κ-deformed gauge theory and θ-deformed gravity

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Abstract

Noncommutative (deformed, quantum) spaces are deformations of the usual commutative space-time. They depend on parameters, such that for certain values of parameters they become the usual space-time. The symmetry acting on them is given in terms of a deformed quantum group symmetry. In this work we discuss two special examples, the θ-deformed space and the κ-deformed space.

In the case of the θ-deformed space we construct a deformed theory of gravity. In the first step the deformed diffeomorphism symmetry is introduced. It is given in terms of the Hopf algebra of deformed diffeomorphisms. The algebra structure is unchanged (as compared to the commutative diffeomorphism symmetry), but the comultiplication changes. In the commutative limit we obtain the Hopf algebra of undeformed diffeomorphisms. Based on this deformed symmetry a covariant tensor calculus is constructed and concepts such as metric, covariant derivative, curvature and torsion are defined. An action that is invariant under the deformed diffeomorphisms is constructed. In the zeroth order in the deformation parameter it reduces to the commutative Einstein-Hilbert action while in higher orders correction terms appear. They are given in terms of the commutative fields (metric, vierbein) and the deformation parameter enters as the coupling constant. One special example of this deformed symmetry, the θ-deformed global Poincaré symmetry, is also discussed.

In the case of the κ-deformed space our aim is the construction of noncommutative gauge theories. Starting from the algebraic definition of the κ-deformed space, derivatives and the deformed Lorentz generators are introduced. Choosing one particular set of derivatives, the κ-Poincaré Hopf algebra is defined. The algebraic setting is then mapped to the space of commuting coordinates. In the next step, using the enveloping algebra approach and the Seiberg-Witten map, a general nonabelian gauge theory on this deformed space is constructed. As a consequence of the deformed Leibniz rules for the derivatives used in the construction, the gauge field is derivative-valued. As in the θ-deformed case, in the zeroth order of the deformation parameter the theory reduces to its commutative analog and the higher order corrections are given in terms of the usual (commutative) fields. In this way the field content of the theory is unchanged, but new interactions appear. The deformation parameter takes the role of the coupling constant. For the special case of $U(1)$ gauge theory the action for the gauge field coupled to fermionic matter is formulated and the equations of motion and the conserved current(s) are calculated. The ambiguities in the Seiberg-Witten map are discussed and partially fixed, and an effective action (up to first order in the deformation parameter) which is invariant under the usual Poincaré symmetry is obtained.

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Introduction

Since the formulations of General Relativity [1] (GR) and Quantum Field Theory (QFT) [2], [3] in the early decades of the last century, the nature of space-time at small distances has become one of the fundamental problems in physics. The first idea of a discrete space-time was formulated by Heisenberg [4]. His motivation was the regularisation of the divergent electron self-energy. However, he soon gave up this idea, regarding it as too radical. In the attempt to eliminate the ultra-violet (UV) divergences in QFT, Snyder [5] in 1947 proposed a way to obtain a discrete space-time replacing the usual coordinates by the operators satisfying nontrivial commutation relations. This was the first time that noncommutative spaces appeared in physics.

However, Snyder's idea was not accepted at that time. One reason was that the renormalisation theory came out to be very successful in eliminating the divergences in QFT. The second reason was the mathematical complexity of noncommutative spaces. It took some time until the mathematical structure was formulated and the first physical models were derived [6], [7], [8].

The mathematical structure of noncommutative spaces became more clear in the eighties and the nineties of the last century. One of the main results was the Gel'fand-Naimark theorem [9]. It basically states that it is possible to describe a manifold by (an appropriately restricted) algebra of functions on the manifold. The space behind can be ignored completely since all the important informations are now contained in the algebra of functions. This theorem can be generalised in different ways. For example, the algebra of functions does not have to be commutative, it can be a deformation of the commutative one. If the deformation is continuous, then there exists a set of continuous parameters that control the noncommutativity. The usual commutative space-time (manifold) is obtained for special values of these parameters. The deformed algebra of functions is not the algebra of functions on a manifold but on a "noncommutative space". The main notion that is lost in this generalisation is that of a point, "noncommutative geometry is pointless geometry".

The deformation quantisation [10], [11], [12] provides a setting for connecting deformed and undeformed spaces. It allows one to describe the properties of a noncommutative space in a perturbative way, order by order in the deformation parameter. In the zeroth order the commutative space-time is obtained. The main idea of the deformation quantisation is to represent a noncommutative space on the space of commuting coordinates. Then the noncommutative multiplication of two functions is given in terms of the $\star$-product of the functions, which is defined as a formal power series of bidifferential operators acting on the functions.

Another important concept in the noncommutative geometry is that of Hopf algebras [13] and quantum groups [14], [15]. Generalising manifolds to noncommutative spaces one looses the usual space-time symmetries (Lorentz, Poincaré,...). However, is some cases there exists a deformed symmetry acting on a noncommutative space and it is given in terms of a deformed Hopf algebra [16], [17]. This enables one to discuss symmetries of noncommutative spaces.

But not only the mathematical structure of noncommutative spaces became clearer in the last years. Some recent important physical results and observations renewed the interest for the noncommutative geometry. In the following we name some of them. For the proper list of examples and motivation see for example [18], [19], [20].

We start with one classical effect. Consider an electron moving in a homogeneous and constant magnetic field $B^{\mu\nu}$. In the limit of strong magnetic field and small electron mass (restriction to the lowest Landau level), the classical Poisson bracket is

$$\{\pi^\mu, x^\nu\} \to \{x^\mu, x^\nu\} = \frac{(B^{-1})^{\mu\nu}}{e}.$$

We see that in this limit, coordinates perpendicular to the magnetic field do not commute. This ideas are relevant for the theory of the quantum Hall effect [21], [22].

Another motivation comes from the attempts to construct a quantum theory of gravity. It is believed that the space-time must change its nature at distances comparable to the Planck scale. From GR and QFT we know that an object with a given energy E has two lengths associated with it. One is the Compton wavelength, $\lambda = \frac{\hbar}{E}$ and the other one is its Schwarzschild radius $R_s = G_N E$. As the energy E grows, the point is reached when the Schwarzschild radius becomes bigger than the Compton wavelength. At this point our standard knowledge of QFT does not apply anymore. Also, measuring positions to better accuracies than the Plank length is not possible, since the energy required for such a measurement modifies the geometry at this scales [23].

A very strong argument in favour of noncommutative theories came recently from the string theory [24], [25], [26]. In [26] it was shown that a noncommutative field theory is obtained in a particular limit of the open string theory on D-brane backgrounds in the presence of a constant NS-NS B-field. The end points of open strings behave as electric charges in the presence of an external magnetic field $B_{\mu\nu}$, which results in a polarisation of the open strings. Seiberg and Witten proposed a low-energy limit (different than the usual one) in which the separation between the string endpoints becomes

$$\Delta X^i = \theta^{ij} G_{jk} p^k.$$

Here G^{ij} is the open string metric, p^i is the momentum of the string and the index $i = 1, \dots, p$ labels the D-branes directions. This limit makes the string rigid, of the finite length which depends on the momentum. The resulting low-energy effective theory is a noncommutative field theory, the constant parameter $\theta^{ij} = (B^{-1})^{ij}$ measures the noncommutativity. This suggests that certain properties of string theories could be obtained studying simpler (compared with the string theory) noncommutative field theories.

Finally, the original motivation of Heisenberg and Snyder was revisited in the last years. Namely, using the methods of the deformation quantisation we learned how to formulate noncommutative field theories. It is possible to write down the action and calculate the Feynman rules for certain noncommutative field theories. Then one calculates the divergences and compares them with the divergences in the corresponding commutative field theory. In the case of scalar ϕ^4 θ-deformed field theory [19], due to the noncommutativity (θ

is the constant deformation parameter which controls noncommutativity) both planar and nonplanar diagrams are relevant. The UV behaviour of the planar diagrams is the same as in the corresponding commutative theory, so noncommutativity does not change (improve) anything there. In the case of nonplanar diagrams, the ultra-violet/infra-red (UV/IR) mixing appears [27], [28]. Namely, one-loop diagrams turn out to be finite for arbitrary values of the external momenta p^μ. However, in the limit $p^\mu \to 0$ divergences reappear. This can be interpreted as a mixing between UV and IR divergences; noncommutativity ($\theta^{ij} \neq 0$) replaces the UV divergences with the $p^\mu \to 0$ IR divergences. Also, the commutative limit $\theta^{ij} \to 0$ is not smooth. From all this it is obvious that the noncommutativity does not solve the problem of divergences in QFT, but it rather introduces some new effects.

The aim of this thesis is to formulate (gauge) field theories on noncommutative spaces, such that they are consistent with the deformed symmetry of the space in question. We study two special examples, the θ-deformed space [24], [25] and the κ-deformed space [29], [30]. The general strategy is to start from an abstract algebra of coordinates which defines our noncommutative space [31], [32]. The derivatives are then introduced as maps on this abstract algebra [33], [34]. They cannot be uniquely defined and one has to find arguments to single out one specific set of them. One way to do this is to construct the deformed symmetry. In the case of the κ-deformed space [35], we recover the κ-deformed Poincaré algebra [29], [30]. In the θ-deformed space we construct the deformed diffeomorphism symmetry [36]. Then the θ-deformed global Poincaré symmetry [37], [38], [39], [40] is the subalgebra (sub Hopf algebra) of this deformed symmetry.

In the next step, we represent the abstract noncommutative space (together with the derivatives and the deformed symmetry) on the space of commuting coordinates. This gives us the playground for construction of the noncommutative (gauge) field theories.

In the case of the κ-deformed space we concentrate on the noncommutative $SU(N)$ theories. Using the enveloping algebra approach and the Seiberg-Witten map [32], [41], the noncommutative gauge theory is constructed perturbatively order by order in the deformation parameter. In this way we obtain an effective theory which provides corrections to the commutative theory up to first order in the deformation parameter. These corrections are given in terms of the commutative fields, so the field content of the theory is not changed. However, new interactions arise and the deformation parameter enters as a coupling constant. This approach has been used to construct the noncommutative gauge theory on the θ-deformed space [41], [42], as well as the generalisation of the Standard Model [43], [44]. Using these results some new effects which do not appear in the commutative Standard Model were calculated in [45], [46]. Also, it was shown that the theories obtained in this perturbative way are anomaly free [47], [48], [49]. It is interesting to note that cutting the theory at some order in the deformation parameter one avoids the UV/IR mixing. It only appears in the "summed-up" theories, that is theories to all orders in the deformation parameter. Also, the "summed-up" models allow generalisation of the $U(N)$ gauge theories only, with some exceptions [50], [51].

In the θ-deformed case we turn to the local space-time symmetries and their (possible) deformations. It is well known that the gravity can be seen as a gauge theory where the gauge symmetry is the Poincaré symmetry of the space-time. However, it was not clear if it is possible to generalise it to the noncommutative spaces in the same way one generalises the "usual" gauge theories. This problem was analysed in the previous years in [52], [53], [54] . Here we construct the deformed diffeomorphism symmetry and use it to formulate a gravity theory on the θ-deformed space [36].

The structure of the thesis is the following: In the first chapter we shortly review the definition of a noncommutative space in terms of the abstract algebra of coordinates. Then we discuss derivatives and symmetries introduced as maps on this space. In order to obtain a theory which gives some predictions (numbers finally) we represent the abstract algebra formalism on the space of commuting coordinates and introduce the $\star$-product approach. At the end of the chapter, as an illustration, the described method is applied to one special example, the θ-deformed space.

Chapters 2, 3 and 4 concern the κ-deformed space and the formulation of gauge theory on it. We start with the abstract algebra of coordinates and introduce derivatives and the κ-deformed Poincaré algebra as the deformed symmetry of this space. Aiming at the construction of the noncommutative gauge theory, we represent everything on the space of commuting coordinates. The gauge theory is then constructed using the enveloping algebra approach and the Seiberg-Witten map. The gauge field becomes derivative valued due to the nontrivial Leibniz rules of the derivatives used in the construction. To define an action a "good" integral is needed. This problem is shortly discussed in the beginning of Chapter 4. After defining the integral, the $U(1)$ gauge theory is derived. Using the freedom in the Seiberg-Witten map one obtains the effective action which is explicitly x-independent and invariant under the commutative Poincaré symmetry. Also, the ambiguity in the conserved $U(1)$ current is discussed. This work is done in collaboration with Larisa Jonke, Frank Meyer, Lutz Möller, Efrossini Tsouchnika, Julius Wess and Michael Wohlgenannt and the results are published in [35], [55], [56], [57].

In the last chapter we turn to the problem of defining a gravity theory on noncommutative spaces. We choose to work with the θ-deformed space because of its simplicity. The starting point is the construction of the Hopf algebra of deformed diffeomorphisms. Using this result fields and tensor calculus are introduced. Following the same steps as in the commutative case, one constructs covariant derivative, curvature tensor, torsion,. . . . The final result is the deformed Einstein-Hilbert action and the equation of motion coming from it. Expansions of some results up to first order in the deformation parameter are given. Most of the results presented here are obtained together with Paolo Aschieri, Christian Blohmann, Frank Meyer, Peter Schupp and Julius Wess and are published in [58], [36].

1

Noncommutative spaces

The notion of noncommutative (deformed) spaces is based on the simple idea of replacing ordinary (commutative) coordinates

$$[x^\mu, x^\nu] = 0, \qquad \mu = 0, \dots n,$$

with noncommutative operators

$$[\hat{x}^\mu, \hat{x}^\nu] \neq 0.$$

Since operators $\hat{x}^\mu$ do not commute they can not be diagonalised simultaneously, similarly as in quantum mechanics where operators of coordinates and momenta do not commute. Space-time is then given by the collection of the eigenvalues of the operators $\hat{x}^\mu$. If the spectrum is discrete then space-time will also be discrete. In the case of commuting coordinates we obtain a continuous spectrum leading to the continuous space-time we are familiar with.

In this chapter we recall the definition of noncommutative spaces and some of their properties. Differential calculus as well as the concept of deformed symmetry (in terms of Hopf algebras) are introduced. We start with the abstract algebra approach, but also formulate the representation on the space of commuting coordinates, the so-called $\star$-product representation. In order to illustrate the abstract mathematical formalism, at the end of the chapter one simple example of a deformed space is presented.

1.1 Definition

Noncommutative (deformed)[1] space is generated by $n+1$ abstract coordinates $\hat{x}^\mu$ which fulfil

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x}), \qquad \mu = 0, \dots n, \tag{1.1.1}$$

where $\Theta^{\mu\nu}(\hat{x})$ is an arbitrary polynomial of coordinates [59], [38]. More precisely, the noncommutative space $\hat{\mathcal{A}}_{\hat{x}}$ is the associative algebra, freely generated by $\hat{x}^\mu$ coordinates and divided by the ideal generated by (1.1.1)

$$\hat{\mathcal{A}}_{\hat{x}} = \frac{\mathbb{C}[[\hat{x}^{\mu_0}, \dots, \hat{x}^{\mu_n}]]}{\big([\hat{x}^\mu, \hat{x}^\nu] - \Theta^{\mu\nu}(\hat{x})\big)}. \tag{1.1.2}$$

[1] "Noncommutative" and "deformed" will be used as synonyms from now on, whereas "classical", "undeformed" and "usual" will be synonyms for "commutative".

The elements of this space are all possible polynomials in the coordinates $\hat{x}^\mu$. If one element can be transformed into the other one using (1.1.1) then this two elements are considered equal. Before proceeding further, we clarify the notation we use. Coordinates $\hat{x}^\mu$ generate the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$, while the operators $\hat{\partial}_\rho$, $\hat{L}_{\alpha\beta}$, ... are maps of the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$ into itself. Variables without the hat symbol, like x^μ, ∂_ρ, ... are usual commutative variables. Sometimes we use $\mathcal{A}_x$ to denote the space of commuting coordinates[2].

The defining relation of the deformed space (1.1.1) is very general since on the right hand side we have an arbitrary polynomial of coordinates. Usually one consideres some special examples of it. Among them there are three very important ones

$$\text{Canonical or } \theta\text{-deformed spaces} \quad [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.1.3}$$

$$\text{Lie algebra deformed spaces} \quad [\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda, \tag{1.1.4}$$

$$q\text{-deformed spaces} \quad \hat{x}^\mu \hat{x}^\nu = \frac{1}{q} R^{\mu\nu}{}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma. \tag{1.1.5}$$

In the case of θ-deformed spaces [24], [25], $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is an antisymmetric constant matrix of mass dimension -2. For Lie algebra deformed spaces [29], [30], [32], $C^{\mu\nu}_\lambda$ are Lie algebra structure constants of mass dimension -1. And finally, $R^{\mu\nu}{}_{\rho\sigma}$ is the dimensionless R-matrix of the quantum space [60], [61], [62].

This three examples fulfil the Poincaré-Birkoff-Witt (PBW) property. PBW property has been first developed for Lie algebras [63], but it applies to the other deformed algebras as well. It states that the finite dimensional vector spaces, spanned by the homogeneous polynomials of degree r, have the same dimension as the corresponding vector spaces of commuting variables. In the case of examples (1.1.3)-(1.1.5) PBW property allows us to introduce a basis of ordered monomials in $\hat{\mathcal{A}}_{\hat{x}}$. In the following we always restrict the general $\Theta^{\mu\nu}(\hat{x})$ to one of the three cases (1.1.3)-(1.1.5), that is $\hat{\mathcal{A}}_{\hat{x}}$ denotes canonical, Lie algebra or q-deformed spaces.

In order to write the elements of $\hat{\mathcal{A}}_{\hat{x}}$ in a unique way one imposes an ordering prescription. There are many possible orderings for a given abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$. The most oftenly used ones are the symmetric and the normal ordering.

If we chose the symmetric ordering, the basis in the algebra is given by

$$\begin{aligned} :\hat{x}^\mu: &= \hat{x}^\mu, \\ :\hat{x}^\mu \hat{x}^\nu: &= \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu), \\ &\dots . \end{aligned} \tag{1.1.6}$$

The vector space spanned by this basis we denote by $\hat{V}$

$$\hat{V} = \sum_r \oplus \hat{V}_r, \tag{1.1.7}$$

where $\hat{V}_r$ is the vector space spaned by homogeneous polynomials of degree r in the coordinates $\hat{x}^\mu$. The corresponding vector spaces of commuting coordinates we denote by V and V_r

[2]In the notation of (1.1.2) the space of commuting coordinates is the associative algebra freely generated by the commutative coordinates x^μ

$$\mathcal{A}_x = \mathbb{C}[[x^{\mu_0}, \dots, x^{\mu_n}]].$$

respectively. An arbitrary element of $\hat{\mathcal{A}}_{\hat{x}}$ is then written as an expansion in the basis (1.1.6)

$$\begin{aligned}\hat{f}(\hat{x}) &= \sum_{j=0}^{\infty} C_{\mu_1\dots\mu_j} : \hat{x}^{\mu_1} \dots \hat{x}^{\mu_j} : \\ &= C_0 + C_{1\mu} : \hat{x}^{\mu} : + C_{2\mu\nu} : \hat{x}^{\mu}\hat{x}^{\nu} : + \dots \end{aligned} \tag{1.1.8}$$

and it is fully characterised by the expansion coefficients $C_{\mu_1\dots\mu_j}$. The power series expansion (1.1.8) we call the formal power series expansion since we do not say anything about its convergence.

In the case of normal ordering the "order" of coordinates is specified. For example, let $\hat{x}^n$ stand to the furthest left of the expression, then let $\hat{x}^{n-1}$ come after it and so on until $\hat{x}^0$ which then stands on the furthest right. The basis is given by

$$\begin{aligned} : \hat{x}^{\mu} : &= \hat{x}^{\mu}, \\ : \hat{x}^{\mu}\hat{x}^{\nu} : &= \hat{x}^{\mu}\hat{x}^{\nu}, \quad \mu \geq \nu, \\ & \dots \end{aligned} \tag{1.1.9}$$

and every element of $\hat{\mathcal{A}}_{\hat{x}}$ can be expanded in this basis. Of course, this is not the only possibility for normal ordering but just an example.

However, multiplying two arbitrary functions $\hat{f}$, $\hat{g} \in \hat{\mathcal{A}}_{\hat{x}}$ gives the result which is no longer written as an expansion in basis and the elements have to be reordered. For example, we take the symmetric ordering and multiply two basis elements

$$\begin{aligned} : \hat{x}^{\mu} : \cdot : \hat{x}^{\nu} : &= \hat{x}^{\mu}\hat{x}^{\nu} \\ &= \frac{1}{2}(\hat{x}^{\mu}\hat{x}^{\nu} + \hat{x}^{\nu}\hat{x}^{\mu}) + \frac{1}{2}(\hat{x}^{\mu}\hat{x}^{\nu} - \hat{x}^{\nu}\hat{x}^{\mu}) \\ &= : \hat{x}^{\mu}\hat{x}^{\nu} : + \frac{i}{2}\Theta^{\mu\nu}(\hat{x}). \end{aligned} \tag{1.1.10}$$

In the first line we obtain a result which is not written in terms of basis elements, then we rewrite it differently. Using relations (1.1.1) in the last line, the result expressed in terms of basis elements follows. Once again we mention that $\Theta^{\mu\nu}(\hat{x})$ is restricted to one of the three examples (1.1.3)-(1.1.5) that fulfil PBW property. We come back to this result in Section 1.4 when we consider a representation on the space of commuting coordinates.

1.2 Derivatives

Having defined our framework, we now introduce the concept of derivatives on a deformed space.

Derivatives are maps of the deformed space into itself [33], [34]

$$\begin{aligned} \hat{\partial}_{\rho} : \quad \hat{\mathcal{A}}_{\hat{x}} &\rightarrow \hat{\mathcal{A}}_{\hat{x}} \\ \hat{f}(\hat{x}) &\mapsto (\hat{\partial}_{\rho}\hat{f})(\hat{x}). \end{aligned} \tag{1.2.1}$$

They are usually defined by the action on the coordinates. This action is extended to the free algebra of coordinates. To define a map on $\hat{\mathcal{A}}_{\hat{x}}$, derivatives have to be consistent with the defining relations (1.1.1). Also, we demand that they are a deformation of the usual partial derivatives.

Having all this in mind, we make a very general ansatz for the commutator of derivatives and coordinates

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho + \sum_j A^{\mu\lambda_1\dots\lambda_j}_\rho \hat{\partial}_{\lambda_1}\dots\hat{\partial}_{\lambda_j}. \tag{1.2.2}$$

The coefficients $A^{\mu\lambda_1\dots\lambda_j}_\rho$ are complex numbers. Demanding that (1.2.2) is consistent with the relations (1.1.1) leads to conditions on these coefficients. Ansatz (1.2.2) is not the most general one, since we do not allow the right hand side of (1.2.2) to depend on coordinates. However, in all the examples considered here this will be sufficient. Let us just sketch the idea here. The explicit examples will be given in Sections 1.5 and 2.1. One demands that

$$\hat{\partial}_\rho\Big([\hat{x}^\mu, \hat{x}^\nu] - i\Theta^{\mu\nu}(\hat{x})\Big) = 0, \tag{1.2.3}$$

and commutes the derivative $\hat{\partial}_\rho$ to the furthest right using Ansatz (1.2.2). The final result has to be zero, otherwise new commutation relations on coordinates arise. This gives some conditions on the coefficients $A^{\nu\rho_1\dots\rho_j}_\mu$, but in most of the cases these conditions are not sufficient to determine the coefficients uniquely.

Using (1.2.2) one calculates the Leibniz rule by applying derivatives on ordered monomials and generalising the result to the product of arbitrary functions. The presence of nonzero additional terms in (1.2.2) leads to a deformed Leibniz rule

$$\hat{\partial}_\rho(\hat{f}\cdot\hat{g}) = (\hat{\partial}_\rho\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{\partial}_\rho\hat{g}) + \text{ additional terms.} \tag{1.2.4}$$

1.3 Symmetries

The concept of symmetry is very important in physics. Classically, symmetries are described by Lie groups or Lie algebras and the physical space is the representation space of the symmetry algebra. For example, the commutative Minkowski space-time is the representation space of the Poincaré algebra. Therefore, the question arises if one can introduce deformed spaces as representation spaces of some symmetry algebras. At first sight it looks as this will not be possible. If one looks at (1.1.3), that is the θ-deformed space, it is obvious that the Lorentz invariance is broken since the left-hand side transforms like a tensor while the right-hand side is constant[3]. However, it turns out that it is possible to deform the concept of symmetry such that it can be applied to deformed spaces as well. This is done in the framework of Hopf algebras [60].

It is well known that the function algebra over a classical Lie group $\mathcal{F}(\mathcal{G})$ is a Hopf algebra. The deformation of this classical function algebra to the respective quantum (deformed) group $\mathcal{F}(\mathcal{G})_h$ is well defined. The deformed function algebra $\mathcal{F}(\mathcal{G})_h$ is again a Hopf algebra and it depends on parameter h. In the limit $h \to 0$ $\mathcal{F}(\mathcal{G})_h$ reduces to the classical function algebra $\mathcal{F}(\mathcal{G})$. It is very important that this deformation does not lead out of the category of Hopf algebras. This motivates studying Hopf algebras in more detail.

As mentioned above, the function algebra over a Lie group and the enveloping algebra of a Lie algebra are examples of Hopf algebras. In general, a Hopf algebra $\mathcal{A}$ consists of an

[3]Note that there are authors who treat $\theta^{\mu\nu}$ as a tensor [64], [65].

algebra and a coalgebra structure which are compatible with each other. Additionally, there is a map called antipode which coresponds to the inverse of a group. In the following we give the precise definition of Hopf algebras.

We start repeating the definition of an algebra and some related concepts. An algebra (associative algebra with unit) is a vector space $\mathcal{A}$ over a field F, with two linear maps, multiplication or product $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit $\eta: F \to \mathcal{A}$ such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m), \tag{1.3.1}$$

$$m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta). \tag{1.3.2}$$

Here id is the identity map on $\mathcal{A}$. If we have two algebras $\mathcal{A}$ and $\mathcal{B}$ we can define an algebra homomorphism. It is a F-linear map $\varphi: \mathcal{A} \to \mathcal{B}$ such that $\varphi(aa') = \varphi(a)\varphi(a')$ for all $a, a' \in \mathcal{A}$ and $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Also, one defines a tensor product algebra $\mathcal{A} \otimes \mathcal{B}$. Its vector space is the tensor product of vector spaces of $\mathcal{A}$ and $\mathcal{B}$ and the multiplication is defined by $m_{\mathcal{A}\otimes\mathcal{B}} = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$ where τ is the so-called flip operator, $\tau(a \otimes b) = b \otimes a$, that is

$$(a \otimes b)(a' \otimes b') \stackrel{\mathrm{def}}{=} aa' \otimes bb', \quad a, a' \in \mathcal{A} \text{ and } b, b' \in \mathcal{B}. \tag{1.3.3}$$

Now we introduce the concept of coalgebra. A coalgebra is defined as a vector space $\mathcal{A}$ over F with two linear maps, comultiplication or coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon: \mathcal{A} \to F$, such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{1.3.4}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{1.3.5}$$

Equation (1.3.4) is referred to as the coassociativity of the comultiplication Δ and it is dual to the associativity of the multiplication m (1.3.1). In the same way as for algebras, one defines the coalgebra homomorphism and the tensor product coalgebra. A coalgebra homomorphism is defined as F-linear map $\varphi: \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are coalgebras, such that

$$\Delta_{\mathcal{B}} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{\mathcal{A}}, \quad \varepsilon_{\mathcal{A}} = \varepsilon_{\mathcal{B}} \circ \varphi. \tag{1.3.6}$$

The tensor product coalgebra $\mathcal{A} \otimes \mathcal{B}$ is the coalgebra built on the vector space $\mathcal{A} \otimes \mathcal{B}$ with comultiplication $\Delta_{\mathcal{A}\otimes\mathcal{B}} = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_{\mathcal{A}} \otimes \Delta_{\mathcal{B}})$ and counit $\varepsilon_{\mathcal{A}\otimes\mathcal{B}} = \varepsilon_{\mathcal{A}} \otimes \varepsilon_{\mathcal{B}}$. Coalgebra is cocommutative if $\tau \circ \Delta = \Delta$.

If we have an algebra $\mathcal{A}$ that is a coalgebra at the same time and if algebra and coalgebra structures are compatible, we speak about bialgebras. Compatibility is defined in the following way

$$\Delta(a\,a') = \Delta(a)\Delta(a'), \quad \varepsilon(a\,a') = \varepsilon(a)\varepsilon(a'), \quad a, a' \in \mathcal{A} \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(1) = 1, \tag{1.3.7}$$

that is the coproduct and counit are algebra homomorphisms.

Finally, to come from a bialgebra to a Hopf algebra we need one additional structure. It is a linear map called antipode or coinverse $S: \mathcal{A} \to \mathcal{A}$ such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{1.3.8}$$

The antipode is an algebra antihomomorphism as well as a coalgebra antihomomorphism, that is

$$S(aa') = S(a')S(a), \quad a, a' \in \mathcal{A}, \quad S(1) = 1, \tag{1.3.9}$$
$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \quad \varepsilon \circ S = \varepsilon. \tag{1.3.10}$$

To get used to these mathematical concepts we present one example, namely the Hopf algebra of usual partial derivatives on $n+1$ dimensional Minkowski space-time. Although very simple, it will be useful later when we generalise the concept of derivatives to deformed spaces. Generators of the algebra[4] are ∂_ρ, $\rho = 0, \dots n$ and they fulfil

$$[\partial_\rho, \partial_\sigma] = 0. \tag{1.3.11}$$

Multiplication and the unit element are $m(\partial_\rho \otimes \partial_\sigma) = \partial_\rho \partial_\sigma$ and $\eta = 1$ respectively.

The coalgebra sector is given by comultiplication

$$\Delta(\partial_\rho) = \partial_\rho \otimes 1 + 1 \otimes \partial_\rho, \tag{1.3.12}$$

and counit $\varepsilon(\partial_\rho) = 0$. We calculate

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \Delta\partial_\rho &= (\mathrm{id} \otimes \Delta)(\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) = \partial_\rho \otimes 1 \otimes 1 + 1 \otimes (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) \\ (\Delta \otimes \mathrm{id}) \circ \Delta\partial_\rho &= (\Delta \otimes \mathrm{id})(\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) = (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) \otimes 1 + 1 \otimes 1 \otimes \partial_\rho,\end{aligned}$$

and comparing this two lines see that coassociativity (1.3.4) is fulfilled. From

$$\begin{aligned}(\varepsilon \otimes \mathrm{id}) \circ \Delta(\partial_\rho) &= (\varepsilon \otimes \mathrm{id}) \circ (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) = \partial_\rho = \mathrm{id} \circ \partial_\rho, \\ (\mathrm{id} \otimes \varepsilon) \circ \Delta(\partial_\rho) &= (\mathrm{id} \otimes \varepsilon) \circ (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) = \partial_\rho = \mathrm{id} \circ \partial_\rho\end{aligned}$$

we see that (1.3.5) is fulfilled as well. The abstract concept of comultiplication encodes the well known concept of Leibniz rule. For example, from (1.3.12) we have

$$\partial_\rho(fg) = (\partial_\rho f)g + f(\partial_\rho g). \tag{1.3.13}$$

Having comultiplication one can always deduce the Leibniz rule. The other way, abstracting the coproduct from a given Leibniz rule, does not lead to an unique result (in most cases). This is because the Leibniz rule is representation-dependent, while the comultiplication is representation-independent and therefore it is a more general concept.

To be able to speak about the bialgebra of derivatives we have to check if

$$[\Delta(\partial_\rho), \Delta(\partial_\sigma)] = 0 \;\text{ and }\; \varepsilon(\partial_\rho\partial_\sigma) = \varepsilon(\partial_\rho)\varepsilon(\partial_\sigma). \tag{1.3.14}$$

The second relation is obvious. The first one we write explicitly

$$\begin{aligned}[\Delta(\partial_\rho), \Delta(\partial_\sigma)] &= \Delta(\partial_\rho)\Delta(\partial_\sigma) - (\rho \leftrightarrow \sigma) \\ &= (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho)(\partial_\sigma \otimes 1 + 1 \otimes \partial_\sigma) - (\rho \leftrightarrow \sigma) \\ &= \partial_\rho\partial_\sigma \otimes 1 + \partial_\rho \otimes \partial_\sigma + \partial_\sigma \otimes \partial_\rho + 1 \otimes \partial_\rho\partial_\sigma - (\rho \leftrightarrow \sigma) = 0,\end{aligned}$$

[4]Actually, one is here working with the universal enveloping algebra of ∂_ρ derivatives.

where we used that derivatives commute, that is (1.3.11). Looking at (1.3.12) we see that this bialgebra is cocommutative.

Finally, in order to obtain the Hopf algebra of derivatives, we add the antipode $S(\partial_\rho) = -\partial_\rho$ and check if it fulfils (1.3.8)

$$\begin{aligned} m \circ (S \otimes \mathrm{id}) \circ \Delta(\partial_\rho) &= m \circ (S \otimes \mathrm{id}) \circ (\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) \\ &= m(-\partial_\rho \otimes 1 + 1 \otimes \partial_\rho) = -\partial_\rho + \partial_\rho = 0 = \eta \circ \varepsilon(\partial_\rho). \end{aligned}$$

The second part of (1.3.8) one proves analogously.

1.4 Representation on the space of commuting coordinates

So far our analysis of deformed spaces was given in terms of the abstract algebra. But we would like to have a theory which could give some predictions (numbers finally) that might be experimentally checked. The deformation quantisation provides the way to connect deformed and undeformed spaces. It allows us to to describe the properties of a noncommutative space in a perturbative way, order by order in the deformation parameter. In the zeroth order the commutative space-time is obtained.

The main idea of the deformation quantisation is to represent a noncommutative space on the space of commuting coordinates. Remember that we consider only the deformed spaces $\hat{\mathcal{A}}_{\hat{x}}$ which fulfil PBW property and in which a basis can be introduced. Therefore, we can map the basis in $\hat{\mathcal{A}}_{\hat{x}}$ to the basis of monomials of commuting coordinates

$$\begin{aligned} :\hat{x}^\mu: \quad &\mapsto \quad x^\mu, \\ :\hat{x}^\mu\hat{x}^\nu: \quad &\mapsto \quad x^\mu x^\nu, \\ &\dots . \end{aligned}$$

This enables us to map an element $\hat{f}(\hat{x})$ of $\hat{\mathcal{A}}_{\hat{x}}$ to the space of commuting coordinates. We expand it in terms of basis elements and then map every element to the space of commuting coordinates

$$\begin{aligned} &\hat{f}(\hat{x}) = C_0 + C_{1\mu} :\hat{x}^\mu: + C_{2\mu\nu} :\hat{x}^\mu\hat{x}^\nu: + \dots \\ &\quad\downarrow \\ &f(x) = C_0 + C_{1\mu}x^\mu + C_{2\mu\nu}x^\mu x^\nu + \dots . \end{aligned} \tag{1.4.1}$$

The function $f(x)$ is a function of commuting coordinates and is a representation of the abstract function $\hat{f}(\hat{x})$. This establishes the isomorphism between the vector space $\hat{V}$ defined in (1.1.7) and the corresponding vector space of commuting coordinates V.

The next step is to extend this vector space isomorphism to an algebra morphism. To do this one has to map the multiplication in the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$ to the space of commuting coordinates. We start with two elements of $\hat{\mathcal{A}}_{\hat{x}}$, $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$. Their product is an element of $\hat{\mathcal{A}}_{\hat{x}}$:

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{f} \cdot \hat{g}(\hat{x}) \in \hat{\mathcal{A}}_{\hat{x}}. \tag{1.4.2}$$

This element can be expanded in the chosen basis and mapped to the algebra of commuting variables $\mathcal{A}_x$[5]

$$\hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x) \in \mathcal{A}_x. \tag{1.4.3}$$

Its image we label $f \star g(x)$ and it defines the star product ($\star$-product) of two functions. The algebra of noncommuting coordinates $\hat{\mathcal{A}}_{\hat{x}}$ is then isomorphic to the algebra of commuting variables with the $\star$-product as multiplication.

Before discussing some properties of this product, we give one simple example. In Section 1.1, using the symmetric ordering, we found that

$$:\hat{x}^\mu : \cdot : \hat{x}^\nu := : \hat{x}^\mu \hat{x}^\nu : + \frac{i}{2}\Theta^{\mu\nu}(\hat{x}). \tag{1.4.4}$$

This can be mapped to the space of commuting coordinates

$$\begin{array}{c} :\hat{x}^\nu : \cdot : \hat{x}^\mu :=: \hat{x}^\mu \hat{x}^\nu : + \frac{i}{2}\Theta^{\mu\nu}(\hat{x}) \\ \downarrow \\ x^\mu \star x^\nu = x^\mu x^\nu + \frac{i}{2}\Theta^{\mu\nu}(x). \end{array} \tag{1.4.5}$$

Also,

$$:\hat{x}^\nu : \cdot : \hat{x}^\mu := \hat{x}^\nu \hat{x}^\mu \mapsto x^\nu \star x^\mu = x^\nu x^\mu - \frac{i}{2}\Theta^{\mu\nu}(x) \tag{1.4.6}$$

and therefore

$$[\hat{x}^\mu, \hat{x}^\nu] \mapsto [x^\mu \stackrel{\star}{,} x^\nu] = i\Theta^{\mu\nu}(x). \tag{1.4.7}$$

Now we come back to some of the properties of $\star$-products[6]. A general $\star$-product is an associative and noncommutative product and can be written as an expansion in the deformation parameter h

$$f \star g := B_0(f,g) + hB_1(f,g) + h^2 B_2(f,g) + \ldots = B(f,g), \tag{1.4.8}$$

where $B = \sum_k h^k B_k$. The bilinear operators B_k are not independent, since the condition for associativity of (1.4.8) yields an infinite number of equations quadratic in the B's. In addition to associativity, it is useful to require the product to be unital, $f \star 1 = f = 1 \star f$. In the commutative limit ($h \to 0$) a $\star$-product should reproduce the pointwise multiplication of functions

$$B_0(f,g) = (fg)(x) = f(x)g(x). \tag{1.4.9}$$

The product (1.4.8) can be written as

$$f \star g = B(f,g) = \sum_{k=0}^{\infty} h^k B_k^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s}(x)(\partial_{\mu_1} \ldots \partial_{\mu_r} f)(\partial_{\nu_1} \ldots \partial_{\nu_s} g). \tag{1.4.10}$$

[5] Algebra $\mathcal{A}_x$ can be also seen as the vector space V equipped with the usual pointwise multiplication.

[6] Using PBW property is not the only way one can construct $\star$-products, one can use the Weyl quantisation [66], [31], for example.

The series in (1.4.10) we regard as the formal power series. The deformation property and associativity of the $\star$-product imply that

$$\begin{aligned}
\{f, g\}_\star &\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{1}{ih}(f \star g - g \star f) \\
&= \lim_{h\to 0} \frac{1}{ih}[f \stackrel{\star}{,} g] = \big(B_1^{\mu\nu}(x) - B_1^{\nu\mu}(x)\big)(\partial_\mu f)(\partial_\nu g) \\
&= \Theta^{\mu\nu}(x)(\partial_\mu f)(\partial_\nu g) \qquad (1.4.11)
\end{aligned}$$

defines a Poisson bracket. The antisymmetric tensor $\Theta^{\mu\nu} = \Theta^{\mu\nu}(x)$ in the last line represents the Poisson structure. The Jacobi identity which this Poisson structure satisfies is expressed in terms of $\Theta^{\mu\nu}$ as

$$\Theta^{\mu\lambda}\partial_\lambda\Theta^{\nu\rho} + \Theta^{\nu\lambda}\partial_\lambda\Theta^{\rho\mu} + \Theta^{\rho\lambda}\partial_\lambda\Theta^{\mu\nu} = 0. \qquad (1.4.12)$$

Conversely, given a Poisson tensor $\Theta^{\mu\nu}$, we can always find a $\star$-product such that (1.4.11) holds by the Kontsevich construction [12]. We mention that for $\Theta^{\mu\nu}$ more general than in the examples (1.1.3)-(1.1.5), relation (1.4.12) might not be fulfilled and then we can not define a Poison bracket (1.4.11). To first order, the Kontsevich $\star$-product reads

$$f \star g = fg + \frac{ih}{2}\Theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) + \mathcal{O}(h^2), \qquad (1.4.13)$$

and the first order term B_1 of the $\star$-product is proportional to the Poisson bracket. Applied to coordinates, (1.4.13) gives

$$x^\mu \star x^\nu = x^\mu x^\nu - \frac{ih}{2}\Theta^{\mu\nu}(x),$$

that is

$$[x^\mu \stackrel{\star}{,} x^\nu] = ih\Theta^{\mu\nu}(x)$$

which is exactly (1.4.7). In the next section we give one concrete example of the $\star$-product for the θ-deformed space.

So far we have learned how to map functions from the abstract algebra to the space of commuting coordinates and how to multiply them. But this is not enough to formulate a field theory on a deformed space[7]. One has also to learn how to map derivatives $\hat{\partial}_\mu$ from the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$ to the space of the commuting coordinates.

The principle how to do it is given by the following diagram

$$\begin{array}{ccc}
\hat{f}(\hat{x}) & \longmapsto & f(x) \\
\Big\downarrow{\scriptstyle \hat{\partial}_\mu} & & \Big\downarrow{\scriptstyle \partial^\star_\mu} \\
(\hat{\partial}_\mu \hat{f})(\hat{x}) & \longmapsto & (\partial^\star_\mu \star f)(x)\,.
\end{array} \qquad (1.4.14)$$

[7] In Section 1.1 we defined deformed space in terms of the abstract algebra. In this section we represent it on the space of commuting coordinates, and this representation we also call deformed space. It is the usual commutative space, equipped with a $\star$-product, $\star$-derivatives,....

In words, applying $\hat{\partial}_\mu$ to a function $\hat{f}(\hat{x})$ in $\hat{\mathcal{A}}_{\hat{x}}$ gives a new function $(\hat{\partial}\hat{f})(\hat{x})$. This function, as well as $\hat{f}(\hat{x})$, can be mapped to the space of commuting coordinates. The images we call $(\partial^\star_\mu \star f)(x)$ and $f(x)$ respectively. From this two results one "reads off" the operator $\partial^\star_\mu$. In practice, one applies $\hat{\partial}_\mu$ on symmetrised polynomials (if the ordering in $\hat{\mathcal{A}}_{\hat{x}}$ is chosen to be symmetric one) using (1.2.2) and finds a perturbative expression for $\partial^\star_\mu$. This result is then generalised to a closed formula when possible. The same procedure can be used to find $\star$-representations of other operators defined in the abstract algebra (like generators of deformed symmetries, see Section 2.6).

1.5 An example, the θ-deformed space

In this section we present one special example, the θ-deformed space defined by relation (1.1.3). Compared to the other examples of deformed spaces, this is a very simple deformation since the right-hand side of (1.1.3) does not depend on the coordinates $\hat{x}^\mu$. Therefore, some of the properties of the θ-deformed space will be simpler compared to the other more general examples.

As outlined in Section 1.2 we now define derivatives on this space. Since they should be a deformation of the usual derivatives we make the following ansatz

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta). \qquad (1.5.1)$$

Additionally, the derivatives $\hat{\partial}_\rho$ are maps of the θ-deformed space into itself. Therefore, the relation (1.5.1) has to be consistent with (1.1.3). We obtain

$$\begin{aligned}
\hat{\partial}_\rho \hat{x}^\mu \hat{x}^\nu &= ([\hat{\partial}_\rho, \hat{x}^\mu] + \hat{x}^\mu \hat{\partial}_\rho)\hat{x}^\nu \\
&= (\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta))\hat{x}^\nu + \hat{x}^\mu([\hat{\partial}_\rho, \hat{x}^\nu] + \hat{x}^\nu \hat{\partial}_\rho) \\
&= (\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta))\hat{x}^\nu + \hat{x}^\mu(\delta^\nu_\rho + f^\nu_\rho(\hat{\partial}, \theta) + \hat{x}^\nu \hat{\partial}_\rho), \\
\hat{\partial}_\rho \hat{x}^\nu \hat{x}^\mu &= \ldots \\
&= (\delta^\nu_\rho + f^\nu_\rho(\hat{\partial}, \theta))\hat{x}^\mu + \hat{x}^\nu(\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta) + \hat{x}^\mu \hat{\partial}_\rho)
\end{aligned}$$

and

$$\hat{\partial}_\rho \theta^{\mu\nu} = \theta^{\mu\nu} \hat{\partial}_\rho.$$

Adding these three terms together we see that

$$\hat{\partial}_\rho\big([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\big) = \big([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\big)\hat{\partial}_\rho,$$

is fulfilled for $f^\mu_\rho(\hat{\partial}, \theta) = 0$. Therefore,

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho. \qquad (1.5.2)$$

In contrast to the expectations, there are no additional terms in (1.5.2). This is due to the fact that the right hand side of (1.1.3) is constant. In the next chapter we study the κ-deformed space which is an example for a Lie algebra deformation. There it is not possible to set $f^\mu_\rho(\hat{\partial}, \kappa) = 0$ and additional terms arise.

One can check that

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \tag{1.5.3}$$

is consistent with (1.5.2). From (1.5.2) the Leibniz rule for derivatives follows,

$$\hat{\partial}_\rho(\hat{f}\hat{g}) = (\hat{\partial}_\rho\hat{f})\hat{g} + \hat{f}(\hat{\partial}_\rho\hat{g}). \tag{1.5.4}$$

It is undeformed, as expected from (1.5.2). In terms of the comultiplication we have

$$\Delta\hat{\partial}_\rho = \hat{\partial}_\rho \otimes 1 + 1 \otimes \hat{\partial}_\rho. \tag{1.5.5}$$

We define the counit $\varepsilon(\hat{\partial}_\rho) = 0$ and check that (1.3.4) and(1.3.5) are fulfilled. Since this is the case, we speak about the coalgebra of derivatives.

To be able to define the bialgebra of derivatives, we have to check whether the coproduct (1.5.5) is compatible with the algebra of derivatives (1.5.3)

$$[\Delta\hat{\partial}_\rho, \Delta\hat{\partial}_\sigma] = 0, \tag{1.5.6}$$

and whether

$$\varepsilon(\hat{\partial}_\rho\hat{\partial}_\sigma) = \varepsilon(\hat{\partial}_\rho)\varepsilon(\hat{\partial}_\sigma). \tag{1.5.7}$$

It is not difficult to see that both (1.5.6) and (1.5.7) are fulfilled, calculation is the same as in the example given in Section 1.3, so we have the bialgebra of derivatives. Adding the antipode $S(\hat{\partial}_\rho) = -\hat{\partial}_\rho$ this becomes the Hopf algebra of derivatives on the θ-deformed space.

Following the logic of the previous sections one should proceed with the analysis of the deformed symmetries on this space. However, we postpone this problem until Chapter 5. Instead, we continue our analysis in the $\star$-product representation. Since the θ-deformed space has PBW property, one can map functions from the abstract algebra to the space of commuting coordinates, one only has to specify the ordering (basis) in $\hat{\mathcal{A}}_{\hat{x}}$. We chose the symmetric ordering.

For the θ-deformed space the symmetrically ordered $\star$-product is the Moyal-Weyl $\star$-product [66], [67]

$$\begin{aligned} f \star g\,(x) &= \lim_{x\to y} e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}} f(x)g(y) \qquad (1.5.8)\\ &= \sum_{n=1}^{\infty}\Big(\frac{i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}f(x)\Big)\Big(\partial_{\sigma_1}\dots\partial_{\sigma_n}g(x)\Big). \end{aligned}$$

When applied on coordinates, (1.5.8) gives

$$[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}. \tag{1.5.9}$$

Note that one can start from (1.5.9) with the $\star$-product given by (1.5.8) and formulate a theory based on this relation, forgeting about the abstract algebra all together.

The $\star$-product (1.5.8) respects the usual complex conjugation[8]

$$\overline{f \star g\,(x)} = \bar{g} \star \bar{f}\,(x) \tag{1.5.10}$$

[8]One can show that all the symmetrically ordered $\star$-products respect the usual complex conjugation.

and this is the reason why we continue working with it in Chapter 5.

Now we have to introduce the $\star$-representation of the derivatives (1.5.2) following the prescription given in Section 1.4. It is not suprising that

$$\hat{\partial}_\rho \mapsto \partial^\star_\rho = \partial_\rho \tag{1.5.11}$$

and

$$\begin{aligned}\big(\partial^\star_\rho \star (f \star g)\big) = \partial^\star_\rho \triangleright (f \star g) &= \big(\partial^\star_\rho \star f\big) \star g + f \star \big(\partial^\star_\rho \star g\big)\\ &= \big(\partial^\star_\rho \triangleright f\big) \star g + f \star \big(\partial^\star_\rho \triangleright g\big).\end{aligned} \tag{1.5.12}$$

In the last line we have introduced a new notation concerning $\partial^\star_\rho$ derivatives. In order to distinguish between derivatives acting on a function and derivative as an operator multiplied with a function

$$\partial^\star_\rho \star f = (\partial^\star_\rho \star f) + f \star \partial^\star_\rho, \tag{1.5.13}$$

we introduce a new symbol "$\triangleright$" which stands for "a derivative acting on a function", that is

$$\partial^\star_\rho \triangleright f = (\partial^\star_\rho \star f). \tag{1.5.14}$$

Obviously, this symbol replaces the usual bracket notation. However, in what follows we will use both "$\triangleright$" and the bracket notation. Also, for the usual partial derivatives we often omit the bracket notation when it is not necessary and write

$$\partial_\rho f = (\partial_\rho f). \tag{1.5.15}$$

With (1.5.8), (1.5.11) and (1.5.12) one has enough (basic) information to formulate field theories on this space [19], [41], [43].

2

The κ-deformed space

After the general introduction in the previous chapter we continue with one special example of noncommutative spaces. In this and in the next two chapters we study the κ-deformed space and formulation of the gauge field theory on it. The reason why the κ-deformed space has been studied in the last years is that there is a quantum group symmetry acting on it[1]. It is the so-called κ-Poincaré group. Historically, it was first obtained by Lukierski et al. [29], [30] contracting the q-anti-de Sitter Hopf algebra $SO_q(3,2)$. The κ-Poincaré algebra was introduced in [68] as a dual symmetry structure to the κ-Poincaré group. Then the κ-deformed space is introduced as a module of this algebra. This space plays also an important role in the Doubly Special Relativity (DSR) theories [69], [70], [71], which are introduced as a possible generalisation of Special Relativity.

Here we start from the definition of the κ-deformed space in terms of the abstract algebra of coordinates. As it was outlined in the previous chapter, we introduce derivatives and symmetry generators as maps in the abstract algebra and represent obtained results on the space of commuting coordinates.

2.1 Quantum space and derivatives

Algebraically, the $n+1$-dimensional κ-deformed space can be introduced [35] as the algebra freely generated by coordinates $\hat{x}^\mu$ and divided by the ideal generated by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho, \tag{2.1.1}$$

where

$$C^{\mu\nu}_\rho = a(\delta^\mu_n \delta^\nu_\rho - \delta^\nu_n \delta^\mu_\rho), \quad \mu = 0, \dots, n. \tag{2.1.2}$$

Latin indices denote the undeformed dimensions, n denotes the deformed dimension and Greek indices refer to all $n+1$ dimensions. Indices are raised and lowered by the (formal) metric $\eta^{\mu\nu} = diag(1, -1, \dots, -1)$. We have chosen that the constant deformation vector a^μ of length a points in the n-th spacelike direction, $a^n = a$. Different choices are discussed in the literature [72], [73]. In Appendix C we analyse the general κ-deformed space, imposing no restriction on the deformation parameter a^μ. The parameter a is related to the frequently

[1]It has been believed until recently that a quantum group symmetry for the θ-deformed space does not exist. Therefore, the κ-deformed space has been considered to be one of the simplest examples of possible deformations of the usual Minkowski space.

used parameter κ as $a = 1/\kappa$. Using (2.1.2) the commutation relation (2.1.1) is written more explicitly

$$[\hat{x}^n, \hat{x}^l] = ia\hat{x}^l, \quad [\hat{x}^k, \hat{x}^l] = 0; \quad k,l = 0,1,\dots,n-1. \tag{2.1.3}$$

Derivatives are introduced as maps on the abstract algebra of coordinates. One can start from ansatz (1.2.2); demanding the consistency with (2.1.3) one obtains conditions on the coefficients $A_\rho^{\mu\lambda_1\dots\lambda_j}$. However, we start here with a less general ansatz

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu + iA_\mu^{\nu\rho}\hat{\partial}_\rho, \tag{2.1.4}$$

where we supposed that the right hand side is at most linear in derivatives. Looking at the index structure and from dimensional reasons ansatz (2.1.4) can be written in the following way

$$\begin{aligned}
[\hat{\partial}_n, \hat{x}^n] &= 1 + iac_1\hat{\partial}_n, \\
[\hat{\partial}_n, \hat{x}^l] &= iac_2\eta^{lm}\hat{\partial}_m = iac_2\hat{\partial}^l, \\
[\hat{\partial}_j, \hat{x}^n] &= iac_3\hat{\partial}_j, \\
[\hat{\partial}_j, \hat{x}^l] &= \delta_j^l(1 + iac_4\hat{\partial}_n).
\end{aligned} \tag{2.1.5}$$

Then one calculates

$$\begin{aligned}
\hat{\partial}_n\big(\hat{x}^k\hat{x}^l - \hat{x}^l\hat{x}^k\big) &= 0, \\
\hat{\partial}_n\big(\hat{x}^n\hat{x}^l - \hat{x}^l\hat{x}^n - ia\hat{x}^l\big) &= 0, \\
\hat{\partial}_j\big(\hat{x}^k\hat{x}^l - \hat{x}^l\hat{x}^k\big) &= 0, \\
\hat{\partial}_j\big(\hat{x}^n\hat{x}^l - \hat{x}^l\hat{x}^n - ia\hat{x}^l\big) &= 0
\end{aligned} \tag{2.1.6}$$

using (2.1.5). The calculation is the same as in Section 1.5 and it gives the following conditions on the constants c_i

$$\begin{aligned}
&\text{1st equation:} && \text{no conditions,} \\
&\text{2nd equation:} && c_2 = 0 \ \vee \ c_1 - c_3 - 1 = 0, \\
&\text{3rd equation:} && c_2 = 0 \ \vee \ c_4 = 0, \\
&\text{4th equation:} && c_3 - c_4 - 1 = 0 \ \wedge \ \big(c_4 = 0 \ \vee \ c_3 - c_1 - 1 = 0\big).
\end{aligned} \tag{2.1.7}$$

Analysing these conditions we obtain three one parameter families of derivatives

$$\begin{aligned}
[\hat{\partial}_n^{c_1}, \hat{x}^n] &= 1 + iac_1\hat{\partial}_n^{c_1}, & [\hat{\partial}_n^{c_2}, \hat{x}^n] &= 1 + iac_2\hat{\partial}_n^{c_2}, & [\hat{\partial}_n^{c_3}, \hat{x}^n] &= 1 + 2ia\hat{\partial}_n^{c_3}, \\
[\hat{\partial}_n^{c_1}, \hat{x}^l] &= 0, & [\hat{\partial}_n^{c_2}, \hat{x}^l] &= 0, & [\hat{\partial}_n^{c_3}, \hat{x}^l] &= iac_3\hat{\partial}^{c_3\,l}, \\
[\hat{\partial}_j^{c_1}, \hat{x}^n] &= ia\hat{\partial}_j^{c_1}, & [\hat{\partial}_j^{c_2}, \hat{x}^n] &= ia(1+c_2)\hat{\partial}_j^{c_2}, & [\hat{\partial}_j^{c_3}, \hat{x}^n] &= ia\hat{\partial}_j^{c_3}, \\
[\hat{\partial}_j^{c_1}, \hat{x}^l] &= \delta_j^l, & [\hat{\partial}_j^{c_2}, \hat{x}^l] &= \delta_j^l(1 + iac_2\hat{\partial}_n^{c_2}), & [\hat{\partial}_j^{c_3}, \hat{x}^l] &= \delta_j^l.
\end{aligned} \tag{2.1.8}$$

The constants c_1, c_2 and c_3 remain arbitrary and can not be fixed by the consistency condition. One can check that derivatives of all three families commute

$$[\hat{\partial}_\mu^{c_i}, \hat{\partial}_\nu^{c_i}] = 0, \quad i = 1,2,3. \tag{2.1.9}$$

In the following sections we work with one specific choice, namely we take $c_1 = 0$

$$
\begin{aligned}
[\hat{\partial}_n, \hat{x}^\mu] &= \delta^\mu_n, \\
[\hat{\partial}_j, \hat{x}^\mu] &= \delta^\mu_j - ia\eta^{\mu n}\hat{\partial}_j,
\end{aligned}
\tag{2.1.10}
$$

where we denoted $\hat{\partial}^{c_1=0}_\mu$ as $\hat{\partial}_\mu$. But it is always possible to map $\hat{\partial}_\mu$ derivatives to any other $\hat{\partial}^{c_i}_\mu$. The explicit maps are given by

$$
\hat{\partial}^{c_1}_\mu : \qquad \hat{\partial}^{c_1}_j = \hat{\partial}_j, \qquad \hat{\partial}^{c_1}_n = \frac{e^{iac_1\hat{\partial}_n} - 1}{iac_1}, \tag{2.1.11}
$$

$$
\hat{\partial}^{c_2}_\mu : \qquad \hat{\partial}^{c_2}_n = \frac{e^{iac_2\hat{\partial}_n} - 1}{iac_2}, \qquad \hat{\partial}^{c_2}_j = \hat{\partial}_j e^{iac_2\hat{\partial}_n}, \tag{2.1.12}
$$

$$
\hat{\partial}^{c_3}_\mu : \qquad \hat{\partial}^{c_3}_n = \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} + \frac{iac_3}{2}\hat{\partial}_k\hat{\partial}^k, \qquad \hat{\partial}^{c_3}_j = \hat{\partial}_j. \tag{2.1.13}
$$

Next step is to calculate the Leibniz rule for $\hat{\partial}_\mu$ derivatives. Applying $\hat{\partial}_\mu$ on ordered polynomials using (2.1.10) gives

$$
\begin{aligned}
\hat{\partial}_n\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_n\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{\partial}_n\hat{g}), \\
\hat{\partial}_j\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_j\hat{f})\cdot\hat{g} + (e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{\partial}_j\hat{g}),
\end{aligned}
\tag{2.1.14}
$$

or written in terms of coproduct

$$
\begin{aligned}
\Delta\hat{\partial}_n &= \hat{\partial}_n\otimes 1 + 1\otimes\hat{\partial}_n, \\
\Delta\hat{\partial}_j &= \hat{\partial}_j\otimes 1 + e^{ia\hat{\partial}_n}\otimes\hat{\partial}_j.
\end{aligned}
\tag{2.1.15}
$$

Leibniz rules and coproducts for $\hat{\partial}^{c_i}_\mu$ derivatives are

$$
\begin{aligned}
\hat{\partial}^{c_1}_n(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_1}_n\hat{f})\cdot\hat{g} + ((1+ic_1a\hat{\partial}^{c_1}_n)\hat{f})\cdot\hat{\partial}^{c_1}_n\hat{g}, \\
\hat{\partial}^{c_1}_j(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_1}_j\hat{f})\cdot\hat{g} + ((1+ic_1a\hat{\partial}^{c_1}_n)^{1/c_1}\hat{f})\cdot\hat{\partial}^{c_1}_j\hat{g},
\end{aligned}
\tag{2.1.16}
$$

$$
\begin{aligned}
\hat{\partial}^{c_2}_n(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_2}_n\hat{f})\cdot\hat{g} + ((1+ic_2a\hat{\partial}^{c_2}_n)\hat{f})\cdot\hat{\partial}^{c_2}_n\hat{g}, \\
\hat{\partial}^{c_2}_j(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_2}_j\hat{f})\cdot((1+ic_2a\hat{\partial}^{c_2}_n)\hat{g}) + \Big((1+iac_2\hat{\partial}^{c_2}_n)^{(c_2+1)/c_2}\hat{f}\Big)\cdot(\hat{\partial}^{c_2}_j\hat{g}),
\end{aligned}
\tag{2.1.17}
$$

$$
\begin{aligned}
\hat{\partial}^{c_3}_n(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_3}_n\hat{f})\cdot\hat{g} + ((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_m\hat{\partial}^{c_3\,m})\hat{f})\cdot\hat{\partial}^{c_3}_n\hat{g} \\
&\quad + iac_3((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_m\hat{\partial}^{c_3\,m})^{1/2}\hat{\partial}^{c_3}_l\hat{f})\cdot\hat{\partial}^{c_3\,l}\hat{g}, \\
\hat{\partial}^{c_3}_j(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_3}_j\hat{f})\cdot\hat{g} + ((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_m\hat{\partial}^{c_3\,m})^{1/2}\hat{f})\cdot(\hat{\partial}^{c_3}_j\hat{g})
\end{aligned}
\tag{2.1.18}
$$

and

$$\begin{aligned}
\Delta\hat{\partial}_n^{c_1} &= \hat{\partial}_n^{c_1} \otimes 1 + (1 + ic_1 a\hat{\partial}_n^{c_1}) \otimes \hat{\partial}_n^{c_1}, \\
\Delta\hat{\partial}_j^{c_1} &= \hat{\partial}_j^{c_1} \otimes 1 + (1 + ic_1 a\hat{\partial}_n^{c_1})^{1/c_1} \otimes \hat{\partial}_j^{c_1},
\end{aligned} \tag{2.1.19}$$

$$\begin{aligned}
\Delta\hat{\partial}_n^{c_2} &= \hat{\partial}_n^{c_2} \hat{f}) \otimes 1 + (1 + ic_2 a\hat{\partial}_n^{c_2}) \otimes \hat{\partial}_n^{c_2}, \\
\Delta\hat{\partial}_j^{c_2} &= \hat{\partial}_j^{c_2} \otimes (1 + ic_2 a\hat{\partial}_n^{c_2}) + (1 + iac_2\hat{\partial}_n^{c_2})^{(c_2+1)/c_2} \otimes \hat{\partial}_j^{c_2},
\end{aligned} \tag{2.1.20}$$

$$\begin{aligned}
\Delta\hat{\partial}_n^{c_3} &= \hat{\partial}_n^{c_3} \otimes 1 + (1 + 2ia\hat{\partial}_n^{c_3} + a^2 c_3 \hat{\partial}_m^{c_3}\hat{\partial}^{c_3\,m}) \otimes \hat{\partial}_n^{c_3} \\
&\quad + iac_3(1 + 2ia\hat{\partial}_n^{c_3} + a^2 c_3\hat{\partial}_m^{c_3}\hat{\partial}^{c_3\,m})^{1/2}\hat{\partial}_l^{c_3} \otimes \hat{\partial}^{c_3\,l}, \\
\Delta\hat{\partial}_j^{c_3} &= \hat{\partial}_j^{c_3} \otimes 1 + (1 + 2ia\hat{\partial}_n^{c_3} + a^2 c_3\hat{\partial}_m^{c_3}\hat{\partial}^{c_3\,m})^{1/2} \otimes \hat{\partial}_j^{c_3}.
\end{aligned} \tag{2.1.21}$$

One can check that these coproducts are coassociative and that they are consistent with the algebra (2.1.3).

Adding counit and antipode to equations (2.1.9) and (2.1.15) we obtain a Hopf algebra of $\hat{\partial}_\mu$ derivatives on the κ-deformed space

$$\begin{aligned}
\varepsilon(\hat{\partial}_n) = 0, &\qquad S(\hat{\partial}_n) = -\hat{\partial}_n, \\
\varepsilon(\hat{\partial}_j) = 0, &\qquad S(\hat{\partial}_j) = -\hat{\partial}_j e^{-ia\hat{\partial}_n}.
\end{aligned} \tag{2.1.22}$$

Analogously, for $\hat{\partial}_\mu^{c_i}$ derivatives counits and antipodes are

$$\begin{aligned}
\hat{\partial}_\mu^{c_1}: \qquad & \varepsilon(\hat{\partial}_n^{c_1}) = 0, \quad S(\hat{\partial}_n^{c_1}) = \frac{\hat{\partial}_n^{c_1}}{1 + iac_1\hat{\partial}_n^{c_1}}, \\
& \varepsilon(\hat{\partial}_j^{c_1}) = 0, \quad S(\hat{\partial}_j^{c_1}) = -\hat{\partial}_j^{c_1}\left(1 + iac_1\hat{\partial}_n^{c_1}\right)^{-1/c_1},
\end{aligned} \tag{2.1.23}$$

$$\begin{aligned}
\hat{\partial}_\mu^{c_2}: \qquad & \varepsilon(\hat{\partial}_n^{c_2}) = 0, \quad S(\hat{\partial}_n^{c_2}) = \frac{\hat{\partial}_n^{c_2}}{1 + iac_2\hat{\partial}_n^{c_2}}, \\
& \varepsilon(\hat{\partial}_j^{c_2}) = 0, \quad S(\hat{\partial}_j^{c_2}) = -\hat{\partial}_j^{c_2}\left(1 + iac_a\hat{\partial}_n^{c_2}\right)^{-1/c_2-2},
\end{aligned} \tag{2.1.24}$$

$$\begin{aligned}
\hat{\partial}_\mu^{c_3}: \qquad & \varepsilon(\hat{\partial}_n^{c_3}) = 0, \quad S(\hat{\partial}_n^{c_3}) = \frac{\hat{\partial}_n^{c_3} - \frac{ia}{2}c_3\hat{\partial}_l^{c_3}\hat{\partial}^{c_3\,l}}{1 + 2iac_3\hat{\partial}_n^{c_3} + a^2 c_3\hat{\partial}_m^{c_3}\hat{\partial}^{c_3\,m}} + \frac{ia}{2}c_3\hat{\partial}_l^{c_3}\hat{\partial}^{c_3\,l}, \\
& \varepsilon(\hat{\partial}_j^{c_3}) = 0, \quad S(\hat{\partial}_j^{c_3}) = -\hat{\partial}_j^{c_3}\left(1 + 2iac_3\hat{\partial}_n^{c_3} + a^2 c_3\hat{\partial}_l^{c_3}\hat{\partial}^{c_3\,l}\right)^{-1/2}.
\end{aligned} \tag{2.1.25}$$

2.2 Symmetry generators

In our algebraic approach symmetry of the κ-deformed space is given in terms of the symmetry algebra[2] generators $M^{\mu\nu}$. Just like derivatives, they are maps in the abstract algebra. Additional condition on the generators $M^{\mu\nu}$ is that they have to be a deformation of the

[2]This means that we only consider infinitesimal transformations and not the finite ones.

usual (commutative) Lorentz generators. This means that in zeroth order of the deformation parameter a they have to coincide with the generators of the commutative Lorentz transformations

$$[M^{\mu\nu}, \hat{x}^\lambda] = \eta^{\mu\lambda}\hat{x}^\nu - \eta^{\nu\lambda}\hat{x}^\mu + \mathcal{O}(a). \tag{2.2.1}$$

The additional terms on the right hand side of (2.2.1) can be calculated using the consistency conditions

$$\begin{aligned} M^{\mu\nu}\big(\hat{x}^n\hat{x}^l - \hat{x}^l\hat{x}^n - ia\hat{x}^l\big) &= 0, \\ M^{\mu\nu}\big(\hat{x}^k\hat{x}^l - \hat{x}^l\hat{x}^k\big) &= 0. \end{aligned} \tag{2.2.2}$$

The calculation can be done in the same way as for the derivatives in the previous section. However, there are some additional restrictions one can impose on the right hand side of (2.2.1). First of all, the generators $M^{\mu\nu}$ should appear at most linearly on the right hand side of (2.2.1) so that in first order in a only terms with $M^{\mu\nu}$ appear. This follows from dimensional arguments since $M^{\mu\nu}$ are dimensionless[3]. In higher order in a we would have to include derivatives as well, again because of dimensional reasons. Since we want (2.2.1) to close only in coordinates and $M^{\mu\nu}$ generators we stop at first order in a. Of course, the indices on both sides have to match. Using these restrictions one commutes $M^{\mu\nu}$ to the furthest right in equations (2.2.2) and finds conditions on the additional terms. The unique solution[4] is given by

$$\begin{aligned} [M^{ij}, \hat{x}^\mu] &= \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j, \\ [M^{in}, \hat{x}^\mu] &= \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + iaM^{i\mu}. \end{aligned} \tag{2.2.3}$$

We see that M^{ij} commute with coordinates as in the undeformed algebra, while the generators M^{in} have deformed commutation relations with coordinates. We do not refer to M^{in} as boost generators, since n is not the time direction, M^{in} include both boosts M^{0n} and rotations M^{an}, $a = 1, \dots n-1$.

Although from (2.2.3) it follows that $M^{\mu\nu}$ act in a deformed way on coordinates, one can check that the generators $M^{\mu\nu}$ themselves close the undeformed Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}. \tag{2.2.4}$$

But we are also interested in the coalgebra structure. Therefore, one calculates the Leibniz rules for the generators $M^{\mu\nu}$ by applying them to ordered polynomials using (2.2.3) and generalising the result to arbitrary functions. This gives

$$\begin{aligned} M^{ij}(\hat{f}\cdot\hat{g}) &= (M^{ij}\hat{f})\cdot\hat{g} + \hat{f}\cdot(M^{ij}\hat{g}), \\ M^{in}(\hat{f}\cdot\hat{g}) &= (M^{in}\hat{f})\cdot\hat{g} + (e^{ia\hat{\partial}_n}\hat{f})\cdot(M^{in}\hat{g}) + ia(\hat{\partial}_k\hat{f})\cdot(M^{ik}\hat{g}). \end{aligned} \tag{2.2.5}$$

It is not difficult to see that this Leibniz rules come from the following coproduct

$$\begin{aligned} \Delta M^{ij} &= M^{ij}\otimes 1 + 1\otimes M^{ij}, \\ \Delta M^{in} &= M^{in}\otimes 1 + e^{ia\hat{\partial}_n}\otimes M^{in} + ia\hat{\partial}_k\otimes M^{ik}. \end{aligned} \tag{2.2.6}$$

[3]Compare with the undeformed angular momentum $L^{\alpha\beta} = x^\alpha\partial^\beta - x^\beta\partial^\alpha$.

[4]Note the discussion in Appendix B.

Looking at (2.2.6) one sees that the coproduct for the generators $M^{\mu\nu}$ does not close in the algebra of the deformed Lorentz generators, one has to include the derivatives $\hat{\partial}_\mu$ introduced in the previous section as well. In order to have the full κ-deformed Poincaré algebra one needs besides (2.1.9) and (2.2.4) also the commutator between Lorentz generators and derivatives

$$\begin{aligned}
[M^{ij}, \hat{\partial}_\mu] &= \delta^j_\mu \hat{\partial}^i - \delta^i_\mu \hat{\partial}^j, \\
[M^{in}, \hat{\partial}_n] &= \hat{\partial}^i, \\
[M^{in}, \hat{\partial}_j] &= \delta^i_j \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} - \frac{ia}{2}\delta^i_j \hat{\partial}^l \hat{\partial}_l + ia\hat{\partial}^i \hat{\partial}_j.
\end{aligned} \tag{2.2.7}$$

The coproducts are given by (2.1.15) and (2.2.6), counit and antipode for $\hat{\partial}_\mu$ are given by (2.1.22) and counit and antipode for $M^{\mu\nu}$ read

$$\begin{aligned}
&\epsilon(M^{ij}) = 0, \qquad S(M^{ij}) = -M^{ij}, \\
&\epsilon(M^{in}) = 0, \qquad S(M^{in}) = -M^{in} e^{-ia\hat{\partial}_n} + iaM^{ik}\hat{\partial}_k e^{-ia\hat{\partial}_n} + ia(n-1)\hat{\partial}^i e^{-ia\hat{\partial}_n}.
\end{aligned} \tag{2.2.8}$$

One checks that the conditions for a Hopf algebra introduced in Section 1.3 are fulfilled. The Lorentz part of the algebra sector is undeformed (as well as translation part itself), but we have nontrivial commutation relations between Lorentz generators and derivatives. The coalgebra sector is deformed for both Lorentz generators and derivatives. In the next section we define a new set of derivatives such that we obtain the complete algebra sector of the κ-deformed Poincaré Hopf algebra undeformed.

In analogy with the undeformed space one can represent the generators $M^{\mu\nu}$ in terms of coordinates and derivatives

$$\begin{aligned}
M^{ij} &= \hat{x}^i \hat{\partial}^j - \hat{x}^j \hat{\partial}^i \stackrel{\text{def}}{=} \hat{L}^{ij}, \\
M^{in} &= \hat{x}^i \frac{1 - e^{2ia\hat{\partial}_n}}{2ia} - \hat{x}^n \hat{\partial}^i + \frac{ia}{2}\hat{x}^i \hat{\partial}^l \hat{\partial}_l \stackrel{\text{def}}{=} \hat{L}^{in}.
\end{aligned} \tag{2.2.9}$$

2.3 Dirac derivative

We have seen that there is no unique derivative for the κ-deformed space and that one can always relate one set of derivatives with the other one. Now we use this freedom to find new set of derivatives, such that they commute with the generators $M^{\mu\nu}$ in the undeformed way. We call this derivative the Dirac derivative [74], [75] and denote it with $\hat{D}_\rho$. Then

$$[M^{\mu\nu}, \hat{D}_\rho] = \delta^\nu_\rho \hat{D}^\mu - \delta^\mu_\rho \hat{D}^\nu \tag{2.3.1}$$

has to be fulfilled. Using the derivatives $\hat{\partial}_\rho$ and (2.2.7) one shows that the definition

$$\begin{aligned}
\hat{D}_n &= \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{ia}{2}\hat{\partial}^l \hat{\partial}_l e^{-ia\hat{\partial}_n}, \\
\hat{D}_j &= \hat{\partial}_j e^{-ia\hat{\partial}_n},
\end{aligned} \tag{2.3.2}$$

leads to (2.3.1). Since $\hat{D}_\rho$ is the linear combination of $\hat{\partial}_\rho$ derivatives, it is obvious that

$$[\hat{D}_\rho, \hat{D}_\sigma] = 0. \tag{2.3.3}$$

From (2.3.1) it follows that if one uses $\hat{D}_\rho$ instead of $\hat{\partial}_\rho$ as the generators of translations one obtains the undeformed algebra sector of the κ-deformed Poincaré algebra. But in order to have the full Hopf algebra in terms of the generators $M^{\mu\nu}$ and $\hat{D}_\rho$ one has to calculate the coproduct for the derivatives $\hat{D}_\rho$ and add counit and antipode as well. Also, one has to express the derivatives $\hat{\partial}_\rho$ appearing in (2.2.6) in terms of the derivatives $\hat{D}_\rho$.

We start from the commutation relations of $\hat{D}_\rho$ with coordinates. Knowing (2.3.2) and (2.1.10) it is not difficult to calculate them, but at the end the result has to be expressed in terms of the derivatives $\hat{D}_\rho$ and coordinates only. To be able to do this we have to invert the relations (2.3.2). Starting form

$$\begin{aligned} \hat{\partial}_j &= \hat{D}_j e^{ia\hat{\partial}_n} \quad \Rightarrow \quad \hat{\partial}_l\hat{\partial}^l = \hat{D}_l\hat{D}^l e^{2ia\hat{\partial}_n}, \\ \hat{D}_n &= \frac{1}{2ia}\left(e^{ia\hat{\partial}_n} - e^{-ia\hat{\partial}_n}\right) - \frac{ia}{2}\hat{D}_l\hat{D}^l e^{ia\hat{\partial}_n} \end{aligned} \tag{2.3.4}$$

and multiplying equation (2.3.4) by $e^{-ia\hat{\partial}_n}$ leads to the quadratic equation for $e^{-ia\hat{\partial}_n}$

$$e^{-2ia\hat{\partial}_n} + 2ia\hat{D}_n e^{-ia\hat{\partial}_n} - a^2\hat{D}_l\hat{D}^l - 1 = 0.$$

The solution of this equation is

$$e^{-ia\hat{\partial}_n} = -ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}, \tag{2.3.5}$$

where the sign of the square root is determined by the limit $a \to 0$. On the other hand, multiplying equation (2.3.4) by $e^{ia\hat{\partial}_n}$ we find a quadratic equation for $e^{ia\hat{\partial}_n}$ with the solution

$$e^{ia\hat{\partial}_n} = \frac{1}{1 + a^2\hat{D}_l\hat{D}^l}\left(ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}\right). \tag{2.3.6}$$

It is easy to verify that (2.3.6) is the inverse of (2.3.5). Now we invert (2.3.2)

$$\begin{aligned} \hat{\partial}_i &= \frac{\hat{D}_i}{1 + a^2\hat{D}_l\hat{D}^l}\left(ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}\right), \\ \hat{\partial}_n &= -\frac{1}{ia}\ln\left(-ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}\right). \end{aligned} \tag{2.3.7}$$

This result was independently obtained in [76]. Using this result the commutator of the derivatives $\hat{D}_\rho$ and coordinates is obtained

$$\begin{aligned} [\hat{D}_n, \hat{x}^n] &= \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}, \\ [\hat{D}_n, \hat{x}^l] &= ia\hat{D}^l, \\ [\hat{D}_j, \hat{x}^n] &= 0, \\ [\hat{D}_j, \hat{x}^l] &= \delta_j^l\left(-ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}\right). \end{aligned} \tag{2.3.8}$$

It is obvious that the Dirac derivative is not a linear derivative (in the sense of (2.1.4)), the right hand side of (2.3.8) being a complicated function of $\hat{D}_\rho$. Also, the Leibniz rule following from (2.3.8) is not simple

$$
\begin{aligned}
\hat{D}_n\,(\hat{f}\cdot\hat{g}) &= (\hat{D}_n\hat{f})\cdot(e^{-ia\hat{\partial}_n}\hat{g}) + (e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{D}_n\hat{g}) - ia(\hat{D}_l e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{D}^l\hat{g}),\\
\hat{D}_j\,(\hat{f}\cdot\hat{g}) &= (\hat{D}_j\hat{f})\cdot(e^{-ia\hat{\partial}_n}\hat{g}) + \hat{f}\cdot(\hat{D}_j\hat{g}).
\end{aligned}
\tag{2.3.9}
$$

For $e^{\pm ia\hat{\partial}_n}$ the expressions (2.3.5) and (2.3.6) have to be inserted. The comultiplication from which (2.3.9) follows is

$$
\begin{aligned}
\Delta\hat{D}_n &= \hat{D}_n\otimes\Big(-ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big) + \frac{ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l}\otimes\hat{D}_n\\
&\quad +ia\frac{\hat{D}_k}{1+a^2\hat{D}_l\hat{D}^l}\Big(ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big)\otimes\hat{D}^k,\\
\Delta\hat{D}_j &= \hat{D}_j\otimes\Big(-ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big)+1\otimes\hat{D}_j,
\end{aligned}
\tag{2.3.10}
$$

where we have used equations (2.3.5) and (2.3.6). One can check that (2.3.10) is coassociative and that it is consistent with the algebra (2.3.3). Counit and antipode are

$$
\begin{aligned}
\varepsilon(\hat{D}_n) &= 0, \qquad S(\hat{D}_n) = -\hat{D}_n + ia\hat{D}_k\hat{D}^k\frac{ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l},\\
\varepsilon(\hat{D}_j) &= 0, \qquad S(\hat{\partial}_j) = -\hat{D}_j\frac{ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l}.
\end{aligned}
\tag{2.3.11}
$$

Finally, with all this relations one can introduce the κ-deformed Poincaré Hopf algebra in terms of the generators $M^{\mu\nu}$ and $\hat{D}_\rho$. All the necessary relations have already been written, but just for the completeness we collect them all at one place here.

Algebra sector

$$
\begin{aligned}
[M^{\mu\nu},M^{\rho\sigma}] &= \eta^{\mu\sigma}M^{\nu\rho}+\eta^{\nu\rho}M^{\mu\sigma}-\eta^{\mu\rho}M^{\nu\sigma}-\eta^{\nu\sigma}M^{\mu\rho},\\
[\hat{D}_\rho,\hat{D}_\sigma] &= 0,\\
[M^{\mu\nu},\hat{D}_\rho] &= \delta^\nu_\rho\hat{D}^\mu-\delta^\mu_\rho\hat{D}^\nu.
\end{aligned}
\tag{2.3.12}
$$

Coproducts

$$
\begin{aligned}
\Delta M^{ij} &= M^{ij}\otimes 1+1\otimes M^{ij},\\
\Delta M^{in} &= M^{in}\otimes 1+\frac{ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l}\otimes M^{in}\\
&\quad +\frac{ia\hat{D}_k}{1+a^2\hat{D}_l\hat{D}^l}\Big(ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big)\otimes M^{ik},\\
\Delta\hat{D}_n &= \hat{D}_n\otimes\Big(-ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big) + \frac{ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l}\otimes\hat{D}_n\\
&\quad +ia\frac{\hat{D}_k}{1+a^2\hat{D}_l\hat{D}^l}\Big(ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big)\otimes\hat{D}^k,\\
\Delta\hat{D}_j &= \hat{D}_j\otimes\Big(-ia\hat{D}_n+\sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}\Big)+1\otimes\hat{D}_j.
\end{aligned}
\tag{2.3.13}
$$

Counits and antipodes

$$\epsilon(M^{ij}) = 0, \qquad S(M^{ij}) = -M^{ij},$$
$$\epsilon(M^{in}) = 0, \qquad S(M^{in}) = -M^{in}e^{-ia\hat{\partial}_n} + iaM^{ik}\hat{\partial}_k e^{-ia\hat{\partial}_n} + ia(n-1)\hat{\partial}^i e^{-ia\hat{\partial}_n}$$
$$\varepsilon(\hat{D}_n) = 0, \qquad S(\hat{D}_n) = -\hat{D}_n + ia\hat{D}_k\hat{D}^k \frac{ia\hat{D}_n + \sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l},$$
$$\varepsilon(\hat{D}_j) = 0, \qquad S(\hat{\partial}_j) = -\hat{D}_j \frac{ia\hat{D}_n + \sqrt{1+a^2\hat{D}_\mu\hat{D}^\mu}}{1+a^2\hat{D}_l\hat{D}^l}. \qquad (2.3.14)$$

One sees from (2.3.12) that the algebra sector is undeformed (as it has been demanded in the beginning of this section), but the coalgebra sector (2.3.13) is deformed for both $M^{\mu\nu}$ and $\hat{D}_\rho$ generators. To be more precise, there is no deformation for the generators M^{ij}, since they are Lorentz generators for the undeformed dimensions.

Like in the classical case one is also interested in the invariants (Casimir operators) of this algebra. For the usual four dimensional Minkowski space-time there are two invariants, the square of momenta $P^2 = P_\mu P^\mu$ ($P_\mu = i\partial_\mu$) and the square of the Pauli-Lubanski vector $W^2 = W_\mu W^\mu$ ($W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma$). We would like to generalise this to the $n+1$ dimensional κ-deformed space. From (2.3.1) follows

$$[M^{\mu\nu}, \hat{D}_\rho\hat{D}^\rho] = 0, \qquad (2.3.15)$$

that is $\hat{D}_\rho\hat{D}^\rho$ can be considered as a generalisation of P^2. However, this is not the only possibility. One can show that the lowest order invariant in terms of the derivatives $\hat{\partial}_\rho$ is [29], [30]

$$\hat{\Box} = e^{-ia\hat{\partial}_n}\hat{\partial}_l\hat{\partial}^l - \frac{2}{a^2}\Big(1-\cos(a\hat{\partial}_n)\Big) \qquad (2.3.16)$$

and it fulfils

$$[M^{\mu\nu}, \hat{\Box}] = 0. \qquad (2.3.17)$$

The operator $\hat{\Box}$ defined by (2.3.16) we call the deformed d'Alembert operator and we use it to construct the κ-deformed Klein-Gordon equation in Section 2.5. Unlike in the classical case, we have

$$\hat{D}_\mu\hat{D}^\mu = \hat{\Box}\Big(1+\frac{a^2}{4}\hat{\Box}\Big), \qquad (2.3.18)$$

the square of the Dirac derivative is not the d'Alembert operator.

Concerning the second invariant, we first introduce the generalisation of the Pauli-Lubanski vector in $d = n+1 = (2k+1)+1$ dimensions [77]

$$W_{\mu_1\ldots\mu_{2i-1}} = \epsilon_{\mu_1\ldots\mu_n}M^{\mu_{2i}\mu_{2i+1}}\ldots M^{\mu_{n-2}\mu_{n-1}}\hat{D}^{\mu_n}. \qquad (2.3.19)$$

There are $(d-2)/2$ invariants given by

$$W_{i+1}^2 = W_{\mu_1\ldots\mu_{2i-1}}W^{\mu_1\ldots\mu_{2i-1}}, \qquad i = 1,\ldots,\frac{d-2}{2}. \qquad (2.3.20)$$

Note that since the algebra sector (2.3.12) is not deformed, the invariants are just the straight-forward generalisation of the commutative ones. The exception is of course the deformed d'Alembert operator (2.3.16).

2.4 Representation on the space of commuting coordinates

In this section we construct the $\star$-product representation of the abstract algebra and maps on it (derivatives, Lorentz generators,...) defined in previous sections. The general procedure has been outlined in Chapter 1, here we just apply it to the κ-deformed space.

We choose to work with the symmetric ordering[5]. In that case the $\star$-product is given by [56]

$$f \star g(z) = \lim_{\substack{x \to z \\ y \to z}} \exp\left(z^j \partial_{x^j} \left(\frac{\partial_n}{\partial_{x^n}} e^{-ia\partial_{y^n}} \frac{1 - e^{-ia\partial_{x^n}}}{1 - e^{-ia\partial_n}} - 1 \right) + z^j \partial_{y^j} \left(\frac{\partial_n}{\partial_{y^n}} \frac{1 - e^{-ia\partial_{y^n}}}{1 - e^{-ia\partial_n}} - 1 \right) \right) f(x)g(y), \tag{2.4.1}$$

where we have used the abbreviations

$$\partial_{x^n} = \frac{\partial}{\partial x^n}, \quad \partial_{y^n} = \frac{\partial}{\partial y^n}, \quad \partial_n = \frac{\partial}{\partial x^n} + \frac{\partial}{\partial y^n}. \tag{2.4.2}$$

Expanding this to second order in a we obtain:

$$\begin{aligned} f \star g\,(x) = {} & f(x)g(x) + \frac{ia}{2} x^j \Big(\partial_n f(x) \partial_j g(x) - \partial_j f(x) \partial_n g(x) \Big) \\ & - \frac{a^2}{12} x^j \Big(\partial_n^2 f(x) \partial_j g(x) - \partial_j \partial_n f(x) \partial_n g(x) \\ & \qquad - \partial_n f(x) \partial_j \partial_n g(x) + \partial_j f(x) \partial_n^2 g(x) \Big) \\ & - \frac{a^2}{8} x^j x^k \Big(\partial_n^2 f(x) \partial_j \partial_k g(x) - 2 \partial_j \partial_n f(x) \partial_n \partial_k g(x) \\ & \qquad + \partial_j \partial_k f(x) \partial_n^2 g(x) \Big) + \mathcal{O}(a^3). \end{aligned} \tag{2.4.3}$$

$$\begin{aligned} = {} & f(x)g(x) + \frac{i}{2} C^{\rho\sigma}_{\lambda} x^\lambda (\partial_\rho f(x))(\partial_\sigma g(x)) \\ & - \frac{1}{8} C^{\rho_1\sigma_1}_{\lambda_1} C^{\rho_2\sigma_2}_{\lambda_2} x^{\lambda_1} x^{\lambda_2} (\partial_{\rho_1}\partial_{\rho_2} f(x))(\partial_{\sigma_1}\partial_{\sigma_2} g(x)) \\ & + \frac{1}{12} C^{\rho_1\sigma_1}_{\lambda_1} C^{\rho_2\sigma_2}_{\rho_1} x^{\lambda_1} \Big((\partial_{\sigma_1}\partial_{\rho_2} f(x))(\partial_{\sigma_2} g(x)) \\ & \qquad - (\partial_{\rho_2} f(x))(\partial_{\sigma_1}\partial_{\sigma_2} g(x)) \Big) + \mathcal{O}(a^3). \end{aligned} \tag{2.4.4}$$

One can check that this $\star$-product respects the usual complex conjugation,

$$\overline{f \star g\,(x)} = \bar{g} \star \bar{f}\,(x) \tag{2.4.5}$$

Applied to coordinates, (2.4.1) gives

$$[x^n \stackrel{\star}{,} x^l] = iax^l, \quad [x^k \stackrel{\star}{,} x^l] = 0, \tag{2.4.6}$$

[5] The results for the normal ordering have been given in [56].

as expected. Also,

$$\begin{aligned}
x^l\star f(x) &= x^l\frac{ia\partial_n}{e^{ia\partial_n}-1}f(x), \qquad f(x)\star x^l = x^l\frac{-ia\partial_n}{e^{-ia\partial_n}-1}f(x),\\
x^n\star f(x) &= \Big(x^n-\frac{x^k\partial_k}{\partial_n}(\frac{ia\partial_n}{e^{ia\partial_n}-1}-1)\Big)f(x),\\
f(x)\star x^n &= \Big(x^n-\frac{x^k\partial_k}{\partial_n}(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}-1)\Big)f(x), \qquad (2.4.7)
\end{aligned}$$

where $f(x)$ is an arbitrary function.

As the next step we map the derivatives $\hat{\partial}_\rho$ to the operators $\partial^\star_\rho$ acting on the space of commuting coordinates. From (2.1.10) it follows that the derivative $\hat{\partial}_n$ has the undeformed commutation relations with coordinates and therefore

$$\hat{\partial}_n \mapsto \partial^\star_n = \partial_n, \qquad (2.4.8)$$

where ∂_n is the usual partial derivative.

In order to find the $\star$-product representation of $\hat{\partial}_j$ we apply it to symmetrically ordered polynomials. From

$$\hat{\partial}_j\hat{x}^\mu = \hat{x}^\mu\hat{\partial}_j + \delta^\mu_j - ia\eta^{\mu n}\hat{\partial}_j \qquad (2.4.9)$$

we read off

$$\hat{\partial}_j \mapsto \partial^\star_j = \partial_j + \mathcal{O}(a), \qquad (2.4.10)$$

where ∂_j is the usual partial derivative. As the next step one uses (2.1.10) to calculate

$$\begin{aligned}
\hat{\partial}_j\frac{1}{2}\Big(\hat{x}^n\hat{x}^l+\hat{x}^l\hat{x}^n\Big) &= \dots \qquad (2.4.11)\\
&= \frac{1}{2}\Big(\hat{x}^n(\hat{x}^l\hat{\partial}_j+\underline{\delta^l_j})+ia(\hat{x}^l\hat{\partial}_j+\underline{\delta^l_j})+\hat{x}^l(\hat{x}^n\hat{\partial}_j+ia\hat{\partial}_j)+\underline{\delta^l_j\hat{x}^n}\Big).
\end{aligned}$$

The underlined terms in (2.4.11) give

$$\hat{\partial}_j \mapsto \partial^\star_j = \partial_j + \frac{ia}{2}\partial_j\partial_n + \mathcal{O}(a^2). \qquad (2.4.12)$$

Continuing in the similar way leads to the higher order terms. The final result is given by

$$\hat{\partial}_j \mapsto \partial^\star_j = \partial_j\frac{e^{ia\partial_n}-1}{ia\partial_n}. \qquad (2.4.13)$$

The Leibniz rule for the derivatives $\partial^\star_\rho$ follows from (2.1.14)

$$\begin{aligned}
\partial^\star_n\triangleright(f\star g) &= \big(\partial^\star_n\triangleright f\big)\star g + f\star\big(\partial^\star_n\triangleright g\big),\\
\partial^\star_j\triangleright(f\star g) &= \big(\partial^\star_j\triangleright f\big)\star g + (e^{ia\partial^\star_n}\triangleright f)\star\big(\partial^\star_j\triangleright g\big), \qquad (2.4.14)
\end{aligned}$$

where the notation $\partial^\star_\mu\triangleright f$ was introduced in (1.5.14).

Using (2.3.2), (2.4.8) and (2.4.13) the $\star$-product representation of the derivatives $\hat{D}_\rho$ is calculated

$$\begin{aligned}
\hat{D}_n \mapsto D^\star_n &= \frac{1}{a}\sin(a\partial_n) + \frac{i}{a\partial_n^2}(\cos(a\partial_n)-1)\partial_l\partial^l,\\
\hat{D}_j \mapsto D^\star_j &= \partial_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}, \qquad (2.4.15)
\end{aligned}$$

as well as the Leibniz rules

$$\begin{aligned} D_n^\star \triangleright (f \star g) &= (D_n^\star \triangleright f) \star (e^{-ia\partial_n^\star} \triangleright g) + (e^{ia\partial_n^\star} \triangleright f) \star (D_n^\star \triangleright g) \\ &\quad -ia\big(D_l^\star e^{ia\partial_n^\star} \triangleright f\big) \star (D^{\star l} \triangleright g), \end{aligned} \tag{2.4.16}$$

$$D_j^\star \triangleright (f \star g) = (D_j^\star \triangleright f) \star (e^{-ia\partial_n^\star} \triangleright g) + f \star (D_j^\star \triangleright g). \tag{2.4.17}$$

Finally, for the d'Alembert operator we find

$$\hat{\Box} \mapsto \Box^\star = \frac{2}{a^2\partial_n^2}\Big(1 - \cos(a\partial_n)\Big)\partial_\mu\partial^\mu. \tag{2.4.18}$$

To find the $\star$-product representation of the generators $M^{\mu\nu}$ one proceeds in the same way like in the case of the derivatives $\hat{\partial}_\rho$, applying them on symmetrically ordered polynomials. The result is

$$\begin{aligned} \hat{L}^{ij} \mapsto L^{\star\, ij} &= x^i\partial^j - x^j\partial^i, \\ \hat{L}^{in} \mapsto L^{\star\, in} &= x^i\partial^n - x^n\partial^i + x^i\partial_\mu\partial^\mu \frac{e^{ia\partial_n} - 1}{2\partial_n} - x^\nu\partial_\nu\partial^i \frac{e^{ia\partial_n} - 1 - ia\partial_n}{ia\partial_n^2}. \end{aligned} \tag{2.4.19}$$

For the Leibniz rule (2.2.5) we obtain

$$\begin{aligned} L^{\star\, ij} \triangleright \big(f \star g\big) &= (L^{\star\, ij} \triangleright f) \star g + f \star (L^{\star\, ij} \triangleright g), \\ L^{\star\, in} \triangleright \big(f \star g\big) &= \big(L^{\star\, in} \triangleright f\big) \star g + \big(e^{ia\partial_n^\star} \triangleright f\big) \star \big(L^{\star\, in} \triangleright g\big) \\ &\quad + ia\big(\partial_k^\star \triangleright f\big) \star \big(L^{\star ik} \triangleright g\big). \end{aligned} \tag{2.4.20}$$

At the end of this section we mention one more set of nonlinear derivatives, $\hat{\delta}_\rho$. They are given by

$$\begin{aligned} [\hat{\delta}_n, \hat{x}^n] &= 1, \\ [\hat{\delta}_n, \hat{x}^l] &= 0, \\ [\hat{\delta}_j, \hat{x}^n] &= \Big(1 - \frac{ia\hat{\delta}_n}{e^{ia\hat{\delta}_n} - 1}\Big)\frac{\hat{\delta}_j}{\hat{\delta}_n}, \\ [\hat{\delta}_j, \hat{x}^l] &= \delta_j^l \frac{ia\hat{\delta}_n}{e^{ia\hat{\delta}_n} - 1}. \end{aligned} \tag{2.4.21}$$

The Leibniz rule for for $\hat{\delta}_n$ is trivial, while for $\hat{\delta}_j$ it is very complicated and no closed form has been obtained so far

$$\begin{aligned} \hat{\delta}_n(\hat{f} \cdot \hat{g}) &= (\hat{\delta}_n\hat{f}) \cdot \hat{g} + \hat{f} \cdot (\hat{\delta}_n\hat{g}), \\ \hat{\delta}_j(\hat{f} \cdot \hat{g}) &= (\hat{\delta}_j\hat{f}) \cdot \big(1 - \frac{ia}{2}\hat{\delta}_n - \frac{a^2}{12}\hat{\delta}_n\hat{\delta}_n\big)\hat{g} + \Big(\big(1 + \frac{ia}{2}\hat{\delta}_n - \frac{a^2}{12}\hat{\delta}_n\hat{\delta}_n\big)\hat{f}\Big) \cdot (\hat{\delta}_j\hat{g}) \\ &\quad + \frac{a^2}{12}\Big((\hat{\delta}_n\hat{f}) \cdot (\hat{\delta}_n\hat{\delta}_j\hat{g}) + (\hat{\delta}_n\hat{\delta}_j\hat{f}) \cdot (\hat{\delta}_n\hat{g})\Big) + \mathcal{O}(a^3). \end{aligned} \tag{2.4.22}$$

For the completeness we also give the commutation relations between the derivatives $\hat{\delta}_\rho$ and the Lorentz generators

$$\begin{aligned}
[M^{ij},\hat{\delta}_n] &= 0,\\
[M^{ij},\hat{\delta}_l] &= \delta^i_l\hat{\delta}^j - \delta^j_l\hat{\delta}^i,\\
[M^{ln},\hat{\delta}_n] &= -\hat{\delta}^l\frac{e^{ia\hat{\delta}_n}-1}{ia\hat{\delta}_n},\\
[M^{ln},\hat{\delta}_j] &= \frac{1}{2}\delta^l_j\hat{\delta}_n\Big(e^{ia\hat{\delta}_n}+1+\big(e^{ia\hat{\delta}_n}-1\big)\frac{\hat{\delta}_m\hat{\delta}^m}{\hat{\delta}_n\hat{\delta}_n}\Big)\\
&\quad+\frac{e^{ia\hat{\delta}_n}-ia\hat{\delta}_n-1}{ia\hat{\delta}_n\hat{\delta}_n}\hat{\delta}^l\hat{\delta}_j.
\end{aligned}\tag{2.4.23}$$

The reason why these derivatives are interesting is their $\star$-product representation

$$\begin{aligned}
\hat{\delta}_n &\to \delta^\star_n = \partial_n,\\
\hat{\delta}_j &\to \delta^\star_j = \partial_j,
\end{aligned}\tag{2.4.24}$$

so they are represented by the usual partial derivatives acting on the space of commuting coordinates. We use this result in Appendix B to discuss an alternative way of obtaining symmetry of the κ-deformed space.

2.5 Fields and equations of motion

Since we are interested in defining a field theory on the κ-deformed space, we need a definition of fields. Also, we formulate covariant equations of motion for a free scalar field and a free Dirac spinor.

2.5.1 Fields

Under the classical Lorentz transformations

$$x^\lambda \to x'^\lambda = x^\lambda + x^\mu {\omega_\mu}^\lambda, \qquad \omega^{\mu\nu} = -\omega^{\nu\mu} = \text{ const. }, \tag{2.5.1}$$

a scalar field $\phi^0(x)$[6] transforms like

$$\phi'^0(x') = \phi^0(x). \tag{2.5.2}$$

For an infinitesimal parameter ${\omega_\mu}^\lambda$ this transformation reads

$$\delta^{cl}_\omega\phi^0(x) = \phi'^0(x) - \phi^0(x) = -x^\mu{\omega_\mu}^\lambda\partial_\lambda\phi^0(x) = -\frac{1}{2}\omega^{\alpha\beta}L_{\alpha\beta}\phi^0(x). \tag{2.5.3}$$

Here $L_{\alpha\beta}$ is the orbital part of Lorentz generator

$$L_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha. \tag{2.5.4}$$

[6]We write $\phi^0(x)$ in order to distinguish classical fields from the noncommutative ones.

This transformation law we generalise to

$$\delta_\omega \phi(x) = -\frac{1}{2}\omega^{\alpha\beta}(L^\star_{\alpha\beta} \star \phi(x)) = -\frac{1}{2}\omega^{\alpha\beta} L^\star_{\alpha\beta} \triangleright \phi(x), \tag{2.5.5}$$

where $L^\star_{\alpha\beta}$ are generators of the κ-deformed Lorentz transformations given by (2.4.19). Note that $L^\star_{\alpha\beta} \triangleright \phi(x)$ is just the usual action of a differential operator, for example

$$L^\star_{in} \triangleright \phi = x_i(\partial_n \phi) - x_n(\partial_i \phi) - x_i\Big(\partial_\mu \partial^\mu \frac{e^{ia\partial_n} - 1}{2\partial_n}\phi\Big) + x^\nu\Big(\partial_\nu \partial_i \frac{e^{ia\partial_n} - 1 - ia\partial_n}{ia\partial_n^2}\phi\Big). \tag{2.5.6}$$

This is because all the $\star$-product contribution have already been included (expanded) in writing down the expressions (2.4.19) for the generators $L^\star_{\alpha\beta}$.

Classically, the transformation law of a covariant vector field can be obtained considering the transformation law of the derivative of a scalar field $\partial_\mu \phi^0$. However, we have seen in Section 2.1 and later in Section 2.3 that there is no unique derivative on the κ-deformed space. Depending on the choice of derivatives we will obtain different transformation laws of a vector field. To stay as close as possible to the classical transformation laws we look at the Dirac derivative of a scalar field[7]

$$\begin{aligned}\delta_\omega(D^\star_\mu \triangleright \phi) &= D^\star_\mu \triangleright (\delta_\omega \phi) = -\frac{1}{2}\omega^{\alpha\beta} D^\star_\mu \triangleright (L^\star_{\alpha\beta} \triangleright \phi) \\ &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} \triangleright (D^\star_\mu \triangleright \phi) - \eta_{\beta\mu}(D^\star_\alpha \triangleright \phi) + \eta_{\alpha\mu}(D^\star_\beta \triangleright \phi)\Big).\end{aligned} \tag{2.5.7}$$

This leads to

$$\delta_\omega V_\mu = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} \triangleright V_\mu - \eta_{\beta\mu} V_\alpha + \eta_{\alpha\mu} V_\beta\Big). \tag{2.5.8}$$

The first term is the transformation of the argument, while the other two stand for the index transformation. This can be written as

$$\delta_\omega V_\mu = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} + \Sigma_{\alpha\beta}\Big) \triangleright V_\mu, \tag{2.5.9}$$

where $\Sigma_{\alpha\beta}$ is the constant matrix[8] in the index space of fields. Transformation law of an arbitrary covariant tensor is

$$\begin{aligned}\delta_\omega T_{\mu_1\ldots\mu_r} &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} \triangleright T_{\mu_1\ldots\mu_r} - \eta_{\beta\mu_1} T_{\alpha\mu_2\ldots\mu_r} + \eta_{\alpha\mu_1} T_{\beta\mu_2\ldots\mu_r} \\ &\qquad - \cdots - \eta_{\beta\mu_r} T_{\mu_1\ldots\mu_{r-1}\alpha} + \eta_{\alpha\mu_r} T_{\mu_1\ldots\mu_{r-1}\beta}\Big) \\ &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} + \Sigma_{\alpha\beta}\Big) \triangleright T_{\mu_1\ldots\mu_r}.\end{aligned} \tag{2.5.10}$$

Because of the deformed Leibniz rules (2.4.20) we have that the $\star$-product of two scalar fields is a scalar field again

$$\delta_\omega(\phi_1 \star \phi_2) = -\frac{1}{2}\omega^{\alpha\beta} L^\star_{\alpha\beta} \triangleright (\phi_1 \star \phi_2). \tag{2.5.11}$$

[7]Although this sentence appears not to make any sense, we repeat that "Dirac derivative" is just the name we use for one specific choice of derivatives.

[8]Notation $\Sigma_{\alpha\beta} \triangleright V_\mu$ stands just for the usual multiplication, $\Sigma_{\alpha\beta}$ being a constant matrix.

In the case of vector fields we have

$$\delta_\omega(V_\mu\star V_\nu) = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta}\triangleright(V_\mu\star V_\nu) - \big(\eta_{\beta\mu}V_\alpha - \eta_{\alpha\mu}V_\beta\big)\star V_\nu - V_\mu\star\big(\eta_{\beta\nu}V_\alpha - \eta_{\alpha\nu}V_\beta\big)\Big), \quad (2.5.12)$$

that is the $\star$-product of two vector fields transforms like a second rank tensor. Writing this in a different way

$$\begin{aligned}\delta_\omega(V_\mu\star V_\nu) &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} + \Sigma_{\alpha\beta}\Big)\triangleright(V_\mu\star V_\nu)\\ &= -\frac{1}{2}\omega^{\alpha\beta}L^\star_{\alpha\beta}\triangleright(V_\mu\star V_\nu) - \frac{1}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta}\triangleright(V_\mu\star V_\nu), \end{aligned}\quad (2.5.13)$$

and comparing with (2.5.12) we see that

$$\Sigma_{\alpha\beta}\triangleright(V_\mu\star V_\nu) = (\Sigma_{\alpha\beta}\triangleright V_\mu)\star V_\nu + V_\mu\star(\Sigma_{\alpha\beta}\triangleright V_\nu), \quad (2.5.14)$$

the index part of the Lorentz transformation has the undeformed Leibniz rule.

To calculate the transformation law of a contravariant vector field we use (2.5.12). Demanding that $V^\mu\star V_\mu$ transforms like a scalar field

$$\delta_\omega(V^\mu\star V_\mu) = -\frac{1}{2}\omega^{\alpha\beta}L^\star_{\alpha\beta}\triangleright(V^\mu\star V_\mu), \quad (2.5.15)$$

we find

$$\delta_\omega V^\mu = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta}\triangleright V^\mu + \delta^\mu_\beta\eta_{\alpha\lambda}V^\lambda - \delta^\mu_\alpha\eta_{\beta\lambda}V^\lambda\Big). \quad (2.5.16)$$

This can be generalised to an arbitrary contravariant tensor

$$\begin{aligned}\delta_\omega T^{\mu_1\dots\mu_r} &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta}\triangleright T^{\mu_1\dots\mu_r} - \delta^{\mu_1}_\beta\eta_{\alpha\lambda}T^{\lambda\mu_2\dots\mu_r} + \delta^{\mu_1}_\alpha\eta_{\beta\lambda}T^{\lambda\mu_2\dots\mu_r}\\ &\quad - \dots - \delta^{\mu_r}_\beta\eta_{\alpha\lambda}T^{\mu_1\dots\mu_{r-1}\lambda} + \delta^{\mu_r}_\alpha\eta_{\alpha\lambda}T^{\mu_1\dots\mu_{r-1}\lambda}\Big)\\ &= -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} + \Sigma_{\alpha\beta}\Big)\triangleright T^{\mu_1\dots\mu_r}. \end{aligned}\quad (2.5.17)$$

Also, using (2.4.20) and (2.5.14) one shows that the $\star$-product of two arbitrary tensors is a tensor again.

2.5.2 Covariant equations of motion

Having defined fields, we now formulate the covariant equations of motion. First we consider the equation for a free scalar field, that is a generalisation of the Klein-Gordon equation. In Section 2.3 we found the deformed d'Alembert operator. Its representation on the space of commuting coordinates is given by (2.4.18). Using this it is not difficult to write the deformed Klein-Gordon equation

$$(\Box^\star + m^2)\phi(x) = 0. \quad (2.5.18)$$

Because of (2.3.17) and (2.5.5) this equation is covariant. One can continue and analyse the solutions of this equation, dispersion relation and even proceed towards quantisation. Part of that has been done in [78].

Next, we look at the κ-deformed equation for a free spinor field ψ, that is the κ-deformed Dirac equation. Using the derivatives (2.4.15) one writes

$$(i\gamma^\mu D^\star_\mu - m)\psi = 0, \tag{2.5.19}$$

where γ^μ are the usual γ matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.5.20}$$

Using (2.3.1),

$$[M_{\alpha\beta}, \gamma^\mu] = \delta^\mu_\beta \eta_{\alpha\lambda}\gamma^\lambda - \delta^\mu_\alpha \eta_{\beta\lambda}\gamma^\lambda \tag{2.5.21}$$

and

$$\delta_\omega \psi = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} + \Sigma_{\alpha\beta}\Big) \triangleright \psi, \tag{2.5.22}$$

where $\Sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$, one checks that equation (2.5.19) is covariant under the κ-deformed Lorentz transformations. Next step, the analysis of possible solutions has been partially done in [78].

One can use both (2.5.18) and (2.5.19) to try out the quantisation procedure [79]. However, this problem is still not very well understood.

3

Construction of gauge theories on the κ-deformed space

The best known description of fundamental interactions is given in terms of gauge theories. Electromagnetic, weak and strong interactions are obtained by localising internal symmetries while gravity can be understood as the gauge theory with the Poincaré group as the gauge group. Therefore, it is of importance to generalise this concept to deformed spaces as well. In this chapter we construct a general nonabelian gauge theory on the κ-deformed space. However, the construction is done in a very general way so that it can be applied to other deformed spaces as well. In our approach we have to introduce enveloping algebra-valued quantities (noncommutative gauge parameter, noncommutative gauge field, ...) [32]. This leads to (apparently) infinitely many degrees of freedom in the theory. The problem is solved in terms of the Seiberg-Witten map [26]. This map allows to express noncommutative variables in terms of the corresponding commutative ones and this reduces the number of degrees of freedom to the commutative ones.

We start with reviewing the commutative gauge theory. Then we define the noncommutative gauge transformations, see how the enveloping algebra comes in the play and explicitly construct solutions of the Seiberg-Witten map for the noncommutative variables. As a consequence of the nontrivial Leibniz rules for the derivatives we use in the construction, the noncommutative gauge field becomes derivative valued.

3.1 Commutative gauge theory

Gauge theories were first introduced by C. N. Yang and R. L. Mills in 1954 [80] and they became very important tool in particle physics. Before introducing the concept of gauge theory on noncommutative spaces, we shortly repeat basic steps in the construction of gauge theories on the commutative space.

The nonabelian gauge group is generated by the hermitian generators T^a that fulfil

$$[T^a, T^b] = if^{abc}T^c, \quad a = 1, \ldots, n \tag{3.1.1}$$

where f^{abc} are structure constants of the group and the sum over repeated index (here c) is understood. From the Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \tag{3.1.2}$$

it follows

$$f^{abc}f^{dce} + f^{ace}f^{bdc} + f^{dac}f^{bce} = 0. \tag{3.1.3}$$

The matter field[1] $\psi^0(x)$ is in a certain irreducible representation (fundamental for example) of this group. Under the gauge transformations[2] we have

$$\psi^0(x) \to \psi'^0(x) = e^{i\alpha^a(x)T^a}\psi^0(x) \overset{\text{def}}{=} U_\alpha(x)\psi^0(x), \tag{3.1.4}$$

or infinitesimally

$$\delta_\alpha\psi^0(x) = i\alpha^a(x)T^a\psi^0(x) \overset{\text{def}}{=} i\alpha(x)\psi^0(x). \tag{3.1.5}$$

Since the parameter $\alpha(x)$ is x-dependent,[3] the derivative of a field does not transform like the field itself

$$\delta_\alpha(\partial_\mu\psi^0) = i\alpha(\partial_\mu\psi^0) + i(\partial_\mu\alpha)\psi^0. \tag{3.1.6}$$

As a consequence of (3.1.6) the action

$$S_m = \int \mathrm{d}^4x\, \bar\psi^0(i\gamma^\mu\partial_\mu - m)\psi^0$$

for the free spinor field (for example) is not invariant under (3.1.5). The symmetry can be restored introducing the covariant derivative

$$\mathcal{D}^0_\mu\psi = \partial_\mu\psi - iA^{0a}_\mu T^a\psi \overset{\text{def}}{=} \partial_\mu\psi - iA^0_\mu\psi, \tag{3.1.7}$$

such that

$$\delta_\alpha(\mathcal{D}_\mu\psi^0) = i\alpha\mathcal{D}_\mu\psi^0. \tag{3.1.8}$$

In (3.1.7) we introduced the Lie algebra-valued gauge field A^0_μ in order to achieve (3.1.8). The transformation law of the gauge field A^0_μ follows from (3.1.8)

$$\delta_\alpha A^0_\mu = \partial_\mu\alpha + i[\alpha, A^0_\mu], \tag{3.1.9}$$

or in terms of the finite transformations

$$A^0_\mu \to A^{0'}_\mu = UA^0_\mu U^{-1} + iU(\partial_\mu U^{-1}). \tag{3.1.10}$$

To construct the action for the field ψ which is invariant under (3.1.5) and (3.1.9) one uses the minimal coupling prescription, that is in the ordinary action one replaces all the partial derivatives with the covariant ones

$$S_m = \int \mathrm{d}^4x\, \bar\psi^0(i\gamma^\mu\partial_\mu - m)\psi^0 \to S_m = \int \mathrm{d}^4x\, \bar\psi^0(i\gamma^\mu(\partial_\mu - iA^0_\mu) - m)\psi^0. \tag{3.1.11}$$

[1]We write the all the variables with the upper index 0 to distinguish them from the noncommutative ones which will be introduced in the next section.

[2]In this and the next chapter we treat only internal symmetries, that is the transformations that do not change coordinates, $\delta_\alpha x^\mu = 0$. However, one can analyse the global Poincaré symmetry in the same way. This is how gravity arises as a gauge theory. We will follow this approach in the last chapter when we try to construct a theory of gravity on a deformed space.

[3]From now on we will keep this dependence implicit just to simplify the formulas.

Using (3.1.5) and (3.1.9) one checks explicitly that this action is gauge invariant. Note that although we have started from the free field theory, the action (3.1.11) describes an interacting theory. The interaction comes from coupling of the matter field with the gauge field. In this way the electroweak and strong interactions arise, the gauge groups being $SU(2)_{\rm I} \times U(1)_{\rm Y}$ and $SU(3)_{\rm C}$ respectively.

In order for A^0_μ not to be only external but also a dynamical field one has to introduce the kinetic term for it. Therefore, we define the field-strength tensor $F^0_{\mu\nu} = F^{0a}_{\mu\nu}T^a$ as

$$F^0_{\mu\nu} = i[\mathcal{D}^0_\mu, \mathcal{D}^0_\nu]. \tag{3.1.12}$$

Applying (3.1.12) to the field ψ^0 gives

$$\begin{aligned} F^0_{\mu\nu} = F^{0a}_{\mu\nu}T^a &= \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu] && (3.1.13)\\ &= \big(\partial_\mu A^{0a}_\nu - \partial_\nu A^{0a}_\mu + f^{abc}A^{0b}_\mu A^{0c}_\nu\big)T^a && (3.1.14) \end{aligned}$$

and

$$\delta_\alpha F^0_{\mu\nu} = i[\alpha, F^0_{\mu\nu}]. \tag{3.1.15}$$

The covariant derivative fulfils the Jacobi identity

$$[\mathcal{D}^0_\mu, [\mathcal{D}^0_\nu, \mathcal{D}^0_\rho]] + [\mathcal{D}^0_\nu, [\mathcal{D}^0_\rho, \mathcal{D}^0_\mu]] + [\mathcal{D}^0_\rho, [\mathcal{D}^0_\mu, \mathcal{D}^0_\nu]] = 0. \tag{3.1.16}$$

Using (3.1.12), from (3.1.16) the Bianchi identity for the field-strength tensor follows

$$\mathcal{D}^0_\mu F^0_{\nu\rho} + \mathcal{D}^0_\nu F^0_{\rho\mu} + \mathcal{D}^0_\rho F^0_{\mu\nu} = 0. \tag{3.1.17}$$

The restriction that the action for the gauge field has to be gauge invariant and the renormalisability properties of the theory fix the kinetic term for the gauge field uniquely. The mass term $m^2 A^0_\mu A^{0\mu}$ is not allowed to appear in the action since it is not gauge invariant. Historically, this was the problem with the theory of Yang and Mills. Namely, it was known that weak and strong interactions are of short-range type so they were supposed to be carried by massive particles. But the Yang-Mills theory gave interactions carried by massless particles. The problem was solved finally with the discovery of the mechanism of spontaneously broken symmetry [81], [82]. This made it possible for gauge fields to became massive and the Yang-Mills theory was finally accepted as the theory which describes the fundamental interactions.

The gauge invariant action for the gauge field A^0_μ (the Yang-Mills action) is

$$S_g = \int \mathrm{d}^4x \,\Big(-\frac{1}{4}\mathrm{Tr}(F^0_{\mu\nu}F^{0\mu\nu})\Big), \tag{3.1.18}$$

where the factor $-1/4$ is chosen in analogy with the electrodynamics.

Finally, the complete action is

$$\begin{aligned} S &= S_m + S_g \\ &= \int \mathrm{d}^4x\, \bar\psi^0(i\gamma^\mu(\partial_\mu - iA^0_\mu) - m)\psi^0 + \int \mathrm{d}^4x\,\Big(-\frac{1}{4}\mathrm{Tr}(F^0_{\mu\nu}F^{0\mu\nu})\Big). && (3.1.19) \end{aligned}$$

Having (3.1.19) one has enough information to analyse the theory, calculate the equations of motion, conserved quantities....

As an example we present the equation of motion and the conserved current for the theory given by (3.1.19). Varying the action (3.1.19) with respect to $\bar\psi^0$, ψ^0 and A_ρ^{0a} gives the following equations of motion

$$\begin{aligned}
\delta\bar\psi^0 : &\quad i\gamma^\mu(\partial_\mu\psi^0) + \gamma^\mu A_\mu^0\psi^0 - m\psi^0 = 0, && (3.1.20)\\
\delta\psi^0 : &\quad -i(\partial_\mu\bar\psi^0)\gamma^\mu + \bar\psi^0\gamma^\mu A_\mu^0 - m\bar\psi^0 = 0, && (3.1.21)\\
\delta A_\rho^0 : &\quad \partial_\mu F^{0\mu\rho\, a} + f^{abc}A_\mu^{0b}F^{0\mu\rho\, c} + \bar\psi^0\gamma^\rho T^a\psi = 0. && (3.1.22)
\end{aligned}$$

The current $J^{\rho a}$ is introduced as

$$J^{\rho a} = f^{abc}A_\mu^{0b}F^{0\mu\rho\, c} + \bar\psi^0\gamma^\rho T^a\psi. \tag{3.1.23}$$

Due to the antisymmetry of $F^{0\mu\rho\, a}$ we have $\partial_\mu\partial_\rho F^{0\mu\rho\, a} = 0$ and this, together with (3.1.22) and (3.1.23) gives

$$\partial_\rho J^{\rho a} = 0, \tag{3.1.24}$$

that is $J^{\rho a}$ is the conserved current. In order to check if our theory is consistent, we prove explicitly (using the equations of motion for the fields ψ, $\bar\psi$ and A_μ^{0a}) that $J^{\rho a}$ is conserved

$$\begin{aligned}
\partial_\rho J^{\rho a} &= \partial_\rho\Big(f^{abc}A_\mu^{0b}F^{0\mu\rho\, c} + \bar\psi^0\gamma^\rho T^a\psi^0\Big)\\
&= f^{abc}\big((\partial_\rho A_\mu^{0b})F^{0\mu\rho\, c} + A_\mu^{0b}(\partial_\rho F^{0\mu\rho\, c})\big) + (\partial_\rho\bar\psi^0)\gamma^\rho T^a\psi^0 + \bar\psi^0\gamma^\rho T^a(\partial_\rho\psi^0)\\
&= -\frac{1}{2}f^{abc}\big(\partial_\rho A_\mu^{0b} - \partial_\mu A_\rho^{0b} + f^{bde}A_\rho^{0d}A_\mu^{0e}\big)F^{0\rho\mu\, c} + \frac{1}{2}f^{abc}f^{bde}A_\rho^{0d}A_\mu^{0e}F^{0\rho\mu\, c}\\
&\quad + f^{abc}A_\mu^{0b}\big(f^{cde}A_\rho^{0d}F^{0\rho\mu\, e} + \bar\psi^0\gamma^\mu T^c\psi^0\big)\\
&\quad - i(\bar\psi^0\gamma^\rho A_\rho^0 - m\bar\psi^0)T^a\psi^0 + i\bar\psi^0 T^a(\gamma^\rho A_\rho^0\psi^0 - m\psi^0) = 0.
\end{aligned} \tag{3.1.25}$$

We can cancel the first term in the third line due to the antisymmetry of the structure constants f^{abc} and also the terms proportional to m in the last line. We obtain

$$\begin{aligned}
\partial_\rho J^{\rho a} &= \frac{1}{2}f^{abc}f^{bde}A_\rho^{0d}A_\mu^{0e}F^{0\rho\mu\, c} + \frac{1}{2}f^{abc}f^{cde}\big(A_\mu^{0b}A_\rho^{0d} - A_\rho^{0b}A_\mu^{0d}\big)F^{0\rho\mu\, e}\\
&\quad + f^{abc}A_\rho^{0b}\bar\psi^0\gamma^\rho T^c\psi^0 + i\bar\psi^0\gamma^\rho A_\rho^{0b}\big(T^aT^b - T^bT^a\big)\psi^0\\
&= \frac{1}{2}A_\rho^{0d}A_\mu^{0e}F^{0\rho\mu\, c}\big(f^{abc}f^{bde} + f^{aeb}f^{bdc} - f^{adb}f^{bec}\big)\\
&\quad + f^{abc}A_\rho^{0b}\bar\psi^0\gamma^\rho T^c\psi^0 - f^{abc}A_\rho^{0b}\bar\psi^0\gamma^\rho T^c\psi^0\\
&= 0,
\end{aligned} \tag{3.1.26}$$

where we have used the Jacobi identity (3.1.3) to cancel the terms in the third line.

3.2 Noncommutative gauge theory, setting

In the following we generalise the concepts introduced in the previous section to deformed spaces. Although we are especially interested in the gauge theory on the κ-deformed space, we try to keep the analysis as general as possible, such that it can be applied to other deformed spaces as well. The construction of the noncommutative gauge theory is done in the $\star$-product approach. However, one can equally well define noncommutative gauge transformations in the abstract algebra and perform the analysis there. Our motivation for doing everything in the $\star$-product approach is that in this way one can obtain results which might be experimentally checked.

The noncommutative gauge transformation of a noncommutative field $\psi(x)$ is defined to be

$$\delta_\Lambda \psi(x) = i\Lambda(x) \star \psi(x), \tag{3.2.1}$$

where $\Lambda(x)$ is the noncommutative gauge parameter. This is an infinitesimal transformation, for finite transformations one has

$$\psi(x) \to \psi'(x) = e_\star^{i\Lambda(x)} \star \psi(x), \tag{3.2.2}$$

where $e_\star^{i\Lambda(x)}$ is the $\star$-exponential function. This function is the formal power series, the ordinary multiplication in every summand is replaced by the $\star$-multiplication

$$e_\star^{i\Lambda(x)} = 1 + i\Lambda - \frac{1}{2}\Lambda \star \Lambda - \frac{i}{6}\Lambda \star \Lambda \star \Lambda + \dots. \tag{3.2.3}$$

One can check that the equality $e_\star^{i\Lambda(x)} \star e_\star^{-i\Lambda(x)} = 1$ is fulfilled.

With this definition of a gauge transformation one can proceed in two ways. One is the so-called "covariant coordinate approach" [83], [84], [85]. This approach is closely related with the appearance of noncommutative gauge theory in the framework of string theory. Mathematically, it is based on the inner derivations on a deformed space and therefore is convenient if one does not know the exterior derivatives on the deformed space in question. The basic idea comes from the observation that the $\star$-multiplication of a field with a coordinate is no longer a gauge covariant operation,

$$\delta_\Lambda(x^\mu \star \psi) = x^\mu \star \delta_\Lambda \psi = ix^\mu \star \Lambda \star \psi \neq i\Lambda \star (x^\mu \star \psi), \tag{3.2.4}$$

since the $\star$-product is noncommutative. The problem is solved by introducing the covariant coordinate $X^\mu = x^\mu + A^\mu$, such that

$$\delta_\Lambda(X^\mu \star \psi) = i\Lambda \star (X^\mu \star \psi). \tag{3.2.5}$$

Here A^μ is the noncommutative gauge potential; its transformation law follows from (3.2.5). Then one proceeds with defining the field-strength tensor as in the commutative case. However, we do not wish to follow this approach here. Like we said, it is convenient since one introduces the gauge potential without having to fix a differential calculus on a deformed space first; one uses only inner derivations. In this way a variety of concepts on a deformed space can be introduced without knowing much about its additional algebraic or geometric structure.

In the previous chapter we analysed in detail the differential calculus on the κ-deformed space. Therefore, we have enough information not to follow the approach of covariant coordinates here. Instead, we construct covariant derivatives and proceeded in the full analogy with the commutative theory.

Our starting point is the infinitesimal noncommutative transformation of a noncommutative field ψ

$$\delta_\Lambda \psi(x) = i\Lambda(x) \star \psi(x). \tag{3.2.6}$$

The field ψ belongs to a certain irreducible representation (fundamental for example) of the gauge group. We consider a general nonabelian gauge group generated by the generators T^a which fulfil (3.1.1). In order to check if our definition of the gauge transformation (3.2.6) is good, we remember that the commutative gauge transformations close in the algebra

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha,\beta]}. \tag{3.2.7}$$

In words, the commutator of two gauge transformations δ_α and δ_β is again a gauge transformation with the parameter $-i[\alpha, \beta]$. Now we check if the same holds for the transformations (3.2.6)

$$(\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1})\psi(x) = (\Lambda_1 \star \Lambda_2 - \Lambda_2 \star \Lambda_1) \star \psi. \tag{3.2.8}$$

If we take $\Lambda_{1,2} = \Lambda^a_{1,2}T^a$, that is the Lie algebra-valued noncommutative gauge parameters, we obtain

$$\begin{aligned}(\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1})\psi(x) = & \frac{1}{2}(\Lambda^a_1 \star \Lambda^b_2 + \Lambda^b_2 \star \Lambda^a_1)[T^a, T^b] \star \psi \\ & + \frac{1}{2}(\Lambda^a_1 \star \Lambda^b_2 - \Lambda^b_2 \star \Lambda^a_1)\{T^a, T^b\} \star \psi. \end{aligned} \tag{3.2.9}$$

The first term in (3.2.9) is again Lie algebra-valued because of (3.1.1). However, the second term is not Lie algebra-valued in general. In the special case of $U(N)$ gauge theories one can express the anticommutator $\{T^a, T^b\}$ in the generators T^a only (no products of generators) [26]. In the case of $SU(N)$ groups, which are needed for the construction of a noncommutative generalisation of the Standard Model, this is not possible. There are two ways to solve this problem. One is to look only at the $U(N)$ gauge groups (and also $SO(N)$ and $Sp(N)$ groups [50]). Then the commutator of two gauge transformations is again in the algebra of transformations. Second possibility is to give up the Lie algebra-valued gauge parameter and define the enveloping algebra-valued gauge transformations [32]. This approach we follow here.

3.3 Enveloping algebra approach

The enveloping algebra of the Lie algebra (3.1.1) is the algebra freely generated by the generators T^a and divided by the ideal generated by (3.1.1)

$$\mathcal{A}_T = \frac{\mathbb{C}[[T^1, \ldots, T^n]]}{([T^a, T^b] - i f^{ab}{}_c T^c)}. \tag{3.3.1}$$

It is infinite dimensional and its elements are all possible products of the generators modulo the commutation relations. Especially, both $[T^a, T^b]$ and $\{T^a, T^b\}$ are in the enveloping algebra. Therefore, the transformations (3.2.6) close in the enveloping algebra and we continue our analysis there.

A basis in the enveloping algebra can be chosen by specifying the ordering. We choose the symmetric ordering because of its invariance under the conjugation, for example

$$\overline{:T^aT^b:} = \frac{1}{2}\overline{(T^aT^b + T^bT^a)} = \frac{1}{2}(T^bT^a + T^aT^b) =: T^aT^b : .$$

The basis is then

$$\begin{aligned} :T^a: &= T^a, \\ :T^aT^b: &= \frac{1}{2}(T^aT^b + T^bT^a), \\ &\dots \\ :T^{a_1}\dots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)}). \end{aligned} \tag{3.3.2}$$

Since the noncommutative gauge parameter is enveloping algebra-valued it can be expanded in the basis (3.3.2)

$$\begin{aligned} \Lambda(x) &= \sum_{l=1}^{\infty}\sum_{\text{basis}}\Lambda_l^{a_1\dots a_l}(x) : T^{a_1}\dots T^{a_l} \\ &= \Lambda^a(x) : T^a : + \ \Lambda_2^{a_1a_2}(x) : T^{a_1}T^{a_2} : + \dots . \end{aligned} \tag{3.3.3}$$

Now we define the covariant derivative $\mathcal{D}_\mu\psi(x) = \partial^\star_\mu \triangleright \psi(x) - iV_\mu \star \psi(x)$ by its transformation law

$$\delta_\Lambda(\mathcal{D}_\mu\psi(x)) = i\Lambda \star \mathcal{D}_\mu\psi(x). \tag{3.3.4}$$

Before we continue, one remarks is in order. The derivative $\partial^\star_\mu$ is not specified here. The choice of $\partial^\star_\mu$ depends on the choice of a deformed space on which we want to construct the gauge theory (and not only on that, as we shall see later). The reason for not specifying $\partial^\star_\mu$ immediately is that we are trying to be as general as possible and first discuss some problems characteristic for the noncommutative gauge theories in general. We come back to this problem in the next section.

The transformation law for the noncommutative gauge field V_μ, just like in the commutative case, follows from (3.3.4)

$$(\delta_\Lambda V_\mu) \star \psi = \partial^\star_\mu \triangleright (\Lambda \star \psi) - \Lambda \star (\partial^\star_\mu \triangleright \psi) + i\Lambda \star V_\mu \star \psi - iV_\mu \star \Lambda \star \psi. \tag{3.3.5}$$

Note that before we have specified the derivatives $\partial^\star_\mu$ we can not write (3.3.5) more explicitly since we do not know the Leibniz rules for the derivatives $\partial^\star_\mu$. It is important that from (3.3.5) and (3.3.3) it follows that V_μ has to be enveloping algebra-valued as well

$$\begin{aligned} V_\mu &= \sum_{l=1}^{\infty}\sum_{\text{basis}} V^l_{\mu\, a_1\dots a_l} : T^{a_1}\dots T^{a_l} : \\ &= V^a_\mu(x) : T^a : + \ V^{a_1a_2}_\mu : T^{a_1}T^{a_2} : + \dots . \end{aligned} \tag{3.3.6}$$

From (3.3.3) and (3.3.6) looks like we have formulated a theory with infinitely many degrees of freedom which is an unphysical situation. The way to solve this problem is to demand that all higher orders degrees of freedom are not independent, but that they can be expressed in terms of the finitely many degrees of freedom. The most natural then is to demand that all higher order degrees of freedom can be expressed in terms of the zeroth order degrees of freedom, that is the Lie algebra-valued quantities. If this is possible (and that we can only determine by explicit calculation, there is no principle to determine a priori if this reduction is possible or not) then we reduce the number of degrees of freedom to the classical one. That means that our noncommutative quantities (gauge parameter, gauge fields,...) will be functions of the classical ones. The explicit construction is know as the Seiberg-Witten map and it was first used by N. Seiberg and E. Witten in [26]. In the next section we continue with the construction of the Seiberg-Witten map.

3.4 Seiberg-Witten map

We start with the noncommutative gauge parameter Λ and suppose that it can be written as a function of the commutative gauge parameter $\alpha = \alpha^a T^a$ and the commutative gauge field $A^0_\mu = A^{0a}_\mu T^a$. Therefore, we introduce the following notation

$$\Lambda(x) = \Lambda_\alpha(A^0_\mu; x) \stackrel{\text{def}}{=} \Lambda_\alpha(x) \stackrel{\text{def}}{=} \Lambda_\alpha, \tag{3.4.1}$$

and keep the dependence on the commutative variables implicit as well as the x-dependence, unless we want to stress something. Also

$$\delta_\Lambda \psi = i\Lambda \star \psi = i\Lambda_\alpha \star \psi = \delta_\alpha \psi. \tag{3.4.2}$$

Here $\delta_\alpha\psi$ means that having the expression for the noncommutative field (here ψ) in terms of the commutative variables (ψ^0 and A^0_μ in this case, as we shall see later), noncommutative gauge transformation (3.4.2) is given by the commutative gauge transformation of the expanded noncommutative field. From (3.4.2) we have

$$\begin{aligned} \delta_\alpha\delta_\beta\psi &= \delta_\alpha(i\Lambda_\beta \star \psi) \\ &= i(\delta_\alpha\Lambda_\beta)\star\psi + i\Lambda_\beta \star (i\Lambda_\alpha \star \psi) = i(\delta_\alpha\Lambda_\beta)\star\psi - \Lambda_\beta\star\Lambda_\alpha\star\psi. \end{aligned} \tag{3.4.3}$$

The variation $\delta_\alpha\Lambda_\beta$ is nonzero since Λ_β depends on the commutative gauge field A^0_μ and $\delta_\alpha A^0_\mu = \partial_\mu\alpha - i[A^0_\mu, \alpha]$. The consistency condition (3.2.7) then gives

$$\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha - i\Lambda_\alpha\star\Lambda_\beta + i\Lambda_\beta\star\Lambda_\alpha = \Lambda_{-i[\alpha,\beta]}, \tag{3.4.4}$$

where we have omitted $\star\psi$ in each term since the equation must be true for an arbitrary field ψ. Since (3.4.4) is an equation in Λ_α only, we can use it to solve the Seiberg-Witten map for Λ_α. We solve it perturbatively, that is we expand Λ_α in the deformation parameter a[4]

$$\Lambda_\alpha = \Lambda^0_\alpha + a\Lambda^1_\alpha + \cdots + a^k\Lambda^k_\alpha + \dots. \tag{3.4.5}$$

But also the $\star$-product in (3.4.4) has to be expanded. Therefore, this is the place at which we leave the general discussion and specialise to the κ-deformed space. For the $\star$-product we use (2.4.4), that is the symmetrically ordered $\star$-product. Expanding equation (3.4.4) up to first order in a gives

$$a^0: \quad \delta_\alpha\Lambda^0_\beta - \delta_\beta\Lambda^0_\alpha - i[\Lambda^0_\alpha, \Lambda^0_\beta] = \Lambda^0_{-i[\alpha,\beta]} \tag{3.4.6}$$

$$\begin{aligned} a^1: \quad a\delta_\alpha\Lambda^1_\beta - a\delta_\beta\Lambda^1_\alpha &- ia[\Lambda^0_\alpha \stackrel{\star^1}{,} \Lambda^0_\beta] \\ &- ia[\Lambda^0_\alpha, \Lambda^1_\beta] - ia[\Lambda^1_\alpha, \Lambda^0_\beta] = a\Lambda^1_{-i[\alpha,\beta]}, \end{aligned} \tag{3.4.7}$$

[4]Different expansions are possible, for example the expansion in the enveloping algebra basis (3.3.3), or the expansion in the number of factors of gauge field A^0_μ. Of course, these expansions will not coincide.

where we introduced $\star^1$ for the first order term of the $\star$-product,

$$a\Lambda^0_\alpha \star^1 \Lambda^0_\beta = \frac{i}{2} C^{\rho\sigma}_\lambda x^\lambda (\partial_\rho \Lambda^0_\alpha)(\partial_\sigma \Lambda^0_\beta).$$

Looking at (3.4.6) we see that it will be fulfilled with the choice $\Lambda^0_\alpha = \alpha$, that is in zeroth order noncommutative gauge transformation reduces to the classical one[5].

Equation (3.4.7) is an inhomogeneous linear equation in Λ^1_α with the solution

$$a\Lambda^1_\alpha = -\frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \{A^0_\rho, \partial_\sigma \alpha\}. \tag{3.4.8}$$

Calculating explicitly $\delta_\alpha \Lambda^1_\beta$, using (3.1.9) one can check that (3.4.7) is fulfilled. However, this solution is not unique, one can add to it solutions of the homogeneous equation

$$a\delta_\alpha \Lambda^1_\beta - a\delta_\beta \Lambda^1_\alpha - i[\alpha, \Lambda^1_\beta] - i[\Lambda^1_\alpha, \beta] - a\Lambda^1_{-i[\alpha,\beta]} = 0. \tag{3.4.9}$$

The analysis of ambiguities of the Seiberg-Witten map was done in detail in [42] for the θ-deformed space. Most of the things said there can be applied here as well and we will not go into details. In the next chapter we use the freedom in the Seiberg-Witten map when discussing the ambiguity of the conserved current in the case of $U(1)$ gauge theory.

Now one can solve the second order for Λ_α and so on. The second order solution is given in [86]. Here we do not go into second order analysis. Just for completeness we write the solution for Λ_α up to first order in a

$$\Lambda_\alpha = \alpha - \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \{A^0_\rho, \partial_\sigma \alpha\}. \tag{3.4.10}$$

Before proceeding further, we have two comments. When solving (3.4.7) it was supposed that Λ^1_α is not derivative-valued, that is it is a function not a differential operator. Nevertheless, the different approach is possible and we describe it shortly at the end of the last chapter. If one compares (3.4.10) with the solution for Λ^1_α in the case of θ-deformed space [41] one sees that they are the same, replaced $\theta^{\rho\sigma}$ with $C^{\rho\sigma}_\lambda x^\lambda$. This is the consequence of the first order similarity of the symmetrically ordered $\star$-product for the κ-deformed space and the Moyal-Weyl $\star$-product (1.5.8)

$$f \star g = fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) + \dots, \tag{3.4.11}$$

$$f \star g = fg + \frac{i}{2} C^{\rho\sigma}_\lambda x^\lambda (\partial_\rho f)(\partial_\sigma g) + \dots. \tag{3.4.12}$$

However, this analogy only applies to the first order. Already in the second order the $\star$-products (3.4.11) and (3.4.12) are different and also the explicit x-dependence of Λ_α in (3.4.10) produces new terms in second order in the case of κ-deformed space, compared with the θ-deformed space [86].

Having the solution of the Seiberg-Witten map for the noncommutative gauge parameter Λ_α we can now solve the Seiberg-Witten map for the noncommutative matter field ψ using equation (3.4.2). Again, ψ has to be expanded in the deformation parameter a

$$\psi = \psi^0 + a\psi^1 + \dots. \tag{3.4.13}$$

[5]Note that $\delta_\alpha \Lambda^0_\beta$=0.

Inserting this into (3.4.2) and expanding the $\star$-product to first order in a gives

$$a^0: \quad \delta_\alpha\psi^0 = i\alpha\psi^0, \tag{3.4.14}$$

$$a^1: \quad a\delta_\alpha\psi^1 = ia\alpha\psi^1 + ia\Lambda^1_\alpha\psi^0 - \frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda(\partial_\rho\alpha)(\partial_\sigma\psi^0). \tag{3.4.15}$$

The zeroth order equation is fulfilled if ψ^0 is the commutative matter field, since $\delta_\alpha\psi^0 = i\alpha\psi^0$. The first order solution is

$$a\psi^1 = -\frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda A^0_\rho(\partial_\sigma\psi^0) + \frac{i}{8}C^{\rho\sigma}_\lambda x^\lambda[A^0_\rho, A^0_\sigma]\psi^0. \tag{3.4.16}$$

The comments about the ambiguities of the Seiberg-Witten map and about the similarity of the first order solution with the solution in the case of the θ-deformed space also apply here. Again for the completeness we write the solution for the field ψ up to first order in a

$$\psi = \psi^0 - \frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda A^0_\rho(\partial_\sigma\psi^0) + \frac{i}{8}C^{\rho\sigma}_\lambda x^\lambda[A^0_\rho, A^0_\sigma]\psi^0. \tag{3.4.17}$$

3.5 Covariant derivative and the gauge field

The covariant derivative was introduced in (3.3.4). However, there we have not specified the derivatives $\partial^\star_\mu$. Now we do that first and then solve the Seiberg-Witten map for the noncommutative gauge field V_μ.

As it was shown in Chapter 2, there is no unique derivative on the κ-deformed space and any of derivatives obtained there is a good candidate for $\partial^\star_\mu$. But if we look at the transformation laws of different derivatives under the κ-deformed Lorentz transformations, we see that all but one have very complicated commutation relations with the κ-Lorentz generators $M^{\mu\nu}$. The one that is different is the Dirac derivative $D^\star_\mu$ introduced in (2.4.15). It fulfils

$$[L^{\star\,\mu\nu}, D^\star_\rho] = \delta^\nu_\rho D^{\star\,\mu} - \delta^\mu_\rho D^{\star\,\nu}, \tag{3.5.1}$$

where $L^{\star\,\mu\nu}$ are given in (2.4.19). Therefore, the covariant derivative defined as $\mathcal{D}_\mu = D^\star_\mu - iV_\mu$ transforms as

$$\delta_\omega(\mathcal{D}_\rho\psi) = -\frac{1}{2}\omega^{\alpha\beta}\Big(L^\star_{\alpha\beta} \triangleright (\mathcal{D}_\rho\psi) - \eta_{\beta\rho}(\mathcal{D}_\alpha\psi) + \eta_{\alpha\rho}(\mathcal{D}_\beta\psi)\Big). \tag{3.5.2}$$

This means that the gauge field V_ρ transforms as a covariant vector (2.5.8).

Having fixed the derivative we want to covariantise, we proceed in the familiar way, by calculating the transformation law of the gauge filed V_μ. From

$$\delta_\alpha(\mathcal{D}_\mu\psi) = i\Lambda_\alpha \star \mathcal{D}_\mu\psi = i\Lambda_\alpha \star \left(D^\star_\mu \triangleright \psi - iV_\mu \star \psi\right) \tag{3.5.3}$$

it follows

$$(\delta_\alpha V_\mu) \star \psi = D^\star_\mu \triangleright (\Lambda_\alpha \star \psi) - \Lambda_\alpha \star (D^\star_\mu \triangleright \psi) + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu] \star \psi. \tag{3.5.4}$$

Now one has to be careful since the derivatives $D^\star_\mu$ have the nontrivial Leibniz rules (2.4.16) and (2.4.17). It is convenient to analyse separately the nth and jth components.

First let us look at the jth component of (3.5.4). Using the Leibniz rule for $D^\star_j$ (2.4.17), we obtain

$$\delta_\alpha V_j \star \psi = (D^\star_j \triangleright \Lambda_\alpha) \star \left(e^{-ia\partial_n} \triangleright \psi\right) + i\Lambda_\alpha \star V_j \star \psi - iV_j \star (\Lambda_\alpha \star \psi)\,. \qquad (3.5.5)$$

If we demand that V_j is a function and not a differential operator, we see from (3.5.5) that we can not solve this equation. The problematic term is the first term on the right-hand side; it is the consequence of the nontrivial Leibniz rule for $D^\star_j$[6]. The only way to solve this is to allow that V_j is a differential operator, that is it is derivative-valued. Looking at (3.5.5) we make the following ansatz

$$V_j = A_j \star e^{-ia\partial_n} \qquad (3.5.6)$$

and insert it in (3.5.5). After using $e^{-ia\partial_n} \triangleright (f \star g) = (e^{-ia\partial_n} \triangleright f) \star (e^{-ia\partial_n} \triangleright g)$ and omitting $e^{-ia\partial_n} \triangleright \psi$ on the right-hand side we arrive at

$$\delta_\alpha A_j = (D^\star_j \triangleright \Lambda_\alpha) + i\Lambda_\alpha \star A_j - iA_j \star \left(e^{-ia\partial_n} \triangleright \Lambda_\alpha\right). \qquad (3.5.7)$$

This equation can now be used to solve the Seiberg-Witten map for the field A_j. As before, A_j is expanded in the deformation parameter a:

$$A_j = A^0_j + aA^1_j + \dots. \qquad (3.5.8)$$

Expanding (3.5.7) gives

$$a^0: \quad \delta_\alpha A^0_j = \partial_j\alpha + i[\alpha, A^0_j] \qquad (3.5.9)$$

$$\begin{aligned} a^1: \quad a\delta_\alpha A^1_j &= a\partial_j\Lambda^1 - \frac{ia}{2}\partial_j\partial_n\alpha + ia[\alpha, A^1_j] + ia[\Lambda^1, A^0_j] \\ &\quad + ia[\alpha \stackrel{\star^1}{,} A^0_j] - aA^0_j(\partial_n\alpha). \end{aligned} \qquad (3.5.10)$$

The zeroth order solution is obviously the commutative gauge field A^0_j. The first order solution is given by

$$aA^1_j = -\frac{ia}{2}\partial_n A^0_j - \frac{a}{4}\{A^0_n, A^0_j\} + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(\{F^0_{\rho j}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_j\}\Big). \qquad (3.5.11)$$

That this is really a solution of (3.5.10) one can check explicitly using (3.1.9). For completeness we write the solution for the field V_j up to first order in a

$$V_j = A^0_j - iaA^0_j\partial_n - \frac{ia}{2}\partial_n A^0_j - \frac{a}{4}\{A^0_n, A^0_j\} + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(\{F^0_{\rho j}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_j\}\Big). \qquad (3.5.12)$$

Comparing this solution with the solution for the noncommutative gauge field in the θ-deformed case, we see that now they differ already in the first order, namely the second, third and fourth term in (3.5.12) do not appear in the θ-deformed case.

Before going to the nth component of (3.5.4) we remark one more thing. Solving for V_j we made ansatz (3.5.6). However, this is not the most general ansatz; we could have started with

$$V_j = A_{1j} \star e^{-ia\partial_n} + A_{2j} \star D^\star_n + A_3 \star D^\star_j + A^l_{4j} \star D^\star_l. \qquad (3.5.13)$$

[6] Choosing the different set of derivatives [87] one can avoid the derivative-valued gauge fields, but then one loses the symmetry properties.

Inserting this into (3.5.5) gives

$$\begin{aligned}\delta_\alpha A_{1j} &= (D^\star_j \triangleright \Lambda_\alpha) + i\Lambda_\alpha \star A_{1j} - iA_{1j} \star (e^{-ia\partial_n} \triangleright \Lambda_\alpha) \\ &\quad -iA_{2j} \star (D^\star_n \triangleright \Lambda_\alpha) - iA_3 \star (D^\star_j \triangleright \Lambda_\alpha) - iA^l_{4j} \star (D^\star_l \triangleright \Lambda_\alpha), \qquad (3.5.14)\\ \delta_\alpha A_{2j} &= i\Lambda_\alpha \star A_{2j} - iA_{2j} \star \big(e^{ia\partial_n} \triangleright \Lambda_\alpha\big), \qquad (3.5.15)\\ \delta_\alpha A_3 &= i\Lambda_\alpha \star A_3 - iA_3 \star \Lambda_\alpha, \qquad (3.5.16)\\ \delta_\alpha A^l_{4j} &= i\Lambda_\alpha \star A^l_{4j} - iA^l_{4j} \star \Lambda_\alpha - aA_{2j} \star (\partial^{\star\, l} \triangleright \Lambda_\alpha), \qquad (3.5.17)\end{aligned}$$

where we have collected terms proportional to $\star e^{-ia\partial_n} \triangleright \psi$, $\star D^\star_n \triangleright \psi$, $\star D^\star_j \triangleright \psi$ and $\star D^\star_l \triangleright \psi$ respectively. To solve for A_{1j} we have to solve first the equations for A_{2j}, A_3 and A^l_{4j}. Let us start from A_{2j}. Expanding (3.5.15) gives

$$\begin{aligned} a^0: &\quad \delta_\alpha A^0_{2j} = i[\alpha, A^0_{2j}] \qquad (3.5.18)\\ a^1: &\quad a\delta_\alpha A^1_{2j} = ia[\alpha, A^1_{2j}] + ia[\Lambda^1, A^0_{2j}] + ia[\alpha \stackrel{\star}{,}{}^1 A^0_{2j}] + aA^0_{2j}(\partial_n\alpha). \qquad (3.5.19)\end{aligned}$$

Looking at the zeroth order equation we see that there are no inhomogeneous terms at the right-hand side, so $A^0_{2j} = 0$ is a solution. Equally well, the choice $A^0_{2j} = c_1 F^0_{nj}$ where c_1 is an arbitrary constant is a possible solution of (3.5.18). But this would mean that the noncommutative gauge potential starts like

$$V_j = A^0_j + F^0_{nj}\partial_n + \mathcal{O}(a)$$

which in the limit $a \to 0$ does not reduce to the commutative gauge potential. Since we always insist on the commutative limit of our theory this solution is not what we want. The only possibility is therefore to put $A^0_{2j} = 0$. For the first order equation this then gives

$$a\delta_\alpha A^1_{2j} = i[\alpha, A^1_{2j}]. \qquad (3.5.20)$$

Again, this is a homogeneous equation and we can choose $A^1_{2j} = 0$. Also, $A^1_{2j} = c_2 a F^0_{nj}$ is allowed by (3.5.20) but this time there is no commutative limit restriction. Since so far we have been ignoring the terms that were the solutions of homogeneous equations (when solving for Λ_α, ψ, ...); we do the same here, that is we choose $A^1_{2j} = 0$. One should keep in mind that different choices are possible, but being the solutions of homogeneous equations they reduce to the freedom in the Seiberg-Witten map.

For A_3 we repeat the previous analysis with the same result, $A_3 = 0$ to first order in a. With this solution the same follows for A^l_{4j}: $A^l_{4j} = 0$. Inserting this in (3.5.14) gives (3.5.10) again. Therefore, the solution for V_j is the same as before, (3.5.12). To conclude this remark, we could have started from a more general ansatz for V_j leading to a different solution. However, the difference beetwen (3.5.12) and the new solution are just the terms which are solutions of the homogeneous equations and we say that this two solutions are equivalent up to the freedom in the Seiberg-Witten map.

Now, we look at the nth component of equation (3.5.4). Using the Leibniz rule for $D^\star_n$ (2.4.16) leads to

$$\begin{aligned}\delta_\alpha V_n \star \psi &= (D^\star_n \triangleright \Lambda_\alpha) \star (e^{-ia\partial_n} \triangleright \psi) + \Big((e^{ia\partial_n} - 1) \triangleright \Lambda_\alpha\Big) \star D^\star_n \triangleright \psi \qquad (3.5.21)\\ &\quad -ia\big(D^\star_j \triangleright (e^{ia\partial_n} \triangleright \Lambda_\alpha)\big) \star (D^{\star\, j} \triangleright \psi) + i\Lambda_\alpha \star V_n \star \psi - iV_n \star (\Lambda_\alpha \star \psi)\,.\end{aligned}$$

The ansatz for V_n is

$$V_n = A_{1n} \star e^{-ia\partial_n} + A_{2n} \star D^\star_n + A^l_{3n} \star D^\star_l, \tag{3.5.22}$$

but this time it is the most general ansatz we could think of. It gives (after collecting terms proportional to $\star e^{-ia\partial_n} \triangleright \psi$, $\star D^\star_n \triangleright \psi$ and $\star D^\star_l \triangleright \psi$) the transformation laws for the fields A_{1n}, A_{2n} and A^l_{3n} respectively:

$$\begin{aligned}\delta_\alpha A_{1n} &= (D^\star_n \triangleright \Lambda_\alpha) + i\Lambda_\alpha \star A_{1n} - iA_{1n} \star (e^{-ia\partial_n} \triangleright \Lambda_\alpha) \\ &\quad - iA_{2n} \star (D^\star_n \Lambda_\alpha) - iA^l_{3n} \star (D^\star_l \triangleright \Lambda_\alpha), \end{aligned}\tag{3.5.23}$$

$$\delta_\alpha A_{2n} = \big((e^{ia\partial_n} - 1) \triangleright \Lambda_\alpha\big) + i\Lambda_\alpha \star A_{2n} - iA_{2n} \star \big(e^{ia\partial_n} \triangleright \Lambda_\alpha\big), \tag{3.5.24}$$

$$\delta_\alpha A^l_{3n} = -ia\big(\partial^{\star\, l} \triangleright \Lambda_\alpha\big) + i\Lambda_\alpha \star A^l_{3n} - iA^l_{3n} \star \Lambda_\alpha - aA_{2n} \star (\partial^{\star\, l}\Lambda_\alpha). \tag{3.5.25}$$

We have to solve A_{2n} first. We expand (3.5.24)

$$a^0: \quad \delta_\alpha A^0_{2n} = i[\alpha, A^0_{2n}] \tag{3.5.26}$$

$$\begin{aligned} a^1: \quad a\delta_\alpha A^1_{2n} &= ia\partial_n\alpha + ia[\alpha, A^1_{2n}] + ia[\Lambda^1, A^0_{2n}] \\ &\quad + ia[\alpha \stackrel{\star}{;}^1 A^0_{2n}] + aA^0_{2n}(\partial_n\alpha). \end{aligned}\tag{3.5.27}$$

With the same arguments as before (proper commutative limit), the zeroth order solution is $A^0_{2n} = 0$. This then gives

$$a\delta_\alpha A^1_{2n} = ia\partial_n\alpha + ia[\alpha, A^1_{2n}],$$

and the solution for A_{2n} up to first order in a is

$$A_{2n} = iaA^0_n. \tag{3.5.28}$$

Again we have ignored the solutions of the homogeneous equation (for example, we could have added the first order term $c_3 C^{\rho\sigma}_\lambda x^\lambda F^0_{\rho\sigma}$ to the solution (3.5.28)).

Expanding (3.5.25) gives

$$a^0: \quad \delta_\alpha A^{0l}_{3n} = i[\alpha, A^{0l}_{3n}] \tag{3.5.29}$$

$$\begin{aligned} a^1: \quad a\delta_\alpha A^{0l}_{3n} &= -ia\partial^l\alpha + ia[\alpha, A^{1l}_{3n}] + ia[\Lambda^1, A^{0l}_{3n}] \\ &\quad + ia[\alpha \stackrel{\star}{;}^1 A^{0l}_{3n}] - aA^0_{2n}(\partial^l\alpha). \end{aligned}\tag{3.5.30}$$

The solution up to first order is

$$A^l_{3n} = -iaA^{0l}. \tag{3.5.31}$$

Finally we come to (3.5.23)

$$a^0: \quad \delta_\alpha A^0_{1n} = \partial_n\alpha + i[\alpha, A^0_{1n}] \tag{3.5.32}$$

$$\begin{aligned} a^1: \quad a\delta_\alpha A^1_{1n} &= a\partial_n\Lambda^1 - \frac{ia}{2}\partial_l\partial^l\alpha + ia[\alpha, A^1_{1n}] + ia[\Lambda^1, A^0_{1n}] \\ &\quad + ia[\alpha \stackrel{\star}{;}^1 A^0_{1n}] - iaA^1_{2n}(\partial_n\alpha) - iaA^{1l}_{3n}(\partial_l\alpha). \end{aligned}\tag{3.5.33}$$

The zeroth order solution is just the commutative field $A^0_{1n} = A^0_n$. Inserting this and the solutions for A_{2n} and A^l_{3n} we find

$$aA^1_{1n} = -\frac{a}{2}\Big(i\partial_j A^{0j} + A^0_j A^{0j}\Big) + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(\{F^0_{\rho n}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_n\}\Big). \tag{3.5.34}$$

Summing up all the results leads to the solution for V_n up to first order in a

$$V_n = A^0_n - iaA^{0j}\partial_j - \frac{ia}{2}\partial_j A^{0j} - \frac{a}{2}A^0_j A^{0j} + \frac{1}{4}C^{\rho\sigma}_{\ \ \lambda} x^\lambda\Big(\{F^0_{\rho n}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_n\}\Big). \quad (3.5.35)$$

With this result we end our analysis of the Seiberg-Witten map on the κ-deformed space. Once again we mention that the problem of ambiguities has not been discussed here. A part of it we will analyse in the next chapter and the rest is left for the future work.

3.6 Gauge covariant Lagrangians

In this section we want to construct the gauge covariant (invariant) Lagrangians for the matter and gauge field. To write down the Lagrangian for the matter field we already have all that we need. For the gauge field Lagrangian we first have to calculate the field-strength tensor. It is defined as

$$\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu] \quad (3.6.1)$$

and from here it follows

$$\delta_\alpha \mathcal{F}_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} \mathcal{F}_{\mu\nu}]. \quad (3.6.2)$$

Applying this to the field ψ (calculating $\mathcal{F}_{\mu\nu}\star\psi$) gives

$$\begin{aligned}
\mathcal{F}_{ij} &= \Big((D^\star_i \triangleright V_j) - (D^\star_j \triangleright V_i) - iV_i \star (e^{-ia\partial_n} \triangleright V_j) + iV_j \star (e^{-ia\partial_n} \triangleright V_i)\Big) \star e^{-2ia\partial_n} \\
&= F'_{ij} \star e^{-2ia\partial_n}, \quad (3.6.3)\\
\mathcal{F}_{nj} &= F'_{nj1} \star e^{-2ia\partial_n} + F'_{nj2} \star e^{-ia\partial_n} D^\star_n + F'^{l}_{nj3} \star e^{-ia\partial_n} D^\star_l, \quad (3.6.4)
\end{aligned}$$

where we have used the Leibniz rules for $D^\star_\mu$ derivatives and

$$\begin{aligned}
F'_{nj1} &= (D^\star_n \triangleright V_j) - (D^\star_j \triangleright V_{n1}) - iV_{n1} \star (e^{-ia\partial_n} \triangleright V_j) + iV_j \star (e^{-ia\partial_n} \triangleright V_{n1}) \\
&\quad -iV^l_{n3} \star (D^\star_l \triangleright V_j) - iV_{n2} \star (D^\star_n \triangleright V_j), \quad (3.6.5)\\
F'_{nj2} &= \Big((e^{ia\partial_n} - 1) \triangleright V_j\Big) - (D^\star_j \triangleright V_{n2}) - iV_{n2} \star (e^{ia\partial_n} \triangleright V_j) \\
&\quad +iV_j \star (e^{-ia\partial_n} \triangleright V_{n2}), \quad (3.6.6)\\
F'^{l}_{nj3} &= -ia(\partial^{\star l} \triangleright V_j) - (D^\star_j \triangleright V^l_{n3}) - aV_{n2} \star (\partial^{\star l} \triangleright V_j) \\
&\quad -iV^l_{n3} \star V_j + V_j \star (e^{-ia\partial_n} \triangleright V^l_{n3}). \quad (3.6.7)
\end{aligned}$$

From (3.6.3) and (3.6.4) it is obvious that $\mathcal{F}_{\mu\nu}$ is derivative-valued. Therefore, the term $\mathcal{F}_{\mu\nu}\star\mathcal{F}^{\mu\nu}$ in the Lagrangian (or in the action afterwards) does not make sense. This problem can be solved using the analogy with the gravity theory. There one also calculates the commutator of two covariant derivatives and expresses the result in terms of the curvature tensor and torsion. We try to do the same here.

Let us look first at the component $\mathcal{F}_{ij}$. We expand $e^{-2ia\partial_n}$ up to first order in a

$$\begin{aligned}
\mathcal{F}_{ij} &= F'_{ij} \star e^{-2ia\partial_n} = F'_{ij} \star (1 - 2ia\partial_n) \\
&= F'_{ij} \star \Big(1 - 2ia(\partial_n - iA^0_n) + 2aA^0_n\Big) = F'_{ij} \star \Big(1 + 2aA^0_n - 2ia\mathcal{D}^0_n\Big) \\
&= F_{ij} + T^n_{ij} \star \mathcal{D}^0_n.
\end{aligned}$$

Here we have covariantised the derivative ∂_n by adding the field A_n^0 to it (since we are interested only in the theory up to first order in a) and subtracting the same term. Note that F_{ij} is different from F'_{ij}, the difference is in the first order term coming from covariantising ∂_n. One can continue this separation on terms proportional to the covariant derivatives to higher orders and for $\mathcal{F}_{nj}$ as well. We conclude that the field-strength tensor $\mathcal{F}_{\mu\nu}$ can be written as

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^{\rho}_{\mu\nu}\mathcal{D}_\rho + \dots + T^{\rho_1\dots\rho_l}_{\mu\nu} : \mathcal{D}_{\rho_1}\dots\mathcal{D}_{\rho_l} : + \dots\ , \tag{3.6.8}$$

where the colons denote a basis in the enveloping algebra of covariant derivatives[7]. The term which is not derivative-valued we call curvature-like term, while the other terms $T^{\rho_1\dots\rho_l}_{\mu\nu}$ are torsion-like terms. When writing the Lagrangian (action) we use only the curvature-like terms and ignore all the torsion-like terms.

Each of the terms in the expansion transforms covariantly again and that is why we define the torsion-like terms proportional to the covariant and not to the usual derivatives. To make this clearer we start form the definition (3.6.1). Then

$$\begin{aligned} \delta_\alpha(\mathcal{F}_{\mu\nu}\star\psi) &= i\Lambda_\alpha\star\mathcal{F}_{\mu\nu}\star\psi \\ \delta_\alpha\Big(F_{\mu\nu}\star\psi + T^{\rho}_{\mu\nu}\star(\mathcal{D}_\rho\psi)+\dots\Big) &= i\Lambda_\alpha\star\Big(F_{\mu\nu}\star\psi + T^{\rho}_{\mu\nu}\star(\mathcal{D}_\rho\psi)+\dots\Big). \end{aligned}$$

From here we have

$$\delta_\alpha F_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\mu\nu}], \tag{3.6.9}$$

$$\delta_\alpha T^{\rho}_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} T^{\rho}_{\mu\nu}]. \tag{3.6.10}$$

$$\dots .$$

If one would expand $\mathcal{F}_{\mu\nu}$ in terms of the usual derivatives, a term like $T^{\rho}_{\mu\nu}\star(\partial_\rho\psi)$ would appear. Then

$$\begin{aligned} \delta_\alpha\big(T^{\rho}_{\mu\nu}\star(\partial_\rho\psi)\big) &= (\delta_\alpha T^{\rho}_{\mu\nu})\star(\partial_\rho\psi) + T^{\rho}_{\mu\nu}\star\big(\delta_\alpha(\partial_\rho\psi)\big) \\ &\stackrel{\text{def}}{=} i\Lambda_\alpha\star T^{\rho}_{\mu\nu}\star(\partial_\rho\psi). \end{aligned}$$

The last term in the first line produces an additional term (since $\partial_\rho\psi$ does not transform covariantly) spoiling the covariant transformation of $T^{\rho}_{\mu\nu}$.

Rewriting (3.6.3) and (3.6.4) in the form (3.6.8) and expanding everything up to first order in a gives

$$F_{ij} = F^0_{ij} - ia\mathcal{D}^0_n F^0_{ij} + \frac{1}{4}C^{\rho\sigma}_{\lambda}x^\lambda\Big(2\{F^0_{\rho i}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{ij}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{ij}\}\Big), \tag{3.6.11}$$

$$T^{\mu}_{ij} = -2ia\delta^\mu_n F^0_{ij}, \tag{3.6.12}$$

$$\begin{aligned} F_{nj} &= F^0_{nj} - \frac{ia}{2}\mathcal{D}^{\mu 0}F^0_{\mu j} \\ &\quad + \frac{1}{4}C^{\rho\sigma}_{\lambda}x^\lambda\Big(2\{F^0_{\rho n}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{nj}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{nj}\}\Big), \end{aligned} \tag{3.6.13}$$

$$T^{\mu}_{nj} = -ia\eta^{\mu l}F^0_{lj} - ia\delta^\mu_n F^0_{nj}. \tag{3.6.14}$$

[7] The enveloping algebra of covariant derivatives is the algebra freely generated by derivatives $\mathcal{D}_\rho$ and divided by the ideal generated by (3.6.1).

Now we have all the ingredients to write both Lagrangians for the matter and gauge field.

The dynamics of the gauge field can be formulated using the tensor $F^{\mu\nu}$

$$\mathcal{L}_{\text{gauge}} = c\ \mathrm{Tr}\left(F^{\mu\nu} \star F_{\mu\nu}\right). \tag{3.6.15}$$

Note, however, that $\mathrm{Tr}\left(F_{\mu\nu} \star F^{\mu\nu}\right)$ is not invariant because of the $\star$-product in (3.6.15) but

$$\delta_\alpha \mathcal{L}_{\text{gauge}} = i[\Lambda_\alpha \stackrel{\star}{,} \mathcal{L}_{\text{gauge}}]. \tag{3.6.16}$$

The Lagrangian $\mathcal{L}_{\text{gauge}}$ gives the action (formulated with an integral with the trace property, see the next chapter) which is gauge invariant. The trace will depend on the representation of the generators T^a because higher products of the generators will enter through the enveloping algebra. Expanding (3.6.15) up to first order in a and choosing, in analogy with the undeformed theory $c = -\frac{1}{4}$, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} = & -\frac{1}{4}\mathrm{Tr}(F^0_{\mu\nu}F^{0\mu\nu}) \\
& -\frac{i}{8}C_\lambda^{\rho\sigma}x^\lambda \mathrm{Tr}\Big((\mathcal{D}^0_\rho F^{0\mu\nu})(\mathcal{D}^0_\sigma F^0_{\mu\nu}) + \frac{i}{2}\{A^0_\rho, (\partial_\sigma + \mathcal{D}^0_\sigma)(F^{0\mu\nu}F^0_{\mu\nu})\} \\
& -i\{F^{0\mu\nu}, \{F^0_{\mu\rho}, F^0_{\nu\sigma}\}\}\Big) && (3.6.17) \\
& +\frac{ia}{4}\mathrm{Tr}\Big(\mathcal{D}^0_n(F^{0\mu\nu}F^0_{\mu\nu}) - \{(\mathcal{D}^0_\mu F^{0\mu j}), F^0_{nj}\}\Big), && (3.6.18)
\end{aligned}$$

where $\mathcal{D}^0_\mu\psi = \partial_\mu\psi - iA^0_\mu\psi$ and $\mathcal{D}^0_\mu F^0_{\alpha\beta} = \partial_\mu F^0_{\alpha\beta} - i[A^0_\mu, F^0_{\alpha\beta}]$.

The Lagrangian for the matter field is

$$\mathcal{L}_{\text{matter}} = \bar{\psi} \star (i\gamma^\mu \mathcal{D}_\mu - m)\,\psi \tag{3.6.19}$$

and

$$\delta_\alpha \mathcal{L}_{\text{matter}} = 0. \tag{3.6.20}$$

Expanded up to first order in a (3.6.19) gives

$$\begin{aligned}
\mathcal{L}_{\text{matter}} = & \bar{\psi}^0(i\gamma^\mu(\partial_\mu - iA^0_\mu) - m)\psi^0 \\
& +\frac{i}{2}C_\lambda^{\rho\sigma}x^\lambda(\overline{\mathcal{D}^0_\rho\psi^0})\mathcal{D}^0_\sigma\left(i\gamma^\mu\mathcal{D}^0_\mu - m\right)\psi^0 - \frac{i}{2}x^\nu C_\nu^{\rho\sigma}\bar{\psi}^0\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0). \\
& +\frac{a}{2}\bar{\psi}^0\gamma^j(\mathcal{D}^0_n\mathcal{D}^0_j\psi^0) + \frac{a}{2}\bar{\psi}^0\gamma^n(\mathcal{D}^0_j\mathcal{D}^{0j}\psi^0). && (3.6.21)
\end{aligned}$$

These are the Lagrangians which define the dynamics on the κ-deformed space. In the next chapter we formulate the action from these Lagrangians.

4

$U(1)$ gauge theory on the κ-deformed space

In this chapter we focus on the $U(1)$ gauge theory. We would like to come from the Lagrangians obtained in the previous chapter to the action which is necessary if one wants to analyse the properties of the given theory. To be able to write the action one needs a definition of an integral. Therefore, we first define an integral for the κ-deformed space (we continue to work in the $\star$-product approach) and formulate the variational principle. Having these two things at hand we write the action for the $U(1)$ gauge theory coupled to fermions, κ-deformed electrodynamics, and analyse some of its properties. The $U(1)$ gauge theory has been chosen for its simplicity, but the same approach is possible for nonabelian gauge theories as well. Calculations are presented up to first order in the deformation parameter but in some cases the results can be generalised to all orders. We say explicitly which results are and which are not known to all orders.

4.1 Integral and the variational principle

An integral is a linear map of the algebra $\mathcal{A}_x$ into complex numbers

$$\int : \quad \mathcal{A}_x \longrightarrow \mathbb{C}, \tag{4.1.1}$$

$$\int (c_1 f + c_2 g) = c_1 \int f + c_2 \int g, \quad \forall f, g \in \mathcal{A}_x, \quad c_i \in \mathbb{C}. \tag{4.1.2}$$

However, there is one additional property we demand, the cyclicity. The motivation for this requirement can be given easily. The transformation law of the field-strength tensor $F_{\mu\nu} = F^a_{\mu\nu} T^a$ (for generality we are still considering nonabelian gauge theory) is

$$\delta_\alpha F_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\mu\nu}]. \tag{4.1.3}$$

The Yang-Mills action is given by

$$S_{YM} = -\frac{1}{4}\mathrm{Tr} \int \mathrm{d}^{n+1}x \, F_{\mu\nu} \star F^{\mu\nu}, \tag{4.1.4}$$

where for the moment $\int \mathrm{d}^{n+1}x$ is the usual commutative integral. Using (4.1.3) we obtain

$$\begin{aligned}\delta_\alpha S_{YM} &= \delta_\alpha \Big(-\frac{1}{4}\mathrm{Tr}\int \mathrm{d}^{n+1}x\, F_{\mu\nu}\star F^{\mu\nu}\Big)\\ &= -\frac{1}{4}\mathrm{Tr}\int \mathrm{d}^{n+1}x\, \left(i\Lambda_\alpha\star F_{\mu\nu}\star F^{\mu\nu} - iF_{\mu\nu}\star F^{\mu\nu}\star\Lambda_\alpha\right).\end{aligned} \tag{4.1.5}$$

If we would have the usual pointwise product in (4.1.5), then cyclicly permuting under the trace would lead to $\delta_\alpha S_{YM} = 0$. Unfortunately, we cannot do the same here since the $\star$-product is noncommutative. One way to repair this (if it is the only way we are not sure) is to demand one additional property of the integral called cyclicity

$$\int \mathrm{d}^{n+1}x\,(f(x)\star g(x)) = \int \mathrm{d}^{n+1}x\,(g(x)\star f(x)) = \int \mathrm{d}^{n+1}x\,(f(x)g(x)). \tag{4.1.6}$$

From (4.1.6) it follows

$$\int \mathrm{d}^{n+1}x\,(f_1\star f_2\star\cdots\star f_k) = \int \mathrm{d}^{n+1}x\,(f_k\star f_1\star f_2\star\cdots\star f_{k-1}), \tag{4.1.7}$$

that is cyclic permutations under the integral are allowed. Then $\delta_\alpha S_{YM} = 0$ follows form (4.1.5).

So far the discussion has been general, the $\star$-product in (4.1.6) has not been specified. Since we are interested in the κ-deformed space, we now use the $\star$-product (2.4.4), expand it up to first order in a and check if (4.1.6) is fulfilled

$$\begin{aligned}\int \mathrm{d}^{n+1}x\,(f\star g) &= \int \mathrm{d}^{n+1}x\,\Big(fg + \frac{i}{2}C^{\alpha\beta}_{\lambda}x^\lambda(\partial_\alpha f)(\partial_\beta g)\Big)\\ &= \int \mathrm{d}^{n+1}x\,\Big(fg + \frac{i}{2}C^{\alpha\beta}_{\lambda}\partial_\beta\big(x^\lambda(\partial_\alpha f)g\big) - \frac{i}{2}C^{\alpha\beta}_{\beta}(\partial_\alpha f)g - \frac{i}{2}C^{\alpha\beta}_{\lambda}x^\lambda(\partial_\alpha\partial_\beta f)g\Big)\\ &= \int \mathrm{d}^{n+1}x\,\Big(fg - \frac{i}{2}C^{\alpha\beta}_{\beta}(\partial_\alpha f)g\Big) \neq \int \mathrm{d}^{n+1}x\, fg.\end{aligned}$$

We see that the usual integral is not suitable for defining the action[1]. Expanding the $\star$-product to higher orders only gives new terms on the left-hand side in the last line and the conclusion stays the same. Nevertheless, one can modify the usual integral by adding the measure function [88], [89] such that

$$\int \mathrm{d}^{n+1}x\,\mu(x)\,(f\star g) = \int \mathrm{d}^{n+1}x\,\mu(x)\,(g\star f) = \int \mathrm{d}^{n+1}x\,\mu(x)\,(fg). \tag{4.1.8}$$

Note that $\mu(x)$ is not $\star$-multiplied with other functions, it is a part of the volume element. Expanding (4.1.8) up to first order in the deformation parameter gives

$$\begin{aligned}\int \mathrm{d}^{n+1}x\,\mu\,(f\star g) &= \int \mathrm{d}^{n+1}x\,\mu\,\Big(fg + \frac{i}{2}C^{\alpha\beta}_{\lambda}x^\lambda(\partial_\alpha f)(\partial_\beta g)\Big)\\ &= \int \mathrm{d}^{n+1}x\,\Big(\mu fg - \frac{i}{2}C^{\alpha\beta}_{\lambda}x^\lambda(\partial_\beta\mu)(\partial_\alpha f)g - \frac{i}{2}C^{\alpha\beta}_{\beta}\mu(\partial_\alpha f)g\Big)\\ &\overset{\text{def}}{=} \int \mathrm{d}^{n+1}x\,\mu\, fg.\end{aligned}$$

[1] In the case of the θ-deformed space, that is the Moyal-Weyl $\star$-product the usual integral fulfils (4.1.6). We use this in the last chapter when constructing the deformed Einstein-Hilbert action on this space.

Using the explicit form for the structure constants $C_\lambda^{\alpha\beta}$ (2.1.2) from the last line follows

$$\partial_n \mu(x) = 0, \qquad x^j \partial_j \mu(x) = -n\mu(x). \tag{4.1.9}$$

Expanding the $\star$-product up to second order one sees that there are no new conditions on μ coming from the second order; this is valid for all orders. This result was also obtained in [90].

The solution of equations (4.1.9) is not unique. Some of the possibilities are

$$\mu = \frac{1}{x^0 \ldots x^{n-1}}, \quad \mu = \frac{1}{\big((x^0)^2 \ldots (x^{n-1})^2\big)^{\frac{n-1}{2}}}, \quad \ldots . \tag{4.1.10}$$

The second problem that is obvious from (4.1.10) is that the measure function is singular at 0. However, after defining the Lagrangian density in such a way that it vanishes at zero we can choose a positive-definite measure function. Note also that the explicit form of μ is not required in any of the calculations later on, we use only relations (4.1.9) so the non-uniqueness of μ does not affect our results.

With this integral we define the action as follows:

$$S = \int d^{n+1}x \, \mu(x)\mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian density. From (4.1.9) we see that the measure function is a-independent and therefore it does not vanish in the $a \to 0$ limit. Since we want that our theory gives corrections to the classical one, we have to define the Lagrangian density such that

$$\lim_{a\to 0} \mu\mathcal{L} = \mathcal{L}^0.$$

Here $\mathcal{L}$ is the effective Lagrangian density expanded in powers of the deformation parameter a, and $\mathcal{L}^0$ is the Lagrangian density of the corresponding undeformed field theory.

Having found the integral that fulfils (4.1.6) (or (4.1.8) equivalently), we define the variational principle. Namely, we can always bring the function to be varied to one side of the product under the integral and then vary it

$$\frac{\delta}{\delta g(x)} \int \mathrm{d}^{n+1}x \, \mu \, f \star g \star h = \frac{\delta}{\delta g(x)} \int \mathrm{d}^{n+1}x \, \mu \, g(h \star f) = \mu \, h \star f. \tag{4.1.11}$$

There is one more thing to be clarified before writing down the action. We know that the usual derivative is an antihermitian operator

$$\int \mathrm{d}^{n+1}x f(\partial_\sigma g) = -\int \mathrm{d}^{n+1}x (\partial_\sigma f) g$$

provided that functions $f, g \to 0$ at the boundary surface at infinity. Using the definition of the integral (4.1.8) however,

$$\int \mathrm{d}^{n+1}x\mu f(\partial_\sigma g) = -\int \mathrm{d}^{n+1}x\big((\partial_\sigma \mu) f + \mu(\partial_\sigma f)\big) g \neq -\int \mathrm{d}^{n+1}x\mu(\partial_\sigma f) g,$$

since we have introduced the x-dependent measure function in the definition of the integral. If $\sigma = n$ then because of (4.1.9) there is no additional term and ∂_n is antihermitian. When $\sigma = j$ we introduce $\tilde{\partial}_j = \partial_j + \rho_j(x)$ such that

$$\int \mathrm{d}^{n+1}x\mu f(\tilde{\partial}_j g) = -\int \mathrm{d}^{n+1}x\mu(\tilde{\partial}_j f)g \tag{4.1.12}$$

and from this requirement we calculate $\rho_j(x)$,

$$\begin{aligned}\int \mathrm{d}^{n+1}x\mu f(\tilde{\partial}_j g) &= \int \mathrm{d}^{n+1}x\mu f(\partial_j g + \rho_j g)\\ &= -\int \mathrm{d}^{n+1}x\mu(\tilde{\partial}_j f)g - \int \mathrm{d}^{n+1}x\big(\partial_j\mu - 2\mu\rho_j\big)fg.\end{aligned}$$

From here it follows

$$\rho_j(x) = \frac{\partial_j\mu}{2\mu}. \tag{4.1.13}$$

In this way we obtain antihermitian derivatives compatible with the integral (4.1.8). Substituting

$$\partial_j \longrightarrow \tilde{\partial}_j = \partial_j + \frac{\partial_j\mu}{2\mu},\ \partial_n \longrightarrow \tilde{\partial}_n = \partial_n$$

in (2.4.15) one obtains

$$\begin{aligned}\tilde{D}^\star_n &= \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\tilde{\partial}_l\tilde{\partial}^l\right),\\ \tilde{D}^\star_j &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\tilde{\partial}_j\end{aligned} \tag{4.1.14}$$

and

$$\int \mathrm{d}^{n+1}x\mu\, \bar{f}\star(\tilde{D}^\star_\sigma \triangleright g) = -\int \mathrm{d}^{n+1}x\mu\, \overline{(\tilde{D}^\star_\sigma \triangleright f)}\star g. \tag{4.1.15}$$

Now one has to recalculate the Leibniz rules for this modified derivatives. We do this explicitly for $\tilde{D}^\star_j$; for $\tilde{D}^\star_n$ the calculation is analogous. From (4.1.13) follows

$$\partial_n\rho_j = 0, \quad x^l(\partial_l\rho_j) = -\rho_j \tag{4.1.16}$$

and from here and (2.4.7)

$$\rho_j\star f = \rho_j\frac{e^{ia\partial_n}-1}{ia\partial_n}f, \quad f\star\rho_j = \rho_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}f. \tag{4.1.17}$$

Then

$$\tilde{D}^\star_j\triangleright(f\star g) = \frac{e^{-ia\partial_n}-1}{-ia\partial_n}(\partial_j+\rho_j)(f\star g) = D^\star_j\triangleright(f\star g) + \rho_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}(f\star g). \tag{4.1.18}$$

Using (4.1.17) we have

$$\begin{aligned}\rho_j \frac{e^{-ia\partial_n}-1}{-ia\partial_n}(f\star g) &= f\star g\star\rho_j \\ &= f\star\rho_j\star g + f\star[g\stackrel{\star}{,}\rho_j] \\ &= f\star\rho_j\star g + f\star\rho_j\star\Big(\frac{2-e^{ia\partial_n}-e^{-ia\partial_n}}{e^{ia\partial_n}-1}g\Big) \\ &= f\star\rho_j\star(e^{-ia\partial_n}g) = f\star\Big(\rho_j\frac{1-e^{-ia\partial_n}}{ia\partial_n}g\Big).\end{aligned}$$

Inserting this result in (4.1.18) gives the Leibniz rule for $\tilde{D}^\star_j$

$$\tilde{D}^\star_j\,(f\star g) = (D^\star_j\triangleright f)\star(e^{-ia\partial_n}\triangleright g) + f\star(\tilde{D}^\star_j\triangleright g). \tag{4.1.19}$$

The Leibniz rule for $\tilde{D}^\star_n$ is obtained in the similar way

$$\begin{aligned}\tilde{D}^\star_n\triangleright(f\star g) = &\,(D^\star_n\triangleright f)\star(e^{-ia\partial_n}\triangleright g) + (e^{ia\partial_n}\triangleright f)\star(\tilde{D}^\star_n\triangleright g) \\ &-ia\,\big(D^\star_j\triangleright(e^{ia\partial_n}\triangleright f)\big)\star(\tilde{D}^{\star j}\triangleright g).\end{aligned} \tag{4.1.20}$$

It is interesting to note that $\tilde{D}^\star_\sigma$ always acts on the last term in the product, function g in this case. We use this result in the next section when discussing the field-strength tensor.

4.2 Modified Seiberg-Witten map

In the previous chapter we have already solved the Seiberg-Witten map for a general non-abelian gauge theory on the κ-deformed space. Therefore, the solutions for the $U(1)$ gauge theory immediately follow. However, our solution for the gauge field depends on the definition of the covariant derivative and this in turn on the choice of the derivative we want to gauge. Since we prefer to work with $\tilde{D}^\star_\mu$ instead of $D^\star_\mu$ because of (4.1.15), we have to modify the solutions for the gauge field V_μ obtained in the previous chapter.

Once again we start from the covariant derivative $\tilde{\mathcal{D}}_\mu = \tilde{D}^\star_\mu - i\tilde{V}_\mu$ and its transformation law

$$\delta_\alpha\Big(\tilde{\mathcal{D}}_\mu\psi(x)\Big) = i\Lambda_\alpha(x)\star\tilde{\mathcal{D}}_\mu\psi(x). \tag{4.2.1}$$

From (4.2.1) it follows

$$\delta\tilde{V}_\mu\star\psi = \tilde{D}^\star_\mu\triangleright(\Lambda_\alpha\star\psi) - \Lambda_\alpha\star\Big(\tilde{D}^\star_\mu\triangleright\psi\Big) + i\Lambda_\alpha\star\tilde{V}_\mu\star\psi - i\tilde{V}_\mu\star(\Lambda_\alpha\star\psi)\,. \tag{4.2.2}$$

Again, it is more convenient to separate the nth and the jth components of (4.2.2).

First we look at the jth component. Using the Leibniz rule for $\tilde{D}^\star_j$ (4.1.19), we obtain

$$\delta_\alpha\tilde{V}_j\star\psi = (D^\star_j\triangleright\Lambda_\alpha)\star\big(e^{-ia\partial_n}\triangleright\psi\big) + i\Lambda_\alpha\star\tilde{V}_j\star\psi - i\tilde{V}_j\star(\Lambda_\alpha\star\psi)\,. \tag{4.2.3}$$

Inserting the ansatz

$$\tilde{V}_j = \tilde{A}_j\star e^{-ia\partial_n} \tag{4.2.4}$$

in (4.2.3) leads to

$$\delta_\alpha \tilde{A}_j = (D_j^\star \triangleright \Lambda_\alpha) + i\Lambda_\alpha \star \tilde{A}_j - i\tilde{A}_j \star \big(e^{-ia\partial_n} \triangleright \Lambda_\alpha\big). \tag{4.2.5}$$

Expanding $\tilde{A}_j$ field in the deformation parameter a:

$$\tilde{A}_j = \tilde{A}_j^0 + \tilde{A}_j^1 + \dots$$

and using the solution for the gauge parameter Λ_α (3.4.10) gives the solution for $\tilde{V}_j$ up to first order in a

$$\tilde{V}_j = A_j^0 - iaA_j^0\partial_n - \frac{ia}{2}\partial_n A_j^0 - \frac{a}{4}\{A_n^0, A_j^0\} + \frac{1}{4}C_\lambda^{\rho\sigma}x^\lambda\Big(\{F_{\rho j}^0, A_\sigma^0\} - \{A_\rho^0, \partial_\sigma A_j^0\}\Big). \tag{4.2.6}$$

One notices that this solution is the same as (3.5.12). This is to be expected, because the $\tilde{V}_j$ field is only ∂_n derivative-valued and ∂_n is not modified.

Next, we look at the nth component of equation (4.2.2). Using the Leibniz rule for $\tilde{D}_n^\star$ (4.1.20) leads to

$$\begin{aligned}\delta_\alpha \tilde{V}_n \star \psi = {} & (D_n^\star \triangleright \Lambda_\alpha) \star (e^{-ia\partial_n} \triangleright \psi) + \Big((e^{ia\partial_n} - 1) \triangleright \Lambda_\alpha\Big) \star (\tilde{D}_n^\star \triangleright \psi) \\ & -ia(D_j^\star \triangleright (e^{ia\partial_n} \triangleright \Lambda_\alpha)) \star (\tilde{D}^{\star j} \triangleright \psi) + i\Lambda_\alpha \star \tilde{V}_n \star \psi - i\tilde{V}_n \star (\Lambda_\alpha \star \psi)\,.\end{aligned} \tag{4.2.7}$$

We make the following ansatz

$$\tilde{V}_n = \tilde{A}_{1n} \star e^{-ia\partial_n} + \tilde{A}_{2n} \star \tilde{D}_n^\star + \tilde{A}_{3n}^l \star \tilde{D}_l^\star \tag{4.2.8}$$

and insert it in equation (4.2.7). Collecting terms proportional to $\star e^{-ia\partial_n}\psi$, $\star\tilde{D}_n^\star\psi$ and $\star\tilde{D}_l^\star\psi$ we obtain the transformation laws for the fields $\tilde{A}_{1n}$, $\tilde{A}_{2n}$ and $\tilde{A}_{3n}^l$ respectively

$$\begin{aligned}\delta_\alpha \tilde{A}_{1n} &= (D_n^\star \triangleright \Lambda_\alpha) + i\Lambda_\alpha \star \tilde{A}_{1n} - i\tilde{A}_{1n} \star (e^{-ia\partial_n} \triangleright \Lambda_\alpha) \\ &\quad -i\tilde{A}_{2n} \star (D_n^\star \triangleright \Lambda_\alpha) - i\tilde{A}_{3n}^l \star (D_l^\star \triangleright \Lambda_\alpha), \\ \delta_\alpha \tilde{A}_{2n} &= \big((e^{ia\partial_n} - 1) \triangleright \Lambda_\alpha\big) + i\Lambda_\alpha \star \tilde{A}_{2n} - i\tilde{A}_{2n} \star \big(e^{ia\partial_n} \triangleright \Lambda_\alpha\big), \\ \delta_\alpha \tilde{A}_{3n}^l &= -ia\big(\partial^{\star l} \triangleright \Lambda_\alpha\big) + i\Lambda_\alpha \star \tilde{A}_{3n}^l - i\tilde{A}_{3n}^l \star \Lambda_\alpha - a\tilde{A}_{2n} \star (\partial^{\star l} \triangleright \Lambda_\alpha).\end{aligned} \tag{4.2.9}$$

Up to first order in a the solutions of these equations are

$$\begin{aligned}\tilde{A}_{1n} &= A_n^0 - \frac{a}{2}\Big(i\partial_j A^{0j} + A_j^0 A^{0j}\Big) + \frac{1}{4}C_\lambda^{\rho\sigma}x^\lambda\Big(\{F_{\rho n}^0, A_\sigma^0\} - \{A_\rho^0, \partial_\sigma A_n^0\}\Big), \\ \tilde{A}_{2n} &= iaA_n^0, \\ \tilde{A}_{3n}^j &= -iaA^{0j},\end{aligned}$$

and

$$\tilde{V}_n = A_n^0 - iaA^{0j}\tilde{\partial}_j - \frac{ia}{2}\partial_j A^{0j} - \frac{a}{2}A_j^0 A^{0j} + \frac{1}{4}C_\lambda^{\rho\sigma}x^\lambda\Big(\{F_{\rho n}^0, A_\sigma^0\} - \{A_\rho^0, \partial_\sigma A_n^0\}\Big). \tag{4.2.10}$$

Comparing this solution with the solution for V_n (3.5.35), one sees that the only difference is in the term $-iaA^{0j}\tilde{\partial}_j$. This is obviously the consequence of modifying derivatives.

As the next step we construct the field-strength tensor. It is defined as

$$\mathcal{F}_{\mu\nu} = i[\tilde{\mathcal{D}}_\mu \stackrel{\star}{,} \tilde{\mathcal{D}}_\nu]. \tag{4.2.11}$$

Applying this to the field ψ gives

$$\begin{aligned}\mathcal{F}_{ij} &= \Big((D^\star_i \triangleright V_j) - (D^\star_j \triangleright V_i) \\ &\quad -iV_i \star (e^{-ia\partial_n} \triangleright V_j) + iV_j \star (e^{-ia\partial_n} \triangleright V_i)\Big) \star e^{-2ia\partial_n}, &(4.2.12)\\ \mathcal{F}_{nj} &= F_{nj1} \star e^{-2ia\partial_n} + F_{nj2} \star e^{-ia\partial_n} \tilde{D}^\star_n + F^l_{nj3} \star e^{-ia\partial_n} \tilde{D}^\star_l, &(4.2.13)\end{aligned}$$

where

$$\begin{aligned}F_{nj1} &= (D^\star_n \triangleright V_j) - (D^\star_j \triangleright V_{n1}) - iV_{n1} \star (e^{-ia\partial_n} \triangleright V_j) + iV_j \star (e^{-ia\partial_n} \triangleright V_{n1}) \\ &\quad -iV^l_{n3} \star (D^\star_l \triangleright V_j) - iV_{n2} \star (D^\star_n \triangleright V_j), \\ F_{nj2} &= \big((e^{ia\partial_n} - 1) \triangleright V_j\big) - (D^\star_j \triangleright V_{n2}) - iV_{n2} \star (e^{ia\partial_n} \triangleright V_j) + iV_j \star (e^{-ia\partial_n} \triangleright V_{n2}), \\ F^l_{nj3} &= -ia(\partial^{\star\, l} \triangleright V_j) - (D^\star_j \triangleright V^l_{n3}) - aV_{n2} \star (\partial^{\star\, l} \triangleright V_j) - iV^l_{n3} \star V_j + V_j \star (e^{-ia\partial_n} \triangleright V^l_{n3}),\end{aligned}$$

just like in (3.6.5)-(3.6.7).

Again we split $\mathcal{F}_{\mu\nu}$ into the curvature-like terms and torsion-like terms [2]

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu} \tilde{\mathcal{D}}_\rho + \dots + T^{\rho_1 \dots \rho_l}_{\mu\nu} : \tilde{\mathcal{D}}_{\rho_1} \dots \tilde{\mathcal{D}}_{\rho_l} : + \dots . \tag{4.2.14}$$

Expanding (4.2.12) and (4.2.13) up to first order in a and rewriting them in form (4.2.14) gives

$$\begin{aligned}F_{ij} &= F^0_{ij} - ia\mathcal{D}^0_n F^0_{ij} + \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \Big(2\{F^0_{\rho i}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{ij}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{ij}\}\Big), \\ T^\mu_{ij} &= -2ia\delta^\mu_n F^0_{ij}, \\ F_{nj} &= F^0_{nj} - \frac{ia}{2} \mathcal{D}^{\mu 0} F^0_{\mu j} \\ &\quad + \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \Big(2\{F^0_{\rho n}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{nj}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{nj}\}\Big), \\ T^\mu_{nj} &= -ia\eta^{\mu l} F^0_{lj} - ia\delta^\mu_n F^0_{nj}. &(4.2.15)\end{aligned}$$

These results are the same as (3.6.11)-(3.6.14). Actually, we checked this up to second order in a, and because of the structure of equations (4.2.12), (4.2.13), (4.1.19) and (4.1.20), we expect that this holds to all orders in a. As in the action for the gauge field only the curvature-like term is used[3], from equations (4.2.15) we see that the modification of the derivatives does not affect the gauge part of the action.

[2] Note that now the torsion-like terms are defined to be coefficients in front of the modified covariant derivatives.

[3] That we use only the curvature-like term in the action is the matter of choice. In principle, one can also include terms depending on the torsion-like terms.

4.3 The action for the κ-deformed electrodynamics

Now we concentrate on the $U(1)$ gauge theory coupled to fermions. For convenience we list the solutions of the Seiberg-Witten map for the gauge parameter Λ_α, gauge field $\tilde{V}_\mu$ and matter filed ψ in the case of $U(1)$ gauge group

$$\Lambda_\alpha = \alpha - \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda A_\rho^0(\partial_\sigma\alpha), \tag{4.3.1}$$

$$\psi = \psi^0 - \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda A_\rho^0(\partial_\sigma\psi^0), \tag{4.3.2}$$

$$\begin{aligned}\tilde{V}_j &= A_j^0 - iaA_j^0\partial_n - \frac{ia}{2}\partial_n A_j^0 - \frac{a}{2}A_n^0A_j^0 + \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda\Big(2(\partial_\rho A_j^0) - (\partial_j A_\rho^0)\Big)A_\sigma^0,\\ \tilde{V}_n &= A_n^0 - iaA^{0j}\tilde{\partial}_j - \frac{ia}{2}\partial_j A^{0j} - \frac{a}{2}A_j^0A^{0j} + \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda\Big(2(\partial_\rho A_n^0) - (\partial_n A_\rho^0)\Big)A_\sigma^0,\end{aligned} \tag{4.3.3}$$

$$\begin{aligned}F_{ij} &= F_{ij}^0 - ia\partial_n F_{ij}^0 + C_\lambda^{\rho\sigma}x^\lambda\Big(F_{\rho i}^0F_{\sigma j}^0 + (\partial_\rho F_{ij}^0)A_\sigma^0\Big),\\ F_{nj} &= F_{nj}^0 - \frac{ia}{2}\partial^\mu F_{\mu j}^0 + C_\lambda^{\rho\sigma}x^\lambda\Big(F_{\rho n}^0F_{\sigma j}^0 + (\partial_\rho F_{nj}^0)A_\sigma^0\Big).\end{aligned} \tag{4.3.4}$$

4.3.1 Matter field action

First we look at the action for the matter field without gauge symmetry. A proper action for the spinor field $\tilde{\psi}$ would be:

$$S_m = \int \mathrm{d}^{n+1}x\, \mu\, \overline{\tilde{\psi}} \star \Big(i\gamma^\sigma\tilde{D}_\sigma^\star \triangleright \tilde{\psi} - m\tilde{\psi}\Big), \tag{4.3.5}$$

where $\tilde{D}_\sigma^\star$ is given in (4.1.14). Varying (4.3.5) with respect to $\overline{\tilde{\psi}}$ using (4.1.11) we obtain

$$\mu\Big(i\gamma^\sigma\tilde{D}_\sigma^\star \triangleright \tilde{\psi} - m\tilde{\psi}\Big) = 0. \tag{4.3.6}$$

The classical limit of this equation is

$$\mu(i\gamma^\sigma\tilde{\partial}_\sigma - m)\tilde{\psi} = 0,$$

since μ and ρ_j are a-independent. In order to correct this (like we have said before, we want to have a theory with the good classical limit) we notice that

$$\tilde{\partial}_j(\mu^{-\frac{1}{2}}f) = (\partial_j + \rho_j)(\mu^{-\frac{1}{2}}f) = \mu^{-\frac{1}{2}}(\partial_j f) \tag{4.3.7}$$

and as a consequence

$$\tilde{D}_\sigma^\star\mu^{-\frac{1}{2}} = \mu^{-\frac{1}{2}}D_\sigma^\star. \tag{4.3.8}$$

This suggests that we can rescale the field $\tilde{\psi}$

$$\tilde{\psi} = \mu^{-\frac{1}{2}}\psi. \tag{4.3.9}$$

Inserting (4.3.9) into (4.3.6) gives, after using (4.3.8)

$$i\gamma^\sigma D_\sigma^\star \triangleright \psi - m\psi = 0, \tag{4.3.10}$$

which has the proper classical limit.

The other way to obtain (4.3.10) is to insert (4.3.9) in the action (4.3.5) directly

$$\begin{aligned} S_m &= \int \mathrm{d}^{n+1}x\, \mu\, (\overline{\mu^{-\frac{1}{2}}\psi}) \star \Big(i\gamma^\sigma \tilde{D}^\star_\sigma \triangleright (\mu^{-\frac{1}{2}}\psi) - m\mu^{-\frac{1}{2}}\psi \Big) \\ &= \int \mathrm{d}^{n+1}x\, \mu\, (\mu^{-\frac{1}{2}}\overline{\psi}) \Big(i\mu^{-\frac{1}{2}}\gamma^\sigma (D^\star_\sigma \triangleright \psi) - m\mu^{-\frac{1}{2}}\psi \Big) \\ &= \int \mathrm{d}^{n+1}x\, \overline{\psi} \big(i\gamma^\lambda D^\star_\lambda \triangleright \psi - m\psi \big), \end{aligned} \tag{4.3.11}$$

where coming from the first to the second line we used (4.1.8) and (4.3.8). The equation of motion following from this action is exactly (4.3.10).

Now we write the gauge covariant version of (4.3.5)

$$\begin{aligned} S_m &= \int \mathrm{d}^{n+1}x\, \mu\, \Big(\bar{\tilde{\psi}} \star (i\gamma^\mu \tilde{\mathcal{D}}_\mu - m)\tilde{\psi} \Big) \\ &= \int \mathrm{d}^{n+1}x\, \mu\, \Big(\bar{\tilde{\psi}} \star \big(i\gamma^\mu \tilde{D}^\star_\mu \triangleright \tilde{\psi} + \gamma^\mu \tilde{V}_\mu \star \tilde{\psi} - m\tilde{\psi} \big) \Big). \end{aligned} \tag{4.3.12}$$

Using the variational principle we obtain the equation of motion for the matter field $\tilde{\psi}$:

$$\mu \big(i\gamma^\mu \tilde{D}^\star_\mu \triangleright \tilde{\psi} + \gamma^\mu \tilde{V}_\mu \star \tilde{\psi} - m\tilde{\psi} \big) = 0. \tag{4.3.13}$$

Again we have the same problem, equation (4.3.13) does not have the proper classical limit. Unfortunately, we can not use (4.3.9) now since this rescaling is not compatible with the Seiberg-Witten map. Namely, if $\delta_\alpha \psi = i\Lambda_\alpha \star \psi$ then

$$\delta_\alpha \tilde{\psi} = \delta_\alpha (\mu^{-\frac{1}{2}}\psi) = i\mu^{-\frac{1}{2}}(\Lambda_\alpha \star \psi) \neq i\Lambda_\alpha \star \tilde{\psi},$$

since the $\star$-product is noncommutative and the action (4.3.12) will not be gauge invariant.

Nevertheless, demanding

$$\delta_\alpha \tilde{\psi} = i\Lambda_\alpha \star \tilde{\psi}$$

one redoes the Seiberg-Witten map for the field $\tilde{\psi}$, but this time taking the solution in $a \to 0$ limit to be $\tilde{\psi}^0 = \mu^{-\frac{1}{2}}\psi^0$ instead of ψ^0. This is allowed by the transformation law $\delta_\alpha \tilde{\psi}^0 = i\alpha\tilde{\psi}^0$. Repeating the same calculation as in the previous chapter[4] we find the following solution

$$\tilde{\psi} = \mu^{-\frac{1}{2}}\psi^0 - \mu^{-\frac{1}{2}}\frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda A^0_\rho \partial_\sigma \psi^0 - \frac{na}{4}\mu^{-\frac{1}{2}}A^0_n \psi^0. \tag{4.3.14}$$

The additional term arises as the consequence of requesting that the $a \to 0$ limit of the solution for $\tilde{\psi}$ field is $\mu^{-\frac{1}{2}}\psi^0$. This we might call the "covariant rescaling".

[4]In (4.3.14) we wrote the solution for $\tilde{\psi}$ field in the case of $U(1)$ gauge group. Nevertheless, one can calculate the solution in the case of an arbitrary nonabelian gauge group, first order result is

$$\psi = \mu^{-\frac{1}{2}}\psi^0 - \frac{1}{2}\mu^{-\frac{1}{2}}C^{\rho\sigma}_\lambda x^\lambda A^0_\rho \partial_\sigma \psi^0 + \frac{i}{8}\mu^{-\frac{1}{2}}C^{\rho\sigma}_\lambda x^\lambda [A^0_\rho, A^0_\sigma]\psi^0 - \frac{na}{4}\mu^{-\frac{1}{2}}A^0_n \psi^0.$$

Now we insert (4.3.14) and (4.3.3) in (4.3.13) and expand the Dirac derivatives up to first order in a as well. The equation of motion up to first order in a follows

$$(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{2}C^{\rho\sigma}_\lambda\gamma_\rho\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0 - \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) - \frac{i}{4}C^{\rho\sigma}_\sigma\gamma^\mu F^0_{\mu\rho}\psi^0 = 0. \tag{4.3.15}$$

The equation of motion for $\bar\psi$ is obtained analogously, one varies the action with respect to the field $\tilde\psi$, rescales the field $\tilde{\bar\psi}$ and expands the Dirac derivatives. The result is

$$- i\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\mu - m\bar\psi^0 - \frac{1}{2}C^{\rho\sigma}_\lambda\overline{\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0}\gamma_\rho + \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\overline{\mathcal{D}^0_\sigma\psi^0}\gamma^\mu F^0_{\mu\rho} + \frac{i}{4}C^{\rho\sigma}_\sigma\bar\psi^0\gamma^\mu F^0_{\mu\rho} = 0. \tag{4.3.16}$$

We see that (4.3.16) is the hermitian conjugate of (4.3.15) (and vice versa) as expected.

But we are also interested in the effective action for fermions up to first order in a. Let us write the action (4.3.12) with all the derivatives explicitly, using (4.2.4) and (4.2.8):

$$S_m = \int \mathrm{d}^{n+1}x\,\mu\,\Big(\tilde{\bar{\tilde\psi}}\star(i\gamma^\mu\tilde D^\star_\mu\triangleright\tilde\psi - m\tilde\psi) + \tilde{\bar{\tilde\psi}}\star\gamma^\mu V_{\mu 1}\star(e^{-ia\partial_n}\triangleright\tilde\psi) + \tilde{\bar{\tilde\psi}}\star\gamma^n V_{n2}\star(\tilde D^\star_n\triangleright\tilde\psi) + \tilde{\bar{\tilde\psi}}\star\gamma^n V^j_{n3}\star(\tilde D^\star_j\triangleright\tilde\psi)\Big). \tag{4.3.17}$$

Now we repeat the calculation leading to (4.3.11). We omit one $\star$ in (4.3.17), rescale the fermionic fields using (4.3.14) and finally insert the solutions for the Seiberg-Witten map for the gauge field and obtain up to first order in a[5]

$$\begin{aligned} S_m = \int \mathrm{d}^{n+1}x\Big(&\bar\psi^0(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\bar\psi^0F^0_{\rho\sigma}(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 \\ &-\frac{1}{2}C^{\rho\sigma}_\lambda\bar\psi^0\gamma_\rho\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0 - \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\bar\psi^0\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) - \frac{i}{4}C^{\rho\sigma}_\sigma\bar\psi^0\gamma^\mu F^0_{\mu\rho}\psi^0\Big). \end{aligned} \tag{4.3.18}$$

It might not be obvious but the action (4.3.18) is hermitian.

Since the integral in (4.3.18) is the usual integral, applying the variational principle to (4.3.18) leads to the usual Euler-Lagrange equation of motion[6]

$$\partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\psi)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} + \frac{\partial\mathcal{L}}{\partial\psi} = 0. \tag{4.3.19}$$

Using (4.3.19) the equations of motion for the fields $\bar\psi$ and ψ follow from (4.3.18). They are the same as (4.1.14) and (4.3.16) so we do not write them again.

[5]The covariant derivative $\mathcal{D}^0_\mu$ is the usual covariant derivative for the undeformed $U(1)$ gauge field theory, $\mathcal{D}^0_\mu = \partial_\mu - iA^0_\mu$.

[6]Note that in (4.3.18) the Lagrangian density depends on the second derivatives of fields as well.

4.3.2 Gauge field action

Our first guess for the gauge field action is

$$S_g = -\frac{1}{4}\int \mathrm{d}^{n+1}x\ \mu\ \mathrm{Tr}\left(F_{\mu\nu} \star F^{\mu\nu}\right). \tag{4.3.20}$$

To check if this is a good guess we look shortly at the equations of motion following from (4.3.20).

If the matter field is not present, the equation of motion for the gauge field is

$$\mu\Big(\partial_\mu F^{0\mu\rho} + \text{ higher order terms }\Big) = 0, \tag{4.3.21}$$

that is

$$\partial_\mu F^{0\mu\rho} + \text{ higher order terms } = 0, \tag{4.3.22}$$

since μ is a nonzero function. Equation (4.3.22) has the proper classical limit.

Now we add matter field to this action, that is we consider the theory described by the action

$$S = S_m + S_g, \tag{4.3.23}$$

where S_m is given by (4.3.18). The equation of motion for the gauge field follows from

$$\frac{\delta(S_g + S_m)}{\delta A^0_\rho} = 0, \tag{4.3.24}$$

$$\mu\Big(\partial_\mu F^{0\mu\rho} + \text{ higher order terms }\Big) = -(\bar\psi\gamma^\rho\psi + \text{ higher order terms }). \tag{4.3.25}$$

Looking at the classical limit of this equation one finds

$$\partial_\mu F^{0\mu\rho} = -\frac{1}{\mu}\bar\psi\gamma^\rho\psi, \tag{4.3.26}$$

which is obviously not the classical equation. Therefore, we have to cancel the measure[7] under the integral in (4.3.20). The covariant rescaling will not work here. To be more precise, it might work for the $U(1)$ gauge theory since $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$. Then one would rescale the gauge field in a proper way and obtain the rescaling of the field-strength tensor such that it cancels the measure under the integral. But for the general nonabelian gauge theory we have $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu]$. Since $F^0_{\mu\nu}$ consists of both linear and quadratic terms in A^0_μ, one cannot rescale the gauge field and obtain the rescaling of the field-strength tensor as well. Since we would like to have a procedure how to write the action for a general nonabelian gauge theory on the κ-deformed space, we do not consider the possibility of covariant rescaling of the gauge field, but try to do something else instead.

[7]Leaving μ in (4.3.20) but also in (4.3.17) will not give the requested result since then the equations of motion for ψ and $\bar\psi$ will not have the proper limit because of the derivatives $\tilde\partial_j$ appearing.

In [91] the similar problem was analysed, namely the construction of the Yang-Mills action on the $E_q(2)$-covariant plane. The solution was the following. One can write the action for the gauge field as

$$S_g = -\frac{1}{4}\int d^{n+1}x\ \mu(x)\ \mathrm{Tr}\,(X\star F_{\mu\nu}\star F^{\mu\nu})\,, \tag{4.3.27}$$

where X is the gauge covariant expression (so that the gauge invariance of the action is not spoiled)

$$\delta_\alpha X = i[\Lambda_\alpha \stackrel{\star}{,} X]. \tag{4.3.28}$$

In the $a\to 0$ limit X should cancel the measure μ under the integral, leading to the equation of motion with the good classical limit.

We take that approach here and calculate X from (4.3.28) in the case of $U(1)$ gauge theory, having in mind that the generalisation to the nonabelian gauge theory is straightforward[8]. We obtain up to first order in a,

$$X = (1 - anA^0_n)\mu^{-1}. \tag{4.3.29}$$

Expanding (4.3.27) up to first order in a and using the solutions for the Seiberg-Witten map (4.3.4) and (4.3.29), the effective action for the gauge field follows

$$S_g = -\frac{1}{4}\int \mathrm{d}^{n+1}x\ \Big(F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu}F^0_{\mu\nu}F^0_{\rho\sigma} + 2C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu}F^0_{\mu\rho}F^0_{\nu\sigma}\Big). \tag{4.3.30}$$

4.3.3 Conserved currents

The complete action for the $U(1)$ gauge theory coupled with matter is $S = S_m + S_g$. The equations of motion for the matter fields are given by (4.3.15) and (4.3.16). For the gauge field we have from (4.3.24)

$$\begin{aligned}
-J^\rho = {} & \partial_\mu F^{0\mu\rho} - \frac{1}{2}C^{\alpha\beta}_\mu F^{0\mu\rho}F^0_{\alpha\beta} - \frac{1}{4}C^{\mu\rho}_\mu F^{0\alpha\beta}F^0_{\alpha\beta} + C^{\alpha\beta}_\mu F^{0\mu}_{\ \alpha}F^{0\rho}_{\ \beta}\\
& - C^{\alpha\rho}_\mu F^{0\beta\mu}F^0_{\alpha\beta} + C^{\alpha\mu}_\mu F^{0\beta\rho}F^0_{\alpha\beta}\\
& + C^{\alpha\beta}_\lambda x^\lambda\partial_\mu\big(F^{0\mu}_{\ \alpha}F^{0\rho}_{\ \beta} - \frac{1}{2}F^{0\mu\rho}F^0_{\alpha\beta}\big) - \frac{1}{4}C^{\mu\rho}_\lambda x^\lambda\partial_\mu(F^{0\alpha\beta}F^0_{\alpha\beta})\\
& - C^{\alpha\rho}_\lambda x^\lambda\partial_\mu(F^{0\beta\mu}F^0_{\alpha\beta}) + C^{\alpha\mu}_\lambda x^\lambda\partial_\mu(F^{0\beta\rho}F^0_{\alpha\beta}).
\end{aligned} \tag{4.3.31}$$

The current J^μ is given by

$$\begin{aligned}
J^\rho = {} & \frac{\delta S_m}{\delta A^0_\rho} = -\partial_\mu\frac{\partial\mathcal{L}_m}{\partial(\partial_\mu A^0_\rho)} + \frac{\partial\mathcal{L}_m}{\partial A^0_\rho}\\
= {} & \bar\psi^0\gamma^\rho\psi^0 - \frac{1}{2}C^{\alpha\beta}_\lambda x^\lambda\bar\psi^0\gamma^\rho F^0_{\alpha\beta}\psi^0 - C^{\alpha\rho}_\lambda x^\lambda\bar\psi^0\gamma^\mu F^0_{\mu\alpha}\psi^0 - \frac{i}{2}C^{\alpha\mu}_\lambda\eta^{\lambda\rho}\overline{\mathcal{D}^0_\mu\psi^0}\gamma_\alpha\psi^0\\
& - \frac{i}{2}C^{\alpha\rho}_\mu\bar\psi^0\gamma^\mu(\mathcal{D}^0_\alpha\psi^0) + \frac{i}{2}C^{\alpha\rho}_\mu\bar\psi^0\gamma_\alpha(\mathcal{D}^{0\mu}\psi^0) + \frac{i}{4}C^{\alpha\mu}_\alpha\Big(\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\rho\psi^0 - \bar\psi^0\gamma^\rho\mathcal{D}^0_\mu\psi^0\Big)\\
& + \frac{i}{2}C^{\alpha\mu}_\lambda x^\lambda\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\rho(\mathcal{D}^0_\alpha\psi^0).
\end{aligned} \tag{4.3.32}$$

[8] The solution for X up to first order in a in the case of an arbitrary nonabelian group is the same as (4.3.29), the difference arises in the second and higher orders in a.

Using the equations of motion (4.3.15) and (4.3.16) one can show that the current (4.3.32) is conserved, $\partial_\rho J^\rho = 0$.

In the undeformed gauge theory the existence and the conservation of the current J^ρ are the consequences of the symmetry of the action with respect to the gauge transformations. One expects that the same applies here. To check this we calculate the variation of the action (4.3.18) for

$$\delta_\alpha \psi^0 = i\alpha\psi^0, \quad \delta_\alpha \bar\psi^0 = -i\alpha\bar\psi^0, \quad \delta_\alpha A^0_\mu = \partial_\mu \alpha \quad \text{and } \delta_\alpha F^0_{\mu\nu} = 0. \tag{4.3.33}$$

Since the action (4.3.18) depends on the second derivatives of the field ψ^0 and $\bar\psi^0$ and on the first derivatives of the gauge field A^0_ρ (unlike in the classical case), we have to derive the Nöther theorem from the beginning. We do it for the $U(1)$ gauge symmetry.

$$\begin{aligned} \delta_\alpha S_m &= \delta_\alpha \int \mathrm{d}^{n+1}x\, \mathcal{L}\Big(\psi^0, \bar\psi^0, \partial\psi^0, \partial\bar\psi^0, \partial^2\psi^0, \partial^2\bar\psi^0, A^0_\rho, \partial A^0_\rho\Big) \\ \delta_\alpha \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \psi^0}\delta_\alpha\psi^0 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^0)}\delta_\alpha(\partial_\mu\psi^0) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu\psi^0)}\delta_\alpha(\partial_\mu\partial_\nu\psi^0) \\ &\quad + (\delta_\alpha\bar\psi^0)\frac{\partial \mathcal{L}}{\partial\bar\psi^0} + (\delta_\alpha(\partial_\mu\bar\psi^0))\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi^0)} + (\delta_\alpha(\partial_\mu\partial_\nu\bar\psi^0))\frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu\bar\psi^0)} \\ &\quad + \frac{\partial \mathcal{L}}{\partial A^0_\mu}\delta_\alpha A^0_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^0_\mu)}\delta_\alpha(\partial_\nu A^0_\mu). \end{aligned} \tag{4.3.34}$$

Inserting (4.3.33) into (4.3.34), using that $\delta_\alpha(\partial_\mu\psi^0) = \partial_\mu(\delta_\alpha\psi^0)$ and the equations of motion for the fields ψ^0 and $\bar\psi^0$ (4.3.19) gives (after partial integration)

$$\delta_\alpha S_m = \int \mathrm{d}^{n+1}x\, \Big(\alpha(\partial_\rho j^\rho) + (\partial_\rho\alpha)M^\rho + (\partial_\rho\partial_\mu\alpha)M^{\rho\mu}\Big), \tag{4.3.35}$$

where

$$\begin{aligned} j^\rho &= i\Bigg(\frac{\partial \mathcal{L}}{\partial(\partial_\rho\psi^0)}\psi^0 - \bar\psi^0\frac{\partial \mathcal{L}}{\partial(\partial_\rho\bar\psi^0)} - \Big(\partial_\nu\frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\psi^0)}\Big)\psi^0 + \bar\psi^0\Big(\partial_\nu\frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\bar\psi^0)}\Big) \\ &\quad + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\psi^0)}(\partial_\nu\psi^0) - (\partial_\nu\bar\psi^0)\frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\bar\psi^0)}\Bigg), \end{aligned} \tag{4.3.36}$$

$$\begin{aligned} M^\rho &= i\Bigg(\frac{\partial \mathcal{L}}{\partial(\partial_\rho\psi^0)}\psi^0 - \bar\psi^0\frac{\partial \mathcal{L}}{\partial(\partial_\rho\bar\psi^0)} \\ &\quad + 2\frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\psi^0)}(\partial_\nu\psi^0) - 2(\partial_\nu\bar\psi^0)\frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\rho\bar\psi^0)} - i\frac{\partial \mathcal{L}}{\partial A^0_\rho}\Bigg), \end{aligned} \tag{4.3.37}$$

$$M^{\rho\mu} = i\Bigg(\frac{\partial \mathcal{L}}{\partial(\partial_\rho\partial_\mu\psi^0)}\psi^0 - \bar\psi^0\frac{\partial \mathcal{L}}{\partial(\partial_\rho\partial_\mu\bar\psi^0)} - \frac{i}{2}\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^0_\rho)} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho A^0_\mu)}\Big)\Bigg). \tag{4.3.38}$$

Since $\alpha(x)$, $\partial_\rho\alpha(x)$ and $\partial_\rho\partial_\mu\alpha(x)$ are independent parameters (functions), then from the requirement that $\delta_\alpha S_m = 0$ we have

$$\partial_\rho j^\rho = 0, \tag{4.3.39}$$

$$M^\rho = 0, \tag{4.3.40}$$

$$M^{\rho\mu} = 0. \tag{4.3.41}$$

Using the explicit form of the Lagrangian (4.3.18) one calculates that indeed $M^\rho = 0$, $M^{\rho\mu} = 0$, while for j^ρ

$$\begin{aligned} j^\rho &= \bar{\psi}^0\gamma^\rho\psi^0 - \frac{1}{4}C^{\alpha\beta}_\lambda x^\lambda\bar{\psi}^0\gamma^\rho F^0_{\alpha\beta}\psi^0 - \frac{1}{2}C^{\alpha\rho}_\lambda x^\lambda\bar{\psi}^0\gamma^\mu F^0_{\mu\alpha}\psi^0 \\ &-\frac{i}{4}C^{\alpha\mu}_\lambda\eta^{\lambda\rho}\Big(\overline{\mathcal{D}^0_\mu\psi^0}\gamma_\alpha\psi^0 - \bar{\psi}^0\gamma_\alpha\mathcal{D}^0_\mu\psi^0\Big) - \frac{i}{4}C^{\alpha\rho}_\lambda\Big(\overline{\mathcal{D}^{0\lambda}\psi^0}\gamma_\alpha\psi^0 - \bar{\psi}^0\gamma_\alpha\mathcal{D}^{0\lambda}\psi^0\Big). \end{aligned} \tag{4.3.42}$$

Comparing this result with (4.3.32) it looks as there are two conserved currents in the theory. We first check if the difference $\Delta^\rho = J^\rho - j^\rho$ is a topological current, that is one proves that it is conserved without using the equations of motion (off-shell). To do this, we rewrite the difference of (4.3.32) and (4.3.42) more conveniently

$$\begin{aligned} \Delta^\rho = J^\rho - j^\rho &= \frac{i}{4}\Big(C^{\alpha\rho}_\lambda\eta^{\lambda\mu} - C^{\alpha\mu}_\lambda\eta^{\lambda\rho}\Big)\partial_\mu(\bar{\psi}^0\gamma_\alpha\psi^0) + \frac{1}{2}C^{\alpha\beta}_\lambda x^\lambda\partial_\beta(\bar{\psi}^0\gamma^\rho A^0_\alpha\psi^0) \\ &-\frac{1}{2}C^{\alpha\rho}_\lambda x^\lambda\bar{\psi}^0\gamma^\mu(\partial_\mu A^0_\alpha)\psi^0 + \frac{i}{2}C^{\alpha\beta}_\lambda x^\lambda\partial_\beta(\bar{\psi}^0\gamma^\rho(\partial_\alpha\psi^0)) + \frac{1}{2}C^{\alpha\rho}_\lambda x^\lambda\bar{\psi}^0\gamma^\mu(\partial_\alpha A^0_\mu)\psi^0 \\ &-\frac{i}{2}C^{\alpha\rho}_\mu\bar{\psi}^0\gamma^\mu(\partial_\alpha\psi^0 - iA^0_\alpha\psi^0) + \frac{1}{2}C^{\alpha\beta}_\beta\bar{\psi}^0\gamma^\rho A^0_\alpha\psi^0 \\ &-\frac{i}{4}C^{\alpha\beta}_\beta\partial_\alpha(\bar{\psi}^0\gamma^\rho\psi^0) + \frac{i}{2}C^{\alpha\mu}_\mu\bar{\psi}^0\gamma^\rho(\partial_\alpha\psi^0). \end{aligned} \tag{4.3.43}$$

If one wants to check now if $\partial_\rho\Delta^\rho = 0$,[9] one observes the following

$$\begin{aligned} \partial_\rho\Delta^\rho &= \frac{i}{2}C^{\alpha\beta}_\lambda x^\lambda\partial_\rho\partial_\beta\Big(\bar{\psi}^0\gamma^\rho(\partial_\alpha\psi^0)\Big) + \frac{1}{2}C^{\alpha\rho}_\lambda x^\lambda\partial_\rho\Big(\bar{\psi}^0\gamma^\mu(\partial_\alpha A^0_\mu)\psi^0\Big) \\ &+\frac{1}{2}C^{\alpha\rho}_\rho\bar{\psi}^0\gamma^\mu(\partial_\alpha A^0_\mu)\psi^0 + \frac{i}{2}C^{\alpha\rho}_\rho\partial_\mu\Big(\bar{\psi}^0\gamma^\mu(\partial_\alpha\psi)\Big). \end{aligned} \tag{4.3.44}$$

Here we have cancelled all the terms we could without using explicitly the equations of motion. First term in (4.3.43) is a real topological term; to cancel the others we used that $\partial_\rho(\bar{\psi}^0\gamma^\rho\psi^0) = 0$. To cancel the terms in (4.3.44) we have to use the zeroth order equations of motion for the fields ψ^0 and $\bar{\psi}^0$, but not the equation of motion for the field A^0_μ. Therefore, Δ^ρ is not really a topological current.

To have two different conserved currents in the theory with only one symmetry is an unexpected result. One possible interpretation is that it is actually a manifestation of the freedom of the Seiberg-Witten map. However, there might be some other reasons for the appearance of two conserved currents in our theory, but so far we were not able to answer this question properly.

4.4 Seiberg-Witten map and the gauge symmetry

It is well known that the Seiberg-Witten map is not unique [92], [93], [94], [95]. The detailed analysis of the ambiguities in the Seiberg-Witten map in the θ-deformed space was given in [42] and we adopt this analysis for the problem at hand. Of course, we only look at the first order contributions and the $U(1)$ gauge theory.

[9]Since $\Delta^\rho = J^\rho - j^\rho$, $\partial_\rho\Delta^\rho = 0$ is certainly fulfilled. The only question is whether one can prove it explicitly without using the equations of motion.

Let us start from the solution for the gauge parameter Λ_α (4.3.1). The homogeneous equation expanded up to first order is

$$\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha = 0. \tag{4.4.1}$$

If we restrict ourselves to Λ^1_α which is hermitian and do not allow for a derivative valued gauge parameter, then there are no terms coming from the homogeneous equation, $\Delta\Lambda^1_\alpha = 0$.

The solution of the Seiberg-Witten map for the fermions (4.3.14) allows an additional term

$$\Delta\tilde{\psi} = b_1\mu^{-1/2}C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\rho\sigma}\psi^0, \tag{4.4.2}$$

which does affect the action, as we shall see later.

When analysing possible additional terms in the solution of the Seiberg-Witten map for the gauge filed in [42] the restriction was made that one does not allow for the derivative valued terms to appear. However, in our setting the derivative-valued gauge fields appear naturally, as the consequence of the nontrivial Leibniz rules for the Dirac derivatives (4.1.19) and (4.1.20). Therefore, the most general solution of the homogeneous equation is given by

$$\begin{aligned}\Delta\tilde{V}_\mu =\ & ib_2C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\mu\rho}\tilde{\mathcal{D}}^0_\sigma + ib_3C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\rho\sigma}\tilde{\mathcal{D}}^0_\mu + \\ & + ib_4C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\sigma F^0_{\mu\rho}) + ib_5C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\mu F^0_{\rho\sigma}) + ib_6C^{\rho\sigma}_{\sigma}F^0_{\mu\rho} + ib_7C^{\rho\sigma}_{\mu}F^0_{\rho\sigma}.\end{aligned} \tag{4.4.3}$$

Terms $ib_4C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\sigma F^0_{\mu\rho})$ and $ib_5C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\mu F^0_{\rho\sigma})$ are related by the Bianchi identity

$$C^{\rho\sigma}_{\lambda}x^\lambda\Big((\mathcal{D}^0_\sigma F^0_{\mu\rho}) + (\mathcal{D}^0_\mu F^0_{\rho\sigma}) + (\mathcal{D}^0_\rho F^0_{\sigma\mu})\Big) = 0.$$

Changing ρ and σ indices in the last term, we see

$$C^{\rho\sigma}_{\lambda}x^\lambda\Big(2(\mathcal{D}^0_\sigma F^0_{\mu\rho}) + (\mathcal{D}^0_\mu F^0_{\rho\sigma})\Big) = 0,$$

that is the terms proportional to b_4 and b_5 are not independent, so it is enough to write one of them.

If in the last two terms in (4.4.3) we use the explicit formula for $C^{\rho\sigma}_{\lambda}$, we obtain

$$iab_6nF^0_{\mu n} + iab_7(F^0_{n\mu} - F^0_{\mu n}) = ia(b_6n - 2b_7)F^0_{\mu n}.$$

In this way we reduce the number of arbitrary constants from 6 to 4,

$$\begin{aligned}\Delta\tilde{V}_\mu =\ & ib_2C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\mu\rho}\tilde{\mathcal{D}}^0_\sigma + ib_3C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\rho\sigma}\tilde{\mathcal{D}}^0_\mu + \\ & + ib_4C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\sigma F^0_{\mu\rho}) + iab_5F^0_{\mu n}.\end{aligned} \tag{4.4.4}$$

Now we demand that $\Delta\tilde{V}_\mu$ is hermitian (that is natural, since $\tilde{V}_\mu$ is also hermitian and we do not want to spoil that). This fixes the constants b_4 and b_5 and finally

$$\begin{aligned}\Delta\tilde{V}_\mu =\ & ib_2C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\mu\rho}\tilde{\mathcal{D}}^0_\sigma + ib_3C^{\rho\sigma}_{\lambda}x^\lambda F^0_{\rho\sigma}\tilde{\mathcal{D}}^0_\mu + \\ & + \frac{i}{2}(b_2 - 2b_3)C^{\rho\sigma}_{\lambda}x^\lambda(\mathcal{D}^0_\sigma F^0_{\mu\rho}) + \frac{ia}{2}(nb_2 - 2b_3)F^0_{\mu n}.\end{aligned} \tag{4.4.5}$$

Note also that $\mathcal{D}^0_\sigma F^0_{\mu\rho} = \partial^0_\sigma F^0_{\mu\rho}$ since we work with the $U(1)$ gauge theory.

The new terms from (4.4.5) lead to the modification of the curvature-like terms

$$\Delta F_{\mu\nu} = 2b_2 C^{\rho\sigma}_{\lambda} x^{\lambda} F_{\mu\sigma} F_{\nu\rho} + 2b_3 C^{\rho\sigma}_{\lambda} x^{\lambda} F_{\mu\nu} F_{\rho\sigma} + \\ -\frac{i}{2}(b_2 - 2b_3) C^{\rho\sigma}_{\lambda} x^{\lambda}\Big(\mathcal{D}^0_\nu \mathcal{D}^0_\sigma F^0_{\mu\rho} - \mathcal{D}^0_\mu \mathcal{D}^0_\sigma F^0_{\nu\rho}\Big) - \frac{ia}{2}(n-1) b_2 \mathcal{D}^0_n F^0_{\mu\nu} \quad (4.4.6)$$

and the torsion-like terms

$$\Delta T^{\sigma}_{\mu\nu} = -ib_2\Big(C^{\rho\sigma}_{\nu} F^0_{\mu\rho} - C^{\rho\sigma}_{\mu} F^0_{\nu\rho}\Big) - ib_2 C^{\rho\sigma}_{\lambda} x^{\lambda}(\mathcal{D}^0_\nu F^0_{\mu\rho} - \mathcal{D}^0_\mu F^0_{\nu\rho}) \\ -ib_3\Big(\delta^\sigma_\mu (C^{\rho\alpha}_{\nu} F^0_{\rho\alpha} + C^{\rho\alpha}_{\lambda} x^{\lambda}(\mathcal{D}^0_\nu F^0_{\rho\alpha})) - \delta^\sigma_\nu (C^{\rho\alpha}_{\mu} F^0_{\rho\alpha} + C^{\rho\alpha}_{\lambda} x^{\lambda}(\mathcal{D}^0_\mu F^0_{\rho\alpha}))\Big) \quad (4.4.7)$$

of the field strength (4.2.15).

Taking into consideration all the additional terms (4.4.2), (4.4.5) and (4.4.6) we obtain a more general effective action:

$$S = \int \mathrm{d}^{n+1}x \,\Big(\bar\psi^0(i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} \\ -\frac{1}{4}C^{\rho\sigma}_{\lambda}\big(\bar\psi^0\gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0}\gamma_\rho\psi^0\big) \\ -\frac{1}{4}(1-8b_1)C^{\rho\sigma}_{\lambda} x^{\lambda}\bar\psi^0 F^0_{\rho\sigma}(i\gamma^\mu(\mathcal{D}^0_\mu\psi^0) - m\psi^0) - \frac{i}{2}(1-2b_2)C^{\rho\sigma}_{\lambda} x^{\lambda}\bar\psi^0\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) \\ +ib_3 C^{\rho\sigma}_{\lambda} x^{\lambda}\bar\psi^0\gamma^\mu F^0_{\rho\sigma}(\mathcal{D}^0_\mu\psi^0) - 2i(b_1 - \frac{1}{4}(b_2 - 2b_3))C^{\rho\sigma}_{\lambda} x^{\lambda}\bar\psi^0\gamma^\mu(\mathcal{D}^0_\sigma F^0_{\mu\rho})\psi^0 \\ -\frac{ia}{4}(n + 8b_1 - 2nb_2 + 4b_3 - 1)\bar\psi^0\gamma^\mu F^0_{\mu n}\psi^0 \\ -\frac{1}{2}(1-2b_2)C^{\rho\sigma}_{\lambda} x^{\lambda} F^{0\mu\nu} F^0_{\mu\rho} F^0_{\nu\sigma} + \frac{1}{8}(1-8b_3)C^{\rho\sigma}_{\lambda} x^{\lambda} F^{0\mu\nu} F^0_{\mu\nu} F^0_{\rho\sigma}\Big). \quad (4.4.8)$$

All constants b_i are so far completely undetermined and they have been all set to zero in the previous chapter. The reason for this particular choice has been a technical simplicity in the construction of the Seiberg-Witten map. However, we have another interesting possibility. There exists a particular choice of the constants b_i such that all the ambiguous, undetermined terms in the action (4.4.8) are set to zero.

For the massless fermions we choose $b_1 = 1/16$, $b_2 = 1/2$ and $b_3 = 1/8$. The effective action (4.4.8) up to first order in the deformation parameter a is then

$$S = \int \mathrm{d}^{n+1}x \,\Big(i\bar\psi^0\gamma^\mu \mathcal{D}^0_\mu\psi^0 - \frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} \\ -\frac{1}{4}C^{\rho\sigma}_{\lambda}\big(\bar\psi^0\gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi}^0\gamma_\rho\psi^0\big)\Big). \quad (4.4.9)$$

The corresponding equations of motion are

$$i\gamma^\mu \mathcal{D}^0_\mu\psi^0 - \frac{1}{2}C^{\rho\sigma}_{\lambda}\gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 - \frac{i}{4n}C^{\rho\sigma}_{\sigma}\gamma^\mu F^0_{\mu\rho}\psi^0 = 0, \\ -i\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\mu - \frac{1}{2}C^{\rho\sigma}_{\lambda}\overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0}\gamma_\rho + \frac{i}{4n}C^{\rho\sigma}_{\sigma}\bar\psi^0\gamma^\mu F^0_{\mu\rho} = 0, \quad (4.4.10)$$

$$\partial_\mu F^{0\rho\mu} = J^\rho = \bar\psi^0\gamma^\rho\psi^0 \quad (4.4.11) \\ -\frac{i}{4}\left(C^{\alpha\mu}_{\lambda}\eta^{\lambda\rho}\Big(\overline{\mathcal{D}^0_\mu\psi^0}\gamma_\alpha\psi^0 - \bar\psi^0\gamma_\alpha \mathcal{D}^0_\mu\psi^0\Big) + C^{\alpha\rho}_{\lambda}\Big(\overline{\mathcal{D}^{0\lambda}\psi^0}\gamma_\alpha\psi^0 - \bar\psi^0\gamma_\alpha \mathcal{D}^{0\lambda}\psi^0\Big)\right).$$

If one calculates the current j^ρ (4.3.36), coming from the variation of the action (4.4.9) one finds $j^\rho = J^\rho$.

For the massive fermions we have to choose $b_1 = 1/2$, $b_2 = 1/2$ and $b_3 = 0$. This leads to the effective action up to first order in a

$$S = \int \mathrm{d}^{n+1}x \, \Big(\bar\psi^0 (i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4} F^0_{\mu\nu}F^{0\mu\nu} \\ -\frac{1}{4} C^{\rho\sigma}_\lambda \big(\bar\psi^0 \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi}^0 \gamma_\rho \psi^0 \big) + \frac{1}{8} C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu} F^0_{\mu\nu} F^0_{\rho\sigma} \Big). \tag{4.4.12}$$

The equations of motion for the matter field which follow from (4.4.12) are

$$(i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{2} C^{\rho\sigma}_\lambda \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 - \frac{i}{4n} C^{\rho\sigma}_\sigma \gamma^\mu F^0_{\mu\rho}\psi^0 = 0, \\ -i\overline{\mathcal{D}^0_\mu \psi^0}\gamma^\mu - m\bar\psi^0 - \frac{1}{2} C^{\rho\sigma}_\lambda \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0}\gamma_\rho + \frac{i}{4n} C^{\rho\sigma}_\sigma \bar\psi^0 \gamma^\mu F^0_{\mu\rho} = 0. \tag{4.4.13}$$

The conserved current is again only one and it is given by (4.4.11). However, the equation of motion for the gauge field will have additional terms on the left-hand side of (4.4.11) and they will be explicitly x-dependent. It looks like the Yang-Mills part of the action "feels" if the fermionic fields are massive or massless, which is unusual. Also, the explicit x-dependence in the action (4.4.12) is not something that one expects to have. Fortunately, there is a freedom in the Seiberg-Witten map which we have not used so far. Namely, X was also obtained by solving the Seiberg-Witten map (4.3.28). To the solution (4.3.29) we add

$$\Delta X = b_4 \mu^{-1} C^{\rho\sigma}_\lambda x^\lambda F^0_{\rho\sigma}. \tag{4.4.14}$$

This leads to the effective action (for the massive fermions, for the massless fermions we set $b_4 = 0$)

$$S = \int \mathrm{d}^{n+1}x \, \Big(\bar\psi^0 (i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4} F^0_{\mu\nu}F^{0\mu\nu} \tag{4.4.15} \\ -\frac{1}{4} C^{\rho\sigma}_\lambda \big(\bar\psi^0 \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi}^0 \gamma_\rho \psi^0 \big) + \frac{1}{8}(1-2b_4) C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu} F^0_{\mu\nu} F^0_{\rho\sigma} \Big).$$

Choosing $b_4 = 1/2$ we obtain

$$S = \int \mathrm{d}^{n+1}x \, \Big(\bar\psi^0 (i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4} F^0_{\mu\nu}F^{0\mu\nu} \\ -\frac{1}{4} C^{\rho\sigma}_\lambda \big(\bar\psi^0 \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi}^0 \gamma_\rho \psi^0 \big) \Big), \tag{4.4.16}$$

which is exactly (4.4.9) (up to $m = 0$ or $m \neq 0$ for the fermions).

In this way we have obtained the same action, equations of motion and conserved current in the case of $U(1)$ gauge theory coupled to the massive or massless fermions. Furthermore, we have obtained the action and the equations of motion that are both gauge invariant and invariant under the classical Poincaré transformations. Note that no explicitly x-dependent terms appear in the action (4.4.16). This is the reason why we would prefer this choice of the Seiberg-Witten solutions. Since we are not sure if the requirement that the two conserved currents should be equal is correct, we do not want to use it to fix the ambiguities in the Seiberg-Witten map. However, we can demand that the action is explicitly x-independent and this then gives the preferred choice of the Seiberg-Witten map. In addition, we say that this choice gives the unique conserved current.

We end this analysis with a few comments. First note that the fermionic action we have constructed here up to first order in a is

$$S_m = \frac{1}{2}\int \mathrm{d}^{n+1}x \left(\bar{\psi}^0(i\gamma^\mu \mathcal{D}_\mu - m)\psi^0 + (-i\overline{\mathcal{D}_\mu\psi}\gamma^\mu - m\bar{\psi})\psi\right). \tag{4.4.17}$$

Here the operator $\mathcal{D}_\mu$ is Dirac operator (4.1.14) expanded in a in which partial derivatives are covariantised by minimal substitution $\partial_\alpha \to \partial_\alpha - iA^0_\alpha$. We conjecture that this is also valid for higher orders, but one needs to be careful in ordering the derivatives in the expansion of the Dirac operator. Namely, in the second order in a a term like $\bar{\psi}^0\partial_n^2\partial_j\psi^0$ appears. Therefore, it will not be the same if one writes $\bar{\psi}^0\mathcal{D}^0_n\mathcal{D}^0_n\mathcal{D}^0_j\psi^0$ or $\bar{\psi}^0\mathcal{D}^0_n\mathcal{D}^0_j\mathcal{D}^0_n\psi^0$ or $\bar{\psi}^0\mathcal{D}^0_j\mathcal{D}^0_n\mathcal{D}^0_n\psi^0$ since the covariant derivatives do not commute. But analysis here is far from being complete, this is to be investigated in future.

Note also that if one allows for the derivative-valued gauge fields in the θ-deformed case, one can construct the effective $U(1)$ action with no additional terms in the first order with respect to the undeformed $U(1)$ gauge theory, compare with [41].

5

Gravity on the θ-deformed space

In the previous two chapters we were analysing the noncommutative gauge theories. In this chapter we continue with this subject, only now we turn to the local space-time symmetries. Our aim is the construction of a gravity theory on deformed spaces. For simplicity we work with the θ-deformed space which has been introduced in Chapter 1. Nevertheless, the method we use is a rather general one and it can be applied to other deformed spaces as well.

Since our approach is based on a deformation of the commutative diffeomorphism symmetry, we first rewrite some of the well known properties of the commutative diffeomorphisms in a more mathematical way. Then we derive the Hopf algebra of deformed diffeomorphisms and using this result introduce the concept of fields. Repeating the steps one does in the commutative case, we obtain the deformed Einstein-Hilbert action and derive the equation of motion from it. At the end of the chapter we make two remarks. The first one concerns the θ-deformed global Poincaré symmetry which can be viewed as a special case of the diffeomorphism symmetry. The second one concerns a different approach to noncommutative gauge theories.

5.1 Commutative diffeomorphisms

Under the general coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{5.1.1}$$

a scalar field $\phi^0(x)$[1] transforms as

$$\phi'^0(x') = \phi^0(x), \tag{5.1.2}$$

or in the language of infinitesimal transformations

$$\delta_\xi^{cl}\phi^0(x) = \phi'^0(x) - \phi^0(x) = -\xi^\mu\partial_\mu\phi^0(x). \tag{5.1.3}$$

Here $\xi^\mu(x)$ is an arbitrary function of coordinates. These transformations close in the algebra

$$[\delta_\xi^{cl}\delta_\eta^{cl} - \delta_\eta^{cl}\delta_\xi^{cl}] = \delta_{[\xi,\eta]}^{cl}, \tag{5.1.4}$$

[1]We write (almost) all commutative fields with the upper index 0 to distinguish them from their noncommutative analogs.

where $\delta^{cl}_{[\xi,\eta]}\phi^0 = -\big(\xi^\nu(\partial_\nu\eta^\mu) - \eta^\nu(\partial_\nu\xi^\mu)\big)(\partial_\mu\phi^0)$. Algebra (5.1.4) is the algebra of commutative (classical, undeformed) diffeomorphisms.

The product of two scalar fields is a scalar field again

$$\begin{aligned}\delta^{cl}_\xi\Big(\phi^0_1(x)\phi^0_2(x)\Big) &= \Big(\delta^{cl}_\xi\phi^0_1(x)\Big)\phi^0_2(x) + \phi^0_1(x)\Big(\delta^{cl}_\xi\phi^0_2(x)\Big)\\ &= -\xi^\mu\partial_\mu\Big(\phi_1(x)\phi_2(x)\Big) \qquad (5.1.5)\end{aligned}$$

and we see that the Leibniz rule for the operator δ^{cl}_ξ is undeformed. This Leibniz rule follows from the coproduct

$$\Delta\delta^{cl}_\xi = \delta^{cl}_\xi \otimes 1 + 1 \otimes \delta^{cl}_\xi. \qquad (5.1.6)$$

It is not difficult to check that (5.1.6) is coassociative and that it is consistent with the algebra (5.1.4). Adding the counit and antipode

$$\varepsilon(\delta^{cl}_\xi) = 0, \qquad S(\delta^{cl}_\xi) = -\delta^{cl}_\xi \qquad (5.1.7)$$

we define the Hopf algebra of undeformed diffeomorphisms[2].

Under the transformations (5.1.1) an arbitrary tensor transforms as

$$\begin{aligned}\delta^{cl}_\xi T^{0\,\nu_1\dots\nu_s}_{\mu_1\dots\mu_r} &= -\xi^\lambda(\partial_\lambda T^{0\,\nu_1\dots\nu_s}_{\mu_1\dots\mu_r}) - (\partial_{\mu_1}\xi^\lambda)T^{0\,\nu_1\dots\nu_s}_{\lambda\mu_2\dots\mu_r} - \dots - (\partial_{\mu_r}\xi^\lambda)T^{0\,\nu_1\dots\nu_s}_{\mu_1\dots\mu_{r-1}\lambda}\\ &+(\partial_\lambda\xi^{\nu_1})T^{0\,\lambda\nu_2\dots\nu_s}_{\mu_1\dots\mu_r} + \dots + (\partial_\lambda\xi^{\nu_s})T^{0\,\nu_1\dots\nu_{s-1}\lambda}_{\mu_1\dots\mu_r}. \qquad (5.1.8)\end{aligned}$$

Specially, for a covariant vector we have

$$\delta^{cl}_\xi V^0_\mu = -\xi^\lambda(\partial_\lambda V^0_\mu) - (\partial_\mu\xi^\lambda)V^0_\lambda, \qquad (5.1.9)$$

and for a contravariant one

$$\delta^{cl}_\xi V^{0\,\mu} = -\xi^\lambda(\partial_\lambda V^{0\,\mu}) + (\partial_\lambda\xi^\mu)V^{0\,\lambda}. \qquad (5.1.10)$$

Since (5.1.9) is a local transformation (the parameter ξ^λ being x-dependent), one proceeds like in the usual gauge theory, observing that the partial derivative of a vector field does not transform like a second rank tensor. This is repaired by introducing the covariant derivative

$$D^0_\mu V^0_\nu = (\partial_\mu V^0_\nu) - \Gamma^{0\,\alpha}_{\mu\nu}V^0_\alpha, \qquad (5.1.11)$$

where $\Gamma^{0\,\alpha}_{\mu\nu}$ is the commutative connection. Its transformation law follows from the requirement

$$\delta^{cl}_\xi(D^0_\mu V^0_\nu) = -\xi^\lambda(\partial_\lambda D^0_\mu V^0_\nu) - (\partial_\mu\xi^\lambda)D^0_\lambda V^0_\nu - (\partial_\nu\xi^\lambda)D^0_\mu V^0_\lambda, \qquad (5.1.12)$$

and it is given by

$$\delta^{cl}_\xi\Gamma^{0\,\alpha}_{\mu\nu} = -\xi^\lambda(\partial_\lambda\Gamma^{0\,\alpha}_{\mu\nu}) - (\partial_\mu\xi^\lambda)\Gamma^{0\,\alpha}_{\lambda\nu} - (\partial_\nu\xi^\lambda)\Gamma^{0\,\alpha}_{\mu\lambda} + (\partial_\lambda\xi^\alpha)\Gamma^{0\,\lambda}_{\mu\nu} - (\partial_\mu\partial_\nu\xi^\alpha). \qquad (5.1.13)$$

[2] To be more precise, in order to speak about the Hopf algebra of diffeomorphisms one has to go to the enveloping algebra of (5.1.4) [96]. It is the associative algebra freely generated by δ^{cl}_ξ elements and divided by the ideal generated by (5.1.4) relations.

Because of the last term in (5.1.13) $\Gamma^{0\,\alpha}_{\mu\nu}$ is not a tensor. One generalises (5.1.11) to an arbitrary tensor

$$\begin{aligned} D^0_\rho T^{0\,\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_r} &= (\partial_\rho T^{0\,\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_r}) - \Gamma^{0\,\alpha}_{\rho\mu_1} T^{0\,\nu_1\ldots\nu_s}_{\alpha\mu_2\ldots\mu_r} - \cdots - \Gamma^{0\,\alpha}_{\rho\mu_r} T^{0\,\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_{r-1}\alpha} \\ &\quad + \Gamma^{0\,\nu_1}_{\rho\alpha} T^{0\,\alpha\nu_2\ldots\nu_s}_{\mu_1\ldots\mu_r} + \cdots + \Gamma^{0\,\nu_s}_{\rho\alpha} T^{0\,\nu_1\ldots\nu_{s-1}\alpha}_{\mu_1\ldots\mu_r}. \end{aligned} \tag{5.1.14}$$

The commutator of two covariant derivatives defines the curvature tensor $R^0_{\mu\nu\rho}{}^\sigma$ and the torsion $T^{0\,\alpha}_{\mu\nu}$

$$[D_\mu, D_\nu]V^0_\rho = R^0_{\mu\nu\rho}{}^\sigma V^0_\sigma + T^{0\,\alpha}_{\mu\nu} D^0_\alpha V^0_\rho, \tag{5.1.15}$$

with

$$R^0_{\mu\nu\rho}{}^\sigma = (\partial_\nu \Gamma^{0\,\sigma}_{\mu\rho}) - (\partial_\mu \Gamma^{0\,\sigma}_{\nu\rho}) + \Gamma^{0\,\beta}_{\nu\rho}\Gamma^{0\,\sigma}_{\mu\beta} - \Gamma^{0\,\beta}_{\mu\rho}\Gamma^{0\,\sigma}_{\nu\beta}, \tag{5.1.16}$$

$$T^{0\,\alpha}_{\mu\nu} = \Gamma^{0\,\alpha}_{\nu\mu} - \Gamma^{0\,\alpha}_{\mu\nu}. \tag{5.1.17}$$

One also introduces the metric tensor $g_{\mu\nu}$ as a symmetric tensor of the second rank. Together with its inverse $g^{\mu\nu}$, it is used to raise and lower indices. Although metric and connection are independent objects, they can be related introducing the metricity condition

$$D^0_\rho g_{\mu\nu} = (\partial_\rho g_{\mu\nu}) - \Gamma^{0\,\alpha}_{\rho\mu} g_{\alpha\nu} - \Gamma^{0\,\alpha}_{\rho\nu} g_{\mu\alpha} = 0. \tag{5.1.18}$$

This condition enables us to calculate the symmetric part of $\Gamma^{0\,\alpha}_{\mu\nu}$ in terms of the metric tensor and its inverse. If the space is torsion-free, $T^{0\,\alpha}_{\mu\nu} = 0$, then from (5.1.17) it follows that the connection is symmetric in lower indices and it is then given entirely in terms of the metric and its inverse.

Using the metric one defines the Ricci tensor and scalar curvature as

$$R^0_{\mu\nu} = R^0_{\mu\rho\nu}{}^\rho = R^0_{\nu\mu}, \tag{5.1.19}$$

$$R^0 = g^{\mu\nu} R^0_{\mu\nu}. \tag{5.1.20}$$

Finally, the Einstein-Hilbert action[3] is given by

$$S = \int \mathrm{d}^4x\, \sqrt{-g} R^0, \tag{5.1.21}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. In order to have an action which is invariant under the undeformed diffeomorphisms, one has to introduce the measure function $\sqrt{-g}$. Its transformation law

$$\delta^{cl}_\xi \sqrt{-g} = -(\partial_\lambda(\xi^\lambda\sqrt{-g})) = -(\partial_\lambda \xi^\lambda)\sqrt{-g} - \xi^\lambda(\partial_\lambda\sqrt{-g}) \tag{5.1.22}$$

ensures that the action (5.1.21) is invariant. Varying (5.1.21) with respect to the metric gives the equations of motion.

In the next sections we generalise these concepts to the θ-deformed space. The starting point is the Hopf algebra of undeformed diffeomorphisms given in (5.1.4), (5.1.6) and (5.1.7).

[3]Of course, one can define more general actions than (5.1.21) using not only curvature scalar but also curvature tensor, Ricci tensor and, if it is different from zero, torsion.

5.2 Deformed diffeomorphisms

In this section we introduce the mathematical tools necessary to derive a gravity theory on the θ-deformed space. We work in the $\star$-product formalism. For convenience some of the important formulas from Section 1.5 are repeated.

The θ-deformed space is defined by (1.1.3). Functions in the abstract algebra are represented by functions of the commutative coordinates, while the abstract algebra multiplication is represented by the Moyal-Weyl $\star$-product

$$f \star g\,(x) = e^{\frac{i}{2}\frac{\partial}{\partial x^\rho}\theta^{\rho\sigma}\frac{\partial}{\partial y^\sigma}} f(x)g(y)\Big|_{y\to x}, \tag{5.2.1}$$

or

$$\begin{aligned} f \star g\,(x) &= \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}f(x)\Big)\Big(\partial_{\sigma_1}\dots\partial_{\sigma_n}g(x)\Big) \\ &= fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) - \frac{1}{8}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}(\partial_{\rho_1}\partial_{\rho_2}f)(\partial_{\sigma_1}\partial_{\sigma_2}g) + \dots, \end{aligned} \tag{5.2.2}$$

with $\theta^{\rho\sigma} = -\theta^{\sigma\rho} = const.$. The derivatives in the abstract algebra $\hat{\partial}_\lambda$ are represented by the $\star$-derivatives $\partial^\star_\lambda$

$$\partial^\star_\lambda = \partial_\lambda, \tag{5.2.3}$$

where ∂_λ are the usual partial derivatives. In the following we will mainly write ∂_λ, only when we want to stress something we write explicitly $\partial^\star_\lambda$. Because of (5.2.3) this makes no difference to our results. The Leibniz rule for the derivatives (5.2.3) is

$$\begin{aligned} \partial^\star_\lambda \star \big(f \star g\big) &= \big(\partial^\star_\lambda \star f\big)\star g + f\star\big(\partial^\star_\lambda \star g\big) \\ &= \big(\partial^\star_\lambda \triangleright f\big)\star g + f\star\big(\partial^\star_\lambda \triangleright g\big), \end{aligned} \tag{5.2.4}$$

where the "$\triangleright$" notation was introduced in (1.5.14).

5.2.1 Inversion of the $\star$-product

In order to proceed towards a deformed theory of gravity we have to introduce a few more concepts. To start with, we define the $\star$-action[4] of a vector field[5] $\xi = \xi^\mu\partial_\mu$ on a function $f(x)$

$$\xi \triangleright f = \xi^\mu \star (\partial_\mu f). \tag{5.2.5}$$

Expanding the $\star$-product in (5.2.5) gives

$$\begin{aligned} \xi\triangleright f &= \xi^\mu(x)\star\partial_\mu f(x) \\ &= \sum_n \left(\frac{i}{2}\right)^n\frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}\xi^\mu(x)\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n}\partial_\mu f(x) \\ &= \xi^\mu(\partial_\mu f(x)) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\xi^\mu)(\partial_\sigma\partial_\mu f(x)) + \dots \\ &\stackrel{\text{def}}{=} (\Xi f(x)). \end{aligned} \tag{5.2.6}$$

[4]Just like for derivatives, one also makes the difference between the $\star$-product and the $\star$-action of a differential operator on a function. The first one is $\xi\star f = \xi^\mu\star(\partial_\mu f) + (\xi^\mu\star f)\partial_\mu$ while the second is $\xi\triangleright f = \xi^\mu\star(\partial_\mu f)$.

[5]Note that ξ is not a vector field in the sense of (5.1.9) or (5.1.10).

We see that the $\star$-action of a vector field on a function is given in terms of the higher order differential operators acting in the usual way on a function. In this way a map from vector fields to higher order differential operators is defined. Before we continue, we mention why it is important to have this $\star$-action. In the abstract algebra one has $\hat{\xi} = \hat{\xi}^\mu \hat{\partial}_\mu$ as a vector field. When this is mapped to the space of commuting coordinates

$$\hat{\xi}\hat{f} = \hat{\xi}^\mu \hat{\partial}_\mu \hat{f} \mapsto \xi^\mu(x) \star \partial^\star_\mu \star f = \xi^\mu(x) \star (\partial^\star_\mu \triangleright f + f \star \partial^\star_\mu), \tag{5.2.7}$$

the first term in (5.2.7) is exactly (5.2.5).

Since the map (5.2.6) starts with the identity it is possible to invert it, that is express the usual action of a higher order differential operator on a function in terms of the $\star$-action. We are interested in the deformation of the commutative diffeomorphisms and therefore we look at[6]

$$(\xi f(x)) = \xi^\mu(\partial_\mu f(x)) = X^\star_\xi \triangleright f(x). \tag{5.2.8}$$

The operator $X^\star_\xi$ is constructed perturbatively from the above requirement using the $\star$-product (5.2.2)

$$\begin{aligned} X^\star_\xi &= X^{\star 0}_\xi + X^{\star 1}_\xi + \dots \\ X^\star_\xi \triangleright f &= \left(X^{\star 0}_\xi \star f\right) + \left(X^{\star 1}_\xi \star f\right) + \dots \\ &= \left(X^{\star 0}_\xi f\right) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho X^{\star 0}_\xi)(\partial_\sigma f) + \left(X^{\star 1}_\xi f\right) + \dots \\ &\overset{\text{def}}{=} \xi^\mu(\partial_\mu f). \end{aligned}$$

This leads to

$$X^{\star 1}_\xi = -\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho X^{\star 0}_\xi)\partial_\sigma = -\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \xi^\mu)\partial_\sigma \partial_\mu,$$

so the solution up to first order is

$$X^\star_\xi = \xi^\mu \partial_\mu - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \xi^\mu)\partial_\sigma \partial_\mu. \tag{5.2.9}$$

It is not difficult to generalise this to all orders

$$X^\star_\xi = \sum_n \left(\frac{-i}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}\xi^\mu\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n}\partial_\mu. \tag{5.2.10}$$

Before giving the physical meaning to this result we make a few comments. Everything that has been done for a vector field ξ can be generalised to higher order differential operators. Also, generalisation to more general $\star$-products (more complicated deformed spaces) is possible but it will not be analysed here. An important property of (5.2.10) is that it has a meaning in the abstract algebra as well (unlike $\xi^\mu \partial_\mu$)

$$\begin{aligned} X^\star_\xi \triangleright \phi &= \xi^\mu \star (\partial_\mu \phi) - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \xi^\mu) \star (\partial_\sigma \partial_\mu \phi) + \dots \\ &\downarrow \\ \hat{X}_\xi \hat{\phi} &= \hat{\xi}^\mu(\hat{\partial}_\mu \hat{\phi}) - \frac{i}{2}\theta^{\rho\sigma}(\hat{\partial}_\rho \hat{\xi}^\mu)(\hat{\partial}_\sigma \hat{\partial}_\mu \hat{\phi}) + \dots . \end{aligned} \tag{5.2.11}$$

[6] Although it looks like we have too many brackets in (5.2.8), it is convenient to write them all in order to know how to understand expressions like (5.2.8).

5.2.2 Hopf algebra of deformed diffeomorphisms

The transformation law of a scalar field $\phi^0(x)$ under the commutative diffeomorphisms is given by (5.1.3). We define the transformation law of a noncommutative scalar field[7] $\phi(x)$ to be

$$\delta_\xi \phi(x) = \phi'(x) - \phi(x) = -\xi^\mu \partial_\mu \phi(x) = -X^\star_\xi \triangleright \phi(x) \tag{5.2.12}$$

and this we call the deformed transformation law of a scalar field. To see if this transformations close in the algebra, one calculates

$$\delta_\xi \delta_\eta \phi(x) = \delta_\xi(-X^\star_\eta \triangleright \phi(x)) = X^\star_\eta \triangleright X^\star_\xi \triangleright \phi(x).$$

Form here it follows

$$\delta_\xi \delta_\eta - \delta_\eta \delta_\xi = \delta_{[\xi,\eta]}. \tag{5.2.13}$$

However, this result was expected.

What has been done so far is just rewriting the classical transformation law (5.1.3) in a rather complicated way (so no reason to call it "deformed"). But now we remember one more property of the classical diffeomorphisms. Namely, (5.1.4) tells us that the usual product of two scalar fields transforms as a scalar field. This we generalise by demanding that the $\star$-product of two scalar fields is a scalar field again

$$\delta_\xi(\phi_1 \star \phi_2) = -X_\xi \triangleright (\phi_1 \star \phi_2). \tag{5.2.14}$$

The right-hand side of (5.2.14), written more explicitly using (5.2.12), reads

$$\begin{aligned}\delta_\xi(\phi_1 \star \phi_2) &= -\xi^\mu \big(\partial_\mu(\phi_1 \star \phi_2)\big) = -\xi^\mu \Big((\partial_\mu \phi_1) \star \phi_2 + \phi_1 \star (\partial_\mu \phi_2)\Big) \\ &\neq -(\xi^\mu(\partial_\mu \phi_1)) \star \phi_2 - \phi_1 \star (\xi^\mu(\partial_\mu \phi_2)),\end{aligned}$$

since the $\star$-product is noncommutative. Commuting ξ^μ through the $\star$-product gives additional terms

$$\delta_\xi(\phi_1 \star \phi_2) = (\delta_\xi \phi_1) \star \phi_2 + \phi_1 \star (\delta_\xi \phi_2) + \text{ additional terms.} \tag{5.2.15}$$

and this is where the difference between (5.2.12) and the classical transformation law (5.1.3) arises. The Leibniz rule for the transformations (5.2.12) is not (5.1.5) but it has to be deformed to (5.2.15).

In order to find the additional terms we expand the $\star$-product and the operators $X^\star_\xi$ in (5.2.15). Expanding the left-hand side of (5.2.15) up to first order in the deformation parameter θ gives

$$\begin{aligned}\delta_\xi(\phi_1 \star \phi_2) &= -\xi^\mu \Big((\partial_\mu \phi_1) \star \phi_2 + \phi_1 \star (\partial_\mu \phi_2)\Big) \qquad (5.2.16)\\ &= -\xi^\mu \Big((\partial_\mu \phi_1)\phi_2 + \phi_1(\partial_\mu \phi_2) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \partial_\mu \phi_1)(\partial_\sigma \phi_2) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \phi_1)(\partial_\sigma \partial_\mu \phi_2)\Big).\end{aligned}$$

[7]The definitions of fields and tensor calculus will be the subject of the next section. However, in order to derive the results that follow we introduce the notion of a scalar field here.

From the right-hand side of (5.2.15) follows

$$\begin{aligned}(\delta_\xi\phi_1)\star\phi_2+\phi_1\star(\delta_\xi\phi_2)+F(\phi_1,\phi_2,\theta,\partial,\xi) &= -\xi^\mu(\partial_\mu\phi_1)\phi_2-\xi_\mu\phi_1(\partial_\mu\phi_2)\\ &-\frac{i}{2}\theta^{\rho\sigma}\Big((\partial_\rho\xi^\mu)(\partial_\mu\phi_1)+\xi^\mu(\partial_\rho\partial_\mu\phi_1)\Big)(\partial_\sigma\phi_2)\\ &-\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\phi_1)\Big((\partial_\sigma\xi^\mu)(\partial_\mu\phi_2)+\xi^\mu(\partial_\sigma\partial_\mu\phi_2)\Big)\\ &+F(\phi_1,\phi_2,\theta,\partial,\xi), \qquad (5.2.17)\end{aligned}$$

where we labeled the unknown additional terms as $F(\phi_1,\phi_2,\theta,\partial,\xi)$. Comparing (5.2.16) and (5.2.17) we find

$$F(\phi_1,\phi_2,\theta,\partial)=\frac{i}{2}\theta^{\rho\sigma}\Big((\partial_\rho\xi^\mu)(\partial_\mu\phi_1)(\partial_\sigma\phi_2)+(\partial_\rho\phi_1)(\partial_\sigma\xi^\mu)(\partial_\mu\phi_2)\Big).$$

Finally, we write the Leibniz rule for the transformation δ_ξ up to first order in the deformation parameter θ

$$\delta_\xi(\phi_1\star\phi_2)=(\delta_\xi\phi_1)\star\phi_2+\phi_1\star(\delta_\xi\phi_2)-\frac{i}{2}\theta^{\rho\sigma}\Big(\big((\partial_\rho\delta_\xi)\phi_1\big)\partial_\sigma\phi_2+(\partial_\rho\phi_1)\big((\partial_\sigma\delta_\xi)\phi_2\big)\Big), \qquad (5.2.18)$$

with $(\partial_\rho\delta_\xi)\phi_1\stackrel{\text{def}}{=}-(\partial_\rho\xi^\mu)\partial_\mu\phi_1$.

This Leibniz rule follows from the abstract comultiplication, which up to first order in θ reads

$$\Delta\delta_\xi=\delta_\xi\otimes 1+1\otimes\delta_\xi-\frac{i}{2}\theta^{\rho\sigma}\Big((\partial_\rho\delta_\xi)\otimes\partial_\sigma+\partial_\rho\otimes(\partial_\sigma\delta_\xi)\Big). \qquad (5.2.19)$$

It is not difficult to check that (5.2.19) is coassociative. The counit is defined in the following way

$$\varepsilon(\delta_\xi)=0 \qquad (5.2.20)$$

and it fulfils (1.3.5). Therefore, we have the coalgebra of deformed diffeomorphisms.

For a bialgebra we have to check whether

$$\begin{aligned}[\Delta\delta_\xi,\Delta\delta_\eta] &= \Delta\delta_{[\xi,\eta]}, \qquad (5.2.21)\\ \varepsilon(\delta_\xi\delta_\eta) &= \varepsilon(\delta_\xi)\varepsilon(\delta_\eta). \qquad (5.2.22)\end{aligned}$$

We prove (5.2.21), (5.2.22) is obvious.

$$\begin{aligned}\Delta\delta_\xi\cdot\Delta\delta_\eta &= \Big(\delta_\xi\otimes 1+1\otimes\delta_\xi-\frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\xi)\otimes\partial_\sigma+\partial_\rho\otimes(\partial_\sigma\delta_\xi)\big)\Big)\\ &\quad\cdot\Big(\delta_\eta\otimes 1+1\otimes\delta_\eta-\frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\eta)\otimes\partial_\sigma+\partial_\rho\otimes(\partial_\sigma\delta_\eta)\big)\Big)\\ &= \delta_\xi\delta_\eta\otimes 1+\delta_\xi\otimes\delta_\eta-\frac{i}{2}\theta^{\rho\sigma}\big(\delta_\xi(\partial_\rho\delta_\eta)\otimes\partial_\sigma+\delta_\xi\partial_\rho\otimes(\partial_\sigma\delta_\eta)\big)\\ &\quad+\delta_\eta\otimes\delta_\xi+1\otimes\delta_\xi\delta_\eta-\frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\eta)\otimes\delta_\xi\partial_\sigma+\partial_\rho\otimes\delta_\xi(\partial_\sigma\delta_\eta)\big)\\ &\quad-\frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\xi)\otimes\partial_\sigma\delta_\eta+\partial_\rho\otimes(\partial_\sigma\delta_\xi)\delta_\eta+(\partial_\rho\delta_\xi)\delta_\eta\otimes\partial_\sigma+\partial_\rho\delta_\eta\otimes(\partial_\sigma\delta_\xi)\big)\end{aligned}$$

$$
\begin{aligned}
-\Delta\delta_\eta\cdot\Delta\delta_\xi = \dots\\
&= -\delta_\eta\delta_\xi\otimes 1 - \delta_\eta\otimes\delta_\xi + \frac{i}{2}\theta^{\rho\sigma}\big(\delta_\eta(\partial_\rho\delta_\xi)\otimes\partial_\sigma + \delta_\eta\partial_\rho\otimes(\partial_\sigma\delta_\xi)\big)\\
&\quad -\delta_\xi\otimes\delta_\eta - 1\otimes\delta_\eta\delta_\xi + \frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\xi)\otimes\delta_\eta\partial_\sigma + \partial_\rho\otimes\delta_\eta(\partial_\sigma\delta_\xi)\big)\\
&\quad +\frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\delta_\eta)\otimes\partial_\sigma\delta_\xi + \partial_\rho\otimes(\partial_\sigma\delta_\eta)\delta_\xi + (\partial_\rho\delta_\eta)\delta_\xi\otimes\partial_\sigma + \partial_\rho\delta_\xi\otimes(\partial_\sigma\delta_\eta)\big)
\end{aligned}
$$

$$
\Delta\delta_{[\xi,\eta]} = \delta_{[\xi,\eta]}\otimes 1 + 1\otimes\delta_{[\xi,\eta]} - \frac{i}{2}\theta^{\rho\sigma}\Big((\partial_\rho\delta_{[\xi,\eta]})\otimes\partial_\sigma + \partial_\rho\otimes(\partial_\sigma\delta_{[\xi,\eta]})\Big) \tag{5.2.23}
$$

Adding these three expressions and using (5.2.13) gives (5.2.21) only if

$$
\begin{aligned}
\frac{i}{2}\theta^{\rho\sigma}\Big(&\,(\partial_\rho\delta_\xi)\otimes\partial_\sigma\delta_\eta + \partial_\rho\otimes(\partial_\sigma\delta_\xi)\delta_\eta\\
&+(\partial_\rho\delta_\xi)\delta_\eta\otimes\partial_\sigma + \partial_\rho\delta_\eta\otimes(\partial_\sigma\delta_\xi) - (\xi\leftrightarrow\eta)\Big) = 0.
\end{aligned} \tag{5.2.24}
$$

By using $\partial_\rho\delta_\eta\otimes(\partial_\sigma\delta_\xi) = \big((\partial_\rho\delta_\eta)+\delta_\eta\partial_\rho\big)\otimes(\partial_\sigma\delta_\xi)$, one proves the last equation; then (5.2.21) is proven as well.

Adding the antipode $S(\delta_\xi) = -\delta_\xi$ leads to a Hopf algebra. The condition for the antipode (1.3.8) is not difficult to check, but in order to do that we first rewrite (5.2.19) in a different way

$$
\Delta\delta_\xi = \delta_\xi\otimes 1 + 1\otimes\delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\Big(\delta_{(\partial_\rho\xi)}\otimes\partial_\sigma + \partial_\rho\otimes\delta_{(\partial_\sigma\xi)}\Big), \tag{5.2.25}
$$

using $\delta_{(\partial_\rho\xi)} = (\partial_\rho\delta_\xi)$. With (5.2.25)

$$
\begin{aligned}
m\circ(S\otimes\mathrm{id})\circ\Delta(\delta_\xi) &= m\circ(S\otimes\mathrm{id})\circ\Big(\delta_\xi\otimes 1 + 1\otimes\delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\big(\delta_{(\partial_\rho\xi)}\otimes\partial_\sigma + \partial_\rho\otimes\delta_{(\partial_\sigma\xi)}\big)\Big)\\
&= m\circ\Big(-\delta_\xi\otimes 1 + 1\otimes\delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\big(-\delta_{(\partial_\rho\xi)}\otimes\partial_\sigma - \partial_\rho\otimes\delta_{(\partial_\sigma\xi)}\big)\Big)\\
&= -\delta_\xi + \delta_\xi + \frac{i}{2}\theta^{\rho\sigma}\big(\delta_{(\partial_\rho\xi)}\partial_\sigma - \partial_\sigma\delta_{(\partial_\rho\xi)}\big)\\
&= +\frac{i}{2}\theta^{\rho\sigma}\big(\delta_{(\partial_\rho\xi)}\partial_\sigma - (\partial_\sigma\partial_\rho\delta_\xi) - \delta_{(\partial_\rho\xi)}\partial_\sigma\big) = 0 = \eta\circ\varepsilon(\delta_\xi),
\end{aligned}
$$

where we have used $S(\partial_\rho) = -\partial_\rho$, $(\partial_\sigma\delta_{(\partial_\rho\xi)}) = (\partial_\sigma\partial_\rho\delta_\xi)$ and the antisymmetry of $\theta^{\rho\sigma}$.

We have shown that the Hopf algebra of deformed diffeomorphisms exists and that it is given by

$$
\begin{aligned}
\delta_\xi\delta_\eta - \delta_\eta\delta_\xi &= \delta_{[\xi,\eta]},\quad \varepsilon(\delta_\xi) = 0,\quad S(\delta_\xi) = -\delta_\xi,\\
\Delta\delta_\xi = \delta_\xi\otimes 1 + 1\otimes\delta_\xi &- \frac{i}{2}\theta^{\rho\sigma}\Big(\delta_{(\partial_\rho\xi)}\otimes\partial_\sigma + \partial_\rho\otimes\delta_{(\partial_\sigma\xi)}\Big).
\end{aligned} \tag{5.2.26}
$$

The algebra sector is undeformed, while the coalgebra sector becomes deformed. In the limit $\theta\to 0$ (commutative space), this Hopf algebra reduces to the Hopf algebra of undeformed diffeomorphisms (5.1.4), (5.1.6) and (5.1.7).

All the calculations done here are up to first order in θ. However, the result up to all orders in θ is known [36] and we cite it here for completeness

$$\begin{aligned}
\delta_\xi\delta_\eta\, &- \,\delta_\eta\delta_\xi = \delta_{[\xi,\eta]},\\
\Delta\delta_\xi &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\Big(\delta_\xi\otimes 1 + 1\otimes\delta_\xi\Big)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\\
&= \delta_\xi\otimes 1 + 1\otimes\delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\Big(\delta_{(\partial_\rho\xi)}\otimes\partial_\sigma + \partial_\rho\otimes\delta_{(\partial_\sigma\xi)}\Big) + \dots,\\
\varepsilon(\delta_\xi) &= 0, \quad S(\delta_\xi) = -\delta_\xi.
\end{aligned} \tag{5.2.27}$$

As in the case of classical diffeomorphisms, one has to consider the full enveloping algebra of diffeomorphisms in order to be (mathematically) precise.

5.2.3 Consequences of the deformed coproduct

In (5.2.18) the Leibniz rule for δ_ξ acting on the $\star$-product of two scalar fields is given. When writing down effective actions one expands all the $\star$-products between fields. Therefore, we should also know how to transform expanded expressions like

$$\delta_\xi(\phi_1\star\phi_2) = \delta_\xi\Big(\phi_1\phi_2 + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\phi_1)(\partial_\sigma\phi_2) + \dots\Big). \tag{5.2.28}$$

Let us for the moment forget about the nontrivial Leibniz rule for the operator δ_ξ and calculate (5.2.28) as we would do it in the classical case, by using $\delta_\xi(\partial_\rho\phi) = \partial_\rho(\delta_\xi\phi)$. We obtain

$$\begin{aligned}
\delta_\xi(\phi_1\star\phi_2) &= (\delta_\xi\phi_1)\phi_2 + \phi_1(\delta_\xi\phi_2) + \frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho(\delta_\xi\phi_1))(\partial_\sigma\phi_2) + (\partial_\rho\phi_1)(\partial_\sigma(\delta_\xi\phi_2)\big) + \dots\\
&= -\xi^\mu\partial_\mu(\phi_1\star\phi_2) - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\xi^\mu)\big((\partial_\mu\phi_1)(\partial_\sigma\phi_2) - (\partial_\sigma\phi_1)(\partial_\mu\phi_2)\big),
\end{aligned} \tag{5.2.29}$$

which is not the transformation law of a scalar field since there is one additional term compared to (5.2.14). But we know that $\phi_1\star\phi_2$ is a scalar field so we must have done something wrong calculating this. The answer is the following. The nontrivial Leibniz rule for the operator δ_ξ will also affect the expanded expressions like (5.2.28). The classical transformations do not apply there anymore: one has to change them in a way "dictated" by the deformed comultiplication. Coming back to the concrete example one sees that the problem arises when commuting $\delta_\xi(\partial_\rho\phi) = \partial_\rho(\delta_\xi\phi)$, or equivalently when treating $\partial_\rho\phi$ as a vector field. This has to be changed to

$$\delta_\xi(\partial_\rho\phi) = -\xi^\mu\partial_\mu\partial_\rho\phi. \tag{5.2.30}$$

One says that either δ_ξ and ∂_ρ do not commute anymore, or that the derivatives contracted with $\theta^{\rho\sigma}$ do not transform like the usual derivatives. However this might look unnatural, we always remember that this is the rule dictated by the comultiplication.

Having this in mind we analyse the transformation properties of objects (fields, equations, Lagrangians,...) before expanding the $\star$-product, when possible. Analysis can also be done after the expansion but then one has to be very careful about the rules one uses.

5.3 Tensor calculus

In the previous section we defined a scalar field by its transformation law (5.2.12). In this section we generalise this transformation law to vector and tensor fields. As an example of a tensor of the second rank we discuss the properties of the metric tensor. Once again we mention that all calculations are done up to first order in θ, but some of them can be written to all orders [36].

5.3.1 Fields

In analogy with (5.2.12), the transformation law of a covariant vector field is given by

$$\begin{aligned}\delta_\xi V_\mu &= -\xi^\lambda(\partial_\lambda V_\mu) - (\partial_\mu\xi^\lambda)V_\lambda = -X^\star_\xi \triangleright V_\mu - X^\star_{(\partial_\mu\xi^\lambda)} \triangleright V_\lambda \\ &= -\xi^\lambda \star (\partial_\lambda V_\mu) + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\xi^\lambda) \star (\partial_\sigma\partial_\lambda V_\mu) \\ &\quad -(\partial_\mu\xi^\lambda) \star V_\lambda + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\partial_\mu\xi^\lambda) \star (\partial_\sigma V_\lambda) + \dots,\end{aligned} \tag{5.3.1}$$

where in the last two lines $X^\star_\xi$ and $X^\star_{(\partial_\mu\xi^\lambda)}$ are expanded. This we generalise to the transformation law of an arbitrary covariant tensor

$$\begin{aligned}\delta_\xi T_{\mu_1\dots\mu_r} &= -\xi^\lambda(\partial_\lambda T_{\mu_1\dots\mu_r}) - (\partial_{\mu_1}\xi^\lambda)T_{\lambda\mu_2\dots\mu_r} - \dots - (\partial_{\mu_r}\xi^\lambda)T_{\mu_1\dots\mu_{r-1}\lambda} \\ &= -X^\star_\xi \triangleright T_{\mu_1\dots\mu_r} - X^\star_{(\partial_{\mu_1}\xi^\lambda)} \triangleright T_{\lambda\mu_2\dots\mu_r} - \dots - X^\star_{(\partial_{\mu_r}\xi^\lambda)} \triangleright T_{\mu_1\dots\mu_{r-1}\lambda}.\end{aligned} \tag{5.3.2}$$

It is not difficult to check that this transformation close in the algebra (5.2.13).

By demanding that the $\star$-product of two scalar fields should be a scalar field again we derived the Leibniz rule for the operator δ_ξ (5.2.18) and then abstracted the coproduct (5.2.19). Now we check if this was the correct thing to do. Namely, the coproduct is a representation-independent concept so our result (5.2.19) should also apply to vector and tensor fields. For example, the $\star$-product of two covariant vector fields should transform as a second rank tensor if we use (5.2.19)

$$\delta_\xi(V_\mu \star V_\nu) = (\delta_\xi V_\mu) \star V_\nu + V_\mu \star (\delta_\xi V_\nu) - \frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{(\partial_\rho\xi)}V_\mu)(\partial_\sigma V_\nu) + (\partial_\rho V_\mu)(\delta_{(\partial_\sigma\xi)}V_\nu)\Big).$$

Expanding the $\star$-product in first two terms and cancelling some of the terms coming from that expansion with the nontrivial terms in the coproduct leads to

$$\begin{aligned}\delta_\xi(V_\mu \star V_\nu) &= -\xi^\lambda\partial_\lambda(V_\mu V_\nu) - (\partial_\mu\xi^\lambda)V_\lambda V_\nu - (\partial_\nu\xi^\lambda)V_\mu V_\lambda \\ &\quad -\frac{i}{2}\theta^{\rho\sigma}\Big(\xi^\lambda\partial_\lambda\big((\partial_\rho V_\mu)(\partial_\sigma V_\nu)\big) + (\partial_\mu\xi^\lambda)(\partial_\rho V_\lambda)(\partial_\sigma V_\nu) + (\partial_\nu\xi^\lambda)(\partial_\rho V_\mu)(\partial_\sigma V_\lambda)\Big) \\ &= -\xi^\lambda\partial_\lambda(V_\mu \star V_\nu) - (\partial_\mu\xi^\lambda)(V_\lambda \star V_\nu) - (\partial_\nu\xi^\lambda)(V_\mu \star V_\lambda) \\ &= -X^\star_\xi \triangleright (V_\mu \star V_\nu) - X^\star_{(\partial_\mu\xi^\lambda)} \triangleright (V_\lambda \star V_\nu) - X^\star_{(\partial_\nu\xi^\lambda)} \triangleright (V_\mu \star V_\lambda),\end{aligned} \tag{5.3.3}$$

which we wanted to prove. This means that (5.2.19) is the correct coproduct.

All that has been done for the covariant vectors (tensors) can also be done for the contravariant ones. We just summarise the results

$$\delta_\xi V^\mu = -X^\star_\xi \triangleright V^\mu + X^\star_{(\partial_\lambda\xi^\mu)} \triangleright V^\lambda, \tag{5.3.4}$$

$$\delta_\xi T^{\mu_1\dots\mu_r} = -X^\star_\xi \triangleright T^{\mu_1\dots\mu_r} + X^\star_{(\partial_\lambda\xi^{\mu_1})} \triangleright T^{\lambda\mu_2\dots\mu_r} + \dots + X^\star_{(\partial_\lambda\xi^{\mu_r})} \triangleright T^{\mu_1\dots\mu_{r-1}\lambda}. \tag{5.3.5}$$

Again, $V^\mu \star V^\nu$ transforms like a second rank tensor due to the coproduct (5.2.19). Also, having covariant and contravariant vectors and tensors one can construct invariants. For example,

$$\begin{aligned}\delta_\xi(V_\mu \star V^\mu) &= (\delta_\xi V_\mu) \star V^\mu + V_\mu \star (\delta_\xi V^\mu) - \frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{(\partial_\rho\xi)}V_\mu)(\partial_\sigma V^\mu) + (\partial_\rho V_\mu)(\delta_{(\partial_\sigma\xi)}V^\mu)\Big)\\ &= \dots\\ &= -\xi^\lambda\partial_\lambda(V_\mu \star V^\mu) = -X^\star_\xi \triangleright (V_\mu \star V^\mu). \end{aligned} \tag{5.3.6}$$

To summarise, we know now how to multiply vector and tensor fields and how to construct invariants under the transformations (5.2.26).

5.3.2 Metric tensor

An important example of a tensor is the metric tensor. Classically, it is a symmetric tensor of rank two

$$\delta^{cl}_\xi g_{\mu\nu} = -\xi^\rho(\partial_\rho g_{\mu\nu}) - (\partial_\mu\xi^\rho)g_{\rho\nu} - (\partial_\nu\xi^\rho)g_{\mu\rho}. \tag{5.3.7}$$

Its inverse $g^{\mu\nu}$ is defined by

$$g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho. \tag{5.3.8}$$

In analogy to the classical case, we define the noncommutative metric tensor $G_{\mu\nu}$ as a symmetric tensor of rank two

$$\hat{\delta}_\xi G_{\mu\nu} = -X^\star_\xi \triangleright G_{\mu\nu} - X^\star_{(\partial_\mu\xi^\rho)} \triangleright G_{\rho\nu} - X^\star_{(\partial_\nu\xi^\rho)} \triangleright G_{\mu\rho}, \tag{5.3.9}$$

with the condition that it reduces to the classical metric tensor in the $\theta \to 0$ limit,

$$G_{\mu\nu}\Big|_{\theta=0} = g_{\mu\nu}. \tag{5.3.10}$$

However, these conditions do not determine $G_{\mu\nu}$ uniquely and in the following we present a few different solutions.

Looking at the transformation law of $G_{\mu\nu}$ we see that the choice $G_{\mu\nu} = g_{\mu\nu}$, that is the noncommutative metric equals the classical metric, is consistent with (5.3.9). The condition (5.3.10) is automatically fulfilled and we obtain the θ-independent metric tensor. Our final aim is the construction of the deformed Einstein-Hilbert action. Varying this action with respect to the metric one should obtain deformed equations of motion. By solving these equations we should obtain the noncommutative metric in terms of the classical one and the θ-dependent corrections. Therefore, starting with the commutative metric and saying later that it becomes θ-dependent might look a little odd[8]. Instead, one can choose from the beginning a θ-dependent metric tensor. Then one expands it in orders of the deformation parameter θ

$$G_{\mu\nu} = g_{\mu\nu} + G^1_{\mu\nu} + \dots, \tag{5.3.11}$$

where $G^1_{\mu\nu}$ is the first order correction which one calculates by solving the equations of motion.

[8] However, this is just the problem of interpretation. Starting with the classical fields and finding out later that they have to have θ-dependent corrections is normally done in the framework of the Seiberg-Witten map, see Chapter 3 and Chapter 4.

On the other hand, we remember that the classical metric tensor can be expressed in terms of the vierbein $e_\mu{}^a$

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b, \tag{5.3.12}$$

where η_{ab} is the flat Minkowski metric and a and b are local Lorentz indices. This we generalise to the noncommutative metric tensor

$$G_{\mu\nu} = \frac{1}{2}\Big(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^a \star E_\mu{}^b\Big)\eta_{ab}, \tag{5.3.13}$$

where $E_\mu{}^a$ is the noncommutative vierbein. In order to fulfil (5.3.9), $E_\mu{}^a$ has to transform as a vector field (5.3.1) and the coproduct (5.2.19) has to be used. Because of (5.3.10) in the limit $\theta \to 0$ it has to reduce to the classical vierbein

$$E_\mu{}^a = e_\mu{}^a + E_\mu{}^{a\,1} + \dots. \tag{5.3.14}$$

Note that one can also start with the classical vierbein (it is consistent with both (5.3.9) and (5.3.10)) and after solving the equations of motion obtain that it becomes θ-dependent. The arguments pro and contra are the same as for choosing $g_{\mu\nu}$ as the noncommutative metric and we do not repeat them.

For the moment we do not specify the metric tensor. Instead, we look at the inverse metric. Starting with the noncommutative metric tensor $G_{\mu\nu}$, one can introduce two inverses. The inverse with respect to the pointwise multiplication (classical inverse) we denote by $G^{\mu\nu}$

$$G_{\mu\nu} \cdot G^{\nu\rho} = \delta_\mu^\rho, \tag{5.3.15}$$

and the inverse with respect to the $\star$-multiplication with $G^{\mu\nu\star}$

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta_\mu^\rho. \tag{5.3.16}$$

Expanding $G^{\nu\rho\star}$ in the deformation parameter θ and inserting the expansion in (5.3.16) gives the $\star$-inverse in terms of the classical inverse

$$\begin{aligned} G^{\mu\nu\star} &= G^{\mu\nu} + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho G^{\mu\alpha})(\partial_\sigma G_{\alpha\beta})G^{\beta\nu} \\ &= 2G^{\mu\nu} - G^{\mu\alpha} \star G_{\alpha\beta} \star G^{\beta\nu}. \end{aligned} \tag{5.3.17}$$

This result is valid up to first order in θ. The exact result will of course depend on the choice of $G_{\mu\nu}$. From (5.3.16), using the comultiplication (5.2.19), it follows that $G^{\mu\nu\star}$ transforms like a tensor of rank two

$$\delta_\xi G^{\mu\nu\star} = -X^\star_\xi \triangleright G^{\mu\nu\star} + X^\star_{(\partial_\rho\xi^\mu)} \triangleright G^{\rho\nu\star} + X^\star_{(\partial_\rho\xi^\nu)} \triangleright G^{\mu\rho\star}. \tag{5.3.18}$$

Although $G_{\mu\nu}$ is a symmetric tensor, its $\star$-inverse is not symmetric

$$G^{\mu\nu\star} \neq G^{\nu\mu\star}. \tag{5.3.19}$$

5.4 Curvature and torsion

In this section we define geometrical objects like curvature tensor, torsion,.... They do not have the geometrical interpretation like in the commutative case, but we use them to obtain the deformed Einstein-Hilbert action in the next section.

5.4.1 Covariant derivative

Let us proceed as in the commutative case, by observing that the partial derivative of a vector field transforms as

$$\begin{aligned}\delta_\xi(\partial_\mu V_\nu) &= (\partial_\mu \delta_\xi V_\nu) \\ &= -X^\star_\xi \star\triangleright(\partial_\mu V_\nu) - X^\star_{(\partial_\mu\xi^\lambda)} \triangleright (\partial_\lambda V_\nu) - X^\star_{(\partial_\nu\xi^\lambda)} \triangleright (\partial_\mu V_\lambda) - X^\star_{(\partial_\mu\partial_\nu\xi^\lambda)} \triangleright V_\lambda \\ &= -\xi^\lambda\partial_\lambda(\partial_\mu V_\nu) - (\partial_\mu\xi^\lambda)(\partial_\lambda V_\nu) - (\partial_\nu\xi^\lambda)(\partial_\mu V_\lambda) - (\partial_\mu\partial_\nu\xi^\lambda)V_\lambda. \qquad (5.4.1)\end{aligned}$$

Here we have used

$$(\partial_\mu X^\star_\xi) = \sum_n \left(\frac{-i}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}\partial_\mu\xi^\lambda\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n}\partial_\lambda = X_{(\partial_\mu\xi^\lambda)}\partial_\lambda \qquad (5.4.2)$$

and similarly $\partial_\mu X_{(\partial_\nu\xi^\lambda)} = X_{(\partial_\mu\partial_\nu\xi^\lambda)}$. Because of the last term in (5.4.1) this is not the transformation law of a tensor. To repair this we introduce the covariant derivative

$$D_\mu V_\nu = (\partial_\mu V_\nu) - \Gamma^\alpha_{\mu\nu}\star V_\alpha, \qquad (5.4.3)$$

where $\Gamma^\alpha_{\mu\nu}$ is the noncommutative connection. From the demand that (5.4.3) transforms as a tensor of rank two

$$\delta_\xi(D_\mu V_\nu) = -X^\star_\xi \star\triangleright(D_\mu V_\nu) - X^\star_{(\partial_\mu\xi^\lambda)} \triangleright (D_\lambda V_\nu) - X^\star_{(\partial_\nu\xi^\lambda)} \triangleright (D_\mu V_\lambda) \qquad (5.4.4)$$

one calculates the transformation law of the connection $\Gamma^\alpha_{\mu\nu}$

$$\delta_\xi(D_\mu V_\nu) = \partial_\mu(\delta_\xi V_\nu) - \delta_\xi(\Gamma^\alpha_{\mu\nu}\star V_\alpha). \qquad (5.4.5)$$

To calculate the last term in (5.4.5) we have to use the coproduct[9] (5.2.19). We write this term explicitly

$$\begin{aligned}\delta_\xi(\Gamma^\alpha_{\mu\nu}\star V_\alpha) &= (\delta_\xi\Gamma^\alpha_{\mu\nu})\star V_\alpha + \Gamma^\alpha_{\mu\nu}\star(\delta_\xi V_\alpha) \\ &\quad -\frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{(\partial_\rho\xi)}\Gamma^\alpha_{\mu\nu})(\partial_\sigma V_\alpha) + (\partial_\rho\Gamma^\alpha_{\mu\nu})(\delta_{(\partial_\sigma\xi)}V_\alpha)\Big). \qquad (5.4.6)\end{aligned}$$

To calculate $\delta_\xi\Gamma^\alpha_{\mu\nu}$ we proceed in the usual way. We expand $\delta_\xi\Gamma^\alpha_{\mu\nu}$ in orders of the deformation parameter and expand all $\star$-products in (5.4.6). This gives

$$\delta_\xi\Gamma^\alpha_{\mu\nu} = -X^\star_\xi\triangleright\Gamma^\alpha_{\mu\nu} - X^\star_{(\partial_\mu\xi^\lambda)}\triangleright\Gamma^\alpha_{\lambda\nu} - X^\star_{(\partial_\nu\xi^\lambda)}\triangleright\Gamma^\alpha_{\mu\lambda} + X^\star_{(\partial_\lambda\xi^\alpha)}\triangleright\Gamma^\lambda_{\mu\nu} - \partial_\mu\partial_\nu\xi^\alpha. \qquad (5.4.7)$$

Expanding all $\star$-products and the operators $X^\star$ gives the classical transformation law, as expected.

In analogy with (5.4.3) one defines the covariant derivative of a contravariant vector and of an arbitrary tensor

$$D_\mu V^\nu = (\partial_\mu V^\nu) + \Gamma^\nu_{\mu\alpha}\star V^\alpha, \qquad (5.4.8)$$

$$\begin{aligned}D_\lambda T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_p} &= (\partial_\lambda T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_p}) - \Gamma^\alpha_{\lambda\mu_1}\star T^{\nu_1\dots\nu_r}_{\alpha\mu_2\dots\mu_p} - \dots - \Gamma^\alpha_{\lambda\mu_p}\star T^{\nu_1\dots\nu_r}_{\mu_1\dots\mu_{p-1}\alpha} \\ &\quad +\Gamma^{\nu_1}_{\lambda\alpha}\star T^{\alpha\nu_2\dots\nu_r}_{\mu_1\dots\mu_p} + \dots + \Gamma^{\nu_r}_{\lambda\alpha}\star T^{\nu_1\dots\nu_{r-1}\alpha}_{\mu_1\dots\mu_p}. \qquad (5.4.9)\end{aligned}$$

[9]This step is not straightforward, since the coproduct applies to tensors, but we know that $\Gamma^\alpha_{\mu\nu}$ is not a tensor.

5.4.2 Curvature tensor, Ricci tensor and scalar curvature

The $\star$-commutator of two covariant derivatives applied on a vector field gives the curvature tensor and torsion

$$[D_\mu \stackrel{\star}{,} D_\nu] \star V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T^\alpha_{\mu\nu} \star D_\alpha V_\rho. \tag{5.4.10}$$

By using (5.4.9) one obtains

$$\begin{aligned} D_\mu D_\nu V_\rho &= (\partial_\mu(D_\nu V_\rho)) - \Gamma^\alpha_{\mu\nu} \star D_\alpha V_\rho - \Gamma^\alpha_{\mu\rho} \star D_\nu V_\alpha \\ &= \dots \end{aligned}$$

and finally

$$R_{\mu\nu\rho}{}^\sigma = (\partial_\nu \Gamma^\sigma_{\mu\rho}) - (\partial_\mu \Gamma^\sigma_{\nu\rho}) + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \tag{5.4.11}$$

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}. \tag{5.4.12}$$

From (5.4.11) it follows

$$R_{\mu\nu\rho}{}^\sigma = -R_{\nu\mu\rho}{}^\sigma \tag{5.4.13}$$

like in the commutative case, but

$$R_{\mu\nu\rho\sigma} \stackrel{\text{def}}{=} R_{\mu\nu\rho}{}^\alpha \star G_{\alpha\sigma} \neq R_{\mu\nu\sigma\rho}, \tag{5.4.14}$$

$$R_{\mu\nu\rho\sigma} \neq R_{\rho\sigma\mu\nu}. \tag{5.4.15}$$

This is a consequence of having the $\star$-product in (5.4.11).

The Ricci tensor is defined as

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma. \tag{5.4.16}$$

Contracting the first and the fourth index gives the same result because of (5.4.13). Unlike in the classical case, here it is also possible to contract the third and the fourth index since the curvature tensor is not antisymmetric with respect to these two indices. However, the commutative limit of this result[10] will not give the commutative Ricci tensor, so we do not consider this possibility. From this analysis it follows that we can define the Ricci tensor uniquely. One should also note that it is not symmetric

$$R_{\mu\nu} \neq R_{\nu\mu}. \tag{5.4.17}$$

However, there are more possible definitions of the scalar curvature. Some of them are

$$R = G^{\mu\nu\star} \star R_{\nu\mu}, \tag{5.4.18}$$

$$R = R_{\nu\mu} \star G^{\mu\nu\star}, \tag{5.4.19}$$

$$R = \frac{1}{2}\big(G^{\mu\nu\star} \star R_{\nu\mu} + R_{\nu\mu} \star G^{\mu\nu\star}\big). \tag{5.4.20}$$

We choose (5.4.18) to be our working definition, but one should keep in mind that there are other possibilities.

[10]In the deformed case from (5.4.14) we have $R_{\mu\nu\sigma}{}^\sigma = \mathcal{O}(\theta)$ and in the limit $\theta \to 0$, $R_{\mu\nu\sigma}{}^\sigma \to 0$.

Finally, from (5.4.12) we see that if the connection is symmetric, the torsion vanishes. In the following we analyse only the torsion-free case, that is

$$\Gamma_{\nu\mu}^{\alpha} = \Gamma_{\mu\nu}^{\alpha}. \tag{5.4.21}$$

In order to relate the connection with the metric tensor in the commutative case one imposes the metricity condition (5.1.18). We generalise this construction to the θ-deformed case. Namely, we demand that

$$D_\alpha G_{\beta\gamma} = (\partial_\alpha G_{\beta\gamma}) - \Gamma_{\alpha\beta}^{\rho} \star G_{\rho\gamma} - \Gamma_{\alpha\gamma}^{\rho} \star G_{\beta\rho} = 0. \tag{5.4.22}$$

By writing this equation two more times, cyclicly permuting the indices and adding all three equations we obtain

$$2\Gamma_{\alpha\beta}^{\rho} \star G_{\rho\gamma} = (\partial_\alpha G_{\beta\gamma}) + (\partial_\beta G_{\alpha\gamma}) - (\partial_\gamma G_{\alpha\beta}). \tag{5.4.23}$$

Now it is clear why we insisted on having the $\star$-inverse of $G_{\mu\nu}$. Using the classical inverse we can not extract $\Gamma_{\alpha\beta}^{\rho}$ from (5.4.23), we have to use the $\star$-inverse. Then the unique result follows

$$\Gamma_{\alpha\beta}^{\sigma} = \frac{1}{2}\Big((\partial_\alpha G_{\beta\gamma}) + (\partial_\beta G_{\alpha\gamma}) - (\partial_\gamma G_{\alpha\beta})\Big) \star G^{\gamma\sigma\star}. \tag{5.4.24}$$

To obtain this result we have used that the metric tensor and connection are symmetric. In analogy with the commutative case, we call the connection (5.4.24) Christoffel symbol. Using the transformation properties of $G_{\mu\nu}$ and $G^{\mu\nu\star}$, (5.3.9) and (5.3.18) respectively, and the coproduct (5.2.19), from (5.4.24) the transformation law (5.4.7) of the Christoffel symbol follows.

Using the result (5.4.24) one expresses the curvature tensor, Ricci tensor and scalar curvature in terms of the metric tensor and its inverse.

5.5 Deformed Einstein-Hilbert action

In the commutative case, the Einstein-Hilbert action is given by (5.1.21). In the following we generalise this to the θ-deformed space. Our aim is to construct an action invariant under the deformed diffeomorphisms which in the zeroth order limit reduces to (5.1.21).

We need an integral with the cyclic property (see also Section 4.1),

$$\int \mathrm{d}^4x\, (f_1 \star f_2 \star \cdots \star f_k) = \int \mathrm{d}^4x\, (f_k \star f_1 \star f_2 \star \cdots \star f_{k-1}). \tag{5.5.1}$$

Fortunately, the θ-deformed space is simple enough and the usual commutative integral has this property. In the previous section we obtained the scalar curvature, so the only thing left to generalise is the density $\sqrt{-g}$.

We need a $\star$-density $E^\star$ that transforms like

$$\delta_\xi E^\star = -X_\xi^\star \triangleright E^\star - X_{(\partial_\lambda \xi^\lambda)}^\star \triangleright E^\star. \tag{5.5.2}$$

This gives

$$\delta_\xi (E^\star \star R) = -\partial_\mu^\star \triangleright \Big(X_{\xi^\mu}^\star \triangleright (E^\star \star R)\Big) \tag{5.5.3}$$

and the action

$$S = \int \mathrm{d}^4x\, E^\star \star R \tag{5.5.4}$$

is invariant under the deformed diffeomorphisms

$$\delta_\xi\Big(\int \mathrm{d}^4x\, E^\star \star R\Big) = 0. \tag{5.5.5}$$

The problem with this so far undetermined $\star$-density is that the transformation law (5.5.2) does not give enough conditions to fix $E^\star$ uniquely. Adding the requirement of the proper commutative limit does not help. For example, we can take

$$E^\star = \sqrt{-g}, \tag{5.5.6}$$

that is the classical object. It transforms like (5.5.2) and (of course) has the good commutative limit. Also, following the arguments from Section 5.2, one can take

$$E^\star = \sqrt{-G}, \quad G = \det G_{\mu\nu} = \det(g_{\mu\nu} + G^1_{\mu\nu} + \dots), \tag{5.5.7}$$

where $G^1_{\mu\nu}$ is the first order correction to the noncommutative metric tensor.

Let us consider this possibility in more detail. The deformed Einstein-Hilbert action is then

$$S = \int \mathrm{d}^4x\, \sqrt{-G} \star R \tag{5.5.8}$$

and we suppose that R is expressed in terms of $G_{\mu\nu}$ and its inverse using (5.4.11), (5.4.18) and (5.4.24). Varying the action (5.5.8) with respect to $G_{\mu\nu}$ leads to

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\, \Big((\delta\sqrt{-G}) \star R + \sqrt{-G} \star (\delta R)\Big)\\ &= \int \mathrm{d}^4x\, \Big((\frac{1}{2}\sqrt{-G}G^{\mu\nu}(\delta G_{\nu\mu})) \star R + \sqrt{-G} \star (\delta R)\Big).\end{aligned} \tag{5.5.9}$$

Here we have used that

$$\delta G = \delta(\det(G_{\mu\nu})) = G G^{\mu\nu}(\delta G_{\nu\mu}). \tag{5.5.10}$$

The problem with (5.5.9) is that in the first term we have a mixture of the-$\star$-product and the usual pointwise product. This makes it difficult (impossible) to write (5.5.9) in the following form

$$\delta S = \int \mathrm{d}^4x\, \delta G_{\mu\nu} \star \Big(\dots\Big). \tag{5.5.11}$$

In order to have $\star$-product everywhere one should know how to define $(\sqrt{-G})_\star$. Unfortunately, the square root is a concept which is hard to generalise, so we have to try something else.

The classical action (5.1.21) can be written in a different way

$$S = \int \mathrm{d}^4x\, eR^0, \tag{5.5.12}$$

where $e = \det e_\mu{}^a$ is the determinant of the classical vierbein. In Section 5.2 we have already introduced the noncommutative vierbein and we have to generalise the concept of a determinant. This is not too difficult, we define the $\star$-determinant as

$$E^\star = \det{}_\star E_\mu{}^a = \frac{1}{4!}\varepsilon^{\mu_1\ldots\mu_4}\varepsilon_{a_1\ldots a_4}E_{\mu_1}^{\;a_1}\star\cdots\star E_{\mu_4}^{\;a_4}, \tag{5.5.13}$$

where $\varepsilon^{\mu_1\ldots\mu_4}$ is the totally antisymmetric tensor of rank 4. By using the comultiplication (5.2.19) one checks that (5.5.13) has the right transformation property (5.5.2) and (5.3.14) ensures the good commutative limit.

Finally, the deformed Einstein-Hilbert action we define as

$$\begin{aligned} S &= \int \mathrm{d}^4x\, (E^\star \star R + \text{ c.c. }), \\ &= \int \mathrm{d}^4x\, (E^\star \star R + \bar{R}\star\overline{E^\star}). \end{aligned} \tag{5.5.14}$$

In order to have a real action we added the complex conjugated part also. The action (5.5.14) can be varied with respect to $E_\mu{}^a$ to give the equations of motion. Of course, this fixes our choice of the noncommutative metric tensor to (5.3.13) and all the quantities like $R_{\mu\nu}$, R, ... have to be expressed in terms of $E_\mu{}^a$.

5.6 Equations of motion

In this section we calculate the equations of motion coming from the action (5.5.14). We vary the action only with respect to $E_\mu{}^a$, equations coming from the variation with respect to $\bar{E}_\mu{}^a$ are the complex conjugate of the equation coming from variation with respect to $E_\mu{}^a$. Therefore, we ignore the second term in (5.5.14),

$$\begin{aligned} \delta S &= \int \mathrm{d}^4x\, \Big((\delta E^\star)\star R + E^\star\star(\delta R)\Big) \\ &= I + II. \end{aligned} \tag{5.6.1}$$

The first term is not difficult to calculate, it follows from (5.5.13)

$$\begin{aligned} I &= \int \mathrm{d}^4x\, (\delta E^\star)\star R \\ &= \frac{1}{4!}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{a_1a_2a_3a_4}\int \mathrm{d}^4x\, (\delta E_{\mu_1}^{\;a_1})\star\Big(E_{\mu_2}^{\;a_2}\star E_{\mu_3}^{\;a_3}\star E_{\mu_4}^{\;a_4}\star R \\ &\qquad + E_{\mu_3}^{\;a_3}\star E_{\mu_4}^{\;a_4}\star R\star E_{\mu_2}^{\;a_2} + E_{\mu_4}^{\;a_4}\star R\star E_{\mu_2}^{\;a_2}\star E_{\mu_3}^{\;a_3} + R\star E_{\mu_2}^{\;a_2}\star E_{\mu_3}^{\;a_3}\star E_{\mu_4}^{\;a_4}\Big). \end{aligned} \tag{5.6.2}$$

The second term is more complicated and we do it in a few steps. We write it in terms of the variation of the metric tensor,

$$\begin{aligned} II &= \int \mathrm{d}^4x\, E^\star\star(\delta R) = \ldots \\ &= \int \mathrm{d}^4x\, (\delta G_{\alpha\beta})\star\mathcal{M}^{\alpha\beta}, \end{aligned} \tag{5.6.3}$$

where $\mathcal{M}^{\alpha\beta}$ is a complicated function of $G_{\mu\nu}$ (or of $E_\mu{}^a$) which we have to calculate. Finally, one inserts $\delta G_{\alpha\beta}$ in terms of $\delta E_\mu{}^a$ which follows from (5.3.13).

From (5.3.16) and (5.4.24) it follows

$$\begin{aligned} \delta G^{\mu\nu\star} &= -G^{\mu\alpha\star} \star (\delta G_{\alpha\beta}) \star G^{\beta\nu\star}, && (5.6.4)\\ \delta\Gamma_{\mu\nu}^{\ \ \alpha} &= \frac{1}{2}\Big((\partial_\mu \delta G_{\nu\gamma}) + (\partial_\nu \delta G_{\mu\gamma}) - (\partial_\gamma \delta G_{\mu\nu})\Big) \star G^{\gamma\alpha\star} - \Gamma_{\mu\nu}^{\ \ \lambda} \star (\delta G_{\lambda\omega}) \star G^{\omega\alpha\star}. && (5.6.5) \end{aligned}$$

By using this results we obtain

$$\begin{aligned} \delta R = &-G^{\mu\alpha\star} \star (\delta G_{\alpha\beta}) \star G^{\beta\nu\star} \star R_{\nu\mu} + G^{\mu\nu\star} \star \Big((\partial_\alpha \delta\Gamma_{\mu\nu}^{\ \ \alpha}) - (\partial_\nu \delta\Gamma_{\mu\alpha}^{\ \ \alpha})\\ &+(\delta\Gamma_{\alpha\mu}^{\ \ \beta}) \star \Gamma_{\beta\nu}^{\ \ \alpha} + \Gamma_{\alpha\mu}^{\ \ \beta} \star (\delta\Gamma_{\beta\nu}^{\ \ \alpha}) - (\delta\Gamma_{\mu\nu}^{\ \ \beta}) \star \Gamma_{\beta\alpha}^{\ \ \alpha} - \Gamma_{\mu\nu}^{\ \ \beta} \star (\delta\Gamma_{\beta\alpha}^{\ \ \alpha})\Big). \qquad (5.6.6) \end{aligned}$$

Inserting this into (5.6.3) gives (after some cyclic permutation and partial integration)

$$\begin{aligned} II = \int \mathrm{d}^4x\, (\delta G_{\beta\gamma}) \star \Bigg(&- G^{\gamma\nu\star} \star R_{\nu\mu} \star E^\star \star G^{\mu\beta\star} + G^{\gamma\alpha\star} \star E^\star \star G^{\mu\nu\star} \star R_{\alpha\nu\mu}{}^\beta\\ &+\Big(G^{\gamma\lambda\star} \star \Gamma_{\lambda\alpha}^{\ \ \alpha} - (\partial_\alpha G^{\gamma\alpha\star})\Big) \star E^\star \star G^{\mu\nu\star} \star \Gamma_{\mu\nu}^{\ \ \beta}\\ &-\Big(G^{\gamma\lambda\star} \star \Gamma_{\lambda\nu}^{\ \ \alpha} - (\partial_\nu G^{\gamma\alpha\star})\Big) \star E^\star \star G^{\mu\nu\star} \star \Gamma_{\alpha\mu}^{\ \ \beta}\\ &+\frac{1}{2}\partial_\alpha\partial_\mu\Big(G^{\gamma\alpha\star} \star E^\star \star G^{\mu\beta\star} - G^{\mu\alpha\star} \star E^\star \star G^{\gamma\beta\star}\\ &\qquad -G^{\gamma\beta\star} \star E^\star \star G^{\mu\alpha\star} + G^{\mu\beta\star} \star E^\star \star G^{\gamma\alpha\star}\Big)\\ &-\frac{1}{2}\partial_\mu\Big(\big((\partial_\alpha G^{\gamma\alpha\star}) - G^{\gamma\lambda\star} \star \Gamma_{\lambda\alpha}^{\ \ \alpha}\big) \star E^\star \star (G^{\beta\mu\star} + G^{\mu\beta\star})\\ &\qquad -\big((\partial_\alpha G^{\mu\alpha\star}) - G^{\mu\lambda\star} \star \Gamma_{\lambda\alpha}^{\ \ \alpha}\big) \star E^\star \star G^{\gamma\beta\star}\\ &\qquad -\big((\partial_\lambda G^{\gamma\mu\star}) - G^{\gamma\alpha\star} \star \Gamma_{\lambda\alpha}^{\ \ \mu}\big) \star E^\star \star G^{\beta\lambda\star}\\ &\qquad -\big((\partial_\lambda G^{\gamma\beta\star}) - G^{\gamma\alpha\star} \star \Gamma_{\lambda\alpha}^{\ \ \beta}\big) \star E^\star \star G^{\mu\nu\star}\\ &\qquad -\big((\partial_\lambda G^{\mu\beta\star}) - G^{\mu\alpha\star} \star \Gamma_{\lambda\alpha}^{\ \ \beta}\big) \star E^\star \star G^{\gamma\lambda\star}\\ &\qquad -\big(3G^{\gamma\mu\star} - G^{\mu\gamma\star}\big) \star E^\star \star G^{\alpha\lambda\star} \star \Gamma_{\lambda\alpha}^{\ \ \beta}\\ &\qquad -G^{\mu\alpha\star} \star E^\star \star G^{\lambda\gamma\star} \star \Gamma_{\lambda\alpha}^{\ \ \beta} - G^{\gamma\beta\star} \star E^\star \star G^{\alpha\lambda\star} \star \Gamma_{\lambda\alpha}^{\ \ \mu}\\ &\qquad +G^{\gamma\alpha\star} \star E^\star \star \big(G^{\lambda\mu\star} \star \Gamma_{\lambda\alpha}^{\ \ \beta} + G^{\lambda\beta\star} \star \Gamma_{\lambda\alpha}^{\ \ \mu}\big)\Big)\Bigg) \qquad (5.6.7)\\ \stackrel{\text{def}}{=} \int \mathrm{d}^4x\, &(\delta G_{\beta\gamma}) \star \mathcal{M}^{\beta\gamma}. \end{aligned}$$

Inserting

$$\delta G_{\alpha\beta} = \frac{1}{2}\eta_{ab}\Big((\delta E_\alpha{}^a) \star E_\beta{}^b + E_\alpha{}^a \star (\delta E_\beta{}^b) + (\delta E_\beta{}^a) \star E_\alpha{}^b + E_\beta{}^a \star (\delta E_\alpha{}^b)\Big) \qquad (5.6.8)$$

into (5.6.7) and adding (5.6.2) to it, gives finally

$$\delta S = \int \mathrm{d}^4x\, (\delta E_\beta{}^a) \star \Big(\frac{1}{4!}\epsilon^{\beta\mu_2\mu_3\mu_4}\epsilon_{aa_2a_3a_4}\Big(E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4} \star R + E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4} \star R \star E_{\mu_2}^{\ a_2} + E_{\mu_4}^{\ a_4} \star R \star E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} + R \star E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4}\Big) + \frac{1}{2}\eta_{ab}\Big(E_\gamma{}^b \star (\mathcal{M}^{\beta\gamma} + \mathcal{M}^{\gamma\beta}) + (\mathcal{M}^{\beta\gamma} + \mathcal{M}^{\gamma\beta}) \star E_\gamma{}^b\Big)\Big). \tag{5.6.9}$$

The equation of motion is then

$$\frac{1}{4!}\epsilon^{\beta\mu_2\mu_3\mu_4}\epsilon_{aa_2a_3a_4}\Big(E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4} \star R + E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4} \star R \star E_{\mu_2}^{\ a_2} + E_{\mu_4}^{\ a_4} \star R \star E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} + R \star E_{\mu_2}^{\ a_2} \star E_{\mu_3}^{\ a_3} \star E_{\mu_4}^{\ a_4}\Big) + \frac{1}{2}\eta_{ab}\Big(E_\gamma{}^b \star (\mathcal{M}^{\beta\gamma} + \mathcal{M}^{\gamma\beta}) + (\mathcal{M}^{\beta\gamma} + \mathcal{M}^{\gamma\beta}) \star E_\gamma{}^b\Big) = 0. \tag{5.6.10}$$

Unfortunately, this result is very complicated and it is difficult to make any conclusions just by looking at it. The expansion in the deformation parameter might be of some help here, so we proceed in that way in the next section.

5.7 Expansion in the deformation parameter

In this section we expand some of the results from the previous sections up to first order in the deformation parameter θ and in terms of the classical fields, vierbein $e_\mu{}^a$, metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$[11] . We start with the basic object, the vierbein. It is given by

$$E_\mu{}^a = e_\mu{}^a + E_\mu{}^{a\,1} + \dots, \tag{5.7.1}$$

where $E_\mu{}^{a\,1}$ is linear in $\theta^{\rho\sigma}$. Note that this differs from the approach that was taken in the paper [36]. There the vierbein is taken to be the classical object, keeping in mind that after solving the equations of motion it becomes θ-dependent. Here we start from the beginning with the θ-dependent object. Using (5.7.1) and (5.5.13) one calculates $E^\star$

$$E^\star = e + \frac{1}{3!}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{a_1a_2a_3a_4}E_{\mu_1}^{\ a_1\,1}e_{\mu_2}^{\ a_2}e_{\mu_3}^{\ a_3}e_{\mu_4}^{\ a_4}. \tag{5.7.2}$$

From (5.3.13) and (5.3.16) it follows

$$G_{\mu\nu} = g_{\mu\nu} + \eta_{ab}\big(E_\mu{}^{a\,1}e_\nu{}^b + e_\mu{}^a E_\nu{}^{b\,1}\big), \tag{5.7.3}$$

$$G^{\mu\nu\star} = g^{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}g^{\mu\alpha}(\partial_\rho g_{\alpha\beta})(\partial_\sigma g^{\beta\nu}) - \eta_{ab}g^{\mu\alpha}\big(E_\alpha{}^{a\,1}e_\beta{}^b + e_\alpha{}^a E_\beta{}^{b\,1}\big)g^{\beta\nu}. \tag{5.7.4}$$

[11]Since one can express $g_{\mu\nu}$ and $g^{\mu\nu}$ in terms of $e_\mu{}^a$, what we obtain might not be the final form for the results.

For the Christoffel symbol from (5.4.24) we obtain

$$\begin{aligned}
\Gamma_{\mu\nu}^{\alpha} &= \Gamma_{\mu\nu}^{0\,\alpha} + \frac{i}{4}\theta^{\rho\sigma}\big(\partial_\rho\partial_\mu g_{\gamma\nu} + \partial_\rho\partial_\nu g_{\gamma\mu} - \partial_\rho\partial_\gamma g_{\mu\nu}\big)(\partial_\sigma g^{\gamma\alpha}) \\
&\quad -\frac{i}{4}\Gamma_{\mu\nu}^{0\,\gamma}\Big(\theta^{\rho\sigma}(\partial_\rho g_{\gamma\omega})(\partial_\sigma g^{\omega\alpha}) - 2i\eta_{ab}\big(E_\gamma{}^{a\,1}e_\omega{}^b + e_\gamma{}^a E_\omega{}^{b\,1}\big)g^{\omega\alpha}\Big) \\
&\quad +\eta_{ab}\Big(\partial_\mu\big(E_\nu{}^{a\,1}e_\gamma{}^b + e_\nu{}^a E_\gamma{}^{b\,1}\big) \\
&\qquad +\partial_\nu\big(E_\mu{}^{a\,1}e_\gamma{}^b + e_\mu{}^a E_\gamma{}^{b\,1}\big) - \partial_\gamma\big(E_\mu{}^{a\,1}e_\nu{}^b + e_\mu{}^a E_\nu{}^{b\,1}\big)\Big)g^{\gamma\alpha} \\
&= \Gamma_{\mu\nu}^{0\,\alpha} + \Gamma_{\mu\nu}^{\ \alpha\,1}.
\end{aligned} \tag{5.7.5}$$

We see that already this result is long and not very readable. Therefore, we just give the implicit result for the curvature tensor

$$\begin{aligned}
R_{\mu\nu\alpha}{}^{\beta} &= R^{0}_{\mu\nu\alpha}{}^{\beta} + (\partial_\nu\Gamma_{\mu\alpha}^{\beta\,1}) - (\partial_\mu\Gamma_{\nu\alpha}^{\beta\,1}) + \frac{i}{2}\theta^{\rho\sigma}\big((\partial_\rho\Gamma_{\nu\alpha}^{0\,\lambda})(\partial_\sigma\Gamma_{\lambda\mu}^{0\,\beta}) - (\partial_\rho\Gamma_{\mu\alpha}^{0\,\lambda})(\partial_\sigma\Gamma_{\lambda\nu}^{0\,\beta})\big) \\
&\quad +\Gamma_{\nu\alpha}^{\lambda\,1}\Gamma_{\lambda\mu}^{0\,\beta} + \Gamma_{\nu\alpha}^{0\,\lambda}\Gamma_{\lambda\mu}^{\beta\,1} - \Gamma_{\mu\alpha}^{\lambda\,1}\Gamma_{\lambda\nu}^{0\,\beta} - \Gamma_{\mu\alpha}^{0\,\lambda}\Gamma_{\lambda\nu}^{\beta\,1}.
\end{aligned} \tag{5.7.6}$$

One can continue like this and calculate $R_{\mu\nu}$ and R in terms of the classical fields and corrections. This results can be inserted into the equation of motion (5.6.10) obtained in the previous section. Solving that equation one finds the corrections to the classical vierbein and sees how the noncommutativity influences the classical solutions. However, we are not going to do these calculations here, they will be the subject of future research.

5.8 The θ-deformed Poincaré algebra

In this section we consider one special example of the deformed diffeomorphisms. Namely, we take the vector field ξ to be linear in the coordinates

$$\xi_\omega = x^\mu\omega_\mu{}^\lambda\partial_\lambda, \tag{5.8.1}$$

where $\omega_{\mu\nu}$ is a constant antisymmetric matrix. One obtains in this way the global θ-deformed Lorentz transformations.

The transformation law of a scalar field is

$$\delta_\omega\phi = -X_\omega^\star \triangleright \phi = -x^\mu\omega_\mu{}^\lambda(\partial_\lambda\phi), \tag{5.8.2}$$

where[12]

$$X_\omega^\star = x^\mu\omega_\mu{}^\lambda\partial_\lambda^\star - \frac{i}{2}\theta^{\rho\sigma}\omega_\rho{}^\lambda\partial_\lambda^\star\partial_\sigma^\star. \tag{5.8.3}$$

Since ξ_ω is linear in x, formula (5.8.3) is already the exact expression to all orders in θ. We rewrite (5.8.2) in a more familiar way

$$\delta_\omega\phi = -\frac{1}{2}\omega^{\alpha\beta}L_{\alpha\beta}\phi, \tag{5.8.4}$$

[12]In this section we write $\partial_\mu^\star$ to stress that the derivatives are the elements of the θ-deformed Poincaré algebra. However, one can also use the ∂_μ notation, results are the same.

where $L_{\alpha\beta}$ is the orbital part of the Lorentz generator $M_{\alpha\beta}$ given by

$$\begin{aligned} L_{\alpha\beta} &= x_\alpha \partial_\beta - x_\beta \partial_\alpha, \\ &= x_\alpha \star \partial^\star_\beta - x_\beta \star \partial^\star_\alpha + \frac{i}{2}\theta^{\rho\sigma}\big(\eta_{\rho\alpha}\partial^\star_\beta - \eta_{\rho\beta}\partial^\star_\alpha\big)\partial^\star_\sigma. \end{aligned} \tag{5.8.5}$$

The first line of (5.8.5) tells us that $L_{\alpha\beta}$ is the same like in the classical case. In the second line this result is rewritten in terms of the $\star$-product such that it also has a meaning in the abstract algebra[13] . In order to include the spin part of $M_{\alpha\beta}$ we look at the transformation law of a vector field

$$\begin{aligned} \delta_\omega V_\mu &= -X^\star_\omega \triangleright V_\mu - X^\star_{\omega_\mu{}^\alpha} \triangleright V_\alpha \\ &= -\frac{1}{2}\omega^{\alpha\beta}(x_\alpha\partial_\beta - x_\beta\partial_\alpha)V_\mu - \frac{1}{2}\omega^{\alpha\beta}(\eta_{\mu\alpha}V_\beta - \eta_{\mu\beta}V_\alpha) \\ &= -\frac{1}{2}\omega^{\alpha\beta}(L_{\alpha\beta} + \Sigma_{\alpha\beta})V_\mu, \end{aligned} \tag{5.8.6}$$

where $\Sigma_{\alpha\beta}$ is the constant matrix in the index space of fields. Again, the obtained result is equal to the classical one.

From (5.2.13) it follows

$$[\delta_\omega, \delta_{\omega'}] = \delta_{[\omega,\omega']}, \tag{5.8.7}$$

or in terms of the generators $M_{\alpha\beta}$

$$[M_{\rho\sigma}, M_{\alpha\beta}] = \eta_{\rho\beta}M_{\sigma\alpha} + \eta_{\sigma\alpha}M_{\rho\beta} - \eta_{\rho\alpha}M_{\sigma\beta} - \eta_{\sigma\beta}M_{\rho\alpha}. \tag{5.8.8}$$

If derivatives are included as well,

$$[\delta_\omega, \partial^\star_\mu] = \omega_\mu{}^\alpha \partial^\star_\alpha, \tag{5.8.9}$$

or

$$[M_{\alpha\beta}, \partial^\star_\mu] = \eta_{\mu\alpha}\partial^\star_\beta - \eta_{\mu\beta}\partial^\star_\alpha \quad \text{and} \quad [\partial^\star_\mu, \partial^\star_\nu] = 0 \tag{5.8.10}$$

we see that the algebra sector of the θ-deformed Poincaré transformations is undeformed.

Let us now look at the coprodut for this transformations. From (5.2.19) it follows

$$\Delta\delta_\omega = \delta_\omega \otimes 1 + 1 \otimes \delta_\omega - \frac{i}{2}\theta^{\rho\sigma}\Big((\partial^\star_\rho \star \delta_\omega) \otimes \partial^\star_\sigma + \partial^\star_\rho \otimes (\partial^\star_\sigma \star \delta_\omega)\Big), \tag{5.8.11}$$

or

$$\Delta M_{\alpha\beta} = M_{\alpha\beta} \otimes 1 + 1 \otimes M_{\alpha\beta} + \frac{i}{2}\theta^{\rho\sigma}\Big((\eta_{\rho\alpha}\partial^\star_\beta - \eta_{\rho\beta}\partial^\star_\alpha) \otimes \partial^\star_\sigma + \partial^\star_\rho \otimes (\eta_{\sigma\alpha}\partial^\star_\beta - \eta_{\sigma\beta}\partial^\star_\alpha)\Big). \tag{5.8.12}$$

[13]In the abstract algebra $L_{\alpha\beta}$ reads

$$L_{\alpha\beta} = \hat{x}_\alpha\hat{\partial}_\beta - \hat{x}_\beta\hat{\partial}_\alpha + \frac{i}{2}\theta^{\rho\sigma}(\eta_{\rho\alpha}\hat{\partial}_\beta - \eta_{\rho\beta}\hat{\partial}_\alpha)\hat{\partial}_\sigma.$$

Splitting $M_{\alpha\beta}$ into orbital and spin parts gives (looking at (5.8.11) and noticing that $\Sigma_{\alpha\beta}$ is constant matrix)

$$\begin{aligned}\Delta L_{\alpha\beta} &= L_{\alpha\beta}\otimes 1 + 1\otimes L_{\alpha\beta} \\ &\quad + \frac{i}{2}\theta^{\rho\sigma}\Big((\eta_{\rho\alpha}\partial^\star_\beta - \eta_{\rho\beta}\partial^\star_\alpha)\otimes\partial^\star_\sigma + \partial^\star_\rho\otimes(\eta_{\sigma\alpha}\partial^\star_\beta - \eta_{\sigma\beta}\partial^\star_\alpha)\Big), \qquad (5.8.13)\\ \Delta\Sigma_{\alpha\beta} &= \Sigma_{\alpha\beta}\otimes 1 + 1\otimes\Sigma_{\alpha\beta}. \qquad (5.8.14)\end{aligned}$$

We see that the coproduct for the orbital part of the generator $M_{\alpha\beta}$ is deformed, while for the spin part we obtain the undeformed coproduct. The coassociativity follows from the coassociativity of (5.2.19), as well as the consistency with the algebra. Adding counits and antipods defines the θ-deformed Poncaré Hopf algebra

$$\begin{aligned}[M_{\rho\sigma}, M_{\alpha\beta}] &= \eta_{\rho\beta}M_{\sigma\alpha} + \eta_{\sigma\alpha}M_{\rho\beta} - \eta_{\rho\alpha}M_{\sigma\beta} - \eta_{\sigma\beta}M_{\rho\alpha},\\ [M_{\alpha\beta}, \partial^\star_\mu] &= \eta_{\mu\alpha}\partial^\star_\beta - \eta_{\mu\beta}\partial^\star_\alpha, \qquad [\partial^\star_\mu, \partial^\star_\nu] = 0,\\ \Delta M_{\alpha\beta} &= M_{\alpha\beta}\otimes 1 + 1\otimes M_{\alpha\beta} + \frac{i}{2}\theta^{\rho\sigma}\Big((\eta_{\rho\alpha}\partial^\star_\beta - \eta_{\rho\beta}\partial^\star_\alpha)\otimes\partial^\star_\sigma + \partial^\star_\rho\otimes(\eta_{\sigma\alpha}\partial^\star_\beta - \eta_{\sigma\beta}\partial^\star_\alpha)\Big),\\ \Delta\partial^\star_\mu &= \partial^\star_\mu\otimes 1 + 1\otimes\partial^\star_\mu,\\ \varepsilon(\partial^\star_\mu) &= 0, \qquad \varepsilon(M_{\alpha\beta}) = 0,\\ S(\partial^\star_\mu) &= -\partial^\star_\mu, \qquad S(M_{\alpha\beta}) = -M_{\alpha\beta}. \qquad (5.8.15)\end{aligned}$$

One should notice that the generators $M_{\alpha\beta}$ do not close the Hopf algebra themselves since in (5.8.12) derivatives appear. Using different approaches, this result was obtained in [37] [39], [40] also.

Having the θ-deformed Poincaré symmetry at hand, one can construct theories that are invariant under this symmetry and analyse their properties. We give one very simple example. Let us consider ϕ^3 theory

$$\mathcal{L} = \frac{1}{2}(\partial^\star_\mu\star\phi)\star(\partial^{\star\mu}\star\phi) - \frac{m^2}{2}\phi\star\phi - \lambda\phi\star\phi\star\phi. \qquad (5.8.16)$$

One checks that under (5.8.15) this Lagrangian density transforms as

$$\delta_\omega\mathcal{L} = -X^\star_\omega \triangleright \mathcal{L} = -x^\alpha\omega_\alpha{}^\lambda(\partial_\lambda\mathcal{L}). \qquad (5.8.17)$$

To construct the action we use the usual integral as in Section 5.5 and obtain

$$S = \int \mathrm{d}^4x\,\Big(\frac{1}{2}(\partial^\star_\mu\star\phi)\star(\partial^{\star\mu}\star\phi) - \frac{m^2}{2}\phi\star\phi - \lambda\phi\star\phi\star\phi\Big). \qquad (5.8.18)$$

From (5.8.17) it follows that this action is invariant. Using the variational principle as in Section 5.6

$$\begin{aligned}\delta S &= \delta\Bigg(\int \mathrm{d}^4x\,\Big(-\frac{1}{2}\phi\star(\partial^{\star\mu}\partial^\star_\mu\phi) - \frac{m^2}{2}\phi\star\phi - \lambda\phi\star\phi\star\phi\Big)\Bigg)\\ &= \int \mathrm{d}^4x\,\delta\phi(x)\star\Big(-2\frac{1}{2}(\partial^{\star\mu}\partial^\star_\mu\phi) - 2\frac{m^2}{2}\phi - 3\lambda\phi\star\phi\Big) \qquad (5.8.19)\end{aligned}$$

we obtain the following equation of motion

$$(\partial^{\star\mu}\partial^\star_\mu\phi) + m^2\phi + 3\lambda\phi\star\phi = 0. \tag{5.8.20}$$

The other way to obtain (5.8.20) (expanded in the deformation parameter) is to first expand the $\star$-products in the action (5.8.18) and then vary it with respect to the field ϕ.

Since we have an action invariant under the θ-deformed Poincaré symmetry, the next step is to look at the conserved quantities. Unfortunately, this does not seem to be straightforward. It looks as the energy momentum tensor is either conserved or symmetric and not both. This remains as an open question and will be considered in the future.

5.9 Noncommutative gauge theory, revisited

At the end of this chapter we return to the noncommutative gauge theory.

The starting point for our construction of the noncommutative gauge theory in Chapter 3 was the consistency condition for the noncommutative gauge parameter

$$\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha - i\Lambda_\alpha\star\Lambda_\beta + i\Lambda_\beta\star\Lambda_\alpha = \Lambda_{-i[\alpha,\beta]}, \tag{5.9.1}$$

or expanded up to first order in the deformation parameter

$$\begin{aligned} \theta^0: &\quad -i[\Lambda^0_\alpha,\Lambda^0_\beta] = \Lambda^0_{-i[\alpha,\beta]} \quad \Rightarrow \Lambda^0_\alpha = \alpha \\ \theta^1: &\quad \delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha - i[\Lambda^1_\alpha,\beta] - i[\alpha,\Lambda^1_\beta] - i[\alpha \overset{\star^1}{,} \beta] = \Lambda^1_{-i[\alpha,\beta]}. \end{aligned} \tag{5.9.2}$$

Because of the inhomogeneous term $-i[\alpha \overset{\star^1}{,} \beta]$ in the last equation we said that $\delta_\alpha\Lambda^1_\beta$ must be different from zero in order to cancel this term and that led to the Seiberg-Witten map construction. However, this is true only if we do not allow Λ_α to be derivative-valued, that is a differential operator. But we can try to follow the strategy from the previous sections now. Namely, one can lift the commutative gauge transformations to the $\star$-product representation as we have done for the diffeomorphisms.

The transformation

$$\delta_\alpha\psi = i\alpha\psi \tag{5.9.3}$$

can be written as

$$\delta_\alpha\psi = iX^\star_\alpha \triangleright \psi, \tag{5.9.4}$$

where

$$\begin{aligned} X^\star_\alpha &= \sum_n \Big(\frac{-i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}\alpha\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n} \\ &= \alpha - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\alpha)\partial_\sigma + \dots . \end{aligned} \tag{5.9.5}$$

By inserting $\Lambda^1_\alpha = -\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\alpha)\partial_\sigma$ in (5.9.2) we see that, although now $\delta_\alpha\Lambda^1_\beta = 0$, this equation is satisfied due to the derivative valuedness of Λ^1_α. Both

$$\Lambda^1_\alpha = -\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\alpha)\partial_\sigma \overset{\text{def}}{=} \Lambda^{1\ dv}_\alpha \tag{5.9.6}$$

$$\Lambda^1_\alpha = -\frac{1}{4}\theta^{\rho\sigma}\{A^0_\rho,\partial_\sigma\alpha\} \overset{\text{def}}{=} \Lambda^{1\ ea}_\alpha \tag{5.9.7}$$

are the first order hermitian solutions of the consistency condition (5.9.2). While (5.9.7) is enveloping algebra-valued, (5.9.6) is derivative-valued. One can always change from one to the other adding terms which are solutions of the homogeneous equation

$$\delta_\alpha \Lambda^1_\beta - \delta_\beta \Lambda^1_\alpha - i[\Lambda^1_\alpha, \beta] - i[\alpha, \Lambda^1_\beta] = \Lambda^1_{-i[\alpha,\beta]}. \tag{5.9.8}$$

One checks explicitly that

$$\Lambda^{1\ ea}_\alpha = \Lambda^{1\ dv}_\alpha + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \alpha)\mathcal{D}^0_\sigma - \frac{1}{4}\theta^{\rho\sigma}[\partial_\rho\alpha, A^0_\sigma]. \tag{5.9.9}$$

But there is an important difference. With the enveloping algebra-valued gauge parameter we have

$$\begin{aligned}\delta_\alpha(\bar\psi \star \psi) &= (\delta_\alpha \bar\psi)\star\psi + \bar\psi \star (\delta_\alpha \psi)\\ &= -i\bar\psi\star\Lambda_\alpha\star\psi + i\bar\psi\star\Lambda_\alpha\star\psi = 0,\end{aligned}$$

that is $\bar\psi\star\psi$ is invariant under the noncommutative gauge transformation. The situation is different with the derivative-valued gauge parameter

$$\begin{aligned}\delta_\alpha(\bar\psi \star \psi) &= (\delta_\alpha \bar\psi)\star\psi + \bar\psi \star (\delta_\alpha \psi)\\ &= -i(\bar\psi \triangleleft X^\star_\alpha)\star\psi + i\bar\psi\star(X^\star_\alpha \triangleright \psi),\end{aligned}$$

where $(\bar\psi \triangleleft X^\star_\alpha)$ means that the derivatives from $X^\star_\alpha$ act on $\bar\psi$. Expanding this up to first order in θ

$$\begin{aligned}\delta_\alpha(\bar\psi \star \psi) &= -i\Big(\bar\psi\star\alpha - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\bar\psi)\star(\partial_\sigma\alpha)\Big)\star\psi\\ &\quad + i\bar\psi\star\Big(\alpha\star\psi - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho\alpha)\star(\partial_\sigma\psi)\Big) \neq 0\end{aligned}$$

one sees that the first order terms do not cancel and $\bar\psi\star\psi$ is not invariant. Continuing to higher orders does not improve this. We had the same problem when analysing the diffeomorphisms. There we deformed the coproduct of transformations in order to achieve the invariance of the $\star$-product of two scalar fields for example. It is possible to do the same here. However, we do not want to change the coproduct of the gauge transformations since that is not in the agreement with the philosophy of the Seiberg-Witten map (the commutative gauge transformations induce the noncommutative ones). Instead, one observes that under the integral the term $\bar\psi\star\psi$ becomes invariant

$$\begin{aligned}\delta_\alpha \int \mathrm{d}^4x\, \bar\psi\star\psi &= \dots\\ &= \int \mathrm{d}^4x\, \Big(-\frac{1}{2}\theta^{\rho\sigma}(\partial_\rho\bar\psi)(\partial_\sigma\alpha)\psi + \frac{1}{2}\theta^{\rho\sigma}\bar\psi(\partial_\rho\alpha)(\partial_\sigma\psi)\Big)\\ &= \frac{1}{2}\int \mathrm{d}^4x\, \theta^{\rho\sigma}\Big((\partial_\rho\bar\psi)(\partial_\sigma\alpha)\psi + \partial_\sigma(\bar\psi(\partial_\rho\alpha))\psi)\\ &\qquad -(\partial_\sigma\bar\psi)(\partial_\rho\alpha))\psi - \bar\psi(\partial_\rho\partial_\sigma\alpha))\psi\Big) = 0.\end{aligned}$$

We partially integrated the second term in the first line, dropped one surface term, cancelled $\theta^{\rho\sigma}\bar{\psi}(\partial_\rho\partial_\sigma\alpha))\psi$ term due to the antisymmetry of $\theta^{\rho\sigma}$ and by using again the antisymmetry of $\theta^{\rho\sigma}$ we changed indices ρ and σ in the third term.

One can continue to higher orders and the conclusion is the same, $\bar{\psi}\star\psi$ is invariant under the integral only. That means that the Lagrangian

$$\mathcal{L}_m = \bar{\psi}\star\Big(i\gamma^\mu\mathcal{D}_\mu\psi - m\psi\Big)$$

is not gauge invariant as before (with the enveloping algebra-valued transformations), but the action

$$S_m = \int \mathrm{d}^4x\, \bar{\psi}\star\Big(i\gamma^\mu\mathcal{D}_\mu\psi - m\psi\Big)$$

is gauge invariant. This situation we have already met in the case of the Lagrangian (action) for the gauge field. Namely,

$$\mathcal{L}_g = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu})$$

is only gauge covariant, $\delta_\alpha\mathcal{L}_g = i[\Lambda_\alpha \stackrel{\star}{,} \mathcal{L}_g]$, but the action

$$S_g = -\frac{1}{4}\int \mathrm{d}^4x\, \mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu})$$

is gauge invariant provided that the integral has the cyclic property (see Section 4.1).

From this perspective it looks like there is no need to deform the coproduct for (5.9.4) transformations. One can do that of course, but we do not consider that possibility here. Instead we continue as usual, by introducing the covariant derivative.

From the definition of the covariant derivative it follows

$$\begin{aligned}\delta_\alpha(\mathcal{D}_\mu\psi) &= iX^\star_\alpha \triangleright(\mathcal{D}_\mu\psi) = iX^\star_\alpha\triangleright(\partial_\mu\psi - iV_\mu\triangleright\psi),\\ \delta_\alpha V_\mu\triangleright\psi &= (\partial_\mu X^\star_\alpha)\triangleright\psi + i\Big(X^\star_\alpha\star V_\mu - V_\mu\star X^\star_\alpha\Big)\triangleright\psi,\end{aligned}$$

or

$$\delta_\alpha V_\mu = (\partial_\mu X^\star_\alpha) + i\Big(X^\star_\alpha\star V_\mu - V_\mu\star X^\star_\alpha\Big). \tag{5.9.10}$$

Here V_μ is the noncommutative gauge field and writing $V_\mu\triangleright\psi$ we are taking into account that it might be derivative valued. Expanding (5.9.10) to first order gives

$$\theta^0 : \quad \delta_\alpha V^0_\mu = \partial_\mu\alpha + i[\alpha, V^0_\mu] \tag{5.9.11}$$

$$\begin{aligned}\theta^1 : \quad \delta_\alpha V^1_\mu &= -\frac{i}{2}\theta^{\rho\sigma}(\partial_{\rho\partial_\mu}\alpha)\partial_\sigma + i[\alpha, V^1_\mu]\\ &+\frac{1}{2}\theta^{\rho\sigma}\Big((\partial_\rho\alpha)V^0_\mu - (\partial_\rho V^0_\mu)\alpha + \alpha(\partial_\rho V^0_\mu) - V^0_\mu(\partial_\rho\alpha)\Big)\partial_\sigma.\end{aligned} \tag{5.9.12}$$

The zeroth order solution is the commutative gauge field $V^0_\mu = A^0_\mu$, the first order solution (up to the solutions of the homogeneous equation) is

$$V^1_\mu = -\frac{i}{2}\theta^{\rho\sigma}(\partial_\rho A^0_\mu)\partial_\sigma. \tag{5.9.13}$$

Continuing to higher orders leads to the following result

$$\begin{aligned} V_\mu &= A^0_\mu - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho A^0_\mu)\partial_\sigma + \dots \\ &= \sum_n \Big(\frac{-i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}A^0_\mu\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n} = X^\star_{A^0_\mu}. \end{aligned} \tag{5.9.14}$$

The field-strength tensor is

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= i[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu] = F^0_{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho F^0_{\mu\nu})\partial_\sigma + \dots \\ &= \sum_n \Big(\frac{-i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n}F^0_{\mu\nu}\Big)\partial_{\sigma_1}\dots\partial_{\sigma_n} = X^\star_{F^0_{\mu\nu}}. \end{aligned} \tag{5.9.15}$$

Its transformation law is given by

$$\delta_\alpha \mathcal{F}_{\mu\nu} = i[X_\alpha \stackrel{\star}{,} \mathcal{F}_{\mu\nu}] = X_{i[\alpha, F^0_{\mu\nu}]}. \tag{5.9.16}$$

From (5.9.15) it is obvious that $\mathcal{F}_{\mu\nu}$ is derivative-valued, just as in the case of the κ-deformed space. There we split it into the curvature-like and torsion-like terms and used only the curvature-like terms to construct the action for the gauge field. One would expect that the same could be done here. Unfortunately, this seems not to be the case. One can do the splitting, up to first order the result is

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= F^0_{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}(\partial_\rho F^0_{\mu\nu})A^0_\sigma - \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho F^0_{\mu\nu})\mathcal{D}^0_\sigma + \dots \\ &= F_{\mu\nu} + T^\rho_{\mu\nu}\mathcal{D}_\rho + \dots . \end{aligned} \tag{5.9.17}$$

The problem with this result is that $F_{\mu\nu}$ (or equivalently $T^\rho_{\mu\nu}$) does not transform covariantly,

$$\delta_\alpha F_{\mu\nu} \neq i[X_\alpha \stackrel{\star}{,} F_{\mu\nu}].$$

This is far from being understood properly. However, one has to find the way to solve this since we can not construct the Lagrangian (action) for the gauge field otherwise.

Concerning the action for the matter field we can speculate that it will remain classical (commutative) since

$$\begin{aligned} S_m &= \int \mathrm{d}^4x\, \bar\psi \star \Big(i\gamma^\mu \mathcal{D}_\mu\psi - m\psi\Big) \\ &= \int \mathrm{d}^4x\, \bar\psi\Big(i\gamma^\mu(\partial_\mu\psi - iV_\mu \triangleright \psi) - m\psi\Big) \\ &= \int \mathrm{d}^4x\, \bar\psi\Big(i\gamma^\mu(\partial_\mu\psi - iA^0_\mu\psi) - m\psi\Big), \end{aligned}$$

where we have used the cyclicity property to remove one $\star$ and the solution (5.9.14) for the V_μ field.

Another interesting problem in relation to the enveloping algebra-valued vs. derivative-valued gauge transformations is the application to the diffeomorphisms. There we have the reversed situation: the derivative-valued parameter and the connection come out naturally.

The question would be if one can find the enveloping algebra analogue of these solutions. Additional difficulty is that in the case of the diffeomorphisms partial derivatives are at the same time the generators of transformations (they play the role of the gauge group generators T^a) so solution (5.9.14) is in a way already enveloping algebra-valued. However, better understanding of this is needed.

Additional motivation for studying this problem comes from the fact that the local Lorentz transformations can be treated as the usual gauge transformations. If one wants to study deformation of the classical local Lorentz transformations (and that is necessary in order to have a more general theory of gravity, not only General Relativity), the question is which approach to take. In some recent papers [53], [97] the local Lorentz transformations were analysed using the enveloping algebra approach. On the other hand, if one wants to treat the deformed local Lorentz transformations and the deformed diffeomorphisms in the same way, then to use the derivative-valued transformations looks like a natural thing to do. All this will be the subject of further research.

Appendix A

Vector fields in the κ-deformed space

In Section 2.5 we have defined the transformation law of a covariant vector field using the transformation law of the Dirac derivative of a scalar field. It was also said that this is not the only way to define vector fields since derivatives on the κ-deformed space are not unique. Here we continue this discussion and as an example derive the transformation law of a vector field related with the derivative $\hat{\partial}_\rho$ introduced in (2.1.10).

For the convenience we use the abstract algebra approach. The transformation law of a scalar field (2.5.5), rewritten in the abstract algebra notation, is

$$\delta_\omega \hat{\phi}(\hat{x}) = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{\phi}(\hat{x}), \tag{A.1}$$

where $\hat{L}^{\alpha\beta}$ is given by (2.2.9). The transformation law of a vector field V_μ (2.5.8) has been obtained by generalising the transformation law of the Dirac derivative of a scalar field. In the abstract algebra notation (2.5.8) reads

$$\begin{aligned}\delta_\omega \hat{V}_\mu &= -\frac{1}{2}\omega^{\alpha\beta}\Big(\hat{L}_{\alpha\beta}\hat{V}_\mu - \eta_{\beta\mu}\hat{V}_\alpha + \eta_{\alpha\mu}\hat{V}_\beta\Big)\\ &= -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{V}_\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \hat{V}_\mu].\end{aligned} \tag{A.2}$$

Now we introduce a vector field $\hat{A}_\mu$ associated with the derivatives $\hat{\partial}_\mu$ (2.1.10). The transformation law of the $\hat{\partial}_\mu$ derivative of a scalar field is

$$\delta_\omega \hat{\partial}_\mu \hat{\phi} = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{\partial}_\mu\hat{\phi} + \frac{1}{2}\omega^{\alpha\beta}[\hat{L}_{\alpha\beta}, \hat{\partial}_\mu]\hat{\phi}, \tag{A.3}$$

where for $[\hat{L}_{\alpha\beta}, \hat{\partial}_\mu]$ (2.2.7) should be inserted. The transformation law of a vector field $\hat{A}_\mu$ should be such that it reproduces (A.3) when $\hat{A}_\mu$ is replaced with $\hat{\partial}_\mu\hat{\phi}$. The central assumption is that in this transformation law vector fields appear linearly on the right hand side. Also, we choose that derivatives are always to the left of the vector field $\hat{A}_\mu$. Since on the right hand side of (2.2.7) complicated expression of derivatives appears, the generalisation of (2.2.7) to the transformation law of a vector field $\hat{A}_\mu$ is not straightforward. However, demanding that this transformations close in the algebra

$$\delta_\omega\delta_{\omega'} - \delta_{\omega'}\delta_\omega = \delta_{[\omega,\omega']} \tag{A.4}$$

one finds the following solution

$$\delta_\omega \hat{A}_\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{A}_\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \hat{A}_\mu], \tag{A.5}$$

with

$$\begin{aligned}
[M_{ij}, \hat{A}_l] &= \eta_{jl}\hat{A}_i - \eta_{il}\hat{A}_j, \qquad\qquad [M_{ij}, \hat{A}_n] = 0,\\
[M_{in}, \hat{A}_l] &= \eta_{il}\frac{e^{2ia\hat{\partial}_n}-1}{2ia\hat{\partial}_n}\hat{A}_n - \frac{ia}{2}\eta_{il}\hat{\partial}_m\hat{A}^m + \frac{ia}{2}\big(\hat{\partial}_l\hat{A}_i + \hat{\partial}_i\hat{A}_l\big)\\
&\quad -\eta_{il}\frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\big(\hat{\partial}_n\hat{\partial}_m\hat{A}^m - \hat{\partial}_m\hat{\partial}^m\hat{A}_n\big)\\
&\quad +\Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\big(\hat{\partial}_n\hat{\partial}_i\hat{A}_l + \hat{\partial}_n\hat{\partial}_l\hat{A}_i - 2\hat{\partial}_l\hat{\partial}_i\hat{A}_n\big),\\
[M_{in}, \hat{A}_n] &= -\hat{A}_i.
\end{aligned} \tag{A.6}$$

To calculate (A.6) one makes an ansatz in terms of the power series expansion in a and inserts it into (A.4). This gives a recursion formula which can be solved and the solution[1] is given by (A.6).

The square of the vector field corresponding to the Dirac derivative $\hat{V}^\mu\hat{V}_\mu$ is an invariant under the κ-deformed Poincaré transformations. To form an invariant from the vector field $\hat{A}_\mu$, we have to define a vector field $\breve{A}^\mu$ such that

$$\delta_\omega(\hat{A}_\lambda\breve{A}^\lambda) = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}(\hat{A}_\lambda\breve{A}^\lambda). \tag{A.7}$$

Using (A.6) we construct the transformation law for $\breve{A}^\mu$

$$\delta_\omega\breve{A}^\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\breve{A}^\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \breve{A}^\mu] \tag{A.8}$$

and

$$\begin{aligned}
[M_{ij}, \breve{A}^l] &= \delta^l_j\breve{A}_i - \delta^l_i\breve{A}_j, \qquad\qquad [M_{ij}, \breve{A}^n] = 0,\\
[M_{in}, \breve{A}^l] &= -\delta^l_i\breve{A}_n + \frac{ia}{2}\breve{A}_i\hat{\partial}^l - \frac{ia}{2}\breve{A}^l\hat{\partial}_i - \frac{ia}{2}\delta^l_i\breve{A}^m\hat{\partial}_m\\
&\quad +\frac{a}{2}\breve{A}_i\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}^l - (\delta^l_i\breve{A}^m\hat{\partial}_m + \breve{A}^l\hat{\partial}_i)\Big(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big),\\
[M_{in}, \breve{A}_n] &= \breve{A}_i\frac{1-e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n} - \breve{A}_i\frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_m\hat{\partial}^m + 2\breve{A}^m\Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\hat{\partial}_i\hat{\partial}_m.
\end{aligned} \tag{A.9}$$

The transformation (A.8) represents the algebra (A.4).

All the relations considered up to now are invariant under the conjugation

$$\begin{aligned}
\bar{\hat{x}}^\mu &= \hat{x}^\mu, &\qquad \bar{\hat{\partial}}_\mu &= -\hat{\partial}_\mu,\\
\overline{M}^{ij} &= -M^{ij}, &\qquad \overline{M}^{in} &= -M^{in}.
\end{aligned} \tag{A.10}$$

[1] This solution is not unique. If the symmetrisation in the third term of $[M_{in}, \hat{A}_l]$ is not performed, the last term of $[M_{in}, \hat{A}_l]$ vanishes.

Comparing (A.6) and (A.9), we see that $\breve{A}^\mu$ transforms with the derivatives on the right hand side, but $\overline{\breve{A}}^\mu \neq \hat{A}^\mu$, they transform differently. The transformation law of $\overline{\hat{A}}_\mu$ is:

$$\delta_\omega \overline{\hat{A}}_\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\overline{\hat{A}}_\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \overline{\hat{A}}_\mu], \tag{A.11}$$

with

$$\begin{aligned}
[M_{ij}, \overline{\hat{A}}_l] &= \eta_{jl}\overline{\hat{A}}_i - \eta_{il}\overline{\hat{A}}_j, \qquad\qquad [M_{ij}, \overline{\hat{A}}_n] = 0, \\
[M_{in}, \overline{\hat{A}}_l] &= \overline{\hat{A}}_n \eta_{il}\frac{e^{2ia\hat{\partial}_n}-1}{2ia\hat{\partial}_n} - \frac{ia}{2}\eta_{il}\overline{\breve{A}}^m\hat{\partial}_m + \frac{ia}{2}\big(\overline{\hat{A}}_i\hat{\partial}_l + \overline{\hat{A}}_l\hat{\partial}_i\big) \\
&\quad -\eta_{il}\big(\overline{\breve{A}}^m\hat{\partial}_n\hat{\partial}_m - \overline{\hat{A}}_n\hat{\partial}_m\hat{\partial}^m\big)\frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big) \\
&\quad +\big(\overline{\hat{A}}_l\hat{\partial}_n\hat{\partial}_i + \overline{\hat{A}}_i\hat{\partial}_n\hat{\partial}_l - 2\overline{\hat{A}}_n\hat{\partial}_l\hat{\partial}_i\big)\Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big), \\
[M_{in}, \overline{\hat{A}}_n] &= -\overline{\hat{A}}_i.
\end{aligned} \tag{A.12}$$

The dual of $\overline{\hat{A}}_\mu$ is $\overline{\breve{A}}^\mu$,

$$\delta_\omega(\overline{\hat{A}}_\lambda\overline{\breve{A}}^\lambda) = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}(\overline{\hat{A}}_\lambda\overline{\breve{A}}^\lambda). \tag{A.13}$$

Its transformation law is

$$\delta_\omega\overline{\breve{A}}^\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\overline{\breve{A}}^\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \overline{\breve{A}}^\mu] \tag{A.14}$$

and

$$\begin{aligned}
[M_{ij}, \overline{\breve{A}}^l] &= \delta^l_j\overline{\breve{A}}_i - \delta^l_i\overline{\breve{A}}_j, \qquad\qquad [M_{ij}, \overline{\breve{A}}^n] = 0, \\
[M_{in}, \overline{\breve{A}}^l] &= -\delta^l_i\overline{\breve{A}}_n + \frac{ia}{2}\hat{\partial}^l\overline{\breve{A}}_i - \frac{ia}{2}\hat{\partial}_i\overline{\breve{A}}^l - \frac{ia}{2}\delta^l_i\hat{\partial}_m\overline{\breve{A}}^m \\
&\quad +\frac{a}{2}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}^l\overline{\breve{A}}_i - \Big(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)(\delta^l_i\hat{\partial}_m\overline{\breve{A}}^m + \hat{\partial}_i\overline{\breve{A}}^l),
\end{aligned} \tag{A.15}$$

$$[M_{in}, \overline{\breve{A}}_n] = \frac{1-e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n}\overline{\breve{A}}_i - \frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_m\hat{\partial}^m\overline{\breve{A}}_i + 2\Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\hat{\partial}_i\hat{\partial}_m\overline{\breve{A}}^m.$$

All the vector fields introduced so far $\hat{A}_\mu$, $\breve{A}^\mu$, $\overline{\hat{A}}_\mu$ and $\overline{\breve{A}}^\mu$ can be obtained from the vector field $\hat{V}_\mu$ corresponding to the Dirac derivative via the derivative-valued map

$$\hat{V}_\mu = E_\mu{}^\nu\hat{A}_\nu, \qquad \hat{A}_\mu = (E^{-1})_\mu{}^\nu\hat{V}_\nu. \tag{A.16}$$

The matrix $E_\mu{}^\nu = E_\mu{}^\nu(\partial)$ in (A.16) depends only on derivatives. To find its explicit form one expands the transformation laws of $\hat{V}_\mu$, $\hat{A}_\mu$ and $\hat{\partial}_\mu$, (A.2), (A.6) and (2.2.7) in powers of a. In the zeroth order we assume that

$$\hat{V}_\mu\big|_{\mathcal{O}(a^0)} = \hat{A}_\mu\big|_{\mathcal{O}(a^0)}. \tag{A.17}$$

In this way a recursion formula in a is obtained and its solution is given by

$$\begin{aligned}
E_n{}^n &= \frac{1}{a\hat{\partial}_n}\sin(a\hat{\partial}_n) - e^{-ia\hat{\partial}_n}\Big(\frac{ia}{2} + \frac{i}{\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\frac{\hat{\partial}_m\hat{\partial}^m}{\hat{\partial}_n},\\
E_n{}^j &= \frac{i}{\hat{\partial}_n}e^{-ia\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}^j,\\
E_j{}^n &= \Big(e^{-ia\hat{\partial}_n} - \frac{1-e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}\Big)\frac{\hat{\partial}_j}{\hat{\partial}_n},\\
E_i{}^j &= \delta_i^j\frac{1-e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}.
\end{aligned} \tag{A.18}$$

One can also construct the inverse of this map. Here we write result expanded up to second order in the deformation parameter, the full expression is given in [56]

$$\begin{aligned}
(E^{-1})_n{}^n &= 1 - \frac{(ia)^2}{6}\hat{\partial}_n^2 - \frac{(ia)^2}{4}\hat{\partial}_m\hat{\partial}^m + \mathcal{O}(a^3),\\
(E^{-1})_n{}^j &= -\frac{ia}{2}\hat{\partial}^j + \frac{(ia)^2}{4}\hat{\partial}_n\hat{\partial}^j + \mathcal{O}(a^3),\\
(E^{-1})_j{}^n &= \frac{ia}{2}\hat{\partial}_j - \frac{(ia)^2}{12}\hat{\partial}_n\hat{\partial}_j + \mathcal{O}(a^3),\\
(E^{-1})_i{}^j &= \delta_i^j\Big(1 + \frac{ia}{2}\hat{\partial}_n + \frac{(ia)^2}{12}\hat{\partial}_n\hat{\partial}_n\Big) - \frac{(ia)^2}{4}\hat{\partial}_i\hat{\partial}^j + \mathcal{O}(a^3).
\end{aligned} \tag{A.19}$$

Appendix B

The κ-deformed symmetry from the inversion of the $\star$-product

Here we apply the technique from Chapter 5 to construct a deformed symmetry for the κ-deformed space introduced in Chapter 2. The underlying idea is to compare the symmetry obtained in this way with the already known κ-Poincaré symmetry analysed in Chapter 2. We use the symmetrically ordered $\star$-product (2.4.4) and derivatives defined in (2.4.21) and (2.4.24).

The transformation law of a scalar field up to second order in the deformation parameter a is

$$\begin{aligned}\delta_\xi\phi &= -\xi^\mu\partial_\mu\phi = -X^\star_\xi \triangleright \phi \\ &= -\Big(\xi^\mu\partial_\mu - \frac{i}{2}C_\lambda^{\rho\sigma}x^\lambda(\partial_\rho\xi^\mu)\partial_\sigma\partial_\mu - \frac{1}{8}C_\lambda^{\rho\sigma}C_\gamma^{\alpha\beta}x^\lambda x^\gamma(\partial_\rho\partial_\alpha\xi^\mu)\partial_\sigma\partial_\beta\partial_\mu \\ &\quad -\frac{1}{12}C_\lambda^{\rho\sigma}C_\rho^{\alpha\beta}x^\lambda(\partial_\sigma\partial_\alpha\xi^\mu)\partial_\beta\partial_\mu - \frac{1}{6}C_\lambda^{\rho\sigma}C_\rho^{\alpha\beta}x^\lambda(\partial_\alpha\xi^\mu)\partial_\sigma\partial_\beta\partial_\mu + \mathcal{O}(a^3)\Big)\triangleright\phi. \end{aligned} \tag{B.1}$$

For the special case of translations, $\xi^\mu = b^\mu = const.$ (B.1) gives

$$\delta^t_\xi\phi = -b^\mu \star(\partial_\mu\phi) = -b^\mu(\partial_\mu\phi). \tag{B.2}$$

For Lorentz rotations, $\xi^\mu = x^\nu{\omega_\nu}^\mu$ we have

$$\begin{aligned}\delta^l_\xi\phi &= -x^\lambda{\omega_\lambda}^\mu\star(\partial_\mu\phi) + \frac{i}{2}C_\lambda^{\rho\sigma}x^\lambda{\omega_\rho}^\mu\star(\partial_\sigma\partial_\mu\phi) + \frac{1}{6}C_\lambda^{\rho\sigma}C_\rho^{\alpha\beta}x^\lambda{\omega_\alpha}^\mu\star(\partial_\sigma\partial_\beta\partial_\mu\phi) \\ &= -\frac{1}{2}\omega^{\alpha\beta}(L_{\alpha\beta}\phi),\end{aligned} \tag{B.3}$$

where $L_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha$.

Transformations (B.2) and (B.3) close in the undeformed algebra

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}, \\ [\partial_\rho, \partial_\sigma] &= 0, \\ [L_{\mu\nu}, \partial_\rho] &= \eta_{\nu\rho}\partial_\mu - \eta_{\mu\rho}\partial_\nu.\end{aligned} \tag{B.4}$$

Their coproducts are

$$\begin{aligned}\Delta\partial_n &= \partial_n \otimes 1 + 1 \otimes \partial_n,\\ \Delta\partial_j &= \partial_j \otimes \big(1 - \frac{ia}{2}\partial_n - \frac{a^2}{12}\partial_n\partial_n\big) + \big(1 + \frac{ia}{2}\partial_n - \frac{a^2}{12}\partial_n\partial_n\big) \otimes \partial_j\\ &\quad + \frac{a^2}{12}\big(\partial_n \otimes \partial_n\partial_j + \partial_n\partial_j \otimes \partial_n\big) + \mathcal{O}(a^3).\end{aligned} \tag{B.5}$$

$$\begin{aligned}\Delta L_{\alpha\beta} &= L_{\alpha\beta} \otimes 1 + 1 \otimes L_{\alpha\beta}\\ &\quad - \Bigg(\frac{ia}{2}\delta^n_\alpha(\partial_\beta \otimes x^\lambda\partial_\lambda - x^\lambda\partial_\lambda \otimes \partial_\beta)\\ &\qquad - \frac{5a^2}{12}\delta^n_\alpha(\partial_n\partial_\beta \otimes x^\lambda\partial_\lambda + x^\lambda\partial_\lambda \otimes \partial_n\partial_\beta)\\ &\qquad + \frac{a^2}{4}\delta^n_\alpha\big(x^\lambda\partial_\lambda\partial_n \otimes x^\lambda\partial_\lambda\partial_\beta - \partial_n\partial_\beta \otimes (x^\lambda\partial_\lambda)^2\\ &\qquad - (x^\lambda\partial_\lambda)^2 \otimes \partial_n\partial_\beta + x^\lambda\partial_\lambda\partial_\beta \otimes x^\lambda\partial_\lambda\partial_n\big)\\ &\qquad + \frac{a^2}{6}\delta^n_\alpha\big(x^\lambda\partial_\lambda\partial_n \otimes \partial_\beta + \partial_\beta \otimes x^\lambda\partial_\lambda\partial_n + \partial_n \otimes x^\lambda\partial_\lambda\partial_\beta\\ &\qquad + x^\lambda\partial_\lambda\partial_\beta \otimes \partial_n\big) - \alpha \longleftrightarrow \beta \Bigg) + \mathcal{O}(a^3).\end{aligned} \tag{B.6}$$

From (B.6) it is obvious that $\Delta L_{\alpha\beta}$ does not close in the algebra of derivatives and Lorentz generators (Poincaré algebra). Therefore, we have to enlarge the algebra and include coordinates as well. Before proceeding further we make one remark concerning the uniqueness of solution (2.2.3) for the commutator of Lorentz generators and coordinates. From equations (B.3) and (B.6) we see that the solution

$$[M^{\rho\sigma}, \hat{x}^\mu] = \eta^{\mu\sigma}\hat{x}^\rho - \eta^{\mu\rho}\hat{x}^\sigma \tag{B.7}$$

is also possible, that is it fulfils (2.2.2) conditions. However, it leads to the coproduct of $M^{\rho\sigma}$ generators that does not close in $M^{\rho\sigma}$ and $\hat{\partial}_\mu$ only, but also includes the coordinates. This is the reason why this solution has not been considered in Chapter 2.

The way that coordinates appear in (B.6) suggests introducing the generator of dilatations. Inserting $\xi^\mu = \epsilon x^\mu$ with ϵ real constant in (B.1) gives for infinitesimal dilatations

$$\delta^d_\xi\phi = -\epsilon x^\mu \star (\partial_\mu\phi) = -\epsilon x^\mu(\partial_\mu\phi) = -\epsilon D\phi. \tag{B.8}$$

As the next step we check that generators ∂_μ, $L_{\alpha\beta}$ and D close in the undeformed algebra[1]. In addition to (B.4) we obtain

$$\begin{aligned}[][D, D] &= 0,\\ [D, \partial_\mu] &= \partial_\mu,\\ [D, L_{\mu\nu}] &= 0.\end{aligned} \tag{B.9}$$

[1]This step is obvious. The transformations (B.2), (B.3) and (B.8) are classical transformations and therefore the algebra is undeformed.

Coproduct of the generator of dilatations is

$$\Delta D = D \otimes 1 + 1 \otimes D - \frac{ia}{2}\big(\partial_n \otimes D - D \otimes \partial_n\big) + \frac{a^2}{12}\big(\partial_n^2 \otimes D + D \otimes \partial_n^2\big) + \frac{a^2}{4}\big(\partial_n^2 \otimes D^2 + D^2 \otimes \partial_n^2 - 2D\partial_n \otimes D\partial_n\big) + \frac{a^2}{6}\big(D\partial_n \otimes \partial_n + \partial_n \otimes D\partial_n\big) + \mathcal{O}(a^3). \tag{B.10}$$

Coproduct of the Lorentz generators (B.6) can now be rewritten as

$$\Delta L_{\alpha\beta} = L_{\alpha\beta} \otimes 1 + 1 \otimes L_{\alpha\beta} - \Big(\frac{ia}{2}\delta_\alpha^n(\partial_\beta \otimes D - D \otimes \partial_\beta) - \frac{5a^2}{12}\delta_\alpha^n(\partial_n\partial_\beta \otimes D + D \otimes \partial_n\partial_\beta) + \frac{a^2}{4}\delta_\alpha^n(D\partial_n \otimes D\partial_\beta - \partial_n\partial_\beta \otimes D^2 - D^2 \otimes \partial_n\partial_\beta + D\partial_\beta \otimes D\partial_n) + \frac{a^2}{6}\delta_\alpha^n(D\partial_n \otimes \partial_\beta + \partial_\beta \otimes D\partial_n + \partial_n \otimes D\partial_\beta + D\partial_\beta \otimes \partial_n) - \alpha \longleftrightarrow \beta\Big) + \mathcal{O}(a^3). \tag{B.11}$$

From (B.11) we see that $\Delta L_{\alpha\beta}$ closes in the algebra of ∂_μ, $L_{\alpha\beta}$ and D generators. Adding counits and antipodes

$$\varepsilon(\partial_\mu) = \varepsilon(D) = \varepsilon(L_{\alpha\beta}) = 0. \tag{B.12}$$

$$S(\partial_\mu) = -\partial_\mu, \tag{B.13}$$

$$S(D) = -D + \frac{ia}{2}\partial_n - \frac{5a^2}{12}a^2\partial_n^2 + \mathcal{O}(a^3), \tag{B.14}$$

$$S(L_{\alpha\beta}) = -L_{\alpha\beta} - \frac{ia}{2}(\eta_{\alpha n}\partial_\beta - \eta_{\beta n}\partial_\alpha) - \frac{a^2}{3}\partial_n(\eta_{\alpha n}\partial_\beta - \eta_{\beta n}\partial_\alpha) + \mathcal{O}(a^3) \tag{B.15}$$

we obtain the κ-deformed Weil Hopf algebra.

In this way we have constructed another deformed symmetry for the κ-deformed space. Comparing this result with the κ-Poincaré Hopf algebra discussed in Chapter 2, we see that this two quantum symmetries are not equal. The problem is that in the "$\star$-product inversion" approach coordinates naturally appear and one is forced to replace the Poincaré algebra with a larger one (in this case with the deformed Weil algebra).

Appendix C

The general κ-deformed space

In this appendix we concentrate on the general κ-deformed space, that is we do not specify $a^n = a$ and $a^j = 0$ in equation (2.1.1).

C.1 Derivatives

Starting from the defining relation for the general κ-deformed space

$$[\hat{x}^\mu, \hat{x}^\nu] = ia^\mu \hat{x}^\nu - ia^\nu \hat{x}^\mu, \qquad \mu = 0, \dots n, \tag{C.1.1}$$

we obtain three families of the linear derivatives

$$\begin{aligned} [\hat{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu + ia^\nu \hat{\partial}_\mu, \\ [\check{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu (1 - ia^\lambda \check{\partial}_\lambda), \\ [\tilde{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu + ia^\nu \tilde{\partial}_\mu + i\eta_{\mu\rho}\eta^{\nu\sigma} a^\rho \tilde{\partial}_\sigma. \end{aligned} \tag{C.1.2}$$

Here $\eta^{\mu\nu} = diag(1, -1, \dots, -1)$ is the formal metric. The last two solutions can be expressed in terms of $\hat{\partial}_\mu$ derivative[1]

$$\begin{aligned} \check{\partial}_\mu &= \frac{\hat{\partial}_\mu}{1 + ia^\rho \hat{\partial}_\rho}, \\ \tilde{\partial}_\mu &= \hat{\partial}_\mu + \frac{i}{2}\eta_{\mu\rho}\eta^{\alpha\beta} a^\rho \hat{\partial}_\alpha \hat{\partial}_\beta. \end{aligned} \tag{C.1.3}$$

Derivatives commute among themselves

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \tag{C.1.4}$$

as well as for the other two solutions. Leibniz rules for (C.1.2) derivatives are given by

$$\begin{aligned} \hat{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\hat{\partial}_\mu \hat{f}) \cdot \hat{g} + ((1 + ia^\nu \hat{\partial}_\nu)\hat{f}) \cdot (\hat{\partial}_\mu \hat{g}), \\ \check{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\check{\partial}_\mu \hat{f}) \cdot ((1 - ia^\nu \check{\partial}_\nu)\hat{g}) + \hat{f} \cdot (\check{\partial}_\mu \hat{g}), \\ \tilde{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\tilde{\partial}_\mu \hat{f}) \cdot \hat{g} + ((1 + ia^\nu \tilde{\partial}_\nu)\hat{f}) \cdot (\tilde{\partial}_\mu \hat{g}) + ia^\rho \eta_{\rho\mu}\eta^{\alpha\beta} (\tilde{\partial}_\alpha \hat{f})(\tilde{\partial}_\beta \hat{g}), \end{aligned} \tag{C.1.5}$$

[1] One can chose any of this three solutions to work with and express the other two in terms of it. Here we have chosen to work with the first one for no special reason but the simplicity of some formulas.

or in terms of the coproduct

$$\begin{aligned}
\Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + (1 + ia^\nu\hat{\partial}_\nu) \otimes \hat{\partial}_\mu, \\
\Delta\check{\partial}_\mu &= \check{\partial}_\mu \otimes (1 - ia^\nu\check{\partial}_\nu) + 1 \otimes \check{\partial}_\mu, \\
\Delta\tilde{\partial}_\mu &= \tilde{\partial}_\mu \otimes 1 + (1 + ia^\nu\tilde{\partial}_\nu) \otimes \tilde{\partial}_\mu + ia^\rho\eta_{\rho\mu}\eta^{\alpha\beta}\tilde{\partial}_\alpha \otimes \tilde{\partial}_\beta.
\end{aligned} \tag{C.1.6}$$

C.2 Deformed symmetry

Deformed symmetry is introduced like the map on the abstract algebra given by generators $M^{\mu\nu}$. The commutation relation between the generators and coordinates is given by

$$[M^{\rho\sigma}, \hat{x}^\mu] = \eta^{\sigma\mu}\hat{x}^\rho - \eta^{\rho\mu}\hat{x}^\sigma - ia^\rho M^{\sigma\mu} + ia^\sigma M^{\rho\mu}. \tag{C.2.1}$$

Using this result one checks that $M_{\mu\nu}$ generators close the undeformed Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}. \tag{C.2.2}$$

From (C.2.1) the Leibniz rule follows

$$M^{\rho\sigma}(\hat{f} \cdot \hat{g}) = (M^{\rho\sigma}\hat{f}) \cdot \hat{g} + \hat{f} \cdot (M^{\rho\sigma}\hat{g}) + (ia^\rho\hat{\partial}_\lambda\hat{f}) \cdot (M^{\lambda\sigma}\hat{g}) - (ia^\sigma\hat{\partial}_\lambda\hat{f}) \cdot (M^{\lambda\rho}\hat{g}), \tag{C.2.3}$$

or in the terms of the abstract comultiplication

$$\Delta M^{\rho\sigma} = M^{\rho\sigma} \otimes 1 + 1 \otimes M^{\rho\sigma} + ia^\rho\hat{\partial}_\lambda \otimes M^{\lambda\sigma} - ia^\sigma\hat{\partial}_\lambda \otimes M^{\lambda\rho}. \tag{C.2.4}$$

Again we see that the comultiplication does not close on the generators $M^{\rho\sigma}$, derivatives have to be added. Therefore, we have to define the commutator of $M^{\rho\sigma}$ with derivatives. This can be done by first representing the generators $M^{\rho\sigma}$ by coordinates and derivatives. Just like in Section 2.2 we call the generators $M^{\rho\sigma}$ in this realisation $\hat{L}^{\rho\sigma}$

$$\hat{L}^{\rho\sigma} = \eta^{\rho\lambda}\hat{x}^\sigma\hat{\partial}_\lambda - \eta^{\sigma\lambda}\hat{x}^\rho\hat{\partial}_\lambda + \frac{i}{2}(a^\rho\hat{x}^\sigma - a^\sigma\hat{x}^\rho)\eta^{\kappa\lambda}\hat{\partial}_\kappa\hat{\partial}_\lambda. \tag{C.2.5}$$

From (C.2.5) the action of $M^{\rho\sigma}$ on the derivatives follows immediately

$$[M^{\rho\sigma}, \hat{\partial}_\mu] = \delta^\sigma_\mu\eta^{\rho\lambda}\hat{\partial}_\lambda - \delta^\rho_\mu\eta^{\sigma\lambda}\hat{\partial}_\lambda + i(a^\sigma\eta^{\rho\lambda} - a^\rho\eta^{\sigma\lambda})\hat{\partial}_\lambda\hat{\partial}_\mu + \frac{i}{2}(\delta^\sigma_\mu a^\rho - \delta^\rho_\mu a^\sigma)\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda. \tag{C.2.6}$$

The κ-Poincaré Hopf algebra is defined by the relations (C.1.4), (C.2.2), (C.2.6), (C.2.4), (C.1.5) and we have to add the counit and antipode:

$$\begin{aligned}
&\varepsilon(M^{\rho\sigma}) = 0, \qquad \varepsilon(\hat{\partial}_\mu) = 0, \\
&S(M^{\rho\sigma}) = -M^{\rho\sigma} + ia^\rho M^{\lambda\sigma}\frac{\hat{\partial}_\lambda}{1 + ia^\nu\hat{\partial}_\nu} - ia^\sigma M^{\lambda\rho}\frac{\hat{\partial}_\lambda}{1 + ia^\nu\hat{\partial}_\nu} - i(n-1)\frac{(a^\rho\hat{\partial}_\sigma - a^\sigma\hat{\partial}_\rho)}{1 + ia^\nu\hat{\partial}_\nu}, \\
&S(\hat{\partial}_\mu) = -\frac{\hat{\partial}_\mu}{1 + ia^\nu\hat{\partial}_\nu}.
\end{aligned} \tag{C.2.7}$$

C.3 Dirac derivative, invariants

Like in Section 2.3 we introduce the Dirac derivative and the deformed D'Alambert operator. The relevant formulas are

$$[M^{\rho\sigma}, \hat{D}_\mu] = (\delta^\sigma_\mu \eta^{\rho\nu} - \delta^\rho_\mu \eta^{\sigma\nu})\hat{D}_\nu, \qquad \hat{D}_\mu = \frac{\hat{\partial}_\mu + \frac{i}{2} a^\mu \eta^{\kappa\lambda}\hat{\partial}_\kappa \hat{\partial}_\lambda}{1 + ia^\rho \hat{\partial}_\rho}, \tag{C.3.1}$$

$$[M^{\rho\sigma}, \hat{\Box}] = 0, \qquad \hat{\Box} = \frac{\eta^{\kappa\lambda}\hat{\partial}_\kappa\hat{\partial}_\lambda}{1 + ia^\rho\hat{\partial}_\rho}, \tag{C.3.2}$$

$$\eta^{\mu\nu}\hat{D}_\mu\hat{D}_\nu = \hat{\Box}(1 - \frac{\eta_{\mu\nu}a^\mu a^\nu}{4}\hat{\Box}). \tag{C.3.3}$$

The map from $\hat{\partial}_\mu$ to $\hat{D}_\mu$ is invertible:

$$\hat{\partial}_\mu = \frac{\hat{D}_\mu - \frac{i\eta_{\mu\rho}a^\rho}{\eta_{\alpha\beta}a^\alpha a^\beta}\Big(1 - \sqrt{1 - \eta_{\sigma\tau}a^\sigma a^\tau \eta^{\lambda\omega}\hat{D}_\lambda\hat{D}_\omega}\Big)}{-ia^\gamma\hat{D}_\gamma + \sqrt{1 - \eta_{\gamma\delta}a^\gamma a^\delta \eta^{\lambda\nu}\hat{D}_\lambda\hat{D}_\nu}}. \tag{C.3.4}$$

From the commutation relations

$$[\hat{D}_\mu, \hat{x}^\nu] = \delta^\nu_\mu\Big(-ia^\rho\hat{D}_\rho + \sqrt{1 - \eta_{\sigma\tau}a^\sigma a^\tau\eta^{\lambda\omega}\hat{D}_\lambda\hat{D}_\omega}\Big) + i\eta_{\mu\rho}\eta^{\nu\sigma}a^\rho\hat{D}_\sigma \tag{C.3.5}$$

the Leibniz rule and comultiplication for the Dirac derivative follow

$$\begin{aligned}
\hat{D}_\mu(\hat{f}\cdot\hat{g}) &= (\hat{D}_\mu\hat{f})\cdot\frac{1}{1+ia^\nu\hat{\partial}_\nu}\hat{g} + ((1+ia^\nu\hat{\partial}_\nu)\hat{f})\cdot(\hat{D}_\mu\hat{g}) \\
&\quad + (i\eta_{\mu\rho}a^\rho\eta^{\kappa\lambda}\hat{\partial}_\kappa\hat{f})\cdot\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu}\hat{g} - (ia^\lambda\hat{\partial}_\lambda\hat{f})\cdot\frac{\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu}\hat{g}, \\
\Delta\hat{D}_\mu &= \hat{D}_\mu\otimes\frac{1}{1+ia^\nu\hat{\partial}_\nu} + (1+ia^\nu\hat{\partial}_\nu)\otimes\hat{D}_\mu \\
&\quad + i\eta_{\mu\rho}a^\rho\eta^{\kappa\lambda}\hat{\partial}_\kappa\otimes\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu} - ia^\lambda\hat{\partial}_\lambda\otimes\frac{\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu}.
\end{aligned} \tag{C.3.6}$$

Equation (C.3.4) has to be used to express $\hat{\partial}_\mu$ by $\hat{D}_\mu$ on the righthand side of (C.3.6).

It is now convenient to define the κ-Poincaré Hopf algebra in terms of the Dirac derivatives and $M^{\mu\nu}$ generators.

Algebra sector

$$\begin{aligned}
[\hat{D}_\mu, \hat{D}_\nu] &= 0, \\
[M^{\rho\sigma}, \hat{D}_\mu] &= (\delta^\sigma_\mu\eta^{\rho\nu} - \delta^\rho_\mu\eta^{\sigma\nu})\hat{D}_\nu \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}.
\end{aligned} \tag{C.3.7}$$

Coproducts

$$\begin{aligned}
\Delta\hat{D}_\mu &= \hat{D}_\mu\otimes\frac{1}{1+ia^\nu\hat{\partial}_\nu} + (1+ia^\nu\hat{\partial}_\nu)\otimes\hat{D}_\mu \\
&\quad + ia^\mu\eta^{\kappa\lambda}\hat{\partial}_\kappa\otimes\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu} - ia^\lambda\hat{\partial}_\lambda\otimes\frac{\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu}, \\
\Delta M^{\rho\sigma} &= M^{\rho\sigma}\otimes 1 + 1\otimes M^{\rho\sigma} + ia^\rho\hat{\partial}_\lambda\otimes M^{\lambda\sigma} - ia^\sigma\hat{\partial}_\lambda\otimes M^{\lambda\rho}.
\end{aligned} \tag{C.3.8}$$

Counits and antipodes

$$
\begin{aligned}
\varepsilon(\hat{D}_\mu) &= 0, \\
\varepsilon(M^{\rho\sigma}) &= 0 \\
S(\hat{D}_\mu) &= -\hat{D}_\mu + \frac{ia^\mu \eta^{\kappa\lambda}\hat{\partial}_\kappa \hat{\partial}_\lambda - ia^\lambda \hat{\partial}_\lambda \hat{\partial}_\mu}{1 + ia^\nu \hat{\partial}_\nu}, \\
S(M^{\rho\sigma}) &= -M^{\rho\sigma} + ia^\rho M^{\lambda\sigma} \frac{\hat{\partial}_\lambda}{1 + ia^\nu \hat{\partial}_\nu} \\
&\quad -ia^\sigma M^{\lambda\rho} \frac{\hat{\partial}_\lambda}{1 + ia^\nu \hat{\partial}_\nu} - i(n-1)\frac{(a^\rho \hat{\partial}_\sigma - a^\sigma \hat{\partial}_\rho)}{1 + ia^\nu \hat{\partial}_\nu}.
\end{aligned} \tag{C.3.9}
$$

Casimir operators are again given by the deformed d'Alembert operator (C.3.2) and the generalisation of the square of the Pauli-Lubanski vector

$$
\begin{aligned}
W_{i+1}^2 &= W_{\mu_1 \ldots \mu_{2i-1}} W^{\mu_1 \ldots \mu_{2i-1}}, \qquad i = 1, \ldots \frac{d-2}{2}, \quad d = n+1 = (2k+1)+1, \\
W_{\mu_1 \ldots \mu_{2i-1}} &= \epsilon_{\mu_1 \ldots \mu_n} M^{\mu_{2i}\mu_{2i+1}} \ldots M^{\mu_{n-2}\mu_{n-1}} \hat{D}^{\mu_n}.
\end{aligned} \tag{C.3.10}
$$

C.4 Fields

Transformation law of a scalar field under the κ-deformed Lorentz transformations is given by

$$
\delta_\omega \hat{\phi}(\hat{x}) = -\frac{1}{2}\omega^{\alpha\beta} \hat{L}_{\alpha\beta} \hat{\phi}(\hat{x}), \tag{C.4.1}
$$

where $\hat{L}^{\alpha\beta}$ is given by (C.2.5). For a vector field $\hat{V}_\mu$ associated with the Dirac derivative (C.3.1) we have

$$
\begin{aligned}
\delta_\omega \hat{V}_\mu &= -\frac{1}{2}\omega^{\alpha\beta}\Big(\hat{L}_{\alpha\beta}\hat{V}_\mu - \eta_{\beta\mu}\hat{V}_\alpha + \eta_{\alpha\mu}\hat{V}_\beta\Big) \\
&= -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{V}_\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \hat{V}_\mu].
\end{aligned} \tag{C.4.2}
$$

Also, for a vector field $\hat{A}_\mu$ related with the derivative $\hat{\partial}_\mu$ we have

$$
\delta_\omega \hat{A}_\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{A}_\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \hat{A}_\mu], \tag{C.4.3}
$$

with

$$
\begin{aligned}
[M_{\alpha\beta}, \hat{A}_\mu] &= \eta_{\beta\mu}\hat{A}_\alpha - \eta_{\alpha\mu}\hat{A}_\beta + \frac{i}{2}(a_\beta \hat{\partial}_\alpha - a_\alpha \hat{\partial}_\beta)\hat{A}_\mu + \frac{i}{2}\hat{\partial}_\mu(a_\beta \hat{A}_\alpha - a_\alpha \hat{A}_\beta) \\
&\quad + \frac{i}{2}(\eta_{\beta\mu}a_\alpha - \eta_{\alpha\mu}a_\beta)\hat{\partial}_\lambda \hat{A}^\lambda + \frac{1}{4}(\eta_{\beta\mu}a_\alpha - \eta_{\alpha\mu}a_\beta)\frac{a^\lambda(\hat{\partial}_\lambda \hat{\partial}_\omega \hat{A}^\omega - \hat{\partial}_\omega \hat{\partial}^\omega \hat{A}_\lambda)}{1 + \frac{i}{2}a^\gamma \hat{\partial}_\gamma} \\
&\quad + \frac{a^\lambda}{ia^\gamma \hat{\partial}_\gamma}\Big(1 + \frac{i}{2}a^\omega \hat{\partial}_\omega - \frac{ia^\omega \hat{\partial}_\omega}{\log(1 + ia^\tau \hat{\partial}_\tau)}\Big) \\
&\quad \Big(a_\alpha(2\hat{\partial}_\beta \hat{\partial}_\mu \hat{A}_\lambda - \hat{\partial}_\lambda \hat{\partial}_\mu \hat{A}_\beta - \hat{\partial}_\lambda \hat{\partial}_\beta \hat{A}_\mu) \\
&\quad - a_\beta(2\hat{\partial}_\alpha \hat{\partial}_\mu \hat{A}_\lambda - \hat{\partial}_\lambda \hat{\partial}_\mu \hat{A}_\alpha - \hat{\partial}_\lambda \hat{\partial}_\alpha \hat{A}_\mu)\Big).
\end{aligned} \tag{C.4.4}
$$

The last two terms vanish if $\hat{A}_\mu$ is replaced by $\hat{\partial}_\mu \hat{\phi}$. In (C.4.4) they are needed for the representation property (A.4).

The transformation law of a vector field $\hat{B}^\mu$ dual to $\hat{A}_\mu$ vector field follows from

$$\delta_\omega(\hat{A}_\lambda \hat{B}^\lambda) = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}(\hat{A}_\lambda \hat{B}^\lambda) \tag{C.4.5}$$

and it is given by

$$\delta_\omega \hat{B}^\mu = -\frac{1}{2}\omega^{\alpha\beta}\hat{L}_{\alpha\beta}\hat{B}^\mu + \frac{1}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, \hat{B}^\mu], \tag{C.4.6}$$

and

$$\begin{aligned}
[\hat{M}_{\alpha\beta}, \hat{B}^\mu] = {} & \delta^\mu_\beta \hat{B}_\alpha - \delta^\mu_\alpha \hat{B}_\beta + \frac{i}{2}(a_\beta \hat{B}_\alpha - a_\alpha \hat{B}_\beta)\hat{\partial}^\mu - \frac{a^\mu \hat{B}^\lambda \hat{\partial}_\lambda}{i a^\gamma \hat{\partial}_\gamma}(a_\beta \hat{\partial}_\alpha - a_\alpha \hat{\partial}_\beta) \\
& + \frac{1}{4}(a_\beta \hat{B}_\alpha - a_\alpha \hat{B}_\beta)\frac{a^\lambda \hat{\partial}_\lambda \hat{\partial}^\mu - a^\mu \hat{\partial}_\lambda \hat{\partial}^\lambda}{1 + \frac{i}{2} a^\gamma \hat{\partial}_\gamma} \\
& + \frac{1}{(i a^\gamma \hat{\partial}_\gamma)^2}\Big(1 - \frac{i a^\nu \hat{\partial}_\nu}{\log(1 + i a^\tau \hat{\partial}_\tau)}\Big) \\
& \Big(2 a^\mu \hat{B}^\lambda \hat{\partial}_\lambda (a_\beta \hat{\partial}_\alpha - a_\alpha \hat{\partial}_\beta) - \hat{B}^\mu a^\lambda \hat{\partial}_\lambda (a_\beta \hat{\partial}_\alpha - a_\alpha \hat{\partial}_\beta) \\
& - \hat{B}^\omega a^\lambda \hat{\partial}_\omega \hat{\partial}_\lambda (\delta^\mu_\alpha a_\beta - \delta^\mu_\beta a_\alpha)\Big).
\end{aligned} \tag{C.4.7}$$

The derivatives $\hat{D}_\mu$ and $\hat{\partial}_\mu$ can be transformed into each other using (C.3.1) and (C.3.4). Such a map exists between $\hat{A}_\mu$ and $\hat{V}_\mu$ as well. We demand that

$$\hat{V}_\mu = E_\mu{}^\rho \hat{A}_\rho, \qquad \hat{A}_\rho = (E^{-1})_\rho{}^\mu \hat{V}_\mu, \tag{C.4.8}$$

such that it is consistent with (C.4.2) and (C.4.3). The solution is given by

$$\begin{aligned}
E_\alpha{}^\mu = {} & \delta^\mu_\alpha \frac{i a^\omega \hat{\partial}_\omega}{(1 + i a^\rho \hat{\partial}_\rho)\log(1 + i a^\sigma \hat{\partial}_\sigma)} - a_\alpha \hat{\partial}^\mu \frac{a^\omega \hat{\partial}_\omega}{(1 + i a^\rho \hat{\partial}_\rho)(2 + i a^\gamma \hat{\partial}_\gamma)\log(1 + i a^\lambda \hat{\partial}_\lambda)} \\
& + \frac{i a^\mu \hat{\partial}_\alpha}{1 + i a^\rho \hat{\partial}_\rho}\Big(\frac{1}{i a^\omega \hat{\partial}_\omega} - \frac{1}{\log(1 + i a^\omega \hat{\partial}_\omega)}\Big) \\
& - \frac{a_\alpha a^\mu \hat{\partial}_\rho \hat{\partial}^\rho}{(1 + i a^\sigma \hat{\partial}_\sigma)(2 + i a^\gamma \hat{\partial}_\gamma)}\Big(\frac{1}{2} + \frac{1}{i a^\omega \hat{\partial}_\omega} - \frac{1}{\log(1 + i a^\omega \hat{\partial}_\omega)}\Big).
\end{aligned} \tag{C.4.9}$$

It depends only on derivatives and not on coordinates.

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