Compactifications of the Heterotic String with Unitary Bundles

Timo Weigand

Munich 2006

Compactifications of the Heterotic String with Unitary Bundles

Dissertation
an der Fakultät für Physik
Ludwig-Maximilians-Universität München

vorgelegt von
Timo Weigand
aus Freiburg im Breisgau

München, den 23. Mai 2006

Gutachter

---

1. Gutachter: Prof. Dr. Dieter Lüst
2. Gutachter: Prof. Dr. Ivo Sachs

Datum der mündlichen Prüfung: 14. Juli 2006

To the memory of my father and to my mother

Zusammenfassung

In dieser Dissertation untersuche ich eine große neue Klasse vierdimensionaler supersymmetrischer Stringvakua, definiert als Kompaktifizierungen des $E_8 \times E_8$ und des $SO(32)$ heterotischen Strings auf glatten komplex-dreidimensionalen Calabi-Yau-Mannigfaltigkeiten mit unitären Eichbündeln und heterotischen Fünfbranen. Dies ermöglicht die Konstruktion phänomenologisch interessanter Stringkompaktifizierungen auf einfach zusammenhängenden Mannigfaltigkeiten insofern die konventionelle Eichbrechung mittels Wilsonlinien ersetzt wird durch die Einbettung nicht-trivialer Linienbündel in die zehndimensionale Eichgruppe.

Im ersten Teil der Arbeit wird die Anwendung dieser Idee auf den $E_8 \times E_8$ heterotischen String diskutiert. Auf die Definition einer großen Klasse gruppentheoretischer Einbettungen mit unitären Bündeln folgt die Analyse der effektiven vierdimensionalen $\mathcal{N} = 1$ Supergravitationstheorie. Das gleichzeitige Auftreten von Fünfbranen und abelschen Eichfeldern erfordert die Einführung neuer anomaliekürzender Gegenterme in die effektive Wirkung. Diese werden ferner mithilfe einer M-Theorierechnung hergeleitet. Die vollständigen Green-Schwarz-Terme ermöglichen es, die Ein-Loop-Korrekturen der Eichkopplungen zu berechnen. Aus dem eichinvarianten Kählerpotential der Modulifelder leite ich eine perturbative Ein-Loop-Modifizierung des Fayet-Iliopoulos D-Termes ab. Darauf aufbauend schlage ich eine Deformation der hermiteschen Yang-Mills-Gleichung in erster Ordnung Störungstheorie vor und führe außerdem die Idee der λ-Stabilität als das perturbativ exakte Stabililtätskonzept ein, welches die in nullter Ordnung gültige Mumford-Stabilität ersetzt.

Im folgenden definiere ich eine Klasse $SO(32)$ heterotischer Vakua mittels unitärer Bündel und heterotischer Fünfbranen. Das sich ergebende Spektrum steht im Einklang mit der S-dualen Typ-I- Theorie bzw. den Typ-IIB-Orientifolds. Im Rahmen einer analogen Analyse der vierdimensionalen Supergravitation findet die vorgeschlagene Ein-Loop-Korrektur der Stabilitätsbedingung weitere Untermauerung, indem die Korrekturen im heterotischen Bild als das S-duale Analogon des perturbativen Anteils der Π-Stabilitätsbedingung identifiziert werden. Letztere ist als das korrekte Stabilitätskonzept in der Typ-IIB-Theorie bekannt.

Es folgt eine Darstellung der Konstruktion stabiler holomorpher Vektorbündel auf elliptisch gefaserten Calabi-Yau-Mannigfaltigkeiten mit Hilfe der Methode spektraler Überdeckungen. Daraufhin präsentiere ich semirealistische Beispiele $SO(32)$ heterotischer Vakua mit Pati-Salam und MSSM-ähnlichen Eichsektoren. Diese verallgemeinern, im S-dualen Bild, das Konzept von magnetisierten $D9$-Branen auf toroidalen Hintergründen zu nicht-abelschen Braneworlds auf echten Calabi-Yau-Mannigfaltigkeiten.

Den Abschluss der Arbeit bildet die Konstruktion realistischer Vakua mit flipped $SU(5)$ GUT und MSSM Eichgruppe im Rahmen der $E_8 \times E_8$-Theorie und auf der Grundlage der Einbettung von Linienbündeln in beide E_8-Faktoren. Einige der phänomenologisch attraktiven Eigenschaften der stringtheoretischen Realisierung von flipped $SU(5)$ Modellen, insbesondere die Stabilität des Protons, werden diskutiert. MSSM-artige Eichkopplungsvereinheitlichung ist für die auf Ein-Loop-Ebene korrigierten Eichkopplungen möglich. Ich konstruiere einige explizite supersymmetrische Stringvakua, sowohl mit GUT als auch direkt mit Standardmodelleichgruppe, die genau die beobachteten drei Generationen chiraler Materie ohne weitere exotische chirale Fermionen zeigen.

Abstract

In this thesis we investigate a large new class of four-dimensional supersymmetric string vacua defined as compactifications of the $E_8 \times E_8$ and the $SO(32)$ heterotic string on smooth Calabi-Yau threefolds with unitary gauge bundles and heterotic five-branes. This opens up the way for phenomenologically interesting string compactifications on simply connected manifolds in that the conventional gauge symmetry breaking via Wilson lines is replaced by the embedding of non-flat line bundles into the ten-dimensional gauge group.

The first part of the thesis discusses the implementation of this idea into the $E_8 \times E_8$ heterotic string. After specifying a large class of group theoretic embeddings featuring unitary bundles, we analyse the effective four-dimensional $\mathcal{N} = 1$ supergravity upon compactification. The simultaneous presence of five-branes and abelian gauge groups requires the introduction of new anomaly cancelling counter terms into the effective action. These are also derived by an M-theory computation. The full set of Green-Schwarz terms allows for the extraction of the threshold corrections. From the gauge invariant Kähler potential for the moduli fields we derive a modification of the Fayet-Iliopoulos D-terms arising at one-loop in string perturbation theory. From this we conjecture a one-loop deformation of the Hermitian Yang-Mills equation and introduce the idea of λ-stability as the perturbatively correct stability concept generalising the notion of Mumford stability valid at tree-level.

We then proceed to a definition of $SO(32)$ heterotic vacua with unitary gauge bundles in the presence of heterotic five-branes and find agreement of the resulting spectrum with the S-dual framework of Type I/Type IIB orientifolds. A similar analysis of the effective four-dimensional supergravity is performed. Further evidence for the proposed one-loop correction to the stability condition is found by identifying the heterotic corrections as the S-dual of the perturbative part of Π-stability as the correct stability concept in Type IIB theory.

After reviewing the construction of holomorphic stable vector bundles on elliptically fibered Calabi-Yau manifolds via spectral covers, we provide semi-realistic examples for $SO(32)$ heterotic vacua with Pati-Salam and MSSM-like gauge sectors. These can be viewed, by S-duality, as the generalisation of toroidal magnetized $D9$-branes to non-abelian braneworlds on genuine Calabi-Yau manifolds.

We finally discuss the construction of realistic vacua with flipped $SU(5)$ GUT and MSSM gauge group within the $E_8 \times E_8$ framework, based on the embedding of line bundles into both E_8 factors. Some of the appealing phenomenological properties of this stringy realisation of flipped $SU(5)$ models, in particular stability of the proton, are discussed. MSSM-like gauge coupling unification is possible for the threshold corrected gauge couplings. We explicitly construct a couple of supersymmetric string vacua in both setups with precisely the three observed chiral matter generations and without any exotic chiral states.

Acknowledgements

I would like to thank all members of the String Theory Group of Professor Dieter Lüst at the Max-Planck-Institut and Ludwig-Maximilans-Universität, München, as well as the High Energy Group at the Department of Applied Mathematics and Theoretical Physics, Cambridge, for providing such an outstanding working atmosphere which I had the privilege to enjoy during the period of my PhD.

I am indebted to Ralph Blumenhagen for sharing with me his insights into String Theory in the course of innumerable and time-consuming discussions, for his steady support and the very stimulating and intensive supervision of my work. Likewise, I am indebted to Dieter Lüst for his continuous, reliable support throughout all stages of my PhD period and for many very challenging, intensive and fruitful discussions. I thank both of them for the big investments they make into their students, who always have the feeling of being taken seriously as colleagues.

I am very grateful to Gabriele Honecker for the highly enjoyable and fruitful collaboration in several projects throughout my time in Munich.
I would like to thank Florian Gmeiner for being such a good PhD colleague, in particular for many important discussions and for collaboration. I am also very grateful to him for patiently explaining to me the mysteries of my computer.

I thank Joe Conlon, Johanna Erdmenger, Claus Jeschek, Dan Oprisa, Hendryk Pfeiffer, Erik Plauschinn, Giuseppe Policastro, Fernando Quevedo, Sakura Schäfer-Nameki, Maximilian Schmidt-Sommerfeld, Aninda Sinha, Stefan Stieberger and Dimitrios Tsimpis for many interesting discussions. I am grateful to Sebastian Moster for the fruitful collaboration begun in the last stages of my PhD.
Special thanks to Gabriele Honecker and Maximilian Schmidt-Sommerfeld for many important remarks on parts of this manuscript.

I am grateful to Christopher Padfield and Tom Körner, Trinity Hall, as well as Fernando Quevedo for support during the first stages of my PhD in Cambridge.

I thank the German Academic Exchange Service, the German National Academic Foundation, Trinity Hall Cambridge, EPSRC and above all the Maximilianeum, Munich, for financial support during my studies.

Finally, I am indebted to my parents, my family and my friends, for all. Words cannot account for that.

Contents

Chapter 1

Introduction

1.1 Prologue: An invitation to String Theory

The quest for a fundamental theory of the observed gravitational, electro-weak and strong interactions is one of the most pressing intellectual challenges of our time. Among the heritage of the past century are two beautiful, complementary and intriguingly successful attempts to describe particular corners of the physical world we observe - General Relativity and Quantum Field Theory. It is well-known that they both reproduce and predict a huge amount of empirical data with breath-taking accuracy. It is equally well-known, however, that they are both unacceptable as *fundamental* physical theories. They carry inside themselves the seed for their eventual incompleteness in the disguise of unphysical infinities which signal the inevitable breakdown of their validity.

General Relativity collapses as a well-defined theory whenever a massive object with a radius smaller than its Schwarzschild radius collapses under its self-gravitation to form a black hole. What is puzzling is that even though the initial conditions involve a well-defined extended object, like a sufficiently heavy star undergoing the final stages of its evolution, the dynamical laws of gravity force this mass to contract to a pointlike massive object with a formally infinite density. One might argue that the very concept of pointlike objects, familiar from classical mechanics, is merely an idealisation and no reason to worry, but the situation just described is of a totally different genre. We start with very physical and sensible initial conditions, and are inevitably driven, by the equations of motion, into a regime where some of the most fundamental assumptions of the theory such as the notion of spacetime as a smooth manifold break down. Clearly, as a pragmatic outside observer we will never be affected by the unphysical center of the black hole due to the event horizon surrounding it. But the theory is incomplete in the sense that there exist situations inside its domain of regime to which it cannot be sensibly applied. Apparently, at some stages of such a pathological process, Nature obeys different laws of gravity.

Quantum Field Theory breaks down when a charged matter particle interacts with the vector bosons coupling to the, say, electro-magnetic field it sources -

even the first loop diagram in Quantum Electrodynamics related to the self-energy of the electron formally diverges. Again we can - and do - hide the infinity for practical purposes by introducing a cutoff, and the fact that it is possible to extract non-trivial information using this technique of regularisation and renormalisation at all is certainly a miracle by itself. Still, the need for such a procedure is unsatisfactory because it indicates the breakdown of the dynamical laws at high energies. In both cases we face the paradox that we have at our disposal a powerful *formalism* in triumphant agreement with experiments and observations which at the same time is incomplete as a physical *theory*. It yields an empirically successful *effective* description of certain phenomena after we agree to integrate out those high energy degrees of freedom which are apparently not accounted for correctly.

The situation is not ameliorated if one takes into account the mutual incompatibility between the classical, deterministic character of General Relativity and the intrinsically probabilistic nature of Quantum Mechanics in its conventional interpretation as the conceptual foundation of Quantum Field Theory. At this stage by the very latest one cannot close one's eyes any longer since physical processes at such high energies that the gravitational interaction cannot be consistently neglected require, and be it merely for the sake of an effective approach, a genuinely quantum description of gravity together with the other forces.

Apart from these indisputable conceptual issues there is an aesthetic one. It is often stated that the Standard Model of Particle Physics contains at least 19 free parameters in the form of the masses and couplings of the observed particles. This is an optimistic point of view, because, if one wants to be malicious, it actually involves an infinite number of free parameters. A theory should not only explain what we observe, but also what we do not observe[1], and Quantum Field Theory knows of no underlying intrinsic principle whatsoever which singles out the Standard Model inside the moduli space of anomaly-free gauge theories - except that we happen to observe it.

The ultimate goal of String Theory [1–7][2] is none less than to overcome all these difficulties and to provide a consistent ultra-violet completion of both Quantum Field Theory and General Relativity. What is remarkable is that one and the same concept appears to have the potential to tackle both challenges simultaneously. The basic idea is to avoid the infinities of Quantum Field Theory by smoothening the apparently unphysical interaction vertices, thus leading to ultra-violet finite loop amplitudes. This is the purpose of introducing the notion of one-dimensional extended objects as the fundamental entities. Everything else is forced upon us by requiring a consistent quantisation of the classical theory of the string propagating in spacetime. *Kinematically*, this is a very conservative approach in that it rests upon the well-established principle of general covariance

[1] We are aware that, depending on their epistemological background, the reader may or may not agree with this argument.

[2] Classic textbooks include [8–12].

of spacetime and assumes the standard axioms and methods of Quantum Mechanics[3]. What makes the theory revolutionary are rather the *dynamical* laws it predicts in the genuinely stringy regime and even more so the way how these laws are *derived* just from requiring consistency of the theory. Basically without any further input than the kinematical pillars just quoted the two dynamical sanctuaries of modern physics inevitably follow in the low-energy limit: Einstein's gravitational equations and the concept of gauge interactions.

It is important to stress that the structure of the fundamental laws governing the low-energy phenomenology of the universe comes out almost as a byproduct. The peaceful coexistence of gravity and Yang-Mills theory at the quantum level in String Theory is an immediate consequence of the presence of closed and open strings as the only two topologies which a one-dimensional object can exhibit. The role of the graviton is played by the massless spin two excitations of the closed string, and Einstein's equations follow by requiring Weyl invariance of the non-linear σ-model describing the string propagation on a (curved) background manifold. The latter is equivalent to the conformal symmetry of the two-dimensional string worldsheet to be anomaly-free, which is one of the consistency conditions for the theory to make sense, more precisely for the absence of negative norm states in the Fock space. The Yang-Mills gauge bosons, by contrast, are furnished by the massless open strings or, in a dual description, particular massless excitations of the closed heterotic string. In any case, once we observe in our theory Yang-Mills interactions, we automatically observe gravity as well, because a theory of open strings necessarily requires the presence of closed strings. This is dictated by another consistency condition, namely the cancellation of certain infrared divergences in the one-loop amplitude which are related to the presence of a tadpole. Ironically, whereas in conventional Quantum Field Theory it seems impossible to describe both Yang-Mills theory and gravity at the quantum level, in String Theory, it is impossible to observe Yang-Mills theory *without* incorporating gravity.

The way how the dynamical laws of gravity are modified at higher energies or at smaller distances makes it furthermore conceivable that the drastic curvature singularities of black holes or the Big Bang might be resolved [13]. These questions are related to the emergence of stringy or quantum geometric properties of spacetime as seen by suitable string probes [14]. In switching the point of view from target space to the string worldsheet, the fundamental physical concept is no longer classical spacetime but the way how the string propagates along it. In this picture classically unacceptable singularities are no conceptual issue provided they leave the theory of the string probing it well-defined. The implementation of a holographic principle [15] in the context of the AdS/CFT conjecture [16, 17] and the spectacular microscopic computation of the internal degrees of freedom of (at least BPS) black holes [18], in perfect agreement with their thermodynamical entropy, are further pieces of evidence that String Theory really includes the cor-

[3]It has therefore in its present formulation nothing to say about conceptual issues of the interpretation of Quantum Mechanics and related questions.

rect number of degrees of freedom to yield a consistent description of Quantum Gravity.

At the same time, the theory gives rise to certain general features which are not necessarily forced upon us just from the current low-energy experiments and observations, but nonetheless enjoy popularity among many phenomenologists. The most prominent example is the prediction of extra dimensions - based on the renowned theorem that String Theory is well-defined only if the target space is ten-dimensional[4]. Furthermore, every consistent, i.e. tachyon-free and stable string theory in ten dimensions is automatically supersymmetric - out of the four possible definitions of a modular invariant one-loop amplitude two lead to a stable and supersymmetric spectrum, the remaining ones suffering from the presence of tachyons in ten dimensions. Both these features - extra dimensions and supersymmetry - are of course often considered for purely phenomenological reasons in bottom-up approaches - e.g. in Randall-Sundrum-like brane-world scenarios [20] or to account, among several other things, for the weak hierarchy problem by means of low-energy supersymmetry. In String Theory, by contrast, there is nothing ad hoc about the emergence of this extra structure which has so far not been observed in experiments - it is a logical consequence[5] of the string consistency conditions.

The crucial test which String Theory has to pass in the long run is whether it can make more explicit contact with the low-energy physics of the Standard Model than to account merely for the structural foundations of gravity and Yang-Mills theory. To appreciate what a difficult endeavour this may be, we should keep in mind that the Standard Model in its present version could only be formulated with the help of huge amounts of data just around the weak scale, i.e. at distances of 10^{-18} meters, where it is a good description of Nature. We would not have the least idea of the existence of QCD or the details of the weak sector if all our experiments were restricted to the scale of, say, some meters. Unfortunately, this is precisely the situation we face today in trying to reconstruct the physics at the Planck scale of 10^{-35} meters just from our empirical data. One single collider experiment at these energies would certainly be enough to decide immediately whether or not String Theory is realized in Nature. It is thus obviously wrong to claim that String Theory is in principle not falsifiable as a physical theory. After all it is as big a *conceptual* shortcoming of String Theory not to lead to unique predictions at the TeV scale as it is a *conceptual* shortcoming of Quantum Chromodynamics to make no predictions which Kopernikus could have falsified

[4]This is actually an oversimplification since what is really predicted is the total conformal anomaly of the worldsheet fields which has to cancel that of the Faddeev-Popov ghosts. Attempts to include fields different from additional spacetime coordinates lead to so-called non-critical String Theory in lower dimensions [19]. Their use for phenomenological applications is yet to be understood. The 26-dimensional bosonic string, by contrast, is unstable due to the presence of a closed tachyon, and it is still unclear if it might be related to a lower-dimensional string theory upon tachyon condensation.

[5]For the case of extra dimensions this is true modulo the remark in footnote 4.

with the help of his telescope (or at most a magnifying glass). Even more remarkable is it that there exist important theoretical arguments of the type just reviewed that String Theory might well account for Nature's ultra-violet degrees of freedom.

The standard approach towards describing our four-dimensional world from the point of view of String Theory is to describe the extra dimensions as compactified on a small six-dimensional space. The idea is that the infinite tower of Kaluza-Klein modes decouples from the four-dimensional theory at low energies and only the massless modes give rise to the observed matter. This logic leads to a geometrisation of the laws of four-dimensional physics which are encapsulated in the topological and geometric details of the compactification manifold. The background manifold itself and the values of the background fields, i.e. the possible vacuum expectation values of the internal components of the string fields, are subject to strong string theoretic consistency conditions which define the resulting four-dimensional effective theory as a *solution* of the equations of motion.

It is in this sense that String Theory overcomes the arbitrariness inherent to any phenomenologically motivated bottom-up approach like the Standard Model: There exists a single underlying theory with a number of effectively four-dimensional groundstates. The phenomenon that a physical theory admits more than one solution to its equations of motion is of course well familiar. Clearly, General Relativity does not predict the specific distance between the earth and the sun. Rather, this is the phenomenological input required in order to identify the specific solution to Einstein's equations compatible with these initial conditions, on the basis of which we then extract all further information. Nobody would claim that this justifies discarding the laws of General Relativity.

To keep the analogy, a question of prime importance in String Theory is thus to determine which of its solutions are compatible with the properties of our vacuum at all energies up to which we can rely on experimental input. More clearly: Are there realistic four-dimensional string vacua and, if so, how dense do they lie in the total solution space of String Theory? Up to which energy do we have to measure such that there is only one vacuum left compatible with all data up to that point? And finally, given that hypothetical vacuum, does it make further predictions (possibly at higher energies) which we can verify or falsify? Or is there a dynamical mechanism, probably non-perturbative in nature, which singles out some stable solutions over others?

At the moment we are far from a definite answer to any of these questions. The number of meta-stable four-dimensional string vacua making out the string landscape [21–23] is currently estimated to be of the order of 10^{500} [24] (see also [25] for an early estimate), which seems computationally out of any reach [26]. At least, the number of stable vacua appears to be finite. This is already a big success as compared to the even vaster space of anomaly-free and renormalisable effective quantum field theories which can be constructed without a consistent coupling to gravity [27]. We are by now not aware of a genuinely non-perturbative formulation of the theory, and most investigations are tied to highly non-generic perturbative corners of the moduli space of the hypothetical underlying M-theory. Our available techniques are restricted to the computation of the very basic low-energy properties of a given vacuum. In short, we need to understand the theory better. But we can nonetheless start and investigate some relevant features of at least those domains in the moduli space which are accessible to us at this stage. This is the objective of String Phenomenology.

1.2 Classic heterotic model building

Historically, the earliest attempts of string model building focused on the heterotic string [7]. Its worldsheet theory contains different fields in the left- and right-moving sector. In its fermionic formulation this is easily understood as follows: The right-moving fields are the same as in the corresponding sector of the superstring, i.e. ten worldsheet scalars X_-^μ transforming as $\mathbf{8}_V$ under the little group $SO(8)$ in ten dimensions and their superpartners, the worldsheet Majorana-Weyl spinors ψ_-^μ. Together with the superconformal ghost system, the right-moving conformal anomaly is cancelled. The left-moving sector, by contrast, comprises, apart from the left-moving X_+^μ, another 32 worldsheet Majorana-Weyl spinors λ_+^A which are singlets under $SO(1,9)$. Since the left-moving system is not supersymmetric, again the critical number of now 26 bosonic degrees of freedom is present to cancel the ghost conformal anomaly. The physical states arise as the tensor product of the right-moving and the left-moving excitations. There are two fully consistent choices to assign periodic or antiperiodic boundary conditions to the λ_+^A. If all of them carry the same boundary conditions, the left-moving sector exhibits an $SO(32)$ global symmetry which is actually promoted to a gauge symmetry. This can be most easily understood already from the appearance of a massless state in the $\mathbf{8}_V$ of $SO(8)$ and carrying antisymmetric indices A, B under $SO(32)$ - the gauge boson. Since the full spectrum contains states in the even-rank tensor representations and those related to one of the two spinor representations of $Spin(32)$, the gauge symmetry is actually not $SO(32)$ but rather $Spin(32)/\mathbb{Z}_2$[6]. If by contrast, the λ_+^A pair into two groups, each with the same boundary conditions, the naive gauge symmetry $Spin(16) \times Spin(16)$ is in fact further enhanced to $E_8 \times E_8$ upon performing a GSO projection.

In both cases, the massless bosonic sector comprises, in addition to the vector bosons, gauge singlets which decompose under $SO(1,9)$ into the spin two symmetric traceless representation, the graviton, furthermore the antisymmetric representation, yielding the Kalb-Ramond B-field and finally a scalar, the dilaton. The spacetime theory is $\mathcal{N}=1$ supersymmetric and therefore contains likewise the fermionic superpartners of all bosonic states.

At energies much smaller than the lowest lying massive states, the effective

[6] In standard abuse of notation we will, however, stick to the misnomer $SO(32)$ heterotic string.

theory is dominated by the massless modes we have just reviewed. In particular, one can think of appropriate coherent states of the massless fields as determining the background configuration probed by the string. In that sense, the background metric of the spacetime manifold on which the string propagates is to be viewed as a non-trivial vacuum expectation value for the graviton. Similarly, we can think of background values for the field strength of the antisymmetric tensor field, for the dilaton and the Yang-Mills gauge field. The background fields are subject to a number of strong consistency conditions since they have to be solutions to the stringy equations of motion. These will be reviewed extensively in chapter 2. Suffice it here to recall that in the simplest case, where the dilaton field is constant and the three-form field strength vanishes, the six-dimensional manifold on which we compactify has to be Calabi-Yau to ensure $\mathcal{N} = 1$ supersymmetry and therefore physical stability at the compactification scale [28][7].

In the presence of background values for the massless string fields, the world-sheet action describing the propagation of the string is the $(0, 2)$ σ-model [29, 30], which in favourable circumstances can be rephrased in terms of a linear σ-model [31]. The resulting conformal field theory is a highly complicated and non-trivially coupled system which, up to now, has not been solved for the generic case.

There are in principle two different approaches to bypass this technical difficulty. One can either focus on very special background manifolds on which the worldsheet theory is still exactly solvable as a conformal field theory (CFT). Cases where this is feasible are toroidal orbifold compactifications [32–35], or very symmetric points in the moduli space of genuine Calabi-Yau manifolds corresponding to exactly solvable abstract CFTs such as Gepner models [36, 37]. Slightly different CFT methods include free fermionic [38] and free bosonic [25] constructions. The advantage of the CFT approach is that whenever we have an exactly solvable conformal field theory at our disposal, its information is exact both perturbatively and non-perturbatively in α'. Unfortunately this technology is currently applicable to only a small fraction of relevant background configurations. Alternatively, one can try to analyse directly the spacetime effective field theory in the zero mode approximation. This approach is valid only in the strictly perturbative regime, i.e. for the typical radius of the background manifold much bigger than the string length and for sufficiently small string coupling. In other words, it is in a way insensitive to many genuinely stringy elements of the theory, but it is sufficiently powerful as far as an analysis of the vacuum states is concerned[8].

This geometric approach was pioneered in [28, 39] soon after the formulation of the heterotic theory. What makes the $E_8 \times E_8$ string so attractive for model building is the natural way how the standard semi-simple GUT gauge groups E_6,

[7]Extended supersymmetry in four dimensions would of course also lead to stable configurations.

[8]We will describe the methods of this latter effective or geometric approach in great detail in chapter 2.

$SO(10)$ and $SU(5)$ arise as subgroups of E_8. Consequently, the task is to break E_8 down to one of these GUT groups by giving VEVs to the internal field strengths in the commutant of the final gauge group. For the cases just listed these are $SU(3)$, $SU(4)$ and $SU(5)$, respectively. Accordingly, the **248** representation of E_8 splits into the respective GUT multiplets which incorporate the chiral fermions of the Standard Model. Consistent E_6 GUT models, for example, are especially straightforward to obtain by identifying the $SU(3)$ field strength with non-trivial background value with the curvature of the tangent bundle of the Calabi-Yau manifold. In that case the supersymmetry conditions for the gauge fields implying in particular the Yang-Mills equation of motion are automatically satisfied. The number of **27** and $\overline{\mathbf{27}}$ are simply counted by the Kähler and complex structure moduli of the Calabi-Yau and one might think that all one needs to do is search for appropriate geometric configurations. Unfortunately, E_6 is not very attractive as a GUT group from the phenomenological point of view since its fundamental representation **27** decomposes into $\mathbf{16} + \mathbf{10} + \mathbf{1}$ upon breaking E_6 to $SO(10)$ so that one GUT generation of E_6 yields not only one full generation of MSSM matter in form of the **16**, but additional chiral exotics.

To arrive at the phenomenologically more appealing $SO(10)$ and $SU(5)$ scenarios, one has to construct stable holomorphic vector bundles with structure group $SU(4)$ and $SU(5)$ respectively [39]. The mathematical property of stability essentially guarantees that the bundle allows for a connection which is a supersymmetric solution to the Yang-Mills equations. To prove stability for a bundle is already a very challenging task from the mathematical point of view and it took until 1997 that a sufficiently general procedure was found to construct such stable $SU(N)$ bundles on a large class of Calabi-Yau manifolds, the spectral cover construction [40, 41]. However, in conventional stringy GUT scenarios it is impossible to realize the GUT breaking further down to $SU(3) \times SU(2) \times U(1)_Y$ via a field theoretic Higgs mechanism, simply because the required vector-like pairs from which the GUT Higgs could arise are not present in the particle spectrum[9]. To break $SU(5)$ down to the Standard Model group, for example, the Higgs field must transform in the adjoint representation of $SU(5)$, but we will see that the four-dimensional bosonic particle spectrum contains only one vector multiplet in the **24**, the gauge multiplet, and no further such states. To our rescue comes the use of Wilson lines as an alternative GUT breaking mechanism. Wilson lines are globally non-trivial background values of the gauge connection which locally are pure gauge and therefore induce a vanishing background field strength.

This considerably complicates the construction of heterotic Standard Model vacua. The point is that in order to have these Wilson lines at our disposal, we need non-trivial elements in the first cohomology group of the internal manifold, i.e. homotopically non-trivial one-cycles along which the connection one-form can take a non-zero VEV. Now on general grounds, a Calabi-Yau can never admit

[9]Note, however, that in the context of higher-level Kac-Moody algebras GUT Higgses *can* be realized.

continuous Wilson lines, i.e. elements of $H^1(\mathcal{M}, \mathbb{R})$, but at most torsional ones as non-trivial elements of $H^1(\mathcal{M}, \mathbb{Z})$. This means that we have to construct non-simply connected Calabi-Yau manifolds such that their Wilson lines are just right to break the GUT group to the MSSM gauge group. For example, $\mathbb{Z}_2$-valued Wilson lines break $SU(5)$ down to $SU(3) \times SU(2) \times U(1)_Y$, whereas $\mathbb{Z}_2 \times \mathbb{Z}_2$-valued ones produce one additional abelian gauge factor $U(1)_{B-L}$ [42]. While this gauged $U(1)_{B-L}$ helps to suppress proton decay, it poses the problem that different effects have to be invoked in order to break it to a global symmetry. The same holds for $SO(10)$, which requires at least $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines and likewise ends up with an additional $U(1)_{B-L}$.

Finding Calabi-Yau manifolds with such first fundamental groups is once more a highly non-trivial task, and it has been one of the recent triumphs of string model building to provide classes of such Calabi-Yau manifolds as quotients of manifolds under an appropriate freely-acting orbifold group and to construct non-abelian vector bundles on them [43–48]. Globally defined realistic models from $SU(5)$ GUT on manifolds with $\mathbb{Z}_2$ Wilson lines in this context have been provided in [49]. For non-supersymmetric models from $SO(10)$ using $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines see [50, 51]. A recent construction of promising models in the setup of heterotic toroidal orbifolds can be found in [52].

1.3 Unitary bundles in heterotic compactifications

Independently of the heterotic model building industry, the discovery of D-branes [53] has opened up a complementary - or rather dual - path to incorporating gauge interactions into String Theory, more precisely the Type II theory or orientifolds thereof. A stack of N coincident D-branes accommodates a $U(N)$ gauge field in form of the massless modes of the open strings whose both ends are attached to the brane. Soon it was realized that two stacks of such branes intersecting at a non-trivial angle feature chiral fermions in the bifundamental representation of the two unitary groups [54,55]. This had the prospect of constructing MSSM-like models from type IIA orientifolds which live at the four-dimensional overlap of several stacks of D6-branes wrapping in addition special Lagrangian three-cycles on the internal Calabi-Yau and intersecting at supersymmetric angles [56][10]. On the other hand, it turns out extremely difficult to extend this class of constructions to non-toroidal backgrounds. What hampers progress into this direction is the special Lagrangian condition for supersymmetric three-cycles. Being real in nature, this constraint cannot be tackled with the help of complex geometry and is rather challenging to cope with. Instead one might try to invoke abstract CFT methods and consider rational conformal field theories corresponding to orientifolds at the Gepner point of certain Calabi-Yau

[10] For a complete list of references exploiting this idea see e.g. the most recent review [57].

manifolds[11], but again this strategy is not applicable to more generic situations.

The architecture of the Intersecting Brane World models differs from the $E_8 \times E_8$ approach in that, instead of starting from one unifying group and then accomplishing favourable gauge breaking, one combines a number of separate $U(N)$ modules given by the various brane stacks to mimic the product structure of the MSSM gauge group or modifications thereof like Pati-Salam or left-right symmetric models. But are the constructions really so different? The objects mirror dual to D6-branes at angles in Type IIA theory are spacefilling D9-branes in Type I theory, endowed with non-trivial background field strengths for the abelian diagonal of the $U(N)$ gauge group. These magnetized branes in turn are S-dual to abelian background bundles in the $SO(32)$ heterotic theory. The natural subgroups of $SO(32)$ are indeed just $U(N)$ groups, and we can therefore interpret the intersecting brane picture as the geometric realisation of the breaking of $SO(32)$ into its $U(N)$ subgroups via abelian background bundles.

It is thus of obvious relevance to explore the usually neglected use of non-trivial line bundles[12] in heterotic compactifications with the hope of extending our model building possibilities beyond the classic embedding of vector bundles with vanishing first Chern class only. Likewise, one might wonder if turning on also non-abelian gauge bundles on D9-branes wrapping genuine Calabi-Yau manifolds in Type I leads to interesting constructions. Since the supersymmetry condition on the gauge bundles is holomorphic, there is reason to hope that this bypasses the technical difficulty which the construction of special Lagrangian submanifolds poses on the Type IIA side.

It is the aim of this thesis to investigate these questions.

Our main motivation stems from the interpretation of discrete Wilson lines as *flat* abelian bundles which are embedded into the ten-dimensional gauge group. As we pointed out, the construction of Calabi-Yau manifolds with non-trivial first fundamental class is very involved. In fact, the only known example featuring e.g. $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines necessary for $SO(10)$ GUT breaking is the one constructed in [46]. Arbitrary line bundles, by contrast, are comparatively straightforward objects - on Calabi-Yau manifolds they are simply determined by specifying their first Chern class as an element in $H^2(\mathcal{M}, \mathbb{Z})$. If it were possible to replace the GUT breaking through Wilson lines by the embedding of *non-flat* line or more general unitary bundles, this would open up the very interesting prospect of heterotic string model building on simply-connected manifolds.

The relevance of progress into this direction becomes even more obvious if one takes into account the following crucial aspect: Eventually all realistic model building activities have to be extended beyond the special case that the internal manifold is Calabi-Yau. The underlying rationale is that the geometric moduli of the internal manifold as well as the dilaton appear as massless fields in the four-dimensional field theory and are as such unacceptable from the phenomenological

[11] Recent progress in the construction of Type II orientifolds of Gepner models has been made in [58–63] and our own work [64, 65].

[12] For some early references see [30, 66–68] and more recently [69].

point of view. In configurations with non-trivial form field fluxes in addition to gauge instantons, the moduli are generically rendered massive via a superpotential generated by these fluxes and can therefore be removed from the low-energy spectrum. The resulting background manifold, however, is in general no longer Calabi-Yau as a consequence of the modified Killing spinor equations and the gravitational backreaction of the fluxes. In the case of heterotic compactifications with non-trivial three-form flux [70–74], it is not even Kähler, and certainly not simply a toroidal orbifold. All methods which are restricted to one of these two properties have therefore no chance to yield completely realistic models in the end. The lesson we learn is that in engineering the gauge sector we should rely as little as possible on the particular non-generic structure of our concrete background manifold. This, however, is just what we are doing in pursuing the Wilson line approach to GUT breaking - after all one needs to identify very specific elements in the first homotopy group, which in more general situations may be extremely hard to compute.

Let us outline the structure of this thesis. Before getting started, chapter 2 reminds the reader of the basic concepts and technical details of Calabi-Yau compactifications of the heterotic string. Also, we will take this opportunity to introduce our conventions and field normalisations. The highlighted string theoretic consistency conditions are the basis of the whole subsequent analysis.

In chapter 3 we discuss the general theory of $E_8 \times E_8$ string compactifications featuring unitary gauge instantons. The group theory of the associated embedding gives rise to an unexpectedly rich structure of possible low-energy gauge groups including in particular flipped $SU(5) \times U(1)_X$ GUT [75] and just the MSSM gauge group. In addition we allow for heterotic five-branes, in which case we are actually in the strongly coupled Horava-Witten regime [76, 77]. The presence of abelian gauge factors requires a careful study of possible anomalies and the associated generalised four-dimensional Green-Schwarz mechanism. We will see that consistency of the vacua calls for new anomaly cancelling counter terms in the presence of abelian gauge fields and five-branes. These counter terms will furthermore be derived explicitly by dimensional reduction of eleven-dimensional heterotic M-theory to ten dimensions. Apart from the issue of anomaly cancellation, the Green-Schwarz mechanism yields important terms in the low-energy effective action which arise at one loop in string perturbation theory. Specifically, we will analyse the gauge threshold corrections, find a new contribution to the D-term scalar potential for five-branes and identify a one-loop correction to the Fayet-Iliopoulos term associated with the abelian gauge fields. We will argue that it represents actually a perturbative correction to the Donaldson-Uhlenbeck-Yau supersymmetry condition on the gauge fields and conjecture a corresponding deformation of the local Hermitian Yang-Mills equation as the perturbatively exact generalisation of the string tree-level supersymmetry condition.

An analogous investigation is possible also for the $SO(32)$ heterotic string with unitary bundles and five-branes and is the subject of chapter 4. The analysis of the breaking of $SO(32)$ into its unitary subgroups and the associated decomposition of the adjoint representation will reveal a gauge sector and spectrum which exactly mimic that in the S-dual/T-dual framework of intersecting branes, as anticipated already. The details of the Green-Schwarz mechanism are different to what we encountered in the $E_8 \times E_8$ theory, in particular as far as the five-brane contributions are concerned, but again we will find loop corrections to the gauge couplings and the Donaldson-Uhlenbeck-Yau condition. In the S-dual Type I framework, these one-loop terms become perturbative α'-corrections which are well-known to affect also the *local* supersymmetry equations and the resulting stability condition. In fact, they make out just the perturbative part of the full Π-stability condition in the derived bounded category of coherent sheaves [78]. This serves as further support for our conjecture about the modified supersymmetry condition for the $E_8 \times E_8$ string.

To apply the results of chapter 3 and 4 to concrete model building it is necessary to have control over the moduli space of stable holomorphic unitary vector bundles. In chapter 5 we therefore review the spectral cover construction [40, 41] for $SU(N)$ bundles over elliptically fibered Calabi-Yau manifolds. By twisting the $SU(N)$ bundles with an additional line bundle, we can construct bundles with unitary gauge groups. For special classes of twist bundles this procedure is equivalent to a subclass of the bundles provided by the generalisation of the original spectral cover method due to [69].

In chapter 6 we provide two examples of semi-realistic vacua of the $SO(32)$ heterotic theory with Pati-Salam and MSSM-like gauge group respectively. They illustrate the general architecture of this type of vacua and its similarity to the intersecting brane framework. This is a direct consequence of the group structure of $SO(32)$. Generically, as we will see, the generic quiver structure of the models makes it hard to suppress chiral exotic matter in supersymmetric configurations. These vacua can likewise be interpreted as arising from D9-branes in the Type I with non-abelian gauge field VEVs.

Chapter 7 introduces a setup for the construction of realistic flipped $SU(5) \times U(1)_X$ GUT and $SU(3) \times SU(2) \times U(1)_Y$ MSSM vacua from the $E_8 \times E_8$ string. The key to keeping the respective $U(1)$ potential massless is to embed the same line bundle into both E_8 factors. The flipped $SU(5)$ models are phenomenologically particularly attractive due to the absence of operators triggering proton decay. Gauge coupling unification in both scenarios holds at the level of the threshold corrected gauge couplings. As far as concrete phenomenological applications are concerned, the main result of this thesis is the construction of four-dimensional vacua with flipped $SU(5)$ and Standard Model gauge group featuring precisely three chiral generations and no further chiral exotics on simply-connected manifolds. A collection of these vacua will be presented in the remainder of chapter 7 and in appendix D.

Finally, we conclude with a on outlook to the most pressing questions to be investigated in the future.

Supplementary material is provided in the appendices. Some useful definitions and formulae regarding the topological invariants of holomorphic vector bundles can be found in appendix A, together with a couple of trace identities which are frequently used throughout this thesis. In appendix B we collect the Kähler cone constraints for elliptically fibered Calabi-Yau manifolds over del Pezzo surfaces. These are relevant when it comes to checking the supersymmetry conditions on the gauge bundles. For the convenience of the reader, we have chosen to include in appendix C a discussion of the transformation rules for multiple $U(1)$ factors which, though elementary, might give rise to some confusion.

Chapter 2

The vacuum structure of heterotic compactifications

2.1 On the heterotic low-energy effective field theory

The low-energy effective theory of the heterotic string is given by ten-dimensional $\mathcal{N} = 1$ supergravity coupled to super Yang-Mills theory. Depending on which of the two heterotic theories we consider, the original ten-dimensional gauge group is $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$[1] and will be referred to as $\widetilde{G}$. The low-energy dynamics of both theories only differs in the gauge sector as long we restrict ourselves to the perturbative limit of weak string coupling. The bosonic degrees of freedom comprise the ten-dimensional metric, the dilaton ϕ_{10}, the Kalb-Ramond two-form $B^{(2)}$ and the gauge potential A with field strength $F = dA - iA \wedge A$. At lowest order in the string coupling, the bosonic part of the string frame Lagrangian takes the following form

$$\begin{aligned} S_{het} &= \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} \left[R + 4\, d\phi_{10} \wedge \star d\phi_{10} - \frac{1}{2} H \wedge \star H \right] \\ &- \frac{1}{2g_{10}^2} \int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} \mathrm{tr}(F \wedge \star F). \end{aligned} \tag{2.1}$$

We will stick throughout this thesis to the conventions of [12]. In this normalisation the relative size of the gravitational and the Yang-Mills interaction is set by $\kappa_{10}^2 = \frac{1}{2}(2\pi)^7\,(\alpha')^4$ and $g_{10}^2 = 2\,(2\pi)^7(\alpha')^3$. We adopt the standard notation that 'tr' denotes the trace in the vector representation of the gauge group and 'Tr' formally refers to the trace over the adjoint representation. In particular the two are related via $\mathrm{Tr}F^2 = 30\,\mathrm{tr}F^2$ (see also appendix A.2).

An important role will be played by the heterotic three-form field strength

$$H = dB^{(2)} - \frac{\alpha'}{4}(\omega_{YM} - \omega_L), \tag{2.2}$$

[1] Nonetheless, the latter case is usually denoted as the $SO(32)$ theory, cf. section 1.2 .

which involves the gauge and gravitational Chern-Simons three-forms defined in terms of the gauge potential A and the spin connection Ω by

$$\begin{aligned} \omega_{YM} &= \mathrm{tr} A \wedge dA - \tfrac{2i}{3}\,\mathrm{tr} A \wedge A \wedge A, & d\omega_{YM} &= \mathrm{tr} F^2, \\ \omega_L &= \mathrm{tr}\,\Omega \wedge d\Omega - \tfrac{2}{3}\,\mathrm{tr}\,\Omega \wedge \Omega \wedge \Omega, & d\omega_L &= \mathrm{tr} R^2. \end{aligned} \tag{2.3}$$

Note that in the last line, the trace $\mathrm{tr}R^2$ is over the fundamental representation of the tangent bundle of spacetime, which, for flat ten-dimensional space, has structure group $SO(1,9)$. A crucial point to take into account is that $B^{(2)}$ is not a globally defined two-form. This is because it is not invariant under a combined gauge transformation of the Yang-Mills potential and the spin connection

$$\begin{aligned} \delta A &= d\chi - i[A,\chi], & \delta\omega_{YM} &= d\,\mathrm{tr}(\chi \wedge dA), \\ \delta\Omega &= d\theta + [\Omega,\theta], & \delta\omega_L &= d\,\mathrm{tr}(\theta \wedge d\Omega), \end{aligned} \tag{2.4}$$

but likewise transforms as

$$\delta B^{(2)} = \frac{\alpha'}{4}\left[\mathrm{tr}(\chi \wedge dA) - \mathrm{tr}(\theta \wedge d\Omega)\right]. \tag{2.5}$$

The definition (2.2) makes clear that the gauge invariant and therefore globally defined object is the three-form field strength H.

The chiral massless fermionic spectrum consists of the gravitino in the **56** representation of $SO(1,9)$, the **8'** dilatino, both interacting only gravitationally, and the **8** gaugino[2] in the adjoint of the gauge group. The ten-dimensional theory therefore exhibits gravitational, gauge and mixed gauge-gravitational anomalies resulting from anomalous hexagon diagrams at one-loop in string perturbation theory. It is of course among the renowned peculiarities of the gauge groups $E_8 \times E_8$ and $SO(32)$ that the non-factorisable anomalies vanish by themselves and the factorisable ones can be cast into a form suitable to be cancelled by adding a one-loop counter term. This counter term involves the two-form potential $B^{(2)}$ and is therefore, according to (2.4), not gauge invariant. The resulting classical anomalies absorb the one-loop field theoretic anomalies, thus rendering the theory well-defined. Since we will make heavy use of it in the sequel, let us display the Green-Schwarz anomaly cancelling one-loop counter term [79, 80],

$$S_{GS} = \frac{1}{24\,(2\pi)^5\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge X_8, \tag{2.6}$$

where the eight-form X_8 reads

$$X_8 = \frac{1}{24}\mathrm{Tr}F^4 - \frac{1}{7200}\left(\mathrm{Tr}F^2\right)^2 - \frac{1}{240}\left(\mathrm{Tr}F^2\right)\left(\mathrm{tr}R^2\right) + \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}\left(\mathrm{tr}R^2\right)^2. \tag{2.7}$$

[2]The $\mathbf{8}'$ and the $\mathbf{8}$ are of opposite chirality.

Ten-dimensional Hodge duality relates the Kalb-Ramond two-form to a six-form $B^{(6)}$ via

$$\star_{10}\, dB^{(2)} = e^{2\phi_{10}}\, dB^{(6)}. \tag{2.8}$$

This suggests the existence of a five-dimensional object as the source for $B^{(6)}$ and therefore as the magnetic dual of the fundamental string to which $B^{(2)}$ couples. These heterotic five-branes are genuinely non-perturbative objects. The natural framework to study them is consequently the strong coupling limit of the heterotic theory. In this regime the parallels between the $E_8 \times E_8$ and the $SO(32)$ theory come to an end and we need to distinguish as to which theory we are referring to.

For gauge group $E_8 \times E_8$ the strong coupling limit is given by Horava-Witten theory [77], which can be viewed, in the low-energy approximation, as eleven-dimensional supergravity on the interval $S^1/\mathbb{Z}_2$. We will discuss some aspects of this theory relevant for our purposes in detail later on in section (3.4.4). The object which reduces to the heterotic five-brane in ten dimensions upon compactification of Horava-Witten theory along the eleventh dimension is known as the M5-brane. It represents the magnetic dual of the membranes as the fundamental entities in M-theory. The world volume Γ_a of the M5-brane supports a self-dual tensor field $\widetilde{B}_a$, which will play a role of similar importance as its cousin $B^{(2)}$ in section (3.4.4). The effective action governing the five-brane dynamics in ten dimensions can be inferred by dimensionally reducing the known Pasti-Sorokin-Tonin action for the corresponding M5-brane in heterotic M-theory. For the details of the full PST action we refer to [81], and for the parts of prime interest to us again to section (3.4.4).

By contrast, the $SO(32)$ heterotic string reduces in the limit of strong string coupling to the weakly coupled Type I theory [82]. The low-energy degrees of freedom of both theories are related to one another by S-duality. Now the Type I theory, too, involves a five-brane, the D5-brane, which is therefore S-dual to the $SO(32)$ heterotic five-brane. As a result, the dynamics of the latter differs considerably from the one of its counterpart in the $E_8 \times E_8$ theory in that it supports symplectic gauge fields on its worldvolume and gives rise to chiral fermions charged under this symplectic group [83]. Again, we postpone a more detailed discussion to section (4.1).

Having recalled the different strong coupling origins of the $E_8 \times E_8$ and the $SO(32)$ five-brane, we stress that in both cases their role as magnetic sources for the Kalb-Ramond field is encoded in the coupling to $B^{(6)}$

$$S_5^{WZ} \;=\; -\sum_a N_a T_5 \int_{\Gamma_a} B^{(6)} = -\sum_a N_a T_5 \int_{\mathcal{M}^{(10)}} B^{(6)} \wedge \delta(\Gamma_a), \tag{2.9}$$

where we consider stacks of N_a five-branes with worldvolume Γ_a and $\delta(\Gamma_a)$ denotes the four-form Poincaré dual to Γ_a. The five-brane tension as appearing aboove is $T_5 = ((2\pi)^5 \alpha'^3)^{-1}$. Note however the implicit factor of $e^{-2\phi_{10}}$ present in $B^{(6)}$

as a consequence of the relation (2.8) so that effectively, the five-brane tension is of order $\frac{1}{g_s^2}$.

Since it will be of great importance for our purposes later on, let us take a closer look at the action for $B^{(2)}$ respectively $B^{(6)}$. Dualizing the kinetic action of H and extracting all terms involving $B^{(6)}$ leads us to

$$-\frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} e^{2\phi_{10}} dB^{(6)} \wedge \star dB^{(6)} + \frac{\alpha'}{8\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} B^{(6)} \wedge \Big(\mathrm{tr}F^2 - \mathrm{tr}R^2 - 4(2\pi)^2 \sum_a N_a\, \delta(\Gamma_a)\Big). \quad (2.10)$$

The equation of motion after variation of $B^{(6)}$ follows as

$$d(e^{2\phi_{10}} \star dB^{(6)}) = \frac{\alpha'}{4}\Big(\mathrm{tr}F^2 - \mathrm{tr}R^2 - 4(2\pi)^2 \sum_a N_a\, \delta(\Gamma_a)\Big). \quad (2.11)$$

In view of (2.8) and (2.2), the left-hand side is of course nothing other than dH [3], and (2.11) constitutes its modified Bianchi identity. Since dH is an exact form, so must be the expression inside the brackets on the right. This statement is the so-called Green-Schwarz anomaly cancellation or tadpole cancellation condition in the presence of five-branes,

$$\frac{1}{4(2\pi)^2}(\mathrm{tr}F^2 - \mathrm{tr}R^2) - \sum_a N_a\, \delta(\Gamma_a) = [0], \quad (2.12)$$

which requires that the left-hand side has to vanish in cohomology.

2.2 Calabi-Yau compactification

Our chief interest is in compactifications of the ten-dimensional string theory down to four dimensions [28]. From now on, we will therefore consider the topology of ten-dimensional spacetime to be given by the direct product[4]

$$\mathcal{M}^{(10)} = \mathbb{R}^{(1,3)} \times \mathcal{M}. \quad (2.13)$$

For stability reasons we insist that supersymmetry be unbroken at the compactification scale, in which case the internal six-dimensional manifold has to

[3] One should definitely resist the temptation of equating the left-hand side simply to zero, using that $d(e^{2\phi_{10}} \star dB^{(6)}) = d(dB^{(2)})$. Recall that $dB^{(2)}$ is not globally defined and therefore is not an exact form, so $d(dB^{(2)})$ need not vanish.

[4] We will not consider the general case of warped products in this thesis. Also we will simply write $\mathbb{R}^{1,3}$ for the external space although we will at no place discuss issues like the cosmological constant etc. Our focus will be exclusively on the gauge sector.

admit a globally defined Killing spinor ϵ. By standard arguments this reduces the structure group of its tangent bundle to $SU(3)$ (cf. [84] for a formulation in the modern language of G-structures, for a recent review of related ideas and more references see also [85]). Unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions amounts to a solution of the Killing spinor equations, i.e. vanishing of the supervariation of the gravitino ψ, the dilatino λ and the gaugino χ as the fermionic superpartners of the bosonic fields entering the action (2.1). The supervariations relate the fermionic zero-modes to the bosonic ones and in a given vacuum state depend on the expectation values of the latter. In order to keep four-dimensional Lorentz invariance, only the internal components of the bosonic fields may take a non-trivial vacuum expectation value. Schematically[5], the Killing equations, at string tree-level and at lowest order in α', read [70]

$$\begin{aligned} 0 &= \delta\psi = \nabla\epsilon + \tfrac{1}{4}\,\mathbf{H}\epsilon, \\ 0 &= \delta\lambda = \partial\!\!\!/\phi_{10}\epsilon + \tfrac{1}{2}\,\mathbf{H}\epsilon, \\ 0 &= \delta\chi = 2\,\mathbf{F}\epsilon. \end{aligned} \tag{2.14}$$

Here $\mathbf{H}$ and $\mathbf{F}$ denote a suitable Gamma matrix contraction with the internal background values for the three-form and Yang-Mills field strength, respectively.

Clearly, in the absence of a vacuum expectation value (VEV) for the background field strength H, the first equation implies that the Killing spinor be covariantly constant with respect to the Levi-Civita connection. It follows that $\mathcal{M}$ is to be of $SU(3)$ holonomy, i.e. a Calabi-Yau manifold. We restrict all our considerations to this special case, together with a constant dilaton in order to satisfy also the dilatino equation. More precisely, we do not consider background values for H at *zeroeth* order in α'. Nonetheless, the Bianchi identity (2.11) for H relates a non-trivial VEV for the internal curvature as well as for the Yang-Mills fields to a VEV for H, which, however, arises at linear order in α'. As reviewed e.g. in [9], corrections to the Calabi-Yau condition at this order do not break supersymmetry spontaneously, but can be accounted for by correcting the vacuum order by order. Note also that the gravitational backreaction of the gauge flux is likewise of order α', as can be seen by comparing the different orders of α' of the Einstein-Hilbert term and the Yang-Mills kinetic term in the action (2.1). Consequently, at zeroeth order in α', the Calabi-Yau indeed solves the six-dimensional Einstein equations. As long as we are in the genuine supergravity regime, where the typical length scale of the internal manifold is much bigger that $\sqrt{\alpha'}$, it is therefore justified to neglect both these effects. The more general case in the context of heterotic compactifications was already pioneered in [70] and has recently enjoyed revived interest among physicists and mathematicians, see e.g. [71–74]. It will require some more sophisticated analysis in the case of in-

[5] Note that this simple form of the Killing spinor equations involves some rescaling of the bosonic and fermionic fields which is detailed in [70] and which we do not display here since it will play no role in the sequel.

terest to us and will be the subject of future work. The supersymmetry condition for the Yang-Mills field strength will be discussed in detail in the next section.

If supersymmetry is preserved, the effective theory upon compactification is given, again in the zero-mode approximation, by four-dimensional $\mathcal{N} = 1$ supergravity. Most remarkably, the characteristics of the four-dimensional effective dynamics is entirely captured by the topology and geometry of the internal manifold together with a consistent choice of vacuum expectation values for the bosonic zero modes encountered in the previous section.

We will extensively exploit this fact in order to describe the dynamics of the gauge sector. A priori, if we simply compactify the theory on a Calabi-Yau manifold without extra structure, the four-dimensional gauge fields transform in the adjoint representation of the original heterotic gauge group $\widetilde{G}$. In general, however, the internal space may carry background gauge flux. This means that some of the gauge bosons corresponding to the generators of some subgroup $G \subset \widetilde{G}$ may take a non-trivial vacuum expectation value on $\mathcal{M}$. Of course not any arbitrary configuration of gauge fluxes is allowed: The background values of the field strength are subject to the Bianchi identity and the Yang-Mills equations of motion, together with additional constraints if they are to preserve supersymmetry. Pure field theoretic considerations imply that the four-dimensional gauge group is broken to the commutant H of G in the original gauge group $\widetilde{G}$,

$$G \subset \widetilde{G} \longrightarrow H = \widetilde{G}/G. \tag{2.15}$$

In more mathematical terms, the effective gauge sector is therefore governed by the suitable embedding of a background gauge bundle W over $\mathcal{M}$ with structure group G into the full $E_8 \times E_8$ or $SO(32)$ bundle [39]. Note that the requirement that the background gauge field satisfy the Bianchi identity is automatically fulfilled if it arises as the connection of a vector bundle whereas the Yang-Mills equations of motion have to be imposed separately. Remarkably, a large amount of physical information is present already in the purely topological part of the bundle data, most notably its various characteristic classes (see appendix A.1 for a collection of some of their properties). This is true in particular as far as the emergence of chiral fermions in four dimensions is concerned, as we now review.

The ten-dimensional massless fermions charged under the Yang-Mills sector are the gauginos as the fermionic superpartners of the gauge bosons and transform in the 496-dimensional adjoint representation of $\widetilde{G}$. The embedding (2.15) induces the decomposition of this adjoint into the various irreducible representations of the four-dimensional gauge group H and the structure group G of the internal bundle,

$$\mathbf{496} \longrightarrow \bigoplus_j (R_j, r_j). \tag{2.16}$$

That is, each four-dimensional massless fermion in representation R_j of the unbroken gauge group carries specific charges, encoded in r_j, also under the structure group of the background bundle. Let us state that to each r_j we can associate a corresponding internal bundle U_j which is essentially some tensor product bundle of W or its subbundles. We will explain how to determine U_j when discussing the concrete embeddings we are interested in. This entanglement between the four-dimensional properties R_j of a massless state and its internal origin is the basis for determining the massless spectrum of a compactification from the geometry of the internal background bundles. In view of the splitting of the ten-dimensional Dirac operator $\not{D}_{10} = \not{D}_4 + \not{D}_6$ under compactification on $\mathcal{M}$, it is furthermore clear that the fermionic zero modes in four dimensions are given by the kernel of the internal Dirac operator. Furthermore, the splitting of the ten-dimensional chirality operator into the four- and six-dimensional ones is such that the four-dimensional chirality of the fermion equals its six-dimensional one. As a matter of fact, on a Calabi-Yau manifold the positive (negative) chirality subspace of the kernel of the Dirac operator is isomorphic to the even (odd) degree subspace of the Dolbeault cohomology. Since it would lead too far to detail the derivation of this standard theorem, we refer e.g. to [9] for an account. Taking this for granted, we conclude that the fermionic zero modes in the representation R_j under H are given by the Dolbeault cohomology $H^*(\mathcal{M}, U_j)$ of the internal bundle U_j which is associated to the representation r_j under G. Of course, if $\mathcal{N} = 1$ supersymmetry is unbroken each fermion appears with a complex bosonic superpartner to form a chiral supermultiplet. Most importantly, if the representation r_j is complex, the fermionic spectrum is chiral and the net-number of chiral matter multiplets is given by the index of the Dolbeault complex twisted by the respective bundle U_j. It is the content of the Riemann-Roch-Hirzebruch theorem that this index can be computed as the Euler number

$$\begin{aligned} \chi(\mathcal{M}, U_j) &= \sum_{i=0}^{3} (-1)^i \dim(H^i(\mathcal{M}, U_j) \\ &= \int_{\mathcal{M}} \left[\mathrm{ch}_3(U_j) + \frac{1}{12}\, c_2(T\mathcal{M})\, c_1(U_j) \right]. \end{aligned} \tag{2.17}$$

To be crystal clear, $H^i(\mathcal{M}, U_j)$ denotes the cohomology group of U_j-valued $(0,i)$-forms on $\mathcal{M}$ under the Dolbeault operator $\overline{\partial}$. In fact, for a holomorphic bundle U_j over a complex n-fold, by Serre duality not all cohomology classes are independent due to the relation

$$H^i(\mathcal{M}, U_j) \simeq H^{n-i}(\mathcal{M}, U_j^* \otimes \mathcal{K}_{\mathcal{M}}), \tag{2.18}$$

where U_j^* denotes the complex conjugate bundle to U_j and $\mathcal{K}_{\mathcal{M}}$ is the canonical bundle of $\mathcal{M}$ with $c_1(\mathcal{K}_{\mathcal{M}}) = -c_1(T\mathcal{M})$. Clearly, $\mathcal{K}_{\mathcal{M}}$ is trivial for Calabi-Yau manifolds.

We state at this stage already that for a non-trivial μ-stable bundle of zero slope necessarily $H^0(\mathcal{M}, U_j) = 0 = H^3(\mathcal{M}, U_j)$ and the same holds true for the conjugate bundle U_j^*. Fermions transforming in the representations R_j corresponding to a non-trivial internal r_j and thus to a non-trivial U_j are therefore counted precisely by $H^1(\mathcal{M}, U_j)$ and $H^2(\mathcal{M}, U_j) \simeq H^1(\mathcal{M}, U_j^*)$ as long as U_j is stable. For the bundles which count the chiral part of the spectrum, this will always be the case. In view of the described relation between the four- and six-dimensional chirality and the Dolbeault degree, the first cohomology group counts the left-handed and the latter the right-handed chiral multiplets.

On the other hand, as follows from the group theoretic decomposition of the **496**, the four-dimensional gauge bosons transform in the trivial representation under G [6], and the cohomology of the trivial bundle $\mathcal{O}$ on a Calabi-Yau is simply $\dim H^*(\mathcal{M}, \mathcal{O}) = (1, 0, 0, 1)$. This is obvious if one recalls that $H^i(\mathcal{M}, \mathcal{O}) = H^{(0,i)}(\mathcal{M})$ and the Hodge numbers of a Calabi-Yau are given by $h^{(0,0)} = 1 = h^{(0,3)}$ and $h^{(0,1)} = 0 = h^{(0,2)}$. H^0 and H^3 therefore count vector multiplets, which will be of use later on when we detect possible gauge enhancements by searching for additional cohomology groups of the trivial bundle.

Another generic feature is the appearance of singlets under the four-dimensional gauge groups, but transforming in the adjoint representation of the internal gauge group. These singlets are the moduli fields associated to the deformations of the internal bundle. For $SU(N)$ bundles V, the adjoint is simply the trace free part of $V \otimes V^*$. Stability of V implies that $H^0(\mathcal{M}, V \otimes V^*) = 1 = H^3(\mathcal{M}, V \otimes V^*)$. Subtracting this single element, which corresponds precisely to the trace part, we find that the bundle moduli are counted by $H^1(\mathcal{M}, V \otimes V^*)$.

Finally, we will be interested in compactifications featuring also the presence of non-perturbative five-branes. In those cases we leave, strictly speaking, the regime of exactly zero string coupling, $g_s = 0$, since the tension of the five-branes scales like $\frac{1}{g_s^2}$ and we cannot accept for their mass to diverge, of course. Even though $g_s > 0$, this does not imply, however, that we are inevitably beyond the perturbative framework since we can still constrain ourselves to small non-vanishing g_s such that all perturbative effects higher than the one-loop level and even more so additional non-perturbative corrections can consistently be neglected. In the case of the $E_8 \times E_8$ heterotic string, the strong coupling limit of the theory was pointed out already to be given by eleven-dimensional M-theory on $S^1/\mathbb{Z}_2$, with the two E_8 factors arising from the two orbifold fixed planes at the opposite ends of the interval. We will always assume that the heterotic five-branes, if present, are localised in the eleven-dimensional bulk between the E_8-planes so that they do not interfere with the geometry of the gauge bundles, possibly leading to chirality or gauge group changing small instanton transitions [86]. This assumption is standard in all heterotic compactifications with five-branes in the literature and should of course be eventually justified by explicitly computing the effects fixing the five-brane position along the eleventh dimension for concrete models. As stated already, we will, in this work, not be concerned with any issues of geometric moduli fixing, postponing this important, but involved

[6] This is true as long as the gauge group is not enhanced due to degeneracies of the embedding of the internal bundles. The class of $SO(32)$ vacua we will analyse in chapter 4 is precisely of that form.

question for a future analysis.

2.3 Consistency conditions for model building

The high degree of consistency of String Theory in its fundamental σ-model formulation on the worldsheet translates itself into severe constraints which the geometric data in the effective description have to satisfy in order to define a consistent supersymmetric string vacuum. These can be summarized as follows:

- At tree-level, the gauge bundles have to be holomorphic, μ-stable and satisfy the Donaldson-Uhlenbeck-Yau equation.
- The five-branes have to wrap holomorphic two-cycles on the internal manifold $\mathcal{M}$.
- The gauge bundle and five-branes are subject to the anomaly cancellation condition.
- The second Stiefel-Whitney class of the gauge bundle has to vanish.

Let us turn to a detailed discussion of these constraints.

The gauge degrees of freedom of the background bundle are subject to the Yang-Mills equation of motion and the Bianchi identity. Moreover, as we noted already, we insist on unbroken supersymmetry at the compactification scale to guarantee physical stability of the vacuum. Recall from (2.14) that the supersymmetry condition on the gauge degrees of freedom is determined by demanding that the variation of the gaugino vanish in the vacuum, $\delta\chi = 0$. At string tree-level, this yields the following two equations in terms of holomorphic coordinates on $\mathcal{M}$ involving the field strength of the background gauge fields (see e.g. [9]),

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{a\bar{b}} F_{a\bar{b}} = 0. \tag{2.19}$$

The first equation implies that W has to be a *holomorphic* vector bundle, i.e that it has to admit a holomorphic connection. Due to its holomorphicity, this constraint can only arise as an F-term in the effective $\mathcal{N} = 1$ supergravity description and therefore does not receive any perturbative corrections in α' or the string loop expansion [87].

The second equation in (2.19) can be conveniently rewritten as $J \wedge J \wedge F = 0$ by taking the Hodge dual. This is actually the zero-slope limit of the general Hermitian Yang-Mills (HYM) equation

$$J \wedge J \wedge F = 2\pi\, \mu(W)\, \mathrm{vol}_{\mathcal{M}}\, \mathrm{id}, \tag{2.20}$$

where id denotes the identity matrix acting on the fibre and J represents the Kähler form of the internal Calabi-Yau. As the name suggests, in combination with holomorphicity and the Bianchi identity for F, this condition automatically implies that the Yang-Mills equation of motion is satisfied. In (2.20) the slope μ of a vector bundle $\mathcal{V}$ with respect to the Kähler form J on a manifold $\mathcal{M}$ is defined as[7]

$$\mu(\mathcal{V}) = \frac{1}{\mathrm{rk}(\mathcal{V})} \int_{\mathcal{M}} J \wedge J \wedge c_1(\mathcal{V}). \tag{2.21}$$

According to a theorem by Donaldson [88] and by Uhlenbeck and Yau [89], (2.20) has a unique solution if and only if the vector bundle W in question is μ-stable[8], i.e. if for each subbundle $\mathcal{V}$ of W with $0 < \mathrm{rk}(\mathcal{V}) < \mathrm{rk}(W)$ one has

$$\mu(\mathcal{V}) < \mu(W). \tag{2.22}$$

Consequently, the zero-slope limit of the Hermitian Yang-Mills equations (2.19) relevant at tree level is satisfied precisely by holomorphic μ-stable bundles which meet in addition the integrability condition

$$\int_{\mathcal{M}} J \wedge J \wedge c_1(W) = 0. \tag{2.23}$$

In case the bundle W is the Whitney sum of several bundles, as it will be in the case of interest to us, each summand bundle has to be stable and satisfy (2.23). We will refer to the latter constraints in the following as the tree-level Donaldson-Uhlenbeck-Yau (DUY) equation. It is important to realize that the condition of μ-stability is completely independent of the actual numerical value which the slope μ takes. The latter is encoded in the DUY equation, which insists on $\mu(W) = 0$ and therefore makes clear that the supersymmetry condition at tree-level is not merely (2.20), but a forteriori $J \wedge J \wedge F = 0$. Consider for example a complex line bundle L, i.e. a complex vector bundle with structure group $U(1)$. The Bianchi identity $dF = 0$ implies in this case that $J \wedge J \wedge F$, together with $dJ = 0$ for Kähler manifolds, is automatically a constant multiple of the volume form so that the local HYM equation (2.20) is trivially satisfied. This is in agreement with the DUY theorem since a line bundle over a Calabi-Yau manifold is also trivially stable. The tree-level supersymmetry condition is thus merely given by the DUY equation (2.23). Clearly this is no more true for non-abelian bundles.

We stress that the Hermitian Yang-Mills and also the DUY condition in the form above are valid only at tree-level and were derived for situations where no other fields besides the gauge fields take a non-zero vacuum expectation value. As given in (2.23), the DUY condition puts a constraint on the Kähler form of the

[7] The factor of 2π in the Hermitian Yang-Mills equation is just a consequence of the definition of $c_1(V) = \frac{1}{2\pi}\mathrm{tr}F$. Furthermore we have normalized the volume of $\mathcal{M}$ to one.

[8] To be precise, it is sufficient that the bundle be μ-semistable. In that case, however, it may split into subbundles such that the resulting structure group is a subgroup of the original one. The commutant of the structure group in $\widehat{G}$, and thus the visible gauge group, would therefore get enhanced during this process, which we would clearly like to avoid in well-defined physical vacua.

internal manifold, which after all cannot take arbitrary values but has to lie inside the so-called Kähler cone. We will analyse these constraints in great detail in the sequel and derive perturbative corrections both to the stability condition and to the DUY equation. Besides we will see explicitly how the DUY equation emerges also as a D-term constraint from the four-dimensional effective supergravity.

Let us turn to the supersymmetry condition for the heterotic five-branes. In order to keep Lorentz invariance in four dimensions, we only allow for situations where the worldvolume Γ_a of the five-brane fills the four large dimensions and therefore wraps in addition an internal two-cycle, denoted by γ_a [90]. The standard arguments involving κ-symmetry on the worldvolume of the five-brane yield that for unbroken supersymmetry the two-cycle γ_a has to be holomorphic [91]. All configurations considered henceforth will be of this type. Put differently, the cohomology class associated with the two-cycle γ_a must be effective[9]. The set of effective classes forms a cone, the so-called Mori cone, in $H^2(\mathcal{M},\mathbb{Z})$. This is due to the fact that a linear combination of two-forms with positive integer coefficients again corresponds to an effective class if the original two-forms do. It is convenient to introduce furthermore the notation $\overline{\gamma}_a$ for the element in $H^4(\mathcal{M},\mathbb{Z})$ Poincaré dual to γ_a.

We have already encountered the anomaly cancellation condition (2.12) which translates into a constraint to be satisfied by the internal gauge bundle W, the tangent bundle $T\mathcal{M}$ of the internal space and the configuration of heterotic five-branes. As we recall, it arises simply as the Bianchi identity for the three-form field strength H. Its violation results in the appearance of gauge and gravitational anomalies in the effective theory, since (2.12) is a necessary and sufficient condition for the ten-dimensional anomaly cancellation mechanism to work. Turning the arguments around we can - and will - read (2.12) as the constraint that the cohomology class $[W]$[10] defined by

$$[W] = \left[\frac{1}{4(2\pi)^2}\,\mathrm{tr}\overline{F}^2\right] - \left[\frac{1}{4(2\pi)^2}\mathrm{tr}\overline{R}^2\right] \tag{2.24}$$

must admit the interpretation as the class Poincaré dual to the homology class of a sum of holomorphic curves. Here $\overline{F}$ and $\overline{R}$ denote the internal background field strength with values in G and the curvature two-form on $T\mathcal{M}$, respectively. According to what we just said this translates into the requirement that the Hodge dual class of $[W]$ be effective. That is, we insist that the tadpole of the gauge instantons and the Calabi-Yau tangent bundle can just be cancelled by a system of supersymmetric five-branes. Failure of effectiveness of $[W]$ (or more precisely its Hodge dual class) means that the five-branes, which we can always

[9]Recall that in general, effectiveness of a cohomology class of two-forms just states that its representatives are indeed dual to a smooth holomorphic curve, as required.

[10]We trust that it does not confuse the reader that we stick to the standard notation in the literature and denote the five-brane class as $[W]$. It will always be clear if W refers to the internal gauge bundle or the five-brane class.

introduce, are non-supersymmetric and in particular non-BPS with respect to the gauge sector. Due to potential instabilities, we do not consider such situations in this work[11].

There is a slightly more subtle topological condition on the gauge bundles which states that the second Stiefel-Whitney class of W has to vanish. This requirement was originally derived from the absence of world-sheet anomalies in the two-dimensional non-linear sigma model and we refer to [94, 95] for more details. Since the second Stiefel-Whitney class of a holomorphic bundle is isomorphic to the $\mathbb{Z}_2$-restriction of its first Chern class [30], the condition is satisfied precisely if

$$c_1(W) \in H^2(\mathcal{M}, 2\mathbb{Z}). \tag{2.25}$$

In the case of the $SO(32)$ string we will find a simple spacetime interpretation for (2.25) as being equivalent to the absence of a global Witten anomaly on the five-branes in every topological sector of the vacuum. Due to its role as the cancellation condition for the torsion K-theory charges of non-BPS D7-branes in the S-dual Type I framework [96], we will sometimes refer to (2.25) as the K-theory constraint. We are not aware of a similar spacetime interpretation for the $E_8 \times E_8$ theory.

[11] See, however, [92, 93] for a proposal of supersymmetry breaking vacua in the presence of anti-five-branes.

Chapter 3

The $E_8 \times E_8$ Heterotic string with unitary bundles

The vacuum structure of perturbative four-dimensional heterotic compactifications is, as we reviewed in the previous chapter, specified by a stable, holomorphic vector bundle W over the internal Calabi-Yau manifold $\mathcal{M}$ together with an embedding of its structure group G into the original ten-dimensional heterotic gauge group $\widetilde{G}$. By an appropriate choice of G and the bundle data, one can thereby try and construct four-dimensional vacua with phenomenologically appealing gauge group and matter content. As we also recalled in section 1.2, the standard realisation of GUT groups in this context is to embed an an $SU(4)$ or $SU(5)$ bundle into one of the two E_8 factors leading to $SO(10)$ and $SU(5)$, respectively, as the resulting observable gauge groups. The chiral matter arising in these scenarios transforms in the $(\mathbf{16})$ or $(\mathbf{10}) + (\overline{\mathbf{5}})$ representation of the gauge group. The spectrum does not provide any appropriate vector-like matter, i.e. Higgs fields, required to break the GUT group down to the Standard Model. This drawback is overcome by breaking $SO(10)$ or $SU(5)$ via non-trivial discrete Wilson lines, which in general can only exist if the first homotopy group of the Calabi-Yau is non-trivial. Such Calabi-Yau threefolds can be constructed by taking free discrete quotients of a Calabi-Yau with vanishing fundamental group. The electroweak Higgs can appear from the $(\mathbf{10})$ or the $(\mathbf{5}) + (\overline{\mathbf{5}})$ representations. From the physical point of view, this is a very simple and compelling picture and recently models whose particle spectrum is quite close to the Standard Model have been constructed [44, 49, 51].

The starting point for our investigations is the following fact: The described breaking of the GUT gauge symmetry down to the Standard Model via discrete Wilson lines involves, in more mathematical terms, *flat* abelian bundles. This, however, is not the most general type of construction. An obvious question is to explore whether one can use also *non-flat* line bundles to obtain phenomenologically interesting GUT or MSSM-like models from the $E_8 \times E_8$ string. The content of this chapter is a thorough and systematic analysis of this idea, based on [97, 98].[1] We will first have to understand the group theoretic embedding of vector bundles with non-semisimple structure group and the resulting matter content upon decomposition of the adjoint representation of $E_8 \times E_8$. We will then proceed to a detailed analysis of the low-energy effective theory in four dimensions. The presence of anomalous $U(1)$ factors in the visible gauge group necessitates a careful study of the anomaly cancellation mechanism, which is particularly subtle in the presence of non-perturbative five-branes. We will derive new anomaly cancelling terms upon reduction of the five-brane action from heterotic M-theory down to ten dimensions. The importance of these terms is obvious only in the presence of $U(1)$ groups and has therefore been overlooked previously. Most importantly, the various one-loop terms provided by the full Green-Schwarz mechanism will further lead us to the discovery of perturbative corrections to the D-term superymmetry conditions affecting in particular the relevant stability condition for the background bundles. We will conclude our analysis of the general features of the $E_8 \times E_8$ heterotic string with unitary bundles by exemplifying the rich embedding patterns leading to flipped $SU(5)$ GUT models or directly to the Standard Model gauge symmetry even on manifolds without Wilson lines. Further phenomenological applications of the ideas presented in this chapter are postponed to chapter 7.

3.1 Group theoretic embedding

The vector bundles we consider are of the following generic form

$$W = W_1 \oplus W_2, \tag{3.1}$$

where the structure group G_i of W_i is embedded into the first and second factor of $E_8^{(1)} \times E_8^{(2)}$, respectively, with commutant H_i,

$$G_1 \times G_2 \subset E_8^{(1)} \times E_8^{(2)} \to H_1 \times H_2. \tag{3.2}$$

For each building block W_i we consider the Whitney sum of $SU(N_i)$ or $U(N_i)$ bundles. They are chosen such that the structure group of W_i contains at least one abelian factor. In order to determine the unbroken gauge group H_i relevant for the physics in the string vacuum, we need to recall some group theoretic generalities concerning the embedding of non-semisimple $G_i \subset E_8^{(i)}$.

As a matter of fact, it is not possible to directly embed the unitary group $U(N)$ into E_8 because all subgroups of the latter are semi-simple. One therefore has to take a detour by first choosing some auxiliary semi-simple subgroup $SU(\mathcal{N}_i) \subset$

[1] A study of $U(N)$ bundles in the framework of the spectral cover construction has appeared recently in [69]. Besides that, to our knowledge, the only constructions prior to our analysis [97] are some scattered results on aspects of four-dimensional models [30, 99, 100] and a few papers on five- and six-dimensional models [66–68, 70, 101]. Our analysis differs considerably from some of the conclusions in [100] and [68]. Recently, more aspects of the framework of [97] have been analysed in [102] and [103].

$E_8^{(i)}$.[2] Of course, we are very familiar with the embedding of this $SU(\mathcal{N}_i)$ into $E_8^{(i)}$ by considering the usual branching rules for $E_8^{(i)}$(see e.g. [104]). Let us collectively denote the commutant of $\mathrm{SU}(\mathcal{N}_i)$ in $E_8^{(i)}$ as $E_{9-\mathcal{N}_i}$. Concretely, for $\mathcal{N}_i = 7, 6, \ldots, 2$ it is known to be given by $SU(2)$, $SU(3) \times SU(2)$, $SU(5)$, $Spin(10)$, E_6 and E_7, respectively.

What may be not so familiar is the second step, the embedding of the non-semisimple structure group G_i into this auxiliary $SU(\mathcal{N}_i)$. It can be accomplished in two distinct ways.

The first type of construction - dubbed of type A in the sequel - is based on the embedding $SU(N_i) \times U(1)^{M_i} \subset SU(N_i + M_i)$ and invokes in its most elementary version the Whitney sum

$$W_i = V_{N_i} \oplus \bigoplus_{m_i=1}^{M_i} L_{m_i} \qquad \text{(Type A)}. \tag{3.3}$$

Here, the vector bundle V_{N_i} has structure group $SU(N_i) \subset SU(N_i + M_i)$ and the field strengths of the line bundles L_{m_i} are identified with the specific $U(1)$ generators in $SU(N_i + M_i)$ which commute with the generators of the chosen $SU(N_i)$. To be more precise, the $U(1)$ generators are determined iteratively by following the stepwise decomposition

$$SU(N_i + M_i) \to SU(N_i + M_i - 1) \times U(1)_1 \to \ldots \to SU(N_i) \times \prod_{m_i=1}^{M_i} U(1)_{m_i}. \tag{3.4}$$

Clearly, in each step the new $U(1)_{k_i}$ generator T_{k_i} can be represented by the diagonal $SU(N_i + M_i)$ matrix

$$T_{k_i} = \mathrm{diag}_{N_i+M_i}(\underbrace{1, \ldots, 1}_{N_i+M_i-k_i \text{ times}}, -(N_i + M_i - k_i), 0, \ldots, 0). \tag{3.5}$$

This realizes the promised embedding of the structure group $SU(N_i) \times U(1)^{M_i}$ of the bundle W_i into $SU(N_i+M_i)$. We anticipate that the states in the fundamental representation of the line bundle L_{m_i} can be taken to carry unit $U(1)_{m_i}$ charge, thus fixing the otherwise arbitrary $U(1)$ charge normalization. The various line bundles are not correlated among one another and in particular V_{N_i} gives no contribution to the $U(1)$ charges. For later purposes, we summarize this by writing

$$Q_{k_i}(L_{m_i}) = \delta_{k_i,m_i}, \qquad Q_{k_i}(V_{N_i}) = 0. \tag{3.6}$$

The relevance of this $U(1)_{m_i}$ charge which we thereby attribute to the line and vector bundles will become clear when we discuss the cohomology groups (3.18)

[2] For the moment, let us concentrate on the case where we really have only one factor of $SU(\mathcal{N}_i)$. Generalizations are obvious and will be sketched at the end of this section.

counting the massless spectrum.

Example:

We illustrate this Type A embedding by a simple example. Consider only one E_8 factor. In the first step of our construction, take $\mathcal{N}=4$, corresponding to the embedding $SU(4) \subset E_8 \to SO(10)$. Now we decompose the internal $SU(4)$ as $SU(4) \to SU(3) \times U(1)$. This is accomplished by means of a bundle $W = V \oplus L$, where V is a rank three bundle with $c_1(V)=0$ and L a complex line bundle. The structure group $SU(3)\times U(1)$ of W is embedded into this $SU(4)$ by identifying the field strength of the connection of L with the $SU(4)$ generator $T = \mathrm{diag}(1,1,1,-3)$. L is assigned $U(1)$ charge 1. In all, this realizes the embedding

$$SU(3)\times U(1) \subset SU(4) \subset E_8 \longrightarrow SO(10)\times U(1). \tag{3.7}$$

As an alternative to the type A construction, one can embed $U(N_i)$ bundles V_{N_i} into $E_8^{(i)}$ by means of a particular procedure where one actually starts with a $U(N_i)\times U(1)^{M_i}$ bundle W_i with $c_1(W_i)=0$. To emphasize the difference from the ansatz (3.3) for $SU(N_i)\times U(1)$ bundles, let us adopt the notation

$$W_i = V_{N_i} \oplus \bigoplus_{m_i=1}^{M_i} L_{m_i}^{-1} \qquad \text{with} \quad c_1(W_i)=0 \qquad\qquad \text{(Type B)} \tag{3.8}$$

for $U(N_i)\times U(1)^{M_i}$ bundles.

What distinguishes the two constructions is that in (3.8) the line bundles are no more independent, but are chosen just to absorb the diagonal $U(1)$-charge of $U(N_i)$ in the splitting $SU(N_i+M_i) \to U(N_i)\times U(1)^{M_i}$. At the level of the bundles, this means that, as indicated, the first Chern classes of the various summand bundles add to zero. Group theoretically, one has to fix the embedding of the $U(1)$ part of the structure group into $SU(N_i+M_i)$. For $k_i = 1,\ldots,M_i$ this can be described by the charges

$$Q_{k_i} = (\underbrace{Q_{k_i}(V_{N_i}),\ldots,Q_{k_i}(V_{N_i})}_{N_i \text{ times}}, Q_{k_i}(L_1^{-1}),\ldots,Q_{k_i}(L_{m_i}^{-1})) \tag{3.9}$$

with

$$N_i\, Q_{k_i}(V_{N_i}) + \sum_{m_i=1}^{M_i} Q_{k_i}(L_{m_i}^{-1}) = 0. \tag{3.10}$$

The concrete charge assignment is again found iteratively by invoking the decomposition (3.4), where in each step we can use the freedom to choose a normalization of the new abelian charge in order to write

$$Q_{k_i} = (\underbrace{1,\ldots,1}_{N_i+M_i-k_i \text{ times}}, -(N_i+M_i-k_i), 0,\ldots,0), \tag{3.11}$$

which clearly differs from its previous analogue (3.6). Note that as a consequence of the correlation between the $U(1)$ part of the structure group of V_{N_i} and that of line bundles, the bundle W_i has structure group $SU(N_i) \times U(1)^{M_i}$. For the detailed computation of the various anomalies associated with the $U(1)$-factors, it will turn out to be convenient to introduce the matrix

$$\mathcal{Q}_{k_i m_i} = Q_{k_i}(V_{N_i}) + Q_{k_i}(L_{m_i}). \tag{3.12}$$

Example:

Applying this construction to our toy $SO(10)$ chain (3.7) we now take $W = V \oplus L^{-1}$, with V a $U(3)$ bundle and the line bundle L chosen such that $c_1(W) = c_1(V) - c_1(L) = 0$. Clearly, L can be attributed $U(1)$ charge 3, V carries unit charge, and (3.10) is satisfied with $Q = (1, 1, 1, -3)$, see (3.9) and (3.11). Note also that $\mathcal{Q} = 4$.

Both constructions (3.3) and (3.8) admit obvious generalizations: Instead of considering only *one* non-abelian bundle V_{N_i} per $E_8^{(i)}$, we can, of course, allow for several suitable $SU(N_i^{k_i})$ or $U(N_i^{k_i})$ factors and embed them into $SU(\sum_{k_i} N_i^{k_i} + M_i)$. The point is that when embedding $U(1)_{m_i}$ into $SU(N_i)$, we can alternatively identify its generator T_{m_i} with any other diagonal $SU(N_i)$ generator, inducing thereby the branching $U(1)_{m_i} \subset SU(N_i) \longrightarrow SU(A_i) \times SU(B_i) \times U(1)$ with $A_i + B_i = N_i$. As far as the type B construction is concerned, the generalisation of the above is to realise the breaking $U(N_i) \longrightarrow U(A_i) \times U(B_i)$, $A_i + B_i = N_i$. A systematic description of the latter type of embeddings has recently been given in [105]. Arbitrary iterations and combinations are obvious.

Let us summarize the systematics: As described, the unbroken gauge group in four dimensions is given by the commutant $H_1 \times H_2$ of the structure group $G_1 \times G_2 \subset E_8^{(1)} \times E_8^{(2)}$. In particular, its *non-abelian* part is determined - leaving aside the issue of additional enhancements for the moment - by the standard commutant of the $SU(\mathcal{N}_i)$ in $E_8^{(i)}$. The detailed form of how the $SU(N_i^{k_i})$ or $U(N_i^{k_i})$ groups are embedded into the $SU(\mathcal{N}_i)$ decides on the additional *abelian* group factors which can potentially occur. It is clear that the abelian part of the structure group is contained in H ($U(1)$ factors of type (i) according to [30, 68, 106]), because the $U(1)$s commute with themselves. There might also be additional $U(1)$ factors in H not contained in the structure group ($U(1)$ factors of type (ii)). Finally, we anticipate that, depending now on the particular topological properties of the vector bundles we choose, the gauge group can be further enhanced or $U(1)$ factors can become massive due to the Green-Schwarz mechanism. These two issues will be explored more extensively in the subsequent sections.

In view of the above, a complete and systematic classification of all possible embeddings and the resulting gauge groups is in principle possible, but not very illuminating. Of potential phenomenological interest is the embedding of those $SU(N_i+M_i)$ factors leading either directly to $SU(3)\times SU(2)$ as the non-abelian part of the commutant in E_8 or to appealing GUT groups such as $SO(10)$, $SU(5)$ or the Pati-Salam $SU(4)\times SU(2)\times SU(2)$. On simply-connected Calabi-Yau manifolds, the need to realize the final gauge group breaking down to the MSSM without the aid of Wilson lines further eliminates $SO(10)$ and Georgi-Glashow $SU(5)$ since the GUT Higgs states required in these scenarios are absent in the massless spectrum. Since a general feature of our approach is the appearance of at least one $U(1)$ factor in the gauge group, we are very naturally lead to all those scenarios where such abelian groups occur. Besides the direct realisation of the MSSM gauge sector this is most prominently the so-called flipped GUT framework, in particular the flipped $SU(5)\times U(1)_X$ model [75]. We anticipate that - unlike the conventional GUT models - the GUT Higgsing merely requires scalars in much smaller representations which *are* present in the spectrum. This yields the important prospect of bypassing the need of Wilson lines and therefore non-simply connected background manifolds.

In all concrete examples we will restrict ourselves to (at most) one non-abelian bundle per $E_8^{(i)}$ factor[3]. We will therefore stick in our notation to this case.

3.2 Massless spectrum and cohomology classes

To determine the massless spectrum, one analyses, as in (2.16), the splitting of the adjoint representation of $E_8\times E_8$ into irreducible representations $R_{x_i}^{(i)}$ under the four-dimensional group and the internal one, denoted as $r_{x_i}^{(i)}$,

$$\mathbf{248}\times\mathbf{248}\rightarrow\sum_{x_1}(R_{x_1}^{(1)},r_{x_1}^{(1)};1,1)+\sum_{x_2}(1,1;R_{x_2}^{(2)},r_{x_2}^{(2)}). \tag{3.13}$$

From the structure of (3.13) it appears at first sight that the two $E_8^{(i)}$ sectors are hidden to each other in the sense that all states charged under, say, $E_8^{(2)}$ are singlets under $E_8^{(1)}$ and vice versa. This is definitely true for the non-abelian part of the representations, which arises after embedding the $SU(N_i+M_i)$ into $E_8^{(i)}$. However, in the presence of *abelian* gauge group factors, this picture changes. In the original, diagonal basis of $U(1)_{m_i}$ generators, it still holds true that the states in representation $R_{x_1}^{(1)}$ are uncharged under the abelian group factors embedded into $E_8^{(2)}$ and vice versa. But we are free to perform a change of basis and consider arbitrary linear combinations of $U(1)$ generators from both $E_8^{(i)}$.[4] In particular, states in the representation, say, $(1,1;R_k^{(2)},r_k^{(2)})$, though coming as singlets under H_1, may carry non-trivial charges under the $U(1)$ group generated by the linear

[3] As it turns out, these are precisely the phenomenologically appealing ones.

[4] In fact, these may be just the massless combinations surviving the Green-Schwarz mechanism. Our favourite construction in chapter 7 will be precisely of this form.

combination $a_{m_1}T_{m_1} + b_{n_2}T_{n_2}$ of generators T_{m_i} of $U(1)_{m_i}$. As a consequence of the embedding of $U(N)$ bundles, the two $E_8^{(i)}$ are no more completely hidden to each other.

In the class of models based on the splitting $SU(N_i+M_i) \to SU(N_i)\times U(1)^{M_i}$ for the internal bundle, we can give a rather general closed expression for the representations $r_{x_i}^{(i)}$ which occur. It is based on the elementary observation that under $SU(N+1) \to SU(N) \times U(1)$ we have the following decomposition of the lowest irreducible representations

$$\begin{array}{rcc} \mathbf{Adj(N+1)} & \to & \mathbf{Adj(N)}_0 + (1)_0 + (\mathbf{N})_{N+1} + (\overline{\mathbf{N}})_{-(N+1)}, \\ (\mathbf{N+1}) & \to & (\mathbf{N})_1 + 1_{-N}, \\ \mathbf{\Lambda^2(N+1)} & \to & \mathbf{\Lambda^2(N)}_2 + (\mathbf{N})_{-(N-1)}, \\ \mathbf{\Lambda^3(N+1)} & \to & \mathbf{\Lambda^3(N)}_3 + \mathbf{\Lambda^2(N)}_{-(N-2)}. \end{array} \tag{3.14}$$

For the various antisymmetric tensor representations we write more suggestively

$$\binom{N+1}{k} \to \binom{N}{k}_k + \binom{N}{k-1}_{-(N+1-k)}. \tag{3.15}$$

One can now follow the various steps in the full decomposition $SU(N+M) \to SU(N) \times U(1)^M$ for each of the two $E_8^{(i)}$ as in (3.4) and prove by induction the following decomposition of the lowest representations which we will encounter in our applications

$$\begin{aligned} \mathbf{Adj(N+M)} &\to \mathbf{Adj(N)}_{(0,\dots,0)} + M \times (1)_{(0,\dots,0)} + \\ &\quad \left(\sum_{k=0}^{M-1} (\mathbf{N})_{\vec{\mathcal{Q}}_k^1} + c.c\right) + \left(\sum_{j=0}^{M-2}\sum_{k=0}^{M-j-2} (1)_{\vec{\mathcal{Q}}_{j,k}^2} + c.c\right), \\ (\mathbf{N+M}) &\to (\mathbf{N})_{(1,\dots,1)} + \sum_{j=0}^{M-1} (1)_{\vec{\mathcal{Q}}_j^3}, \\ \mathbf{\Lambda^2(N+M)} &\to \mathbf{\Lambda^2(N)}_{(2,\dots,2)} + \sum_{k=0}^{M-1} (\mathbf{N})_{\vec{\mathcal{Q}}_k^4} + \sum_{j=0}^{M-2}\sum_{k=0}^{M-j-2} (1)_{\vec{\mathcal{Q}}_{j,k}^5}, \\ \mathbf{\Lambda^3(N+M)} &\to \mathbf{\Lambda^3(N)}_{(3,\dots,3)} + \sum_{k=0}^{M-1} \mathbf{\Lambda^2(N)}_{\vec{\mathcal{Q}}_k^6} + \sum_{k=0}^{M-2}\sum_{l=0}^{M-k-2} (\mathbf{N})_{\vec{\mathcal{Q}}_{k,l}^7} + \\ &\quad \sum_{l=0}^{M-3}\sum_{j=0}^{M-l-3}\sum_{k=0}^{M-j-l-3} (1)_{\vec{\mathcal{Q}}_{l,j,k}^8}. \end{aligned} \tag{3.16}$$

The various $U(1)$ charge vectors of the states are given by

$$\vec{\mathcal{Q}}_k^1 = (\underbrace{1,\dots,1}_{k}, (N+k+1), (N+k+2), \dots, (N+M)),$$

$$
\begin{aligned}
\vec{\mathcal{Q}}^2_{j,k} &= (\underbrace{0,\ldots,0}_{j},\, (-N+j), \underbrace{1,\ldots,1}_{k},\, (N+j+k+2), (N+j+k+3),\\
&\quad \ldots,\, (N+M)),\\
\vec{\mathcal{Q}}^3_{j} &= (\underbrace{0,\ldots,0}_{j},\, -(N+j),\, 1,\ldots,1),\\
\vec{\mathcal{Q}}^4_{k} &= (\underbrace{1,\ldots,1}_{k},\, (-N-k+1),\, 2,\ldots,2),\\
\vec{\mathcal{Q}}^5_{j,k} &= (\underbrace{0,\ldots,0}_{j},\, -(N+j), \underbrace{1,\ldots,1}_{k},\, -(N+k+j),\, 2,\ldots,2),\\
\vec{\mathcal{Q}}^6_{k} &= (\underbrace{2,\ldots,2}_{k},\, -(N+k-2),\, 3,\ldots,3),\\
\vec{\mathcal{Q}}^7_{k,l} &= (\underbrace{1,\ldots,1}_{k},\, -(N+k-1),\, \underbrace{2,\ldots,2}_{l},\, -(N+k+l-1),\, 3,\ldots,3),\\
\vec{\mathcal{Q}}^8_{l,j,k} &= (\underbrace{0,\ldots,0}_{l},\, -(N+l),\, \underbrace{1,\ldots,1}_{j},\, -(N+l+j),\, \underbrace{2,\ldots,2}_{k}, -(N+l+j+k),\\
&\quad 3,\ldots,3).
\end{aligned}
\tag{3.17}
$$

Following the discussion in section (2.2), thanks to the non-trivial internal gauge background we find four-dimensional *chiral* matter in representations $R^{(i)}_{x_i}$ specified by the cohomology class $H^*(\mathcal{M}, U^{(i)}_{x_i})$. What we can say at the general level is that the fields in representation $R^{(i)}_{x_i}$ will be counted by cohomology groups of the form

$$
H^*\left(\mathcal{M}, \bigwedge^{\alpha^i_{x_i}} V_{N_i} \otimes \bigotimes_{m_i=1}^{M_i} (\underbrace{L_{m_i} \otimes \ldots \otimes L_{m_i}}_{\beta^{m_i}_{x_i}-\text{times}})\right). \tag{3.18}
$$

From the decomposition (3.16) we immediately identify the $\alpha^i_{x_i}$ as the rank of the tensor representations of $SU(N_i)$ occurring in the corresponding internal $r^{(i)}_{x_i}$. The powers $\beta^{m_i}_{x_i}$ of the line bundle are determined by demanding that the $U(1)_{k_i}$ charges $q^{k_i}_{x_i}$ of the representation $R^{(i)}_{x_i}$ be correctly reproduced. Very generally, they are found by solving

$$
q^{k_i}_{x_i} = \alpha^i_{x_i}\, Q_{k_i}(V_{N_i}) + \sum_{m_i} \beta^{m_i}_{x_i}\, Q_{k_i}(L_{m_i}). \tag{3.19}
$$

As we described, for embeddings of Type A, (3.3), the abelian charges of the occurring representations are entirely due to the respective line bundles, see (3.6). Thus the powers $\beta^{m_i}_{x_i}$ in (3.18) can simply be read off from the entries in the charge vectors specified in (3.16) and (3.17), since after all $\beta^{m_i}_{x_i} = q^{m_i}_{x_i}$. By contrast, for Type B embeddings, (3.8), the various line bundles and the vector bundle are interrelated, and we need to take into account the different $U(1)$ charges (3.11) carried by the bundles to determine the $\beta_{x_i}^{m_i}$. In the explicit examples we will discuss in the sequel this is straightforwardly accomplished.

Example:

We again conclude these general remarks by exemplifying the procedure for our simple model defined in (3.7). The first embedding, $SU(4) \subset E_8 \to SO(10)$ induces the familiar decomposition

$$\mathbf{248} \longrightarrow (\mathbf{15},\mathbf{1}) + (\mathbf{1},\mathbf{45}) + (\mathbf{4},\mathbf{16}) + (\overline{\mathbf{4}},\overline{\mathbf{16}}) + (\mathbf{6},\mathbf{10}). \tag{3.20}$$

Now we decompose the internal $SU(4)$ representations under $SU(4) \to SU(3) \times U(1)$ according to (3.14) as

$$\begin{aligned} \mathbf{15} &\longrightarrow \mathbf{8}_0 + \mathbf{1}_0 + \mathbf{3}_4 + \overline{\mathbf{3}}_{-4}, \\ \mathbf{4} &\longrightarrow \mathbf{3}_1 + \mathbf{1}_{-3}, \\ \mathbf{6} &\longrightarrow \overline{\mathbf{3}}_2 + \mathbf{3}_{-2}. \end{aligned} \tag{3.21}$$

Combining these two steps leads to the spectrum[5]

$$\mathbf{248} \stackrel{SU(3)\times SO(10)\times U(1)}{\longrightarrow} \left\{ \begin{array}{c} (\mathbf{1},\mathbf{45})_0 \\ (\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{1})_0 + (\mathbf{3},\mathbf{1})_4 + (\overline{\mathbf{3}},\mathbf{1})_{-4} \\ (\mathbf{3},\mathbf{16})_1 + (\mathbf{1},\mathbf{16})_{-3} \\ (\overline{\mathbf{3}},\overline{\mathbf{16}})_{-1} + (\mathbf{1},\overline{\mathbf{16}})_3 \\ (\overline{\mathbf{3}},\mathbf{10})_2 + (\mathbf{3},\mathbf{10})_{-2} \end{array} \right\}. \tag{3.22}$$

As a straightforward application of the prescription (3.19) we find furthermore the cohomology groups listed in table 3.1 counting the massless spectrum. In evaluating (3.19) we used that for Type A constructions, $Q(V) = 0$ and $Q(L) = 1$, whereas for Type B the charge assignments are normalized such that $Q(V) = 1$ and $Q(L) = 3$. In addition to the spectrum tabulated there we find of course the vector multiplets containing the gauge bosons of $SO(10)$ and of the $U(1)$ factor and which are counted by $H^*(\mathcal{M},\mathcal{O})$ with $\dim H^*(\mathcal{M},\mathcal{O}) = (1,0,0,1)$ due to the absence of continuous Wilson lines on a Calabi-Yau manifold. Note also the additional singlets under the four-dimensional gauge group counted by $H^*(\mathcal{M}, adj(V))$. These correspond to the vector bundle moduli of V and describe the possible deformations of its geometry.

3.3 Global consistency conditions

We have seen that the background bundles are subject to two topological constraints, (2.24) and (2.25), in order that the resulting string vacuum be globally well-defined. Now that we have specified the concrete embeddings, it is time

[5]Note that in the last line we used that the antisymmetric of $SU(4)$ is given by the $\overline{\mathbf{3}}$.

reps.	Cohomology (Type A)	Cohomology (Type B)
$\mathbf{16}_1$	$H^*(\mathcal{M}, V\otimes L)$	$H^*(\mathcal{M}, V)$
$\mathbf{16}_{-3}$	$H^*(\mathcal{M}, L^{-3})$	$H^*(\mathcal{M}, L^{-1})$
$\mathbf{10}_{-2}$	$H^*(\mathcal{M}, V\otimes L^{-2}) =$ $H^*(\mathcal{M}, (\bigwedge^2 V\otimes L^2)^*)$	$H^*(\mathcal{M}, V\otimes L^{-1}) =$ $H^*(\mathcal{M}, (\bigwedge^2 V)^*)$
$\mathbf{1}_4$	$H^*(\mathcal{M}, V\otimes L^4)$	$H^*(\mathcal{M}, V\otimes L)$

Table 3.1: Massless spectrum of $H = SO(10)\times U(1)$ models.

to evaluate their implications. For this purpose, let us establish the following notation which will be used extensively in the subsequent discussions. The ten-dimensional field strengths $F^{10} = F^{10}_1 + F^{10}_2$ are written, upon compactification, as $F^{10}_i = F_i + \overline{F}_i$, where F_i is the external four dimensional part taking values in H_i and $\overline{F}_i$ denotes the internal six-dimensional part, which takes values in the structure group G_i of the bundle. Recall that the $U(1)$ factors of type (i) are special in that they appear both in G_i and H_i. We denote the four-dimensional $U(1)$ two-form field strengths as f_{m_i} and the internal ones as $\overline{f}_{m_i}$.

It will furthermore turn out useful to relate the traces appearing in expressions like (2.12) to the Chern classes of the background gauge bundle and the tangent bundle of the internal manifold. This can be accomplished with the help of identities of the type

$$
\begin{aligned}
\mathrm{tr}_{E_8^{(i)}}\overline{F}_i^2 &= \frac{1}{30}\sum_{x_i} 2\,(2\pi)^2\Big(\mathrm{ch}_2(U^{(i)}_{x_i})\,\times\,\dim(R^{(i)}_{x_i})\Big)\\
&= 4\,(2\pi)^2\Big[\mathrm{ch}_2(V_{N_i}) + \sum_{m_i,n_i=1}^{M_i}\epsilon_{m_i,n_i}\,c_1(L_{m_i})\wedge c_1(L_{n_i})\Big], \qquad (3.23)\\
\mathrm{tr}(\overline{R}^2) &= \mathrm{tr}_f^{SO(6)}(\overline{R}^2) = 2\,\mathrm{tr}_f^{SU(3)}\overline{R}^2 = -4\,(2\pi)^2 c_2(T). \qquad (3.24)
\end{aligned}
$$

For constructions of type A, the parameters ϵ_{m_i,n_i} depend on the concrete embedding; for type B, by contrast, we will see in the explicit examples that in fact $\epsilon_{m_i,n_i} = \frac{1}{2}\delta_{m_i,n_i}$. Similarly we introduce the expansion coefficients κ_{m_i,n_i} and η_{m_i,n_i}, which will be important later on and which are defined by evaluating the following traces over the concrete spectrum,

$$
\begin{aligned}
\mathrm{tr}_{E_8^{(i)}}(F_i\overline{F}_i) &= \frac{1}{30}\sum_{x_i} 2\pi\Big(\mathrm{ch}_1(U^{(i)}_{x_i})\,\times\,\dim(R^{(i)}_{x_i})\,\times(\sum_{m_i=1}^{M_i} q^{m_i}_{x_i}\,f_{m_i})\Big)\\
&= \sum_{m_i,n_i=1}^{M_i}\kappa_{m_i,n_i}\,f_{m_i}\wedge\overline{f}_{n_i},
\end{aligned}
$$

$$\begin{aligned} \mathrm{tr}_{E_8^{(i)}}(F_i^2) &= \frac{1}{30}\sum_{x_i}\dim(r^i_{x_i})\left(\mathrm{tr}^{E_{9-N_i-M_i}}_{R^{(i)}_{x_i}}(F_i)^2+\dim(R^i_{x_i})\sum_{m_i,n_i} q^{m_i}_{x_i}\,q^{n_i}_{x_i}\,f_{m_i}\wedge f_{n_i}\right) \\ &= 2\,\mathrm{tr}_f^{E_{9-N_i-M_i}}(F_i^2)+\sum_{m_i,n_i=1}^{M_i}\eta_{m_i,n_i}\,f_{m_i}\wedge f_{n_i}. \end{aligned} \tag{3.25}$$

By $q^{m_i}_{x_i}$ we denote again the charge of the representation $R^{(i)}_{x_i}$ under $U(1)_{m_i}$. In fact for decompositions of the type specified in the previous section, $\eta_{m_i,n_i}=0=\kappa_{m_i,n_i}=\epsilon_{m_i,n_i}$ for $m_i\neq n_i$. This is a consequence of the fact the $U(1)_{m_i}$ arise from the embedding into some $SU(\mathcal{N}_i)$: In each line of the decomposition (3.16), the separate trace over the individual $U(1)_{m_i}$ vanishes.

Finally, the tadpole condition (2.12) can be cast into the form

$$\sum_{i=1}^{2}\left(\mathrm{ch}_2(V_{N_i})+\sum_{m_i,n_1=1}^{M_i}\epsilon_{m_i,n_i}c_1(L_{m_i})\wedge c_1(L_{n_i})\right)-\sum_a N_a\overline{\gamma}_a=-c_2(T). \tag{3.26}$$

Recall that $\overline{\gamma}_a$ denotes the internal four-form Poincaré dual to the holomorphic two-cycle γ_a wrapped by the five-branes.

The second global consistency condition, the K-theory constraint (2.25), is seen to be non-trivial only for embeddings of type A, in which case it reads

$$\sum_{m_1=1}^{M_1}c_1(L_{m_1})+\sum_{m_2=1}^{M_2}c_1(L_{m_2})\in H^2(\mathcal{M},2\mathbb{Z}). \tag{3.27}$$

Clearly for embeddings of type B, (3.8), with $c_1(W_i)=0$, it is automatically satisfied.

3.4 Anomaly cancellation

In String Theory, all irreducible anomalies cancel directly due to the string consistency constraints [107] such as tadpole cancellation. The factorisable ones, by contrast, do not. For the four-dimensional effective theory resulting from string compactifications this means that all non-abelian cubic gauge anomalies do cancel, whereas the mixed abelian-nonabelian, the mixed abelian-gravitational and the cubic abelian ones do not. Since each $U(1)$ bundle in the structure group of the bundle implies a $U(1)$ gauge symmetry in four dimensions, all these latter three anomalies appear. For the string vacuum to be consistent, they have to be cancelled by a generalised Green-Schwarz mechanism[6]. This section is devoted to a detailed study of the factorisable anomalies due to the embedding of non-semisimple gauge bundles in the $E_8\times E_8$ theory and the associated anomaly cancellation mechanism. The latter is by no means just of academic interest,

[6]The Green-Schwarz mechanism for several $U(1)$ symmetries in $E_8\times E_8$ heterotic compactifications has also been discussed in [68], but their results differ from our conclusions.

but allows us to extract crucial information about the effective four-dimensional field theory. The point is that the Green-Schwarz mechanism provides certain terms in the effective action which arise at one-loop in string perturbation theory. Apart from the issue of anomaly cancellation, these terms will be the basis for determining the threshold corrections of the gauge kinetic functions and one-loop corrections to the Donaldson-Uhlenbeck-Yau supersymmetry condition for the gauge bundles. Even more fundamentally, the detailed form of the Green-Schwarz terms decides upon which of the abelian gauge factors become massive via a Stückelberg-type mechanism and thus only survive as global symmetries. A careful study of the Green-Schwarz mechanism is therefore of immediate relevance even if we were only interested in the most basic physical properties of the string vacua.

After presenting in section (3.4.1) the field theoretic anomalies, we will thoroughly explain the generalized Green-Schwarz mechanism, focusing in section (3.4.2) on the case without five-branes. It will turn out that the inclusion of five-branes requires additional Green-Schwarz terms, as becomes obvious only in the context of abelian gauge bundles. These modifications will be discussed in (3.4.2) and derived from Horava-Witten theory in (3.4.4). We will conclude this section by summarizing the axion-gauge boson mass terms in (3.4.5) which are important for concrete model building.

3.4.1 Field theoretic anomalies

We restrict the detailed discussion for brevity to the case that V_{N_i} has structure group $SU(N_i)$, i.e. embeddings of Type A; we will indicate the modifications in the otherwise largely analogous analysis of $U(N_i)$ bundles at the end of this section.

The field theoretic mixed $U(1)_{m_i}$-$E^2_{9-\mathcal{N}_j}$ and mixed $U(1)_{m_i}$-$G^2_{\mu\nu}$ anomalies for $m_i \in \{1, \ldots, M_i\}, i, j \in 1, 2$ can be computed by considering the chiral particle spectrum resulting from the concrete embedding. Mathematically, anomalies in four dimensions are characterised by their anomaly six-forms [108], which in our case are given by

$$\begin{aligned} A_{U(1)_{m_i}-E^2_{9-\mathcal{N}_i}} &\sim f_{m_i} \wedge \mathrm{tr}_f^{E_{9-\mathcal{N}_i}} F_i^2 \left[\sum_{x_i} C^{(2)}(R^{(i)}_{x_i})\, q^{m_i}_{x_i} \chi(\mathcal{M}, U_{x_i}) \right], \\ A_{U(1)_{m_i}-G^2_{\mu\nu}} &\sim f_{m_i} \wedge \mathrm{tr} R^2 \left[\sum_{x_i} q^{m_i}_{x_i}\, \dim(R^{(i)}_{x_i})\, \chi(\mathcal{M}, U_{x_i}) \right], \qquad (3.28) \\ A_{U(1)_{m_i}-U(1)_{n_i}-U(1)_{p_i}} &\sim f_{m_i} \wedge f_{n_i} \wedge f_{p_i} \left[\sum_{x_i} q^{m_i}_{x_i}\, q^{n_i}_{x_i}\, q^{p_i}_{x_i} \dim(R^{(i)}_{x_i})\, \chi(\mathcal{M}, U_{x_i}) \right]. \end{aligned}$$

Here, $C^{(2)}(R^{(i)}_{x_i})$ relates the traces over the representation $R^{(i)}_{x_i}$ of $E_{9-\mathcal{N}_j}$ and the fundamental representation via

$$\mathrm{tr}_{R^{(i)}_{x_i}} F_i^2 = C^{(2)}(R^{(i)}_{x_i})\, \mathrm{tr}_f F_i^2, \tag{3.29}$$

and its value for the relevant representations is listed in appendix A.2, whereas the $q^{m_i}_{x_i}$ constitute, as we recall, the $U(1)_{m_i}$ charge of the representation $R^{(i)}_{x_i}$. Note that in this diagonal basis of $U(1)$ generators, the anomalies involving $U(1)_{m_i}$ stem exclusively from the states charged under the same $E_8^{(i)}$, and there exist no $U(1)_{m_1} - E^2_{9-\mathcal{N}_2}$ anomalies.

In view of the slightly cumbersome general form of the occurring representations (3.16), (3.17), it is not very illuminating to perform this field theoretic computation for the most general embedding possible. On the other hand, it is a simple task to do so for a specific model. The results are compatible with the following universal expression for the anomaly six-forms:

$$\begin{aligned} A_{U(1)_{m_i}-E^2_{9-\mathcal{N}_i}} &\sim f_{m_i} \wedge \mathrm{tr} F_1^2 \left[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(\mathrm{tr}\overline{F}_i^2 - \frac{1}{2} \mathrm{tr}\overline{R}^2 \right) \right], & (3.30) \\ A_{U(1)_{m_i}-G^2_{\mu\nu}} &\sim f_{m_i} \wedge \mathrm{tr} R^2 \left[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(12\, \mathrm{tr}\overline{F}_i^2 - 5\, \mathrm{tr}\overline{R}^2 \right) \right]. & (3.31) \end{aligned}$$

To arrive at expressions of this type we will have to use (3.23) in order to relate the Chern classes arising in the formula (2.17) for the net chirality of the representations to the traces over the field strengths appearing in (3.30) and (3.31). The $U(1)_{m_i}$-$U(1)_{n_i}$-$U(1)_{p_i}$ anomalies are slightly more complicated and can be summarized in the following general form

$$\begin{aligned} A_{U(1)_{m_i}-U(1)_{n_i}-U(1)_{p_i}} \sim\ & f_{m_i} \wedge f_{n_i} \wedge f_{p_i} \left[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \delta_{n_i p_i} \left(\mathrm{tr}\overline{F}_i^2 - \frac{1}{2} \mathrm{tr}\overline{R}^2 \right) + \right. \\ & \left. c_{m_i n_i p_i}\, \overline{f}_{m_i} \wedge \overline{f}_{n_i} \wedge \overline{f}_{p_i} \right]. \end{aligned} \tag{3.32}$$

Here we have assumed that for at least two $U(1)$s being identical, the single one is $U(1)_{m_i}$. For $m_i \neq n_i \neq p_i$ the first term in (3.32) is absent. For $n_i = p_i$ the relative factor between the first and the second term in (3.32) can be expressed as

$$c_{m_i n_i n_i} = \frac{8}{3}\, \epsilon_{n_i, n_i}\, \sigma_{m_i n_i n_i}. \tag{3.33}$$

$\sigma_{m_i n_i n_i}$ denotes the symmetry factor of the anomalous diagram, i.e. $\sigma_{m_i n_i n_i} = 3$ for $m_i \neq n_i$ and $\sigma_{m_i m_i m_i} = 1$. The parameter ϵ_{n_i, n_i} was defined in (3.23).

For embeddings of Type B, the concrete expressions get slightly modified as a consequence of the different powers of line bundles appearing in the chiral index $\chi(\mathcal{M}, U_{x_j})$. As it turns out, we need to introduce the linear combination

$$\widehat{\overline{f}}_{m_i} = \sum_{k_i=1}^{M_i} \mathcal{Q}_{m_i k_i}\, \overline{f}_{k_i} \tag{3.34}$$

in terms of the charge matrix (3.12). The mixed abelian-nonabelian and gravitational anomaly six-forms in this case differ from the ones displayed in (3.30) only by the replacement $\overline{f}^{m_i} \to \widehat{\overline{f}}^{m_i}$, whereas the cubic abelian anomalies are now best summarized by

$$A_{U(1)_{m_i}-U(1)_{n_i}-U(1)_{p_i}} \sim f_{m_i} \wedge f_{n_i} \wedge f_{p_i} \Big[\int_{\mathcal{M}} \hat{c}_{m_i n_i p_i} \widehat{\overline{f}}_{m_i} \wedge \delta_{n_i p_i} \left(\mathrm{tr} \overline{F}_1^2 - \frac{1}{2} \mathrm{tr} \overline{R}^2 \right) + \widehat{\overline{f}}_{m_i} \wedge \widehat{\overline{f}}_{n_i} \wedge \widehat{\overline{f}}_{p_i} \Big] \tag{3.35}$$

with

$$\hat{c}_{m_i n_i p_i} = \frac{3}{8} \frac{\kappa_{m_i, m_i}}{\sigma_{m_i n_i p_i}}. \tag{3.36}$$

3.4.2 The four-dimensional Green-Schwarz mechanism without five-branes

Since the ten-dimensional string theory is anomaly-free, there must exist a mechanism which cancels the above field theoretic (mixed-) abelian anomalies which occur in the four-dimensional field theory. This is, of course, none other than the four-dimensional analogue of the Green-Schwarz mechanism. As in ten dimensions, it provides certain counter terms in the low-energy effective action leading to anomalous couplings between the involved gauge fields. The point is that the thereby induced anomaly six-form is just of the right form to cancel the one-loop field theoretic anomalies.

Before analysing the explicit form of the counter terms involved, we make a slight digression to discuss the general field theoretic features of the mechanism. A key role is played by certain four-dimensional two-form and scalar fields (axions). Concretely, they arise upon dimensional reduction of the Kalb-Ramond two-form $B^{(2)}$ and the self-dual tensor fields on the worldvolume of the five-branes. Suppose we have a collection $b_j^{(2)}, b_j^{(0)}$ of such fields, with the superscripts denoting their respective rank in four dimensions. As we will see, the two-form fields and scalars are Hodge dual to each other, satisfying

$$db_j^{(0)} = \beta_j \star_4 db_j^{(2)} \tag{3.37}$$

for some β_j to be determined later. This relation allows us to write the kinetic action for the $b_j^{(2)}$ as

$$S_{kin}^j = \alpha_j \int_{\mathbb{R}_{1,3}} db_j^{(2)} \wedge \star_4 db_j^{(2)} = \frac{\alpha_j}{\beta_j} \int_{\mathbb{R}_{1,3}} db_j^{(2)} \wedge db_j^{(0)}. \tag{3.38}$$

As a dynamical input, we will find the following two types of couplings,

$$S_{vertex} = \sum_j \mathcal{A}_j \int_{\mathbb{R}_{1,3}} b_j^{(0)} \wedge \mathrm{tr} \mathcal{F}^2, \tag{3.39}$$

$$S_{mass} = \sum_j \int_{\mathbb{R}_{1,3}} b_j^{(2)} \wedge \sum_m \mathcal{M}_{jm} f_m. \tag{3.40}$$

The coupling constants $\mathcal{A}_j, \mathcal{M}_{jm}$ will follow from the concrete Lagrangian and are just some parameters for the time being. The index m takes values in $1, \ldots, M_1, M_1+1, \ldots, M_1+M_2$ and labels the $U(1)$ groups stemming from both E_8 factors. $\mathcal{F}$ stands for one of the fields F_i or R with appropriate Chern-Simons form ω such that $d\omega = \mathrm{tr}\mathcal{F}^2$, and $f_m = dA_m$ denotes the field strength of the $U(1)_m$ gauge symmetry, under which A_m transforms as $\delta A_m = d\lambda_m$.

We can now straightforwardly integrate S_{mass} by parts and combine it with S^j_{kin} to integrate out the axions, writing schematically

$$db_j^{(0)} = \frac{\beta_j}{\alpha_j} \sum_m \mathcal{M}_{jm} A_m. \tag{3.41}$$

If we insert this back into S_{vertex} after integrating the latter by parts, we find the couplings

$$S_{coup} = -\sum_j \frac{\beta_j}{\alpha_j} \mathcal{A}_j \sum_m \mathcal{M}_{jm} \int_{\mathbb{R}_{1,3}} A_m \wedge \omega. \tag{3.42}$$

These terms are clearly not invariant under the abelian gauge transformations. With respect to, say, the $U(1)_n$ symmetry they transform as

$$\delta_{U(1)_n} S_{coup} = -\int_{\mathbb{R}_{1,3}} (\hat{I}_4)_n \quad \text{with} \quad (\hat{I}_4)_n = \sum_j \frac{\beta_j}{\alpha_j} \mathcal{A}_j \, \mathcal{M}_{jn} \, (d\lambda_n \wedge \omega)\,. \tag{3.43}$$

$\hat{I}_4$ therefore defines an anomalous six-form $(\hat{I}_6)_n$ via the chain [108]

$$(\hat{I}_6)_n = d(\hat{I}_5)_n, \qquad \delta_{U(1)_n}(\hat{I}_5)_n = d(\hat{I}_4)_n, \tag{3.44}$$

and we conclude that we indeed arrive at the anomaly six-form for the mixed $U(1)_n - \mathcal{F}^2$ anomaly

$$A^{GS}_{U(1)_n - \mathcal{F}^2} \sim \sum_j \frac{\beta_j}{\alpha_j} \mathcal{A}_j \; \mathcal{M}_{jn} \; \left(f_n \wedge \mathrm{tr}\mathcal{F}^2\right). \tag{3.45}$$

The corresponding anomalous diagram therefore hinges both upon the presence of the mass term S_{mass} and of the vertex coupling S_{vertex}. By contrast, even if the latter is absent, S_{mass} induces a Stückelberg-type mass term for some of the abelian gauge fields. This is immediately clear if we plug (3.41) back into (3.40). After integrating by parts we identify the following mass term for the abelian gauge fields

$$S_{Stuckelberg} = -\sum_{m,n=1}^{M_1+M_2} (\mathrm{M})^2_{m,n} \; (A_m \wedge \star_4 A_n) \tag{3.46}$$

with the squared mass matrix given by

$$(\mathrm{M})^2_{m,n} = \sum_j \frac{1}{\alpha_j} \mathcal{M}_{jm} \, \mathcal{M}_{jn}. \tag{3.47}$$

To determine the massless abelian gauge factors we therefore need to find the zero eigenvectors of the mass matrix $\mathrm{M}^2_{m,n}$. It will be more convenient to work instead with the coupling matrix $\mathcal{M}_{jn}$ because it can be read off directly from the effective action without further manipulations. By elementary linear algebra one can convince oneself[7], after performing a suitable basis transformation, that the massless abelian gauge factors are precisely those linear combinations of $U(1)_m$ whose gauge potential $A_f = \sum_m a_m A_m$ lies in the kernel of $\mathcal{M}_{jm}$, i.e

$$U(1)_f = \sum_m a_m U(1)_m \quad \text{is massless} \quad \Longleftrightarrow \quad \sum_m \mathcal{M}_{jm}\, a_m = 0. \tag{3.48}$$

We stress in particular that the various abelian factors from the two different E_8 may combine into a massless $U(1)$. The number of massive $U(1)$s is given by the rank of the matrix $\mathcal{M}_{jm}$ and is always at least as big as the number of anomalous $U(1)$s. However, since the mass generating terms are independent of the existence of additional vertex couplings S_{vertex}, an abelian factor can well acquire mass without being anomalous, i.e. without participating in the actual Green-Schwarz mechanism. This phenomenon is familiar already from the cancellation pattern of abelian anomalies in Type I/ Type II orientifolds (see e.g. [109]).

After these general remarks, we can now identify the relevant terms in the four-dimensional effective action. For the $E_8 \times E_8$ theory, there are altogether three different contributions to the counter terms: The actual Green-Schwarz terms, the kinetic action for the three-form field strength and, in the presence of heterotic five-branes, additional couplings which are non-vanishing only if the gauge bundle contains abelian factors. For this reason, the latter are not considered in the classic compactification with $SU(N)$ bundles only.

The four-dimensional Green-Schwarz terms arise upon dimensional reduction from their ten-dimensional parents given in (2.6) and (2.7). If we explicitly take care of the two E_8 factors by writing $F = F_1 + F_2$, we get for the anomaly eight-form (2.7)

$$\begin{aligned} X_8 \;=\; & \frac{1}{4}\left(\mathrm{tr}F_1^2\right)^2 + \frac{1}{4}\left(\mathrm{tr}F_2^2\right)^2 - \frac{1}{4}\left(\mathrm{tr}F_1^2\right)\left(\mathrm{tr}F_2^2\right) - \frac{1}{8}\left(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2\right)\left(\mathrm{tr}R^2\right) + \\ & \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}\left(\mathrm{tr}R^2\right)^2 . \end{aligned} \tag{3.49}$$

To arrive at this result we have to take into account that $\mathrm{Tr}_{E_8\times E_8}(F_1^q F_2^r) = 0$ (for simultaneously non-vanishing q and r) and furthermore use the trace identities (A.16) in appendix A.2. With the help of the tadpole cancellation condition (2.12), we dimensionally reduce this term to

$$S_{GS} \;=\; \sum_{i=1}^{2}\Big\{ \frac{1}{8\,(2\pi)^3\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left(\mathrm{tr}F_i^2\right) \left[\frac{1}{4(2\pi)^2}\left(\mathrm{tr}\overline{F}_i^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2\right) - \frac{1}{3}[W]\right]$$

[7]This is spelled out in appendix C.

$$(3.50)$$

$$+ \quad \frac{1}{4\,(2\pi)^3\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \mathrm{tr}(F_i \overline{F}_i) \left[\frac{1}{4(2\pi)^2}\left(\mathrm{tr}\overline{F}_i^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2\right) - \frac{1}{3}[W]\right] \tag{3.51}$$

$$+ \quad \frac{1}{24\,(2\pi)^5\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left[\mathrm{tr}(F_i \overline{F}_i)\right]^2 \Big\} \tag{3.52}$$

$$- \quad \frac{1}{96\,(2\pi)^3\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left[\frac{1}{4(2\pi)^2}\left(\mathrm{tr}R^2\right)\left(\mathrm{tr}\overline{R}^2\right) - 2[W]\right] \tag{3.53}$$

$$- \quad \frac{1}{24\,(2\pi)^5\,\alpha'} \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \mathrm{tr}(F_1 \overline{F}_1)\,\mathrm{tr}(F_2 \overline{F}_2). \tag{3.54}$$

Note the explicit dependence on the heterotic five-branes present in the most general case via the terms involving $[W] = \sum_a N_a \,\overline{\gamma}_a$. We will discuss the consequences of their contributions momentarily; for the time being, let us consider the special case without five-branes, i.e. where $[W] = 0$.

In this situation, the only missing ingredient is the kinetic term

$$S_{kin} = -\frac{1}{4\kappa_{10}^2} \int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}}\, H \wedge \star_{10} H. \tag{3.55}$$

For the purpose of the dimensional reduction it is convenient to make use of a basis of two-forms ω_k, $k = 1, \ldots, h_{11}$ and their Hodge dual four-forms[8] $\widehat{\omega}^k$ with the property

$$\int_{\mathcal{M}} \omega_k \wedge \widehat{\omega}_{k'} = \delta_{kk'}. \tag{3.56}$$

In terms of the string length $\ell_s = 2\pi\sqrt{\alpha'}$ we now expand

$$\begin{aligned} B^{(2)} &= b_0^{(2)} + \ell_s^2 \sum_{k=1}^{h_{11}} b_k^{(0)}\,\omega_k, \quad B^{(6)} = \ell_s^6\, b_0^{(0)}\,\mathrm{vol}_6 + \ell_s^4 \sum_{k=1}^{h_{11}} b_k^{(2)}\,\widehat{\omega}_k, \\ \mathrm{tr}\overline{F}_1^2 &= (2\pi)^2 \sum_{k=1}^{h_{11}} (\mathrm{tr}\overline{F}_1^2)_k\,\widehat{\omega}_k, \quad \mathrm{tr}\overline{R}^2 = (2\pi)^2 \sum_{k=1}^{h_{11}} (\mathrm{tr}\overline{R}^2)_k\,\widehat{\omega}_k, \\ \overline{f}_m &= 2\pi \sum_{k=1}^{h_{11}} (\overline{f}_m)_k\,\omega_k, \end{aligned} \tag{3.57}$$

where for dimensional reasons we have introduced appropriate powers of α' and vol_6 is the volume form on $\mathcal{M}$ normalized such that $\int_{\mathcal{M}} \mathrm{vol}_6 = 1$. Note that

[8] One might wonder at first sight why we only take the even cohomology into account. The point is that even if the internal manifold exhibited elements in $H^1(\mathcal{M}, \mathbb{Z})$ we would not pick up any four-dimensional contributions from the Green-Schwarz terms corresponding to the expansion of $B^{(2)}$ into internal and external one-forms. The same applies to the potential expansion of $B^{(6)}$ into internal and external 3-forms.

$\overline{f}_k^m \in \mathbb{Z}$ due to the integrality of $c_1(L) \in H^2(\mathcal{M}, 2\mathbb{Z})$. Let us anticipate that the universal axion $b_0^{(0)}$ complexifies the dilaton to form the complex scalar of a chiral supermulitplet in the $\mathcal{N}=1$ supergravity theory, whereas the $b_k^{(0)}$ pair with the Kähler moduli. As a consequence of the duality between $B^{(2)}$ and $B^{(6)}$, both types of two-forms $b_j^{(2)}$ are related to their axionic counterparts by $\star_4\, db_j^{(2)} = e^{2\phi_{10}}\, db_j^{(0)}$ for all $j \in \{0, 1, \ldots, h_{11}\}$, as promised in (3.37).

The general strategy is clear: Insert the expansions (3.57) into (3.50) - (3.54) as well as (3.55) and organize the surviving contributions as vertex (3.39) and mass terms (3.40). For simplicity, we focus now on the mixed abelian-nonabelian and abelian-gravitational anomalies. The GS-terms (3.50) and (3.53) give rise to the following vertex terms in four dimensions

$$S_{GS} \;=\; \sum_{i=1}^{2}\Big\{\frac{1}{32\,(2\pi)}\int_{\mathbb{R}_{1,3}}\sum_{k=1}^{h_{11}}\Big(b_k^{(0)}\,\mathrm{tr}F_1^2\Big)\;(\mathrm{tr}\overline{F}_1^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2)_k\Big\} \tag{3.58}$$

$$-\frac{1}{384\,(2\pi)}\int_{\mathbb{R}_{1,3}}\sum_{k=1}^{h_{11}}\Big(b_k^{(0)}\,\mathrm{tr}R^2\Big)\;(\mathrm{tr}\overline{R}^2)_k. \tag{3.59}$$

By contrast, from (3.51) we yield a mass term for the four-dimensional two-form field $b_0^{(2)}$

$$S^0_{mass} \;=\; \frac{1}{16\,(2\pi)^5\alpha'}\int_{\mathbb{R}_{1,3}}\sum_{m_1=1}^{M_1}\Big(b_0^{(2)}\wedge f_{m_1}\Big)\,\kappa_{m_1,m_1}\int_{\mathcal{M}}\overline{f}_{m_1}\wedge(\mathrm{tr}\overline{F}_1^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2)$$
$$+(1\leftrightarrow 2), \tag{3.60}$$

where we have used that $\kappa_{m_i,n_i} = 0$ for $m_i \neq n_i$ (see (3.25)). This mass term for the universal axion is obviously only present for $U(1)$ symmetries of type (i), reflecting the fact that for the $E_8 \times E_8$ heterotic string $U(1)$ factors of type (ii) are always non-anomalous.

To cancel the anomalies we also need a GS-term for the external axion $b_0^{(0)}$ and mass terms for the Kähler axions $b_k^{(2)}$. They emerge from (3.55), which contains, apart from the kinetic action for $B^{(2)}$, the cross term

$$S_{\mathrm{kin}} \;=\; \frac{\alpha'}{8\kappa_{10}^2}\int(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2)\wedge B^{(6)}. \tag{3.61}$$

On the one hand, this gives rise to a four-dimensional GS-term

$$S^0_{GS} = \frac{1}{8\pi}\int_{\mathbb{R}_{1,3}} b_0^{(0)}\wedge(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2). \tag{3.62}$$

In addition, reducing $\mathrm{tr}F_i \wedge F_i$ such that one factor takes values in the external $U(1)$s and the other in the internal ones, we find mass terms for the $b_k^{(2)}$. After dimensional reduction one eventually arrives at four-dimensional couplings of the form

$$S_{\text{mass}} = \sum_{i=1}^{2} \Big\{ \frac{1}{2\ell_s^2} \int_{\mathbb{R}_{1,3}} \sum_{m_i=1}^{M_i} \sum_{k=1}^{h_{11}} \left(f_{m_i} \wedge b_k^{(2)} \right) \, \kappa_{m_i,m_i} \, (\overline{f}_{m_i})_k \Big\}. \tag{3.63}$$

The GS-couplings (3.58),(3.62) and the mass terms (3.60), (3.63) have precisely the structure of the general coupling and mass terms considered in (3.39) and (3.40), which, as we showed, lead to appropriate anomaly six-forms and cancel the field theoretic anomalies. In other words, they generate tree-level graphs of the form displayed in figure 3.1, which provide couplings of the same type as the ones appearing in the mixed gauge anomalies. For the mixed abelian-nonabelian GS contribution we get, according to the foregoing discussion,

$$A^{GS}_{U(1)_{m_i} - E^2_{9-\mathcal{N}_i}} \sim \frac{\kappa_{m_i,m_i}}{32(2\pi)^6 \, \alpha'} \, f_{m_i} \wedge \text{tr} F_1^2 \, \left[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(\text{tr} \overline{F}_1^2 - \frac{1}{2} \text{tr} \overline{R}^2 \right) \right]. \tag{3.64}$$

For the mixed abelian-gravitational anomaly the contributions from internal axions and the four-dimensional one add up to

$$\begin{aligned} A^{GS}_{U(1)_{m_i} - G^2_{\mu\nu}} &\sim -\frac{\kappa_{m_i,m_i}}{64(2\pi)^6 \, \alpha'} \, f_{m_i} \wedge \text{tr} R^2 \, \Big[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(\text{tr} \overline{F}_1^2 - \frac{1}{2} \text{tr} \overline{R}^2 \right) \\ &\qquad + \frac{1}{12} \int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(\text{tr} \overline{R}^2 \right) \Big] \\ &= -\frac{\kappa_{m_i,m_i}}{64(2\pi)^6 \, \alpha'} \, f_{m_i} \wedge \text{tr} R^2 \, \left[\int_{\mathcal{M}} \overline{f}_{m_i} \wedge \left(\text{tr} \overline{F}_1^2 - \frac{5}{12} \text{tr} \overline{R}^2 \right) \right]. \end{aligned} \tag{3.65}$$

Along the same lines, one can also show that the mixed $U(1)^3$ anomalies cancel. Now also the Green-Schwarz couplings (3.52) contribute.

3.4.3 The generalized Green-Schwarz mechanism including five-branes

The inclusion of heterotic five-branes complicates the story of anomaly cancellation and leads to interesting new phenomena. The point is that in order to generate the correct anomaly cancelling couplings from the Green-Schwarz terms, we have to assume tadpole cancellation to organize the various contributions as in (3.50) - (3.54). This leads, in the presence of five-branes, to additional five-brane dependent contributions which yield anomalous diagrams in the effective theory, but without there existing any one-loop anomalies which would have to be cancelled by them.

Let us go back to (3.50) - (3.54) and collect the terms involving the five-brane class $[W]$. From these we can, following the analogous steps performed in the previous section, construct an anomaly six-form. The result is

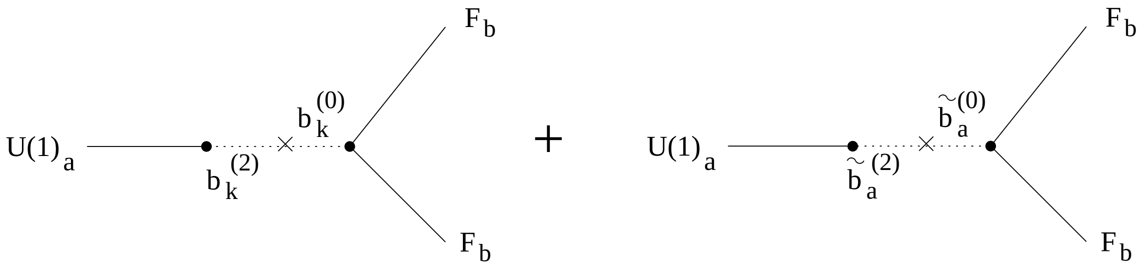

Figure 3.1: Green-Schwarz counter term for the mixed gauge anomaly.

$$A^{M5} \sim -\frac{1}{24(2\pi)^4\alpha'} \sum_a N_a \int_{\gamma_a} \mathrm{tr}(F_1\overline{F}_1) \Big[\frac{1}{4}\left(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2\right) + \frac{3}{4}\left(\mathrm{tr}F_1^2 - \mathrm{tr}F_2^2\right)\Big] + (1 \leftrightarrow 2). \quad (3.66)$$

Since there does not exist any chiral matter from the M5-branes, the only way to compensate the anomaly from (3.66) is by additional Green-Schwarz terms from the M5-branes. In the next section, we will provide a rigorous derivation of the presence of these terms independently of the requirement of anomaly cancellation. Here we will anticipate their form and discuss their role played in the Green-Schwarz mechanism.

Let us start by observing that the first term in (3.66) can precisely be cancelled by introducing the additional coupling

$$S^{(1)}_{GS} = \frac{1}{96(2\pi)^3\alpha'} \sum_a N_a \int_{\Gamma_a} B^{(2)} \wedge \left(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2\right) \quad (3.67)$$

in the effective action. To show this we simply have to perform dimensional reduction and follow the steps detailed at the beginning of the previous section and construct the anomaly six-form induced by the coupling (3.67).

To cope with the second contribution in (3.66), we recall from the general discussion in section (2.1) that on the six-dimensional world-volume of an M5-brane there lives a tensor field $\widetilde{B}_a$ which is self-dual with respect to the metric on the six-dimensional worldvolume of the five-brane,

$$d\widetilde{B}_a = \star_a d\widetilde{B}_a. \quad (3.68)$$

Note that the corresponding Hodge star operator factorizes as $\star_a = \star_4 \otimes \star_{2_a}$ into the external four-dimensional piece and the one defined with respect to the metric of the two-cycle wrapped by the five-brane. By dimensional reduction $\widetilde{B}_a$ gives rise to a two-form and a dual scalar

$$\widetilde{B}_a \;=\; \widetilde{b}^{(2)}_a + \ell_s^2\, \widetilde{b}^{(0)}_a\, \widehat{\gamma}_a \qquad \text{with} \quad d\widetilde{b}^{(0)}_a = \star_4\, d\widetilde{b}^{(2)}_a. \quad (3.69)$$

Here have introduced $\widehat{\gamma}_a$ as the Hodge dual of $\overline{\gamma}_a$ such that it satisfies $\ell_s^2 \star_{2_a} \widehat{\gamma}_a = 1$. For completeness, we point out that if the five-brane wraps a holomorphic curve of genus g, then taking one leg of $\widetilde{B}_a$ to be along one of its $2g$ one-cycles gives rise to $2g$ additional vector fields in four dimensions, only g of which carry independent degrees of freedom due to the self-duality of $\widetilde{B}_a$ [110]. Consequently, we encounter an additional gauge group of $U(1)^g$ in four dimensions, possibly enhanced if certain components of the holomorphic curve coincide. Since there exists no chiral matter charged under this gauge group, and even more so no matter charged simultaneously under the visible gauge group resulting from the E_8, it is veritably hidden and will not affect us any more in the sequel.

The extra pair of dual $\widetilde{b}_a^{(0)} - \widetilde{b}_a^{(2)}$ can generate additional Green-Schwarz counter terms, again completely in the spirit of the previous section. More precisely, one can apply the by now familiar strategy and convince oneself that the following coupling term

$$S_{GS}^{(2)} = \frac{1}{8(2\pi)^3 \alpha'} \sum_a N_a \int_{\Gamma_a} \widetilde{B}_a \wedge \left(\mathrm{tr} F_1^2 - \mathrm{tr} F_2^2 \right) \tag{3.70}$$

provides just the right counter terms to cancel the second five-brane dependent part in (3.66).

In fact, (3.70) can be viewed as arising from the cross terms in the kinetic action for the three-forms $\widetilde{H}_a$

$$S_{kin} = -\frac{1}{2\,(2\pi)^3 (\alpha')^2} \int_{\Gamma_a} \widetilde{H}_a \wedge \star_a \widetilde{H}_a \tag{3.71}$$

with

$$\widetilde{H}_a = d\widetilde{B}_a - \frac{\alpha'}{8} \left(\omega_{Y,1} - \omega_{Y,2} \right). \tag{3.72}$$

Note that we are free to choose some normalisation of $\widetilde{H}_a$ and correspondingly also of its kinetic action. What is fixed by requiring anomaly cancellation is, as we recall from the discussion around (3.45), merely the ratio of the prefactor of the kinetic term for the two-form fields (3.71) and of the Green-Schwarz like coupling (3.70). One can easily check that the normalisations of (3.67),(3.70) and (3.71) are indeed consistent with the anomaly six-form (3.66) if we take into account that $d\widetilde{B}_a$ is self-dual with respect to $\star_a$. As a general remark, it is known that due to the self-duality of $\widetilde{H}_a$, we should actually stick to the M-theory five-brane action [81], as will be done in section (3.4.4).

To conclude, both the terms (3.67) and (3.70) must indeed be present in the ten-dimensional effective action of the E_8 heterotic string for a consistent five-brane coupling. Even though the requirement of these terms by anomaly cancellation is manifest only once we allow for background bundles with non-zero first Chern class, their presence cannot depend on the gauge instanton background, of course. In particular, they have an effect on the gauge kinetic function also of the field strength associated with the semi-simple part of the gauge group, as we will see in section (3.4.5). It is reassuring to note that both new contributions to the effective action are also consistent with the analogous Green-Schwarz mechanism in six-dimensional compactifications, as analysed recently in [105]. Still, as a non-trivial consistency check for our setup, it is highly desirable to provide an independent derivation of the unfamiliar couplings from the viewpoint of heterotic M-theory. We will endeavour to do so in the next section.

3.4.4 M-theory origin of new GS-terms

The presence of the counter terms (3.67) and (3.70) can indeed be derived directly from Horava-Witten theory. The logic is very similar to that leading to the usual Green-Schwarz terms from heterotic M-theory, as first described in [111, 112]. Here we will extend the analysis to the five-brane dependent terms.[9]

As pointed out several times, Horava-Witten theory is eleven-dimensional supergravity plus higher derivative Chern-Simons couplings compactified on the circle S^1 and modded out further by a $\mathbb{Z}_2$ involution acting on the eleventh dimension. Horava and Witten found [76] that the two ten-dimensional fixed planes under the orbifold $\mathbb{Z}_2$ action give rise to anomalies which can only be cancelled by postulating the existence of an E_8 gauge theory on each of these planes. The two ten-dimensional E_8 gauge theories are identified with the gauge sector charged under the two factors in the heterotic $E_8 \times E_8$ theory. As it will turn out, the ten-dimensional dilaton is related to the size of the eleventh dimension and thus to the separation of the two E_8 sectors along the interval $S^1/\mathbb{Z}_2$. As always when dealing with orbifold theories one has the choice to work either "downstairs" on the space modded out by the geometric orbifold action and after projecting out all states not invariant under it, or in the "upstairs" picture. This means in our case that we consider the action on the circle S^1, bearing in mind, however, that we will eventually identify two opposite points on the circle and keep only those terms in the action invariant under the induced $\mathbb{Z}_2$ action.

The effective action of heterotic M-theory in the upstairs picture is given by an eleven-dimensional bulk part on $\mathcal{M}_u^{11}$, the ten-dimensional gauge actions defined on $\mathcal{M}^{(10)}$ and in addition the contribution from possible M5-branes. Concretely we use the conventions that [76, 77, 114]

$$
\begin{aligned}
S &= S_{kin} + S_{CS} + S_{curv} + S_{YM} + S_{M5}, \qquad (3.73)\\
S_{kin} &= \frac{1}{2\,\overline{\kappa}^2}\int_{\mathcal{M}_u^{11}} R\,\Omega - \frac{1}{2} G \wedge \star G,
\end{aligned}
$$

[9] Our derivation was done independently from [113], where a similar analysis has been performed. Note that this reference does not use the resulting Green-Schwarz terms for cancellation of abelian anomalies and does not consider the terms (3.70) arising from the M5-brane action. Also, to the best of our knowledge, the connection between the new GS terms and the FI-D-terms in section 2.6 has not been explored previously.

$$
\begin{aligned}
S_{CS} &= \frac{1}{2\,\overline{\kappa}^2}\int_{\mathcal{M}_u^{11}}\frac{1}{6}C\wedge G\wedge G,\\
S_{curv} &= \frac{1}{48(2\pi)^3\overline{\kappa}^2 T_5}\int_{\mathcal{M}_u^{11}} C\wedge\left(\frac{1}{8}\mathrm{tr}\,R^4-\frac{1}{32}(\mathrm{tr}\,R^2)^2\right),\\
S_{YM} &= -\sum_{i=1}^{2}\frac{1}{2\lambda^2}\int_{\mathcal{M}^{(10)}}\mathrm{tr}\,(F^i\wedge\star F^i)-\frac{1}{2}\mathrm{tr}\,(R\wedge\star R),
\end{aligned}
$$

where $\mathcal{M}_u^{11}=\mathcal{M}^{(10)}\times S^1$. The compact eleventh dimension takes values in the range $-\pi\rho < x^{11}\leq \pi\rho$ and the gauge fields are localized at $x^{11}=0,\pi\rho$. The part of M5-brane action S_{M5} [81] relevant for our purposes will be given at the end of this section. The presence of a five-brane at position y along x^{11} requires that we also include its $\mathbb{Z}_2$ image at $-y$ with which the original brane will eventually be identified. Eleven-dimensional indices will be denoted by $I,J,K,\ldots$ and ten-dimensional ones by $A,B,C,\ldots$. The ten-dimensional gauge couplings are related to $\overline{\kappa}$ via $\lambda^2=(4\pi)(2\pi\overline{\kappa}^2)^{2/3}$ and the tension of the five-brane is given by $T_5=(\frac{2\pi}{\overline{\kappa}^4})^{1/3}$ [114]. Finally, under the orbifold action $x^{11}\mapsto -x^{11}$, C_{AB11}, G_{ABC11} and the components $g^{(11)}_{AB}$ and $g^{(11)}_{11\,11}$ of the eleven-dimensional metric are even, but C_{ABC} and G_{ABCD} are odd [76] .

Supersymmetry conservation requires the inclusion of particular combinations of the gauge field strengths and the curvature into the Bianchi identity for the field strength $G=dC$ [76]. Following the intuition that five-branes effectively contribute to the action like gauge instantons[10], this Bianchi identity is modified further by M5-contributions and takes the general form [110]

$$
\begin{aligned}
(dG)_{11ABCD} &= -\frac{\overline{\kappa}^2}{\lambda^2}\Big(J_1\delta(x^{11})+J_2\delta(x^{11}-\pi\rho)\\
&\quad+\frac{1}{2}J_5\left(\delta(x^{11}-y)+\delta(x^{11}+y)\right)\Big)_{ABCD}.
\end{aligned}
\tag{3.74}
$$

Note that we take into account the contribution from the five-brane at $x^{11}=y$ and its mirror brane at $x^{11}=-y$ such that together their effect is that of one unit of gauge instanton (thus the factor $\frac{1}{2}$). The generalisation to the case of several five-branes is obvious. The gauge and curvature sources at the orbifold fixed planes are given by $J_i=\mathrm{tr}\,F_i\wedge F_i-\frac{1}{2}\mathrm{tr}\,R\wedge R=d\omega_i$ for $i=1,2$, while the five-brane contributes $J_5=-4(2\pi)^2\delta(\Gamma)$. Here $\delta(\Gamma)$ is the four-form Poincaré dual to the worldvolume of the five-brane in $\mathcal{M}^{(10)}$.[11] In analogy with the Yang-Mills and Lorentz Chern-Simons forms we also introduce the ten-dimensional three-form ω_5 satisfying $J_5=d\omega_5$.

Being interested in the ten-dimensional theory after Kaluza-Klein reduction on S^1, we now focus on the situation where the eleventh dimension is much smaller

[10] Alternatively, we can derive this contribution from the CS coupling of the M5-brane to the dual six-form potential, essentially along the lines of the derivation of equ. (2.11) reviewed in section 2.1.

[11] When we further compactify $\mathcal{M}^{(10)}=\mathbb{R}^{(1,3)}\times CY_3$ we have the obvious decomposition $\delta(\Gamma)=\delta(\mathbb{R}^{(1,3)})\wedge\overline{\gamma}$ for a five-brane wrapping the two-cycle dual to the four-form $\overline{\gamma}$ on CY_3.

than the ten-dimensional space. This is the limit in which the effective action of the ten-dimensional weakly coupled heterotic string arises [111, 112]. In this regime ten-dimensional derivatives of gauge and curvature terms can be neglected as compared to field variations along x^{11}. Hence, one can give an approximate solution for G and C to the above Bianchi identity and the equations of motion $D^I G_{IJKL} = 0$ by splitting the fields into their zero-mode and a background part as $C = C^{(0)} + C^{(1)}$ and $G = G^{(0)} + G^{(1)}$. Including also the five-brane sources, we get

$$
\begin{aligned}
C_{ABC} &= C^{(1)}_{ABC}, \qquad C_{AB11} = B^{(2)}_{AB}, \\
G_{ABCD} &= G^{(1)}_{ABCD}, \qquad G_{ABC11} = (dB)_{ABC} + G^{(1)}_{ABC11},
\end{aligned}
$$

$$
\begin{aligned}
C^{(1)}_{ABC} &= -\frac{\overline{\kappa}^2}{2\lambda^2}\Big(\omega_1\epsilon(x^{11}) + \frac{1}{2}\omega_5(\epsilon(x^{11}-y)+\epsilon(x^{11}+y)) \\
&\qquad -\frac{x^{11}}{\pi\rho}(\omega_1+\omega_2+\omega_5)\Big)_{ABC} \\
G^{(1)}_{ABCD} &= -\frac{\overline{\kappa}^2}{2\lambda^2}\Big(J_1\epsilon(x^{11}) + \frac{1}{2}J_5(\epsilon(x^{11}-y)+\epsilon(x^{11}+y)) \\
&\qquad -\frac{x^{11}}{\pi\rho}(J_1+J_2+J_5)\Big)_{ABCD} \\
G^{(1)}_{ABC11} &= -\frac{\overline{\kappa}^2}{2\lambda^2\pi\rho}(\omega_1+\omega_2+\omega_5)_{ABC}. \qquad (3.75)
\end{aligned}
$$

$\epsilon(x^{11})$ denotes the step function, i.e. $\epsilon(x^{11}) = +1$ for x^{11} positive and -1 otherwise. We have introduced also the ten-dimensional two-form field $B^{(2)}$ which arises as the $\mathbb{Z}_2$ invariant components of C. Note that $G^{(1)}_{ABCD}$ is not continuous at $x^{11} = \pi\rho \cong -\pi\rho$ but rather takes the limiting values

$$
G^{(1)}_{ABCD}|_{\pi\rho,<} = \frac{\overline{\kappa}^2}{2\lambda^2}J_2, \qquad G^{(1)}_{ABCD}|_{-\pi\rho,>} = -\frac{\overline{\kappa}^2}{2\lambda^2}J_2 \qquad (3.76)
$$

on both sides of the second orbifold plane. When we take the exterior derivative dG, this gives a δ-function localized at $\pi\rho$ and proportional to $2J_2$,

$$
\begin{aligned}
(dG)_{11ABCD} &= \partial_{11}G^{(1)}_{[ABCD]} - 4\partial_{[A}G^{(1)}_{|11|BCD]} \qquad (3.77) \\
&= -\frac{\overline{\kappa}^2}{2\lambda^2}\Big[2J_1\delta(x^{11}) + J_5((\delta(x^{11}-y)+\delta(x^{11}+y))) \\
&\qquad +2J_2\delta(x^{11}-\pi\rho) - \frac{1}{\pi\rho}(J_1+J_2+J_5)\Big]_{ABCD} \\
&\quad -\frac{4\overline{\kappa}^2}{2\lambda^2\pi\rho}\left[\frac{1}{4}(J_1+J_2+J_5)_{ABCD}\right],
\end{aligned}
$$

so that the field configuration (3.75) indeed solves the Bianchi identity (3.74). Similarly, one may convince oneself that the equations of motion for the field strength G are satisfied up to terms proportional to ∂J_i, which are assumed to be negligible in the limit we are considering [111, 112].

The ten-dimensional weakly coupled heterotic string theory is recovered by compactification on S^1 according to the standard ansatz

$$ds_{11}^2 = e^{-2\phi_{10}/3}\, g_{AB}^{(10)}\, dx^A\, dx^B + e^{(4\phi_{10}/3)}(dx^{11})^2, \tag{3.78}$$

where we keep only those parts of the action which are invariant under $x^{11} \mapsto -x^{11}$. In particular, the kinetic term for G contains a part involving the combination $G_{11ABC}G^{11ABC}$. Inserting the solution (3.75), integrating over S^1 and focussing only on terms not involving ω_5 due to the five-branes precisely yields the familiar kinetic term

$$S_{kin}^H = -\frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} H \wedge \star H \tag{3.79}$$

for the ten-dimensional three-form field strength $H = dB^{(2)} - \frac{\alpha'}{4}(\omega_1 + \omega_2)$ after setting

$$\frac{1}{\kappa_{10}^2} = \frac{2\pi\rho}{\overline{\kappa}^2}, \qquad\qquad \alpha' = \frac{4\overline{\kappa}^2}{2\lambda^2\pi\rho} = \frac{2^{-1/3}}{\pi^2\rho}\left(\frac{\overline{\kappa}}{4\pi}\right)^{2/3}. \tag{3.80}$$

We are now ready to investigate the origin of the complete Green-Schwarz counter terms including the contribution from the five-branes. They arise at order $(\frac{\overline{\kappa}^2}{2\lambda^2})^2$ after inserting the above solution for C and G into the Chern-Simons terms S_{CS} in (3.73) as

$$\begin{aligned} S_{CS}|_{(\frac{\overline{\kappa}^2}{2\lambda^2})^2} &= \frac{3}{12\,\overline{\kappa}^2}\int_{\mathcal{M}^{(10)}}\int_{S^1} B^{(2)} \wedge G^{(1)} \wedge G^{(1)} \wedge dx^{11} \qquad (3.81)\\ &= \frac{\pi\rho}{4\overline{\kappa}^2}(\frac{\overline{\kappa}^2}{2\lambda^2})^2 \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left(\frac{2}{3}(J_1^2 + J_2^2 - J_1 J_2) - \frac{1}{6}J_5(J_1 + J_2)\right)\end{aligned}$$

plus additional terms proportional to $\int B^{(2)} \wedge J_5^2$, which however vanish after performing the integral. To arrive at this expression we place the five-brane and its mirror symmetrically at $y = \pm\frac{\pi\rho}{2}$ between the two orbifold fixed-planes. Note that the combination $C_{[AB11}G^{(1)}_{CDEF}G^{(1)}_{GHIJ]}$ is indeed even under the orbifold action and therefore survives in ten dimensions. Additional contributions from the higher curvature corrections S_{curv} are

$$\begin{aligned} S_{curv} &= \frac{1}{48(2\pi)^3\overline{\kappa}^2 T_5}\int_{\mathcal{M}_u^{11}} C \wedge \left(\frac{1}{8}\mathrm{tr}\, R^4 - \frac{1}{32}(\mathrm{tr}\, R^2)^2\right)\\ &= \frac{1}{24(2\pi)^5\alpha'}\int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left(\frac{1}{8}\mathrm{tr}\, R^4 - \frac{1}{32}(\mathrm{tr}\, R^2)^2\right). \qquad (3.82)\end{aligned}$$

The part $\frac{2}{3}(J_1^2 + J_2^2 - J_1 J_2)$ in (3.81) combines with (3.82) into the standard Green-Schwarz eight-form X_8 [111, 112].

The additional counter terms (3.67) we are after now arise from $J_5(J_1 + J_2) = -4(2\pi)^2\delta(\Gamma) \wedge (\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2)$. In summary, (3.81) and (3.82) yield in the ten-dimensional limit

$$S_{GS} = c \int_{\mathcal{M}^{(10)}} B^{(2)} \wedge \left(X_8 + \frac{(2\pi)^2}{4}\, \delta(\Gamma) \wedge (\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 - \mathrm{tr}R^2) \right) \tag{3.83}$$

with

$$c = \frac{8}{3}\frac{\pi\rho}{4\overline{\kappa}^2}\left(\frac{\overline{\kappa}^2}{2\lambda^2}\right)^2 = \frac{1}{24(2\pi)^5\alpha'}, \tag{3.84}$$

as postulated in (3.67).

The origin of the second five-brane dependent counter term (3.70) lies in the M5-brane action. With the normalisations of [81] (see e.g. also [115]), the part relevant for our analysis is given by

$$S_{M5} = -\frac{T_5}{2}\sum_a N_a \int_{\Gamma_a \cup \Gamma_a'} \left(\frac{1}{4}\widetilde{F}_a \wedge \star \widetilde{F}_a + \widetilde{C} + \frac{1}{2} d\widetilde{B}_a \wedge C \right), \tag{3.85}$$

again summing over all branes and their mirrors. Here $\widetilde{F}_a = d\widetilde{B}_a - C$ is the modified field strength of the self-dual tensor field $\widetilde{B}_a$ living on the five-brane and $\widetilde{C}$ is the bulk six-form potential dual to C. The contribution from (3.85) we are interested in is the topological coupling $d\widetilde{B}_a \wedge C$. Following the general strategy we insert again the appropriate background solution for C and place brane and mirror brane at $y = \pm\frac{\pi\rho}{2}$ respectively to find

$$\begin{aligned} S_{top} &= -\frac{T_5}{4}\sum_a N_a \left(\int_{\Gamma_a} \widetilde{B}_a \wedge dC^{(1)} + \int_{\Gamma_a'} \widetilde{B}_a \wedge dC^{(1)} \right) = \\ &= \frac{T_5}{4}\frac{\overline{\kappa}^2}{2\lambda^2}\sum_a N_a \int_{\Gamma_a} \widetilde{B}_a \wedge (\mathrm{tr}F_1^2 - \mathrm{tr}F_2^2). \end{aligned} \tag{3.86}$$

It can be checked that, together with the kinetic term for $\widetilde{B}_a$, this coupling indeed yields precisely the required counter terms to cancel the contribution to the five-brane anomaly in the second line of (3.66). In the standard ten-dimensional normalisation of the kinetic action for $\widetilde{B}_a$ which we used in (3.71) one eventually recovers the counter term (3.70). Note that we are always free to change the normalization of $\widetilde{B}_a$. What goes into the induced anomaly six-form is the merely the relative normalisation of the above vertex coupling and the kinetic term for $\widetilde{B}_a$ and unaffected by such trivial field redefinitions.

3.4.5 Gauge-axion masses from the Stückelberg mechanism

A central question we need to address is which of the abelian gauge factors remain massless after the Green-Schwarz mechanism cancels potential anomalies.

We recall from the discussion around (3.46) that the coupling terms S_{mass} involved in the anomaly cancellation process induce a Stückelberg-like mechanism for the abelian gauge factors which is specified by the mass matrix $\mathrm{M}^2_{m,n}$ in $S_{Stuckelberg} = -\sum_{m,n=1}^{M_i} \mathrm{M}^2_{m,n}\ (A_m \wedge \star_4 A_n)$. We now collect all contributions to these axion-gauge boson mass terms from the universal axion, $b_0^{(0)}$, the Kähler-axions, $b_k^{(0)}$, and finally the five-brane axions $\widetilde{b}_a^{(0)}$. For later purposes it is convenient to display the results directly in terms of the Chern characters of the background bundles (cf. (3.23)). This will allow us to identify the massless $U(1)$ combinations by inspecting the topological data of the bundles.

The mass term involving the universal axion reads

$$\begin{aligned} S^{0,m_i}_{mass} \;=\;& \frac{1}{4(2\pi)^2\alpha'} \int_{\mathbb{R}^{(1,3)}} b_0^{(2)} \wedge f_{m_i} \Bigg[\sum_{n_i=1}^{M_i} \kappa_{m_i,n_i} \int_{\mathcal{M}} c_1(L_{n_i}) \wedge \Bigg(\mathrm{ch}_2(V_{N_i}) + \\ & \sum_{k_i,l_i=1}^{M_i} \epsilon_{m_i,l_i}\, c_1(L_{k_i}) \wedge c_1(L_{l_i}) + \frac{1}{2}\, c_2(T) - \frac{1}{4} \sum_a N_a \overline{\gamma}_a \Bigg) \Bigg]. \end{aligned} \quad (3.87)$$

It arises as the sum of (3.51) and the extra counter term (3.67).

For the Kähler axions the kinetic term for H_3 induces the mass terms,

$$S^{k,m_i}_{mass} \;=\; \frac{1}{2(2\pi)^2\alpha'} \int_{\mathbb{R}^{(1,3)}} b_k^{(2)} \wedge f_{m_i} \Bigg[\sum_{n_i=1}^{M_i} \kappa_{m_i,n_i} \int_{\mathcal{M}} c_1(L_{n_i}) \wedge \widehat{\omega}_k \Bigg], \quad (3.88)$$

as we recall from (3.63), and the five-brane Green-Schwarz term (3.70) yields the mass term

$$S^{a,m_i}_{mass} \;=\; \pm \frac{1}{4\,(2\pi)^2\alpha'} \int_{\mathbb{R}^{(1,3)}} \widetilde{b}_a^{(2)} \wedge f_{m_i} \Bigg[\sum_{n_i=1}^{M_i} \kappa_{m_i,n_i} \int_{\mathcal{M}} c_1(L_{n_i}) \wedge \overline{\gamma}_a \Bigg] \quad (3.89)$$

for the 5-brane axions. The plus sign holds for the abelian field strengths arising from $E_8^{(1)}$ and the minus sign for $E_8^{(2)}$.

From these expressions one can immediately identify the matrix $\mathcal{M}_{jm}$ of equ. (3.40), with j running over all bulk and brane axion labels. We recall that the kernel of $\mathcal{M}_{jm}$ is related to the massless combinations of abelian gauge fields or axions, respectively, as described in equ. (3.48). Finally, let us point out that the mass terms are all of the same order in both string and sigma model perturbation theory. It is noteworthy that, though all mass terms are of order M_s^2, the mass eigenstates of the gauge bosons can in principle have masses significantly lower than the string scale at least in situations with multiple abelian factors.

3.5 Gauge couplings

In this section we extract the holomorphic gauge kinetic functions for the non-abelian and abelian gauge groups [80, 116–119]. Recall that the gauge kinetic functions f_a are encoded in the four-dimensional Yang-Mills Lagrangian, which, up to second order and in our sign conventions, takes the form (cf. e.g. [120])

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Re}(f_a)\,\mathrm{tr}(F \wedge \star F) + \frac{1}{2}\,\mathrm{Im}(f_a)\,\mathrm{tr}(F \wedge F). \tag{3.90}$$

In particular, the gauge coupling g, defined by

$$\mathcal{L}_{kin} = -\frac{1}{4g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}), \tag{3.91}$$

is seen to be given by $\mathrm{Re}(f_a)$ in this normalisation, possibly up to a multiplicative constant which takes account of the proper normalisation of the trace and which will be fixed later. Dimensional reduction of the ten-dimensional tree-level term

$$S^{(10)}_{YM} = -\frac{1}{2\kappa^2_{10}} \int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}}\, \frac{\alpha'}{4}\left(\mathrm{tr}(F_1 \wedge \star_{10} F_1) + \mathrm{tr}(F_2 \wedge \star_{10} F_2)\right) \tag{3.92}$$

reveals the tree-level gauge coupling as appearing in

$$S^{(4)}_{YM} = -\frac{1}{2\pi} \int_{\mathbb{R}_{1,3}} \frac{\mathrm{Vol}(\mathcal{M})}{\ell_s^6} e^{-2\phi_{10}}\, \frac{1}{4}\Big(\mathrm{tr}(F_1 \wedge \star_4 F_1) + \mathrm{tr}(F_2 \wedge \star_4 F_2)\Big). \tag{3.93}$$

The traces are, at this stage, still formally taken over the two E_8 factors without differentiating between the actual gauge groups in four dimensions. For later purposes we note also that the compact volume is computed from

$$\mathrm{Vol}(\mathcal{M}) = \frac{1}{6}\int_{\mathcal{M}} J \wedge J \wedge J = \frac{\ell_s^6}{6} \sum_{i,j,k} d_{ijk}\, \alpha_i\, \alpha_j\, \alpha_k, \tag{3.94}$$

where $d_{ijk} = \int_{\mathcal{M}} \omega_i \wedge \omega_j \wedge \omega_k$ are the triple intersection numbers of the basis of two-forms and the Kähler form is expanded as $J = \ell_s^2 \sum_{i=1}^{h_{11}} \alpha_i \omega_i$.

The axionic coupling involving $\mathrm{Im}(f_a)$, by contrast, is contained in the cross term (3.62) emerging from the kinetic action for H,

$$S^0_{GS} = \frac{1}{8\pi} \int_{\mathbb{R}_{1,3}} b_0^{(0)} \wedge \Big(\mathrm{tr}(F_1 \wedge F_1) + \mathrm{tr}(F_2 \wedge F_2)\Big). \tag{3.95}$$

Consequently the full tree level gauge kinetic function is simply $f = \frac{1}{2}S$ [12] with the complexified dilaton defined as

$$S \;=\; \frac{1}{2\pi}\left[e^{-2\phi_{10}}\frac{\mathrm{Vol}(\mathcal{M})}{\ell_s^6} + i\, b_0^{(0)}\right]. \tag{3.96}$$

However, in the course of the discussion of the Green-Schwarz mechanism we have encountered further axionic couplings similar to (3.95) but involving

[12] To be quite pedantic, there arise additional normalisation constants related to the precise definition of the traces over the gauge factors. We will discuss them momentarily for the non-abelian and abelian factors in four dimensions.

the Kähler and the five-brane axions. These stem from the conventional Green-Schwarz terms (3.50) and the new five-brane dependent couplings (3.67), (3.70). In the effective four-dimensional $\mathcal{N}=1$ supergravity, these axions are not arbitrary fields but form the imaginary part of the lowest lying component in a chiral superfield [121][13]. The full complex bosonic part of these superfields is given by

$$T_k = \frac{1}{2\pi}\left[-\frac{1}{\ell_s^2}\int_{\mathcal{M}} J\wedge\widehat{\omega}_k + ib_k^{(0)}\right], \tag{3.97}$$

$$\Lambda_a = \frac{1}{2\pi}\left[-\lambda_a\frac{\mathrm{Vol}(\Gamma_a)}{\ell_s^2} + i\widetilde{b}_a^{(0)}\right]. \tag{3.98}$$

The λ_a denote the scalars which together with the self-dual two-forms $\widetilde{B}_a$ combine into tensor multiplets on the six-dimensional world-volume of the five-branes. In the strong coupling Horava-Witten model these scalars are nothing else than the position of the respective five-branes along the eleventh direction. The normalisation of the real versus the imaginary parts of (3.97) and (3.98) is such that the kinetic terms for all scalars is incorporated correctly in a suitable Kähler potential. The Kähler potential consistent with the above choice will be given in the next section.

Due to these axionic couplings which involve the imaginary parts of the superfields (3.97) and (3.98), $\mathrm{Im}f_a$ receives additional contributions. The $\mathcal{N}=1$ supergravity formalism dictates that the full gauge kinetic function is a holomorphic quantity, and therefore a modification of its imaginary part cannot leave its real part inert. Rather, it must be that the full complex correction term is again proportional to the bosonic part of an $\mathcal{N}=1$ superfield[14].

The gauge kinetic function for the field strengths of the non-abelian gauge groups which we collectively denoted as $E_{9-\mathcal{N}_i}$ can therefore be written, in the large radius regime, as

$$f_{E_{9-\mathcal{N}_i}} = S + \frac{1}{8}\sum_{k=1}^{h_{11}} T_k\left(\mathrm{tr}\overline{F}_{1,2}^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2 - \sum_a N_a\overline{\gamma}_a\right)_k \pm \frac{1}{2}\sum_a N_a\Lambda_a. \tag{3.99}$$

This precise normalisation arises when we express the trace over the E_8 in terms of the trace over the actual gauge group in four dimensions. From equation (3.25) we recover a factor of 2 in front of the non-abelian traces which we have included in (3.99). The upper sign of the last term involving the superfields Λ_a is for the first E_8, the lower one for the second. This is an immediate consequence of the the form of the five-brane dependent counter term (3.70). We have furthermore introduced the notation

$$\overline{\gamma}_a = \sum_{k=1}^{h_{11}}(\overline{\gamma}_a)_k\;\widehat{\omega}_k. \tag{3.100}$$

[13]In abuse of notation, we will sometimes also refer to the complex bosonic component as the superfield, just for brevity. It will always be clear from the context what is meant.

[14]And mutatis mutandis for the fermionic terms if we consider f_a as a veritable superfield instead of focusing just on its bosonic part.

The physical quantities we are interested in are the gauge couplings as the real part of f_a, for which one gets at linear order in λ_a

$$\begin{aligned}\frac{4\pi}{g^2_{E_{9-\mathcal{N}_i}}} &= \frac{e^{-2\phi_{10}}}{3\ell_s^6}\int_{\mathcal{M}} J\wedge J\wedge J - \frac{1}{\ell_s^2}\int_{\mathcal{M}} J\wedge\frac{1}{4(2\pi)^2}\left(\mathrm{tr}\overline{F}^2_{1,2} - \frac{1}{2}\mathrm{tr}\overline{R}^2\right)\\ &+\frac{1}{\ell_s^2}\sum_a N_a\left(\frac{1}{4}\mp\lambda_a\right)\int_{\Gamma_a} J. \end{aligned} \tag{3.101}$$

This makes it clear how the first term, the tree-level gauge coupling, receives one-loop threshold corrections depending both on the Kähler moduli of the Calabi-Yau and the five-brane moduli λ_a (see also [113]). If we set all five-brane moduli to zero, then we nevertheless get a five-brane contribution of 1/4 to the one-loop gauge couplings in both the first and the second E_8. From the Horava-Witten point of view this means that for $\lambda_a = 0$, the five-brane is placed exactly in the middle between the two end-of-the-world nine-branes and λ_a is measured with respect to this symmetric configuration (see figure 3.2). We will give further evidence for this interpretation momentarily.

The next-to-leading order M-theory computation carried out in [122,123] provides an $\mathcal{O}(\lambda^2)$ correction to the real part of the dilaton superfield

$$S = \frac{1}{2\pi}\left[e^{-2\phi_{10}}\frac{\mathrm{Vol}(\mathcal{M})}{\ell_s^6} + \sum_a N_a\frac{\lambda_a^2}{2\ell_s^2}\int_{\Gamma_a} J + i\, b_0^{(0)}\right]. \tag{3.102}$$

This correction was derived in [123] essentially by requiring that the kinetic terms for the self-dual two-form on the M5-brane can indeed be correctly incorporated into an appropriate Kähler potential. Using this result and holomorphicity of the gauge kinetic function leads to the gauge couplings

$$\begin{aligned}\frac{4\pi}{g^2_{E_{9-\mathcal{N}_i}}} &= \frac{1}{3\ell_s^6\, g_s^2}\int_{\mathcal{M}} J\wedge J\wedge J\\ &-\frac{1}{\ell_s^2}\int_{\mathcal{M}} J\wedge\left(\mathrm{ch}_2(V_{N_i}) + \sum_{m_i,n_i=1}^{M_i}\epsilon_{m_i,n_i}c_1(L_{m_i})\wedge c_1(L_{n_i}) + \frac{1}{2}\,c_2(T)\right)\\ &+\frac{1}{\ell_s^2}\sum_a N_a\left(\frac{1}{2}\mp\lambda_a\right)^2\int_{\Gamma_a} J. \end{aligned} \tag{3.103}$$

For $\lambda_a = -\frac{1}{2}$, the contribution of the five-brane to the threshold corrections from $E_8^{(1)}$ is precisely that of a small instanton inside $E_8^{(1)}$ [83]. This unambiguously identifies λ_a as the relative position of the five-brane measured with respect to the middle of the interval between the orbifold planes, as suggested already. Different normalisations of the counter terms (3.70) would have resulted in a corresponding redefinition of λ_a. As expected, if one places the five-brane inside the $E_8^{(2)}$ wall, its gauge threshold corrections to the gauge couplings from $E_8^{(1)}$ vanish and vice versa.

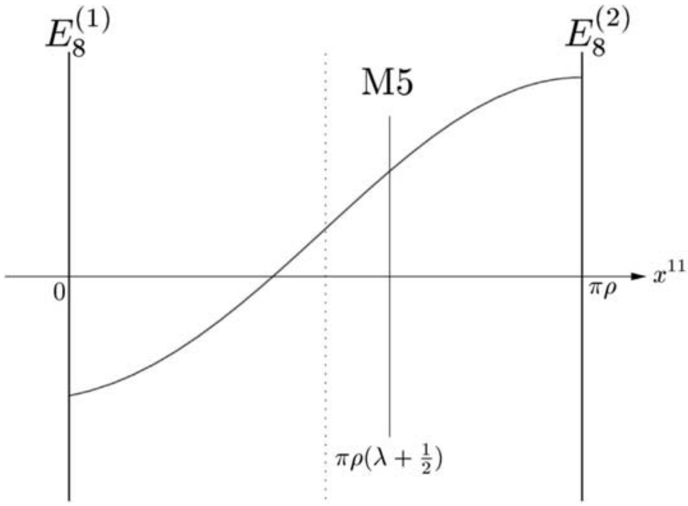

Figure 3.2: M5-brane potential in Horava-Witten theory on the Quintic induced by abelian gauge flux on $E_8^{(1)}$.

For the abelian gauge groups things are slightly different. Now also the Green-Schwarz terms (3.52) and (3.54) lead to axionic couplings besides the ones we have encountered already. The resulting gauge couplings are in general non-diagonal and are readily found to be given by

$$\begin{aligned}\frac{4\pi}{g^2_{m_i,n_i}} &= \frac{\eta_{m_i,n_i}}{12\ell_s^6 g_s^2}\int_{\mathcal{M}} J\wedge J\wedge J \\ &-\frac{\eta_{m_i,n_i}}{4\ell_s^2}\int_{\mathcal{M}} J\wedge\left(\mathrm{ch}_2(V_{N_i})+\sum_{m_i,n_i=1}^{M_i}\epsilon_{m_i,n_i}c_1(L_{m_i})\wedge c_1(L_{n_i})+\frac{1}{2}\,c_2(T)\right) \\ &-\frac{1}{12\ell_s^2}\int_{\mathcal{M}} J\wedge\left(\sum_{p_i,q_i=1}^{M_i}\kappa_{m_i,p_i}\kappa_{n_i,q_i}\,c_1(L_{p_i})\,c_1(L_{q_i})\right) \\ &+\frac{\eta_{m_i,n_i}}{4\ell_s^2}\sum_a N_a\left(\frac{1}{2}\mp\lambda_a\right)^2\int_{\Gamma_a} J \qquad (3.104)\end{aligned}$$

for both $U(1)$ factors from the same E_8 factor and by

$$\frac{4\pi}{g^2_{m_1,n_2}} = \frac{1}{24\ell_s^2}\int_{\mathcal{M}} J\wedge\left(\sum_{p_1=1}^{M_1}\sum_{q_2=1}^{M_2}\kappa_{m_1,p_1}\kappa_{n_2,q_2}\,c_1(L_{p_1})\,c_1(L_{q_2})\right) \qquad (3.105)$$

for one $U(1)$ from the first and one $U(1)$ from the second E_8. Apparently, only for trivial line bundles, i.e. Wilson lines, do the extra threshold corrections vanish. The normalisation relative to the expression for the non-abelian gauge groups arises as follows: First we have to remember once more how to express the trace over E_8 in terms of the four-dimensional gauge groups, see equation (3.25). In addition, the generators of the non-abelian groups are canonically normalized as $\mathrm{tr} T_a T_b = \frac{1}{2}\delta_{ab}$, and we need to adjust the normalisation of the abelian gauge factors by explicitly including this factor of $\frac{1}{2}$ into the gauge coupling.

We conclude the present discussion with an important remark. As is obvious from the explicit expressions (3.103), (3.104), the tree-level contribution to the real part of the gauge kinetic function is always positive, as it must; after all, $\mathrm{Re}(f)$ is just the inverse square of the gauge couplings. Clearly, positivity of $\mathrm{Re}(f)$ must still hold after subtracting the threshold corrections, at least in the regime of small string coupling, where all potential higher corrections are negligible compared to the one-loop thresholds. A violation of this bound would indicate severe inconsistencies in the effective field theory, possibly in the sense that the four-dimensional supergravity we have written down does not follow as the consistent truncation of the full ten-dimensional theory. In any case, we insist on positivity of the real part of the threshold corrected gauge kinetic functions as an effective supersymmetry condition. Since the threshold corrections manifestly depend on the Kähler moduli, the five-brane position moduli and the dilaton, this condition imposes constraints on the involved moduli fields. In short, in a supersymmetric vacuum we must ensure that

$$\mathrm{Re}(f_{E_{9-\mathcal{N}_i}}) > 0, \qquad \mathrm{Re}(f_{U(1)}) > 0, \tag{3.106}$$

for the two non-abelian gauge sectors and for all unbroken, i.e. anomaly-free and massless abelian gauge groups.

3.6 D-terms and supersymmetry constraints

The Green-Schwarz counter terms have provided us with important non-trivial information about the four-dimensional low-energy effective action, notably the gauge threshold corrections. The couplings between the abelian gauge fields and the axions have furthermore produced mass terms not only for the Kähler axions, but also for the universal axio-dilaton and the axions emerging from the five-branes, if present. In four-dimensional $\mathcal{N} = 1$ supergravity, theses axions form the imaginary part of the bosonic component of chiral superfields. The real parts are, as we have seen, given by the Kähler moduli, the dilaton and the moduli parameterising, in the M-theory limit, the position of the branes along the eleventh dimension. In supersymmetry preserving vacua, there must thus exist a mechanism which likewise renders the corresponding partners of the axions massive since a splitting of the mass terms within one supermultiplet is incompatible with supersymmetry. At string tree level, the Donaldson-Uhlenbeck-Yau equation is precisely of the right form to yield the required mass terms for the Kähler moduli. We therefore need to find analogous mass terms for the dilaton and the five-brane moduli. It is natural to expect that the violation of the equal-mass-condition for all components of a supermultiplet is manifestly correlated with the supersymmetry condition. On the other hand, we know that in theories with massive abelian gauge factors, Fayet-Iliopoulos (FI) D-terms signal a possible supersymmetry breakdown (e.g. [124]). This is therefore the starting point for our investigations. We will make heavy use of the standard fact that the FI terms can be computed from the Kähler potential $\mathcal{K}$ with the help of the supersymmetric field theory formula (e.g. [120])

$$D_m \frac{\xi_m}{g_m^2} = D_m \frac{\partial\mathcal{K}}{\partial V_m}\Big|_{V_m=0}, \tag{3.107}$$

where V_m constitutes the abelian vector superfields associated with the abelian gauge symmetry $U(1)_m$. After deriving the gauge invariant Kähler potential, it will be straightforward to extract the FI terms. We will find an intriguing relation between the FI terms and the DUY equation which allows us to identify one-loop corrections to the latter involving the dilaton and the five-brane moduli. They will indeed solve the puzzle about the missing mass terms. They also imply a modification of the stability condition on the gauge bundles arising at one-loop. Finally, we will comment on a new D-term contribution to the scalar potential of the M5-brane in heterotic M-theory in the presence of abelian gauge flux on the end-of-the-world branes which may be of significance in cosmological applications.

3.6.1 Gauge invariant Kähler potential

In four-dimensional $\mathcal{N}=1$ supergravity, the Kähler potential $\mathcal{K}$ is determined by requiring that it reproduces the various kinetic terms in the four-dimensional action in the Einstein frame. Recall that the latter is obtained from the four-dimensional string frame action (i.e. the one after compactifying (2.1)) via the redefinition [12]

$$G_S^{(4)} = e^{2\phi_{10}}\, G_E^{(4)} \Longrightarrow R_S = e^{-2\phi_{10}}\left[R_E - 6\nabla^2\phi_{10} - 6(\partial\phi_{10})^2\right]. \tag{3.108}$$

In particular, under this transformation the string frame kinetic terms for the dilaton and its axion $b_0^{(0)}$ become in Einstein frame

$$\begin{aligned}
&\frac{\mathrm{vol}(\mathcal{M})}{2\kappa_{10}^2}\int_{\mathbb{R}^{1,3}}(-G_S^{(4)})^{\frac{1}{2}}e^{-2\phi_{10}}\left[-R_S + 4\,\partial_\mu\phi_{10}\partial^\mu\phi_{10} - \frac{e^{4\phi_{10}}}{2}\,\partial_\mu b_0^{(0)}\partial^\mu b_0^{(0)}\right] \longrightarrow \\
&\frac{\mathrm{vol}(\mathcal{M})}{2\kappa_{10}^2}\int_{\mathbb{R}^{1,3}}(-G_E^{(4)})^{\frac{1}{2}}\left[-R_E - 2\,\partial_\mu\phi_{10}\partial^\mu\phi_{10} - \frac{e^{4\phi_{10}}}{2}\,\partial_\mu b_0^{(0)}\partial^\mu b_0^{(0)}\right].
\end{aligned} \tag{3.109}$$

Note that the factor of $e^{4\phi_{10}}$ in front of the axionic kinetic term in the first line arises after dualizing the kinetic term for $dB^{(2)}$ in (2.1) with the help of $dB^{(2)} = e^{2\phi_{10}}dB^{(6)}$ and then extracting the four-dimensional axion.

For the heterotic string without abelian gauge factors, the part of $\mathcal{K}$ relevant for our present purposes is very well-known and given by the expression

$$\mathcal{K} \;=\; -\frac{M_{pl}^2}{8\pi}\ln\left[\widetilde{S} + \widetilde{S}^* + \sum_a \frac{N_a}{2}\frac{(\widetilde{\Lambda}_a + \widetilde{\Lambda}_a^*)^2}{(\overline{\gamma}_a)_k(\widetilde{T}_k + \widetilde{T}_k^*)}\right]$$

$$-\frac{M_{pl}^2}{8\pi}\ln\left[-\sum_{i,j,k=1}^{h_{11}}\frac{d_{ijk}}{6}\left(\widetilde{T}_i+\widetilde{T}_i^*\right)\left(\widetilde{T}_j+\widetilde{T}_j^*\right)\left(\widetilde{T}_k+\widetilde{T}_k^*\right)\right]. \qquad (3.110)$$

Here $\frac{M_{pl}^2}{8\pi}=\kappa_{10}^{-2}\,\mathrm{Vol}(\mathcal{M})$, and the superfields $\widetilde{S},\widetilde{T}_k,\widetilde{\Lambda}_a$ have as their bosonic components the complex scalars defined in (3.102), (3.97) and (3.98) respectively. The quadratic part involving the five-brane supermultiplets $\widetilde{\Lambda}_a$ is non-standard and will be commented on momentarily. Ignoring it for a second, we can readily convince ourselves that this Kähler potential encodes the correct kinetic terms for the various scalars in the Einstein frame. To demonstrate this standard computation for the case of the dilaton we adopt the notation of [120] and define the complete $\mathcal{N}=1$ superfield $\widetilde{S}$ as

$$\widetilde{S}=S+\sqrt{2}\theta\psi+i\,\theta\sigma^\mu\overline{\theta}\,\partial_\mu S+\ldots \qquad (3.111)$$

with S given by (3.96). The kinetic term for the dilaton and its axionic partner in the Einstein frame then follows upon performing the Grassmann integral $\int d^2\theta d^2\overline{\theta}\,\mathcal{K}$ and extracting the term

$$\begin{aligned} S_{kin}^{(E)} &= \int_{\mathbb{R}_{1,3}}\frac{\partial^2\mathcal{K}}{\partial S\,\partial S^*}\Big|_{S=0}\partial_\mu S\,\partial^\mu S^* \qquad (3.112)\\ &= -\frac{\mathrm{Vol}(\mathcal{M})}{\kappa_{10}^2}\left(\int_{\mathbb{R}_{1,3}}\partial_\mu\,\phi_{10}\partial^\mu\phi_{10}+\int_{\mathbb{R}_{1,3}}e^{4\phi_{10}}\,\frac{1}{4}\partial_\mu\phi_{10}\,\partial^\mu\phi_{10}\right). \end{aligned}$$

A similar computation can of course be performed for the Kähler superfields $\widetilde{T}_k$.

If we include heterotic five-branes, the Kähler potential has to be adjusted such that it also yields the kinetic terms for the brane position moduli λ_a and their axionic partners $\widetilde{b}_a^{(0)}$. They can be deduced from the Pasti-Sorokin-Tonin action for the M5-brane [81]. We pointed out already that, following this logic, the authors of [122, 123] derived a correction quadratic in Λ_a in the definition of the superfield S which we have displayed in (3.102). This correction indeed incorporates the correct kinetic action if in addition one supplements the standard contribution $-\ln(\widetilde{S}+\widetilde{S}^*)$ to $\mathcal{K}$ by a term quadratic in $\widetilde{\Lambda}_a+\widetilde{\Lambda}_a^*$ resulting in

$$-\ln(\widetilde{S}+\widetilde{S}^*)\longrightarrow-\ln\left[\widetilde{S}+\widetilde{S}^*+\sum_a\frac{N_a}{2}\frac{(\widetilde{\Lambda}_a+\widetilde{\Lambda}_a^*)^2}{\sum_{k=1}^{h_{1,1}}(\overline{\gamma}_a)_k(\widetilde{T}_k+\widetilde{T}_k^*)}\right]. \qquad (3.113)$$

For a detailed derivation of these terms in the dilatonic Kähler potential we refer to [122, 123], but the computation is similar in spirit to the one sketched above.

The presence of massive $U(1)$ factors in the four-dimensional gauge group modifies $\mathcal{K}$ further in a very important manner. This is due to the fact that in the resulting supergravity theory, the mass terms between the abelian gauge fields and the axions enforce the gauging of the axionic shift symmetry. Quite generally, if the standard kinetic Lagrangian for some scalar field $b^{(0)}$ is supplemented by the coupling to an abelian gauge field[15] as in

$$S_{axion} = \int_{\mathbb{R}_{1,3}} \partial_\mu b^{(0)} \partial^\mu b^{(0)} + Q_m \, b^{(0)} \, (\partial_\mu A^\mu_m), \tag{3.114}$$

then unbroken $U(1)_m$ gauge symmetry requires that under

$$A^\mu_m \longrightarrow A^\mu_m + \partial^\mu \chi_m \tag{3.115}$$

the axion transforms as

$$b^{(0)} \longrightarrow b^{(0)} + \frac{Q_m}{2} \, \chi_m. \tag{3.116}$$

This is readily verified by considering the transformation

$$\delta S_{axion} = \int_{\mathbb{R}_{1,3}} 2 \, \partial_\mu b^{(0)} \partial^\mu (\frac{Q_m}{2} \chi_m) + Q_m b^{(0)} \, \partial_\mu \partial^\mu \chi_m + \mathcal{O}(Q_m^2) = 0. \tag{3.117}$$

To put it differently, the global abelian symmetry $b^{(0)} \to b^{(0)} + \text{const}$ is promoted to a local symmetry. In slightly more technical supergravity language, this is just the simplest version of the gauging of one of the global isometries of the scalar Kähler manifold. These gauged isometries need not be restricted to abelian shift symmetries. For a discussion of the most general case we refer e.g. to [121]. Upon gauging, the Kähler potential has to be modified by appropriate counter terms in order to remain gauge invariant. This procedure is comparatively easy in our abelian case. Introducing the abelian vector superfield V_m and, respectively, chiral superfield Φ_m and $\widetilde{B}$ with lowest components as in

$$\begin{aligned} V_m &= \theta \sigma_\mu \overline{\theta} A^\mu_m + \ldots, & \Phi_m = \frac{i}{2} \chi_m + \ldots, \\ \widetilde{B} &= (r + i b^{(0)}) + \ldots, \end{aligned} \tag{3.118}$$

we note that the required gauge transformation translates as follows into superfield language [120]

$$\begin{matrix} A^\mu_m \to A^\mu_m + \partial^\mu \chi_\mu \\ b^{(0)} \to b^{(0)} + \frac{Q_m}{2} \chi_m \end{matrix} \quad \longleftrightarrow \quad \left\{ \begin{matrix} V_m \to V_m + \Phi_m + \Phi_m^* \\ \widetilde{B} \to \widetilde{B} + Q_m \, \Phi_m \end{matrix} \right\} \tag{3.119}$$

Applying all this to our specific case at hand, it is clear that the Kähler potential (3.110) is rendered gauge invariant by a suitable subtraction of the abelian vector superfields multiplied by the respective charges occurring in the axionic couplings. Concretely, this results in the following gauge invariant Kähler potential

[15] Note that this coupling is precisely of the form of the mass terms (3.87),(3.88),(3.89). Just use Hodge duality to rewrite $\int b^{(2)} \wedge f \sim \int b^{(0)} \wedge d \star_4 A$.

$$
\begin{aligned}
\mathcal{K} \;=\; & -\frac{M_{pl}^2}{8\pi}\ln\Big[S+S^* - \sum_m Q_0^m\, V_m + \sum_a \frac{N_a}{2}\frac{(\Lambda_a+\Lambda_a^* - \sum_m Q_a^m\, V_m)^2}{(\overline{\gamma}_a)_k (T_k + T_k^* - \sum_a Q_k^m\, V_m)}\Big] \\
& -\frac{M_{pl}^2}{8\pi}\ln\Big[-\sum_{i,j,k=1}^{h_{11}} \frac{d_{ijk}}{6}\Big(T_i+T_i^* - \sum_m Q_i^m\, V_m\Big)\Big(T_j+T_j^* - \sum_m Q_j^m\, V_m\Big) \\
& \qquad\qquad \Big(T_k+T_k^* - \sum_m Q_k^m\, V_m\Big)\Big]
\end{aligned}
\tag{3.120}
$$

with appropriately defined superfields V_m. The charges Q_k^m can be identified as the couplings in the mass terms (3.87),(3.88),(3.89) using the definition

$$
S_{mass} = \sum_{m=1}^{M}\sum_{k=0}^{h_{11}} \frac{Q_k^m}{2\pi\alpha'}\int_{\mathbb{R}_{1,3}} f_m \wedge b_k^{(2)} + \sum_{m=1}^{M}\sum_a \frac{Q_a^m}{2\pi\alpha'}\int_{\mathbb{R}_{1,3}} f_m \wedge \widetilde{b}_a^{(2)}. \tag{3.121}
$$

Indeed it can be checked explicitly that this Kähler potential correctly reproduces also the various gauge-axion coupling terms by a Grassmann integral similar to that performed in (3.112).

3.6.2 Fayet-Iliopoulos terms and D-term constraints

We are finally in a position to come back to our initial goal, the computation of the FI terms defined by (3.107). What we obtain after some algebra from the Kähler potential (3.120) and the charges (3.121) is

$$
\begin{aligned}
\frac{\xi_{m_i}}{g_{m_i}^2} = -\frac{1}{8\ell_s^6}\sum_{n_i=1}^{M_i} \kappa_{m_i,n_i}\Big[& \int_{\mathcal{M}} J\wedge J\wedge \frac{\overline{f}_{n_i}}{2\pi} \\
& - e^{2\phi_{10}}\,\ell_s^4 \int_{\mathcal{M}} \frac{\overline{f}_{n_i}}{2\pi}\wedge \frac{1}{4(2\pi)^2}\Big(\mathrm{tr}\overline{F}_i^2 - \frac{1}{2}\mathrm{tr}\overline{R}^2\Big) \\
& + e^{2\phi_{10}}\,\ell_s^4 \sum_a N_a \Big(\frac{1}{2}\mp\lambda_a\Big)^2 \int_{\gamma_a} \frac{\overline{f}_{n_i}}{2\pi}\Big].
\end{aligned}
\tag{3.122}
$$

Obviously, the first term in (3.122) appears at string tree-level, whereas the second and third terms arise at one-loop in string perturbation theory. The reason that we have been able to derive these perturbative corrections just from the effective field theory lies of course once again in the one-loop nature of the Green-Schwarz terms which are responsible for the gauging of the supergravity.

The presence of one-loop corrections to the FI terms indicates important modifications of the D-term supersymmetry condition on the gauge bundles, as we now discuss. By definition, the FI parameters for the various $U(1)_{m_i}$ gauge groups in the effective four-dimensional $\mathcal{N} = 1$ supergravity are related to the scalar D-term potential via

$$V_D = \frac{1}{2} \sum_{m_i} V_D^{m_i} = \sum_{m_i} \frac{1}{2(g_{YM}^{m_i})^2} \left| \sum_\alpha q_\alpha^{m_i} |\phi_\alpha|^2 + \xi_{m_i} \right|^2, \qquad (3.123)$$

where the ϕ_α denote scalar fields with charge $q_\alpha^{m_i}$ under the $U(1)_{m_i}$. Note that there might exist additional contributions not involving the gauge bundles such as terms purely quadratic in the matter fields (see e.g. [125] and references therein). The vacuum of the theory is of course determined by minimizing the *complete* scalar potential including in particular the F-terms. A necessary condition for the vacuum to be supersymmetric is that the positive semi-definite quantity $V_D^{m_i}$ has to vanish for each $U(1)_{m_i}$ separately[16]. Now $V_D^{m_i}$ contains two qualitatively very different contributions: $\sum_\alpha q_\alpha^{m_i} |\phi_\alpha|^2$, which involves the vacuum expectation value of the charged matter fields, and the FI term ξ_{m_i}. The latter depends on the topological data of the background gauge bundles including the five-branes, the Kähler moduli and, by the one-loop correction, on the dilaton. A non-vanishing FI parameter does not necessarily indicate a breaking of supersymmetry as long as the VEVs of the charged matter fields can be chosen in a supersymmetric manner as to compensate ξ_{m_i} such that $V_D^{m_i} = 0$. Obviously, this is possible at most for multiplets with non-zero Euler characteristic since each field and its complex conjugate contribute with opposite signs in the D-term. Whether or not this can happen depends crucially on the structure of the additional ϕ_α-dependent terms in the scalar potential. In cases where there are no such terms which independently force ϕ_α to be zero, the D-term merely constrains a combination of the charged matter fields on the one hand and of the Kähler and brane moduli and the dilaton on the other. If, by contrast, there were, say, a mass term of the form $V_\phi = m_\alpha \phi_\alpha^2$, a non-vanishing FI parameter would clearly be incompatible with supersymmetry [125].

As an upshot of this discussion, the effective supergravity analysis results in the following D-term supersymmetry constraint on the gauge bundles,

$$\xi_{m_i}(g_s, J, \lambda_a) = \Delta_{m_i}(\phi_\alpha) \qquad (3.124)$$

for some function Δ_{m_i} depending on the charged matter fields. If we can ignore the term $\Delta_{m_i}(\phi_\alpha)$, for reasons of the type discussed above, then the gauge bundles are subject to the supersymmetry constraints $\xi_{m_i} = 0$, i.e.

[16] In addition, of course, also the Kähler covariant derivative of the F-term superpotential has to be zero, $DW = 0$. Together, these two constraints are necessary and sufficient for the theory to be in a supersymmetric minimum.

$$\int_{\mathcal{M}} J \wedge J \wedge c_1(L_{m_i})$$
$$-\ell_s^4\, g_s^2 \int_{\mathcal{M}} c_1(L_{m_i}) \wedge \Big(\mathrm{ch}_2(V_{N_i}) + \sum_{m_i,n_i=1}^{M_i} \epsilon_{m_i,n_i} c_1(L_{m_i}) \wedge c_1(L_{n_i}) + \frac{1}{2}\, c_2(T)\Big)$$
$$+\ell_s^4\, g_s^2 \sum_a N_a \left(\frac{1}{2} \mp \lambda_a\right)^2 \int_{\gamma_a} c_1(L_{m_i}) = 0. \tag{3.125}$$

In these cases, the conditions (3.125) provide constraints fixing, in principle, combinations of the Kähler moduli, the dilaton and five-brane moduli. Therefore, the constraint $\xi_{m_i} = 0$ effectively renders a particular combination of the moduli fields massive. This is just what has to happen in supersymmetric vacua, well in accord with the fact that the axionic partners of these moduli likewise receive a mass due to the coupling to $U(1)_{m_i}$. In particular, if we did not include the one-loop correction involving the dilaton and the brane moduli, this would be in direct conflict with the mass terms induced for the axions $b_0^{(0)}$ and $\widetilde{b}_a^{(0)}$. After all, in supersymmetric configurations the whole supermultiplet has to become massive, not just some of its components.

Note that the Kähler form J as appearing above is not dimensionless, but implicitly contains a factor of α'. Therefore, the perturbative corrections effectively depend only on g_s^2. In principle, a cancellation of the tree-level against the one-loop term can be achieved in the perturbative regime of large internal radii and small g_s provided that the tree-level term can be arranged to be sufficiently small by itself. On manifolds with several Kähler moduli this is clearly possible, depending on the details of the intersection form, of course.

We conclude this section with a side remark on what happens when we cancel a non-vanishing Fayet-Iliopoulos term against the VEV of a charged scalar as in (3.124) (see also [102]). From the field theory analysis, what we expect in such a situation is that the scalar VEV induces the breaking of part of the four-dimensional gauge symmetry. There is a very neat way how to understand this Higgsing of the observable gauge group from the point of view of the internal bundles. To illustrate the idea, consider the easiest case with just one abelian gauge factor, i.e. suppose that the internal bundle is given by the direct sum $W_i = V_{N_i} \oplus L^{-1}$ with structure group $SU(N_i) \times U(1)$. For simplicity, assume furthermore that the charged scalar in question corresponds to the internal bundle $U_{x_i}^{(i)} = V_{N_i} \otimes L$, in the notation of (2.17). Giving a VEV to this scalar means that we turn on an element in the first cohomology group $H^{(1)}(\mathcal{M}, U_{x_i}^{(i)})$ [17]. Now, as a mathematical fact, turning on an element in $H^{(1)}(\mathcal{M}, V_{N_i} \otimes L)$ implies a deformation of the internal bundle W such that it no longer splits into a direct sum but rather is given by the extension of L^{-1} by V_{N_i} [40], i.e. it fits into the

[17] As we will discussed, the internal bundles have to be stable in the mathematical sense, in which case $H^{(0)}(\mathcal{M}, U_{x_i}^{(i)})$ and $H^{(3)}(\mathcal{M}, U_{x_i}^{(i)})$ vanish and all matter comes from $H^{(1)}$ or $H^{(2)}$. W.l.o.g we assume that $H^{(1)}(\mathcal{M}, U_{x_i}^{(i)}) \neq 0$, otherwise just switch to the complex conjugate representation using Serre duality.

short exact sequence

$$0 \longrightarrow V_{N_i} \longrightarrow \widetilde{W} \longrightarrow L^{-1} \longrightarrow 0. \qquad (3.126)$$

The bundle $\widetilde{W}$ hereby defined has in fact structure group $SU(N_i+1)$, which contains $SU(N_i) \times U(1)$, the structure group of $V_{N_i} \oplus L^{-1}$. The visible gauge group, being the respective commutant in $E_8^{(i)}$, therefore gets reduced, in this case precisely by the abelian factor which is Higgsed away in the field theoretic picture.

What this tells us is that a cancellation of a non-vanishing FI term against matter field contributions is only possible at the cost of a severe deformation of the geometry of our gauge bundle. If we want to stick to our initial framework of Whitney sums of internal $SU(N)$ or $U(N)$ bundles, this means that we really have to insist on a vanishing FI term as the D-term supersymmetry condition.

3.6.3 Loop-corrected Hermitian Yang-Mills equation and the concept of λ- stability

In the previous section, we have derived the supersymmetry condition on the gauge bundles by a purely field theoretic analysis of the D-term in the effective four-dimensional supergravity. A priori, we cannot exclude that this approach misses certain subtleties. The point is that we have assumed from the very beginning that the effective theory in four dimensions can be described within the framework of $\mathcal{N}=1$ supergravity, whose properties we have used heavily in deriving the supersymmetry constraints for the ground state of the theory.

To see that these supersymmetry conditions may not be the whole story, consider as an example the requirement that the internal manifold be Calabi-Yau, as dictated by the Killing spinor equation for the gravitino in the absence of H-flux. Once we assume the Calabi-Yau constraint and therefore trust the machinery of four-dimensional $\mathcal{N}=1$ supergravity, we do not recover it from the field theory analysis any more. We rather have to consult the ten-dimensional theory. All we can expect from the four-dimensional analysis is that we identify potential sources for spontaneous supersymmetry breakdown within an in principle supersymmetric theory.

Let us therefore compare the four-dimensional results to the direct analysis of the ten-dimensional Killing spinor equation for the gaugino.

As we recall from the discussion in section (2.3), at tree level each summand bundle of W has to be holomorphic and μ-stable with respect to zero slope. The latter means that the each of the stable summand bundles needs to satisfy the DUY equation

$$\int_{\mathcal{M}} J \wedge J \wedge c_1(V_{n_i}) = 0, \qquad \int_{\mathcal{M}} J \wedge J \wedge c_1(L_{m_i}) = 0, \qquad (3.127)$$

to be satisfied for all n_i, m_i.

Evidently, the left-hand side of (3.127) is just the tree-level part of the FI term (3.122). We realize that our concerns were justified in that the supersymmetry condition revealed by the four-dimensional analysis is incomplete: it is blind to the local supersymmetry equation, encoded in the requirement of stability, and only yields the associated integrability condition. Nonetheless, in view of the agreement at tree-level between the DUY equation and the FI term, it is most natural to interpret the one-loop correction of the latter as nothing other than a one-loop correction of the DUY equation. But since the DUY is the integrability condition for a more fundamental local constraint, the Hermitian Yang-Mills equation, this suggests that the latter is likewise corrected at one-loop. In fact, it is consistent to propose the following

Conjecture 1:

The perturbatively exact supersymmetry condition on the gauge bundle is given by the one-loop deformed Hermitian Yang-Mills equation

$$J \wedge J \wedge F_{k_i} - (2\pi\alpha')^2 \, \frac{g_s^2}{4} \, F_{k_i} \wedge d\left(\omega_{YMi} - \frac{1}{2}\omega_L\right) = 2\pi\, \lambda\left(V_{k_i}, \alpha' g_s\right) \, \mathrm{vol}_{\mathcal{M}} \, \mathrm{id} \tag{3.128}$$

together with

$$\lambda(V_{k_i}, \alpha' g_s) = \frac{1}{\mathrm{rk}(V_{k_i})} \Delta_{k_i}(\phi_\alpha). \tag{3.129}$$

Here V_{k_i} represents any of the bundles V_{N_i}, L_{m_i} in $E_8^{(i)}$ and F_{k_i} the corresponding field strength. The deformed slope $\lambda(V_{k_i}, \alpha' g_s)$ is defined as the integral over the left-hand side of (3.128) divided by the rank of V_{k_i},

$$\begin{aligned} \lambda(V_{k_i}, \alpha' g_s) \;\; \equiv \;\; & \frac{1}{\mathrm{rk}(V_{k_i})} \int_{\mathcal{M}} J \wedge J \wedge c_1(V_{k_i}) \\ & -(2\pi\alpha')^2 \, \frac{g_s^2}{4} \, c_1(V_{k_i}) \wedge d\left(\omega_{YMi} - \frac{1}{2}\omega_L\right), \end{aligned} \tag{3.130}$$

in precise analogy with (2.21). The notation ω_{YMi} refers to the complete Chern-Simons three-form of the bundle W_i satisfying $d\omega_{YMi} = \mathrm{tr}F_i^2$. We formally subsumed the contributions from the five-branes into this quantity since, as we observed in section (3.5), their effect is precisely that of a gauge instanton after a small instanton transition.

We recall from the previous section that, taking the implicit factor of $(\alpha')^2$ in the tree-level part $J \wedge J \wedge c_1(V_{k_i})$ into account, the perturbative correction of the slope arises of course precisely at order g_s^2 relative to the tree-level part. The reason why we chose to write the modified slope as $\lambda(V_{k_i}, \alpha' g_s)$ is to remind us that the correction becomes small as compared to the tree-level term if g_s is small and/or we are in the large radius regime, where integrals involving J dominate. This will be important momentarily.

Mimicking the situation at tree-level, the supersymmetry condition comes in two parts: The local constraint is the deformed Hermitian Yang-Mills equation (3.128). In addition we have to specify which value the deformed slope has to take. This latter piece of information is all we find from the four-dimensional D-term constraint (3.124) upon identifying the deformed slope with the loop-corrected FI term. Note that equation (3.129) is just a reformulation of this D-term constraint[18].

Strictly speaking, we cannot rigorously exclude the appearance of additional cohomologically trivial forms on the left-hand side of (3.128) which vanish upon integration and whose effect cannot simply be detected in the supergravity analysis. After all, the latter only provides us with the integrated version of the Hermitian Yang-Mills equation. To be completely precise we should therefore add the exterior derivative of some potential globally defined five-form. Irrespective of this subtlety, the definition of $\lambda(V_{k_i}, \alpha' g_s)$ as the integral over the left-hand side of (3.128) is independent of such terms, of course.

In view of the deformation of the HYM equation at one loop in string perturbation theory, also the stability condition on the gauge bundles must be modified appropriately. So which is the stability condition guaranteeing a solution to (3.128)?

Let us neglect for the moment the D-term constraint on λ, which relates the tree-level and the one-loop piece in λ, and focus solely on the deformed HYM equation (3.128) for arbitrary λ. To find the correct notion of stability in this less constrained situation, we rely on some inspiration from an analogous problem in the mathematical literature, as studied by Leung [126]. He considers a different deformation of the HYM equation, namely

$$e^{tJ+\frac{1}{2\pi}F}\,\mathrm{Td}(\mathcal{M}) = \gamma(V,t)\,\mathrm{id}, \quad \text{where} \quad \gamma(V,t) = \frac{1}{\mathrm{rk}V}\int_{\mathcal{M}} e^{tJ}\,\mathrm{ch}(F)\,\mathrm{Td}(\mathcal{M}). \tag{3.131}$$

The quantity $\gamma(V,t)$ is known as the Gieseker slope of V. The important point is that the term at highest order in t is just the familiar $t^2\, J\wedge J\wedge F$, whereas the deformations are of lower order. In this sense equ. (3.131) is perturbative in t since it reduces to the undeformed HYM equation for $t\to\infty$. What Leung proved is the following theorem: For every vector bundle V there exists a $T_V > 0$ such that for all $t > T_V$ V admits a connection whose field strength is a solution of equ.(3.131) (for this t) if and only if V is $\gamma(V,t)$-stable, i.e. if each subsheaf $\mathcal{W}$ of V is of smaller $\gamma(\mathcal{W},t)$-slope than V.

To make the analogy to our situation crystal clear, we divide equ.(3.128) by $(\alpha' g_s)^2$ and identify $(\alpha' g_s)^{-1}$ with t. As in Leung's case, for large t the tree-level

[18] In Type IIB theory, as will be discussed, this equation defines which $\mathcal{N}=1$ subalgebra of the bulk $\mathcal{N}=2$ supersymmetry algebra the gauge instantons on the D-branes have to respect.

part both in the HYM equation and in the associated slope dominates over the loop correction. Clearly, what we mean by small α' is that we are in the large radius regime. All that differs in our case is the precise form of this perturbative correction, but this is irrelevant for Leung's argument to work.
We are thus lead to the following

Conjecture 2:

Given a holomorphic vector bundle V, then there exists a value of $\alpha' g_s$, depending on V, such that for all $\alpha' g_s$ smaller than this critical value V admits a connection whose field strength satisfies the one-loop deformed Hermitian Yang-Mills equation (3.128) iff each subbundle $\mathcal{W}$ with $\mathrm{rk}(\mathcal{W}) < \mathrm{rk}(V)$ satisfies $\lambda(\mathcal{W}, \alpha' g_s) < \lambda(V, \alpha' g_s)$.

This proposal receives convincing support from the corresponding phenomena occurring in the context of the $SO(32)$ heterotic string, as we will discuss in section (4.7.3). There we will be able to identify the one-loop corrected stability condition on the bundles as the S-dual version of the perturbative part of the Π-stability condition as formulated in the context of the derived bounded category of coherent sheaves [78] in type II B string theory. Indeed, on the Type I side, the perturbatively exact stability condition *is* just given by replacing the familiar μ-slope with the λ-slope in the above perturbative sense. A mathematical proof of this statement can be found in [127] and more details will be provided in section (4.7.3).

On the other hand one can easily convince oneself that perturbatively every μ-stable bundle is also λ-stable in the following sense: Given a μ-stable vector bundle V, then there exists a value of $\alpha' g_s$ (depending on V) such that for all $\alpha' g_s$ smaller than that critical value V is $\lambda(V, \alpha' g_s)$-stable (with respect to these values of $\alpha' g_s$). This follows from the fact that for $\alpha' g_s$ sufficiently small, the dominant part in the λ-slope of V and of each of its finitely many subsheaves $\mathcal{W}$ is the tree-level part, which is just the μ-slope. The perturbative corrections therefore do not spoil the fact that $\lambda(\mathcal{W}, \alpha' g_s) < \lambda(V, \alpha' g_s)$ since $\mu(\mathcal{W}) < \mu(V)$ for all $\mathcal{W}$ by assumption.

The situation changes drastically if we now take into account also the D-term condition (3.129), i.e. if we pose additional constraints on the value which the slope of V is to take. Assume for simplicity that we do not turn on any charged matter fields so that the slope is simply equated to zero according to equ. (3.129). If the one-loop contribution in the λ-slope for V does not happen to vanish by itself, this implies that the tree-level and the one-loop piece have to cancel each other and must therefore be of the same order of magnitude. The above arguments concerning our simple version of λ-stability and its relation to μ-stability, however, only work if the tree-level part dominates arbitrarily over the loop-correction for $\alpha' g_s$ small enough. As a result, for a non-vanishing one-loop term, we cannot simply infer that a μ-stable bundle solves the deformed HYM equation. This does not mean that the one-loop term necessarily has to vanish for supersymmetry to be preserved, but in case it does not, we do not yet have an appropriate stability concept guaranteeing a solution to the HYM, and a more sophisticated mathematical analysis is required. Let us emphasize at this stage already that the concrete applications we will present are not in conflict with this subtlety since the one-loop contribution to the DUY equation will vanish by construction in all cases of interest.

We stress furthermore that although the one-loop part of the slope $\lambda(V, \alpha' g_s)$ is clearly present only if $c_1(V) \neq 0$, this does not mean that the above analysis is relevant only if we embed a $U(N)$ as opposed to an $SU(N)$ bundle into E_8. Rather, the one-loop terms in the local Hermitian Yang-Mills equation are in general non-vanishing also for $SU(N)$ bundles. In this case, however, thanks to the foregoing arguments, μ-stability is always sufficient for supersymmetry in the same way as it is sufficient for $U(N)$ bundles for which the correction in $\lambda(V, \alpha' g_s)$ vanishes. In both cases, there must not exist an a priori lower bound on $\alpha' g_s$ since in relating μ-stability to λ-stability, we do not know the critical value of g_s below which the first implies the latter.

Which further corrections to the DUY condition and to the Hermitian Yang-Mills equation do we expect? From the supergravity analysis of the D-term and the usual non-renormalisation arguments, it is clear that there cannot exist any higher perturbative string-loop contributions. Moreover, it is known [128] that there are no one-loop Fayet-Iliopoulos terms in the Type I string theory. Consequently, S-duality dictates that the DUY equation is also exact in sigma-model perturbation theory since it maps expressions at one-loop order in g_s to perturbative α' corrections. However, there might, and most probably will be additional non-perturbative corrections in g_s and α' which are beyond the scope of this analysis. After all, it is the appearance of non-perturbative α' corrections to the D-term supersymmetry conditions in Type IIB which requires the introduction of the concept of full Π-stability [78].

3.6.4 D-term potential for M5-branes

Let us go back to the Fayet-Iliopoulos term (3.122) and discuss possible conclusions about the D-terms arising from the five-branes. Apparently, a flux through the two-cycle γ_a of a five-brane on the wall $E_8^{(i)}$ generates a one-loop D-term potential for the five-brane modulus λ_a. From (3.122) it seems at first sight that this D-term repels the five-brane from the wall and vanishes only if the five-brane lies on top of the other wall. However, recall from (3.123) that the D-term scalar potential for a massive $U(1)$ actually involves the quotient of the FI-term and the gauge coupling, which, too, depends on the five-brane modulus in a non-trivial manner.

In order to get a qualitative idea of the combined effect of the FI terms and the threshold corrected gauge coupling, it is instructive to analyse a simple toy example. Consider the Quintic Calabi-Yau manifold, which has only one Kähler modulus, and assume that we have chosen a vector bundle $V \oplus L^{-1}$ embedded into the first E_8 wall without any matter charged under the $U(1)$. Then the D-term potential arising from the FI-term of the $U(1)$ is simply

$$V_D = \frac{1}{2} g^2 \left(\frac{\xi}{g^2} \right)^2 , \tag{3.132}$$

where g denotes the gauge coupling of the $U(1)$. For the Quintic one has $c_2(T) = 10\eta^2$ and $J = \ell_s^2 r\, \eta$ with $r > 0$ in terms of the single $(1,1)$-form η. Moreover, we write $\mathrm{ch}_2(V) = -v\eta^2 + \frac{1}{2} l^2 \eta^2$ and $\mathrm{ch}_2(L) = \frac{1}{2} l^2 \eta^2$ and introduce one five-brane wrapping the class γ. The tadpole cancellation condition then reads

$$-v + l^2 - \gamma^2 = -10. \tag{3.133}$$

The relevant D-term potential takes the form

$$V_D \simeq \frac{\left(\frac{r^2}{g_s^2} - (\gamma^2 - 5) + \left(\frac{1}{2} - \lambda\right)^2 \gamma^2 \right)^2}{\left(\frac{r^2}{g_s^2} - 3(\gamma^2 - 5) + 3\left(\frac{1}{2} - \lambda\right)^2 \gamma^2 - \frac{\kappa_{1,1}^2}{\eta_{1,1}}\, l^2 \right)}. \tag{3.134}$$

For fixed string coupling $g_s = 0.5$, radius $r = 2$ and a choice of parameters $\gamma = l = 2$, $\kappa_{1,1}^2/\eta_{1,1} = 1/10$, this potential for the five-brane modulus λ has the characteristic shape shown in figure 3.2. Naively, as pointed out, from the FI-term one might have expected that the five-brane is repelled by the E_8 walls carrying a non-trivial line bundle. However, the contribution of the g^2 term multiplying the FI-term in the scalar potential changes this picture and leads to an attractive potential between the five-brane and the E_8 wall carrying the bundle.

How can we understand the physics behind this attractive interaction? Arising at one loop in the weakly coupled heterotic string, it is expected to be due to appropriate amplitudes from membranes after unfolding the wrapped eleventh dimension in the strongly-coupled Horava-Witten regime. In fact, as derived in [123], there are non-perturbative contributions to the F-term superpotential from open membranes stretching between one of the orbifold fixed planes and the M5-brane provided that the worldvolume of the membranes is precisely of the form $I \times \gamma_a$. Here I simply denotes the interval along the eleventh dimension between the orbifold plane and five-brane. We see that, apparently, such configurations also contribute to the D-term potential if the membrane can couple to some abelian background gauge flux on the orbifold plane. As is manifest in (3.125), this can only happen if the five-brane wraps a two-cycle which, pulled back to the end of the world, carries non-vanishing gauge flux. In particular, this interpretation explains why the five-brane is sensitive to the presence of the gauge flux along γ_a even though it may be placed at an arbitrary position along the eleventh dimension: The presence of the gauge flux is communicated by the exchange of appropriate open membranes.

This interpretation of the D-term potential as being due to open membranes stretching between the orbifold fixed plane and the M5-brane is well in agreement with the generic form of the potential found in (3.134): The contribution of the membranes is of course minimized precisely if the interval along which they wrap between the end of the world and the five-brane is vanishing.

3.7 Example (I): Breaking E_8 to flipped $SU(5) \times U(1)_X$

It is high time to illustrate the hitherto studied framework by means of concrete examples. The number of possible embeddings is extremely high if we take into account all conceivable combinations of the various building blocks at our disposal. In the next two sections, we will therefore restrict our attention to realistic four-dimensional gauge groups, focusing on the detailed application of the technical aspects presented by now. Phenomenological considerations and concrete model building are postponed to chapter 7.

As a warm-up we exemplify the breaking of the E_8 group down to the flipped $SU(5)$ gauge group based on the branching

$$SU(4) \times U(1)_{X'} \subset SU(5) \subset E_8 \longrightarrow SU(5) \times U(1)_{X'}. \tag{3.135}$$

The embedding $SU(5) \subset E_8 \to SU(5)$ induces the familiar decomposition

$$\mathbf{248} \longrightarrow (\mathbf{24},\mathbf{1}) + (\mathbf{1},\mathbf{24}) + (\mathbf{5},\mathbf{10}) + (\overline{\mathbf{5}},\overline{\mathbf{10}}) + (\mathbf{10},\overline{\mathbf{5}}) + (\overline{\mathbf{10}},\mathbf{5}). \tag{3.136}$$

Next we decompose the internal $SU(5)$ representations under $SU(5) \to SU(4) \times U(1)_{X'}$ according to (3.14) as

$$\begin{aligned} \mathbf{24} &\longrightarrow \mathbf{15}_0 + \mathbf{1}_0 + \mathbf{4}_5 + \overline{\mathbf{4}}_{-5}, \\ \mathbf{5} &\longrightarrow \mathbf{4}_1 + \mathbf{1}_{-4}, \\ \mathbf{10} &\longrightarrow \mathbf{6}_2 + \mathbf{4}_{-3}. \end{aligned} \tag{3.137}$$

In combination these two steps lead to the spectrum[19]

$$\mathbf{248} \overset{SU(4)\times SU(5)\times U(1)_{X'}}{\longrightarrow} \left\{ \begin{array}{c} (\mathbf{15},\mathbf{1})_0 \\ (\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{10})_{-4} + (\mathbf{1},\overline{\mathbf{10}})_4 + (\mathbf{1},\mathbf{24})_0 \\ (\mathbf{4},\mathbf{1})_5 + (\mathbf{4},\overline{\mathbf{5}})_{-3} + (\mathbf{4},\mathbf{10})_1 \\ (\overline{\mathbf{4}},\mathbf{1})_{-5} + (\overline{\mathbf{4}},\mathbf{5})_3 + (\overline{\mathbf{4}},\overline{\mathbf{10}})_{-1} \\ (\mathbf{6},\mathbf{5})_{-2} + (\mathbf{6},\overline{\mathbf{5}})_2 \end{array} \right\}. \tag{3.138}$$

We point out, at this stage merely as an appetizer, that the abelian charges of the spectrum are proportional to the $U(1)_X$ in the flipped $SU(5)$ model, thus justifying the notation. This crucial fact will be heavily exploited in the context of the phenomenological adventures of chapter 7.

[19] Note that in the last line we used that $\mathbf{6} = \overline{\mathbf{6}}$ for the antisymmetric of $SU(4)$.

Let us now turn to the explicit bundles which realize this breaking of E_8. Starting with constructions of type A, we choose the Whitney sum

$$W = V \oplus L \qquad\qquad \text{such that} \quad c_1(V) = 0 \tag{3.139}$$

with structure group $G = SU(4) \times U(1)$. The embedding of the line bundle is accomplished by identifying its field strength with the diagonal $SU(5)$ generator

$$T_{X'} = (1,1,1,1,-4). \tag{3.140}$$

As shown in table 3.2, the decomposition (3.138) allows one immediately to read off the cohomology classes determining the massless spectrum.

reps.	Cohomology (Type A)
$\mathbf{10}_1$	$H^*(\mathcal{M}, V \otimes L)$
$\mathbf{10}_{-4}$	$H^*(\mathcal{M}, L^{-4})$
$\overline{\mathbf{5}}_{-3}$	$H^*(\mathcal{M}, V \otimes L^{-3})$
$\overline{\mathbf{5}}_2$	$H^*(\mathcal{M}, \bigwedge^2 V \otimes L^2)$
$\mathbf{1}_5$	$H^*(\mathcal{M}, V \otimes L^5)$

Table 3.2: Massless spectrum of $H = SU(5) \times U(1)_{X'}$ models.

From this embedding of the structure group, we can determine the resulting tadpole cancellation condition (3.26) by computing the traces as spelled out in (3.23),

$$\begin{aligned} \mathrm{tr}(\overline{F}^2) &= \frac{1}{30}\,\mathrm{Tr}(\overline{F}^2) = \frac{1}{30}\sum_x 2(2\pi)^2(\mathrm{ch}_2(U_x) \times \dim(R_x)) \\ &= 2\,\mathrm{tr}_f^{SU(4)}(\overline{F}^2_{SU(4)}) + 40\,\overline{F}^2_{U(1)} = 4\,(2\pi)^2\,(-c_2(V) + 10\,c_1^2(L)), \\ \mathrm{tr}(\overline{R}^2) &= 2\,\mathrm{tr}_f^{SU(3)}(\overline{R}^2) = -4\,(2\pi)^2\,c_2(T). \end{aligned} \tag{3.141}$$

This yields the tadpole cancellation condition

$$c_2(V) - 10\,c_1^2(L) = c_2(T). \tag{3.142}$$

The net-number of chiral multiplets is given by the Euler characteristic of the various bundles in table 3.2. Note that extra gauge bosons are counted by $H^*(\mathcal{M}, \mathcal{O})$, which can only appear if L^4 is the trivial bundle $\mathcal{O}$, i.e. $c_1(L) = 0$. Clearly in this case the gauge symmetry is extended to $SO(10)$, which is precisely the commutant of $SU(4)$ in E_8. Another way to see this is that the 20 additional vector multiplets from the $(\mathbf{1}, \mathbf{10})_{-4}$ and its conjugate arising when L^4 gets trivial precisely fill out, together with the $\mathbf{24}+\mathbf{1}$ in the adjoint of $SU(5)\times U(1)_{X'}$, the 45-dimensional adjoint representation of $SO(10)$. We will encounter much more intricate patterns of gauge symmetry enhancement for the case that more $U(1)$ bundles are involved in the next section.

It is now a straightforward exercise to compute the four-dimensional gauge anomalies from the general expressions given in equation (3.28), using also the trace identities of appendix A.2.

• The non-abelian $SU(5)^3$ anomaly is proportional to

$$A_{SU(5)^3} = \chi(\mathcal{M}, V\otimes L) + \chi(\mathcal{M}, L^{-4}) - \chi(\mathcal{M}, V\otimes L^{-3}) - \chi(\mathcal{M}, \textstyle\bigwedge^2 V\otimes L^2) \tag{3.143}$$

and vanishes identically even without invoking the tadpole cancellation condition.

• The mixed abelian-gravitational anomaly $U(1)_{X'} - G^2_{\mu\nu}$ however does not directly vanish and is given by

$$\begin{aligned} A_{U(1)-G^2_{\mu\nu}} &= 10\,\chi(\mathcal{M}, V\otimes L) - 40\,\chi(\mathcal{M}, L^{-4}) - 15\,\chi(\mathcal{M}, V\otimes L^{-3}) + \\ &\quad 10\,\chi(\mathcal{M}, \textstyle\bigwedge^2 V\otimes L^2) + 5\,\chi(\mathcal{M}, V\otimes L^5) \\ &= 10\int_{\mathcal{M}} c_1(L)\,\left[12(-c_2(V)+10\,c_1^2(L)) + 5\,c_2(T)\right]. \end{aligned} \tag{3.144}$$

• Similarly the mixed abelian-non-abelian anomaly $U(1)_{X'} - SU(5)^2$ takes the form

$$\begin{aligned} A_{U(1)-SU(5)^2} &= 3\,\chi(\mathcal{M}, V\otimes L) - 12\,\chi(\mathcal{M}, L^{-4}) - 3\,\chi(\mathcal{M}, V\otimes L^{-3}) + \\ &\quad 2\,\chi(\mathcal{M}, \textstyle\bigwedge^2 V\otimes L^2) \\ &= 10\int_{\mathcal{M}} c_1(L)\,\left[2(-\,c_2(V)+10\,c_1^2(L)) + c_2(T)\right]. \end{aligned} \tag{3.145}$$

• Finally for the $U(1)^3_{X'}$ anomaly one obtains

$$\begin{aligned} A_{U(1)^3} &= 10\,\chi(\mathcal{M}, V\otimes L) - 640\,\chi(\mathcal{M}, L^{-4}) - 135\,\chi(\mathcal{M}, V\otimes L^{-3}) + \\ &\quad 40\,\chi(\mathcal{M}, \textstyle\bigwedge^2 V\otimes L^2) + 125\,\chi(\mathcal{M}, V\otimes L^5) \qquad (3.146) \\ &= 200\int_{\mathcal{M}} c_1(L)\,\left[6(-c_2(V)+10\,c_1^2(L)) + 40c_1^2(L) + 3\,c_2(T)\right]. \end{aligned}$$

These results are in complete agreement with the general expressions (3.30) - (3.32) if one uses (3.141) to rewrite them in terms of traces. Note that the integrands only vanish if $c_1(L)=0$, in which case the gauge group is enhanced to $SO(10)$. In this simple construction, the $U(1)_{X'}$ is therefore massive and only present as a global symmetry. We will find a way to circumvent this apparent drawback in chapter 7 when it comes to the construction of realistic flipped $SU(5)\times U(1)_X$ vacua.

For embeddings of Type B, one starts with a bundle

$$W = V\oplus L^{-1}, \quad \text{with } c_1(V)=c_1(L),\ \ \text{rank}(V)=4, \tag{3.147}$$

which has structure group $SU(4) \times U(1)$. This bundle W can now be embedded into an $SU(5)$ subgroup of E_8 so that the commutant is again $SU(5) \times U(1)_{X'}$. We embed the $U(1)$ bundle such that

$$Q_{X'} = (1,1,1,1,-4), \tag{3.148}$$

implying that the matrix $\mathcal{Q}$ defined in (3.12) is simply

$$\mathcal{Q} = Q_{X'}(V) + Q_{X'}(L) = 5. \tag{3.149}$$

The massless spectrum is given by the cohomology classes listed in Table 3.3.

reps.	Cohom.
$\mathbf{10}_1$	$H^*(\mathcal{M}, V)$
$\mathbf{10}_{-4}$	$H^*(\mathcal{M}, L^{-1})$
$\overline{\mathbf{5}}_{-3}$	$H^*(\mathcal{M}, V \otimes L^{-1})$
$\overline{\mathbf{5}}_2$	$H^*(\mathcal{M}, \bigwedge^2 V)$
$\mathbf{1}_5$	$H^*(\mathcal{M}, V \otimes L)$

Table 3.3: Massless spectrum of $H = SU(5) \times U(1)_{X'}$ models.

An explicit evaluation of the traces (see again (3.23)) as

$$\begin{aligned} \mathrm{tr}(\overline{F}^2) &= \frac{1}{30}\,\mathrm{Tr}(\overline{F}^2) = \frac{1}{30}\sum_x 2(2\pi)^2(\mathrm{ch}_2(U_x) \times \dim(R_x)) \\ &= 4(2\pi)^2(\mathrm{ch}_2(V) + \mathrm{ch}_2(L)) \end{aligned} \tag{3.150}$$

convinces us that the tadpole cancellation condition reads

$$c_2(V) - c_1^2(V) = c_2(T). \tag{3.151}$$

Similarly to the type A case, one can show that all non-abelian gauge anomalies cancel and that the abelian ones,

$$\begin{aligned} A_{U(1)-G^2_{\mu\nu}} &= \frac{5}{2}\int_{\mathcal{M}} c_1(L)\ \Big[12\Big(-c_2(V) + c_1^2(L)\Big) + 5\,c_2(T)\Big]\,, \\ A_{U(1)-SU(5)^2} &= \frac{5}{2}\int_{\mathcal{M}} c_1(L)\ \Big[2\Big(-c_2(V) + c_1^2(L)\Big) + c_2(T)\Big]\,, \\ A_{U(1)^3} &= 25\int_{\mathcal{M}} c_1(L)\ \Big[12\Big(-c_2(V) + c_1^2(L)\Big) + 5c_1^2(L) + 6\,c_2(T)\Big]\,, \end{aligned} \tag{3.152}$$

being consistent with the general result displayed at the end of section (3.4.1), are cancelled by a Green-Schwarz mechanism. Note in particular that $\eta_{X',X'} = 40$, see (3.25).

3.8 Example (II): Breaking E_8 to $SU(3) \times SU(2) \times U(1)_Y$

Our model building possibilities are not limited to the construction of GUT group vacua. In this section, we exemplify the breaking of E_8 directly down to the Standard Model gauge group based on the branching

$$SU(6) \subset E_8 \longrightarrow SU(3) \times SU(2). \tag{3.153}$$

The general strategy presented in section (3.1) allows us to iteratively incorporate additional line bundles and thus to introduce various abelian gauge factors into the visible gauge group. This is at the cost of lowering the rank of the non-abelian bundle V_{N_i}. In the presence of several $U(1)$ factors an extremely rich pattern emerges with numerous ways to obtain the Standard Model gauge group and spectrum. In this section, we merely focus on one of the two E_8 factors in order to explain the building blocks for the phenomenological applications to be discussed later.

As far as the resulting spectrum is concerned, we first note that the embedding (3.153) induces the following decomposition of the adjoint representation of E_8

$$\begin{aligned} \mathbf{248} \quad \longrightarrow \quad & (\mathbf{35};\mathbf{1},\mathbf{1}) + (\mathbf{1};\mathbf{8},\mathbf{1}) + (\mathbf{1};\mathbf{1},\mathbf{3}) + \\ & (\mathbf{20};\mathbf{1},\mathbf{2}) + ((\mathbf{6};\mathbf{3},\mathbf{2}) + (\mathbf{15};\overline{\mathbf{3}},1) + c.c.). \end{aligned} \tag{3.154}$$

We now decompose the internal $SU(6)$ following the steps spelled out in section (3.1). Specifically, we perform the decompositions

$$\begin{aligned} SU(6) \quad \longrightarrow \quad & SU(5) \times U(1)_{Y'} \longrightarrow SU(4) \times U(1)_{X'} \times U(1)_{Y'} \\ \longrightarrow \quad & SU(3) \times U(1)_Z \times U(1)_{X'} \times U(1)_{Y'}. \end{aligned} \tag{3.155}$$

3.8.1 Bundles with structure group $SU(5) \times U(1)$

To realize the first step in the sequence (3.155), we choose a bundle of type A with structure group $SU(5) \times U(1)_{Y'}$, i.e. we consider the configuration

$$W_1 = V \oplus L, \quad \text{with} \quad \text{rank}(V) = 5. \tag{3.156}$$

Clearly, the commutant in $E_8^{(1)}$ is $SU(3) \times SU(2) \times U(1)_{Y'}$. The abelian charges of the states follow from the embedding of $U(1)_{Y'}$ into $SU(6)$ such that the abelian generator is identified with the diagonal element

$$T_{Y'} = (1,1,1,1,1,-5) \tag{3.157}$$

in $SU(6)$. We decompose the various $SU(6)$ representations under the splitting $SU(6) \longrightarrow SU(5) \times U(1)_{Y'}$,

$$\begin{aligned}
\mathbf{35} &\longrightarrow \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{5}_6 + \overline{\mathbf{5}}_{-6}, \\
\mathbf{6} &\longrightarrow \mathbf{5}_1 + \mathbf{1}_{-5}, \\
\mathbf{15} &\longrightarrow \mathbf{10}_2 + \mathbf{5}_{-4}, \\
\mathbf{20} &\longrightarrow \overline{\mathbf{10}}_3 + \mathbf{10}_{-3}.
\end{aligned} \tag{3.158}$$

One may convince oneself that this is in agreement with the general branching rule (3.14), taking into account in particular that the third rank antisymmetric representation of $SU(5)$ is the $\overline{\mathbf{10}}$. Combining (3.158) with (3.154) eventually leads to the decomposition of the adjoint representation of E_8 as

$$\mathbf{248} \overset{SU(5)\times SU(3)\times SU(2)\times U(1)_{Y'}}{\longrightarrow} \left\{ \begin{array}{c} (\mathbf{24};\mathbf{1},\mathbf{1})_0 + (\mathbf{1};\mathbf{1},\mathbf{1})_0 + (\mathbf{1};\mathbf{8},\mathbf{1})_0 + (\mathbf{1};\mathbf{1},\mathbf{3})_0 \\ (\mathbf{5};\mathbf{3},\mathbf{2})_1 + (\mathbf{1};\mathbf{3},\mathbf{2})_{-5} + c.c. \\ (\mathbf{10};\overline{\mathbf{3}},\mathbf{1})_2 + (\mathbf{5};\overline{\mathbf{3}},\mathbf{1})_{-4} + c.c. \\ (\overline{\mathbf{10}};\mathbf{1},\mathbf{2})_3 + (\mathbf{5};\mathbf{1},\mathbf{1})_6 + c.c. \end{array} \right\}. \tag{3.159}$$

As becomes obvious after redefining the visible $U(1)$ charges as

$$Q_Y = \frac{1}{3} Q_{Y'}, \tag{3.160}$$

(3.159) apparently contains states with just the Standard Model quantum numbers, as displayed in table (3.4). The expressions for the cohomology classes counting the chiral fermions follow from the general considerations at the end of section (3.2) and are listed in the second column of table (3.4).

$SU(3) \times SU(2) \times U(1)_Y$	cohom. (type A)	cohom. (type B)	SM part.
$(\mathbf{3},\mathbf{2})_{\frac{1}{3}}$	$\chi(V \otimes L)$	$\chi(V)$	q_L
$(\mathbf{3},\mathbf{2})_{-\frac{5}{3}}$	$\chi(L^{-5})$	$\chi(L^{-1})$	$-$
$(\overline{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$\chi(\bigwedge^2 V \otimes L^2)$	$\chi(\bigwedge^2 V)$	d_R^c
$(\overline{\mathbf{3}},\mathbf{1})_{-\frac{4}{3}}$	$\chi(V \otimes L^{-4})$	$\chi(V \otimes L^{-1})$	u_R^c
$(\mathbf{1},\mathbf{2})_{-1}$	$\chi(\bigwedge^2 V \otimes L^{-3})$	$\chi(\bigwedge^2 V \otimes L^{-1})$	l_L
$(\mathbf{1},\mathbf{1})_2$	$\chi(V \otimes L^6)$	$\chi(V \otimes L)$	e_R^c

Table 3.4: Massless spectrum of $H = SU(3) \times SU(2) \times U(1)_Y$ models from internal $SU(5) \times U(1)$ bundles.

To study the gauge enhancement pattern, we recall that additional gauge bosons (respectively their fermionic superpartners) in the visible spectrum, which would indicate the enhancement of the original gauge group, are counted by $H^*(\mathcal{M},\mathcal{O})$. Inspection of the appearing cohomology groups reveals that this is only possible when $c_1(L)=0$, in which case $H^*(\mathcal{M},L^{-5})$ degenerates. The appearance of a trivial bundle therefore enlarges the number of gauge bosons from $8+3+1$ by the vector-like pair $(\mathbf{3},\mathbf{2})_{-\frac{5}{3}}$ to yield precisely the 24 generators of $SU(5)$. This is just what we expect, since the commutant of $SU(5)$ is of course simply $SU(5)$ to which the visible gauge group must get enhanced.

The tadpole cancellation condition follows from the by now well-familiar evaluation of the traces over the spectrum[20]

$$\begin{aligned} \mathrm{tr}(\overline{F}^2) &= \frac{1}{30}\,\mathrm{Tr}(\overline{F}^2) = \frac{1}{30}\sum_x 2(2\pi)^2(\mathrm{ch}_2(U_x)\times\dim(R_x)) \\ &= 2\,\mathrm{tr}_f^{SU(5)}(\overline{F}^2_{SU(5)}) + 60\overline{F}^2_{U(1)_{Y'}} = 4\,(2\pi)^2\,(-c_2(V)+15c_1^2(L)), \\ \mathrm{tr}(\overline{R}^2) &= 2\,\mathrm{tr}_f^{SU(3)}(\overline{R}^2) = -4\,(2\pi)^2\,c_2(T). \end{aligned} \tag{3.161}$$

The tadpole cancellation condition (3.26) consequently takes the form

$$c_2(V) - 15\,c_1^2(L) = c_2(T). \tag{3.162}$$

We now proceed to the computation of the field-theoretic anomalies with the help of (3.28).

- The non-abelian $SU(3)^3$ anomaly is proportional to
$$A_{SU(3)^3} = 2\,(\chi(V\otimes L)+\chi(L^{-5})) - \chi(\Lambda^2V\otimes L^2) - \chi(V\otimes L^{-4}) \tag{3.163}$$
and vanishes even without invoking the tadpole cancellation condition. Of course there are no $SU(2)^3$ anomalies anyway.

- For the mixed abelian-gravitational $U(1)_{Y'}-G^2_{\mu\nu}$ anomaly, we find the in general non-vanishing expression
$$\begin{aligned} A_{U(1)_{Y'}-G^2} &= 6\,\chi(V\otimes L) - 30\,\chi(L^{-5}) + 6\,\chi(\Lambda^2V\otimes L^2) - 12\chi(V\otimes L^{-4}) \\ &\quad -6\,\chi(\Lambda^2V\otimes L^{-3}) + 6\,\chi(V\otimes L^6) \\ &= 180\int_{\mathcal{M}} c_1(L)\,\left[(-c_2(V)+15\,c_1^2(L)) + \frac{5}{12}\,c_2(T)\right]. \end{aligned} \tag{3.164}$$

- Similarly the mixed abelian-non-abelian anomaly $U(1)_{Y'}-SU(3)^2$ takes the form
$$\begin{aligned} A_{U(1)-SU(3)^2} &= 2\,\chi(V\otimes L) - 10\,\chi(L^{-5}) + 2\,\chi(\Lambda^2V\otimes L^2) - 4\,\chi(V\otimes L^{-4}) \\ &= 30\int_{\mathcal{M}} c_1(L)\,\left[(-\,c_2(V)+15\,c_1^2(L)) + \frac{1}{2}c_2(T)\right], \end{aligned} \tag{3.165}$$

[20]Note that we keep the original normalisation of $U(1)_{Y'}$ which differs from that of the visible hypercharge by a factor of 3.

and the mixed abelian-non-abelian anomaly $U(1)_{Y'} - SU(2)^2$ follows likewise as

$$\begin{aligned} A_{U(1)-SU(2)^2} &= 3\,\chi(V \otimes L) - 15\,\chi(L^{-5}) - 3\,\chi(\Lambda^2 V \otimes L^{-3}) \\ &= 30 \int_{\mathcal{M}} c_1(L) \left[(-\, c_2(V) + 15\, c_1^2(L)) + \frac{1}{2} c_2(T) \right]. \end{aligned} \tag{3.166}$$

- Finally, we obtain the following cubic abelian $U(1)^3_{Y'}$ anomaly

$$\begin{aligned} A_{U(1)^3_{Y'}} &= 6\,\chi(V \otimes L) - 750\,\chi(L^{-5}) + 24\,\chi(\Lambda^2 V \otimes L^2) - 144\,\chi(V \otimes L^{-4}) \\ &\quad -54\,\chi(\Lambda^2 V \otimes L^{-3}) + 216\,\chi(V \otimes L^6) \\ &= 2700 \int_{\mathcal{M}} c_1(L) \left[(-c_2(V) + 15\, c_1^2(L)) + \frac{1}{2}\, c_2(T) + 10 c_1(L)^2 \right]. \end{aligned} \tag{3.167}$$

It is satisfactory to note that these anomalies are in agreement with the general formulae (3.31), (3.30) and (3.32). As a result, unless the line bundle is trivial, i.e. $c_1(L) = 0$, the $U(1)_Y$ symmetry is anomalous and its anomaly has the right form to be cancelled by the Green-Schwarz mechanism. From the general form of the axion-boson mass terms (3.87) and (3.88), we convince ourselves that the $U(1)_Y$ is indeed massive whenever $c_1(L) \neq 0$.

Having discussed the details of the type A construction, let us start alternatively with a bundle of type B, i.e.

$$W = V \oplus L^{-1}, \quad \text{with } c_1(V) = c_1(L), \ \ \text{rank}(V) = 5, \tag{3.168}$$

and embed the $U(1)_{Y'}$ bundle such that

$$Q_{Y'} = (1, 1, 1, 1, 1, -5). \tag{3.169}$$

The massless spectrum is now counted by the cohomology groups summarized in the third column of table (3.4). Explicit computation yields

$$\begin{aligned} \text{tr}(\overline{F}^2) &= \frac{1}{30}\,\text{Tr}(\overline{F}^2) = \frac{1}{30} \sum_x 2(2\pi)^2 (\text{ch}_2(U_x) \times \dim(R_x)) \\ &= 4(2\pi)^2 (\text{ch}_2(V) + \text{ch}_2(L)) \end{aligned} \tag{3.170}$$

and confirms the assertion made earlier that the tadpole condition for type B bundles takes the form

$$c_2(V) - c_1^2(V) = c_2(T). \tag{3.171}$$

Again, the resulting anomalies are in agreement with the general expression displayed in section (3.4.1).

If we are interested in phenomenological applications, we must therefore find a mechanism how to keep the $U(1)_Y$ massless. What rescues us is that for suitably chosen bundle data the Stückelberg mechanism only yields masses for particular combinations of $U(1)$ factors. Let us therefore proceed and include another line bundle.

3.8.2 Bundles with structure group $SU(4) \times U(1)^2$

By means of a second $U(1)_{X'}$ bundle, we can further break the internal $SU(5)$ to $SU(4) \times U(1)_{X'}$ while keeping the non-abelian part of the visible Standard Model gauge symmetry. Concretely, we now consider an $SU(4) \times U(1)_{X'} \times U(1)_{Y'}$ bundle of type A à la

$$W = V \oplus L_1 \oplus L_2 \tag{3.172}$$

or of type B, i.e,

$$W = V \oplus L_1^{-1} \oplus L_2^{-1} \qquad \text{with} \qquad c_1(W) = 0, \tag{3.173}$$

respectively. In this latter case, the embedding of the two $U(1)$ bundles into $SU(6)$ is given by

$$Q_{X'} = (1, 1, 1, 1, -4, 0), \quad Q_{Y'} = (1, 1, 1, 1, 1, -5). \tag{3.174}$$

The for later use we note that the traces (3.25) yield $\eta_{X',X'} = 40$ and $\eta_{Y,Y} = 60$. For the type B construction, the charge matrix becomes

$$\mathcal{Q} = \begin{pmatrix} 5 & 1 \\ 0 & 6 \end{pmatrix}. \tag{3.175}$$

The visible gauge group is $H = SU(3) \times SU(2) \times U(1)_{X'} \times U(1)_{Y'}$ and the resulting decomposition of the adjoint representation of E_8 reads

$$\mathbf{248} \overset{SU(4)\times SU(3)\times SU(2)\times U(1)^2}{\longrightarrow} \left\{ \begin{array}{c} (\mathbf{15},\mathbf{1},\mathbf{1})_{0,0} \\ 2\times(\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0} \\ (\mathbf{1},\mathbf{3},\mathbf{2})_{0,-5} + c.c. \\ (\mathbf{1},\mathbf{3},\mathbf{2})_{-4,1} + (\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_{-4,-4} + (\mathbf{1},\mathbf{1},\mathbf{1})_{-4,6} + c.c. \\ (\mathbf{4},\mathbf{3},\mathbf{2})_{1,1} + (\mathbf{4},\overline{\mathbf{3}},\mathbf{1})_{1,-4} + (\mathbf{4},\mathbf{1},\mathbf{1})_{1,6} + c.c. \\ (\mathbf{4},\overline{\mathbf{3}},\mathbf{1})_{-3,2} + (\mathbf{4},\mathbf{1},\mathbf{2})_{-3,-3} + (\mathbf{4},\mathbf{1},\mathbf{1})_{5,0} + c.c \\ (\mathbf{6},\overline{\mathbf{3}},\mathbf{1})_{2,2} + (\mathbf{6},\mathbf{1},\mathbf{2})_{2,-3} + c.c. \end{array} \right\}.$$

The (possibly anomalous) hypercharge $U(1)_Y$ and the $U(1)_{B-L}$ charge are given by the linear combinations

$$Q_Y = -\frac{1}{15}\, Q_{Y'} + \frac{2}{5}\, Q_{X'}, \qquad Q_{B-L} = \frac{2}{15}\, Q_{Y'} + \frac{1}{5}\, Q_{X'}. \tag{3.176}$$

The massless spectrum is counted by the cohomology classes in table 3.5. As far as the interpretation of the states as Standard Model particles is concerned, a comparison of the spectrum in table 3.5 and the one in table 3.4 reveals a general feature: The inclusion of several $U(1)$ factors in the same E_8 factor, which seems to be required in order to keep the $U(1)_Y$ massless, gives rise to a number of (unwanted) chiral exotic states whose cohomology is counted just

reps.	cohom. (type A)	cohom. (type B)	SM part.
$(\mathbf{3},\mathbf{2})_{1,1}$	$H^*(\mathcal{M}, V\otimes L_1\otimes L_2)$	$H^*(\mathcal{M}, V)$	q_L
$(\overline{\mathbf{3}},\mathbf{1})_{1,-4}$	$H^*(\mathcal{M}, V\otimes L_1\otimes L_2^{-4})$	$H^*(\mathcal{M}, V\otimes L_2^{-1})$	d_R^c
$(\mathbf{1},\mathbf{1})_{1,6}$	$H^*(\mathcal{M}, V\otimes L_1\otimes L_2^{6})$	$H^*(\mathcal{M}, V\otimes L_2)$	ν_R
$(\overline{\mathbf{3}},\mathbf{1})_{-3,2}$	$H^*(\mathcal{M}, V\otimes L_1^{-3}\otimes L_2^{2})$	$H^*(\mathcal{M}, V\otimes L_1^{-1})$	u_R^c
$(\mathbf{1},\mathbf{2})_{-3,-3}$	$H^*(\mathcal{M}, V\otimes L_1^{-3}\otimes L_2^{-3})$	$H^*(\mathcal{M}, V\otimes L_1^{-1}\otimes L_2^{-1})$	l_L
$(\mathbf{1},\mathbf{1})_{5,0}$	$H^*(\mathcal{M}, V\otimes L_1^{5})$	$H^*(\mathcal{M}, V\otimes L_1)$	e_R^c
$(\overline{\mathbf{3}},\mathbf{1})_{2,2}$	$H^*(\mathcal{M}, \bigwedge^2 V\otimes L_1^{2}\otimes L_2^{2})$	$H^*(\mathcal{M}, \bigwedge^2 V)$	(d_R^c)
$(\mathbf{1},\mathbf{2})_{2,-3}$	$H^*(\mathcal{M}, \bigwedge^2 V\otimes L_1^{2}\otimes L_2^{-3})$	$H^*(\mathcal{M}, \bigwedge^2 V\otimes L_2^{-1})$	(l_L^c)
$(\mathbf{3},\mathbf{2})_{-4,1}$	$H^*(\mathcal{M}, L_1^{-4}\otimes L_2)$	$H^*(\mathcal{M}, L_1^{-1})$	-
$(\overline{\mathbf{3}},\mathbf{1})_{-4,-4}$	$H^*(\mathcal{M}, L_1^{-4}\otimes L_2^{-4})$	$H^*(\mathcal{M}, L_1^{-1}\otimes L_2^{-1})$	-
$(\mathbf{1},\mathbf{1})_{-4,6}$	$H^*(\mathcal{M}, L_1^{-4}\otimes L_2^{6})$	$H^*(\mathcal{M}, L_1^{-1}\otimes L_2)$	-
$(\mathbf{3},\mathbf{2})_{0,-5}$	$H^*(\mathcal{M}, L_2^{-5})$	$H^*(\mathcal{M}, L_2^{-1})$	-

Table 3.5: Massless spectrum of $H = SU(3)\times SU(2)\times U(1)_{X'}\times U(1)_{Y'}$ models. The last column gives the interpretation as SM particles with correct Q_Y and Q_{B-L}. Brackets denote that only the hypercharge of the state is the SM one.

by tensor products of the line bundles. We will find a way how to avoid this drawback later on.

The resulting tadpole cancellation condition reads

$$c_2(V) - 10\, c_1^2(L_1) - 15\, c_1^2(L_2) = c_2(T) \tag{3.177}$$

for the type A bundle and

$$-\mathrm{ch}_2(V) - \frac{1}{2}\sum_{i=1}^{2} c_1^2(L_i) = c_2(T) \tag{3.178}$$

for the type B bundle. For generic first Chern classes $c_1(L_1)$ and $c_1(L_2)$, the two $U(1)$ gauge symmetries are anomalous and gain a mass via the Green-Schwarz mechanism. Therefore, the generic unbroken gauge symmetry is $SU(3)\times SU(2)$. By computing the various anomalies, one finds that the linear combination

$$U(1)_f \simeq \kappa_1\, U(1)_{X'} + \kappa_2\, U(1)_{Y'} \tag{3.179}$$

is anomaly-free precisely if the first Chern classes of the two line bundles for the $SU(4)\times U(1)^2$ case satisfy the relation

$$2\kappa_1\, c_1(L_1) + 3\kappa_2\, c_1(L_2) = 0 \tag{3.180}$$

and for the $U(4)\times U(1)^2$ case

$$5\kappa_1\, c_1(L_1) + (6\kappa_2 - \kappa_1)\, c_1(L_2) = 0. \tag{3.181}$$

A detailed analysis of the relevant mass matrix shows that in these situations the anomaly-free $U(1)_f$ is also massless and therefore unbroken.

In the $SU(4)\times U(1)^2$ case, for certain values of the parameters κ_1, κ_2 some of the line bundles $L_1^{-4}\otimes L_2$, $L_1^{-4}\otimes L_2^{-4}$, $L_1^{-4}\otimes L_2^{6}$ and L_2^{-5} appearing in Table 3.5 become trivial and signal a non-abelian enhancement of the gauge symmetry. For the $U(4)\times U(1)^2$ bundles the situation is of course completely similar. The five[21] possible non-abelian enhancements of $SU(3)\times SU(2)$ are depicted in figure (3.3). The easiest way to find the enhanced gauge groups is to count the number of additional gauge bosons arising when one of the tensor products of line bundles becomes trivial. For example, when $L_1^{-4}\otimes L_2^{6}$ is trivial, i.e. $c_1(L_1) = \frac{3}{2} c_1(L_2)$, we find two additional vector multiplets (from $(\mathbf{1},\mathbf{1})_{-4,6}$ and its conjugate) which enhance the $SU(3)\times SU(2)\times U(1)$ to $SU(3)\times SU(2)\times SU(2)$. Likewise, one may check that indeed the chiral spectrum organizes into corresponding multiplets of the enhanced gauge group by computing explicitly the various Euler characters of the representations. This reveals that not only the expected $SO(10)$ and $SU(5)$ gauge groups are possible, but also other gauge groups containing $SU(3)\times SU(2)\times U(1)^2$ as a subgroup.

Another way to understand these gauge symmetry enhancements is by observing that the linear relations (3.180), (3.181) for the two line bundles imply a reduction of the structure group to $SU(4)\times U(1)$, which of course enhances the commutant. Its precise form depends on how the $U(1)$ is embedded into $SO(10)$, but such a group theoretic analysis is not necessary as one can read off the enhanced gauge symmetries simply from Table 3.5.

3.8.3 Bundles with structure group $SU(3)\times U(1)^3$

Let us explore further the model building possibilities several line bundles bring about and consider the embedding of a bundle of the type

$$W = V \oplus L_1 \oplus L_2 \oplus L_3 \tag{3.182}$$

with structure group $G = SU(3)\times U(1)\times U(1)\times U(1)$. We thus break E_8 down to $H = SU(3)\times SU(2)\times U(1)_Z\times U(1)_{X'}\times U(1)_{Y'}$ by replacing the internal $SU(4)$ bundle of the previous example by an $SU(3)\times U(1)_Z$ bundle. Alternatively, one can again choose the bundle W to be of the form

$$W = V \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1} \tag{3.183}$$

and the structure group of V to be $U(3)$. In this latter case, the embedding of the three $U(1)$ bundles into $SU(6)$ is given by

$$Q_1 = (1,1,1,-3,0,0),\; Q_2 = (1,1,1,1,-4,0),\; Q_3 = (1,1,1,1,1,-5) \tag{3.184}$$

[21] Including the case that all line bundles are trivial.

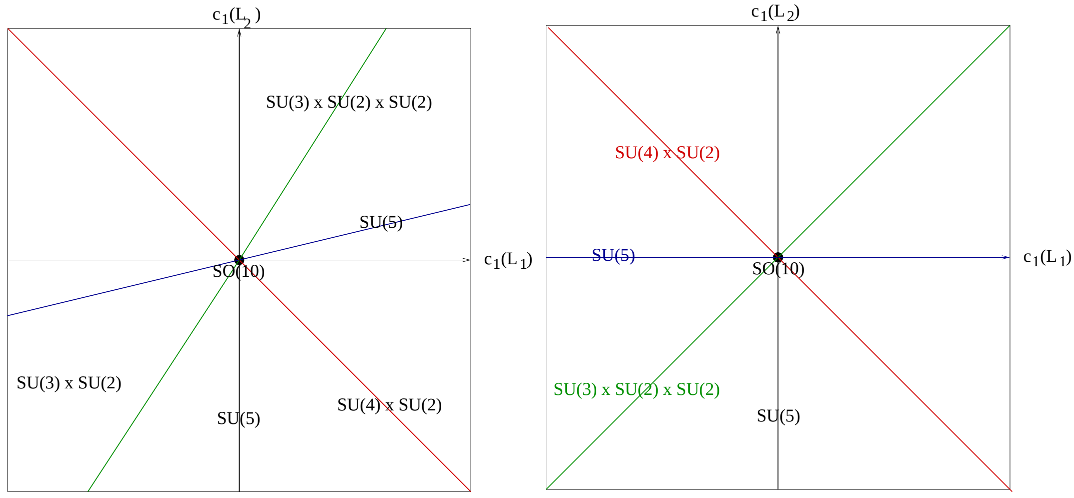

Figure 3.3: Gauge symmetry enhancement for bundles with structure group $SU(4) \times U(1)^2$. On generic lines through the origin the gauge symmetry is enhanced to $SU(3) \times SU(2) \times U(1)$ while for the specific values shown one gets even non-abelian enhancement. The left image shows the loci of non-abelian enhancement in the $(c_1(L_1), c_1(L_2))$-plane for Type A bundles and the right image for Type B.

with $\eta_{Z,Z} = 24$, $\eta_{X',X'} = 40$ and $\eta_{Y,Y} = 60$. This leads to

$$\mathcal{Q} = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{pmatrix} . \tag{3.185}$$

The massless spectrum for both cases is counted by the respective cohomology classes in Table 3.6.

The resulting tadpole cancellation condition reads

$$c_2(V) - 6\, c_1^2(L_1) - 10\, c_1^2(L_2) - 15\, c_1^2(L_3) = c_2(T) \tag{3.186}$$

for the $SU(3) \times U(1)^3$ bundle and

$$-\mathrm{ch}_2(V) - \frac{1}{2} \sum_{i=1}^{3} c_1^2(L_i) = c_2(T) \tag{3.187}$$

for the $U(3) \times U(1)^3$ bundle.

For generic first Chern classes $c_1(L_1)$, $c_1(L_2)$ and $c_1(L_3)$ the three $U(1)$ gauge symmetries are anomalous and gain a mass via the Green-Schwarz mechanism, resulting as before in $SU(3) \times SU(2)$ as the generic gauge symmetry. However, for particular choices of the bundle data we encounter a rich pattern of gauge enhancements, as we will now discuss systematically.

The computation of the various anomalies for the $SU(3) \times U(1)^3$ case reveals that the linear combination

$$U(1)_f = \kappa_1 \, U(1)_Z + \kappa_2 \, U(1)_{X'} + \kappa_3 \, U(1)_{Y'} \tag{3.188}$$

is anomaly-free precisely if the first Chern classes of the line bundles satisfy

$$6\kappa_1 \, c_1(L_1) + 10\kappa_2 \, c_1(L_2) + 15\kappa_3 \, c_1(L_3) = 0. \tag{3.189}$$

The corresponding constraint for the $U(3) \times U(1)^3$ case reads

$$4\kappa_1 c_1(L_1) - (\kappa_1 - 5\kappa_2) \, c_1(L_2) + (6\kappa_3 + \kappa_1 - \kappa_2) \, c_1(L_3) = 0. \tag{3.190}$$

For linearly independent first Chern classes, the respective equation cannot be satisfied other than trivially, of course, and we are left with gauge group $SU(3) \times SU(2)$. If, however, the $c_1(L_i)$ span a two- or one-dimensional subspace of their cohomology class, we can find – modulo rescaling – precisely one or, respectively, two non-anomalous $U(1)_f$. These $U(1)$ symmetries remain indeed massless.

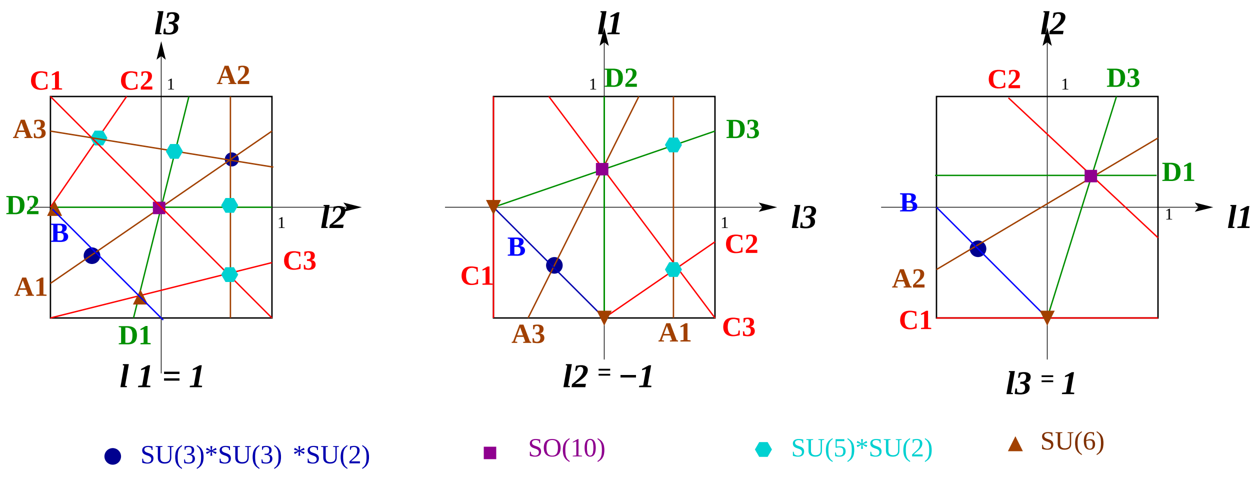

Figure 3.4: Gauge symmetry enhancement for $SU(3) \times U(1)^3$ bundles of Type A. The picture shows the projection of the various planes defined in Table 3.7 into the planes $l_i \equiv c_1(L_i) = 1$. At the point $l_i = 0$ for $i = 1, 2, 3$, the observable gauge group is E_6.

A closer look at Table 3.6 reveals a large number of possibilities for further non-abelian gauge enhancements for those choices of $c_1(L_1), c_1(L_2), c_1(L_3)$ where additional gauge bosons in the $H^*(\mathcal{M}, \mathcal{O})$ representation arise. In fact, one can verify that the spectrum then organises itself into multiplets of the corresponding gauge group, as listed in Table 3.7. We arrive at even higher rank gauge groups if several of the states transform in the trivial bundle simultaneously. The resulting enhancement pattern is plotted schematically in Figure 3.4 for the case that V

has structure group $SU(3)$. An analogous pattern can of course be derived for the $U(3)$ bundle construction.

Independently of the concrete bundle data, one can check that quite a few values of $\kappa_1, \kappa_2, \kappa_3$ admit an interpretation of the corresponding abelian factor, if massless, as the MSSM hypercharge $U(1)_Y$. We list them in Table 3.8 and Table 3.9 together with the respective candidates for MSSM fermions exhibiting the required $SU(3) \times SU(2) \times U(1)_Y$ (but not necessarily $U(1)_{B-L}$) quantum numbers.

class	reps.	cohom. (type A)	cohom. (type B)
$D1$	$(\mathbf{1};\mathbf{3},\mathbf{2})_{0,-4,1}$	$H^*(\mathcal{M}, L_2^{-4} \otimes L_3)$	$H^*(\mathcal{M}, L_2^{-1})$
$D2$	$(\mathbf{1};\mathbf{3},\mathbf{2})_{0,0,-5}$	$H^*(\mathcal{M}, L_3^{-5})$	$H^*(\mathcal{M}, L_3^{-1})$
$D3$	$(\mathbf{1};\mathbf{3},\mathbf{2})_{-3,1,1}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2 \otimes L_3)$	$H^*(\mathcal{M}, L_1^{-1})$
$D4$	$(\mathbf{3};\mathbf{3},\mathbf{2})_{1,1,1}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2 \otimes L_3)$	$H^*(\mathcal{M}, V)$
$B1$	$(\mathbf{1};\mathbf{1},\mathbf{2})_{-3,-3,-3}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2^{-3} \otimes L_3^{-3})$	$H^*(\mathcal{M}, L_1^{-1} \otimes L_2^{-1} \otimes L_3^{-1})$
$B2$	$(\mathbf{3};\mathbf{1},\mathbf{2})_{-2,2,-3}$	$H^*(\mathcal{M}, V \otimes L_1^{-2} \otimes L_2^2 \otimes L_3^{-3})$	$H^*(\mathcal{M}, V \otimes L_1^{-1} \otimes L_3^{-1})$
$B3$	$(\mathbf{3};\mathbf{1},\mathbf{2})_{-2,-2,3}$	$H^*(\mathcal{M}, V \otimes L_1^{-2} \otimes L_2^{-2} \otimes L_3^3)$	$H^*(\mathcal{M}, V \otimes L_1^{-1} \otimes L_2^{-1})$
$B4$	$(\mathbf{3};\mathbf{1},\mathbf{2})_{1,-3,-3}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2^{-3} \otimes L_3^{-3})$	$H^*(\mathcal{M}, V \otimes L_2^{-1} \otimes L_3^{-1})$
$C1$	$(\mathbf{1};\overline{\mathbf{3}},\mathbf{1})_{0,-4,-4}$	$H^*(\mathcal{M}, L_2^{-4} \otimes L_3^{-4})$	$H^*(\mathcal{M}, L_2^{-1} \otimes L_3^{-1})$
$C2$	$(\mathbf{1};\overline{\mathbf{3}},\mathbf{1})_{-3,-3,2}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2^{-3} \otimes L_3^2)$	$H^*(\mathcal{M}, L_1^{-1} \otimes L_2^{-1})$
$C3$	$(\mathbf{1};\overline{\mathbf{3}},\mathbf{1})_{-3,1,-4}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2 \otimes L_3^{-4})$	$H^*(\mathcal{M}, L_1^{-1} \otimes L_3^{-1})$
$C4$	$(\mathbf{3};\overline{\mathbf{3}},\mathbf{1})_{-2,2,2}$	$H^*(\mathcal{M}, V \otimes L_1^{-2} \otimes L_2^2 \otimes L_3^2)$	$H^*(\mathcal{M}, V \otimes L_1^{-1})$
$C5$	$(\overline{\mathbf{3}};\overline{\mathbf{3}},\mathbf{1})_{2,2,2}$	$H^*(\mathcal{M}, \bigwedge^2 V \otimes L_1^2 \otimes L_2^2 \otimes L_3^2)$	$H^*(\mathcal{M}, \bigwedge^2 V)$
$C6$	$(\mathbf{3};\overline{\mathbf{3}},\mathbf{1})_{1,-3,2}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2^{-3} \otimes L_3^2)$	$H^*(\mathcal{M}, V \otimes L_2^{-1})$
$C7$	$(\mathbf{3};\overline{\mathbf{3}},\mathbf{1})_{1,1,-4}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2 \otimes L_3^{-4})$	$H^*(\mathcal{M}, V \otimes L_3^{-1})$
$A1$	$(\mathbf{1};\mathbf{1},\mathbf{1})_{0,-4,6}$	$H^*(\mathcal{M}, L_2^{-4} \otimes L_3^6)$	$H^*(\mathcal{M}, L_2^{-1} \otimes L_3)$
$A2$	$(\mathbf{1};\mathbf{1},\mathbf{1})_{-3,5,0}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2^5)$	$H^*(\mathcal{M}, L_1^{-1} \otimes L_2)$
$A3$	$(\mathbf{1};\mathbf{1},\mathbf{1})_{-3,1,6}$	$H^*(\mathcal{M}, L_1^{-3} \otimes L_2 \otimes L_3^6)$	$H^*(\mathcal{M}, L_1^{-1} \otimes L_3)$
$A4$	$(\mathbf{3};\mathbf{1},\mathbf{1})_{1,5,0}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2^5)$	$H^*(\mathcal{M}, V \otimes L_2)$
$A5$	$(\mathbf{3};\mathbf{1},\mathbf{1})_{1,1,6}$	$H^*(\mathcal{M}, V \otimes L_1^1 \otimes L_2 \otimes L_3^6)$	$H^*(\mathcal{M}, V \otimes L_3)$
$A6$	$(\mathbf{3};\mathbf{1},\mathbf{1})_{4,0,0}$	$H^*(\mathcal{M}, V \otimes L_1^4)$	$H^*(\mathcal{M}, V \otimes L_1)$

Table 3.6: Massless spectrum of $H = SU(3) \times SU(2) \times U(1)^3$ models.

	rep.	Type A	Type B	gauge group
A1	$(1,1,1)_{0,-4,6}$	$-2l_2+3l_3=0$	$-l_2+l_3=0$	$SU(3)\times SU(2)^2$
A2	$(1,1,1)_{-3,5,0}$	$3l_1-5l_2=0$	$l_1-l_2=0$	$SU(3)\times SU(2)^2$
A3	$(1,1,1)_{-3,1,6}$	$3l_1-l_2-6l_3=0$	$l_1-l_3=0$	$SU(3)\times SU(2)^2$
B1	$(1,1,\mathbf{2})_{-3,-3,-3}$	$l_1+l_2+l_3=0$	$l_1+l_2+l_3=0$	$SU(3)\times SU(3)$
C1	$(1,\overline{\mathbf{3}},1)_{0,-4,-4}$	$l_2+l_3=0$	$l_2+l_3=0$	$SU(4)\times SU(2)$
C2	$(1,\overline{\mathbf{3}},1)_{-3,-3,-2}$	$3l_1+2l_2+3l_3=0$	$l_1+l_2=0$	$SU(4)\times SU(2)$
C3	$(1,\overline{\mathbf{3}},1)_{-3,1,-4}$	$3l_1-l_2+4l_3=0$	$l_1+l_3=0$	$SU(4)\times SU(2)$
D1	$(1,\mathbf{3},\mathbf{2})_{0,-4,1}$	$-4l_2+l_3=0$	$l_2=0$	$SU(5)$
D2	$(1,\mathbf{3},\mathbf{2})_{0,0,-5}$	$l_3=0$	$l_3=0$	$SU(5)$
D3	$(1,\mathbf{3},\mathbf{2})_{-3,1,1}$	$3l_1-l_2-l_3=0$	$l_1=0$	$SU(5)$

Table 3.7: Generic enhancement of $SU(3)\times SU(2)$ by additional non-chiral degrees of freedom for both the Type A and Type B embedding. We use the notation $l_i=c_1(L_i)$.

part.	class	$\begin{pmatrix}\frac{1}{2}\\ \frac{1}{10}\\ -\frac{1}{15}\end{pmatrix}$	$\begin{pmatrix}-\frac{5}{14}\\ \frac{1}{14}\\ -\frac{13}{21}\end{pmatrix}$	$\begin{pmatrix}\frac{3}{2}\\ -\frac{1}{10}\\ \frac{1}{15}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}\\ \frac{33}{30}\\ -\frac{1}{15}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{10}\\ -\frac{7}{15}\end{pmatrix}$
Q_L	D	$1,2,4$	$1,3$	1	$2,3$	4	4
$\overline{U}_R$	C	$2,3,4$	$4,6$	$6,7$	$4,7$	$4,7$	$4,6$
$\overline{D}_R$	C	$1,5,6,7$	2	1	3	$1,2,5$	$1,3,5$
L	B	$1,2,3,\overline{4}$	3	$\overline{4}$	2	$1,\overline{3},\overline{4}$	$1,\overline{2},\overline{4}$
$\overline{E}_R$	A	$\overline{2},\overline{3},6$	$\overline{4},\overline{6}$	$\overline{4},\overline{5}$	$\overline{5},\overline{6}$	$\overline{4},5,6$	$\overline{4},5,6$
$\overline{\nu}_R$	A	$1,4,5$	2	1	3	3	1

Table 3.8: MSSM particle candidates for choices of $(\kappa_1,\kappa_2,\kappa_3)$, part I. The labels of the representations refer to the position in the respective sections of Table 3.6 with bars denoting hermitian conjugation.

part.	class	$\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{4} \\ \frac{3}{20} \\ -\frac{4}{15} \end{pmatrix}$	$\begin{pmatrix} -1 \\ \frac{1}{5} \\ -\frac{7}{15} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{12} \\ \frac{7}{60} \\ -\frac{1}{15} \end{pmatrix}$	$\begin{pmatrix} -1 \\ \frac{3}{5} \\ -\frac{1}{15} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} \\ \frac{7}{10} \\ -\frac{7}{15} \end{pmatrix}$
Q_L	D	4	$1, 3$	1	$2, 3$	2	3
$\overline{U}_R$	C	$6, 7$	5	6	5	7	4
$\overline{D}_R$	C	$2, 3, 5$	$2, 7$	$4, 7$	$3, 6$	$6, 4$	$6, 7$
L	B	$1, \overline{2}, \overline{3}$	$\overline{2}, \overline{4}$	$\overline{3}, \overline{4}$	$\overline{3}, \overline{4}$	$\overline{2}, \overline{4}$	$\overline{2}, \overline{3}$
$\overline{E}_R$	A	$4, 5, \overline{6}$	$\overline{5}$	$\overline{1}, 2, \overline{4}, \overline{5}$	$\overline{4}$	$1, 3, \overline{5}$	$\overline{2}, \overline{3}, \overline{6}$
$\overline{\nu}_R$	A	1	2	3	3	2	1

Table 3.9: MSSM particle candidates for choices of $(\kappa_1, \kappa_2, \kappa_3)$, part II.

Chapter 4

The SO(32) heterotic string with unitary bundles and five-branes

In view of the rich structure we have encountered in the $E_8 \times E_8$ string with unitary bundles, it is natural to try and follow a similar strategy in the heterotic theory with gauge group $SO(32)$. The differences in the perturbative sector will be entirely due to the peculiarities of $SO(32)$ as opposed to $E_8 \times E_8$. We will review momentarily that $SO(32)$ possesses a very natural embedding of gauge bundles with unitary structure group. In fact, its decomposition into products of $U(N)$ subgroups will reproduce exactly the massless spectrum we are familiar with in the S-dual Type I framework with magnetized D9-branes. The dynamics of five-branes differs considerably from the $E_8 \times E_8$ case in that now the five-branes also contribute chiral fermions and additional symplectic gauge factors. Consequently, the Green-Schwarz anomaly cancellation pattern has to be reconsidered. It hinges, as far as the five-branes are concerned, on an anomalous coupling of the heterotic five-brane to the bulk, analogously to the anomaly inflow arguments for D-branes. As an important aspect we will compare the low-energy effective action, notably the Fayet-Iliopoulos terms and the resulting one-loop corrected Donaldson-Uhlenbeck-Yau equation, to known results on the Type I/ Type IIB side. This will serve as evidence for our interpretation of the correction terms in the DUY constraint as the four-dimensional shadow of a modified stability condition.

Since, despite all the differences in the details, the general strategy is very close to the procedure in the $E_8 \times E_8$ case, we will often be rather brief as far the explanation of the conceptual background is concerned in order to avoid redundancies. In those cases, the required material has already been covered in chapter 3 to which we refer for additional details. The contents of this chapter is based on [129–131].

4.1 A class of $SO(32)$ heterotic string vacua

We compactify the $SO(32)$ heterotic string on a Calabi-Yau manifold $\mathcal{M}$ and consider decompositions of the gauge group $SO(32)$ into its unitary subgroups. Our strategy is to invoke the Whitney sum of internal vector bundles

$$W = \bigoplus_{i=1}^{K} V_i. \tag{4.1}$$

Each V_i denotes a rank n_i unitary bundle, i.e.it has structure group $U(n_i)$. The group theoretic embedding is again accomplished in a two-step process, similarly to the $E_8 \times E_8$ construction. The first step involves the natural $U(M_i)$ subgroups of $SO(32)$ via the embedding

$$U(M_i) \subset SO(32) \longrightarrow SO(32 - 2M_i) \times U(1)_i. \tag{4.2}$$

Into this $U(M_i)$, we diagonally embed the structure group $U(n_i)$ of the bundle V_i such that $M_i = n_i\, N_i$, i.e.

$$U(n_i) \subset U(n_i N_i) \longrightarrow U(N_i). \tag{4.3}$$

The emergence of the non-abelian group $U(N_i)$ can be understood as the non-abelian enhancement of the naive commutant $U(1)^{N_i}$. We just observed similar phenomena in the $E_8 \times E_8$ theory, where non-abelian enhancement was tied to the degeneracy of some of the internal bundles.

In all, this accomplishes the embedding

$$\prod_{i=1}^{K} U(n_i) \subset \prod_{i=1}^{K} U(n_i N_i) \subset SO(32) \tag{4.4}$$

and the resulting observable non-abelian gauge group is

$$H = SO(2M) \times \prod_{i=1}^{K} U(N_i) \qquad \text{with } M + \sum_{i=1}^{K} M_i = 16. \tag{4.5}$$

As we will discuss, maximally only the anomaly-free part of the $U(1)^K$ gauge factors remains in the low energy gauge group - a feature which we are by now well familiar with from the discussion of the $E_8 \times E_8$ theory.

In addition to this perturbative sector we take into account the possible contribution from heterotic five-branes [83, 132–135], which we will denote as H5-branes to distinguish them from their cousins in the $E_8 \times E_8$ theory. In contrast to the situation encountered there, the inclusion of H5-branes does affect also the gauge sector of the compactifications. We noted already in section (2.1) that the worldvolume of an $SO(32)$ H5-brane accommodates a massless gauge field. To be more precise, let us recall from section (2.3) that for supersymmetry each

H5-brane has to wrap an (in general reducible) holomorphic cycle γ on $\mathcal{M}$. This means that the associated cohomology class $[\widehat{\gamma}] \in H^2(\mathcal{M}, 2\mathbb{Z})$ is effective, i.e. lies inside the Mori cone of $\mathcal{M}$. If γ is irreducible, this really corresponds to a single H5-brane and gives rise to an additional $Sp(2)$ gauge group in the effective action. The appearance of these symplectic gauge degrees of freedom was derived in [83] by virtue of S-duality between the H5-brane and the D5-brane in Type I theory. The latter, in turn, is known to carry symplectic gauge groups [136]. If γ is reducible, we decompose it into the irreducible generators of the Mori cone γ_a, $\gamma = \sum_{a=1}^{L} N_a \gamma_a$, $N_a \in \mathbb{Z}_0^+$. Due to the multiple wrapping around each irreducible curve γ_a, the additional gauge group in the effective action gets enhanced to $\prod_a Sp(2N_a)$. The decomposition into generators may not be unique and the gauge group may therefore vary in the different regions of the associated moduli space. However, its total rank and the total number of chiral degrees of freedom charged under the symplectic groups are only dependent on γ, of course.

By heterotic-Type I duality, one can infer that the effective low energy action on the H5-branes has to have the usual Chern-Simons form

$$S_{H5_a} = -\mu_5 \int_{\mathbb{R}_{1,3}\times\gamma_a} \sum_{n=0}^{1} B^{(4n+2)} \wedge \left(N_a + \frac{\ell_s^4}{2(2\pi)^2} \mathrm{tr}_{Sp(2N_a)} F_a^2 \right) \wedge \frac{\sqrt{\hat{\mathcal{A}}(\mathrm{T}\gamma_a)}}{\sqrt{\hat{\mathcal{A}}(\mathrm{N}\gamma_a)}}, \quad (4.6)$$

with the H5-brane tension $\mu_5 = \frac{1}{(2\pi)^5 (\alpha')^3}$. $\mathrm{T}\gamma_a$ and $\mathrm{N}\gamma_a$ denote the tangent bundle and the normal bundle, respectively, of the 2-cycle γ_a, which for concreteness we take to be irreducible from now on and wrapped by a stack of N_a H5-branes. The curvature occurring in the definition of the $\hat{A}$- genus $\hat{A}(\mathcal{M}) = 1 + \frac{1}{48}\frac{1}{(2\pi)^2}\mathrm{tr}\mathcal{R}^2 + \cdots$ is defined as $\mathcal{R} = -i\sqrt{2}\,\ell_s^2 R$ ($\ell_s \equiv 2\pi\sqrt{\alpha'}$ as before). This type of anomalous coupling of the five-brane to the bulk is required in order to cancel the gravitational anomalies on the $SO(32)$ H5-brane world-volume. Strictly speaking, the well-known anomaly-inflow arguments leading to (4.6) were applied in the S-dual Type I framework [137], but the structure of gravitational anomalies is not affected by S-duality and therefore the full Wess-Zumino coupling is given by (4.6) also on the heterotic side.[1] The sign of the Chern-Simons action is dictated by supersymmetry: Jumping ahead a little, we state that the choice in (4.6) guarantees that the real part of the gauge kinetic function for the $Sp(2N_a)$-group is indeed positive, as we demonstrate in section 4.5. Note that (4.6) implies both the usual magnetic coupling to $B^{(6)}$ and a coupling to $B^{(2)}$. The latter will be essential in section (4.4) when it comes to cancelling the mixed abelian-gravitational and abelian-symplectic anomalies by the generalized Green-Schwarz mechanism.

For our upcoming purposes it is useful to recall the somewhat complementary interpretation of the $SO(32)$ five-brane as an instanton of zero size [83]. In

[1] The normalisations of $\mathcal{R}$ and of the term involving $\mathrm{tr}_{Sp(2N_a)} F_a^2$ differ from what one might naively expect in view of the CS action of a $D5$-brane in Type II B by a factor of $\sqrt{2}$ and 2, respectively. This is a consequence of a corresponding redefinition of α' in the context of the S-duality transformation to be discussed further in section 4.7.

intuitive terms, we can think of it as a gauge instanton background which, unlike the holomorphic bundle W, is not spread out along the entire internal manifold, but which has support only on the two-cycle γ_a. Mathematically, such an object is defined as the skyscraper sheaf $\mathcal{O}|_{\gamma_a}$, which is the restriction of the trivial sheaf on $\mathcal{M}$ to γ_a. Being a coherent sheaf, $\mathcal{O}|_{\gamma_a}$ admits a locally free resolution, given by an appropriate Koszul sequence. For details on Koszul sequences we refer to the mathematical literature, e.g. [138, 139]. Suffice it here to recall that the general Koszul sequence is an exact sequence which provides the resolution for the restriction of a vector bundle to some codimension k hypersurface Y as [30]

$$0 \to V \otimes \wedge^k N^* \to V \otimes \wedge^{k-1} N^* \to \ldots V \otimes N^* \to V \to V|_Y \to 0, \tag{4.7}$$

where the hypersurface Y emerges as the zero locus of a holomorphic section of N. This determines the total Chern character of $V|_Y$ as

$$\mathrm{ch}(V|_Y) \quad = \quad \mathrm{ch}(V) - \mathrm{ch}(V \otimes N^*) + \mathrm{ch}(V \otimes \wedge^2 N^*) + \ldots + (-1)^k \mathrm{ch}(V \otimes \wedge^k N^*). \tag{4.8}$$

Heuristically, we can think of γ_a as the complete intersection of two generic divisors D_1 and D_2, $\gamma_a = D_1 \cap D_2$. This means that the Poincaré dual four-form, $\overline{\gamma}_a$, is given by the cohomological intersection $\overline{\gamma}_a = D_1 \cdot D_2$. In this case we can take for the rank two holomorphic bundle N simply the direct sum $\mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$. Recall that $\mathcal{O}(D_1)$ is the line bundle on $\mathcal{M}$ with first Chern class $c_1(\mathcal{O}(D_1)) = D_1$. Furthermore $\wedge^2 N = \mathcal{O}(D_1 + D_2)$, as follows already from the computation of the Chern classes (see also appendix A.1). In all, we take as the defining sequence for $\mathcal{O}(\gamma_a)$

$$0 \to \mathcal{O}(-D_1 - D_2) \to \mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \to \mathcal{O}_{\mathcal{M}} \to \mathcal{O}|_{\gamma_a} \to 0. \tag{4.9}$$

It follows from equation (4.8) that the Chern characters of the sheaf $\mathcal{O}|_{\gamma_a}$ can readily be computed as $\mathrm{ch}(\mathcal{O}|_{\gamma_a}) = (0, 0, D_1 \cdot D_2, 0)$. In deriving this we have assumed that the divisors D_1 and D_2 are in generic position so that in particular $D_1 \cdot D_1 \cdot D_2 = 0 = D_2 \cdot D_2 \cdot D_1$.

Due to the overall minus sign in the Chern-Simons coupling of the five-brane to the bulk, we have to include an extra sign into the Chern character. As a conclusion, the five-brane has as its defining Chern character

$$\mathrm{ch}(\mathcal{O}|_{\gamma_a}) = (0, 0, -\overline{\gamma}_a, 0). \tag{4.10}$$

This is precisely what we expect from its interpretation as an instanton of zero size: its "instanton number", i.e. $c_2(\mathcal{O}|_{\gamma_a})$, is given simply by the effective class Poincaré dual to the class of the two-cycle it wraps.

4.2 The massless spectrum

The perturbative spectrum can be determined from the decomposition of the adjoint representation of $SO(32)$ into representations of $SO(2M) \times \prod_i U(N_i) \times U(n_i)$,

$$
\mathbf{496} \to \left(\begin{array}{c} (\mathbf{Anti}_{SO(2M)}; \mathbf{1}; \mathbf{1}) \\ \sum_{j=1}^{K} (\mathbf{1}; \mathbf{Adj}_{U(N_j)}; \mathbf{Adj}_{U(n_j)}) \\ \sum_{j=1}^{K} (\mathbf{1}; \mathbf{Anti}_{U(N_j)}; \mathbf{Sym}_{U(n_j)}) + (\mathbf{1}; \mathbf{Sym}_{U(N_j)}; \mathbf{Anti}_{U(n_j)}) + h.c. \\ \sum_{i<j} (\mathbf{1}; \mathbf{N}_i, \mathbf{N}_j; \mathbf{n}_i, \mathbf{n}_j) + (\mathbf{1}; \mathbf{N}_i, \overline{\mathbf{N}}_j; \mathbf{n}_i, \overline{\mathbf{n}}_j) + h.c. \\ \sum_{j=1}^{K} (2M; \mathbf{N}_j; \mathbf{n}_j) + h.c. \end{array} \right).
$$

The internal cohomology groups counting the various states are listed in table 4.1. It is most striking that we encounter the same massless spectrum as for the perturbative Type I string on a smooth Calabi-Yau space with magnetized B-type D9-branes[2]. A prominent role is played by the chiral matter in the bifundamental representations of pairs of observable $U(N_i)$ factors. Correspondingly, in the framework of intersecting D-branes T-dual to the Type I string with magnetized D9-branes, chiral matter is localized at the intersection of two stacks of D6-branes and likewise transforms in the bifundamental of the two gauge groups realized on the respective worldvolumes. Apparently, on the S-dual heterotic side, this typical structure emerges automatically due to the natural $U(N)$ subgroups of $SO(32)$ and the associated decomposition of the adjoint representation. It will therefore come as no surprise that the architecture of the concrete models we will present in chapter 6 is very reminiscent of the multiple stack constructions known from the intersecting brane picture.

The appearance of massless states in the adjoint of $U(N_i)$ and counted by $H^*(\mathcal{M}, V_i \otimes V_i^*)$ deserves some further comments. The element[3] in $H^0(\mathcal{M}, V_i \otimes V_i^*)$ counts the vector multiplet of the $U(N_i)$ group which contains its gauge bosons. The elements in $H^1(\mathcal{M}, V_i \otimes V_i^*)$, by contrast, correspond to the moduli fields associated with the bundle deformations. In the special case that the internal bundle is abelian, $V_i \otimes V_i^* = \mathcal{O}$ and we find $h^1(\mathcal{M}, \mathcal{O})$ massless chiral multiplets transforming in the *adjoint* representation of a $U(N_i)$ observable gauge factor, just as in the Type I framework and for intersecting branes. On genuine Calabi-Yau manifolds, there do not exist any homologically non-trivial one-cycles, of course, and this fits with the fact that on a Calabi-Yau a line bundle has no continuous moduli - it is defined once and for all by its first Chern class as an element in $H^2(\mathcal{M}, \mathbb{Z})$. On the torus, however, one has $H^1(T^6, \mathcal{O}) = 3$, and the complex adjoint scalars correspond to the continuous Wilson lines on $\mathcal{M}$ which parameterise the continuous deformations of a line bundle respectively the

[2]Note, however, the recent investigation [140] of toroidal *orbifold* compactifications of the $SO(32)$ heterotic string where models are found featuring e.g. the **16** spinor representation. Such spinor representations are not present in our $SO(32)$ heterotic context. We stress that our results are valid for the case of *smooth* background manifolds.

[3]Recall that due to stability of V, $H^0(\mathcal{M}, V_i \otimes V_i^*) = 1$.

deformations of the intersecting branes. Analogously, turning on non-abelian bundles $U(n_i)$ on the Type I D9-branes gives rise to $H^1(\mathcal{M}, V_i \otimes V_i^*)$ moduli corresponding to the deformations of the $U(n_i)$ bundle.

reps.	$H = \prod_{i=1}^{K} SU(N_i) \times U(1)_i \times SO(2M) \times \prod_{a=1}^{L} Sp(2N_a)$
$(\mathbf{Adj}_{U(N_i)})_{0(i)}$	$H^*(\mathcal{M}, V_i \otimes V_i^*)$
$(\mathbf{Sym}_{U(N_i)})_{2(i)}$	$H^*(\mathcal{M}, \bigwedge^2 V_i)$
$(\mathbf{Anti}_{U(N_i)})_{2(i)}$	$H^*(\mathcal{M}, \bigotimes_s^2 V_i)$
$(\mathbf{N}_i, \mathbf{N}_j)_{1(i),1(j)}$	$H^*(\mathcal{M}, V_i \otimes V_j)$
$(\mathbf{N}_i, \overline{\mathbf{N}}_j)_{1(i),-1(j)}$	$H^*(\mathcal{M}, V_i \otimes V_j^*)$
$(\mathbf{Adj}_{SO(2M)})$	$H^*(\mathcal{M}, \mathcal{O})$
$(\mathbf{2M}, \mathbf{N}_i)_{1(i)}$	$H^*(\mathcal{M}, V_i)$
$(\mathbf{Anti}_{Sp(2N_a)})$	$\mathrm{Ext}^*_{\mathcal{M}}(\mathcal{O}\vert_{\gamma_a}, \mathcal{O}\vert_{\gamma_a})$
$(\mathbf{N}_i, \mathbf{2N}_a)_{1(i)}$	$\mathrm{Ext}^*_{\mathcal{M}}(V_i, \mathcal{O}\vert_{\gamma_a})$
$(\mathbf{2N}_a, \mathbf{2N}_b)$	$\mathrm{Ext}^*_{\mathcal{M}}(\mathcal{O}\vert_{\gamma_a}, \mathcal{O}\vert_{\gamma_b})$

Table 4.1: Massless spectrum with the structure group taken to be $G = \prod_{i=1}^{K} U(n_i)$. The subscripts in the first column denote the charges under decomposition $U(N_i) \to SU(N_i) \times U(1)_i$.

Additional chiral matter appears from the non-perturbative H5-branes (see the three last lines of table 4.1), which is absent for the M5-branes in $E_8 \times E_8$ heterotic string compactifications [110]. In the latter case this is in accord with the possibility of moving the five-branes into the eleven-dimensional bulk in the Horava-Witten theory. For the $SO(32)$ theory, by contrast, the description of the H5-brane as the skyscraper sheaf $\mathcal{O}|_{\gamma_a}$ makes it clear that the brane should be treated on the same footing as the smooth gauge instantons given by the bundle W, and this analogy must be taken even more seriously when it comes to the zero modes of the Dirac operator.

The matter arising in the H5-brane sector is described by appropriate extension groups. Following for instance [141], the global extension groups $\mathrm{Ext}^*_{\mathcal{M}}(\mathcal{E}, \mathcal{F})$ of two coherent sheaves on $\mathcal{M}$ give the sheaf theoretic generalisation of the cohomology groups $H^*(\mathcal{M}, \mathcal{E} \otimes \mathcal{F}^*)$ for vector bundles on smooth manifolds. The cohomology groups in table 4.1 counting the zero modes in the bifundamental of one $Sp(2N_a)$ and one $U(N_i)$ factor are therefore the straightforward sheaf theoretic generalisation of the Dolbeault cohomology groups in case only smooth vector bundles are involved.

In particular, it is shown in [141] that

$$\mathrm{Ext}^1_{\mathcal{M}}(\mathcal{O}|_{\gamma_a},\mathcal{O}|_{\gamma_a}) = H^1(\gamma_a,\mathcal{O}) + H^0(\gamma_a,\mathrm{N}\gamma_a), \tag{4.11}$$

where the first term contains the possible Wilson line moduli on the H5-brane and the second term the geometric deformations of the two-cycles $\gamma_a \subset \mathcal{M}$. All these chiral supermultiplets transform in the antisymmetric representation of the symplectic gauge factor.

The chirality index of the perturbative spectrum can be determined from the Euler characteristics (2.17) of the various bundles U_i occurring in the decomposition of $SO(32)$. This is true also for the matter arising from the H5-branes or rather the coherent sheaves $\mathcal{O}|_{\gamma_a}$. Namely, for general coherent sheaves the righthand side in (2.17) measures the alternating sum of the dimensions of the global extensions. It follows that in the non-perturbative sector, the H5-branes give rise to *chiral* matter in the bifundamental $(\mathbf{N}_i, 2\mathbf{N}_a)_{1(i)}$, which is counted by the index

$$\chi(\mathcal{M}, V_i \otimes \mathcal{O}|^*_{\gamma_a}) = -\int_{\mathcal{M}} c_1(V_i) \wedge \overline{\gamma}_a. \tag{4.12}$$

The righthand side of (4.12) is an immediate consequence of the formula for the Euler characteristic (2.17) once we remember that with the help of (4.10) $\mathrm{ch}_3(V_i \otimes \mathcal{O}|^*_{\gamma_a}) = -c_1(V_i) \wedge \overline{\gamma}_a$ and $\mathrm{ch}_1((V_i \otimes \mathcal{O}|^*_{\gamma_a}) = 0$. In agreement with the absence of chiral matter for symplectic gauge groups only, for two H5-branes wrapping 2-cycles γ_a and γ_b one gets $\chi(\mathcal{M}, \mathcal{O}|_{\gamma_a} \otimes \mathcal{O}|^*_{\gamma_b}) = 0$.

For later use we point out that the requisite formulae to compute the Euler characteristics of products of bundles $V_i \otimes V_j$ and the (anti)-symmetric product bundle, $\bigwedge^2 V$ and $\bigotimes^2_s V$ respectively, appearing in Table (4.1) can be found in appendix A.1.

4.3 Global consistency conditions

We can proceed to a detailed analysis of the topological consistency conditions our internal bundles have to satisfy.

In order to evaluate the tadpole cancellation condition for our spectrum we need, as in the $E_8 \times E_8$ case, to express the formal trace over the internal Yang-Mills field strength in (2.24) by the topological data of W and the manifold $\mathcal{M}$. With the help of table 4.1 we can convince ourselves that

$$\begin{aligned} \mathrm{tr}\overline{F}^2 &= \frac{1}{30}\sum_x 2(2\pi)^2\,(\mathrm{ch}_2(U_x) \times \dim(R_x)) = \\ &= 4(2\pi)^2 \sum_i N_i\,\mathrm{ch}_2(V_i). \end{aligned} \tag{4.13}$$

For later use we note that similar trace identities of this type are collected in appendix A.3.

Consequently, the tadpole condition takes the simple form

$$\sum_{i=1}^{K} N_i \, \mathrm{ch}_2(V_i) - \sum_{a=1}^{L} N_a \gamma_a = -c_2(T), \tag{4.14}$$

to be satisfied in cohomology.

In the presence of symplectic gauge group factors due to the H5-branes we need to worry about potential global $Sp(2N_a)$ anomalies. As we know from [142] this Witten anomaly is absent precisely if the number of chiral fermions in the fundamental of the $Sp(2N_a)$ group is even. Clearly, for a stack of N_a five-branes wrapping the cycle γ_a, the chiral index of the $Sp(2N_a)$ is given by

$$\mathrm{index}_{Sp(2N_a)} = -\sum_i N_i \int_{\mathcal{M}} c_1(V_i) \wedge \overline{\gamma}_a = -\int_{\mathcal{M}} c_1(W) \wedge \overline{\gamma}_a. \tag{4.15}$$

So apparently, the K-theory condition

$$c_1(W) = \sum_i N_i \, c_1(V_i) \in H^2(\mathcal{M}, 2\mathbb{Z}) \tag{4.16}$$

ensures the absence of a Witten anomaly for every probe five-brane and has therefore the field theoretic interpretation as a global consistency condition for every topological sector of the theory. Recall from section 2.3 that from the point of view of the underlying $(0,2)$ model, the rationale behind (4.16) is actually the requirement of absence of worldsheet anomalies [94,95]. The connection between these two different arguments leading to (4.16) is comparable to the situation in Type I string theory, where the analogue of (4.16) corresponds, microscopically, to the torsion K-theory constraint for the non-BPS D7-brane [96]. Alternatively, this condition can likewise be derived by requiring the absence of global Witten anomalies on D5-branes for every possible probe brane and not just for the concrete vacuum under consideration.

4.4 Anomaly cancellation

4.4.1 Field theoretic anomalies

Now let us discuss the resulting anomalies. The expressions for the field theoretic anomalies follow immediately from the chiral spectrum in table (4.1). For the cubic non-abelian anomalies we obtain[4] from

$$\begin{aligned} \mathcal{A}_{SU(N_i)^3} \;\sim\;& (N_i - 4)\, \chi(\textstyle\bigotimes_s^2 V_i)) + (N_i + 4)\chi(\textstyle\bigwedge^2 V_i)) + 2M\, \chi(V_i) \\ &+ \sum_{j \neq i} N_j \left(\chi(V_i \otimes V_j) + \chi(V_i \otimes V_j^*)\right) + \sum_a 2\, N_a\, \chi(V_i \otimes \mathcal{O}|^*_{\gamma_a}) \end{aligned} \tag{4.17}$$

[4] This uses once again the trace identities listed in appendix A.2.

the expression in terms of Chern characters,

$$\mathcal{A}_{SU(N_i)^3} \sim 2\int_{\mathcal{M}} c_1(V_i) \times \text{Tad}. \tag{4.18}$$

Here

$$\text{Tad} = c_2(T) + \sum_{j=1}^{K} N_j \, \text{ch}_2(V_j) - \sum_a N_a \overline{\gamma}_a = 0 \tag{4.19}$$

in cohomology thanks to tadpole cancellation (4.14). Thus in contrast to the $E_8 \times E_8$ examples, the cubic non-abelian anomalies vanish only if the Bianchi identity for H is satisfied [107].

The explicit expressions for all mixed and cubic abelian anomalies can readily be computed along the same lines. Here we only state the result in terms of the various Chern characters up to tadpole cancellation

$$\begin{aligned}
\mathcal{A}_{U(1)_i - SU(N_j)^2} &\sim 2N_i \int_{\mathcal{M}} n_j \, \text{ch}_3(V_i) + 2N_i \int_{\mathcal{M}} c_1(V_i) \wedge \Big(\text{ch}_2(V_j) + \frac{n_j}{12} c_2(T)\Big), \\
\mathcal{A}_{U(1)_i - U(1)_j^2} &\sim N_j \, \mathcal{A}_{U(1)_i - SU(N_j)^2}, \\
\mathcal{A}_{U(1)_i - G_{\mu\nu}^2} &\sim \frac{1}{2}\int_{\mathcal{M}} N_i \, c_1(V_i) \, c_2(T) + 24 \int_{\mathcal{M}} N_i \, \text{ch}_3(V_i), \\
\mathcal{A}_{U(1)_i - SO(2M)^2} &\sim \frac{1}{12}\int_{\mathcal{M}} N_i \, c_1(V_i) \, c_2(T) + \int_{\mathcal{M}} N_i \, \text{ch}_3(V_i), \\
\mathcal{A}_{U(1)_i - Sp(2N_a)^2} &\sim -N_i \int_{\mathcal{M}} c_1(V_i) \wedge \overline{\gamma}
\end{aligned} \tag{4.20}$$

For the first two anomalies we assumed that $i \neq j$, with straightforward generalisations.

4.4.2 Green Schwarz mechanism including five-branes

The Green-Schwarz mechanism cancelling the cubic abelian and mixed abelian anomalies works in principle in a manner very similar to what we encountered in the context of the $E_8 \times E_8$ string with $U(N)$ bundles. The details of the four-dimensional counter terms, however, are quite different for the following two reasons: Firstly $SO(32)$ possesses, unlike $E_8 \times E_8$, an independent fourth-order Casimir. Secondly the five-brane part in the anomaly cancellation pattern is quite different in that the five-branes do not only affect the tadpole condition but also yield explicit contributions to the anomalies themselves via the $Sp(2N_a)$ valued chiral fermions. At the same time, we encounter no self-dual tensor fields on their world-volume which, in the context of the E_8 string, lead to new vertex and mass terms. There are, however, five-brane dependent vertex couplings, but no such mass terms, emerging from the Wess-Zumino coupling (4.6) to the bulk two-form $B^{(2)}$.

Since the knowledge of the dimensionally reduced Green-Schwarz and mass terms bore such rich fruit in the previous case and was essential far beyond the issue of anomaly cancellation, we will now present the resulting expressions, sticking closely to the philosophy and the notation of section (3.4.2).

In the $SO(32)$ case, dimensional reduction of the GS counter term (2.6) and (2.7) to four dimensions gives, upon splitting again the gauge field into a four-dimensional part F and the internal part $\overline{F}$,

$$S_{GS} = \frac{1}{(2\pi)^3\ell_s^2}\int B^{(2)} \wedge \frac{1}{144}\mathrm{Tr}(F\overline{F}^3) \tag{4.21}$$

$$- \frac{1}{(2\pi)^3\ell_s^2}\int B^{(2)} \wedge \frac{1}{2880}\mathrm{Tr}(F\overline{F}) \wedge \left(\frac{1}{15}\mathrm{Tr}\overline{F}^2 + \mathrm{tr}\overline{R}^2\right) \tag{4.22}$$

$$+ \frac{1}{(2\pi)^3\ell_s^2}\int B^{(2)} \wedge \left(\frac{1}{96}\mathrm{Tr}(F^2\overline{F}^2) - \frac{1}{43200}[\mathrm{Tr}(F\overline{F})]^2\right) \tag{4.23}$$

$$- \frac{1}{(2\pi)^3\ell_s^2}\int B^{(2)} \wedge \frac{1}{5760}\mathrm{Tr}(F^2) \wedge \left(\frac{1}{15}\mathrm{Tr}\overline{F}^2 + \mathrm{tr}\overline{R}^2\right) \tag{4.24}$$

$$+ \frac{1}{(2\pi)^3\ell_s^2}\int B^{(2)} \wedge \frac{1}{384}\mathrm{tr}R^2 \wedge \left(\mathrm{tr}\overline{R}^2 - \frac{1}{15}\mathrm{Tr}\overline{F}^2\right). \tag{4.25}$$

The specific prefactors of the traces follow from the general trace identities listed in appendix A.2.

The expressions (4.21), (4.22) are mass terms for the $U(1)$ gauge factors. (4.23) and (4.24) lead to vertex couplings of the axions with two gauge fields and finally the expression (4.25) gives rise to vertex couplings of the axions and two gravitons.

There are, of course, additional mass terms and vertex couplings originating in the cross kinetic term for H (3.61) in the ten-dimensional effective action as well as vertex couplings from the H5-brane action (4.6).

The traces occurring in the kinetic and counter terms are evaluated for the spectrum in table 4.1 in appendix A.3. With these results at hand, it is a simple task to collect the explicit mass and GS terms.

From (4.21) and (4.22) we find that the four-dimensional two-form field $b_0^{(2)}$ is rendered massive by the coupling to the abelian gauge fields given by

$$S^0_{mass} = \frac{1}{3(2\pi)^5\alpha'}\sum_{i=1}^{K} N_i \int_{\mathbb{R}_{1,3}} b_0^{(2)} \wedge f_i \int_X \left(\mathrm{tr}_{U(n_i)}\overline{F}^3 - \frac{1}{16}\mathrm{tr}_{U(n_i)}\overline{F} \wedge \mathrm{tr}\overline{R}^2\right). \tag{4.26}$$

In addition, (3.61) yields mass terms for the internal two-forms $b_k^{(2)}$,

$$S_{mass} = \frac{1}{(2\pi)^2\alpha'}\sum_{i=1}^{K}\sum_{k=1}^{h_{11}} N_i \int_{\mathbb{R}_{1,3}} (b_k^{(2)} \wedge f_i)\,[\mathrm{tr}_{U(n_i)}\overline{F}]_k. \tag{4.27}$$

The GS counter terms (4.23) and (4.24) provide the anomalous couplings of the axions to the external gauge fields and curvature,

$$\begin{aligned} S_{GS} &= \frac{1}{2\pi}\sum_{k=1}^{h_{11}}\int_{\mathbb{R}_{1,3}} b_k^{(0)} \wedge \Big\{\sum_{i=1}^{K}\left(\mathrm{tr}_{SU(N_i)}F^2 + N_i(f_i)^2\right)\left[\frac{1}{2}\,\mathrm{tr}_{U(n_i)}\overline{F}^2 - \frac{n_i}{96}\mathrm{tr}\overline{R}^2\right]_k \\ &\qquad -\frac{1}{192}\mathrm{tr}_{SO(2M)}F^2\,[\mathrm{tr}\overline{R}^2]_k \\ &\qquad +\frac{1}{384}\mathrm{tr}R^2\Big[\mathrm{tr}\overline{R}^2 - 4\sum_{i=1}^{K}N_i\mathrm{tr}_{U(n_i)}\overline{F}^2\Big]_k\Big\}. \end{aligned} \tag{4.28}$$

These are supplemented by couplings to the symplectic gauge fields and the curvature present in the H5-brane action (4.6),

$$S_{GS}^{H5} = -\frac{1}{4\pi}\sum_{k=1}^{h_{11}}\int_{\mathbb{R}_{1,3}}[\gamma_a]_k\; b_k^{(0)} \wedge \left(\mathrm{tr}_{Sp(2N_a)}F_a^2 - \frac{N_a}{24}\,\mathrm{tr}R^2\right) \tag{4.29}$$

with $[\gamma_a]_k = \int_{\gamma_a}\omega_k$.

Last but not least, from the kinetic term (3.61) for H we inherit the axio-dilaton vertex

$$S_{GS}^{0} = \frac{1}{8\pi}\int_{\mathbb{R}_{1,3}} b_0^{(0)} \wedge \Big(2\sum_{i=1}^{K}n_i\left(\mathrm{tr}_{SU(N_i)}F^2 + N_i(f_i)^2\right) + \mathrm{tr}_{SO(2M)}F^2 - \mathrm{tr}R^2\Big).$$

We can now follow the steps spelled out in section (3.4.2) and derive the various anomaly six-forms. For the mixed $U(1)_i - SU(N_j)$ anomaly, for instance, we find

$$\begin{aligned} \mathcal{A}_{U(1)_i-SU(N_j)^2} &\sim \frac{1}{6(2\pi)^6\alpha'}\,f_i \wedge \mathrm{tr}_{SU(N_j)}F^2 \\ &\quad \int_{\mathcal{M}}\Big(n_j\mathrm{tr}_{U(n_i)}\overline{F}^3 + 3\,\mathrm{tr}_{U(n_i)}\overline{F}\wedge\mathrm{tr}_{U(n_j)}\overline{F}^2 - \frac{n_j}{8}\mathrm{tr}_{U(n_i)}\overline{F}\wedge\mathrm{tr}\overline{R}^2\Big), \end{aligned} \tag{4.30}$$

which is just tailor-made to cancel the mixed $U(1)_i - SU(N_j)^2$ anomaly. The cancellation pattern for the remaining abelian-non-abelian, cubic abelian and mixed abelian-gravitational anomalies follows the same lines. Let us just list the resulting anomaly six-forms

$$\begin{aligned} \mathcal{A}_{U(1)_i-SO^2} &\sim \frac{1}{12(2\pi)^6\alpha'}\,f_i\wedge\mathrm{tr}_{SO(2M)}F^2\int_{\mathcal{M}}\Big(\mathrm{tr}_{U(n_i)}\overline{F}^3 - \frac{1}{8}\mathrm{tr}_{U(n_i)}\overline{F}\wedge\mathrm{tr}\overline{R}^2\Big), \\ \mathcal{A}_{U(1)_i-G_{\mu\nu}^2} &\sim -\frac{1}{12(2\pi)^6\alpha'}f_i\wedge\mathrm{tr}R^2\int_{\mathcal{M}}\Big(\mathrm{tr}_{U(n_i)}\overline{F}^3 - \frac{1}{16}\mathrm{tr}_{U(n_i)}\overline{F}\wedge\mathrm{tr}\overline{R}^2\Big), \\ \mathcal{A}_{U(1)_i-Sp(2N_a)} &\sim -\frac{1}{2(2\pi)^4\alpha'}\,f_i\wedge\mathrm{tr}_{SO(2N_a)}F^2\int_{\mathcal{M}}\mathrm{tr}\overline{F}\wedge\overline{\gamma}_a, \end{aligned}$$

$$\mathcal{A}_{U(1)_i - U(1)_j^2} \sim \frac{1}{6(2\pi)^6\alpha'} f_i \wedge f_j^2 \int_{\mathcal{M}} \Big(N_j \, (n_j \, \mathrm{tr}_{U(n_i)} \overline{F}_i^3 - \frac{n_j}{8} \mathrm{tr}_{U(n_i)} \overline{F} \wedge \mathrm{tr} \overline{R}^2) + \mathrm{tr}_{U(n_i)} \overline{F} \wedge \mathrm{tr}_{U(n_j)} \overline{F}^2 \Big) \qquad (4.31)$$

and point out that they are in perfect agreement with the field theoretic anomalies given in the previous section. As usual, the anomalous $U(1)$s are rendered massive and therefore remain in the low-energy domain as perturbative global symmetries. The situation parallels that in Type I [109] and heterotic $E_8 \times E_8$-theory, where the number of massive abelian factors is at least as large as that of the anomalous ones and in general given by the rank of the mass matrix $\mathcal{M}_{ki}$, as defined in (3.39),

$$\mathcal{M}_{ki} = \begin{cases} \frac{1}{(2\pi)^2\alpha'} (\mathrm{tr}_{U(n_i)} \overline{F})_k & \text{for } k \in \{1, \dots, h_{11}\} \\ \frac{1}{3(2\pi)^5\alpha'} \int_{\mathcal{M}} \Big(\mathrm{tr}_{U(n_i)} \overline{F}^3 - \frac{1}{16} \mathrm{tr}_{U(n_i)} \overline{F} \wedge \mathrm{tr} \overline{R}^2 \Big) & \text{for } k = 0. \end{cases} \qquad (4.32)$$

We stress once more that in contrast to the M5-brane of the $E_8 \times E_8$ theory, the H5-branes clearly do not contribute any mass terms due to the absence of additional tensor fields emerging from their worldvolume.

4.5 Non-universal gauge kinetic functions

Let us now derive the gauge kinetic functions [80, 116, 118, 119] as introduced in section (3.5), to which we refer for further conceptual details. With the definition of the complexified dilaton (3.96) and Kähler moduli (3.97) the full gauge kinetic functions for the $SU(N_i)$, $U(1)_i$ and $SO(2M)$ groups can be read off from their imaginary parts in (4.28) and (4.30) to be

$$\begin{aligned} \mathrm{f}_{SU(N_i)} &= n_i \, S + \sum_{k=1}^{h_{11}} T_k \Big(\mathrm{tr}_{U(n_i)} (\overline{F}^2)_k - \frac{n_i}{48} (\mathrm{tr} \overline{R}^2)_k \Big), \\ \mathrm{f}_{U(1)_i} &= \frac{1}{2} \, N_i \mathrm{f}_{SU(N_i)}, \\ \mathrm{f}_{SO(2M)} &= \frac{1}{2} S - \frac{1}{96} \sum_{k=1}^{h_{11}} T_k \, (\mathrm{tr} \overline{R}^2)_k. \end{aligned} \qquad (4.33)$$

As in the $E_8 \times E_8$ case the relative normalisations for the different gauge groups are a consequence of the trace identities, see in this case appendix A.3. Again, the abelian gauge couplings receive an extra factor of $\frac{1}{2}$ as compared to the non-abelian ones due to the canonical normalisation of the non-abelian second order Casimir. In addition, the gauge kinetic functions for the symplectic factors are

$$\mathrm{f}_{Sp(2N_a)} = \frac{1}{2\pi \ell_s^2} \int_{\gamma_a} (J - iB) \,, \qquad (4.34)$$

as we find from (4.29).

Note that the real part of the gauge kinetic function are positive definite by definition. Therefore, as for the $E_8 \times E_8$ theory, requiring positivity of the expressions (4.33) in the perturbative regime, $g_s \ll 1$ and internal radii much bigger than the string scale, imposes extra conditions on the allowed bundles. Concretely, reality of the one-loop corrected $SU(N_i)$ and $U(1)_i$ gauge couplings is guaranteed provided that in this regime

$$\frac{n_i}{3!} \int_X J \wedge J \wedge J - 2\, g_s^2\, \ell_s^4 \int_X J \wedge \Big(\mathrm{ch}_2(V_i) + \frac{n_i}{24}\, c_2(T) \Big) > 0. \qquad (4.35)$$

The analogous constraint for the $SO(2M)$ group, where the term $\mathrm{ch}(V_i)$ is absent, is normally trivially satisfied, since for all manifolds we will encounter $\int_{\mathcal{M}} J \wedge c_2(T) < 0$. The real part of (4.34) is always positive as long as the Kähler form J lies in the Kähler cone. This is a consequence of the minus sign in the Wess-Zumino coupling (4.6) and actually serves as its justification.

Away from the small coupling and large radii limit one expects both world-sheet and stringy instanton corrections to the gauge kinetic functions [118].

In contrast to the $E_8 \times E_8$ construction, no off-diagonal couplings among abelian factors occur. Even more strikingly, the tree-level and one-loop corrected non-abelian and abelian gauge couplings of an observable $SU(N_i)$ and $U(1)_i$ gauge factor only depend on the internal gauge flux in the corresponding $U(n_i)$. Since we used the same decomposition of $SO(32)$ that naturally appears for intersecting D-branes, S-duality tells us that after all this result is not surprising. There, each stack of D-branes comes with its own gauge coupling determined by the size of the three-cycle the D6-branes are wrapping around.

4.6 Fayet-Iliopoulos terms

We conclude our general discussion of the $SO(32)$ theory with the derivation of the Fayet-Iliopoulos terms generated by the massive $U(1)$ symmetries. Our methods largely parallel the ones applied in the context of the $E_8 \times E_8$ theory. We will therefore be comparatively brief and refer to section (3.6) for more information. Suffice it here to recall that the starting point for the derivation of the FI terms is the the gauge invariant Kähler potential

$$\begin{aligned} \mathcal{K} \;=\; \frac{M_{pl}^2}{8\pi} \Big[&- \ln\Big(S + S^* - \sum_x Q_0^x\, V_x \Big) - \ln\Big(- \sum_{i,j,k=1}^{h_{11}} \frac{d_{ijk}}{6} \Big(T_i + T_i^* - \sum_x Q_i^x\, V_x \Big) \\ & \Big(T_j + T_j^* - \sum_x Q_j^x\, V_x \Big) \Big(T_k + T_k^* - \sum_x Q_k^x\, V_x \Big) \Big) \Big]. \end{aligned} \qquad (4.36)$$

This is precisely as for the E_8 string, see (3.120), except the fact that there are no contributions from tensor fields living on the five-brane, of course. The charges

Q_k^x are again defined via

$$S_{mass} = \sum_{x=1}^{K} \sum_{k=0}^{h_{11}} \frac{Q_k^x}{2\pi\alpha'} \int_{\mathbb{R}_{1,3}} f_i \wedge b_k^{(2)} \tag{4.37}$$

and are encoded in the mass terms (4.26) and (4.27).

We can therefore straightforwardly derive the coefficients ξ_x of the FI-terms from the gauge invariant Kähler potential $\mathcal{K}$ via the relation

$$\frac{\xi_x}{g_x^2} = \frac{\partial \mathcal{K}}{\partial V_x}\bigg|_{V=0}. \tag{4.38}$$

Inserting the concrete expressions for the charges eventually leads to the conclusion that the FI terms vanish if and only if

$$\frac{1}{2}\int_{\mathcal{M}} J \wedge J \wedge \mathrm{tr}_{U(n_i)}\overline{F} - \frac{2\, g_s^2\, \ell_s^4}{3!}\int_{\mathcal{M}} \left(\mathrm{tr}_{U(n_i)}\overline{F}^3 - \frac{1}{16}\mathrm{tr}_{U(n_i)}\overline{F} \wedge \mathrm{tr}\overline{R}^2\right) = 0 \tag{4.39}$$

for each external $U(1)_i$ factor separately. It is intriguing that, as expected from the intersecting D-brane picture, the FI-term for $U(1)_i$ only depends on the corresponding internal vector bundle with structure group $U(n_i)$. This is to be contrasted with the analogous expression (3.122) for the $E_8 \times E_8$ string, where the one-loop correction of the FI term involves the second Chern classes of all vectors bundles embedded into the same E_8 factor as the abelian gauge group under investigation. Note that the one-loop correction in (4.39) involves the cubic term $\mathrm{tr}_{U(n_i)}\overline{F}^3$. This can be traced back to the fact that in contrast to E_8 the group $SO(32)$ has an independent fourth order Casimir operator. It implies the well-known result that for the $SO(32)$ heterotic string a bundle with structure group $SU(N)$ generates a non-vanishing one-loop FI-term [124][5]. Again, away from the small string coupling and large radii limit one expects additional non-perturbative world-sheet and string instanton contributions to (4.39). We will further investigate the implications of the supersymmetry condition (4.39) of a vanishing FI term in section (4.7.3).

4.7 S-duality to the Type I string

An immediate question concerns the relation between the phenomena studied in the context of the $SO(32)$ heterotic and the S-dual Type I framework. Our aim is therefore to apply Heterotic-Type I S-duality to the equations derived by now and to shed new light on their significance by comparison with known results on the Type I side. The main conclusion of this analysis will be the identification of

[5] There exist $SU(N)$ bundles, however, with vanishing FI terms if the bundle data happen to be such that $\mathrm{ch}_3(V) = 0$.

the supersymmetry conditions (4.39) and (4.35) as the integrability condition for a deformed Hermitian Yang-Mills equation. The corresponding statement for the $E_8 \times E_8$ string has been conjectured in section (3.6.3) and is further supported by this observation. Before we can tackle this issue in section (4.7.3), however, it is indispensable to derive the precise form of the higher-order counter terms in the Type I effective action. In particular, we need to investigate the full set of S-duality transformation rules which relate the gauge kinetic functions and FI terms to their Type I/Type II B counterparts. As a subtlety arising in the Type I effective action, we are always free to absorb an additive shift in the dilaton by a redefinition of α'. For the purpose of quantitative statements we need to make sure that all terms in the kinetic action on the Type I and heterotic side are canonically normalized before they are transformed into one another by S-duality. We therefore cannot help it but pause for a moment and first derive the S-dual Type I action together with its precise relationship to the heterotic action presented in (2.1). Although the contents of this section is well-known in principle, we consider it enlightening to present the arguments leading to the final Type I action (4.50) - not only in view of the remarkable confusion in the literature about the proper normalisation of the Green-Schwarz terms. Along the way, we will also provide the justification for the $SO(32)$ H5-brane action postulated in (4.6) as well as for our normalisation (2.6) of the Green-Schwarz counter terms.

4.7.1 The Type I effective action

We take as our starting point the relevant bosonic parts of the ten-dimensional Type IIB effective action including the Chern-Simons terms of a stack of M D9-branes [12],

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} R - \frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} G_3 \wedge \star G_3 \\ &\quad - \frac{1}{2\, g_Y^2}\int_{\mathcal{M}^{(10)}} e^{-\phi_{10}}\, \mathrm{tr}_{U(M)}\left[F\wedge \star F\right] + \mu_9 \int_{\mathcal{M}^{(10)}} \sum_n C_{2n+2}\wedge \mathrm{ch}(i\mathcal{F})\wedge \sqrt{\hat{A}}, \end{aligned} \tag{4.40}$$

where $\kappa_{10} = \frac{1}{2}(2\pi)^7(\alpha')^4$, $\mu_9 = \frac{1}{(2\pi)^9(\alpha')^5}$, $\frac{1}{g_Y^2} = (2\pi\alpha')^2\mu_9$, $\mathcal{R} = -i\ell_s^2 R$ and

$$\begin{aligned} \mathrm{ch}_k(i\mathcal{F}) &= \frac{\ell_s^{2k}}{k!\,(2\pi)^k}\,\mathrm{tr}_{U(M)} F^k, \\ \sqrt{\hat{\mathcal{A}}\,(\mathcal{R})} &= 1 - \frac{\ell_s^4}{96\,(2\pi)^2}\,\mathrm{tr}R^2 + \frac{\ell_s^8}{18432\,(2\pi)^4}\left(\mathrm{tr}R^2\right)^2 + \\ &\quad \frac{\ell_s^8}{11520\,(2\pi)^4}\left(\mathrm{tr}R^4\right). \end{aligned} \tag{4.41}$$

The traces are over the fundamental representation of the $U(M)$ gauge theory living on the D9-branes and of $SO(1,9)$, respectively. $G_3 = dC_2$ denotes the

Ramond-Ramond (RR) three-form field strength. Its magnetic dual is the six-form potential C_6 satisfying $\star_{10} dC_6 = dC_2$. Note that in contrast to the heterotic string, there are no factors of $e^{2\phi_{10}}$ affecting this magnetic-electric duality transformation. In (4.40) and in the definition of G_3 we omitted all additional kinetic and Chern-Simons terms involving the RR forms C_0 and C_4 of the full Type IIB action.

In compactifying the ten-dimensional theory on $\mathbb{R}^{1,3} \times \mathcal{M}$, we allow in addition for stacks of N_a $D5$-branes wrapping the holomorphic 2-cycles γ_a on $\mathcal{M}$. They, too, give rise to $U(N_a)$ gauge groups on their worldvolume. The Chern-Simons action on the D5-branes reads

$$S^{CS}_{D5_a} = -\mu_5 \int_{\mathbb{R}^{1,3}\times\gamma_a} \left(\sum_{n=0}^{1} C_{4n+2}\right) \wedge \left(N_a + \frac{\ell_s^4}{2\,(2\pi)^2}\mathrm{tr}_{U(N_a)}(F_a^2)\right) \wedge \frac{\sqrt{\hat{\mathcal{A}}\,(\mathrm{T}\gamma_a)}}{\sqrt{\hat{\mathcal{A}}\,(\mathrm{N}\gamma_a)}} \tag{4.42}$$

with $\mu_5 = \frac{1}{(2\pi)^5\alpha'^3}$. Here $\mathrm{T}\gamma_a$ denotes the tangent bundle and $\mathrm{N}\gamma_a$ the normal bundle of the D5-brane in $\mathcal{M}$.

The type I theory emerges after modding out the Type IIB string by the world-sheet parity transformation $\Omega : (\sigma,\tau) \to (-\sigma,\tau)$. At the level of the effective action, this first of all means that we project out the anti-invariant RR potentials C_0 and C_4 and introduce the Ω image of the stack of branes, i.e a stack of M $D9$-branes and stacks of N_a D5-branes, each with the negative respective field strength $-F$.

To keep further track of the projection, we divide the resulting action by a factor of two. Next we need to take into account that the orientifold projection results in a tadpole for the Ramond-Ramond ten-form, C_{10}, and, since the Calabi-Yau is generically curved, an induced tadpole for the six-form C_6.

Quantitatively, these tadpoles are given by the CS-terms on the $O9$-plane [137, 143]

$$S^{CS}_{O9} = -32\,\mu_9 \int_{\mathcal{M}^{(10)}} \left(\sum_{n=0}^{2} C_{4n+2}\right) \wedge \sqrt{\hat{\mathcal{L}}\left(\frac{\mathcal{R}}{4}\right)}. \tag{4.43}$$

The Hirzebruch genus $\hat{\mathcal{L}}$ is defined as

$$\sqrt{\hat{\mathcal{L}}\left(\frac{\mathcal{R}}{4}\right)} = 1 + \frac{\ell_s^4}{192\,(2\pi)^2}\,\mathrm{tr}R^2 + \frac{\ell_s^8}{73728\,(2\pi)^4}\,\left(\mathrm{tr}R^2\right)^2 - \frac{\ell_s^8}{92160\,(2\pi)^4}\,\left(\mathrm{tr}R^4\right). \tag{4.44}$$

In particular, extracting the top form contributions both from the Wess-Zumino coupling of the $D9$-brane and of the orientifold,

$$S_{C_{10}} = \mu_9 \int_{\mathcal{M}^{(10)}} \left(\frac{1}{2}\,2M - 32\right) C_{10}, \tag{4.45}$$

clearly shows that the $D9$-brane tadpole is cancelled precisely for $M = 16$.

The preliminary Type I action therefore becomes[6]

$$
\begin{aligned}
S_I \;=\;& \frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} R - \frac{1}{8\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} G_3\wedge\star G_3 \\
& -\frac{1}{2\,g_Y^2}\int_{\mathcal{M}^{(10)}} e^{-\phi_{10}}\mathrm{tr}_{U(16)}\left[F\wedge\star F\right] + \mu_9\int_{\mathcal{M}^{(10)}}\sum_n C_{4n+2}\wedge \mathrm{ch}(i\mathcal{F})\wedge\sqrt{\hat{A}} \\
& -32\,\mu_9\int_{\mathcal{M}^{(10)}}\left(\sum_{n=0}^{2} C_{4n+2}\right)\wedge\sqrt{\hat{\mathcal{L}}\left(\frac{\mathcal{R}}{4}\right)} \qquad (4.46)\\
& -\mu_5\int_{\mathbb{R}^{1,3}\times\gamma_a}\left(\sum_{n=0}^{1} C_{4n+2}\right)\wedge\left(N_a + \frac{\ell_s^4}{2\,(2\pi)^2}\mathrm{tr}_{U(N_a)}(F_a^2)\right)\wedge\frac{\sqrt{\hat{A}\,(\mathrm{T}\gamma_a)}}{\sqrt{\hat{A}\,(\mathrm{N}\gamma_a)}}.
\end{aligned}
$$

For brevity we have omitted the kinetic term for the gauge fields on the five-branes.

Now from a detailed worldsheet analysis, we know that due to the Ω-projection the gauge group on the D9-branes is actually no more $U(16)$ but rather $SO(32)$ and likewise the D5-branes carry gauge group $SP(2N_a)$ instead of $U(N_a)$ [136]. We therefore re-express the traces over the fundamental representation of the unitary groups by the ones over $SO(32)$ and $Sp(2N_a)$, respectively, with the help of the trace identities

$$
\begin{aligned}
\mathrm{tr}_{U(16)}[F^2] &= \frac{1}{2}\mathrm{tr}_{SO(32)}[F^2], \qquad \mathrm{tr}_{U(16)}[F^4] = \frac{1}{48}\mathrm{Tr}_{SO(32)}[F^4], \\
\mathrm{tr}_{U(N_a)}[F^2] &= \frac{1}{2}\mathrm{tr}_{Sp(2N_a)}[F^2], \qquad (4.47)
\end{aligned}
$$

with $\mathrm{Tr}_{SO(32)}$ denoting, as always, the trace in the adjoint representation.

We see, however, that the kinetic terms, including the ones for the Yang-Mills fields, are not yet canonically normalized. This can be remedied by rescaling

$$
C_2 \to 2\sqrt{2}\,C_2, \quad \alpha' \to \sqrt{2}\,\alpha', \quad e^{\phi_{10}} \to \frac{1}{2\sqrt{2}}\,e^{\phi_{10}}. \qquad (4.48)
$$

By Hodge duality this also implies

$$
C_6 \to 2\sqrt{2}\,C_6. \qquad (4.49)
$$

After this redefinition we carefully collect all the Chern-Simons terms and eventually arrive at the action

$$
S_I \;=\; \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} e^{-2\phi_{10}} R - \frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}^{(10)}} G_3\wedge\star G_3
$$

[6]Note that at this stage the D-brane action is formally unaltered as compared to the original Type IIB action. This is a consequence of dividing the latter by a factor of 2 *after* adding the Ω-image of the branes and furthermore identifying the branes with their orientifold image.

$$
\begin{aligned}
&-\frac{1}{2\,\overline{g}_Y^2}\int_{\mathcal{M}^{(10)}} e^{-\phi_{10}}\,\mathrm{tr}_{SO(32)}\,[F\wedge\star F]\\
&+\frac{2}{4\kappa_{10}^2}\frac{\alpha'}{4}\int_{\mathcal{M}^{(10)}} C_6\wedge\left(\mathrm{tr}_{SO(32)}[F^2]-\mathrm{tr}[R^2]-4(2\pi)^2\sum_a N_a\overline{\gamma}_a)\right)\\
&-\mu_5\int_{\mathbb{R}^{1,3}\times\gamma_a} C_2\wedge\left(\frac{\ell_s^4}{2\,(2\pi)^2}\mathrm{tr}_{Sp(2N_a)}(F_a^2)\right)\wedge\frac{\sqrt{\hat{\mathcal{A}}\,(\mathrm{T}\gamma_a)}}{\sqrt{\hat{\mathcal{A}}\,(\mathrm{N}\gamma_a)}}\\
&+\frac{1}{24\,(2\pi)^5\,\alpha'}\int_{\mathcal{M}^{(10)}} C_2\wedge X_8,
\end{aligned}
\tag{4.50}
$$

where in the expressions involving $\hat{\mathcal{A}}\,(\mathrm{T}\gamma_a)$ and $\hat{\mathcal{A}}\,(\mathrm{N}\gamma_a)$ we now define $\mathcal{R} = -i\sqrt{2}\ell_s^2 R$ to keep track of the rescaling of α'. Also, we introduced the Type I gauge coupling $\frac{1}{\overline{g}_Y^2} = \frac{1}{2(2\pi)^7(\alpha')^3}$. The anomaly eight-from X_8 is indeed just the one we encountered in the Green-Schwarz mechanism in the heterotic theory and given by equation (2.7).

This action is really S-dual to the heterotic string action (2.1) by an application of the transformation rules

$$
\begin{aligned}
g_s^I &= (g_s^H)^{-1},\\
J^I &= (g_s^H)^{-1}J^H
\end{aligned}
\tag{4.51}
$$

and letting $C^{(2)}\to B^{(2)}$.

In particular, this justifies the concrete form and normalisation (4.6) of the anomalous Wess-Zumino coupling of the $SO(32)$ heterotic five-brane, which after all was essential to derive the correct Green-Schwarz terms. Moreover, we have explicitly convinced ourselves how on the Type I side the anomaly cancelling Green-Schwarz counter terms arise from the Chern-Simons couplings of the $D9$- and $D5$-branes and the $O9$-planes. They appear at first order in open string perturbation theory, as we see by comparison with the Yang-Mills kinetic terms at order $e^{-\phi_{10}} = g_{open}^{(-1)}$. Along the way, this supports the normalisation (2.6) of the one-loop GS-terms with respect to the tree-level effective action on the heterotic side.

It is clear that we can proceed *precisely* as for the $SO(32)$ heterotic string and consider gauge background fields of the form (4.1) on the internal part of the spacetime-filling $D9$-branes such that the original $SO(32)$ gauge symmetry is broken correspondingly. This is, of course, nothing other than the introduction of magnetized $D9$-branes. The resulting global consistency conditions for the internal gauge fields, the spectrum and cohomology groups as well as the details of the GS mechanism follow by copying the steps spelled out for the heterotic setup. Note in particular that the requirement that the rank of the heterotic gauge group be 16 translates into the cancellation of the $D9$-tadpole, whereas the Bianchi identity for H or anomaly cancellation condition in the heterotic theory corresponds to $D5$-tadpole cancellation in Type I. In all, this certainly puts the framework of Type IIB magnetized D-branes conceptually on just the same footing as the dual heterotic model building with gauge instanton backgrounds. We anticipated these parallels already in section 4.2 when pointing out that the massless spectrum of the $SO(32)$ string with unitary bundles and that of the Type I/IIB framework with magnetized D9-branes are in one-to-one correspondence. It is furthermore clear that the magnetized D-brane picture is by no means restricted to turning on just the diagonal abelian part of the gauge fields on the worldvolume of the branes. All statements about the $SO(32)$ heterotic string with unitary bundles should therefore also be read as the generalisation of the setup of magnetized D-branes to non-abelian background bundles on their worldvolume.

4.7.2 The gauge couplings for Type I

After this little exercise, we are finally in a position to take a fresh look at the supersymmetry conditions (4.39) and (4.35) by analysing them in the S-dual Type I setup. To do so, we can either perform the analogous computation of the gauge kinetic function and FI terms as they follow from dimensional reduction of the Type I action (4.50) - or simply apply the S-duality transformation rules (4.51) to the heterotic results. We go for the second option and write the expression for the gauge couplings in a way which is more suitable for the S-duality transformation. The real part of the holomorphic gauge kinetic function $\mathrm{f}_{SU(N_i)}$ can be cast into the form

$$\mathrm{Re}(\mathrm{f}^H_{SU(N_i)}) = \frac{1}{\pi \ell_s^6}\left[\frac{n_i}{3!}\, g_s^{-2}\int_{\mathcal{M}} J\wedge J\wedge J - (2\pi\alpha')^2\int_{\mathcal{M}} J\wedge\left(\mathrm{tr}_{U(n_i)}\overline{F}^2 - \frac{n_i}{48}\mathrm{tr}\overline{R}^2\right)\right]. \tag{4.52}$$

For reasons which will become clear momentarily, we will actually be interested in the S-dual expressions normalized with respect to the original Type IIB theory from which Type I arises after the orientifold projection. As we have just discussed this requires that we rescale, after applying (4.51),

$$\alpha' \to \frac{1}{\sqrt{2}}\,\alpha', \quad e^{\phi_{10}} \to 2\sqrt{2}\, e^{\phi_{10}}. \tag{4.53}$$

The resulting Type I expressions are to be read as defined with respect to the canonically normalized Type IIB action. In this sense, the gauge couplings S-dual to (4.52) are

$$\mathrm{Re}(\mathrm{f}^I_{SU(N_i)}) = \frac{1}{\pi \ell_s^6 g_s}\left[\frac{n_i}{3!}\int_{\mathcal{M}} J\wedge J\wedge J - \frac{(2\pi\alpha')^2}{2}\int_{\mathcal{M}} J\wedge\left(\mathrm{tr}_{U(n_i)}\overline{F}^2 - \frac{n_i}{48}\mathrm{tr}\overline{R}^2\right)\right] \tag{4.54}$$

on the Type I/IIB side. Most importantly, the one-loop term has now become a perturbative α'-correction to the tree-level gauge coupling.

4.7.3 The non-abelian MMMS condition

The same S-duality relations (4.51), (4.53) applied to the FI-terms (4.39) yield

$$\frac{1}{2}\int_{\mathcal{M}} J\wedge J\wedge \mathrm{tr}_{U(n_i)}\overline{F} - \frac{(2\pi\alpha')^2}{3!}\int_{\mathcal{M}}\left(\mathrm{tr}_{U(n_i)}\overline{F}^3 - \frac{1}{16}\mathrm{tr}_{U(n_i)}\overline{F}\wedge \mathrm{tr}\overline{R}^2\right) = 0 \tag{4.55}$$

on the Type I/ IIB side, where the second term is again a perturbative α'-correction. We can combine the gauge kinetic function and the FI-term into a single complex quantity, the central charge

$$\mathcal{Z} = \int_{\mathcal{M}} \mathrm{tr}_{U(n)}\left[e^{-i\frac{\pi}{2}}\left(e^{-iJ\,\mathrm{id}+\mathcal{F}}\sqrt{\hat{A}(\mathcal{M})}\right)\right], \tag{4.56}$$

defined in terms of $\mathcal{F} = 2\pi\alpha'\overline{F}$. The gauge coupling and the FI-term are seen to be proportional to the real and imaginary part, respectively, of $\mathcal{Z}$.

In the case of abelian D9-branes in Type IIB we know that one can introduce an additional phase parameterising which $\mathcal{N}=1$ supersymmetry of the underlying $\mathcal{N}=2$ bulk supersymmetry is preserved by the brane. Therefore, the general Type IIB supersymmetry condition is

$$\begin{aligned} \mathrm{Im}\left(\int_{\mathcal{M}} \mathrm{tr}_{U(n)}\left[e^{-i\varphi}\, e^{-iJ\,\mathrm{id}+\mathcal{F}}\sqrt{\hat{A}(\mathcal{M})}\right]\right) &= 0, \\ \mathrm{Re}\left(\int_{\mathcal{M}} \mathrm{tr}_{U(n)}\left[e^{-i\varphi}\, e^{-iJ\,\mathrm{id}+\mathcal{F}}\sqrt{\hat{A}(\mathcal{M})}\right]\right) &> 0. \end{aligned} \tag{4.57}$$

As usual in Type IIB theory coupled to a brane, we have now defined $\mathcal{F} = 2\pi\alpha'\overline{F} + B\,\mathrm{id}$, thus taking into account the fact that for open strings only this combination is a gauge invariant quantity. Clearly, on the right-hand side of the first equation in (4.57), there might appear a non-vanishing function of the charged matter fields as previously in (3.124), but having discussed these terms at length in section (3.6.2) we can here just assume them to vanish for simplicity.

Note that (4.56) is precisely the perturbative part of the expression for the central charge as it appears in the Π-stability condition [78] for general B-type branes[7]. To our knowledge the form of this expression has never been derived from first principles. Rather, we understand that the central charge has been designed in such a way as to keep in analogy with the well-known RR-charge of the B-type-brane as seen in the Chern-Simons action - it is simply assumed that in the geometric limit, the two quantities coincide [144].

We find it quite interesting though not unexpected that, starting from the well-known Green-Schwarz anomaly terms, our four-dimensional effective field

[7] This is true at least for space filling branes in case we consider also non-abelian fields. Of course our analysis has nothing to say about lower-dimensional non-abelian branes.

theory analysis leads precisely to the perturbative part of the Π-stability condition for B-type branes.

Equation (4.57) is also the integrability condition for the non-abelian generalisation of the MMMS equation for D9-branes in a curved background. The abelian version of this equation has been proven (without the curvature terms) in [145] starting from the DBI action of a single D-brane and it has been confirmed by a world-sheet calculation in [146]. Up to now it is strictly speaking only a conjecture that it can easily be generalised to (4.57) [127, 147]. However, our analysis relies exclusively on quantities of the four-dimensional $\mathcal{N} = 1$ effective supergravity theory, the one-loop FI-term and the holomorphic gauge kinetic function. In particular, the non-renormalization theorems guarantee the absence of further perturbative corrections, thus dictating (4.57) as the perturbatively exact integrability condition at least for D9-branes. The absence of a stringy one-loop correction was shown in [128]. Of course, there will be additional non-perturbative corrections, which in the $g_s \to 0$ limit make out the complete Π-stability expression [78].

As we discussed in detail in section (3.6.3) in the context of the E_8-string, the integrability condition (4.57) is not yet sufficient for supersymmetry preservation, but has to be supplemented by the correct stability condition. This will be the direct generalisation of μ-stability, which is the valid notion of stability only at leading order in α' and g_s.

We can now largely repeat the analysis of section (3.6.3): First, we have to know the local supersymmetry equation for non-abelian D9-branes underlying (4.57). All we can say for sure starting from (4.57) is that the local SUSY condition for D9-branes has to be of the form

$$\left[\mathrm{Im}\left(e^{-i\varphi}\, e^{-iJ\,\mathrm{id}+\mathcal{F}}\,\sqrt{\hat{A}(\mathcal{M})}\right)\right]_{top} + d\alpha_5 = 0,$$

where α_5 is a globally defined 5-form so that $d\alpha_5$ is gauge covariant. At least for compactifications on genuine Calabi-Yau manifolds, where $dJ = 0$ and $dH = 0$, we cannot find any 5-form of this type which is also invariant under the axionic $U(1)$ gauge symmetry $B \to B + d\chi$, $A \to A - \chi$ and does lead to a non-vanishing $d\alpha_5$.

Therefore, we conclude that the possible correction $d\alpha_5$ is absent and that indeed the local supersymmetry condition is given by

$$\left[\mathrm{Im}\left(e^{-i\varphi}\, e^{-iJ\,\mathrm{id}+\mathcal{F}}\,\sqrt{\hat{A}(\mathcal{M})}\right)\right]_{top} = \pi(V)\mathrm{id}\,\mathrm{vol}\mathcal{M} \qquad (4.58)$$

and in addition

$$\pi(V) = 0 \qquad (4.59)$$

or suitable generalisations if one allows for a cancellation of the FI terms against chiral charged matter fields. This is just the counterpart of the full Hermitian

Yang-Mills equation (3.128) we proposed in the context of the $E_8 \times E_8$ theory. Likewise, the π-slope is now defined as

$$\pi(V) \equiv \frac{1}{\mathrm{rk}(V)} \mathrm{Im} \left(\int_{\mathcal{M}} \mathrm{tr}_{U(n)} \left[e^{-i\varphi} e^{-iJ\,\mathrm{id} + \mathcal{F}} \sqrt{\hat{A}(\mathcal{M})} \right] \right) . \tag{4.60}$$

A strictly perturbative (in the sense explained in section 3.128) notion of stability relevant for (4.58) has been analysed in [127] and been called π-stability (to stress that it is only the perturbative part of Π-stability). In particular, the authors have shown that for α' smaller than a critical value depending on the bundle V, equation (4.58) has a unique solution precisely if the bundle is stable with respect to the deformed slope $\pi(V)$. This actually serves as additional support for our corresponding conjecture regarding λ-stability in section (3.6.3). As the authors of [127] have also shown, in this perturbative sense μ-stability implies π-stability. However, we face the same problem that this notion of stability assumes that the terms in $\pi(V)$ at zeroth order in α' dominate over the higher order corrections in the extreme perturbative regime. This may be in conflict with the DUY equation (4.59). For a detailed discussion of this point we refer back to section (3.128). We hasten to anticipate in this context that all concrete examples we will construct in the sequel are not affected by this caveat since the deformation of the slope vanishes and are therefore supersymmetric provided they are μ-stable. To prove supersymmetry of non-abelian bundles in the more general situation it is necessary to find a stability criterion which is not only valid for arbitrarily small higher order corrections.

Chapter 5

Stable holomorphic $U(n)$ bundles on elliptically fibered Calabi-Yau manifolds

We have by now made extensive use of the equivalence of the following two types of objects: solutions of the Hermitian Yang-Mills equations for a connection with values in the gauge group G on the one hand and holomorphic stable bundles with structure group G (or, rather, the complexification of G) on the other. Our interest has been in $G = SU(n)$ or $U(n)$, but the correspondence is not restricted to this choice. We have seen that the Hermitian Yang-Mills equation for both heterotic theories receives perturbative corrections arising precisely at one-loop in string perturbation theory. In Type I/IIB theory, by contrast, the corrections are perturbative in α'. In any case, the stability condition constraining the holomorphic bundles is modified and no longer given by μ-stability, but by λ- and π-stability, respectively. Since μ-stability implies λ-stability in the perturbative limit, we can therefore, as far as concrete applications in model building are concerned, stick to the more familiar μ-stability constraint. As a result, the question of prime importance both to heterotic and Type I/IIB model building in this context concerns the construction of suitable stable holomorphic vector bundles over a Calabi-Yau threefold $\mathcal{M}$. The classification and construction of the most general such bundles is a challenging and unsolved mathematical problem. Luckily, for the special case that the Calabi-Yau manifold is elliptically fibered, a large class of μ-stable holomorphic G-bundles is at our disposal thanks to the spectral cover construction, pioneered by Friedman, Morgan and Witten (FMW) in [40] and Donagi [41] and further developed by several authors [47, 69, 103, 148–150]. This will be the playground to provide concrete examples of the general theory presented in the previous chapter, the main focus being eventually on phenomenologically interesting model building. In order to make this work as self-contained as possible and to introduce our notation, we will first review very briefly the main ingredients of this mathematical construction relevant for our applications. In doing so, we will rely on the original literature [40, 69, 149] to which we refer for further details.

5.1 Elliptically fibered Calabi-Yau manifolds

An elliptically fibered complex three-fold $\mathcal{M}$ is given by a complex two-surface B, the base space, together with an analytic map

$$\pi : \mathcal{M} \to B, \tag{5.1}$$

where fibers over each point b in the base,

$$E_b = \pi^{-1}(b), \tag{5.2}$$

are elliptic curves. Recall that an elliptic curve is a two-torus with a complex structure inducing an abelian group law. In particular it contains a distinguished point p acting as the zero element in this group.
We require the fibration $\mathcal{M}$ to admit a global section $\sigma : B \to \mathcal{M}$, assigning to every point in the base $b \in B$ the zero element $\sigma(b) = p \in E_b$ on the fiber[1]. This section embeds the base as a submanifold into $\mathcal{M}$ and we will often not distinguish between B as a complex two-fold and $\sigma(B)$ as a four-cycle in $\mathcal{M}$. The associated homology class in $H_4(\mathcal{M}, \mathbb{Z})$ then intersects the fibre class precisely once. It will be useful to introduce also the class in $H^2(\mathcal{M}, \mathbb{Z})$ Poincaré dual to the class of $\sigma(B)$. In slight abuse of notation, it will also be referred to as σ. The respective meaning will hopefully always be clear from the context. Its cohomological self-intersection can be proven to be [40]

$$\sigma \cdot \sigma = -\sigma \cdot \pi^*(c_1(B)). \tag{5.3}$$

Likewise, we introduce $F \in H^4(\mathcal{M}, \mathbb{Z})$ as the Poincaré dual to the fibre class. The fact that the base class intersects the class of the generic fibre once is reflected in the cohomological intersection form

$$\sigma \cdot F = 1. \tag{5.4}$$

This shows that F is actually the Hodge dual to the two-form σ. Now that we are at it, we state for later purposes the simple fact that the intersection form of the pull-back to $\mathcal{M}$ of two classes α and β in $H^2(B, \mathbb{Z})$ is given by the pull-back of the intersection on B,

$$\pi^*(\alpha) \cdot \pi^*(\beta) = \pi^*(\alpha \cdot \beta) = (\alpha \cdot \beta)\, F. \tag{5.5}$$

Often we will simply omit the π^* when talking about the pull-back of two-forms to $\mathcal{M}$ and likewise the F in expressions of the form above.

[1] See, however, [47, 48, 103] for the spectral cover construction on elliptically fibered three-folds which admit *two* sections.

Let us now turn our attention to the elliptic fibre. Elliptic curves can be described as the hyperplane in $\mathbb{CP}^2$ defined by the homogeneous Weierstrass equation

$$zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3, \tag{5.6}$$

where x, y, z are homogeneous coordinates on $\mathbb{CP}^2$ and g_2 and g_3 define the complex structure. When we fiber the elliptic curve over the base, this means that the x, y, z and likewise g_2 and g_3 must be promoted to global sections of a line bundle $\mathcal{L}$ on B, and the choice of $\mathcal{L}$ defines the fibration.

We can actually take $\mathcal{L}$ to be the conormal bundle to the section $\sigma(B)$ so that the fibration is now defined by the specific choice of σ . Then x, y, z are sections of $\mathcal{L}^2$, $\mathcal{L}^3$ and $\mathcal{O}$ whereas g_2 and g_3 appear as sections of $\mathcal{L}^4$ and $\mathcal{L}^6$, respectively. If the fibration $\mathcal{M}$ is to be Calabi-Yau, the first Chern class of the tangent bundle T must vanish,

$$c_1(T) = 0. \tag{5.7}$$

As shown e.g. in [149], this implies $\mathcal{L} = K_B^{-1}$, where K_B is the canonical bundle of the base space. It follows that K_B^{-4} and K_B^{-6} must have sections g_2 and g_3, respectively. The surfaces compatible with this condition are found to be del Pezzo, Hirzebruch, Enriques and blow-ups of Hirzebruch surfaces [151]. Note, however, that the construction of stable holomorphic bundles on elliptically fibered three-folds does not hinge upon the Calabi-Yau property. In order to simplify the mathematical apparatus, we nonetheless assume (5.7) in the sequel.

FMW showed that on such spaces the Chern classes of the tangent bundle of the total space follow from the Chern classes of the base space. Especially, we state for later purposes that the second Chern class of the tangent bundle can be computed as

$$c_2(T) = 12\sigma \cdot \pi^*(c_1(B)) + \big(11c_1(B)^2 + c_2(B)\big)\, F. \tag{5.8}$$

5.2 The spectral cover construction

The basic idea of the spectral cover method is to first construct a stable $U(n)$ or $SU(n)$ bundle on the elliptic fibre over each point of the base, which is then extended over the whole manifold by gluing the data together suitably. Recall that in general, a $U(n)$ or $SU(n)$ bundle defines a rank n complex vector bundle. Such a rank n bundle over an an elliptic curve must, in order to satisfy the Hermitian Yang-Mills equation, be of degree zero. Note that this is still true after taking into account the one-loop corrections which vanish trivially upon restriction to a complex curve. More precisely, a rank n bundle can be shown to be isomorphic to the direct sum of n complex line bundles

$$\mathcal{V}|_{E_b} = \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_n, \tag{5.9}$$

each of which has to be of zero degree. If $G = SU(n)$ as opposed to $U(n)$, $\mathcal{V}|_{E_b}$ must in addition be of trivial determinant, i.e. $\bigotimes_{i=1}^n \mathcal{N}_i = \mathcal{O}_{E_b}$. The zero degree condition on $\mathcal{N}_i$ implies that there exists for each $\mathcal{N}_i$ a meromorphic section with precisely one zero at some Q_i and a pole at p, i.e. $\mathcal{N}_i = \mathcal{O}_{E_b}(Q_i - p)$. Consequently, stable $(S)U(n)$ bundles on an elliptic curve are in one-to-one correspondence with the unordered n-tuple of points Q_i, and the reduction of $U(N)$ to $SU(n)$ is encoded in the additional requirement that $\sum_i (Q_i - p) = 0$ in the group law of the elliptic curve.

Having understood the restriction of a rank n bundle $\mathcal{V}$ to each elliptic fibre, we can now proceed to constructing the whole of $\mathcal{V}$. In intuitive terms, the above implies that over an elliptically fibered manifold a $U(n)$ vector bundle determines a set of n points, varying over the base. More precisely, the bundle $\mathcal{V}$ over $\mathcal{M}$ with the property

$$\mathcal{V}|_{E_b} = \bigoplus_{i=1}^{n} \mathcal{O}(Q_i - p) \tag{5.10}$$

uniquely defines an n-fold ramified cover C of B, the spectral cover. It is defined by a projection

$$\pi_C : C \to B \qquad \text{and} \qquad C \cap E_b = \pi_C^{-1}(b) = \bigcup_i Q_i. \tag{5.11}$$

C is conveniently described, as a hypersurface in $\mathcal{M}$, by its Poincaré dual two-form $n\sigma + \ldots$. The first part is due to the fact that C is an n-fold cover of B. As discussed in [149], if we insist that $\mathcal{V}|_{E_b}$ be an $SU(n)$ bundle[2] then the additional terms in the definition of C must emerge from the pull-back of a two-form on B, i.e.

$$[C] = n\sigma + \pi^*(\eta) \in H^2(\mathcal{M}, \mathbb{Z}) \tag{5.12}$$

for η some effective class in $H^2(B, \mathbb{Z})$. We will henceforth assume this to be the case.

Several distinct bundles over $\mathcal{M}$ may well give rise to the same spectral cover C since the latter only encodes the information about the restriction of $\mathcal{V}$ to the fibre E_b. To recover $\mathcal{V}$ from the spectral data we need to specify in addition how it varies over the base, i.e. $\mathcal{V}|_B$. As discussed in [40] this is uniquely accomplished by the so-called spectral line bundle $\mathcal{N}$ on C with the property

$$\pi_{C*}\mathcal{N} = \mathcal{V}|_B. \tag{5.13}$$

We can formalise these results by introducing the notion of the Poincaré line bundle $\mathcal{P}$. For this purpose, consider the fibre product $\mathcal{M} \times_B \mathcal{M}$ as the

[2] This only means that the part of V over the elliptic fibre is of trivial determinant. Nonetheless, the full $\mathcal{V}$ can have a non-vanishing first Chern-class, which, however, does not receive contributions from the fibre. This will become clear shortly.

set of pairs $(z_1, z_2) \in \mathcal{M} \times \mathcal{M}$ with $\pi(z_1) = \pi(z_2)$. Furthermore we need to introduce π_1 and π_2 as the projections on the first and second factor, respectively. Moreover, σ_1 denotes the section $\sigma_1 : B \to X \to X \times_B X'$ and σ_2 the section $\sigma_2 : B \to X' \to X \times_B X'$. Then $\mathcal{P}$ is defined as the bundle over $\mathcal{M} \times_B \mathcal{M}$ with the two properties

$$\mathcal{P}|_{E_b \times x} \simeq \mathcal{P}|_{x \times E_b} \simeq \mathcal{O}_{E_b}(x - p), \qquad (\pi_{1*}(\mathcal{P}))\,|_B = \mathcal{O}_B. \tag{5.14}$$

Introducing the diagonal divisor Δ, the first Chern class of the Poincaré line bundle is [40]

$$c_1(\mathcal{P}) = \Delta - \sigma_1 - \sigma_2 - c_1(B). \tag{5.15}$$

We will denote by $\mathcal{P}_B$ the restriction of $\mathcal{P}$ to $\mathcal{M} \times_B C$. Now by definition, $\pi_{1*}(\mathcal{P}_B)|_{E_b} = \bigoplus_i \mathcal{O}(Q_i - p)$, as is clear from the fact that $C \cap E_b = \bigcup_i Q_i$ and the first property in (5.14). This remains true if we tensor $\mathcal{P}_B$ with $\pi_2^* \mathcal{N}$ for some line bundle $\mathcal{N}$ on C. After all, $\pi^* \mathcal{N}$ as a bundle on $\mathcal{M}$ is trivial when restricted to the fibre E_b. On the other hand, $\mathcal{P}|_{\sigma \times_B E_b}$ is likewise trivial due to the second property in (5.14), and so $\pi_{1*}(\pi_2^* \mathcal{N} \otimes \mathcal{P}_B)|_B$ is simply given by $\pi_{C*}\mathcal{N}$. In other words, the bundle

$$\mathcal{V} = \pi_{1*}(\pi_2^* \mathcal{N} \otimes \mathcal{P}_B) \tag{5.16}$$

indeed exhibits the two defining properties (5.10) and (5.13). This establishes the definition of an $(S)U(n)$ bundle on the elliptically fibered Calabi-Yau threefold in terms of the spectral data $(C, \mathcal{N})$. We reiterate that we will only consider the case that the restriction of the bundle to the elliptic fibre is an $SU(n)$ bundle, i.e. that C is as in (5.12).

The bundles constructed so far are only μ-semi-stable. It has been shown in [152], Theorem 7.1, that the spectral cover must be irreducible in order to obtain a μ-stable one, which imposes two more conditions to the curve η [153]:

- The linear system $|\eta|$ has to be base point free.
- The class $\eta - nc_1(B)$ has to be effective.

We will be more specific about their implications when it comes to a discussion of the properties of the basis. In fact, the proof guarantees stability of the bundle with respect to an ample class, i.e. a Kähler class, $J = \epsilon\sigma + J_B$ such that the Kähler parameter of the fiber lies in a certain range near the boundary of the Kähler cone, that is for sufficiently small ϵ. Since the value of ϵ is not known, in all models involving the spectral cover constructions it is therefore a subtle issue if the region of stability overlaps with the perturbative regime, which is needed to have control over non-perturbative effects. In all examples which will be relevant for us, the constraints will leave us enough freedom to go to regions of the Kähler cone where ϵ is much smaller than J_B.

We now give the topological invariants of the bundle $\mathcal{V}$ defined by (5.16). The working horse for this computation is the Grothendieck-Riemann-Roch (GRR) theorem stating that, for a coherent sheaf V over a variety Y with a smooth projection $\pi : Y \to X$, the Chern characters of the push-forward sheaf $\pi_* W$ over X can be computed from

$$\mathrm{ch}\Big(\pi_!(W)\Big)\,\mathrm{Td}(X) = \pi_*\Big(\mathrm{ch}(W)\,\mathrm{Td}(Y)\Big), \tag{5.17}$$

with the operation π_* on the right being essentially integration along the fibre of π. For completeness we note that $\pi_!(W)$, appearing on the left, is the K-theoretic Gysin map which is defined as $\pi_!(W) = \sum_i (-1) R^i \pi_*(W)$ in terms of the higher direct image sheaves $R^i\pi_*(W)$. The latter can be thought of as the sheaf over X whose stalk over $U \subset X$ is given by the cohomology group $H^i(\pi^{-1}(U), W|_{\pi^{-1}(U)})$ and the alternating sum is to be understood in the K-theoretic sense. More information can be found e.g. in [138].

The idea is now to apply this theorem to the projection $\pi_1 : \mathcal{M} \times_B C \to \mathcal{M}$ and with W given by $\pi_2^*\mathcal{N} \otimes \mathcal{P}_B$. In this case, the fiber of π_1 over a point $\sigma(b)$ in $\mathcal{M}$ consists simply of the n points in the n-fold cover C which project to b under $\pi_C : C \to B$. Since the fiber is zero-dimensional, all direct images $R^i\pi_*(V)$ higher than $R^0\pi_{1*}(W) = \pi_{1*}(\pi_2^*\mathcal{N} \otimes \mathcal{P}_B)$ vanish. The latter is just the definition of $\mathcal{V}$ and this allows us to compute the Chern classes of $\mathcal{V}$ from

$$\mathrm{ch}(\mathcal{V})\mathrm{Td}(\mathcal{M}) = \pi_{1*}\left(e^{c_1(\pi_2^*\mathcal{N}\otimes\mathcal{P}_B)}\,\mathrm{Td}(\mathcal{M}\times C)\right). \tag{5.18}$$

As discussed in [40], this relates, after additional manipulations, in particular $c_1(\mathcal{N})$ and $c_1(\mathcal{V})$ as

$$c_1(\mathcal{N}) = \frac{1}{n}\,\pi_C^* c_1(\mathcal{V})|_B - \frac{1}{2}\,c_1(TC) + \frac{1}{2}\,\pi_C^* c_1(B) + \gamma \tag{5.19}$$

in terms of the cohomology class γ satisfying

$$\pi_{C*}\gamma = 0. \tag{5.20}$$

One can prove that γ can in general be written as

$$\gamma = \lambda(n\sigma - \pi_C^*\eta + n\pi_C^* c_1(B)), \tag{5.21}$$

where $\lambda \in \mathbb{Q}$. Note furthermore that $c_1(TC)$ is minus the first Chern class of the canonical bundle $K_C = \mathcal{O}(C)$ on C, i.e. $c_1(TC) = -n\sigma - \pi_C^*(\eta)$.

We now parameterise $c_1(\mathcal{V})$ by some element $c_1(\zeta) \in H^2(B, \mathbb{Z})$ to be specified momentarily,

$$c_1(\mathcal{V}) = \pi^* c_1(\zeta). \tag{5.22}$$

Putting everything together, we have

$$c_1(\mathcal{N}) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi_C^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi_C^* c_1(B) + \frac{1}{n}\,\pi_C^* c_1(\zeta) \tag{5.23}$$

Since $c_1(\mathcal{N})$ must be an integer class, not every value of $\lambda \in \mathbb{Q}$ and $c_1(\zeta) \in H^2(B,\mathbb{Z})$ is allowed in the ansatz for $c_1(\mathcal{V})$. Rather they are subject to the constraints

$$\begin{aligned} n\left(\frac{1}{2}+\lambda\right) &\in \mathbb{Z}, \\ \left(\frac{1}{2}-\lambda\right)\eta+\left(n\lambda+\frac{1}{2}\right)c_1(B)+\frac{1}{n}c_1(\zeta) &\in H^2(B,\mathbb{Z}), \end{aligned} \tag{5.24}$$

but can otherwise be chosen arbitrarily. Note that if we are interested in $SU(n)$ bundles as e.g. in [40], then simply $c_1(\zeta) = 0$ so that $c_1(\mathcal{V}) = 0$. All other consistent choices yield $U(n)$ bundles. Allowing non-trivial values for $c_1(\mathcal{V})$ was first considered in [69] and motivated by the relative Fourier-Mukai transform, but we will not invoke this picture here[3] . Further applications of the GRR theorem lead, after considerable work, to the following expressions for the second and third Chern classes [40, 69, 148]

$$\begin{aligned} \mathrm{ch}_2(\mathcal{V}) &= -\sigma\cdot\pi^*\eta+\left(\frac{1}{2n}c_1(\zeta)^2-\omega\right)F, \\ \mathrm{ch}_3(\mathcal{V}) &= \lambda\eta\cdot(\eta-nc_1(B))-\frac{1}{n}c_1(\zeta)\cdot\eta, \end{aligned} \tag{5.25}$$

where

$$\omega = -\frac{1}{24}c_1(B)^2(n^3-n)+\frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)n\eta\cdot(\eta-nc_1(B)). \tag{5.26}$$

Note that $\mathrm{ch}_3(V)$ has already been integrated over the fiber.

As we emphasized several times, this kind of construction only gives bundles with trivial first Chern class as restricted to the elliptic fibres. To be more general, we can however twist the bundle $\mathcal{V}$ defined via the spectral cover construction with an additional line bundle $\mathcal{Q}$ on X with [131]

$$c_1(\mathcal{Q}) = q\sigma+\pi^*(c_1(\zeta_Q)), \tag{5.27}$$

where $\pi^*(c_1(\zeta_Q)) \in H^2(X,\mathbb{Z})$. The resulting $U(n)$ bundle

$$V = \mathcal{V}\otimes\mathcal{Q} \tag{5.28}$$

is μ-stable precisely if the original bundle $\mathcal{V}$ is [30]. The Chern classes for V are straightforwardly computed from the ones of $\mathcal{V}$ and from $c_1(\mathcal{Q})$ (see also appendix A.1). Note that the contribution form $\pi^*(c_1(\zeta_Q))$ can be absorbed into an additive shift of $c_1(\zeta)$ by $nc_1(\zeta_Q)$. W.l.o.g. we will henceforth assume that $c_1(\zeta_Q) = 0$.

The Chern characters of V then read

[3] To recover their expressions, simply set $c_1(\zeta) = \eta_E - \frac{n}{2}c_1(B)$ in the notation of [69].

$$\begin{aligned}
\mathrm{ch}_1(V) &= nq\sigma + c_1(\zeta), && (5.29)\\
\mathrm{ch}_2(V) &= \left[-\eta + \frac{q}{2}(2c_1(\zeta) - n\,qc_1(B))\right]\sigma + a_F, && (5.30)\\
\mathrm{ch}_3(V) &= \lambda\eta\cdot(\eta - nc_1(B)) - \frac{1}{n}\eta\cdot c_1(\zeta) + q\left(\frac{1}{2n}c_1(\zeta)^2 - \omega\right) + && (5.31)\\
& qc_1(B)\left(\eta - \frac{q}{2}c_1(\zeta) + \frac{nq^2}{6}c_1(B)\right),
\end{aligned}$$

where

$$a_F = \frac{1}{2n}c_1(\zeta)^2 - \omega. \tag{5.32}$$

For later purposes we also list the Chern classes,

$$\begin{aligned}
c_1(V) &= nq\sigma + c_1(\zeta), && (5.33)\\
c_2(V) &= \left[\eta + q(n-1)\left(c_1(\zeta) - \frac{q}{2n}c_1(B)\right)\right]\sigma + \frac{1}{2}c_1(\zeta)^2 - a_F, && (5.34)\\
c_3(V) &= \frac{q^2}{6}(n^2-3n+2)\left(nqc_1(B)^2 - 3c_1(\zeta)\cdot c_1(B)\right), && (5.35)\\
&+\frac{q}{2n}(n^2-2n+2)c_1(\zeta)^2 + (2q - nq - 2n\lambda)\,\eta\cdot c_1(B)\\
&+\frac{n-2}{n}\,\eta\cdot c_1(\zeta) + 2\lambda\,\eta^2 - nq\,a_F - 2q\,\omega.
\end{aligned}$$

To summarize, this class of $U(n)$ bundles is completely specified by the rational number λ, the integer q and the classes η and $c_1(\zeta)$.

5.3 del Pezzo base manifolds

As alluded to already, the Calabi-Yau condition imposes severe constraints on which complex two-surfaces are eligible as base manifolds of our elliptic fibration. Among the possibilities classified in [151] we can choose as the base manifold one of the del Pezzo surfaces dP_r with $r = 0, \ldots, 9$. The surface dP_r is defined by blowing up r points in generic position on $\mathbb{P}_2$. This means that $H^2(\mathrm{dP_r})$ is generated by the $r+1$ elements $l, E_1, \ldots, E_r$, where l is the hyperplane class inherited from $\mathbb{P}_2$ and the E_m denote the r exceptional cycles introduced by the blow-ups. The intersection form can be computed as

$$l\cdot l = 1, \quad l\cdot E_m = 0, \quad E_m\cdot E_n = -\delta_{m,n}. \tag{5.36}$$

The first equation follows from the fact that two representatives of the class l define two complex lines in generic position which clearly intersect precisely once. The self-intersection for the blow-ups is the usual one for exceptional cycles. Furthermore, a complex line in generic position does not pass through any of the blow-ups, thus $l \cdot E_m = 0$.

The Chern classes read

$$c_1(dP_r) = 3l - \sum_{m=1}^{r} E_m, \qquad c_2(dP_r) = 3 + r. \tag{5.37}$$

We clearly recover the part involving l as simply the first Chern class of the anti-canonical bundle of the parent $\mathbb{P}_2$. For the second Chern class of the elliptic threefold $\mathcal{M}$ we obtain, applying (5.8),

$$c_2(T\mathcal{M}) = 12\sigma c_1(B) + (102 - 10r)\, F. \tag{5.38}$$

Now for a vector bundle V_i we can expand η_i and $c_1(\zeta_i)$ in a cohomological basis

$$\eta_i = \eta_i^{(0)}\, l + \sum_{m=1}^{r} \eta_i^{(m)}\, E_m, \qquad c_1(\zeta_i) = \zeta_i^{(0)}\, l + \sum_{m=1}^{r} \zeta_i^{(m)}\, E_m. \tag{5.39}$$

As mentioned before we have to require that η is effective and that for stability $\eta - n\, c_1(B)$ is effective as well. Fortunately, the generating system for the cone of effective curves on dP_r has been given in [154] and we list the reformulated result of [153] in Table 5.1 for completeness. Recall that a general effective class can be expanded into a linear combination of these Mori cone generators with non-negative integer coefficients.

Moreover, $|\eta|$ is known to be base point free if $\eta \cdot E \geq 0$ for every curve E with $E^2 = -1$ and $E \cdot c_1(B) = 1$. Such curves are precisely given by the generators of the Mori cone listed in Table 5.1.

r	Generators	#
1	$E_1,\, l - E_1$	2
2	$E_i,\, l - E_1 - E_2$	3
3	$E_i,\, l - E_i - E_j$	6
4	$E_i,\, l - E_i - E_j$	10
5	$E_i,\, l - E_i - E_j,\, 2l - E_1 - E_2 - E_3 - E_4 - E_5$	16
6	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m$	27
7	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m,$ $3l - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o$	56
8	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m,$ $3l - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o,$ $4l - 2(E_i + E_j + E_k) - \sum_{i=1}^{5} E_{m_i},$ $5l - 2\sum_{i=1}^{6} E_{m_i} - E_k - E_l,\, 6l - 3E_i - 2\sum_{i=1}^{7} E_{m_i}$	240
9	$f = 3 - \sum_{i=1}^{9} E_i$, and $\{y_a\}$ with $y_a^2 = -1$, $y_a \cdot f = 1$	∞

Table 5.1: Generators for the Mori cone of each dP_r, $r = 1, \ldots, 9$. All indices $i, j, \ldots \in \{1, \ldots, r\}$ in the table are distinct. The effective classes can be written as linear combinations of the generators with integer non-negative coefficients.

Chapter 6

Semi-realistic $SO(32)$ string vacua

We have finally collected all the relevant material we need in order to discuss the applications of the novel embedding of $U(n)$ bundles to string model building in either heterotic theory. In this chapter, based on [131] , we start with the $SO(32)$ heterotic corner. From our discussion in chapter 4 it is clear that the parameter space of potentially consistent vacua is extremely huge. A systematic search for interesting models, let alone a complete classification of the associated landscape[1], therefore appears challenging and is far beyond the scope of this work. The large number of a priori possibilities is due to two independent sources.

First we need to specify a concrete embedding of the type discussed in section (4.1). Even if we restrict all considerations from the beginning to a phenomenologically appealing visible gauge sector - e.g. such that it reproduce the Pati-Salam or MSSM gauge group - we have the choice of the intermediate group $U(M_i)$. Basically this amounts to the "internal" integer degree of freedom n_i in equation (4.4) for each visible group factor. The effective tadpole has to be cancelled by introducing an appropriate hidden sector consisting of hidden gauge bundles and/or five-branes. The combinatorics governing this problem renders a classification of all possibilities highly non-trivial.

All this is of course completely independent of the question on which concrete background manifold one endeavours to construct suitable vector bundles. For reasons of practicability we will focus on the class of stable holomorphic bundles on elliptically fibered Calabi-Yau manifolds the essential properties of which we have just reviewed in chapter 5. Any alternative methods to construct stable bundles over more general Calabi-Yau threefolds serve, in principle, as equally good starting points for model building. The discrete parameter space even for the special set of bundles based on the spectral cover construction is enormous. In this chapter we present two semi-realistic examples which our very preliminary and restrictive survey has produced and whose properties are typical of a large set of solutions that can easily be generated. In fact, we have only covered a tiny fraction of the solution space of vector bundles on elliptic fibrations over dP_3 and

[1]See [155,156] for a treatment of the landscape of string vacua in the S-dual framework of magnetized D9-branes with abelian bundles respectively intersecting branes.

dP$_4$.

We have emphasized several times by now the one-to-one correspondence between the architecture of the $SO(32)$ heterotic theory with $U(n)$ bundles and the structure known from the context of intersecting D-brane model building. Taking this duality at face value we therefore advocate the following examples alternatively as Type I vacua with non-abelian magnetized D9-branes on non-toroidal three-folds including D5-branes.

Before digging into the details of the models, it only remains to evaluate the loop-corrected DUY condition (4.39) for this class of vector bundles. With the help of the Chern characters as given in equation (5.29), we obtain the DUY equation

$$\begin{aligned} & \frac{1}{2} r_\sigma \Big(2J_B - r_\sigma\, c_1(B)\Big)\, \left(c_1(\zeta) - nq\, c_1(B)\right) + \frac{nq}{2} J_B^2 \\ = \;& 2\, g_s^2 \left[\chi(V) - \frac{1}{2} c_1(\zeta)\, c_1(B) - \frac{nq}{24}\left(c_2(B) - c_1(B)^2\right)\right] \end{aligned} \tag{6.1}$$

after expressing $J = \ell_s^2 (r_\sigma\, \sigma + J_B)$ in terms of J_B, the Kähler form on the base B. This equation has to be satisfied inside the Kähler cone for the model to be well-defined. The constraints on the Kähler moduli resulting from this requirement are collected in appendix B.

The positivity condition (4.35) on the real part of the gauge kinetic function for a $U(N)$ factor leads to the second constraint

$$\begin{aligned} & \frac{n}{3!} r_\sigma \left(r_\sigma^2 c_1(B)^2 - 3 r_\sigma c_1(B) J_B + 3 J_B^2\right) \\ - \;& 2\, g_s^2 \left[\left(r_\sigma c_1(B) - J_B\right) \Big(\eta - \frac{q}{2}\left(2 c_1(\zeta) - nq\, c_1(B)\right)\Big) + r_\sigma a_F\right] \\ - \;& g_s^2\, n \left[c_1(B) J_B + \frac{r_\sigma}{12}\left(c_2(B) - c_1(B)^2\right)\right] > 0. \end{aligned} \tag{6.2}$$

These conditions impose strong constraints on the bundles to be put simultaneously on the manifold $\mathcal{M}$. We recall that in general each $U(n)$ bundle freezes one combination of the dilaton and the $b_2(B) + 1$ radii.

6.1 A four-generation Pati-Salam model on dP$_3$

As a first example we choose the basis of the elliptic fibration to be the del Pezzo surface dP$_3$. Then we embed a bundle with structure group $U(1) \times U(2)^2$ into $U(4)^3$ yielding the observable group

$$H = U(4) \times U(2)^2 \times SO(8). \tag{6.3}$$

The data for the twisted bundles are given in Table 6.1.

It can be checked explicitly from (5.24) that this data results in well-defined spectral bundles $\mathcal{N}$. Furthermore, η_b and η_c as well as

$$\eta_b - 2c_1(B) = 5l - E_1 - 3E_2 - E_3, \quad \eta_c - 2c_1(B) = l - E_1 + E_2 - E_3 \tag{6.4}$$

$U(n_i)$	λ_i	η_i	q_i	ζ_i
$U(1)_a$	0	0	0	$-2l+3E_2+3E_3$
$U(2)_b$	0	$11l-3E_1-5E_2-3E_3$	0	$-2l+2\sum_{m=1}^{3}E_m$
$U(2)_c$	0	$7l-3E_1-E_2-3E_3$	0	$-8l+8\sum_{m=1}^{3}E_m$

Table 6.1: Defining data for a $U(1)\times U(2)^2$ bundle.

are effective and the linear systems $|\eta_b|$, $|\eta_c|$ are base-point free, i.e. all intersections with the basis of the Mori cone listed in Table 5.1 are non-negative. Therefore, the constructed bundles are indeed μ-stable.

Finally, the tadpole

$$c_2(T) = 12\left[3l - \sum_{m=1}^{3} E_m\right]\sigma + 72 \tag{6.5}$$

is cancelled without adding H5-branes due to

$$\begin{aligned} \mathrm{ch}_2(V_a) &= -7, \\ \mathrm{ch}_2(V_b) &= [-11l+3E_1+5E_2+3E_3]\,\sigma+8, \\ \mathrm{ch}_2(V_c) &= [-7l+3E_1+E_2+3E_3]\,\sigma-30. \end{aligned} \tag{6.6}$$

The resulting chiral spectrum is displayed in Table 6.2. Observe in particular that there is no chiral state charged under $SO(8)$ due to $\chi(V_i)=0$ and that there are no symmetric or antisymmetric chiral states since in addition $\zeta_i\cdot\mathrm{ch}_2(V_i) = \zeta_i\cdot c_2(T) = 0$ for all i.

The analysis of the chiral spectrum shows that all three $U(1)$ factors are anomaly-free. However, the mass matrix (4.32) has rank two, and only the linear combination $4U(1)_b - U(1)_c$ remains massless.

$U(4)_a\times U(2)_b\times U(2)_c$	mult.
$(\overline{\mathbf{4}},\overline{\mathbf{2}},\mathbf{1})_{-1,-1,0}$	2
$(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})_{-1,1,0}$	2
$(\mathbf{4},\mathbf{1},\overline{\mathbf{2}})_{1,0,-1}$	2
$(\mathbf{4},\mathbf{1},\mathbf{2})_{1,0,1}$	2

Table 6.2: Chiral spectrum of a four generation Pati-Salam model on dP_3.

The resulting DUY conditions are very simple in this configuration since all one-loop contributions cancel,

$$\begin{aligned} r_\sigma\,(3r_2+3r_3+2r_0) &= 0, \\ r_\sigma\left(r_0+\sum_{m=1}^{3} r_m\right) &= 0. \end{aligned} \tag{6.7}$$

According to our discussion in section 4.7.3 this ensures that μ-stability is just the right criterion for the bundle to satisfy the Hermitian Yang-Mills equation. Positivity of the gauge kinetic functions requires

$$\begin{aligned} r_\sigma\left(2r_\sigma^2-r_\sigma(3r_0+\sum_{m=1}^{3} r_m)+r_0^2-\sum_{m=1}^{3} r_m^2\right)-2\,g_s^2\left(-14r_\sigma+3r_0+\sum_{m=1}^{3} r_m\right) &> 0, \\ r_\sigma\left(2r_\sigma^2-r_\sigma(3r_0+\sum_{m=1}^{3} r_m)+r_0^2-\sum_{m=1}^{3} r_m^2\right)-2\,g_s^2\,(30r_\sigma-8r_0-2r_1-4r_2-2r_3) &> 0, \\ r_\sigma\left(2r_\sigma^2-r_\sigma(3r_0+\sum_{m=1}^{3} r_m)+r_0^2-\sum_{m=1}^{3} r_m^2\right)+2\,g_s^2\,(16r_\sigma+4r_0+2r_1+2r_3) &> 0. \end{aligned}$$

These conditions can be fulfilled in the perturbative regime inside the Kähler cone, e.g. for arbitrary r_σ and $g_s < 0.11\,r_\sigma$, $r_0 = 1.8\,r_\sigma$, $r_1 = r_2 = r_3 = -0.6\,r_\sigma$.

6.2 A three-generation Standard-like model on $\mathbf{dP}_4$

This section is devoted to a three-generation Standard-like model involving four vector bundles, where we now take the base manifold to be dP_4. It can be regarded as the generalized S-dual version of the four-stack models which have become popular in the framework of intersecting branes. Our aim is therefore to obtain a visible gauge group $U(3)_a\times U(2)_b\times U(1)_c\times U(1)_d$ and realize the quarks and leptons as appropriate bifundamentals. A possible choice of the hypercharge as a (massless) combination of the abelian factors is given by $Q_Y=\frac{1}{6}Q_a+\frac{1}{2}(Q_c+Q_d)$. In this case, also some of the (anti-)symmetric representations carry MSSM quantum numbers . The details of the chiral MSSM spectrum we try to reproduce can be found in Table 6.4.

Among the many possibilities we consider the simple embedding of the structure group $G=U(1)\times U(1)\times U(2)\times U(1)$ into $U(3)\times U(2)\times U(2)\times U(1)$. This leads to

$$H=U(3)\times U(2)\times U(1)\times U(1)\times SO(16) \tag{6.8}$$

modulo the issue of anomalous abelian factors. We choose the bundles characterized in Table 6.3.

$U(n_i)$	λ_i	η_i	q_i	ζ_i
$U(1)_a$	0	0	1	$5l - 3E_1 - 5E_2 - E_3$
$U(1)_b$	0	0	1	$-3l + 5E_1 + 2E_2 - E_3 + E_4$
$SU(2)_c$	0	$7l - 3E_1 - 3E_2 - E_3 - E_4$	0	0
$U(1)_d$	0	0	- 1	$-5l + 3E_1 + 5E_2 + E_3$

Table 6.3: Defining data for a $U(1) \times U(1) \times SU(2) \times U(1)$ bundle.

Note that V_c actually has structure group $SU(2)$ rather than $U(2)$ since its first Chern class vanishes, which however makes no difference in the group theoretic decomposition of $SO(32)$. Again, one may verify explicitly that the conditions for μ-stability are satisfied. Let us also point out that the requirement (4.16) of cancellation of the Witten anomaly, which is non-trivial for odd N_a, is satisfied by the configuration. Furthermore, the $U(1)_Y$ hypercharge is indeed massless as desired (see (4.32)). However, since the rank of the mass matrix is two, we get another massless $U(1)$ in the four-dimensional gauge group, which is readily identified as $U(1)_c$. The perturbative low energy gauge group is therefore

$$H = SU(3) \times SU(2) \times U(1)_Y \times U(1)' \times SO(16). \tag{6.9}$$

The degeneracy of the bundle V_a and $V_d = V_a^*$ leads to a gauge enhancement of the $U(3)_a$ and the $U(1)_d$ to a $U(4)$. Apart from these drawbacks, the configuration indeed gives rise to three families of the MSSM chiral spectrum as listed in Table 6.4.

In addition, we get some chiral exotic matter in the antisymmetric of the $U(2)$ and in the bifundamental of the $SO(16)$ with the $U(3)$ and $U(2)$, respectively (see Table 6.5).

In contrast to the previous example, the chosen bundles alone do not satisfy the tadpole cancellation condition. However, the resulting tadpole can be cancelled by including H5-branes, which demonstrates the importance of allowing for these non-perturbative objects. From the general form of the tadpole equation we find the four-form characterizing this tadpole to be

$$[W] = c_2(T) + \sum_{i=1}^{4} N_i \, \mathrm{ch}_2(V_i) = 22\, F + (34l - 8E_1 - 22E_2 - 14E_3 - 6E_4)\, \sigma. \tag{6.10}$$

Its Poincaré dual class $[\Gamma] = 22\sigma + 34l - 8E_1 - 22E_2 - 14E_3 - 6E_4$ lies inside the Mori cone, i.e. is effective, and can thus be regarded as the homology class associated to a (reducible) holomorphic curve around which we may wrap a system of H5-branes. To determine the detailed spectrum and gauge group

$U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d \times SO(16) \times \prod_a Sp(2N_a)$				
MSSM particle	repr.	index	mult.	total
Q_L	$(\mathbf{3},\overline{\mathbf{2}};1,1)_{(1,-1,0,0)}$	$\chi(X,V_a\otimes V_b^*)$	8	
Q_L	$(\mathbf{3},\mathbf{2};1,1)_{(1,1,0,0)}$	$\chi(X,V_a\otimes V_b)$	-11	-3
u_R	$(\overline{\mathbf{3}},1;1,1)_{(-1,0,-1,0)}$	$\chi(X,V_a^*\otimes V_c^*)$	-3	
u_R	$(\overline{\mathbf{3}},1;1,1)_{(-1,0,0,-1)}$	$\chi(X,V_a^*\otimes V_d^*)$	0	-3
d_R	$(\overline{\mathbf{3}},1;1,1)_{(-1,0,1,0)}$	$\chi(X,V_a^*\otimes V_c)$	-3	
d_R	$(\overline{\mathbf{3}},1;1,1)_{(-1,0,0,1)}$	$\chi(X,V_a^*\otimes V_d)$	45	
d_R	$(\overline{\mathbf{3}}_A,1;1,1)_{(2,0,0,0)}$	$\chi(X,\bigotimes_s^2 V_a)$	-45	-3
L	$(1,\mathbf{2};1,1)_{(0,1,-1,0)}$	$\chi(X,V_b\otimes V_c^*)$	-7	
L	$(1,\mathbf{2};1,1)_{(0,1,0,-1)}$	$\chi(X,V_b\otimes V_d^*)$	-11	
L	$(1,\overline{\mathbf{2}};1,1)_{(0,-1,-1,0)}$	$\chi(X,V_b^*\otimes V_c^*)$	7	
L	$(1,\overline{\mathbf{2}};1,1)_{(0,-1,0,-1)}$	$\chi(X,V_b^*\otimes V_d^*)$	8	-3
e_R	$(1,1;1,1)_{(0,0,2,0)}$	$\chi(X,\bigwedge^2 V_c)$	0	
e_R	$(1,1;1,1)_{(0,0,0,2)}$	$\chi(X,\bigwedge^2 V_d)$	0	
e_R	$(1,1;1,1)_{(0,0,1,1)}$	$\chi(X,V_c\otimes V_d)$	-3	-3
ν_R	$(1,1;1,1)_{(0,0,-1,1)}$	$\chi(X,V_c^*\otimes V_d)$	-3	-3

Table 6.4: Chiral MSSM spectrum for a four-stack model with $Q_Y = \frac{1}{6}Q_a + \frac{1}{2}(Q_c + Q_d)$.

supported by the branes we must choose a decomposition of $[\Gamma]$ into irreducible effective classes around each of which we can wrap one H5-brane. These are given precisely by the generators of the Mori cone in Table 5.1. Note that the decomposition is not unique and constitutes (part of) the moduli space of our model; what is universal is the total number of chiral degrees of freedom charged under the symplectic sector (see Table 6.5) and its total rank. In our case, the latter is easily found to be 74. For instance, the decomposition

$$[\Gamma] = 22\,\sigma + 22(l - E_2 - E_3) + 12(l - E_1 - E_4) + 4E_1 + 8E_3 + 6E_4 \qquad (6.11)$$

results in the symplectic gauge group $Sp(44)\times Sp(44)\times Sp(24)\times Sp(8)\times Sp(16)\times Sp(12)$. The bifundamental exotics between the MSSM group and this symplectic gauge sector can be determined with the help of (4.12). Ideally, this group would be hidden, of course.

$U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d \times SO(16) \times \prod_a Sp(2N_a)$				
MSSM particle	repr.	index	mult.	total
-	$(1, \mathbf{1}_A; 1, 1)_{(0,2,0,0)}$	$\chi(X, \bigotimes_s^2 V_b)$	-77	-77
-	$(\mathbf{3}, 1; \mathbf{16}, 1)_{(1,0,0,0)}$	$\chi(X, V_a)$	-1	-1
-	$(1, \mathbf{2}; \mathbf{16}, 1)_{(0,1,0,0)}$	$\chi(X, V_b)$	-11	-11
-	$(1, 1; \mathbf{16}, 1)_{(0,0,1,0)}$	$\chi(X, V_c)$	0	0
-	$(1, 1; \mathbf{16}, 1)_{(0,0,0,1)}$	$\chi(X, V_d)$	1	1
-	$\sum_a (\mathbf{3}, 1; 1, \mathbf{2N_a})_{(1,0,0,0)}$	$\chi(X, V_a \otimes \mathcal{O}\|_\gamma)$	8	8
-	$\sum_a (1, \mathbf{2}; 1, \mathbf{2N_a})_{(0,1,0,0)}$	$\chi(X, V_b \otimes \mathcal{O}\|_\Gamma)$	56	56
-	$\sum_a (1, 1; 1, \mathbf{2N_a})_{(0,0,1,0)}$	$\chi(X, V_c \otimes \mathcal{O}\|_\Gamma)$	0	0
-	$\sum_a (1, 1; 1, \mathbf{2N_a})_{(0,0,0,1)}$	$\chi(X, V_d \otimes \mathcal{O}\|_\Gamma)$	-8	-8

Table 6.5: Chiral exotic spectrum for the four-stack model with $Q_Y = \frac{1}{6}Q_a + \frac{1}{2}(Q_c + Q_d)$. In the second column, the first two entries refer to the $U(3)$ and $U(2)$ factors, the third to the SO(16) group and the fourth collectively represents the symplectic charges. The $U(1)$ charges are read off from the lower-case entries.

The only independent DUY equations are those for V_a and V_b

$$\frac{1}{2}(r_0^2 - \sum_{m=1}^{4} r_m^2) + r_\sigma(2r_0 + 2r_1 + 4r_2 - r_4 - \frac{1}{2}r_\sigma) = -\frac{49}{6}g_s^2, \quad (6.12)$$

$$\frac{1}{2}(r_0^2 - \sum_{m=1}^{4} r_m^2) + r_\sigma(-6r_0 - 6r_1 - 3r_2 - 2r_4 + \frac{7}{2}r_\sigma) = -\frac{121}{6}g_s^2, \quad (6.13)$$

and only fix two of the Kähler moduli. Note that V_a and V_b, being line bundles, automatically satisfy the Hermitian Yang-Mills equations. The reason is that their field strength is constant over the manifold as a consequence of the Bianchi identity, which in the abelian case implies $dF = 0$.

The $SU(2)$-bundle V_c, by contrast, is such that its one-loop part in the DUY correction vanishes, so that for V_c μ-stability is sufficient for supersymmetry. Therefore, the supersymmetry condition reduces entirely to the DUY equation and no further stability analysis is required.

A solution to (6.12) for which the real part of the various gauge kinetic functions is positive can well be found inside the Kähler cone and in the perturbative regime. E.g. by taking $r_2 = -2.5\, r_\sigma$, $r_3 = -1.1\, r_\sigma$, $r_4 = -\, r_\sigma$ and $g_s < 0.41\, r_\sigma$ for arbitrary r_σ, the solution for r_0 and r_1 satisfies all Kähler cone constraints. We can therefore always choose r_σ and g_s such that the model is indeed in the perturbative regime.

Chapter 7

GUT and Standard Model vacua from $E_8 \times E_8$

Our ultimate goal is to find a new framework for the construction of realistic string vacua. Concretely, we have already described two very promising scenarios how to arrive at phenomenologically appealing gauge groups and a realistic particle spectrum in the framework of the $E_8 \times E_8$ string. As one of its virtues the method of embedding $U(N)$ bundles has the potential to yield just the right gauge groups without relying on the use of Wilson lines on the Calabi-Yau manifold, which would restrict the choice of the background geometry considerably. Recall that the Wilson lines as flat abelian gauge bundles inherited from the geometry are replaced by veritable line bundles with non-vanishing first Chern class. In other words, we have the freedom to put extra structure on our internal manifold instead of having to take from it what we get.

The first example we encountered in section 3.7 was the breaking of E_8 down to flipped $SU(5) \times U(1)_X$ via an $SU(4) \times U(1)$ gauge instanton, the second one being the breaking $SU(5) \times U(1) \subset E_8 \rightarrow SU(3) \times SU(2) \times U(1)_Y$, see section 3.8. Provided that we can ensure that the abelian gauge factor remains massless, both models therefore succeed in yielding the right gauge group in four dimensions. In the second case, this is obvious as we obtain the MSSM gauge group directly. In the GUT $SU(5) \times U(1)_X$ framework, by contrast, we have to rely in addition on a field theoretic Higgs mechanism in order to break the GUT group down to the Standard Model group. Unlike in the Georgi-Glasham $SU(5)$ one arrives at by invoking just conventional $SU(5)$ instantons on the Calabi-Yau, the spectrum in our model flipped $SU(5)$ model indeed provides a GUT Higgs field suitable to accomplish this task.

The question of primary importance is therefore how to keep the $U(1)$ massless. One possibility, explored already in section (3.8) for the $SU(3) \times SU(2)$ setup, is to reduce the rank of the non-abelian instanton by embedding several $U(1)$ bundles into the same E_8 factors such that the right linear combination of $U(1)$s remains massless. While this is possible in principle and indeed gives rise to an extremely rich vacuum structure, we witnessed how the additional line bundles inevitably produced exotic matter. One might try to find explicit bundle configurations such that the cohomology groups counting this matter are trivial, but we follow here an easier and more natural solution by embedding the extra line bundle not into the same, but rather into the *second* E_8. This leaves the gauge group and matter from the first E_8 intact while it allows nonetheless for a massless combination of the two $U(1)$s. In both cases, the requirement that the $U(1)_X$ and $U(1)_Y$, respectively, do not acquire a mass automatically leads to a spectrum with precisely g generations of flipped $SU(5)$/MSSM matter and no further chiral exotics. The phenomenology of the flipped $SU(5)$ model is particularly attractive due to the absence of dangerous proton decay operators. We will furthermore see that the predictions of both scenarios for gauge coupling unification are compatible with the Standard Model running of the coupling constants once we take threshold corrections into account. We have found several three-generation realisations of both the flipped $SU(5)$ and the MSSM construction which are listed in an appendix. The contents on this chapter is based on [98].

7.1 Flipped $SU(5) \times U(1)_X$

7.1.1 $SU(4) \times U(1)$ bundles

The technical details of the breaking of E_8 down to $SU(5) \times U(1)_X$ have been discussed at length in section (3.7). For convenience we repeat in table 7.1 merely the visible spectrum resulting from the first E_8 factor upon embedding the $SU(4) \times U(1)$ bundle $W = V \oplus L^{-1}$ (see the discussion after equation (3.147)).

$SU(5) \times U(1)_{X'}$	cohomology (type B)	SM part.
$\mathbf{10}_1$	$H^*(V)$	$(q_L, d^c_R, \nu^c_R) + [H_{10} + \overline{H}_{10}]$
$\mathbf{10}_{-4}$	$H^*(L^{-1})$	$-$
$\overline{\mathbf{5}}_{-3}$	$H^*(V \otimes L^{-1})$	(u^c_R, l_L)
$\overline{\mathbf{5}}_2$	$H^*(\bigwedge^2 V)$	$[(h_3, h_2) + (\overline{h}_3, \overline{h}_2)]$
$\mathbf{1}_5$	$H^*(V \otimes L)$	e^c_R

Table 7.1: Massless spectrum of $H = SU(5) \times U(1)_{X'}$ models.

The massless fields precisely carry, up to a common factor, the $U(1)_X$ charges as appearing in the flipped $SU(5)$ GUT model [75, 157], $Q_X = \frac{1}{2} Q_{X'}$.[1] Recall that this model differs from the conventional Georgi-Glashow GUT scenario [158]

[1]Note that the normalisation of Q_X, as chosen here, differs from the one in [75] by a factor of $-\frac{1}{2}$.

in that the MSSM $U(1)_Y$ is not entirely contained in the $SU(5)$, but arises as the specific linear combination

$$\frac{1}{2}Q_Y = -\frac{1}{5}Q_Z + \frac{2}{5}Q_X, \tag{7.1}$$

where Z is the generator of $SU(5)$ commuting with the generators of the Standard Model $SU(3) \times SU(2)$. In the normalisation of [75] Z is given by $Z = \mathrm{diag}(-1/3, -1/3, -1/3, 1/2, 1/2)$. The way how the MSSM matter is organized into flipped $SU(5)$ multiplets is related to the Georgi-Glashow scenario by the flip

$$d_R^c \leftrightarrow u_R^c, \qquad e_R^c \leftrightarrow \nu_R^c. \tag{7.2}$$

Most importantly, the $(\mathbf{10})_1$ contains the right-handed neutrino as a particle uncharged under the MSSM $SU(3) \times SU(2) \times U(1)_Y$, and giving it a VEV can therefore serve as the Higgs effect which breaks the GUT group down to the Standard Model one. It is this peculiarity of flipped $SU(5)$ which at first sight allows us to work on manifolds without Wilson lines. However, if we only consider the bundle (3.147) inside the first E_8 with $c_1(L) \neq 0$, one Kähler/dilaton modulus receives a mass from the DUY constraint and therefore also one axion in combination with the $U(1)_X$ gauge boson. We explicitly demonstrated this in section (3.7) by showing that the $U(1)_{X'}$ is anomalous. Therefore, after GUT Higgsing by H_{10} the resulting $U(1)_Y$ would also be massive. This seems to bring us back into the old situation that we are forced to consider manifolds with non-vanishing fundamental group to allow for non-trivial flat bundles[2].

Alternatively, here we propose to embed another line bundle into the second E_8 such that a linear combination of the two observable $U(1)$'s remains massless. A priori, one might think that we can take any other line bundle L_2. However, from the form of the mass terms, in particular (3.88), for the two abelian gauge factors we see immediately that the first Chern classes of the abelian bundles in both E_8s must be linearly dependent. The free overall factor relating them can of course be absorbed into the linear combination of the two $U(1)$s which remains massless. Therefore, we take $L_2 = L$ and embed the direct sum

$$W_2 = L \oplus L^{-1} \tag{7.3}$$

into the second E_8, where it leads to the breaking $E_8 \to E_7 \times U(1)_2$ and the decomposition

$$\mathbf{248} \overset{E_7 \times U(1)}{\longrightarrow} \left\{ \ (\mathbf{133})_0 + (\mathbf{1})_0 + (\mathbf{56})_1 + (\mathbf{56})_{-1} + (\mathbf{1})_2 + (\mathbf{1})_{-2} \ \right\}. \tag{7.4}$$

Note that we prefer to invoke the embedding of type B rather than type A also in the second E_8 factor so that the K-theory constraint $c_1(W) \in H^2(\mathcal{M}, 2\mathbb{Z})$ is trivially satisfied. The resulting massless spectrum is displayed in Table 7.2.

$E_7 \times U(1)_2$	cohomology (type B)
$\mathbf{56}_1$	$H^*(L)$
$\mathbf{1}_2$	$H^*(L^2)$

Table 7.2: Massless spectrum of $H = E_7 \times U(1)_2$ models.

Clearly, this is just the simplest possible choice for the "hidden" bundle. It is straightforward to consider additional non-abelian summand bundles, but we will not do so here[3].

It is needless to state that the trace over the second E_8 factor yields

$$\mathrm{tr}_{E_8^{(2)}}(\overline{F}^2) = 4(2\pi)^2\,(2\,\mathrm{ch}_2(L)). \tag{7.5}$$

In combination with the corresponding expressions (3.150) for the bundle in $E_8^{(1)}$, the tadpole cancellation condition for this model, including possible five-brane contributions, reads

$$\mathrm{ch}_2(V) + 3\,\mathrm{ch}_2(L) - \sum_a N_a \overline{\gamma}_a = -c_2(T). \tag{7.6}$$

Let us now take a closer look at the conditions for masslessness of a linear combination of the two $U(1)$s. Clearly, all three kinds of mass terms (3.87), (3.88) and (3.89) for $U(1)_{X'}$ and $U(1)_2$ must be related by the same constant factor if such a combination is to exist. We anticipated already that the contributions from the Kähler axions can vanish for a linear linear combination only if the first Chern classes of the line bundles in both E_8 factors are linearly dependent. More precisely, taking into account that

$$\kappa_{X',X'} = 10, \qquad \kappa_{2,2} = 4, \tag{7.7}$$

as can be computed via equ.(3.25), one realizes that precisely the linear combination

$$U(1)_X = \frac{1}{2}\left(U(1)_{X'} - \frac{5}{2}\,U(1)_2\right) \tag{7.8}$$

has a chance to remain massless. From (3.89) we find that in the presence of five-branes, this requires the absence of mass terms from the axions $\tilde{b}_a$ stemming

[2] For $\pi_1(X) = 0$, a line bundle with $c_1(L) = 0$ is always trivial and the observable gauge group gets enhanced to $SO(10)$.

[3] The reason is that they would produce additional matter charged under $U(1)_2$ in the second E_8 which will therefore appear as exotic electrically charged, but otherwise neutral fields from the point of view of the "visible" sector. The only exception is the embedding of an $SU(2)\times U(1)$ into the second E_8, in which case the analysis goes through almost identically.

from the self-dual tensors on their worldvolume since these terms contribute with opposite signs in the two E_8 sectors. Going now back to the mass term involving the universal axio-dilaton, we conclude that the combination (7.8) indeed remains massless if and only if the following conditions are satisfied

$$\int_{\mathcal{M}} c_1(L) \wedge c_2(V) = 0, \qquad \int_{\gamma_a} c_1(L) = 0 \quad \text{for all M5 branes.} \tag{7.9}$$

In this case the resulting chiral massless spectrum simplifies considerably and is given in table 7.3 .

$SU(5) \times U(1)_X \times E_7$	chirality	SM part.
$(\mathbf{10},\mathbf{1})_{\frac{1}{2}}$	$\chi(V) = g$	$(q_L, d^c_R, \nu^c_R) + [H_{10} + \overline{H}_{10}]$
$(\mathbf{10},\mathbf{1})_{-2}$	$\chi(L^{-1}) = 0$	–
$(\overline{\mathbf{5}},\mathbf{1})_{-\frac{3}{2}}$	$\chi(V \otimes L^{-1}) = g$	(u^c_R, l_L)
$(\overline{\mathbf{5}},\mathbf{1})_{1}$	$\chi(\bigwedge^2 V) = 0$	$[(h_3, h_2) + (\overline{h}_3, \overline{h}_2)]$
$(\mathbf{1},\mathbf{1})_{\frac{5}{2}}$	$\chi(V \otimes L) + \chi(L^{-2}) = g$	e^c_R
$(\mathbf{1},\mathbf{56})_{\frac{5}{4}}$	$\chi(L^{-1}) = 0$	–

Table 7.3: Massless spectrum of $H = SU(5) \times U(1)_X$ models with $g = \frac{1}{2}\int_X c_3(V)$.

Remarkably, just the requirement that the $U(1)_X$ be massless automatically leads to precisely g generations of flipped $SU(5)$ matter and no further chiral exotic states. This is straightforward to see: Just take the wedge product of the tadpole equation (7.6) with $c_1(L)$, integrate over $\mathcal{M}$ and use (7.9) to find

$$\begin{aligned} &\int_{\mathcal{M}} c_1(L)^3 = -\frac{1}{2}\int_{\mathcal{M}} c_2(T) \wedge c_1(L) \\ &\Longrightarrow \chi(L^{\pm 1}) = 0, \quad \chi(V \otimes L^{-1}) = \chi(V \otimes L) + \chi(L^{-2}) = \chi(V). \end{aligned} \tag{7.10}$$

One important and very attractive consequence of the breaking of E_8 to $SU(5)$ via a non-trivial line bundle is that the electroweak Higgs carries different quantum numbers than the lepton doublets, as is obvious from table 7.1. The consequences of this peculiarity, which distinguishes the spectrum of our flipped models from that emerging from conventional Wilson line breaking, for the absence of proton decay operators will be discussed in the next section.

Note that in general the right-handed electrons receive contributions from both the first and the second E_8. From a phenomenological point of view, we need to circumvent these latter in order to avoid non-MSSM like selection rules for their Yukawa couplings. They are absent if additionally one requires

$$(7.6),\ (7.9) \text{ and } \chi(L^{-2}) = 0 \Longrightarrow \int_{\mathcal{M}} c_1^3(L) = 0 = \int_{\mathcal{M}} c_1(L) \wedge c_2(T) = 0. \tag{7.11}$$

With these extra conditions, the generalized DUY condition for the bundle L simplifies considerably,

$$\int_{\mathcal{M}} J \wedge J \wedge c_1(L) = 0, \tag{7.12}$$

and contains only the tree-level part. The same holds for V, of course. We recall the crucial observation made in section 3.6.3 that it is precisely in such a situation that μ-stability of V guarantees a solution to the deformed Hermitian Yang-Mills equation for sufficiently small g_s. Also, equation 7.12 "freezes" only one of the h_{11} Kähler moduli. By contrast, the threshold corrections to the gauge kinetic functions will be non-vanishing. For consistency of the low-energy effective theory we need to ensure that the DUY can actually be solved in a regime inside the Kähler cone where the real part of the threshold corrected gauge kinetic functions is positive, at least for the unbroken gauge symmetries. Apart form the $SU(5)$ and the hidden E_7 symmetry, we will therefore have to check this condition for the gauge kinetic function of the generator of $U(1)_X$, which is given by[4]

$$f_{X,X} = \frac{1}{4}\left(f_{X',X'} + \left(\frac{5}{2}\right)^2 f_{2,2} - 5\, f_{X',2}\right) \tag{7.13}$$

in terms of the corresponding quantities for $U(1)_{X'}$ and $U(1)_2$.

7.1.2 Yukawa couplings and proton decay

This string theory realization of flipped $SU(5) \times U(1)_X$ exhibits many of the characteristic features of the field theory GUT model. For their details we refer to [75, 157, 159, 160].

The GUT breaking is naturally accomplished via a non-vanishing vacuum expectation value of the singlet component in $H_{10} + \overline{H}_{10}$. This leads to a natural solution of the doublet-triplet splitting problem via a missing partner mechanism in the superpotential coupling

$$\mathbf{10}^H_{\frac{1}{2}}\, \mathbf{10}^H_{\frac{1}{2}}\, \mathbf{5}_{-1}. \tag{7.14}$$

The reason is that after GUT breaking all components of $H_{10} + \overline{H}_{10}$ acquire a GUT scale mass except for a singlet and a triplet which combine, via the above coupling, with the triplet h_3 in the $\overline{\mathbf{5}}_1$, i.e. the electro-weak Higgs, in just the right way as to make it heavy. More details are given in [159].

This has very attractive consequences for proton stability since problematic dimension-five operators involving the otherwise present h_3 component and which would mediate proton decay can be suppressed. Furthermore, as shown in [161], flipped $SU(5)$ differs from the Georgi-Glashow model in that also the dimension-six proton decay operators, emerging after integrating out the off-diagonal gauge

[4]See appendix C for some remarks on this point.

bosons in the $(\mathbf{3},\mathbf{2})$, can be completely eliminated. Additional details and more references can also be found in [162].

Moreover, the gauge invariant Yukawa couplings

$$\mathbf{10}^i_{\frac{1}{2}}\,\mathbf{10}^j_{\frac{1}{2}}\,\mathbf{5}_{-1},\quad \mathbf{10}^i_{\frac{1}{2}}\,\overline{\mathbf{5}}^j_{-\frac{3}{2}}\,\overline{\mathbf{5}}_{1},\quad \overline{\mathbf{5}}^i_{-\frac{3}{2}}\,\mathbf{1}^j_{\frac{5}{2}}\,\mathbf{5}_{-1}, \tag{7.15}$$

lead to Dirac mass-terms for the d, (u,ν) and e quarks and leptons after electroweak symmetry breaking. If there exist additional gauge singlets ϕ_{10}, then couplings of the form $\mathbf{10}^i_{\frac{1}{2}}\,\overline{\mathbf{10}}^H_{-\frac{1}{2}}\,\phi_{10}$ can give rise to Majorana type neutrino masses and therefore to a see-saw mechanism. These gauge singlets are certainly present in our set-up in the form of the vector bundle moduli, i.e. non-chiral matter counted by $H^*(\mathcal{M}, V\otimes V^*)$.

Since the electroweak Higgs carries different quantum numbers than the lepton doublet, the dangerous dimension-four proton decay operators

$$\mathbf{l\,l\,e}\quad\in\quad \overline{\mathbf{5}}^i_{-\frac{3}{2}}\,\mathbf{1}^j_{\frac{5}{2}}\,\overline{\mathbf{5}}^k_{-\frac{3}{2}},\qquad\qquad \mathbf{q\,d\,l},\quad \mathbf{u\,d\,d}\quad\in\quad \mathbf{10}^i_{\frac{1}{2}}\,\mathbf{10}^j_{\frac{1}{2}}\,\overline{\mathbf{5}}^k_{-\frac{3}{2}} \tag{7.16}$$

are not gauge invariant and thus absent. A detailed discussion of this peculiar property of heterotic constructions with line bundles has recently been given in [102] in the context of Georgi-Glashow $SU(5)$.

7.1.3 Gauge coupling unification

We now discuss the issue of gauge coupling unification in detail.

The basis of the subsequent analysis is the well-known logarithmic running of the coupling constants for the gauge factors, labelled by i, in some low-energy effective field theory,

$$\frac{1}{\alpha(\mu)_i} = \frac{k_i}{\alpha_{GUT}} + \frac{b_i}{2\pi}\log\left(\frac{\mu}{M_{GUT}}\right). \tag{7.17}$$

Here, α_{GUT} represents the values of the inverse squared gauge coupling (times 4π) of a hypothetical GUT gauge group at the unification scale M_{GUT}. The coefficients b_i parameterise the field theoretic running of the couplings due to one-loop graphs. Their value is of course set by the charged particle content up to the GUT scale. The well-known observation for the Standard Model is that, given the values for α_3, α_Y and α_2 measured at the weak scale and under the assumptions of just the MSSM matter up to M_{GUT}, the system of three equations (7.17) is satisfied with $M_{GUT} = 2\cdot 10^{16}$ GeV and $k_3 = k_2 = \frac{3}{5}k_Y$ [163, 164].

Now if one breaks a stringy $SU(5)$ or $SO(10)$ GUT model down to the Standard Model via discrete Wilson lines, then the underlying string theory already makes a definite prediction for the parameters k_i which relate the gauge couplings at M_{GUT}. These are indeed the usual ones as for $SU(5)$ or $SO(10)$ GUT theories, i.e.

$$\alpha_3 = \alpha_2 = \frac{5}{3}\alpha_Y = \alpha_{GUT}. \tag{7.18}$$

Consequently, for consistency with the observed MSSM couplings at the weak scale, one can deduce from (7.17) that $\alpha_{GUT} \simeq \frac{1}{24}$.

As we have seen, in String Theory, the gauge couplings comprise, beyond their tree-level part, additional string one-loop threshold corrections. Under the phenomenological assumption that up to α_{GUT} the MSSM spectrum is not augmented by additional light fields, a phenomenologically acceptable string vacuum must therefore reproduce the relations (7.18) for the full, possibly threshold corrected, gauge couplings. If we are in a regime where the threshold corrections are negligible, then (7.18) must hold at string tree-level; otherwise the threshold corrections must be such that (7.18) is satisfied for the complete couplings.

An additional complication arises due to the observation that for the weakly coupled heterotic string, the prediction for the Planck scale is too low. The reason is that for small string coupling, $g_s \leq 1$, the theory relates the four-dimensional Newton's constant and the unification scale via

$$G_N \geq \frac{\alpha_{GUT}^{\frac{4}{3}}}{M_{GUT}^2}. \tag{7.19}$$

For the details of the derivation see e.g. [90]. Assuming the quoted values for M_{GUT} and α_{GUT}, the lower bound on G_N is too large by a factor of 400 [90]. This can be remedied in the strong coupling Horava-Witten theory [76, 77, 90]. Here it turns out that the values of the eleven-dimensional Planck mass M_{11}, ρ and $r_{CY} = M_{GUT}^{-1}$ have to lie within a particular range in order to be compatible both with the GUT relations and the Planck scale [5]. It is noteworthy that already the standard Wilson line approach to GUT breaking requires a tuning of the parameters of the internal manifold and the size of the eleventh dimension in order to predict correctly the observationally inferred GUT scale and Planck mass.

Let us now analyse the gauge coupling behaviour in our models. Clearly, if we consider Higgs breaking of the flipped $SU(5)$ GUT model down to the MSSM, then the prediction for the MSSM tree-level couplings α_3 and α_2 at the GUT scale is simply $\alpha_3 = \alpha_2 = \alpha_5$, since they both emerge from the same $SU(5)$. What is special is that the $U(1)_X$ and therefore also the final $U(1)_Y$ gauge symmetry, by contrast, have their origin in both E_8 walls. Recall the definitions of the various abelian charges as

$$\frac{1}{2}Q_Y = -\frac{1}{5}Q_Z + \frac{2}{5}Q_X, \qquad Q_X = \frac{1}{2}\left(Q_{X'} - \frac{5}{2}Q_2\right) \tag{7.20}$$

so that the gauge kinetic functions satisfy the relation

$$f_{YY} = \frac{4}{25}\left(f_{Z,Z} + f_{X',X'} + \left(\frac{5}{2}\right)^2 f_{2,2} - 5\, f_{X',2}\right). \tag{7.21}$$

[5] Very qualitatively, this means that $1 \ll r_{CY} \ll \rho$ in string units. The precise constraints can be found in [90].

Since Q_Z is the diagonal $U(1)$ generator within $SU(5)$, the gauge couplings are identical up to the normalisation

$$f_{Z,Z} = \frac{5}{12} f_{SU(5)}. \tag{7.22}$$

The non-abelian gauge coupling of the $SU(5)$ including the one-loop contribution follows from (3.103) as

$$\begin{aligned} \frac{1}{\alpha_5} &= \frac{1}{3\ell_s^6 g_s^2}\int_{\mathcal{M}} J\wedge J\wedge J - \frac{1}{\ell_s^2}\int_{\mathcal{M}} J\wedge\left[-c_2(V)+c_1^2(L)+\frac{1}{2}c_2(T)\right] \\ &\quad +\frac{1}{\ell_s^2}\sum_a N_a\left(\frac{1}{2}-\lambda_a\right)^2\int_{\gamma_a} J. \end{aligned} \tag{7.23}$$

Using

$$\eta_{X',X'} = 40, \quad \eta_{2,2} = 4, \quad \kappa_{1,1} = 10, \quad \kappa_{2,2} = 4, \tag{7.24}$$

we can likewise read off the expressions for $f_{X',X'}$, $f_{2,2}$ and $f_{X',2}$ from (3.104) and (3.105). In view of the relations (7.21) and (7.22) we eventually conclude that

$$\frac{1}{\alpha_Y} = \frac{8}{3}\frac{1}{\alpha_5} - \frac{1}{\ell_s^2}\int_{\mathcal{M}} J\wedge\left[c_2(V)+4\,c_1^2(L)\right] + \frac{2}{\ell_s^2}\sum_a N_a\lambda_a\int_{\gamma_a} J. \tag{7.25}$$

Note that the second and third summands in (7.25) arise at one-loop as compared to the lowest order contribution in $\frac{1}{\alpha_5}$. As we see, these string models do not give rise to the usual GUT tree level relation $\alpha_{GUT} = \frac{5}{3}\alpha_Y$, but instead to $\alpha_{GUT} = \frac{8}{3}\alpha_Y$. Therefore, if we assume just the Standard Model spectrum up to the unification scale (i.e. no additional vector-like matter like Higgs pairs) and if we are in a situation where the threshold corrections present in (7.25) are negligible, the gauge couplings do not unify at M_{GUT}. This is, however, not compelling once we give up one of the two stated assumptions. As far as the threshold corrections are concerned, depending on their precise value in the vacuum under consideration, they can eventually give a unified gauge coupling picture again. Defining

$$\frac{1}{\alpha_Y} = \frac{8}{3}\frac{1}{\alpha_{GUT}} + \Delta \tag{7.26}$$

we see that the threshold correction must take the value $\Delta = -\frac{1}{\alpha_{GUT}} \sim -24$, i.e.

$$\left.\frac{1}{\alpha_Y}\right|_{1-\text{loop}} = -\frac{3}{8}\left.\frac{1}{\alpha_Y}\right|_{\text{tree}}. \tag{7.27}$$

For $\alpha_{GUT} = 1/24$, such a relation can just be satisfied with $g_s < 1$ and $r_{CY} > \sqrt{\alpha'}$ for large enough Chern classes of the vector bundles. We will see in the next section that for our explicit models this is indeed possible. Of course, in the weakly coupled heterotic framework, the Planck scale still comes out too low and one must consider Horava-Witten theory, where now the next-to-leading order corrections to the gauge couplings are to be taken into account.

To conclude, what distinguishes our models from the standard Wilson line approach to GUT breaking is the appearance of one further constraint on the geometry of the compactifications. We reiterate that in the standard scenario, too, the condition that the four-dimensional Planck mass come out correctly reduces the predictive power of the setup in that it involves additional tuning of the geometric parameters of the background. In that respect, including also (7.27) into the model building wish-list is conceptually just along the lines of the standard procedure.

Alternatively, one can contemplate that extra light Higgs fields, if present in the non-chiral spectrum, might lead to gauge coupling unification at a different scale. However, this scale is necessarily lower than the usual GUT scale, which worsens the mismatch of the Planck scale.

7.1.4 An example on dP_4

Having discussed the chief phenomenological aspects of our heterotic flipped $SU(5)$ construction, we now prove that it is indeed possible to find explicit realisations in our framework which meet all the string consistency conditions and give rise to precisely the chiral MSSM spectrum. We choose as our background manifold elliptically fibered Calabi-Yau threefolds over the base dP_4 (see section 5.3 for a summary of their properties). We recall in particular that the second Chern class of the tangent bundle is given by (5.8),

$$c_2(T) = [36l - 12\sum_{i=1}^{4} E_i]\,\sigma + 62F, \tag{7.28}$$

where $c_1(\mathrm{dP}_4)$ is expanded in the cohomological basis and F is the class of the fiber. The Mori cone is generated by the 10 effective classes E_i, $l - E_i - E_j$, $i, j = 1, \ldots, 4$, $i \neq j$.

We have found a couple of three-generation flipped $SU(5)$ vacua satisfying all the required constraints. They are displayed in table D.1 of appendix D. We choose the following example to demonstrate their properties. The $U(4)$ bundle is given by the data

$$\begin{aligned} \lambda &= \frac{1}{4}, \quad q = 0, \\ \eta &= 14l - 2E_1 - 6E_2 - 6E_3 - 2E_4, \\ c_1(\zeta) &= -4l + 4E_2 + 4E_3 + 4E_4. \end{aligned} \tag{7.29}$$

Note that the first Chern class of the line bundle $\mathcal{N}$ in the spectral cover construction (5.23) is an integer class, as required:

$$c_1(\mathcal{N}) = 3\sigma + \pi_C^* \left(8l - 2E_1 - 3E_2 - 3E_3 - 2E_4\right). \tag{7.30}$$

It is easy to see that $|\eta|$ is base point free, since its intersection with the generators of the Mori cone is always positive. One can also easily show that η is effective as well as $\eta - 4c_1(\mathrm{dP}_4) = 2l + 2E_1 - 2E_2 - 2E_3 + 2E_4$. Thus, this bundle is μ-stable. The resulting Chern classes are

$$\begin{aligned} c_1(V) &= -4l + 4E_2 + 4E_3 + 4E_4, && (7.31) \\ c_2(V) &= [14l - 2E_1 - 6E_2 - 6E_3 - 2E_4]\,\sigma - 29F. && (7.32) \end{aligned}$$

In our setup, the first Chern class of the line bundle must be equal to the first Chern class of the vector bundle (see (3.147)), thus

$$c_1(L) = -4l + 4E_2 + 4E_3 + 4E_4. \tag{7.33}$$

To find a solution to the tadpole condition, we also include M5-branes. Their combined associated cohomology class is

$$[W] = 27F + (22l - 10E_1 - 6E_2 - 6E_3 - 10E_4)\,\sigma. \tag{7.34}$$

To make physical sense, $[W]$ must be Poincaré dual to the homology class of a curve γ in $\mathcal{M}$, and must be therefore effective. $[W]$ is effective if its part on the fiber is greater than or equal to zero and its part on the base is effective in B. Therefore, we rewrite $[W]$ in terms of generators of the Mori cone,

$$\begin{aligned} [W] &= \sum_a N_a \overline{\gamma}_a = 27F + [12E_1 + 6(l - E_1 - E_2) && (7.35) \\ &\quad +6(l - E_1 - E_3) + 10(l - E_1 - E_4)]\,\sigma. \end{aligned}$$

The generators of the Mori cone, being irreducible as effective classes, represent the classes dual to the irreducible curves γ_a around which we wrap N_a five-branes. In general, this decomposition is not unique. However, we also have to satisfy the constraint $\int_{\gamma_a} c_1(L) = 0$ for a massless $U(1)_X$, and (7.35) is the only remaining decomposition compatible with this requirement. The tadpole cancellation condition for this setup, written in terms of Chern classes, takes the form

$$-c_2(V) + 2c_1^2(L) - [W] = -c_2(T) \tag{7.36}$$

and is indeed satisfied. It is a simple calculation to show that the conditions to keep the $U(1)_X$ in the flipped $SU(5)$ model massless hold

$$\int_{\mathcal{M}} c_1(L) \wedge c_2(V) = 0, \quad \int_{\gamma_a} c_1(L) = \int_{\mathcal{M}} c_1(L) \wedge \overline{\gamma}_a = 0. \tag{7.37}$$

Since the Chern class of the line bundle has no part in the fiber, the integral over its third power trivially vanishes,

$$\int_{\mathcal{M}} c_1^3(L) = 0, \tag{7.38}$$

and thus a contribution to the right-handed electrons from the second E_8 factor is prevented. The number of generations in our example is given by

$$\chi(V) = \frac{1}{2}\int_{\mathcal{M}} c_3(V) = 3 \tag{7.39}$$

since $\int_{\mathcal{M}} c_1(V) \wedge c_2(V) = \int_{\mathcal{M}} c_1(L) \wedge c_2(V) = 0$.

Expanding the Kähler class in the cohomological basis,

$$J = l_s^2(r_\sigma \sigma + r_0 l + \sum_{m=1}^{4} r_m E_i), \tag{7.40}$$

the DUY-equation (7.12)

$$\int_{\mathcal{M}} J \wedge J \wedge c_1(L) = -8 l_s^4 r_\sigma (r_0 + r_2 + r_3 + r_4) = 0 \tag{7.41}$$

fixes one Kähler modulus. There exist solutions inside the Kähler cone. Take as an example

$$0 < r_\sigma < 2\rho, \quad r_0 = 3\rho, \quad r_m = -\rho, \quad m = 1, \ldots, 4. \tag{7.42}$$

With this choice, equation (7.41) holds and the Kähler class lies inside the Kähler cone for every $\rho \in \mathbb{R}^+$.

The universal gauge coupling for the non-abelian visible gauge group (3.103) can be computed as[6]

$$\frac{4\pi}{g_1^2} = \frac{1}{3g_s^2}\left(5r_\sigma^3 - 15r_\sigma^2\rho + 15r_\sigma\rho^2\right) - 24r_\sigma - 4\rho - (\frac{1}{2} - \lambda_5)^2(7r_\sigma - 34\rho), \tag{7.43}$$

which is positive for a suitable choice of parameters. The abelian gauge couplings are given by (3.104,3.105)

$$\begin{aligned} 4\pi \mathrm{Re}\,(f_{i,i}) &= \frac{\eta_{i,i}}{4}\left(\frac{1}{3g_s^2}(5r_\sigma^3 - 15r_\sigma^2\rho + 15r_\sigma\rho^2)\right. && (7.44) \\ &\left. -24r_\sigma - 4\rho - (\frac{1}{2} - \lambda_5)^2(7r_\sigma - 34\rho)\right) + \frac{320}{3}r_\sigma, \\ 4\pi \mathrm{Re}\,(f_{X',2}) &= -\frac{160}{3}r_\sigma && (7.45) \end{aligned}$$

with $\eta_{X',X'} = 40$ and $\eta_{2,2} = 4$. The resulting gauge coupling (7.13) for the $U(1)_X$ is then positive again:

$$\begin{aligned} 4\pi \mathrm{Re}\, f_{X,X} &= \frac{65}{16}\left(\frac{1}{3g_s^2}(5r_\sigma^3 - 15r_\sigma^2\rho + 15r_\sigma\rho^2) - 24r_\sigma - 4\rho\right. && (7.46) \\ &\left. -(\frac{1}{2} - \lambda_5)^2(7r_\sigma - 34\rho)\right) + 260\, r_\sigma. \end{aligned}$$

[6]Note that in the following equations, λ_5 is the five-brane modulus and not the parameter belonging to the bundle data.

In view of the discussion of possible gauge coupling unification, we note that the threshold correction as defined in 7.27 is, assuming for simplicity that $\lambda_a = 0$ for all five-branes,

$$\Delta = -\frac{1}{\ell_s^2}\int_{\mathcal{M}} J \wedge \left[c_2(V) + 4\, c_1^2(L)\right] = 183 r_\sigma - 26\rho \tag{7.47}$$

and has the correct sign if $r_\sigma < \frac{26}{183}\rho$.

Note that with this choice for r_σ, the positivity of the gauge couplings can still be achieved and, equally importantly, it is consistent with the requirement that $r_\sigma \leq \rho$ in order that the proof of μ-stability of the bundles can be trusted.

To summarize, this example with three chiral generations satisfies the tadpole condition (7.6) as well as the constraints (7.9) guaranteeing a massless $U(1)_X$. We have no non-MSSM like selection rules for the Yukawa couplings of the right-handed electrons since there are indeed no contributions from the second E_8 (7.11). Furthermore, the Kähler moduli can be chosen such that the DUY equation (7.12) holds and the gauge couplings are positive.

In appendix D, we list all three-generation models we have found on dP_4 by a computer search which likewise satisfy all these conditions. We have also found three-generation examples for a scenario directly giving rise to the Standard Model gauge symmetry, to be discussed in the next section.

7.2 Just the $SU(3) \times SU(2) \times U(1)_Y$ gauge symmetry

7.2.1 $SU(5) \times U(1)$ bundles

As we have spelled out in section 3.8.1, the direct breaking of E_8 to the Standard Model group is possible by choosing a bundle with structure group $SU(5) \times U(1)_{Y'}$, resulting in gauge group $SU(3) \times SU(2) \times U(1)_{Y'}$. Similarly to the flipped $SU(5)$ construction, we embed a bundle of type B,

$$W = V \oplus L^{-1}, \quad \text{with } c_1(V) = c_1(L),\ \text{rank}(V) = 5 \tag{7.48}$$

into the first E_8.

We have seen that again the $U(1)_{Y'}$ by itself cannot remain massless so that we will perform the same construction as for the flipped $SU(5)$ model. We can therefore be comparatively brief about the details of the largely analogous construction. We embed the line bundle L, or rather $W_2 = L \oplus L^{-1}$, also in the second E_8 and realize that here the linear combination

$$U(1)_Y = \frac{1}{3}\left(U(1)_{Y'} - 3\, U(1)_2\right) \tag{7.49}$$

remains massless if again the conditions

$$\int_{\mathcal{M}} c_1(L) \wedge c_2(V) = 0, \qquad \int_{\gamma_a} c_1(L) = 0 \tag{7.50}$$

are satisfied. The resulting chiral massless spectrum takes the simple form given in table 7.4.

$SU(3)\times SU(2)\times U(1)_Y\times E_7$	chirality	SM part.
$(\mathbf{3},\mathbf{2},\mathbf{1})_{\frac{1}{3}}$	$\chi(V)=g$	q_L
$(\mathbf{3},\mathbf{2},\mathbf{1})_{-\frac{5}{3}}$	$\chi(L^{-1})=0$	–
$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{\frac{2}{3}}$	$\chi(\bigwedge^2 V)=g$	d^c_R
$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-\frac{4}{3}}$	$\chi(V\otimes L^{-1})=g$	u^c_R
$(\mathbf{1},\mathbf{2},\mathbf{1})_{-1}$	$\chi(\bigwedge^2 V\otimes L^{-1})=g$	l_L
$(\mathbf{1},\mathbf{1},\mathbf{1})_{2}$	$\chi(V\otimes L)+\chi(L^{-2})=g$	e^c_R
$(\mathbf{1},\mathbf{1},\mathbf{56})_{1}$	$\chi(L^{-1})=0$	–

Table 7.4: Massless spectrum of $H = SU(3)\times SU(2)\times U(1)_Y$ models with $g = \frac{1}{2}\int_{\mathcal{M}} c_3(V)$.

Therefore, one gets precisely g generations of Standard Model matter without a right-handed neutrino. The right-handed electrons have contributions from both the first and the second E_8. The latter are again absent if additionally one requires

$$\int_{\mathcal{M}} c_1^3(L) = 0. \tag{7.51}$$

In this model, there are no additional gauge or obvious discrete symmetries carried by the Standard Model particles, so that the dangerous dimension four proton decay operators are not necessarily vanishing. We refer to table D.2 in appendix D for a couple of examples with just the Standard Model chiral matter which we have found in this setup using the spectral cover method over dP_4 fibered Calabi-Yau threefolds.

7.2.2 Gauge coupling unification

The issue of gauge coupling unification is precisely the same as what we have encountered in the flipped $SU(5)$ context. Now the gauge kinetic function for

$$U(1)_Y = \frac{1}{3}\left(U(1)_{Y'} - 3\,U(1)_2\right) \tag{7.52}$$

follows as

$$f_{Y,Y} = \frac{1}{9}\left(f_{Y',Y'} - 6 f_{Y',2} + 9 f_{2,2}\right). \tag{7.53}$$

Each individual term above can be computed from the general expressions (3.104) and (3.105) with the help of the trace parameters

$$\eta_{1,1} = 60, \quad \eta_{2,2} = 4, \quad \kappa_{1,1} = 12, \quad \kappa_{2,2} = 4, \tag{7.54}$$

and the gauge couplings for $SU(3)$ and $SU(2)$ equal the expression (7.23). One eventually concludes that again

$$\frac{1}{\alpha_Y} \;=\; \frac{8}{3}\frac{1}{\alpha_{3,2}} - \frac{1}{\ell_s^2}\int_X J \wedge \left[c_2(V) + 4\, c_1^2(L)\right] + \frac{2}{\ell_s^2}\sum_a N_a \lambda_a \int_{\gamma_a} J. \tag{7.55}$$

We therefore find ourselves exactly in the same situation as in section (7.1.3), to which we refer for a discussion of the significance of this result for gauge coupling unification.

Chapter 8

Conclusions and Outlook

The embedding of unitary bundles into the ten-dimensional gauge group of the heterotic string reveals a remarkably rich and hitherto neglected structure. At the conceptual level, the starring role in geometric string compactifications is played by holomorphic stable bundles - both in the heterotic and the Type I/ Type IIB orientifold corner of the M-theory moduli space. Despite the differences in the fundamental worldsheet formulation of these dual theories, we can therefore apply basically the same techniques to an investigation of their perturbative four-dimensional vacua. The differences in the structure of the emerging gauge sector in this setup has been identified as being primarily due to the group theoretic features of $E_8 \times E_8$ on the one hand and $SO(32)$ on the other, most notably the respective natural subgroups including the decomposition of the adjoint representation. The identical massless spectrum emerging from the $SO(32)$ heterotic and the Type I string on D9-branes with unitary gauge flux is satisfactory in view of the conjectured S-duality relating both descriptions, but not completely trivial - after all S-duality is a non-perturbative symmetry and interchanges, at the microscopic level, the fundamental strings of one theory with the solitonic, non-perturbative objects of the other. In that respect we point out that although we found complete agreement in our specific setup, there is still a puzzle remaining how the recent emergence of various spinor representations in the context of $SO(32)$ heterotic orbifold models [140] can be understood from the point of the view of our bundle constructions. An answer to this question might well follow from a better understanding of the general relation between orbifold constructions and smooth Calabi-Yau compactifications.

Focusing again on the latter, it perfectly fits into the picture just sketched that the well-established α'-corrections to the supersymmetry condition for background gauge fields translate into string-loop corrections on the heterotic side. For the $SO(32)$ theory the expressions we found for the integrated supersymmetry condition are in one-to-one correspondence with the Type IIB MSSM equation [145] and only depend on the information of the individual $U(N)$ gauge factor under consideration. Clearly this just what we expect from the S-dual picture of independent magnetized D9-brane stacks. For the $E_8 \times E_8$ theory, by contrast, the one-loop correction involves contributions from all background instantons. On the Type IIB side the perturbative α'-corrections are known to affect not only the integrated supersymmetry equation, but also the local Hermitian Yang-Mills equations and therefore modify the stability condition from μ-stability, valid at tree-level, to π-stability. This inspired us to conjecture a corresponding modification of the stability condition on the bundles also on the heterotic side which we called λ-stability. Both λ- and π-stability seem to be the right criterion only in the strict perturbative sense and applicable only under the assumption that the tree-level part in the respective slope dominates in a well-defined manner over the string-loop or α'-correction. In addition, the non-perturbative contributions induced by worldsheet instantons in Type IIB make out the full Π-stability condition in the derived bounded category of coherent sheaves and are expected to have a heterotic counterpart in the form of spacetime instantons. A detailed study of these effects including the precise mathematical definition of heterotic λ-stability is to follow. Independently of this mathematical question it would be important to justify the proposed deformation of the Hermitian Yang-Mills equation by an analysis of the ten-dimensional Killing spinor equations at the one-loop level.

In practical terms, the supersymmetry and thus stability condition on the heterotic/Type IIB side appears to be more approachable than in the mirror dual framework of Type IIA orientifolds. The reason is that the special Lagrangian condition on supersymmetric three-cycles for A-branes is beyond the regime of complex geometry, whose powerful technology, on the other hand, enables one to construct quite general supersymmetric holomorphic bundles as the dual objects. In this way, we can view the embedding of unitary bundles into the $SO(32)$ heterotic/Type I string as bypassing the unsolved mathematical problem of identifying special Lagrangian three-cycles on general Calabi-Yau manifolds.

As far as the model building prospects are concerned, the most prominent advantage of the embedding of unitary bundles into the $E_8 \times E_8$ string is the "decoupling" of the gauge bundles from the topology and geometry of the background manifold in that we do no more depend on the presence of a non-trivial first fundamental group. We expect this to be of crucial assistance when it comes to extending heterotic model building to the more realistic framework of non-Kähler compactifications with non-vanishing form field fluxes. This will eventually be inevitable in order to tackle such pressing problems as moduli stabilisation and dynamical supersymmetry breaking with nonetheless realistic gauge sectors.

As a first step, however, we have restricted our explicit model search to the standard framework of elliptically fibered Calabi-Yau backgrounds where we can rely on the spectral cover construction of stable holomorphic bundles. Even a very preliminary search has revealed a number of vacua with flipped $SU(5) \times U(1)_X$ and MSSM gauge group and precisely the observed three generations of chiral matter. From the phenomenological point of view, this is just the very first step. A computation of the cohomology groups which count the charged matter will also reveal the amount of vector-like matter pairs which cannot be deduced just from the Euler characteristic of the gauge bundles. In particular, we need to determine the number of electro-weak Higgs pairs and, in the case of the flipped $SU(5) \times U(1)_X$ models, the number of GUT Higgses which are required for the vacua to give rise to realistic models at the weak scale. A derivation of the mathematical methods required for this computation is beyond the scope of this thesis and is postponed to the forthcoming publication [165], where we will also exploit the framework of stable bundle extensions for our model search. Let us merely anticipate here that this technique seems to provide us with a surprisingly large number of models with a very realistic spectrum including the appearance of precisely three families of quarks and leptons.

An even more challenging task will be the computation of the Yukawa couplings and μ-terms, possibly along the lines of [166–168]. As we briefly outlined, there seem to exist no a priori selection rules in our case which forbid any of the phenomenologically required Yukawas, but the explicit computation of the physical couplings is only possible once we know the Kähler potential for the charged matter fields in order to normalise their kinetic terms appropriately.

Our entire analysis has focused on the perturbative, large volume regime and avoided an explicit worldsheet formulation. It is not only of academic interest, though, to clarify the status of the underlying $(0,2)$ non-linear σ-model and whether or not it admits a description in terms of a Landau-Ginzburg [169] or gauged linear σ-model [31]. In such situations, the theory can be shown to be free of potentially destabilising worldsheet instantons [170–172].

In the absence of a deeper understanding of the structure principles behind the vast landscape of string vacua the fate of all string model building attempts is to resemble the search for the famous needle in a hay stack. Unless this situation changes drastically due to some revolutionary insights, it appears therefore reasonable to supplement the concrete model-by-model search by a statistical analysis of the distribution of the characteristic features in the moduli space of vacua. In view of the conceptual similarities of the gauge sectors arising on the Type II and the heterotic side, the statistical approach performed in [155, 156, 173, 174] for Type IIA orientifolds or of [63] for models at the Gepner point seems within reach also for the heterotic string. Such an analysis of a special class of non-supersymmetric four-dimensional heterotic vacua has recently appeared in [175]. After all, the aim of String Theory is none less than to determine the status of the observed laws of Nature within the set of thinkable worlds.

Appendix A

Some useful mathematical facts

A.1 Topological invariants of vector bundles

Throughout this thesis we have made extensive use of various topological invariants of vector bundles. For convenience of the reader we collect here some useful definitions and identities. Much more information can be found e.g. in [176].

Let V be a complex rank r vector bundle over a complex d-dimensional manifold with field strength F. Then the total Chern character $\mathrm{ch}(V)$ is defined as

$$\begin{aligned}
\mathrm{ch}(F) &= \mathrm{tr}\, e^{\frac{1}{2\pi}F} = \sum_{k=1}^{d} \mathrm{ch}_k(V) \\
\mathrm{ch}_k(V) &= \frac{1}{k!\,(2\pi)^k} \mathrm{tr}\, F^k.
\end{aligned} \tag{A.1}$$

Note that $\mathrm{ch}_0(V) = r$. Furthermore the Chern characters of the complex conjugate bundle V^* are

$$\mathrm{ch}_k(V^*) = (-1)^k \mathrm{ch}_k(V). \tag{A.2}$$

The Chern character of the tensor product and the Whitney sum of two vector bundles V_a and V_b of rank r_a and r_b respectively can be found from the relation.

$$\begin{aligned}
\mathrm{ch}(V_a \otimes V_b) &= \mathrm{ch}(V_a) \wedge \mathrm{ch}(V_b), \\
\mathrm{ch}(V_a \oplus V_b) &= \mathrm{ch}(V_a) + \mathrm{ch}(V_b).
\end{aligned} \tag{A.3}$$

In particular,

$$\begin{aligned}
\mathrm{ch}_0(V_a \otimes V_b) &= r_a\, r_b \\
\mathrm{ch}_1(V_a \otimes V_b) &= r_b\, \mathrm{ch}_1(V_a) + r_a\, \mathrm{ch}_1(V_b), \qquad \text{(A.4)} \\
\mathrm{ch}_2(V_a \otimes V_b) &= r_b\, \mathrm{ch}_2(V_a) + \mathrm{ch}_1(V_a) \wedge \mathrm{ch}_1(V_b) + r_a\, \mathrm{ch}_2(V_b) \\
\mathrm{ch}_3(V_a \otimes V_b) &= r_b\, \mathrm{ch}_3(V_a) + \mathrm{ch}_1(V_a) \wedge \mathrm{ch}_2(V_b) + \mathrm{ch}_2(V_a) \wedge \mathrm{ch}_1(V_b) + r_a\, \mathrm{ch}_3(V_b).
\end{aligned}$$

It immediately follows that the Chern characters of the "adjoint" $V \otimes V^*$ bundle read

$$\begin{aligned} \mathrm{ch}_0(V \otimes V^*) &= 2r, \\ \mathrm{ch}_1(V \otimes V^*) &= 0, \\ \mathrm{ch}_2(V \otimes V^*) &= 2r\,\mathrm{ch}_2(V) - (\mathrm{ch}_1(V))^2, \\ \mathrm{ch}_3(V \otimes V^*) &= 0. \end{aligned} \tag{A.5}$$

For the Chern characters of the antisymmetric and symmetric tensor products one can prove that (see e.g. [153])

$$\begin{aligned} \mathrm{ch}_1(\bigwedge^2 V) &= (r-1)\,\mathrm{ch}_1(V), \\ \mathrm{ch}_2(\bigwedge^2 V) &= (r-2)\,\mathrm{ch}_2(V) + \frac{1}{2}\,\mathrm{ch}_1^2(V), \\ \mathrm{ch}_3(\bigwedge^2 V) &= (r-4)\,\mathrm{ch}_3(V) + \mathrm{ch}_2(V)\,\mathrm{ch}_1(V). \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \mathrm{ch}_1(\bigotimes^2 V) &= (r+1)\,\mathrm{ch}_1(V), \\ \mathrm{ch}_2(\bigotimes^2 V) &= (r+2)\,\mathrm{ch}_2(V) + \frac{1}{2}\,\mathrm{ch}_1^2(V), \\ \mathrm{ch}_3(\bigotimes^2 V) &= (r+4)\,\mathrm{ch}_3(V) + \mathrm{ch}_2(V)\,\mathrm{ch}_1(V). \end{aligned} \tag{A.7}$$

By contrast, the total Chern class $c(V)$ of a vector bundle V is defined as

$$c(V) = \det(1 + \frac{1}{2\pi}F) = \sum_{k=1}^{\min(r,d)} c_k(V) \tag{A.8}$$

and satisfies

$$c(V_a \oplus V_b) = c(V_a) \wedge c(V_b). \tag{A.9}$$

In particular $c_0(V) = 1$ and for a line bundle L all Chern classes higher than $k = 1$ vanish identically, $c(L) = 1 + c_1(L)$.

The first three Chern classes and Chern characters are related as

$$\begin{aligned} \mathrm{ch}_1(V) &= c_1(V), \\ \mathrm{ch}_2(V) &= -c_2(V) + \frac{1}{2}c_1^2(V), \\ \mathrm{ch}_3(V) &= \frac{1}{2}c_3(V) - \frac{1}{2}\,c_1(V) \wedge c_2(V) + \frac{1}{6}c_1^3(V). \end{aligned} \tag{A.10}$$

The relevance of the Chern characters is obvious from their appearance in the Hirzebruch-Riemann-Roch index theorem, which counts, as we recall from section 2.2, the alternating Hodge numbers of the twisted Dolbeault complex,

$$\begin{aligned}\chi(\mathcal{M},V) &= \sum_{i=0}^{3}(-1)^i \dim(H^i(\mathcal{M},V) = \int_{\mathcal{M}} \mathrm{ch}(V)\wedge \mathrm{Td}(T\mathcal{M}) \\ &= \int_{\mathcal{M}} \left[\mathrm{ch}_3(V) + \frac{1}{12}\, c_2(T\mathcal{M})\, c_1(V)\right]. \end{aligned} \tag{A.11}$$

The last line is valid only if the manifold has complex dimension 3. The other lowest dimensional cases follow from the definition of the Todd classes

$$\begin{aligned}\mathrm{Td}_0(V) &= 1, \\ \mathrm{Td}_1(V) &= \frac{1}{2}c_1(V), \\ \mathrm{Td}_2(V) &= \frac{1}{12}(c_1^2(V) + c_2(V)) \\ &\cdots \end{aligned} \tag{A.12}$$

Restricting ourselves again to the case that $\dim(\mathcal{M}) = 3$, we can compute the Euler characteristics of products of bundles $V_a \otimes V_b$ with the help of the formula

$$\chi(V_a \otimes V_b) = r_a\, \chi(V_b) + r_b\, \chi(V_a) + c_1(V_a)\, \mathrm{ch}_2(V_b) + \mathrm{ch}_2(V_b)\, c_1(V_a). \tag{A.13}$$

Finally, for the Euler characteristic of the antisymmetric product bundle $\bigwedge^2 V$ one obtains

$$\chi(\textstyle\bigwedge^2 V) = (r-4)\, \chi(V) + c_1(V) \left(\mathrm{ch}_2(V) + \frac{1}{4}c_2(T\mathcal{M})\right) \tag{A.14}$$

and for the symmetric product bundle $\bigotimes_s^2 V$

$$\chi(\textstyle\bigotimes_s^2 V) = (r+4)\, \chi(V) + c_1(V) \left(\mathrm{ch}_2(V) - \frac{1}{4}c_2(T\mathcal{M})\right). \tag{A.15}$$

A.2 Some general trace identities

We now display some useful trace identities for $E_8 \times E_8$, $SO(32)$ and unitary groups which we have used in various places of this work. A more complete account can also be found e.g. in [177].

The symbol tr denotes, unless we explicitly specify the representation otherwise, the trace over the fundamental representation of a gauge group, while Tr refers to the adjoint. The two objects are related as follows for the cases relevant for our purposes:

$$\begin{aligned}\mathrm{Tr}_{SU(N)}F^2 &= 2N\, \mathrm{tr}_{SU(N)}F^2, \\ \mathrm{Tr}_{SO(N)}F^2 &= (N-2)\, \mathrm{tr}_{SO(N)}F^2, \end{aligned} \tag{A.16}$$

$$\mathrm{Tr}_{E_8} F^2 = 30\,\mathrm{tr}_{E_8} F^2,$$

$$\begin{aligned}
\mathrm{Tr}_{SU(N)} F^4 &= 2N\,\mathrm{tr}_{SU(N)} F^4 + 6\,(\mathrm{tr}_{SU(N)} F^2)^2 \\
\mathrm{Tr}_{SO(N)} F^4 &= (N-8)\,\mathrm{tr}_{SO(N)} F^4 + 3\,(\mathrm{tr}_{SO(N)} F^2)^2 \\
\mathrm{Tr}_{E_8} F^4 &= 9\,(\mathrm{tr}_{E_8} F^2)^2.
\end{aligned} \tag{A.17}$$

In evaluating the field theoretic anomaly six-forms we also encounter traces over the symmetric and antisymmetric representations. For $SU(N)$ the ones relevant for us are given by

$$\begin{aligned}
\mathrm{tr}^{\mathrm{Anti}}_{SU(N)} F^2 &= (N-2)\,\mathrm{tr}_{SU(N)} F^2, \\
\mathrm{tr}^{\mathrm{Sym}}_{SU(N)} F^2 &= (N+2)\,\mathrm{tr}_{SU(N)} F^2,
\end{aligned} \tag{A.18}$$

$$\begin{aligned}
\mathrm{tr}^{\mathrm{Anti}}_{SU(N)} F^3 &= (N-4)\,\mathrm{tr}_{SU(N)} F^3, \\
\mathrm{tr}^{\mathrm{Sym}}_{SU(N)} F^3 &= (N+4)\,\mathrm{tr}_{SU(N)} F^3.
\end{aligned} \tag{A.19}$$

The second order Casimir for $SO(N)$ is of course just

$$\mathrm{tr}^{\mathrm{Anti}}_{SO(N)} F^2 = \mathrm{Tr}_{SO(N)} F^2 = (N-2)\,\mathrm{tr}_{SO(N)} F^2. \tag{A.20}$$

A.3 Trace identities for the $SO(32)$ heterotic string

We collect here some useful trace identities for the spectrum of the $SO(32)$ heterotic string $U(n_i)$ factors diagonally embedded into $U(n_i N_i)$ as displayed in table (4.1).

$$\begin{aligned}
\mathrm{Tr} F\overline{F}^3 &= 12 \sum_{j=1}^{K} N_j\, f_j \wedge \left(4\mathrm{tr}_{U(n_j)} \overline{F}^3 + \mathrm{tr}_{U(n_j)} \overline{F} \sum_{i=1}^{K} N_i \mathrm{tr}_{U(n_i)} \overline{F}^2 \right), \\
\mathrm{Tr} F^2\overline{F}^2 &= 4 \sum_{j=1}^{K} \left(\mathrm{tr}_{SU(N_j)} F^2 + N_j\, (f_j)^2 \right) \wedge \left(12\, \mathrm{tr}_{U(n_j)} \overline{F}^2 + n_j \sum_{i=1}^{K} N_i\, \mathrm{tr}_{U(n_i)} \overline{F}^2 \right) \\
&+ 8 \sum_{i,j=1}^{K} N_i N_j\ f_i\, f_j \wedge \mathrm{tr}_{U(n_i)} \overline{F}\, \mathrm{tr}_{U(n_j)} \overline{F} + 2\, \mathrm{tr}_{SO(2M)} F^2 \wedge \sum_{j=1}^{K} N_j\, \mathrm{tr}_{U(n_j)} \overline{F}^2, \\
\mathrm{Tr} F^2 &= 30\, \mathrm{tr}_{SO(2M)} F^2 + 60 \sum_{j=1}^{K} n_j \left(\mathrm{tr}_{SU(N_j)} F^2 + N_j (f_j)^2 \right), \\
\mathrm{Tr} F\overline{F} &= 60 \sum_{j=1}^{K} N_j f_j \wedge \mathrm{tr}_{U(n_j)} \overline{F}, \\
\mathrm{Tr} \overline{F}^2 &= 60 \sum_{j=1}^{K} N_j\, \mathrm{tr}_{U(n_j)} \overline{F}^2.
\end{aligned} \tag{A.21}$$

Appendix B

Kähler cone constraints on Calabi-Yau's with base dP_r

The DUY equations have to admit solutions for the Kähler parameters inside the Kähler cone, i.e. such that the integral of powers of the Kähler form over all appropriate cycles are positive,

$$\int_{2-cycle} J > 0, \qquad \int_{4-cycle} J^2 > 0, \qquad \int_{\mathcal{M}} J^3 > 0. \tag{B.1}$$

We expand the Kähler form on the elliptically fibered Calabi-Yau as $J = l_s^2\,(r_\sigma\,\sigma + J_B)$ with $J_B = r_0\,l + \sum_{m=1}^{r} r_m E_m$ being the Kähler form on the base manifold dP_r in terms of the canonical basis.

From the first constraint we read immediately that the radii must satisfy

$$r_\sigma > 0, \qquad r_0 > 0, \qquad r_m < 0 \quad \text{for } m \in \{1, ..., r\}. \tag{B.2}$$

The second inequality, $\int J^2 > 0$, holds precisely if in addition

$$r_0^2 - \sum_{m=1}^{r} r_m^2 > 0, \quad r_\sigma < \frac{2}{3}\,r_0, \quad r_\sigma < -2r_m \quad \text{for } m \in \{1, .., r\}. \tag{B.3}$$

Finally positivity of the volume of the Calabi-Yau necessitates that also

$$r_\sigma^3\,(9 - r) - 3r_\sigma^2\,(3r_0 + \sum_{m=1}^{r} r_m) + 3r_\sigma\,(r_0^2 - \sum_{m=1}^{r} r_m^2) > 0. \tag{B.4}$$

Appendix C

Transformation rules for multiple $U(1)$ factors

In this appendix we recall, using elementary linear algebra, the rules for the basis transformation occurring when we define specific linear combinations of abelian gauge factors.

Suppose we are given a Lagrangian invariant under the abelian gauge symmetries $U(1)_m$, $m \in \{1, \dots, M\}$, each with generator T_m, gauge potential A_m and field strength F_m. The covariant derivative of the combined system of $U(1)$s is written as $D_\mu = \partial_\mu + i(\vec{A_\mu})^T \vec{T}$, where we have introduced an obvious vector notation for the various $U(1)$s. Consider now an orthogonal basis transformation in the $U(1)$-space such that the charge vector $\vec{Q}$ of a particle is transformed as

$$\vec{Q} \longrightarrow \vec{\tilde{Q}} = X\,\vec{Q}, \qquad X^T = X^{-1}. \tag{C.1}$$

Clearly this transforms the generators $\vec{T} \longrightarrow \vec{\tilde{T}} = X\,\vec{T}$ and thus

$$\vec{A} \longrightarrow \vec{\tilde{A}} = X\,\vec{A}, \tag{C.2}$$

so that the covariant derivative remains unchanged as it must.

Now suppose furthermore that the Lagrangian contains mass terms for the abelian gauge potentials, written schematically

$$\mathcal{L}_{mass} = \vec{A}^T M^2 \vec{A}, \qquad M^2 = \mathcal{M}^T \mathcal{M} \tag{C.3}$$

for some mass matrix M^2. We recover furthermore the $(k \times m)$ coupling matrix $\mathcal{M}$ introduced in equ.(3.40), where the index k labels the various axions to which the abelian field strengths couple via $\mathcal{M}$. Written in terms of the new gauge fields $\widetilde{A}$ the mass Lagrangian reads

$$\mathcal{L}_{mass} = (\vec{\tilde{A}})^T (X M^2 X^T) \vec{\tilde{A}} = (\vec{\tilde{A}})^T D \vec{\tilde{A}} = \sum_m \widetilde{A}_m d_{m,m} \widetilde{A}_m, \tag{C.4}$$

where we have assumed that the transformation is such that it diagonalizes the mass matrix M^2. To find the massless combination of $U(1)$ potentials just in terms of the matrix $\mathcal{M}$ we stress the obvious fact that

$$D = X\mathcal{M}^T\mathcal{M}X^T = (\mathcal{M}X^T)^T\mathcal{M}X^T. \tag{C.5}$$

The gauge potential $\widetilde{A}_m$ is massless iff $0 = d_{m,m}$, which is equivalent to requiring that the vector $\mathcal{M}\vec{X}^{(m)} = 0$, where $\vec{X}^{(m)} = (a_1, \ldots, a_m)$ represents the m-th column of X written as an m- vector. We have therefore convinced ourselves of the elementary fact that

$$\widetilde{A}_m = \sum_m a_m A_m \quad \text{is massless} \Longleftrightarrow \sum_k \mathcal{M}_{km}\, a_m = 0. \tag{C.6}$$

Precisely the same lines of reasoning apply, of course, to the transformation of the gauge kinetic function responsible for the coupling of the field strengths via

$$\mathcal{L}_{coup} = (\vec{F})^T f \vec{F} = (\vec{\widetilde{F}})^T (X f X^T)\vec{\widetilde{F}}. \tag{C.7}$$

Concretely, in section 7.1.1 we define

$$U(1)_X = \frac{1}{2}\left(U(1)_{X'} - \frac{5}{2}U(1)_2\right), \tag{C.8}$$

with the orthogonal $U(1)$ given by

$$U(1)_{\widetilde{X}} = \frac{1}{2}\left(\frac{5}{2}U(1)_{X'} + \frac{5}{2}U(1)_2\right). \tag{C.9}$$

This yields the transformation matrix $X = \frac{1}{2}\begin{pmatrix} 1 & -\frac{5}{2} \\ \frac{5}{2} & 1 \end{pmatrix}$, which is orthogonal up to normalisation. In all, we find indeed that

$$f_{X,X} = \frac{1}{4}\left(f_{X',X'} + \frac{5}{2}f_{2,2} - 5 f_{X',2}\right), \tag{C.10}$$

as stated in equ.(7.13).

Appendix D

Three-generation models

We list all consistent, supersymmetric three-generation models we have found by a computer search on elliptically fibered Calabi-Yau spaces with base spaces $\mathrm{dP_r}$, $r = 1, \ldots, 4$ and the Hirzebruch surfaces F_r in a range from $-10, \ldots, 10$ for all parameters. We have found three-generation models only on dP_4. Table D.1 contains the three-generation examples for the flipped $SU(5)$ model discussed in section 7.1, whereas in table D.2 we list all three-generation vacua directly with MSSM gauge group (see section 7.2) which we have found.

λ	η	q	$c_1(\zeta)$	$[W]$
$\frac{1}{4}$	$14l - 2E_1 - 6E_2 - 6E_3 - 2E_4$	0	$-4l + 4E_2 + 4E_3 + 4E_4$	$27F + (22l - 10E_1 - 6E_2 - 6E_3 - 10E_4)\sigma$
$\frac{1}{4}$	$18l - 10E_1 - 6E_2 - 6E_3 - 6E_4$	0	$-4l + 4E_2 + 4E_3 + 4E_4$	$27F + (18l - 2E_1 - 6E_2 - 6E_3 - 6E_4)\sigma$
$\frac{1}{4}$	$14l - 6E_1 - 2E_2 - 2E_3 - 6E_4$	0	$-4E_1 + 4E_4$	$27F + (22l - 6E_1 - 10E_2 - 10E_3 - 6E_4)\sigma$
$\frac{1}{4}$	$14l - 2E_1 - 6E_2 - 6E_3 - 2E_4$	0	$-4E_1 + 4E_4$	$27F + (22l - 10E_1 - 6E_2 - 6E_3 - 10E_4)\sigma$
$\frac{1}{4}$	$18l - 6E_1 - 10E_2 - 6E_3 - 6E_4$	0	$-4E_1 + 4E_4$	$27F + (18l - 6E_1 - 2E_2 - 6E_3 - 6E_4)\sigma$
$\frac{1}{4}$	$14l - 2E_1 - 6E_2 - 6E_3 - 2E_4$	0	$4l - 4E_1 - 4E_2 - 4E_3$	$27F + (22l - 10E_1 - 6E_2 - 6E_3 - 10E_4)\sigma$
$\frac{1}{4}$	$18l - 6E_1 - 6E_2 - 6E_3 - 10E_4$	0	$4l - 4E_1 - 4E_2 - 4E_3$	$27F + (18l - 6E_1 - 6E_2 - 6E_3 - 2E_4)\sigma$

Table D.1: Flipped $SU(5) \times U(1)_X$ models on dP_4.

λ	η	q	$c_1(\zeta)$	$[W]$
$\frac{1}{2}$	$15l - 3E_1 - 5E_2 - 5E_3 - 5E_4$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (21l - 9E_1 - 7E_2 - 7E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$15l - 2E_1 - 5E_2 - 5E_3 - 5E_4$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (21l - 10E_1 - 7E_2 - 7E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_1 - 7E_2 - 5E_3 - 5E_4$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (19l - 5E_1 - 5E_2 - 7E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_1 - 8E_2 - 5E_3 - 5E_4$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (18l - 4E_1 - 4E_2 - 7E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$20l - 3E_1 - 10E_2 - 10E_3$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (16l - 9E_1 - 2E_2 - 2E_3 - 12E_4)\sigma$
$\frac{1}{2}$	$20l - 2E_1 - 10E_2 - 10E_3$	0	$-5l + 5E_2 + 5E_3 + 5E_4$	$7F + (16l - 10E_1 - 2E_2 - 2E_3 - 12E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 5E_2 - 3E_3 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (21l - 7E_1 - 7E_2 - 9E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 5E_2 - 2E_3 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (21l - 7E_1 - 7E_2 - 10E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 3E_2 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (21l - 7E_1 - 9E_2 - 12E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 2E_2 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (21l - 7E_1 - 10E_2 - 12E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_2 - 3E_3$	0	$-5E_1 + 5E_4$	$7F + (21l - 12E_1 - 7E_2 - 9E_3 - 12E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_1 - 5E_2 - 5E_3 - 7E_4$	0	$-5E_1 + 5E_4$	$7F + (19l - 5E_1 - 7E_2 - 7E_3 - 5E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_1 - 5E_2 - 7E_4$	0	$-5E_1 + 5E_4$	$7F + (19l - 5E_1 - 7E_2 - 12E_3 - 5E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_1 - 7E_4$	0	$-5E_1 + 5E_4$	$7F + (19l - 5E_1 - 12E_2 - 12E_3 - 5E_4)\sigma$
$\frac{1}{2}$	$17l - 5E_1 - 7E_2 - 7E_3 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (19l - 7E_1 - 5E_2 - 5E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_2 - 7E_3$	0	$-5E_1 + 5E_4$	$7F + (19l - 12E_1 - 5E_2 - 5E_3 - 12E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_1 - 5E_2 - 5E_3 - 8E_4$	0	$-5E_1 + 5E_4$	$7F + (18l - 4E_1 - 7E_2 - 7E_3 - 4E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_1 - 5E_2 - 8E_4$	0	$-5E_1 + 5E_4$	$7F + (18l - 4E_1 - 7E_2 - 12E_3 - 4E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_1 - 8E_4$	0	$-5E_1 + 5E_4$	$7F + (18l - 4E_1 - 12E_2 - 12E_3 - 4E_4)\sigma$
$\frac{1}{2}$	$18l - 5E_1 - 8E_2 - 8E_3 - 5E_4$	0	$-5E_1 + 5E_4$	$7F + (18l - 7E_1 - 4E_2 - 4E_3 - 7E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_2 - 8E_3$	0	$-5E_1 + 5E_4$	$7F + (18l - 12E_1 - 4E_2 - 4E_3 - 12E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 5E_2 - 3E_3 - 10E_4$	0	$-5E_1 + 5E_4$	$7F + (16l - 2E_1 - 7E_2 - 9E_3 - 2E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 5E_2 - 2E_3 - 10E_4$	0	$-5E_1 + 5E_4$	$7F + (16l - 2E_1 - 7E_2 - 10E_3 - 2E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 3E_2 - 10E_4$	0	$-5E_1 + 5E_4$	$7F + (16l - 2E_1 - 9E_2 - 12E_3 - 2E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 2E_2 - 10E_4$	0	$-5E_1 + 5E_4$	$7F + (16l - 2E_1 - 10E_2 - 12E_3 - 2E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 5E_2 - 5E_3 - 3E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (21l - 7E_1 - 7E_2 - 7E_3 - 9E_4)\sigma$
$\frac{1}{2}$	$15l - 5E_1 - 5E_2 - 5E_3 - 2E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (21l - 7E_1 - 7E_2 - 7E_3 - 10E_4)\sigma$
$\frac{1}{2}$	$17l - 7E_1 - 5E_2 - 5E_3 - 7E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (19l - 5E_1 - 7E_2 - 7E_3 - 5E_4)\sigma$
$\frac{1}{2}$	$18l - 8E_1 - 5E_2 - 5E_3 - 8E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (18l - 4E_1 - 7E_2 - 7E_3 - 4E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 10E_2 - 3E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (16l - 2E_1 - 2E_2 - 12E_3 - 9E_4)\sigma$
$\frac{1}{2}$	$20l - 10E_1 - 10E_2 - 2E_4$	0	$5l - 5E_1 - 5E_2 - 5E_3$	$7F + (16l - 2E_1 - 2E_2 - 12E_3 - 10E_4)\sigma$

Table D.2: $SU(3) \times SU(2) \times U(1)$ models on dP$_4$.

Bibliography

[1] P. Ramond, "Dual theory for free fermions," *Phys. Rev.* **D3** (1971) 2415–2418.

[2] A. Neveu and J. H. Schwarz, "Factorizable dual model of pions," *Nucl. Phys.* **B31** (1971) 86–112.

[3] M. B. Green and J. H. Schwarz, "Supersymmetrical dual string theory," *Nucl. Phys.* **B181** (1981) 502–530.

[4] M. B. Green and J. H. Schwarz, "Supersymmetrical dual string theory. 2. Vertices and trees," *Nucl. Phys.* **B198** (1982) 252–268.

[5] M. B. Green and J. H. Schwarz, "Supersymmetrical dual string theory. 3. Loops and renormalisation," *Nucl. Phys.* **B198** (1982) 441–460.

[6] M. B. Green and J. H. Schwarz, "Supersymmetrical string theory," *Phys. Lett.* **B109** (1982) 444–448.

[7] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The heterotic string," *Phys. Rev. Lett.* **54** (1985) 502–505.

[8] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring Theory. Vol. 1: Introduction,". Cambridge, Uk: Univ. Pr. (1987) 469 p.

[9] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring Theory. Vol. 2: Loop amplitudes, anomalies and phenomenology,". Cambridge, Uk: Univ. Pr. (1987) 596 p.

[10] D. Lust and S. Theisen, "Lectures on string theory," *Lect. Notes Phys.* **346** (1989) 1–346.

[11] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,". Cambridge, UK: Univ. Pr. (1998) 402 p.

[12] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond,". Cambridge, UK: Univ. Pr. (1998) 531 p.

[13] A. M. Polyakov, "A Few projects in string theory," `hep-th/9304146`.

[14] B. R. Greene, "String theory on Calabi-Yau manifolds," hep-th/9702155.

[15] G. 't Hooft, "Dimensional reduction in quantum gravity," gr-qc/9310026.

[16] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.

[17] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, hep-th/9802150.

[18] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," *Phys. Lett.* **B379** (1996) 99–104, hep-th/9601029.

[19] D. Kutasov and N. Seiberg, "Noncritical superstrings," *Phys. Lett.* **B251** (1990) 67–72.

[20] L. Randall and R. Sundrum, "A large mass hierarchy from a small extra dimension," *Phys. Rev. Lett.* **83** (1999) 3370–3373, hep-ph/9905221.

[21] A. N. Schellekens, "The landscape 'avant la lettre'," physics/0604134.

[22] L. Susskind, "The anthropic landscape of string theory," hep-th/0302219.

[23] M. R. Douglas, "The statistics of string / M theory vacua," *JHEP* **05** (2003) 046, hep-th/0303194.

[24] R. Bousso and J. Polchinski, "Quantization of four-form fluxes and dynamical neutralization of the cosmological constant," *JHEP* **06** (2000) 006, hep-th/0004134.

[25] W. Lerche, D. Lust, and A. N. Schellekens, "Chiral four-dimensional heterotic strings from self-dual lattices," *Nucl. Phys.* **B287** (1987) 477.

[26] F. Denef and M. R. Douglas, "Computational complexity of the landscape. I," hep-th/0602072.

[27] C. Vafa, "The string landscape and the swampland," hep-th/0509212.

[28] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum configurations for superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[29] C. M. Hull and E. Witten, "Supersymmetric sigma models and the heterotic string," *Phys. Lett.* **B160** (1985) 398–402.

[30] J. Distler and B. R. Greene, "Aspects of (2,0) string compactifications," *Nucl. Phys.* **B304** (1988) 1.

[31] E. Witten, "Phases of N = 2 theories in two dimensions," *Nucl. Phys.* **B403** (1993) 159–222, `hep-th/9301042`.

[32] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on orbifolds," *Nucl. Phys.* **B261** (1985) 678–686.

[33] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on orbifolds. 2," *Nucl. Phys.* **B274** (1986) 285–314.

[34] L. E. Ibanez, H. P. Nilles, and F. Quevedo, "Orbifolds and Wilson lines," *Phys. Lett.* **B187** (1987) 25–32.

[35] L. E. Ibanez, J. E. Kim, H. P. Nilles, and F. Quevedo, "Orbifold compactifications with three families of SU(3)x SU(2) x U(1)*n," *Phys. Lett.* **B191** (1987) 282–286.

[36] D. Gepner, "Exactly solvable string compactifications on manifolds of SU(N) holonomy," *Phys. Lett.* **B199** (1987) 380–388.

[37] R. Blumenhagen and A. Wisskirchen, "Exactly solvable (0,2) supersymmetric string vacua with GUT gauge groups," *Nucl. Phys.* **B454** (1995) 561–586, `hep-th/9506104`.

[38] I. Antoniadis, C. P. Bachas, and C. Kounnas, "Four-dimensional superstrings," *Nucl. Phys.* **B289** (1987) 87.

[39] E. Witten, "New issues in manifolds of SU(3) holonomy," *Nucl. Phys.* **B268** (1986) 79.

[40] R. Friedman, J. Morgan, and E. Witten, "Vector bundles and F theory," *Commun. Math. Phys.* **187** (1997) 679–743, `hep-th/9701162`.

[41] R. Donagi, "Principal bundles on elliptic fibrations," *Asian J. Math.* **1** (1997) 214–223, `alg-geom/9702002`.

[42] E. Witten, "Symmetry breaking patterns in superstring models," *Nucl. Phys.* **B258** (1985) 75.

[43] R. Donagi, B. A. Ovrut, T. Pantev, and D. Waldram, "Standard-model bundles on non-simply connected Calabi-Yau threefolds," *JHEP* **08** (2001) 053, `hep-th/0008008`.

[44] R. Donagi, B. A. Ovrut, T. Pantev, and D. Waldram, "Standard-model bundles," *Adv. Theor. Math. Phys.* **5** (2002) 563–615, `math.ag/0008010`.

[45] R. Donagi, B. A. Ovrut, T. Pantev, and D. Waldram, "Spectral involutions on rational elliptic surfaces," *Adv. Theor. Math. Phys.* **5** (2002) 499–561, `math.ag/0008011`.

[46] V. Braun, B. A. Ovrut, T. Pantev, and R. Reinbacher, "Elliptic Calabi-Yau threefolds with Z(3) x Z(3) Wilson lines," *JHEP* **12** (2004) 062, hep-th/0410055.

[47] B. Andreas, G. Curio, and A. Klemm, "Towards the standard model spectrum from elliptic Calabi- Yau," *Int. J. Mod. Phys.* **A19** (2004) 1987, hep-th/9903052.

[48] G. Curio, "Standard model bundles of the heterotic string," hep-th/0412182.

[49] V. Bouchard and R. Donagi, "An SU(5) heterotic standard model," *Phys. Lett.* **B633** (2006) 783–791, hep-th/0512149.

[50] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "A heterotic standard model," *Phys. Lett.* **B618** (2005) 252–258, hep-th/0501070.

[51] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "The exact MSSM spectrum from string theory," hep-th/0512177.

[52] W. Buchmuller, K. Hamaguchi, O. Lebedev, and M. Ratz, "The supersymmetric standard model from the heterotic string," hep-ph/0511035.

[53] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, hep-th/9510017.

[54] M. Berkooz, M. R. Douglas, and R. G. Leigh, "Branes intersecting at angles," *Nucl. Phys.* **B480** (1996) 265–278, hep-th/9606139.

[55] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti, and Y. S. Stanev, "Chiral asymmetry in four-dimensional open- string vacua," *Phys. Lett.* **B385** (1996) 96–102, hep-th/9606169.

[56] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, "Noncommutative compactifications of type I strings on tori with magnetic background flux," *JHEP* **10** (2000) 006, hep-th/0007024.

[57] R. Blumenhagen, M. Cvetic, P. Langacker, and G. Shiu, "Toward realistic intersecting D-brane models," hep-th/0502005.

[58] R. Blumenhagen and A. Wisskirchen, "Spectra of 4D, N = 1 type I string vacua on non-toroidal CY threefolds," *Phys. Lett.* **B438** (1998) 52–60, hep-th/9806131.

[59] G. Aldazabal, E. C. Andres, M. Leston, and C. Nunez, "Type IIB orientifolds on Gepner points," *JHEP* **09** (2003) 067, hep-th/0307183.

[60] R. Blumenhagen, "Supersymmetric orientifolds of Gepner models," *JHEP* **11** (2003) 055, hep-th/0310244.

[61] I. Brunner, K. Hori, K. Hosomichi, and J. Walcher, "Orientifolds of Gepner models," hep-th/0401137.

[62] G. Aldazabal, E. C. Andres, and J. E. Juknevich, "Particle models from orientifolds at Gepner-orbifold points," *JHEP* **05** (2004) 054, hep-th/0403262.

[63] T. P. T. Dijkstra, L. R. Huiszoon, and A. N. Schellekens, "Supersymmetric standard model spectra from RCFT orientifolds," *Nucl. Phys.* **B710** (2005) 3–57, hep-th/0411129.

[64] R. Blumenhagen and T. Weigand, "Chiral supersymmetric Gepner model orientifolds," *JHEP* **02** (2004) 041, hep-th/0401148.

[65] R. Blumenhagen and T. Weigand, "A note on partition functions of Gepner model orientifolds," *Phys. Lett.* **B591** (2004) 161–169, hep-th/0403299.

[66] M. B. Green, J. H. Schwarz, and P. C. West, "Anomaly free chiral theories in six dimensions," *Nucl. Phys.* **B254** (1985) 327–348.

[67] G. Aldazabal, A. Font, L. E. Ibanez, and A. M. Uranga, "New branches of string compactifications and their F-theory duals," *Nucl. Phys.* **B492** (1997) 119–151, hep-th/9607121.

[68] A. Lukas and K. S. Stelle, "Heterotic anomaly cancellation in five dimensions," *JHEP* **01** (2000) 010, hep-th/9911156.

[69] B. Andreas and D. Hernandez Ruiperez, "U(n) vector bundles on Calabi-Yau threefolds for string theory compactifications," hep-th/0410170.

[70] A. Strominger, "Superstrings with torsion," *Nucl. Phys.* **B274** (1986) 253.

[71] G. Lopes Cardoso *et al.*, "Non-Kaehler string backgrounds and their five torsion classes," *Nucl. Phys.* **B652** (2003) 5–34, hep-th/0211118.

[72] J.-X. Fu and S.-T. Yau, "Existence of supersymmetric Hermitian metrics with torsion on non-Kaehler manifolds," hep-th/0509028.

[73] J.-X. Fu and S.-T. Yau, "The theory of superstring with flux on non-Kaehler manifolds and the complex Monge-Ampere equation," hep-th/0604063.

[74] K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng, and S.-T. Yau, "Anomaly Cancellation and Smooth Non-Kahler Solutions in Heterotic String Theory," hep-th/0604137.

[75] S. M. Barr, "A new symmetry breaking pattern for SO(10) and proton decay," *Phys. Lett.* **B112** (1982) 219.

[76] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven dimensions," *Nucl. Phys.* **B460** (1996) 506–524, hep-th/9510209.

[77] P. Horava and E. Witten, "Eleven-Dimensional Supergravity on a Manifold with Boundary," *Nucl. Phys.* **B475** (1996) 94–114, hep-th/9603142.

[78] M. R. Douglas, B. Fiol, and C. Romelsberger, "Stability and BPS branes," hep-th/0002037.

[79] M. B. Green and J. H. Schwarz, "Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory," *Phys. Lett.* **B149** (1984) 117–122.

[80] L. E. Ibanez and H. P. Nilles, "Low-energy remnants of superstring anomaly cancellation terms," *Phys. Lett.* **B169** (1986) 354.

[81] P. Pasti, D. P. Sorokin, and M. Tonin, "Covariant action for a D = 11 five-brane with the chiral field," *Phys. Lett.* **B398** (1997) 41–46, hep-th/9701037.

[82] M. J. Duff, R. Minasian, and E. Witten, "Evidence for Heterotic/Heterotic Duality," *Nucl. Phys.* **B465** (1996) 413–438, hep-th/9601036.

[83] E. Witten, "Small Instantons in String Theory," *Nucl. Phys.* **B460** (1996) 541–559, hep-th/9511030.

[84] J. P. Gauntlett, D. Martelli, and D. Waldram, "Superstrings with intrinsic torsion," *Phys. Rev.* **D69** (2004) 086002, hep-th/0302158.

[85] M. Grana, "Flux compactifications in string theory: A comprehensive review," *Phys. Rept.* **423** (2006) 91–158, hep-th/0509003.

[86] B. A. Ovrut, T. Pantev, and J. Park, "Small instanton transitions in heterotic M-theory," *JHEP* **05** (2000) 045, hep-th/0001133.

[87] M. Dine and N. Seiberg, "Nonrenormalisation theorems in superstring theories," *Phys. Rev. Lett.* **57** (1986) 2625.

[88] S. Donaldson, "Antiselfdual Yang-Mills connections on complex algebraic surfaces and stable bundles," *Proc. London Soc.* **3** (1985) 1.

[89] K. Uhlenbeck and S. T. Yau, "On the existence of Hermitian-Yang-Mills connectionsin stable vector bundles," *Comm. Pure Appl. Math.* **39** (1986).

[90] E. Witten, "Strong Coupling Expansion Of Calabi-Yau Compactification," *Nucl. Phys.* **B471** (1996) 135–158, `hep-th/9602070`.

[91] K. Becker, M. Becker, and A. Strominger, "Five-branes, membranes and nonperturbative string theory," *Nucl. Phys.* **B456** (1995) 130–152, `hep-th/9507158`.

[92] E. I. Buchbinder and B. A. Ovrut, "Vacuum stability in heterotic M-theory," *Phys. Rev.* **D69** (2004) 086010, `hep-th/0310112`.

[93] V. Braun and B. A. Ovrut, "Stabilizing moduli with a positive cosmological constant in heterotic M-theory," `hep-th/0603088`.

[94] E. Witten, "Global Anomalies In String Theory," `Print-85-0620 (Princeton)`.

[95] D. S. Freed, "Determinants, Torsion And Strings," *Commun. Math. Phys.* **107** (1986).

[96] A. M. Uranga, "D-brane probes, RR tadpole cancellation and K-theory charge," *Nucl. Phys.* **B598** (2001) 225–246, `hep-th/0011048`.

[97] R. Blumenhagen, G. Honecker, and T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles," *JHEP* **06** (2005) 020, `hep-th/0504232`.

[98] R. Blumenhagen, S. Moster, and T. Weigand, "Heterotic GUT and standard model vacua from simply connected Calabi-Yau manifolds," `hep-th/0603015`.

[99] E. R. Sharpe, "Boundary superpotentials," *Nucl. Phys.* **B523** (1998) 211–231, `hep-th/9611196`.

[100] Z. Lalak, S. Pokorski, and S. Thomas, "Beyond the standard embedding in M-theory on S(1)/Z(2)," *Nucl. Phys.* **B549** (1999) 63–97, `hep-ph/9807503`.

[101] P. Berglund, A. Klemm, P. Mayr, and S. Theisen, "On type IIB vacua with varying coupling constant," *Nucl. Phys.* **B558** (1999) 178–204, `hep-th/9805189`.

[102] R. Tatar and T. Watari, "Proton decay, Yukawa couplings and underlying gauge symmetry in string theory," `hep-th/0602238`.

[103] B. Andreas and G. Curio, "Standard Models from Heterotic String Theory," `hep-th/0602247`.

[104] R. Slansky, "Group theory for unified model building," *Phys. Rept.* **79** (1981) 1–128.

[105] G. Honecker, "Massive U(1)s and heterotic five-branes on K3," `hep-th/0602101`.

[106] M. Dine, N. Seiberg, X. G. Wen, and E. Witten, "Nonperturbative effects of the string worldsheet. 2," *Nucl. Phys.* **B289** (1987) 319.

[107] E. Witten, "Some properties of O(32) superstrings," *Phys. Lett.* **B149** (1984) 351–356.

[108] B. Zumino, Y.-S. Wu, and A. Zee, "Chiral anomalies, higher dimensions, and differential geometry," *Nucl. Phys.* **B239** (1984) 477–507.

[109] L. E. Ibanez, F. Marchesano, and R. Rabadan, "Getting just the standard model at intersecting branes," *JHEP* **11** (2001) 002, `hep-th/0105155`.

[110] A. Lukas, B. A. Ovrut, and D. Waldram, "Non-standard embedding and five-branes in heterotic M- theory," *Phys. Rev.* **D59** (1999) 106005, `hep-th/9808101`.

[111] A. Lukas, B. A. Ovrut, and D. Waldram, "On the four-dimensional effective action of strongly coupled heterotic string theory," *Nucl. Phys.* **B532** (1998) 43–82, `hep-th/9710208`.

[112] A. Lukas, B. A. Ovrut, and D. Waldram, "The ten-dimensional effective action of strongly coupled heterotic string theory," *Nucl. Phys.* **B540** (1999) 230–246, `hep-th/9801087`.

[113] L. Carlevaro and J.-P. Derendinger, "Five-brane thresholds and membrane instantons in four- dimensional heterotic M-theory," *Nucl. Phys.* **B736** (2006) 1–33, `hep-th/0502225`.

[114] J. O. Conrad, "Brane tensions and coupling constants from within M-theory," *Phys. Lett.* **B421** (1998) 119–124, `hep-th/9708031`.

[115] M. Brandle and A. Lukas, "Five-branes in heterotic brane-world theories," *Phys. Rev.* **D65** (2002) 064024, `hep-th/0109173`.

[116] J. P. Derendinger, L. E. Ibanez, and H. P. Nilles, "On the low-energy limit of superstring theories," *Nucl. Phys.* **B267** (1986) 365.

[117] K. Choi and J. E. Kim, "Compactification and axions in E(8)xE(8) superstring models," *Phys. Lett.* **B165** (1985) 71.

[118] H. P. Nilles and S. Stieberger, "String unification, universal one-loop corrections and strongly coupled heterotic string theory," *Nucl. Phys.* **B499** (1997) 3–28, `hep-th/9702110`.

[119] S. Stieberger, "(0,2) heterotic gauge couplings and their M-theory origin," *Nucl. Phys.* **B541** (1999) 109–144, hep-th/9807124.

[120] A. Bilal, "Introduction to supersymmetry," hep-th/0101055.

[121] J. Wess and J. Bagger, "Supersymmetry and supergravity,". Princeton University Press (1992) 259 p.

[122] J.-P. Derendinger and R. Sauser, "A five-brane modulus in the effective N = 1 supergravity of M-theory," *Nucl. Phys.* **B598** (2001) 87–114, hep-th/0009054.

[123] G. W. Moore, G. Peradze, and N. Saulina, "Instabilities in heterotic M-theory induced by open membrane instantons," *Nucl. Phys.* **B607** (2001) 117–154, hep-th/0012104.

[124] M. Dine, N. Seiberg, and E. Witten, "Fayet-Iliopoulos terms in string theory," *Nucl. Phys.* **B289** (1987) 589.

[125] M. P. Garcia del Moral, "A new mechanism of Kahler moduli stabilization in type IIB theory," hep-th/0506116.

[126] N. Leung, "Einstein type metrics and stability on vector bundles," *J.Differential Geometry* **45** (1997) 514.

[127] H. Enger and C. A. Lutken, "Non-linear Yang-Mills instantons from strings are pi-stable D-branes," *Nucl. Phys.* **B695** (2004) 73–83, hep-th/0312254.

[128] E. Poppitz, "On the one loop Fayet-Iliopoulos term in chiral four dimensional type I orbifolds," *Nucl. Phys.* **B542** (1999) 31–44, hep-th/9810010.

[129] R. Blumenhagen, G. Honecker, and T. Weigand, "Supersymmetric (non-)abelian bundles in the type I and SO(32) heterotic string," *JHEP* **08** (2005) 009, hep-th/0507041.

[130] R. Blumenhagen, G. Honecker, and T. Weigand, "Non-abelian brane worlds: The open string story," hep-th/0510050.

[131] R. Blumenhagen, G. Honecker, and T. Weigand, "Non-abelian brane worlds: The heterotic string story," *JHEP* **10** (2005) 086, hep-th/0510049.

[132] A. Strominger, "Heterotic solitons," *Nucl. Phys.* **B343** (1990) 167–184.

[133] M. J. Duff and J. X. Lu, "Remarks on string / five-brane duality," *Nucl. Phys.* **B354** (1991) 129–140.

[134] M. J. Duff and J. X. Lu, "Strings from five-branes," *Phys. Rev. Lett.* **66** (1991) 1402–1405.

[135] M. J. Duff and J. X. Lu, "A Duality between strings and five-branes," *Class. Quant. Grav.* **9** (1992) 1–16.

[136] E. G. Gimon and J. Polchinski, "Consistency Conditions for Orientifolds and D-Manifolds," *Phys. Rev.* **D54** (1996) 1667–1676, `hep-th/9601038`.

[137] M. B. Green, J. A. Harvey, and G. W. Moore, "I-brane inflow and anomalous couplings on D-branes," *Class. Quant. Grav.* **14** (1997) 47–52, `hep-th/9605033`.

[138] R. Hartshorne, "Algebraic geometry,". Springer-Verlag.

[139] P. Griffiths and J. Harris, "Principles of Algebraic geometry,". Wiley NY.

[140] H. P. Nilles, S. Ramos-Sanchez, P. K. S. Vaudrevange, and A. Wingerter, "Exploring the SO(32) heterotic string," `hep-th/0603086`.

[141] S. Katz, T. Pantev, and E. Sharpe, "D-branes, orbifolds, and Ext groups," *Nucl. Phys.* **B673** (2003) 263–300, `hep-th/0212218`.

[142] E. Witten, "An SU(2) anomaly," *Phys. Lett.* **B117** (1982) 324–328.

[143] R. Blumenhagen, V. Braun, B. Körs, and D. Lüst, "Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes," *JHEP* **07** (2002) 026, `hep-th/0206038`.

[144] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," `hep-th/0403166`.

[145] M. Marino, R. Minasian, G. W. Moore, and A. Strominger, "Nonlinear instantons from supersymmetric p-branes," *JHEP* **01** (2000) 005, `hep-th/9911206`.

[146] A. Kapustin and Y. Li, "Stability conditions for topological D-branes: A worldsheet approach," `hep-th/0311101`.

[147] R. Minasian and A. Tomasiello, "Variations on stability," *Nucl. Phys.* **B631** (2002) 43–65, `hep-th/0104041`.

[148] G. Curio, "Chiral matter and transitions in heterotic string models," *Phys. Lett.* **B435** (1998) 39–48, `hep-th/9803224`.

[149] R. Donagi, A. Lukas, B. A. Ovrut, and D. Waldram, "Holomorphic vector bundles and non-perturbative vacua in M- theory," *JHEP* **06** (1999) 034, `hep-th/9901009`.

[150] B. Andreas, "On vector bundles and chiral matter in N = 1 heterotic compactifications," *JHEP* **01** (1999) 011, `hep-th/9802202`.

[151] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," *Nucl. Phys.* **B476** (1996) 437–469, hep-th/9603161.

[152] R. Friedman, J. W. Morgan, and E. Witten, "Vector Bundles over Elliptic Fibrations," alg-geom/9709029.

[153] R. Donagi, Y.-H. He, B. A. Ovrut, and R. Reinbacher, "The particle spectrum of heterotic compactifications," *JHEP* **12** (2004) 054, hep-th/0405014.

[154] P. e. T. Demazure, "Seminaire sur les Singularities des Surfaces," *Ecole Polytechnique* (1976-1977).

[155] R. Blumenhagen, F. Gmeiner, G. Honecker, D. Lust, and T. Weigand, "The statistics of supersymmetric D-brane models," *Nucl. Phys.* **B713** (2005) 83–135, hep-th/0411173.

[156] F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lust, and T. Weigand, "One in a billion: MSSM-like D-brane statistics," *JHEP* **01** (2006) 004, hep-th/0510170.

[157] J. P. Derendinger, J. E. Kim, and D. V. Nanopoulos, "Anti - SU(5)," *Phys. Lett.* **B139** (1984) 170.

[158] H. Georgi and S. L. Glashow, "Unity of all elementary particle forces," *Phys. Rev. Lett.* **32** (1974) 438–441.

[159] I. Antoniadis, J. R. Ellis, J. S. Hagelin, and D. V. Nanopoulos, "Supersymmetric flipped SU(5) revitalized," *Phys. Lett.* **B194** (1987) 231.

[160] I. Antoniadis, J. R. Ellis, J. S. Hagelin, and D. V. Nanopoulos, "The flipped SU(5) X U(1) string model revamped," *Phys. Lett.* **B231** (1989) 65.

[161] I. Dorsner and P. Fileviez Perez, "Could we rotate proton decay away?," *Phys. Lett.* **B606** (2005) 367–370, hep-ph/0409190.

[162] P. Nath and P. F. Perez, "Proton stability in grand unified theories, in strings, and in branes," hep-ph/0601023.

[163] J. R. Ellis, S. Kelley, and D. V. Nanopoulos, "Precision LEP data, supersymmetric GUTs and string unification," *Phys. Lett.* **B249** (1990) 441–448.

[164] U. Amaldi, W. de Boer, and H. Furstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP," *Phys. Lett.* **B260** (1991) 447–455.

[165] R. Blumenhagen, S. Moster, R. Reinbacher, and T. Weigand *to appear.*

[166] V. Braun, Y.-H. He, and B. A. Ovrut, "Yukawa couplings in heterotic standard models," hep-th/0601204.

[167] V. Bouchard, M. Cvetic, and R. Donagi, "Tri-linear couplings in an heterotic minimal supersymmetric standard model," hep-th/0602096.

[168] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "Moduli dependent mu-terms in a heterotic standard model," hep-th/0510142.

[169] J. Distler and S. Kachru, "(0,2) Landau-Ginzburg theory," *Nucl. Phys.* **B413** (1994) 213–243, hep-th/9309110.

[170] E. Silverstein and E. Witten, "Criteria for conformal invariance of (0,2) models," *Nucl. Phys.* **B444** (1995) 161–190, hep-th/9503212.

[171] A. Basu and S. Sethi, "World-sheet stability of (0,2) linear sigma models," *Phys. Rev.* **D68** (2003) 025003, hep-th/0303066.

[172] C. Beasley and E. Witten, "Residues and world-sheet instantons," *JHEP* **10** (2003) 065, hep-th/0304115.

[173] F. Gmeiner, "Standard model statistics of a type II orientifold," hep-th/0512190.

[174] F. Gmeiner and M. Stein, "Statistics of SU(5) D-brane models on a type II orientifold," hep-th/0603019.

[175] K. R. Dienes, "Statistics on the Heterotic Landscape: Gauge Groups and Cosmological Constants of Four-Dimensional Heterotic Strings," hep-th/0602286.

[176] M. Nakahara, "Geometry, topology and physics,". Adam Hilger (1990).

[177] J. Erler, "Anomaly cancellation in six-dimensions," *J. Math. Phys.* **35** (1994) 1819–1833, hep-th/9304104.

Curriculum Vitae

Timo Weigand

born 28/02/1979 in Freiburg im Breisgau, Germany

Academic Background

09/2004 - 07/2006	Ph.D. student at the Max-Planck-Institut (MPI) für Physik, München, Germany Supervisor: Prof. Dr. Dieter Lüst, and
10/2003 - 07/2006	Ph.D. student at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, United Kingdom Supervisor: Dr. Ralph Blumenhagen
10/2002 - 07/2003	Part III of the Mathematical Tripos, University of Cambridge
11/1999 - 07/2002	Undergraduate Studies in Physics at Ludwig-Maximilians-Universität (LMU), München
09/1989 - 07/1998	Grammar School at Josef-Effner-Gymnasium, Dachau

Grants and Scholarships

08/2004 - 07/2006	Ph.D. Scholarship of the German National Academic Foundation
10/2003 - 08/2004	Research Scholarship of the German Exchange Service (DAAD) Trinity Hall Research Scholarship
05/2001 - 08/2003	Scholarship of the German National Academic Foundation
09/1999 - 08/2003	Scholarship according to the Act of Bavarian Parliament on the Funding of Highly Talented Students
Since 09/1998	Scholar of the Maximilianeum Foundation, München