Existence of Spontaneous Pair Creation

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1 Introduction

Pair creation - particularly electron-positron-pair creation - is a often discussed phenomenon in modern physics. In this work we want to restrict ourselves to the creation of electron positron pairs from the vacuum in strong external fields which vary slowly in time.

It is a well known fact (see [10] for example) that pair creation is not possible in static fields, so at first sight it would be natural to assume, that the probability of creating a pair from the vacuum goes to zero as the time derivative of the external field goes to zero, too. As we will see later, this is not true in general. If the external field strength exceeds a critical threshold, there will be a sudden jump in the probability of adiabatic pair creation. This sudden jump led to the notion of "spontaneous pair creation".

The problem of spontaneous pair creation has been discussed in the physical literature (see for example [2]). Several attempts have been made to verify or falsify the existence of spontaneous pair creation by experiments. So far there exist no clear experimental data neither in favor nor against the existence of spontaneous par creation. The reason for the lack of such experiments is that the critical strength of the field which is needed to create pairs spontaneously is very big. One possible way to produce such strong potentials is by colliding heavy ions. The total charge of the collided ions has to be greater than 180e - so one has to collide two atoms not much smaller than Uranium. Such a collision leads two many other reactions in the nuclei, so it is not easy to distinguish where the detected electrons and positrons come from.

In the physical literature the Dirac equation with the so called "Dirac sea" interpretation is used to describe pair creation. We will do the same since this interpretation gives a picturesque description of the situation which leads us also through the mathematical argument. Therefore we will formulate the theorem of spontaneous pair creation as a one particle Dirac problem. Non rigourously the problem can be described as follows: Is it possible to get transitions between the negative and the positive continuous spectrum by adiabatically turning on an external potential beyond the critical value and adiabatically turning it off again? After that we introduce the Dirac equation in second quantization and give a heuristical connection between the Dirac equation with "Dirac sea" interpretation and the Dirac equation in second quantization, which gives us also the right heuristics how to formulate and prove a Corollary which translates the Theorem of spontaneous pair creation into the formalism of second quantized Dirac equation. Then we will prove the Theorem of spontaneous pair creation itself.

For this proof we will need some spectral properties of the Dirac operators and properties of its bound states and generalized eigenfunctions. Increasing the coupling constant of the potential one will see, that any bound state vanishes in the positive continuous spectrum. For the proof we will need that the time derivative of the energy of the bound state is non zero as it reaches m - the threshold of the positive continuous spectrum. In this case there always exists a bound state with energy equal to m. In our first attempt to prove the existence of spontaneous pair creation we made the mistake in assuming, that there exists no bound state with energy m, hence our old proof was worthless.

That mistake was revealed by Gheorge Nenciu, who also suggested some literature that was very helpful in understanding the true situation.

1.1 The free Dirac equation

The Dirac equation was one of the first equations to describe particle-antiparticle-creation and -annihilation effects.

Originally the goal of the Dirac equation was to have a covariant relativistic wave equation - generalizing the non-relativistic Schrödinger equation.

$$i\frac{\partial \psi_t}{\partial t} = -i\sum_{l=1}^{3} \alpha_l \partial_l \psi_t + \beta m \psi_t \equiv D^0 \psi_t \tag{1}$$

With complex valued α_l and β this is obviously not possible as, to get the right "dispersion relation" (i.e. the relativistic energy-momentum relation $E^2 = k^2 + m^2$ under Fourier transforming $\psi_t(x)$ in $\mathbf{t}$ and $\mathbf{x}$), the α_l and β may not commute, but with matrix valued α_l and β it is. One possible choice is

$$\alpha_l = \begin{pmatrix} 0 & \sigma_l \\ \sigma_l & 0 \end{pmatrix}; \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}; l = 1, 2, 3 \tag{2}$$

with σ_l being the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This choice is called the "standard representation" and was introduced by Dirac.

So ψ is not a complex valued function, but a 4-vector valued function and the underlying Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3)^4 = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.

The "generalized" Eigenfunctions ϕ of D^0

$$D^0\phi = E_k\phi = \pm\sqrt{m^2+k^2}\phi$$

are of the form $e^{i\mathbf{k}\cdot\mathbf{x}}\gamma$, where gamma is a (complex valued) 4-vector. For any $\mathbf{k}$ four different choices of γ are possible, so for each $\mathbf{k}$ we get four different Eigenfunctions. Two of them have positive energy, two of them negative energy. So we denote the Eigenfunctions of D^0 by $\phi^{\pm,j,\mathbf{k}}$, where the sign stands for the sign of the energy, $j \in \{1,2\}$ for the two different spins.

As the sign of the energy will play an important role in the following sections, we define the subspaces $\mathcal{H}_+$ which is the span of the eigenfunctions with positive energy, $\mathcal{H}_-$ which is the span of the eigenfunctions with negative energy and the projectors P^+ and P^- into these spaces. As these subspaces are orthogonal we can write

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

So in contrast to the free Schrödinger Hamiltonian H_0, the essential spectrum of D^0 is not bounded from below. It consists of two absolutely continuous parts, a positive part reaching from m to ∞ and a negative part reaching from $-m$ to $-\infty$. In the case of static external fields this does not lead to any problems, though the physical interpretation of electrons with negative energy may be difficult. But introducing coupling of the electron to a radiation field would lead to a radiation catastrophe: Any electron would fall down the negative energy continuum emitting radiation.

This radiation catastrophe can be heuristically overcome by using antisymmetrized wave functions. The wave function Ω describing the "vacuum" is the antisymmetrized product of all the eigenfunctions of D^0 with negative energy (the so called Dirac sea)

$$\Omega = \prod_{k\in\mathbb{R};\, j=1;2}^{antisym} \phi^{\pm,j,\mathbf{k}} \; .$$

The antisymmetrized product of Ω and an additional wave function $\psi \in \mathcal{H}_+$ describes an electron with position probability density $\mid \psi \mid^2$. Introducing an external interaction which - to keep things simple for the moment - is supposed to have no influence on Ω, any one electron wave function $\psi_i \times \Omega$ with $\psi_i \in \mathcal{H}_+$ will after the interaction stay a wave function of the form $\psi_f \times \Omega$ with $\psi_f \in \mathcal{H}_+$ since all possible one particle states in Ω are occupied and adding one more state would by antisymmetrization yield zero. This phenomenon is called Pauli's exclusion principle. Although this heuristic picture has not been made rigorous we use it anyhow, as it is very descriptive.

Starting with the vacuum Ω, it is now possible that a "photon" with energy bigger than $2m$ is absorbed by one of the electrons with negative energy, which is thus lifted into the positive energy spectrum leaving a "hole" in the Dirac sea behind. We end up with a wave function, which is the antisymmetrized product of all eigenfunctions with negative energy except one and some $\psi \in \mathcal{H}_+$. The hole propagates - the propagation is given indirectly by the propagation of all the other electrons in the Dirac sea - like a particle with the same mass but opposite charge as the electron and is physically interpreted as a positron - the antiparticle of the electron. Therefore we interpret the factors of our multi particle wave functions with positive energy as electrons, missing factors with negative energy ("holes in the Dirac sea") as positrons. This is the so called "sea interpretation" of the Dirac equation, originally found by Dirac.

1.2 The Dirac sea under influence of an external potential

We shall now give a more quantitative description. To keep the description of the interaction as simple as possible we neglect interaction between the different particles - so the differential equations describing our multi particle wave function decouple and we can use the Dirac equation for each factor ψ_s^r of our multi particle wave function $\Psi_t = \prod_{r\in\mathbb{R}}^{antisym} \psi_t^r$ separately.

The one particle Dirac equation with external potential reads:

$$i\frac{\partial\psi_t}{\partial t} = -i\sum_{l=1}^{3}\alpha_l\partial_l\psi_t + A\!\!\!/\psi_t + \beta m\psi_t \equiv (D^0 + A\!\!\!/)\psi_t \tag{3}$$

where

$$A\!\!\!/ = \mathbf{1}A_0 + \sum_{l=1}^{3}\alpha_l A_l \ . \tag{4}$$

To get a physical interpretation of Ψ_t (for example to calculate the amplitude of pair creation) one has to write it down in the "free eigenbasis" or more to say as linear combination of products of eigenfunctions of the free Dirac equation, though it may make sense to use another basis calculating the ψ_t than the set of eigenfunctions of the free Dirac equation.

2 Spontaneous pair creation in Dirac theory

2.1 Formulation of the problem

One way of phrasing our result is that we show the existence of slowly varying potentials which create pairs. A very simple way to describe such a potential is the so called adiabatic switching formalism: We consider a potential which can be factorized into a purely space-dependent part and a purely time dependent part - the so called switching factor.

$$\not{A}^{\lambda}_{s}(\mathbf{x}) \;=\; \lambda\varphi(s)\not{A}(\mathbf{x})\,. \tag{5}$$

Here $A_l \in C^{\infty}$ (see connection between $\not{A}(\mathbf{x})$ and A_l in (4)) for all $l \in \{0,1,2,3\}$, $\not{A}(\mathbf{x})$ has compact support $\mathcal{C}$ and can (due to CPT-symmetry without loss of generality) be defined to be repulsive for electrons. For φ we assume: $\varphi(s) \in C^1$ with

$$\begin{aligned}
\lim_{s\to\pm\infty}\varphi(s) &= 0 \\
\partial_s\varphi(s) &< 0 \quad \text{for } s<0 \\
\partial_s\varphi(s) &> 0 \quad \text{for } s>0 \\
\partial_s\varphi(s) &= 0 \quad \text{for } s=0 \\
\mid\partial_s\varphi(s)\mid &< C \quad \text{for some } C\in\mathbb{R}^+ \\
\varphi_0 &= 1\,.
\end{aligned} \tag{6}$$

We consider now a slowly varying function $A_{t\varepsilon}$ in (3). Then going to the macroscopic time scale $s = t\varepsilon$ and introducing $\psi^{\varepsilon}_{s} = \psi_{\frac{s}{\varepsilon}}$ we obtain

$$i\varepsilon\frac{\partial\psi^{\varepsilon}_{s}}{\partial s} = (D^0 + \not{A}^{\lambda}_{s}(\mathbf{x}))\psi^{\varepsilon}_{s} =: D_s\psi^{\varepsilon}_{s} \tag{7}$$

By our description of the Dirac sea we expect (see [2]) that for sufficiently strong potentials the probability of creating (at least) one pair from the vacuum is one in the adiabatic limit ($\lim_{\varepsilon\to 0}$).

Heuristically to get a multi particle wave function Ψ^{ε}_{s} which describes adiabatic pair creation from the vacuum with probability one (which means $\lim_{\varepsilon\to 0}\lim_{s\to-\infty}\Psi^{\varepsilon}_{s} = \Omega$ and $\lim_{\varepsilon\to 0}\lim_{s\to+\infty}\langle\Psi^{\varepsilon}_{s}\mid\Omega\rangle = 0$) at least one of the factors $\psi^{\varepsilon,r}_{s}$ of Ψ^{ε}_{s} has to lie in the positive energy subspace $\mathcal{H}_+$ in the limit $\lim_{\varepsilon\to 0}\lim_{s\to+\infty}$:

$$\lim_{\varepsilon\to 0}\lim_{s\to+\infty} \parallel P^+\psi_s^{\varepsilon,r} \parallel = 1 \ . \tag{8}$$

Since $\lim_{\varepsilon\to 0}\lim_{s\to-\infty}\Psi_s^\varepsilon = \Omega$ all the $\psi_s^{\varepsilon,r}$ lie in the negative energy subspace $\mathcal{H}_-$ in the limit $\lim_{\varepsilon\to 0}\lim_{s\to-\infty}$:

$$\lim_{\varepsilon\to 0}\lim_{s\to-\infty} \parallel P^+\psi_s^{\varepsilon,r} \parallel = 0 \ . \tag{9}$$

So heuristically spontaneous pair creation can be treated as a problem of the one particle Dirac equation: Find solutions of the one particle Dirac equation with external potential (7) which satisfy (8) and (9).

We shall prove that such solutions exist. We shall also show that this result holds correspondingly in the so called second quantized Dirac field with external field setting, whenever the latter makes sense, i.e. that pair creation holds with exactly the same parameters in Fock space.

2.2 SPC as one particle Dirac problem

Dealing with the Dirac equation with static external potentials it is helpful to choose the eigenbasis of the Dirac operator $D^0 + A\!\!\!/$.

Since the time variation of the potentials is very slow, presenting the wave function in the eigenbasis of the Dirac operator will still be helpful. But we have a family of Dirac operators indexed by s and thus a family of eigenenbasis'. One expects that under fairly general conditions on the potential, the eigenfunctions of the Dirac operator $D^0 + \mu A\!\!\!/$ and their eigenvalues vary smoothly with μ, thus in the adiabatic case the wave function representation in the "time dependent basis" will remain invariant (up to a certain error). In particular no jumping over spectral gaps is possible.

The only way to get transition from the negative to the positive continuous spectrum is by "lifting" bound states. Take a potential without bound states which is repulsive for electrons. Increasing μ at least one bound state emerges from the negative continuous spectrum, is transported through the spectral gap between the positive and negative continuous spectrum and vanishes - let us say at $\mu = \mu_1$ - in the positive continuous spectrum. An example for such a potential is a scalar potential

$$A(\mathbf{x}) = \begin{cases} 1 & \text{for} \quad x < R \\ 0 & \text{for} \quad x > R \end{cases}$$

for some $R \in \mathbb{R}$ (see [6]).

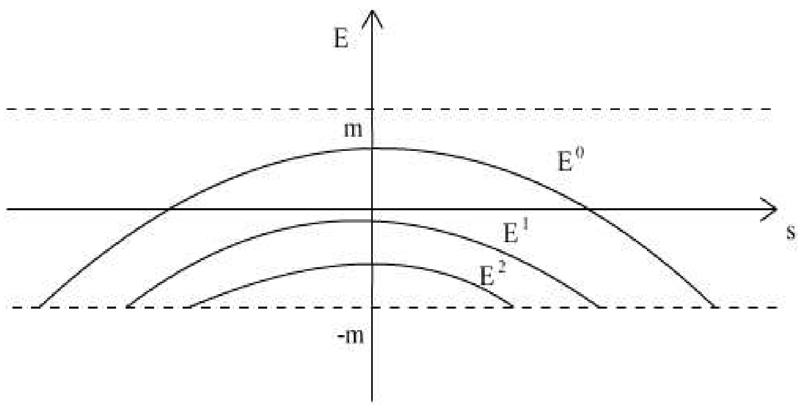

Figure 1: Bound spectrum of the Dirac operator in the undercritical case

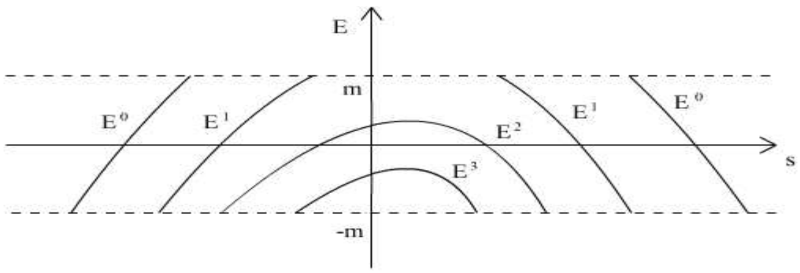

Figure 2: Bound spectrum of the Dirac operator in the overcritical case

With this picture in mind, we are able to understand the cause of spontaneous pair creation: As long as λ is smaller than μ_1 (undercritical regime), the positive continuous energy spectrum is isolated from the rest of the spectrum for all times and no transition from negative to positive energies is possible.

Choosing $\lambda > \mu_1$ (overcritical regime) transitions might be possible.

So under fairly general conditions on the potential $\not{A}(\mathbf{x})$ we expect a sudden change in the probability of creating a pair in the adiabatic limit: For any switching factor satisfying (6) one expects, that the probability of creating a pair from the vacuum is zero as long as λ is smaller than μ_1 and one for $\lambda > \mu_1$. The critical value μ_1 for λ is referred to as λ_c in the literature.

3 Existence of spontaneous pair creation

Let $\lambda > \lambda_c$. As λ will be fixed in the following, we drop the index "λ" for the potential. We assume that only one eigenvalue E_s - which may be degenerated - disappears in the upper continuous spectrum. Let $\mathcal{N}_s$ denote the respective set of bound states, i.e. for all $\phi_s \in \mathcal{N}_s$

$$D_s\phi_s := (D^0 + \not{A}_s(\mathbf{x}))\phi_s = E_s\phi_s \ .$$

Let $s_{m1} < 0$ the time the bound states disappear in the continuous positive spectrum and $s_{m2} > 0$ the time, the bound states evolve again

$$\begin{aligned} \lim_{s \nearrow s_{m1}} E_s &= m \\ \lim_{s \searrow s_{m2}} E_s &= m \ . \end{aligned} \tag{10}$$

Any normalized bound state $\phi_s \in \mathcal{N}_s$ could in principle lead to a pair creation. In the following these bound states will be called "overcritical".

Definition 3.1 *We use the notation "properly dives into the positive continuous spectrum" for overcritical bound states if there exists a $s_0 < s_{m1}$ such that*

$$0 < \partial_s E_s < C \tag{11}$$

for all $s_0 \leq s \leq s_{m1}$.

For the heuristic treatment we assume that E_s is not degenerated, i.e. that (up to a phase factor) only one bound state is overcritical. A generalization to overcritical function which do not dive into the positive continuous spectrum properly is possible, too, but laborious and - as the class of overcritical functions of bound states which do not dive into the positive continuous spectrum probably is very special - is left out in this work.

The theorem later will be formulated for an arbitrary number of overcritical functions.

Let s_{m1} and s_{m2} denote the values of s where the bound state disappears in the positive continuous spectrum and where it evolves again

$$\lim_{s \nearrow s_{m1}} E_s = m \qquad\qquad \lim_{s \searrow s_{m2}} E_s = m \ . \tag{12}$$

$s_i < s_{m1}$ and $s_f > s_{m2}$ be values of s where the overcritical bound state already / still exist. Furthermore we choose s_i and s_f so that there is no crossing of the energy of the eigenstates with the energy of other eigenstates for $s_i < s < s_f$.

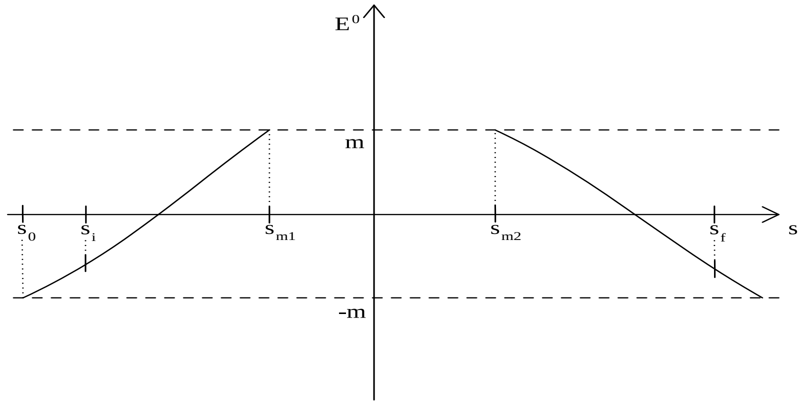

Figure 3: Energy eigenvalue of $\phi_{\varphi(s)}$

Let us consider a wave function ψ_s^ε which is solution of the Dirac equation (7) with potential $A\!\!\!/_s(\mathbf{x})$ and which is at time $t_i = \frac{s_i}{\varepsilon}$ equal (up to a phase factor) to the overcritical bound state of the Dirac operator with potential $A\!\!\!/_{s_i}(\mathbf{x})$.

$$\psi_{s_i}^\varepsilon = \phi_{s_i} \tag{13}$$

Due to the adiabatic switching formalism, ψ_s^ε will lie in the negative continuous energy spectrum for all times $s < s_0$, so $\psi_{s_i}^\varepsilon$ will satisfy (8). Assume, that $\psi_{s_f}^\varepsilon$ is orthogonal to ϕ_{s_f} (so ψ_s^ε "missed" the only way, that would lead it back into the negative continuous spectrum). Such a ψ_s^ε will satisfy (9), too. As we heuristically assumed that wave functions which satisfy (8) and (9) describe the creation of a pair, we define the probability of adiabatic pair creation as

$$p(A\!\!\!/) = 1 - \lim_{\varepsilon \to 0} \langle \psi_{s_f}^\varepsilon \mid \phi_{s_f} \rangle \ . \tag{14}$$

($\langle f \mid g \rangle$ denotes the scalar product $\int f^t g d^3x$. In our case f and g are vector valued.)

We shall show later on, that this is in fact equal to the probability of adiabatic pair creation in a Fock space setting.

3.1 The Theorem

For technical reasons we will restrict ourselves to pure electric potentials (in this case $A\!\!\!/$ is a multiple of the unit matrix, so we can treat it as a scalar and write A instead of $A\!\!\!/$) of the form defined in (5) where $A(\mathbf{x})$ is such that

$$\int A(\mathbf{x})\phi_{\lambda_c}(\mathbf{x})d^3x \neq 0 \tag{15}$$

or

$$(1-i\beta)\int A(\mathbf{x})\phi_{\lambda_c}(\mathbf{x})\mathbf{x}d^3x \neq 0 \tag{16}$$

Remark 3.2 *The set of potentials A which satisfy (15) or (16) is not empty. One can easily prove that the s-wave of a spherical step potential satisfies (15): In this case ϕ_{λ_c} can be split into two two-spinors which are both eigenstates of the Schrödinger equation with certain potential and energy. The lowest energy state of the Schrödinger equation is always positive, so is A and (15) follows.*

Theorem 3.3 *Let $\lambda A_s(\mathbf{x})$ be of the form defined in (5) with $A_l = 0$ for $l \neq 0$ and $A > 0$, where*

- *$A(\mathbf{x}) \in C^\infty$ is compactly supported and satisfies (15) or (16),*
- *λ is such that only one eigenvalue E_s disappears in the upper continuous spectrum and*
- *$\varphi(s)$ is such that E_s dives properly into the positive continuous spectrum.*

Let ϕ_s and $\widetilde{\phi}_s$ be overcritical bound states of $A_s(\mathbf{x})$, let $s_i < s_{m1}$ and $s_f > s_{m2}$ be such that ϕ_{s_i} and $\widetilde{\phi}_{s_f}$ already / still exist. Let ψ_s^ε be solution of the Dirac equation (7) with $\psi_{s_i}^\varepsilon = \phi_{s_i}$, then

$$\lim_{\varepsilon\to 0}\langle\psi_{s_f}^\varepsilon \mid \widetilde{\phi}_{s_f}\rangle = 0\ . \tag{17}$$

One direct result of Theorem 3.3 and the time adiabatic theorem is

Corollary 3.4 *Let $\widetilde{\psi}_s^\varepsilon$ be any solution of the dirac equation where $\widetilde{\psi}_{s_i}^\varepsilon$ is orthogonal to all overcritical bound states ϕ_{s_i} and $\lim_{s\to-\infty} \parallel P^+\widetilde{\psi}_s^\varepsilon \parallel = 0$. Then under the conditions of Theorem 3.3*

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{s\to\pm\infty} \parallel P^\pm\psi_s^\varepsilon \parallel &= 1\\ \lim_{\varepsilon\to 0}\lim_{s\to+\infty} \parallel P^+\widetilde{\psi}_s^\varepsilon \parallel &= 0\end{aligned}$$

To prove the existence of spontaneous pair creation we follow the propagation of ψ_s^ε. For sufficiently small ε, ψ_s^ε follows more or less the bound states ϕ_s. Reaching the time s_{m1} the bound state "vanishes" in the positive continuous spectrum of the Hamiltonian. It is left to assure, that ψ_s^ε will stay in the positive continuous spectrum after removing the potential again and not fall back via the overcritical bound state into the negative energy spectrum.

3.2 Nenciu's contribution

The picture we gave so far has been developed by Nenciu [5]. He did not prove all of Theorem 3.3 but conjectured it. Instead he used a switching factor with a jump of height δ at $s = s_{m1} - \mathcal{O}(\delta)$ chosen in a way that for $s < s_{m1} - \mathcal{O}(\delta)$ the bound state is isolated from the positive energy spectrum and for $s_{m1} - \mathcal{O}(\delta) \leq s < s_{m2}$ the bound state disappears. This jump of course is a violation of the adiabatic idea. But it gives a regime with a sudden change in the pair creation amplitude when the coupling constant reaches the critical value.

For $\delta \ll 1$ the part of the wave function which does not lie in the upper continuous spectrum is negligible. So for any fixed $0 < \delta \ll 1$ the wave function will show a typical scattering state behavior and thus propagate away from the range of the potential. Hence it is orthogonal to any bound state which may reappear at times $s > s_{m2}$. Hence we have pair creation with probability one in the limit $\lim_{\delta\to 0} \lim_{\varepsilon\to 0}$.

4 Fock space formulation of the theorem

The usually accepted way of describing pair creation and annihilation effects is by virtue of the so called Second Quantized Dirac Equation with external potential.

We shall show that our main Theorem 3.3 yields as corollary the corresponding statement in the setting of second quantized formulation.

4.1 Heuristic connection between Fock space and Dirac sea

The Dirac sea uses the ideas of a wave function describing an infinite number of particles. This idea has not been made mathematically rigorous. We want to give a phenomenological description: Any heuristically antisymmetrized multi particle wave function $\Psi_t = \prod_{r\in\mathbb{R}}^{antisym} \psi_t^r$ where all the ψ_t^r are eigenfunctions of the free Dirac operator can be clearly characterized by giving the number of states with positive energy and the number of holes in the Dirac sea and their spins and momenta. Generalization to wave functions $\Psi_t = \prod_{r\in\mathbb{R}}^{antisym} \psi_t^r$ which are not products of eigenfunctions is possible via linear combination of the Ψ_t.

Let us introduce the Fock space $\mathcal{F}$. This space essentially focuses on the arbitrary number of electron positron **pairs** which may be present. One takes the direct sum of all spaces $\mathcal{F}^{(n)}$ describing n electron-positron pairs

$$\mathcal{F} = \oplus_{n=0}^{\infty}\mathcal{F}^{(n)}$$

where the wave functions in $\mathcal{F}^{(n)}$ describe n particles with positive energy and n holes in the Dirac sea (see the interpretation of the multi particle wave functions given in section 1.1):

$$\mathcal{F}^{(n)} = \mathcal{F}_+^{(n)} \otimes \mathcal{F}_-^{(n)} \ .$$

$\mathcal{F}_\pm^{(n)}$ are the antisymmetrized tensor products of n copies of $\mathcal{H}_\pm$.

Here we have defined the Fock space for zero total charge. It is possible to allow more electrons than positrons or vice versa. But as we are dealing with pair creation from the vacuum (the state where no electrons and positrons are present), the total charge of our system is always zero.

4.2 Spontaneous pair creation in Second Quantized Dirac Theory

What happens in the presence of a compactly supported time dependent potential $\not{A}_t$? In contrast to the free case the vacuum is in general not stable anymore (see discussion in section 1.1). Electrons with negative energy may propagate into the positive energy spectrum, leaving holes in the Dirac sea, so pair creation may occur.

We want to estimate the amplitude of pair creations and the wave functions of the created electrons and positrons during a small time interval Δt.

Let us start at time t with an arbitrary number n of particles with positive energy. As we only deal with an uncharged system, there are also n holes in the Dirac sea - or in other words all but n negative energy states are occupied. Assume that at time $t = 0$ all the wave functions ψ_t^r are eigenfunctions of D^0. In general $\mid \Psi_t\rangle$ will of course be a superposition of such wave functions, also with different numbers of pairs, but due to linearity the generalization is easy. Since there is no interaction between the particles, the differential equations for the different particles decouple and the propagation of the system is given by the propagation of each single particle. In general there might be transition from $\mathcal{H}^+$ to $\mathcal{H}^-$, so pair creation may occur.

To calculate the amplitude of the pair creations and the wave functions of the electrons and positrons at time $t + \Delta t$, one has to observe the propagation of all particles which are - as mentioned above - of infinite number. For each factor ψ_t^r of the multi particle wave function $\Psi_t = \prod_{r\in\mathbb{R}}^{antisym} \psi_t^r$ the propagation is given by the Dirac equation (3). So for small Δt we get (using the notation D_t for the Dirac operator $D^0 + \not{A}_t$)

$$\psi_{t+\Delta t}^r = (1 - i\Delta t D_t)\psi_t^r$$

as approximation.

Look at the situation in Fock space, identifying particles with positive energy as electrons and holes in the Dirac sea as positrons.

Let us start with a vector in Fock space describing n electrons and n positrons whose wave functions are eigenfunctions of D^0.

From the propagation of the particles which are present in the Dirac-sea picture we construct the new Fock space vector. We separate each Dirac particle, calculate its propagation and put it back again. Taking away a particle with negative energy leaves a hole in the Dirac sea. In Fock-space language, taking away a particle with negative energy means to create a positron, taking away a particle with positive energy means to annihilate an electron.

Therefore we define the operators $a_{\pm,j,\mathbf{k}}$ as the creators of a electron/positron with spin j and momentum $\mathbf{k}$, $a^\dagger_{\pm,j,\mathbf{k}}$ as the annihilators. As the multi particle wave function was antisymmetrized, these operators satisfy the anti-commutation relations

$$\begin{aligned} a_{\pm,j,\mathbf{k}}a^\dagger_{\pm,j',\mathbf{k}'} + a^\dagger_{\pm,j',\mathbf{k}'}a_{\pm,j,\mathbf{k}} &= \delta(\mathbf{k},\mathbf{k}')\delta(s,j') \\ a_{\mp,j,\mathbf{k}}a^\dagger_{\pm,j',\mathbf{k}'} + a^\dagger_{\pm,j',\mathbf{k}'}a_{\mp,j,\mathbf{k}} &= 0 \end{aligned} \tag{18}$$

Otherwise it would be possible to create two electrons or two positrons with same spin and same momentum, which is a violation of Pauli's exclusion principle.

We start with the propagation of the particles with negative energy. As described above, we take away a particle with negative energy and "quantum numbers" j and $\mathbf{k}$ for spin and momentum. In Fock space language, taking away a electron with negative energy is **creation** of a positron

$$\mid \nu_t\rangle = a_{-,j,\mathbf{k}} \mid \Psi_t\rangle \tag{19}$$

Then we calculate the propagation of the particle with the given spin and momentum

$$\varphi_{t+\Delta t} = (1 - i\Delta t D_t)\phi^{-,j,\mathbf{k}}$$

We calculate for each "quantum number" $\pm$, j' and $\mathbf{k}'$ the scalar product of $\phi^{\pm,j',\mathbf{k}'}$ and $\varphi_{t+\Delta t}$

$$\widehat{\varphi}^{\pm,j',\mathbf{k}'}_{t+\Delta t} = \langle \phi^{\pm,j',\mathbf{k}'} \mid (1 - i\Delta t D_t)\phi^{-,j,\mathbf{k}} \mid\rangle$$

and create/annihilate all the electrons/positrons with these quantum numbers in the given amplitude to ν (see 19):

$$\begin{aligned} \Big(\sum_{j'=1,2} \int_{\mathbf{k}'} a^\dagger_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid\rangle d^3k' \\ + \sum_{j'=1,2} \int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid\rangle d^3k' \ \Big) \ a_{-,j,\mathbf{k}} \mid \Psi_t\rangle \ . \end{aligned}$$

We do this procedure for any occupied state of the Dirac sea. Since $a_{-,j,\mathbf{k}} \mid \Psi\rangle = 0$ if there is a positron with spin j and $\mathbf{k}$ present in $\mid \Psi\rangle$ we can generalize this formula to all states - not caring whether they are occupied or not. This leads us to:

$$\sum_{j=1,2}\int_{\mathbf{k}} \quad (\quad \sum_{j'=1,2}\int_{\mathbf{k}'} a^{\dagger}_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k'$$

$$+ \sum_{j'=1,2}\int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \quad) \quad a_{-,j,\mathbf{k}} d^3k \mid \Psi_t \rangle \; .$$

Observing the propagation of the particles with positive energy we have to use - as described above - the annihilator of electrons. Now we use the fact, that $a^{\dagger}_{+,j,\mathbf{k}} \mid \Psi\rangle = 0$ if the electron with the given quantum numbers is not present in $\mid \Psi\rangle$ to generalize the formula to all j and $\mathbf{k}$ without caring, whether the particles with the given quantum numbers are present in our system.

So observing the propagation of all particles we get for small Δt:

$$\begin{aligned}
\mid \Psi_{t+\Delta t}\rangle \quad = \quad & (\sum_{j=1,2}\int_{\mathbf{k}} (\sum_{j'=1,2}\int_{\mathbf{k}'} a^{\dagger}_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \\
& + \sum_{j'=1,2}\int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{+,j,\mathbf{k}} \mid \rangle d^3k' \quad) \quad a^{\dagger}_{+,j,\mathbf{k}} d^3k \\
& + \sum_{j=1,2}\int_{\mathbf{k}} (\sum_{j'=1,2}\int_{\mathbf{k}'} a^{\dagger}_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \\
& + \sum_{j'=1,2}\int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid 1 - i\Delta t D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \quad) \quad a_{-,j,\mathbf{k}} d^3k \quad) \quad \mid \Psi_t \rangle \; .
\end{aligned}$$

and

$$\begin{aligned}
\mid \Psi_{t+\Delta t}\rangle - \mid \Psi_t\rangle \quad = \quad & -i\Delta t \quad (\quad \sum_{j=1,2}\int_{\mathbf{k}} (\sum_{j'=1,2}\int_{\mathbf{k}'} a^{\dagger}_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \\
& + \sum_{j'=1,2}\int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid D_t \phi^{+,j,\mathbf{k}} \mid \rangle d^3k' \quad) \quad a^{\dagger}_{+,j,\mathbf{k}} d^3k \\
& + \sum_{j=1,2}\int_{\mathbf{k}} (\sum_{j'=1,2}\int_{\mathbf{k}'} a^{\dagger}_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \\
& + \sum_{j'=1,2}\int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid \rangle d^3k' \quad) \quad a_{-,j,\mathbf{k}} d^3k \quad) \quad \mid \Psi_t \rangle \; .
\end{aligned}$$

Thus dividing by Δt we get with $\Delta t \to 0$

$$
\begin{aligned}
i\partial_t \mid \Psi_t\rangle \;=\; & \Big(\sum_{j=1,2} \int_{\mathbf{k}} \Big(\sum_{j'=1,2} \int_{\mathbf{k}'} a^\dagger_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid\rangle d^3k' \\
& + \sum_{j'=1,2} \int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid D_t \phi^{+,j,\mathbf{k}} \mid\rangle d^3k' \Big) \; a^\dagger_{+,j,\mathbf{k}} d^3k \\
& + \sum_{j=1,2} \int_{\mathbf{k}} \Big(\sum_{j'=1,2} \int_{\mathbf{k}'} a^\dagger_{-,j,\mathbf{k}} \langle \phi^{-,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid\rangle d^3k' \\
& + \sum_{j'=1,2} \int_{\mathbf{k}'} a_{+,j,\mathbf{k}} \langle \phi^{+,j',\mathbf{k}'} \mid D_t \phi^{-,j,\mathbf{k}} \mid\rangle d^3k' \Big) \; a_{-,j,\mathbf{k}} d^3k \Big) \quad \mid \Psi_t\rangle \; .
\end{aligned}
$$

Defining the field operators as

$$
\widehat{\chi} := \sum_{j=1}^{2} \int \phi_k^{+,j} a_{+,j,k} + \phi_k^{-,j} a^\dagger_{-,j,k} d^3k \tag{20}
$$

we get

$$
i\partial_t \mid \Psi_t\rangle = \int d^3x \widehat{\chi}^\dagger D_t \widehat{\chi} \quad \mid \Psi_t\rangle \tag{21}
$$

as second quantized Dirac equation with $\int d^3x \widehat{\chi}^\dagger D \widehat{\chi}$ as Dirac field Hamiltonian.

We want to draw from Corollary 3.4 another Corollary asserting the pair creation in the Second Quantized Dirac Equation. That equation is not always well depending on the choice of A. We assume that A be such that the second quantized Dirac equation makes sense (see [10] for references). We call such A's good. Using the notation $\mid \Omega\rangle$ for the vacuum and having the ideas of the previous sections in mind, we have

Corollary 4.1 *Let $A_s(\mathbf{x})$ be a good potential of the form defined (5) with (at least) one overcritical bound state. Let this overcritical bound state dive properly into the positive continuous spectrum. Let $\mid \Psi_t^\varepsilon\rangle$ be a solution of the Second Quantized Dirac Equation with potential $A_{\frac{t}{\varepsilon}}(\mathbf{x})$ with $\lim_{t\to-\infty} \mid \Psi_t^\varepsilon\rangle = \mid \Omega\rangle$.*

Then

$$
\lim_{\varepsilon\to 0} \lim_{t\to\infty} \langle \Omega \mid \Psi_t^\varepsilon\rangle = 0 \; .
$$

Proof of Corollary 4.1

We shall prove this Corollary rigorously, following the intuition given by the sea picture. "Adding" an electron with positive energy to the multi particle wave function $\Psi_t = \prod_{r\in\mathbb{R}}^{antisym} \psi_t^r$ corresponds to the creation of an electron in Fock space, "adding" an electron with negative energy corresponds to the annihilation of a positron.

"Subtracting" an electron with positive energy corresponds to the annihilation of an electron, "subtracting" an electron with negative energy corresponds to the creation of a positron.

Hence we define for any $\omega \in L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ the operators in Fock space

$$\begin{aligned}\widehat{\omega}^\dagger &:= \sum_{j=1}^{2}\int \langle\omega, \phi_k^{+,j}\rangle a_{+,j,k}^\dagger d^3k + \sum_{j=1}^{2}\int \langle\omega, \phi_k^{-,j}\rangle a_{-,j,k} d^3k \\ \widehat{\omega} &:= \sum_{j=1}^{2}\int \langle\omega, \phi_k^{+,j}\rangle a_{+,j,k} d^3k + \sum_{j=1}^{2}\int \langle\omega, \phi_k^{-,j}\rangle a_{-,j,k}^\dagger d^3k\ .\end{aligned} \tag{22}$$

whereas $\phi_k^{+,j}$ and $\phi_k^{-,j}$ are the solutions of the free Dirac equation with momentum $\mathbf{k}$ spin $j \in \{1;2\}$ and positive and negative energy respectively.

Following the ideas above we get

Lemma 4.2 *Let* $\mid \Psi_t\rangle$ *be solution of the Second Quantized Dirac Equation (21),* $\xi_t \in L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ *be solution of the Dirac equation (7).*

Then

$$\begin{aligned}\mid \widetilde{\Psi}_t\rangle &:= \widehat{\xi_t}^\dagger \mid \Psi_t\rangle \\ \mid \widetilde{\widetilde{\Psi}}_t\rangle &:= \widehat{\xi_t} \mid \Psi_t\rangle\end{aligned}$$

are solutions of the Second Quantized Dirac Equation (21).

The proof of this Lemma is given below.

We use this Lemma on $\mid \Psi_t^\varepsilon\rangle$, the special $\lim_{t\to-\infty} \mid \Psi_t^\varepsilon\rangle = \mid \Omega\rangle$ solution of the second quantized Dirac equation with potential $A_{\frac{t}{\varepsilon}}(\mathbf{x})$ and on $\psi^\varepsilon_{\frac{t}{\varepsilon}}$. It follows that

$$\mid \overline{\Psi}_t^\varepsilon\rangle := \widehat{\psi^\varepsilon_{\frac{t}{\varepsilon}}} \mid \Psi_t^\varepsilon\rangle \tag{23}$$

is solution of the Second Quantized Dirac Equation. Furthermore one can show by direct calculation (using the commutator relations of the $a_{\pm,j,\mathbf{k}}$ and $a^\dagger_{\pm,j,\mathbf{k}}$, the fact, that $a^\dagger_{\pm,j,\mathbf{k}} \mid \Omega\rangle = 0$ and (9)) that

$$
\begin{aligned}
\lim_{t\to\infty}\widehat{\psi^{\varepsilon}_{\frac{t}{\varepsilon}}}^{\dagger}\mid\overline{\Psi^{\varepsilon}_{t}}\rangle &= \lim_{t\to\infty}\widehat{\psi^{\varepsilon}_{\frac{t}{\varepsilon}}}^{\dagger}\widehat{\psi^{\varepsilon}_{\frac{t}{\varepsilon}}}\mid\Psi^{\varepsilon}_{t}\rangle \\
&= \lim_{t\to\infty}\widehat{\psi}^{\varepsilon\dagger}_{t}\Big(\sum_{l=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{+,j}_{k}\rangle\ \ a_{+,j,k}d^{3}k\mid\Psi^{\varepsilon}_{t}\rangle \\
&\quad+\sum_{j=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ a^{\dagger}_{-,j,k}d^{3}k\mid\Psi^{\varepsilon}_{t}\rangle\ \ \Big) \\
&= \lim_{t\to\infty}\widehat{\psi}^{\varepsilon\dagger}_{t}\Big(\sum_{j=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ a^{\dagger}_{-,j,k}d^{3}k\mid\Psi^{\varepsilon}_{t}\rangle\ \ \Big) \\
&= \lim_{t\to\infty}\Big(\sum_{l=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{+,j'}_{k}\rangle\ \ a^{\dagger}_{+,l,k}d^{3}k \\
&\quad+\sum_{l=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j'}_{k}\rangle\ \ a_{-,l,k}d^{3}k\Big)\sum_{j=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ a^{\dagger}_{-,j,k}d^{3}k\mid\Psi^{\varepsilon}_{t}\rangle \\
&= \lim_{t\to\infty}\sum_{l,j=1}^{2}\int\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j'}_{k'}\rangle\ \ \langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ a_{-,l,k'}a^{\dagger}_{-,j,k}\mid\Psi^{\varepsilon}_{t}\rangle\ \ d^{3}k'd^{3}k \\
&= \lim_{t\to\infty}\Big(\sum_{l,j=1}^{2}\int\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j'}_{k'}\rangle\ \ \langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ \delta(j,l)\delta(\mathbf{k},\mathbf{k}')\mid\Psi^{\varepsilon}_{t}\rangle\ \ d^{3}k'd^{3}k \\
&\quad-\sum_{l,j=1}^{2}\int\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j'}_{k'}\rangle\ \ \langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ a^{\dagger}_{-,j,k}a_{-,l,k'}\mid\Psi^{\varepsilon}_{t}\rangle\ \ d^{3}k'd^{3}k\Big) \\
&= \lim_{t\to\infty}\sum_{j=1}^{2}\int\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{\mathbf{k}}\rangle\ \ \langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle\ \ \ \mid\Psi^{\varepsilon}_{t}\rangle\ \ d^{3}k \\
&= \lim_{t\to\infty}\int\mid P^{-}\mathcal{F}_{0}(\psi^{\varepsilon}_{\frac{t}{\varepsilon}})\mid^{2}\ d^{3}k\mid\Psi^{\varepsilon}_{t}\rangle \\
&= \lim_{t\to\infty}\mid\Psi^{\varepsilon}_{t}\rangle\quad . \qquad (24)
\end{aligned}
$$

This equation, $\lim_{t\to\infty}\langle\psi^{\varepsilon}_{\frac{t}{\varepsilon}}\mid\phi^{-,j}_{k}\rangle=0$ (which follows directly from Corollary 3.4) and $a^{\dagger}_{\pm,j,\mathbf{k}}\mid\Omega\rangle=0$ yield

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{t\to\infty}\langle\Omega\mid\Psi_t^{\varepsilon}\rangle &= \lim_{\varepsilon\to 0}\lim_{t\to\infty}\langle\Omega\mid\widehat{\psi_{\frac{t}{\varepsilon}}^{\varepsilon}}^{\dagger}\mid\overline{\Psi_t^{\varepsilon}}\rangle\\
&= \lim_{\varepsilon\to 0}\lim_{t\to\infty}\langle\widehat{\psi_{\frac{t}{\varepsilon}}^{\varepsilon}}\Omega\mid\overline{\Psi_t^{\varepsilon}}\rangle\\
&= \lim_{\varepsilon\to 0}\lim_{t\to\infty}\langle\Big(\sum_{j=1}^{2}\int\langle\psi_{\frac{t}{\varepsilon}}^{\varepsilon}\mid\phi_k^{+,j}\rangle a_{+,j,k}^{\dagger}d^3k\\
&\quad+\sum_{j=1}^{2}\int\langle\psi_{\frac{t}{\varepsilon}}^{\varepsilon}\mid\phi_k^{-,j}\rangle a_{-,j,k}d^3k\Big)\Omega\mid\overline{\Psi_t^{\varepsilon}}\rangle\\
&= \lim_{\varepsilon\to 0}\lim_{t\to\infty}\langle\Big(\sum_{j=1}^{2}\int\langle\psi_{\frac{t}{\varepsilon}}^{\varepsilon}\mid\phi_k^{+,j}\rangle a_{+,j,k}^{\dagger}d^3k\Big)\Omega\mid\overline{\Psi_t^{\varepsilon}}\rangle\\
&= 0
\end{aligned}
$$

which proves Corollary 4.1.

□

We shall now give the proof of Lemma 4.2. For ease of reference we recall what the Lemma says.

Lemma 4.2

Let $\mid\Psi_t\rangle$ *be solution of the Second Quantized Dirac Equation (21),* $\xi_t\in L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^4$ *be solution of the Dirac equation (7) with* $\widehat{\xi_t}$ *defined as in (22).*
Then

$$
\begin{aligned}
\mid\widetilde{\Psi}_t\rangle &:= \widehat{\xi_t}^{\dagger}\mid\Psi_t\rangle \qquad (25)\\
\mid\widetilde{\widetilde{\Psi}}_t\rangle &:= \widehat{\xi_t}\mid\Psi_t\rangle \qquad (26)
\end{aligned}
$$

are solutions of the Second Quantized Dirac Equation (21).

Proof

We will only prove (25). (26) follows analogously. We shall leave out the index t.

We need to show that

$$\partial_t\big(\widehat{\xi}^{\dagger}\mid\Psi\rangle\big)=H\big(\widehat{\xi}^{\dagger}\mid\Psi\rangle\big) \qquad (27)$$

We start with the right hand side:

$$H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \;\; \widehat{\xi}^\dagger \mid \Psi_t\rangle \;.$$

Inserting the field operator(20) leads to

$$H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; \int \widehat{\psi}^\dagger D \sum_{s=1}^{2} \int \phi_k^{+,s} a_{+,s,k} + \phi_k^{-,s} a^\dagger_{-,s,k} d^3k d^3x \;\; \widehat{\xi}^\dagger \mid \Psi_t\rangle \;.$$

Using the definition (22) of the operator $\widehat{\xi}^\dagger$ yields

$$\begin{aligned} H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; & \int \widehat{\psi}^\dagger D \sum_{s=1}^{2} \int \phi_k^{+,s} a_{+,s,k} + \phi_k^{-,s} a^\dagger_{-,s,k} d^3k \sum_{s'=1}^{2} \int \langle \xi, \phi_{k'}^{+,s'} \rangle a^\dagger_{+,s',k'} d^3k' \\ & + \int \langle \xi, \phi_k^{-,s'} \rangle a_{-,s',k'} d^3k' d^3x \quad \mid \Psi_t\rangle \;. \end{aligned}$$

Next we use the commutation relations for the $a^{(\dagger)}_{\pm,s,k}$ (18). This leads to

$$\begin{aligned} H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; & \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle \\ & + \int \widehat{\psi}^\dagger D \sum_{s,s'=1}^{2} \int \int \langle \xi, \phi_{k'}^{+,s'} \rangle \phi_k^{+,s} \delta(k,k')\delta(s,s') \\ & \qquad + \langle \xi, \phi_{k'}^{-,s'} \rangle \phi_k^{-,s} \delta(k,k')\delta(s,s') d^3k d^3k' d^3x \quad \mid \Psi_t\rangle \end{aligned}$$

Executing the d^3k'-integration gives us

$$\begin{aligned} H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; & \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle + \int \widehat{\psi}^\dagger D \sum_{s=1}^{2} \int \langle \xi, \phi_k^{+,s} \rangle \phi_k^{+,s} \\ & \qquad + \langle \xi, \phi_k^{-,s} \rangle \phi_k^{-,s} d^3k d^3x \quad \mid \Psi_t\rangle \\ =\; & \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle + \int \widehat{\psi}^\dagger D\xi d^3x \quad \mid \Psi_t\rangle \;. \end{aligned}$$

As ξ is by definition solution of the Dirac equation (7) we can write

$$H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) \;=\; \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle + \int \widehat{\psi}^\dagger \partial_t \xi d^3x \quad \mid \Psi_t\rangle \;.$$

We again use (20), now for $\widehat{\psi}^\dagger$ and get

$$\begin{aligned} H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) &= \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle \\ &\quad + \int \sum_{s=1}^{2} \int \phi_k^{+,s} a^\dagger_{+,s,k} + \phi_k^{-,s} a_{-,s,k} d^3k \partial_t \xi d^3x \quad \mid \Psi_t\rangle \ . \end{aligned}$$

Executing the d^3x-integration gives us

$$\begin{aligned} H\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) &= \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle \\ &\quad + \sum_{s=1}^{2} \int \langle \phi_k^{+,s}, \partial_t \xi\rangle a^\dagger_{+,s,k} + \langle \phi_k^{-,s}, \partial_t \xi\rangle a_{-,s,k} d^3k \quad \mid \Psi_t\rangle \ . \end{aligned}$$

For the left hand side of (27) we have

$$\begin{aligned} \partial_t\big(\widehat{\xi}^\dagger \mid \Psi_t\rangle\big) &= \partial_t \widehat{\xi}^\dagger \mid \Psi_t\rangle + \widehat{\xi}^\dagger \partial_t \mid \Psi_t\rangle \\ &= \sum_{s=1}^{2} \int \langle \phi_k^{+,s}, \partial_t \xi\rangle a^\dagger_{+,s,k} + \langle \phi_k^{-,s}, \partial_t \xi\rangle a_{-,s,k} d^3k \quad \mid \Psi_t\rangle \\ &\quad + \widehat{\xi}^\dagger \int \widehat{\psi}^\dagger D\widehat{\psi} d^3x \quad \mid \Psi_t\rangle \end{aligned}$$

and (27) follows.

□

5 The "Critical" Bound State

We consider the Dirac operator

$$D_\mu := D^0 + \mu A(\mathbf{x}) \ . \tag{28}$$

We call a coupling constant λ_c critical, if and only if there exists a solution $\phi_{\lambda_c} \in L^2$ with

$$(D_{\lambda_c} - m)\,\phi_{\lambda_c} \equiv 0\ . \tag{29}$$

Note that by our choice of the potential $A \in C^\infty$ the solution ϕ_{λ_c} is also C^∞. To see whether a critical state exists we invert (29) and get formally

$$\phi_{\lambda_c}(\mathbf{x}) = (m - D^0)^{-1}\lambda_c A(\mathbf{x})\phi_{\lambda_c}(\mathbf{x})\ , \tag{30}$$

and replacing the $(m - D^0)^{-1} = \lim_{\delta\to 0}(m - D^0 + i\delta)^{-1}$ by the integral kernel $G^+_{k=0} = G^+$:

$$(m - D^0)G^+(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')\ . \tag{31}$$

where [10]

$$G^+(\mathbf{x}) = \frac{1}{4\pi}\left(-x^{-1}(m + \beta m) - ix^{-2}\sum_{j=1}^{3}\alpha_j\frac{x_j}{x}\right)\ , \tag{32}$$

we obtain the Lippmann Schwinger equation

$$\begin{aligned}
\phi_{\lambda_c}(\mathbf{x}) &= \int G^+(\mathbf{x}')\lambda_c A(\mathbf{x} - \mathbf{x}')\phi_{\lambda_c}(\mathbf{x} - \mathbf{x}')d^3x' \qquad (33)\\
&= -\int \frac{1}{4\pi}x'^{-1}(m + \beta m)\lambda_c A(\mathbf{x} - \mathbf{x}')\phi_{\lambda_c}(\mathbf{x} - \mathbf{x}')d^3x'\\
&\quad -\int \frac{1}{4\pi}x'^{-2}\sum_{j=1}^{3}\alpha_j\frac{x'_j}{x'}\lambda_c A(\mathbf{x} - \mathbf{x}')\phi_{\lambda_c}(\mathbf{x} - \mathbf{x}')d^3x'\\
&=: \phi_{c,1}(\mathbf{x}) + \phi_{c,2}(\mathbf{x})\ . \qquad (34)
\end{aligned}$$

Since A has compact support, $\phi_{c,2}(\mathbf{x})$ decays like x^{-2} and thus is in L^2. For $\phi_{c,1}(\mathbf{x})$ we can write

$$\begin{aligned}
\phi_{c,1}(\mathbf{x}) &= -x^{-1}\int \frac{1}{4\pi}(m + \beta m)\lambda_c A(\mathbf{x} - \mathbf{x}')\phi_{\lambda_c}(\mathbf{x} - \mathbf{x}')d^3x'\\
&\quad -\int \frac{1}{4\pi}(x'^{-1} - x^{-1})(m + \beta m)\lambda_c A(\mathbf{x} - \mathbf{x}')\phi_{\lambda_c}(\mathbf{x} - \mathbf{x}')d^3x'\\
&=: \phi_{c,3}(\mathbf{x}) + \phi_{c,4}(\mathbf{x})\ . \qquad (35)
\end{aligned}$$

Using

$$x'^{-1} - x^{-1} = \frac{x - x'}{xx'}$$

we see that for large x and $\mathbf{x}-\mathbf{x}' \in \mathcal{S}_A$, the compact support of A,

$$| x'^{-1} - x^{-1} | \leq \operatorname{diam}(\mathcal{S}_A) \frac{1}{xx'}$$

is of order x^{-2} and thus $\phi_{c,4}$ of order x^{-2}. Hence $\phi_{c,4} \in L^2$.

The decay of $\phi_{c,3}(\mathbf{x})$ depends on the spinor components of $\phi_{\lambda_c}(\mathbf{y})$. There are two possibilities:

Either the spinor components of $\phi_{\lambda_c}(\mathbf{y})$ are such that

$$\int (1+\beta) A(\mathbf{y}) \phi_{\lambda_c}(\mathbf{y}) d^3y \neq 0$$

and thus $\phi_{c,1}(\mathbf{x})$ is of order x^{-1} and thus not in L^2 or such that the spinor

$$\int (1+\beta) A(\mathbf{y}) \phi_{\lambda_c}(\mathbf{y}) d^3y = 0 \tag{36}$$

and thus $\phi_{c,1}(\mathbf{x})$ is of order x^{-2} and thus in L^2. The identity (36) will play a crucial role later on.

It has been proven by Klaus [4] that for "critical" bound states that dive into the positive continuous spectrum properly ϕ_{λ_c} is in L^2. Thus (36) holds in our case and we have that $\phi_{\lambda_c} = \phi_{c,2} + \phi_{c,4}$, and thus

$$| \phi_{\lambda_c} | \leq C x^{-2} \tag{37}$$

for some appropriate $C < \infty$.

Notation 5.1 *In what follows the letters C and C_n, $n \in \mathbb{N}_0$ will be used for various constants that need not be identical even within the same equation.*

In the following we denote the set of bound states present at $\mu = \lambda_c$ by $\mathcal{N}$:

$$\mathcal{N} := \{\phi_{\lambda_c} \in L^2 : (D_{\lambda_c} - m)\phi_{\lambda_c} = 0\} \,. \tag{38}$$

6 Generalized Eigenfunctions

In the following we will use with slight abuse of notation the spin index $j \in \{1,2,3,4\}$, where $j = 1,3$ stands for the different spins of the eigenfunctions with negative energy, $j = 2,4$ stands for the different spins of the eigenfunctions with positive energy.

The generalized eigenfunctions $\phi^j(\mathbf{k}, \mu, \mathbf{x})$ are solutions of

$$(-1)^j E_k \phi^j(\mathbf{k}, \mu, \mathbf{x}) = D_\mu \phi^j(\mathbf{k}, \mu, \mathbf{x}) \tag{39}$$

with $E_k = \sqrt{m^2 + k^2}$.

We change to the Lippmann Schwinger equation

$$\phi^j(\mathbf{k}, \mu, \mathbf{x}) = \phi^j(\mathbf{k}, 0, \mathbf{x}) + \int G_k^+(\mathbf{x} - \mathbf{x}')\mu A(\mathbf{x}')\phi^j(\mathbf{k}, \mu, \mathbf{x}')d^3x' \tag{40}$$

with $\phi^j(\mathbf{k}, 0, \mathbf{x})$ being the well known generalized eigenfunctions of the free Dirac operator D^0 [10] and G_k^+ being the kernel of $(E_k - D^0)^{-1} = \lim_{\delta\to 0}(E_k - D^0 + i\delta)^{-1}$

$$G_k^+(\mathbf{x}) = \frac{1}{4\pi} e^{ikx} \left(-x^{-1}(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - ix^{-2} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} \right) . \tag{41}$$

Lemma 6.1 *Let $\not{A} \in C^\infty$ be compactly supported, $A > 0$, $\overline{\mu} > \lambda_c$ be such, that λ_c is the only critical coupling constant in $[\lambda_c, \overline{\mu}]$. Let $\mathcal{P} := \mathbb{R}^3 \times [\lambda_c, \overline{\mu}] \backslash (0, \lambda_c)$, $j = 2, 4$.*

Then

(a) *there exist unique solutions $\phi^j(\mathbf{k}, \mu, \cdot)$ of (40) in L^∞ for all $(\mathbf{k}, \mu) \in \mathcal{P}$ such that*

(b) *for any $(\mathbf{k}, \mu) \in \mathcal{P}$, these solutions $\phi^j(\mathbf{k}, \mu, \cdot)$ are Hölder continuous of degree 1 in $\mathbf{x}$,*

(c) *any such solution $\phi^j(\mathbf{k}, \mu, \cdot)$ satisfies (39),*

(d) *for any $\mu \in [\lambda_c, \overline{\mu}]$ the set of $\{\phi^j(\mathbf{k}, \mu, \cdot)\}$ defines a generalized Fourier transform in the space of scattering states in the positive continuous subspace by*

$$\mathcal{F}_\mu(\psi)(\mathbf{k}, j) := \int (2\pi)^{-\frac{3}{2}} \langle \phi^j(\mathbf{k}, \mu, \mathbf{x}), \psi(\mathbf{x}) \rangle d^3x \tag{42}$$

and

$$\psi(\mathbf{x}) = \sum_{j=1}^{4} \int (2\pi)^{-\frac{3}{2}} \phi^j(\mathbf{k}, \mu, \mathbf{x}) \mathcal{F}_\mu(\psi)(\mathbf{k}, j) d^3k . \tag{43}$$

The so defined $\mathcal{F}_\mu(\psi)$ is isometric to ψ, i.e.

$$\| \psi \| := \left(\int \mid \psi(\mathbf{x}) \mid^2 d^3x \right)^{\frac{1}{2}} = \sum_{j=1}^{4} \left(\int \mid \mathcal{F}_\mu(\psi)(\mathbf{k}, j) \mid^2 d^3k \right)^{\frac{1}{2}} =: \| \mathcal{F}_\mu(\psi) \|$$

(e) *the functions $\phi^j(\mathbf{k}, \mu, \mathbf{x})$ are infinitely often continuously differentiable with respect to k for $(\mathbf{k}, \mu) \in \mathcal{P}$. Furthermore there exists $0 < \alpha < \infty$ uniform in $(\mathbf{k}, \mu) \in \mathcal{P}$ and for all $n \in \mathbb{N}_0$ constants $C_n < \infty$ uniform in $(\mathbf{k}, \mu) \in \mathcal{P}$ and functions $f^n(\mathbf{k}, \mu, \mathbf{x}) \in \mathcal{N}$ (see (38)) with ($\| f(\cdot) \|_\infty := \sup_{\mathbf{x} \in \mathbb{R}^3} \mid f(\mathbf{x}) \mid$)*

$$\| f^n(\mathbf{k}, \mu, \cdot) \|_\infty \quad < \quad C_n \left(1 + k^n \left(\mid \mu - \lambda_c - \alpha k^2 \mid + k^3 \right)^{-n-1} \right) \qquad (44)$$

such that

$$\begin{aligned} &\| (\mid \cdot \mid + 1)^{-n} \left(\partial_k^n \phi^j(\mathbf{k}, \mu, \cdot) - f^n(\mathbf{k}, \mu, \cdot) \right) \|_\infty \\ &\qquad < C_n \left(1 + (\mu - \lambda_c + k^2) \| f^n(\mathbf{k}, \mu, \cdot) \|_\infty \right) . \end{aligned} \qquad (45)$$

Remark 6.2 *For $j = 1, 3$ we can use the results in [1].*

The divergent behavior of the generalized eigenfunctions expressed by (44) is related to the fact, that there exist solutions ϕ_{λ_c} of (29). Non rigorously such solutions can be seen as solutions of (40) with $\| \phi_{\lambda_c} \|_\infty \gg 1$ so that $\phi^j(\mathbf{k}, 0, \mathbf{x})$ becomes negligible. Since the generalized eigenfunctions are continuous in μ and $\mathbf{k}$ *it is reasonable to assume that the divergent part of $\phi^j(\mathbf{k}, \mu, \mathbf{x})$ as $\mu \to \lambda_c$ and $k \to 0$ lies in $\mathcal{N}$.*

Equation (44) and (45) give us a quantitative estimate of the divergent behavior of $\phi^j(\mathbf{k}, \mu, \cdot)$ and its derivatives with respect to k for small k and μ close to λ_c.

Observe for example the case $n = 0$. Adding (44) and (45) yields, that $\phi^j(\mathbf{k}, \mu, \cdot)$ diverges like the right hand side of (44) (note, that for small k and small $\mu - \lambda_c$ the right hand side of (45) is much smaller than the right hand side of (44)).

Hence subtracting a sufficient multiple of the critical bound states $f^0(\mathbf{k}, \mu, \mathbf{x})$ the divergence of the generalized eigenfunctions becomes weaker, as can be seen on (45).

Proof of Lemma 6.1

For (a) - (d) with $\mu \neq \lambda_c$ one can use Lemma 3.4 in [1]. The $\mu = \lambda_c$ with $\mathbf{k} \neq 0$ can be proven equivalently (one needs the invertibility of $1 - \mu T_k$ which is given in that case).

We shall therefore only prove

Part (e) of the Lemma

We start with a short summary of proof of part (a) of the Lemma, following the proof in [1].

We first show that for any $(\mathbf{k}, \mu) \in \mathcal{P}$ there exists a unique solution $\phi^j(\mathbf{k}, \mu, \cdot)$ of (48).

Let $\mathcal{B}$ be the Banach space of all continuous functions tending uniformly to zero as $x \to \infty$ (equipped with the supremum norm).

Defining the family of operators $T_k : L^\infty \to \mathcal{B}$ by

$$T_k f(\mathbf{x}) := \int G_k^+(\mathbf{x}') A(\mathbf{x} - \mathbf{x}') f(\mathbf{x} - \mathbf{x}') d^3x' \quad (46)$$

$$= \int G_k^+(\mathbf{x} - \mathbf{x}') A(\mathbf{x}') f(\mathbf{x}') d^3x' \quad (47)$$

(40) can be written as

$$\phi^j(\mathbf{k}, \mu, \mathbf{x}) = \phi^j(\mathbf{k}, 0, \mathbf{x}) + \mu T_{\mathbf{k}} \phi^j(\mathbf{k}, \mu, \mathbf{x}) \ . \quad (48)$$

The proof that T_k maps C^∞ into $\mathcal{B}$ can be found in [1]. Note that the definition of T_k yields that $\| T_k \|_\infty^{op}$ exists. Using the continuity of T_k it follows that

$$\sup_{k < k_0} \| T_k \|_\infty < \infty \quad (49)$$

for any $k_0 < \infty$. In view of (32) and (41) we have that

$$T_0 \phi_{\lambda_c} = \int G^+(\mathbf{x}') A(\mathbf{x} - \mathbf{x}') \phi_{\lambda_c}(\mathbf{x} - \mathbf{x}') d^3x'$$

It follows with (33) that

$$\lambda_c T_0 \phi_{\lambda_c} = \phi_{\lambda_c} \quad (50)$$

Furthermore defining

$$\zeta^j(\mathbf{k},\mu,\mathbf{x}) := \phi^j(\mathbf{k},\mu,\mathbf{x}) - \phi^j(\mathbf{k},0,\mathbf{x}) \tag{51}$$

and

$$g^j(\mathbf{k},\mu,\mathbf{x}) := -\mu T_{\mathbf{k}}\phi^j(\mathbf{k},0,\mathbf{x}) \tag{52}$$

(48) becomes

$$\zeta^j(\mathbf{k},\mu,\mathbf{x}) \;\; = \;\; \mu T_{\mathbf{k}}\zeta^j(\mathbf{k},\mu,\mathbf{x}) + g^j(\mathbf{k},\mu,\mathbf{x})\ . \tag{53}$$

We wish to show that (53) has a unique solution in $\mathcal{B}$ for any $(\mathbf{k},\mu) \in \mathcal{P}$. For the Schrödinger Greens-function, this has been proven by Ikebe [3]. We want to proceed in the same way.

Note that by [1] $g(\mathbf{k},\mu,\cdot) \in \mathcal{B}$ for any $(\mathbf{k},\mu) \in \mathcal{P}$. Ikebe uses the Riesz-Schauder theory of completely continuous operators in a Banach space [7]:

If T is a completely continuous operator in $\mathcal{B}$, then for any given $g \in \mathcal{B}$ the equation

$$f = g + Tf \tag{54}$$

has a unique solution in $\mathcal{B}$ if $\widetilde{f} = T\widetilde{f}$ implies that $\widetilde{f} \equiv 0$.

Since T_k is a "nice" integral operator it is completely continuous. We wish to assert that

$$\widetilde{f}(\mathbf{x}) = -\mu T_{\mathbf{k}}\widetilde{f}(\mathbf{x}) \tag{55}$$

has for $(\mathbf{k},\mu) \in \mathcal{P}$ only the trivial solution. Due to [4] there are no zero energy resonances for the class of Dirac operators we consider, so the only non trivial solutions are the bound states with energy m. Since we assumed that there is no bound state with energy m in $\mu \in]\lambda_c, \overline{\mu}]$, (55) has for $(\mathbf{k},\mu) \in \mathcal{P}$ the unique solution $\widetilde{f} \equiv 0$.

Now we are in the position to prove (e). We formulate (e) for $\zeta^j(\mathbf{k},\mu,\cdot)$, which can straight forwardly be done.

Lemma 6.3 *Let $A \in C^\infty$ be compactly supported, $A > 0$, $\overline{\mu}$ be such, that λ_c is the only critical coupling constant in $[\lambda_c, \overline{\mu}]$. Let $\mathcal{B}$ be the Banach space of all continuous functions tending uniformly to zero as $x \to \infty$ (equipped with the supremum norm). Then on $\mathcal{P}$ the functions $\zeta^j(\mathbf{k},\mu,\cdot) \in \mathcal{B}$ are infinitely often continuously differentiable with respect to k, furthermore there exists a $0 < \alpha < \infty$ and for every $n \in \mathbb{N}_0$ a constant $C^n < \infty$ uniform in $(\mathbf{k},\mu) \in \mathcal{P}$ and a function $f^n(\mathbf{k},\mu,\mathbf{x}) \in \mathcal{N}$ (see (38)) with*

$$\| f^n(\mathbf{k},\mu,\cdot) \|_\infty < C^n \left(1 + k^n \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-n-1}\right) \tag{56}$$

such that

$$\begin{aligned} &\| (x+1)^{-n} \left(\partial_k^n \zeta^j(\mathbf{k},\mu,\cdot) - f^n(\mathbf{k},\mu,\cdot)\right) \|_\infty \\ &\qquad < C^n \left(1 + (\mu - \lambda_c + k^2) \| f^n(\mathbf{k},\mu,\cdot) \|_\infty\right) . \end{aligned} \tag{57}$$

From (57)

$$\begin{aligned} &\| (| \cdot | + 1)^{-n} \left(\partial_k^n \phi^j(\mathbf{k},\mu,\cdot) - f^n(\mathbf{k},\mu,\cdot)\right) \|_\infty \\ &\quad \leq \| (| \cdot | + 1)^{-n} \left(\partial_k^n \zeta^j(\mathbf{k},\mu,\cdot) - f^n(\mathbf{k},\mu,\cdot)\right) \|_\infty \\ &\quad\quad + \| (| \cdot | + 1)^{-n} \left(\partial_k^n \phi^j(\mathbf{k},0,\cdot) - f^n(\mathbf{k},\mu,\cdot)\right) \|_\infty \\ &\quad < C_n \left(1 + (\mu - \lambda_c + k^2) \| f^n(\mathbf{k},\mu,\cdot) \|_\infty\right) + C_n \end{aligned}$$

Lemma 6.1 (e) follows.

□

The main difficulty in proving Lemma 6.3 arises from small k. Therefore we show first

Lemma 6.4 *Under the conditions of Lemma 6.3 there exists a $k_0 > 0$ such that on $\mathcal{P}_{k_0} := \{(\mathbf{k},\mu) \in \mathbb{R}^3 \times [\lambda_c; \overline{\mu}] : k < k_0\} \backslash (0,\lambda_c)$ the functions $\zeta^j(\mathbf{k},\mu,\mathbf{x})$ are infinitely often continuously differentiable with respect to k, furthermore there exists a $0 < \alpha < \infty$ and for every $n \in \mathbb{N}_0$ a constant $C < \infty$ uniform in $(\mathbf{k},\mu) \in \mathcal{P}_{k_0}$ and a function $f^n(\mathbf{k},\mu,\mathbf{x}) \in \mathcal{N}$ with*

$$\| f^n(\mathbf{k},\mu,\cdot) \|_\infty \;<\; C^n \left(1 + k^n \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-n-1}\right)$$

such that

$$\| (x+1)^{-n} \left(\partial_k^n \zeta^j(\mathbf{k},\mu,\cdot) - f^n(\mathbf{k},\mu,\cdot)\right) \|_\infty < C^n \left(1 + (\mu - \lambda_c + k^2) \| f^n \|_\infty\right) .$$

Remark 6.5 *From the proof we shall see that k_0 is small.*

We first prove Lemma 6.4, later we will show that Lemma 6.4 implies Lemma 6.3.

Proof of Lemma 6.4

For ease of notation we shall drop the spin index j.
We define a split of $\mathcal{B}$ into a direct sum of two "orthogonal" linear subspaces.

Definition 6.6 *Set*

$$\mathcal{M} := A\mathcal{N} = \{f \mid f = A\phi, \phi \in \mathcal{N}\} \ . \tag{58}$$

and let $\mathcal{M}^\perp \subset \mathcal{B}$ be the set of functions in $\mathcal{B}$ which are "orthogonal" to $\mathcal{M}$ in the sense that

$$f \in \mathcal{M}^\perp \Leftrightarrow \langle Af, \phi\rangle = 0 \qquad \forall \phi \in \mathcal{N} \ .$$

Remark 6.7 *Since f need not be in L^2 we nevertheless write with slight abuse of notation $\langle f, A\phi\rangle$ for $\langle Af, \phi\rangle$.*

Lemma 6.8

$$\mathcal{B} = \mathcal{M} \oplus \mathcal{M}^\perp \ , \tag{59}$$

i.e. every $f \in \mathcal{B}$ can be uniquely decomposed in $f^\parallel \in \mathcal{M}$ and $f^\perp \in \mathcal{M}^\perp$ such that

$$f = f^\parallel + f^\perp \ . \tag{60}$$

Proof

Let $f \in \mathcal{B}$, $1_{\mathcal{S}_A}$ be the characteristic function of the support of A. Set

$$\begin{aligned} f_1 &:= 1_{\mathcal{S}_A} f \\ f_2 &:= (1 - 1_{\mathcal{S}_A}) f \ . \end{aligned}$$

f_2 is zero on the support of A hence it follows trivially that $f_2 \in \mathcal{M}^\perp$. $f_1 \in \mathcal{B}$, being compactly supported, is in $L^2 \cap \mathcal{B}$.

Since $\mathcal{M}$ is a linear subspace of $L^2 \cap \mathcal{B}$ and $\mathcal{M}^\perp \cap L^2$ its orthogonal complement it follows that

$$L^2 \cap \mathcal{B} = \mathcal{M} \oplus (\mathcal{M}^\perp \cap L^2) \; .$$

Hence there exists a $f^\parallel \in \mathcal{M}$ and a $f_3 \in \mathcal{M}^\perp \cap L^2$ with $f_1 = f^\parallel + f_3$. Setting $f^\perp := f_2 + f_3$ (60) follows.

□

We introduce now for any $\mathcal{A} \subseteq \mathcal{B}$ and any $k_0 > 0$ the sets $\widetilde{\mathcal{A}}_{k_0}$ of functions $f(\mathbf{k}, \mu, \mathbf{x}) : \mathcal{P} \times \mathbb{R}^3 \to \mathbb{C}^4$:

Definition 6.9 *Let* $\mathcal{P}_{k_0} := \{(\mathbf{k}, \mu) \in \mathbb{R}^3 \times [\lambda_c; \overline{\mu}] : k < k_0\} \backslash (0, \lambda_c)$, *then*

$$\begin{aligned} f(\mathbf{k},\mu,\mathbf{x}) \in \widetilde{\mathcal{A}}_{k_0} \Leftrightarrow \quad & (a) \quad f(\mathbf{k},\mu,\cdot) \in \mathcal{A} \quad \textit{for any } (\mathbf{k},\mu) \in \mathcal{P}_{k_0} \\ & (b) \quad f(\mathbf{k},\mu,\mathbf{x}) \textit{ is for any } \mathbf{k} \in \mathbb{R}^3 \textit{ with } k \le k_0 \textit{ continuous in } \mu \\ & \qquad \textit{with respect to the supremum norm in } \mathcal{B} \, , \\ & (c) \quad \| f \|_\infty := \sup_{(\mathbf{k},\mu,\mathbf{x}) \in \mathcal{P}_{k_0} \times \mathbb{R}^3} \{ | f(\mathbf{k},\mu,\mathbf{x}) | \} < \infty \; . \end{aligned}$$

We shall first prove Lemma 6.4 for $n = 0$. Choose $(\mathbf{k}, \mu) \in \mathcal{P}$ and recall equation (53)

$$\zeta(\mathbf{k}, \mu, \mathbf{x}) = (1 - \mu T_{\mathbf{k}})^{-1} \, g(\mathbf{k}, \mu, \mathbf{x}) \; . \tag{61}$$

We wish to estimate the "close to $k = 0$" behavior of $\zeta^j(\mathbf{k}, \mu, \mathbf{x})$. For that we will split $g(\mathbf{k}, \mu, \cdot)$ into two parts

$$g(\mathbf{k}, \mu, \cdot) =: g(\mathbf{k}, \mu, \cdot)^\parallel + g^\perp(\mathbf{k}, \mu, \cdot) \tag{62}$$

where $g(\mathbf{k}, \mu, \cdot)^\parallel \in \mathcal{M}$ and $g^\perp(\mathbf{k}, \mu, \cdot) \in \mathcal{M}^\perp$, i.e. $g(\mathbf{k}, \mu, \cdot)^\parallel$ can be written as (note that A is positive and ϕ is nonzero in the range of the potential, thus $\| A\phi_{\lambda_c} \|^2 \neq 0$)

$$g(\mathbf{k}, \mu, \cdot)^\parallel \;\; = \;\; A\phi_{\lambda_c} \frac{\langle \phi_{\lambda_c} \mid A g(\mathbf{k}, \mu, \cdot) \rangle}{\| A\phi_{\lambda_c} \|^2} \tag{63}$$

for some $\phi_{\lambda_c} \in \mathcal{N}$. We choose the normalization $\| \phi_{\lambda_c} \|_\infty = 1$.

Letting now $(\mathbf{k}, \mu)$ vary in $\mathcal{P}_{k_0}$ we shall show that $g(\mathbf{k}, \mu, \cdot)^\parallel(\mathbf{x}) \in \widetilde{\mathcal{M}}_{k_0}$ and $g^\perp(\mathbf{k}, \mu, \cdot)(\mathbf{x}) \in \widetilde{\mathcal{M}}^\perp_{k_0}$ for any $k_0 > 0$.

Since $\| T_k \|_\infty^{op}$ is bounded uniformly in $k \leq k_0$ for any $k_0 < \infty$ (see (49)) it follows that $\| g(\mathbf{k}, \mu, \cdot) \|_\infty$ (see (52)) is bounded uniformly in any $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$. Hence

$$\begin{aligned}
\langle \phi_{\lambda_c} \mid A g(\mathbf{k}, \mu, \mathbf{x}) \rangle &= \int \phi_{\lambda_c}(\mathbf{x}) A(\mathbf{x}) g(\mathbf{k}, \mu, \mathbf{x}) d^3x \\
&\leq \| g(\mathbf{k}, \mu, \cdot) \|_\infty \int \phi_{\lambda_c}(\mathbf{x}) A(\mathbf{x}) d^3x \\
&\leq C \| g(\mathbf{k}, \mu, \cdot) \|_\infty
\end{aligned}$$

is bounded uniformly in $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ for any $k_0 < \infty$ with an appropriate $C < \infty$ since ϕ_{λ_c} is bounded uniformly in $(\mathbf{k}, \mu) \in \mathcal{P}$.

Hence we have for $g^{\|}(\mathbf{k}, \mu, \cdot)$ and $g^{\perp}(\mathbf{k}, \mu, \cdot) := g(\mathbf{k}, \mu, \cdot) - g^{\|}(\mathbf{k}, \mu, \cdot)$ that on $\mathcal{P}_{k_0}$

$$\| g^{\|} \|_\infty < \infty \tag{64}$$

$$\| g^{\perp} \|_\infty < \infty \, . \tag{65}$$

The continuity of the scalar product and the continuity of $g(\mathbf{k}, \mu, \cdot)$ in μ yield that $g^{\|}(\mathbf{k}, \mu, \cdot)$ and $g^{\perp}(\mathbf{k}, \mu, \cdot)$ are continuous in μ for any $\mathbf{k} \in \mathbb{R}^3$, hence $g^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ and $g^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ for any $k_0 > 0$.

We now return to (61). We get with (62)

$$\zeta^j(\mathbf{k}, \mu, \cdot) = (\mu T_k - 1)^{-1} g^{\|}(\mathbf{k}, \mu, \cdot) + (\mu T_k - 1)^{-1} g^{\perp}(\mathbf{k}, \mu, \cdot) \, . \tag{66}$$

We shall determine $\zeta^j(\mathbf{k}, \mu, \cdot)$ now more precisely by the following iteration procedure

Lemma 6.10 *Let $k_0 > 0$. Then there exits a constant $0 < C < \infty$ such that for any $h_0^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ and for any $h_0^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ there exist $f_1 \in \widetilde{\mathcal{N}}_{k_0}$, $\omega_1 \in \widetilde{\mathcal{B}}_{k_0}$, $h_1^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ and $h_1^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ with*

$$\begin{aligned}
& (\mu T_k - 1)^{-1} h_0^{\|} + (\mu T_k - 1)^{-1} h_0^{\perp} \\
= \; & f_1 + \omega_1 + (\mu T_k - 1)^{-1} h_1^{\|} + (\mu T_k - 1)^{-1} h_1^{\perp}
\end{aligned} \tag{67}$$

and

$$
\begin{aligned}
\| f_1(\mathbf{k},\mu,\cdot) \|_\infty &\le C \| h_0^{\|}(\mathbf{k},\mu,\cdot) \|_\infty \left(| \mu - \lambda_c - \alpha k^2 | + k^3 \right)^{-1} \\
\| \omega_1(\mathbf{k},\mu,\cdot) \|_\infty &\le C k^2 \| f_1(\mathbf{k},\mu,\cdot) \|_\infty + C \| h_0^{\perp}(\mathbf{k},\mu,\cdot) \|_\infty \\
\| h_1^{\|}(\mathbf{k},\mu,\cdot) \|_\infty &\le C k \| h_0^{\|}(\mathbf{k},\mu,\cdot) \|_\infty + C k^2 \| h_0^{\perp}(\mathbf{k},\mu,\cdot) \|_\infty \\
\| h_1^{\perp}(\mathbf{k},\mu,\cdot) \|_\infty &\le C \| h_1^{\|}(\mathbf{k},\mu,\cdot) \|_\infty + C k \| h_0^{\perp}(\mathbf{k},\mu,\cdot) \|_\infty \ .
\end{aligned}
$$

The proof of this Lemma is the heart of this section and will be done later. Iterating this Lemma p-times yields that

$$
\begin{aligned}
&(\mu T_k - 1)^{-1} h_0^{\|} + (\mu T_k - 1)^{-1} h_0^{\perp} \\
&\qquad = \sum_{j=1}^{p} f_j + \sum_{j=1}^{p} \widetilde{\omega}_j + (\mu T_k - 1)^{-1} h_p^{\|} + (\mu T_k - 1)^{-1} h_p^{\perp} \qquad (68)
\end{aligned}
$$

It follows iteratively that

$$
\begin{aligned}
\| h_j^{\|}(\mathbf{k},\mu,\cdot) \|_\infty &\le C(3Ck)^j && (69) \\
\| h_j^{\perp}(\mathbf{k},\mu,\cdot) \|_\infty &\le C(3Ck)^{j-1} && (70) \\
\| f_j(\mathbf{k},\mu,\cdot) \|_\infty &\le \left(| \mu - \lambda_c - \alpha k^2 | + k^3 \right)^{-1} C^{j+1} k^{j-1} && (71) \\
\| \omega_j(\mathbf{k},\mu,\cdot) \|_\infty &\le C k^2 \| f_j(\mathbf{k},\mu,\cdot) \|_\infty + C^{j+1} k^{j-1} \\
&\le \left(C^2 \left(| \mu - \lambda_c - \alpha k^2 | + k^3 \right)^{-1} C^j k^2 + C^{j+1} \right) k^{j-1} \ . && (72)
\end{aligned}
$$

Next we prove the convergence of the four summands on the right hand side of (68). Choose k_0 such that $Ck_0 < 1$. Note that in fact C does depend on k_0. But since $\mathcal{P}_{k_0} \subset \mathcal{P}_{k_1}$ for all $k_0 < k_1$, it follows that the C one gets for k_0 is smaller or equal to the C one gets for k_1, hence k_0 can in fact be chosen such that $Ck_0 < 1$. Then it follows with (69) and (70) that

$$
\begin{aligned}
\lim_{j\to\infty} \| h_j^{\perp} \|_\infty &= 0 \\
\lim_{j\to\infty} \| h_j^{\|} \|_\infty &= 0 \ .
\end{aligned}
$$

Since the operator $(\mu T_k - 1)^{-1}$ is for fixed $(\mathbf{k},\mu) \neq (0,\lambda_c)$ a bounded operator in $\mathcal{B}$ it follows that

$$
\lim_{p\to\infty} \| (\mu T_k - 1)^{-1} h_p^{\|} + (\mu T_k - 1)^{-1} h_p^{\perp} \| = 0 \ . \qquad (73)
$$

Since $\sum_{j=1}^{N} f_j$ is geometric it is a Cauchy series for all $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$. $\mathcal{N}$ is a finitely dimensional vector space hence it is complete. It follows that for any $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ there exists a $f(\mathbf{k}, \mu, \cdot)$ such that

$$\lim_{N\to\infty} \| \sum_{j=1}^{N} f_j(\mathbf{k}, \mu, \cdot) - f(\mathbf{k}, \mu, \cdot) \|_\infty = 0 \ .$$

Using the completeness of $\mathcal{B}$ we get similarly with (72) the convergence of $\sum_{j=1}^{N} \omega_j$, i.e. for any $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ there exists a $\omega(\mathbf{k}, \mu, \cdot) \in \widetilde{\mathcal{B}}_{k_0}$ such that

$$\lim_{N\to\infty} \| \sum_{j=1}^{N} \omega_j(\mathbf{k}, \mu, \cdot) - \omega(\mathbf{k}, \mu, \cdot) \|_\infty = 0 \ .$$

Using (71) we have that

$$\begin{aligned}
\| f(\mathbf{k}, \mu, \cdot) \|_\infty &= \| \sum_{j=1}^{\infty} f_j(\mathbf{k}, \mu, \cdot) \|_\infty \\
&\leq \sum_{j=1}^{\infty} \| f_j(\mathbf{k}, \mu, \cdot) \|_\infty \\
&\leq \sum_{j=1}^{\infty} \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1} C^{j+1} k^{j-1} \\
&= \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1} \sum_{j=0}^{\infty} C^{j+1} k^j \\
&= C \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1} \frac{1}{1 - Ck} \ , \qquad (74)
\end{aligned}$$

using (72)

$$\begin{aligned}
\| \omega(\mathbf{k}, \mu, \cdot) \|_\infty &= \| \sum_{j=1}^{\infty} \omega_j(\mathbf{k}, \mu, \cdot) \|_\infty && (75) \\
&\leq C_1 k^2 \sum_{j=1}^{\infty} \| f_j(\mathbf{k}, \mu, \cdot) \|_\infty + \sum_{j=1}^{\infty} C^{j+1} k^{j-1} && (76) \\
&= C k^2 \sum_{j=1}^{\infty} \| f_j(\mathbf{k}, \mu, \cdot) \|_\infty + \sum_{j=0}^{\infty} C^{j+1} k^j && (77) \\
&= C k^2 \sum_{j=1}^{\infty} \| f_j(\mathbf{k}, \mu, \cdot) \|_\infty + \frac{C}{1 - Ck} \ . && (78)
\end{aligned}$$

So taking the limit $p \to \infty$ on the right hand side of (68) and using (73) yields

$$(\mu T_k - 1)^{-1} h_0^{\|} + (\mu T_k - 1)^{-1} h_0^{\perp} = f + \omega \, .$$

We apply this to $h_0^{\|} = g^{\|}$ and $h_0^{\perp} = g^{\perp}$ observing (64) and (65). With (62) and (61)

$$\zeta^j = (\mu T_k - 1)^{-1} g = f + \omega \, . \tag{79}$$

(75), (79) and (74) yield Lemma 6.4 for $n = 0$.

□

Proof of Lemma 6.10

Let $h_0^{\|} \in \widetilde{\mathcal{M}}_{k_0}$, $h_0^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$. We denote

$$\| h_0^{\|} \|_\infty =: C^{\|} \tag{80}$$

$$\| h_0^{\perp} \|_\infty =: C^{\perp} \, . \tag{81}$$

To prove the Lemma we first control the term $(\mu T_k - 1)^{-1} h_0^{\|}$. Since the control of this term is involved we give the result in

Lemma 6.11 *There exist $\alpha, k_0, C_1, C_2 \in \mathbb{R}$, $C_3 > 0$ such that for any $h^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ there exists $f \in \widetilde{\mathcal{N}}_{k_0}$ and $h^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ so that*

$$(\mu T_k - 1)^{-1} h^{\|} = f + (\mu T_k - 1)^{-1} h^{\perp}$$

and

$$\| f(\mathbf{k}, \mu, \cdot) \|_\infty \leq C_1 \| h^{\|}(\mathbf{k}, \mu, \cdot) \|_\infty \left(| \mu - \lambda_c - \alpha k^2 | + C_3 k^3 \right)^{-1}$$

and

$$\| h^{\perp}(\mathbf{k}, \mu, \cdot) \|_\infty \leq C_2 (\mu - \lambda_c + k^2) \| f(\mathbf{k}, \mu, \cdot) \|_\infty$$

which we will prove later on.

Using Lemma 6.11 for $h^{\|} = h_0^{\|}$ we get that there exist a $f \in \widetilde{\mathcal{N}}_{k_0}$ and $h^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ with

$$(\mu T_k - 1)^{-1} h_0^{\|} = f + (\mu T_k - 1)^{-1} h^{\perp}$$

and with (80)

$$\begin{aligned} \| f(\mathbf{k}, \mu, \cdot) \|_\infty &\leq C_1 C^\| \left(| \mu - \lambda_c - \alpha k^2 | + C_3 k^3 \right)^{-1} & (82) \\ \| h^\perp(\mathbf{k}, \mu, \cdot) \|_\infty &\leq C_2 \| f(\mathbf{k}, \mu, \cdot) \|_\infty (\mu - \lambda_c + k^2) \; . & (83) \end{aligned}$$

Hence setting

$$\widetilde{h}^\perp = h^\perp + h_0^\perp$$

we have by (80) and (83)

$$\| \widetilde{h}^\perp(\mathbf{k}, \mu, \cdot) \|_\infty \leq C_2 \| f(\mathbf{k}, \mu, \cdot) \|_\infty (\mu - \lambda_c + k^2) + C^\perp \tag{84}$$

and we obtain

$$(\mu T_k - 1)^{-1} \left(h_0^\| + h_0^\perp \right) - f = (\mu T_k - 1)^{-1} \widetilde{h}^\perp \; , \tag{85}$$

(85) is almost of the form of (67) in Lemma 6.10. To obtain the desired result we need to estimate now $(\mu T_k - 1)^{-1}\widetilde{h}^\perp$. For that we consider first the operator $(\mu T_k - 1)^{-1}$ for $k = 0$.

Our proof is based on the insight that $\lambda_c T_0 - 1$ is invertible on $\mathcal{M}^\perp$ which is spelled out in Lemma 6.12 below. For this we note that $\lambda_c T_0$ is symmetric with respect to the scalar product $\langle f, Ag \rangle$, f, $g \in \mathcal{B}$ (see (130) below). This explains a posteriori that $\mathcal{M}$ and $\mathcal{M}^\perp$ are the relevant spaces and not $\mathcal{N}$ and $\mathcal{N}^\perp$ as one might think at first sight.

Furthermore $\mathcal{M}^\perp \cap \mathcal{N} = \{0\}$, which is immediate from (remember that $A > 0$)

$$\langle A\phi_{\lambda_c} \mid \phi_{\lambda_c} \rangle = \int A(\mathbf{x}) \mid \phi_{\lambda_c}(\mathbf{x}) \mid^2 d^3x > 0 \; . \tag{86}$$

Lemma 6.12 *(a) For any $\mu \in [\lambda_c, \overline{\mu}]$ we have that*

$$h^\perp \in \mathcal{M}^\perp \Leftrightarrow (\mu T_0 - 1) h^\perp \in \mathcal{M}^\perp \; .$$

(b) For any $\mu \in [\lambda_c, \overline{\mu}]$ the map $\mu T_0 - 1 : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is invertible.

(c) There exists a $C < \infty$ such that for all $h^\perp \in \mathcal{M}^\perp$ and all $\mu \in [\lambda_c, \overline{\mu}]$

$$\| (\mu T_0 - 1)^{-1} h^\perp \|_\infty \leq C \| h^\perp \|_\infty \; . \tag{87}$$

which will be proven below.
Setting now

$$\omega(\mathbf{k},\mu,\mathbf{x}) := (\mu T_0 - 1)^{-1}\widetilde{h}^{\perp} \ . \tag{88}$$

we get with (84) in (87) for $h^{\perp} = \widetilde{h}^{\perp}$

$$\| \omega(\mathbf{k},\mu,\cdot) \|_{\infty} \leq CC_2 \| f(\mathbf{k},\mu,\cdot) \|_{\infty} (\mu - \lambda_c + k^2) + CC^{\perp} \ . \tag{89}$$

Thus $\omega(\mathbf{k},\mu,\cdot) \in \mathcal{B}$.
Now we can estimate $(\mu T_k - 1)^{-1}\widetilde{h}^{\perp}$. Writing

$$(\mu T_k - 1)\omega(\mathbf{k},\mu,\cdot) \ = \ \widetilde{h}^{\perp} + \mu(T_k - T_0)\omega(\mathbf{k},\mu,\cdot) \ ,$$

we obtain

$$\begin{aligned}(\mu T_k - 1)^{-1}\widetilde{h}^{\perp} \ &= \ \omega(\mathbf{k},\mu,\cdot) - (\mu T_k - 1)^{-1}\mu(T_k - T_0)\omega(\mathbf{k},\mu,\cdot) \\ &=: \ \omega(\mathbf{k},\mu,\cdot) + (\mu T_k - 1)^{-1}h_1 \end{aligned} \tag{90}$$

with

$$h_1(\mathbf{k},\mu,\cdot) := -\mu(T_k - T_0)\omega(\mathbf{k},\mu,\cdot) \ . \tag{91}$$

We now consider the usual splitting (62)

$$h_1 \ =: \ h_1^{\|} + h_1^{\perp}$$

and estimate the $\| h_1^{\|}(\mathbf{k},\mu,\cdot) \|_{\infty}$ and $\| h_1^{\perp}(\mathbf{k},\mu,\cdot) \|_{\infty}$ separately. This can be done using

Lemma 6.13 *There exists a $C \in \mathbb{R}$ such that for all $\omega \in \mathcal{B}$*

$$\| (T_k - T_0)\omega \|_{\infty} < Ck \| \omega \|_{\infty} \ ,$$

and

$$| \langle A(T_k - T_0)\omega \mid \phi_{\lambda_c} \rangle | < Ck^2 \| \omega \|_{\infty}$$

for all $\phi_{\lambda_c} \in \mathcal{N}$ with $\| \phi_{\lambda_c} \|_{\infty} = 1$.

The proof will be given later.
We use the Lemma in (91) for $\omega = \omega(\mathbf{k}, \mu, \cdot)$ and get with (89) that

$$\begin{aligned}
\| h_1(\mathbf{k}, \mu, \cdot) \|_\infty &= \| \mu (T_k - T_0) \omega(\mathbf{k}, \mu, \cdot) \|_\infty \\
&< \mu C C_2 k \left(\| f(\mathbf{k}, \mu, \mathbf{x}) \|_\infty (\mu - \lambda_c + k^2) + C^\perp \right) \\
\| h_1^\parallel(\mathbf{k}, \mu, \cdot) \|_\infty &\leq \sup_{\phi_{\lambda_c} \in \mathcal{N}} | \langle A h_1(\mathbf{k}, \mu, \cdot) \mid \phi_{\lambda_c} \rangle | \\
&< \mu C C_2 k^2 \left(\| f(\mathbf{k}, \mu, \mathbf{x}) \|_\infty (\mu - \lambda_c + k^2) + C^\perp \right)
\end{aligned}$$

It follows with (82) that

$$\| h_1(\mathbf{k}, \mu, \cdot) \|_\infty < \mu C C_2 k \left(\frac{C_1 C^\parallel (\mu - \lambda_c + k^2)}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} + C^\perp \right) \tag{92}$$

$$\| h_1^\parallel(\mathbf{k}, \mu, \cdot) \|_\infty < \mu C C_2 k^2 \left(\frac{C_1 C^\parallel (\mu - \lambda_c + k^2)}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} + C^\perp \right) \tag{93}$$

Since for $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ ($\Rightarrow k \leq k_0 < 1$).

$$\begin{aligned}
\left| \frac{k(\mu - \lambda_c + k^2)}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} \right| &= \left| \frac{k(\mu - \lambda_c - \alpha k^2) + \alpha k^3 + k^3}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} \right| \\
&= \left| \frac{k(\mu - \lambda_c - \alpha k^2)}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} + \frac{k^3 (1 + \alpha)}{| \mu - \lambda_c - \alpha k^2 | + C_3 k^3} \right| \\
&\leq 1 + \frac{(1 + \alpha)}{C_3}
\end{aligned}$$

it follows with (92) and (93) that there exists a $C < \infty$ such that

$$\| h_1^\parallel(\mathbf{k}, \mu, \cdot) \|_\infty \leq C \left(C^\parallel k^2 + C^\perp k \right) \tag{94}$$

$$\| h_1^\perp(\mathbf{k}, \mu, \cdot) \|_\infty := \| h_1(\mathbf{k}, \mu, \cdot) - h_1^\parallel(\mathbf{k}, \mu, \cdot) \|_\infty \leq C \left(C^\parallel k + C^\perp \right) . \tag{95}$$

With (82) and (89) Lemma 6.10 follows.

□

We shall now give the proofs of Lemma 6.11, 6.12 and 6.13 . For ease of reference we recall each time what the Lemmata say.

Lemma 6.11

There exist $\alpha, k_0, C_1, C_2 \in \mathbb{R}$,$C_3 > 0$ *such that for any* $h^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ *there exists* $f \in \widetilde{\mathcal{N}}_{k_0}$ *and* $h^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ *so that*

$$(\mu T_k - 1)^{-1} h^{\|} = f + (\mu T_k - 1)^{-1} h^{\perp} \tag{96}$$

and

$$\| f(\mathbf{k}, \mu, \cdot) \|_\infty \le C_1 \| h^{\|}(\mathbf{k}, \mu, \cdot) \|_\infty \left(| \mu - \lambda_c - \alpha k^2 | + C_3 k^3\right)^{-1} \tag{97}$$

and

$$\| h^{\perp}(\mathbf{k}, \mu, \cdot) \|_\infty \quad \le \quad C_2(\mu - \lambda_c + k^2) \| f(\mathbf{k}, \mu, \cdot) \|_\infty \tag{98}$$

Proof

Let $h^{\|} \in \widetilde{\mathcal{M}}_{k_0}$. Let the degeneracy of the critical bound state be n.
(96) is equivalent to

$$(\mu T_k - 1) f(\mathbf{k}, \mu, \cdot) = h^{\|} + h^{\perp}(\mathbf{k}, \mu, \cdot) \ .$$

While the logic here is that $h^{\|}$ is given and f, $h^{\perp}$ are to be found, we turn the argument around. We start with controlling $(\mu T_k - 1) \widetilde{f}$ for arbitrary $\widetilde{f} \in \widetilde{\mathcal{N}}_{k_0}$, and show that there exists a $\widetilde{h}^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ and a $\widetilde{h}^{\perp} \in \widetilde{\mathcal{M}}_{k_0}^{\perp}$ such that

$$(\mu T_k - 1) \widetilde{f} = \widetilde{h}^{\|} + \widetilde{h}^{\perp} \ .$$

Since $\mu T_k - 1$ is a linear operator and since the projectors from $\mathcal{B}$ onto $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are linear, it follows that for any $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ there exists linear operators $B(\mathbf{k}, \mu)$ from $\mathcal{N} \to \mathcal{M}$ and $B^{\perp}(\mathbf{k}, \mu)$ from $\mathcal{N} \to \mathcal{M}^{\perp}$ such that for any $\widetilde{f} \in \widetilde{\mathcal{N}}_{k_0}$

$$\widetilde{h}^{\|}(\mathbf{k}, \mu, \cdot) = B(\mathbf{k}, \mu) \widetilde{f}(\mathbf{k}, \mu, \cdot) \tag{99}$$

$$\widetilde{h}^{\perp}(\mathbf{k}, \mu, \cdot) = B^{\perp}(\mathbf{k}, \mu) \widetilde{f}(\mathbf{k}, \mu, \cdot) \tag{100}$$

Note that $\mathcal{N}$ and $\mathcal{M}$ have finite dimension, so B is a mapping between finite dimensional vector spaces.

We now give some properties of $B(\mathbf{k}, \mu)$ and $B^{\perp}(\mathbf{k}, \mu)$. We will first show how they imply the Lemma and then prove them one after another.

(a)

$$B(\mathbf{k},\mu) \text{ is invertible for any } (\mathbf{k},\mu) \in \mathcal{P}_{k_0} \tag{101}$$

(b)

$$\| B^{-1}(\mathbf{k},\mu) \|_2^{op} < C_1 \left(| \mu - \lambda_c - \alpha k^2 | + C_3 k^3 \right)^{-1} \tag{102}$$

(c)

$$\| B^{\perp}(\mathbf{k},\mu) \|_{\infty}^{op} < \widetilde{C}_2(\mu - \lambda_c + k^2) \tag{103}$$

for appropriate $C_1 < \infty$, $C_2 < \infty$, $0 < C_3 < \infty$ uniform in $(\mathbf{k},\mu) \in \mathcal{P}_{k_0}$.

Assume that (a) holds, i.e. we can find for any $h^{\|} \in \widetilde{\mathcal{M}}_{k_0}$ a $f \in \widetilde{\mathcal{N}}_{k_0}$ such that the projection of $(\mu T_k - 1)f$ onto $\widetilde{\mathcal{M}}_{k_0}$ is equal to $h^{\|}$. Let $h^{\perp}$ be the projection of $(\mu T_k - 1)f$ onto $\widetilde{\mathcal{M}}_{k_0}^{\perp}$. It follows that (96) is satisfied.

Assume that furthermore (b) holds. Using the equivalence of all norms in a finitely dimensional vector space (i.e. replacing $\| \cdot \|_{\infty}$ by $\| \cdot \|$ in the n-dimensional spaces $\mathcal{M}$ and $\mathcal{N}$) it follows that $\| B^{-1}(\mathbf{k},\mu) \|_2^{op} \leq C \| B^{-1}(\mathbf{k},\mu) \|_{\infty}^{op}$. Since

$$\begin{aligned} \| \widetilde{f}(\mathbf{k},\mu,\cdot) \|_{\infty} &= \| B(\mathbf{k},\mu)^{-1} \widetilde{h}^{\|}(\mathbf{k},\mu,\cdot) \|_{\infty} \\ &\leq \| B(\mathbf{k},\mu)^{-1} \|_{\infty}^{op} \| \widetilde{h}^{\|}(\mathbf{k},\mu,\cdot) \|_{\infty} \end{aligned}$$

(97) follows.

Assume that furthermore (c) holds. Since $\widetilde{h}^{\perp}(\mathbf{k},\mu,\cdot) = B^{\perp}(\mathbf{k},\mu)\widetilde{f}(\mathbf{k},\mu,\cdot)$ (98) follows.

It is left to prove that (a)-(c) hold to verify Lemma 6.11.

(a) holds if $B(\mathbf{k},\mu)f = 0$ has no non trivial solution.

Furthermore for (b) if B were invertible

$$\| B^{-1}(\mathbf{k},\mu) \|_2^{op} := \sup_{\widetilde{h}^{\|} \in \mathcal{M} \setminus \{0\}} \frac{\| B^{-1} \widetilde{h}^{\|} \|}{\| \widetilde{h}^{\|} \|} = \sup_{\widetilde{f} \in \mathcal{N} \setminus \{0\}} \frac{\| \widetilde{f} \|}{\| B\widetilde{f} \|} = \left(\inf_{\widetilde{f} \in \mathcal{N}, \|\widetilde{f}\|=1} \| B\widetilde{f} \| \right)^{-1}$$

Hence (a) and (b) follow from

$$\inf_{\phi_{\lambda_c} \in \mathcal{N}, \|\phi_{\lambda_c}\|=1} \| B\phi_{\lambda_c} \| \geq C \left(| \mu - \lambda_c - \alpha k^2 | + C_3 k^3 \right) \tag{104}$$

with $C > 0$ uniform in $(\mathbf{k},\mu) \in \mathcal{P}_{k_0}$.

Using Schwartz inequality we have that

$$\begin{aligned}\| B\phi_{\lambda_c} \| &\geq \frac{1}{\| A\phi_{\lambda_c} \|} |\langle B\phi_{\lambda_c}, A\phi_{\lambda_c}\rangle| = \frac{1}{\| A\phi_{\lambda_c} \|} |\langle P_{\mathcal{M}}(\mu T_k - 1)\phi_{\lambda_c}, A\phi_{\lambda_c}\rangle| \\ &= \frac{1}{\| A\phi_{\lambda_c} \|} |A(\mu T_k - 1)\phi_{\lambda_c}, \phi_{\lambda_c}\rangle|\end{aligned}$$

where $P_{\mathcal{M}}$ is the projector on $\mathcal{M}$. Since A is bounded it follows, that

$$\sup_{\phi_{\lambda_c}\in\mathcal{N}, \|\phi_{\lambda_c}\|=1} \| A\phi_{\lambda_c} \| < \infty$$

hence (104) holds if there exists a $C > 0$ uniform in $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ with

$$\inf_{\phi_{\lambda_c}\in\mathcal{N}, \|\phi_{\lambda_c}\|=1} \{|\langle A(\mu T_k - 1)\phi_{\lambda_c}, \phi_{\lambda_c}\rangle|\} \geq C \mid \mu - \lambda_c - \alpha k^2 \mid + k^3 \ . \tag{105}$$

We shall show this now.

For this let $\phi_{\lambda_c} \in \mathcal{N}\backslash\{0\}$. We shall use Taylors formula to estimate $(\mu T_k - 1)\,\phi_{\lambda_c}$. In view of (46)

$$(\mu T_{\mathbf{k}} - 1)\phi_{\lambda_c} = \int \left(G_k^+(\mathbf{x}') - G_0^+(\mathbf{x}')\right) A(\mathbf{x} - \mathbf{x}') f(\mathbf{x} - \mathbf{x}') d^3x' \ ,$$

i.e. we develop G_k^+ around $k = 0$, so we need the following derivatives

$$\begin{aligned}\partial_k G_k^+ &= \partial_k \left(\frac{1}{4\pi} e^{ikx} \left(-x^{-1}(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - i x^{-2} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} \right) \right) \\ &= \frac{1}{4\pi} e^{ikx} \left(-i(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) + x^{-1} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} - x^{-1}(\frac{k}{E_k} + \sum_{j=1}^{3} \alpha_j \frac{x_j}{x}) \right) \ . \\ &= \frac{1}{4\pi} e^{ikx} \left(-i(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - x^{-1} \frac{k}{E_k} \right) \ , \qquad (106)\end{aligned}$$

$$\begin{aligned}\partial_k^2 G_k^+ &= \partial_k \frac{1}{4\pi} e^{ikx} \left(-i(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - x^{-1} \frac{k}{E_k} \right) \\ &= \frac{1}{4\pi} e^{ikx} \left(x(E_k + i \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - i \frac{k}{E_k} - i \frac{k}{E_k} - i \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} - x^{-1} \frac{m^2}{E_k^3} \right) \\ &= \frac{1}{4\pi} e^{ikx} \left(x(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - 2i \frac{k}{E_k} - i \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} - x^{-1} \frac{m^2}{E_k^3} \right) \quad (107)\end{aligned}$$

and

$$
\begin{aligned}
\partial_k^3 G_k^+ &= \partial_k \frac{1}{4\pi} e^{ikx} \left(x(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - 2i\frac{k}{E_k} - i\sum_{j=1}^{3} \alpha_j \frac{x_j}{x} - x^{-1}\frac{m^2}{E_k^3} \right) \\
&= \frac{1}{4\pi} e^{ikx} \big(ix^2 (E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) + 2x\frac{k}{E_k} - i\frac{m^2}{E_k^3} + \sum_{j=1}^{3} \alpha_j x_j \\
&\quad + x\frac{k}{E_k} + \sum_{j=1}^{3} \alpha_j x_j - 2i\frac{m^2}{E_k^3} + 3x^{-1}\frac{km^2}{E_k^5} \big) \\
&= \frac{1}{4\pi} e^{ikx} \big(ix^2 (E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) + 3x\frac{k}{E_k} \\
&\quad + 2\sum_{j=1}^{3} \alpha_j x_j - 3i\frac{m^2}{E_k^3} + 3x^{-1}\frac{km^2}{E_k^5} \big) \qquad (108)
\end{aligned}
$$

By Taylors formula there exists a $k_0 < k$ such that

$$
\begin{aligned}
\langle A\,(1-\mu T_k)\,\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle &= \langle A\,(1-\mu T_0)\,\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle + k\,(\partial_k \langle A\mu T_k \phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle\,|_{k=0}) \\
&\quad + \frac{1}{2}k^2 \left(\partial_k^2 \langle A\mu T_k \phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle\,|_{k=0}\right) \\
&\quad + \frac{1}{6}k^3 \left(\partial_k^3 \langle A\mu T_k \phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle\,|_{k=0}\right) \\
&\quad + \frac{1}{24}k^4 \left(\partial_k^4 \langle A\mu T_k \phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle\,|_{k=k_0}\right) \\
&=: S_0 + S_1 + S_2 + S_3 + S_4\ . \qquad (109)
\end{aligned}
$$

We estimate these terms separately. For the first term we have using (50) that

$$
S_0 = \langle A\,(1-\mu T_0)\,\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle = \frac{\lambda_c - \mu}{\lambda_c} \langle A\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle > 0 \qquad (110)
$$

for $\lambda_c > \mu$ since A is positive.
For S_1 we obtain with (106) that

$$
\mu \partial_k T_k \,|_{k=0}\, \phi_{\lambda_c} = -i \int \frac{1}{4\pi}\,(m + \beta m)\, A(\mathbf{x}-\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}-\mathbf{x}') d^3x'\ .
$$

Hence by (36)

$$S_1 = 0\ . \tag{111}$$

For S_3 we obtain by (108)

$$\begin{aligned}
S_3 &= \frac{1}{6}k^3\langle A\mu \int \partial_k^3 G_k^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')d^3x' \mid \phi_{\lambda_c}\rangle \\
&= \frac{1}{6}k^3\langle A\mu \frac{1}{4\pi}\int i(\mathbf{x}-\mathbf{x}')^2(m+\beta m)A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')d^3x' \mid \phi_{\lambda_c}\rangle \\
&\quad +\frac{1}{6}k^3\langle A\mu \frac{1}{4\pi}\int 2\sum_{j=1}^{3}\alpha_j(x_j-x_j')A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')d^3x' \mid \phi_{\lambda_c}\rangle \\
&\quad +\frac{1}{6}k^3\langle A\mu \frac{1}{4\pi}\int -3i\frac{m^2}{m^3}A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')d^3x' \mid \phi_{\lambda_c}\rangle \\
&= \frac{\mu m i}{24\pi}k^3\int\int A(\mathbf{x})(\mathbf{x}-\mathbf{x}')^2A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')(1+\beta)\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x \\
&\quad +\frac{\mu}{12\pi}k^3\int\int A(\mathbf{x})\sum_{j=1}^{3}A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')\alpha_j(x_j-x_j')\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x \\
&\quad -\frac{\mu i}{8\pi m}k^3\int\int A(\mathbf{x})A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x \\
&=: S_{3,1}+S_{3,2}+S_{3,3}\ .
\end{aligned} \tag{112}$$

For $S_{3,1}$ we can write

$$\begin{aligned}
S_{3,1} &= \frac{\mu m i}{24\pi}k^3\int\int A(\mathbf{x})(\mathbf{x}^2+\mathbf{x}'^2)A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')(1+\beta)\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x \\
&\quad -\frac{\mu m i}{12\pi}k^3\int\int A(\mathbf{x})\mathbf{x}\cdot\mathbf{x}'A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')(1+\beta)\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x\ .
\end{aligned} \tag{113}$$

Using symmetry in exchanging $\mathbf{x}$ with $\mathbf{x}'$ on the first term it becomes

$$\begin{aligned}
&\frac{\mu m i}{12\pi}k^3\int\int A(\mathbf{x})\mathbf{x}^2A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')(1+\beta)\phi_{\lambda_c}^\dagger(\mathbf{x})d^3x'd^3x \\
&= \frac{\mu m i}{12\pi}k^3\int A(\mathbf{x})\mathbf{x}^2\phi_{\lambda_c}(\mathbf{x})\int(1+\beta)A(\mathbf{x}')\phi_{\lambda_c}^\dagger(\mathbf{x}')d^3x'd^3x = 0
\end{aligned}$$

by (36). Thus

$$S_{3,1} = -\frac{\mu m i}{12\pi}k^3\int A(\mathbf{x})\phi_{\lambda_c}^\dagger(\mathbf{x})\mathbf{x}d^3x(1+\beta)\cdot\int \mathbf{x}'A(\mathbf{x}')\phi_{\lambda_c}(\mathbf{x}')d^3x'\ .$$

Setting

$$\xi := \sqrt{\frac{\mu m}{12\pi}} \int A(\mathbf{x})\phi_{\lambda_c}(\mathbf{x})\mathbf{x} d^3x \ . \tag{114}$$

we obtain

$$S_{3,1} \quad = \quad -ik^3\langle \xi(1-\beta)\xi\rangle \ , \tag{115}$$

Since (β) is self adjoint it follows that $\langle\xi(1-\beta)\xi\rangle \in \mathbb{R}$, since $\parallel \beta \parallel = 1$ it follows that $\langle\xi(1-\beta)\xi\rangle \geq 0$ hence there exists a $C_4 \in \mathbb{R}_0^+$ such that

$$S_{3,1} = -ik^3C_4 \ . \tag{116}$$

Due to symmetry in exchanging $\mathbf{x}$ with $\mathbf{x}'$ we have that

$$S_{3,2} = -S_{3,2} = 0 \ . \tag{117}$$

For $S_{3,3}$ we can write

$$S_{3,3} \quad = \quad -\frac{\mu i}{8\pi m}k^3 \left| \int A(\mathbf{x})\phi_{\lambda_c}(\mathbf{x})d^3x \right|^2 \tag{118}$$

it follows that there exists a $C_5 \geq 0$ with

$$S_{3,3} = -ik^3C_5 \ . \tag{119}$$

This (116) and (117) in (112) yield that there exists a $C_3 \geq 0$ such that

$$S_3 = -ik^3C_3 \ . \tag{120}$$

Since A was defined to satisfy either (15) or (16) it follows taking note of (114) and (115) as well as (118) that C_4 or $C_5 > 0$.

For S_4 (see (109)) we have that there exists a $C \in \mathbb{R}$ such that

$$\mid S_4 \mid \leq k^4 C \ . \tag{121}$$

(110), (111) and (120) in (109) yield that

$$\langle A\left(1-\mu T_k\right)\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle = \frac{\mu-\lambda_c}{\lambda_c}\langle A\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle + \frac{1}{2}\partial_k^2\langle A\mu T_k\phi_{\lambda_c} \mid \phi_{\lambda_c}\rangle \mid_{k=0} k^2 - ik^3C_3 + S_4 \ .$$

We split the equation into real and imaginary part (observing that A is positive)

$$\begin{aligned}\Re(\langle A\,(1-\mu T_k)\,\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle) &= \frac{\mu-\lambda_c}{\lambda_c}\langle A\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle \\ &\quad +\Re(\frac{1}{2}\partial_k^2\langle A\mu T_k\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle\mid_{k=0})k^2+\Re(S_4)\quad(122)\\ \Im(\langle A\,(1-\mu T_k)\,\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle) &= \Im(\frac{1}{2}\partial_k^2\langle A\mu T_k\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle\mid_{k=0}k^2)\\ &\quad -k^3C_5+\Im(S_4)\qquad(123)\end{aligned}$$

Since the first summand in (122) is positive and the second summand in (122)is of order k^2 it follows with (121) that there exists a $C>0$ and a $\alpha\in\mathbb{R}$ such that

$$\mid\Re(\langle A\,(1-\mu T_k)\,\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle)\mid\geq C\left|\frac{\mu-\lambda_c}{\lambda_c}-\alpha k^2\right|\ . \qquad(124)$$

For (123) observe that since $k^3<k^2$ for $k\to 0$ it follows that there exists a $C>0$ such that for $k<k_0$ with appropriate k_0

$$\mid\Im(\langle A\,(1-\mu T_k)\,\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle)\mid\geq Ck^3\ . \qquad(125)$$

Remark 6.14 *(125) seems to be not optimal since the right hand side of (123) seems to be of order of order k^2. But*

$$\Im(\frac{1}{2}\partial_k^2\langle A\mu T_k\phi_{\lambda_c}\mid\phi_{\lambda_c}\rangle\mid_{k=0}k^2)=0\ ,$$

the proof of which is not given since (125) suffices.

Using now the fact that the absolute value of a complex number $z=u+iv$

$$\mid z\mid=\frac{1}{2}\mid z\mid+\frac{1}{2}\mid z\mid\geq\frac{1}{2}\mid u\mid+\frac{1}{2}\mid v\mid$$

with (124) and (125) equation (105) follows.

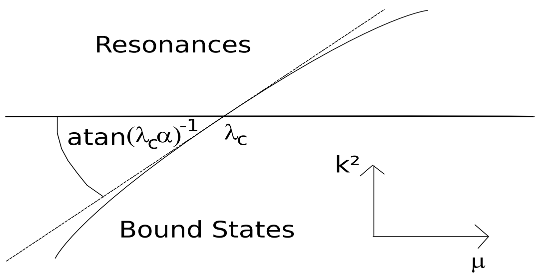

Figure 4: Resonances and eigenvalues. For $\frac{\mu-\lambda_c}{\lambda_c} - \alpha k^2 \approx 0$ we have eigenvalues for $\mu \leq \lambda_c$ and resonances in the continuous spectrum for $\mu > \lambda_c$.

Remark 6.15 *Note that α is in fact greater than zero: For $\mu < \lambda_c$ there exist imaginary k (in this case we in fact have no imaginary part c.f. (123): The imaginary k makes all imaginary parts which are of odd order in k real) - thus negative k^2 - such that $B(\mathbf{k}, \mu)\widetilde{f}(\mathbf{k}, \mu, \cdot) = 0$ (namely the respective eigenvalues). Hence we can find $\mu < \lambda_c$ and $k^2 < 0$ such that $\frac{\mu-\lambda_c}{\lambda_c} - \alpha k^2 \approx 0$, hence $\alpha \geq 0$. Since ϕ_μ dives properly into the continuous spectrum, hence $\frac{\mu-\lambda_c}{\lambda_c} - \alpha k^2 \approx 0$ for $\frac{\mu-\lambda_c}{\lambda_c}$ proportional to $E_k \approx \frac{k^2}{2m}$ it follows that $\alpha \neq 0$.*

Also in the case $\mu > \lambda_c$, $k \in \mathbb{R}$ it may happen that $\frac{\mu-\lambda_c}{\lambda_c} - \alpha k^2 = 0$. This case is called "resonance" in the physical literature. Around the resonance the norm of B is governed by the imaginary part above which is of order k^3. Varying k the real part above changes its sign when crossing the resonance, so does the divergent part of the generalized eigenfunctions.

We thus have proven (101) and (102) (recalling (105) yields (104); (104) yields (101) and (102)). It is left to prove (103).

Let $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$, $f \in \mathcal{N}$. Similar as above we have using Taylors formula that

$$(\mu T_k - 1)f \;=\; (\mu T_0 - 1)f + k\partial_k(\mu T_k)f \mid_{k=0} + \mathcal{O}(k^2) \parallel f(\mathbf{k}, \mu, \cdot) \parallel_\infty \;.$$

Since $f \in \widetilde{\mathcal{N}}$

$$(\lambda_c T_0 - 1)f = 0$$

and thus

$$(\mu T_0 - 1)f = (\frac{\mu}{\lambda_c} - 1)f = \frac{\mu - \lambda_c}{\lambda_c} f \;.$$

It follows that

$$(\mu T_k - 1)f = \frac{\mu - \lambda_c}{\lambda_c} f + k\partial_k(\mu T_k - 1)f \mid_{k=0} + \mathcal{O}(k^2) \parallel f \parallel_\infty \ .$$

and

$$\begin{aligned} \parallel (\mu T_k - 1)f \parallel_\infty &\leq \frac{\mu - \lambda_c}{\lambda_c} \parallel f \parallel_\infty + k \parallel [\partial_k(\mu T_k - 1)]_{k=0} f \mid_{k=0} \parallel_\infty \\ &\quad + \mathcal{O}(k^2) \parallel f \parallel_\infty \ . \end{aligned} \tag{126}$$

With (106) and (46) we have that by virtue of (36)

$$[\partial_k(\mu T_k - 1)]_{k=0} f = \frac{-im}{4\pi} \int (1+\beta) A(\mathbf{x}') f(\mathbf{x}') d^3x' \ .$$

Using (36) it follows that

$$[\partial_k(\mu T_k - 1)]_{k=0} f = 0 \ .$$

Thus we can estimate (126) by

$$\parallel (\mu T_k - 1)f \parallel_\infty < C \frac{\mu - \lambda_c}{\lambda_c} \parallel f \parallel_\infty + Ck^2 \parallel f \parallel_\infty \tag{127}$$

with appropriate $C < \infty$ uniform in $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$.
Hence in view of (100)

$$\begin{aligned} \parallel B^\perp(\mathbf{k}, \mu) \parallel_\infty^{op} &= \sup_{f \in \mathcal{N}, \|f\|_\infty = 1} \parallel B^\perp(\mathbf{k}, \mu) f \parallel_\infty \\ &= \sup_{f \in \mathcal{N}, \|f\|_\infty = 1} \parallel (1 - P^\parallel)(\mu T_k - 1) f \parallel_\infty \\ &\leq \sup_{f \in \mathcal{N}, \|f\|_\infty = 1} \parallel (\mu T_k - 1) f \parallel_\infty \\ &\quad + \sup_{f \in \mathcal{N}, \|f\|_\infty = 1} \parallel P^\parallel (\mu T_k - 1) f \parallel_\infty \end{aligned} \tag{128}$$

Using the equivalence of all norms in the finite dimensional vector space $\mathcal{M}$ it follows that there exists a $C < \infty$ uniform in $(\mathbf{k}, \mu) \in \mathcal{P}_{k_0}$ such that

$$
\begin{aligned}
\sup_{f\in\mathcal{N},\|f\|_\infty=1} \| P^{\|}(\mu T_k - 1)f \|_\infty &\leq C \sup_{f\in\mathcal{N},\|f\|_\infty=1} \| P^{\|}(\mu T_k - 1)f \| \\
&\leq C \sup_{f\in\mathcal{N},\|f\|_\infty=1} \| (\mu T_k - 1)f \| \\
&\leq C\frac{\mu-\lambda_c}{\lambda_c} \| f \|_\infty + C_3 k^2 \| f \|_\infty
\end{aligned}
$$

by (127).
This and (127) in (128) yield (103).

□

We turn now to the proof of Lemma 6.12, and we recall the Lemma for convenience.

Lemma 6.12

(a) For any $\mu \in [\lambda_c, \overline{\mu}]$ we have that

$$h^\perp \in \mathcal{M}^\perp \Leftrightarrow (\mu T_0 - 1)h^\perp \in \mathcal{M}^\perp \ .$$

(b) For any $\mu \in [\lambda_c, \overline{\mu}]$ the map $\mu T_0 - 1 : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is invertible.

(c) There exists a $C < \infty$ such that for all $h^\perp \in \mathcal{M}^\perp$ and all $\mu \in [\lambda_c, \overline{\mu}]$

$$\| (\mu T_0 - 1)^{-1} h^\perp \|_\infty \leq C \| h^\perp \|_\infty \ .$$

Proof of part a) of Lemma 6.12

Let $\mu \in [\lambda_c, \overline{\mu}]$.
We show first that for $h \in \mathcal{B}$ and $g \in \mathcal{B} \cap L_2$ with $T_0 g \in L_2$

$$\langle Ah, T_0 g\rangle = \langle AT_0 h, g\rangle \qquad (129)$$

by computing

$$\begin{aligned}
\langle Ah \mid T_0 g\rangle &= \int A(\mathbf{x})h^*(\mathbf{x})T_0 g(\mathbf{x})d^3x \\
&= \int A(\mathbf{x})h^*(\mathbf{x})\int G_0^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')g(\mathbf{x}')d^3x'd^3x \\
&= \int\int A(\mathbf{x})h^*(\mathbf{x})G_0^+(\mathbf{x}-\mathbf{x}')d^3xA(\mathbf{x}')g(\mathbf{x}')d^3x' \\
&= \langle AT_0 h \mid g\rangle \ . \qquad (130)
\end{aligned}$$

We may apply this to $h \in \mathcal{B}$ and $g = \phi \in \mathcal{N}$ to obtain

$$\begin{aligned}
\langle Ah, \phi\rangle &= \langle Ah, \lambda_c T_0 \phi\rangle \\
&= \lambda_c \langle AT_0 h, \phi\rangle
\end{aligned}$$

and thus

$$\frac{\mu}{\lambda_c}\langle Ah, \phi\rangle = \mu\langle AT_0 h, \phi\rangle$$

and hence

$$\langle A\left(1-\mu T_0\right)h \mid \phi\rangle = \langle A\frac{\lambda_c-\mu}{\lambda_c}h \mid \phi\rangle = \frac{\lambda_c-\mu}{\lambda_c}\langle Ah, \phi\rangle \ . \qquad (131)$$

This equation directly implies part a) of Lemma 6.12: If $h \in \mathcal{M}^\perp$ (which means that $\langle Ah, \phi_{\lambda_c}\rangle = 0$) it follows that $(1-\mu T_0)h \in \mathcal{M}^\perp$ (which means $\langle A\left(1-\mu T_0\right)h \mid \phi\rangle = 0$) and vice versus.

Proof of part b) of Lemma 6.12 for $\mu \neq \lambda_c$

First observe that there exists no bound state or resonance for $\mu \in]\lambda_c, \overline{\mu}]$ hence following the proof of Lemma 3.4. in [1] $(\mu T_0 - 1)^{-1} f$ exists for all $f \in \mathcal{B}$, so $\mu T_0 - 1$ is invertible on $\mathcal{M}^\perp \subset \mathcal{B}$.

Proof of part c) of Lemma 6.12 for $\mu \neq \lambda_c$

Let $\mu \in]\lambda_c, \overline{\mu}]$, $f \in \mathcal{M}^\perp$. Set

$$h := \left(1-\mu T_0\right)^{-1} f \ . \qquad (132)$$

Then by Lemma 6.12 (a)

$$h \in \mathcal{M}^{\perp} \; . \tag{133}$$

We now show that there exists a $C < \infty$ such that for all $h \in \mathcal{M}^{\perp}$ and $\mu \in]\lambda_c, \overline{\mu}]$

$$\| h \|_{\infty} \leq C \| (1 - \mu T_0) \, h \|_{\infty} \tag{134}$$

from which (87) follows.

We will prove (134) by contradiction. Hence assume that for every $C > 0$ there exist a μ_C with $\lambda_c < \mu_C \leq \overline{\mu}$ and a function $h_C \in \mathcal{M}^{\perp}$ such that

$$\| h_C \|_{\infty} > C \| (1 - \mu_C T_0) \, h_C \|_{\infty} \; . \tag{135}$$

Since T_0 is a linear operator we can restrict ourselves to functions h_C with $\| h_C \|_{\infty} = 1$. Hence (135) becomes

$$\| (1 - \mu_C T_0) \, h_C \|_{\infty} < \frac{1}{C} \; . \tag{136}$$

Consider a sequence $C_n \to \infty$. Hence there exists a series of elements $(\mu_n, h_n)_{n \in \mathbb{N}}$ in $]\lambda_c, \overline{\mu}] \times \mathcal{M}^{\perp}$ such that

$$\lim_{n \to \infty} \| (1 - \mu_n T_0) \, h_n \|_{\infty} = 0 \; ,$$

i.e. in sup norm

$$\lim_{n \to \infty} (1 - \mu_n T_0) \, h_n = 0 \; , \tag{137}$$

But the sequence $\mu_n T_0 h_n$ is Arzela-Ascoli compact, since

$$\mathcal{A} := \{ T_0 g \in \mathcal{B}, \| g \|_{\infty} = 1 \} \tag{138}$$

is compact in the Arzela-Ascoli sense, i.e. for any $\delta > 0$ there exists a $\zeta > 0$ such that

$$| f(\mathbf{x}) - f(\mathbf{y}) | < \delta \tag{139}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ with $\| \mathbf{x} - \mathbf{y} \| < \zeta$ and all $f \in \mathcal{A}$.

To prove this let $\delta > 0$, $f \in \mathcal{A}$ and let g be such that $f = T_0 g$,$\| g \|_{\infty} = 1$.

Then

$$
\begin{aligned}
\mid f(\mathbf{x}) - f(\mathbf{y}) \mid &= \mid T_0 g(\mathbf{x}) - T_0 g(\mathbf{y}) \mid \\
&= \Big| \int G_0^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')g(\mathbf{x}')d^3x' \\
&\quad - \int G_0^+(\mathbf{y}-\mathbf{y}')A(\mathbf{y}')g(\mathbf{y}')d^3x' \Big| \\
&= \Big| \int \left(G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \right) A(\mathbf{x}')g(\mathbf{x}')d^3x' \Big| \\
&\leq \parallel A \parallel_\infty \parallel g \parallel_\infty \int_{\mathcal{S}_A} \mid G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \mid d^3x' \\
&= \parallel A \parallel_\infty \int_{\mathcal{S}_A} \mid G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \mid d^3x' \qquad (140)
\end{aligned}
$$

Let $r > 0$ and $K_r(\mathbf{x})$ the open ball around $\mathbf{x}$ with radius r. Since $\mid G_0^+(\mathbf{x}-\mathbf{x}') \mid$ (see (41)) is integrable over any compact set we can choose r so small that for all $\mathbf{x}$ and all $\mathbf{y}$

$$
\int_{K_r(\mathbf{x})} \mid G_0^+(\mathbf{x}-\mathbf{x}') \mid d^3x' < \frac{\delta}{3 \parallel A \parallel_\infty} \qquad (141)
$$

$$
\int_{K_r(\mathbf{x})} \mid G_0^+(\mathbf{y}-\mathbf{x}') \mid d^3x' < \frac{\delta}{3 \parallel A \parallel_\infty} . \qquad (142)
$$

Let

$$
h(\mathbf{x}) := \mid G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \mid
$$

For $\mathbf{y} \in K_{\frac{r}{2}}(\mathbf{x})$ the function is continuous on the compact set $\mathcal{S}_A \backslash K_r(\mathbf{x})$ and thus uniformly continuous on $\mathcal{S}_A \backslash K_r(\mathbf{x})$(recall that $G_0^+(\mathbf{x}')$ is continuous on $\mathbb{R}^3 \backslash \{0\}$), i.e. there exists a $\zeta_1 > 0$ such that

$$
\mid G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \mid < \frac{\delta}{3 \parallel A \parallel_\infty \int_{\mathcal{S}_A \backslash K_r(\mathbf{x})} d^3x}
$$

for all $\mathbf{x}' \in \mathcal{S}_A \backslash K_r(\mathbf{x}) \mid$ and for all $\parallel \mathbf{x} - \mathbf{y} \parallel < \zeta_1$, hence

$$
\parallel A \parallel_\infty \int_{\mathcal{S}_A \backslash K_r(\mathbf{x})} \mid G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') \mid d^3x' \leq \frac{\delta}{3}
$$

for $\parallel \mathbf{x} - \mathbf{y} \parallel < \zeta_1$.
With (141) and (142) we obtain for (140)

$$| f(\mathbf{x}) - f(\mathbf{y}) | \leq \| A \|_\infty \int_{\mathcal{S}_A} | G_0^+(\mathbf{x}-\mathbf{x}') - G_0^+(\mathbf{y}-\mathbf{x}') | \, d^3x' \quad \leq \quad \delta$$

for $\| \mathbf{x} - \mathbf{y} \| < \zeta := \min\{\zeta_1, \frac{r}{2}\}$.

It follows directly that $\widetilde{\mathcal{A}} := [\lambda_c, \overline{\mu}] \times \cup_{\mu \in [\lambda_c, \overline{\mu}]} \mu \mathcal{A}$ is compact.

Thus there exists a convergent subsequence $(\mu_{n(k)}, \mu_{n(k)} T_0 h_{n(k)})_{k \in \mathbb{N}}$ of $(\mu_n, \mu_n T_0 h_n)_{n \in \mathbb{N}}$ with $\lim_{k\to\infty} T_0 h_{n(k)} = \widetilde{h}^\perp \in \mathcal{A}$ and $\lim_{k\to\infty} \mu_{n(k)} = \mu \in [\lambda_c, \overline{\mu}]$.

By virtue of (137) $h_{n(k)}$ converges, hence there exists a $h \in \mathcal{B}$ with $\lim_{k\to\infty} h_{n(k)} = h$. Since T_0 is continuous, $T_0 h = \widetilde{h}^\perp$.

Hence by (137) $(1-\mu T_0)h = 0$. Since $(1-\mu T_0)h = 0$ has nontrivial solutions only for $\mu = \lambda_c$ it follows that $\mu = \lambda_c$ and $h \in \mathcal{N}$. Since A is positive we have that

$$\langle h, Ah \rangle > 0 \ . \tag{143}$$

On the other hand since $h_n \in \mathcal{M}^\perp$

$$\langle h, Ah_n \rangle = 0 \tag{144}$$

for all $n \in \mathbb{N}$. (143) with (144) is contradiction to the continuity of the scalar product. So (134) holds and part c) of the Lemma follows for $\mu \neq \lambda_c$.

Proof of part (b) of Lemma 6.12 for $\mu = \lambda_c$

Let $f \in \mathcal{M}^\perp$. We develop $(1-\mu T_0)^{-1}$ into a Neumann series around $1 + \frac{1}{2C}$, which we show to be convergent on $M^\perp$ if C is chosen such that (87) is satisfied for $\mu \neq \lambda_c$ and $\frac{\lambda_c}{1+\frac{1}{2C}} \leq \overline{\mu}$ (note, that $\lim_{C\to\infty} \frac{\lambda_c}{1+\frac{1}{2C}} = \lambda_c < \overline{\mu}$).

Lemma 6.16 *Let $f \in \mathcal{M}^\perp$ and let*

$$f_n := (1 + \frac{1}{2C} - \lambda_c T_0)^{-1} \sum_{j=0}^{n} \left(\frac{1}{2C} (1 + \frac{1}{2C} - \lambda_c T_0)^{-1} \right)^j f \ . \tag{145}$$

Then there exists a $\overline{f} \in \mathcal{B}$ such that

$$\lim_{n\to\infty} \| f_n - \overline{f} \|_\infty = 0 \ .$$

Proof

Let $f \in \mathcal{M}^\perp$. We have that

$$1 + \frac{1}{2C} - \lambda_c T_0 \;=\; (1 + \frac{1}{2C}) \left(1 - \frac{\lambda_c}{1 + \frac{1}{2C}} T_0 \right) = (1 + \frac{1}{2C}) \, (1 - \mu_0 T_0) \; .$$

where $\mu_0 := \frac{\lambda_c}{1+\frac{1}{2C}}$. Note that $\mu_0 > \lambda_c$ and by our choice of C $\mu_0 \le \overline{\mu}$.
Hence part (c) of the Lemma for $\mu = \mu_0 \in]\lambda_c, \overline{\mu}]$ yields that on $\mathcal{M}^\perp$

$$\sup_{f \in \mathcal{M}^\perp} \frac{\| \, (1 + \frac{1}{2C} - \lambda_c T_0)^{-1} f \, \|_\infty}{\| \, f \, \|_\infty} \le C (1 + \frac{1}{2C})^{-1} \; .$$

It follows that

$$\sup_{f \in \mathcal{M}^\perp} \frac{\| \, (1 + \frac{1}{2C} - \lambda_c T_0)^{-1} f \, \|_\infty}{\| \, f \, \|_\infty} \le \frac{1}{2} (1 + \frac{1}{2C})^{-1} < \frac{1}{2} \; .$$

Using part (a) of the Lemma it follows, that for $f \in \mathcal{M}^\perp$ all the summands in (145) are in $\mathcal{M}^\perp$. Hence

$$\sup_{f \in \mathcal{M}^\perp} \frac{\| \, \left((1 + \frac{1}{2C} - \lambda_c T_0)^{-1} \right)^j f \, \|_\infty}{\| \, f \, \|_\infty} < \frac{1}{2^j} \; .$$

Hence the series on the right hand side of (145) is majorized by a geometrical series and thus it converges and since $\mathcal{B}$ is a banach space, $\overline{f} \in \mathcal{B}$ with

$$\overline{f}(\mathbf{x}) := \left((1 + \frac{1}{2C} - \lambda_c T_0)^{-1} \sum_{j=0}^{\infty} \left(\frac{1}{2C} (1 + \frac{1}{2C} - \lambda_c T_0)^{-1} \right)^j f \right) (\mathbf{x})$$

exists.

$\square$

Furthermore we have for $\overline{f}$ that

$$(1 - \lambda_c T_0) \overline{f} \;=\; f + \frac{1}{2C} \overline{f} - \frac{1}{2C} \overline{f} = f \; .$$

We have proven that for any $f \in \mathcal{M}^\perp$ there exists a $\overline{f} \in \mathcal{B}$ such that $(1 - \lambda_c T_0)\overline{f} = f$, hence $(1 - \lambda_c T_0)$ is invertible on $\mathcal{M}^\perp$ which is part (b) of Lemma 6.12 for $\mu = \lambda_c$.

Proof of part (c) of Lemma 6.12 for $\mu = \lambda_c$

Part c) of Lemma 6.12 for $\mu = \lambda_c$ follows direct from part (b) of the Lemma and part (c) of the Lemma for $\mu \neq \lambda_c$ using the continuity of the operator T_0 (see (61)).

□

Proof of Lemma 6.13

We recall the Lemma for convenience

Lemma 6.13

There exists a $C \in \mathbb{R}$ such that for all $h \in \mathcal{B}$

$$\| (T_k - T_0)h \|_\infty < Ck \| h \|_\infty \ , \tag{146}$$

and

$$| \langle A(T_k - T_0)h \mid \phi_{\lambda_c} \rangle | < Ck^2 \| h \|_\infty \tag{147}$$

for all $\phi_{\lambda_c} \in \mathcal{N}$ with $\| \phi_{\lambda_c} \|_\infty = 1$.

Proof

Let $h \in \mathcal{B}$. We have using (46)

$$(T_k - T_0)h \ = \int \left(G_k^+(\mathbf{x}') - G_0^+(\mathbf{x}') \right) A(\mathbf{x} - \mathbf{x}')h(\mathbf{x} - \mathbf{x}')d^3x' \ .$$

It follows that

$$\begin{aligned} \| (T_k - T_0)h \|_\infty \ &\leq \ \| h \|_\infty \int | G_k^+(\mathbf{x}') - G_0^+(\mathbf{x}') | A(\mathbf{x} - \mathbf{x}')d^3x' \\ &\leq \ \| h \|_\infty \| A \|_\infty \int_{\mathbf{x}-\mathbf{x}' \in \mathcal{S}_A} | G_k^+(\mathbf{x}') - G_0^+(\mathbf{x}') | \, d^3x' \ . \end{aligned}$$

Using the definition of G_k^+ (see (41)) we have that

$$
\begin{aligned}
&\parallel G_k^+(\mathbf{x}) - G_0^+(\mathbf{x}) \parallel_\infty \\
&=\parallel \frac{1}{4\pi} e^{ikx} \left(-x^{-1}(E_k + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x} + \beta m) - i x^{-2} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} \right) \\
&\quad - \frac{1}{4\pi} \left(-x^{-1}(m + \beta m) - i x^{-2} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} \right) \parallel_\infty \\
&\leq \parallel \frac{1}{4\pi} e^{ikx} \left(-x^{-1}(E_k - m + \sum_{j=1}^{3} \alpha_j k \frac{x_j}{x}) \right) \parallel_\infty \\
&\quad + \parallel \frac{1}{4\pi}(e^{ikx} - 1)) \left(-x^{-1}(m + \beta m) - i x^{-2} \sum_{j=1}^{3} \alpha_j \frac{x_j}{x} \right) \parallel_\infty \ .
\end{aligned}
$$

The first summand is of order k. Since $e^{ikx} - 1$ is of order kx, the second summand is of order k and (146) follows.

For the left hand side of (147) we use Taylors formula, i.e. that there exists a k_1 so that

$$
\begin{aligned}
&\langle A(T_k - T_0)h, \phi_{\lambda_c} \rangle \\
&\qquad = k \partial_k \langle A(\mathbf{x})(T_k h)(\mathbf{x}), \phi_{\lambda_c}(\mathbf{x}) \rangle \mid_{k=0} + \frac{1}{2} k^2 \partial_k^2 \langle A(T_k)h, \phi_{\lambda_c} \rangle \mid_{k=k_1} \\
&\qquad = k \langle A(\mathbf{x})(\partial_k T_k \mid_{k=0} h)(\mathbf{x}), \phi_{\lambda_c}(\mathbf{x}) \rangle + \frac{1}{2} k^2 \langle A(\partial_k^2 T_k) \mid_{k=k_1} h \mid \phi_{\lambda_c} \rangle \\
&\qquad =: S_1 + S_2 \qquad (148)
\end{aligned}
$$

For S_1 we have using (46) and (106)

$$
\begin{aligned}
S_1 &= k \int (A(\mathbf{x})(\partial_k T_k \mid_{k=0} h)(\mathbf{x}))^* \phi_{\lambda_c}(\mathbf{x}) d^3x \\
&= k \int \int A(\mathbf{x}) \left(\partial_k G_k(\mathbf{x} - \mathbf{x}') \mid_{k=0} A(\mathbf{x}') h(\mathbf{x}') \right)^* \phi_{\lambda_c}(\mathbf{x}) d^3x' d^3x \\
&= k \int \int A(\mathbf{x}) \frac{1}{4\pi} \left(i(m + \beta m) \right) A(\mathbf{x}') h^*(\mathbf{x}') \phi_{\lambda_c}(\mathbf{x}) d^3x' d^3x \\
&= k \int A(\mathbf{x}) \left(i(m + \beta m) \right) \phi_{\lambda_c}(\mathbf{x}) d^3x \frac{1}{4\pi} A(\mathbf{x}') h^*(\mathbf{x}') d^3x' = 0 \qquad (149)
\end{aligned}
$$

by virtue of (36) (Recall that A is scalar).

For S_2 we have using (46) and (106) that

$$\begin{aligned}
(\partial_k^2 T_k)h \mid_{k=k_1} &= \int \frac{1}{4\pi} e^{ik_1|\mathbf{x}-\mathbf{x}'|} \big(\mid \mathbf{x}-\mathbf{x}' \mid (E_{k_1} + \sum_{j=1}^{3} \alpha_j k_1 \frac{x_j - x_j'}{x} + \beta m) \\
& -2i\frac{k_1}{E_{k_1}} - i\sum_{j=1}^{3} \alpha_j \frac{x_j - x_j'}{\mid \mathbf{x}-\mathbf{x}' \mid} - \mid \mathbf{x}-\mathbf{x}' \mid^{-1} \frac{m^2}{E_{k_1}^3} \big) A(\mathbf{x}')h(\mathbf{x}')d^3x' \\
&\leq \frac{1}{4\pi} \int \mid \mathbf{x}-\mathbf{x}' \mid \ \big| (E_{k_1} + \sum_{j=1}^{3} \alpha_j k_1 \frac{x_j - x_j'}{x} + \beta m) - 2i\frac{k_1}{E_{k_1}} \\
& -i\sum_{j=1}^{3} \alpha_j \frac{x_j - x_j'}{\mid \mathbf{x}-\mathbf{x}' \mid} - \mid \mathbf{x}-\mathbf{x}' \mid^{-1} \frac{m^2}{E_{k_1}^3}) \big| \ A(\mathbf{x}') \parallel h \parallel_\infty d^3x' \ .
\end{aligned}$$

Since $A(\mathbf{x}')$ has compact support $\frac{(\partial_k^2 T_k)h|_{k=k_1}}{\|h\|_\infty}$ is bounded in $\mathbf{x} \in \mathbb{R}^3$. Going back to (148) we see - again using that $A(\mathbf{x}')$ has compact support - that there exists a $C < \infty$ such that

$$S_2 \leq Ck^2 \parallel h \parallel_\infty$$

for all in $h \in \mathcal{B}$ and $\mathbf{k} \in \mathbb{R}^3$.
With (149) (147) follows and Lemma 6.13 is proven.

□

Lemma 6.4 for $n > 0$

We exemplarily prove (e) for $n = 1$.
Heuristically deriving (53) with respect to k will yield $\partial_k \zeta^j(\mathbf{k}, \mu, \cdot)$. We denote the function we get by this formal method by $\dot{\zeta}^j(\mathbf{k}, \mu, \cdot)$.

$$\dot{\zeta}^j(\mathbf{k}, \mu, \mathbf{x}) = \mu \partial_k g(\mathbf{k}, \mu, \mathbf{x}) - \mu(\partial_k T_k)\zeta^j(\mathbf{k}, \mu, \mathbf{x}) - \mu T_k \dot{\zeta}^j(\mathbf{k}, \mu, \mathbf{x}) \ .$$

Using the definition of T_k (46 we get

$$\begin{aligned}
\dot{\zeta}^j(\mathbf{k}, \mu, \mathbf{x}) &= \mu \partial_k g(\mathbf{k}, \mu, \mathbf{x}) - \mu \int A(\mathbf{x}') \partial_k G(\mathbf{k}, \mu, \cdot)^+ (\mathbf{x}-\mathbf{x}') \zeta^j(\mathbf{k}, \mu, \mathbf{x}') d^3x' \\
& -\mu T_k \dot{\zeta}^j(\mathbf{k}, \mu, \mathbf{x}) \ . \qquad (150)
\end{aligned}$$

In [1] it is shown that (150) has a unique solution and that in fact $\dot{\zeta}^j(\mathbf{k},\mu,\cdot) = \partial_k\zeta^j(\mathbf{k},\mu,\cdot)$.

In the present paper we want to go further and also establish the estimates needed in (44) and (45). Set

$$\begin{aligned}\widetilde{g}(\mathbf{k},\mu,\mathbf{x}) \;&:=\; \mu\partial_k g(\mathbf{k},\mu,\mathbf{x}) \\ &\quad -\mu\int A(\mathbf{x}')\zeta^j(\mathbf{k},\mu,\mathbf{x}')\partial_k G(\mathbf{k},\mu,\cdot)^+(\mathbf{x}-\mathbf{x}')d^3x' \;, \qquad (151)\end{aligned}$$

hence

$$\dot{\zeta}^j(\mathbf{k},\mu,\mathbf{x}) = \widetilde{g}(\mathbf{k},\mu,\mathbf{x}) - \mu T_k\dot{\zeta}^j(\mathbf{k},\mu,\mathbf{x}) \;.$$

Note that $\widetilde{g}(\mathbf{k},\mu,\mathbf{x})$ is in general not in $\mathcal{B}$. Hence we proceed as above (see below (48)) and define

$$\overline{\zeta}^j(\mathbf{k},\mu,\mathbf{x}) \;:=\; \dot{\zeta}^j(\mathbf{k},\mu,\mathbf{x}) - \widetilde{g}(\mathbf{k},\mu,\mathbf{x})(\mathbf{x}) \;.$$

$\overline{\zeta}^j(\mathbf{k},\mu,\cdot)$ satisfies:

$$\overline{\zeta}^j(\mathbf{k},\mu,\mathbf{x}) = \mu\overline{g}(\mathbf{k},\mu,\cdot) - \mu T_k\overline{\zeta}^j(\mathbf{k},\mu,\cdot) \qquad (152)$$

with

$$\overline{g}(\mathbf{k},\mu,\cdot) := -T_k\widetilde{g}(\mathbf{k},\mu,\mathbf{x}) \;. \qquad (153)$$

Since T_k maps C^∞ into $\mathcal{B}$ it follows that $\overline{g} \in \mathcal{B}$.

Multiplying both sides with of (152) $\parallel \overline{g}(\mathbf{k},\mu,\cdot) \parallel_\infty^{-1}$ it is formally equivalent to (53). Hence showing that $\overline{g} \in \widetilde{\mathcal{B}}$ and that

$$\parallel \overline{g}(\mathbf{k},\mu,\cdot) \parallel_\infty < C\left(1 + k\left(\mid \mu - \lambda_c - \alpha k^2 \mid + k^3\right)^{-1}\right) \;. \qquad (154)$$

for some appropriate constant $C < \infty$ we get (44) and (45) for $n = 1$.

We need some properties of $\overline{g}$, for that we control $\widetilde{g}$. We define

$$\widetilde{g}_1(\mathbf{k},\mu,\mathbf{x}) \;:=\; \mu\partial_k g(\mathbf{k},\mu,\mathbf{x}) \qquad (155)$$

$$\widetilde{g}_2(\mathbf{k},\mu,\mathbf{x}) \;:=\; \mu\int A(\mathbf{x}-\mathbf{x}')\zeta^j(\mathbf{k},\mu,\mathbf{x}-\mathbf{x}')\partial_k G(\mathbf{k},\mu,\cdot)^+(\mathbf{x}')d^3x' \qquad (156)$$

hence

$$\widetilde{g} = \widetilde{g}_1 + \widetilde{g}_2 \ .$$

Using (52) we have that

$$\begin{aligned}
\widetilde{g}_1(\mathbf{k},\mu,\mathbf{x}) &= -\mu\partial_k T_{\mathbf{k}}\phi^j(\mathbf{k},0,\mathbf{x}) \\
&= -\mu\int A(\mathbf{x}-\mathbf{x}')\phi_0^{j,\mathbf{k}}(\mathbf{x}-\mathbf{x}')\partial_k G_k^+(\mathbf{x}')d^3x' \\
&\quad -\mu\int G_k^+(\mathbf{x}')A(\mathbf{x}-\mathbf{x}')\partial_k\phi_0^{j,\mathbf{k}}(\mathbf{x}-\mathbf{x}')d^3x' \ .
\end{aligned}$$

Using (106) it follows that

$$\begin{aligned}
\widetilde{g}_1(\mathbf{k},\mu,\mathbf{x}) &= -\mu\int A(\mathbf{x}-\mathbf{x}')\phi_0^{j,\mathbf{k}}(\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\quad \left(-i(E_k+\sum_{j=1}^3\alpha_j k\frac{x'_j}{x'}+\beta m) - x'^{-1}\frac{k}{E_k}\right)d^3x' \\
&\quad -\mu\int G_k^+(\mathbf{x}')A(\mathbf{x}-\mathbf{x}')\partial_k\phi_0^{j,\mathbf{k}}(\mathbf{x}-\mathbf{x}')d^3x' \ .
\end{aligned}$$

Since A has compact support it follows that

$$\begin{aligned}
\sup_{\mathbf{x}\in\mathbb{R}^3}\int A(\mathbf{x}-\mathbf{x}')d^3x' &< \infty \\
\sup_{\mathbf{x}\in\mathbb{R}^3}\int A(\mathbf{x}-\mathbf{x}')x'd^3x' &< \infty \\
\sup_{\mathbf{x}\in\mathbb{R}^3}\int A(\mathbf{x}-\mathbf{x}')x'^2d^3x' &< \infty \ .
\end{aligned}$$

Since $\phi^j(\mathbf{k},0,\mathbf{x})$ is normalized it follows in view of (41) that

$$\sup_{(\mathbf{k},\mu)\in\mathcal{P}} \| \widetilde{g}_1(\mathbf{k},\mu,\cdot) \|_\infty < \infty \ . \tag{157}$$

For $\widetilde{g}_2$ we have in view of (41)

$$\begin{aligned}
\widetilde{g}_2(\mathbf{k},\mu,\mathbf{x}) &= \mu\int A(\mathbf{x}-\mathbf{x}')\zeta^j(\mathbf{k},\mu,\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\quad \left(-i(E_k+\sum_{j=1}^3\alpha_j k\frac{x'_j}{x'}+\beta m) - x'^{-1}\frac{k}{E_k}\right)d^3x'
\end{aligned}$$

With (79) it follows that

$$\begin{aligned}
\widetilde{g}_2 &= \mu \int A(\mathbf{x}-\mathbf{x}')\omega^j(\mathbf{k},\mu,\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\qquad \left(-i(E_k+\sum_{j=1}^{3}\alpha_j k\frac{x'_j}{x'}+\beta m)-x'^{-1}\frac{k}{E_k}\right)d^3x' \\
&\quad +\mu \int A(\mathbf{x}-\mathbf{x}')f(\mathbf{k},\mu,\mathbf{x})(\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\qquad \left(-i(E_k+\sum_{j=1}^{3}\alpha_j k\frac{x'_j}{x'}+\beta m)-x'^{-1}\frac{k}{E_k}\right)d^3x' \\
&=: \widetilde{g}_3+\widetilde{g}_4 \ .
\end{aligned}$$

Using Lemma 6.1 (e) for $n=0$ it follows, that

$$\sup_{k\in\mathbb{R}^3;\mu\geq\lambda_c}\{\|\,\widetilde{g}_3(\mathbf{k},\mu,\cdot)\,\|_\infty\}<\infty \ . \qquad (158)$$

For $\widetilde{g}_4$ we rewrite in view of (36)

$$\begin{aligned}
\widetilde{g}_4 &= -i\mu \int A(\mathbf{x}-\mathbf{x}')f^0(\mathbf{k},\mu,\mathbf{x})(\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'}\,(m+\beta m)\,d^3x' \\
&\quad +\mu \int A(\mathbf{x}-\mathbf{x}')f^0(\mathbf{k},\mu,\mathbf{x})(\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\qquad \left(-i(E_k-m+\sum_{j=1}^{3}\alpha_j k\frac{x'_j}{x'})-x'^{-1}\frac{k}{E_k}\right)d^3x' \\
&= \mu \int A(\mathbf{x}-\mathbf{x}')f^0(\mathbf{k},\mu,\mathbf{x})(\mathbf{x}-\mathbf{x}')\frac{1}{4\pi}e^{ikx'} \\
&\qquad \left(-i(E_k-m+\sum_{j=1}^{3}\alpha_j k\frac{x'_j}{x'})-x'^{-1}\frac{k}{E_k}\right)d^3x'
\end{aligned}$$

Note that by relativistic dispersion relation E_k-m is for small k of order k^2. Furthermore A is compactly supported, so there exists a constant $C<\infty$ such that

$$\|\,\widetilde{g}_4(\mathbf{k},\mu,\cdot)\,\|_\infty \;\leq\; Ck\,\|\,f(\mathbf{k},\mu,\mathbf{x})\,\|_\infty$$

using Lemma 6.1 (e) for $n=0$ it follows that

$$\| \widetilde{g}_4 \|_\infty \leq Ck \left(1 + \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1}\right) . \tag{159}$$

With (157) and (158) it follows that there exists a constant $C < \infty$ such that

$$\| \widetilde{g}(\mathbf{k}, \mu, \mathbf{x}) \|_\infty \leq C \left(1 + k \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1}\right) . \tag{160}$$

With (153) we have that

$$\| \overline{g}_{\mathbf{k}} \|_\infty \leq C \left(1 + k \left(| \mu - \lambda_c - \alpha k^2 | + k^3\right)^{-1}\right)$$

$$\overline{g}_{\mathbf{k}} \in \mathcal{B} . \tag{161}$$

Similar as above we can show, that (152) has a unique solution in $\mathcal{B}$ and that

$$\lim_{k \to \infty} \| \partial_k \overline{\zeta}^j (\mathbf{k}, \mu, \cdot) \|_\infty = 0 . \tag{162}$$

Similar as above we are left with controlling

$$\lim_{(\mathbf{k},\mu) \to (\lambda_c, 0)} \| \partial_k \overline{\zeta}_\mu^{j,0} \|_\infty .$$

We proceed as above, using (161). It follows that $\partial_k \zeta^j(\mathbf{k}, \mu, \cdot)$ is of order $k \left(\min\{| \mu - \lambda_c |^{-1}, k^{-2}\}\right)^2$

Similarly we get (e) for $n > 1$. The factors $(x+1)^{-n}$ are needed to keep the functions $\widetilde{g}(\mathbf{k}, \mu, \mathbf{x})$ (see (151) for the function $\widetilde{g}(\mathbf{k}, \mu, \mathbf{x})$ for $n = 1$) and the $\partial_k^n \phi^j(\mathbf{k}, 0, \mathbf{x})$ bounded.

□

Proof of Lemma 6.3

We show that Lemma 6.4 implies Lemma 6.3.

Due to [1] the ζ^j exist and are infinitely often continuously differentiable in k for all $(\mathbf{k}, \mu) \in \mathcal{P}$. Hence the estimates on the generalized eigenfunctions (right hand side of (44) and (45)) follow for any compact subset of $\mathbb{R}^3 \times [\lambda_c, \overline{\mu}] \backslash (0, \lambda_c)$.

We also verified (44) and (45) for some subset of $\mathcal{P}$ "around" $(0, \lambda_c)$ (Lemma 6.4), so it is left to verify the estimates for $k \to \infty$. Hence Lemma 6.3 follows from

$$\lim_{k \to \infty} \| (x+1)^{-n} \partial_k^n \zeta^j \|_\infty < C_n \tag{163}$$

for all $n \in \mathbb{N}_0$ with appropriate constants $C_n < \infty$. In the following we refer to [8] and show only how the proof of Theorem 2.4. in [8] generalizes to our case.

We will show that ϕ^j can for sufficiently large k be written as a Born series, i.e. that there exists a $K < \infty$ such that

$$\phi^j(\mathbf{k}, \mu, \mathbf{x}) := \sum_{j=0}^{\infty} T_k^j \phi^j(\mathbf{k}, 0, \mathbf{x}) \tag{164}$$

exists for all $k > K$. To prove that the right hand side of (164) exists we will first derive a formula for T_k^j.

Using (46) we can write for any $\chi \in L^\infty$

$$\begin{aligned} T_k^2\chi &= T_k \int G_k^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\chi(\mathbf{x}')d^3x' \\ &= \int G_k^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}') \int G_k^+(\mathbf{x}'-\mathbf{x}'')A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x''d^3x' \\ &= \int\int G_k^+(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')G_k^+(\mathbf{x}'-\mathbf{x}'')d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \ . \end{aligned} \tag{165}$$

Using (31) we have that

$$G_k^+(\mathbf{x}) = (E_k + D^0)G_k^{KG,+}(\mathbf{x}) \tag{166}$$

(remember that D^0 is the free dirac operator (1)) where $G_k^{KG,+}(\mathbf{x})$ is the Klein Gordon kernel which solves

$$(E_k^2 - (D^0)^2)G_k^{KG,+}(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \ . \tag{167}$$

(166) in (165) yields in view of (1) that

$$\begin{aligned} T_k^2\chi &= \int\int \Big((E_k + D^0)G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\Big) A(\mathbf{x}') \\ &\qquad \Big((E_k + D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\Big) d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\ &= \int\int \Big((E_k - i\nabla\!\!\!/ + \beta m)G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\Big) A(\mathbf{x}') \\ &\qquad \Big((E_k + D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\Big) d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \ . \end{aligned}$$

One partial integration yields

$$\begin{aligned}
T_k^2\chi &= \int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\left((E_k+i\nabla\!\!\!/+2\beta m)\left(A(\mathbf{x}')(E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\right)\right)d^3x' \\
&\qquad A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&= \int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\left((E_k-D_0+2\beta m)\left(A(\mathbf{x}')(E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\right)\right)d^3x' \\
&\qquad A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&= \int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')2\beta mA(\mathbf{x}')\left((E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\right)d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&\quad+\int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\left((E_k-D_0)\left(A(\mathbf{x}')(E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')\right)\right)d^3x' \\
&\qquad A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x''\ .
\end{aligned}$$

With (167) it follows that

$$\begin{aligned}
T_k^2\chi &= \int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')2\beta mA(\mathbf{x}')(E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&\quad+\int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\left(\nabla\!\!\!/A(\mathbf{x}')\right)(E_k+D^0)G_k^{KG,+}(\mathbf{x}'-\mathbf{x}'')d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&\quad+\int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x}'')d^3x'A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'' \\
&= \int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')2\beta mA(\mathbf{x}')G_k^+(\mathbf{x}'-\mathbf{x}'')A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'd^3x'' \\
&\quad+\int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')\left(\nabla\!\!\!/A(\mathbf{x}')\right)G_k^+(\mathbf{x}'-\mathbf{x}'')A(\mathbf{x}'')\chi(\mathbf{x}'')d^3x'd^3x'' \\
&\quad+\int\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\chi(\mathbf{x}')d^3x'
\end{aligned}$$

Defining the operator $T_k^{KG}(A):L^\infty\to L^\infty$ by

$$T_k^{KG}(A)\chi=\int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\chi(\mathbf{x}')d^3x' \qquad (168)$$

we can write

$$T_k^2=T_k^{KG}(2\beta mA)T_k+T_k^{KG}(\nabla\!\!\!/A)T_k+T_k^{KG}(A) \qquad (169)$$

In a similar manner we write

$$
\begin{aligned}
T_k\phi^j(\mathbf{k},0,\mathbf{x}) &= \int (E_k + D^0) G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\phi^j(\mathbf{k},0,\mathbf{x}')d^3x' \\
&= 2\beta m \int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\phi^j(\mathbf{k},0,\mathbf{x}')d^3x' \\
&\quad + \int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')(E_k - D^0)\phi^j(\mathbf{k},0,\mathbf{x}')d^3x' \\
&= 2\beta m \int G_k^{KG,+}(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\phi^j(\mathbf{k},0,\mathbf{x}')d^3x' \qquad (170) \\
&= T_k^{KG}(2\beta m A)\phi^j(\mathbf{k},0,\mathbf{x}) \qquad (171)
\end{aligned}
$$

(169) and (170) yield, that all the $T_k^j\phi^j(\mathbf{k},0,\mathbf{x})$ can be written as sums of powers of $T_k^{KG}(A)$, $T_k^{KG}(2m\beta A)$ and $T_k^{KG}(\nabla\!\!\!/ A(\mathbf{x}'))$ acting on $\phi^j(\mathbf{k},0,\mathbf{x}')$.

Following the proof in [9] (note, that the Klein Gordon eigenvalue equation and the Schrödinger eigenvalue equation are formally equivalent, with $E_k = \sqrt{m^2+k^2}$ in the Klein Gordon, $E_k = \frac{k^2}{2m}$ in the Schrödinger case) we can conclude that $\sup_{\mathbf{k}\in\mathbb{R}^3\setminus B_1} \| T_k^{KG} \|_\infty^{op} < \infty$ and

$$\lim_{k\to\infty} \| (T_k^{KG})^2 \|_\infty^{op} = 0 \ .$$

Hence there exists a $K>0$ such that $\| T_k^{KG}(A) \|_\infty^{op} < \frac{1}{2}$, $\| T_k^{KG}(2m\beta A) \|_\infty^{op} < \frac{1}{2}$ and $\| T_k^{KG}(\nabla\!\!\!/ A(\mathbf{x}')) \|_\infty^{op} < \frac{1}{2}$. So the right hand side of (164) is bounded by a geometrical series. Furthermore it follows that

$$\| \phi^j(\mathbf{k},\mu,\mathbf{x}) - \phi^j(\mathbf{k},0,\mathbf{x}) \|_\infty := \| \sum_{l=1}^{\infty} T_k^l \phi^j(\mathbf{k},0,\mathbf{x}) \|_\infty$$

is bounded uniformly in $\mathbf{k}\in\mathbb{R}^3; k>K$ and (163) follows for $n=0$.

$\square$

7 Derivative of ϕ_μ

We will need some properties of the critical bound state ϕ_μ. We will only observe bound states which dive into the continuous spectrum properly. Note that the switching factor satisfies (6), so (172) is satisfied if and only if there exists a $\mu_0 < \lambda_c$ such that

$$0 < \partial_\mu E_\mu < C \tag{172}$$

for all $\mu_0 \leq \mu \leq \lambda_c$.

Lemma 7.1 *For every $\phi_{\lambda_c} \in \mathcal{N}$ (see (38)) there exists a $\mu_0 < \lambda_c$ and a $C < \infty$ such that one can find a function ϕ_μ for any $\mu \in [\mu_0, \lambda_c]$ with $D_\mu \phi_\mu = E_\mu \phi_\mu$ such that*

$$\parallel \partial_\mu \phi_\mu \parallel \leq C(\lambda_c - \mu)^{-\frac{25}{28}} \tag{173}$$

for all $\mu \in [\mu_0, \lambda_c[$.

Remark 7.2 *This estimate is not optimal, but sufficient for what is needed later. It seems reasonable to conjecture that the correct exponent is $-\frac{1}{2}$.*

Proof of Lemma 7.1

By assumption we have that only one eigenvalue dives into the upper continuous spectrum, hence there is a gap between m and the next smaller eigenvalue E_{-1}. For transparency of the proof we assume that D_{λ_c} has no further eigenvalues between $-m$ and m (This assumption is merely convenient and can be easily relaxed at the cost of more terms).

Let $\phi_{\lambda_c} \in \mathcal{N}$ with $\parallel \phi_{\lambda_c} \parallel = 1$. Using Gram Schmidt one can find a orthonormal Basis $B_\mathcal{N} := \{\phi_{\lambda_c}, \Phi^2, \Phi^3 \ldots \Phi^n\}$ of $\mathcal{N}$. Let $\mathcal{N}_\mu$ be the set of eigenfunctions of D_μ with energy eigenvalue E_μ.

For any $\mu \leq \lambda_c$ we choose a normalized $\phi_\mu \in \mathcal{N}_\mu$ such that $\phi_\mu \perp \Phi^l$ for all $2 \leq l \leq n$.

We first prove, that such a ϕ_μ exists for any $\mathcal{N}_\mu$.

Let $P_\mathcal{N}$ be the projector onto $\mathcal{N}$, $\{\Phi^l_\mu; l = 1, \ldots, n\}$ be a basis of $\mathcal{N}_\mu$.

1. Case: The vectors $P_\mathcal{N}\Phi^l_\mu; l = 1, \ldots, n$ are linearly independent

Choose a $\widetilde{\phi}_\mu \in \mathcal{N}_\mu$ such that $P_{\mathcal{N}}\widetilde{\phi}_\mu = \phi_{\lambda_c}$ Normalizing $\widetilde{\phi}_\mu$ yields ϕ_μ.

2. Case: The vectors $P_{\mathcal{N}}\Phi^l_\mu; l = 1, \ldots, n$ are linearly dependent

Choose a nontrivial $\widetilde{\phi}_\mu \in \mathcal{N}_\mu$ such that $P_{\mathcal{N}}\widetilde{\phi}_\mu = 0$. Normalizing $\widetilde{\phi}_\mu$ yields ϕ_μ.

Now we show that ϕ_μ satisfies the conditions of the Lemma. Therefore we first define

$$\zeta_\mu := \frac{\phi_\mu}{\langle \phi_\mu, \phi_{\lambda_c}\rangle} ,$$

where by definition of ϕ_μ

$$\lim_{\mu \to \lambda_c} \langle \phi_\mu, \phi_{\lambda_c}\rangle = 1 , \tag{174}$$

so that $\langle \phi_\mu, \phi_{\lambda_c}\rangle \neq 0$ for μ_0 close enough to λ_c, hence ζ_μ is well defined for $\mu \in [\mu_0, \lambda_c]$.

Thus

$$(\varepsilon D_\mu - E_\mu)\zeta_\mu = 0 , \quad \langle \zeta_\mu, \phi_{\lambda_c}\rangle = 1 \tag{175}$$

and using that $\varepsilon(D_\mu - D_\nu) = (\mu - \nu)A$ we get

$$\begin{aligned}
0 &= \langle (\varepsilon D_\mu - E_\mu)\zeta_\mu, \phi_{\lambda_c}\rangle \\
&= \langle (\varepsilon D_{\lambda_c} - E_\mu)\zeta_\mu, \phi_{\lambda_c}\rangle + \langle (\mu - \lambda_c)A\zeta_\mu, \phi_{\lambda_c}\rangle \\
&= \langle \zeta_\mu, (\varepsilon D_{\lambda_c} - E_\mu)\phi_{\lambda_c}\rangle + \langle (\mu - \lambda_c)A\zeta_\mu, \phi_{\lambda_c}\rangle \\
&= (m - E_\mu)\langle \zeta_\mu, \phi_{\lambda_c}\rangle + \langle (\mu - \lambda_c)A\zeta_\mu, \phi_{\lambda_c}\rangle \\
&= (m - E_\mu) + \langle (\mu - \lambda_c)A\zeta_\mu, \phi_{\lambda_c}\rangle
\end{aligned}$$

Hence

$$\begin{aligned}
(m - E_\mu)\phi_{\lambda_c} &= \langle (\lambda_c - \mu)A\zeta_\mu, \phi_{\lambda_c}\rangle\phi_{\lambda_c} \\
&= (\lambda_c - \mu)A\zeta_\mu + (\langle (\lambda_c - \mu)A\zeta_\mu, \phi_{\lambda_c}\rangle\phi_{\lambda_c} - (\lambda_c - \mu)A\zeta_\mu) \\
&= \varepsilon\left(D_{\lambda_c} - D_\mu\right)\zeta_\mu + (\langle (\lambda_c - \mu)A\zeta_\mu, \phi_{\lambda_c}\rangle\phi_{\lambda_c} - (\lambda_c - \mu)A\zeta_\mu)
\end{aligned}$$

Using $m\phi_{\lambda_c} = D_{\lambda_c}\phi_{\lambda_c}$ and $D_\mu\zeta_\mu = E_\mu\zeta_\mu$ we get that

$$\left(D_{\lambda_c} - E_\mu\right)\phi_{\lambda_c} = \left(D_{\lambda_c} - E_\mu\right)\zeta_\mu + (\langle (\lambda_c - \mu)A\zeta_\mu, \phi_{\lambda_c}\rangle\phi_{\lambda_c} - (\lambda_c - \mu)A\zeta_\mu) .$$

Note that $(D_{\lambda_c} - E_\mu)^{-1}$ exists for all $\mu < \lambda_c$, hence

$$\phi_{\lambda_c} \;=\; \zeta_\mu - (\lambda_c - \mu)\,(D_{\lambda_c} - E_\mu)^{-1}\,(A\zeta_\mu - \langle A\zeta_\mu, \phi_{\lambda_c}\rangle\phi_{\lambda_c})\;.$$

This leads us to define for any $\mu \in [\mu_0, \lambda_c[$ the linear operator $R_\mu : L^2 \to L^2$ by

$$R_\mu\chi := (D_{\lambda_c} - E_\mu)^{-1}\,(A\chi - \langle A\chi, \phi_{\lambda_c}\rangle\phi_{\lambda_c}) \tag{176}$$

for all $\chi \in L^2$, so that

$$\begin{aligned}\phi_{\lambda_c} &= (1 - (\lambda_c - \mu)R_\mu)\,\zeta_\mu \quad \text{or}\\ \zeta_\mu &= (1 - (\lambda_c - \mu)R_\mu)^{-1}\,\phi_{\lambda_c}\;.\end{aligned}$$

Below we will show that there exists a $C < \infty$ such that

$$\parallel R_\mu \parallel_2^{op} < C\,(\lambda_c - \mu)^{-\frac{25}{28}} \tag{177}$$

Hence taking μ_0 close enough to λ_c we have that for all $\mu \in [\mu_0, \lambda_c[$ there exist $q < 1$ so that

$$\parallel (\lambda_c - \mu)R_\mu \parallel_2^{op} \leq q < 1\;, \tag{178}$$

and hence

$$L^2 \ni \zeta_\mu \;=\; \sum_{j=0}^{\infty}(\mu - \lambda_c)^j R_\mu^j \phi_{\lambda_c}\;. \tag{179}$$

It is this series which we shall eventually differentiate with respect to μ. First we prove (177).

Let $\chi \in L^2$ with $\parallel \chi \parallel = 1$ and $\chi \perp \Phi^l$ for all $2 \leq l \leq n$. We set

$$\xi := A\chi - \langle A\chi, \phi_{\lambda_c}\rangle\phi_{\lambda_c}\;. \tag{180}$$

Note that by construction $\xi \perp \phi_{\lambda_c}$ hence - since $\chi \perp \Phi^l$ and $\phi_{\lambda_c} \perp \Phi^l$ for all $2 \leq l \leq n$ - that $\xi \perp \mathcal{N}$

$$R_\mu\chi = (D_{\lambda_c} - \mu)^{-1}\,\xi\;. \tag{181}$$

Let $\mathcal{B}(r_\mu)$ be the ball around zero with Radius

$$r_\mu = (\lambda_c - \mu)^{-\frac{3}{14}} \ . \tag{182}$$

r_μ is defined such, that part of χ which lies outside $\mathcal{B}(r_\mu)$ is in L^2 sufficiently small and the part of χ which lies inside $\mathcal{B}(r_\mu)$ is in L^1 sufficiently small. Furthermore $\mathcal{S}_A \subset \mathcal{B}(r_\mu)$ for sufficiently large r_μ, so the part of χ which lies outside $\mathcal{B}(r_\mu)$ is a multiple of ϕ_{λ_c}.Below we will have two different methods in our estimates, using smallness in L^2 and smallness in L^1.

For large enough r_μ we have that the $P_{\mathcal{N}}(1_{\mathcal{B}(r_\mu)}\Phi^l)$ for $1 \leq l \leq n$ are linear independent. Hence we can find a $\widetilde{\phi}_{\lambda_c} \in \mathcal{N}$ such that

$$P_{\mathcal{N}}(1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c}) = P_{\mathcal{N}}(1_{\mathcal{B}(r_\mu)}\xi) \ .$$

Hence

$$\xi_{1,\mu} \ := \ 1_{\mathcal{B}(r_\mu)}\xi - 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c} \tag{183}$$

$$\xi_{2,\mu} \ := \ \xi - \xi_{1,\mu} \ . \tag{184}$$

are orthogonal to $\mathcal{N}$.
$\xi_{1,\mu}$ has compact support $\mathcal{B}(r_\mu)$, so

$$\| \xi_{1,\mu} \|_1 \leq \frac{4}{3}\pi r_\mu^3 \| \xi_{1,\mu} \| \leq C r_\mu^3 \ , \tag{185}$$

for some appropriate $C < \infty$. Introducing (183) and (184) into (181)

$$R_\mu \chi = (D_{\lambda_c} - E_\mu)^{-1} \xi \ = \ (D_{\lambda_c} - E_\mu)^{-1} \xi_{1,\mu} + (D_{\lambda_c} - E_\mu)^{-1} \xi_{2,\mu}$$

we see that (177) holds if for some appropriate $K < \infty$

$$\| (D_{\lambda_c} - E_\mu)^{-1} \xi_{1,\mu} \| < K (\lambda_c - \mu)^{-\frac{25}{28}} \tag{186}$$

$$\| (D_{\lambda_c} - E_\mu)^{-1} \xi_{2,\mu} \| < K (\lambda_c - \mu)^{-\frac{25}{28}} \ . \tag{187}$$

We show (186). By Lemma 6.1

$$\begin{aligned} \widehat{\xi}_{1,\mu}(\mathbf{k}, j) \ &:= \ \mathcal{F}_{\lambda_c}(\xi_{1,\mu})(\mathbf{k}, j) = \int (2\pi)^{-\frac{3}{2}} [\phi^j(\mathbf{k}, \lambda_c, \mathbf{x}), \xi_{1,\mu}(\mathbf{x})] d^3x \\ &= \ \int (2\pi)^{-\frac{3}{2}} [\phi^j(\mathbf{k}, \lambda_c, \mathbf{x}) - f^0(\mathbf{k}, \mu, \mathbf{x}), \xi_{1,\mu}(\mathbf{x})] d^3x \\ &\quad + \int (2\pi)^{-\frac{3}{2}} [f^0(\mathbf{k}, \mu, \mathbf{x}), \xi_{1,\mu}(\mathbf{x})] d^3x \end{aligned}$$

Since $\xi_{1,\mu}$ is orthogonal to ϕ_{λ_c} we have that

$$\begin{aligned}
\mid \widehat{\xi}_{1,\mu}(\mathbf{k},j) \mid &:= \mid \mathcal{F}_{\lambda_c}(\xi_{1,\mu})(\mathbf{k},j) \mid \\
&= \left| \int (2\pi)^{-\frac{3}{2}} [\phi^j(\mathbf{k},\lambda_c,\mathbf{x}) - f^0(\mathbf{k},\mu,\mathbf{x}), \xi_{1,\mu}(\mathbf{x})] d^3x \right| \\
&\leq \parallel \phi^j(\mathbf{k},\lambda_c,\cdot) - f^0(\mathbf{k},\mu,\cdot) \parallel_\infty \parallel \xi_{1,\mu} \parallel_1 \ .
\end{aligned}$$

With Lemma 6.1 (e) and (185) it follows that there exists a $C < \infty$ such that

$$\mid \widehat{\xi}_{1,\mu}(\mathbf{k},j) \mid \ \leq \ C r_\mu^3 \tag{188}$$

for all $\mathbf{k} \in \mathbb{R}^3$, $\mu \in [\mu_0, \lambda_c]$. Thus

$$\begin{aligned}
\parallel (D_{\lambda_c} - E_\mu)^{-1} \xi_{1,\mu} \parallel &= \parallel \frac{1}{(-1)^j E_k - E_\mu} \widehat{\xi}_{1,\mu} \parallel \\
&= \left(\sum_{j=1}^{2} \int \mid \frac{1}{(-1)^j E_k - E_\mu} \widehat{\xi}_{1,\mu}(\mathbf{k},j) \mid^2 d^3k \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j=1}^{2} \int_{k<1} \mid \frac{1}{E_k - E_\mu} \widehat{\xi}_{1,\mu}(\mathbf{k},j) \mid^2 d^3k \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{j=1}^{2} \int_{k>1} \mid \frac{1}{E_k - E_\mu} \widehat{\xi}_{1,\mu}(\mathbf{k},j) \mid^2 d^3k \right)^{\frac{1}{2}} \ .
\end{aligned}$$

Since $E_k = \sqrt{k^2 + m^2} \geq m > E_\mu$ it follows that

$$E_k - E_\mu \geq m - E_\mu > 0 \tag{189}$$

and

$$E_k - E_\mu \geq E_k - m \geq 0 \ .$$

Hence we have with (188)

$$
\begin{aligned}
\| (D_{\lambda_c} - E_\mu)^{-1}\, \xi_{1,\mu} \| \;\; &\leq \;\; C r_\mu^3 \left(\sum_{j=1}^{2} \int_{k<(m-E_\mu)^{\frac{1}{2}}} \mid \frac{1}{m - E_\mu} \mid^2 d^3k \right)^{\frac{1}{2}} \\
&\quad + C r_\mu^3 \left(\sum_{j=1}^{2} \int_{1>k>(m-E_\mu)^{\frac{1}{2}}} \mid \frac{1}{E_k - m} \mid^2 d^3k \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{j=1}^{2} \int_{k>1} \mid \frac{1}{E_k - m} \widehat{\xi}_{1,\mu}(\mathbf{k}, j) \mid^2 d^3k \right)^{\frac{1}{2}}
\end{aligned}
$$

For $k > 1$ we have that $\frac{1}{E_k - m} \leq \frac{1}{E_1 - m}$. Thus

$$
\left(\sum_{j=1}^{2} \int_{k>1} \mid \frac{1}{E_k - m} \widehat{\xi}_{1,\mu}(\mathbf{k}, j) \mid^2 d^3k \right)^{\frac{1}{2}} \;\; \leq \;\; \frac{1}{E_1 - m} \| \xi_{1,\mu} \| \leq \frac{C}{E_1 - m} < \infty \, .
$$

It follows that

$$
\begin{aligned}
\| (D_{\lambda_c} - E_\mu)^{-1}\, \xi_{1,\mu} \| \;\; &\leq \;\; C r_\mu^3 \left(\sum_{j=1}^{2} \int_{k<(m-E_\mu)^{\frac{1}{2}}} \mid \frac{1}{(m - E_\mu)} \mid^2 d^3k \right)^{\frac{1}{2}} \\
&\quad + C r_\mu^3 \left(\sum_{j=1}^{2} \int_{1>k>(m-E_\mu)^{\frac{1}{2}}} \mid \frac{1}{E_k - m} \mid^2 d^3k \right)^{\frac{1}{2}} + C
\end{aligned}
$$

Since $\mid E_k - m \mid$ is for $k < 1$ of order k^2 (non relativistic limit of the kinetic energy), there exists a $C < \infty$ such that

$$
\mid E_k - m \mid \geq \frac{1}{C} k^2 \quad \text{for } 1 > k > (m - E_\mu)^{\frac{1}{2}} \, .
$$

Hence

$$
\begin{aligned}
\| (D_{\lambda_c} - E_\mu)^{-1}\, \xi_{1,\mu} \| \;\; &\leq \;\; C r_\mu^3 (m - E_\mu)^{-1} \left(\frac{4\pi}{3} (m - E_\mu)^{\frac{1}{2}} \right)^{\frac{3}{2}} \\
&\quad + C r_\mu^3 \left(\sum_{j=1}^{2} \int_{k>(m-E_\mu)^{\frac{1}{2}}} k^{-4} d^3k \right)^{\frac{1}{2}} + C \qquad (190) \\
&= \;\; C r_\mu^3 \left(\frac{4\pi}{3} \right)^{\frac{3}{2}} (m - E_\mu)^{-\frac{1}{4}} + C r_\mu^3 \left(4\pi (m - E_\mu) \right)^{-\frac{1}{4}} + C \, .
\end{aligned}
$$

Hence there exists for μ_0 close enough to λ_c a $C<\infty$ such that

$$\| \left(D_{\lambda_c}-E_\mu\right)^{-1}\xi_{1,\mu} \| \leq C r_\mu^3 \left(m-E_\mu\right)^{-\frac{1}{4}}$$

Note that (due to (172)) $m-E_\mu \leq C(\lambda_c-\mu)$. Using this and (182) we obtain (186) with some appropriate constant $K<\infty$.

Next we prove (187). By (183) and (184)

$$\| \xi_{2,\mu} \| = \| \xi - \xi_{1,\mu} \| \leq \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| + \| 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c} \|$$

Using $\xi_{1,\mu}\perp\mathcal{N}$ it we have with (183)

$$\langle 1_{\mathcal{B}(r_\mu)}\xi, \widetilde{\phi}_{\lambda_c}\rangle = \langle 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c}, \widetilde{\phi}_{\lambda_c}\rangle = \| 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c} \|^2 \approx \| \widetilde{\phi}_{\lambda_c} \|^2 \ .$$

and by Schwartz inequality

$$| \langle 1_{\mathcal{B}(r_\mu)}\xi, \widetilde{\phi}_{\lambda_c}\rangle | = | \langle \xi - 1_{\mathcal{B}(r_\mu)}\xi, \widetilde{\phi}_{\lambda_c}\rangle | \leq \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| \ \| \widetilde{\phi}_{\lambda_c} \| \ .$$

hence for R_0 large enough there exists a $C<\infty$ uniform in $r_\mu > R_0$ such that

$$\| 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c} \| \leq C \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| \ .$$

It follows that

$$\begin{aligned} \| \xi_{2,\mu} \| &\leq \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| + \| 1_{\mathcal{B}(r_\mu)}\widetilde{\phi}_{\lambda_c} \| \\ &= 2 \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| \ . \end{aligned} \tag{191}$$

By (180) and the fact that A is compactely supported we have that for large enough r_μ ξ is outside the ball $\mathcal{B}(r_\mu)$ a multiple of ϕ_{λ_c}. Hence $\xi - 1_{\mathcal{B}(r_\mu)}\xi$ is outside the ball $\mathcal{B}(r_\mu)$ a multiple of ϕ_{λ_c}.

From (37) $| \phi_{\lambda_c} | \leq Cx^{-2}$. Hence $\| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| \leq C \left(\int_{x>r_\mu} x^{-4} d^3x\right)^{\frac{1}{2}}$. So there exists a $C<\infty$ such that

$$\| \xi_{2,\mu} \| \leq 2 \| \xi - 1_{\mathcal{B}(r_\mu)}\xi \| \leq C r_\mu^{-\frac{1}{2}} \ .$$

It follows that by (189)

$$\begin{aligned} \| \left(D_{\lambda_c}-E_\mu\right)^{-1}\xi_{2,\mu} \| &= \| \frac{1}{E_k-E_\mu}\widehat{\zeta}_{2,\mu} \| \leq \frac{1}{m-E_\mu} \| \widehat{\xi}_{2,\mu} \| \\ &\leq | m-E_\mu |^{-1} C r_\mu^{-\frac{1}{2}} \end{aligned} \tag{192}$$

Hence (187) follows as above. We have thus established (177) and we turn now to the differentiation of ϕ_μ respectively ζ_μ (179).

Next we estimate $\| \partial_\mu R_\mu \|_2^{op}$. Using the spectral decomposition the differentiation yields

$$\begin{aligned}
\partial_\mu R_\mu \chi &= \partial_\mu \left(D_{\lambda_c} - E_\mu\right)^{-1} \left(A\chi - \langle A\chi, \phi_{\lambda_c}\rangle \phi_{\lambda_c}\right) \\
&= \left(\partial_\mu E_\mu\right) \left(D_{\lambda_c} - E_\mu\right)^{-2} \left(A\chi - \langle A\chi, \phi_{\lambda_c}\rangle \phi_{\lambda_c}\right) \\
&= \left(\partial_\mu E_\mu\right) \left(D_{\lambda_c} - E_\mu\right)^{-1} R_\mu \chi \,.
\end{aligned}$$

It follows using (172) that there exists a $C < \infty$ such that

$$\| \partial_\mu R_\mu \|_2^{op} \leq C(m - E_\mu)^{-1} \| R_\mu \|_2^{op} \tag{193}$$

Next we estimate $\partial_\mu \zeta_\mu$. Using (179) (Note that $\sum_{j=1}^\infty j\left((\mu - \lambda_c) R_\mu\right)^{j-1} \phi_{\lambda_c}$ is majorized by a convergent power series uniformly in $\mu_0 \leq \mu \leq \lambda_c$ (see (178), hence we can exchange limit and differentiation)

$$\begin{aligned}
\partial_\mu \zeta_\mu &= \partial_\mu \sum_{j=0}^\infty \left((\mu - \lambda_c) R_\mu\right)^j \phi_{\lambda_c} \\
&= \left(\partial_\mu (\mu - \lambda_c) R_\mu\right) \sum_{j=1}^\infty j\left((\mu - \lambda_c) R_\mu\right)^{j-1} \phi_{\lambda_c} \\
&= R_\mu \sum_{j=1}^\infty j\left((\mu - \lambda_c) R_\mu\right)^{j-1} \phi_{\lambda_c} \\
&\quad + (\mu - \lambda_c)(\partial_\mu R_\mu) \sum_{j=1}^\infty j\left((\mu - \lambda_c) R_\mu\right)^{j-1} \phi_{\lambda_c}
\end{aligned}$$

By (178) $\| \sum_{j=1}^\infty j(\mu - \lambda_c)^{j-1} R_\mu^j \phi_{\lambda_c} \| \leq C < \infty$ uniform in $\mu_0 \leq \mu < \lambda_c$. Thus by (177) and (193)

$$\| \partial_\mu \zeta_\mu \| \;\leq\; C\left(\| R_\mu \|_2^{op} + \| (\mu - \lambda_c) \partial_\mu R_\mu \|_2^{op}\right) \leq C \| R_\mu \|_2^{op} \tag{194}$$

with appropriate $C < \infty$.

Finally, to prove (173), we observe that

$$\phi_\mu = \frac{\zeta_\mu}{\| \zeta_\mu \|} \,,$$

and thus

$$\partial_\mu \phi_\mu = \frac{\partial_\mu \zeta_\mu}{\parallel \zeta_\mu \parallel} - \frac{\zeta_\mu}{\parallel \zeta_\mu \parallel^2} \partial_\mu \parallel \zeta_\mu \parallel \ .$$

Due to (175) we have that $\parallel \zeta_\mu \parallel \geq 1$ for all $\mu \in [\mu_0, \lambda_c[$. Furthermore by triangle inequality $\partial_\mu \parallel \zeta_\mu \parallel \leq \parallel \partial_\mu \zeta_\mu \parallel$, therefore

$$\parallel \partial_\mu \phi_\mu \parallel \leq 2 \parallel \partial_\mu \zeta_\mu \parallel \ .$$

This, (177) and (194) yield (173).

□

8 Proof of Theorem 3.3

In the following we will set $s_{m1} = 0$. Since we shall employ often eigenfunction expansions we need the following properties of the generalized eigenfunctions. We provide the major results of Lemma 6.1 and Lemma 7.1 in our notation, i.e. $\mu = \lambda\varphi(s)$ with the restriction $0 < \partial_s \varphi(s) < \infty$ (see (6)).

We will slightly abuse notation, writing ϕ_s for $\phi_{\lambda\varphi(s)}$.

Corollary 8.1 *Let A be compactly supported, $A > 0$, $\widetilde{s} > 0$ be such, that λ_c is the only critical coupling constant in $[\lambda_c, \lambda\varphi(\widetilde{s})]$ and $\partial_s \varphi(s) > 0$ in $[\lambda_c, \lambda\varphi(\widetilde{s})]$. Let $\mathcal{B}$ be the Banach space of all continuous functions tending uniformly to zero as $x \to \infty$ equipped with the supremum norm. Then*

(a) *there exist unique solutions $\phi^j(\mathbf{k}, s, \mathbf{x})$ of (29) in $\mathcal{B}$ for all $k \in \mathbb{R}^3$, $s \in (0, \widetilde{s}]$ such that*

(b) *for any $s \in (0, \widetilde{s}]$ the set of $\phi^j(\mathbf{k}, s, \mathbf{x})$ define a generalized Fourier transform in the space of scattering states by*

$$\mathcal{F}_s(\psi)(\mathbf{k}, j) := \int (2\pi)^{-\frac{3}{2}} \langle \phi^j(\mathbf{k}, s, \mathbf{x}), \psi(\mathbf{x}) \rangle d^3x \qquad (195)$$

and

$$\psi(\mathbf{x}) = \sum_{j=1}^{4} \int (2\pi)^{-\frac{3}{2}} \phi^j(\mathbf{k}, s, \mathbf{x}) \mathcal{F}_s(\psi)(\mathbf{k}, j) d^3k \ . \qquad (196)$$

The so defined $\mathcal{F}_s(\psi)$ is isometric to ψ, i.e.

$$\| \psi \| := \left(\int \mid \psi(\mathbf{x}) \mid^2 d^3x \right)^{\frac{1}{2}} = \sum_{j=1}^{4} \left(\int \mid \mathcal{F}_s(\psi)(\mathbf{k}, j) \mid^2 d^3k \right)^{\frac{1}{2}} =: \| \mathcal{F}_s(\psi) \|$$

(c) *the functions $\phi^j(\mathbf{k}, s, \mathbf{x})$ are infinitely often continuously differentiable with respect to k, furthermore there exist $\alpha, C \in \mathbb{R}^+$ uniform in $(\mathbf{k}, s) \in \mathbb{R}^3 \times [0, s_{m2}]$ and for all $n \in \mathbb{N}_0$ functions $f^n(\mathbf{k}, s, \mathbf{x}) \in \mathcal{N}$ (see (38)) with*

$$\| f^n(\mathbf{k}, s, \cdot) \|_\infty < C \left(1 + k^n \left(\mid s - \alpha k^2 \mid + k^3 \right)^{-n-1} \right) \tag{197}$$

such that

$$\zeta^{j,n}(\mathbf{k}, s, \mathbf{x}) := \partial_k^n \phi^j(\mathbf{k}, s, \mathbf{x}) - f^n(\mathbf{k}, s, \mathbf{x}) \phi_0(\mathbf{x}) \tag{198}$$

satisfies

$$\| (x+1)^{-n} \zeta^{j,n}(\mathbf{k}, s, \mathbf{x}) \|_\infty < C \left(1 + (s + k^2) \| f^n(\mathbf{k}, s, \cdot) \|_\infty \right) . \tag{199}$$

in particular for $n = 0$

$$\| \phi^j(\mathbf{k}, s, \mathbf{x}) \|_\infty < C \left(1 + \left(\mid s - \alpha k^2 \mid + k^3 \right)^{-1} \right) . \tag{200}$$

(d) *Let ϕ_s be defined as above. Then there exists $s_{in} < 0$ and $C < \infty$ such that*

$$\| \partial_s \phi_s \| \leq C s^{-\frac{25}{28}}$$

for all $s_{in} \leq s \leq 0$.

To prove Theorem 3.3 we have to control the time propagation of $\phi_{s_{in}}$. This propagation is qualitatively different for $s < 0$ and $s > 0$. Hence we control the propagation for $s < 0$ and $s > 0$ separately.

8.1 Control of ψ^ε_s for $s_{in} \le s \le 0$

The adiabatic theorem yields that for any $s < 0$ the wave function will stay a multiple of the respective bound state as ε goes to zero.

We shall extend the assertion to $s = 0$.

Lemma 8.2 *Let $\psi^\varepsilon_{s,s_{in}}$ with $s \in [s_{in}, 0[$ be solution of the Dirac equation with $\psi^\varepsilon_{s_{in},s_{in}} = \phi_{s_{in}}$.*

Then uniform in $\varepsilon > 0$

$$\lim_{s_{in}\to 0} \mid \langle \psi^\varepsilon_{0,s_{in}}, \phi_0 \rangle \mid = 1 \ .$$

Proof

We introduce

$$\psi^{\varepsilon,1}_{s,s_{in}} := \exp\left(-\frac{i}{\varepsilon}\int_{s_{in}}^{s} E_v dv\right)\phi_s \ . \tag{201}$$

Note, that $\psi^{\varepsilon,1}_{s_{in},sin} = \phi_{s_{in}}$ and $\mid \langle \psi^{\varepsilon,1}_{0,s_{in}}, \phi_0 \rangle \mid = 1$. Thus to prove the Lemma we need only show that $\psi^\varepsilon_{0,s_{in}}$ will be equal to $\psi^{\varepsilon,1}_{0,s_{in}}$ in the limit $\lim_{s_{in}\to 0}$ uniform in ε, i.e.

$$\lim_{s_{in}\to 0} \sup_{0<\varepsilon\le 1} \parallel \psi^\varepsilon_{0,s_{in}} - \psi^{\varepsilon,1}_{0,s_{in}} \parallel = 0 \ . \tag{202}$$

Let now $U^\varepsilon(s,u)$ be the propagator of the Dirac equation, i.e. $i\partial_s U^\varepsilon(s,0) = \frac{1}{\varepsilon} D_s U^\varepsilon(s,0)$. Using that $\psi^\varepsilon_{s,s_{in}}$ is solution of the Dirac equation it follows with (201) that

$$\begin{aligned}
\psi^{\varepsilon,1}_{s,s_{in}} - \psi^\varepsilon_{s,s_{in}} &= \int_{s_{in}}^{s} \partial_u \left(U^\varepsilon(s,u)\psi^{\varepsilon,1}_{u,s_{in}}\right) du \\
&= -i\int_{s_{in}}^{s} U^\varepsilon(s,u)\left(\frac{D_u}{\varepsilon} - i\partial_u\right)\exp\left(-\frac{i}{\varepsilon}\int_{s_{in}}^{u} E_v dv\right)\phi_u du \\
&= -\frac{i}{\varepsilon}\int_{s_{in}}^{s} U^\varepsilon(s,u)\left(D_u - E_u\right)\exp\left(-\frac{i}{\varepsilon}\int_{s_{in}}^{u} E_v dv\right)\phi_u du \\
&\quad +i\int_{s_{in}}^{s} U^\varepsilon(s,u)\exp\left(-\frac{i}{\varepsilon}\int_{s_{in}}^{u} E_v dv\right)\partial_u\phi_u du
\end{aligned}$$

Since $(D_u - E_u)\phi_u = 0$

$$\psi^{\varepsilon,1}_{s,s_{in}} - \psi^{\varepsilon}_{s,s_{in}} = i\int_{s_{in}}^{s} U^{\varepsilon}(s,u)\exp\left(-\frac{i}{\varepsilon}\int_{s_{in}}^{u} E_v dv\right)\partial_u\phi_u du\ .$$

Hence by unitarity of U^{ε} and by Corollary 8.1 (d)

$$\begin{aligned} \| \psi^{\varepsilon}_{0,s_{in}} - \psi^{\varepsilon,1}_{0,s_{in}} \| &\leq \int_{s_{in}}^{0} \| \partial_u\phi_u \| du \\ &\leq C\int_{s_{in}}^{0} u^{-\frac{25}{28}} du = -\frac{25}{28}C[u^{\frac{3}{28}}]^{0}_{s_{in}} = \frac{25}{28}Cs_{in}^{\frac{3}{28}} \end{aligned}$$

and (202) follows.

□

Corollary 8.3 *(Adiabatic Theorem without a gap)*
Let $s_i < s_{m1}$ and $s_f > s_{m2}$ be such that ϕ_{s_i} and ϕ_{s_f} already / still exist. Let $U^{\varepsilon}(s,u)$ be the time evolution operator of the Dirac equation (7) on the adiabatic time scale. Let ϕ_s be an overcritical bound state of the Dirac operator with potential $A_s(\mathbf{x})$ of the form (5). Let ϕ_s dive properly into the positive continuous spectrum (see (11)).

$$\lim_{\varepsilon\to 0} |\langle U^{\varepsilon}(0,s_i)\phi_{s_i},\phi_0\rangle| = 1 \tag{203}$$
$$\lim_{\varepsilon\to 0} |\langle U^{\varepsilon}(s_f,s_{m2})\phi_0,\phi_{s_f}\rangle| = 1\ . \tag{204}$$

Proof

We only prove (203). (204) follows equivalently following the propagation of ϕ_{s_f} backwards in time.

We will show that for any $\delta > 0$ there exists a $\varepsilon_0 > 0$ and phase factors π^{ε}_0 such that

$$\| U^{\varepsilon}(0,s_i)\phi_{s_i} - \pi^{\varepsilon}_0\phi_0 \| < \delta \tag{205}$$

for all $\varepsilon < \varepsilon_0$.

Using Lemma 8.2 we choose $s_{in} > 0$ such that

$$\| \psi^{\varepsilon}_{0,s_{in}} - \pi^{\varepsilon}_1\phi_0 \| < \frac{1}{2}\delta$$

for all $\varepsilon < \varepsilon_0$ with an appropriate phase factor π_1^ε.
Then using the adiabatic Theorem (see [9]) we choose $\varepsilon > 0$ such that

$$\| U^\varepsilon(s_{in}, s_i)\phi_{s_i} - \pi_2^\varepsilon \phi_{s_{in}} \| < \frac{1}{2}\delta$$

for all $\varepsilon < \varepsilon_0$ with an appropriate phase factor π_2^ε.
Using the triangle inequality we get that

$$\begin{aligned}
&\| U^\varepsilon(0, s_i)\phi_{s_i} - \pi_1^\varepsilon\pi_2^\varepsilon\phi_0 \| \\
&\quad\leq \| \pi_2^\varepsilon U^\varepsilon(0, s_{in})\phi_{s_i} - \pi_1^\varepsilon\pi_2^\varepsilon\phi_0 \| + \| U^\varepsilon(s_{in}, s_i)\phi_{s_i} - \pi_2^\varepsilon\phi_{s_{in}} \| \\
&\quad= \| \psi_{0,s_{in}}^\varepsilon - \pi_1^\varepsilon\phi_0 \| + \| U^\varepsilon(0, s_i)\phi_{s_i} - \pi_2^\varepsilon\phi_{s_{in}} \| \\
&\quad< \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned}$$

and (205) follows.

□

8.2 Control of ψ_s^ε for $s > 0$

Due to Corollary 8.3 ψ_0^ε is - in the given limits and up to a phase factor - equal to the bound state ϕ_0. So Theorem 3.3 is a direct result of Corollary 8.3 and

Lemma 8.4 *Let $U^\varepsilon(s, u)$ be the time evolution operator of the Dirac equation (7) on the adiabatic time scale. Let ϕ_s and $\widetilde{\phi}_s$ be overcritical bound states of the Dirac operator with potential $A_s(\mathbf{x})$ of the form (5) that dive properly into the positive continuous spectrum (see (11)). Then (remember that $D_0 = D_{s_{m2}}$, hence $\widetilde{\phi}_{s_{m2}} = \widetilde{\phi}_0$)*

$$\lim_{\varepsilon\to 0}\langle U^\varepsilon(s_{m2}, 0)\phi_0, \widetilde{\phi}_0\rangle = 0 \ . \tag{206}$$

Proof

Set

$$\psi_s^\varepsilon := U^\varepsilon(s, 0)\phi_0 \tag{207}$$

To begin with we introduce the time s_ε which shall be specified later and which should be thought of as a time much larger than ε and smaller than 1 (for example $\varepsilon^{\frac{1}{2000}}$. s_ε will serve for defining an appropriate approximating dynamics: Let $V^\varepsilon_{s_\varepsilon}$ be the time evolution operator of the time independent Dirac operator $\frac{1}{\varepsilon}D_{s_\varepsilon}$. This evolution is controllable since we have by Corollary 8.1 good control of the generalized eigenfunctions and

$$V^\varepsilon_{s_\varepsilon}(s,u)\phi^j(\mathbf{k},s_\varepsilon,\mathbf{x}) = \exp\left(-\frac{i}{\varepsilon}E_k(s-u)\right)\phi^j(\mathbf{k},s_\varepsilon,\mathbf{x}) \tag{208}$$

In the following we will always use the notation

$$\widehat{\psi} := \mathcal{F}_{s_\varepsilon}(\psi) \tag{209}$$

for the generalized Fourier transform of ψ in the $\phi^j(\mathbf{k},s_\varepsilon,\mathbf{x})$ eigenbasis.

We first give some formulas for different propagators, we will need below.

Let $\overline{U}(s,s_{in})$ and $\widetilde{U}(s,s_{in})$ be time propagators, $\overline{D}_s$ and $\widetilde{D}_s$ the respective - in general time dependent - generators, i.e.

$$\partial_s\overline{U}(s,s_{in}) = \overline{D}_sU(s,s_{in}) \quad \partial_s\widetilde{U}(s,s_{in}) = \widetilde{D}_s\widetilde{U}(s,s_{in}) \ .$$

Then

$$\begin{aligned}
\overline{U}(s,s_{in}) - \widetilde{U}(s,s_{in}) &= -\int_{s_{in}}^{s}\partial_u\left(\overline{U}^\varepsilon(s,u)\widetilde{U}(u,s_{in})\right)du \\
&= -i\int_{s_{in}}^{s}\overline{U}^\varepsilon(s,u)\left(\overline{D}_u - i\partial_u\right)\widetilde{U}(u,s_{in})du \qquad (210)\\
&= -i\int_{s_{in}}^{s}\overline{U}^\varepsilon(s,u)\left(\overline{D}_u - \widetilde{D}_u\right)\widetilde{U}(u,s_{in})du \ . \qquad (211)
\end{aligned}$$

We shall use the following identity for the time U^ε, which follows directly from (211) setting $\widetilde{U} = U^\varepsilon$ and $\overline{U} = V^\varepsilon_{s_\varepsilon}$.

$$U^\varepsilon(s,0) = V^\varepsilon_{s_\varepsilon}(s,0) + i\int_0^s V^\varepsilon_{s_\varepsilon}(s,w)\frac{1}{\varepsilon}(D_w - D_{s_\varepsilon})U^\varepsilon(w,0)dw \ . \tag{212}$$

We shall now approximate ψ^ε_s in three steps by a wave function $\psi^{\varepsilon,3}_s$ which is easier to control and such that the difference $\| \psi^{\varepsilon,3}_s - \psi^\varepsilon_s \| \to 0$ as $\varepsilon \to 0$.

1. Step:

We replace ϕ_0 by ϕ^ε given by

$$\phi^\varepsilon(\mathbf{x}) := \phi_0(\mathbf{x})(1 - \rho_{\varepsilon^{-\frac{1}{10000}}}(\mathbf{x})) \tag{213}$$

where $\rho_\kappa \in C^\infty$ is a mollifier given by

$$\rho(\mathbf{x}) := \begin{cases} 0, & \text{for } x \leq 1; \\ 1, & \text{for } x \geq 2. \end{cases} \tag{214}$$

and

$$\rho_\kappa(\mathbf{x}) := \rho(\frac{\mathbf{x}}{\kappa}) \tag{215}$$

for $\kappa > 0$. Hence $\phi^\varepsilon(\mathbf{x}) = 0$ for $\mathbf{x} \geq 2\varepsilon^{-\frac{1}{1000}}$ and $\phi^\varepsilon(\mathbf{x}) = \phi_0(\mathbf{x})$ for $\mathbf{x} \leq \varepsilon^{-\frac{1}{1000}}$. So ϕ^ε has compact support $\mathcal{T}_\varepsilon$ with

$$\mid \mathcal{T}_\varepsilon \mid \leq \frac{4}{3}\pi 8 \varepsilon^{-\frac{3}{10000}} \tag{216}$$

and that

$$\parallel \phi^\varepsilon \parallel \leq 1 \ . \tag{217}$$

We set

$$\psi_s^{\varepsilon,1} := U^\varepsilon(s,0)\phi^\varepsilon \tag{218}$$

and for the error

$$\eta_s^{\varepsilon,1} \quad := \quad \psi_s^\varepsilon - \psi_s^{\varepsilon,1} \ . \tag{219}$$

2. Step:

Observing (212) we obtain for (218)

$$\psi_s^{\varepsilon,1} = V_{s_\varepsilon}^\varepsilon(s,0)\phi^\varepsilon + i\int_0^s V_{s_\varepsilon}^\varepsilon(s,w)\frac{1}{\varepsilon}(D_w - D_{s_\varepsilon})U^\varepsilon(w,0)\phi^\varepsilon dw \tag{220}$$

and setting

$$\zeta^\varepsilon_w(\mathbf{x}) \;:=\; \frac{1}{\varepsilon}(D_w - D_{s_\varepsilon})U^\varepsilon(w,0)\phi^\varepsilon = \frac{1}{\varepsilon}(D_w - D_{s_\varepsilon})\psi^{\varepsilon,1}_w \;, \tag{221}$$

(220) becomes

$$\psi^{\varepsilon,1}_s \;:=\; V^\varepsilon_{s_\varepsilon}(s,0)\phi^\varepsilon + i\int_0^s V^\varepsilon_{s_\varepsilon}(s,w)\zeta^\varepsilon_w(\mathbf{x})dw \tag{222}$$

We write

$$\zeta^\varepsilon_w(\mathbf{x}) \;=\; \sum_{j=1}^{4}\int (2\pi)^{-\frac{3}{2}}\phi^j(\mathbf{k},s_\varepsilon,\mathbf{x})\widehat{\zeta^\varepsilon_w}(\mathbf{k},j)d^3k$$

and replace for $0 \le w \le s_\varepsilon$ $\widehat{\zeta^\varepsilon_w}(\mathbf{k},j)$ - using some

$$\varepsilon \ll k_\varepsilon \ll 1 \ll K_\varepsilon \tag{223}$$

which will be specified later on - by

$$\widehat{\zeta^{\varepsilon,1}_w}(\mathbf{k},j) := \widehat{\zeta^\varepsilon_w}(\mathbf{k},j)\rho_{k_\varepsilon}(1-\rho_{K_\varepsilon}) \tag{224}$$

Furthermore we write (213) as

$$\phi^\varepsilon(\mathbf{x}) \;=\; \sum_{j=1}^{4}\int (2\pi)^{-\frac{3}{2}}\phi^j(\mathbf{k},s_\varepsilon,\mathbf{x})\widehat{\phi^\varepsilon}(\mathbf{k},j)d^3k$$

and replace $\widehat{\phi^\varepsilon}$ by

$$\widehat{\phi^{\varepsilon,1}}(\mathbf{k},j) := \widehat{\phi^\varepsilon}(\mathbf{k},j)\rho_\kappa(\mathbf{k})(1-\rho_{K_\varepsilon}) \;. \tag{225}$$

These replacements define a new wave function, namely

$$\psi^{\varepsilon,2}_s \;:=\; V^\varepsilon_{s_\varepsilon}(s,0)\phi^{\varepsilon,1} + i\int_0^s V^\varepsilon_{s_\varepsilon}(s,w)\zeta^{\varepsilon,1}_w(\mathbf{x})dw \tag{226}$$

for $s \le s_\varepsilon$ and

$$\psi^{\varepsilon,2}_s \;=\; U(s,s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon} \tag{227}$$

for $s > s_\varepsilon$.

We shall need the difference

$$\eta_s^{\varepsilon,2} := \psi_s^{\varepsilon,1} - \psi_s^{\varepsilon,2}$$

only at time $s = s_{m2}$ (see Lemma 8.5 below) and we note here already that (see (218) and (227))

$$\begin{aligned}
\eta_{s_{m2}}^{\varepsilon,2} &= \psi_{s_{m2}}^{\varepsilon,1} - \psi_{s_{m2}}^{\varepsilon,2} \\
&= U^\varepsilon(s_{m2}, s_\varepsilon)\left(\psi_{s_\varepsilon}^{\varepsilon,1} - \psi_{s_\varepsilon}^{\varepsilon,2}\right) .
\end{aligned}$$

With (220) and (226) we get that

$$\begin{aligned}
\eta_{s_{m2}}^{\varepsilon,2} &= U^\varepsilon(s_{m2}, s_\varepsilon) V_{s_\varepsilon}^\varepsilon(s_\varepsilon, 0)\left(\phi^\varepsilon - \phi^{\varepsilon,1}\right) \\
&\quad + iU^\varepsilon(s_{m2}, s_\varepsilon)\int_0^{s_\varepsilon} V_{s_\varepsilon}^\varepsilon(s_\varepsilon, w)\left(\zeta_w^\varepsilon(\mathbf{x}) - \zeta_w^{\varepsilon,1}(\mathbf{x})\right) dw .
\end{aligned} \tag{228}$$

3. Step:

In this last step we more or less assert that the wave function evolution after time s_ε is close to the auxiliary time evolution $V_{s_\varepsilon}^\varepsilon$, namely we replace $\psi_s^{\varepsilon,2}$ by

$$\psi_s^{\varepsilon,3} := \begin{cases} \psi_s^{\varepsilon,2}, & \text{for } s \le s_\varepsilon; \\ V_{s_\varepsilon}^\varepsilon(s, s_\varepsilon)\psi_{s_\varepsilon}^{\varepsilon,2}, & \text{else.} \end{cases} \tag{229}$$

Again we shall need the difference

$$\eta_s^{\varepsilon,3} := \psi_s^{\varepsilon,2} - \psi_s^{\varepsilon,3}$$

only at time $s = s_{m2}$ (see Lemma 8.5 below) and we note here already that (see (227) and (229))

$$\eta_{s_{m2}}^{\varepsilon,3} = \left(U^\varepsilon(s_{m2}, s_\varepsilon) - V_{s_\varepsilon}^\varepsilon(s_{m2}, s_\varepsilon)\right)\psi_{s_\varepsilon}^{\varepsilon,2} . \tag{230}$$

We use (211) setting $\widetilde{U}^\varepsilon = V_{s_\varepsilon}^\varepsilon$ and $\overline{U} = U^\varepsilon$ and get

$$\eta_{s_{m2}}^{\varepsilon,3} = -i\int_{s_\varepsilon}^{s_{m2}} U^\varepsilon(s_{m2}, w)\frac{1}{\varepsilon}(D_w - D_{s_\varepsilon})V_{s_\varepsilon}^\varepsilon(w, s_\varepsilon)\psi_{s_\varepsilon}^{\varepsilon,2} dw . \tag{231}$$

Now Lemma 8.4 follows from

Lemma 8.5 *Let* $\psi_s^{\varepsilon,3}$ *and* $\eta_s^{\varepsilon,l}$ $l = 1,2,3$ *given by (229), (219), (228) and (230). Then for* $s_\varepsilon = \varepsilon^{\frac{1}{2000}}$, $k_\varepsilon = \varepsilon^{\frac{4}{9}+\frac{1}{1000}}$ *and* $K_\varepsilon = \varepsilon^{-4}$

(a)

$$\lim_{\varepsilon\to 0}\langle\psi_{s_{m2}}^{\varepsilon,3}, \widetilde{\phi}_0\rangle = 0$$

(b)

$$\lim_{s_\varepsilon\to 0} \parallel \eta_{s_{m2}}^{\varepsilon,l} \parallel \;=\; 0$$

for $l = 1,2,3$.

Proof

To prove the Lemma we need to control wave functions which have $V_{s_\varepsilon}^\varepsilon$ as time evolution operator. We note that the wave functions above which have $V_{s_\varepsilon}^\varepsilon$ as time evolution operator have nice features, which we shall summarize below. We will use that such wave functions (which are smooth in generalized momentum space and not to heavily peaked around $\mathbf{k} = 0$ as $s_\varepsilon \to 0$) show a typical scattering behavior, i.e. they decay fast enough in time.

Lemma 8.6 *Let* $\mathcal{R}_\mathcal{S}$ *be defined by*

$$\widehat{\xi}(\mathbf{k}, j) \in \mathcal{R}_\mathcal{S} \Leftrightarrow \parallel \xi \parallel = 1 \text{ and } \xi(\mathbf{x}) \text{ has support } \mathcal{S} \tag{232}$$

For any $n \in \mathbb{N}_0$ *there exist* $C_n < \infty$ *such that for any* $1 > \varepsilon, k_\varepsilon, s_\varepsilon, u > 0$, $K_\varepsilon < \infty$ *and any compact set* $\mathcal{S}_\varepsilon \subset \mathbb{R}^3$

(a) for all $\widehat{\chi} \in (1-\rho_{k_\varepsilon})\mathcal{R}_{\mathcal{S}_\varepsilon}$

$$\parallel \chi \parallel \le C_0 \sup_{k<2k_\varepsilon} \left(\left(\mid s_\varepsilon - \alpha k^2 \mid + k^3 \right)^{-1} \right) \sqrt{\mid \mathcal{S}_\varepsilon \mid} k_\varepsilon^{\frac{3}{2}} \; . \tag{233}$$

and

$$\parallel V_{s_\varepsilon}^\varepsilon(u,0)\chi \parallel_\infty \le C_0 \sup_{k<2k_\varepsilon} \left(\left(\mid s_\varepsilon - \alpha k^2 \mid + k^3 \right)^{-2} \right) \sqrt{\mid \mathcal{S}_\varepsilon \mid} k_\varepsilon^3 \; . \tag{234}$$

(b) for all $\chi(\mathbf{x})$ with $\widehat{\chi} \in \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon})\mathcal{R}_{\mathcal{S}_\varepsilon}$

$$\| V^\varepsilon_{s_\varepsilon}(u,0)\chi \|_\infty \leq C_n K_\varepsilon^3 \sqrt{| \mathcal{S}_\varepsilon |}\frac{\varepsilon^n}{u^n} s_\varepsilon^{-3}\left(k_\varepsilon^{-2n} + s_\varepsilon^{-\frac{3}{2}n}\right) . \quad (235)$$

Proof of Lemma 8.6 (a) formula (233)

Let $\widehat{\chi} \in (1-\rho_{k_\varepsilon})\mathcal{R}_{\mathcal{S}_\varepsilon}$. Writing $\widehat{\chi} = \widehat{\eta}(\mathbf{k},j)(1-\rho_{k_\varepsilon})$ with $\widehat{\eta} \in R_{\mathcal{S}_\varepsilon}$ in view of (215) we have

$$\begin{aligned} \| \widehat{\chi} \| &= \left(\int | \widehat{\eta}(\mathbf{k},j)(1-\rho_{k_\varepsilon}) |^2 \, d^3k \right)^{\frac{1}{2}} \\ &\leq \| \widehat{\chi} \| \leq \sup_{k<2k_\varepsilon} \{ | \widehat{\eta}(\mathbf{k},j) | \} \| 1-\rho_{k_\varepsilon} \| \end{aligned} \quad (236)$$

Substituting $p = \frac{k}{k_\varepsilon}$ yields in view of (215)

$$\| \rho_{k_\varepsilon} - 1 \| = k_\varepsilon^{\frac{3}{2}} \left(\int | \rho(p) - 1 |^2 \, d^3p \right)^{\frac{1}{2}} \quad (237)$$

where by (214) $\int | \rho(p) - 1 |^2 \, d^3p$ is bounded.
Furthermore

$$\begin{aligned} \sup_{k<2k_\varepsilon} \{ | \widehat{\eta}(\mathbf{k},j) | \} &\leq \sup_{k<2k_\varepsilon} \{ \int (2\pi)^{-\frac{3}{2}} | \langle \phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}), \eta(\mathbf{x}) \rangle | \, d^3x \} \\ &\leq \sup_{k<2k_\varepsilon} \{ \| \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \| \} \int (2\pi)^{-\frac{3}{2}} | \eta(\mathbf{x}) | \, d^3x \end{aligned} \quad (238)$$

Using Schwartz (observing $\widehat{\eta} \in \mathcal{R}_{\mathcal{S}_\varepsilon}$)

$$\begin{aligned} \| \eta \|_1 &= \left| \int | \eta(\mathbf{x}) | \, d^3x \right| \\ &= \left| \int 1_{\mathcal{S}_\varepsilon} | \eta(\mathbf{x}) | \, d^3x \right| \\ &\leq \| \eta(\mathbf{x}) \| \sqrt{| \mathcal{S}_\varepsilon |} = \sqrt{| \mathcal{S}_\varepsilon |} \end{aligned} \quad (239)$$

For $\sup_{k<2k_\varepsilon}\{ \| \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \|_\infty \}$ we have by Corollary 8.1 (c) formula (200) that

$$\sup_{k<2k_\varepsilon} \{\| \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \|_\infty\} \leq \sup_{k<2k_\varepsilon} \{C \left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-1}\} \,. \tag{240}$$

Hence for (238)

$$\sup_{k<2k_\varepsilon} \{| \widehat{\eta}(\mathbf{k}, j) |\} \leq \sup_{k<2k_\varepsilon} \{C\sqrt{| \mathcal{S}_\varepsilon |} \left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-1}\} \,. \tag{241}$$

This and (237) in (236) yield (233).

Proof of Lemma 8.6 (a) formula (234)

As above and in view of Corollary 8.1 (b)

$$\begin{aligned}
&\| V^\varepsilon_{s_\varepsilon}(u,0)\chi \|_\infty \\
&\leq \| \int (2\pi)^{-\frac{3}{2}} \mid V^\varepsilon_{s_\varepsilon}(u,0)\widehat{\eta}(\mathbf{k}, j)(1-\rho_{k_\varepsilon}(\mathbf{k}))\phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}) \mid d^3k \|_\infty \\
&= \| \int (2\pi)^{-\frac{3}{2}} \mid e^{-\frac{i}{\varepsilon}E_k u}\widehat{\eta}(\mathbf{k}, j)(1-\rho_{k_\varepsilon}(\mathbf{k}))\phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}) \mid d^3k \|_\infty \\
&\leq \sup_{k<2k_\varepsilon} \{| \widehat{\eta}(\mathbf{k}, j) |\} \sup_{k<2k_\varepsilon} \{\| \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \|_\infty\} \int (2\pi)^{-\frac{3}{2}} \mid (1-\rho_{k_\varepsilon}(\mathbf{k})) \mid d^3k \,.
\end{aligned}$$

By (240) and (241) and substituting $p = \frac{k}{k_\varepsilon}$ we can find a $C < \infty$ such that

$$\| V^\varepsilon_{s_\varepsilon}(u,0)\chi \|_\infty \;\; \leq \;\; C \sup_{k<2k_\varepsilon} \{\left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-2}\}\sqrt{| \mathcal{S}_\varepsilon |}k^3_\varepsilon \,. \tag{242}$$

Proof of Lemma 8.6 (b)

Using Corollary 8.1 (b) and (208) we have that

$$\begin{aligned}
V^\varepsilon_{s_\varepsilon}(u,0)\chi(\mathbf{x}) &= V^\varepsilon_{s_\varepsilon}(u,0)\sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}}\phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x})\widehat{\chi}(\mathbf{k}, j)d^3k \\
&= \sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{i}{\varepsilon}uE_k\right)\phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x})\widehat{\chi}(\mathbf{k}, j)d^3k
\end{aligned}$$

We estimate the right hand side via stationary phase method, i.e. we integrate by parts. Using $\frac{i\varepsilon E_k}{ku}\partial_k \exp\left(-\frac{i}{\varepsilon}uE_k\right) = \exp\left(-\frac{i}{\varepsilon}uE_k\right)$ n partial integrations yield - writing

$$\left(\partial_k \frac{E_k}{k}\right)^n := \partial_k \frac{E_k}{k} \partial_k \frac{E_k}{k} \ldots$$

where ∂_k acts on everything to the right -

$$\begin{aligned} V^\varepsilon_{s_\varepsilon}(u,0)\chi(\mathbf{x}) &= (-i\frac{\varepsilon}{u})^n \sum_{j=1}^{4} \int_0^\infty \int (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{i}{\varepsilon} u E_k\right) \\ &\quad \left(\left(\partial_k \frac{E_k}{k}\right)^n \phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}) \widehat{\chi}(\mathbf{k}, j) k^2\right) d\Omega dk \\ &= (-i\frac{\varepsilon}{u})^n \sum_{j=1}^{4} \int k^{-2} (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{i}{\varepsilon} u E_k\right) \\ &\quad \left(\left(\partial_k \frac{E_k}{k}\right)^n \phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}) \widehat{\chi}(\mathbf{k}, j) k^2\right) d^3k \end{aligned}$$

Since $\rho_{k_\varepsilon}(\mathbf{k}) = 0$ for $k \leq k_\varepsilon$ and $k \geq K_\varepsilon$

$$\begin{aligned} &\parallel V^\varepsilon_{s_\varepsilon}(u,0)\chi \parallel_\infty \\ &\leq (\frac{\varepsilon}{u})^n \frac{4}{3} \pi K_\varepsilon^3 \sup_{k > k_\varepsilon} \parallel k^{-2} \left(\left(\partial_k \frac{E_k}{k}\right)^n \sum_{j=1}^{4} \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \widehat{\chi}(\mathbf{k}, j) k^2\right) \parallel_\infty \ . \end{aligned}$$

Since $\widehat{\chi}_\varepsilon \in \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon})\mathcal{R}_{\mathcal{S}_\varepsilon}$ (234) holds, if for any $n \in \mathbb{N}_0$ there exists a $C_n < \infty$ such that

$$\begin{aligned} \sup_{k > k_\varepsilon, \widehat{\eta} \in \mathcal{R}_\mathcal{S}} &\parallel k^{-2} \left(\left(\partial_k \frac{E_k}{k}\right)^n \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon}) \sum_{j=1}^{4} \widehat{\eta}(\mathbf{k}, j) \phi^j(\mathbf{k}, s_\varepsilon, \cdot) k^2\right) \parallel_\infty \\ &\leq C_n \sqrt{\mid \mathcal{S}_\varepsilon \mid} s_\varepsilon^{-3} \left(k_\varepsilon^{-2n} + s_\varepsilon^{-\frac{3}{2}n}\right) \ . \end{aligned} \tag{243}$$

For this we first show that for any $j, l, r \in \mathbb{N}_0$ there exist $C_{j,l,r}$ with

$$\left(\partial_k \frac{E_k}{k}\right)^n k^2 f(k) = \sum_{j+l+r=n} C_{j,l,r} E_k^{n-2r} k^{-n-l+r+2} \partial_k^j f(k) \ . \tag{244}$$

We prove this equation by induction over n. For $n = 0$ (244) follows trivially. Assume that (244) holds for some $n \in \mathbb{N}$. It follows that

$$
\begin{aligned}
\left(\partial_k \frac{E_k}{k}\right)^{n+1} k^2 f(k) &= \partial_k \frac{E_k}{k} \left(\partial_k \frac{E_k}{k}\right)^{n} k^2 f(k) \\
&= \partial_k \frac{E_k}{k} \sum_{j+l+r=n} C_{j,l,r} E_k^{n-2r} k^{-n+2-l+r} \partial_k^j f(k) \\
&= \partial_k \sum_{j+l+r=n} C_{j,l,r} E_k^{n-2r+1} k^{(-n-1+2)-l+r} \partial_k^j f(k) \\
&= \sum_{j+l+r=n} C_{j,l,r} \left(\partial_k E_k^{n-2r+1}\right) k^{(-n-1+2)-l+r} \partial_k^j f(k) \\
&\quad + \sum_{j+l+r=n} C_{j,l,r} E_k^{n-2r+1} \left(\partial_k k^{(-n-1+2)-l+r}\right) \partial_k^j f(k) \\
&\quad + \sum_{j+l+r=n} C_{j,l,r} E_k^{n-2r+1} k^{-n+3-l+r} \partial_k^{j+1} f(k)
\end{aligned}
$$

Using that $E_k = \sqrt{k^2+m^2}$ we have that

$$
\partial_k E_k^n = n E_k^{n-1} \partial_k \sqrt{k^2+m^2} = n E_k^{n-2} k \; .
$$

Setting $\widetilde{n} = n+1$, $\widetilde{j} = j+1$, $\widetilde{l} = l+1$ and $\widetilde{r} = r+1$ yields

$$
\begin{aligned}
\left(\partial_k \frac{E_k}{k}\right)^{n+1} k^2 f(k) &= \sum_{j+l+\widetilde{r}=\widetilde{n}} C_{j,l,r} E_k^{\widetilde{n}-2\widetilde{r}} k^{-\widetilde{n}+2-l+\widetilde{r}} \partial_k^j f(k) \\
&\quad + \sum_{j+l+r=\widetilde{n}} C_{j,l,r} E_k^{\widetilde{n}-2r} k^{-\widetilde{n}+2-\widetilde{l}+r} \partial_k^j f(k) \\
&\quad + \sum_{\widetilde{j}+l+r=\widetilde{n}} C_{\widetilde{j},l,r} E_k^{\widetilde{n}-2r} k^{-\widetilde{n}+2-l+r} \partial_k^{j+1} f(k)
\end{aligned}
$$

for appropriate $C_{\widetilde{j},l,r} < \infty$, $C_{j,\widetilde{l},r} < \infty$ and $C_{j,l,\widetilde{r}} < \infty$, and (244) follows for $\widetilde{n} = n+1$. Induction yields that (244) holds for all $n \in \mathbb{N}_0$.

Note that for $k \to 0$ $k^{-2} E_k^{n-2r} k^{-n+2-l+r}$ is of order k^{-n-l+r}. For $k \to \infty$ E_k is of order k, hence $k^{-2} E_k^{n-2r} k^{-n+2-l+r}$ is of order k^{-l-r} (hence bounded for large k). Since we only observe $k_\varepsilon \to 0$ it follows with (244) that for any $n, j \in \mathbb{N}_0$ there exist $C_{n,j} < \infty$ such that

$$
\mid k^2 \left(\partial_k \frac{E_k}{k}\right)^n k^2 f(k) \mid \;\leq\; \sum_{j=0}^{n} C_{n,j} k^{-2n+j} \mid \partial_k^j f(k) \mid \; . \qquad (245)
$$

Remember that due to Corollary 8.1 (c),

$$\| \partial_k^n \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \|_\infty \leq C_n \left(1 + k^n \left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-n-2}\right) . \tag{246}$$

Next we show that

$$\sup_{k>k_\varepsilon}\{| \partial_k^n \rho_{k_\varepsilon}(\mathbf{k}) |\} \leq C_n k_\varepsilon^n \tag{247}$$

$$\sup_{\widehat{\eta}\in\mathcal{R}_\mathcal{S}} | \partial_k^n \widehat{\eta}(\mathbf{k}, j) | \leq C_n \sqrt{\mathcal{S}_\varepsilon} \left(1 + k^n \left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-n-2}\right) . \tag{248}$$

We start with (247). Using the definition of ρ_{k_ε} (215) and substituting $\mathbf{k} = k_\varepsilon \mathbf{p}$ yields

$$\sup_{k>k_\varepsilon}\{| \partial_k^n \rho_{k_\varepsilon}(\mathbf{k}) |\} \leq k_\varepsilon^{-n} \sup_{\mathbf{p}\in\mathbb{R}^3}\{| \partial_p^n \rho(\mathbf{p}) |\}$$

Since $\rho(p) \in C^\infty$ (247) follows.
it is left to prove (248). Let $\widehat{\eta} \in \mathcal{R}_{\mathcal{S}_\varepsilon}$. Using Corollary 8.1 (b) we have that

$$\begin{aligned} | \partial_k^n \widehat{\eta}(\mathbf{k}, j) | &= \left| \partial_k^n \int (2\pi)^{-\frac{3}{2}} \langle \phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}), \eta(\mathbf{x}) \rangle d^3x \right| \\ &= \left| \int (2\pi)^{-\frac{3}{2}} \langle \partial_k^n \phi^j(\mathbf{k}, s_\varepsilon, \mathbf{x}), \eta(\mathbf{x}) \rangle d^3x \right| \\ &\leq \| \partial_k^n \phi^j(\mathbf{k}, s_\varepsilon, \cdot) \|_\infty \left| \int (2\pi)^{-\frac{3}{2}} | \eta(\mathbf{x}) | d^3x \right| . \end{aligned}$$

Using (239) and (246) (248) follows.
It is left to show how (245) - (248) imply (243).
Using the product rule of differentiation, (246) - (248) yield that for any $n, j \in \mathbb{N}$ there exist $C_{n,j} < \infty$ such that

$$\begin{aligned} &\| \left(\partial_k^n \rho_{k_\varepsilon} \widehat{\eta}(\mathbf{k}, j) \phi^j(\mathbf{k}, s_\varepsilon, \cdot)\right) \|_\infty \\ &\qquad \leq \sum_{l=0}^{n} C_{n,l} \sqrt{| \mathcal{S}_\varepsilon |} \left(1 + k^l \left(| s_\varepsilon - \alpha k^2 | + k^3\right)^{-l-2}\right) k_\varepsilon^{-n+l} . \end{aligned}$$

The sum will now be estimated by n-times the largest summand which however depends on k_ε. Hence there exists for any $n \in \mathbb{N}$ a $C_n < \infty$ such that

$$\sup_{\widehat{\eta}\in\mathcal{R}_{\mathcal{S}}} \| \partial_k^n \rho_{k_\varepsilon}\widehat{\eta}(\mathbf{k},j)\phi^j(\mathbf{k},s_\varepsilon,\cdot) \|_\infty$$
$$\leq C_n\sqrt{\mid \mathcal{S}_\varepsilon \mid}\left(1+\left(k_\varepsilon^{-n}\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-2}\right)+\left(k^n\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-2-n}\right)\right)$$

With (245) it follows that for any $n, j \in \mathbb{N}_0$ there exist $C_{n,j} < \infty$ such that

$$\sup_{k>k_\varepsilon,\widehat{\eta}\in\mathcal{R}_{\mathcal{S}}} \| k^{-2}\left(\partial_k\frac{E_k}{k}\right)^n k^2\rho_{k_\varepsilon}\widehat{\eta}(\mathbf{k},j)\phi^j(\mathbf{k},s_\varepsilon,\cdot)\|_\infty$$
$$\leq \sqrt{\mid \mathcal{S}_\varepsilon \mid}\sum_{j=0}^n C_{n,j}k_\varepsilon^{-2n}\sup_{k>k_\varepsilon}\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-2}$$
$$+\sqrt{\mid \mathcal{S}_\varepsilon \mid}\sum_{j=0}^n C_{n,j}k_\varepsilon^{-2n+2j}\sup_{k>k_\varepsilon}\left(1+\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-2-j}\right) .$$

Note that the supremum $\sup_{k>k_\varepsilon}\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-1}$ is realized at the resonance $\alpha k \approx s_\varepsilon$. Hence there exists a constant $C < \infty$ such that

$$\sup_{k<2k_\varepsilon}\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-1} \leq \sup_{\mathbf{k}\in\mathbb{R}^3}\left(\mid s_\varepsilon-\alpha k^2\mid +k^3\right)^{-1} < Cs_\varepsilon^{-\frac{3}{2}}$$

It follows that

$$\sup_{k>k_\varepsilon,\widehat{\eta}\in\mathcal{R}_{\mathcal{S}}} \| k^{-2}\left(\partial_k\frac{E_k}{k}\right)^n k^2\rho_{k_\varepsilon}\widehat{\eta}(\mathbf{k},j)\phi^j(\mathbf{k},s_\varepsilon,\cdot)\|_\infty$$
$$\leq \sqrt{\mid \mathcal{S}_\varepsilon \mid}\sum_{j=0}^n C_{n,j}\left(1+k_\varepsilon^{-2n}s_\varepsilon^{-3}+k_\varepsilon^{-2n+2j}s_\varepsilon^{-3-\frac{3}{2}j}\right) .$$

As above the sum will be estimated by n-times the largest summand which again depends on k_ε. Using that $1 < s_\varepsilon^{-\frac{3}{2}n}$ (243) follows.

□

Lemma 8.7 *Let $\psi_s^{\varepsilon,1}$ be defined as above (see (218)), $K^\varepsilon = \varepsilon^{-4}$. Then there exists a $C > 0$ such that*

$$\| \mathcal{F}_w(\psi_s^{\varepsilon,1})\rho_{K_\varepsilon} \| < C\varepsilon^2$$

and

$$\| \mathcal{F}_w(A\psi_s^{\varepsilon,1})\rho_{K_\varepsilon} \| < C\varepsilon^2$$

for all $w, s > 0$.

Proof of Lemma 8.7

We first show, that the energy of $\psi_s^{\varepsilon,1}$ is bounded uniform in $\varepsilon > 0$ and $s > 0$. Let B be either 1 or A. We have that

$$\begin{aligned}
\partial_s \langle B\psi_s^{\varepsilon,1}, D_s B\psi_s^{\varepsilon,1}\rangle &= \langle B\left(\partial_s\psi_s^{\varepsilon,1}\right), D_s B\psi_s^{\varepsilon,1}\rangle + \langle B\psi_s^{\varepsilon,1}, \left(\partial_s D_s\right) B\psi_s^{\varepsilon,1}\rangle \\
&\quad + \langle B\psi_s^{\varepsilon,1}, D_s B\partial_s\psi_s^{\varepsilon,1}\rangle \\
&= \langle B\frac{i}{\varepsilon}D_s\psi_s^{\varepsilon,1}, D_s B\psi_s^{\varepsilon,1}\rangle + \langle B\psi_s^{\varepsilon,1}, \left(\partial_s D_s\right) B\psi_s^{\varepsilon,1}\rangle \\
&\quad + \langle B\psi_s^{\varepsilon,1}, D_s B\frac{i}{\varepsilon}D_s\psi_s^{\varepsilon,1}\rangle \\
&= \langle B\psi_s^{\varepsilon,1}, \left(\partial_s D_s\right) B\psi_s^{\varepsilon,1}\rangle \\
&= \langle B\psi_s^{\varepsilon,1}\left(\partial_s\varphi(s)A\right), B\psi_s^{\varepsilon,1}\rangle \\
&\leq C \parallel A \parallel_\infty^3 \partial_s\varphi(s)
\end{aligned}$$

Hence observing $\psi_0^{\varepsilon,1} = \phi^\varepsilon$ and $D_0\phi_0 = m\phi_0$

$$\begin{aligned}
\langle B\psi_s^{\varepsilon,1}, D_s B\psi_s^{\varepsilon,1}\rangle &\leq C \parallel A \parallel_\infty^3 \left(\varphi(s) - \varphi(0)\right) + \langle B\phi^\varepsilon, D_0 B\phi^\varepsilon\rangle \\
&\leq C \parallel A \parallel_\infty^3 \left(\varphi(s) - \varphi(0)\right) + \langle B\phi^\varepsilon, D_0 B(1 - \rho_{\varepsilon\frac{1}{100}})\phi_0\rangle \\
&= C \parallel A \parallel_\infty^3 \left(\varphi(s) - \varphi(0)\right) + \langle B\phi^\varepsilon, B(1 - \rho_{\varepsilon\frac{1}{100}})D_0\phi_0\rangle \\
&\quad + \langle B\phi^\varepsilon, \nabla\!\!\!/ \left(B(1 - \rho_{\varepsilon\frac{1}{100}})\right)\phi_0\rangle \\
&= C \parallel A \parallel_\infty^3 \left(\varphi(s) - \varphi(0)\right) + m \parallel A \parallel_\infty^2 \langle\phi^\varepsilon, \phi^\varepsilon\rangle \\
&\quad + \langle B\phi^\varepsilon, \nabla\!\!\!/ \left(B(1 - \rho_{\varepsilon\frac{1}{100}})\right)\phi_0\rangle\ .
\end{aligned}$$

Since $D_s - D_w = (\varphi(s) - \varphi(w))A$ it follows that

$$\begin{aligned}
\mid \langle B\psi_s^{\varepsilon,1}, D_w B\psi_s^{\varepsilon,1}\rangle \mid &\leq \mid \langle B\psi_s^{\varepsilon,1}(\varphi(s) - \varphi(w))A, B\psi_s^{\varepsilon,1}\rangle \mid + \parallel A \parallel_\infty^3 \left(\varphi(s) - \varphi(0)\right) \\
&\quad + m \parallel A \parallel_\infty^2 \langle\phi^\varepsilon, \phi^\varepsilon\rangle + \langle B\phi^\varepsilon, \nabla\!\!\!/ \left(B(1 - \rho_{\varepsilon\frac{1}{100}})\right)\phi_0\rangle\ .
\end{aligned}$$

Noting that

$$\parallel \nabla\!\!\!/(1 - \rho_{\varepsilon\frac{1}{100}}) \parallel_\infty = \varepsilon^{\frac{1}{100}} \parallel \nabla\!\!\!/(1 - \rho) \parallel_\infty\ ,$$

and that $\parallel \nabla\!\!\!/ A \parallel_\infty < \infty$ we have that

$$\mid \langle B\psi_s^{\varepsilon,1}D_w, B\psi_s^{\varepsilon,1}\rangle \mid < C \ \text{ for } \varepsilon < 1 \ \text{ and for all } s, w\ .$$

We calculate the scalar product on the left hand side in generalized Fourier space using that $E_k = \sqrt{m^2 + k^2} > k$

$$
\begin{aligned}
C > \langle B\psi_s^{\varepsilon,1} D_w, B\psi_s^{\varepsilon,1} \rangle &= \int \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) E_k \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) d^3k \\
&\geq \int \rho_{K_\varepsilon}(\mathbf{k}) \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) E_k \rho_{K_\varepsilon}(\mathbf{k}) \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) d^3k \\
&\geq K_\varepsilon \int \rho_{K_\varepsilon}(\mathbf{k}) \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) \rho_{K_\varepsilon}(\mathbf{k}) \mathcal{F}_w(B\psi_s^{\varepsilon,1})(\mathbf{k}, j) d^3k \\
&= K_\varepsilon \parallel \rho_{K_\varepsilon} \mathcal{F}_w(B\psi_s^{\varepsilon,1}) \parallel^2 \ .
\end{aligned}
$$

Hence $\varepsilon^{-2} \parallel \rho_{K_\varepsilon} \mathcal{F}_w(B\psi_s^{\varepsilon,1}) \parallel$ is bounded and Lemma 8.7 follows.

□

We shall now provide an estimate, how long the wave function $\psi_s^{\varepsilon,1}$ will stay in the range of the potential. That time is roughly of the order of $s = \varepsilon^{\frac{1}{3}-\delta}$ (see below). For times larger than $s = \varepsilon^{\frac{1}{3}-\delta}$ the part of the wave function which is affected by the potential goes to zero with $\varepsilon \to 0$. Note that we establish the estimate (249) only for times $s < \widetilde{s}$. But this suffices already to establish the main result since for times larger than $\widetilde{s}$ the potential has no more influence on the motion of the wave function, i.e. it evolves freely and thus behaves like a scattering state going off to infinity.

Lemma 8.8 *Let $\psi_s^{\varepsilon,1}$ be given by (214), $K^\varepsilon = \varepsilon^{-4}$, $\widetilde{s}$ as in Corollary 8.1. Then there exists a $C > 0$ such that*

$$
\parallel 1_{\mathcal{S}_A} \psi_s^{\varepsilon,1} \parallel \ \leq \ \min\{1, C s^{-\frac{5}{2}} \varepsilon^{\frac{5}{6} - \frac{1}{1000}}\} \tag{249}
$$

for all $0 \leq s \leq \widetilde{s}$.

Proof of Lemma 8.8

We shall use that

$$
\begin{aligned}
\parallel 1_{\mathcal{S}_A} \chi \parallel &\leq \parallel 1_{\mathcal{S}_A} \parallel = \sqrt{\mid \mathcal{S}_A \mid} \ \parallel \chi \parallel_\infty \qquad \text{for } \chi \in L^\infty & (250) \\
\parallel 1_{\mathcal{S}_A} \chi \parallel &\leq \parallel 1_{\mathcal{S}_A} \parallel_\infty \ \parallel \chi \parallel = \parallel \chi \parallel \quad \text{for } \chi \in L^2 \ . & (251)
\end{aligned}
$$

By (213), (214) and the unitarity of U^ε - that

$$\parallel 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \parallel \leq \parallel \psi_s^{\varepsilon,1} \parallel \leq 1 \ . \tag{252}$$

Since for $s < 2\varepsilon^{\frac{1}{3}}$ and ε small enough $s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} \gg 1$ the difficult part is to show that (249) holds for $s \geq 2\varepsilon^{\frac{1}{3}}$.

Let $\widetilde{s} \geq s \geq 2\varepsilon^{\frac{1}{3}}$. Using (208) with $s_\varepsilon = s$ and

$$D_w - D_s = D_0 + \varphi(w)A - (D_0 + \varphi(s)) = (\varphi(w) - \varphi(s)) \tag{253}$$

we have that

$$\begin{aligned} 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} &= 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)\phi^\varepsilon \\ &\quad + i1_{\mathcal{S}_A}\varepsilon^{-1}\int_0^s V_s^\varepsilon(s,w)(\varphi(w)-\varphi(s))A\psi_w^{\varepsilon,1}(\mathbf{x})dw \ . \end{aligned} \tag{254}$$

Now

$$\begin{aligned} \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)\phi^\varepsilon \parallel &\leq \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)(1-\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel \\ &\quad + \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)(1-\rho_{K_\varepsilon})\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel \\ &\quad + \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)\rho_{K_\varepsilon}\mathcal{F}_s(\phi^\varepsilon \parallel) \end{aligned}$$

We use (250) and (251),the unitarity of V_s^ε and the isometry in ordinary and generalized momentum space (see Corollary 8.1 (b)) to obtain

$$\begin{aligned} \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)\phi^\varepsilon \parallel &\leq C \parallel V_s^\varepsilon(s,0)(1-\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel_\infty \\ &\quad + C \parallel V_s^\varepsilon(s,0)(1-\rho_{K_\varepsilon}\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel_\infty + \parallel \rho_{K_\varepsilon}\mathcal{F}_s(\phi^\varepsilon) \parallel \\ &= C \parallel V_s^\varepsilon(s,0)(1-\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel_\infty \\ &\quad + C \parallel V_s^\varepsilon(s,0)(1-\rho_{K_\varepsilon})\rho_{\kappa_{\varepsilon,s}}\mathcal{F}_s(\phi^\varepsilon) \parallel_\infty + \parallel \rho_{K_\varepsilon}\mathcal{F}_s(\phi^\varepsilon) \parallel \ . \end{aligned}$$

Now we use Lemma 8.6 (a) (setting $s_\varepsilon = s$, $k_\varepsilon = \kappa_{\varepsilon,s}$, $u = s$ and $\widehat{\chi} = (1 - \rho_{k_\varepsilon})\frac{\mathcal{F}_s(\phi^\varepsilon)}{\|\phi^\varepsilon\|}$) on the first, Lemma 8.6 (b) (setting $s_\varepsilon = s$, $k_\varepsilon = \kappa_{\varepsilon,s}$, $u = s$ and $\widehat{\chi} = \rho_{k_\varepsilon}(\mathbf{k}))(1-\rho_{K_\varepsilon}(\mathbf{k}))\frac{\phi^\varepsilon}{\|\phi^\varepsilon\|}$) on the second and Lemma 8.7 on the third summand. Hence there exists for any $n \in \mathbb{N}$ a $C_n < \infty$ and a $C < \infty$ such that

$$\begin{aligned} \parallel 1_{\mathcal{S}_A}V_s^\varepsilon(s,0)\phi^\varepsilon \parallel &\leq C \sup_{k<2\kappa_{\varepsilon,s}} \left((\mid s - \alpha k^2 \mid + k^3)^{-2} \right) \sqrt{\mid \mathcal{T}_\varepsilon \mid}\kappa_{\varepsilon,s}^3 \\ &\quad + C_n K_\varepsilon^3 \sqrt{\mid \mathcal{T}_\varepsilon \mid}\frac{\varepsilon^n}{s^n}s^{-3}\left(\kappa_{\varepsilon,s}^{-2n} + s^{-\frac{3}{2}n}\right) + C\varepsilon^2 \ , \end{aligned}$$

where we recall (c.f.(213)) that $\mathcal{T}_\varepsilon$ is the support of $\phi\varepsilon$.
We now choose

$$\kappa_{\varepsilon,s} = \varepsilon^{\frac{4999}{10000}} s^{-\frac{1}{2}} \ . \tag{255}$$

so that

$$\frac{\varepsilon}{s\kappa_{\varepsilon,s}^2} \quad = \quad \varepsilon^{\frac{2}{10000}} \tag{256}$$

and for $s > 2\varepsilon^{\frac{1}{3}}$

$$\frac{\varepsilon}{s} s^{-\frac{3}{2}} \quad = \quad \varepsilon s^{-\frac{5}{2}} < 2^{-\frac{5}{2}} \varepsilon^{\frac{1}{6}} < \varepsilon^{\frac{2}{10000}} \tag{257}$$

$$\kappa_{\varepsilon,s}^2 \quad = \quad \varepsilon^{\frac{9998}{10000}} s^{-1} \ll s \ . \tag{258}$$

Hence there exists a $C < \infty$ such that

$$\inf_{k < 2\kappa_{\varepsilon,s}^2} \left(\mid s - \alpha\kappa_{\varepsilon,s}^2 \mid + s^{\frac{3}{2}} \right) \quad > \quad Cs \tag{259}$$

It follows that there exists for any $n \in \mathbb{N}$ a $C_n < \infty$ and a $C < \infty$ such that

$$\| 1_{\mathcal{S}_A} V_s^\varepsilon(s,0)\phi^\varepsilon \| \quad \leq \quad Cs^{-2}\sqrt{\mid \mathcal{T}_\varepsilon \mid} \kappa_{\varepsilon,s}^3 + C_n K_\varepsilon^3 \sqrt{\mid \mathcal{T}_\varepsilon \mid} s^{-3} \varepsilon^{\frac{2}{10000}n} + C\varepsilon^2 \ .$$

Choosing n large enough the second term decays faster than any polynomial in ε. Using (216) noting that $\sqrt{\mid T_\varepsilon \mid} < \mid T_\varepsilon \mid$ for small enough ε we can find a $C < \infty$ such that

$$\| 1_{\mathcal{S}_A} V_s^\varepsilon(s,0)\phi^\varepsilon \| \leq Cs^{-\frac{7}{2}} \varepsilon^{-\frac{3}{10000}} \varepsilon^{\frac{14997}{10000}} = Cs^{-\frac{7}{2}} \varepsilon^{\frac{14994}{10000}} \tag{260}$$

Next we estimate the second summand in (254). Below we will introduce the $\kappa_{\varepsilon,s-w}$-cutoff. For $w \to s$ $\kappa_{\varepsilon,s-w}$ goes to infinity. So to keep the $\kappa_{\varepsilon,s-w}$ cutoff small we use the above estimate only for sufficiently large $s-w$ and handle $s-w < \sigma_\varepsilon$ for some σ_ε which will be specified later separately. Hence we split

$$\begin{aligned} & 1_{\mathcal{S}_A} \int_0^s V_s^\varepsilon(s,w)(\varphi(w) - \varphi(s))\varepsilon^{-1} A\psi_w^{\varepsilon,1}(\mathbf{x})dw \\ & \leq \varepsilon^{-1} \int_{\sigma_\varepsilon}^s \| 1_{\mathcal{S}_A} V_s^\varepsilon(s,w)(\varphi(w) - \varphi(s)) A\psi_w^{\varepsilon,1} \| \, dw \\ & +\varepsilon^{-1} \int_0^{\sigma_\varepsilon} \| 1_{\mathcal{S}_A} V_s^\varepsilon(s,w)(\varphi(w) - \varphi(s)) A\psi_w^{\varepsilon,1} \| \, dw \\ & =: S_1 + S_2 \ . \end{aligned} \tag{261}$$

For S_1 we have using (251), (6) and (252)

$$\begin{aligned} S_1 &\leq \int_{\sigma_\varepsilon}^{s} \| V_s^\varepsilon(s,w)\varepsilon^{-1}(\varphi(w)-\varphi(s))A\psi_w^{\varepsilon,1} \| \, dw \\ &= \varepsilon^{-1}\int_{\sigma_\varepsilon}^{s} \| (\varphi(w)-\varphi(s))A\psi_w^{\varepsilon,1} \| \, dw \\ &\leq \varepsilon^{-1}\int_{\sigma_\varepsilon}^{s} \| A \|_\infty \| (\varphi(w)-\varphi(s))1_{\mathcal{S}_A}\psi_w^{\varepsilon,1} \| \, dw \\ &\leq C\varepsilon^{-1}\int_{\sigma_\varepsilon}^{s} (s-w) \| 1_{\mathcal{S}_A}\psi_w^{\varepsilon,1} \| \, dw \; . \\ &\leq C\varepsilon^{-1}\int_{\sigma_\varepsilon}^{s}(s-w)dw = \frac{C}{2}\varepsilon^{-1}(s-\sigma_\varepsilon)^2 \end{aligned} \tag{262}$$

$$\begin{aligned} S_2 &\leq \varepsilon^{-1}\int_0^{\sigma_\varepsilon} \| 1_{\mathcal{S}_A}V_s^\varepsilon(s,w)(1-\rho_{K_\varepsilon})\rho_{\kappa_{\varepsilon,s-w}}(\varphi(w)-\varphi(s))\mathcal{F}_s(A\psi_w^{\varepsilon,1}) \| \, dw \\ &\quad +\varepsilon^{-1}\textstyle\int_0^{\sigma_\varepsilon} \| 1_{\mathcal{S}_A}V_s^\varepsilon(s,w)\rho_{K_\varepsilon}(\varphi(w)-\varphi(s))\mathcal{F}_s(A\psi_w^{\varepsilon,1}) \| \, dw \\ &\quad +\varepsilon^{-1}\int_0^{\sigma_\varepsilon} \| 1_{\mathcal{S}_A}V_s^\varepsilon(s,w)(1-\rho_{\kappa_{\varepsilon,s-w}}(\varphi(w)-\varphi(s))\mathcal{F}_s(A\psi_w^{\varepsilon,1}) \| \, dw \\ &=: S_3+S_4+S_5 \; . \end{aligned} \tag{263}$$

For S_3 we have by (250)

$$S_3 \leq \sqrt{| \mathcal{S}_A |}\varepsilon^{-1}\int_0^{\sigma_\varepsilon} \| V_s^\varepsilon(s,w)(1-\rho_{K_\varepsilon})\rho_{\kappa_{\varepsilon,s-w}}(\varphi(w)-\varphi(s))\mathcal{F}_s(A\psi_w^{\varepsilon,1}) \|_\infty \, dw \; .$$

Applying Lemma 8.6 (b) (choosing $k_\varepsilon = \kappa_{\varepsilon,s-w}$, $s_\varepsilon = s$, $u = s-w$ and $\widehat{\chi} = (1-\rho_{K_\varepsilon})\rho_{\kappa_{\varepsilon,s-w}}\frac{\mathcal{F}_s(A\psi_w^{\varepsilon,1})}{\|A\psi_w^{\varepsilon,1}\|}$) there exists a $C<\infty$ so that

$$\begin{aligned} S_3 &\leq C\varepsilon^{-1}\int_0^{\sigma_\varepsilon}(\varphi(w)-\varphi(s))C_nK_\varepsilon^3\sqrt{| \mathcal{S}_A |}\frac{\varepsilon^n}{(s-w)^n}s^{-3} \\ &\qquad \left(\kappa_{\varepsilon,s-w}^{-2n}+s^{-\frac{3}{2}n}\right) \| A\psi_w^{\varepsilon,1} \| \, dw \; . \end{aligned}$$

Now we choose σ_ε such that this integral decays faster than any polynomial in ε, i.e. setting

$$\sigma_\varepsilon := s-\varepsilon^{\frac{9998}{10000}}s^{-\frac{3}{2}} \; , \tag{264}$$

we get for $w \leq \sigma_\varepsilon$

$$s-w \geq \varepsilon^{\frac{9998}{10000}} s^{-\frac{3}{2}} \quad \Rightarrow \quad s^{-\frac{3}{2}} \leq \frac{s-w}{\varepsilon^{\frac{9998}{10000}}} = \kappa_{\varepsilon,s-w}^{-2} \; . \tag{265}$$

So there exists a $C<\infty$ such that

$$S_3 \;\; \leq \;\; C\varepsilon^{-1} \int_0^{\sigma_\varepsilon} (\varphi(w)-\varphi(s)) C_n K_\varepsilon^3 \sqrt{\mid \mathcal{S}_A \mid} \left(\frac{\varepsilon}{(s-w)\kappa_{\varepsilon,s-w}^2} \right)^n s^{-3} \; .$$

Since $\frac{\varepsilon}{(s-w)\kappa_{\varepsilon,s-w}^2} = \varepsilon^{\frac{2}{10000}}$ (see (256)) it follows that the integrand of S_3 decays faster than any polynomial in ε and all $w \leq \sigma_\varepsilon$, hence there exists a $C<\infty$ such that

$$S_3 \leq C\varepsilon \; . \tag{266}$$

Furthermore (264) yields that

$$S_1 \leq \frac{C}{2} \varepsilon^{\frac{9996}{10000}} s^{-3} \tag{267}$$

Next we estimate S_4 (see (263)). Using (250) and unitarity of V_s^ε we have

$$\begin{aligned} S_4 \;\; &\leq \;\; \varepsilon^{-1} \int_0^{\sigma_\varepsilon} \parallel V_s^\varepsilon(s,w) \rho_{K_\varepsilon}(\varphi(w)-\varphi(s)) \mathcal{F}_s(A\psi_w^{\varepsilon,1}) \parallel dw \\ &\leq \;\; \varepsilon^{-1} \int_0^{\sigma_\varepsilon} \parallel \rho_{K_\varepsilon}(\varphi(w)-\varphi(s)) \mathcal{F}_s(A\psi_w^{\varepsilon,1}) \parallel dw \end{aligned}$$

Lemma 8.7 yields that there exists a $C<\infty$ such that

$$\begin{aligned} S_4 \;\; &\leq \;\; C\varepsilon^{-1} \int_0^{\sigma_\varepsilon} (\varphi(s)-\varphi(w)) \varepsilon^2 dw \\ &\leq \;\; C\varepsilon \; . \end{aligned} \tag{268}$$

For S_5 by (250)

$$S_5 \;\; \leq \;\; \sqrt{\mid \mathcal{S}_A \mid} \varepsilon^{-1} \int_0^{\sigma_\varepsilon} \parallel V_s^\varepsilon(s,w)(1-\rho_{\kappa_{\varepsilon,s-w}})(\varphi(w)-\varphi(s)) \mathcal{F}_s(A\psi_w^{\varepsilon,1}) \parallel_\infty dw \; ,$$

and using (6) ($\mid \partial_s \varphi(s) \mid < C$, hence $\mid \varphi(w)-\varphi(s) \mid < C(w-s)$)

$$S_5 \;\; \leq \;\; C\varepsilon^{-1} \int_0^{\sigma_\varepsilon} (s-w) \parallel V_s^\varepsilon(s,w)(1-\rho_{\kappa_{\varepsilon,s-w}}) \mathcal{F}_s(A\psi_w^{\varepsilon,1}) \parallel_\infty dw \; .$$

Applying Lemma 8.6 (a) (with $s_\varepsilon = s$, $k_\varepsilon = \kappa_{\varepsilon,s}$, $u = s - w$ and $\widehat{\chi} = (1 - \rho_{\kappa_{\varepsilon,s-w}} \frac{\mathcal{F}_s(A\psi_w^{\varepsilon,1})}{\|A\psi_w^{\varepsilon,1}\|})$, there exists a $C < \infty$, such that

$$\begin{aligned} S_5 &\leq C\varepsilon^{-1} \int_0^{\sigma_\varepsilon} (s-w) \parallel A\psi_w^{\varepsilon,1} \parallel \\ & \sup_{k<2\kappa_{\varepsilon,s-w}} \left(\left(\mid s - \alpha k^2 \mid + k^3 \right)^{-2} \right) \sqrt{\mid \mathcal{S}_A \mid} \kappa_{\varepsilon,s-w}^3 dw \end{aligned}$$

With (256) it follows that

$$\kappa_{\varepsilon,s-w}^2 \ll s - w < s \ ,$$

hence there exists a $C < \infty$ such that

$$\inf_{k<2\kappa_{\varepsilon,s-w}^2} \left(\mid s - \alpha\kappa_{\varepsilon,s}^2 \mid + s^{\frac{3}{2}} \right) > Cs \ .$$

This and (253) yield that

$$\begin{aligned} S_5 &\leq C\varepsilon^{-1} \int_0^{\sigma_\varepsilon} (s-w) \parallel A\psi_w^{\varepsilon,1} \parallel s^{-2} \kappa_{\varepsilon,s-w}^3 dw \\ &= C \int_0^{\sigma_\varepsilon} \parallel A\psi_w^{\varepsilon,1} \parallel s^{-2} \varepsilon^{\frac{4997}{10000}} (s-w)^{-\frac{1}{2}} dw \\ &\leq C \int_0^{\sigma_\varepsilon} \parallel 1_{\mathcal{S}_A} \psi_w^{\varepsilon,1} \parallel s^{-2} \varepsilon^{\frac{4997}{10000}} (s-w)^{-\frac{1}{2}} dw \end{aligned}$$

Summarizing (254), (260) - (263), (267) - (268) we get that there exists a $C < \infty$ such that for all $2\varepsilon^{\frac{1}{3}} \leq s \leq \widetilde{s}$

$$\begin{aligned} \parallel 1_{\mathcal{S}_A} \psi_s^{\varepsilon,1} \parallel &\leq Cs^{-\frac{7}{2}} \varepsilon^{\frac{14994}{10000}} + C\varepsilon^{\frac{9996}{10000}} s^{-3} \\ & + C \int_0^{\sigma_\varepsilon} \parallel 1_{\mathcal{S}_A} \psi_w^{\varepsilon,1} \parallel s^{-2} \varepsilon^{\frac{4997}{10000}} (s-w)^{-\frac{1}{2}} dw \ . \end{aligned}$$

For $s \geq 2\varepsilon^{\frac{1}{3}}$ we have that $s^{-\frac{7}{2}} \varepsilon^{\frac{14994}{10000}} = (s^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}}) s^{-3} \varepsilon^{\frac{9994}{10000}} \ll s^{-3} \varepsilon^{\frac{9994}{10000}}$. Hence there exists a $C < \infty$ such that

$$\parallel 1_{\mathcal{S}_A} \psi_s^{\varepsilon,1} \parallel \leq Cs^{-3} \varepsilon^{\frac{9994}{10000}} + C \int_0^{s} \parallel 1_{\mathcal{S}_A} \psi_w^{\varepsilon,1} \parallel s^{-2} \varepsilon^{\frac{4997}{10000}} (s-w)^{-\frac{1}{2}} dw \ . \tag{269}$$

Furthermore $\parallel \psi_w^{\varepsilon,1} \parallel \leq 1$, so

$$\begin{aligned}
\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| &\leq C\varepsilon^{\frac{9994}{10000}}s^{-3} + C\int_0^s s^{-2}\varepsilon^{\frac{4997}{10000}}(s-w)^{-\frac{1}{2}}dw \\
&= C\varepsilon^{\frac{9994}{10000}}s^{-3} + 2Cs^{-2}\varepsilon^{\frac{4997}{10000}}[(s-w)^{\frac{1}{2}}]_0^s \\
&= C\varepsilon^{\frac{9994}{10000}}s^{-3} + 2Cs^{-\frac{3}{2}}\varepsilon^{\frac{4997}{10000}}
\end{aligned}$$

Observing that $s > 2\varepsilon^{\frac{1}{3}}$ we have

$$\varepsilon^{\frac{9994}{10000}}s^{-3} = s^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}s^{-\frac{3}{2}}\varepsilon^{\frac{4994}{10000}} < s^{-\frac{3}{2}}\varepsilon^{\frac{4994}{10000}}$$

and hence

$$\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| \leq Cs^{-\frac{3}{2}}\varepsilon^{\frac{4994}{10000}} \tag{270}$$

Once more using (269) yields that

$$\begin{aligned}
\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| \leq\; & Cs^{-3}\varepsilon^{\frac{9994}{10000}} + C\int_0^{\varepsilon^{\frac{1}{3}}} \| 1_{\mathcal{S}_A}\psi_w^{\varepsilon,1} \| s^{-2}\varepsilon^{\frac{4997}{10000}}(s-w)^{-\frac{1}{2}}dw \\
& + C\int_{\varepsilon^{\frac{1}{3}}}^s \| 1_{\mathcal{S}_A}\psi_w^{\varepsilon,1} \| s^{-2}\varepsilon^{\frac{4997}{10000}}(s-w)^{-\frac{1}{2}}dw \; .
\end{aligned}$$

Inserting (252) into the second, (270) into the third summand yields

$$\begin{aligned}
\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| \leq\; & C\varepsilon^{\frac{9994}{10000}}s^{-3} + C\int_0^{\varepsilon^{\frac{1}{3}}} s^{-2}\varepsilon^{\frac{4997}{10000}}(s-w)^{-\frac{1}{2}}dw \\
& + C\int_{\varepsilon^{\frac{1}{3}}}^s s^{-2}\varepsilon^{\frac{4994}{10000}}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}\varepsilon^{\frac{4997}{10000}}dw
\end{aligned}$$

Note that $w \leq \varepsilon^{\frac{1}{3}}$ and $s \geq 2\varepsilon^{\frac{1}{3}}$, hence $s - w \geq \frac{s}{2}$. Hence

$$\begin{aligned}
\int_0^{\varepsilon^{\frac{1}{3}}} s^{-2}\varepsilon^{\frac{4997}{10000}}(s-w)^{-\frac{1}{2}}dw &\leq 2^{-\frac{1}{2}}s^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}}\int_0^{\varepsilon^{\frac{1}{3}}} dw \\
&= 2^{-\frac{1}{2}}s^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}+\frac{1}{3}} \; .
\end{aligned}$$

It follows that

$$
\begin{aligned}
\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| &\leq C\varepsilon^{\frac{9994}{10000}}s^{-3} + Cs^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}+\frac{1}{3}} + Cs^{-2}\varepsilon^{\frac{9991}{10000}}\int_{\varepsilon^{\frac{1}{3}}}^{s}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}dw \\
&= C\varepsilon^{\frac{9994}{10000}}s^{-3} + Cs^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}+\frac{1}{3}} + Cs^{-2}\varepsilon^{\frac{9991}{10000}}\int_{\varepsilon^{\frac{1}{3}}}^{\frac{s}{2}}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}dw \\
&\quad + Cs^{-2}\varepsilon^{\frac{9991}{10000}}\int_{\frac{s}{2}}^{s}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}dw
\end{aligned}
$$

Using that

$$
\int_{\varepsilon^{\frac{1}{3}}}^{\frac{s}{2}}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}dw \leq \sqrt{2}s^{-\frac{1}{2}}\int_{\varepsilon^{\frac{1}{3}}}^{\frac{s}{2}} w^{-\frac{3}{2}}dw \leq 2^{-\frac{1}{2}}s^{-\frac{1}{2}}\varepsilon^{-\frac{1}{6}}
$$

and

$$
\int_{\frac{s}{2}}^{s}(s-w)^{-\frac{1}{2}}w^{-\frac{3}{2}}dw \leq (\frac{s}{2})^{-\frac{3}{2}}\int_{\frac{s}{2}}^{s}(s-w)^{-\frac{1}{2}}dw \leq 2^{\frac{1}{2}}s^{-1}
$$

it follows that there exists a $C < \infty$ such that

$$
\begin{aligned}
\| 1_{\mathcal{S}_A}\psi_s^{\varepsilon,1} \| &\leq C\varepsilon^{\frac{9994}{10000}}s^{-3} + Cs^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}+\frac{1}{3}} + Cs^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{9}{10000}} \\
&\quad + Cs^{-3}\varepsilon^{1-\frac{9}{10000}}
\end{aligned}
$$

For $s > \varepsilon^{\frac{1}{3}}$ we have that

$$
\begin{aligned}
\varepsilon^{\frac{9994}{10000}}s^{-3} &= s^{-\frac{1}{2}}\varepsilon^{\frac{1}{6}+\frac{4}{10000}}s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} < s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} \\
s^{-\frac{5}{2}}\varepsilon^{\frac{4997}{10000}+\frac{1}{3}} &= \varepsilon^{\frac{7}{10000}}s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} < s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} \\
s^{-3}\varepsilon^{1-\frac{9}{10000}} &= s^{-\frac{1}{2}}\varepsilon^{\frac{1}{6}}s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}} < s^{-\frac{5}{2}}\varepsilon^{\frac{5}{6}-\frac{1}{1000}}
\end{aligned}
$$

and (249) and thus Lemma 8.8 follows.

□

Proof of Lemma 8.5 (a)

We need to show that

$$\lim_{\varepsilon\to 0}\langle \psi^{\varepsilon,3}_{s_{m2}}, \widetilde{\phi}_0\rangle = 0 \ .$$

Using (227) and (224) we have that

$$\psi^{\varepsilon,3}_{s_{m2}} = V_{s_\varepsilon}(s_{m2}, s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon} \ = \ V^{\varepsilon}_{s_\varepsilon}(s_{m2}, 0)\phi^{\varepsilon,1} + i\int_0^{s_\varepsilon} V^{\varepsilon}_{s_\varepsilon}(s_{m2}, w)\zeta^{\varepsilon,1}_w(\mathbf{x})dw \ .$$

Note that by construction $\widehat{\zeta}^{\varepsilon,1}_w(\mathbf{k}, j) \in \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon})\mathcal{R}_{\mathcal{S}_A}$ (see (220)) and $\widehat{\phi}^{\varepsilon,1} = \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon})\mathcal{F}_s(\phi^\varepsilon)$ (see (223)). Hence by Lemma 8.6 (c) (with $\chi = \frac{\zeta^{\varepsilon,2}_w(\mathbf{x})}{\|\zeta^{\varepsilon,2}_w(\mathbf{x})\|}$ and $\chi =\| \phi^\varepsilon \|$) we get as above that

$$\| \psi^{\varepsilon,3}_{s_{m2}} \|_\infty \ \leq \ C_n K^3_\varepsilon \sqrt{| \mathcal{T}_\varepsilon |}\frac{\varepsilon^n}{s^n_{m2}} s^3_\varepsilon \left(k^{-2n}_\varepsilon + s_\varepsilon^{-\frac{3}{2}n}\right) \qquad (271)$$

$$+ \int_0^{s_\varepsilon} C_n K^3_\varepsilon \sqrt{| \mathcal{S}_A |}\frac{\varepsilon^n}{(s_{m2}-w)^n} s^3_\varepsilon \left(k^{-2n}_\varepsilon + s_\varepsilon^{-\frac{3}{2}n}\right) dw \qquad (272)$$

Furthermore since $k_\varepsilon = \varepsilon^{\frac{4}{9}+\frac{1}{1000}}$ and $s_\varepsilon = \varepsilon^{\frac{1}{2000}}$ (c.f. Lemma 8.5)

$$\frac{\varepsilon}{s_{m2}-s_\varepsilon}\left(k^{-2}_\varepsilon + s_\varepsilon^{-\frac{3}{2}}\right) < C\varepsilon^{\frac{1}{9}-\frac{2}{1000}} \ .$$

By choosing n large enough

$$\lim_{\varepsilon\to 0} \| \psi^{\varepsilon,3}_{s_{m2}} \|_\infty = 0 \ ,$$

and hence

$$\lim_{\varepsilon\to 0}\langle \psi^{\varepsilon,3}_{s_{m2}}, \widetilde{\phi}_0\rangle = 0 \ .$$

Proof of Lemma 8.5 (b) for $l = 1$

Using unitarity of U^ε we get in view of (214), (203) and (215)

$$\begin{aligned}\lim_{\varepsilon\to 0} \| \eta^{\varepsilon,1}_{sm2} \| \ &= \ \lim_{\varepsilon\to 0} \| \psi^{\varepsilon}_{s_{m2}} - \psi^{\varepsilon,1}_{s_{m2}} \| \\ &= \ \lim_{\varepsilon\to 0} \| \phi^\varepsilon - \phi_0 \| \ .\end{aligned}$$

With (213), (221) and (211) it follows that since $\phi_0 \in L^2$

$$\begin{aligned}\lim_{\varepsilon\to 0} \parallel \eta^{\varepsilon,1}_{sm2} \parallel &= \lim_{\varepsilon\to 0} \parallel \phi_0 \rho_{\varepsilon^{-\frac{1}{10000}}} \parallel \\ &\le \lim_{\varepsilon\to 0} \parallel \phi_0 (1 - 1_{\varepsilon^{-\frac{1}{10000}}}) \parallel = 0 \ .\end{aligned}$$

Proof of Lemma 8.5 (b) for $l = 2$

Using (226) and the unitarity of the propagators U^ε and V_{s_ε} we get that

$$\parallel \eta^{\varepsilon,2}_{s_{m2}} \parallel \ \le \ \parallel \left(\phi^\varepsilon - \phi^{\varepsilon,1}\right) \parallel + \int_0^{s_\varepsilon} \parallel \zeta^\varepsilon_w - \zeta^{\varepsilon,1}_w \parallel dw \ .$$

Using (223) and (220) we have that

$$\begin{aligned}\parallel \eta^{\varepsilon,2}_{s_{m2}} \parallel &\le \parallel \widehat{\phi}^\varepsilon \left(1 - \rho_\kappa(1 - \rho_{K_\varepsilon})\right) \parallel \\ &\quad + \int_0^{s_\varepsilon} \parallel \widehat{\zeta}^\varepsilon_w \left(1 - \rho_{k_\varepsilon}(1 - \rho_{K_\varepsilon})\right) \parallel dw \\ &\le \parallel \widehat{\phi}^\varepsilon (1 - \rho_\kappa) \parallel + \parallel \widehat{\phi}^\varepsilon \rho_{K_\varepsilon} \parallel \\ &\quad + \int_0^{s_\varepsilon} \parallel \widehat{\zeta}^\varepsilon_w (1 - \rho_{k_\varepsilon}) \parallel dw + \int_0^{s_\varepsilon} \parallel \widehat{\zeta}^\varepsilon_w \rho_{K_\varepsilon} \parallel dw \ .\end{aligned}$$

Using Lemma 8.6 (a)(with $\chi = \frac{\phi^\varepsilon}{\|\phi^\varepsilon\|}$) on the first summand and (with $\chi = \frac{\zeta^\varepsilon_w}{\|\zeta^\varepsilon_w\|}$) on the third summand and Lemma 8.7 on the second and the fourth summand it follows that there exists a $C < \infty$ such that

$$\begin{aligned}\parallel \eta^{\varepsilon,2}_{s_{m2}} \parallel &\le C \sup_{k<2k_\varepsilon} \left(\left(\mid s_\varepsilon - \alpha k^2 \mid + k^3\right)^{-1}\right) \sqrt{\mid \mathcal{T}_\varepsilon \mid} k_\varepsilon^{\frac{3}{2}} \parallel \phi^\varepsilon \parallel + C\varepsilon^2 \\ &\quad + C \int_0^{s_\varepsilon} \parallel \zeta^\varepsilon_w \parallel \sup_{k<2k_\varepsilon} \left(\left(\mid s_\varepsilon - \alpha k^2 \mid + k^3\right)^{-1}\right) \sqrt{\mid \mathcal{S}_A \mid} k_\varepsilon^{\frac{3}{2}} dw \\ &\quad + C \int_0^{s_\varepsilon} (\varphi(s_\varepsilon) - \varphi(w)) \varepsilon dw \ .\end{aligned}$$

Using (217) and (6)

$$\parallel \zeta^\varepsilon_w \parallel \le C(s_\varepsilon - w)\varepsilon^{-1} \parallel 1_{\mathcal{S}_A} \psi^\varepsilon_w \parallel \ .$$

Furthermore we have that $k_\varepsilon^2 \ll s_\varepsilon$, hence there exists a $C < \infty$ such that

$$\sup_{k<2k_\varepsilon} \left((| s_\varepsilon - \alpha k^2 | + k^3)^{-1} \right) < C s_\varepsilon^{-1}$$

and it follows that

$$\begin{aligned}
\| \eta^{\varepsilon,2}_{s_{m2}} \| &\leq C s_\varepsilon^{-1} \sqrt{| \mathcal{T}_\varepsilon |} k_\varepsilon^{\frac{3}{2}} \| \phi^\varepsilon \| + C\varepsilon^2 \\
&+ C \int_0^{s_\varepsilon} (s_\varepsilon - w)\varepsilon^{-1} \| 1_{\mathcal{S}_A} \psi^\varepsilon_w \| s_\varepsilon^{-1} \sqrt{| \mathcal{S}_A |} k_\varepsilon^{\frac{3}{2}} dw \\
&+ C \int_0^{s_\varepsilon} (\varphi(s_\varepsilon) - \varphi(w))\varepsilon dw \, .
\end{aligned}$$

Note that $\| \phi_0 \| = 1$, hence with (251) $\| \phi^\varepsilon \| \leq 1$, so

$$\begin{aligned}
\lim_{\varepsilon \to 0} \| \eta^{\varepsilon,2}_{s_{m2}} \| &\equiv \lim_{\varepsilon \to 0} C \int_0^{s_\varepsilon} (s_\varepsilon - w)\varepsilon^{-1} \| 1_{\mathcal{S}_A} \psi^\varepsilon_w \| s_\varepsilon^{-1} \sqrt{| \mathcal{S}_A |} k_\varepsilon^{\frac{3}{2}} dw \\
&+ \lim_{\varepsilon \to 0} C \int_0^{s_\varepsilon} (\varphi(s_\varepsilon) - \varphi(w))\varepsilon dw \\
&= \lim_{\varepsilon \to 0} C \int_0^{s_\varepsilon} (s_\varepsilon - w)\varepsilon^{-1} \| 1_{\mathcal{S}_A} \psi^\varepsilon_w \| s_\varepsilon^{-1} \sqrt{| \mathcal{S}_A |} k_\varepsilon^{\frac{3}{2}} dw \, .
\end{aligned}$$

Next we split the integration and apply on the first integral $\| 1_{\mathcal{S}_A} \psi^\varepsilon_w \| \leq 1$ and on the second integral Lemma 8.8

$$\begin{aligned}
\lim_{\varepsilon \to 0} \| \eta^{\varepsilon,2}_{s_{m2}} \| &= \lim_{\varepsilon \to 0} C \int_0^{\varepsilon^{\frac{1}{3}}} (s_\varepsilon - w)\varepsilon^{-1} s_\varepsilon^{-1} \sqrt{| \mathcal{S}_A |} k_\varepsilon^{\frac{3}{2}} dw \\
&+ \lim_{\varepsilon \to 0} C \int_{\varepsilon^{\frac{1}{3}}}^{s_\varepsilon} (s_\varepsilon - w) w^{-\frac{5}{2}} \varepsilon^{-\frac{1}{1000} - \frac{1}{6}} s_\varepsilon^{-1} \sqrt{| \mathcal{S}_A |} k_\varepsilon^{\frac{3}{2}} dw \\
&\leq \lim_{\varepsilon \to 0} C \int_0^{\varepsilon^{\frac{1}{3}}} \varepsilon^{-1} k_\varepsilon^{\frac{3}{2}} dw + \lim_{\varepsilon \to 0} C [-w^{-\frac{3}{2}} \varepsilon^{-\frac{1}{1000} - \frac{1}{6}} k_\varepsilon^{\frac{3}{2}}]^{s_\varepsilon}_{\varepsilon^{\frac{1}{3}}} \\
&\leq \lim_{\varepsilon \to 0} C \varepsilon^{-\frac{2}{3}} k_\varepsilon^{\frac{3}{2}} + \lim_{\varepsilon \to 0} C \varepsilon^{-\frac{1}{2}} \varepsilon^{-\frac{1}{1000} - \frac{1}{6}} k_\varepsilon^{\frac{3}{2}} \, .
\end{aligned}$$

Using that $k_\varepsilon = \varepsilon^{\frac{4}{9} + \frac{1}{1000}}$ it follows that

$$\varepsilon^{-\frac{2}{3}} k_\varepsilon^{\frac{3}{2}} = \varepsilon^{\frac{3}{2000}}$$

and

$$\varepsilon^{-\frac{1}{2}} \varepsilon^{-\frac{1}{1000} - \frac{1}{6}} k_\varepsilon^{\frac{3}{2}} = \varepsilon^{\frac{1}{2000}} \, ,$$

hence

$$\lim_{\varepsilon\to 0} \parallel \eta^{\varepsilon,2}_{s_{m2}} \parallel = 0 \tag{273}$$

and Lemma 8.5 (b) follows for $l=2$.

Proof of Lemma 8.5 (b) for $l=3$

Using (229) we have

$$\begin{aligned}
\eta^{\varepsilon,3}_{s_{m2}} &= i\varepsilon^{-1}\int_{s_\varepsilon}^{s_{m2}} U^\varepsilon(s_{m2},w)(D_w - D_{s_\varepsilon})V^\varepsilon_{s_\varepsilon}(w,s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon}dw \\
&= i\varepsilon^{-1}\int_{s_\varepsilon}^{s_\varepsilon+\varepsilon^{\frac{1}{10}}} U^\varepsilon(s_{m2},w)(D_w - D_{s_\varepsilon})V^\varepsilon_{s_\varepsilon}(w,s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon}dw \\
&\quad + i\varepsilon^{-1}\int_{s_\varepsilon+\varepsilon^{\frac{1}{10}}}^{s_{m2}} U^\varepsilon(s_{m2},w)(D_w - D_{s_\varepsilon})V^\varepsilon_{s_\varepsilon}(w,s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon}dw \\
&=: S_1 + S_2
\end{aligned}$$

For S_1 we can write (doing the last steps in section "3. Step" backwards)

$$S_1 = \psi^{\varepsilon,2}_{s_\varepsilon+\varepsilon^{\frac{1}{10}}} - \psi^{\varepsilon,3}_{s_\varepsilon+\varepsilon^{\frac{1}{10}}}$$

Using the definition of $\psi^{\varepsilon,3}_s$ (see (227)) yields in view of (224) that

$$\begin{aligned}
\psi^{\varepsilon,3}_{s_\varepsilon+\varepsilon^{\frac{1}{10}}} &= V^\varepsilon_{s_\varepsilon}(s_\varepsilon+\varepsilon^{\frac{1}{10}},s_\varepsilon)\psi^{\varepsilon,2}_{s_\varepsilon} \\
&= V^\varepsilon_{s_\varepsilon}(s_\varepsilon+\varepsilon^{\frac{1}{10}},0)\phi^{\varepsilon,1} + i\int_0^{s_\varepsilon} V^\varepsilon_{s_\varepsilon}(s_\varepsilon+\varepsilon^{\frac{1}{10}},w)\zeta^{\varepsilon,1}_w(\mathbf{x})dw
\end{aligned}$$

Hence with (224)

$$S_1 = i\int_{s_\varepsilon}^{s_\varepsilon+\varepsilon^{\frac{1}{10}}} V^\varepsilon_{s_\varepsilon}(s_\varepsilon+\varepsilon^{\frac{1}{10}},w)\zeta^{\varepsilon,1}_w(\mathbf{x})dw\ .$$

and hence (by unitarity of $V^\varepsilon_{s_\varepsilon}$)

$$\parallel S_1 \parallel \le \int_{s_\varepsilon}^{s_\varepsilon+\varepsilon^{\frac{1}{10}}} \parallel \zeta^{\varepsilon,1}_w \parallel dw\ . \tag{274}$$

Using (220), (217) and Lemma 8.8 we have that

$$\begin{aligned} \| \zeta_w^{\varepsilon,1} \| &= \| \rho_{k_\varepsilon}(1-\rho_{K_\varepsilon})\mathcal{F}_s(\zeta_w^\varepsilon) \| \leq \| \zeta_w^\varepsilon \| \\ &= \| (\varphi(s_\varepsilon)-\varphi(w))\varepsilon^{-1}A\psi_w^{\varepsilon,1} \| \leq C \mid w - s_\varepsilon \mid w^{-\frac{5}{2}}\varepsilon^{-\frac{1}{6}-\frac{1}{1000}} \end{aligned}$$

This in (274) yields that

$$\begin{aligned} \| S_1 \| &\leq C\int_{s_\varepsilon}^{s_\varepsilon+\varepsilon^{\frac{1}{10}}} (w-s_\varepsilon)w^{-\frac{5}{2}}\varepsilon^{-\frac{1}{6}-\frac{1}{1000}}dw \\ &\leq C\varepsilon^{\frac{1}{10}}(\varepsilon^{\frac{1}{10}})s_\varepsilon^{-\frac{5}{2}}\varepsilon^{-\frac{1}{6}-\frac{1}{1000}} . \end{aligned}$$

Since $s_\varepsilon = \varepsilon^{\frac{1}{2000}}$

$$\| S_1 \| \leq C\varepsilon^{\frac{1}{30}-\frac{9}{4000}} ,$$

hence

$$\lim_{\varepsilon\to 0} \| S_1 \| = 0 . \tag{275}$$

It is left to estimate S_2. We write using (227), the triangle inequality and the unitarity of U^ε

$$\begin{aligned} \| S_2 \| &\leq \varepsilon^{-1}\int_{s_\varepsilon+\varepsilon^{\frac{1}{10}}}^{s} \| U^\varepsilon(s,w)(D_w - D_{s_\varepsilon})\psi_w^{\varepsilon,3} \| dw \\ &= \varepsilon^{-1}\int_{s_\varepsilon+\varepsilon^{\frac{1}{10}}}^{s} (\varphi(w)-\varphi(s_\varepsilon)) \| A\psi_w^{\varepsilon,3} \| dw \qquad (276) \\ &\leq \varepsilon^{-1}\int_{s_\varepsilon+\varepsilon^{\frac{1}{10}}}^{s} (\varphi(w)-\varphi(s_\varepsilon)) \| A \| \| \psi_w^{\varepsilon,3} \|_\infty dw \qquad (277) \end{aligned}$$

To control this we estimate $\| \psi_s^{\varepsilon,3} \|_\infty$.
We write using (227) and (224)

$$\psi_s^{\varepsilon,3} = V_{s_\varepsilon}^\varepsilon(s,0)\phi^{\varepsilon,1} + i\int_0^{s_\varepsilon} V_{s_\varepsilon}^\varepsilon(s,w)\zeta_w^{\varepsilon,1}(\mathbf{x})dw$$

Using Lemma 8.6 (b) (with $\chi = \frac{\phi^\varepsilon}{\|\phi^\varepsilon\|}$) on the first summand and (with $\chi = \frac{\zeta_w^\varepsilon}{\|\zeta_w^\varepsilon\|}$) on the second summand yields

$$\begin{aligned} \| \psi_s^{\varepsilon,3} \|_\infty &\leq C_n \| \phi^\varepsilon \| K_\varepsilon^3 \sqrt{| \mathcal{T}_\varepsilon |} \frac{\varepsilon^n}{s^n} s_\varepsilon^{-3} \left(k_\varepsilon^{-2n} + s_\varepsilon^{-\frac{3}{2}n} \right) \\ &+ C_n \int_0^{s_\varepsilon} \| \zeta_w^\varepsilon \| K_\varepsilon^3 \sqrt{| \mathcal{S}_A |} \frac{\varepsilon^n}{(s-w)^n} s_\varepsilon^{-3} \left(k_\varepsilon^{-2n} + s_\varepsilon^{-\frac{3}{2}n} \right) dw \end{aligned}$$

Note that

$$\frac{\varepsilon}{(s-u)k_\varepsilon^2} = \frac{\varepsilon^{\frac{1}{9}-\frac{1}{500}}}{s-u} < \varepsilon^{\frac{1}{90}-\frac{1}{500}}$$

for any $u < s_\varepsilon$ and $s > s_\varepsilon + \varepsilon^{\frac{1}{10}}$ and that $k_\varepsilon^{-2} \gg s_\varepsilon^{-\frac{3}{2}}$. Hence

$$\| \psi_s^{\varepsilon,3} \|_\infty$$

decays for all $s > s_\varepsilon + \varepsilon^{\frac{1}{10}}$ faster than any polynomial in ε.
In view of (276) it follows that

$$\lim_{\varepsilon \to 0} \| S_2 \| = 0 \, . \tag{278}$$

This and (275) yield the Lemma.

□

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