinite time for the particle to fall into the black hole, in the particle's rest frame the measured time interval is finite.

B.3.2 Freely falling observer in light cone coordinates

Alternatively, we can consider the metric in light cone coordinates

$$ds^2 = f(U, V) dU dV \, . \tag{B.41}$$

Then the equation of motion of the radially infalling particle can be derived from the Lagrangian

$$\mathcal{L} = f(U, V) \dot{U} \dot{V} , \tag{B.42}$$

where the dots still denote the derivative with respect to proper time τ. Applying the Euler-Lagrange equations (B.29) we obtain

$$\begin{aligned} \frac{\ddot{V}}{\dot{V}} &= -\frac{(\partial_V f) \dot{V}}{f} \\ \frac{\ddot{U}}{\dot{U}} &= -\frac{(\partial_U f) \dot{U}}{f} . \end{aligned} \tag{B.43}$$

In light cone coordinates the normalization condition of the four-velocity becomes

$$f(U, V) \dot{U} \dot{V} = 1 \, , \tag{B.44}$$

which together with (B.43) yields

$$\frac{\ddot{U}}{\dot{U}} + \frac{\ddot{V}}{\dot{V}} = -\frac{\dot{f}}{f} . \tag{B.45}$$

If we assume that $f = f(U \cdot V)$ the above equations simplify and we obtain

$$\frac{U}{\dot{U}} + k = \frac{V}{\dot{V}} \, , \tag{B.46}$$

where k is a constant of integration. In addition $r = r(U \cdot V)$ satisfies the equation

$$\dot{r}^2 = 2M \left(\frac{1}{r} - \frac{1}{R} \right), \tag{B.47}$$

where R is the point where the observer is at rest. Hence, (B.46) and (B.47) are the controlling equations for a freely falling observer in the black hole space-time.

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