D-branes on Calabi-Yau Spaces

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Zusammenfassung

In dieser Arbeit werden die Eigenschaften von D-Branen auf Calabi-Yau-Räumen untersucht. Kompaktifizierungen von Typ II Stringtheorien auf diesen Räumen, bei denen D-Branen hinzugefügt werden, führen zu $\mathcal{N} = 1$ supersymmetrischen Eichtheorien auf dem Weltvolumen dieser D-Branen.

Sowohl die Calabi-Yau-Räume als auch die D-Branen besitzen im allgemeinen einen Moduliraum. Wir untersuchen die Abhängigkeit der Eichtheorie von der Wahl der Moduli, insbesondere derjenigen der Kählerstrukutur der Calabi-Yau-Mannigfaltigkeit. Dazu wählen wir zwei Punkte in diesem Moduliraum, die dadurch ausgezeichnet sind, dass es eine explizite Beschreibung des Spektrums der D-Branen gibt. Der eine Punkt entspricht einer Mannigfaltigkeit mit grossem Volumen, auf der die D-Branen durch klassische Geometrie von Vektorbündeln beschrieben werden. Am anderen Punkt ist die Ausdehnung der Mannigfaltigkeit kleiner als ihre Quantenfluktuationen, so dass die klassische Geometrie ihre Bedeutung verliert und durch eine konforme Feldtheorie ersetzt werden muss. Der Witten-Index im offenen String-Sektor ist unabhängig von der Variation dieser Moduli und dient, zusammen mit der Mirrorsymmetrie, als Werkzeug um die beiden Beschreibungen zu vergleichen.

Wir geben eine ausführliche und allgemeine Darstellung dieser beiden Beschreibungen für die Klasse der Fermatschen Hyperflächen in gewichtet-projektiven Räumen. Wir führen den Vergleich in vielen, repräsentativen Beispielen explizit durch. Darunter sind Mannigfaltigkeiten mit elliptischen und $K3$-Faserungen und solche, deren Moduliraum sich in einen Moduliraum einer anderen Mannigfaltigkeit einbetten lässt. Ein Schwerpunkt wird dabei auf D4-Branen, insbesondere die Dimension ihrer Moduliräume gelegt.

Mit den entwickelten Methoden können wir die modifizierte geometrische Hypothese von Douglas, die im wesentlichen besagt, dass die Eigenschaften dieser D-branen bzw. dieser Eichtheorien zum einen Teil durch klassiche Geometrie und zum anderen Teil durch die Mirrorsymmetrie bestimmt werden können, durch unsere Resultate weiter bestätigen. Eine Besonderheit dieser Eichtheorien ist das Auftreten von Linien marginaler Stabilität, an denen BPS-Zustände zerfallen können. Wir zeigen die Existenz solcher Linien im Rahmen dieser Klasse von Calabi-Yau-Räumen auf zwei verschiedene Weisen und diskutieren den Zusammenhang zur Bildung gebundener Zustände. Von besonderem Interesse ist die D0-Bran, deren Auftreten in dieser Beschreibung erklärt wird.

Abstract

In this thesis the properties of D-branes on Calabi–Yau spaces are investigated. Compactifications of type II string theories on these spaces to which D-branes are added lead to $\mathcal{N} = 1$ supersymmetric gauge theories on the world-volume of these D-branes.

Both the Calabi–Yau spaces and the D-branes have in general a moduli space. We examine the dependence of the gauge theory on the choice of the moduli, in particular those of the Kähler structure of the Calabi–Yau manifold. For this purpose we choose two points in this moduli space which are distinguished by the fact that there exists an explicit description of the spectrum of the D-branes. One of these points corresponds to a manifold in the large volume limit on which the D-branes are described by classical geometry of vector bundles. At the other points the size of the manifold is smaller than its quantum fluctuations such that the classical geometry looses its meaning and has to be replaced by a conformal field theory. The Witten index in the open string sector is independent of the variation of these moduli and serves, together with mirror symmetry, as a tool to compare the two descriptions.

We give an extensive and general presentation of these two descriptions for the class of Fermat hypersurfaces in weighted projective spaces. We explicitly carry out the comparison in many representative examples. Among them are manifolds admitting elliptic and $K3$-fibrations and manifolds whose moduli space can be embedded into the moduli space of another manifold. One main focus is on D4-branes, in particular on the dimension of their moduli space.

Using the methods developed we are able to further confirm with our results the modified geometric hypothesis by Douglas. It essentially states that the properties of these D-branes or of these gauge theories can be determined partly by classical geometry, partly by mirror symmetry. A peculiarity of these gauge theories is the appearance of lines of marginal stability at which BPS states can decay. We show the existence of such lines in the framework of this class of Calabi–Yau spaces in two different ways and discuss the connection to the formation of bound states. Of particular interest is the D0-brane whose appearance in this framework is explained.

Contents

1. Introduction

The realization that, apart from strings, there are further basic, dynamical objects in string theory, namely D-branes has initiated a revolution in the understanding of this theory [1]. These D-branes are objects on which open strings can end and their nature is non-perturbative in the closed string sector. They are extended objects of any dimension $p < 10$. The massless modes of the open strings ending on them define a gauge theory on their world-volume. The low-energy limit of this world-volume theory is the dimensional reduction of $\mathcal{N} = 1$ Super-Yang-Mills theory in ten dimensions down to $p+1$ dimensions.

One important consequence from their understanding is, in particular, that it opened new ways of constructing supersymmetric gauge theories embedded in a more fundamental theory which can encompass the (minimally supersymmetric) Standard Model. The old method was to take the heterotic string theory in ten dimensions and compactify it on a three-dimensional Calabi–Yau space in order to get an $\mathcal{N} = 1$ $D = 4$ supergravity theory, i.e. a theory with four supercharges. This space is a complex three-dimensional compact manifold admitting a Ricci-flat Kähler metric. However, up to date, a realistic model satisfying all the constraints of the Standard Model has never been achieved.

In global supersymmetry, realistic models are only possible for $\mathcal{N} = 1$. This follows from a basic observation in particle physics: the massless fermions of helicity $\frac{1}{2}$ do not transform under $SU(3) \times SU(2) \times U(1)$ the same way the helicity $-\frac{1}{2}$ fermions transform. By looking at the supersymmetry algebra one sees that in global supersymmetry with $\mathcal{N} > 1$, the helicity $\frac{1}{2}$ and helicity $-\frac{1}{2}$ fermions necessarily transform identically. In other words such theories with $\mathcal{N} > 1$ are non-chiral while the Standard Model is.

One of the central properties of D-branes is the following. Vacuum states containing a single D-brane are not annihilated by all of the supercharges but only by half of them. If we want to construct an $\mathcal{N} = 1$ theory in four dimensions, then we should look for an $\mathcal{N} \geq 2$ theory and add D-branes to it. The simplest possibility is to compactify a type II string theory on a flat six-torus and place several D-branes at specific angles in such a way that only four supercharges remain unbroken. Alternatively, one might use $K3 \times T^2$ as a compactifying space.

While in these cases the D-branes have to be arranged in a particular way, it is possible to compactify the ten-dimensional theory such that the restrictions on the number and angles are less severe, in particular adding a single D-brane is already sufficient. This is realized by starting with a compactification of type II string theory on a Calabi–Yau space resulting in a theory in four dimensions whose low-energy limit is $\mathcal{N} = 2$ supergravity. Such compactifications come in families parametrized by the Kähler and complex structure deformations of the Calabi–Yau manifold. The parameter spaces are generally called moduli spaces, even though they are not always moduli spaces in the strict mathematical sense. We denote the moduli space of a Calabi–Yau manifold by $\mathcal{M}_{\mathrm{CY}}$. The D-branes are then added by wrapping them around submanifolds of the Calabi–Yau space. If such a configuration preserves $\mathcal{N} = 1$ supersymmetry, then the submanifold is called a supersymmetric cycle.

We require that the D-brane fills out the four non-compact space-time dimensions and allow it have p dimensions along the compact Calabi–Yau manifold. Adding D-branes introduces a gauge bundle on the submanifold wrapped by the D-brane. Parallel D-branes do not exert any force onto each other and therefore can be stacked on top of each other. The number r of D-branes in this stack determines the rank of the gauge group and therefore the rank of the gauge bundle. We will assume that, for a fixed Calabi–Yau space, this bundle has a direct product structure inherited from the base which is the product of space-time and the supersymmetric cycle in the Calabi–Yau space. Subsequently, we will only consider the factor belonging to the Calabi–Yau threefold. Decreasing the size of this manifold, the world-volume theory will be an $\mathcal{N} = 1$ supersymmetric gauge theory in four space-time dimensions. This theory comes in a family, too, parametrized by the deformations of the vector bundle and by the deformations of the supersymmetric cycle. The information contained in the gauge bundle on the supersymmetric cycle is then encoded e.g. in the superpotential of this gauge theory.

The data of this theory is a gauge group G, a complex manifold X parametrized by chiral superfields ϕ^i describing these deformations, a Kähler potential K on X, an action by holomorphic isometries of G on X, a superpotential W which is a holomorphic and G-invariant function on X and moment maps μ which are the D-terms. If G contains $U(1)$ factors, each of these can have an associated real constant ζ_a, the Fayet-Illiopoulos parameter. The space $\mathcal{M}_D$ describing families of D-branes is then the solution to $\text{grad}W = 0$ in the symplectic quotient of X by G. However, not all solutions correspond to points in the quotient, only stable objects do, which depends on the specific moment maps. In addition, there is a holomorphic function, called the gauge kinetic function, which determines the coupling constants.

We want to know what are the possible gauge theories that can be described in this way. This is equivalent to looking for a classification of all supersymmetry preserving D-branes at each point in the Calabi–Yau moduli space and finding their world-volume moduli spaces $\mathcal{M}_D$. We are interested in the change of this moduli space $\mathcal{M}_D$ if we vary the parameters of the Calabi–Yau manifold and in the corresponding change of the spectrum of the gauge theory. We want to understand in particular what happens if we include stringy effects. In this thesis we work in the limit $g_s \to 0$ but allow for $\frac{\alpha'}{R^2} \to 1$ where R is a characteristic size of the Calabi–Yau manifold. This means that we allow the Calabi–Yau manifold to become so small that one cannot distinguish the "manifold" from its quantum fluctuations such that any classical notion of geometry looses its meaning. We can achieve this by tuning the Kähler structure parameters to a particular point in their moduli space. At this point we need a quantum description of the Calabi–Yau which is given by a conformal field theory called the Gepner model. Since D-branes are naturally described as boundary conditions in a conformal field theory we obtain two frameworks to study them: classical geometry for large Calabi–Yau spaces and the Gepner model for small Calabi–Yau spaces.

It is obvious to ask to what extent these effects lead to qualitative changes in the description of the physics of D-branes, i.e. of the low-energy effective action, the dimension of the moduli space, the types of singularities, the spectrum and so on. A natural starting point [2] for an answer is to state a geometric hypothesis that all these properties are the same as predicted by naive geometric considerations. This is motivated by the fact that in theories with 16 and 8 supercharges, corresponding to flat, toroidal and $K3$ compactifications mentioned above this hypothesis is essentially true [3] due to the large amount of supersymmetry. However, in theories with four supercharges, i.e. the Calabi–Yau compactifications of our interest, D-branes are much less understood. This makes them very interesting objects to study.

The most important effect on the low-energy theory in the presence of D-branes is the appearance of new massless states [4] at special points. This phenomenon is roughly described as follows. We can tune the Kähler moduli of the Calabi–Yau threefold to another particular point at which the conformal field theory breaks down. At this point a cycle of the Calabi–Yau space vanishes. A D-brane wrapping such a cycle becomes massless and therefore leads to new physical degrees of freedom in the low-energy theory. Taking these degrees of freedom into account in the description of the theory shows it to be well-behaved, i.e. the string theory has no singularity. Moreover, these degrees of freedom correspond to the W-bosons in the gauge theory and therefore lead to an enhancement of gauge symmetry at this point in moduli space.

This is just one example of effects that can appear when we vary the parameters of the Calabi–Yau threefold. However, we are interested in the limit where the Calabi–Yau manifold becomes small. Using the description of D-branes in both of these frameworks – geometry and conformal field theory – we can answer these questions stated above at least partially. This line of research has been initiated by Douglas and his collaborators in [5] where the direct comparison of the two descriptions of D-branes has been performed in a particular example, the quintic threefold. The main tool in this comparison is the Witten index which can be computed in both limits, at small and at large volume. It contains the necessary information about the supersymmetric spectrum. A few other examples have been subsequently studied which shed more and more light on the stringy effects.

It turns out that the changes are not only quantitative, i.e. that the masses and couplings of the D-branes change, but also qualitative: The spectrum and the moduli space undergo radical changes in such a way that e.g. geometric branes are destabilized at small volume. These invalidate the geometric hypothesis. However, we have another important technique at our disposal: mirror symmetry. Almost all Calabi–Yau threefolds have a mirror manifold. It has been known for a long time that some problems are easier to study by mapping them to the mirror manifold, solving them there, and mapping the solution back to the original manifold. Applying this idea to the D-branes on Calabi–Yau spaces leads to a modified geometric hypothesis [2]. This essentially states that some D-brane questions are geometric on the original Calabi–Yau threefold and others are geometric on the mirror manifold. The remaining questions can be answered by applying mirror symmetry. Therefore mirror symmetry provides the second important tool for this work.

Mirror symmetry has turned out to be a very fascinating concept in mathematics. The investigation of D-branes on Calabi–Yau spaces includes other mathematically very interesting problems like the classification of holomorphic vector bundles on a Calabi–Yau threefold. There are almost no results known in the mathematics literature. However there are quite a few results on the classification of holomorphic vector bundles on complex surfaces. Since some of these surfaces appear as hypersurfaces in a Calabi–Yau threefold we can try and use these results and compare them to those obtained in the Gepner model. For this reason we will focus mostly on D4-branes wrapping divisors in the Calabi–Yau manifold although we will include other D-branes in the general analysis.

The first part of this thesis provides the necessary background for the closed string sector of our theories. In Chapter 2 we review the properties of $\mathcal{N} = (2, 2)$ superconformal field theories which we require in order to have an $\mathcal{N} = 2$ supergravity theory in four space-time dimensions. The main points in this chapter are the introduction of the Gepner model and the interpolation between the different descriptions of string compactifications on a Calabi–Yau manifold by tuning the Kähler moduli. This yields a "geographical" map of the moduli space of the Calabi–Yau manifold which helps navigating when moving in this space. Also, the concept of the Witten index and the basic facts leading to mirror symmetry as well as its properties are reviewed.

Chapter 3 is about the geometric side of the closed string sector. A particular class of Calabi–Yau spaces is introduced, namely hypersurfaces in toric varieties for which mirror symmetry is manifest by construction. In this chapter many useful properties of such Calabi–Yau spaces are collected. Our main focus is on hypersurfaces in blown-up weighted projective spaces. We give all the necessary formulas to determine the geometry and apply them to several interesting examples.

In the second part of this thesis we include the open string sector by introducing the D-branes in Chapter 4. There they are described as boundary conditions in a conformal field theory, leading to the concept of boundary conformal field theories. We review the application of this concept to $\mathcal{N} = (2, 2)$ superconformal field theories introduced in Chapter 2 and, in particular, to the Gepner model. The rational boundary states are constructed and their Witten index and number of moduli are computed. This provides the properties and data for one side of the comparison of D-branes on Calabi–Yau spaces.

In Chapter 5 we study the geometry of D-branes on the class of Calabi–Yau manifolds discussed in Chapter 3 by relating their properties to those of vector bundles and sheaves on these manifolds. The main focus lies on the charge lattice, the BPS central charge and on the moduli of vector bundles. The BPS central charge is the quantity which connects the classical geometric side to the conformal field theoretic side. We generalize results that were known for particular examples to the whole class of Calabi–Yau manifolds in Chapter 3. We also point out that we can obtain more information for D4-branes which makes them particularly interesting for the comparison.

The main results are given in Chapter 6 where the connection between the two descriptions of D-branes given in the previous chapters using the Witten index is explained and applied to the examples discussed in Chapter 3. We motivate and state the modified geometric hypothesis and how it can be tested. A number of tests will be carried out by using the results of the comparison. We emphasize on so-called rational D4-branes in order to make the comparison explicit and make quantitative as well as qualitative statements. General observations on D-geometric aspects are also presented. In particular, we extend the comparison to cases where the Kähler moduli space has several large volume limits with different geometries and several Gepner points corresponding to different Gepner models.

In the Appendices A and B we explain in two examples two different methods how we can quantitatively move around in the Calabi–Yau moduli space, i.e. how to transport the information from the point where the classical geometry is valid to the point where the description by a Gepner model is necessary. Finally, Appendix C contains the topological data of those Calabi–Yau spaces we explicitly use for the comparison.

2. $\mathcal{N} = (2,2)$ Superconformal Field Theories

2.1. General facts

$\mathcal{N} = (2,2)$ superconformal field theories lie at the heart of compactifications of type II string theories on Calabi–Yau spaces. In this chapter we review those aspects which will become relevant to the study of D-branes on these spaces later on. For more details see [6], [7] and [8]

The $\mathcal{N} = (2,2)$ superconformal algebra and its representations

Our main interest lies in the $\mathcal{N} = 2$ superconformal algebra [9], [10], [11]. This algebra is generated by the energy-momentum tensor $T(z)$, two weight 3/2 supercurrents $G^+(z)$ and $G^-(z)$ as well as a $U(1)$ current $J(z)$ forming a supermultiplet

$$\begin{array}{c} T(z) \\ G^+(z) \quad G^-(z) \\ J(z) \end{array} \tag{2.1}$$

They have the following operator product expansions

$$\begin{aligned}
T(z)T(w) &= \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.} && \text{(2.2a)}\\
T(z)G^\pm(w) &= \frac{\frac{3}{2}}{(z-w)^2} G^\pm(w) + \frac{\partial_w G^\pm(w)}{z-w} + \text{reg.} && \text{(2.2b)}\\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \text{reg.} && \text{(2.2c)}\\
G^+(z)G^-(w) &= \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} + \text{reg.} && \text{(2.2d)}\\
J(z)G^\pm(w) &= \pm\frac{G^\pm(w)}{z-w} && \text{(2.2e)}\\
J(z)J(w) &= \frac{\frac{c}{3}}{(z-w)^2} + \text{reg.} && \text{(2.2f)}
\end{aligned}$$

Eq. (2.2a) defines the usual ($\mathcal{N} = 0$) conformal algebra with central charge c. Eqns. (2.2b) and (2.2c) imply that $G^\pm(z)$ and $J(z)$ are primary fields of the Virasoro algebra with weight $\frac{3}{2}$ and 1, respectively. (2.2e) implies that $G^\pm(z)$ have the $U(1)$ charges ± 1. From (2.2f) we see that $J(z)$ can be bosonized and written as

$$J(z) = i\sqrt{\frac{c}{3}}\partial_z \varphi(z) \tag{2.3}$$

where $\varphi(z)$ is a free scalar boson. We can re-express this data in terms of modes by writing

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \tag{2.4a}$$

$$G^{\pm}(z) = \sum_{n=-\infty}^{\infty} G^{\pm}_{n\pm\eta\pm\frac{1}{2}} z^{-(n\pm\eta\pm\frac{1}{2})-\frac{3}{2}} \tag{2.4b}$$

$$J(z) = \sum_{n=-\infty}^{\infty} J_n z^{-n-1} \tag{2.4c}$$

The parameter η in the mode expansion (2.4b) lies in the range $0 \leq \eta < 1$. This parameter controls the boundary conditions on the supercurrents. If we change $z \to e^{2\pi i} z$ then

$$G^{\pm}(e^{2\pi i} z) = e^{\mp 2\pi i \eta} G^{\pm}(z) \tag{2.5}$$

where $\eta = 0$ corresponds to the Neveu-Schwarz (NS) sector in which $G^{\pm}$ are periodic and $\eta = \frac{1}{2}$ to the Ramond (R) sector in which $G^{\pm}$ are anti-periodic. In terms of these modes, the $\mathcal{N} = 2$ superconformal algebra takes the form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{m+n,0} \tag{2.6a}$$

$$[L_n, G^{\pm}_s] = \left(\frac{n}{2} - s\right) G^{\pm}_{n+s} \tag{2.6b}$$

$$[L_n, J_m] = -mJ_{m+n} \tag{2.6c}$$

$$\begin{aligned}\{G^{+}_{n+\eta+\frac{1}{2}}, G^{-}_{m-\eta-\frac{1}{2}}\} &= 2L_{m+n} + (n-m+2\eta+1)J_{n+m} \\ &\quad + \frac{c}{3}\left((n+\eta+\frac{1}{2})^2 - \frac{1}{4}\right)\delta_{m+n,0}\end{aligned} \tag{2.6d}$$

$$[J_n, G^{\pm}_{m\pm\eta\pm\frac{1}{2}}] = \pm G^{\pm}_{m+n\pm\eta\pm\frac{1}{2}} \tag{2.6e}$$

$$[J_m, J_n] = \frac{c}{3} m\delta_{m+n,0} \tag{2.6f}$$

Here s can be either integral or half-integral.

Out of the combination $G(z) = G^+(z) + G^-(z)$ and $T(z)$ one can form the $\mathcal{N} = 1$ superconformal algebra with eqs. (2.2a), (2.2b) and

$$G(z)G(w) = \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2T(w)}{z-w} + \text{reg.} \tag{2.7}$$

By adding the right-movers $\overline{T}_n, \overline{G}_s$ and $\overline{J}_m$ we obtain the $\mathcal{N} = (2,2)$ superconformal algebra. The finite-dimensional subalgebra in the NS sector, generated by $L_{0,\pm1}$, J_0 and $G^{\pm}_{\pm\frac{1}{2}}$ is $OSp(2|2)$ and corresponds to the $\mathcal{N} = 2$ supersymmetry algebra, and similarly in the right-moving sector. Its automorphism group is $O(2)$. The Hamiltonian and the momentum are $H = L_0 + \overline{L}_0$ and $P = L_0 - \overline{L}_0$, respectively. We define

$$F_V = J_0 + \overline{J}_0 \qquad\qquad F_A = J_0 - \overline{J}_0 \tag{2.8}$$

to be the generators of the vector and axial R-symmetry, respectively.

The spectrum of the $\mathcal{N} = 2$ superconformal field theory is determined by the representation theory of the $\mathcal{N} = 2$ superconformal algebra. Unitary (irreducible) representations of this algebra are those satisfying the hermiticity conditions

$$L_n^{\dagger} = L_{-n} \qquad\qquad J_n^{\dagger} = J_{-n} \qquad\qquad \left(G^{\pm}_s\right)^{\dagger} = G^{\mp}_{-s} \tag{2.9}$$

and satisfying the requirement that the internal product in the Fock space should be positive definite. They can be built up in a systematic manner using the notion of highest weight states. This is done by dividing the modes L_n, $G_r^{\pm}$ and J_m into raising and lowering operators. The zero modes are used to label the states in a representation. The modes with positive indices can be viewed as lowering operators as they lower the L_0 eigenvalue of a state. Since L_0 is the left-moving part of the Hamiltonian H we can assume that the eigenvalue is bounded from below. In the NS sector, the highest weight state $|\phi\rangle$ is then defined through the following properties

$$\begin{aligned} L_0|\phi\rangle &= h_\phi|\phi\rangle && (2.10a)\\ J_0|\phi\rangle &= q_\phi|\phi\rangle && (2.10b)\\ L_n|\phi\rangle &= 0 \qquad n>0 && (2.10c)\\ G_r^{\pm}|\phi\rangle &= 0 \qquad r>0 && (2.10d)\\ J_m|\phi\rangle &= 0 \qquad m>0 && (2.10e)\end{aligned}$$

If we are in the R sector then we also have to deal with the $G_0^{\pm}$ modes. If a state $|\phi\rangle$ in the Ramond sector satisfies

$$G_0^{\pm}|\phi\rangle = 0 \tag{2.11}$$

then we say it is a Ramond ground state. A representation of the $\mathcal{N}=2$ superconformal algebra can is built by acting on $|\phi\rangle$ with all possible combinations of the raising operators $\prod L_{n_i}J_{m_j}G_{r_k}|\phi\rangle$, that is with modes having negative mode numbers. By the operator-state isomorphism we can think of the state $|\phi\rangle$ as being built from the action of the superconformal primary field $\phi(z)$ according to $|\phi\rangle = \phi(0)|0\rangle$. The constraint that $|\phi\rangle$ be a highest weight state is then equivalent to $\phi(z)$ satisfying

$$\begin{aligned} T(z)\phi(w) &= \frac{h_\phi}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + \text{reg.} && (2.12a)\\ G^{\pm}(z)\phi(w) &= \frac{(G^{\pm}_{-\frac{1}{2}}\phi)(w)}{z-w} + \text{reg.} && (2.12b)\\ J(z)\phi(w) &= \frac{q_\phi}{z-w}\phi(w) + \text{reg.} && (2.12c)\end{aligned}$$

where h_ϕ is the conformal weight and q_ϕ is the $U(1)$ charge of the state $|\phi\rangle$. Eventually, we will be interested in type II string theory. The string consists of left-movers (holomorphic fields) and right-movers (anti-holomorphic fields). The underlying algebra is then a $\mathcal{N}=(2,2)$ superconformal algebra, consisting of a right-moving (holomorphic) $\mathcal{N}=2$ superconformal algebra generated by L_n, G_r and J_m and a left-moving (anti-holomorphic) $\mathcal{N}=2$ superconformal algebra generated by $\bar{L}_n, \bar{G}_r$ and $\bar{J}_m$ subject to the level-matching condition $L_0 = \bar{L}_0$.

Chiral fields

There is a distinguished subset of $\mathcal{N}=2$ superconformal primary fields known as chiral primary fields whose importance will become clear in the following. By definition, a chiral primary field is a primary field ϕ that creates a state $|\phi\rangle$ which is annihilated by the operator $G^+_{-\frac{1}{2}}$, that is

$$G^+_{-\frac{1}{2}}|\phi\rangle = 0 \tag{2.13}$$

In the operator product language, this implies that

$$G^+(z)\phi(w) = \text{reg.} \tag{2.14}$$

that is, there is no singularity in the product. Similarly, an antichiral primary field is defined by

$$G^{-}_{-\frac{1}{2}}|\phi\rangle = 0 \tag{2.15}$$

On the anti-holomorphic side chiral and antichiral primary fields are defined by replacing $G^{\pm}$ with $\overline{G}^{\pm}$. In this way, we obtain four kinds of particular primary fields, the (c,c) fields which are chiral in both the holomorphic and anti-holomorphic sense, the (a,c) fields which are antichiral in the holomorphic sense and chiral in the anti-holomorphic sense, as well as their complex conjugates, the (a,a) fields and the (c,a) fields. They are interesting because of the following three important properties. First, there are a finite number of them in any non-degenerate $\mathcal{N} = 2$ superconformal field theory, for which the spectrum of L_0 is discrete, i.e. for a chiral primary field ϕ

$$h_\phi \leq \frac{c}{6} \tag{2.16}$$

Second, they satisfy

$$h_\phi = \pm\frac{q_\phi}{2} \tag{2.17}$$

where the plus sign refers to chiral fields and the minus sign to antichiral fields. Finally, third, they yield a non-singular and closed ring under the operation of operator product, i.e. for chiral primary fields $\phi(z)$ and $\chi(w)$ we have no singular terms in

$$(\phi\chi)(z) = \lim_{w\to z} \phi(w)\chi(z) \tag{2.18}$$

and $\phi\chi$ is then either again a chiral primary or zero. These properties are all proven by repeated applications of the $\mathcal{N} = 2$ superconformal algebra in (2.6). For the details see [12].

Spectral flow

There is an isomorphism of algebras for all the superconformal algebras parametrized by different values of η [13]. In terms of the modes, it is explicitly given by

$$\begin{aligned} L'_n &= L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0} && \text{(2.19a)}\\ G'^{\pm}_r &= G^{\pm}_{r\pm\eta} && \text{(2.19b)}\\ J'_n &= J_n + \frac{c}{3}\eta\delta_{n,0} && \text{(2.19c)} \end{aligned}$$

This isomorphism can be extended to the representations of the superconformal algebra. Given an (infinite) collection of states $|f\rangle$ with conformal weight h and $U(1)$ charge q providing such a representation with $\eta = 0$ we can construct an isomorphic collection of states $|f_\eta\rangle$ that constitute a representation of the algebra for non-zero η. If U_η is a unitary map which on the level of operator satisfies

$$\begin{aligned} L'_n &= U_\eta L_n U_\eta^{-1} && \text{(2.20a)}\\ G'_r &= U_\eta G_r U_\eta^{-1} && \text{(2.20b)}\\ J'_n &= U_\eta J_n U_\eta^{-1} && \text{(2.20c)} \end{aligned}$$

then at the level of states in the representation of the algebra, the corresponding state is

$$|f_\eta\rangle = U_\eta|f\rangle \tag{2.21}$$

with conformal weight and $U(1)$ charge

$$h_\eta = h - \eta q + \frac{c}{6}\eta^2 \tag{2.22}$$

$$q_\eta = q - \frac{c}{3}\eta \tag{2.23}$$

Using (2.3) any field f which creates the state $|f\rangle$ with $U(1)$ charge q can be written as

$$f(z) = \hat{f}(z) e^{iq\sqrt{\frac{3}{c}}\varphi(z)} \tag{2.24}$$

where $\hat{f}(z)$ is a neutral field. Then the field f_η which creates the state $|f_\eta\rangle$ in the η-twisted sector can be explicitly written as

$$f_\eta(z) = \hat{f}(z) e^{i\sqrt{\frac{3}{c}}\left(q - \frac{c}{3}\eta\right)\varphi(z)} \tag{2.25}$$

i. e. spectral flow is accomplished by shifting the bosonic exponential. From the last two equations we read off that

$$U_\eta = e^{-i\sqrt{\frac{c}{3}}\eta\varphi} \tag{2.26}$$

The GSO projection and modular invariance require that both the NS sector ($\eta = 0$) and the R sector ($\eta = \frac{1}{2}$) be included in the Hilbert space of our theory. As the NS sector gives rise to space-time bosons and the R sector gives rise to space-time fermions, the spectral flow operator $U_{\frac{1}{2}}$ has a space-time interpretation as supersymmetry generator since it takes chiral primary states to states that are annihilated by $G_0^\pm$, i.e. Ramond ground states, and those to antichiral primary states.

The operator $U_{\frac{1}{2}}$ corresponds to the space-time supersymmetry operator which is (at worst) semi-local with respect to all states in the theory. In fact, when $c = 3n$, the ground state $h = q = 0$ of the NS sector is mapped onto states $\Sigma^\pm(z)$ with $h = \frac{n}{8}$, $q = \pm\frac{n}{2}$ in the R sector. From (2.24) we see that an arbitrary field f will have such an operator product expansion with $U_{\frac{1}{2}}$ if its $U(1)$ charge q is an odd integer. Hence, we can conclude that space-time supersymmetry will ensue if we project our theory (in the sense of conformal field theory quotients) onto one with odd integer $U(1)$ charges. This has been established in [14], [15] and [16].

Note that, if we choose $\eta = -1$, we see that the (unique) identity operator flows to a (unique) operator $\Omega(z)$ of charge $q = \frac{c}{3}$ which is a chiral primary operator that saturates the bound (2.16). This shows that the top chiral primary operator is unique.

2.2. Examples

Free field theory

The simplest example is a free field theory consisting of a single complex boson $\phi = \phi^1 + i\phi^2$ and a free complex fermion $\psi = \psi^1 + i\psi^2$. It is relevant for the flat space-time part of the superstring compactification, to be discussed in Section 2.3 and for introducing notation. Being the superpartner of ϕ, ψ splits into a sum of a left-moving (holomorphic) complex fermion $\psi_+(z)$ with complex conjugate $\psi_+^*(z)$ and a right-moving (anti-holomorphic) complex fermion $\psi_-(\bar{z})$ with complex conjugate $\psi_-^*(\bar{z})$. The action then reads

$$S = \int_\Sigma \mathrm{d}^2 z \left(\partial\phi\bar{\partial}\phi^* + \psi_+^*\bar{\partial}\psi_+ + \psi_+\bar{\partial}\psi_+^* + \psi_-^*\partial\psi_- + \psi_-\partial\psi_-^*\right) \tag{2.27}$$

For the holomorphic part of this theory we compute

$$T(z) = -\partial\phi\partial\phi^* + \frac{1}{2}\psi_+^*\partial\psi_+ + \frac{1}{2}\psi_+\partial\psi_+^* \tag{2.28a}$$

$$G^+(z) = \frac{1}{2}\psi_+^*\partial\phi \tag{2.28b}$$

$$G^-(z) = \frac{1}{2}\psi_+\partial\phi^* \tag{2.28c}$$

$$J(z) = \frac{1}{4}\psi_+^*\psi_+ \tag{2.28d}$$

and one can check that these fields satisfy the operator product expansions of the $\mathcal{N} = 2$ superconformal algebra (2.2). This theory has central charge $c = 3$ (in both the holomorphic and the anti-holomorphic sectors) coming from the two bosonic degrees of freedom ($c = 2$) and the two fermionic degrees of freedom ($c = 1$). The (c, c) ring consists of $\{1, \psi_+, \psi_-, \psi_+\psi_-\}$ while the (a, c) ring consists of $\{1, \psi_+^*, \psi_-, \psi_+^*\psi_-\}$. Although a very simple theory, this example does play a key role in string theory as we will see in section 2.3. For further reference we note that the $\mathcal{N} = 1$ supercurrent in (2.7) is $G = G^+ + G^-$.

For later convenience we introduce the superspace formalism which will allow us to simplify the notation. In superspace with coordinates $x^0, x^1, \theta^\pm, \bar{\theta}^\pm$, supersymmetry is realized geometrically by the operators

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{2.29a}$$

$$\overline{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{2.29b}$$

The supersymmetry generators of (2.29a) and (2.29b) anticommute with the operators

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{2.30a}$$

$$\overline{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{2.30b}$$

which will be used in writing Lagrangians. In $\mathcal{N} = (2, 2)$ theories, the simplest type of superfield is a chiral (or (c, c)) superfield Φ which obeys

$$\overline{D}_\pm\Phi = 0 \tag{2.31}$$

and can be expanded as

$$\Phi(x, \theta) = \phi(x) + \sqrt{2}\theta^+\psi_+(x) + \sqrt{2}\theta^-\psi_-(x) + 2\theta^+\theta^-F(x) + \dots \tag{2.32}$$

where the dots involve only the derivatives of $\phi, \psi_\pm$. The hermitian conjugate of Φ is an anti-chiral (or (a, a)) superfield obeying $D_\pm\overline{\Phi} = 0$. A twisted chiral (or (c, a)) superfield Y satisfies [17]

$$\overline{D}_+Y = D_-Y = 0 \tag{2.33}$$

with an expansion

$$Y(x, \theta) = y(x) + \sqrt{2}\theta^+\overline{\chi}_+(x) + \sqrt{2}\bar{\theta}^-\chi_-(x) + 2\theta^+\bar{\theta}^-G(x) + \dots \tag{2.34}$$

In general, the $\theta = 0$ component of a (twisted) chiral field labeled by a capital letter $\Phi, \Sigma, V, Y, \dots$ will be denoted by the corresponding lower case letter $\phi, \sigma, v, y, \dots$.

Supersymmetric Lagrangians are constructed as superspace integrals. Integrating over the full superspace yields the D-terms while the integration over the chiral and twisted chiral subspaces give the F-terms and the twisted F-terms, respectively. In our example, the action (2.27) can be written as a D-term

$$S = \int_\Sigma \mathrm{d}^2x\mathrm{d}^4\theta\overline{\Phi}\Phi \tag{2.35}$$

Non-linear σ-model

As a second example we introduce the $\mathcal{N}=2$ superconformal non-linear σ-model generalizing the previous free field theory example by adding more bosonic fields together with their fermionic superpartners and demanding that the theory need no longer be free. As will see shortly, two copies of it yield a conformal field theory realization of a Calabi–Yau space. The idea is to interpret the bosonic fields as coordinates in a target space which might be a curved Riemannian manifold (X, g). In the previous example one can think of ϕ as a coordinate on the flat manifold $\mathbb{C}$ with trivial Euclidean metric. To define the theory we start with a Riemann surface Σ, a Riemannian manifold (X, g) of dimension n and a map $\phi : \Sigma \mapsto X$. Let $\mathcal{K}$ be the cotangent bundle on Σ. Then the fermions can be viewed as sections of a certain bundle as follows

$$\psi_+ \in \Gamma(\mathcal{K}^{\frac{1}{2}} \otimes \phi^*TX) \qquad \psi_- \in \Gamma(\mathcal{K}^{-\frac{1}{2}} \otimes \phi^*TX) \tag{2.36}$$

The action then is

$$\begin{aligned} S \;=\; \int_\Sigma \mathrm{d}^2z \bigg(&\frac{1}{2}\partial\phi^\mu\bar\partial\phi^\nu\left(g_{\mu\nu}(\phi) + iB_{\mu\nu}(\phi)\right) + \frac{i}{2}\psi_+^\mu(\overline{D}\psi_+)^\nu g_{\mu\nu}(\phi) \\ &+\frac{i}{2}\psi_-^\mu(D\psi_-)^\nu g_{\mu\nu}(\phi) + \frac{1}{4}R_{\mu\nu\rho\sigma}\psi_+^\mu\psi_+^\nu\psi_-^\rho\psi_-^\sigma\bigg) \end{aligned} \tag{2.37}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the metric g of the target space. The B-field $B_{\mu\nu}$ is a harmonic 2-form on X and is related to the winding degrees of freedom of the string. The covariant derivatives are

$$D_\alpha\psi_\pm^\nu = \partial_\alpha\psi_\pm^\nu + \Gamma^\nu_{\rho\sigma}\partial_\alpha X^\rho\psi_\pm^\sigma \tag{2.38}$$

This action has $\mathcal{N}=(1,1)$ supersymmetry. In general, this theory possesses $\mathcal{N}=(2,2)$ supersymmetry only if the target space X is a Kähler manifold. In this case $TX = T^{1,0}X \oplus T^{0,1}X$ and

$$\psi_+ \in \Gamma(K^{\frac{1}{2}} \otimes \phi^*T^{1,0}X) \qquad \psi_- \in \Gamma(K^{-\frac{1}{2}} \otimes \phi^*T^{1,0}X) \tag{2.39a}$$
$$\overline{\psi}_+ \in \Gamma(K^{\frac{1}{2}} \otimes \phi^*T^{0,1}X) \qquad \overline{\psi}_- \in \Gamma(K^{-\frac{1}{2}} \otimes \phi^*T^{0,1}X) \tag{2.39b}$$

and the action reads

$$\begin{aligned} S \;=\; \int_\Sigma \mathrm{d}^2z \Big(&-g_{i\bar{j}}(\phi)\partial^\alpha\phi^i\partial_\alpha\phi^{\bar{j}} + ig_{i\bar{j}}(\phi)\overline{\psi}_+^{\bar{j}}(D_0 - D_1)\psi_+^i \\ &+ ig_{i\bar{j}}(\phi)\overline{\psi}_-^{\bar{j}}(D_0 + D_1)\psi_-^i + R_{i\bar{k}j\bar{l}}\psi_+^i\psi_-^j\psi_-^{\bar{k}}\psi_+^{\bar{l}}\Big) + S_{\text{top}} \end{aligned} \tag{2.40}$$

The term involving the B-field has been written as a topological term $S_{\text{top}} = \int_\Sigma \phi^*(B)$, i.e. it depends only on the cohomology class of B. If the action is normalized such that $\exp(2\pi i S_{\text{top}})$ is single-valued, then the B-field has a discrete symmetry, called Peccei-Quinn symmetry, $B \to B + \delta B$ which must be represented by integer cohomology in order that $\int_\Sigma \phi^*(\delta B)$ will be an integer and hence that $\exp(2\pi i \int_\Sigma \phi^*(\delta B))$ will equal 1. In other words, B is an element of the torus, $B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$. This symmetry will be important in Sections 3.4.2 and 5.6.

In the superspace formalism this action then simplifies to

$$S = \int \mathrm{d}^2 z \mathrm{d}^4\theta K(\Phi^i, \overline{\Phi}^{\bar{\imath}}) + S_{\text{top}} \tag{2.41}$$

where the Φ^i are chiral superfields whose lowest components are the bosonic coordinates above and

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial\phi^i \partial\phi^{\bar{\jmath}}} \tag{2.42}$$

is the Kähler metric. Let us also define the Kähler class $J = i g_{i\bar{\jmath}} \mathrm{d}\phi^i \wedge \mathrm{d}\phi^{\bar{\jmath}} \in H^{1,1}(X)$. J combines with $B = B_{i\bar{\jmath}} \mathrm{d}\phi^i \wedge \mathrm{d}\phi^{\bar{\jmath}}$ to yield the highest component of a complex chiral multiplet. We will denote $\omega = B + iJ$ as the complexified Kähler form although we will often only speak of the Kähler form. For $B_a + iJ_a \in H^{1,1}(X)$ we write $\omega = \sum_{a=1}^{h^{1,1}} t_a(B_a + iJ_a)$ and denote the $t_a \in \mathbb{C}$ as the (complexified) Kähler parameters.

The action (2.37) is conformally invariant only if the β-function of the metric g vanishes. To lowest order in α' this amounts to [18]

$$R_{i\bar{\jmath}} = 0 \qquad\qquad \mathrm{d}B = 0 \tag{2.43}$$

Thus, conformal invariance is achieved by choosing the target manifold to have a Kähler metric with vanishing Ricci tensor. As we will see in Section 3.1 these properties of a manifold define a Calabi–Yau manifold. Hence, a non-linear σ-model with a Calabi–Yau target space leads to a $\mathcal{N} = (2,2)$ superconformal field theory. Note however, that this condition is altered if terms of higher order in α' are included [19]. The spin field $\Sigma(z)$ and the field $\Omega(z)$ from Section 2.1 are the conformal field theory analogues of the covariantly constant spinor and holomorphic $(3,0)$ form on the Calabi–Yau manifold, respectively [20].

The generators T, $G^\pm$ and J of the $\mathcal{N} = 2$ superconformal algebra are as in the free field theory example in (2.28) with the insertion of the metric $g_{i\bar{\jmath}}$, e.g. $J = g_{i\bar{\jmath}}\psi^i_+\psi^{\bar{\jmath}}_+$. In addition the spectral flow operator (2.26) is

$$e^{i\sqrt{3}\varphi} = \Omega_{ijk}\psi^i_+\psi^j_+\psi^k_+ \tag{2.44}$$

where $J = i\sqrt{3}\partial\varphi$ and Ω_{ijk} is the holomorphic $(3,0)$-form of the Calabi–Yau manifold (see again Section 3.1).

Another fact from the superspace formalism that will become important in a moment is that the R-symmetries act on a superfield $\mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm)$ as

$$e^{i\alpha F_V}\mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm) = e^{i\alpha q_V}\mathcal{F}(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \tag{2.45}$$

$$e^{i\alpha F_A}\mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm) = e^{i\alpha q_A}\mathcal{F}(x, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\bar{\theta}^\pm) \tag{2.46}$$

where q_V and q_A are the vector and axial charges of $\mathcal{F}$.

As we have seen at the end of section 2.1 the (c,c) and (a,c) rings can be obtained from the Ramond ground states by spectral flow. Hence, it is sufficient to study these states which are zero energy modes. In a supersymmetric theory this implies that these states necessarily have zero momentum. Zero-momentum states, however, have in the low-energy approximation no spatial dependence on the world-sheet, hence the spatial dependence of these fields can be dropped in the action thereby effectively reducing this example to supersymmetric quantum mechanics on a Kähler manifold [21]. This allows us to find a beautiful characterization of the (c,c) and (a,c) rings of the non-linear σ-model.

The fermionic zero modes satisfy

$$\{\psi^i_{\pm,0}, \psi^j_{\pm,0}\} = \{\overline{\psi}^{\bar{\imath}}_{\pm,0}, \overline{\psi}^{\bar{\jmath}}_{\pm,0}\} = 0 \qquad \{\psi^i_{\pm,0}, \overline{\psi}^{\bar{\jmath}}_{\pm,0}\} = g^{i\bar{\jmath}} \tag{2.47}$$

We therefore see that we can interpret the $\overline{\psi}^{\bar{\jmath}}_{+,0}$ as creation operators and the $\psi^i_{+,0}$ as annihilation operators. Similarly we can choose $\overline{\psi}^{\bar{\jmath}}_{-,0}$ to be creation operators and $\psi^i_{-,0}$ to be annihilation operators. Whether we choose $\overline{\psi}^{\bar{\jmath}}_{+,0}$ or $\psi^i_{+,0}$ to be the creation operators is just a matter of convention. However, once this choice is made, it is of utmost importance which of the right-moving operators is chosen to be the creation operator as will become clear in this and the remaining sections. So, choose a Fock vacuum $|0\rangle$ for the zero mode sector of the Hilbert space of states such that

$$\psi^i_{+,0}|0\rangle = \overline{\psi}^{\bar{\jmath}}_{-,0}|0\rangle = 0 \tag{2.48}$$

Then a general state can be written as

$$|\Phi\rangle = \sum_{p,q} b_{i_1\dots i_p\bar{\jmath}_1\dots\bar{\jmath}_q}\psi^{i_1}_{-,0}\dots\psi^{i_p}_{-,0}\overline{\psi}^{\bar{\jmath}_1}_{+,0}\dots\overline{\psi}^{\bar{\jmath}_q}_{+,0}|0\rangle \tag{2.49}$$

where we also sum over all repeated indices. For a fixed value of the integers p and q the state $|\Phi\rangle$ has $U(1)_V\times U(1)_A$ charges $(-p,q)$.

Because of the anticommuting properties of these Fermi operators, this state is completely antisymmetric under the interchange of any two holomorphic, or any two anti-holomorphic indices. Therefore, the space of such states is isomorphic to the space of (p,q)-forms b on X, $\Lambda^{p,q}(X)$. Since we are interested in the Ramond ground states we are looking for states which are annihilated by the two supercharges $\overline{Q}_+\sim\overline{\psi}^{\bar{\jmath}}_{+,0}D_{\bar{\jmath}}$ and $Q_-\sim\psi^i_{-,0}D_i$ where D_i is the covariant derivative with respect to ϕ^i and this form is valid in the zero mode approximation. The former, acting on $|\Phi\rangle$, is equivalent to the Dolbeault operator $\bar{\partial}$ acting on the corresponding (p,q)-form b, and the latter, acting on $|\Phi\rangle$ is equivalent to the operator $\bar{\partial}^\dagger$ acting on the corresponding (p,q)-form b. Hence, demanding that these operators annihilate the state $|\Phi\rangle$ is mathematically equivalent to finding harmonic (p,q)-forms on X. Therefore, we see that the Ramond ground states are in one-to-one correspondence with the elements of the Dolbeault cohomology on X. Recalling that these Ramond ground states are related to the (a,c) fields by spectral flow and defining $H^*(Q)=\frac{\ker Q}{\operatorname{im} Q}$ with $Q=Q_-+\overline{Q}_+$ we have

$$H^{(a,c)}\equiv H^*(Q)\cong H^{*,*}_{\bar{\partial}}(X) \tag{2.50}$$

Now, remember that we had another choice for the right-moving creation operators. Thus, in addition to (2.48) we should also consider

$$\psi^i_{+,0}|0\rangle = \psi^j_{-,0}|0\rangle = 0 \tag{2.51}$$

In this case, a general state is of the form

$$|\Phi\rangle = \sum_{p,q} b^{i_1\dots i_p}_{\bar{\jmath}_1\dots\bar{\jmath}_q}\psi_{-,0,i_1}\dots\psi_{-,0,i_p}\overline{\psi}^{\bar{\jmath}_1}_{+,0}\dots\overline{\psi}^{\bar{\jmath}_q}_{+,0}|0\rangle \tag{2.52}$$

where $\psi_{-,0,i}=g_{i\bar{\jmath}}\psi^{\bar{\jmath}}_{-,0}$. These states have $U(1)_V\times U(1)_A$ charges (p,q). The same analysis as above shows that these states are in one-to-one correspondence with $(0,q)$-forms taking values in $\Lambda^pT^{1,0}X$. Applying the conditions that the supercharges $\overline{Q}_+\sim\overline{\psi}^{\bar{\jmath}}_{+,0}D_{\bar{\jmath}}$ and $\overline{Q}_-\sim\overline{\psi}^{\bar{\jmath}}_{-,0}D_{\bar{\jmath}}$ annihilate such a state shows it again to be harmonic, and by spectral flow it is related to (c,c) fields. Hence

$$H^{(c,c)}\equiv H^*(Q)\cong H^{0,*}_{\bar{\partial}}(X,\Lambda^*T^{1,0}X) \tag{2.53}$$

Note that by using the spectral flow operator (2.44) we can associate to any element in $H^{0,s}_{\bar{\partial}}(X,\Lambda^rT^{1,0}X)$ a harmonic $(3-r,s)$-form on X which corresponds to the contraction with the holomorphic $(3,0)$-form Ω. Therefore we actually have the isomorphism

$$H^{(c,c)}\cong H^{3-*,*}_{\bar{\partial}}(X) \tag{2.54}$$

A very important observation [22], [12] is that we were completely free to choose the creation operators on the right-moving side which in turn means that we have a freedom in the assignment of the relative sign of the $U(1)_V \times U(1)_A$ charges (p, q). Hence from a conformal field theory point of view there is no distinction in the correspondence of the (a, c)- and (c, c)-rings to the cohomology groups of the Calabi–Yau manifold. We will return to this point in Section 2.6 when we discuss the moduli space of $\mathcal{N} = (2, 2)$ superconformal field theories.

Landau-Ginzburg Models

In the previous example we have generalized the free field theory by adding more fields and interpreting them as coordinates on a curved target space. Now we add the F-terms and twisted F-terms mentioned in the free field theory example. For chiral superfields Φ^i the F-term is

$$\int \mathrm{d}^2\theta W(\Phi) + \text{c.c.} = \frac{1}{2}\int \mathrm{d}\theta^- \mathrm{d}\theta^+ W(\Phi)|_{\bar{\theta}^\pm=0} + \frac{1}{2}\int \mathrm{d}\bar{\theta}^- \mathrm{d}\bar{\theta}^+ \overline{W}(\overline{\Phi})|_{\theta^\pm=0} \tag{2.55}$$

where $W(\Phi)$ is a holomorphic function of the Φ^i's and is called a superpotential. This is invariant under vector and axial R-symmetries only when it is possible to assign R-charges to the Φ^i's such that $W(\Phi)$ has vector and axial charge 2 and 0 respectively. Similarly, for twisted chiral superfields Y^i the twisted F-term is

$$\int \mathrm{d}^2\tilde{\theta}\widetilde{W}(Y) + \text{c.c.} = \frac{1}{2}\int \mathrm{d}\bar{\theta}^- \mathrm{d}\theta^+ \widetilde{W}(Y)|_{\bar{\theta}^+=\theta^-=0} + \frac{1}{2}\int \mathrm{d}\bar{\theta}^+ \mathrm{d}\theta^- \overline{\widetilde{W}}(\overline{Y})|_{\theta^+=\bar{\theta}^-=0} \tag{2.56}$$

where $\widetilde{W}(Y)$ is a holomorphic function of the Y^i's and is called twisted superpotential. For R-invariance, it is required that R-charges can be assigned to the Y^i's so that $\widetilde{W}(Y)$ has vector and axial charge 0 and 2 respectively.

From chiral superfields one can then build an $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg theory by taking an action of the form

$$S = \int \mathrm{d}^2z\mathrm{d}^4\theta K(\Phi^1, \overline{\Phi}^{\bar{1}}, \ldots, \Phi^n, \overline{\Phi}^{\bar{n}}) + \left(\int \mathrm{d}^2z\mathrm{d}^2\theta W(\Phi^1, \ldots, \Phi^n) + \text{h.c.}\right) \tag{2.57}$$

Such a theory is generally not scale-invariant. However, if we let the theory flow under the renormalization group to a non-trivial infrared fixed point (assuming such a point exists), the fixed point theory does not further evolve with changes in scale and hence is a conformally invariant theory. It has been shown [23], [24] at the non-perturbative level that an $\mathcal{N} = 2$ Landau-Ginzburg theory indeed flows to a superconformal field theory at the critical point. All the characteristic features of the superconformal algebras can be read off from the starting Landau-Ginzburg action since they are completely governed by the superpotential W. Indeed, an important property of the renormalization group flow which can be established at least at the level of perturbation theory [25], is that the only renormalization suffered by the superpotential arises from a wavefunction renormalization. If we assume that $W(\Phi^i)$ is a quasi-homogeneous function, i.e. there exist integers k_i, d with

$$W(\lambda^{k_i}\Phi^i) = \lambda^d W(\Phi^i) \tag{2.58}$$

this renormalization is absorbed by an overall rescaling that in effect leaves the superpotential unchanged. This assumption in particular implies that the charge of Φ^i is k_i/d. On the other hand, the kinetic term in (2.57) in general undergoes a substantial renormalization along the flow towards the conformally invariant fixed point. Therefore we can use the superpotential as a renormalization group invariant which describes Landau-Ginzburg models.

Let us consider a few important special cases for $W(\Phi)$ together with the central charges of the corresponding Landau-Ginzburg theories at the conformal point [12]

$$W_{A_{k+1}}(\Phi) = \Phi^{k+2} \qquad c = 3 - \frac{6}{k+2} \tag{2.59a}$$

$$W_{D_k}(\Phi_1, \Phi_2) = \Phi_1^{k-1} + \Phi_1\Phi_2^2 \qquad c = 3 - \frac{6}{2(k-1)} \tag{2.59b}$$

$$W_{E_6}(\Phi_1, \Phi_2) = \Phi_1^3 + \Phi_2^4 \qquad c = 3 - \frac{6}{12} \tag{2.59c}$$

$$W_{E_7}(\Phi_1, \Phi_2) = \Phi_1^3 + \Phi_1\Phi_2^3 \qquad c = 3 - \frac{6}{18} \tag{2.59d}$$

$$W_{E_8}(\Phi_1, \Phi_2) = \Phi_1^3 + \Phi_2^5 \qquad c = 3 - \frac{6}{30} \tag{2.59e}$$

There is a strong and useful connection of these theories to the mathematical theory of singularities [26], [27] as is explained in [28]. For general Landau-Ginzburg theories one can show that

$$H^{(c,c)} = \frac{\mathbb{C}[\Phi^1, \ldots, \Phi^n]}{\partial_{\Phi^j} W(\Phi^1, \ldots, \Phi^n)} \qquad H^{(a,c)} = \{1\} \tag{2.60}$$

i.e. the (c, c) ring is isomorphic to the Jacobian ring consisting of all polynomials in the chiral fields modulo relations of the form $\partial_{\Phi^j} W = 0$.

In Sections 2.4 and 2.5 we will establish a relationship between certain Landau-Ginzburg theories and the non-linear σ-models discussed in the previous example. If there is any relationship between their spectra to hold then at least the dimension of the (c, c) and (a, c) rings should match. Though, in a Landau-Ginzburg theory, we will never get more than one (a, c) field. However, what has not yet been taken into account in the theory is the $U(1)$ charge projection onto odd integral states mentioned at the end of Section 2.1. To compute the relevant spectrum of the Landau-Ginzburg theory this projection has to be implemented by orbifolding by the operators $g = e^{2\pi i J_0}$ and $\bar{g} = e^{2\pi i \bar{J}_0}$. This leads to the Landau-Ginzburg orbifold theories [29], [30]. Since the charges are all multiples of $\frac{1}{d}$, g generates the cyclic group $\mathbb{Z}_d$ of order d. Now there will be contributions to the (a, c) ring from the twisted sectors. The basic observation is that in the lth twisted sector the charges of the states are of the form

$$(Q_l, -Q_l) + (r, r) \tag{2.61}$$

where

$$Q_l = \sum_{lq_i \in \mathbb{Z}} (lq_i - [lq_i] - \frac{1}{2}) \tag{2.62}$$

is the contribution coming from the twisted fields and r is any of the charges generated by the subring of those fields that are invariant under the l-twist. In addition, there can also be contributions from the twisted sectors to the (c, c) ring. This theory has an extra symmetry which will be discussed in detail in Section 2.6.

2.3. Superstring compactifications

Consider the most general superstring compactification to $D = d + 2$ dimensions. We assume that D is even, d is the number of transverse dimensions. The total central charge of the theory in the light cone gauge must be $c = 12$. The flat space-time theory is composed of d free bosons and d free fermions on the world-sheet. We have considered such a theory in the first example of Section 2.2 and seen that its contribution to the trace anomaly of the space-time degrees of freedom is $c_{\text{st}} = \frac{3d}{2}$. The trace anomaly of the internal theory is thus $c_{\text{int}} = 12 - \frac{3d}{2} \equiv 3n$. In particular, in order to compactify to $D = 4$ dimensions we need an internal theory with the trace anomaly $c_{\text{int}} = 9$. We have seen in Section 2.1 that the existence of space-time supersymmetry requires the internal theory to have $\mathcal{N} = 2$ superconformal symmetry. Furthermore, it is important that the restriction on the $U(1)$ charges is on the whole theory including the internal and the four-dimensional part. When we flow by $\eta = \frac{1}{2}$ the charge of an NS state is shifted by $\frac{c_{\text{int}}}{6} = 2$. Thus, if the original state has odd integral charge, so does its image in the R sector. Hence, a string theory of the form $M_4 \times \{c = 9, \mathcal{N} = 2$ superconformal theory $\}$ has space-time supersymmetry if and only if the superconformal field theory has odd integral $U(1)_L$ and $U(1)_R$ charge eigenvalues. Although $\mathcal{N} = (0,2)$ superconformal symmetry is actually sufficient we will focus on $\mathcal{N} = (2,2)$ superconformal field theories in order make contact to compactifications of type II string theories. Hence we build string theory with four extended dimensions by the construction $M_4 \times \{c = 9, \mathcal{N} = 2$ superconformal theory $\}$ where M_4 really refers to a $c = 3$, $\mathcal{N} = 2$ free superconformal theory. The latter theory contains the external fermions which generate a $SO(2)$ current algebra. The field

$$Q = e^{\frac{i}{2}\sqrt{\frac{c}{3}}\varphi} \tag{2.63}$$

where φ is given in terms of the total $U(1)$ current J of the total $c = 12$ theory as in (2.3), is a space time fermion. We denote the $U(1)$ current of the internal theory by J_{int} and the space-time $U(1)$ current by J_{st} which we can express as $J_{\text{int}} = i\sqrt{3}\partial\varphi_{\text{int}}$ and $J_{\text{st}} = i\partial\varphi_{\text{st}}$, respectively. In light-cone gauge the four supersymmetry charges $Q^a, a = 1, \ldots, 4$ can be divided into linearly and non-linearly realized ones [31], [32]. The linear supercharges can be built by substituting into (2.63), noting that all the central charges are divisible by 3, and using $\Sigma^{\pm}(z)$ from Section 2.1

$$Q = \sqrt{p^+} \oint S(z)\Sigma^+(z) \qquad\qquad Q^\dagger = \sqrt{p^+} \oint S^\dagger(z)\Sigma^-(z) \tag{2.64}$$

where $S = e^{\frac{i}{2}\varphi_{\text{st}}}$ is the spin field of the $SO(2)$ current algebra. The nonlinear supercharges are

$$U = \frac{1}{\sqrt{p^+}} \oint \left(\partial_z\phi^1 + i\partial_z\phi^2\right) S(z)\Sigma^+(z) \qquad U^\dagger = \frac{1}{\sqrt{p^+}} \oint \left(\partial_z\phi^1 - i\partial_z\phi^2\right) S^\dagger(z)\Sigma^-(z) \tag{2.65}$$

where ϕ^1 and ϕ^2 are two free transverse bosons and p^+ is the light-cone momentum. There is a second supersymmetry coming from the right movers, i.e. from $\bar{J} = \bar{J}_{\text{int}} + \bar{J}_{\text{st}}$, hence we have $\mathcal{N} = 2$ supersymmetry in $D = 4$. This construction generalizes to compactifications of dimension $D = d + 2$ in which case the current algebra is $SO(d)_1$ at level 1 and the internal part has central charge $c_{\text{int}} = 3n$.

2.4. Gepner Models

Minimal models

The minimal models are important in three ways. First, they are solvable $\mathcal{N} = 2$ superconformal field theories. Second, they are intimately related to the Landau-Ginzburg models and finally, they are the building blocks for the Gepner models.

As we have seen in Section 2.1, an irreducible highest weight representation is specified by the numbers (c, h, q). For values of the central charge in the region $c \leq 3$ the unitarity requirement (2.9) selects a discrete series of theories, the minimal models. They are labeled by three integers (l, m, s) and are characterized by the following values of the central charge, the conformal weight and $U(1)$ charge

$$c = \frac{3k}{k+2} \tag{2.66}$$

$$h^l_{m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \mod 1 \tag{2.67}$$

$$q^l_{m,s} = \frac{m}{k+2} - \frac{s}{2} \mod 1 \tag{2.68}$$

where $k \in \mathbb{Z}_{>0}$ and

$$l = 0, \dots, k \qquad\qquad m = -k-1, -k, \dots, k+2 \qquad\qquad s = -1, 0, 1, 2 \tag{2.69}$$

subject to the condition that $l + m + s \in 2\mathbb{Z}$. The corresponding primary fields are denoted by $\phi^l_{m,s}$. $s = 0, 2$ and $s = \pm 1$ correspond to states in the NS and R sector, respectively. Actually, states with $s = 2$ are not primary; they are included here because they are needed for constructing the massless states in the product theory to be discussed below. The different values of s in each sector denote opposite $\mathbb{Z}_2$ fermion number. There is a field identification between representations satisfying

$$(l, m, s) \equiv (k - l, m + k + 2, s + 2) \tag{2.70}$$

These models can be constructed [33], [34] in terms of unitary irreducible representations of suitable Kac-Moody algebra by applying the super-GKO construction [35] to the quotient $SU(2)_k/U(1)_{2k+4}$ or by adding [36], [37] a free boson to the $\mathbb{Z}_k$ parafermionic field theories [38], [39]. For the description of the $\mathcal{N} = 2$ characters one needs to extend s to take values in $\mathbb{Z}_4$. Given a field $\phi^l_{m,s}$, the corresponding character $\chi^l_{m,s}$ is

$$\chi^l_{m,s}(\tau, z, u) = e^{-2\pi i u}\, \mathrm{tr}_{\mathcal{H}^l_{m,s}}\, e^{2\pi i z J_0} e^{2\pi i \tau (L_0 - \frac{c}{24})} \tag{2.71}$$

where the trace is taken over a projection of $\mathcal{H}^l_{m,s}$ to definite fermion number (mod 2) of a highest weight representation of the (right-moving) $\mathcal{N} = 2$ superconformal algebra with highest weight vector $\phi^l_{m,s}(0)$. Their modular transformations are $\chi^{l'}_{m',s'}(-\frac{1}{\tau}, 0, 0) = S^k_{(l,m,s),(l',m',s')}\chi^l_{m,s}(\tau, 0, 0)$ and $\chi^{l'}_{m',s'}(\tau + 1, 0, 0) = T^k_{(l,m,s),(l',m',s')}\chi^l_{m,s}(\tau, 0, 0)$ with [40]

$$S^k_{(l,m,s),(l',m',s')} = \frac{1}{\sqrt{2}(k+2)} \sin(l, l')_k e^{\pi i \left(\frac{mm'}{k+2} - \frac{ss'}{2}\right)} \tag{2.72}$$

$$T^k_{(l,m,s),(l',m',s')} = e^{\pi i \frac{l(l+2)}{2(k+2)}} e^{\pi i \left(-\frac{m^2}{2(k+2)} + \frac{s^2}{4}\right)} \delta_{l,l'} \delta_{m,m'} \delta_{s,s'} \tag{2.73}$$

where

$$(l, l')_k = \pi \frac{(l+1)(l'+1)}{k+2} \tag{2.74}$$

The modular transformation matrices factor into three pieces, each of which only acts on precisely one of the three indices labeling the characters. The index l transforms under the representation of the modular group carried by level k affine $SU(2)$ characters, while the indices m and s transform under the representations carried by level $-(k + 2)$ and level 2 Θ-functions respectively. Modular invariant combinations of these characters can therefore be constructed by combining known modular invariants for these three types of objects. Up to discrete quotients, the general modular invariant combination takes the form

$$Z^{(k)} = \frac{1}{2} \sum_{\substack{l,\bar{l},m,s \\ l+m+s=0 \mod 2}} A^{(k)}_{l,\bar{l}} \chi^l_{m,s} \chi^{\bar{l}*}_{m,s} \tag{2.75}$$

where $A_{l,\bar{l}}$ is any of the ADE classified modular invariants at level k [41]

$$A_k \qquad \sum_{l=1}^{k+1} |\chi^l|^2 \qquad k \geq 1 \tag{2.76a}$$

$$D_{2j+2} \qquad \sum_{\substack{l=1 \\ l \text{ odd}}}^{2j-1} |\chi^l + \chi^{4j+2-l}|^2 + 2\,|\chi^{2j+1}|^2 \qquad k = 4j,\ j \geq 1 \tag{2.76b}$$

$$D_{2j+1} \qquad \sum_{\substack{l=1 \\ l \text{ odd}}}^{4j-1} |\chi^l|^2 + |\chi^{2j}|^2 + \sum_{\substack{l=2 \\ l \text{ even}}}^{4j-2} \left(\chi^l \chi^{4j-l\,*} + \text{c.c.}\right) \qquad k = 4j-2,\ j \geq 2 \tag{2.76c}$$

$$E_6 \qquad |\chi^1 + \chi^7|^2 + |\chi^4 + \chi^8|^2 + |\chi^5 + \chi^{11}|^2 \qquad k = 10 \tag{2.76d}$$

$$E_7 \qquad |\chi^1 + \chi^{17}|^2 + |\chi^5 + \chi^{13}|^2 + |\chi^7 + \chi^{11}|^2 + |\chi^9|^2 + (\chi^3 + \chi^{15})\chi^{9\,*} + \text{c.c.} \qquad k = 18 \tag{2.76e}$$

$$E_8 \qquad |\chi^1 + \chi^{11} + \chi^{19} + \chi^{29}|^2 + |\chi^7 + \chi^{13} + \chi^{17} + \chi^{23}|^2 \qquad k = 28 \tag{2.76f}$$

In the case of the A_k-, D_{2j+1}- and E_6-type modular invariants the field identifications (2.70) have to be made in both the left and right sector simultaneously while for the D_{2j+2}-, E_7- and E_8-type modular invariants one can apply them independently for the holomorphic and antiholomorphic part. This entails [42], [43] that these theories have a $\mathbb{Z}_n \times \mathbb{Z}_2 \times \overline{\mathbb{Z}}_n \times \overline{\mathbb{Z}}_2$ symmetry with $n = k+2$ for the A_k-, D_{2j+1}- and E_6-type minimal models and $n = \frac{k+2}{2}$ for the D_{2j+2}-, E_7- and E_8-type minimal models which acts as

$$g\phi^l_{m,s} = e^{2\pi i \frac{m}{n}} \phi^l_{m,s} \tag{2.77}$$

$$h\phi^l_{m,s} = (-1)^s \phi^l_{m,s} \tag{2.78}$$

and similarly in the anti-holomorphic sector. We will return to these symmetries in Section 4.3 when discussing the boundary conformal field theory.

That the $\mathcal{N} = 2$ minimal models (2.76) and the Landau-Ginzburg theories in (2.59) are both classified by the ADE groups and that both have for each group the same central charges is not an accident. Indeed one can give strong arguments [44], [45], [12], [46] that the minimal models are the infrared fixed points of Landau-Ginzburg theories. At the conformal point one has the map between chiral primaries of both theories

$$\Phi^l = (l, l, 0) = \phi^l_{l,0} \tag{2.79}$$

and hence the Landau-Ginzburg fields provide a simple representation of the chiral ring.

We can take the orbifold of a minimal model with respect to the diagonal group $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ to obtain a new conformal field theory which is isomorphic to the original one with the sign of the $U(1)_R$ eigenvalue associated with each field being reversed. Therefore, the chiral fields Φ^i are mapped into twisted chiral fields Y^i and W becomes a twisted chiral superpotential $\widetilde{W}$.

Gepner's construction

We have seen in Section 2.3 that conformally invariant non-linear σ-models with Calabi–Yau target spaces have central charge $c = 3n$ where n is the complex dimension of the Calabi–Yau space. Given a collection of r conformal field theories with central charges c_i, $i = 1, \ldots, r$, one can build a new conformal field theory, called the tensor product theory, with central charge $c = \sum_{i=1}^r c_i$. The Hilbert space of this theory is the tensor product of the Hilbert spaces of the constituent theories and the energy-momentum tensor takes the form

$$T = \sum_{i=1}^{r} 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1 \tag{2.80}$$

In fact, since the operation of orbifolding by a finite discrete subgroup does not change the central charge of a conformal field theory, a quotient of the above tensor product will also have central charge $c = \sum_{i=1}^{r} c_i$. Applying this to the minimal models, we see that if we choose a collection of integers $\{k_i | i = 1, \ldots, r\}$ such that

$$c = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 3n \tag{2.81}$$

then the tensor product of these conformal field theorys and orbifolds thereof will have the appropriate central charge for a Calabi–Yau compactification. However, not only the central charge has to match but also the spectrum of this tensor product conformal field theory has to agree with the one from the non-linear σ-model. This has been achieved by Gepner [47], [48] by adjoining external bosons and fermions and finally employing an orbifold-like projection on the $U(1)$ charges enforcing space-time supersymmetry and modular invariance as we are going to review.

In order to label the tensor product representations define

$$\lambda = (l_1, \ldots, l_r) \tag{2.82}$$

$$\mu = (s_0; m_1, \ldots, m_r; s_1, \ldots, s_r) \tag{2.83}$$

where l_j, m_j and s_j take values in the range (2.69) and $s_0 = -1, 0, 1, 2$ characterizes the irreducible representations of the $SO(d)_1$ current algebra that is generated by the external fermions, see Section 2.3. We assume for the moment that $d = 2 \mod 4$. Accordingly, we write

$$\chi^{\lambda}_{\mu}(q) = \chi_{s_0}(q) \prod_{j=1}^{r} \chi^{l_j}_{m_j, s_j}(q) \tag{2.84}$$

where χ_{s_0} is the $SO(d)_1$ character. Gepner introduced special $(2r+1)$-dimensional vectors β_0 with all entries equal to one and β_j, $j = 1, \ldots, r$ having zeroes everywhere except for the first and the $(r+1+j)$th entry which are equal to 2. The scalar product of two vectors μ and μ' is defined by

$$\mu \bullet \mu' = -\frac{d}{8} s_0 s_0' + \frac{1}{2} \sum_{j=1}^{r} \left(\frac{m_j m_j'}{k+2} - \frac{s_j s_j'}{2} \right) \tag{2.85}$$

The total $U(1)$ charge of the highest weight state in $\chi^{\lambda}_{\mu}(q)$ is $q_{\text{tot}} = 2\beta_0 \bullet \mu$, so that the projection onto states with odd $\beta_0 \bullet \mu$ will implement the GSO projection. Similarly, restricting to states with $\beta_i \bullet \mu \in \mathbb{Z}$ ensures that only states in the tensor product of $r+1$ NS sectors (or of $r+1$ R sectors) are admitted. This condition guarantees space-time supersymmetry. Modular invariance of the partition function can be achieved if the above projections are accompanied by adding "twisted" sectors. Set $K = \mathrm{lcm}(4, 2k_j + 4)$ and $b_0 \in \{0, 1, \ldots, K-1\}$, $b_j \in \{0, 1\}$ for $j = 1, \ldots, r$. Then the partition function of a Gepner model describing a superstring compactification to D dimensions is

$$Z^{(r)}_G(\tau, \bar{\tau}) = \frac{1}{2^r} \frac{(\mathrm{Im}\,\tau)^{-\frac{d}{2}}}{|\eta(q)|^2 d} \sum_{b_0=0}^{K-1} \sum_{b_1, \ldots, b_r = 0}^{1} \sum^{\text{ev}}_{\substack{\lambda, \mu,\, \beta_j \bullet \mu \in \mathbb{Z} \\ 2\beta_0 \bullet \mu \in 2\mathbb{Z}+1}} (-1)^{s_0} \prod_{j=1}^{r} A^{(k_j)}_{l_j, \bar{l}_j} \chi^{\lambda}_{\mu}(q) \chi^{\lambda}_{\mu + b \cdot \beta}(\bar{q}) \tag{2.86}$$

where ev means the restriction $l_j + m_j + s_j \in 2\mathbb{Z}$. The summation over $b \cdot \beta = \sum_{j=0}^{r} b_j \beta_j$ introduces the twisted sectors corresponding to the β-restrictions so that, in particular, the Gepner partition function is non-diagonal. The τ-dependent factor in front of the sum accounts for the free bosons associated to the d transversal dimensions of flat external space-time while the $\frac{1}{2^r}$ is due to the field identification (2.70). Furthermore $A^{(k_j)}_{l_j,\bar{l}_j}$ stands for any of the affine modular invariants in (2.76). Using the modular transformation properties of the $SO(d)_1$ characters whose S-matrix is

$$S^{\mathrm{f}}_{s_0,s_0'} = \frac{1}{2} e^{-i\pi \frac{d}{2} \frac{s_0 s_0'}{2}} \tag{2.87}$$

and those of the minimal model characters (2.72) Gepner proved that (2.86) is modular invariant. Note that for $d = 4$ consistency requires to replace d by $d + 2$ in (2.85), (2.86) and (2.87). For an account of all the possible combinations of modular invariants $A^{(k_j)}_{l_j,\bar{l}_j}$ see [43]. We will denote a Gepner model as follows

$$(k_{1G_1}, \ldots, k_{rG_r}) \tag{2.88}$$

where k_j stands for the level and G_j stands for the ADE invariant of the jth subtheory. Since we will be mostly working with A-type invariants we will often drop this subscript. The symmetry group for the Gepner model consists of a semi-direct product of the minimal model symmetries in (2.77) in each subtheory with permutational symmetries S interchanging identical subtheories modded out by the action of the cyclic group generated by $\mu = (2, 2, \ldots, 2)$

$$G = \frac{\prod_{j=1}^{r} \mathbb{Z}_{n_j}}{\mathbb{Z}_{\widetilde{n}}} \rtimes S \tag{2.89}$$

where

$$\widetilde{n} = \begin{cases} \frac{K'}{2} & \text{all } k_i \text{ odd} \\ K' & \text{otherwise} \end{cases} \tag{2.90}$$

with $K' = \mathrm{lcm}(k_j + 2)$. For models with an odd number of factors (in $D = 4$) $\widetilde{n} = K'$ even if not all k_i are odd.

The importance of these models, first pointed out by Gepner [47], [48], is that their massless spectrum is the same as that of a non-linear σ-model on a Calabi–Yau manifold given as a hypersurface in a weighted projective space, or more generally as a complete intersection in a product of weighted projective spaces. Using the correspondence of $\mathcal{N} = 2$ minimal models and Landau-Ginzburg theories discussed in the previous subsection a one-to-one relation between Gepner models and Landau-Ginzburg orbifold theories has been established in [49], [50] and by a path-integral argument it was argued that the following identification holds. Gepner models involving $r = 5$ A-type modular invariants

$$(k_1, \ldots, k_5) \tag{2.91}$$

and non-linear σ-models on degree K' hypersurfaces

$$W(z) = z_1^{k_1+2} + z_2^{k_2+2} + z_3^{k_3+2} + z_4^{k_4+2} + z_5^{k_5+2} = 0 \tag{2.92}$$

in the weighted projective space $\mathbb{P}^4_{\frac{K'}{k_1+2},\ldots,\frac{K'}{k_5+2}}$ (see Section 3.2.2) have the same spectra and symmetries. Note that (2.92) in the Landau-Ginzburg description is the superpotential (2.58) consisting of five terms of the form (2.59a). The coordinate z_i is identified with the chiral primary field $\phi^{(i),1}_{1,0}$ in (2.79). The generalization for the cases with $r \neq 5$ factors as well as models including D- and E-type invariants has been worked out completely in [43]. Note that in the case of $r < 5$ one can always add quadratic terms to the superpotential $W(z)$ since they correspond to $k = 0$ minimal models which have central charge $c = 0$ and therefore no dynamics. A precise relationship between Gepner models and Calabi–Yau manifolds will be given in the following section on gauged linear σ-models.

2.5. Gauged linear σ-model

In this section we show that a Landau-Ginzburg orbifold theory associated with a suitable quasi-homogeneous superpotential $W(\Phi)$ and the non-linear σ-model with target space being the Calabi-Yau manifold defined as the vanishing locus of $W(z)$ in a suitable weighted projective space (or more generally, a toric variety) can be seen as the effective low-energy theories in different phases of the same theory, the gauged linear σ-model. This remarkable connection has been worked out by Witten in [17].

As a first step, we couple the free field theory example of Section 2.2 to abelian gauge fields. For this purpose, we need to introduce the gauge field in the superspace formalism. This is achieved by the vector multiplet which consists of a vector field v_μ, Dirac fermions $\lambda_\pm$, $\overline{\lambda}_\pm$ which are conjugate to each other, and a complex scalar σ. It is represented in a vector superfield V satisfying $V = V^\dagger$ which is expanded in the Wess-Zumino gauge as

$$\begin{aligned} V &= \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ &\quad + \sqrt{2}i\theta^-\theta^+\left(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+\right) + \sqrt{2}i\overline{\theta}^-\overline{\theta}^+\left(\theta^-\lambda_- + \theta^+\lambda_+\right) + 2\theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D \end{aligned} \tag{2.93}$$

where D is a real auxiliary field. Using the gauge covariant derivatives $\mathcal{D}_\pm = e^{-V}D_\pm e^V$, $\overline{\mathcal{D}}_\pm = e^V\overline{D}_\pm e^{-V}$, we can define the field strength as

$$\Sigma = \frac{1}{2}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\} \tag{2.94}$$

$$= \sigma + i\sqrt{2}\left(\theta^+\overline{\lambda}_+ + \overline{\theta}^-\lambda_-\right) + 2\theta^+\overline{\theta}^-(D - iF_{01}) \tag{2.95}$$

where F_{01} is the curvature of v_μ. This is a (covariant) twisted chiral superfield $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$.

The gauged linear σ-model with target space X and gauge group $G = U(1)^{n-d}$ is obtained by coupling n chiral matter multiplets Φ_i with charges Q_i^a under G to the $n-d$ abelian gauge superfields V_a, and introducing Fayet-Illiopoulos terms for the abelian gauge symmetry[1]. The Lagrangian for this theory is

$$\begin{aligned} S &= \int_\Sigma \mathrm{d}^2z\mathrm{d}^4\theta\left(\sum_{i=1}^n \overline{\Phi}_i \exp\left(2\sum_{a=1}^{n-d} Q_i^a V_a\right)\Phi_i - \sum_{a=1}^{n-d}\frac{1}{4e_a^2}\overline{\Sigma}_a\Sigma_a - \sum_{a=1}^{n-d} r_a V_a\right) \\ &\quad + \int_\Sigma \mathrm{d}^2z\sum_{a=1}^{n-d}\frac{\vartheta_a}{2\pi i}F_{a,01} - \int_\Sigma \mathrm{d}^2z\mathrm{d}^2\theta\mathcal{W}(\Phi)|_{\overline{\theta}^+=\overline{\theta}^-=0} + \text{c.c.} \end{aligned} \tag{2.96}$$

The third and fourth terms in (2.96) can be rewritten as

$$S_{D,\theta} = \sum_{a=1}^{n-d}\int_\Sigma \mathrm{d}^2z\left(-r_aD_a + \frac{\vartheta_a}{2\pi i}F_{a,01}\right) = \int_\Sigma \mathrm{d}^2z\mathrm{d}^2\tilde{\theta}\widetilde{\mathcal{W}}(\Sigma)|_{\theta^-=\overline{\theta}^+=0} + \text{c.c.} \tag{2.97}$$

where

$$\widetilde{\mathcal{W}}(\Sigma) = \frac{1}{2\sqrt{2}}\sum_{a=1}^{n-d}\tau_a\Sigma_a \tag{2.98}$$

with

$$\tau_a = ir_a + \frac{\vartheta_a}{2\pi} \tag{2.99}$$

[1] Here n and d are not related to those of the previous sections.

This interaction is a twisted superpotential for the twisted chiral fields Σ_a. ϑ_a is an angular variable and the corresponding term is topological (the analog of the ϑ-angle in $D = 4$ Yang-Mills theory).

The preservation of the R-symmetry group $U(1)_V \times U(1)_A$ at the quantum level is a necessary condition for the emergence of a superconformal theory. $U(1)_V$ is an exact symmetry of the theory but $U(1)_A$ is subject to the chiral anomaly. The axial rotation by $e^{i\alpha}$ shifts the theta angle by $\vartheta_a \to \vartheta_a - 2\alpha \sum_{i=1}^n Q_i^a$. Thus $U(1)_A$ is unbroken if

$$\sum_{i=1}^n Q_i^a = 0 \quad \text{for } a = 1, \ldots, n-d \tag{2.100}$$

Without a superpotential, this theory has additional global symmetries that act on the chiral superfields as

$$\Phi_i \longrightarrow \exp(i\alpha k_i)\,\Phi_i \tag{2.101}$$

with arbitrary k_i and commute with supersymmetry. In the presence of a superpotential $\mathcal{W}$ one must add such a transformation to the right-moving R-symmetry under which $\mathcal{W} \to e^{-i\alpha}\mathcal{W}$ to preserve R-symmetry. The superpotential $\mathcal{W}$ is said to be quasi-homogeneous if k_i exist such that $\mathcal{W}$ transforms in this way, cf. (2.58). The twisted superpotential $\widetilde{\mathcal{W}}(s)$ violates R-symmetry unless it takes the linear form given in (2.98).

We will now show that we can describe Calabi–Yau spaces by means of the gauged linear σ-model. Consider adding an additional chiral superfield Φ_0 with charges $Q_0^a = -\sum_{i=1}^n Q_i^a$. This ensures condition (2.100) for R invariance. We pick the superpotential to be the holomorphic, gauge invariant function

$$\mathcal{W}(\Phi) = \Phi_0 \cdot W(\Phi_1, \ldots, \Phi_n) \tag{2.102}$$

where W has charge Q_0^a under the a^{th} copy of $U(1)^{n-d}$. This charge assignment has to be made in order to preserve R-symmetry. We assume that W be transverse in the sense that the equations

$$\frac{\partial W}{\partial \Phi_i} = 0 \quad \forall i = 1, \ldots, n \tag{2.103}$$

have no common root except at $\Phi_i = 0$. The bosonic potential is

$$U(\phi_0, \ldots, \phi_n, \sigma_a) = \sum_{a=1}^{n-d} \frac{1}{2e_a^2} D_a^2 + \sum_{i=0}^n |F_i|^2 + 2 \sum_{a,b=1}^{n-d} \overline{\sigma}_a \sigma_b \sum_{i=0}^n Q_i^a Q_i^b\, |\phi_i|^2 \tag{2.104}$$

with

$$D_a = -e_a^2 \left(\sum_{i=0}^n Q_i^a\, |\phi_i|^2 - r_a \right) \tag{2.105}$$

and

$$\sum_{i=0}^n |F_i|^2 = |W(\phi_i)|^2 + |\phi_0|^2 \sum_{i=1}^n \left| \frac{\partial W}{\partial \phi_i} \right|^2 \tag{2.106}$$

where we have used the equations of motion for D_a and F_i. The space V of supersymmetric (classical) ground states of this theory is given by the symplectic reduction of $\mathbb{C}^{n+1}$ by G determined by the "moment map" $D : \mathbb{C}^{n+1} \to \mathrm{Lie}(G)^\vee$

$$V(r) = D^{-1}(0)/G \tag{2.107}$$

The space $V(r)$ is not necessarily smooth or of dimension d, there will in general be values of r for which it is altogether empty. If it is of dimension d, it carries a natural complex structure in which the reduced symplectic form becomes a Kähler form ω. Now let us discuss the low-energy physics for various values of r_a. The classical moduli space is the entire complexified Kähler space $\mathbb{C}^{n-d}/\mathbb{Z}^{n-d} = \mathbb{R}^{n-d} \times U(1)^{n-d}$. The classical theory is singular along certain cones in r-space, dividing (real) r-space into regions corresponding to different "phases". These singularities occur whenever there are solutions to $D = 0$ which leave a large continuous subgroup of G unbroken.

If we restrict to those values of r for which $\dim V(r) = d$ then requiring the vanishing of (2.104) will set $\phi_0 = 0$ (restricting to V) and then (2.104) requires that the remaining fields satisfy $W = 0$, in other words that the image of the world-sheet lie in a hypersurface X of V. A closer study of this model shows that the massless modes are precisely the variations of ϕ tangent to X together with their superpartners so we have as the low-energy limit precisely the non-linear σ-model on the Calabi–Yau hypersurface X. This region of r-space in general includes hypersurfaces in various birational models of V, including models with unresolved orbifold singularities.

The other regions of r-space correspond to phases in which the space of vacua is of dimension less than d. In these cases there are massless excitations about these vacua, governed by the superpotential interaction. When the space of vacua is a point the model is a Landau-Ginzburg theory, intermediate cases in which there are massless fluctuations about a non-trivial space of vacua are termed "hybrid" models. In many vacua there are discrete subgroups of G unbroken by the expectation values; the low-energy theory is then a quotient by this subgroup. The physics of the hybrid phases is not well understood.

Let us exhibit these concepts in an example following [51] which will be taken up again in Section 3.5. Let $G = U(1)^2$ act on the chiral superfields $\Phi_0, \ldots, \Phi_6$ as

$$Q_i^a = \begin{pmatrix} -6 & 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{2.108}$$

corresponding to the action of the complexified gauge group $G_{\mathbb{C}}$ on the ϕ_i as

$$g_1(\lambda) : (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \mapsto (\lambda^{-6}\phi_0, \phi_1, \phi_2, \lambda\phi_3, \lambda\phi_4, \lambda^3\phi_5, \lambda\phi_6) \tag{2.109a}$$

$$g_2(\lambda) : (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \mapsto (\phi_0, \lambda\phi_1, \lambda\phi_2, \phi_3, \phi_4, \phi_5, \lambda^{-2}\phi_6) \tag{2.109b}$$

Note that the group element

$$g_1 g_2^2(\lambda) : (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \mapsto (\lambda^{-6}\phi_0, \lambda\phi_1, \lambda\phi_2, \lambda^2\phi_3, \lambda^2\phi_4, \lambda^6\phi_5, \phi_6) \tag{2.109c}$$

defines the $\mathbb{C}^*$ action of a weighted projective space in the variables ϕ_1 to ϕ_5.

For simplicity we choose the couplings to be equal, $e_a = e$ for all a. The D-terms (2.105) are then

$$D_1 = -e^2 \left(|\phi_3|^2 + |\phi_4|^2 + 3\,|\phi_5|^2 + |\phi_6|^2 - 6\,|\phi_0|^2 - r_1 \right) \tag{2.110a}$$

$$D_2 = -e^2 \left(|\phi_1|^2 + |\phi_2|^2 - 2\,|\phi_6|^2 - r_2 \right) \tag{2.110b}$$

The phase boundaries are determined by those values of r for which an unbroken continuous symmetry is consistent with $D_a = 0$ using (2.104). We see from (2.109a) that g_1 is unbroken if $\phi_3 = \phi_4 = \phi_5 = \phi_6 = \phi_0 = 0$, which from (2.110a) can happen at zero energy if $r_1 = 0$, $r_2 \geq 0$. Similarly, because of (2.109b) g_2 is unbroken if $\phi_1 = \phi_2 = \phi_6 = 0$ which implies $r_2 = 0$ but leads in fact to two rays because both signs of r_1 are possible. Finally, if ϕ_6 is the only nonvanishing coordinate, then we see from (2.109c) that $g_1^2 g_2$ is unbroken. This implies $r_1 \geq 0$, $2r_1 + r_2 = 0$. Figure 2.1 shows the structure in r-space. There are four phases, labeled I - IV.

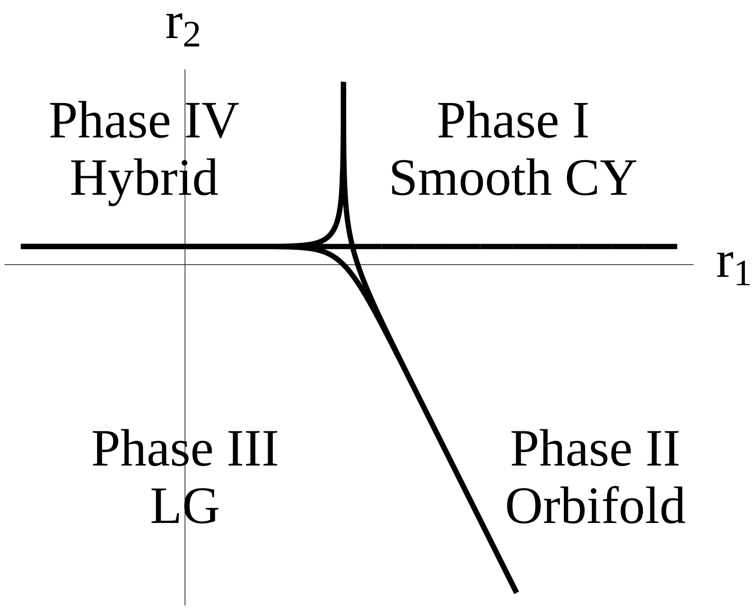

Figure 2.1.: The phase diagram for X taking into account the shift in (2.112).

(I) Requiring the vanishing of (2.104) for $r_1 > 0$, $r_2 > 0$ implies that in each of the sets $\{\phi_1, \phi_2\}$ and $\{\phi_3, \ldots, \phi_6\}$ there must be one nonvanishing ϕ_i. Hence not all $\frac{\partial W}{\partial \phi_j}$ can vanish, implying $\phi_0 = 0$. Since at least one $\phi_i \neq 0$, the σ_a must be zero and so $W = 0$. Therefore the low-energy modes describe a non-linear σ-model on the Calabi–Yau hypersurface X in a space V which can be described as follows

$$V = \frac{\mathbb{C}^6 \setminus F}{(\mathbb{C}^*)^2} \tag{2.111}$$

where F is the excluded set $\{\phi_1 = \phi_2 = 0\} \cup \{\phi_3 = \phi_4 = \phi_5 = \phi_6 = 0\}$ and the $(\mathbb{C}^*)^2$ action is given by (2.109). This is a smooth toric variety describing a blown-up weighted projective space $\mathbb{P}^4_{1,1,2,2,6}$ as a holomorphic quotient, see also [52]. We will return to this model in great detail in Section 3.5.2. Since X is a smooth Calabi–Yau manifold we call this phase the *smooth* or the *Calabi–Yau* phase.

(II) In this phase the excluded regions are $\{\phi_6 = 0\} \cup \{\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0\}$. Again, $U = 0$ implies $\phi_0 = 0$. This corresponds to the original (unresolved) weighted projective space; the low-energy limit is the non-linear σ-model with target space a hypersurface in this space. This is the *orbifold* phase.

(III) In this phase the excluded regions are $\{\phi_0 = 0\} \cup \{\phi_6 = 0\}$. Then $U = 0$ implies the vanishing of all the other coordinates, leading to a unique vacuum configuration given by $W(\phi_i) = 0$ in which G is broken to $\mathbb{Z}_2 \times \mathbb{Z}_6$. Therefore, the fields ϕ_i live in $\mathbb{C}^5/\mathbb{Z}_2 \times \mathbb{Z}_6$. Although their expectation values are set to zero, their quantum fluctuations are massless and governed by the superpotential W. We note that this $\mathbb{Z}_2 \times \mathbb{Z}_6$ action is nothing but the action of $e^{2\pi i J_0}$ and hence constitutes the required $U(1)$ projection discussed in Section 2.2. The region thus corresponds to the *Landau-Ginzburg orbifold* phase.

(IV) In this phase the excluded regions are $\{\phi_0 = 0\} \cup \{\phi_1 = \phi_2 = 0\}$. Here $U = 0$ implies $\phi_3 = \phi_4 = \phi_5 = \phi_6 = 0$, so that g_1 is broken to a discrete subgroup $\mathbb{Z}_6$. The expectation values of ϕ_1, ϕ_2 parametrize (after setting $D_2 = 0$ and taking the G quotient) a moduli space isomorphic to $\mathbb{P}^1$. The fluctuations of $\phi_3\,, \phi_4$ and ϕ_5 are massless; they interact via a superpotential with coefficients depending upon the point in $\mathbb{P}^1$. The model is in a so-called *hybrid* phase combining the properties of a gauged linear σ-model on $\mathbb{P}^1$ with those of a Landau-Ginzburg theory.

The identification with the non-linear σ-model must be made more precise. The metric on X is classically just the restriction of the metric on V. Since this metric is not Ricci-flat the non-linear σ-model is not conformally invariant. This is related to the fact we have not the correct degrees of freedom of the non-linear σ-model. Besides the fields that are constrained to live in the target space X there are additional massive fields in the gauged linear σ-model that are not confined to lie in X. Hence we have to integrate out these massive states and take into account the instanton corrections coming from additional zero size instantons in the gauged linear σ-model. This amounts to the following relation between the parameter τ_a in (2.99) and the Kähler parameter t_a [51]

$$t_a = \tau_a + \Delta_a + \sum_{m=1}^{\infty} K_m e^{2\pi i \tau_m} + \ldots \tag{2.112}$$

where $\Delta_a = \frac{i}{2\pi} \sum_{i=1}^{n} Q_i^a \log Q_i^a$ is the one-loop contribution and K_m represent the first order effect from zero size instantons and $\ldots$ represents the higher orders. Hence, in the classical limit we can identify $r_a \geq 0$ with the Kähler parameter $\operatorname{Im} t_a$.

Similarly, it can be shown that the other phases undergo instanton corrections. Thus, by varying the values of the r_a, we may switch between a target space of a blown-up Calabi–Yau space, a target space of a singular Calabi–Yau space (for which the only massless modes lie within these spaces), a target space which is a point with massless Landau-Ginzburg-type fluctuations about it, and a hybrid model. Each of these theories has instantons and in each case the action of them goes as $|r|$ so that their effects become negligible in the large $|r|$ limit. That is, we have e.g. exactly a theory on a Calabi–Yau space for $r_1 = \infty, r_2 = \infty$ and exactly a Landau-Ginzburg orbifold theory for $r_1 = -\infty, r_2 = -\infty$ between which we can interpolate via the gauged linear σ-model. More generally, the gauged linear σ-model provides us with a technique for interpolating between non-linear σ-models with birationally equivalent target spaces, obtained by varying the D-terms.

Let us say a few words about the phase boundaries where the theory becomes singular. Their locus in t-space is called the discriminant locus. In figure 2.1 we have not indicated the ϑ_a which are related to the B-field. In fact, it was argued in [17], [53] that by a judicious choice of this B-field the boundary set can be avoided while keeping the theory conformal. When the boundary is actually approached then the Calabi–Yau manifold becomes singular. The best-known type of such a singularity is the conifold [54] where the Calabi–Yau manifold acquires a nodal singularity or, in other words, a three-cycle shrinks to zero size. By deforming this singularity one obtains a topologically different Calabi–Yau manifold and it was argued in [4], [55] that the transition from one to the other is physically well behaved. There are other, more complicated types of singularities, e.g. the codimension of the phase boundary can be bigger than one, leading to further kinds of transitions between different Calabi–Yau manifolds [56], [57], [58]. We will discuss some of these singularities in Section 3.3.3.

2.6. Moduli spaces of $\mathcal{N} = (2,2)$ Superconformal Field Theories and Mirror Symmetry

Families of $\mathcal{N} = (2,2)$ Superconformal Field Theories

We can obtain a family of $\mathcal{N} = (2,2)$ superconformal field theories by deforming a given theory by marginal operators, i.e. operators having conformal weight $h + \bar{h} = 2$. Here, we will focus on spinless operators with $h = \bar{h} = 1$ which are truly marginal. Those are marginal operators which continue to be of type $(1, 1)$ after a perturbation of the theory by any other marginal operator. By the superconformal Ward identities [22], [59] it can be shown that among these operators there are two particularly important ones. Let $\phi \in H^{(c,c)}$ with $h = \bar{h} = \frac{1}{2}$, $q = \bar{q} = 1$. Then

$$\Phi_{(1,1)}(w, \bar{w}) = \oint \mathrm{d}^2 z \overline{G}^-(\bar{z}) G^-(z) \phi(w, \bar{w}) \tag{2.113}$$

is a truly marginal operator and corresponds to chiral superfield. If $\phi \in H^{(a,c)}$ with $h = \bar{h} = \frac{1}{2}$, $q = -\bar{q} = 1$ then

$$\Phi_{(-1,1)}(w, \bar{w}) = \oint \mathrm{d}^2 z G^+(z) \overline{G}^-(\bar{z}) \phi(w, \bar{w}) \tag{2.114}$$

is also a truly marginal operator, but corresponds to a twisted chiral superfield. The moduli space of $\mathcal{N} = (2, 2)$ superconformal field theories is then given by all the possible deformations built from these two types of operators. It can be shown [60] that, at least locally, the conformal field theory Zamolodchikov metric on this moduli space is block diagonal between the $\Phi_{(1,1)}$- and the $\Phi_{(-1,1)}$-type marginal operators and hence we can think of this moduli space as being a metric product of two spaces [61] $\mathcal{M}_{\mathrm{CY}}^{\mathrm{SCFT}} = \mathcal{M}_{(1,1)} \times \mathcal{M}_{(-1,1)}$. These two types of marginal operators can be given a geometrical interpretation by recalling the association between the (c, c)- and (a, c)-rings and harmonic differential forms. The (c, c)-fields with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ correspond therefore to harmonic $(2, 1)$-forms while the (a, c)-fields with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ correspond to harmonic $(1, 1)$-forms. We will see in Section 3.1 that the former parametrize the deformations of the complex structure of the Calabi–Yau manifold X while the latter parametrize the deformations of the Kähler structure of X. Hence we can identify the moduli spaces of the $\Phi_{(1,1)}$- and the $\Phi_{(-1,1)}$ operators with the moduli spaces of complex structure of X and of the Kähler structure of X, respectively, and locally write

$$\mathcal{M}_{\mathrm{CY}}^{\mathrm{SCFT}} = \mathcal{M}_{C}^{\mathrm{SCFT}} \times \mathcal{M}_{K}^{\mathrm{SCFT}} \tag{2.115}$$

This result can be shown to be a consequence of the $\mathcal{N} = (2, 2)$ superconformal algebra in (2.6) [62]. In spite of this picture it should be emphasized that the moduli space of $\mathcal{N} = (2, 2)$ superconformal field theories is not a product of the complex structure and Kähler moduli spaces of the Calabi–Yau manifold, not even locally. In fact, the Kähler moduli space of ω can depend on the complex structure of X [56]. However, in the limit of large volume where we have the description in terms of an exact non-linear σ-model the picture persists.

The large volume limit

A discrepancy comes from world-sheet instanton effects [63], [64], [65] and [59]. Non-perturbative corrections arise in the non-linear σ-model because of classical solutions of maps of the string world-sheet into the target space which are not homotopic to a point. For tree-level instantons (recall that we are working at $g_s = 0$) we need to consider algebraic curves of genus zero, i.e. we consider $\pi_2(X)$ which is equivalent to $H_2(X, \mathbb{Z})$ if X is simply connected [66]. We also assume that $h^{2,0}(X) = 0$ (see Section 3.1). It was shown in [63] that a map $S^2 \to X$ given by $\phi^i(\sigma)$ contributes $\exp(-I)$ to the path integral, where

$$I \geq \frac{1}{4\pi\alpha'} \left| \int_{S^2} \mathrm{d}^2\sigma J_{i\bar{\jmath}} \varepsilon^{\alpha\beta} \frac{\partial \phi^i}{\partial \sigma^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial \sigma^\beta} \right| \tag{2.116}$$

and $J_{i\bar{\jmath}}$ is the Kähler form. The equality is satisfied when $\phi^i(\sigma)$ is an algebraic curve. Thus the rational algebraic curves give the instantons we want. In this case, (2.116) can be rewritten as

$$I \propto \int_{\phi(\mathbb{P}^1)} J \tag{2.117}$$

In order to have small instanton contributions the cohomology class of the Kähler form must be such that (2.117) is large for any $C \in H_2(X,\mathbb{Z})$. Since the Kähler form J is a real $(1,1)$-form, its cohomology class can be considered as a point in $\mathbb{R}^{h^{1,1}}$. In order to have a smooth Calabi–Yau manifold X, the Kähler form must satisfy

$$\int_X J \wedge J \wedge J > 0 \qquad \int_D J \wedge J > 0 \qquad \int_C J > 0 \tag{2.118}$$

for homologically non-trivial surfaces D and curves C embedded in X, i.e. J must lie in the Kähler cone. Hence, (2.117) implies that one obtains the results of classical algebraic geometry when the cohomology class of the Kähler form lies in the deep interior of the Kähler cone. This is what we will refer to as the large volume limit. Recall that the FI parameters r_a in the gauged linear σ-model in Section 2.5 are related to the Kähler parameters t_a. Hence we can identify the Kähler cone with the cone $r_1 > 0$, $r_2 > 0$ representing the smooth phase I in figure 2.1.

The Gepner point

There is another important point deep inside phase III which will be referred to as the Gepner point since from Section 2.5 we know that this phase is described by a Landau-Ginzburg orbifold theory which for $r_1 = -\infty$, $r_2 = -\infty$ has no instanton contributions, and from Section 2.4 we know that this description via an exact Landau-Ginzburg orbifold theory is equivalent to the description by the corresponding Gepner model. At this point the residual symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_6$ gets enhanced to $\mathbb{Z}_{12}$. In general, this enhanced symmetry group will be $\mathbb{Z}_d$ where d is the degree of $W(\Phi^i)$. Let us look for the origin and the meaning of this symmetry.

If we start with a conformal field theory having a symmetry group H and take an orbifold by H then the resulting conformal field theory always has some symmetries governed by the existence of the group structure: in the string interactions, strings in sectors g_1, g_2 join to give a string in the sector $g_3 = g_1 g_2$ [67], [68]. For any one-dimensional representation of H, defined by assigning phases $\varepsilon(h)$, to elements $h \in H$, the theory modded out by H has a symmetry which sends a string state in the h twisted Hilbert space to itself times the phase $\varepsilon(h)$. The fact that the interactions respect the group law implies that this is a well defined symmetry of the orbifoldized theory. If the group is e.g. $\mathbb{Z}_d$ the one-dimensional representations form again a $\mathbb{Z}_d$ which is a symmetry of the orbifold theory. The generator of this $\mathbb{Z}_d$ acts by multiplying an element in the r twisted sector by $\exp(2\pi i r/d)$. In this case, twisting the orbifold theory by this $\mathbb{Z}_d$ symmetry returns the original theory we started with [69]. An application of this argument to the $\mathbb{Z}_d$ orbifold of a Landau-Ginzburg theory discussed in Section 2.2. Suppose that there are N elements of the (c,c) ring come from the untwisted sector of the orbifold theory. By the discussion at the beginning of this subsection, there is a N parameter family of complex deformations for which we obtain a theory with an enhanced $\mathbb{Z}_d$ symmetry [70] if we adjust the Kähler parameters suitably. This $\mathbb{Z}_d$ quantum symmetry acts on the chiral primary fields (2.79) in the Landau-Ginzburg orbifold theory or the Gepner model by

$$(\Phi_j)^l = \phi^{(j),l}_{l,0} \longrightarrow e^{\frac{2\pi i}{k_j+2}} (\Phi_j)^l = e^{\frac{2\pi i}{k_j+2}} \phi^{(j),l}_{l,0} \tag{2.119}$$

Note that this is a quantum symmetry of a conformal field theory which has no classical analog even though the underlying conformal field theory is not exactly solvable.

As we try to move from one phase to another conformal perturbation theory about one of these deep interior points breaks down. In the non-linear σ-model region this means that if the Calabi–Yau space gets too small, the expansion parameter $\frac{\alpha'}{r^2}$ gets big and perturbation theory will be invalid. However, using the interpolating gauged linear σ-model we have seen that the conformal field theories corresponding to almost all points in the moduli space are well defined and hence we can smoothly follow a path in the moduli space beyond the smooth region. An important point is that if we allow for analytic continuation, then we can make sense of a perturbative expression about the deep interior point of the $r_1 > 0$, $r_2 > 0$ sector for essentially any point in the moduli space, even with $r_i < 0$ for some i. Thus, in this sense, we can think of the deep interior point in Landau-Ginzburg phase as being the analytic continuation of a Calabi–Yau non-linear σ-model with a particular Kähler class. In terms of the parameters r_i, we see that this special choice seems to require (an analytic continuation to) a negative Kähler class. However, it was shown [71] that the physical parameters $\tilde{r}_i$ (and their analytic continuations) which arise from integrating out massive modes in the gauged linear σ-model are non-trivial functions of the r_i which appear to always be non-negative.

This quantum symmetry will be taken advantage of at several places in this thesis. In Section 3.2.2 it will be related to the symmetry group of the Gepner model and the complex structure deformations of the Calabi–Yau space and further in Section 3.4.2 to the periods of the Calabi–Yau space. In this section we will also make use of the analytic continuation in order to go from the Landau-Ginzburg orbifold phase to the smooth Calabi–Yau phase and back. Finally, the symmetry will reappear in Section 4.3 where the so-called B-type boundary states constructed for the Gepner models come in orbits of this symmetry group.

Mirror symmetry

Here we discuss the consequences of two observations we made on the relative sign of the $U(1)$ charges. At the end of the example of the non-linear σ-model in Section 2.2 we noted that the (c,c)- and the (a,c)-rings differ only by the conventional sign of the relative $U(1)$ charges, while their geometrical counterparts, the cohomology groups $H^{1,1}(X)$ and $H^{2,1}(X)$ differ far more significantly as they are completely different mathematical objects. The resolution of this paradox is given by the claim of [12] that to each Calabi–Yau manifold X there is a second Calabi–Yau manifold X^* corresponding to the same conformal field theory but with the association of $H^{1,1}(X^*)$ and $H^{2,1}(X^*)$ to conformal field theory marginal operators reversed relative to that of X. Since the Hodge diamonds of X and X^* (see Section 3.1) are obtained from each other by a reflection along the diagonal, the pair (X, X^*) is called a mirror pair, and the symmetry relating the two manifolds is called mirror symmetry.

In Section 2.4 we noted that the orbifold of a minimal model with respect to its left-right symmetry group is isomorphic to the original minimal model with the relative sign of the $U(1)$ charges switched. Greene and Plesser [72] realized that this orbifoldizing applied to Gepner models can be used to give an explicit construction of such mirror manifolds. They have shown that

$$(k_1, \dots, k_r) \cong \frac{(k_1, \dots, k_r)}{G} \tag{2.120}$$

where G is the maximal subgroup of $\prod_{j=1}^{r} \mathbb{Z}_{k_j+2}$ by which one can orbifold and preserve the integrality of the $U(1)$ charges of the theory. The isomorphism between the two theories is a reversal of all the $U(1)_R$ eigenvalues of the fields in the left hand side relative to those in the right hand side. If we use the fact from (2.79) that the fields in the Gepner model can be represented as fields in the associated Landau-Ginzburg orbifold theory the action of G is

$$(\Phi_1, \dots, \Phi_r) \mapsto (e^{2\pi i \frac{n_1}{q_1}} \Phi_1, \dots, e^{2\pi i \frac{n_r}{q_r}} \Phi_r) \tag{2.121}$$

for arbitrary integers $(n_1, \dots, n_r)$ such that $\sum_{j=1}^{r} \frac{n_j}{q_j}$ is an integer. Since this operation of orbifolding is independent of the Kähler moduli of the theory, it can be transported from the Landau-Ginzburg phase to the smooth phase where it now acts on the coordinates of the corresponding weighted projective space $\mathbb{P}^n_w$ in the same way. The integrality condition translates into the preservation of the holomorphic 3-form Ω on X.

Mirror symmetry can actually be proven physically in the context of the gauged linear σ-model [73].

Let us briefly review the most important consequence of the existence of such mirror pairs (X, X^*). Consider a (non-vanishing) three-point function of conformal field theory operators corresponding to

$(2,1)$ forms on X. It can be shown [74] that it is given by

$$\mathcal{F}_C^{ijk}(X) = \int_X \Omega^{abc} \tilde{b}_a^{(i)} \wedge \tilde{b}_b^{(j)} \wedge \tilde{b}_c^{(k)} \wedge \Omega \tag{2.122}$$

where the $\tilde{b}_a^{(i)}$ are $(2,1)$-forms expressed as elements of $H^1(X, TX)$ with their subscripts being tangent space indices. Due to a non-renormalization theorem proven in [59] we know that this expression is the exact conformal field theory result. By mirror symmetry the same conformal field theory operators correspond to particular $(1,1)$-forms on the mirror X^* which we can label $b^{(i)}$. Due to the absence of such a non-renormalization theorem the expression for this coupling in terms of geometric quantities on X^* is comparatively complicated [64], [65]

$$\mathcal{F}_K^{ijk}(X^*) = \int_{X^*} b^{(i)} \wedge b^{(j)} \wedge b^{(k)} + \sum_{m,\{u\}} e^{-\int_\Sigma u_m^*(J)} \left(\int_\Sigma u^*(b^{(i)}) \int_\Sigma u^*(b^{(j)}) \int_\Sigma u^*(b^{(k)}) \right) \tag{2.123}$$

where the $b^{(i)} \in H^1(X^*, T^{(1,0)*})$, $\{u\}$ is the set of holomorphic maps $u: \Sigma = \mathbb{P}^1 \to \Gamma$ to rational curves Γ on X^*, π_m is an m-fold cover $\mathbb{P}^1 \to \mathbb{P}^1$ and $u_m = u \circ \pi_m$. The first term is the intersection form on X^* while the second term arises from the infinite series of corrections due to world-sheet instantons (2.117). Since these expressions are the same three-point function in the same conformal field theory, they must be equal [72].

$$\mathcal{F}_C^{ijk}(X) = \mathcal{F}_K^{ijk}(X^*) \tag{2.124}$$

Superconformal Ward identities can be used to show [62] that each factor in (2.115) is a so-called special Kähler manifold (see Section 3.4). The three-point functions in (2.124) are the third derivatives of the prepotentials $\mathcal{F}_C(X)$ of $\mathcal{M}_C^{\mathrm{SCFT}}$ and $\mathcal{F}_K(X^*)$ of $\mathcal{M}_K^{\mathrm{SCFT}}$, respectively.

(2.122) is directly calculable while (2.123) requires the knowledge of the rational curves of every degree on X^*. Turned around, one can use (2.122) to determine the number of rational curves of arbitrary degree on X^*, a question of mathematical interest in the context of enumerative geometry [75], [76].

Since (2.123) contains corrections from world-sheet instantons which vanish only in the large volume limit the Kähler moduli space $\mathcal{M}_K^{\mathrm{SCFT}}$ differs from $\mathcal{M}_K^{\mathrm{geom}}$. However, the moduli space of complex structure deformations $\mathcal{M}_C^{\mathrm{SCFT}}$ coincides with $\mathcal{M}_C^{\mathrm{geom}}$ due to the non-renormalization theorem mentioned above. Using mirror symmetry we can now define $\mathcal{M}_K^{\mathrm{SCFT}}(X^*)$ to be $\mathcal{M}_C^{\mathrm{geom}}(X)$. In particular, we can compute $\mathcal{F}_K(X^*) = \mathcal{F}_C(X)$.

2.7. Witten index

The Witten index will be one of main computational tools in Chapter 6. Since it can be defined in any supersymmetric theory, we will review it in this chapter on superconformal field theories. In any supersymmetric theory there is the operator $(-1)^F$ that distinguishes bosonic from fermionic states in the Hilbert space and anticommutes with the supersymmetry generators Q. The crucial observation [21] is that the states of non-zero energy are paired by the action of Q in two-dimensional supermultiplets while, on the other hand, the zero-energy states form trivial one-dimensional supermultiplets. In general, there may be an arbitrary number $n_B^{E=0}$ of zero-energy bosonic states, and an arbitrary number $n_F^{E=0}$ of zero-energy fermionic states. The difference $n_B^{E=0} - n_F^{E=0}$ does not change under the variation of the parameters of the theory due to the different multiplet structure. Formally, the quantity $n_B^{E=0} - n_F^{E=0}$ may be regarded as the trace of the operator $(-1)^F$. States of non-zero energy do not contribute to $\mathrm{tr}(-1)^F$ because for any bosonic state of non-zero energy that contributes $+1$ to the trace, there is a fermionic state of non-zero energy that contributes -1 and cancels the bosonic contribution. Therefore the Witten index $\mathrm{tr}(-1)^F$ can be evaluated among the zero-energy states only [21]

$$\mathrm{tr}(-1)^F = \mathrm{tr}(-1)^F e^{-\beta H} = n_B^{E=0} - n_F^{E=0} \tag{2.125}$$

The insertion of $e^{-\beta H}$ for arbitrary positive β is necessary to regularize the infinite summation over all states in the Hilbert space which is ill-defined not being absolutely convergent. This is actually independent of β because the states of $E \neq 0$ do not contribute. If the Witten index is non-zero, then supersymmetry is spontaneously broken.

The Witten index can be interpreted mathematically as the index of an operator. If we split the Hilbert space $\mathcal{H}$ of our theory into bosonic and fermionic subspaces, $\mathcal{H}_B$ and $\mathcal{H}_F$, then the supersymmetry charge Q which maps bosons into fermions and vice versa takes the following form

$$Q = \left(\begin{array}{c|c} 0 & M^\dagger \\ \hline M & 0 \end{array} \right) \tag{2.126}$$

where the split of Q corresponds to $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. The zero energy eigenstates are given by the kernels of M and $M^\dagger$ and therefore

$$\mathrm{tr}(-1)^F = \dim\ker M - \dim\ker M^\dagger = \mathrm{ind}\, M \tag{2.127}$$

Hence the independence of the Witten index on the parameters of the theory translates into the fact that it is a topological quantity. Therefore, it can be calculated in a convenient limit. One strategy is to reduce the supersymmetric theory under consideration to a supersymmetric quantum mechanics in the zero momentum limit. This is precisely what we have done in Section 2.2 for the non-linear σ-model with target space a Calabi–Yau manifold. We have seen that the Hamiltonian is $H = Q_+\overline{Q}_+ + \overline{Q}_+Q_+ + Q_-\overline{Q}_- + \overline{Q}_-Q_-$ acting on (r, s)-forms. We can interpret $r + s$ as the number of fermions present, so (r, s)-forms are to be regarded as bosonic or fermionic depending on whether $r + s$ is even or odd. Therefore (anticipating (3.3) and (3.4))

$$\mathrm{tr}(-1)^F = \sum_{r,s=0}^{3} (-1)^{r+s} h^{r,s} = 2(h^{1,1} - h^{2,1}) = \chi(X) \tag{2.128}$$

The computation of the Witten index at the Gepner point has been performed in [77] based on the Witten index for the minimal models and the Landau-Ginzburg theories in [46] and on properties of the characters of the $\mathcal{N} = 2$ superconformal algebra worked out in [78]. The open string version of these computations will be discussed in Sections 4.3.1 and 6.1.

3. Calabi-Yau Spaces

3.1. General properties of Calabi–Yau spaces

There are several ways to define a Calabi–Yau space. The following statements are equivalent and any of them can be taken as the definition of a Calabi–Yau space. A Calabi–Yau manifold X of dimension n

(a) is a compact Kähler manifold of vanishing first Chern class.

(b) admits a Levi-Civita connection with $SU(n)$ holonomy.

(c) admits a nowhere vanishing holomorphic $(n, 0)$-form Ω.

(d) is a compact manifold with a Ricci-flat Kähler metric.

(e) has a trivial canonical bundle $\mathcal{K}_X \cong \mathcal{O}_X$.

That (d) follows from (a) has been conjectured by Calabi [79] and proven by Yau [80]. We assume that the holonomy group is not a subgroup of $SU(n)$ which is equivalent to demanding that [81]

$$h^{p,0}(X) = h^{0,p}(X) = 0 \qquad\qquad p \neq 0, n \tag{3.1}$$

Furthermore, the existence of Ω implies that

$$h^{n,0}(X) = 1 \qquad\qquad h^{p,0}(X) = h^{n-p,0}(X) \quad p = 0, \ldots, n \tag{3.2}$$

By complex conjugation and Poincaré duality, the Hodge diamond for a Calabi–Yau threefold then has the form

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.3}$$

The Euler number of X is

$$\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X)) \tag{3.4}$$

From the exponential cohomology sequence one gets $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ and $\rho(X) = h^{1,1}(X)$, where $\rho(X)$ denotes the Picard number of X. This space is naturally associated with the Kähler deformations of X which are parametrized by $\mathcal{M}_K^{\mathrm{geom}}$. On the other hand, the second non-trivial Hodge number $h^{1,2}(X)$ of X expresses the number of parameters for the complex structure on X. The first order deformations of a Calabi–Yau threefold X are unobstructed and the corresponding local moduli space $\mathcal{M}_C^{\mathrm{geom}}$ of X is smooth and has dimension $\dim \mathcal{M}_C^{\mathrm{geom}}(X) = h^1(X, TX) = h^{1,2}(X)$ [82], [83] and [84]. For a sufficiently generic Calabi–Yau threefold X the Kähler moduli of the complexified Kähler class

$\omega = B + iJ$ is independent of the complex structure of X [56]. So locally, the moduli space of a Calabi–Yau threefold consists of a product of the moduli space of complex structure deformations and the moduli space of Kähler structure deformations

$$\mathcal{M}^{\text{geom}}_{\text{CY}} = \mathcal{M}^{\text{geom}}_{K} \times \mathcal{M}^{\text{geom}}_{C} \tag{3.5}$$

Furthermore, the holomorphic $(3,0)$-form Ω only depends on the complex structure of X. For more details see [85].

The Hodge numbers do not exhaust the topological information available. There is considerable information available in the numbers

$$K_{abc} = \int_X J_a \wedge J_b \wedge J_c \tag{3.6}$$

$$\mathrm{c}_2 \cdot J_a = \int_X \mathrm{c}_2(X) \wedge J_a \tag{3.7}$$

where the J_a are a basis for the harmonic $(1,1)$-forms. These numbers are topological i.e. they do not involve the complex structure in virtue of two facts. The Hodge number $h^{2,0}$ vanishes so $H^{1,1}(X) \cong H^2(X)$ and the Pontrjagin class $\mathrm{p}_1(X) = \mathrm{c}_1(X)^2 - 2\,\mathrm{c}_2(X) = -2\,\mathrm{c}_2(X)$ is proportional to $\mathrm{c}_2(X)$ and is defined for a real manifold independent of any complex structure. A theorem of Wall [86] shows that the data (3.6) and (3.7), together with $b_3 = 2 + 2h^{2,1}$ classify simply connected real six-manifolds. The classification of Calabi–Yau manifolds is more complicated since not every real six-manifold is a Calabi–Yau manifold and a real manifold may admit distinct complex structures in such a way that they may not be continuously deformed into each other. For further properties see [87], [88] and [89]. We will also need singular Calabi–Yau spaces but we defer their introduction to Section 3.2.1.

3.2. Calabi–Yau spaces as hypersurfaces in toric varieties

3.2.1. General facts about toric varieties and dual polyhedra

Although there is no classification of three-dimensional Calabi–Yau spaces yet available, there exist several methods of constructing classes of such spaces and their mirrors [85], [90]. The most prominent ones are Calabi–Yau spaces as hypersurfaces or complete intersections in toric varieties (for which there exists a classification [91]). The reason is that toric varieties have an underlying group structure which allows to reduce almost all of the topological properties of these varieties to calculations of a set of combinatorial data, so-called fans. They provide an elegant framework to carry out the physical ideas by explicit computations. From a physical point of view, the combinatorial data have a direct interpretation in the gauged linear σ-model, as has been discussed in 2.5. Although the formalism applies to general complete intersections in toric varieties, we will restrict ourselves to a simple subclass consisting of Fermat hypersurfaces in weighted projective spaces. As will become clear in later chapters, the reasons are that for Calabi–Yau spaces in weighted projective spaces (as opposed to general toric varieties) there exists a Gepner point in the Kähler moduli space with an enhanced symmetry and that the construction of boundary states in the Gepner model is known only for minimal models with A-type or diagonal invariants which means that we have to take Fermat hypersurfaces.

We will not give an introduction to toric varieties here. Instead we refer to the books by Oda [92] and Fulton [93] and the survey article by Danilov [94]. Mathematical introductions for physicists can be found in [51] and [7]. We will follow mostly [95] and [85].

To describe a toric variety $\mathbb{P}_\Delta$, let us consider an n-dimensional convex integral polyhedron $\Delta \in \mathbb{R}^n$ with vertices ν_i, $i = 1, \ldots, p$ containing the origin $\nu_0 = (0, \ldots, 0)$. An integral polyhedron is a polyhedron whose vertices ν_i are integral with respect to the lattice $M \equiv \mathbb{Z}^n \subset \mathbb{R}^n$. Let $N =$

$\mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be the dual lattice. We define the dual polyhedron by

$$\Delta^* = \{x \in \mathbb{R}^n | (x, y) \geq -1 \quad \forall y \in \Delta\} \subset N_{\mathbb{R}} \tag{3.8}$$

whose vertices are ν_i^*, $i = 1, \ldots, p^*$. In the examples in Section 3.5 and in Appendix C the vertices ν_i^* will be used.

A polyhedron is called reflexive if its dual polyhedron is again an integral polyhedron. We associate to Δ a complete rational fan $\Sigma(\Delta)$ as follows [96]: For every l-dimensional face $\Theta_l \in \Delta$ we define an n-dimensional cone $\sigma(\Theta_l)$ by $\sigma(\Theta_l) = \{\lambda(p' - p) | \lambda \in \mathbb{R}_+, p \in \Delta, p' \in \Theta_l\}$. $\Sigma(\Delta)$ is then given as the collection of $(n - l)$-dimensional dual cones $\sigma^*(\Theta_l)$, $l = 1, \ldots, n$ for all faces of Δ. Similarly, we can associate a fan $\Sigma(\Delta^*)$ to Δ^*. To each pair of reflexive polyhedra (Δ, Δ^*) one can associate a pair of complete fans $(\Sigma(\Delta), \Sigma(\Delta^*))$ and in turn a pair of n-dimensional toric varieties $(\mathbb{P}_{\Delta^*}, \mathbb{P}_\Delta) = (\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{P}_{\Sigma(\Delta)})$. Each toric variety $\mathbb{P}_\Sigma$ contains an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$ whose coordinates we will denote by $X_1, \ldots, X_n$. It admits an action $\mathbb{T} \times \mathbb{P}_\Sigma \to \mathbb{P}_\Sigma$ of $\mathbb{T}$ that extends the natural action of $\mathbb{T}$ on itself.

In each of the toric varieties $\mathbb{P}(\Delta)$, there is a family of Calabi–Yau hypersurfaces given by the closure of the zero section Z_{f_Δ} of the anticanonical bundle of $\mathbb{P}(\Delta)$

$$f_\Delta(X, a) = \sum_{i=0}^{p} a_i X^{\nu_i} \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \tag{3.9}$$

which is a Laurent polynomial in $\mathbb{T}$. The coefficients $a_0, \ldots, a_p$ are coordinates on an affine space $\mathbb{C}^{p+1}$ and $X^{\nu_i} = \prod_{k=1}^n X_k^{\nu_{i,k}}$. f_Δ and Z_{f_Δ} are called Δ-regular if for all $l = 1, \ldots, n$ the f_{Θ_l} and the $X_i \frac{\partial}{\partial X_i} f_{\Theta_l}$, $\forall i = 1, \ldots, n$ do not vanish simultaneously in $\mathbb{T}$. The variation of the parameters a_i under the condition of Δ-regularity leads to a family of Calabi–Yau varieties.

The ambient space $\mathbb{P}_\Delta$ and so Z_{f_Δ} are in general singular. Δ-regularity ensures that the only singularities of Z_{f_Δ} are the ones inherited from the ambient space. Here comes the reflexivity in. In order to obtain a smooth Calabi–Yau manifold we need to resolve the singularities which can be done if and only if $\mathbb{P}_\Delta$ is Gorenstein which is the case if and only if Δ is reflexive [96]. For Gorenstein spaces and singularities see [97], [98] and in the context of toric varieties see [92]. Singularities and their resolution will be discussed in Section 3.3.

We can also define a Calabi–Yau manifold in exactly the same way from the dual polyhedron Δ^*. We denote the families obtained this way by $X_\Delta = \{Z_{f_\Delta} | a \in \mathbb{C}^{p+1}\}$ and by $X_{\Delta^*} = \{Z_{f_{\Delta^*}} | a \in \mathbb{C}^{p+1}\}$. Batyrev showed that from a pair of reflexive polyhedra $(\Sigma(\Delta), \Sigma(\Delta^*))$ one can naturally construct a pair of mirror Calabi–Yau families (X_{Δ^*}, X_Δ). In particular, the mirror map on the Hodge numbers can be explicitly seen through the following formulae [96]

$$\begin{aligned} h^{1,1}(X_\Delta) &= h^{n-2,1}(X_{\Delta^*}) \\ &= l(\Delta^*) - (n+1) - \sum_{\operatorname{codim} \Theta^* = 1} l'(\Theta^*) + \sum_{\operatorname{codim} \Theta^* = 2} l'(\Theta^*) l'(\Theta) \end{aligned} \tag{3.10}$$

$$\begin{aligned} h^{1,1}(X_{\Delta^*}) &= h^{n-2,1}(X_\Delta) \\ &= l(\Delta) - (n+1) - \sum_{\operatorname{codim} \Theta = 1} l'(\Theta) + \sum_{\operatorname{codim} \Theta = 2} l'(\Theta) l'(\Theta^*) \end{aligned} \tag{3.11}$$

where $l(\Theta)$ and $l'(\Theta)$ are the number of integral points on a face $\Theta \in \Delta$ and in its interior, respectively. Recall that the complex parameters $(a_0, \ldots, a_p)$ represent the deformations of the defining equation f. The monomial deformations of f provide the complex structure deformations of X_{Δ^*}, however not all of them. The contribution from the last term in (3.10) and (3.11) can not be associated with a monomial in the Laurent polynomial f. In the language of Landau-Ginzburg theories, if appropriate, they correspond to contributions from twisted sectors, see Section 2.2. They are called non-toric contributions and we denote the Hodge numbers without these contributions by $\tilde{h}^{1,1}(X_\Delta)$ and $\tilde{h}^{n-2,1}(X_\Delta)$. They will play a role in Sections 3.3.3 and 6.2. In general, there is a rather subtle relationship between deformations of the polynomials defining the Calabi–Yau variety X and deformations of the complex structure, see [99] for a thorough treatment of this question.

3.2.2. Calabi–Yau hypersurfaces in weighted projective spaces

As mentioned above we will restrict ourselves to toric ambient varieties which are weighted projective spaces. An n-dimensional weighted projective space is defined as an $(n+1)$-dimensional space modded out by a $\mathbb{C}^*$ action given by the weights $w = (w_1, \dots, w_{n+1})$ as follows

$$\mathbb{P}^n_w = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{(z_1, \dots, z_{n+1}) \sim (\lambda^{w_1} z_1, \dots, \lambda^{w_{n+1}} z_{n+1})}, \quad \lambda \in \mathbb{C}^* \tag{3.12}$$

An extensive study of the properties of such spaces can be found in [100] and [101]. Consider the zero locus of quasi-homogeneous polynomials W_i, $i = 1, \dots, m$ of degree $d_i = \deg(W_i)$

$$X = \{(z_1 : \dots : z_{n+1}) \in \mathbb{P}^n_w | W_i(z_1, \dots, z_{n+1}) = 0, \quad i = 1, \dots, m\} \tag{3.13}$$

In order to ensure that the embedding $X \hookrightarrow \mathbb{P}^n_w$ is smooth, the polynomials W_i must be transversal, i.e. $W_i(z) = 0$ and $\mathrm{d}W_i(z) = 0$ have no simultaneous solution except at $z_0 = \ldots = z_n = 0$. The first Chern class of Xvanishes precisely if $\sum_{i=1}^m d_i = \sum_{j=1}^{n+1} w_j$ and hence X will be a Calabi–Yau variety. We will consider only the case of hypersurfaces where we have a single polynomial W of degree d [101]. We have seen that these varieties come in families which we will denote by $X = \mathbb{P}^n_w[d]$.

We can connect this description to the previous one by noting that one can associate to such a Calabi–Yau hypersurface a reflexive polyhedron if $\mathbb{P}^n_w$ is Gorenstein which is the case if $\mathrm{lcm}(w_1, \dots, w_{n+1})$ divides the degree d [101]. In this case we can define a simplicial reflexive polyhedron $\Delta(w)$ in terms of the weights w as the convex hull of the integral vectors μ of the exponents of all quasi-homogeneous monomials z^μ of degree d shifted by $(-1, \dots, -1)$

$$\Delta(w) = \{x \in \mathbb{R}^{n+1} | (w, x) = 0, \; x_i \geq -1, \; i = 1, \dots, n+1\} \tag{3.14}$$

Note that this implies that the origin is the only point in the interior.

The next restriction we will consider is that of the Fermat hypersurfaces. We call a polynomial W a Fermat polynomial if it consists of monomials z_i^{d/w_i}, $i = 1, \dots, n+1$. In this case $\mathbb{P}^n_w$ is Gorenstein and (Δ, Δ^*) are simplicial. Hence, the toric variety $\mathbb{P}_{\Sigma(\Delta(w))}$ is isomorphic to $\mathbb{P}^d_w$ with X_Δ isomorphic to some $X = \mathbb{P}^n_w[d]$[1]. Then the mirror hypersurface X_{Δ^*} can be understood [96] as an orbifold of X_Δ in $\mathbb{P}_{\Sigma(\Delta^*)}$ giving the orbifold construction of Greene and Plesser [72] explained in Section 2.6.

If furthermore at least one weight is one (say $w_1 = 1$) we may choose $e_i = (-w_i, 0, 0, \dots, 1, \dots, 0)$, $i = 2, \dots, n+1$ as generators for Λ, the lattice induced from the $\mathbb{Z}^{n+1}$ cubic lattice on the hyperplane $H = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} w_i x_i = 0\}$. For this type of models we then always obtain as vertices of $\Delta(w)$

$$\begin{aligned} &\nu_1 = (-1,-1,-1,-1) && \nu_2 = (\frac{d}{w_2} - 1, -1, -1, -1) && \nu_3 = (-1, \frac{d}{w_3} - 1, -1, -1) \\ &\nu_4 = (-1, -1, \frac{d}{w_4} - 1, -1) && \nu_5 = (-1, -1, -1, \frac{d}{w_5} - 1) && \end{aligned} \tag{3.15}$$

and for the vertices of the dual simplex $\Delta^*(w)$ one finds

$$\begin{aligned} &\nu_1^* = (-w_2, -w_3, -w_4, -w_5) && \nu_2^* = (1,0,0,0) && \nu_3^* = (0,1,0,0) \\ &\nu_4^* = (0,0,1,0) && \nu_5^* = (0,0,0,1) && \end{aligned} \tag{3.16}$$

[1]For general hypersurfaces of non-Fermat type, $\mathbb{P}^n_w$ and $\mathbb{P}_{\Delta(w)}$ are only birational. In fact, the fan $\Sigma(\Delta(w)^*)$ is a refinement of the fan of $\mathbb{P}^n_w$. The hypersurfaces $X = \mathbb{P}^n_w[d]$ and $X_{\Delta(w)}$ are related by flop transitions [102].

There is a particular point in the complex structure moduli space of X at which the defining polynomial W exhibits particular symmetries. Let the group in the Greene-Plesser mirror construction (2.120) be G. The most general G-invariant hypersurface X in $\mathbb{P}^4_w$ has an equation of the form

$$W = \sum_{i=1}^{5} a_i z_i^{k_i} + \sum_{j=6}^{p} a_j z^{m_j} \tag{3.17}$$

where $z^{m_j} = \prod z_i^{m_{j,i}}$ are G-invariant monomials given by the monomial-divisor mirror map [103], [96]

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \longmapsto \frac{\left(\prod_{i=1}^{5} z_i\right)^{1+\sum_{j=1}^{4}\mu_j}}{\prod_{i=1}^{4} z_i^{\mu_i d/w_i}} \tag{3.18}$$

There is a natural $(\mathbb{C}^*)^5$ action on the space of a's which allows us to set $a_i = 1$, $i = 1, \ldots, 5$. We can then define new coordinates, generalizing [104], [105] by

$$\psi_j = -\frac{a_{5+j}}{N_j \prod_{i=1}^{5} a_i^{m_{j,i}/k_i}} \qquad j = 1, \ldots, \tilde{h}^{1,1} \tag{3.19}$$

where the N_j are some normalization constants related to the monomials z^{m_j} by $N_j = \frac{d}{m_{j,i} w_i}$ for any i such that $m_{j,i} \neq 0$. Following [105] we extend the action of G on $(z_1, \ldots, z_5)$ to an action of $\widehat{G}$ on $(z_1, \ldots, z_5; \psi_1, \ldots, \psi_{\tilde{h}^{1,1}})$. If we mod the family $\{W = 0\}$ by $\widehat{G}$ then the parameter space $\{(\psi_1, \ldots, \psi_{\tilde{h}^{1,1}})\}$ must be modded out by a $\mathbb{Z}_d$ whose generator g acts by

$$(\psi_1, \ldots, \psi_{\tilde{h}^{1,1}}) \longmapsto (\alpha^{d/N_1}\psi_1, \ldots, \alpha^{d/N_{\tilde{h}^{1,1}}}\psi_{\tilde{h}^{1,1}}) \tag{3.20}$$

where α is a dth root of unity. Now we can define the Gepner point of the complex structure moduli space to be the point which exhibits this additional $\mathbb{Z}_d$ symmetry. Regarding this space as the Kähler moduli space of the mirror X^* then this point corresponds to the one where the description by a Gepner model is valid as was discussed in Section 2.6. So this symmetry is nothing but the quantum symmetry introduced there. We will take up the discussion of the importance of these coordinates in Section 3.4.1 where we study the action of this $\mathbb{Z}_d$ on the periods of the Calabi–Yau manifold.

3.3. Divisors and Curves in Calabi–Yau spaces

Divisors in a three-dimensional toric Calabi–Yau manifold are algebraic surfaces. These are well-studied and classified to a certain extent. Moreover, while there is almost nothing known about stable bundles or sheaves on Calabi–Yau threefolds, there are many results on stable bundles on algebraic surfaces. Since our main interest lies in D-branes which are described by bundles, we will discuss divisors and therefore algebraic surfaces in this subsection in detail. The relation to D-branes and bundles will be elucidated in Chapters 5 and 6.

Recall that an analytic hypersurface $V \subset X$ in a projective variety X is given, for any $p \in V$, in a neighborhood of p as the zero set of a single holomorphic function f. A divisor D on X is a locally finite formal linear combination $D = \sum n_i \cdot V_i$ of irreducible analytic hypersurfaces V_i of X. For example, the Calabi–Yau hypersurface X is a divisor in $\mathbb{P}^n_w$. Such a divisor D is called effective if $n_i \geq 0$ for all i; we then write $D \geq 0$. If the function f is the coefficient of a holomorphic top form on X, then the corresponding divisor is called the canonical divisor and denoted K_X.

There is an important relation between divisors on X and line bundles on X. A line bundle L is characterized by its first Chern class $\mathrm{c}_1(L) = [L] \in H^{1,1}(X)$ which is Poincaré dual to an algebraic submanifold of codimension one, a divisor $L \in H_{2n-2}(X)$. In a coordinate patch U_i the divisor is defined as the zero of a meromorphic function f_i such that on the intersection $U_i \cap U_j$, $g_{ij} = f_i/f_j$ is the transition function of the line bundle L. The trivial line bundle on X will be denoted by $\mathcal{O}_X$. If H is a hyperplane in $\mathbb{P}^n_w$ then the line bundle with Chern class mH on $\mathbb{P}^n_w$ is denoted $\mathcal{O}_{\mathbb{P}^n_w}(mH) = \mathcal{O}_{\mathbb{P}^n_w}(m)$. The line bundle associated to the canonical divisor K_X is called the canonical line bundle $\mathcal{K} = \mathcal{O}_X(K_X)$. In general, the line bundle associated with D will be written as $\mathcal{O}_X(D)$. Note that we will often confuse the divisor D, its homology class $[D]$, its Poincaré dual $[D]$ and its representative and assume that it is clear from the context which notion is appropriate.

As a further example take again the Calabi–Yau manifold X as a hypersurface in $\mathbb{P}^n_w$. The associated line bundle is then $\mathcal{O}_{\mathbb{P}^n_w}(-K_{\mathbb{P}^n_w})$. Such a line bundle is again a toric variety whose vertices are $\bar{\nu}^*_i = (\nu^*_i, 1)$, called the extended vertices. In terms of the gauged linear σ-model in Section 2.5 the total space of this line bundle is the space V of classical ground states (2.107), (2.111). For the computations described in the following subsections it is more convenient to work with these extended vertices.

Toric geometry provides us naturally with a set of divisors. Each integral point ν^*_i, $i = 1, \dots, p$ in $\Delta^* \cap \mathbb{Z}^n$ corresponds to an irreducible $\mathbb{T}$-invariant divisor D_i. For a Calabi–Yau hypersurface X we will denote the restriction of these divisors to the hypersurface by the same letter D_i. There are two main classes of divisors in a Calabi–Yau threefold of our interest, those coming from the resolution of singularities and those defining fibrations. We will discuss them in turn after a short overview over the algebraic surfaces. But first we discuss the Mori cone and its intersection ring in order to be able to compute the properties of the divisors from those of the Calabi–Yau hypersurface.

3.3.1. The Mori cone and the intersection ring

The parameters a_i in (3.9) or in (3.17) can be used to describe $\mathcal{M}^{\mathrm{geom}}_C$ (after subtracting those corresponding to reparametrizations of $\mathbb{P}^n_w$), but they form an affine space which must be compactified. This is achieved via the secondary fan, the central object in the study of mirror symmetry of toric Calabi–Yau spaces, which is roughly defined as follows [102], [85]. The main idea is to compactify $\mathcal{M}^{\mathrm{geom}}_C$ such that it becomes a toric variety where the torus action corresponds to the action of the $U(1)$ gauge groups in the gauged linear σ-model. Let Ξ be the set of the one-dimensional cones of $\Sigma(\Delta^*)$ (i.e. those in (3.16) together with the additional cones coming from the resolution of singularities in Section 3.3.3) and $\Xi^+ = (\Xi \cup \{0\}) \times 1$. Furthermore, if we denote by A_{n-1} the Chow group of Weil divisors modulo linear equivalence then $A(\Xi) = A_{n-1} \otimes \mathbb{R}$ coincides with the affine space of the a_i. Then determine all regular triangulations $\mathcal{T}$ of the convex hull Ξ^+. The vertices of the simplices in $\mathcal{T}$ must be elements of Ξ^+, and regularity means that each of them contains the interior point $\bar{\nu}^*_0$. To $\mathcal{T}$ one can associate in a unique way a cone $\mathcal{C}(\mathcal{T})$ [106], [107]. The cones $\mathcal{C}(\mathcal{T})$ are the maximal cones of the secondary fan, which is a complete fan in $A(\Xi)$.

Its importance lies in the fact that it provides a convenient compactification of both $\mathcal{M}^{\mathrm{geom}}_C$ and $\mathcal{M}^{\mathrm{geom}}_K$. In particular, it is such that the compactification of $\mathcal{M}^{\mathrm{geom}}_C(X)$ is essentially the same as the compactification of $\mathcal{M}^{\mathrm{geom}}_K(X^*)$ [85]. Furthermore, the latter contains the affine toric variety associated to the Kähler cone. From the physical point of view, the secondary fan "is" the phase diagram of the gauged linear σ-model. An example has been given in Figure 2.1. The different phases correspond to the maximal cones $\mathcal{C}(\mathcal{T})$, i.e. the triangulations $\mathcal{T}$. The minimal triangulation, i.e. the one which consists only of the five basic simplices corresponds to the Landau-Ginzburg orbifold phase. A maximal triangulation, i.e. one which uses all the points of Δ^* corresponds to a smooth Calabi–Yau phase.

The basic object to consider in this compactification is the following lattice, the so-called lattice of relations among the vertices

$$L = \left\{ (l_0, \dots, l_p) \in \mathbb{Z}^{p+1} | \sum_{i=0}^{p} l_i \bar{\nu}^*_i = 0 \right\} \tag{3.21}$$

The secondary fan is then a rational polyhedral complete fan in $L^\vee_{\mathbb{R}}$ for the dual lattice $L^\vee$. According to the general construction of toric varieties, the complete fan defines a toric compactification of the torus

$\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}^*)$, which is our compactification of the affine space of a_i's. There is a natural non-degenerate pairing $A^1(\mathbb{P}_{\Sigma(\Delta^*)}) \times L_{\mathbb{R}} \to \mathbb{R}$ which identifies this fan with the one in $A(\Xi)$. Among the cones in the secondary fan, there is a distinguished cone, called the Mori cone, whose geometric meaning is the dual of the Kähler cone of $\mathbb{P}_{\Sigma(\Delta^*)}$. We assume that it is simplicial (for the non-simplicial case see [102] and [85]) and denote the generators of the Mori cone $l^{(1)}, \ldots, l^{(\tilde{h}^{1,1})}$ where $\mathrm{rk}\, H^2(\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{Z}) = \tilde{h}^{1,1}$. Then the Mori cone is $L_{\geq 0} = \mathbb{R}_{\geq 0} l^{(1)} + \cdots + \mathbb{R}_{\geq 0} l^{(\tilde{h}^{1,1})}$.

In order to determine the Mori cone we can use a particular maximal triangulation $\mathcal{T}$. In general this triangulation is not unique. For a chosen $\mathcal{T}$ one proceeds as follows [108]. Consider every pair (S_k, S_l) of four-dimensional simplices in $\mathcal{T}$ which have a common three-dimensional simplex $s_i = S_k \cap S_l$. For all such pairs find the unique linear relation $\sum_{i=1}^6 l_i^{(k,l)} \bar{\nu}_i^* = 0$ among the six points $\bar{\nu}_i^*$ of $S_k \cup S_l$ in which the $l_i^{(k,l)}$ are minimal integers and the coefficients of the two points in $(S_k \cup S_l) \setminus (S_k \cap S_l)$ are non-negative. Finally, find the minimal integer $l^{(a)}$ by which every $l^{(k,l)}$ can be expressed as positive integer linear combination. These are generators of the Mori cone.

The Mori generators define the following linear relations

$$\sum_{i=0}^{p} l_i^{(a)} D_i = 0 \tag{3.22}$$

where D_i is an (overcomplete) basis of $H^2(\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{Z})$ given by the vertices $\bar{\nu}_i^*$. These relations define the ideal $\mathcal{I}_{\mathrm{lin}}$ in the ring $\mathbb{Q}[D_0, D_1, \ldots, D_p]$.

A primitive collection is a collection σ of vertices that do not form a cone but is such that any subset $\sigma' \subset \sigma$ is a cone [109]. For any such primitive collection $\sigma = \{\nu_{i_1}^*, \ldots, \nu_{i_k}^*\}$ we get a non-linear equation for the divisors

$$D_{i_1} \cdot \ldots \cdot D_{i_k} = 0 \tag{3.23}$$

These non-linear relations define the Stanley-Reisner ideal $\mathcal{I}_{SR}$ in the ring $\mathbb{Q}[D_0, D_1, \ldots, D_p]$. Moreover, if the toric variety $\mathbb{P}_{\Sigma(\Delta^*)}$ is non-singular for every collection of k vectors $\sigma = \{\nu_{i_1}^*, \ldots, \nu_{i_k}^*\}$ which does not contain or is not itself a primitive collection, we have

$$D_{i_1} \cdot \ldots \cdot D_{i_k} = 1 \tag{3.24}$$

If the toric variety is singular then the equation is

$$D_{i_1} \cdot \ldots \cdot D_{i_k} = \frac{1}{\left|\det(\nu_{i_1}^*, \ldots, \nu_{i_k}^*)\right|} \tag{3.25}$$

Now let us turn to the description of the intersection ring $A^*(\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{Z})$ which is isomorphic to the cohomology ring $H^{2*}(\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{Z})$ of the nonsingular toric variety $\mathbb{P}_{\Sigma(\Delta^*)}$ in order to determine the necessary topological data of X_Δ. In the case of toric varieties, it has a simple description in terms of the invariant divisors D_i, $i = 0, \ldots, p$

$$A^*(\mathbb{P}_{\Sigma(\Delta^*)}) = \frac{\mathbb{Q}[D_0, D_1, \ldots, D_p]}{\mathcal{I}_{SR} + \mathcal{I}_{\mathrm{lin}}} \tag{3.26}$$

By the non-degenerate pairing $A^1(\mathbb{P}_{\Sigma(\Delta^*)}) \times L_{\mathbb{R}} \to \mathbb{R}$ we see that the dual of the Mori cone $L_{\geq 0}^{\vee}$ lies in $A^1(\mathbb{P}_{\Sigma(\Delta^*)})$, the group of 1-cycles which is dual to $A_{n-1}(\mathbb{P}_{\Sigma(\Delta^*)})$ over $\mathbb{Q}$. In fact, according to the construction of $L_{\geq 0}$, $L_{\geq 0}^{\vee}$ is the Kähler cone of the ambient space $\mathbb{P}_{\Sigma(\Delta^*)}$. We will denote the dual basis to the generators $l^{(1)}, \ldots, l^{(r)}$ by $J_1, \ldots, J_r$.

If D is a hypersurface in X with $\dim X = n$, then the following standard restriction formula [110] relates the intersection form on D to the intersection form on X

$$D_{i_1} \cdot \ldots \cdot D_{i_{n-1}}\big|_D = D_{i_1} \cdot \ldots \cdot D_{i_{n-1}} \cdot D\big|_X \tag{3.27}$$

where D_{i_k} are some divisors on X on the right-hand side of (3.27) and their restrictions to D on the left-hand side of (3.27). We apply this formula to the Calabi–Yau hypersurface in $\mathbb{P}^4_w$. From the Calabi–Yau condition we have $-K_{\mathbb{P}_{\Sigma(\Delta^*)}} = D_0 = \sum_{i=1}^p l_i^{(a)} D_i$ and hence the intersection numbers (3.6) can be written as

$$K_{abc} = -\int_{\mathbb{P}_{\Sigma(\Delta^*)}} K_{\mathbb{P}_{\Sigma(\Delta^*)}} \cdot J_a \cdot J_b \cdot J_c = J_a \cdot J_b \cdot J_c \tag{3.28}$$

where the symbol $\int_{\mathbb{P}_{\Sigma(\Delta^*)}}$ means to take the coefficient of the highest degree element of (3.26) with the normalization determined by the requirement that it gives the Euler number $\chi(X_\Delta)$ from the top Chern class $\mathrm{c}_n(X_\Delta)$ [102]. The toric part of the even cohomology $H^{\mathrm{even}}_{\mathrm{toric}}(X_\Delta, \mathbb{Q})$ may be described by $A^*(\mathbb{P}_{\Sigma(\Delta^*)})_\mathbb{R}/\mathrm{Ann}([X_\Delta])$, where $\mathrm{Ann}([X_\Delta]) = \{v \in A^*(\mathbb{P}_{\Sigma(\Delta^*)}) | [X_\Delta] v = 0\}$.

The Chern classes of X are computed using the adjunction formula [111]

$$\mathrm{c}(X) = \frac{\prod_{i=1}^p (1 - D_i)}{1 - D_0} \tag{3.29}$$

and we will frequently write c_i instead of $\mathrm{c}_i(X)$. Similarly, the Todd class of X is

$$\mathrm{td}(X) = \frac{1 - \exp(-D_0)}{D_0} \prod_{i=1}^p \frac{D_i}{1 - \exp(-D_i)} \tag{3.30}$$

From (3.29) and (3.22) the topological numbers $\mathrm{c}_2 \cdot J_a$ in (3.7) can be determined. In addition, there are useful relations [112] between the intersection numbers (3.28), the linear forms (3.7), the Euler number (3.4) and the Mori generators $l_i^{(a)}$

$$\mathrm{c}_2 \cdot J_a = \frac{1}{2} \sum_{b,c=1}^{\tilde{h}^{1,1}} \left(\sum_{i=1}^p l_i^{(b)} l_i^{(c)} \right) K_{abc} \tag{3.31}$$

$$\chi(X) = \int_X \mathrm{c}_3 = \frac{1}{3} \sum_{a,b,c=1}^{\tilde{h}^{1,1}} \left(\sum_{i=1}^p l_i^{(a)} l_i^{(b)} l_i^{(c)} \right) K_{abc} \tag{3.32}$$

If we describe the divisor D of the Calabi–Yau manifold X by an embedding $i : D \longrightarrow X$ then from the associated exact sequence

$$0 \longrightarrow T_D \longrightarrow T_X|_D \longrightarrow N_{X/D} \longrightarrow 0 \tag{3.33}$$

and $N_{X/D} \cong \mathcal{O}(D)|_D$ we compute

$$(1 + \mathrm{c}_1 + \mathrm{c}_2 + \mathrm{c}_3)|_D = ((1 + \mathrm{c}_1(D) + \mathrm{c}_2(D)) \cdot (1 + D))_D \tag{3.34}$$

where the subscript indicates that the intersection is to be performed on D. Together with the restriction formula (3.27) we obtain

$$\mathrm{c}_1(D) = -D \tag{3.35}$$

$$\mathrm{c}_1(N_{D/X})^2 = D^2 = \mathrm{c}_2(X) - \mathrm{c}_2(D) \tag{3.36}$$

$$\mathrm{c}_2(X) \cdot D + D^3 = \chi(D) \tag{3.37}$$

which gives the relation between the Chern classes of D and the topological numbers of X.

3.3.2. Classification of algebraic surfaces and some of their properties

We will outline here only some properties which are necessary to get a geometric picture of the divisors in a toric Calabi–Yau manifold. For more details and other properties we refer to [113], [114], [98] and [115].

The most important properties of an algebraic surface D are described by its topological and holomorphic invariants. The main topological invariants are its fundamental group $\pi_1(D,*)$, the Betti numbers $b_i(D)$ and the intersection pairing on $H_2(D,\mathbb{Z})$, in particular its signature. The most basic holomorphic invariants are the irregularity $q(D) = h^1(D,\mathcal{O}_D)$ and the geometric genus $p_g(D) = h^2(D,\mathcal{O}_D)$. Additional invariants are given by $h^{1,1}(D)$ and $\mathrm{c}_1(D)^2 = K_D^2$ where K_D is the canonical line bundle of D. The latter is also an important object by itself. Finally, we will need the Euler characteristic $\chi(D) = \sum_i (-1)^i b_i = \int_D \mathrm{c}_2(D)$ and the holomorphic Euler characteristic $\chi(\mathcal{O}_D)$ which by the Hirzebruch-Riemann-Roch theorem (see (5.48)) is

$$\begin{aligned} \chi(D,\mathcal{O}_D) &= 1 - q(D) + p_g(D) \\ &= \int_D \frac{1}{12}(\mathrm{c}_1(D)^2 + \mathrm{c}_2(D)) \end{aligned} \tag{3.38}$$

which is also known as Noether's formula. Since in our case D is a divisor in X, there is a simple way to compute $\chi(\mathcal{O}_D)$ from the data of X. From (3.35) and (3.36) and (3.38) we find

$$\chi(\mathcal{O}_D) = \frac{1}{12}(2D^3 + \mathrm{c}_2 \cdot D) \tag{3.39}$$

This means that we can compute either $q(D)$ or $p_g(D)$ from the toric data of X but not both. Therefore we need more information which will be given in the remainder of this section.

A very useful way to get more information is the Lefschetz theorem for hyperplane sections [116] which states that for an m-dimensional submanifold Y of $\mathbb{P}^n$, $m \geq 2$ and a hyperplane $H \in \mathbb{P}^n$ such that $H \cap Y$ is again a complex manifold, the following inclusion homomorphisms

$$H_i(Y \cap H, \mathbb{Z}) \rightarrow H_i(Y,\mathbb{Z}) \tag{3.40}$$

$$\pi_i(Y \cap H, \mathbb{Z}) \rightarrow \pi_i(Y,\mathbb{Z}) \tag{3.41}$$

are isomorphisms for $0 \leq i \leq n-2$. From this it follows that if Y is a smooth complete intersection of $m-2$ hypersurfaces in $\mathbb{P}^m$ of degree $d_1,\ldots,d_{m-2}$ respectively, then $\pi_1(Y) = 0$. This holds true for general toric varieties. Applying this to the intersection of a $\mathbb{T}$-invariant divisor D with the anticanonical divisor X (the Calabi–Yau hypersurface) gives

$$q(D) = 0 \tag{3.42}$$

and $p_g(D)$ is determined by (3.38).

A divisor D is called nef if $D \cdot C \geq 0$ for all irreducible curves C. A curve E on a smooth surface D such that $E \cong \mathbb{P}^1$ and $E^2 = -1$ is called an exceptional curve. An algebraic surface D is minimal if it contains no exceptional curves, i.e. all exceptional curves have been blown down.

Rational and ruled surfaces

The simplest surface D is the projective plane $\mathbb{P}^2$. It has $\mathrm{c}_1(\mathbb{P}^2) = 3h$, $\mathrm{c}_1(\mathbb{P}^2)^2 = 9$ and $\chi(\mathbb{P}^2) = 3$ where h is a line in $\mathbb{P}^2$. Furthermore $p_g(\mathbb{P}^2) = q(\mathbb{P}^2) = 0$.

A surface D is ruled if there exists a fibration $\pi : D \to C$, where C is a smooth curve, such that the generic fiber of π is isomorphic to $\mathbb{P}^1$. Every ruled surface over $C = \mathbb{P}^1$ is of the form $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for some $n \geq 0$ and is denoted a Hirzebruch surface. Its Picard group is generated by two elements, the class of the fiber f and the class of the zero section C_0 with $C_0 \cdot C_0 = -n$, $f \cdot f = 0$

and $C_0 \cdot f = 1$. There is another section $C_\infty = C_0 + nf$ satisfying $C_\infty \cdot C_\infty = n$, $C_0 \cdot C_\infty = 0$ and $C_\infty \cdot f = 1$. The canonical class is given by $K_{\mathbb{F}_n} = -2C_0 - (n+2)f$. The characteristic numbers of the Hirzebruch surfaces are $\mathrm{c}_1(\mathbb{F}_n)^2 = 8$, $\chi(\mathbb{F}_n) = 4$ and $p_g(\mathbb{F}_n) = q(\mathbb{F}_n) = 0$. Note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. A rational surface is a surface that is birationally equivalent to $\mathbb{P}^2$. A minimal rational surface is either $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \neq 1$.

Let us now consider general ruled surfaces D whose base curve C is a smooth curve of genus g. Their Picard group is $\mathrm{Pic}D \cong \mathrm{Pic}C \oplus \mathbb{Z}$ while the numerical equivalence classes are $\mathrm{Num}(D) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \sigma$ where f is the fiber and σ a section of π. If we write $\mathcal{O}_\sigma(\sigma) = \pi^*\mathcal{O}_C(\mathbf{d})|_\sigma$ for some divisor $\mathbf{d}$ on C, then $K_D = -2\sigma + \pi^*(K_C + \mathbf{d})$. Thus if $\deg \mathbf{d} = d$, then the numerical equivalence class of K_D is $-2\sigma + (2g - 2 + d)f$ and $\mathrm{c}_1(D)^2 = -8(g-1)$. The other invariants are $\chi(D) = 2\chi(C)$, $p_g(D) = 0$ and $q(D) = g$. Finally, given a minimal surface D such that K_D is not nef, then D is rational or ruled.

$K3$ surfaces

A $K3$ surface X is defined to be a surface with $q(X) = 0$ and $K_X = \mathcal{O}_X$, hence $\mathrm{c}_1(X) = 0$. Its Euler number is $\chi(X) = 24$, hence $p_q(X) = 1$. It is the only two-dimensional Calabi–Yau manifold (apart from the complex 2-torus T^4 which has $q(T^4) = 2$ and trivial holonomy group). Algebraic $K3$ surfaces can be constructed as complete intersections of toric varieties in the same way as it has been described in Section 3.2.1 for Calabi–Yau threefolds. They have been classified in [117]. Here we are only interested in hypersurfaces in weighted projective spaces [118], in particular those of Fermat type for which we can use the results of Section 3.2.2 in one dimension less.

The homology group $H_2(X, \mathbb{Z})$ is equipped with the structure of a lattice via the intersection form $I^{(K3)}$. It is even and integral [119]. By Poincaré duality it is unimodular and can be identified with the lattice $H^2(X, \mathbb{Z})$. The Picard group $\mathrm{Pic}\, X$ is naturally identified with the sublattice M of algebraic cycles in $H_2(X, \mathbb{Z})$, called Picard lattice, of signature $(1, \rho-1)$ where the rank $\rho = \mathrm{rk}\, M$ is given by (3.10) for $n = 3$. The periods associated to these cycles vanish. The orthogonal lattice $N = M^\perp$ of the Picard lattice in $H^2(X, \mathbb{Z})$ is the transcendental lattice which is of signature $(2, 20 - \rho)$. We will denote the restriction of $I^{(K3)}$ to $N \otimes \mathbb{R}$ by the same symbol. We decompose $N = U \perp M^*$ where U is the lattice of the hyperbolic plane and has signature $(1, 1)$. If there is a $K3$ surface X^* whose Picard lattice is M^* then X^* is called the mirror surface to X [120]. Mirror symmetry in addition exchanges U with $H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$. This agrees with the mirror symmetry from toric polyhedra Δ, Δ^* associated with $X = X_\Delta$ and $X^* = X_{\Delta^*}$ [117]. The ranks $\rho(X)$ and $\rho(X^*)$ add up to 20 minus the non-toric contributions in (3.11). The subspace $M \otimes \mathbb{R}$ of $H^2(X, \mathbb{R})$ corresponds to the Kähler deformations while the subspace $N \otimes \mathbb{R}$ corresponds to the complex structure deformations of X [121]. The possible lattices M and M^* for hypersurfaces in weighted projective spaces have been studied in [122]. The lattice $M = \langle 2n \rangle \equiv \mathbb{Z} \cdot e$ is generated by e such that $I^{(K3)}(e, e) = 2n$.

Elliptic surfaces

An elliptic surface is a fibration $\pi : S \to C$ from a smooth surface S to a smooth curve C of genus g such that the general fiber is connected and the genus of all smooth fibers is one. Consider the sheaf $R^1\pi_*\mathcal{O}_S$ on C. It can be shown [114] that it is actually a line bundle on C. We denote its dual line bundle by L and set $d = \deg L$. One can show [115] that d is non-negative and that if L is not trivial then $q(S) = g$ and $p_g(S) = d + g - 1$. On the other hand if L is trivial then $q(S) = g + 1$ and $p_g(S) = g$. In both cases we have $\chi(\mathcal{O}_S) = d$. Suppose that all exceptional curves in the fibers have been blown down and that $F_1, \ldots, F_k$ are the multiple fibers of π with multiplicity m_i. The canonical bundle is then $K_S = \pi^*(K_C \otimes L) \otimes O_S(\sum_i (m_i - 1)F_i)$. Furthermore $K_S^2 = 0$ and therefore $\chi(S) = 12\chi(\mathcal{O}_S)$. Since a $K3$ surface can be an elliptic fibration we define properly elliptic surfaces to be those with $K_S \neq 0$.

Surfaces of general type

Surfaces of general type are general in the same sense as are curves of genus ≥ 2. One example of such surfaces which will frequently appear in the examples below are complete intersections in weighted projective spaces $\mathbb{P}^n_w$ of sufficiently high degree. Other examples are products (or more generally fibrations) of curves of genus $g \geq 2$ and ramified double coverings of $\mathbb{P}^2$. For this type of surfaces only some inequalities are known [114]. We have $\mathrm{c}_1(D)^2 > 0$ and $\chi(D) > 0$ as well as $\mathrm{c}_1(D)^2 + \chi(D) = 0$ mod 12. Furthermore there is Noether's inequality $p_g(D) \leq \frac{1}{2}\,\mathrm{c}_1(D)^2 + 2$ and the Miyaoka-Yau inequality $\mathrm{c}_1(D)^2 \leq 3\chi(D)$.

There is a classification for these surfaces due to Enriques and Kodaira, see[114], [123], [115]. Let $P_n(D) = \dim H^0(D, K_D^{\otimes n})$ be the n-th plurigenus. Thus $P_1(D) = p_g(D)$. For each of these surfaces the Kodaira dimension $\kappa(D)$ is then defined as

$$\kappa(D) = \min\{k \in \mathbb{Z} | P_n(D)/n^k \text{ is a bounded function of } n \geq 1\} \tag{3.43}$$

For example, it follows formally that if $P_n(D) = 0$ for all n, then $\kappa(D) = -\infty$. It turns out that the possible values of $\kappa(D)$ are $-\infty, 0, 1, 2$. The classification is then given in table 3.1. We have only

$\kappa(D)$	$q(D)$	$p_g(D)$	K_D	K_D^2	Surface type
$-\infty$	0	0	>0	>0	Rational surface
	g	0	<0	≤ 0	Ruled surface over a curve of genus $g > 0$
0	0	1	0	0	$K3$
1			>0	0	Properly elliptic surface
2			>0	>0	Surface of general type

Table 3.1.: The (partial) Enriques-Kodaira classification of minimal algebraic surfaces

indicated the surfaces that will appear in our discussion.

Finally, let us discuss curves in a surface. Let C be a smooth, irreducible curve on D. By the adjunction formula it follows that

$$g(C) = \frac{K_D \cdot C + C \cdot C}{2} + 1 \tag{3.44}$$

For a general, not necessarily smooth curve C, the arithmetic genus $p_a(C)$ is defined by the expression on the right-hand side of (3.44).

3.3.3. Singularities and their resolutions

We will only consider weighted projective spaces $\mathbb{P}^n_w$ whose weights are relatively prime. If some subset $\{w_i | i \in S\}$ of the weights has a non-trivial common factor N then, due to the $\mathbb{T}$ action, the weighted projective space $\mathbb{P}^n_w$ has singular strata $H_S = \{z \in \mathbb{P}^n_w | z_i = 0 \text{ for } i \notin S\}$. In the case of our interest, the singular locus of a Calabi–Yau hypersurface X in (3.13) which is the intersection $X \cap H_S$ consists of points and curves. For singular points these singularities are locally of the form $\mathbb{C}^3/\mathbb{Z}_N$ while the normal bundle of a singular curve has locally a $\mathbb{C}^2/\mathbb{Z}_N$ singularity [95], also known as cyclic quotient or A_{N-1} singularity (see also [124]). Among the singular points one has to distinguish between isolated points and exceptional points, the latter being singular points on singular curves or the points of intersection of singular curves. The order N of exceptional points exceeds that of the curve. The situation for isolated singular points has been discussed in great detail in [125].

Both types of singularities and their resolution can be described by the methods of toric geometry [92], [93]. The singularities are resolved by the process of blowing them up in the ambient space and taking the proper transform of X [111]. This smooth Calabi–Yau variety will be denoted momentarily $\widetilde{X}$. We will describe this explicitly in an example in Section 3.5.2. Each singular set leads to an exceptional divisor. We will denote by E_i and F_j the proper transforms of the exceptional divisors on $\widetilde{X}$ coming from the resolution of the singular curves and singular points, respectively. H will be the proper transform of the hyperplane class on $\mathbb{P}^4_w$ restricted to X. The Hodge number $h^{1,1}(\widetilde{X})$ is then equal to # exceptional divisors + 1. Furthermore, there are non-toric complex structure deformations of X coming from the blow-ups of curves C with $\mathbb{Z}_N$ singularities whose number is [96], [57]

$$h^{2,1} - \tilde{h}^{2,1} = g(N-1) \tag{3.45}$$

where g is the genus of C. The corresponding exceptional 3-cycles are seen as follows. For each of the exceptional divisors E_i, there is a map $H_1(C) \to H_3(X)$ given by sending a one-cycle γ of the curve C to the three-cycle swept out by the fibers of E_i lying over γ [126].

In toric geometry singularities are described by cones σ which are not basic, i.e. which can not be generated by a basis of the lattice in which σ lies. A standard result now states that a toric variety $\mathbb{P}_\Sigma$ has only quotient singularities if Σ is a simplicial fan, i.e. if all cones in Σ are simplicial. Given a singular cone one resolves the singularities by subdividing the cone into a fan such that each cone in the fan is basic. X has a singular curve precisely when Δ^* has an edge joining two vertices ν_0^*, ν_N^* with $N-1$ equally spaced lattice points $\nu_1^*, \ldots, \nu_{N-1}^*$ in the interior of the edge. The edges corresponding to these lattice points correspond to toric divisors E_i which resolve a surface S of A_{N-1} singularities in $\mathbb{P}^4_w$. Restricting to the hypersurface X, we see that there are $N-1$ divisors E_i in X which resolve the curve C of A_{N-1} singularities. It can be shown [93], [96] that these divisors are locally the product of a curve C and a Hirzebruch-Jung sphere-tree [127]. If the order is $N=1$ then the corresponding exceptional divisor is a ruled surface, otherwise it is a blow-up at $N-1$ points thereof. The genus g can be determined as follows. By duality, the edge $\Theta_1^* = \langle \nu_0^*, \nu_N^* \rangle$ determines a two-dimensional face Θ_2 of Δ. The number of interior points of Θ_2 is equal to g. We refer again to Section 3.5 for explicit examples.

In the case of point singularities the $\mathbb{T}$ action on the normal bundle is $(z_1, z_2, z_3) \to (\lambda z_1, \lambda^a z_2, \lambda^b z_3)$ with $1 + a + b = N$, $a, b \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^*$. The singular cone is generated by e_i, $i = 1,2,3$ in a lattice basis $n_1 = \frac{1}{N}(e_1 + ae_2 + be_3)$, $n_2 = e_2, n_3 = e_3)$. Then all endpoints of the additional vectors generating the nonsingular fan must lie on the plane $\sum_i x_i n_i = 1$. The exceptional divisors are then in one-to-one correspondence with the lattice points inside the cone on this hyperplane. The corresponding divisors can all be described by compact toric surfaces which have been classified [93]. These are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_a$ and their blow-ups at $\mathbb{T}$-fixed points. The resolution is in general not unique if there are more than one lattice points inside this cone. In this case there are several ways to triangulate this cone, each of which leading to a different resolution. The resulting smooth manifolds are all topologically different with the same Hodge numbers but different intersection numbers.

3.3.4. Fibrations

For general properties of fibered Calabi–Yau threefolds see [128]. We summarize here some results which will be useful in Section 3.5 and Chapters 5 and 6.

Elliptic fibrations

The conditions for a Calabi–Yau threefold to admit elliptic or $K3$-fibrations have been found in [129] and are as follows: A Calabi–Yau threefold admits an elliptic fibration $\pi : X \to B$ if there exists an effective divisor S such that

$$S \cdot \Gamma \geq 0 \text{ for all curves } \Gamma \tag{3.46a}$$

$$S^3 = 0 \tag{3.46b}$$

$$S^2 \cdot F \neq 0 \text{ for some divisor } F \neq S \tag{3.46c}$$

In order to have an elliptic fibration with a section one needs in addition the condition that S and F can be chosen such that $S^2 \cdot F$ is a small number.

It is known [130], [131], [129] that for any elliptic fibration $\pi : X \to B$ of a Calabi–Yau threefold X, the base B has at worst orbifold singularities. In fact, the singularities are more constrained than that: together with the collection of curves Σ_i which specify where the elliptic fibration is singular, the singularities have a special property known as log-terminal. Under the assumption (3.1) the base B has the following properties [132], [131]: If the singularities of B are resolved then B is either an Enriques surface, or a blow-up of $\mathbb{P}^2$ or of a Hirzebruch surface $\mathbb{F}_n$ with $n \leq 12$. Note that the latter two appear also in the list of exceptional divisors coming from the resolution of point singularities in the Calabi–Yau space, see Section 3.3.3. The divisor S may be identified with the restriction of $\pi : X \to B$ on the section C_∞ of $\mathbb{F}_n$. Hence, S itself is an elliptic fibration over C_∞. If we represent X in the Weierstrass form [132] the discriminant divisor $\Delta \subset B$ is given by [119] $\Delta = 24C_0 + (24+12n)f$. Hence $\Delta \cdot f = 24$, $\Delta \cdot C_\infty = 24 + 12n$ and therefore $\chi(S) = 24 + 12n = \mathrm{c}_2 \cdot S$.

There are three types of elliptic curves given as hypersurfaces in weighted projective spaces which can appear as generic fibers: $\mathbb{P}^2_{1,a,b}[c]$ with $(a, b) = (1, 1)$, $(1, 2)$ or $(2, 3)$ and $c = 1 + a + b$. Note that a, b, c are the same as those that appeared in Section 3.3.3 when discussing the resolution of the point singularities. If J_B is the dual homology element to the cohomology class of the elliptic fiber and J_i are the remaining basis elements of $H_4(X, \mathbb{Z})$ then the intersection numbers and linear forms can be written as [133]

$$J_B \cdot J_i \cdot J_k = kJ_i \cdot J_k \qquad J_B^2 \cdot J_i = k\,\mathrm{c}_1(B) \cdot J_i \qquad J_B^3 = k\,\mathrm{c}_1(B)^2 \tag{3.47}$$

$$\mathrm{c}_2 \cdot J_i = 12k\,\mathrm{c}_1(B) J_i \qquad \mathrm{c}_2 \cdot J_B = k\,\mathrm{c}_2(B) + k\left(\frac{12}{k} - 1\right)\mathrm{c}_1(B)^2 \tag{3.48}$$

where B is the base of the fibration and on the left-hand side we integrate over X while on the right-hand side we integrate over B. k is the number of sections of the fibration for the various fibers: $k = 3$ for $(a, b) = (1, 1)$, $k = 2$ for $(a, b) = (1, 2)$ and $k = 1$ for $(a, b) = (2, 3)$.

$K3$ fibrations

Similarly, a Calabi–Yau threefold admits a $K3$-fibration $\pi' : X \to \mathbb{P}^1$ if there exists an effective divisor L such that

$$L \cdot \Gamma \geq 0 \text{ for all curves } \Gamma \tag{3.49a}$$

$$L^2 \cdot D = 0 \text{ for all divisors } D \tag{3.49b}$$

Note that the latter implies that $L^3 = 0$. We will assume that all the singular fibers of π' are irreducible.

In order to have both an elliptic and a $K3$-fibration, the fibrations will be compatible if

$$S^2 \cdot L = 0 \tag{3.50}$$

This implies that a generic fiber of π' is an elliptic $K3$ surface. We will denote the divisors defining elliptic and $K3$ fibrations by S and L, respectively.

For K3-fibrations in weighted projective spaces given by Fermat polynomials there is a second way to see the fibration structure. Let the weights of $\mathbb{P}^4_w$ be $w = (1, l-1, lw_2', lw_3', lw_4')$. Then, by the Calabi–Yau condition, the defining polynomial $W(z)$ must have degree $d = ld'$ where $d' = 1 + w_2' + w_3' + w_4'$. Define the divisor L by a parameter $\lambda \in \mathbb{P}^1$ and the hypersurface $z_2 = (\lambda z_1)^{l-1}$ in X. By the scaling properties of $\mathbb{P}^4_{1,l-1,lw_2',lw_3',lw_4'}$ we can set $z_1' = z_1^l$. Then we have

$$W(z) = (1 + \lambda^d) {z_1'}^{d'/(l-1)} + z_3^{d'/w_2'} + z_4^{d'/w_3'} + z_5^{d'/w_4'} = 0 \tag{3.51}$$

But this is precisely the defining equation for a degree d' hypersurface in $\mathbb{P}^3_{1,w_2',w_3',w_4'}$ which is in turn a two-dimensional Calabi–Yau manifold by (3.13), i.e. a $K3$ surface. The base of the fiber is given by the $\mathbb{P}^1$ whose coordinate is λ. A similar argument can be made in order to exhibit an elliptic fibration of a Fermat hypersurface. $K3$ fibrations of this type with $l = 2$ have been studied in [134] and with general l in [135]. If there exists a $K3$ fibration then the reflexive polyhedron Δ_L of the $K3$ surface L is embedded in the reflexive polyhedron Δ_X of the Calabi–Yau threefold X [102]. These toric $K3$ fibrations have been analyzed in detail in [136].

Since the Picard lattice $\mathrm{Pic}(L)$ of a $K3$ surface is an even integral lattice the intersection numbers $K_{Lab} = L \cdot D_a \cdot D_b$ which are equivalent to the intersection form $I_{ab}^{(K3)}$ on $\mathrm{Pic}(L)$ are always even. Combined with (3.49b) we see that

$$K_{abc} \in 2\mathbb{Z} \tag{3.52}$$

if any of the indices a, b or c corresponds to the divisor L.

3.4. Special Geometry

We have mentioned in Section 2.6 that $\mathcal{M}_K^{\mathrm{SCFT}}$ and $\mathcal{M}_C^{\mathrm{SCFT}}$ are both special Kähler manifolds. This property is very important for mirror symmetry and understanding monodromies on the moduli space $\mathcal{M}_{\mathrm{CY}}^{\mathrm{SCFT}}$, so we are briefly reviewing it here. A Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a line bundle $\mathcal{L} \to \mathcal{M}$ such that $\mathrm{c}_1(\mathcal{L}) = [J]$ where J is the Kähler form. If $\mathcal{H}$ is a $Sp(2h^{1,1}+2,\mathbb{R})$ vector bundle over $\mathcal{M}$ and $-i\langle\ |\ \rangle$ the compatible hermitean metric on $\mathcal{H}$ then $\mathcal{M}$ is special Kähler [137] if, for some choice of $\mathcal{H}$, there exists a $\Pi \in \Gamma(\mathcal{M}, \mathcal{H} \otimes \mathcal{L})$ with the property that $J = \frac{i}{2\pi}\partial\bar{\partial}\log\left(-i\langle\Pi|\overline{\Pi}\rangle\right)$. The metric can be defined as $-i\langle\Pi|\overline{\Pi}\rangle = -i\Pi^\dagger\Sigma^{(L)}\Pi$ with $\Sigma^{(L)}$ being the standard symplectic matrix.

Equivalently, $\mathcal{M}$ is special Kähler if locally there exist complex projective coordinates z_a and a homogeneous, degree two holomorphic function $F(z)$ which is related to the Kähler potential K by

$$K = -\log\left(-i\left(z_a\bar{\partial}_a\overline{F} - \bar{z}_a\partial_a F\right)\right) \tag{3.53}$$

The Kähler potential K is related to the norm of Π by $K = -\log||\Pi||^2 \equiv -\log(-i\langle\overline{\Pi}|\Pi\rangle)$. There is a particular choice of coordinates for $\mathcal{M}$, the special coordinates, defined by $t_a = \frac{z_a}{z_0}, a = 1, \ldots, h^{1,1}$. If we define $\mathcal{F}(t) = z_0^{-2}F(z)$ then the Kähler potential is expressed by

$$K(t,\bar{t}) = -\log i\left(2\left(\mathcal{F} - \overline{\mathcal{F}}\right) - \left(\partial_a\mathcal{F} + \partial_{\bar{a}}\overline{\mathcal{F}}\right)(z_a - \bar{z}_{\bar{a}})\right) \tag{3.54}$$

and $\mathcal{F}_{abc} = \partial_a\partial_b\partial_c\mathcal{F}(t)$. The function $\mathcal{F}(t)$ is called the prepotential. Below, we will only work in these special coordinates.

Special geometry also arises naturally in $\mathcal{N} = 2$ supergravity theories in four dimensions [138]. Type IIB string theory compactified on a general Calabi–Yau manifold X has as its low-energy effective theory an $\mathcal{N} = 2$, $D = 4$ supergravity theory with $h^{2,1}(X)+1$ vector fields coming from $h^{2,1}(X)$ vector multiplets and the graviphoton and $h^{1,1}(X)+1$ hypermultiplets (including the dilaton in $D = 4$). In type IIA string theory the identifications for $h^{1,1}$ and $h^{2,1}$ are reversed. The scalars in the vector multiplets parametrize a special Kähler manifold. Hence its geometry is determined by the prepotential $\mathcal{F}_C$ of complex structure moduli in the type IIB case, and by the prepotential $\mathcal{F}_K$ of Kähler moduli in the type IIA case. We have seen in Section 2.6 that $\mathcal{F}_C$ does not get any α' corrections. Since the dilaton sits in a hypermultiplet, $\mathcal{F}_C$ is exact already at string tree level [4], [139] and is therefore entirely computable in terms of classical geometry. On the other hand, $\mathcal{F}_K$ gets quantum corrections because in doing the perturbation expansion around the large volume limit, the expansion parameter $\frac{R^2}{\alpha'}$ is controlled by the Kähler moduli which, being in the vector multiplets, are now varied. However the dilaton is still in a hypermultiplet, so $\mathcal{F}_K$ is still computable at string tree level. For more details about the form of $\mathcal{F}_C$ and $\mathcal{F}_K$ see [140]. In the first case the fibers of the bundle $\mathcal{H}$ over $\mathcal{M}_C^{\mathrm{geom}}$ are given by $H^3(X,\mathbb{Z})$. A given basis in $H^3(X,\mathbb{Z})$ will undergo a monodromy in $Sp(2h^{1,1}+2,\mathbb{Z})$ as it is transported around singularities in $\mathcal{M}_C^{\mathrm{geom}}$.

3.4.1. Periods

Another way of expressing special geometry is the following. Taking subsequent derivatives of the holomorphic $(3,0)$-form Ω with respect to the complex structure moduli a yields elements in $H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$. Since b_3 is finite, there must be linear relations between derivatives of Ω of the form $\mathcal{L}\Omega = \mathrm{d}\eta$ where $\mathcal{L}$ is a linear differential operator whose coefficients depend on the a's. If we integrate this equation over a closed 3-cycle, we will get a differential equation $\mathcal{L}\Pi_i = 0$ satisfied by the periods of Ω. They are defined as

$$\Pi_i(a) = \int_{\Gamma_i} \Omega(a) \qquad\qquad \Gamma_i \in H_3(X, \mathbb{Z}) \tag{3.55}$$

In general we will get a system of coupled linear partial differential equations for the periods of Ω, the so-called extended Δ (or Δ^*) hypergeometric system [141], [142]. These equations are also known as Picard-Fuchs equations. They have only regular singularities. The period integrals for X_{Δ^*} are the most relevant quantities for the application of mirror symmetry to the determination of the quantum geometry of X_Δ. For example [143],

$$\Pi(a) = \frac{1}{(2\pi i)^n} \int_{C_0} \frac{1}{f_{\Delta^*}(X, a)} \prod_{i=1}^{n} \frac{\mathrm{d}X_i}{X_i} \tag{3.56}$$

is the period integral over the torus cycle $C_0 = \{|X_1| = |X_2| = \cdots = |X_n| = 1\}$ in $\mathbb{T}$. For other periods, one has to analyze the differential equation satisfied by (3.56). The Mori cone $L_{\geq 0}$ describes the affine chart $\mathrm{Hom}(L_{\geq 0}, \mathbb{C}^*)$ of the compactification of $\mathcal{M}_C^{\mathrm{geom}}$ given by the secondary fan $A(\Xi)$ as discussed in Section 3.3.1 with coordinates

$$x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}} \qquad k = 1, \ldots, r \tag{3.57}$$

It has been proved in general [144] that the origin $x_1 = \cdots = x_r = 0$ provides a large complex structure limit [145] and there we only have one regular period integral

$$w_0(x) = \sum_{n \in \mathbb{Z}^r_{\geq 0}} \frac{\Gamma\left(1 - \sum_k n_k l_0^{(k)}\right)}{\prod_{i=1}^{p} \Gamma\left(1 + \sum_k n_k l_i^{(k)}\right)} x^n \tag{3.58}$$

the so-called fundamental period [146]. All other period integrals at the large complex structure limit contain logarithmic singularities and can be generated by the classical Frobenius method [142].

Now we describe the local solutions of the Picard-Fuchs equations about the large complex structure limit. To this aim let us introduce in (3.58) the indices $\rho_1, \ldots, \rho_r$

$$w_0(x, \rho) = \sum_{n \in \mathbb{Z}^r_{\geq 0}} c(n + \rho) x^{n+\rho} \tag{3.59}$$

Using the basis $J_1, \cdots, J_r$ of the Kähler cone of $\mathbb{P}_{\Sigma(\Delta^*)}$ and restricting them to the hypersurface X_Δ, there is a convenient way to keep track of the hypergeometric series [147]

$$\begin{aligned} w_0\left(x, \frac{J}{2\pi i}\right) &= w_0(x)\mathbb{1} + \sum_{a=1}^{r} w_a(x) J_a + \\ &\quad + \frac{1}{2!} \sum_{a,b=1}^{r} w_{ab}(x) J_a J_b + \frac{1}{3!} \sum_{a,b,c=1}^{r} w_{abc}(x) J_a J_b J_c \end{aligned} \tag{3.60}$$

where the products of the J_k's are taken in the cohomology ring $H^{\mathrm{even}}_{\mathrm{toric}}(X_\Delta, \mathbb{Q})$. For the remainder of this section we will drop the subscript toric. It's crucial to choose the correct normalization of this solution in order to obtain an integral, symplectic basis for the period integrals [148].

Let us introduce a basis $\mathbb{1}, J_a, J_b^{(2)}, J^{(3)}$ for $H^{\mathrm{even}}(X_\Delta)$ by the property

$$(\mathbb{1}, J^{(3)}) = -1 \qquad\qquad (J_a, J_b^{(2)}) = \delta_{ab} \tag{3.61}$$

with $(A,B) = \int_{X_\Delta} A \wedge B$. For reasons to become clear in section 5.1 we also introduce a skew-symmetric form on $H^{\mathrm{even}}(X,\mathbb{Q})$. First we consider an involution $*$ which acts on $H^{2i}(X,\mathbb{Q})$ by $(-1)^i$. Using this involution we define the Mukai form [149]

$$\begin{aligned} \langle \alpha, \beta \rangle &= -\int_X \alpha \wedge *\beta \wedge \mathrm{td}\, X & (3.62)\\ &= \int_X (\alpha_0\beta_6 - \alpha_2\beta_4 + \alpha_4\beta_2 - \alpha_6\beta_0)\, \mathrm{td}\, X & (3.63) \end{aligned}$$

for $\alpha, \beta \in H^{\mathrm{even}}(X,\mathbb{Q})$. Then there exists [148] a canonical symplectic basis of the skew-symmetric form (3.62) on $H^{\mathrm{even}}(X,\mathbb{Q})$

$$\mathbb{1}, \quad J_a^S, \quad J_b^{(2)}, \quad J^{(3)} \tag{3.64}$$

where

$$J_a^S = \left(J_a - \sum_{b=1}^{r} A_{ab} J_b^{(2)} \right) (\mathrm{td}\, X)^{-1} \tag{3.65}$$

with some rational constants $A_{ab} = A_{ba}$. Corresponding to this basis, we have an integral symplectic basis for the period integrals about the large complex structure limit through

$$w_0(x, \frac{J}{2\pi i}) = w_0(x)\mathbb{1} + \sum_{a=1}^{r} D_a^{(1)} w_0(x) J_a^S + \sum_{b=1}^{r} D_b^{(2)} w_0(x) J_b^{(2)} + D^{(3)} w_0(x) J^{(3)} \tag{3.66}$$

where

$$D_a^{(1)} = \frac{1}{2\pi i}\partial_{\rho_a} \tag{3.67a}$$

$$D_b^{(2)} = \frac{1}{2!(2\pi i)^2} \sum_{c,d=1}^{r} K_{bcd}\partial_{\rho_c}\partial_{\rho_d} + \sum_{a=1}^{r} A_{ab} D_a^{(1)} \tag{3.67b}$$

$$D^{(3)} = -\frac{1}{3!(2\pi i)^3} \sum_{a,b,c=1}^{r} K_{abc}\partial_{\rho_a}\partial_{\rho_b}\partial_{\rho_c} - \sum_{a=1}^{r} \frac{\mathrm{c}_2 \cdot J_a}{12} D_a^{(1)} \tag{3.67c}$$

and the notation $D_a^{(1)} w_0(x)$, for example, means an operation $\lim_{\rho\to 0} D_a^{(1)} w_0(x,\rho)$. We define the periods in the large volume limit to be the coefficients of J in (3.66), i.e. $w_a^{(i)}(x) = D_a^{(i)} w_0(x)$ for $i = 1,2$ and $w^{(3)}(x) = D^{(3)} w_0(x)$. If we evaluate the skew-symmetric form (3.62) on the basis (3.64), we find that it has the standard matrix form of the symplectic form which we denote by $I^{(L)}$. Then the hypergeometric series appearing in the coefficients of (3.64) are integral symplectic with respect to $I^{(L)}$ [148].

3.4.2. The prepotential

The corresponding prepotential has the following form [112]

$$\begin{aligned}\mathcal{F}(t) &= \frac{1}{2}\frac{1}{w_0(x)^2}\left(w_0(x)D^{(3)}w_0(x)+\sum_{a=1}^{r}D_a^{(1)}w_0(x)D_a^{(2)}w_0(x)\right) &(3.68)\\ &= \frac{1}{3!}\sum_{a,b,c=1}^{r}K_{abc}t_at_bt_c+\frac{1}{2}\sum_{a,b=1}^{r}A_{ab}t_at_b-\sum_{a=1}^{r}\frac{c_2\cdot J_a}{24}t_a &(3.69)\\ &\quad+\frac{\zeta(3)}{2(2\pi i)^3}\chi(X)+O(e^{2\pi it})\end{aligned}$$

where

$$t_a=\frac{1}{2\pi i}\frac{D_a^{(1)}w_0(x)}{w_0(x)} \tag{3.70}$$

is the mirror map. The t_a are the coordinates on the Kähler moduli space while the x_i are the coordinates on the complex structure moduli space. The period vector

$$\Pi(t)=(\Pi_0,\Pi_a,\Pi_{h^{1,1}+1},\Pi_{h^{1,1}+a+1})^T=\left(2\mathcal{F}-\frac{\partial\mathcal{F}}{\partial t_a}t_a,\,\frac{\partial\mathcal{F}}{\partial t_a},\,1,\,t_a\right)^T \qquad a=1,\ldots,h^{1,1} \tag{3.71}$$

is then obtained from (3.68)

$$\Pi(t)=\begin{pmatrix}-\frac{1}{6}K_{abc}t_at_bt_c+c_2\cdot J_at_a\\ \frac{1}{2}K_{abc}t_bt_c+A_{ab}t_b+c_2\cdot J_a\\ 1\\ t_a\end{pmatrix} \tag{3.72}$$

The constants A_{ab} have to be fixed such that the basis for the period vectors is integral and symplectic. The integer part is irrelevant as it can be absorbed by an $Sp(2h^{1,1}+2,\mathbb{Z})$ transformation due to the fact that the periods are only defined up to such a transformation. The fractional part can be determined as [112]

$$A_{ab}=\frac{1}{2}K_{aab}\mod\mathbb{Z} \tag{3.73}$$

From (3.52) we see that if any index corresponds to a divisor L representing a $K3$ surface

$$A_{ab}=0 \tag{3.74}$$

If $\Pi(t)$ is a solution to the Picard-Fuchs equations at a point t, then by analytically continuing Π around a singularity t_1 of the equations we arrive at a new solution at t. This must be expressible as linear combination of the basis Π: $\Pi\to A_{t_1}\Pi$ where the $b_3\times b_3$ non-singular matrix A_{t_1} characterizes the monodromy around t_1. If the equation has r singular points we obtain r monodromy matrices $A_{t_1},\ldots,A_{t_r}$. The relation between the monodromy properties of the Picard-Fuchs equations and special geometry has been studied in detail in [150].

As mentioned above, the (quantum) geometry of the Calabi–Yau manifold is encoded in the periods. If we want to make use of the fact discussed in Sections 2.5 and 2.6 that we can relate the description of the Calabi–Yau at the Gepner point to the one in the large volume limit we must have a way to translate the periods from one point to the other. This can be done by analytic continuation as follows. (This is the analytic continuation we referred to in Section 2.6.) We have introduced two sets of local coordinates on the complex structure moduli space that are well-adapted to these two points, respectively. At the Gepner point we have local coordinates $\psi_i, i = 1, \ldots, h^{1,1}$ defined in (3.19). Here and in the following we restrict ourselves to the toric part $\widetilde{H}^*(X, \mathbb{Z})$ of the cohomology $H^*(X, \mathbb{Z})$, see Section 3.2.1. At this point there is a $\mathbb{Z}_d$ monodromy $A : (\psi_1, \ldots, \psi_{\tilde{h}^{1,1}}) \to (\alpha\psi_1, \ldots, \alpha^{n_{\tilde{h}^{1,1}}}\psi_{\tilde{h}^{1,1}})$ induced by the discrete quantum symmetry (2.119) where α is a dth root of unity and n_i are some definite integers depending on the k_j with $n_1 = 1$. We represent the action of the symmetry generator g on the toric part of the even cohomology $\tilde{H}^{\text{even}}(X, \mathbb{Z}) = \tilde{H}^3(X^*, \mathbb{Z})$ by a $\tilde{b}_3 \times \tilde{b}_3$ matrix $A^{(G)}$, $\tilde{b}_3 = 2\tilde{h}^{1,1} + 2$, which is determined as follows. There is a basis of periods on the mirror manifold X^*

$$\varpi^{(G)} = (\varpi_0, \varpi_1, \ldots, \varpi_{\tilde{b}_3 - 1}) \tag{3.75a}$$

defined by

$$\varpi_k(\psi_i) = \varpi_0(\alpha^{kn_i}\psi_i) \qquad k = 1, \ldots, \tilde{b}_3 - 1 \tag{3.75b}$$

which behaves under this monodromy as

$$\varpi^{(G)} \longrightarrow A\varpi^{(G)} \text{ with } A^{(G)} = \begin{pmatrix} 0 & 1 & & & \\ & & & & \text{\Large 0} \\ & \ddots & \ddots & & \\ \text{\Large 0} & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_{\tilde{b}_3 1} & a_{\tilde{b}_3 2} & \cdots & a_{\tilde{b}_3 - 1, p} & a_{\tilde{b}_3\tilde{b}_3} \end{pmatrix} \tag{3.76}$$

satisfying $A^d = 1$. Here $\varpi_0(\psi)$ is the period obtained by analytic continuation of the fundamental period w_0 at large volume; i.e. w_0 is the unique logarithm-free solution of the Picard-Fuchs equations. $A^{(G)}$ is the matrix representing the monodromy A around the Gepner point in the Gepner basis. This choice of basis is indicated by the superscript (G). The entries of the bottom row satisfy $a_{\tilde{b}_3 i} \in \{-1, 0, 1\}, i = 1, \ldots, \tilde{b}_3$. Since $\tilde{b}_3(X) < d$ in general, the periods ϖ_k are not linearly independent. There are relations between them which we will discuss shortly.

On the other hand, in the large complex structure limit we have the local coordinates $x_i, i = 1, \ldots, \tilde{h}^{1,1}$, defined in (3.57). In these coordinates the periods take the natural form (3.66)

$$\varpi^{(L)} = (w_0, w_1^{(1)}, \ldots, w_{\tilde{h}^{1,1}}^{(1)}, w^{(3)}, w_1^{(2)}, \ldots, w_{\tilde{h}^{1,1}}^{(2)}) \tag{3.77}$$

They are related to (3.75b) by a basis transformation M,

$$\varpi^{(L)} = M\varpi^{(G)} \tag{3.78}$$

In Chapter 5 we will need the action of A on the periods expressed in the large complex structure coordinates, i.e. we need the monodromy matrix $A^{(L)} = MA^{(G)}M^{-1}$ in the large volume basis, the latter being indicated by the superscript (L). Therefore we need to know M which can be obtained by analytically continuing either set of periods to the other point. This can in principle be done as is shown in Appendix A in the example $\mathbb{P}^4_{1,1,2,2,6}[12]$ but is very tedious especially for $h^{1,1} > 2$. Using the insights of the study of D-branes on Calabi–Yau spaces [151], [152] it is however possible [153] to reduce the computation to linear algebra. This is shown and explained in the example $\mathbb{P}^4_{1,2,3,3,9}[18]$ in Appendix B. These two examples are discussed in Sections 3.5.2 and C.1.3, respectively. In the process of computing this analytic continuation we also obtain the relations among the periods ϖ_k mentioned above, see (A.28) and (B.12).

Furthermore, we will also need the Mukai intersection form (3.62) which we denoted $I^{(L)}$ in terms of the basis (3.75) which is

$$I^{(G)} = M^{-1}I^{(L)}\left(M^{-1}\right)^T \tag{3.79}$$

The extra $\mathbb{Z}_d$ symmetry at the Gepner point allows us to express $I^{(G)}$ as a polynomial $I^{(G)}(g)$ in the generator g. This generator can be represented on $\widetilde{H}^3(X^*,\mathbb{Z})$ by a $\tilde{b}_3 \times \tilde{b}_3$ matrix obtained from the $d \times d$ shift matrix satisfying $g^d = \mathbb{1}$ subjected to the relations satisfied by the periods. For explicit examples see [154].

In the remainder of this section we consider the effect of two important monodromies in the Kähler moduli space. Let us consider the effect of the Peccei-Quinn symmetry making the replacement on the period vector $\Pi(t)$ [105]

$$t_a \to t_a + \delta_a^b \tag{3.80}$$

This induces a monodromy of the periods about the large complex structure corresponding to an integral matrix S_a

$$\Pi \to S_a \Pi \quad \text{with} \quad S_a = \begin{pmatrix} 1 & -\delta_a^T & \frac{1}{6}K_{aaa} + \frac{1}{12}\,\mathrm{c}_2 \cdot J_a & \frac{1}{2}K_{aa}^T + A_a^T \\ 0 & \mathbb{1} & -\frac{1}{2}K_{aa} + A_a & -K_a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \delta_a & \mathbb{1} \end{pmatrix} \tag{3.81}$$

where we have introduced the vectors $(\delta_a)_b = \delta_a^b$, $(K_{aa})_b = K_{aab}$ and $(A_a)_b = A_{ab}$ as well as the matrix $(K_a)_{bc} = K_{abc}$. If we set $R_a = S_a - \mathbb{1}$ then we observe that

$$\begin{aligned} [R_a, R_b] &= 0 && (3.82)\\ R_a R_b R_c &= K_{abc} Y && (3.83)\\ R_a R_b R_c R_d &= 0 && (3.84) \end{aligned}$$

where Y is a matrix independent of a. These relations give a characterization of the large complex structure limit independent of the choice of basis for the periods. The large complex structure limit consists, in the general case, of $h^{1,1}$ codimension 1 hypersurfaces (i.e. divisors) in the (compactification of the) moduli space given by the secondary fan meeting transversely in a point and such that the monodromies of the period vector about these divisors correspond to the properties (3.82) to (3.84).

Another type of monodromies are those obtained by going around the discriminant locus in the secondary fan. The discriminant locus generally consists of several components. The primary component is the one which separates the smooth Calabi–Yau and the orbifold phases from the remaining ones. In the example in Figure 2.1 it corresponds to the vectors $(0,1)$ and $(1,-2)$. This is also known as the conifold locus. In the basis (3.77) the monodromy matrix takes the following form [104]

$$\Pi \to T\Pi \quad \text{with} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \tag{3.85}$$

where $\mathbb{1}$ is the $h^{1,1} \times h^{1,1}$ unit matrix.

The matrices S_a and T will be reinterpreted in Section 5.6 as natural automorphisms of the Grothendieck group $K_0(X)$ of X.

3.5. Specific examples

There are 7555 weighted projective spaces $\mathbb{P}^4_w$ which admit transverse hypersurfaces. They have been classified in [155] and [156]. We are interested in Fermat hypersurfaces with a few Kähler moduli for computational reasons. In table 3.2 we list all such Calabi–Yau manifolds with $h^{1,1} \leq 6$ together with their Hodge numbers and the corresponding Gepner model. The purpose of this section is twofold.

First, we want to apply the concepts and methods of Sections 3.2 to 3.4 explicitly to some examples. Second, we want to introduce the families of Calabi–Yau hypersurfaces which will be used in the context of the study of D-branes in chapter 6. Furthermore, this section together with Appendix C can also serve as a reference for the toric data of these Calabi–Yau manifolds.

We will now discuss the geometry of some of these spaces more closely. The examples in table 3.2 can be grouped into sets of families having similar geometric properties. The families 1 to 4 do not meet the singularities of their ambient space and therefore only possess one Kähler modulus which is the one inherited from the ambient space. They will be briefly discussed in Section 3.5.1. Then the families 5, 11 and 20 are $K3$-fibrations with fiber $\mathbb{P}^3_{1,1,1,1}[4]$ while 6, 13 and 25 are $K3$-fibrations with fiber $\mathbb{P}^3_{1,1,1,3}[6]$. These are the only one-parameter $K3$ Fermat hypersurfaces. The geometry of these families will be described in detail in Section 3.5.2 and Section C.1. Next, the families 9, 10 and 15 are elliptic fibrations which are not $K3$-fibrations, all with a $\mathbb{P}^2$ base, and with fibers $\mathbb{P}^2_{1,1,1}[3]$, $\mathbb{P}^2_{1,1,2}[4]$ and $\mathbb{P}^2_{1,2,3}[6]$, respectively. Their properties are studied in Section C.2. The families 17, 21, 23 and 26 are also only elliptically fibered but will not be discussed in detail. The families 14, 16 and 18 are both $K3$- and elliptically fibered over the Hirzebruch surface $\mathbb{F}_2$. We will explain their geometry in Section C.3. The families 22 and 24 also admit both an elliptic and $K3$ fibration, but will not be studied further. The remaining ones, 7, 8, 12 and 19 do not have a fibration structure and are not discussed.

No.	Family X	Gepner model	$h^{1,1}$	$\tilde{h}^{1,1}$	$h^{1,2}$	$\tilde{h}^{1,2}$
1	$\mathbb{P}^4_{1,1,1,1,1}[5]$	$(3,3,3,3,3)$	1	1	101	101
2	$\mathbb{P}^4_{1,1,1,1,2}[6]$	$(4,4,4,4,1)$	1	1	103	103
3	$\mathbb{P}^4_{1,1,1,1,4}[8]$	$(6,6,6,6,0)$	1	1	149	149
4	$\mathbb{P}^4_{1,1,1,2,5}[10]$	$(8,8,8,3,0)$	1	1	145	145
5	$\mathbb{P}^4_{1,1,2,2,2}[8]$	$(6,6,2,2,2)$	2	2	86	83
6	$\mathbb{P}^4_{1,1,2,2,6}[12]$	$(10,10,4,4,0)$	2	2	128	126
7	$\mathbb{P}^4_{1,2,2,3,4}[12]$	$(10,4,4,2,1)$	2	2	74	70
8	$\mathbb{P}^4_{1,2,2,2,7}[14]$	$(12,5,5,5,0)$	2	2	122	107
9	$\mathbb{P}^4_{1,1,1,6,9}[18]$	$(16,16,16,1,0)$	2	2	272	272
10	$\mathbb{P}^4_{1,1,1,3,6}[12]$	$(10,10,10,2,0)$	3	2	165	165
11	$\mathbb{P}^4_{1,2,3,3,3}[12]$	$(10,4,2,2,2)$	3	3	69	63
12	$\mathbb{P}^4_{1,3,3,3,5}[15]$	$(13,3,3,3,1)$	3	3	75	63
13	$\mathbb{P}^4_{1,2,3,3,9}[18]$	$(16,7,4,4,0)$	3	3	99	95
14	$\mathbb{P}^4_{1,1,2,8,12}[24]$	$(22,22,10,1,0)$	3	3	243	242
15	$\mathbb{P}^4_{1,1,1,3,3}[9]$	$(7,7,7,1,1)$	4	2	112	112
16	$\mathbb{P}^4_{1,1,2,4,8}[16]$	$(14,14,6,2,0)$	4	3	148	147
17	$\mathbb{P}^4_{1,2,2,10,15}[30]$	$(28,13,13,1,0)$	4	4	208	195
18	$\mathbb{P}^4_{1,1,2,4,4}[12]$	$(10,10,4,1,1)$	5	3	101	100
19	$\mathbb{P}^4_{1,1,3,3,4}[12]$	$(10,10,2,2,1)$	5	2	89	89
20	$\mathbb{P}^4_{1,4,5,5,5}[20]$	$(18,3,2,2,2)$	5	5	65	53
21	$\mathbb{P}^4_{1,1,3,10,15}[30]$	$(28,28,8,1,0)$	5	4	251	251
22	$\mathbb{P}^4_{1,2,3,12,18}[36]$	$(34,16,10,1,0)$	5	5	185	182
23	$\mathbb{P}^4_{1,2,2,5,10}[20]$	$(18,8,8,2,0)$	6	4	120	116
24	$\mathbb{P}^4_{1,2,3,6,12}[24]$	$(22,10,6,2,0)$	6	5	114	111
25	$\mathbb{P}^4_{1,3,4,4,12}[24]$	$(22,6,4,4,0)$	6	5	90	84
26	$\mathbb{P}^4_{1,3,3,14,21}[42]$	$(40,12,12,1,0)$	6	6	180	168

Table 3.2.: Fermat hypersurfaces with $h^{1,1} \leq 6$.

3.5.1. The one-parameter families

The family $\mathbb{P}^4_{1,1,1,1,1}[5] = \mathbb{P}^4[5]$, also known as the quintic in $\mathbb{P}^4$ has served as the most important Calabi–Yau manifold because it is the simplest non-trivial Calabi–Yau space. In [104] Candelas et al have put the mirror symmetry conjecture to work for the first time by explicitly computing the prepotentials on both sides of (2.124) and calculating the instanton contributions. It was also the quintic for which Douglas et al. [5] started the study of D-branes on Calabi–Yau spaces as will be explained in more details in Chapter 6. The other one-parameter families were explored from the point of view of mirror symmetry in analogy to the quintic in [157] and [158]. The cohomology for these models is generated by the restriction of the hyperplane class H of the ambient weighted projective space. Using (3.24), (3.23), (3.28), (3.29), (3.37), (3.38) and (3.42) yields

$$H^3 = 5 \qquad c_2 \cdot H = 50 \qquad \chi(\mathcal{O}_H) = 5 \qquad p_g(H) = 4 \qquad \text{for } \mathbb{P}^4_{1,1,1,1,1}[5] \tag{3.86}$$

$$H^3 = 3 \qquad c_2 \cdot H = 42 \qquad \chi(\mathcal{O}_H) = 4 \qquad p_g(H) = 3 \qquad \text{for } \mathbb{P}^4_{1,1,1,1,2}[6] \tag{3.87}$$

$$H^3 = 2 \qquad c_2 \cdot H = 44 \qquad \chi(\mathcal{O}_H) = 4 \qquad p_g(H) = 3 \qquad \text{for } \mathbb{P}^4_{1,1,1,1,4}[8] \tag{3.88}$$

$$H^3 = 1 \qquad c_2 \cdot H = 34 \qquad \chi(\mathcal{O}_H) = 3 \qquad p_g(H) = 2 \qquad \text{for } \mathbb{P}^4_{1,1,1,2,5}[10] \tag{3.89}$$

The families $\mathbb{P}^4_{1,1,1,1,2}[6]$ and $\mathbb{P}^4_{1,1,1,1,4}[8]$ can be described as triple covering of $\mathbb{P}^3$ branched over a sextic and a double covering of $\mathbb{P}^3$ branched over an octic, respectively [74]. These descriptions can be useful for studying vector bundles on these Calabi-Yau hypersurfaces by relating them to bundles over $\mathbb{P}^3$. The case of rank 2 vector bundles on a double covering of a $\mathbb{P}^3$ branched over a quartic has been extensively studied in [159]. The method used there can be generalized to the octic case, however it seems as if one gets only results for non-generic octics.

Let us explain briefly the Kähler moduli space in these examples [104]. The affine one-dimensional complex structure parameter space of the mirror X^* admits a torus action which can be used to compactify it and obtain a $\mathbb{P}^1$. The manifold degenerates however at three particular points which correspond to the regular singular points of the Picard-Fuchs equation for the periods ϖ: At $\psi = 0$ there is a $\mathbb{Z}_d$ singularity which can be removed by going to the d-fold cover of the moduli space. At $\psi = 1$ there is the conifold singularity where one three-cycle shrinks to zero. And at $\psi = \infty$ the manifold degenerates to five $\mathbb{P}^3$'s intersecting one another in one point. In the Kähler moduli space of X they correspond to the Gepner point, the conifold point and the large volume limit, respectively. This example can serve as a prototype for the general structure of such moduli spaces.

3.5.2. The family $\mathbb{P}^4_{1,1,2,2,6}[12]$

General description of $\mathbb{P}^4_{1,1,2,2,6}[12]$

The geometry of this family has been thoroughly studied in [105] and [142]. In the first part of this subsection we follow closely [105]. In the second we will translate the geometry into the combinatorial data of the corresponding toric variety. The purpose of this redundant description is the following. For more complicated families having, say $h^{1,1} > 2$, it is straightforward to compute all the topological properties. However, these are given as pure numbers and it is generally difficult to recognize known (and simple) topological and geometric structures. Therefore it is useful to have a geometric understanding of the same properties, as well. In the Appendix C we have only collected the toric data for the other families that have been investigated in order to study D-branes on them. In many cases one will have to appeal to the geometric picture in order to get the relevant information. Due to lack of space we restrict ourselves to show this geometric picture explicitly only for the following family.

Consider the Calabi–Yau threefold X which is obtained by resolving the singularities of degree twelve hypersurfaces $\widetilde{X} \subset \mathbb{P}^4_{1,1,2,2,6}$. A typical defining polynomial for such a hypersurface is

$$W(z) = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2. \tag{3.90}$$

Following the general discussion in Section 3.3.3 the singularities occur along the surface $\mathbb{P}^2_{2,2,6} \cong \mathbb{P}^2_{1,1,3}$ defined by $z_1 = z_2 = 0$, where there is a curve C of A_1-singularities which is described by

$$z_1 = z_2 = 0, \qquad z_3^6 + z_4^6 + z_5^2 = 0 \tag{3.91}$$

In general it will be a smooth cubic curve in $\mathbb{P}^2_{1,1,3}$ which has genus $g_C = 2$ [160]. We desingularize by using an auxiliary $\mathbb{P}^1$ with coordinates (y_1, y_2) and define the blow-up $\widetilde{\mathbb{P}}^4 \subset \mathbb{P}^4_{1,1,2,2,6} \times \mathbb{P}^1$ by the equations

$$y_i z_j = y_j z_i, \qquad i, j = 1, 2$$

The exceptional divisor is just $\mathbb{P}^2_{1,1,3} \times \mathbb{P}^1$, where the two projective spaces have coordinates (z_3, z_4, z_5) and (y_1, y_2) respectively. The proper transform of a general degree twelve hypersurface X is seen to intersect the exceptional divisor in a surface defined by a polynomial $g(z_3, z_4, z_5, y_1, y_2)$ which is sextic in the z's and linear in the y's. The fibers of the projection of this surface to the sextic curve $C \subset \mathbb{P}^2_{1,1,3}$ are lines; thus the desingularized Calabi–Yau manifold $\widetilde{X}$ contains a ruled surface with C as its base.

There is a linear system $|L|$ [111] on X generated by polynomials of degree one (i.e. by z_1 and z_2). Every divisor in $|L|$ is the proper transform on X of the zero locus of such a polynomial on $\widetilde{X}$. These divisors are described by means of a parameter $\lambda \in \mathbb{P}^1$ and noting that the weights w_i are of the form discussed in Section 3.3.4 with $l = 2$ the equation of the proper transform of L becomes that of a surface of degree six in $\mathbb{P}^3_{1,1,1,3}$

$$(1 + \lambda^{12}) y_1^6 + z_3^6 + z_4^6 + z_5^2 = 0 \tag{3.92}$$

Note that this is precisely of the form (3.51). Thus, the linear system $|L|$ is thus a one-parameter family of degree six $K3$ surfaces. In other words, this linear system projects X to $\mathbb{P}^1$ with the fibers being $K3$ surfaces [111]. Note that any two distinct members of $|L|$ are disjoint, i.e. $L \cdot L = 0$. There is a second linear system on X which we denote by $|H|$ that is generated by polynomials of degree two (i.e. by linear combinations of $z_1^2, z_1 z_2, z_2^2, z_3$ and z_4). The divisors in $|H|$ are total transforms on X of the zero locus on $\widetilde{X}$ of the corresponding polynomial. A typical polynomial will have non-zero coefficient on z_5, and allows one to solve for z_5 in terms of the other variables, producing a proper transformed equation which defines a surface of degree twelve in $\mathbb{P}^3_{1,1,2,2}$. This is a surface of general type. These two linear systems are related to each other as follows. If we look at $|2L|$, the quadratic polynomials in z_1 and z_2, we get a subsystem of $|H|$ which can be characterized by the property that the polynomials from $|2L|$ vanish on the singular curve C. Interpreted on the resolution X, this means that the total transform of the zero locus of such a polynomial has the form $2L + E$ where $2L$ describes the proper transform and E is the exceptional divisor. Hence, we have

$$|H| = |2L + E| \tag{3.93}$$

We will need the intersection products of these divisors. Since $L \cdot L = 0$, we automatically have

$$H \cdot L^2 = 0, \qquad L^3 = 0 \tag{3.94}$$

Since $|H|$ defines a birational map on X whose image has degree four (the number of common intersection points of three members of $|H|$), we have

$$H^3 = 4 \tag{3.95}$$

When we restrict the linear system $|H|$ to one of the $K3$ surfaces L, we get a quadric linear system on L. It follows that

$$H^2 \cdot L = (H \cap L) \cdot (H \cap L) = 2. \tag{3.96}$$

This tells us that the Picard lattice of L is

$$\mathrm{Pic}(L) = \langle 2 \rangle. \tag{3.97}$$

The intersection numbers with E can be obtained by replacing H by $E + 2L$ in the above which leads to

$$\begin{array}{c} E^3 = -8, \quad E^2 \cdot L = 2, \quad E \cdot L^2 = 0, \\ E^2 \cdot H = -4, \quad E \cdot H^2 = 0, \quad H \cdot E \cdot L = 2 \end{array} \tag{3.98}$$

Next, we consider two classes of some 2-cycles on X. The first class is l, the fiber of the ruling $E \longrightarrow C$. One can identify its cohomology class by noting that $H \cap E$ consists of two fibers lying over the two points of intersection of the hyperplane with C so that

$$l = \frac{1}{2} H \cdot E = \frac{1}{2} H^2 - H \cdot L \tag{3.99}$$

The second class is the intersection of general members of $|H|$ and $|L|$,

$$h = \frac{1}{2} H \cdot L \tag{3.100}$$

The intersection relations between linear systems and curves read

$$L \cdot l = 1 \qquad\qquad L \cdot h = 0 \tag{3.101}$$
$$H \cdot l = 0 \qquad\qquad H \cdot h = 1 \tag{3.102}$$

Finally, from (3.37) and the topological properties of the surfaces E and L given in Section 3.3.2 we find for the second Chern classes

$$c_2 \cdot H = 52, \qquad c_2 \cdot L = 24, \qquad c_2 \cdot E = 4 \tag{3.103}$$

Toric description of $\mathbb{P}^4_{1,1,2,2,6}[12]$

According to (3.16) the extended vertices of the dual polyhedron for this family are

$$\begin{array}{lll} \bar\nu_0^* = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1) & \bar\nu_1^* = (-1,-2,-2,-6,\ \ 1) & \bar\nu_2^* = (\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 1) \\ \bar\nu_3^* = (\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ \ 1) & \bar\nu_4^* = (\ \ 0,\ \ 0,\ \ 1,\ \ 0,\ \ 1) & \bar\nu_5^* = (\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 1) \\ \bar\nu_6^* = (\ \ 0,-1,-1,-3,\ \ 1) & & \end{array} \tag{3.104}$$

where $\bar\nu_6^* = \frac{1}{2}(\bar\nu_1^* + \bar\nu_2^*)$ corresponds to the resolution of the A_1 singularity coming from the weights 2, 2 and 6 as explained in Section 3.3. The dual face Θ_2 to the face $\Theta_1^* = \langle \bar\nu_1^*, \bar\nu_2^* \rangle$ is spanned by (3.15) $\bar\nu_3 = (-1,5,-1,-1)$, $\bar\nu_4 = (-1,-1,5,-1)$ and $\bar\nu_5 = (-1,-1,-1,1)$ and has two interior points $(-1,1,0,0)$ and $(-1,0,1,0)$. Thus the genus of the singular curve is $g = 2$. We have explained in Section 2.6 that such a singularity corresponds to a phase boundary. In the present case this is the horizontal line in Figure 2.1.

The vertices (3.104) satisfy the relations (3.22)

$$\bar\nu_1^* + \bar\nu_2^* - 2\bar\nu_6^* \quad = \quad 0 \tag{3.105a}$$
$$-6\bar\nu_0^* + \bar\nu_3^* + \bar\nu_4^* + 3\bar\nu_5^* + \bar\nu_6^* \quad = \quad 0 \tag{3.105b}$$

which correspond to the D-term equations in (2.110) of the gauged linear σ-model. In this example there is a unique maximal triangulation and it turns out that the Mori generators can be read off from (3.105)

$$l^{(1)} = (\ \ 0,\ \ 1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,-2) \quad l^{(2)} = (-6,\ \ 0,\ \ 0,\ \ 1,\ \ 1,\ \ 3,\ \ 1) \tag{3.106}$$

In general, one has to work through the algorithm given in Section 3.3.1 to obtain the Mori generators. Observe that these are precisely the $U(1)^2$ charge vectors (2.108) of the example we studied in the context of the gauged linear σ-model in Section 2.5. It is useful to arrange this information in a table as follows:

$$\begin{array}{lccccc|cc l}
 & & & & & & C_1 & C_2 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 & K=-6H \\
D_1 & -1 & -2 & -2 & -6 & 1 & 0 & 1 & L \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & H \\
D_4 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & H \\
D_5 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & 3H \\
D_6 & 0 & -1 & -1 & -3 & 1 & 1 & -2 & E=H-2L \\
 & & & & & & h & l &
\end{array} \tag{3.107}$$

The left-hand side of the vertical line is simply an arrangement of the set Ξ of the vertices $\bar{\nu}_i^*$ and we have labelled the corresponding $\mathbb{T}$-invariant divisors by D_i. Let us now explain the right-hand side which corresponds to the two Mori generators written as column vectors. The row vectors $(-6,0),(0,1),(1,0),(3,0),(1,-2)$ span the secondary fan $A(\Xi)$. We can drop $(3,0)$ since it spans the same edge as $(1,0)$ and replace $(-6,0)$ by $(-1,0)$ for the same reason. These are precisely the boundaries of the phases in figure 2.1 we found when analyzing the low-energy effective theory of the gauged linear σ-model in Section 2.5. From the discussion above we see that the divisor L corresponds to $(0,1)$, the divisor E to $(1,-2)$ and the divisor H to $(1,0)$. We also see that there are linear equivalences $D_1 \sim D_2$ and $D_3 \sim D_4$ and $3D_4 \sim D_5$. We have renamed the divisors according to their geometric meaning as explained at several places in Section 3.3.

The Mori generators $l^{(a)}$, $a=1,2$ are dual to curves in $H_2(X,\mathbb{Z})$ which we denote by C_a in the top row. The entries on the right-hand side then correspond to the intersection numbers of these curves with the divisors D_i, or E,H and L. From (3.101) we can identify $l^{(1)}$ with l and $l^{(2)}$ with h which explains the bottom line. Furthermore, the classes $J_1 \equiv H$ and $J_2 \equiv L$ are dual to the Mori generators and hence generate the Kähler cone. It follows that the Kähler cone corresponds to the first quadrant. This is in complete agreement with the corresponding phase in the gauged linear σ-model being the geometric or smooth Calabi-Yau phase.

Next, we want to compute the intersection ring of X. Applying the method from section 3.3.1 we find that there are two primitive collections, $\{\nu_1^*,\nu_2^*\}$ and $\{\nu_3^*,\nu_4^*,\nu_5^*,\nu_6^*\}$, and by (3.23) the Stanley-Reisner ideal is

$$\mathcal{I}_{SR} = \{D_1\cdot D_2 = L^2 = 0, D_3\cdot D_4\cdot D_5\cdot D_6 = 3H^4 - 6H^3\cdot L = 0\} \tag{3.108}$$

Note that the primitive collections also determine the excluded set F in (2.111). This allows us to compute the intersection ring (3.26) of the ambient toric variety $\mathbb{P}^4_{1,1,2,2,6}$ from (3.108) and (3.25) applied to $D_2,\ldots,D_5$ as follows

$$L^4 = H\cdot L^3 = H^2\cdot L^2 = 0, \quad H^3\cdot L = \tfrac{1}{3}, \quad H^4 = \tfrac{2}{3} \tag{3.109}$$

The fractional intersection numbers indicate that we have not blown up the codimension one singularities. The Calabi–Yau hypersurface X is a section of the anti-canonical bundle, i.e. the anti-canonical divisor $-K = \sum_{i=1}^6 D_i = -D_0$ which is indicated in the first row of the table. By using the restriction formula (3.27) we find

$$L^3 = H\cdot L^2 = 0, \quad H^2\cdot L = 2, \quad H^3 = 4 \tag{3.110}$$

which agrees with the results from the previous subsection and further justifies the identification of the divisors D_i with E, H and L.

From the table (3.107) and the intersection ring (3.110) we can get additional information about the geometry of these divisors. First we can apply the criteria for elliptic and $K3$ fibrations. The only effective divisor whose triple self-intersection is zero are multiples of L (which define the $K3$ fibration) and of $3H-2L$. But if we take the curve l then we find that $(3H-2L)\cdot l = -2$ hence condition (3.46a) of the elliptic fibration is not met. From (3.110) we see that the divisor L does not satisfy condition (3.46c). Hence X is not elliptically fibered. On the other hand, L does satisfy conditions (3.49a) and (3.49b) so X is $K3$-fibered in agreement with what we discussed above. We can also study the geometry of E. Since $l \cdot E < 0$, l is contained in E. Furthermore, from the last line of the table, we have a fibration $\pi : E \to \mathbb{P}^2_{113}$. The base intersects X in a curve C of genus 2. From $l \cdot D_2 = l \cdot D_3 = l \cdot D_4 = 0$ we conclude that $l \cap \mathbb{P}^2_{113} = \text{pt}$. Hence l is the fiber of π. The restriction of E to X is therefore a ruled surface over the curve C.

Finally, we need the topological and holomorphic characteristics of the surfaces E, H and L. The second Chern classes are obtained from the degree 2 term in the expansion of (3.29) as well as from the intersection ring (3.110) and agree with (3.103). The Euler characteristics can then be computed from (3.37) and are

$$\chi(E) = -8 \qquad \chi(L) = 24 \qquad \chi(H) = 56 \tag{3.111}$$

The holomorphic Euler characteristic can be calculated from (3.38)

$$\chi(\mathcal{O}_E) = -1 \qquad \chi(\mathcal{O}_L) = 2 \qquad \chi(\mathcal{O}_H) = 5 \tag{3.112}$$

Now, in order to compute $p_g(D)$ and $q(D)$ the toric data is not sufficient, we need to know more. E is a ruled surface over a curve of genus 2. From Section 3.3.2 we find $p_g(E) = 0$ and $q(E) = 3$. H and L are both $\mathbb{T}$-invariant divisors and we can use (3.42) to obtain $q(H) = q(L) = 0$ and by (3.38) $p_g(H) = 4$ and $p_g(L) = 1$. The latter agrees with the fact that L is a $K3$ surface, see Section 3.3.2.

3.6. Nested moduli spaces

In [105], [142] and [161] it was observed that many Calabi–Yau families are birationally equivalent to a different Calabi–Yau manifold when restricted to specific codimension one subspaces in the Kähler moduli space. These are defined by those singularities of the Picard-Fuchs equations where the conformal field theory and therefore the Calabi–Yau space becomes singular. This has been briefly discussed at the end of Section 2.5. We have mentioned there that a singularity occurs if the Kähler class approaches a face of the Kähler cone of X or, in other words, a phase boundary. Each face of the Kähler cone determines a collection of holomorphic 2-spheres whose area shrinks to zero as the face is approached. These 2-spheres can be contracted to points at the expense of introducing singularities into the new space X'. Due to non-perturbative effects the string theory is still well-behaved at these points [4].

There are two cases in which we are interested. In the first case, only a finite number of 2-spheres on X are contracted by this process. This is the mirror description of the conifold singularity [4], [55]. Consider the example $\mathbb{P}^4_{1,1,2,2,6}[12]$ given in Section 2.5 and further studied in Section 3.5.2, now using the language of Mori generators of Section 3.3.1. The conifold transition corresponds to contracting the curve $l = l^{(2)}$ which is dual to the face given by L.

In the second case, a collection of divisors is contracted to a smooth curve. This is precisely the inverse process of the resolution of singular $\mathbb{Z}_N$ curves discussed in Section 3.3.3. It can be shown [57] that in the example above, this corresponds to approaching the phase boundary determined by H, i.e. contracting the Mori generator $l^{(1)}$. In both cases, the transition from X to X' can be studied in detail by analyzing e.g. the Picard-Fuchs operators or the periods.

In general, denote the two spaces in question by $X = \mathbb{P}^4_w[d]$ and by $X' = \mathbb{P}^4_{w'}[d']$. A simple interpretation of the contraction $X \to X'$ is given in terms of toric geometry [108], [102]. Their reflexive polyhedra are nested into each other: $\Delta^*(w') \subset \Delta^*(w)$. This nesting phenomenon is ubiquitous among

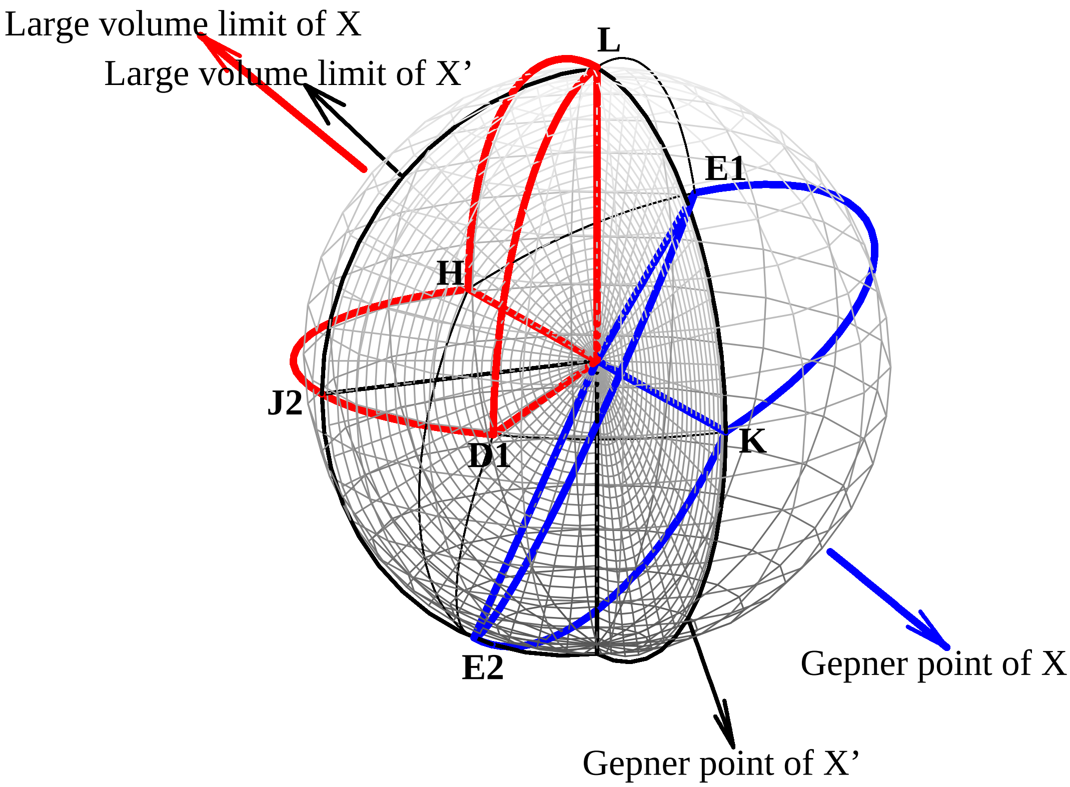

Figure 3.1.: The secondary fans of $X = \mathbb{P}^4_{1,2,3,3,9}[18]$ and $X' = \mathbb{P}^4_{1,1,2,2,6}[12]$.

the reflexive polyhedra and has the following implications for the Kähler and complex structure moduli spaces of the manifolds X, X'.

$$\begin{aligned} &\Delta^*(w') \subset \Delta^*(w) \Longleftrightarrow \Delta(w) \subset \Delta(w') \\ \Longrightarrow &\mathcal{M}_K^{\mathrm{SCFT}}(X') \subseteq \mathcal{M}_K^{\mathrm{SCFT}}(X), \qquad \mathcal{M}_C^{\mathrm{SCFT}}(X) \subseteq \mathcal{M}_C^{\mathrm{SCFT}}(X') \end{aligned} \tag{3.113}$$

We will now discuss this inclusion of Kähler moduli spaces in detail the example of $X = \mathbb{P}^4_{1,2,3,3,9}[18]$ discussed in Section C.1.3 and $X' = \mathbb{P}^4_{1,1,2,2,6}[12]$. The secondary fan describing the Kähler moduli space (see Section 3.3.1) of X has dimension three, therefore we project its real part onto a sphere. This is displayed in figure 3.1. Let us explain in detail the information contained in this figure. We start by noting from (3.107) and (C.24) that the toric polyhedron $\Delta^*(1,1,2,2,6)$ is contained in $\Delta^*(1,2,3,3,9)$ because they differ only by the vertex $\nu^* = (-2,-3,-3,-9)$. Next, we determine the triangulations of $\Delta^*(1,2,3,3,9)$ and find eight of them, one of them being maximal, another one being minimal. Using the algorithm given in Section 3.3.1 the generators of the Mori cone of X, $l^{(1)}$, $l^{(2)}$, and $l^{(3)}$ are computed, the result is also contained in (C.24). The row vectors of the right-hand side of the vertical line in (C.24) are drawn in figure 3.1 as lines starting from the origin and ending on the sphere, and are labeled by the corresponding divisor. Note that the vertex ν^* corresponds to the divisor D_1. By applying the method of [106], [107] to associate a maximal cone of the secondary fan to a triangulation we find the following maximal cones: $\langle 0,H,D_1,L\rangle$, $\langle 0,D_1,L,K\rangle$, $\langle 0,L,K,E_1\rangle$, $\langle 0,H,E_1,L\rangle$, $\langle 0,H,E_1,E_2\rangle$, $\langle 0,H,D_1,E_2\rangle$, $\langle 0,E_2,D_1,K\rangle$ and $\langle 0,K,E_1,E_2\rangle$. The maximal triangulation corresponds to the cone spanned by H, L and D_1 emphasized back on the left which contains the Kähler cone $\langle 0,H,L,J_2 = D_1+H\rangle$. This contains the large volume limit as deep interior point which we have indicated by an arrow pointing from the sphere on outwards. The minimal triangulation corresponds to the cone spanned by K, E_1

and E_2 emphasized in the front on the right. This cone contains the Gepner point which is indicated by another arrow.

Recall that the Mori cone generates the lattice of relations L for X and L' for X'. The Mori cone of X' can be obtained from the one of X by restricting the latter to the sublattice $L' \subset L$. The Mori generators of X' are then related to those of X by $l^{(1)\prime} = l^{(1)} + l^{(2)}$ and $l^{(2)\prime} = l^{(3)}$. We see that in this contraction the entries corresponding to K, H, L and E_2 remain unchanged. Hence, if we identify E_2 in X with E in X', we expect that the secondary fan of X' given in figure 2.1 will be contained in the one of X. To confirm this, we need to take into account that the topology of the divisors H in X and X' is different. We noted, however, in Section C.1.3 that the divisor J_2 in X has the same topological properties as H in X'. We therefore identify the secondary of X' as the two vertical half-disks emphasized in figure 3.1. Unfolding them reproduces precisely the secondary fan in figure 2.1. From the discussion in Section 2.5 we know where the large volume limit and the Gepner point of X' are and we have indicated them again by arrows pointing outwards. It is important to note that the large volume limit of X' is contained in the boundary of the Kähler cone of X and that there is a similar relation between the Gepner points of X and X'.

We will return to this picture in Section 6.4.5 when we discuss the D4-branes wrapping these divisors. As a different kind of inclusion relation it has been observed in [102] that the dual polyhedron $\Delta^*_{K3}(w')$ for some $K3$ hypersurface L sits in the polyhedron $\Delta^*(w)$ for a Calabi–Yau hypersurface $X_{\Delta^*(w)}$ where w and w' are related as in (3.51). In this case $X_{\Delta^*(w)}$ is a $K3$ fibration with fiber L as discussed in Section 3.3.2. This observation has been studied in detail in [162].

4. Boundary Conformal Field Theories

D-branes are defined to be objects on which open strings can end. As often in string theory, there are two descriptions for these objects, one from a geometric point of view and another in terms of conformal field theory. In this chapter we will focus on the latter and describe D-branes in the conformal field theory description of the Calabi–Yau manifold, namely the Gepner model introduced in Section 2.4. The geometry of D-branes will then be the subject of chapter 5. In order to give such a description we need to introduce boundaries into the conformal field theory which leads to so-called boundary conformal field theories.

4.1. Generalities and Definitions

A conformal field theory on a Riemann surface with a boundary requires specifying boundary conditions on the operators. For non-linear σ-models these conditions can be derived by imposing Dirichlet and/or Neumann boundary conditions directly on the σ-model fields. For more general conformal field theories such as the Gepner model we do not have a Lagrangian description, so the construction, classification and interpretation of boundary conditions is not as straightforward.

If the conformal field theory has a chiral symmetry algebra $\mathcal{A}(=\mathcal{A}_L=\mathcal{A}_R)$ one may simplify the problem by demanding that the boundary conditions are invariant under this symmetry. We start with a rational conformal field theory which have a finite set $\mathcal{I}$ of (classes of) irreducible highest weight representations $\mathcal{V}_j$, $j\in\mathcal{I}$. The Hilbert space of the bulk conformal field theory is decomposable into a finite sum of irreducible representations of two copies of $\mathcal{A}$, $\mathcal{H}=\bigoplus_{j,\bar{\jmath}}N_{j\bar{\jmath}}\mathcal{V}_j\otimes\overline{\mathcal{V}}_{\bar{\jmath}}$ with the multiplicity $N_{j\bar{\jmath}}\in\mathbb{N}$ of the left and right copies of $\mathcal{A}$. If χ_j is a character of $\mathcal{A}$, then S_{ij} is the matrix representation of the modular transformation $S:\tau\to-1/\tau$

$$\chi_i(q)=\sum_{j\in\mathcal{I}}S_{ij}\chi_j(\tilde{q}) \tag{4.1}$$

where $q=e^{2\pi i\tau}$ and $\tilde{q}=e^{-\frac{2\pi i}{\tau}}$. The matrix S satisfies $S^T=S$, $S^\dagger=S^{-1}$, $(S_{ij})^*=S_{i^*j}=S_{ij^*}$ and $S^2=C$ where C is the conjugation matrix defined by $C_{ij}=\delta_{ij^*}$. Here i^* denotes the representation conjugate to i under some involution of $\mathcal{I}$, e.g. complex conjugation. The representations $\mathcal{V}_j$ define the fusion algebra

$$\mathcal{V}_i\star\mathcal{V}_j=\sum_k N_{ij}{}^k\mathcal{V}_k \tag{4.2}$$

where the fusion coefficients $N_{ij}{}^k\in\mathbb{N}$ satisfy the Verlinde formula [163]

$$N_{ij}{}^k=\sum_{l\in\mathbb{Z}}\frac{S_{il}S_{jl}\left(S_{kl}\right)^*}{S_{1l}} \tag{4.3}$$

The Virasoro algebra is contained in $\mathcal{A}$ and must be preserved. Let the boundary be at $z=\bar{z}$ in some local coordinates on the Riemann surface corresponding to the half-plane. Reparametrizations should leave the boundary fixed, so we must impose [164]

$$T(z)=\overline{T}(\bar{z})\qquad\text{at } z=\bar{z} \tag{4.4}$$

In other words, no momentum flows across the boundary. If the remaining symmetry algebra is generated by chiral currents $W^{(r)}$ of half-integer or integer conformal dimension h_r, then the boundary conditions are more general [165], [166], [167]

$$W^{(r)}(z) = \Omega \overline{W}^{(r)}(\bar{z})\Omega^\dagger \qquad \text{at } z = \bar{z} \tag{4.5}$$

where Ω is an outer automorphism of $\mathcal{A}$. The action by Ω is allowed since there is no equally fundamental meaning of $W(z) = \overline{W}(\bar{z})$ as in the case of the energy-momentum tensor in (4.4). We can conformally map the (punctured) half-plane to an infinite strip. Due to the boundary conditions, the Hilbert space of states on the boundary decomposes on irreducible representations of a single copy of $\mathcal{A}$ according to $\mathcal{H}_{\beta\alpha} = \bigoplus n_{i\alpha}{}^{\beta}\mathcal{V}_i$ with a new set of multiplicities $n_{i\alpha}{}^{\beta} \in \mathbb{N}$, called annulus coefficients. Here, α and β label some boundary conditions on the left and right boundary of the strip, respectively. Note that these multiplicities satisfy $n_{i\alpha}{}^{\beta} = n_{i^*\beta}{}^{\alpha}$.

Next, we consider a one-loop diagram in the open string channel, i.e. a conformal field theory on an annulus. By world-sheet duality this can also be studied in the closed string channel where time flows from one boundary to the other. The boundaries appear as initial and final conditions on the path integral and are described in the operator formalism by coherent boundary states [168], [169]. By a conformal mapping and world-sheet duality, (4.4) and (4.5) become conditions on the boundary states $|\alpha\rangle_\Omega$

$$\left(L_n - \bar{L}_{-n}\right)|\alpha\rangle_\Omega = 0 \tag{4.6}$$

$$\left(W_n^{(r)} - (-1)^{h_r}\Omega\left(\overline{W}_{-n}^{(r)}\right)\right)|\alpha\rangle_\Omega = 0 \tag{4.7}$$

The solution to these conditions are linear combinations of the Ishibashi states [170], [171]

$$|\, i\, \rangle\rangle_\Omega = \sum_N |i, N\rangle \otimes U\Omega|\bar{\imath}, \overline{N}\rangle \tag{4.8}$$

Here $|i\rangle$ is a highest weight state of the chiral algebra $\mathcal{A}$, the sum is over all descendants of $|i\rangle$; and U is an anti-unitary operator with $U|\bar{\imath}, \overline{0}\rangle = |\bar{\imath}^*, \overline{0}\rangle$ acting only on the right-moving generators as $U\overline{W}_n^{(r)}U^\dagger = (-1)^{h_r}\overline{W}_n^{(r)}$. The Ishibashi states are normalized such that ${}_\Omega\langle\langle j||j'\rangle\rangle_\Omega = \delta_{jj'}S_{1j}$. In terms of this basis, a boundary state $|\alpha\rangle_\Omega$ can be expanded as

$$|\alpha\rangle_\Omega = \sum_{j\in\mathcal{E}} \frac{\psi_\alpha^j}{\sqrt{S_{1j}}}|\, j\, \rangle\rangle_\Omega \tag{4.9}$$

where $\mathcal{E} = \{j \in \mathcal{I} | j = \omega(\bar{\jmath}), N_{\omega(\bar{\jmath})\bar{\jmath}} \neq 0\}$. One defines an involution $\alpha \to \alpha^*$ on the boundary states by $\psi_{\beta^*}^j \equiv \psi_\alpha^{j^*} = \left(\psi_\alpha^j\right)^*$ and the conjugate boundary state [166]

$${}_\Omega\langle\beta| = \sum_{j\in\mathcal{E}} {}_\Omega\langle\langle j| \frac{\psi_{\beta^*}^j}{\sqrt{S_{1j}}} \tag{4.10}$$

World-sheet duality requires that calculations in either channel give the same result. This gives powerful restrictions on possible boundary states. In the closed string channel we find for the tree-level propagation of a closed string from a boundary state $|\alpha\rangle_\Omega$ to a boundary state $|\beta\rangle_\Omega$

$$Z_{\beta\alpha}(\tilde{q}) = {}_\Omega\langle\beta|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\alpha\rangle_\Omega = \sum_{j\in\mathcal{E}} \psi_\alpha^j \left(\psi_\beta^j\right)^* \frac{\chi_j(\tilde{q})}{S_{1j}} \tag{4.11}$$

with $\tilde{q}=e^{-4\pi\frac{L}{T}}$ where L and T are the length and the circumference of the cylinder, respectively. In the open string channel we find for the one-loop evolution of an open string in the Hilbert space $\mathcal{H}_{\beta\alpha}$

$$Z_{\beta\alpha}(q)=\sum_{i\in\mathcal{I}} n_{i\beta}{}^{\alpha}\chi_i(q) \tag{4.12}$$

with $q=e^{-\pi\frac{T}{L}}$. Cardy required that after a modular transformation the two expressions should be the same [164] and obtained a fundamental equation, referred to as the Cardy equation

$$n_{i\beta}{}^{\alpha}=\sum_{j\in\mathcal{E}}\frac{S_{ij}}{S_{1j}}\psi_\alpha^j\left(\psi_\beta^j\right)^* \tag{4.13}$$

In the following, we assume that the boundary states $|\alpha\rangle_\Omega$ in (4.9) are orthonormal and complete [172], [173]. The latter implies that the number of boundary states is equal to the number of independent Ishibashi states and is equal to $|\mathcal{E}|$. One can show that the matrices $(n_i)_\alpha{}^\beta=n_{i\alpha}{}^\beta$ form a representation of the fusion algebra [172], [173]

$$n_i n_j=\sum_{k\in\mathcal{I}} N_{ij}{}^k n_k \tag{4.14}$$

and they thus commute. Moreover, they satisfy $n_1=\mathbb{1}$, $n_i^T=n_{i^*}$.

4.2. Boundary states in $\mathcal{N}=(2,2)$ superconformal field theories

We are interested in describing BPS D-branes which preserve four supercharges, i.e. $\mathcal{N}=1$ space-time supersymmetry in $D=4$. We have seen in Section 2.3 that the closed string sector will have (at least) $\mathcal{N}=(0,2)$ world-sheet supersymmetry. We have required $\mathcal{N}=(2,2)$ supersymmetry in order to have an underlying superconformal field theory for a type II string theory. The requirement of $\mathcal{N}=1$ space-time supersymmetry translates into the condition that the boundary conditions (4.5) must preserve a diagonal $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=(2,2)$ world-sheet supersymmetry [15], [16].

Thus we require the boundary state to be invariant under a linear combination of the left and right $\mathcal{N}=2$ superconformal algebra extended by the spectral flow operators. Consistency restricts the linear combination to correspond to the automorphism group of the algebra which is $O(2)$ for the $\mathcal{N}=2$ superconformal algebra and $\mathbb{Z}_2$ for $\mathcal{N}=1$. Thus [174], the condition (4.5) leads to two classes of boundary conditions: the A-type boundary conditions

$$T=\overline{T} \qquad J=-\overline{J} \qquad G^+=\pm\overline{G}^- \qquad e^{i\phi}=e^{i\bar{\phi}} \tag{4.15}$$

and the B-type boundary conditions

$$T=\overline{T} \qquad J=\overline{J} \qquad G^+=\pm\overline{G}^+ \qquad e^{i\phi}=e^{i\theta}e^{i\bar{\phi}} \tag{4.16}$$

all at $z=\bar{z}$. Both A-type and B-type boundary conditions preserve the $\mathcal{N}=1$ superconformal algebra in (2.7)

$$T=\overline{T} \qquad G=\pm\overline{G} \tag{4.17}$$

at $z=\bar{z}$. These conventions correspond to the open string channel where the boundary propagates in world-sheet time. In the closed string channel, the boundary conditions (4.15) and (4.16) can be rewritten as operator conditions on the boundary states. For the A-type boundary states we have

$$L_n=\overline{L}_{-n} \qquad J_n=\overline{J}_{-n} \qquad G_r^\pm=-i\eta\overline{G}_{-r}^\mp \tag{4.18}$$

and for the B-type boundary states we have

$$L_n = \overline{L}_{-n} \qquad J_n = -\overline{J}_{-n} \qquad G_r^{\pm} = -i\eta\overline{G}_{-r}^{\pm} \tag{4.19}$$

The relative sign change in the $U(1)$ current from (4.15) and (4.16) can be understood as the result of a $\pi/2$ rotation on the components of the spin one current J which corresponds to the open-closed string duality. It is important to observe that mirror symmetry exchanges the A-type and the B-type boundary conditions since it switches the relative sign of the $U(1)$ charges as discussed in Section 2.6. This relative sign is precisely what distinguishes (4.18) from (4.19).

The boundary state can be expanded in terms of the Ishibashi state as in (4.9)

$$|\,\alpha\,\rangle\!\rangle_\Omega = \sum_j B_\alpha^j |\,j\,\rangle\!\rangle_\Omega \tag{4.20}$$

where the sum is over the highest weight states of the $\mathcal{N}=2$ superconformal algebra which appear in the Hilbert space of the non-linear σ-model for the Calabi–Yau space X. They may be chiral primary states or non-chiral primary states. The conditions on the currents in (4.18) and (4.19) at the boundary implies that $q=\bar{q}$ and $q=-\bar{q}$, respectively. This means that the A-type states are charged under (c,c) operators and the B-type states under (a,c) operators. One can show [174] that the B_α^j are independent of the Kähler moduli t_a for the A-type boundary states and that, in the large volume limit, its chiral primary part is completely determined by

$$B_\alpha^0 = \int_{\gamma_\alpha} \Omega \tag{4.21}$$

where γ_α is the supersymmetric 3-cycle on which the corresponding D-brane wraps (see Section 5.2). The other coefficients can be obtained by taking the covariant derivative of $\int_{\gamma_\alpha}\Omega$ on the vacuum line bundle $\mathcal{L}$ over the moduli space of the $\mathcal{N}=2$ superconformal field theories with respect to the complex structure moduli x_j (cf. Section 3.4). Due to the non-renormalization theorem reviewed in Section 2.6 this means in particular that these coefficients are exact.

The coefficients of the B-type boundary states can similarly be shown to be independent of the complex structure moduli x_j but they depend on the Kähler moduli and therefore receive instanton corrections. Similar to the case above, it can be shown that the chiral primary part of the coefficient $B_\alpha^0(\gamma)$ corresponding to the top cohomology $H^{3,3}(X)$ is holomorphic with respect to the Kähler moduli. Furthermore, the other coefficients are again computed by taking derivatives of $B_\alpha^0(\gamma)$ with respect to the t_a. Since it is holomorphic in the t_a the instanton approximation is exact and it can be expressed as a sum over holomorphic maps from the disk to X such that boundary of the disk is mapped to the supersymmetric cycle γ_α it wraps (see Section 5.2)

$$B_\alpha^0 = \int_{\gamma_\alpha} J^p + O(e^{2\pi i t}) \tag{4.22}$$

where $2p$ is the dimension of the cycle γ_α. Note that if $p=0$ or $p=1$ there are no instanton corrections since the image of a holomorphic map of the disc does not intersect with the homology dual to J in these cycles. We will discuss the geometry of these objects more deeply in Section 5.2.

4.3. Boundary states in Gepner models

We have reviewed the Gepner models in Section 2.4. We will use the notation introduced there. We have also discussed in Section 2.5 that they describe Calabi-Yau compactifications at small volume. In this section we will construct the D-branes in a compactification on a Calabi–Yau manifold at the Gepner point.

Since the Gepner model is not rational with respect to the Super-Virasoro algebra, the construction of the most general boundary state has not yet been achieved. Recknagel and Schomerus [166] have found a way to describe a certain subset of boundary states, called rational boundary states, which respect a larger symmetry algebra, namely the $\mathcal{N}=2$ world-sheet algebras of each minimal model factor separately, and can be found by Cardy's technique. We have reviewed this in Section 4.1 and apply it first to a minimal factor theory. Since minimal models are coset theories of $SU(2)_k$ WZW theories, (4.14) yields a recursion relation for the annulus coefficients

$$n_i = n_2 n_{i-1} - n_{i-2}, \qquad i = 3, \ldots, g \tag{4.23}$$

It can be shown [172] that they are classified by the ADE groups. This classification coincides with (2.76) which means that for a given modular invariant for the group G the annulus coefficients are determined by G as follows. If G is the adjacency matrix of the Dynkin diagram of the group G with Coxeter number $g = k+2$, then the boundary conditions α are labeled by the vertices of G. Hence $\mathcal{E} = \mathrm{Exp}(G)$ are the exponents of G and $p \equiv |\mathcal{E}| = \dim n_i = |G|$. Moreover, $n_2 = G$ and $n_g = 0$. All groups G having even exponents, i.e. A_j, D_{2j+1} and E_6 have a $\mathbb{Z}_2$ automorphism γ acting on the nodes of G and preserving G, i.e. ${G_\alpha}^\beta = {G_{\gamma(\alpha)}}^{\gamma(\beta)}$. Choosing $\gamma = \mathrm{id}$ for the other groups, one has

$${n_{g-i,\alpha}}^{\gamma(\beta)} = {n_{i\alpha}}^\beta \tag{4.24}$$

We will need later on an extension of these matrices to values of i up to $2g = 2k+4$. From (4.23) we have the relation

$$n_{g+i} = -n_{g-i} \tag{4.25}$$

Hence the matrices n_i are periodic in i with period $2g$. Finally, the coefficients ψ^j_α in (4.9) are the components of the orthonormal eigenvectors ψ^j of the symmetric matrix G.

Now, we turn to the boundary states of the Gepner models. A priori, they are labeled by

$$|\alpha\rangle\rangle_\Omega = \big|\big(\{L_j\}_{j=1}^r, \{M_j\}_{j=1}^r, \{S_j\}_{j=1}^r\big)\rangle\rangle_\Omega \tag{4.26}$$

where Ω is an outer automorphism of the chiral symmetry algebra, but, as we will see shortly, there are some simplifications. We have seen in Section 4.2 that there are two choices of Ω giving either A- or B-type boundary conditions. In the generic case, i.e. if the levels k_j of the minimal models are pairwise different, the only way to maintain the tensor product symmetry in the presence of a boundary state is to require that Ω have the same action on every factor of the tensor product. In special cases, when $k_{j_1} = k_{j_2}$, there are permutation automorphisms of the tensor product algebra with which one can glue the left-moving generators of the subtheory j_1 to the right-moving generators of subtheory j_2. These will, however, not be considered.

The internal part of these boundary states is [166], [175]

$$|\ \alpha\ \rangle\rangle_\Omega = \frac{1}{\kappa^\Omega_\alpha} \sum_{\lambda+1\in \mathrm{Exp}(G),\mu} \delta_\beta \delta_\Omega B^{\lambda,\mu}_\alpha |\ \lambda,\mu\ \rangle\rangle_\Omega \tag{4.27a}$$

where

$$B^{\lambda,\mu}_\alpha = \prod_{j=1}^r \frac{1}{\sqrt{\sqrt{2}(k_j+2)}} \frac{\psi^{l_j}_{L_j}}{\sqrt{\sin(l_j,0)_{k_j}}} e^{i\pi\frac{m_j M_j}{k_j+2}} e^{-i\pi\frac{s_j S_j}{2}} \tag{4.27b}$$

δ_Ω denotes the constraint that the Ishibashi state $|\ \lambda,\mu\ \rangle\rangle_\Omega$ must appear in the closed string partition function. For A-type boundary conditions this is no constraint as the Ishibashi states are already built on diagonal primary states and δ_β already enforces that the total $U(1)$ charge is integral. However, the B-type Ishibashi states have opposite $U(1)$ charge in the holomorphic and the anti-holomorphic sector, and these only appear as a consequence of the GSO projection; so the δ_β constraint requires that all the m_j are the same modulo $k_j + 2$. The normalization κ_α^Ω is determined later.

It follows from (4.27) that the action of the $\mathbb{Z}_{k_j+2}$ and $\mathbb{Z}_2$ symmetries in (2.77) and (2.78) is $M_j \to M_j + 2$ and $S_j \to S_j + 2$, respectively. As a result of the δ_β constraint, two physically inequivalent choices for S_j are $S = \sum_j S_j = 0, 2 \mod 4$. The $S_j =$ odd case seems to be inconsistent because their RR charges do not fit into a charge lattice together with the $S =$ even states [5]; thus they will violate the charge quantization conditions (5.4) and will not be considered further. In the end, due to the $\mathbb{Z}_2$ symmetry, it is sufficient to consider only boundary states with $S = 0$. A boundary state in the Gepner model can be written as

$$\begin{aligned} | \, \alpha \, \rangle\!\rangle_\Omega &= g_1^{\frac{M_1}{2}} \dots g_r^{\frac{M_r}{2}} h^{\frac{S}{2}} | \, L_1, \dots, L_r; M_1, \dots, M_r; S \, \rangle\!\rangle_\Omega \\ &= g_1^{\frac{M_1 - L_1}{2}} \dots g_r^{\frac{M_r - L_r}{2}} h^{\frac{S}{2}} | \, L_1, \dots, L_r; M_1' = L_1, \dots, M_r' = L_r; S' = 0 \, \rangle\!\rangle_\Omega \end{aligned} \tag{4.28}$$

By the symmetry considerations above, A-type boundary states form representations of the Gepner model group G in (2.89). For B-type boundary states, the δ_β constraint implies in addition that the physically inequivalent choices of M_j can be described by the the single label

$$M = \sum_{j=1}^{r} \frac{K' M_j}{k_j + 2} = \sum_{j=1}^{r} w_j M_j \tag{4.29}$$

where $K' = \mathrm{lcm}(k_j + 2)$ and $w_j = K'/(k_j + 2)$ is the weight of the jth minimal model, cf. (3.12). Hence the B-type boundary states form representations of the quantum symmetry group $\mathbb{Z}_{K'}$ introduced in Section 2.6 and are singlets under G.

Note that for a chiral primary field in a Gepner model $\mu = (0; \lambda; 0, 0, 0, 0, 0)$. Hence the corresponding coefficients in (4.27b) can be identified with (4.21) and (4.22) for A-type and B-type boundary states, respectively. The remaining boundary state coefficients in the Gepner model have no direct geometric interpretation yet.

These boundary states do not include the contribution from the twisted sectors in the corresponding Landau-Ginzburg orbifold theory described in Section 2.2. D-branes at orbifold singularities have been studied in [176] while boundary states at orbifold singularities were discussed in [177] where it was argued that after blowing up the singularity they correspond to D-branes wrapping the exceptional divisor. Boundary states corresponding to branes away from the orbifold fixed points are obtained by summing over the brane's pre-images in the covering space. At the fixed points, however, the expressions for boundary states can involve contributions from the twisted sectors of the theory, leading to a charge under RR potentials coming from these sectors. In [178] it was argued that the boundary states (4.27) described in [166] do not carry charge in the twisted RR sector, in other words they are not elementary. Furthermore, additional, elementary boundary states for the Gepner model which are charged under the twisted sector were given, see Section 4.3.3.

4.3.1. Witten index in Gepner models

To explore the charge lattice of the boundary states, and to find the geometric interpretation of given boundary states, we compute the interaction $I_{\alpha\tilde{\alpha}}$ of two D-brane configurations $| \, \alpha \, \rangle\!\rangle_\Omega$ and $| \, \tilde{\alpha} \, \rangle\!\rangle_\Omega$. We will argue in Section 6.1 that the corresponding conformal field theory quantity is $I^\Omega_{\alpha\tilde{\alpha}} = \mathrm{tr}^\Omega_{\alpha\tilde{\alpha},R}(-1)^F$, i.e. the Witten index in the open string sector [179]. This will be interpreted as an intersection form on the charge lattice of the boundary states. It can be computed by starting in the closed string sector and performing a modular transformation to the open string sector. In the closed string sector this trace corresponds to the amplitude between the RR parts of the boundary states with a $(-1)^{F_L}$ on the world-sheet inserted. For A-type boundary states one obtains [5]

$$I^A_{\alpha\tilde{\alpha}} = \frac{1}{C^A}(-1)^{\frac{S-\tilde{S}}{2}} \sum_{\nu_0=0}^{K-1} (-1)^{\left(\frac{d}{2}+r\right)\nu_0} \prod_{j=1}^{r} n_{L_j,\tilde{L}_j}^{2\nu_0+M_j-\widetilde{M}_j} \tag{4.30}$$

and for the B-type boundary states with $\frac{d}{2}+r$ even

$$I^B_{\alpha\tilde{\alpha}} = \frac{1}{C^B}(-1)^{\frac{S-\tilde{S}}{2}} \sum_{m'_j} \delta^{(K')}_{\frac{M-\widetilde{M}}{2}+\sum_{j=1}^{r}\frac{K'}{2k_j+4}(m'_j+1)} \prod_{j=1}^{r} n_{L_j,\tilde{L}_j}^{m'_j-1} \tag{4.31a}$$

while for $\frac{d}{2}+r$ odd

$$I^B_{\alpha\tilde{\alpha}} = \frac{1}{C^B}(-1)^{\frac{S-\tilde{S}}{2}} \sum_{m'_j} \frac{1}{2}(-1)^{\frac{M-\widetilde{M}}{K'}+\sum_j \frac{m'_j+1}{k'_j+2}} \delta^{(K'/2)}_{\frac{M-\widetilde{M}}{2}+\sum_{j=1}^{r}\frac{K'}{2k_j+4}(m'_j+1)} \prod_{j=1}^{r} n_{L_j,\tilde{L}_j}^{m'_j-1} \tag{4.31b}$$

where

$$\delta^{(n)}_x = \begin{cases} 1 & x = 0 \mod n \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

We choose the normalization $C^A = \kappa^A_\alpha \kappa^A_{\tilde{\alpha}} K$ and $C^B = \kappa^B_\alpha \kappa^B_{\tilde{\alpha}} \prod_{j=1}^{r} \frac{k_j+2}{K}$ in order to satisfy Cardy's condition. The formulas (4.31a) and (4.31b) are valid for $d = 2 \mod 4$. For $d = 4$ they have to be exchanged. The intersection matrix depends only on the differences $M - \widetilde{M}$ which agrees with the discrete symmetry (2.89). We also see that the $\mathbb{Z}_2$ action $S \to S + 2$ changes the orientation of one of the branes. Recall from Section 2.4 the fact that the Ramond ground states are given by $\phi^l_{l+1,1}$ which are identified with $\phi^{k-l}_{-k+l-1,-1}$. Only these states contribute to the Witten index. In deriving this result [5] one then crucially needs the periodic continuation of the annulus coefficients in (4.25).

From these intersection forms we will be able to extract the charges and the open string spectrum for a given brane in Chapter 6. The intersection form can be represented by a matrix I acting on the space of boundary states. Since it commutes with the symmetry group G of the Gepner model, it can be written as a polynomial in the generators g_j of G. For the remainder of this section we restrict ourselves to the A-type modular invariants. In this case, the ψ^l_L in (4.27b) are the modular S-matrix elements S^l_L and from (4.13), (4.3) it follows that $n_{L\widetilde{L}}{}^l = N_{L\widetilde{L}}{}^l$ are the $SU(2)$ fusion coefficients. They are $N_{L\widetilde{L}}{}^l = 1$ for $|L-\widetilde{L}| \leq l \leq \min\{L+\widetilde{L}, 2k-L-\widetilde{L}\}$ with $l+L+\widetilde{L} \in 2\mathbb{Z}$ and $N_{L\widetilde{L}}{}^l = 0$ otherwise. Using their properties, (4.30) and (4.31) can be simplified as follows. In these equations the labels M_j, $\widetilde{M}_j$ can be thought of as indices of a matrix acting on the states. Let us first consider the case of A-type boundary states. Using the action of the $\mathbb{Z}_{k_j}$ symmetry and (4.25) the sum over ν_0 in (4.30) for $L = \tilde{L} = 0$ boundary states can be written as

$$\sum_{\nu_0=0}^{K-1} N_{0,0}^{2\nu_0+M_j-\widetilde{M}_j} = \sum_{\nu_0=0}^{K-1} g^{2\nu_0+M_j-\widetilde{M}_j} = (1-g_j^{-1}) \tag{4.33}$$

so that we can effectively replace

$$\sum_{\nu_0=0}^{K-1} N_{0,0}^{2\nu_0+M_j-\widetilde{M}_j} \to n_{0,0} \equiv (1-g_j^{-1}) \tag{4.34}$$

In the last step in (4.33) we have used the periodicity of the M_j labels and the fact that we sum over the full orbit. The difference $M_j - \widetilde{M}_j$ just indicates the starting point of the summation on the orbit, but due to the symmetry it does not matter where we actually start, hence the dependence on this difference drops out[1].

For the B-type boundary states we have noted in (4.29) that all the m_j have to be identified and hence the M_j reduce to a single label M such that for fixed L the states with different (M, S) form an orbit under the $\mathbb{Z}_{K'}$ quantum symmetry. We will denote the single generator of this symmetry by g. Accordingly, the B-type boundary states are $\mid L; M; S \rangle\rangle_B \equiv \mid L_1, \ldots, L_r; M; S \rangle\rangle_B$ where $0 \leq L_j \leq \lfloor k_j/2 \rfloor$, $0 \leq M \leq K' - 1$ and $S = 0, 2$. The restriction on the L_j is due to the field identification (2.70). Since the two values of S correspond to a brane and its anti-brane, we restrict ourselves to the states with $S = 0$. We denote the set of states obtained from a given state $\mid L_1, \ldots, L_r; 0; 0 \rangle\rangle_B$ by applying g to it as its L-orbit

$$\mid L_1, \ldots, L_r \rangle\rangle_B \equiv \left\{ g^M \mid L_1, \ldots, L_r; 0; 0 \rangle\rangle_B \,\middle|\, M = 0, \ldots, K' - 1 \right\} \tag{4.35}$$

Again, due to the symmetry, the expression in (4.31) can be shortened by noting that the delta function constraint in (4.31) is a shifted $U(1)$ projection we can build a $\mathbb{Z}_{K'}$ invariant polynomial in g such that each factor $N^{m'_j - 1}_{L_j, \tilde{L}_j}$ in (4.31) can be replaced by

$$n_{L,\tilde{L}} = g^{\frac{|L-\tilde{L}|}{2}} + g^{\frac{|L-\tilde{L}|}{2}+1} + \cdots + g^{\frac{|L+\tilde{L}|}{2}} - g^{-1-\frac{|L-\tilde{L}|}{2}} - \cdots - g^{-1-\frac{|L+\tilde{L}|}{2}} \tag{4.36}$$

In particular, for $L = \tilde{L} = 0$ we find

$$N^{m'_j - 1}_{0,0} \rightarrow n_{0,0} = (1 - g_j^{-1}) \tag{4.37}$$

where in this case $g_j = g^{w_j}$ is the generator of the $\mathbb{Z}_{k_j+2}$ subgroup of $\mathbb{Z}_{K'}$. For both types of boundary states there is a linear transformation t_{L_j} which generates the different factors for $L_j \neq 0$ from $n_{0,0}$. In the case of B-type boundary states there is a particularly nice way to represent t_{L_j} [180]

$$t_{L_j} = t^T_{L_j} = \sum_{l=-\frac{L_j}{2}}^{\frac{L_j}{2}} g_j^l \tag{4.38}$$

and therefore

$$n_{L_j,\tilde{L}_j} = t_{L_j} n_{0,0} t_{\tilde{L}_j} \tag{4.39}$$

Hence starting from the boundary state $\mid 0; M; 0 \rangle\rangle_B = \mid 0, 0, 0, 0, 0; M; 0 \rangle\rangle_B$ we can obtain all the other boundary states by

$$\mid L; M; 0 \rangle\rangle_B = \prod_{j=1}^{r} t_{L_j} \mid 0; M; 0 \rangle\rangle_B \tag{4.40}$$

The intersection form for the B-type boundary states then becomes

$$I^B(g) = \prod_{j=1}^{r} n_{L_j,\tilde{L}_j} \tag{4.41}$$

Note that in particular for the $\sum L_j = 0$ states we have

$$I^B_{0,0}(g) = \prod_{j=1}^{r} (1 - g^{-w_j}) \tag{4.42}$$

[1]Note that this $n_{L,\tilde{L}}$ having two indices is not related to the matrix n_L having one index and its entries having three indices

As we have seen, this representation of the intersection form I^B emerges naturally from the boundary conformal field theory through the extension of the annulus coefficients (4.25), but it is highly redundant. There are many non-trivial relations between the boundary states $\mid L; M; S \rangle\rangle_B$ and this redundancy is encoded in I^B.

4.3.2. Number of moduli of boundary states in Gepner models

We are interested in counting the number of moduli for a D-brane state; these will be the massless bosonic i.e. NS open string states. To find their contribution to the open string partition function, it is sufficient to examine the NS-NS part of a transition amplitude in the internal part of space-time. The reason is that the open string NS characters arising from the modular transformations of the RR part of the transition amplitude come with an insertion of $(-1)^F$ [181], [182]. Therefore the transition amplitude between two A-type boundary states is [5]

$$Z^A_{\alpha\tilde{\alpha}} = \frac{1}{C^A} \sum_{\lambda',\mu'}^{\mathrm{NS}} \sum_{\nu_0=0}^{K-1} \prod_{j=1}^{r} n^{l'_j}_{L_j,\tilde{L}_j} \delta^{(2k_j+4)}_{2\nu_0+M_j-\widetilde{M}_j+m'_j} \chi^{\lambda'}_{\mu'}(q) \tag{4.43}$$

and between two B-type boundary states

$$Z^B_{\alpha\tilde{\alpha}} = \frac{1}{C^B} \sum_{\lambda',\mu'}^{\mathrm{NS}} \delta^{(K')}_{\frac{M-\widetilde{M}}{2}+\sum_{j=1}^{r}\frac{K'}{2k_j+4}m'_j} \prod_{j=1}^{r} n^{l'_j}_{L_j,\tilde{L}_j} \chi^{\lambda'}_{\mu'}(q) \tag{4.44}$$

Here, $C^A = \frac{\kappa^A_\alpha \kappa^A_{\tilde{\alpha}} K}{2^{\frac{r}{2}} \prod_{j=1}^{r} k_j+2}$ and $C^B = \frac{\kappa^B_\alpha \kappa^B_{\tilde{\alpha}}}{2^{\frac{r}{2}}}$ where κ^A_α and κ^B_α are again chosen such that Cardy's condition is satisfied. We see that the massless open string spectrum can also be expressed in terms of the annulus coefficients $n^{l'_j}_{L_j,\tilde{L}_j}$.

In the closed string case we have two important conditions that guarantee supersymmetry, namely $\beta_0 \bullet \mu' \in 2\mathbb{Z}+1$ and $\beta_j \bullet \mu' \in \mathbb{Z}$, see Section 2.4, in particular (2.86). By (4.43) and (4.44) this leads to additional conditions on the open string labels α. Assuming that α and $\widetilde{\alpha}$ have the same external part, then two D-brane boundary states $\mid \alpha \rangle\rangle_\Omega$ and $\mid \widetilde{\alpha} \rangle\rangle_\Omega$ preserve the same supersymmetries if [166]

$$Q(\alpha, \widetilde{\alpha}) \equiv -\frac{S-\widetilde{S}}{2} + \sum_{j=1}^{r} \frac{M_j - \widetilde{M}_j}{k_j+2} \in 2\mathbb{Z} \tag{4.45}$$

This condition ensures that there is no tachyon in the open string spectrum such that a single such brane is stable and supersymmetric.

If the two boundary states are the same, there are ν vacuum operators and one spectral flow operator in the open string channel. ν accounts for the fact that in the case that k_j is even the states with $L_j = k_j/2$ appear twice due to the field identification (2.70). If l is the number of L_j which equal $k_j/2$ then ν is determined by

$$\nu = 2^{\tilde{l}}, \qquad \tilde{l} = \begin{cases} l & n+r \text{ odd} \\ l-1 & n+r \text{ even}, \, l>0 \\ 0 & n+r \text{ even}, \, l=0 \end{cases} \tag{4.46}$$

If they are not the same, neither state propagates. If $\nu = 1$ then the unbroken world-volume gauge group is U(1) corresponding to the center-of-mass degree of freedom and the brane can be viewed as single object. If the number of vacua ν is different from one, the boundary state can be thought of as two different D-branes sitting at one point. This would fit with picture of a Coulomb branch in the world-volume theory in which the gauge group is $U(1)^\nu$. Finally, we come to the number of moduli for a D-brane state. The supersymmetry preserving moduli of the D-branes are constructed from chiral vertex operators [5]. The Witten index counts these operators, but with a sign depending on their chirality. We have to remove this sign by hand, and thus the total number of chiral fields is calculated using (4.30) and (4.31) with the fusion matrices replaced by their absolute values. In other words, in contrast to (4.25) we define $|n|_{g+i} = |n_{g-i}|$. We can again write this modified matrix as a polynomial $P^{\Omega}(g_j)$ in the generators g_j. For the remainder of this section we again restrict ourselves to the case with A-type modular invariant. For the B-type boundary states this polynomial then is

$$P^B(g) = \prod_{j=1}^{r} |n_{L_j \tilde{L}_j}| \tag{4.47}$$

where $|n_{L_j L'_j}|$ are the annulus coefficients in (4.39) written out as a polynomial in g and then all minus signs replaced by plus signs. For A-type boundary states one changes the sign in (4.37) and the right hand side of (4.30) yields then the corresponding polynomial $P^A(g_1, ..., g_r)$.

Next, we have to figure out which of the chiral fields are marginal and can be used as a deformation and where they appear in (4.47). If space-time supersymmetry is preserved, the chiral fields have integer $U(1)$ charges. Besides the charge 1 chiral fields one has to take into account charge 2 chiral fields in $Z^{\Omega}_{\alpha\tilde{\alpha}}$ that are related to charge -1 antichiral fields in $Z^{\Omega}_{\alpha\tilde{\alpha}}$ by spectral flow; the latter are the hermitian conjugate of charge 1 fields in $Z^{\Omega}_{\tilde{\alpha}\alpha}$. One can show that therefore $\sum_k m_k$ in the open string channel will be a multiple of K' for marginal, chiral vertex operators. The number of massless chiral superfields is then given [5] by the constant term in

$$m^{(\mathrm{CFT})} = \frac{1}{2} P^B(g) - \nu \tag{4.48}$$

Let us briefly look at the special case of the $\sum L_j = 0$ states. Replacing the plus signs in (4.42) by minus signs there will be exactly one term with $g^{-K'} = 1$ since $\sum w_j = K'$. Together with the constant term 1 and (4.46) we see that $m^{(\mathrm{CFT})} = 0$ for this L-orbit. For a reason to be explained in Section 5.5.3 such boundary states might be called rigid or exceptional states. It has been argued in [152] that these states correspond to the fractional D-brane boundary states in $\mathbb{C}^5/\Gamma$ where $\Gamma \cong \mathbb{Z}_{K'}$ is the discrete subgroup of $SU(5)$ acting as

$$z_i \to e^{\frac{2\pi i w_i}{K'}} z_i \qquad i = 1, \ldots, 5 \tag{4.49}$$

This fact is of central importance for the computation in Appendix B.

4.3.3. Twisted boundary states

The boundary states with at least one label satisfying $L_j = \frac{k_j}{2}$ can be understood in a more precise manner [178], [183] and contain additional information about the D-brane configuration [184]. Since we will make extensive use of this information in Section 6.3, we discuss these results in detail in this section.

The important observation is that $L_j = \frac{k_j}{2}$ is the fixed point of a certain simple current that generically appear in the boundary conformal field theory and not in the bulk conformal field theory of a minimal model and this fixed point must be resolved [178], [183] in order to have a complete description of the boundary conformal field theory. Using the notation of Section 4.1 consider a non-trivial class of irreducible representations $\mathcal{V}_g$, $g \in \mathcal{I}$ such that the fusion product (4.2)

$$\mathcal{V}_g \star \mathcal{V}_j = \mathcal{V}_{g \cdot j} \tag{4.50}$$

gives a single class $\mathcal{V}_{g \cdot j}, g \cdot j \in \mathcal{I}$. Such classes are called simple currents [185], [186] and the set $\mathcal{C}$ of all these simple currents forms an abelian subgroup of $\mathcal{C} \subset \mathcal{I}$. Let Γ be a subgroup of $\mathcal{C}$. Due to (4.2) Γ

acts on the index set $\mathcal{I}$ and splits it into orbits. The length of the orbit of the identity $1 \in \mathcal{I}$ is given by the order of $|\Gamma|$ of the group Γ. Other orbits may be shorter since there can be fixed points, i.e. labels $j \in \mathcal{I}$ for which

$$g \cdot j = j \qquad \text{for some } g \in \Gamma \tag{4.51}$$

The subgroup of all simple currents leaving some $j \in \mathcal{I}$ fixed is called the stabilizer of j

$$\mathcal{S}_j = \{g \in \Gamma | g \cdot j = j\} \tag{4.52}$$

Given a commutator 2-cocycle describing an element of $H^2(\mathcal{S}_j, U(1))$, i.e. a pairing $\epsilon : \mathcal{S}_j \times \mathcal{S}_j \to \mathbb{C}^*$ compatible with the group law and equal to one on the diagonal, one can define the untwisted stabilizer [187]

$$\mathcal{U}_j = \{h \in \mathcal{S}_j | \epsilon(g, h) = 1 \;\forall g \in \mathcal{S}_j\} \tag{4.53}$$

The quantity $\epsilon(g, h)$ is also known as discrete torsion [188]. The most important simple currents in a minimal model are $v = (0, 0, 2)$ (the world-sheet supercurrent), $s = (0, 1, 1)$ (the spectral flow operator), $p = (0, 2, 0)$ (giving the phase symmetries in the Greene-Plesser construction in Section 2.6) and $f = (k, 0, 0)$. Note that the dimensions of these simple currents are generically non-integer and therefore can only appear on the boundary. The last one is the only one with potential fixed points, namely due to (2.70) it is precisely the one mentioned at the beginning of this section. The order of the stabilizer $\mathcal{S}_f$ is exactly ν as in (4.46) [184]. In can be shown in general [189] that the number of L-orbits of independent boundary states associated to a given L is not given by the order ν of the stabilizer $\mathcal{S}_f$ but rather by the order $\widetilde{\nu}$ of the untwisted stabilizer $\mathcal{U}_f$, which differs from ν multiplicatively by a square number

$$\nu = N^2 \widetilde{\nu} \tag{4.54}$$

where $N = 2^{[\frac{\tilde{l}}{2}]}$. This equation means that a fixed point boundary state can be resolved into $\widetilde{\nu}$ independent components that are not further decomposable. A similar relation, $|\Gamma| = \sum_{i=1}^{N_R} (d_{R_i})^2$, was derived for orbifolds with discrete torsion $\mathbb{C}^3/\Gamma$ in [190], [179], [177] (see also [191]). d_{R_i} is the dimension of the irreducible projective representation R_i of Γ. The quantities corresponding to Γ, d_{R_i} and N_R are in the minimal model $\mathcal{S}_f$, N and $\widetilde{\nu}$, respectively. If the discrete torsion $\epsilon(g, h)$ is non-trivial, i.e. $\frac{\nu}{\widetilde{\nu}} = N^2 > 1$, we can only have a projective realization of $\mathcal{S}_f$. We will describe the large volume interpretation of this result in Section 6.3.

The method of simple currents allows to construct new boundary states by considering the combination of currents v_j, s_j and p_j, $j = 1, \ldots, r$ which form the vectors $\beta_0, \ldots, \beta_r$ defined in Section 2.4. Gepner used them to implement the GSO projection on the tensor product of the minimal models. They generate the orbifold group Γ. When analyzing the RR charges of the A-type boundary states (4.27) of Recknagel and Schomerus one notes that they are charged under the untwisted (c, c) fields only [178].

It is possible to take into account some states which are charged under the “twisted” fields that have to be added in order to preserve modular invariance. In addition they have to be in short orbits of the orbifold group Γ. These short orbits appear precisely when k_j is even and $L_j = \frac{k_j}{2}$, hence they are again related to the fixed point of the simple current f. Recall that the quantum symmetry group $\mathbb{Z}_{K'} \subset \Gamma$ acts on the chiral primary fields by multiplication of a phase factor (2.119). Suppose there is a subgroup $\mathbb{Z}_N \subset \mathbb{Z}_{K'}$ for which the $w_j = \frac{K'}{k_j+2}$, $j \in S \subset \{1, \ldots, r\}$ have a non-trivial common factor N. By generalizing the construction of twisted boundary states in flat space [177] it is possible to construct new A-type boundary states arising from the resolution of the fixed point which are linear combinations of states in the untwisted sector and the $\frac{K'}{N}$-twisted sector [178]. If one restricts to the situation that the left- and right-moving charges be the same in all individual models then one can show that $N = 2$ is the only possibility. Other values for N require a more general gluing condition on the factor theories. These boundary states have geometrical interpretation which will be discussed in Section 6.2. A thorough and complete analysis of the A-type boundary states in Gepner models with the simple gluing condition has been given in [183]. In terms of this analysis, the twisted boundary states are called elementary, as opposed to the original unresolved boundary states.

5. D-branes and their Geometry

5.1. General facts on D-branes

D-branes are defined to be objects on which open strings can end. While we have adopted the conformal field theory point of view for their description in the previous chapter, we will study their geometry in the present chapter. D-branes can absorb the momentum of then open strings attached on them and are therefore dynamical objects. Besides this their most important properties are that they are charged under the RR-fields $C^{(i)}$ and that they are BPS saturated states. Furthermore, they are intrinsically non-perturbative objects in closed string theory as their mass goes like $e^{-\frac{1}{g_s}}$. For a detailed account of their properties see [192] and [193].

Since we are interested in D-branes on Calabi–Yau spaces, we assume that space-time M to be of the form $M = X \times \mathbb{R}^{3,1}$ where X is a Calabi–Yau threefold. A configuration of r coincident D-branes with $p+1$ dimensional world-volume $W = \Sigma \times \mathbb{R}$, where the factor $\mathbb{R}$ denotes the time coordinate, is specified by an embedding $f : W \to M$. The ten-dimensional gauge field from the open string sector A^M, $M = 0, \ldots, 9$ becomes in the presence of these D-branes a $U(r)$ gauge field A^μ, $\mu = 0, \ldots, p$ on W, with field strength $F = \mathrm{d}A + [A, A]$ and Higgs fields Φ_i, $i = 1, \ldots, 9-p$ which are $r \times r$ anti-hermitian matrices. The fields from the closed string sector, i.e. the metric g, the 2-form B, the dilaton ϕ and the RR q-forms $C^{(q)}$ are pulled back to W by f. q has to be odd in type IIA string theory and even in type IIB. The tangent bundle of M decomposes as $TM|_W = TW \oplus N_{M/W}$ with curvature tensors $f^* R_T$ and R_N, respectively. The D-brane action is

$$
\begin{aligned}
S \;\; = \;\; & -\tau_p \int_W \mathrm{d}\mu_g e^{-\phi} \mathrm{str} \sqrt{-\det\left(f^*\left(E + E(Q^{-1}-1)E\right) + 2\pi\alpha' F\right)\det(Q)} \\
& \left(1 - \frac{\pi^2}{768}\left(\left|R^{(4)}\right|_g^2 + 2\left|f^* R_T\right|_g^2 - 2\left|R_N\right|_g^2\right) + O(\alpha'^4)\right) \\
& + \mu_p \int_W \mathrm{str}\left(f^*\left(e^{2\pi\alpha' \mathrm{i}_\Phi \mathrm{i}_\Phi} \sum_j C^{(j)}\right) e^{2\pi\alpha' F}\right) \sqrt{\frac{\widehat{\mathrm{A}}(2\pi\alpha' R_T)}{\widehat{\mathrm{A}}(2\pi\alpha' R_N)}}
\end{aligned}
\tag{5.1}
$$

where $E = g + B$, $Q^i_j = \delta^i_j + i[\Phi_i, \Phi_k]E_{kj}$, $|R^{(4)}|_g^2$ is a certain combination of the Riemann tensor [194] and the sum is over odd j in type IIA and over even j in type IIB. str stands for the symmetrized trace over F, $D\Phi_i$, $[\Phi_i, \Phi_j]$ and Φ_i. i_Φ is the interior derivative with respect to the vector $\Phi = (\Phi_1, \ldots, \Phi_{9-p})$. The tension τ_p and the charge μ_p of a Dp-brane are $\tau_p = \mu_p g_s^{-1} = (2\pi)^{-p}\alpha'^{-\frac{p+1}{2}} g_s^{-1}$. In the following we will set $\alpha' = \frac{1}{2\pi}$. The kinetic term has been derived in [1], [195], [194] and [196]. The Wess-Zumino term is due to the fact that the gauge theories on the D-branes can be anomalous. This anomaly can be canceled by an inflow from the bulk theory [197] and a topological argument [198], [199] as well as T-duality [196] yield these Wess-Zumino couplings. The terms involving i_Φ induce couplings to RR fields of higher degree than $p+1$ and will not be considered in the following.

Restricting to D-branes living on the Calabi–Yau part of M, the charges of the unbroken $U(1)$ gauge symmetries are naturally associated with a vector $Q \in H^*(X, \mathbb{Z})$ where $*$ is even or odd depending on the type of string theory and the number of directions in flat space of the D-brane. The reason is that the $U(1)$ gauge fields are obtained by Kaluza-Klein reduction of RR $(p+1)$-form fields $C^{(p+1)}$ and for each homology p-cycle $\Sigma_i \subset X$ we may define a $U(1)$ gauge field $A_i = \int_{\Sigma_i} C^{(p+1)}$. The charge lattice should have a basis dual to the basis of gauge fields and will therefore correspond to the cohomology lattice.

For a single D-brane the second term in (5.1) reduces in this case to

$$S_{WZ} = \mu_p \int_\Sigma \left(\sum_i C^{(i)} e^{F+B} \right) \sqrt{\widehat{\mathrm{A}}(X)} \tag{5.2}$$

from which we can read off the RR charge of a D-brane to be [200]

$$Q(E) = \int_\Sigma \mathrm{ch}(E) \sqrt{\widehat{\mathrm{A}}(X)} = \int_\Sigma \mathrm{ch}(E) \sqrt{\mathrm{td}(X)} \tag{5.3}$$

where the second equality holds for Calabi–Yau spaces using the fact that $\widehat{\mathrm{A}}(X) = e^{\frac{1}{2}\mathrm{c}_1(X)}\,\mathrm{td}(X)$. The expression for $\mathrm{td}(X)$ has been given in (3.30). E is the K-theory class representing the (twisted) gauge bundle with connection A, see Section 5.3.3. (unless stated differently, we will set $B = 0$.) For reasons to become clear in Section 5.3.1 $Q(E)$ is also called the (generalized) Mukai vector. The RR charges of D-branes satisfy a generalization of Dirac's quantization condition

$$Q_{6-p} Q_p = \frac{2\pi}{2\kappa_0^2} \tag{5.4}$$

where $2\kappa_0^2 = 16\pi G_N g_s^{-2}$ and G_N is Newton's constant in ten dimensions.

5.2. Supersymmetric cycles

In this section we will discuss the condition for having supersymmetric cycles. A supersymmetric cycle W is defined by the condition that a world-volume theory on W is supersymmetric [201], [139]. The $(p+1)$-cycle is supersymmetric if the global supersymmetry transformation can be undone by a κ-transformation which implies that $(1-\Gamma)\eta^i = 0$ for the constant spinors η^i on M corresponding to the supersymmetry generators. Γ is a certain combination of $\mathcal{F}$ and the ten-dimensional Γ-matrices [202]. Those η^i which are solutions form the unbroken generators. For D-branes on a Calabi–Yau threefold X whose part of the world-volume inside X is denoted by Σ, there are two types of solutions which will be discussed in turn.

A-type D-branes

An A-type D-brane wraps a three-dimensional special Lagrangian submanifold Σ [203], [139] given by

$$\omega|_\Sigma = 0 \tag{5.5a}$$

$$\mathrm{Re}\, e^{i\theta}\Omega|_\Sigma = 0 \tag{5.5b}$$

$$F = 0 \tag{5.5c}$$

where Ω is the holomorphic $(3,0)$-form, and θ an arbitrary phase. This is the same phase as in the boundary state definition of an A-type D-brane (4.16) and determines which of the original $\mathcal{N} = 2$ supersymmetries is broken. Two branes of different θ together break all supersymmetry. Equivalently to (5.5b) we can require that Ω pulls back to a constant multiple of the volume element on Σ. A nice introduction to the theory of special Lagrangian submanifolds is [204] and [205]. ω^{ij} can be used to get an isomorphism between $T^*\Sigma$ and $N_{X/\Sigma}$ which is the space of deformations of the special Lagrangian submanifold and has real dimension $b^1(\Sigma)$ [206]. The space of flat $U(1)$ connections also has real dimension $b^1(\Sigma)$, thus the deformations of Σ pair up with the Wilson lines to form $b^1(\Sigma)$

complex moduli [207]. There are not many examples of special Lagrangian submanifolds in a Calabi–Yau manifold known. The only general construction known is as the fixed point of an involution, i.e. by taking a real section

$$\operatorname{Im} e^{\frac{2\pi i m_i}{d}} z_i = 0 \tag{5.6}$$

where d is the degree of the Calabi–Yau hypersurface and m_i are integers.

B-type D-branes

B-type D-branes wrap holomorphic cycles and are solutions to the generalized Hitchin equations [208], [3]

$$\begin{aligned} F &\in \Omega^{1,1}(X) && (5.7a)\\ \omega^2 \wedge F &= \lambda \omega^3 \text{ id} && (5.7b)\\ D_\mu \Phi_i &= 0 && (5.7c)\\ [\Phi_i, \Phi_j] &= 0 && (5.7d) \end{aligned}$$

where ω is the complexified Kähler form and $\lambda = \frac{2\pi i \deg(E)}{\mathrm{Vol(X)}}$. In the case of D-branes wrapping the entire Calabi–Yau manifold there are no scalar fields and the system reduces to (5.7a) and (5.7b) which are known as the Hermitian Yang-Mills equations [115] (see also [209]). Connections d_A on a C^∞ bundle E (with a fixed Hermitian structure) that satisfy (5.7a) are in one-to-one correspondence with holomorphic structures on E [210]. Since this holomorphic connection also has to satisfy (5.7b), the corresponding holomorphic vector bundle E will be μ-semi-stable, see Section 5.3.2. In the case that the D-brane does not wrap the entire Calabi–Yau manifold, the Φ_i are sections of the normal bundle with values in $\mathcal{E}nd\, E$ and represent the normal motions of the D-brane in X. Moreover, from (5.7c) we see that if Φ_i is non-diagonal then the vector bundle on X must in general be reducible. In the case where X is a $K3$ surface and Σ is a Riemann surface embedded in X, (5.7) reduces [208] to the system of equations studied by Hitchin [211].

For a fixed RR charge vector Q we will define $\mathcal{M}'_D(Q)$ to be the moduli space of solutions of the system (5.7) modulo the gauge group $U(r)$. By equation (5.7d) $\mathcal{M}'_D(Q)$ has a natural projection to a configuration space of points $\pi : \mathcal{M}'_D(Q) \to \mathrm{Sym}^r(\mathbb{R}^3)$ given by the eigenvalues of the Φ_i, i. e. $\pi : (A_\mu, \Phi_i) \mapsto \{a_i^{(1)}, \dots, a_i^{(r)}\}$ where $\Phi_i \cong \mathrm{diag}(a_i^{(1)}, \dots, a_i^{(r)})$. These give the positions of the r wrapped branes. We will restrict ourselves to the case where all constituents of a D-brane configuration sit at the same point in the non-compact space $\mathbb{R}^3$. From (5.7a) and (5.7b) one can see that over the diagonal $\Delta^{(r)} \subset \mathrm{Sym}^r(\mathbb{R}^3)$ where all points coincide we have the moduli space of solutions to the Hermitian Yang-Mills equations. Hence, the moduli space of D-brane configurations that will be investigated is defined according to [3] as

$$\mathcal{M}_D(Q) = \pi^{-1}(p) \qquad p \in \Delta^{(r)} \tag{5.8}$$

Mirror symmetry

First a point of notation. Since we are interested in $\mathcal{N} = 1$ $D = 4$ supersymmetric gauge theories on the world-volume of a D-brane system it must be extended in the $3+1$ non-compact dimensions. In most of the discussions in this and the next chapter we will however use other realizations of this D-brane system, mostly as particles in $3+1$ dimensions. Hence, we will ignore its space-filling Minkowski dimensions and denote by p only the part in the Calabi–Yau manifold. Therefore, by (5.5a) A-type D-branes are D3-branes, and by (5.7a) B-type D-branes are either D0-, D2-, D4- or D6-branes.

In Section 4.2 we have argued that the A- and B-type boundary states are mirror to each other. Based on the assumption that type IIA string theory on X is really identical to type IIB string theory on X^* Strominger, Yau and Zaslow [212] gave the following interpretation of mirror symmetry in terms of T-duality. Start with a D0-brane in type IIA theory on X. Its moduli space is X itself. By the assumption there must be a D3-brane on X^* wrapping a special Lagrangian submanifold Σ whose moduli space is also X. Therefore, this D3-brane must have $b_1(\Sigma) = 3$ moduli. Furthermore, if we fix a point in the moduli space of the special Lagrangian cycle and only look at the Wilson lines, they give rise to a T^3 factor in the moduli space of the wrapped brane. Therefore, X should be a T^3 fibration $\pi : X \to B$. Repeating the argument with the roles of X and X^* switched yields that X^* must also be a T^3 fibration $\pi^* : X^* \to B^*$. Hence, they conjecture that both X and X^* are fibered by special Lagrangian three-tori, and in particular the mirror of the D0-brane on X is a D3-brane wrapping the fiber T^3 on X^*. Furthermore the D6-brane wrapping X is mapped to D3-brane on the base B^* and vice versa. This can be interpreted as T-duality: Performing a T-duality on the 3 circles of the T^3 turns IIB theory into the IIA theory and change the D3-brane on the T^3 into a D0-brane while the D3-brane wrapping the base becomes a D6-brane. The existence of torus fibrations has been discussed and proven in special cases in [213], [214], [215], [216]. Assuming that X and X^* are mirror T^3-fibrations it was argued in [217] that a real version of the Fourier-Mukai transform [218] carries conditions (5.5) into conditions (5.7). We will return to the argument given above in Section 6.5.1.

It is important to note that the conditions (5.5b) and (5.7b) are not believed to be the correct physical conditions except in the large volume limit. This is related to the fact mentioned in Section 2.2 that Ricci-flatness (2.43) for Calabi–Yau manifolds only holds in the large volume limit. All these equations will be corrected when moving away from this limit in the Kähler moduli space. In a first step, one can replace F by $F + \frac{1}{2\pi\alpha'}B$ in (5.5) and (5.7) and obtain a deformed version of these equations [219]. As mentioned above, we will work however with $B = 0$. In general, one must use the definition of a D-brane as a boundary condition in the conformal field theory of the world-sheet as in Chapter 4. It can be shown that A- and B-type D-branes as defined above appear in the large volume limit interpretation of the boundary non-linear σ-model and the boundary Landau-Ginzburg theory as solutions of the A- and B-type boundary conditions (4.18) and (4.19), respectively [220], [151]. The boundary conditions in the interpolating gauged linear σ-model have been analyzed in [221].

5.3. Vector bundles versus sheaves

5.3.1. Sheaves

The moduli space $\mathcal{M}_D(Q)$ as defined in (5.8) has to be compactified by adding boundary points (more precisely divisors with normal crossings). These correspond to certain singular vector bundles. There are several compactifications known in mathematics and we will discuss in this section the compactification which is chosen by string theory.

First, we include bundles in $\mathcal{M}_D(Q)$ bundles whose connections are reducible [3], [222]. From a qualitative point of view, the reducible connections are the connections for which the gauge field can be made block diagonal

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \tag{5.9}$$

This will happen when we can split the gauge bundle as $E = E' \oplus E''$. On the reducible locus the moduli space is approximately a product of smaller moduli spaces. We will see examples of D-brane configurations corresponding to vector bundles admitting reducible connections in Section 6.4.

The second kind of singular bundles are sheaves. Very roughly speaking, sheaves are vector bundles whose rank can vary over the base space. We will give here a physical approach to sheaves. For general mathematical definitions and statements see [111], [223], [224] and [225]. Particularly useful in the context of stability are [210] and [226]. Let us for the time being take our Calabi–Yau space X to be a

$K3$ surface. The RR charge vector (5.3) then coincides with the Mukai vector [149]

$$v(E) = \left(\mathrm{rk}(E), \mathrm{c}_1(E), \frac{1}{2}\,\mathrm{c}_1(E)^2 - \mathrm{c}_2(E) + \frac{\mathrm{c}_2(X)}{24}\,\mathrm{rk}(E)\right) \tag{5.10}$$

after integration. Consider now a single D4-brane wrapped on X. On flat space it would correspond to a flat $U(1)$ bundle. However, in this case the Mukai vector (5.10) is

$$v(E) = \left(1, 0, \frac{\mathrm{c}_2(X)}{24}\right) \tag{5.11}$$

and hence the D4-brane induces a D0-brane charge via the term $\frac{1}{24}\int_X \mathrm{c}_2(X)C^{(1)}$ from (5.2), where $C^{(1)}$ is the RR 1-form. We will take the convention that the D0-brane charge is $Q(D0) = (0, 0, -1)$. After integrating over X, the RR charge vector for single D4-brane is $(1, 0, 1)$ and induces the D0-brane charge -1. The D-brane moduli space can be viewed as the moduli space of vector bundles E on X as motivated in Section 5.2.

We will now argue that we need not only consider bundles but more generally semi-stable simple coherent sheaves. The definition of semi-stable is deferred to Section 5.3.2. Simple means that the sheaf has no non-trivial automorphisms which is the analog of an irreducible connection. Finally, coherent sheaves will be characterized below. When discussing properties of D-branes, we will simply speak of sheaves, thereby dropping the attributes semi-stable, simple and coherent if they are not necessary in the context. A notable difference between coherent sheaves and vector bundles is that while the dimension of a fiber of a vector bundle is constant as we move along the base X, the dimension of the fiber of a coherent sheaf is allowed to jump.

For illustration, consider a configuration of one D4-brane on X and n D0-branes at points in X. Its charge vector is $(1, 0, 1-n)$[1]. There is no vector bundle whose Mukai vector takes this form, since no line bundle can have non-zero second Chern number $1-n$. But there is indeed such a sheaf. It is the sheaf $\mathcal{I}_{p_1,\ldots,p_n}$ of holomorphic functions on X vanishing at n points $p_1, \ldots, p_n$[2]. This simple example indicates that the use of this generalized notion of a vector bundle enables us to describe the D-brane moduli spaces of various charges on the same footing, including those whose charge vector is not realized as the Mukai vector of a vector bundle. The sheaf $\mathcal{I}_{p_1,\ldots,p_n}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{I}_{p_1,\ldots,p_n} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{p_1,\ldots,p_n} \longrightarrow 0 \tag{5.12}$$

The three objects all have a natural physical interpretation. The first one, as we have just argued, is an ideal sheaf and corresponds to a bound state of a D4-brane with n D0-branes. The second one is the trivial bundle over X, or in the language of sheaves, the structure sheaf of X. The last one is a new object, called a skyscraper sheaf. It corresponds to n D0-branes and its fibers are supported at n points. This is an example of a sheaf whose rank is non-constant. In general, sheaves whose support is a proper subset of X are called torsion sheaves. We see that the language of sheaf theory places configurations with D4-branes on an equal footing with configurations without D4-branes.

A coherent sheaf is essentially any of the sheaves introduced above, i.e. vector bundles (which are also called locally free sheaves), ideal sheaves and torsion sheaves. For a precise definition of coherent sheaves as well as for their properties we refer again to [111] and [223]. One important property which characterizes coherent sheaves nicely is that for a coherent sheaf $\mathcal{F}$ there exists a complex

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{5.13}$$

called a projective resolution, where the $\mathcal{E}_i$ are locally free, i.e. vector bundles. This means that a coherent sheaf can always be described by a finite set of maps between vector bundles.

[1] Such a configuration exists by duality to the heterotic string on T^4.
[2] Such a "$U(1)$ instanton" on X can also be viewed as a non-commutative instanton on X [227].

A D0-brane looks like a zero size instanton on a D4-brane wrapping X [228], [229], [230]. While coherent sheaves are objects of algebraic geometry, instantons are objects of differential geometry. Small instantons are needed for the Donaldson-Uhlenbeck compactification of the instanton moduli space [231], while the coherent sheaves are needed for the Gieseker compactification of the moduli space of stable vector bundles, and on algebraic complex surfaces the two compactifications are related [232]. The relation between small instantons and coherent sheaves on a $K3$ surface can be made rather explicit [233]. For Kähler threefolds there is a natural analogue of the Donaldson-Uhlenbeck compactification [234] which involves ideal instanton singularities along holomorphic curves in the manifold, but also some more complicated codimension 3 singularities.

Everything we have said generalizes to Calabi–Yau threefolds. In particular, the exact sequence (5.12) now describes a configuration of D6- and D0-branes. We will discuss this case in more detail in Section 6.5.1. Another exact sequence of this type which is important for the discussion in this and the next chapter is

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{5.14}$$

Here D is a divisor in X, $\mathcal{O}_D$ is the structure sheaf on D, i.e. the trivial line bundle on D, but viewed from X it is a torsion sheaf with support on D. $\mathcal{O}_X(-D)$ is the ideal sheaf of holomorphic functions vanishing on the divisor D and is actually a line bundle. In the physical language they correspond to a D6-D4-brane bound state, a D6-brane and a D4-brane wrapping the supersymmetric cycle D. All lower-dimensional D-branes can be described in this way. The sequence (5.14) is an example of a projective resolution (5.13) with $\mathcal{F} = \mathcal{O}_D$.

The D4-branes will be of our main interest due to the fact that they wrap complex compact surfaces. The deformation theory of sheaves on those is reasonably well understood and can provide us the necessary information for studying the spectrum of these D4-branes. In addition, since they wrap divisors in the Calabi–Yau threefold they can be related to sheaves thereon via (5.14). This will be done in the following sections and in Chapter 6.

5.3.2. Stability

We have seen in Section 5.2 that as a consequence of the requirement that a D-brane configuration preserve supersymmetry, the sheaf that describes this configuration must be semi-stable. This was encoded in (5.7b). Roughly speaking, if this requirement is not satisfied, the configuration is unstable and will decay into stable, supersymmetric constituents. From this point of view it is very interesting to observe that these two totally different concepts of stability – mathematical and physical – agree. We are therefore led to investigate semi-stable sheaves which will be the content of this section.

We begin with the definition of semi-stable sheaves for which we first need to introduce some technicalities. We assume that all our sheaves are over a toric Calabi-Yau threefold X. For a coherent sheaf $\mathcal{F}$ we define its Chern character $\mathrm{ch}(\mathcal{F})$ by means of a projective resolution (5.13) as follows

$$\mathrm{ch}(\mathcal{F}) = \sum_{i=0}^{n} \mathrm{ch}(\mathcal{E}_i) \tag{5.15}$$

This definition is independent of the choice of the resolution. Furthermore we define the degree of $\mathcal{F}$ to be

$$\deg_\omega(\mathcal{F}) = \int_X \mathrm{c}_1(\mathcal{F}) \wedge \omega^2 \tag{5.16}$$

where ω is the (uncomplexified) Kähler form and we define the normalized degree or the slope of $\mathcal{F}$ to be

$$\mu_\omega(\mathcal{F}) = \frac{\deg_\omega(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \tag{5.17}$$

We will suppress the dependence on ω from now on. A coherent sheaf $\mathcal{E}$ is said to be μ-semi-stable, if for every coherent subsheaf $\mathcal{F}$ with $\mathrm{rk}(\mathcal{F}) > 0$ we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}) \tag{5.18}$$

If strict inequality holds for every subsheaf $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ then we say that $\mathcal{E}$ is μ-stable. If equality holds then we say that $\mathcal{E}$ is strictly μ-semi-stable. Since μ-stability is the only notion of stability we will use, we will drop the μ from now on. A holomorphic vector bundle E is said to be semi-stable (stable) if the sheaf of holomorphic sections $\mathcal{O}(E)$ is semi-stable (stable). Note that even if we are only interested in vector bundles we need to consider not only subbundles but also subsheaves. This is a further motivation why we need to introduce coherent sheaves.

Note that there are different notions of stability in mathematics, e.g. there is also Gieseker stability [235]. It is not yet clear which one is physically relevant, e.g. Gieseker stable objects appeared in [219] and [236]. It is conceivable that string theory needs both of them as limits of Π-stability [237].

Let us give a few examples and simple criteria for stability. A torsion-free coherent sheaf of rank 1 is always stable. If $\mathcal{F}$ is a torsion free coherent sheaf and $\mathcal{L}$ is a line bundle then $\mathcal{F} \otimes \mathcal{L}$ is semi-stable (stable) if and only if $\mathcal{F}$ is semi-stable (stable). $\mathcal{F}$ is semi-stable (stable) if and only if its dual $\mathcal{F}^\vee$ is semi-stable (stable). Furthermore, if

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}_1 \longrightarrow 0 \tag{5.19}$$

is a non-trivial extension with line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ of degree 0 and 1, respectively, then $\mathcal{F}$ is stable. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are torsion-free coherent sheaves then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is semi-stable if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both semi-stable with $\mu(\mathcal{F}_1) = \mu(\mathcal{F}_2)$. However, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonzero, then $\mathcal{F}_1 \oplus \mathcal{F}_2$ can never be stable.

One more notion that we will need is S-equivalence. Suppose that $\mathcal{E}$ is a semi-stable torsion-free sheaf with $\mu(\mathcal{E}) = \mu$. Then there is a filtration $\{0\} = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^k = \mathcal{E}$ such that $\mathcal{F}^i/\mathcal{F}^{i-1}$ is torsion-free and stable for every i and $\mu(\mathcal{F}^i/\mathcal{F}^{i-1}) = \mu$ for all i. Such a (generally non-canonical) filtration is called a Jordan-Hölder filtration of $\mathcal{E}$. The associated graded sheaf $\mathrm{gr}\,\mathcal{E} = \bigoplus_i \mathcal{F}^i/\mathcal{F}^{i-1}$ is independent of the choice of the filtration. Two sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ are S-equivalent if $\mathrm{gr}\,\mathcal{E}_1 = \mathrm{gr}\,\mathcal{E}_2$. This has the following meaning. Points on a moduli space of sheaves that are strictly semi-stable do not necessarily correspond to unique semi-stable sheaves but to S-equivalence classes of strictly semi-stable sheaves. What will be important for us is that each S-equivalence class contains a unique representative that is split, i.e. is a direct sum of stable sheaves [235]. Such a sheaf is also called polystable. The physical relevance of S-equivalence classes has been pointed out in different contexts in [238] and [239].

We have seen that the vector bundles we are interested in satisfy the Hermitian Yang-Mills equation (5.7b). The Donaldson-Uhlenbeck-Yau theorem [240], [241], [115] (see also [209]) now states that if the vector bundle $\mathcal{E}$ admits an irreducible Hermitian Yang-Mills connection then E is μ-stable. Moreover, if the connection is reducible, then $\mathcal{E}$ is strictly semi-stable and is split, i.e. $\mathcal{E} = \bigoplus_i \mathcal{E}_i$ where $\mathcal{E}_i$ admit irreducible Hermitian Yang-Mills connections and are therefore stable. Hence, the representative of the S-equivalence class that is relevant for the physics of D-branes is the split representative. This fact will be often used in Section 6.3.

The most important necessary criterion for stability is the Bogomolov inequality [235]. If $\mathcal{F}$ is a semi-stable torsion free coherent sheaf, then

$$\int_X \Delta(\mathcal{F}) \wedge J \geq 0 \tag{5.20}$$

where $\Delta(\mathcal{F}) \equiv 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1)\,\mathrm{c}_1(\mathcal{F})^2 = \mathrm{c}_2(\mathcal{E}nd\,\mathcal{F})$. On manifolds with $h^{1,1}(X) > 1$ this describes an explicit dependence on the Kähler class ω as described in [239]. Equality in (5.20) defines a boundary within the Kähler cone on which stability degenerates to semi-stability [242]. Physically, this means that the connection on the D-brane becomes reducible, and an enhanced gauge symmetry appears. Furthermore, as sheaves generally do not admit connections, this allows us to define the analog of reducible connections on vector bundles for sheaves and hence to consider those objects which represent both kinds of singularities on $\mathcal{M}_D$ discussed in Section 5.3.1. There is a beautiful relation between these sheaves leading to enhanced gauge symmetry and certain boundary states in the Gepner model [184]. This will be explained in Section 6.3.

5.3.3. The Grothendieck-Riemann-Roch Theorem

Holomorphic K-theory

It was argued in [243] that D-brane charges are actually described by topological K-theory. However, topological K-theory encodes only C^∞ bundles while we have seen in Section 5.2 that our bundles carry a holomorphic structure. Therefore we need a holomorphic version of topological K-theory which is the Grothendieck group [244]. The Grothendieck group $K_0(X)$ [245], [246] is defined to be the quotient of the free abelian group generated by all the coherent sheaves (up to isomorphisms) on X by the subgroup generated by the elements $\mathcal{F} - \mathcal{E} - \mathcal{G}$ for each short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{5.21}$$

of coherent sheaves on X. Note that the main differences between topological and holomorphic K-theory are the following. $K_0(X)$ contains less than $K(X)$ e.g. the non-holomorphic bundles are not in $K_0(X)$ but at the same time it distinguishes objects which in K-theory are the same, namely what we identify with zero are extensions instead of direct sums (topologically they are both the same).

Physically, it allows to interpret tachyon condensation between stable D-branes [247] and descent relations [248] (for the K-theoretic interpretation see also [249]) in terms of projective resolutions and to treat both at the same level [250]. The tachyon condensation can be seen as follows. If we have a brane configuration $\mathcal{E}$ and an anti-brane configuration $\mathcal{F}$ there are open strings with tachyonic modes between them. The low-energy effective field theory has solitonic solutions whose energy is localized around the core of the tachyon, i.e. around the locus where it vanishes. This locus is a source for RR fields of lower dimensions. If the tachyon condenses, i.e. if it reaches its minimal energy, the brane and the anti-brane annihilate and this locus is identified with a new D-brane of lower dimension, supported on the zeroes of the tachyon. The tachyon can be viewed as a section of the sheaf $\mathcal{E} \otimes \mathcal{F}^\vee$ [243].

Let us consider for example the system consisting of $\mathcal{E} = \mathcal{O}_X$ and $\mathcal{F} = \mathcal{O}_X(-D)$ described in (5.14). From this we get that $\mathcal{O}_D = \mathcal{O}_X - \mathcal{O}_X(-D)$ and so the tachyon is a section of $\mathcal{E} \otimes \mathcal{F}^\vee = \mathcal{O}_X(D)$. Now since $\mathcal{O}_X(D)$ are the holomorphic functions on X having a pole on D, the sections of this bundle have a simple zero at D. Hence D is the locus where the tachyon vanishes and, by the preceding discussion, describes a new brane of real codimension 2. This coincides exactly with the interpretation of $\mathcal{O}_D$ as a D4-brane wrapping the cycle D. The D-branes of higher codimension can be obtained [250] from the Koszul complex (a particular projective resolution (5.13) [111]) which gives the analog of the Thom isomorphism and the Atiyah-Bott-Shapiro construction in topological K-theory used by Witten [243]. The holomorphic K-theory also captures the cases of D-branes with lower RR-charges and of stacks of D-branes. Note that when stability issues are taken into account this naive setup must be improved and one has to work with complexes of sheaves and their derived categories instead of sheaves only [251], [252], [253], [254].

The Theorem

We first need to introduce the Gysin homomorphisms f_* in cohomology and $f_!$ in K-theory [255]. If X, Y are compact connected oriented manifolds, and $f : X \to Y$ is a continuous map, there is a homomorphism of $H^*(Y, \mathbb{Z})$-modules $f_* : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$ which maps classes of codimension q in X to classes of codimension q in Y. The action of f_* on $H^*(X, \mathbb{Q})$ is defined as

$$f_*(x) = D_Y^{-1}(f_*^h D_X(x)) \qquad x \in H^*(X, \mathbb{Q}) \tag{5.22}$$

where D_X is the Poincaré duality map $D_X : H^p(X,\mathbb{Q}) \to H_{n-p}(X,\mathbb{Q})$ and f_*^h is the map induced by f on homology. If $g : Y \to Z$ is another continuous map of compact connected oriented manifolds then

$$(f \cdot g)_* = f_* g_* \tag{5.23}$$

Consider the special case in which in Y is point, f is the constant map. Then

$$f_*(v) = \int_X v \cdot 1 \qquad v \in H^*(X,\mathbb{Q}) \tag{5.24}$$

where $1 \in H^0(Y)$ is the identity element. Next, we assume that $f : X \to Y$ is a holomorphic map between algebraic manifolds X, Y and that $b \in K_0(X)$ is represented by a coherent sheaf $\mathcal{F}$. Then one can define a homomorphism $f_! : K_0(X) \to K_0(Y)$ by

$$f_!(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i R^i f_* \mathcal{F} \tag{5.25}$$

where $R^i f_* \mathcal{F}$ are the direct image sheaves [111]. Similar to (5.23) $f_!$ satisfies

$$(f \cdot g)_! = f_! g_! \tag{5.26}$$

and in the special case where Y is a point and $\mathcal{F}$ is locally free it reduces to

$$f_!(\mathcal{F}) = \chi(X,\mathcal{F}) \tag{5.27}$$

where

$$\chi(X,\mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X,\mathcal{F}) \tag{5.28}$$

is the holomorphic Euler characteristic. We can now state the Grothendieck-Riemann-Roch theorem. The equation

$$\mathrm{ch}(f_! b) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(b) \cdot \mathrm{td}(X)) \tag{5.29}$$

holds in $H^*(Y,\mathbb{Q})$ for all $b \in K_0(X)$ [245], [255].

Some Applications

Let us consider two special cases which will be of use in the following sections. Suppose that $i : X \to Y$ is an embedding of X as a submanifold of Y. Then we have the following short exact sequence of bundles (3.33)

$$0 \longrightarrow TX \longrightarrow i^*TY \longrightarrow N_{Y/X} \longrightarrow 0 \tag{5.30}$$

By the multiplicative property of the Todd class, i.e. for an exact sequence of sheaves (5.21) we have $\mathrm{td}(\mathcal{F}) = \mathrm{td}(\mathcal{E})\,\mathrm{td}(\mathcal{G})$ thus

$$\mathrm{td}(X) = (\mathrm{td}(N_{Y/X}))^{-1}\,\mathrm{td}(i^*TY) = (\mathrm{td}(N_{Y/X}))^{-1} i^*\,\mathrm{td}(Y) \tag{5.31}$$

An application of the Gysin homomorphism (5.22) to the embedding $i : X \to Y$ gives

$$i_*(u\, i^* v) = i_*(u) v, \qquad \forall u \in H^*(Y,\mathbb{Q}),\ v \in H^*(X,\mathbb{Q}) \tag{5.32}$$

Therefore (5.29) implies the Riemann-Roch theorem for an embedding [245], [255]

$$\mathrm{ch}(i_! b) = i_* \,\mathrm{ch}(b) \cdot (\mathrm{td}(N_{Y/X}))^{-1} \tag{5.33}$$

We now apply this to the case of curves and divisors in a Calabi–Yau threefold X. Here we generalize [180]. If $i : D \to X$ is an embedding of a divisor D in X, then $i_! = i_*$ [245]. This map is of degree 1 and $i_*1 = D$. Let $b \in K_0(X)$ be the element represented by the coherent sheaf $\mathcal{O}(E)$ of germs of local holomorphic sections of a complex analytic vector bundle E over D. Expansion of eq. (5.33) and comparing terms of the same degree we find

$$\begin{aligned}
\mathrm{rk}(i_*E) &= 0 && (5.34)\\
\mathrm{ch}_1(i_*E) &= \mathrm{rk}(E)D && (5.35)\\
\mathrm{ch}_2(i_*E) &= i_*\left(\mathrm{ch}_1(E) - \tfrac{1}{2}\,\mathrm{rk}(E)\,\mathrm{ch}_1(N_{X/D})\right) && (5.36)\\
\mathrm{ch}_3(i_*E) &= i_*\left(\mathrm{ch}_2(E) + \tfrac{1}{6}\,\mathrm{rk}(E)\,\mathrm{ch}_1(N_{X/D})^2 - \tfrac{1}{2}\,\mathrm{ch}_1(E)\,\mathrm{ch}_1(N_{X/D})\right) && (5.37)
\end{aligned}$$

By using $\mathrm{ch}_1(N_{D/X}) = D$ and (3.36) we can bring eqns. (5.36) and (5.37) into the form

$$\begin{aligned}
\mathrm{ch}_2(i_*E) &= i_*\left(\mathrm{ch}_1(E) - \tfrac{1}{2}\,\mathrm{rk}(E)D\right) && (5.38)\\
\mathrm{ch}_3(i_*E) &= i_*\left(\mathrm{ch}_2(E) + \tfrac{1}{6}\,\mathrm{rk}(E)(\mathrm{c}_2(D) - \mathrm{c}_2(X)) - \tfrac{1}{2}\,\mathrm{ch}_1(E)D\right) && (5.39)
\end{aligned}$$

If $j : C \to X$ is an embedding of a curve C in X, then j_* is of degree 2 and $j_*1 = C$. Repeating the computation above leads to

$$\begin{aligned}
\mathrm{rk}(j_*E) &= 0 && (5.40)\\
\mathrm{ch}_1(j_*E) &= 0 && (5.41)\\
\mathrm{ch}_2(j_*E) &= \mathrm{rk}(E)C && (5.42)\\
\mathrm{ch}_3(j_*E) &= j_*\left(\mathrm{ch}_1(E) - \tfrac{1}{2}\,\mathrm{rk}(E)\,\mathrm{c}_1(N_{X/C})\right) && (5.43)
\end{aligned}$$

By (5.30) and the Calabi–Yau condition we find $j_*\,\mathrm{c}_1(N_{X/C}) = j_*\,\mathrm{c}_1(C) = \deg C = 2p_a(C) - 2$ where the last step follows from Riemann-Roch [110]. Hence

$$\mathrm{ch}_3(j_*E) = j_*\,\mathrm{ch}_1(E) + \mathrm{rk}(E)(1 - p_a(C)) \tag{5.44}$$

We will also need the characteristic classes of i_*E for an embedding $i : C \to X$ of a curve into a $K3$ surface X. These are obtained in the same way

$$\begin{aligned}
\mathrm{rk}(i_*E) &= 0 && (5.45)\\
\mathrm{ch}_1(i_*E) &= \mathrm{rk}(E)C && (5.46)\\
\mathrm{ch}_2(i_*E) &= i_*\,\mathrm{ch}_1(E) + \mathrm{rk}(E)(1 - p_a(C)) && (5.47)
\end{aligned}$$

Everything continues to hold if we replace $\mathcal{O}(E)$ by a general coherent sheaf $\mathcal{F}$ [110].

Let's turn to the second special case in which Y is a point, f is the constant map and $\mathcal{F} = \mathcal{O}(E)$ is the sheaf of holomorphic sections of a complex analytic vector bundle E over X. Then, by (5.24) and (5.27) the Grothendieck-Riemann-Roch theorem (5.29) yields the Hirzebruch-Riemann-Roch theorem [255]

$$\chi(X, E) = \int_X \mathrm{td}(X)\,\mathrm{ch}(E) \tag{5.48}$$

Note that the last equation together with (5.28) give the index theorem for a Dirac operator coupled to the vector bundle E. More generally, for two coherent sheaves $\mathcal{E}$, $\mathcal{F}$ we define [149]

$$\chi(X; \mathcal{E}, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \,\mathrm{ext}^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \tag{5.49}$$

where $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ are the global Ext groups [111] and $\mathrm{ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ denotes their dimension. The following properties of these groups are noteworthy. To a short exact sequence (5.21) two long exact sequences (similar to cohomology) can be associated depending on whether one uses the first or the second argument. Furthermore, if $\mathcal{E}$ is locally free, then

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \cong H^i(X,\mathcal{E}^\vee \otimes \mathcal{F}) \tag{5.50}$$

In particular, for any coherent sheaf $\mathcal{F}$

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{F}) \cong H^i(X,\mathcal{F}) \tag{5.51}$$

Finally, we will also need Serre duality [224], [256], [257]

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \cong \mathrm{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F},\mathcal{E}\otimes\mathcal{K}_X)^\vee \tag{5.52}$$

where $\mathcal{E}$, $\mathcal{F}$ are coherent sheaves and n is the dimension of the smooth variety X.

5.4. The Central Charge of the D-branes

5.4.1. The D-brane charge

In the following we will use the same notation $\mathcal{F}$ for both a coherent sheaf on X and its image in $K_0(X)$. Now note that the Mukai vector

$$v(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}(X)} \tag{5.53}$$

defines a module homomorphism $v : K_0(X) \to H^{\mathrm{even}}(X,\mathbb{Q})$. This definition is such that

$$\chi(X;\mathcal{E},\mathcal{F}) = \langle v(\mathcal{E}), v(\mathcal{F})\rangle \tag{5.54}$$

where the intersection form on the right-hand side was defined in (3.62). Now, recall that for $\mathcal{E}$, $\mathcal{F}$ locally free, i.e. vector bundles, this is just the index of the Dirac operator coupled to $\mathcal{E}^\vee\otimes\mathcal{F}$ ind $i\partial\!\!\!/_{\mathcal{E}^\vee\otimes\mathcal{F}}$. This observation will be taken up in Section 6.1. Note that v induces an isomorphism between $K_0(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $H^{\mathrm{even}}(X,\mathbb{Q})$ [200].

Let $Q : K_0(X) \to H_{\mathrm{even}}(X,\mathbb{Q})$ be defined by

$$\mathcal{E} \mapsto Q(\mathcal{E}) = v(\mathcal{E}) \cap [X] \tag{5.55}$$

By abuse of language we are still using the notion of Chern classes in Q although they are really Chern numbers. We call $Q(\mathcal{E})$ the D-brane charge of $\mathcal{E}$ with its component in H_{2p} representing the D$2p$-brane charge. Since Q is a module homomorphism, it follows that

$$Q(\mathcal{F}) = Q(\mathcal{E}) + Q(\mathcal{G}) \tag{5.56}$$

for each exact sequence (5.21) of coherent sheaves in X. This is interpreted as the charge conservation law when making a D-brane state associated with $\mathcal{F}$ out of those associated with $\mathcal{E}$ and $\mathcal{G}$.

The charge "lattice" $H^{\mathrm{even}}(X,\mathbb{Q})$ has several distinguished isometries. For instance, for an invertible sheaf $\mathcal{L}$ on the Calabi–Yau threefold X, the map $v(\mathcal{F}) \mapsto v'(\mathcal{F})$ with

$$\begin{aligned} v'(\mathcal{F}) &= \mathrm{ch}(\mathcal{L})v(\mathcal{F}) & (5.57)\\ &= v(\mathcal{F}) + \Big(0, \mathrm{rk}(\mathcal{F})\,\mathrm{c}_1(\mathcal{L}), \mathrm{c}_1(\mathcal{F})\,\mathrm{c}_1(\mathcal{L}) + \frac{\mathrm{rk}(\mathcal{F})}{2}\,\mathrm{c}_1(\mathcal{L})^2, & (5.58)\\ &\qquad \mathrm{ch}_2(\mathcal{F})\,\mathrm{c}_1(\mathcal{L}) + \frac{1}{2}\,\mathrm{c}_1(\mathcal{F})\,\mathrm{c}_1(\mathcal{L})^2 + \frac{\mathrm{rk}(\mathcal{F})}{6}\,\mathrm{c}_1(\mathcal{L})^3\Big) \end{aligned}$$

gives an isometry of $H^{\mathrm{even}}(X,\mathbb{Q})$. We will give an interpretation of this isometry in terms of monodromy transformations on the complex structure moduli space of the mirror X^* in Section 5.6.

The $\mathcal{N}=2$ space-time supersymmetry algebra allows for a central charge $Z(\mathcal{E})$ which determines the mass of a BPS state $\mathcal{E}$ as

$$m = |Z(\mathcal{E})| \tag{5.59}$$

Its phase is used to define the grade of a BPS configuration $\mathcal{E}$

$$\phi(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Im}\log Z(\mathcal{E}) \tag{5.60}$$

which is important to study stability questions [237] (see Chapter 6 and in particular (6.1)).

For an A-type D-brane given by a charge vector Q wrapped about the cycle $\Sigma = \sum_{i=0}^{b_3-1} Q_i[\Sigma_i]$ the central charge is [258]

$$Z = \int_\Sigma \Omega = Q_i \Pi_i \tag{5.61}$$

It has been observed that mirror symmetry not only maps $H^{\mathrm{even}}(X,\mathbb{Z})$ to $H^{\mathrm{odd}}(X^*,\mathbb{Z})$, but it does so while respecting the integral structure of the cohomologies [259]. The BPS charge lattice of the low energy effective theory for B-type D-branes is an integral symplectic lattice which can be identified with the middle cohomology lattice of the mirror manifold $H^3(X^*,\mathbb{Z})$. The BPS central charge corresponding to a vector

$$n = \left(n_6, n_4^1, \ldots, n_4^{h^{1,1}}, n_0, n_2^1, \ldots, n_2^{h^{1,1}}\right) \in H^3(X^*,\mathbb{Z}) \cong \mathbb{Z}^{b_3} \tag{5.62}$$

is

$$Z(n) = \Pi_0 n_6 + \sum_{i=1}^{h^{1,1}} \Pi_i n_4^i + \Pi_{h^{1,1}+1} n_0 + \sum_{i=1}^{h^{1,1}} \Pi_{h^{1,1}+i+1} n_2^i \tag{5.63}$$

where $h^{1,1}$ refers to X and the periods are given in (3.72). The n_{2p}^i is a suggestive notation for the number of D2p-branes wrapping a cycle of the ith basis element of $H_{\mathrm{even}}(X,\mathbb{Z})$.

On the other hand, in the large volume limit, the lattice of D-brane charges is an integral quadratic lattice identified with the K-theory lattice $K_0(X)$. The map between these lattices is a non-trivial question in mirror symmetry and is not known in a closed form. In the present case we will construct such a map between the low energy charges n and the topological invariants of the K-theory class $[\mathcal{E}]$ by exploiting the exact form of the D-brane charge $Q(\mathcal{E})$ in (5.55). The central charge associated to a state described by $[\mathcal{E}]$ is then [180]

$$Z(Q) = -\int_X e^{-J} \wedge Q \tag{5.64}$$

where $J = t_a J_a$ is the Kähler form. The factor e^{-J} takes into account the normalization by the volume, where the volume of a $2p$-cycle Σ is determined by Wirtinger's theorem to be $\frac{1}{p!}\int_\Sigma J^p$.

The comparison of (5.63) and (5.64) gives the relation between the low energy charges and the topological invariants of $[\mathcal{E}]$. We derive explicit formulae for the cases when $[\mathcal{E}]$ describes either D6-branes wrapped on X or D4-branes and D2-branes wrapped on holomorphic submanifolds of X.

Recall from Section 2.6 and Section 3.4 that the prepotential in compactified type IIB theory is classically exact and depends only on the complex structure moduli. This is the most basic quantity as it determines the central charges (5.61) and (5.63) through the periods (3.72). Hence, since A-type

D-branes (having an even number of non-compact space dimensions) are controlled by the complex structure moduli, their central charges and masses are exact while those of the B-type D-branes (having an odd number of non-compact space dimensions) receive world-sheet instanton corrections in agreement with the discussion in Section 4.3. The latter can be computed by invoking mirror symmetry. We will take this up when we discuss D-geometry and stability in Chapter 6.

5.4.2. D6-branes

We now consider systems with non-zero D6-brane charge $n_6 \neq 0$ which can be represented by coherent sheaves $\mathcal{F}$ on X. Recall from section 5.2 that the corresponding D-brane configuration is stable only if $\mathcal{F}$ is stable. Expanding (5.53) gives

$$Q = \left(\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \mathrm{ch}_2\,\mathcal{F} + \frac{\mathrm{rk}\,\mathcal{F}}{24}\,\mathrm{c}_2(X), \mathrm{ch}_3(\mathcal{F}) + \frac{1}{24}\,\mathrm{c}_1(\mathcal{F})\,\mathrm{c}_2(X)\right) \tag{5.65}$$

Q is interpreted as a vector of electric and magnetic charges. The shift by $\frac{\mathrm{c}_2}{24}$ is a geometric version of the Witten effect [260]. Indeed, choosing an electric/magnetic polarization so that $H^0 \oplus H^2$ is the lattice of magnetic charges one observes a shift in the electric vector [3]: $q_e \to q_e + \frac{\mathrm{c}_2}{24}$. Expanding (5.64) gives for the associated central charge

$$Z(Q) = \frac{\mathrm{rk}(\mathcal{F})}{6} J^3 - \frac{1}{2}\,\mathrm{ch}_1(\mathcal{F}) \cdot J^2 + \left(\mathrm{ch}_2(\mathcal{F}) + \frac{\mathrm{rk}(\mathcal{F})}{24}\,\mathrm{c}_2(X)\right) J - \left(\mathrm{ch}_3(\mathcal{F}) + \frac{1}{24}\,\mathrm{c}_1(\mathcal{F})\,\mathrm{c}_2(X)\right) \tag{5.66}$$

By a direct comparison of (5.63) using (3.72) and (5.66) using $J = J_i t_i$ we obtain the Chern characters of $\mathcal{F}$

$$\mathrm{rk}(\mathcal{F}) = n_6 \tag{5.67}$$

$$\mathrm{ch}_1(\mathcal{F}) = \sum_{i=1}^{h^{1,1}} n_4^{J_i}\ J_i \tag{5.68}$$

$$\mathrm{ch}_2(\mathcal{F}) = \sum_{i=1}^{h^{1,1}} \left(n_2^{C_i} + A_{ij} n_4^{J_j}\right)\ C_i \tag{5.69}$$

$$\mathrm{ch}_3(\mathcal{F}) = -n_0 - \frac{1}{12} \sum_{i=1}^{h^{1,1}} n_4^{J_i}\ \mathrm{c}_2 \cdot J_i \tag{5.70}$$

where the J_i and the C_i form a basis for $H_4(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ respectively, satisfying $J_i \cdot C_j = \delta_{ij}$. The A_{ij} have been defined in (3.73). The Chern classes will be more convenient for the discussion in chapter 6. We will explicitly give them for the families studied in Section 3.5 and Appendix C .

In these terms the Bogomolov inequality (5.20) becomes

$$\Delta(\mathcal{F}) \cdot J = n_4^{J_i} n_4^{J_j} t_k K_{ijk} - 2n_6 \left(n_2^{C_i} + A_{ij} n_4^{J_j}\right) t_i \geq 0 \tag{5.71}$$

5.4.3. D4-branes

A different class of D-brane states can be obtained by wrapping D4-branes on divisors $i : D \to X$. D-brane configurations are described as above by a coherent sheaf $\mathcal{F}$ of rank r on D which is required to be stable. The associated K-theory class in $K_0(X)$ is defined as the torsion sheaf $i_!\mathcal{F}$ which is the extension of $\mathcal{F}$ by zero to X. Then, the Mukai vector can be computed by an application of the Grothendieck-Riemann-Roch theorem (5.33). Indeed, (5.34), (5.35), (5.38), (5.39) and (5.53) yield

$$Q = \left(0, rD, i_*\,\mathrm{c}_1(\mathcal{F}) + \frac{r}{2} i_*\,\mathrm{c}_1(D), \mathrm{ch}_2(\mathcal{F}) + \frac{1}{2}\,\mathrm{c}_1(\mathcal{F})\,\mathrm{c}_1(D) + \frac{r}{8}\,\mathrm{c}_1(D)^2 + \frac{r}{24}\,\mathrm{c}_2(D)\right) \tag{5.72}$$

Therefore, in the large volume limit, the associated central charge reads

$$\begin{aligned} Z(Q) &= -\frac{r}{2}J^2\cdot D + \Big(i_*\,\mathrm{c}_1(\mathcal{F}) + \frac{r}{2}i_*\,\mathrm{c}_1(D)\Big)\,J \\ &\quad - \mathrm{ch}_2(\mathcal{F}) - \frac{1}{2}\,\mathrm{c}_1(\mathcal{F})\,\mathrm{c}_1(D) - \frac{r}{8}\,\mathrm{c}_1(D)^2 - \frac{r}{24}\,\mathrm{c}_2(D) \end{aligned} \tag{5.73}$$

We again compare (5.63) using (3.72) and (5.73) using $J = J_i t_i$ to obtain the Chern characters of $\mathcal{F}$. If we assume that $D = m_i J_i$ where J_i is a basis of $H_4(X,\mathbb{Z})$ and $m_i \in \mathbb{Z}$ then

$$\mathrm{rk}(\mathcal{F}) = \frac{n_4^{J_i}}{m_i} \qquad \forall\, i \tag{5.74}$$

$$\mathrm{c}_1(\mathcal{F}) = \sum_{i=1}^{h^{1,1}}\left(n_2^{C_i} + A_{ij}n_4^{J_j} + \frac{n_4^{J_{i_0}}}{2m_{i_0}}m_j m_k K_{ijk}\right) C_i \tag{5.75}$$

$$\begin{aligned} \mathrm{ch}_2(\mathcal{F}) &= -n_0 + \frac{n_4^{J_{i_0}}}{12m_{i_0}}m_i m_j m_k K_{ijk} + \frac{1}{2}n_2^{C_i}m_i \\ &\quad - \left(\frac{n_4^{J_{i_0}}}{m_{i_0}}m_i + n_4^{J_i}\right)\frac{\mathrm{c}_2\cdot J_i}{24} + \frac{1}{2}m_i A_{ij} n_4^{J_j} \end{aligned} \tag{5.76}$$

A few comments are in order here. If $m_i = 0$ then obviously also $n_4^{J_i} = 0$. The index i_0 can be chosen from any of the i for which $m_i \neq 0$. The formulas become simplest if one chooses it to be such that $m_{i_0} = 1$, if possible. If one chooses another basis for $H_4(X,\mathbb{Z})$ than J_i one has to transform the $n_4^{J_i}$ correspondingly. It can happen that some of the curves C_i in (5.75) do not appear. This has to be analyzed separately before the use of the formulas for $\mathrm{c}_1(\mathcal{F})$ and $\mathrm{ch}_2(\mathcal{F})$. In this case, we have to set the corresponding coefficient to zero which gives $h^2(X) - h^2(D)$ relations between the corresponding $n_2^{C_i}$ and some of the $n_4^{J_k}$. We will see this explicitly in the examples in chapter 6.

The case of a $K3$ fibration can be worked out in general. Assume that $J_1 = L$ is the $K3$ fiber, and that $i_*\,\mathrm{c}_1(\mathcal{F})$ can be written as $i_*\,\mathrm{c}_1(\mathcal{F}) = \alpha_i J_i|_L$ for some α_i. From the fact that $J_i|_L = \beta_{i1}l + \sum_{j>1}\beta_{ij}C_j$ and $0 = J_i\cdot L|_L = \beta_{i1}l\cdot L = \beta_{i1}$ for all i we see that the curve l does not appear in (5.75). Hence its coefficient must vanish. Using that $n_4^{J_i} = 0$ for $i \neq 1$ and that the $A_{1j} = 0$ for $K3$-fibrations (3.74) one finds

$$\mathrm{rk}(\mathcal{F}) = n_4^L \tag{5.77}$$

$$\mathrm{c}_1(\mathcal{F}) = \sum_{i=2}^{h^{1,1}} n_2^{C_i} C_i \tag{5.78}$$

$$\mathrm{ch}_2(\mathcal{F}) = -n_0 - 2n_4^L \tag{5.79}$$

5.4.4. D2-branes

We can repeat this computation also for $D2$-branes. Let $i : C \hookrightarrow X$ be a curve of arithmetic genus $p_a(C)$ embedded in a Calabi–Yau threefold X. Then (5.42) and (5.44) give for the Mukai vector

$$Q = (0, 0, \mathrm{rk}(\mathcal{F})C, i_*\,\mathrm{c}_1(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1 - p_a(C))) \tag{5.80}$$

Therefore, in the large volume limit, the associated central charge reads

$$Z(Q) = \mathrm{rk}(\mathcal{F})J\cdot C - i_*\,\mathrm{c}_1(\mathcal{F}) - \mathrm{rk}(\mathcal{F})(1 - p_a(C)) \tag{5.81}$$

If we assume that $C = m_i C_i$ where C_i is a basis of $H_2(X,\mathbb{Z})$ and $m_i \in \mathbb{Z}$ then by repeating the comparison of $Z(n)$ and $Z(Q)$ one more time we get

$$\mathrm{rk}(\mathcal{F}) = \frac{n_2^{C_i}}{m_i} \qquad \forall\, i \tag{5.82}$$

$$\mathrm{c}_1(\mathcal{F}) = -n_0 + (p_a(C)-1)\,\frac{n_2^{C_{i_0}}}{m_{i_0}} \tag{5.83}$$

If $m_i = 0$ then obviously also $n_2^{C_i} = 0$. The index i_0 can be chosen from any of the i for which $m_i \neq 0$. The formulas become simplest if one chooses it to be such that $m_{i_0} = 1$, if possible.

5.5. Moduli of D-branes

Let us now turn to the dimension of the moduli space of these sheaves which we will compare in Section 6.3 with the predictions from conformal field theory in Section 4.3.2. As mentioned in the previous sections, our focus lies on D4-branes wrapping a divisor D. Here we have to consider two problems. First, we can study and compute the dimension of the moduli space of the sheaves $\mathcal{F}$ as sheaves on the complex surface D. Second, we have to take into account the changes when we consider them as torsion sheaves $i_*\mathcal{F}$ on the Calabi–Yau X, supported on the divisor D. Roughly speaking, what we have in addition to bear in mind is the possibility that the surface D can move inside X. As we will see, the answer to the first question is rather easy while the second is more difficult. Let us discuss them in turn.

5.5.1. Deformations of vector bundles and sheaves

In this subsection we will see how to describe the deformations of vector bundles or in other words, how to describe the moduli space of vector bundles.

Before treating the general case, let us first look at line bundles. By means of the exponential sequence, the set $\mathrm{Pic}^{\mathrm{c}_1}(X)$ of all line bundles $\mathcal{L}$ with fixed first Chern class c_1 can be identified with the Abelian variety $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$. Therefore it will have dimension

$$m^{(\mathrm{geom})}(\mathcal{L}) = h^{0,1}(X) \tag{5.84}$$

Hence, all line bundles on a Calabi–Yau manifold X will have a zero dimensional moduli space. For line bundles on a surface D the dimension of its moduli space will be $q(D)$ and on a curve C of genus g it will be g.

Next, we are going to review deformations of bundles and sheaves, mainly following [261], [262], [263], [264] and [115]. We start with a family of simple vector bundles $\mathcal{E}$ of rank r over some parameter space T, or in other words a single vector bundle $\mathscr{E}$ over $X \times T$. We assume that T has a distinguished point $t_0 = 0$ and that we are given a fixed isomorphism from the restriction of $\mathscr{E}$ to the slice $X \times \{0\}$ to $\mathcal{E}$. For simplicity we assume that T is smooth of dimension 1, with coordinate t. We denote the set of isomorphism classes of simple vector bundles on X by $\mathcal{M}_D$. Let $g_{ij} : U_i \cap U_j \to GL(r,\mathbb{C})$ be the transition functions for $\mathcal{E}$ with respect to some open cover $\{U_i\}$ of X. The g_{ij} can be viewed as sections of $GL(r,\mathcal{O}_X)$ and satisfy

$$g_{ij}g_{jk}g_{ki} = 1 \qquad \forall\, U_i \cap U_j \cap U_k \tag{5.85}$$

Hence the set $\{g_{ij}\}$ is a multiplicative 1-cocycle with values in $GL(r,\mathcal{O}_X)$. (For definition see [263].) The transition functions for $\mathcal{E}$ can be taken of the form

$$G_{ij}(t) = g_{ij}(1 + t a_{ij}) + O(t^2) \tag{5.86}$$

where $a_{ij} \in GL(r, \mathcal{O}_X)$. The main idea of deformation theory is to consider the 1-cochain $\{G_{ij}(t)\}$ as a first order deformation of $\{g_{ij}\}$. It is a 1-cocycle if and only if

$$g_{jk}^{-1} a_{ij} g_{jk} + a_{jk} = a_{ik} \tag{5.87}$$

This condition says [264] that a_{ij} is an additive 1-cocycle with values in the bundle (more generally sheaf) $\mathcal{E}nd\,\mathcal{E}$ of (local) endomorphisms of $\mathcal{E}$. One can show that different choices of a_{ij} lead to a 1-coboundary for $\mathcal{E}nd\,\mathcal{E}$, so that we have intrinsically defined an element in $H^1(X, \mathcal{E}nd\,\mathcal{E})$. This element is the Kodaira-Spencer class of the family $\mathscr{E} \mapsto X \times T$ [263]. This can be reformulated as follows [264]. The Zariski tangent space of $\mathcal{M}_D$ at the point $[\mathcal{E}]$ is canonically isomorphic to the cohomology group $H^1(X, \mathcal{E}nd\,\mathcal{E})$. Hence the dimension of $\mathcal{M}_D$ is $h^1(X, \mathcal{E}nd\,\mathcal{E})$.

In general, the infinitesimal deformations $\alpha^{(\nu)} = \frac{\partial \mathcal{E}_t}{\partial t_\nu}|_{t=0}$, $\nu = 1, \ldots, N$, along t_ν at $t = (t_1, \ldots, t_N) = 0$ form a basis of $H^1(X, \mathcal{E}nd\,\mathcal{E})$ and can be represented by 1-cocycles $\{a_{ij}^{(1)}\}, \ldots, \{a_{ij}^{(N)}\}$. In this case

$$G_{ij}(t) \equiv \sum_{\mu=(\mu_1,\ldots,\mu_N)} g_{ij}^{(\mu)} t^\mu = g_{ij}^{(0)} (1 + \sum_{\nu=1}^{N} a_{ij}^{(\nu)}) t_\nu + O(t^2) \tag{5.88}$$

where we use the multi-index notation for μ and t. Define the 2-cocycles $\{\mathrm{ob}_{ijk}^{(\mu)}\}$ with coefficients in the sheaf $\mathcal{E}nd\,\mathcal{E}$ by

$$\mathrm{ob}_{ijk}^{(\mu)} = a_{ij}^{(\mu)} g_{ij} a_{jk}^{(\mu)} g_{ij}^{-1} \tag{5.89}$$

Their cohomology classes are denoted by $\mathrm{ob}^{(\mu)}$ and are the obstructions to finding $\{g_{ij}^{(\mu)}\}$ of order $|\mu| = n+1$ so that $\{G_{ij}(t)\}$ satisfies (5.85) up to order $n+2$ provided that the $\{g_{ij}^{(\mu)}\}$ are defined up to order $|\mu| = n$ such that $\{G_{ij}(t)\}$ satisfies (5.85) up to order $n+1$ [263]. It is possible to find such $\{g_{ij}^{(\mu)}\}$ if and only if $\mathrm{ob}^{(\mu)}$ vanishes for every μ with $|\mu| = n+1$. In particular, $\mathcal{M}_D$ is smooth at $[\mathcal{E}]$ if

$$H^2(X, \mathcal{E}nd\,\mathcal{E}) = 0 \tag{5.90}$$

This condition can be improved by noting that if $\{G_{ij}(t)\}$ is a 1-cocycle up to order $n+1$, then so is $\{\det G_{ij}(t)\}$. Let us define $\mathrm{ad}\,\mathcal{E}$ to be the kernel of the trace map $\mathrm{tr} : \mathcal{E}nd\,\mathcal{E} \to \mathcal{O}_X$. Then it can be shown [264] that every obstruction for the moduli space $\mathcal{M}_D$ to be smooth at $[\mathcal{E}]$ lies in the kernel of the trace map $H^2(\mathrm{tr}) : H^2(X, \mathcal{E}nd\,\mathcal{E}) \to H^2(X, \mathcal{O}_X)$. From the exact sequence

$$0 \longrightarrow \mathrm{ad}\,\mathcal{E} \longrightarrow \mathcal{E}nd\,\mathcal{E} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{5.91}$$

we can determine $h^0(X, \mathcal{E}nd\,\mathcal{E}) = h^0(X, \mathrm{ad}\,\mathcal{E}) + h^{0,0}(X)$ and $h^2(X, \mathcal{E}nd\,\mathcal{E}) = h^2(X, \mathrm{ad}\,\mathcal{E}) + h^{0,2}(X)$. In this terminology, $\mathcal{E}$ being simple means

$$H^0(X, \mathrm{ad}\,\mathcal{E}) = 0 \tag{5.92}$$

Note that stability of a vector bundle implies simplicity [265].

All these statements can be generalized to the case where $\mathcal{E}$ is a sheaf [262], [149], [235]. One can similarly define a trace map $\mathrm{tr}^i : \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to H^i(\mathcal{O}_X)$ induced from tr as above. Let $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0$ denote the kernel of tr^i, and let $\mathrm{ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0$ be its dimension. In this case, the tangent space to the moduli space $\mathcal{M}_D$ at $[\mathcal{F}]$ is given by $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0$ and the smoothness condition translates into $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 = 0$. Note that in the case where $\mathcal{F}$ corresponds to a singular point in the moduli space, i.e. when $\dim \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 > 0$ then (5.99) gives an upper bound on the dimension. More precisely, one can show [235] that

$$\mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 \leq \dim \mathcal{M}_D(\mathcal{F}) \leq \mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 - \mathrm{ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_0 \tag{5.93}$$

The right hand side is called the expected dimension of the moduli space $\mathcal{M}_D$.

In general, the dimension of the moduli space is not necessarily constant – the moduli space can have branches of different dimension, and can depend on the moduli of the Calabi–Yau manifold as well. In particular, the expected dimension for a sheaf on a Calabi–Yau threefold is always zero. From a physical point of view, the moduli space $\mathcal{M}_D$ is the moduli space of D-branes wrapping a cycle X introduced in Section 5.2. In [5], [266] it was argued that obstructions to deformations of curves C in a Calabi–Yau threefold X will appear as higher order terms in the superpotential for a D-brane wrapped around this curve C. Analogously, the obstructions in (5.89) will appear as higher-order terms in the superpotential of the $\mathcal{N} = 1$ $D = 4$ world-volume theory of the D-brane described by $\mathcal{E}$ and will lift $M = h^2(X, \mathcal{E}nd\,\mathcal{E})$ of the flat directions. r Dp-branes wrapping a single supersymmetric cycle $\Sigma \subset X$ extended in flat space-time have a $U(r)$ vector multiplet arising from massless open string excitations polarized completely in in $R^{3,1}$. Massless open string excitations polarized in X form adjoint $U(r)$ chiral multiplets Φ^i. The first-order deformations in (5.88) are the scalars in the chiral multiplets, i.e. they parametrize $\mathcal{M}_D$. We claim that if there exists a local obstruction of the form

$$\sum_{|\mu|=2n} \mathrm{ob}^{(\mu)} t^\mu \sim \sum_{i_1,\dots,i_M} W_{i_1,\dots,i_M} t_{i_1}^{\mu_1} \dots t_{i_M}^{\mu_M} \tag{5.94}$$

in (5.89) it will contribute to the superpotential of the world-volume theory a term of the form

$$W(\Phi) \sim \sum_{i_1,\dots,i_M} W_{i_1,\dots,i_M} \left(\Phi^{i_1}\right)^{\mu'_1} \dots \left(\Phi^{i_M}\right)^{\mu'_M} \tag{5.95}$$

with $|\mu'| = n + 1$. For related ideas see [203] and [253].

5.5.2. Bundles and sheaves on algebraic surfaces

In this section we consider the extension the previous general statements to the special case of a algebraic surface D. The computation of the dimension of the moduli space of sheaves $\mathcal{F}$ on D is a simple application of the Hirzebruch-Riemann-Roch theorem [210] as we will review shortly. Let us assume for the moment that $\mathcal{F}$ is simply a vector bundle. Then the dimension is given by $h^1(D, \mathcal{E}nd\,\mathcal{F})$. This is part of the holomorphic Euler characteristic of $\mathcal{F}$ (5.28)

$$\chi(D, \mathcal{E}nd\,\mathcal{F}) = \sum_{i=0}^{2} (-1)^i h^i(D, \mathcal{E}nd\,\mathcal{F}) \tag{5.96}$$

The Hirzebruch-Riemann-Roch theorem (5.48) tells us that this can be determined from the Chern classes of $\mathcal{F}$.

$$\chi(D, \mathcal{E}nd\,\mathcal{F}) = \int_D \mathrm{td}(D)\,\mathrm{ch}(\mathcal{F})\,\mathrm{ch}(\mathcal{F}^\vee) \tag{5.97}$$

$$= \int_D \frac{\mathrm{rk}(\mathcal{F})^2}{12}(\mathrm{c}_1(D)^2 + \mathrm{c}_2(D)) + (\mathrm{rk}(\mathcal{F}) - 1)\,\mathrm{c}_1(\mathcal{F})^2 - 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{c}_2(\mathcal{F}) \tag{5.98}$$

Collecting (5.96), (5.97) and (3.38) we obtain

$$h^1(D, \mathcal{E}nd\,\mathcal{F}) = \mathrm{c}_1(\mathcal{F})^2 - 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{ch}_2(\mathcal{F}) - \mathrm{rk}(\mathcal{F})^2 \chi(\mathcal{O}_D) + 1 + p_g(D) \tag{5.99}$$

where we have expressed c_2 by ch_2 since the latter expression simpler in terms of the charges n (5.76). Recall that for a surface D embedded as a divisor in a Calabi–Yau X, its holomorphic Euler characteristic $\chi(\mathcal{O}_D)$ can be computed in terms of the toric data of X using (3.39). While this discussion applies to bundles one can show with more work that the same formula also holds for sheaves $\mathcal{F}$ when we replace

$h^1(D, \mathcal{E}nd\,\mathcal{F})$ by $\mathrm{ext}^1_{\mathcal{O}_D}(\mathcal{F},\mathcal{F})$, see [262]. We denote this dimension by $m^{(\mathrm{geom},D)}(\mathcal{F})$. According to [210] the moduli space of Hermitian vector bundles admitting an irreducible Einstein-Hermitian connection over a compact Kähler surface D is a non-singular Kähler manifold of dimension $h^1(D, \mathcal{E}nd\,\mathcal{F})$ if

$$\deg K_D = -\int_D \mathrm{c}_1(D) \wedge J^{(D)} = -D^2 \cdot J|_X \le 0 \tag{5.100}$$

By the Donaldson-Uhlenbeck-Yau theorem [240], [241] such bundles are stable. The condition on the degree of the canonical bundle allows to distinguish four cases. First, if $\deg K_D < 0$ then K_D has no holomorphic sections, i.e. $p_g(D) = 0$. If $\deg K_D = 0$ and $K_D \neq \mathcal{O}_D$ then every holomorphic section is parallel so that $p_g(D) = 0$, too. If $K_D = \mathcal{O}_D$, i.e. if D is a torus or a $K3$ surface then $p_g(D) = 1$. Finally, if $\mathrm{c}_1(D) > 0$, then $q(D) = p_g(D) = 0$.

In the special case of the $K3$ surface L, the moduli space carries a holomorphic symplectic structure and the dimension formula reduces to the well-known Mukai formula [149], [264]

$$m^{(\mathrm{geom},L)}(\mathcal{F}) = \mathrm{c}_1(\mathcal{F})^2 - 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{ch}_2(\mathcal{F}) - 2\,\mathrm{rk}(\mathcal{F})^2 + 2 \tag{5.101}$$

or in terms of the D-brane charge vector n

$$m^{(\mathrm{geom},L)}(\mathcal{F}) = n_2^{C_i} n_2^{C_j} I^{(K3),ij} + n_4^L(2n_4^L + n_0) + 2 \tag{5.102}$$

where $I^{(K3),ij}$ is the inverse of the intersection matrix $I^{(K3)}_{ij}$ on the Picard lattice $\mathrm{Pic}\,L$, see Section 3.3.4.

In general, i.e. without the assumption on the degree of the canonical bundle, the moduli space will be singular and reducible. However, one might hope that at least for comparison to the results from conformal field theory the dimension formula will continue to hold for some non-singular subset, say a point, of a component of the moduli space.

5.5.3. Sheaves on Calabi–Yau threefolds

The tangent bundle

It has been proven [241] that the holomorphic tangent bundle $T^{1,0}(X)$ of a Calabi–Yau manifold X with $h^{1,0}(X) = 0$ is stable. For the tangent bundle there is a general method using spectral sequences and the Koszul complex to compute $h^1(X, \mathrm{ad}\,T^{1,0})$. This is explained in detail in [88], following [267] and [268]. This dimension can, in principle, also be computed by counting E_6 singlets in a Gepner model compactification of the heterotic string as in [40] and [43].

Torsion sheaves

With the knowledge of the dimension of the moduli space of sheaves $\mathcal{F}$ on the surface D we can now turn to the second question, the dimension of the moduli space of these sheaves considered as torsion sheaves $i_*\mathcal{F}$ supported on a divisor D inside the Calabi–Yau X. We denote this dimension by $m_X^{(\mathrm{geom},D)}(i_*\mathcal{F}) = \mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{F}, i_*\mathcal{F})$ which we have to compare to $\mathrm{Ext}^1_{\mathcal{O}_D}(\mathcal{F},\mathcal{F})$. This can actually be done. For coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on a divisor $D \in X$ with normal bundle $N_{X/D} = \mathcal{O}_D(D)$ there exists a long exact sequence [269]

$$\longrightarrow \mathrm{Ext}^i_{\mathcal{O}_D}(\mathcal{E},\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(i_*\mathcal{E}, i_*\mathcal{F}) \longrightarrow \mathrm{Ext}^{i-1}_{\mathcal{O}_D}(\mathcal{E},\mathcal{F}(D)) \xrightarrow{\delta} \mathrm{Ext}^{i+1}_{\mathcal{O}_D}(\mathcal{E},\mathcal{F}) \longrightarrow \tag{5.103}$$

Assuming that $\mathcal{F}$ is a simple vector bundle, it can be shown [270] that

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_D}(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{F}, i_*\mathcal{F}) \longrightarrow H^0(D, \mathcal{E}nd(\mathcal{F}) \otimes N_{X/D}) \longrightarrow 0 \tag{5.104}$$

which using $H^0(D, N_{X/D}) = H^0(D, K_D) = H^2(D, \mathcal{O}_D)^\vee$ simplifies to

$$0 \longrightarrow H^1(D, \mathcal{E}nd\,\mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{F}, i_*\mathcal{F}) \longrightarrow H^0(D, \mathcal{E}nd(\mathcal{F})) \otimes H^2(D, \mathcal{O}_D)^\vee \longrightarrow 0 \tag{5.105}$$

so that

$$\mathrm{ext}^1_{\mathcal{O}_X}(i_*\mathcal{F}, i_*\mathcal{F}) = h^1(D, \mathcal{E}nd(\mathcal{F})) + p_g(D) \tag{5.106}$$

or, in general,

$$m_X^{(\mathrm{geom},D)}(i_*\mathcal{F}) = m^{(\mathrm{geom},D)}(\mathcal{F}) + p_g(D) \tag{5.107}$$

We will see in Section 6.4, that there is a high agreement with the result from conformal field theory (4.48).

The FMW construction

Let us say a few words about sheaves supported on the whole Calabi–Yau manifold. One would like to compute the dimension of the moduli space also in this case. If one tries to repeat the Hirzebruch-Riemann-Roch computation in subsection 5.5.2 then one runs immediately into the difficulty that because of (5.51) and Serre duality (5.52)

$$\sum_{i=0}^{3} (-1)^i \dim H^i(X, \mathcal{E}nd\,V) = 0 \tag{5.108}$$

One way out, presented in [271], is to use the fact that elliptically fibered Calabi–Yau manifolds $\pi : X \to B$ with a section σ have an additional symmetry τ. This $\mathbb{Z}_2$ symmetry is generated by an involution that leaves σ invariant and acts as -1 on each fiber. In terms of a Weierstrass model $y^2 = 4x^3 - g_2 x - g_3$, τ is just the operation $y \to -y$ with fixed x. Hence we will assume that we are in a component of the moduli space where τ lifts to an action on the bundles on X and the condition for τ-invariance of a bundle V becomes $\tau^*(V) = V^\vee$. This allows us to compute the $\mathbb{Z}_2$ index and therefore the dimension of the moduli space with the help of a character-valued index theorem [272] applied to the group $\mathbb{Z}_2$.

$$\sum_{i=0}^{3} (-1)^i \operatorname{tr}_{H^i(X, \mathrm{ad}\,V)} \tau = \sum_j \int_{U_j} \frac{\mathrm{ch}(\mathrm{ad}\,V_e|_{U_j}) - \mathrm{ch}(\mathrm{ad}\,V_o|_{U_j})}{1 + e^{\mathrm{c}_1(N_{X/U_j})}} \,\mathrm{td}(U_j) \tag{5.109}$$

where the subscripts e and o correspond to the subbundles on which τ acts by 1 and -1 respectively, and the U_j are the components of the fixed point set of τ. If we write H^i_e and H^i_o for the subspaces of H^i that are even or odd under τ and assume that the bundle is simple, i.e. $H^0_e = H^0_o = 0$ then the dimension of the moduli space is

$$I = h^1(X, \mathrm{ad}\,V_e) - h^2(X, \mathrm{ad}\,V_e) = -\frac{1}{2} \sum_{i=0}^{3} (-1)^i \operatorname{tr}_{H^i(X, \mathrm{ad}\,V)} \tau \tag{5.110}$$

where we have used (5.108). In [271] I was evaluated for an $SU(n)$ bundle V to be

$$I = n - 1 - \int_{U_1} \mathrm{c}_2(V) - \int_{U_2} \mathrm{c}_2(V) \tag{5.111}$$

The two components of the fixed point set are U_1 which is the section σ and hence its cohomology class is that of σ and U_2 which is given by $y = 0$. Since y is a section of $\mathcal{O}(\sigma)^3 \otimes K_B^{-3}$ the cohomology class of U_2 is $3\sigma - 3\,\mathrm{c}_1(K_B)$. The authors of [271] have also determined $\mathrm{c}_2(V)$ to be

$$\mathrm{c}_2(V) = \eta\sigma - \frac{\mathrm{c}_1(K_B)(n^3 - n)}{24} - \frac{n\eta(\eta + n\,\mathrm{c}_1(K_B))}{8} \tag{5.112}$$

where $\eta = \mathrm{c}_1(B) \mod 2$. We will check in Section 6.3 for the elliptic fibrations discussed in Appendix C whether such bundles can be found in the boundary state construction of Section 5.4.2.

Exceptional Sheaves

A class of coherent sheaves with very interesting properties with respect to mirror symmetry are the exceptional sheaves [273], [151]. An exceptional sheaf $\mathcal{E}$ on a weighted projective space Y has $\mathrm{Ext}^0_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) = \mathbb{C}$ and $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}) = 0$ for $i > 0$. An exceptional collection $\mathscr{E}$ of sheaves is an ordered collection of exceptional sheaves $\mathscr{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ such that $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{E}_a, \mathcal{E}_b) = 0$ for $a > b$ and for $a < b$ except at most for a single degree $i = i_0$. In particular, the index (5.49) has at most one non-trivial term equal to $(-1)^{i_0}\,\mathrm{Ext}^{i_0}_{\mathcal{O}_Y}(\mathcal{E}_a, \mathcal{E}_b)$. The computation in Appendix B relies heavily on the concept of helices of sheaves. A helix is defined as an infinite collection $\mathcal{H}_{\mathcal{E}} = \{\mathcal{E}_a\}$ such that the set $\{\mathcal{E}_{n_0+1}, \ldots, \mathcal{E}_{n_0+N}\}$ is an exceptional collection for any n_0. It can be shown that one can obtain a helix starting from a particular exceptional collection which is called the foundation of the helix. We will briefly use the restriction of exceptional sheaves to Calabi–Yau hypersurfaces in Section 6.4.

A comment on sheaves supported on curves

Everything we have said for divisors in a Calabi–Yau threefold X carries in principle over to curves $j : C \to X$ embedded in X. However, there are important difficulties. As in the case of D4-branes we can decompose the question about the dimension of the moduli space of sheaves $\mathcal{F}$ on C into two parts. First, we can restrict ourselves to the moduli space of the sheaves $\mathcal{F}$ as sheaves on the curve C. This has been studied in e.g. [226], [115]. We have computed the dimension of the moduli space of a line bundle $\mathcal{L}$ on a curve C of genus g in (5.84) to be $h^{0,1}(\mathcal{O}_C) = g$. For genus $g \geq 2$, the higher rank case can be reduced to line bundles by the following observation. A rank r vector bundle $\mathcal{E}$ over C describes a collection of r D-branes wrapped on the curve C which should physically be equivalent to a single D-brane wrapped on a curve C' in the homology class of rC [208]. By the Riemann-Hurwitz theorem [111] the genus g' of C' is related to the genus g of C by $g' = r^2(g-1)+1$. This is then the dimension of the moduli space of $\mathcal{E}$. The same result is obtained by applying the Hirzebruch-Riemann-Roch theorem (5.48) to $\mathcal{E}nd\,\mathcal{E}$ assuming that $\mathcal{E}$ is simple.

$$m^{(\mathrm{geom})}(\mathcal{E}) = \mathrm{rk}(\mathcal{E})^2(g-1)+1 \tag{5.113}$$

Second, we have to take into account the possibility that the curve C can move inside X, i.e. consider them as torsion sheaves $j_*\mathcal{F}$ on the Calabi–Yau X, supported on the curve C. This may lead to additional contributions to the dimension of the moduli space of the torsion sheaf $j_*\mathcal{F}$. Infinitesimal supersymmetric deformations of the cycle C are holomorphic sections of the normal bundle $N_{X/C}$ whose number is given by the dimension of the space of these sections, $H^0(X, N_{X/C})$ [263]. These first-order deformations of the cycle can be obstructed either at higher order or by deformations of the complex structure. This obstruction space is $H^1(X, N_{X/C})$ and we need to know its dimension as well. Since $\mathrm{rk}\,N_{X/C} = 2$ this is in general a difficult question, even for rational curves [274]. For further discussion see [105] and [266]. For this reason, we will not say much about D2-branes here and in Chapter 6.

5.6. D-branes and Monodromies

Transporting a D-brane configuration about closed, non-trivial cycles of $\mathcal{M}_K$ will induce an associated $Sp(2h^{1,1}+2,\mathbb{Z})$ monodromy on the B-type branes. In the following we will consider the effect of some of these monodromies on the charges of D-brane configurations.

5.6.1. Monodromy transformation about the large complex structure limit

Since the B-field is an element of the torus $H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$ it induces a monodromy about the large complex structure limit: $B \to B + D$ where D is any of the divisors corresponding to the large complex structure limit of the mirror X^* as explained in Section 3.4.2. Therefore we get an isometry on the Mukai lattice (5.57) with $\mathcal{L} = \mathcal{O}(D)$. For a related discussion see [275]. Hence, $v(\mathcal{F}) \to v(\mathcal{F})\,\mathrm{ch}(\mathcal{O}(D))$ [180]. There it was shown in an example how this transformation acts on the charges n. Here we show the general case. By using (5.67) to (5.70) we can show that for $D = J_a$ the linear transformation $v(\mathcal{F}) \to v(\mathcal{F})\,\mathrm{ch}(\mathcal{O}(J_a))$ acts on n by the matrix $M(J_a)$

$$n \to M(J_a)n \quad \text{with} \quad M(J_a) = \begin{pmatrix} 1 & \delta_a^T & \frac{1}{6}K_{aaa} + \frac{1}{12}\,\mathrm{c}_2 \cdot J_a & \frac{1}{2}K_{aa}^T + A_a^T \\ 0 & \mathbb{1} & \frac{1}{2}K_{aa} + A_a & K_a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \delta_a & \mathbb{1} \end{pmatrix} \tag{5.114}$$

which is precisely the inverse of the monodromy matrix S_a in (3.81)

$$M(J_a) = S_a^{-1} \tag{5.115}$$

This therefore proves that the monodromy transformations S_a on $H^3(X^*, \mathbb{Z})$ of the mirror X^* are converted into automorphisms of $K_0(X)$ of the form

$$\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_X(J_a) \tag{5.116}$$

This preserves stability and the dimension of the moduli space. This is a simple particular case of a more ambitious program initiated by Kontsevich [276] which proposes an interpretation of mirror symmetry as an equivalence between the derived category of complexes of coherent sheaves on X and the Fukaya-Floer category of isotopy classes of graded special Lagrangian submanifolds of the mirror X^* and elaborated systematically in [107].

5.6.2. Monodromy transformation about the discriminant locus

Here we check that all bundles constructed from boundary states satisfy another monodromy transformation proposed by Kontsevich. He suggested that the monodromy T about the conifold locus, the primary component of the discriminant locus, of the mirror $\widehat{X}$ corresponds to the automorphism $K_0(X)$ whose effect on the cohomology can be described by

$$\mathcal{S} : \gamma \longrightarrow \gamma - \left(\int_X \gamma \wedge \mathrm{td}(X) \right) \cdot 1_X \tag{5.117}$$

which corresponds to a change in the topological invariants of the sheaf $\mathcal{F}$

$$\mathrm{ch}(\mathcal{F}) \longrightarrow \mathrm{ch}(\mathcal{F}) - \frac{\mathrm{ch}_1(\mathcal{F})\,\mathrm{c}_2(X)}{12} + \mathrm{ch}_3(\mathcal{F}) \tag{5.118}$$

Now from (5.67) to (5.70) we have

$$\frac{\mathrm{ch}_1(\mathcal{F})\,\mathrm{c}_2(X)}{12} + \mathrm{ch}_3(\mathcal{F}) = \frac{n_4^{J_i}\,\mathrm{c}_2 \cdot J_i}{12} - n_0 - \frac{n_4^{J_i}\,\mathrm{c}_2 \cdot J_i}{12} = -n_0 \tag{5.119}$$

hence we get that

$$n_6 \longrightarrow n_6 + n_0 \tag{5.120}$$

Now recall from (3.85) that the monodromy matrix about the conifold locus in this basis is

$$T = \mathbb{1} - E_{1,h^{1,1}+2} \tag{5.121}$$

where E_{ij} is the matrix with zeroes everywhere except at the (i,j)-th entry. Comparing the last two results, we see that

$$\mathcal{S} = T^{-1} \tag{5.122}$$

This can be generalized to the other components of the discriminant locus. In general, given a spherical sheaf $\mathcal{E}$ [277], one can define a twisted sheaf $T_{\mathcal{E}}(\mathcal{F})$ whose Chern character is

$$\mathrm{ch}(T_{\mathcal{E}}(\mathcal{F})) = \mathrm{ch}(\mathcal{F}) - \langle \mathcal{E}, \mathcal{F} \rangle \, \mathrm{ch}(\mathcal{E}) \tag{5.123}$$

Any line bundle is spherical. Note that in general this transformation does not preserve the dimension of the moduli space. For the primary component, we need to choose $\mathcal{E} = \mathcal{O}_X$ then we obtain (5.119). If the component corresponds to an exceptional divisor $i : F \to X$ shrinking down to a point, then $\mathcal{E} = i_* \mathcal{O}_F$ is spherical. If $\mathcal{O}_p$ represents a D0-brane, then $\mathrm{ch}(\mathcal{O}_p) = p \in H^6(X, \mathbb{Z})$ is the Poincaré dual to a point p. Hence $\langle \mathcal{O}_Y, \mathcal{O}_p \rangle \neq 0$ if and only if $Y = X$. The D0-brane therefore only undergoes a monodromy if we transport it around the primary component of the discriminant locus [278]. For the component corresponding to the contraction of an exceptional divisor to a curve, one has to use spherical complexes instead of only sheaves [277], [278].

6. D-geometry

It is important to note that the geometrical structure which one sees emerging from some particular situation can depend in part on precisely which probe one uses to study it. If one uses a string probe one quantum geometrical structure will be accessed while if one uses for instance a D-brane of a particular dimension, another geometry will become manifest. D-geometry [279] is the study of geometry of M-theory or string theory compactifications as seen by a D-brane. This geometry can differ from the conventional, classical geometry which describes e.g. D-branes as solution of the supergravity equations of motions. "Unconventional" geometry will appear if e.g. we include stringy (α') and quantum (g_s) corrections. The world-volume actions for D-branes get affected by these corrections as can be seen e.g. from (5.1). Other qualitative effects visible at finite α' include T-duality and mirror symmetry as has been discussed in 2.6. There are further effects [279].

In this chapter we will see that we need to modify the geometric hypothesis stated in the Introduction. The core of this chapter will then consist of checking this modified hypothesis in many examples. Hereby we will focus mainly on two aspects, the spectrum of the D-branes and the dimension of their moduli space.

As an example of how the geometry can change consider a supergravity 0-brane at a point in the Calabi–Yau threefold X. Its moduli space is X itself and the metric on the moduli space is just the Ricci-flat metric on X. Now due to α' corrections the moduli space metric of a D0-brane at a point in X will provide a canonical non-Ricci-flat metric for each point in the Calabi–Yau moduli space [279]. We will briefly discuss D0-branes in the Gepner model description in Section 6.4.8.

Another example provides us with the most important difference to theories satisfying the geometric hypothesis. Consider the fact that D-branes are BPS states in string theory and reduce to BPS states in the supersymmetric gauge theory in four dimensions. By the work of Seiberg and Witten [280], [281] it is known that in pure $\mathcal{N}=2$ $SU(2)$ gauge theory the strong coupling spectrum is very different from the semi-classical spectrum. There are lines of marginal stability defined by the condition

$$\mathrm{Im}\,\frac{Z(Q_1)}{Z(Q_2)}=0 \tag{6.1}$$

for the central charges (5.63) of two BPS configurations with charges Q_1 and Q_2. The purely electric W-bosons which are the lightest states in the semi-classical regime are not present in the strong coupling spectrum. In that spectrum the magnetic monopoles are the lightest states. One can try to compare the situation of pure $\mathcal{N}=2$ $SU(2)$ gauge theory with the $\mathcal{N}=1$ gauge theory obtained from a compactification of Type II string theory on a quintic threefold with D-branes since the Seiberg-Witten moduli space and the Kähler moduli space of the quintic have the same form: they are both a $\mathbb{P}^1$ with three points removed. There are however some important differences. In pure $SU(2)$ $\mathcal{N}=2$ gauge theory there is a single line of marginal stability which goes through the massless monopole and dyon points and separates strong and weak coupling limits. On the other hand, at the conifold point of the quintic moduli space it is the D6-brane which becomes massless [4] and there seem to be infinitely many lines in the D-brane world-volume theory [236]. In addition, there can be BPS states whose mass appears to go to zero at a non-singular point in the moduli space [279]. Nevertheless, one concludes that the spectrum of the D-branes depends on the particular point in the moduli space of the Calabi–Yau manifold, but the story is not as simple as in the Seiberg-Witten theory. We will discuss this further in Sections 6.3.1 and 6.5.

6.1. Witten index and intersection matrix

We want to determine the D-brane spectrum at various points of the Calabi–Yau moduli space. With the obvious candidate for a comparison of the spectra, namely the D-brane charges, one runs into the difficulty of normalization of the charges. A better quantity to use is the intersection form governing the Dirac charge quantization condition (5.4) which is canonically normalized, as already noted in [192].

Checking the Dirac quantization condition (5.4) for a D0-brane requires introducing a D6-brane and computing their interaction from an annulus diagram. From the open string point of view, restricting to the massless sector, this computation can be done as follows. It was argued in [279], [179] that by carrying over the results of [282] to the magnetic monopole interaction between a Dp- and a D$(6-p)$-brane, the D0-brane sees the magnetic RR potential of the D6-brane as a Berry phase [283] associated with the Hamiltonian

$$H = E_0 + \sum_{i=1}^{3} \bar{\chi}\sigma^i \chi X^i \tag{6.2}$$

describing fermionic strings stretched between these objects. The fermions χ are a doublet of the $SO(3)$ transverse to both branes and E_0 is a constant shift of energy coming from the massive open strings. The massive string modes will always come in pairs with canceling Berry phase. Thus the interaction relevant for the Dirac quantization condition can be computed by counting fermionic open strings. The matrix $I_{\alpha\beta} = \mathrm{tr}_{\alpha\beta,\,R}(-1)^F$ counting massless Ramond doublets (with chirality) between the D-brane α and the D-brane β is then the conformal field theory analog of the intersection form in a geometric compactification. It is the Witten index in the Ramond sector and from Section 2.7 we know that it does not vary under continuous deformations. In the geometric case, if we consider a set of branes wrapping an integral homology basis this form must be integral and unimodular by Poincaré duality, proving that (5.4) is satisfied [179]. By computing $I_{\alpha\beta}$ in the conformal field theory as in Section 4.3.1 one can check that a particular set of D-branes also satisfies (5.4).

In the nonlinear σ-model introduced in Section 2.2 we can give another argument [151]. Assume for simplicity that there is a single D-brane wrapping X entirely. We can couple the left and right boundaries of the world-sheet to $U(1)$ Chan-Paton gauge fields $A^{(\alpha)}$ and $A^{(\beta)}$ respectively that define holomorphic line bundles E_α and E_β on X. We use that the boundary term is

$$\int_{\partial\Sigma} \mathrm{d}x^0 \left\{\partial_0\phi^\mu A_\mu + iF_{\mu\nu}\psi^\mu\psi^\nu\right\} \tag{6.3}$$

where μ, ν are real coordinates as in (2.37) and the boundary condition is

$$\partial_1\phi^\mu = 0 \qquad\qquad \psi^\mu_- - \psi^\mu_+ = 0 \tag{6.4}$$

The theory is invariant under B-type supersymmetry generated by $Q = \overline{Q}_+ + \overline{Q}_-$ and $Q^\dagger = Q_+ + Q_-$. Since the boundary term (6.3) includes the time derivatives of the fields, the Noether charges are modified. Thus the supercharge Q is expressed as

$$\begin{aligned} Q \;=\; & \sqrt{2}\left(\int_0^\pi \mathrm{d}x^1 \left\{g_{i\bar{\jmath}}\left(\overline{\psi}^{\bar{\jmath}}_+ + \overline{\psi}^{\bar{\jmath}}_-\right)\partial_0\phi^i - g_{i\bar{\jmath}}\left(\overline{\psi}^{\bar{\jmath}}_+ - \overline{\psi}^{\bar{\jmath}}_-\right)\partial_1\phi^i\right\}\right. \\ & \left. + \left(\overline{\psi}^{\bar{\jmath}}_+ + \overline{\psi}^{\bar{\jmath}}_-\right) A^{(\beta)}_{\bar{\jmath}}\Big|_{x^1=\pi} - \left(\overline{\psi}^{\bar{\jmath}}_+ + \overline{\psi}^{\bar{\jmath}}_-\right) A^{(\alpha)}_{\bar{\jmath}}\Big|_{x^1=0}\right) \end{aligned} \tag{6.5}$$

As in the argument given in Section 2.2 we can focus on the zero modes. Then from the boundary condition (6.4), the left and right fermionic zero modes are related as $\psi^i_{-,0} = \psi^i_{+,0}$ and $\overline{\psi}^{\bar{\imath}}_{-,0} = \overline{\psi}^{\bar{\imath}}_{+,0}$. We can identify the quantum mechanical Hilbert space as the space of sections of the bundle $\wedge T^{(0,1)*}X \otimes$

$E_\alpha^* \otimes E_\beta$, on which the fermionic zero modes act as

$$\frac{1}{\sqrt{2}}\left(\overline{\psi}^{\bar{\imath}}_{+,0} + \overline{\psi}^{\bar{\imath}}_{-,0}\right) \quad \longleftrightarrow \quad \mathrm{d}\bar{z}^{\bar{\imath}} \wedge \tag{6.6}$$

$$\frac{1}{\sqrt{2}} g_{i\bar{\jmath}}\left(\psi^{i}_{+,0} + \psi^{i}_{-,0}\right) \quad \longleftrightarrow \quad i_{\frac{\partial}{\partial \bar{z}^{\bar{\imath}}}} \tag{6.7}$$

Then the supercharge Q corresponds to the Dolbeault operator on the bundle $E_\alpha^\vee \otimes E_\beta$:

$$Q \longleftrightarrow 2\bar{\partial}_A = 2\mathrm{d}\bar{z}^{\bar{\imath}}\left(\bar{\partial}_{\bar{\imath}} + A^{(\beta)}_{\bar{\imath}} - A^{(\alpha)}_{\bar{\imath}}\right) \tag{6.8}$$

Thus the Witten index is in this case equal to the index of this Dolbeault operator. By the standard index theorem (5.48), we obtain

$$I_{\alpha\beta} = \chi(E_\alpha, E_\beta) = \int_X \mathrm{ch}(E_\alpha^\vee \otimes E_\beta)\,\mathrm{td}(X) \tag{6.9}$$

It is easy to extend this analysis to the case where the bundles E_α and E_β have higher ranks. By (5.54) this is the intersection form on the Mukai lattice.

It is important to mention a subtlety here. Strictly speaking, the Dirac operator is only defined when coupled to a locally free sheaf, i.e. a vector bundle. It is only in this case that we can define a connection and the covariant derivative $D = \partial + A$. Hence the Dirac index is only defined when the Dirac operator is coupled to a vector bundle. We are, however, interested in general coherent sheaves. We can circumvent this point by counting the chiral fermions directly instead of computing the Dirac index. For two sheaves $\mathcal{E}$ and $\mathcal{F}$ this is achieved by $\chi(\mathcal{E}, \mathcal{F})$ as defined in (5.49). The index theorem (Hirzebruch-Riemann-Roch theorem (5.48)) is then replaced by the more general Grothendieck-Riemann-Roch theorem (5.29) which reduces in the case of locally free sheaves to the former as discussed in Section 5.3.3.

Combining the two arguments we can say that the conformal field theory analog of the intersection number of geometric branes is the overlap integral of the corresponding boundary states weighted with $(-1)^F$. Furthermore, since the Witten index is a topological invariant, it is in particular independent of the Kähler moduli. As mentioned in Section 2.7 we can choose convenient limits to compute it which are of course the Gepner point and the large volume limit. We will turn to the comparison of the spectra of B-type branes at these points in Section 6.3. But first, let us make a few remarks on the A-type boundary states.

6.2. A-type boundary states

The intersection matrix for A-type boundary states with $\sum L_j = 0$ in the Gepner model $(k_1, \ldots, k_5)$ is

$$I^A(g_2, g_3, g_4, g_5) = \left(1 - \prod_{j=2}^{5} g_j\right) \prod_{j=2}^{5}\left(1 - g_j^{k_j+1}\right) \tag{6.10}$$

where g_i, $i = 2, \ldots, 5$ are the generators of the symmetry group of the Gepner model (2.89) satisfying $g_i^{k_i+2} = 1$. We have used the diagonal identification in this group $\prod g_j = 1$ to eliminate g_1. To determine the rank of I_A we can count the number of nonzero eigenvalues [5]. The g_j can be diagonalized as $g_j = \mathrm{diag}(1, e^{\frac{2\pi i}{k_j+2}}, \ldots, e^{\frac{2\pi i(k_j+1)}{k_j+2}})$. Zero eigenvalues appear if a $g_j = 1$ or if $g_2 g_3 g_4 g_5 = 1$. The combinatorics lead to the following result [154]. The rank of the intersection matrix can be related to the following quantity

$$\mathrm{rk}(I^A) = \tilde{b}_3(X) \tag{6.11}$$

where $\tilde{b}_3(X)$ denotes the third Betti number of the corresponding Calabi–Yau family without the contributions from non-polynomial deformations of the complex structure, see Section 3.2.1 and table 3.2. It can be checked that this holds for any Fermat hypersurface X irrespective of $h^{1,1}(X)$. This means that the rank of this intersection matrix counts the number of independent 4-cycles on the mirror Calabi–Yau manifold X^* except those which are coming from a non-toric blow-up or in other words the non-toric complex structure deformations of X whose number was given in (3.45). D-branes wrapping the non-toric divisors have been studied in [57] and [284]. It has been argued in [266] that these complex structure deformations can lead to a superpotential in the non-compact space-time.

The result (6.11) can be viewed as a reflection of the fact mentioned in Section 4.3.3 that the boundary states used to compute (4.30) and hence (6.10) came only from the untwisted sector. What seems to be missing in (6.11) are the contributions from the twisted sector in the Landau-Ginzburg orbifold theory, or in other words, the contributions from exceptional 3-cycles. Recall from Section 3.3.3 that there can be singular $\mathbb{Z}_N$ curves whose resolution contributes twice (3.45) to b_3. This group $\mathbb{Z}_N$ is precisely the group assumed in Section 4.3.3. It was argued in [178] that the boundary states coming from the K/N-twisted sector should be charged under the twisted (c, c) fields that arise from resolving a $\mathbb{Z}_N$ singularity along the curve C. However, as noted in Section 4.3.3 this identification works presently only for $N = 2$, and seems to be unclear [183]. The contribution of these twisted boundary states to the Witten index presumably accounts for the difference between the rank of I^A and the number of independent 3-cycles b_3 according to (3.45).

Recall from Section 5.4.1 that the central charge (5.61), i.e. the masses and the grades of A-type D-branes are exact. Comparing with (6.1) this means that lines of marginal stability are everywhere in Kähler moduli space the same as in the large volume limit. In Section 5.5 we have studied rather extensively the deformations of B-type D-branes. Let us add here a few words about deformations of A-type D-branes. Apart from those which are intrinsic to the special Lagrangian cycle and have already been considered in Section 5.2 there are deformations coming from the variation of the moduli of the Calabi–Yau manifold X. They have been studied by Joyce [285] and physically interpreted in [286] and [2]. The result is that two intersecting D3-branes Σ_1, Σ_2 can intercommute to produce a single D3-brane, or the reverse. There is a criterion, called the angle theorem [206], which says which of the two configurations is stable, and furthermore says that the decay takes place when $Z(Q_1)$ and $Z(Q_2)$ are collinear, i.e. when (6.1) is satisfied, i.e. at a line of marginal stability. We will come back to this point in the next section as well as in Section 6.5.1.

6.3. B-type boundary states in some specific examples

6.3.1. The mirror geometric hypothesis

We have almost everything said in order to state the modified or mirror geometric hypothesis of Douglas [2]. The last ingredient we need are some general considerations on the world-volume effective action of a D-brane. The simplest quantities to look at are the superpotential and the gauge kinetic term because they are holomorphic. Since we are working at tree level and with $\mathrm{d}B = 0$ (see (2.43)) the gauge kinetic term is trivial. However, we can have a non-trivial superpotential W at $g_s \sim 0$. In this case, the bosonic potential then gets a contribution this F-term. Douglas has argued in [5], [2] that there might be a counterpart for the non-renormalization theorem which protects $\mathcal{F}_C$ from corrections from the Kähler moduli t_a of X. The superpotential W should depend only on the t_a for A-type branes and not on the complex structure moduli x_i of X. A concrete realization of W has been given in [207], [266] and [287] which supports this claim.

On the other hand, for a B-type D-brane W should be independent of the t_a and only depend on the x_i, and furthermore be exact, i.e. equal to the large volume result. This is also referred to as the decoupling statement. More precisely the statement is

$$W = m(t_a)W(x_i, v_r) \tag{6.12}$$

where v_r are the intrinsic D-brane moduli and $m(t_a)$ is the mass of the D-brane (5.59). As a consequence of this statement, the moduli space $\mathcal{M}_D$ of B-type D-branes would everywhere in Kähler moduli space be the same as in the large volume limit. In particular, one could compute W at the Gepner point using the methods developed in [288] and would know the result in the large volume limit. As mentioned at the end of Section 4.3.2, the $\sum L_j = 0$ boundary states in the Gepner model are equivalent [152] to the fractional D-brane states in a corresponding Landau-Ginzburg orbifold theory. These can be translated into a so-called quiver gauge theory [176]. In this setup it is possible to write down an ansatz for the superpotential [236], [152] which is to be compared with the results from the Gepner model and the ansatz (5.95) for obstructions to deformations of sheaves.

The superpotential can also contain Fayet-Illiopoulos (FI) D-terms which naturally depend on the Kähler moduli. For A-type D-branes these D-terms are related to stability and the Joyce transitions in Section 6.2 [286]. For the B-type D-branes they are also related to stability [2], more precisely to the phenomenon of enhanced gauge symmetry that appears at the walls in the Kähler cone mentioned in Section 5.3.2. This will be discussed in the next section in more detail.

Having said all this, we can now state the mirror geometric hypothesis of Douglas [2]. Some properties of A-type D-branes, and others of B-type D-branes are determined by geometry. The remaining properties of A-type D-branes can then be determined via mirror symmetry by those of the B-type D-branes, and vice versa. More precisely, the D-brane spectrum, i.e. the central charges and stability can be understood geometrically for A-type D-branes. The D-brane moduli spaces, i.e. their dimension, the superpotential can be studied classically for B-type D-branes. Finally, it may be possible to determine the D-terms, i.e. singularities in the moduli space, for A-type D-branes by geometry. Mirror symmetry should then allow us to compute the remaining properties of the respective type of D-branes. This hypothesis is another, strong manifestation of D-geometry.

In the remainder of this chapter we will explicitly test part of this conjectured hypothesis as well as the underlying conjectured decoupling statement. We will compare the moduli spaces of B-type D-branes, in particular their dimension, at the Gepner point and in the large volume limit. According to the conjecture they should coincide because they are supposed to be independent of the Kähler moduli. We will also compare the spectra at both points in Kähler moduli space and we expect to find discrepancies due to world-sheet instanton effects. We will do this for a representative subset of the families discussed in the examples in Section 3.5 and in Appendix C.

6.3.2. The comparison

We compute and compare the charges not only to know the spectrum but also to work out the stability issue. Remember that a D-brane boundary state is stable if it satisfies (4.45) which is in principle expressed through the charges of the boundary states. On the other hand, we have a notion of stability in the large volume limit in terms of stable sheaves. The Bogomolov criterion (5.20) is expressed in terms of their Chern classes, i.e. of the D-brane charges.

For this comparison we need to express the charges of the B-type boundary states in terms of the large volume charges. A precise form of this comparison is to choose a path in Kähler moduli space and use the flat $Sp(2\tilde{h}^{1,1}+2,\mathbb{Z})$ connection provided by special geometry (see Section 3.4) to transport the charge lattices between the two regimes. This has been performed in the previous chapters, mainly in Section 5.4. In Sections 2.6, 3.3.1, 3.4 and 3.6 we have provided the results necessary to understand the Kähler moduli space and the prepotential. In particular, we can transport the information from the Gepner point to the large volume limit by the analytic continuation matrix M in (3.78). This matrix can be determined by the methods shown in the Appendices A and B.

However, as noted in Section 3.4.2 there is an $Sp(2\tilde{h}^{1,1}+2,\mathbb{Z})$ ambiguity in the periods. Most of it can be resolved by using an important result from mirror symmetry computations on the quintic [104], [289] and [290]. We mentioned in Section 3.5.1 that at the conifold point of the mirror quintic a three-cycle degenerates, i.e. has vanishing period. It turns out that the mirror cycle on the original quintic is the quintic itself, hence the central charge (5.63) of this cycle corresponds to the "pure" D6-brane, i.e. with large volume charges $n(\mathrm{D6})^{(L)} = (1, 0, \ldots, 0)$ in (5.62). This allows us to fix the ambiguity in such a way that the matrix M remains unchanged [5]. Now, that we have a relation between the large volume limit and the conifold point, we need a relation between the latter and the Gepner point. This is given by the fact [104] that this vanishing period can be computed in terms of the periods at the Gepner point (3.75) to be $\Pi = \varpi_1 - \varpi_0$. Hence, the charge vector of this state in the Gepner basis reads $n(\mathrm{D6})^{(G)} = (-1, 1, 0, \ldots, 0)$.

Next, we need to figure out which boundary state corresponds to the state with charges $n(\mathrm{D6})^{(G)}$. As explained in Section 6.1 due to difficulties in normalization it is more convenient to compute the Witten index instead of comparing the charges directly. In that section we also reviewed the argument that the Witten index corresponds to the intersection form I^B, see (6.9). Now at the Gepner point, the left-hand side of this equation is the intersection form (4.41), while the right-hand side is by (5.49) and (3.62) related to $I^{(G)}$ in (3.79). Both expressions can be written as polynomials in the generator g of the quantum symmetry $\mathbb{Z}_d$ in (2.119), where $d = K'$ in order to connect with the notation used in Section 4.3. Comparing the polynomials we see that the one for the $\sum_j L_j = 0$ boundary states in (4.42) exactly matches the one from (3.79). Therefore we can associate the state $\mid 0; M; 0 \rangle\!\rangle_B$ for, say $M = 0$, with the state $n(\mathrm{D6})^{(G)}$[1].

We can obtain the charges for different M by acting with $A^{(G)}$ on $n(\mathrm{D6})^{(G)}$, or equivalently with $A^{(L)}$ on $n(\mathrm{D6})^{(L)}$ which implements the action of $g : M \to M + 2$. The action of $h : S \to S + 2$ is similarly implemented by $n^{(L)} \to -n^{(L)}$. The charges of the states with $\sum_j L_j > 0$ can be obtained from $n(\mathrm{D6})^{(G)}$ by replacing g in (4.40) with $A^{(L)}$. When $\sum_j L_j$ is odd, we also have to multiply the boundary states by $g^{\frac{1}{2}}$. The number of moduli $m^{(\mathrm{CFT})}$ is obtained from (4.48) by the same replacement. Note that the number of marginal operators we obtain are only upper bounds for the dimension of the moduli space as in general these theories will have superpotentials (see also Section 5.5.1).

We have one more important information at our disposal, namely the numbers ν and $\widetilde{\nu}$ in (4.46) and (4.54), respectively. It was suggested in [180], observed in [291] for D4-branes and in [154] for D0-branes that boundary states with $\nu > 1$ may not describe single branes but collections of several branes, and this would correspond to reducible sheaves. In [184] it was shown that each of the $\widetilde{\nu}$ vertex operators that describe the emission of $U(1)$ gauge bosons on the boundary in fact has additional degrees of freedom in terms of $N \times N$ matrices, i.e. each of these vertex operators is associated with a $U(N)$ gauge symmetry. The relation (4.54) therefore reflects the splitting of the collection of ν gauge fields into $\widetilde{\nu}$ separate families, each containing N^2 gauge fields carrying the adjoint representation of $U(N)$. In other words, the D-brane boundary states at the fixed point split into $\widetilde{\nu}$ independent N-fold bound states. Furthermore, consistency requires that the allowed charges Q are an integral multiple of some minimal charge $Q_{\min}$. Recall that (4.54) is due to the analog of non-trivial discrete torsion in the minimal models. It is known [190] that for orbifolds with discrete torsion the RR charge is a multiple of a minimal charge determined by the orbifold group, $Q = \frac{d_R}{|\Gamma|}$. The analog for Gepner models is [184]

$$Q = N Q_{\min} \tag{6.13}$$

The appearance of an enhanced gauge symmetry is a hallmark of a singularity. As mentioned in Section 4.3.3 the simple current responsible for this effect appears generically on the boundary, hence the singularity should lie in the sheaf describing the D-brane. Indeed [184], the sheaves corresponding to these N-fold bound states are precisely the properly semi-stable sheaves $\mathcal{E}$ discussed in Section 5.3.2. If $\mathcal{E}$ is a bundle then it has a reducible connection since the gauge group is $U(N)^{\widetilde{\nu}}$. For such sheaves the geometry of the moduli space is more intricate than for irreducible ones as the relative positions of the configurations correspond to Coulomb branches of additional moduli [222]. This agreement is a first non-trivial test of the mirror geometric hypothesis which will be used to the describe the sheaves below.

In the following sections we give the results of this comparison.

[1] Note that in [5], [180], [291], [154] and [175] the two polynomials were related by an additional basis transformation: $I^B_{00} = (1-g)I^{(G)}(1-g^{-1})$. In the method described in Appendix B this is automatically taken into account, see (B.13).

6.3.3. D-branes on $\mathbb{P}^4_{1,1,2,2,2}[8]$

D6-branes

Here $n = (n_6, n_4^E, n_4^L, n_0, n_2^h, n_2^l)$ and we need the intersections numbers in (C.1), (C.2) and (C.8)[2]

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_6 && (6.14)\\
\mathrm{c}_1(\mathcal{F}) &= n_4^E E + n_4^L L && (6.15)\\
\mathrm{c}_2(\mathcal{F}) &= \left(2n_4^E(n_4^L - n_4^E) - n_2^h\right) h + \left((n_4^E)^2 - n_2^l\right) l && (6.16)\\
\mathrm{c}_3(\mathcal{F}) &= \frac{2}{3}\left(n_4^E\right)^2\left(3n_4^L - 4n_4^E\right) + n_4^E\left(2n_2^l - n_2^h\right) - n_4^L n_2^l && (6.17)\\
&\quad -2n_0 - 4n_4^L + \frac{1}{3}\chi_C n_4^E
\end{aligned}$$

where $\chi_C = -4$ for $\mathbb{P}^4_{1,1,2,2,2}[8]$ (see Section C.1.1). Let us shortly explain the appearance of χ_C in (6.17). The terms linear in the n's come from $\mathrm{ch}_3(\mathcal{F})$ with an additional factor of 2 since $\mathrm{c}_3 = 2\,\mathrm{ch}_3 - \mathrm{ch}_1\mathrm{ch}_2 + \frac{1}{6}\mathrm{ch}_1^3$. Furthermore, by using $\chi(E) = 4(1-g_C)$ for a ruled surface (see Section 3.3.2) and (3.37) as well as $E^3 = 4\chi_C$ [142] the contribution of the divisor E to $\mathrm{c}_3(\mathcal{F})$ is

$$-\frac{1}{2}n_4^E\ \mathrm{c}_2\cdot E = -\frac{1}{2}(\chi(E) - E^3)n_4^E = (-2 + 2g_C + 2\chi_C)n_4^E = \chi_C n_4^E \qquad (6.18)$$

D4-branes

- **The divisor L, a K3 surface**
  For this divisor we have $n = (0, 0, n_4^L, n_0, n_2^h, 0)$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this family become

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^L && (6.19)\\
\mathrm{c}_1(\mathcal{F}) &= n_2^h h && (6.20)\\
\mathrm{ch}_2(\mathcal{F}) &= -2n_4^L - n_0 && (6.21)
\end{aligned}$$

  From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\mathrm{geom},\ L)}(\mathcal{F}) = \frac{1}{4}\left(n_2^h\right)^2 + 2n_4^L\left(n_4^L + n_0\right) + 2 \qquad (6.22)$$

  The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.1. We observe that

$$\Delta^{(L)} = m^{(\mathrm{CFT},L)} - m^{(\mathrm{geom},L)} = 3\nu - 2 \qquad (6.23)$$

  Since the quantum symmetry group of the $K3$ surface is $\mathbb{Z}_4$ we would expect four states per orbit. Generally, the states come in pairs of a brane and its anti-brane and we indicate only the brane. Here, however, in the first orbit, the states do not come in pairs which is due to the fact that the matrix A does not satisfy $A^4 = -\mathbb{1}$. We will come back to this observation in Section 6.5.

- **The divisor E, a ruled surface over a $g = 3$ curve**
  For this divisor we have $n = (0, n_4^E, 0, n_0, n_2^h, n_2^l)$ since $D\cdot E = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^E && (6.24)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - 4n_4^E\right) h + \left(n_2^l + 2n_4^E\right) l && (6.25)\\
\mathrm{ch}_2(\mathcal{F}) &= -2n_2^E + \frac{1}{2}n_2^h - n_2^l - n_0 && (6.26)
\end{aligned}$$

[2] Note that in Sections 6.3.3, 6.3.4, and 6.3.5 we use a non-canonical symplectic intersection form $I^{(L)}$ in (3.79), see [154] and [180]

L-orbit	Mukai vector $v=(n_4^L, n_2^h, -n_4^L-n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\mid 1,0,0,0,0\,\rangle\!\rangle_B$	$(3,-4,\ \ 1)$ $(3,-8,\ \ 3)$ $(1,-4,\ \ 3)$ $(1,\ \ 0,\ \ 1)$	1	1	0	1
$\mid 3,0,0,0,0\,\rangle\!\rangle_B$	$(0,\ \ 4,-2)$ $(2,-4,\ \ 0)$	7	1	6	1
$\mid 3,0,1,0,0\,\rangle\!\rangle_B$	$(2,-8,\ \ 2)$ $(2,\ \ 0,-2)$	14	2	10	4
$\mid 3,0,1,1,0\,\rangle\!\rangle_B$	$(4,-8,\ \ 0)$ $(0,\ \ 8,-4)$	28	4	18	10
$\mid 3,0,1,1,1\,\rangle\!\rangle_B$	$(4,\ \ 0,-4)$ $(4,-16,\ \ 4)$	56	8	34	22

Table 6.1.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E

$$m^{(\mathrm{geom},E)} = 2\left(n_4^E\right)^2 + 2n_0 n_4^E + \frac{1}{2} n_2^h n_2^l + 1 \tag{6.27}$$

The boundary states with $n_4^E \neq 0$ corresponding to D4-branes wrapped on E are displayed in table 6.2. We note that

$$\Delta^{(E)} = m^{(\mathrm{CFT},E)} - m^{(\mathrm{geom},E)} = \nu - 1 \tag{6.28}$$

The rank one bundles in the first line are $\mathcal{O}_E$ and $\mathcal{O}_E(-4h)$. The rank two bundles in the second line are topologically equivalent to $\mathcal{O}_E^{\oplus 2}$ and $\mathcal{O}_E^{\oplus 2}(-4h)$. According to the discussion in Section 6.3.2 they have $\widetilde{\nu}=1$ and $N=2$ and are therefore $U(2)$ bundles and not holomorphic direct sums.

L-orbit	$n=(n_4^E, n_0, n_2^h, n_2^l)$	$m^{(\mathrm{CFT},E)}$	ν	$m^{(\mathrm{geom},E)}$	$\Delta^{(E)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 2,\ \ 4,-2)$ $(1,\ \ 0,\ \ 0,-2)$	3	1	3	0
$\mid 0,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 4,\ \ 8,-4)$ $(2,\ \ 0,\ \ 0,-4)$	12	4	9	3

Table 6.2.: The boundary states corresponding to D4-branes wrapped on the ruled surface E

- **The divisor H**
For this divisor we have $n=(0,n_4^E,2n_4^E,n_0,n_2^h,n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^E && (6.29)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + 4n_4^E\right) h + \left(n_2^l + 2n_4^E\right) l && (6.30)\\ \mathrm{ch}_2(\mathcal{F}) &= -4n_4^E + \frac{1}{2} n_2^h - n_0 && (6.31)\end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = 4\left(n_4^E\right)^2 + 2n_4^E n_0 + \frac{1}{2} n_2^l\left(n_2^h - n_2^l\right) + 6 \tag{6.32}$$

The boundary states with $n_4^E \neq 0$ corresponding to D4-branes wrapped on the divisor H are displayed in table 6.3. Here there is no obvious relationship between $m^{(\mathrm{CFT},H)}$ and $m^{(\mathrm{geom},H)}$. However, we like to point out the fact that in the cases where there are several sheaves in one L-orbit, all of them have the same dimension of the geometric moduli space. Note also that the states in the two orbits, $\mid 3,3,0,0,0\,\rangle\!\rangle_B$ and $\mid 2,0,1,1,1\,\rangle\!\rangle_B$, which have the same charges and twice the charges of the states in the orbit $\mid 2,0,1,0,0\,\rangle\!\rangle_B$ but differ in the number of moduli and vacua. We will come back to this observation in Section 6.4.

L-orbit	$n = (n_4^E, n_2^h, n_2^l, n_0)$	$m^{(\mathrm{CFT},H)}$	ν	$m^{(\mathrm{geom},H)}$	$\Delta^{(H)}$
$\vert\, 2,0,1,0,0\,\rangle\!\rangle_B$	$(1,-4,-2,-2)$ $(1,\ \ 0,-2,\ \ 0)$	11	1	8	3
$\vert\, 3,2,0,0,0\,\rangle\!\rangle_B$	$(2,-4,-4,-2)$	23	1	14	9
$\vert\, 3,3,0,0,0\,\rangle\!\rangle_B$	$(2,-8,-4,-4)$ $(2,\ \ 0,-4,\ \ 0)$	30	2	14	16
$\vert\, 2,0,1,1,1\,\rangle\!\rangle_B$	$(2,-8,-4,-4)$ $(2,\ \ 0,-4,\ \ 0)$	44	4	14	30
$\vert\, 3,2,1,1,0\,\rangle\!\rangle_B$	$(4,-8,-8,-4)$	92	4	38	54
$\vert\, 3,3,1,1,0\,\rangle\!\rangle_B$	$(4,-16,-8,-8)$ $(4,\ \ 0,-8,\ \ 0)$	120	8	38	82

Table 6.3.: The boundary states corresponding to D4-branes wrapped on the divisor H

6.3.4. D-branes on $\mathbb{P}^4_{1,1,2,2,6}[12]$

D6-branes

Due to the great similarity of $\mathbb{P}^4_{1,1,2,2,6}[12]$ to $\mathbb{P}^4_{1,1,2,2,2}[8]$, the formulae (6.14) to (6.17) hold also in this case. We only need from Section 3.5.2 that $\chi_C = -2$.

D4-branes

- **The divisor L, a K3 surface**
  For this divisor we have $n = (0, 0, n_4^L, n_0, n_2^h, 0)$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\mathrm{rk}(\mathcal{F}) = n_4^L \tag{6.33}$$

$$\mathrm{c}_1(\mathcal{F}) = n_2^h h \tag{6.34}$$

$$\mathrm{ch}_2(\mathcal{F}) = -2n_4^L - n_0 \tag{6.35}$$

From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\mathrm{geom},\ L)}(\mathcal{F}) = \frac{1}{2}\left(n_2^h\right)^2 + 2n_4^L\left(n_4^L + n_0\right) + 2 \tag{6.36}$$

The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.4. We observe that

$$\Delta^{(L)} = m^{(\mathrm{CFT},L)} - m^{(\mathrm{geom},L)} = 3\nu - 2 \tag{6.37}$$

L-orbit	Mukai vector $v = (n_4^L, n_2^h, -n_4^L - n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\vert\, 1,0,0,0,0\,\rangle\!\rangle_B$	$(2,-2,\ \ 1)$ $(1,\ \ 0,\ \ 1)$ $(1,-2,\ \ 2)$	1	1	0	1
$\vert\, 3,0,0,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-1)$ $(0,\ \ 2,-1)$ $(1,-2,\ \ 0)$	5	1	4	1
$\vert\, 3,0,1,0,0\,\rangle\!\rangle_B$	$(1,-4,\ \ 1)$ $(2,-2,-1)$ $(1,\ \ 2,-2)$	9	1	8	1
$\vert\, 3,0,1,1,0\,\rangle\!\rangle_B$	$(0,\ \ 6,-3)$ $(3,-6,\ \ 0)$ $(3,\ \ 0,-3)$	21	1	20	1
$\vert\, 5,0,0,0,0\,\rangle\!\rangle_B$	$(2,\ \ 0,\ \ 0)$ $(0,\ \ 0,\ \ 2)$ $(2,-4,\ \ 2)$	6	2	2	4
$\vert\, 5,0,1,0,0\,\rangle\!\rangle_B$	$(2,-4,\ \ 0)$ $(2,\ \ 0,-2)$ $(0,\ \ 4,-2)$	14	2	10	4
$\vert\, 5,0,1,1,0\,\rangle\!\rangle_B$	$(2,-8,\ \ 2)$ $(4,-4,-2)$ $(2,\ \ 4,-4)$	30	2	26	4
$\vert\, 5,0,2,0,0\,\rangle\!\rangle_B$	$(0,\ \ 4,\ \ 0)$ $(4,-4,\ \ 0)$ $(0,\ \ 4,-4)$	20	4	10	10
$\vert\, 5,0,2,1,0\,\rangle\!\rangle_B$	$(4,-8,\ \ 0)$ $(4,\ \ 0,-4)$ $(0,\ \ 8,-4)$	44	4	34	10
$\vert\, 5,0,2,2,0\,\rangle\!\rangle_B$	$(4,-12,\ \ 4)$ $(4,-4,-4)$ $(4,\ \ 4,-4)$	64	8	42	22

Table 6.4.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

- **The divisor E, a ruled surface over a $g = 2$ curve**
  For this divisor we have $n = (0, n_4^E, 0, n_0, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^E && (6.38)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - 2n_4^E\right) h + \left(n_2^l + n_4^E\right) l && (6.39)\\ \mathrm{ch}_2(\mathcal{F}) &= -n_2^E + \frac{1}{2} n_2^h - n_2^l - n_0 && (6.40) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E

$$m^{(\mathrm{geom},E)} = \left(n_4^E\right)^2 + 2n_0 n_4^E + n_2^h n_2^l + 1 \tag{6.41}$$

The boundary states with $n_4^E \neq 0$ corresponding to D4-branes wrapped on E are displayed in table 6.5. We note that except for the last boundary state we have

$$\Delta^{(E)} = m^{(\mathrm{CFT},E)} - m^{(\mathrm{geom},E)} = \nu - 1 \tag{6.42}$$

The two line bundles in the first line are $\mathcal{O}_E$ and $\mathcal{O}_E(-2h)$. The rank two bundles in the second line are topologically equivalent to $\mathcal{O}_E^{\oplus 2}$ and $\mathcal{O}_E^{\oplus 2}(-2h)$. According to the discussion in Section 6.3.2 they have $\widetilde{\nu} = 2$ and $N = 1$ and are therefore $U(1) \times U(1)$ bundles and also holomorphic direct sums. The rank two sheaves in the third line have Chern classes $(-4h \pm 2l, \mp 2)$ and $(\mp 2l, 0)$. Finally, the rank two sheaves in the last line have Chern classes $(-8h - 2l, 8)$ and $(4h + 2l, 6)$. Both of them have $\widetilde{\nu} = 1$ and $N = 2$ and hence gauge group $U(2)$.

L-orbit	$n = (n_4^E, n_0, n_2^h, n_2^l)$	$m^{(\mathrm{CFT},E)}$	ν	$m^{(\mathrm{geom},E)}$	$\Delta^{(E)}$
$\mid 0,0,1,0,0 \rangle\!\rangle_B$	$(1, 1, 2, -1)$ $(1, 0, 0, -1)$	2	1	2	0
$\mid 0,0,2,1,0 \rangle\!\rangle_B$	$(2, 2, 4, -2)$ $(2, 0, 0, -2)$	6	2	5	1
$\mid 0,0,2,2,0 \rangle\!\rangle_B$	$(2, 0, 0, -4)$ $(2, 0, 0, 0)$ $(2, 4, 4, -4)$ $(2, 0, 4, 0)$	8	4	5	3
$\mid 2,0,2,2,0 \rangle\!\rangle_B$	$(2, 0, -4, -4)$ $(2, 4, 8, 0)$	32	4	21	11

Table 6.5.: The boundary states corresponding to D4-branes wrapped on the ruled surface E

- **The divisor H**
  For this divisor we have $n = (0, n_4^E, 2n_4^E, n_0, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^E && (6.43)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + 2n_4^E\right) h + \left(n_2^l + n_4^E\right) l && (6.44)\\ \mathrm{ch}_2(\mathcal{F}) &= -4n_4^E + \frac{1}{2} n_2^h - n_0 && (6.45) \end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = 4\left(n_4^E\right)^2 + 2n_4^E n_0 + n_2^l\left(n_2^h - n_2^l\right) + 5 \tag{6.46}$$

The boundary states with $n_4^E \neq 0$ corresponding to D4-branes wrapped on the divisor H are displayed in table 6.6. Here there is no obvious relationship between $m^{(\mathrm{CFT},H)}$ and $m^{(\mathrm{geom},H)}$. There is a boundary state for which there are more geometric moduli than conformal field theory moduli. In the cases where there are several sheaves in one L-orbit, all of them have the same dimension of the geometric moduli space.

L-orbit	$n = (n_4^E, n_0, n_2^h, n_2^l)$	$m^{(\mathrm{CFT},H)}$	ν	$m^{(\mathrm{geom},H)}$	$\Delta^{(H)}$
$\mid 3,1,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-2,-1)\ (1,\ 0,\ 0,-1)$	10	1	8	2
$\mid 4,0,1,0,0\,\rangle\!\rangle_B$	$(1,-1,-2,-1)\ (1,\ 0,\ 0,-1)$	12	1	8	4
$\mid 2,2,1,0,0\,\rangle\!\rangle_B$	$(2,-2,-4,-2)\ (2,\ 0,\ 0,-2)$	23	1	17	6
$\mid 4,3,0,0,0\,\rangle\!\rangle_B$	$(2,-1,-2,-2)$	25	1	17	8
$\mid 3,3,1,0,0\,\rangle\!\rangle_B$	$(3,-3,-6,-3)\ (3,\ 0,\ 0,-3)$	42	1	32	10
$\mid 5,3,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-4,-2)\ (2,\ 0,\ 0,-2)$	30	2	17	13
$\mid 2,2,2,0,0\,\rangle\!\rangle_B$	$(2,-4,-8,-4)\ (2,\ 0,-4,-4)$ $(2,\ 0,\ 0,\ 0)\ (2,\ 0,\ 4,\ 0)$	34	2	21	13
$\mid 5,3,1,1,0\,\rangle\!\rangle_B$	$(6,-6,-12,-6)\ (6,\ 0,\ 0,-6)$	126	2	113	13
$\mid 2,2,2,1,0\,\rangle\!\rangle_B$	$(4,-4,-8,-4)\ (4,\ 0,\ 0,-4)$	70	2	53	17
$\mid 4,0,2,1,0\,\rangle\!\rangle_B$	$(2,-2,-4,-2)\ (2,\ 0,\ 0,-2)$	38	2	17	21
$\mid 5,4,1,0,0\,\rangle\!\rangle_B$	$(4,-2,-4,-4)$	78	2	53	25
$\mid 4,4,2,0,0\,\rangle\!\rangle_B$	$(4,-2,-8,-6)\ (4,\ 0,\ 0,-2)$	98	2	65	33
$\mid 2,0,2,2,0\,\rangle\!\rangle_B$	$(2,-4,-4,\ 0)\ (2,\ 0,\ 0,-4)$	32	4	5	27
$\mid 5,2,2,0,0\,\rangle\!\rangle_B$	$(0,\ 4,12,\ 4)\ (4,-4,-4,-4)$	68	4	37	31
$\mid 4,0,2,2,0\,\rangle\!\rangle_B$	$(2,-4,-8,-4)\ (2,\ 0,-4,-4)$ $(2,\ 0,\ 0,\ 0)\ (2,\ 0,\ 4,\ 0)$	56	4	21	35
$\mid 5,5,1,0,0\,\rangle\!\rangle_B$	$(4,-4,-8,-4)\ (4,\ 0,\ 0,-4)$	92	4	53	39
$\mid 5,4,2,1,0\,\rangle\!\rangle_B$	$(8,-4,-8,-8)$	236	4	197	39
$\mid 4,2,2,2,0\,\rangle\!\rangle_B$	$(6,-4,-12,-8)\ (6,\ 0,\ 0,-4)$	176	4	133	43
$\mid 5,4,2,0,0\,\rangle\!\rangle_B$	$(4,\ 0,\ 4,\ 0)\ (4,\ 0,-4,-4)$ $(4,-4,-12,-8)$	116	4	69	47
$\mid 5,5,2,0,0\,\rangle\!\rangle_B$	$(4,-8,-16,-8)\ (4,\ 0,-8,-8)$ $(4,\ 0,\ 0,\ 0)\ (4,\ 0,\ 8,\ 0)$	136	4	69	67
$\mid 5,5,2,1,0\,\rangle\!\rangle_B$	$(8,-8,-16,-8)\ (8,\ 0,\ 0,-8)$	280	8	197	83
$\mid 4,3,1,1,0\,\rangle\!\rangle_B$	$(6,-3,-6,-6)$	105	1	113	-8

Table 6.6.: The boundary states corresponding to D4-branes wrapped on the divisor H

6.3.5. D-branes on $\mathbb{P}^4_{1,1,1,6,9}[18]$

D6-branes

Here $J_1 = H$, $J_2 = S$ but we use F and S as basis, hence $n = (n_6, n_4^F, n_4^S, n_0, n_2^h, n_2^l)$ and we need the intersections numbers in (C.64), (C.66) to (C.69) and (C.73).

$$\mathrm{rk}(\mathcal{F}) = n_6 \tag{6.47}$$

$$\mathrm{c}_1(\mathcal{F}) = n_4^F F + n_4^S S \tag{6.48}$$

$$\mathrm{c}_2(\mathcal{F}) = \frac{1}{2}\left(\left(n_4^S\right)^2 + n_4^F + n_4^S - 2n_2^h\right) h + \frac{1}{2}\left(-3\left(n_4^F\right)^2 + 2n_4^F n_4^S + n_4^F - n_2^l\right) l \tag{6.49}$$

$$\begin{aligned}\mathrm{c}_3(\mathcal{F}) = {} & \frac{1}{2}\left(3\left(n_4^F\right)^3 - 3\left(n_4^F\right)^2 n_4^S + n_4^F\left(n_4^S\right)^2\right) + \frac{n_4^F}{2}\left(n_4^F + 3n_4^S\right) \\ & - n_4^F n_2^h - n_4^S n_2^l + n_4^F - 6n_4^S - 2n_0\end{aligned} \tag{6.50}$$

The FMW construction

Here we repeat the argument given in [270] and check whether corresponding boundary states exist. We have seen in Section C.2 that the base of the elliptic fibration $\pi : X \to B$, and therefore the section σ, is a $\mathbb{P}^2$. Furthermore the curve l can be identified with a degree 1 curve in $\mathbb{P}^2$, i.e. a $\mathbb{P}^1$. Recall from Section 3.3.2 that the canonical bundle of $\mathbb{P}^1$ is $K_{\mathbb{P}^2} = -3l = -3S|_F$. A general bundle V will have topological invariants (6.47) to (6.50). Setting $n_4^F = 0$ and using (C.66), (6.49) and (5.111) yields for the dimension of the moduli space [270]

$$n_6 - 3n_2^l + 6n_4^S + 4n_2^h - 2\left(n_4^S\right)^2 \tag{6.51}$$

On the other hand, the bundles constructed in [271] satisfy $\mathrm{c}_1(V) = \mathrm{c}_3(V) = 0$, $\eta = a\,\mathrm{c}_1(\mathbb{P}^2)$ for a odd and n has to be even. In this case, (5.112) gives

$$\mathrm{c}_2(V) = 3al - \frac{3}{8}\left((n^3 - n) + 3a(a-n)n\right)h \tag{6.52}$$

and by comparison with (6.49) one finally obtains

$$\begin{aligned} n_6 &= n && (6.53)\\ n_2^h &= \frac{3}{8}\left(n^3 - n + 3a(a-n)n\right) && (6.54)\\ n_2^l &= -3a && (6.55)\\ n_0 &= n_4^F = n_4^S = 0 && (6.56) \end{aligned}$$

We do not find any boundary states leading to these charges n, because a is never integral and odd. We note that all bundles with n_6 even and $n_4^S = n_4^F = 0$ have the property that $n_2^l = 3n_2^h$. This equation has no real solutions for a given n an even integer.

D4-branes

- **The divisor S, an elliptic surface**
  For this divisor we have $n = (0, 0, n_4^S, n_0, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^S && (6.57)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + 2n_4^S\right)h + n_2^l l && (6.58)\\ \mathrm{ch}_2(\mathcal{F}) &= -3n_4^S + \frac{1}{2}n_2^l - n_0 && (6.59) \end{aligned}$$

  as well as (5.99) yields for the dimension of the moduli space of the sheaves on S

$$m^{(\mathrm{geom},S)} = 3\left(n_4^S\right)^2 + 2n_4^S n_0 + n_2^l\left(3n_4^S + 2n_2^h - 3n_2^l\right) + 3 \tag{6.60}$$

  The boundary states corresponding to D4-branes wrapped on S are displayed in table 6.7. Here we have for the sheaves in the first part of the table

$$\Delta_1^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = 5(\nu - 1) + 2 \tag{6.61}$$

  for the sheaves in the second part

$$\Delta_2^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = 7(\nu - 1) + 4 \tag{6.62}$$

  while for the sheaves in the last part we don't see an obvious relation.

- **The divisor F, a $\mathbb{P}^2$**
  For this divisor we have $n = (0, n_4^F, 0, n_0, 0, n_2^l)$ since $H \cdot F = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^F && (6.63)\\ \mathrm{c}_1(\mathcal{F}) &= n_2^l l && (6.64)\\ \mathrm{ch}_2(\mathcal{F}) &= -n_4^F - \frac{3}{2}n_2^l - n_0 && (6.65) \end{aligned}$$

  as well as (5.99) yields for the dimension of the moduli space of the sheaves on F

$$m^{(\mathrm{geom},F)} = n_4^F\left(2n_0 + n_4^F\right) + n_2^l\left(3n_4^F + n_2^l\right) + 1 \tag{6.66}$$

L-orbit	$n=(n_4^S,n_0,n_2^h,n_2^l)$	$m^{(\mathrm{CFT},S)}$	ν	$m^{(\mathrm{geom},S)}$	$\Delta^{(S)}$
$\mid 7,0,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-1,0)\ (0,0,2,1)\ (1,-1,-3,-1)$	6	1	4	2
$\mid 5,1,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-5,-2)\ (2,-2,-4,-1)\ (1,-1,1,1)$	8	1	6	2
$\mid 6,1,0,0,0\,\rangle\!\rangle_B$	$(1,1,0,0)\ (0,1,4,1)\ (1,0,-4,-1)$	10	1	8	2
$\mid 2,2,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-4,-1)\ (1,0,-3,-1)\ (1,0,-1,0)$ $(1,0,0,0)\ (0,1,3,1)\ (0,0,3,1)$	8	1	6	2
$\mid 4,2,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-6,-2)\ (2,-1,-4,-1)\ (1,0,2,1)$	12	1	10	2
$\mid 5,2,0,0,0\,\rangle\!\rangle_B$	$(2,0,-3,-1)\ (1,1,2,1)\ (1,0,3,1)$ $(1,-2,-7,-2)\ (1,0,-6,-2)\ (2,-1,-5,-1)$	14	1	12	2
$\mid 6,2,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-7,-2)\ (2,0,-4,-1)\ (1,1,3,1)$	16	1	14	2
$\mid 4,3,0,0,0\,\rangle\!\rangle_B$	$(2,0,-1,0)\ (0,1,6,2)\ (2,-1,-7,-2)$	17	1	15	2
$\mid 5,3,0,0,0\,\rangle\!\rangle_B$	$(1,-1,-8,-2)\ (2,1,-4,-1)\ (1,2,4,1)$	20	1	18	2
$\mid 5,4,0,0,0\,\rangle\!\rangle_B$	$(3,-1,-2,0)\ (0,1,8,3)\ (3,-2,-10,-3)$	26	1	24	2
$\mid 5,5,0,0,0\,\rangle\!\rangle_B$	$(3,-3,-12,-3)\ (3,0,-9,-3)\ (3,0,-3,0)$ $(3,0,0,0)\ (0,3,9,3)\ (0,0,9,3)$	32	1	30	2
$\mid 6,5,0,0,0\,\rangle\!\rangle_B$	$(3,1,-1,0)\ (0,2,10,3)\ (3,-1,-11,-3)$	38	1	36	2
$\mid 8,1,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-2,0)\ (0,0,4,2)\ (2,-2,-6,-2)$	14	2	7	7
$\mid 8,2,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-8,-2)\ (2,0,-6,-2)\ (2,0,-2,0)$ $(2,0,0,0)\ (0,2,6,2)\ (0,0,6,2)$	22	2	15	7
$\mid 8,3,0,0,0\,\rangle\!\rangle_B$	$(2,2,0,0)\ (0,2,8,2)\ (2,0,-8,-2)$	30	2	23	7
$\mid 8,4,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-12,-4)\ (4,-2,-8,-2)\ (2,0,4,2)$	38	2	31	7
$\mid 8,5,0,0,0\,\rangle\!\rangle_B$	$(2,0,6,2)\ (2,-4,-14,-4)\ (2,0,-12,-4)$ $(4,-2,-10,-2)\ (4,2,-6,-2)\ (2,2,4,2)$	49	2	39	7
$\mid 8,6,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-14,-4)\ (4,0,-8,-2)\ (2,2,6,2)$	54	2	47	7
$\mid 8,8,0,0,0\,\rangle\!\rangle_B$	$(4,-4,-16,-4)\ (4,0,-12,-4)\ (4,0,-4,0)$ $(4,0,0,0)\ (0,4,12,4)\ (0,0,12,4)$	68	4	51	17
$\mid 7,2,0,0,0\,\rangle\!\rangle_B$	$(2,0,-1,0)\ (0,1,6,2)\ (2,-1,-7,-2)$	19	1	15	4
$\mid 7,5,0,0,0\,\rangle\!\rangle_B$	$(2,1,5,2)\ (2,-2,-13,-4)\ (4,-1,-8,-2)$	43	1	39	4
$\mid 7,8,0,0,0\,\rangle\!\rangle_B$	$(4,0,-2,0)\ (0,2,12,4)\ (4,-2,-14,-4)$	62	2	51	11
$\mid 3,3,1,0,0\,\rangle\!\rangle_B$	$(3,1,-1,0)\ (0,2,10,3)\ (3,-1,-11,-3)$	28	1	36	-8
$\mid 5,2,1,0,0\,\rangle\!\rangle_B$	$(2,1,5,2)\ (2,-2,-13,-4)\ (4,-1,-8,-2)$	29	1	39	-10
$\mid 4,4,1,0,0\,\rangle\!\rangle_B$	$(5,-1,-3,0)\ (0,2,14,5)\ (5,-3,-17,-5)$	44	1	68	-24
$\mid 5,5,1,0,0\,\rangle\!\rangle_B$	$(6,0,-3,0)\ (0,3,18,6)\ (6,-3,-21,-6)$	65	1	111	-46
$\mid 8,2,1,0,0\,\rangle\!\rangle_B$	$(4,0,-2,0)\ (0,2,12,4)\ (4,-2,-14,-4)$	46	2	51	-5
$\mid 8,5,1,0,0\,\rangle\!\rangle_B$	$(4,2,10,4)\ (8,-2,-16,-4)\ (4,-4,-26,-8)$	94	2	147	-53
$\mid 8,8,1,0,0\,\rangle\!\rangle_B$	$(8,0,-4,0)\ (0,4,24,8)\ (8,-4,-28,-8)$	140	4	195	-55

Table 6.7.: The boundary states corresponding to D4-branes wrapped on the elliptic fibration S

The boundary states with $n_4^F \neq 0$ corresponding to D4-branes wrapped on F are displayed in table 6.8. We note that

$$\Delta^{(F)} = m^{(\mathrm{CFT},F)} - m^{(\mathrm{geom},F)} = \nu - 1 \tag{6.67}$$

The sheaves in the first line correspond to $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(-l)$ and $\Omega_{\mathbb{P}^2}(l)$ where $\Omega_{\mathbb{P}^2}$ is the cotangent bundle on $\mathbb{P}^2$. These are exactly the exceptional sheaves that have been found in the study [177], [236] of D-branes on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ whose large volume limit is the blow-up $\mathcal{O}_{\mathbb{P}^2}(-3l)$ which is a non-compact Calabi–Yau space. In the second line we have $\mathcal{I}_p$ and $\mathcal{I}_p(-l)$ where $\mathcal{I}_p$ is the ideal sheaf of a point p. These can be interpreted as bound states of the boundary states in the orbit $\mid 0,0,0,0,0\,\rangle\!\rangle_B$ with the D0-brane which lives in the orbit $\mid 2,0,0,0,0\,\rangle\!\rangle_B$. The fact that $m^{(\mathrm{CFT},F)}$ and $m^{(\mathrm{geom},F)}$ are both two and not three supports the interpretation that this D0-brane is constrained to live on the divisor $F=\mathbb{P}^2$. Finally, in the last line we have a sheaf which is topologically equivalent to $(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-l)) \otimes \mathcal{I}_p$. However, since $\widetilde{\nu}=1$ it cannot be the direct sum. Hence, it could correspond to a non-trivial extension of $\mathcal{I}_p(-l)$ by $\mathcal{I}_p$ or vice versa.

L-orbit	$n = (n_4^F, n_0, n_2^l)$	$m^{(\mathrm{CFT},F)}$	ν	$m^{(\mathrm{geom},F)}$	$\Delta^{(F)}$
$\mid 0,0,0,0,0 \rangle\rangle_B$	$(1,-1,\ \ 0)\ (1,\ \ 0,-1)\ (2,\ \ 0,-1)$	0	1	0	0
$\mid 1,0,0,0,0 \rangle\rangle_B$	$(1,\ \ 1,-1)\ (1,\ \ 0,\ \ 0)$	2	1	2	0
$\mid 1,1,0,0,0 \rangle\rangle_B$	$(2,\ \ 1,-1)$	4	1	4	0

Table 6.8.: The boundary states corresponding to D4-branes wrapped on $\mathbb{P}^2$

- **The divisor H**
For this divisor we have $n = (0, n_4^F, 3n_4^F, n_0, 3n_2^l, n_2^l)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^F \tag{6.68}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^l + 3n_4^F\right)(3h + l) \tag{6.69}$$

$$\mathrm{ch}_2(\mathcal{F}) = -\frac{11}{2}n_4^F + \frac{1}{2}n_2^h - n_0 \tag{6.70}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = 10\left(n_4^F\right)^2 + 2n_4^F n_0 + n_2^l\left(3n_4^F + n_2^l\right) + 10 \tag{6.71}$$

The boundary states with $n_4^F \neq 0$ corresponding to D4-branes wrapped on the divisor H are displayed in table 6.9. Here there is no obvious relationship between $m^{(\mathrm{CFT},H)}$ and $m^{(\mathrm{geom},H)}$. In the cases where there are several sheaves in one L-orbit, all of them have the same dimension of the geometric moduli space.

L-orbit	$n = (n_4^F, n_0, n_2^l)$	$m^{(\mathrm{CFT},H)}$	ν	$m^{(\mathrm{geom},H)}$	$\Delta^{(H)}$
$\mid 7,3,0,0,0 \rangle\rangle_B$	$(1,-1,-1)$	27	1	16	11
$\mid 6,4,0,0,0 \rangle\rangle_B$	$(1,\ \ 0,-2)\ (1,\ \ 0,-1)$ $(1,-1,-3)$	31	1	18	13
$\mid 8,3,1,0,0 \rangle\rangle_B$	$(2,-2,-4)$	62	2	34	28
$\mid 6,4,1,0,0 \rangle\rangle_B$	$(2,-1,-5)\ (2,\ \ 0,-3)$	62	1	41	21
$\mid 6,2,2,0,0 \rangle\rangle_B$	$(2,-1,-4)$	53	1	38	15
$\mid 8,2,2,0,0 \rangle\rangle_B$	$(2,\ \ 0,-4)\ (2,\ \ 0,-2)$ $(2,-2,-6)$	70	2	42	28
$\mid 6,4,2,0,0 \rangle\rangle_B$	$(3,-1,-6)$	93	1	76	17
$\mid 5,5,2,0,0 \rangle\rangle_B$	$(3,\ \ 0,-6)\ (3,\ \ 0,-3)$ $(3,-3,-9)$	97	1	82	15
$\mid 8,6,2,0,0 \rangle\rangle_B$	$(4,-2,-8)$	166	2	122	44
$\mid 7,7,2,0,0 \rangle\rangle_B$	$(4,-1,-8)$	171	1	130	41
$\mid 8,7,2,0,0 \rangle\rangle_B$	$(4,-2,-10)\ (4,\ \ 0,-6)$	190	2	134	56
$\mid 8,8,2,0,0 \rangle\rangle_B$	$(4,\ \ 0,-8)\ (4,\ \ 0,-4)$ $(4,-4,-12)$	212	4	138	74
$\mid 8,6,4,0,0 \rangle\rangle_B$	$(6,-2,-12)$	278	2	274	4
$\mid 6,5,5,0,0 \rangle\rangle_B$	$(6,-3,-12)$	221	1	262	−41
$\mid 7,5,5,0,0 \rangle\rangle_B$	$(6,-3,-15)\ (6,\ \ 0,-9)$	254	1	289	−35
$\mid 8,5,5,0,0 \rangle\rangle_B$	$(6,\ \ 0,-12)\ (6,\ \ 0,-6)$ $(6,-6,-18)$	286	2	298	−12
$\mid 6,6,6,0,0 \rangle\rangle_B$	$(7,-3,-14)$	301	1	360	−59
$\mid 8,8,6,0,0 \rangle\rangle_B$	$(8,-4,-16)$	500	4	458	42
$\mid 8,7,7,0,0 \rangle\rangle_B$	$(8,-2,-16)$	510	2	490	20
$\mid 8,8,7,0,0 \rangle\rangle_B$	$(8,-4,-20)\ (8,\ \ 0,-12)$	572	4	506	66
$\mid 8,8,8,0,0 \rangle\rangle_B$	$(8,\ \ 0,-16)\ (8,\ \ 0,-8)$ $(8,-8,-24)$	640	8	522	118

Table 6.9.: The boundary states corresponding to D4-branes wrapped on the divisor H

6.3.6. D-branes on $\mathbb{P}^4_{1,1,1,3,6}[12]$

- **The divisor S, an elliptic surface**
For this divisor we have $n = (0, 0, n_4^S, n_0, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^S && (6.72)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + n_4^S\right) h + n_2^l l && (6.73)\\ \mathrm{ch}_2(\mathcal{F}) &= -3n_4^S + \tfrac{1}{2} n_2^l - n_0 && (6.74) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on S

$$m^{(\mathrm{geom},S)} = 3\left(n_4^S\right)^2 + 2n_4^S n_0 + \tfrac{1}{2} n_2^l \left(2n_2^h - 3n_2^l\right) + 3 \qquad (6.75)$$

The boundary states corresponding to D4-branes wrapped on S are displayed in table 6.10. Here we have for the sheaves in the first part of the table

$$\Delta_1^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = 5(\nu - 1) + 2 \qquad (6.76)$$

while the sheaves in the second part satisfy

$$\Delta_2^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = 7(\nu - 1) + 4 \qquad (6.77)$$

The last three boundary states satisfy

$$\Delta_3^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = -(\nu - 1) - 4 \qquad (6.78)$$

Furthermore, one can check that there are no line bundles.

- **The divisor F, a sum of two $\mathbb{P}^2$'s**
For this divisor we have $n = (0, n_4^F, -3n_4^F, n_0, 0, n_2^l)$ since $H \cdot F = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^F && (6.79)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^l - 3n_4^E\right) l && (6.80)\\ \mathrm{ch}_2(\mathcal{F}) &= \tfrac{5}{2} n_4^F - \tfrac{3}{2} n_2^l - n_0 && (6.81) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on F

$$m^{(\mathrm{geom},F)} = -\tfrac{5}{2}\left(n_4^F\right)^2 + 2n_0 n_4^F + \tfrac{1}{2}\left(n_2^l\right)^2 + 3 \qquad (6.82)$$

The boundary states corresponding to D4-branes wrapped on F are displayed in table 6.11. We note that

$$\Delta^{(F)} = m^{(\mathrm{CFT},F)} - m^{(\mathrm{geom},F)} = \nu - 1 \qquad (6.83)$$

Recall from Section C.2 that F is not irreducible but consists of two $\mathbb{P}^2$'s. Denote the degree 1 curves in each $\mathbb{P}^2$ by l_1 and l_2. By analogy to the family $\mathbb{P}^4_{1,1,1,6,9}[18]$ we interpret the sheaves in the first row as $\mathcal{O}_F$ and $\mathcal{O}_F(-l_1 - l_2)$ and $\Omega_F(-l_1 - l_2)$ where Ω_F is the cotangent bundle of F. In the second line we then have $\mathcal{I}_{p_1+p_2}$ and $\mathcal{I}_{p_1+p_2}(-l_1 - l_2)$ where p_1, p_2 are points on each of the $\mathbb{P}^2$'s. Finally, in the last line we have a sheaf which is topologically equivalent to $(\mathcal{O}_F \oplus \mathcal{O}_F(-l_1 - l_2)) \otimes \mathcal{I}_{p_1+p_2}$. This interpretation is supported by the fact that although we have $\nu = 2$ which would indicate that we have $U(1) \times U(1)$ bundles, the invertible sheaves found above contradict this. Rather we should think in this case of ν as counting the number of components

L-orbit	$n=(n_4^S,n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT},S)}$	ν	$m^{(\mathrm{geom},S)}$	$\Delta^{(S)}$
$\vert\, 4,0,0,0,0\,\rangle\!\rangle_B$	$(1,-3,-2,-1)$ $(1,\ 1,\ 0,-1)$	6	1	4	2
$\vert\, 2,1,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-2,-2)$ $(0,\ 4,\ 2,\ 0)$	6	1	4	2
$\vert\, 3,1,0,0,0\,\rangle\!\rangle_B$	$(1,-5,-2,-1)$ $(1,\ 3,\ 0,\ 1)$	10	1	8	2
$\vert\, 2,2,0,0,0\,\rangle\!\rangle_B$	$(2,-4,-2,-2)$ $(2,\ 0,-2,\ 0)$ $(0,\ 6,\ 2,\ 2)$ $(0,\ 6,\ 2,\ 0)$	11	1	9	2
$\vert\, 3,2,0,0,0\,\rangle\!\rangle_B$	$(2,-2,-2,\ 0)$ $(0,\ 8,\ 2,\ 2)$	15	1	13	2
$\vert\, 4,1,1,0,0\,\rangle\!\rangle_B$	$(3,-11,-6,-3)$ $(3,\ 5,\ 0,-1)$	26	1	24	2
$\vert\, 4,0,0,1,0\,\rangle\!\rangle_B$	$(2,-2,-2,-2)$ $(0,\ 4,\ 2,\ 0)$	12	2	5	7
$\vert\, 5,1,0,0,0\,\rangle\!\rangle_B$	$(2,-6,-4,-2)$ $(2,\ 2,\ 0,-2)$	14	2	9	7
$\vert\, 3,1,0,1,0\,\rangle\!\rangle_B$	$(2,-2,-2,\ 0)$ $(0,\ 8,\ 2,\ 2)$	20	2	13	7
$\vert\, 5,2,0,0,0\,\rangle\!\rangle_B$	$(2,-6,-4,\ 0)$ $(2,-10,-4,-4)$ $(2,\ 2,\ 0,\ 0)$ $(2,\ 6,\ 0,\ 0)$	22	2	15	7
$\vert\, 5,3,0,0,0\,\rangle\!\rangle_B$	$(2,-10,-4,-2)$ $(2,\ 6,\ 0,\ 2)$	30	2	23	7
$\vert\, 4,1,1,1,0\,\rangle\!\rangle_B$	$(6,-6,-6,-4)$ $(0,16,\ 6,\ 2)$	52	2	45	7
$\vert\, 5,1,0,1,0\,\rangle\!\rangle_B$	$(4,-4,-4,-4)$ $(0,\ 8,\ 4,\ 0)$	28	4	11	17
$\vert\, 5,2,0,0,0\,\rangle\!\rangle_B$	$(4,-8,-4,-4)$ $(4,\ 0,-4,\ 0)$ $(0,12,\ 4,\ 4)$ $(0,12,\ 4,\ 0)$	44	4	27	17
$\vert\, 5,3,0,1,0\,\rangle\!\rangle_B$	$(4,-4,-4,\ 0)$ $(0,16,\ 4,\ 4)$	60	4	43	17
$\vert\, 5,5,1,0,0\,\rangle\!\rangle_B$	$(4,-20,-8,-8)$ $(4,-12,-8,\ 0)$ $(4,\ 4,\ 0,\ 0)$ $(4,12,\ 0,\ 0)$	88	8	51	37
$\vert\, 4,2,0,0,0\,\rangle\!\rangle_B$	$(2,-8,-4,-2)$ $(2,\ 4,\ 0,\ 0)$	19	1	15	4
$\vert\, 4,5,0,0,0\,\rangle\!\rangle_B$	$(4,-4,-4,-2)$ $(0,12,\ 4,\ 2)$	38	2	27	11
$\vert\, 4,5,0,1,0\,\rangle\!\rangle_B$	$(4,-16,-8,-4)$ $(4,\ 8,\ 0,\ 0)$	76	4	51	25
$\vert\, 2,2,1,0,0\,\rangle\!\rangle_B$	$(4,-4,-4,-2)$ $(0,12,\ 4,\ 2)$	23	1	27	-4
$\vert\, 5,2,1,0,0\,\rangle\!\rangle_B$	$(4,-16,-8,-4)$ $(4,\ 8,\ 0,\ 0)$	46	2	51	-5
$\vert\, 5,5,1,0,0\,\rangle\!\rangle_B$	$(8,-8,-8,-4)$ $(0,24,\ 8,\ 4)$	92	4	99	-7
$\vert\, 5,5,1,1,0\,\rangle\!\rangle_B$	$(8,-32,-16,-8)$ $(8,16,\ 0,\ 0)$	184	8	195	-11

Table 6.10.: The boundary states corresponding to D4-branes wrapped on the elliptic fibration S

in F. Hence we find a new interpretation of ν which differs from the one given in [184] and used so far for the irreducible divisors, namely that it is related to the reducibility of the bundle. It is also related to the reducibility of the divisor. It would be interesting to study a reducible divisor which supports reducible bundles. Having clarified this point, we need of course to modify table 6.11. $m^{(\mathrm{geom},F)}$ should really be 0, 4 and 8 and hence $\Delta^{(F)}=\nu-2=0$. With this mind we can interpret the last bundle in the same way as for $\mathbb{P}^4_{1,1,1,6,9}[18]$, namely that it could be a non-trivial extension of $\mathcal{I}_{p_1+p_2}$ by $\mathcal{I}_{p_1+p_2}(-l_1-l_2)$ or vice versa.

L-orbit	$n=(n_4^F,n_2^l,n_0)$	$m^{(\mathrm{CFT},F)}$	ν	$m^{(\mathrm{geom},F)}$	$\Delta^{(F)}$
$\vert\, 0,0,0,1,0\,\rangle\!\rangle_B$	$(1,\ 3,-2)$ $(1,\ 1,\ 0)$ $(2,\ 4,\ 0)$	0	2	-1	1
$\vert\, 1,0,0,1,0\,\rangle\!\rangle_B$	$(1,\ 1,\ 2)$ $(1,\ 3,\ 0)$	4	2	3	1
$\vert\, 1,1,0,1,0\,\rangle\!\rangle_B$	$(2,\ 4,\ 2)$	8	2	7	1

Table 6.11.: The boundary states corresponding to D4-branes wrapped on $\mathbb{P}^2$

- **The divisor H**
  For this divisor we have $n=(0,n_4^H,0,n_0,3n_2^l,n_2^l)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^H \tag{6.84}$$
$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^l+3n_4^H\right)(3h+l) \tag{6.85}$$
$$\mathrm{ch}_2(\mathcal{F}) = -\tfrac{13}{2}n_4^H+\tfrac{3}{2}n_2^h-n_0 \tag{6.86}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = \tfrac{13}{2}\left(n_4^H\right)^2 + 2n_4^H n_0 + \tfrac{1}{2}\left(n_2^l\right)^2 + 11 \tag{6.87}$$

The boundary states corresponding to D4-branes wrapped on the divisor H are displayed in table 6.12. Here there is no obvious relationship between $m^{(\mathrm{CFT},H)}$ and $m^{(\mathrm{geom},H)}$. In the cases where there are several sheaves in one L-orbit, all of them have the same dimension of the geometric moduli space.

L-orbit	$n=(n_4^F,n_0,n_2^l)$	$m^{(\mathrm{CFT},H)}$	ν	$m^{(\mathrm{geom},H)}$	$\Delta^{(H)}$
$\vert 3,1,1,0,0\rangle\rangle_B$	$(1,-3,-1,-1)$	20	1	16	4
$\vert 4,3,0,0,0\rangle\rangle_B$	$(1,-3,-1,-1)$	26	1	16	10
$\vert 4,2,2,0,0\rangle\rangle_B$	$(2,-12,-4,-2)$ $(2,\ 0,\ 0,\ 0)$	59	1	37	22
$\vert 3,3,3,0,0\rangle\rangle_B$	$(3,-9,-3,-3)$	84	1	56	28
$\vert 3,2,2,0,0\rangle\rangle_B$	$(2,-6,-2,-2)$	47	2	31	16
$\vert 5,3,1,0,0\rangle\rangle_B$	$(2,-6,-2,-2)$	62	2	31	31
$\vert 5,2,2,0,0\rangle\rangle_B$	$(2,-18,-6,-4)$ $(2,-6,-2,\ 0)$ $(2,\ 6,\ 2,\ 0)$	70	2	39	31
$\vert 5,4,4,0,0\rangle\rangle_B$	$(4,-12,-4,-2)$	198	2	107	91
$\vert 5,5,3,0,0\rangle\rangle_B$	$(4,-12,-4,-4)$	188	4	91	97
$\vert 5,5,4,0,0\rangle\rangle_B$	$(4,-24,-8,-4)$ $(4,\ 0,\ 0,\ 0)$	236	4	115	121
$\vert 5,5,5,0,0\rangle\rangle_B$	$(4,-36,-12,-8)$ $(4,-12,-4,\ 0)$ $(4,\ 12,\ 4,\ 0)$	280	8	123	157

Table 6.12.: The boundary states corresponding to D4-branes wrapped on the divisor H

6.3.7. D-branes on $\mathbb{P}^4_{1,1,1,3,3}[9]$

- **The divisor S, an elliptic surface**
  For this divisor we have $n=(0,0,n_4^S,n_0,n_2^h,n_2^l)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^S \tag{6.88}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^h+n_4^S\right)h + n_2^l l \tag{6.89}$$

$$\mathrm{ch}_2(\mathcal{F}) = -3n_4^S + \frac{1}{2}n_2^l - n_0 \tag{6.90}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on S

$$m^{(\mathrm{geom},S)} = 3\left(n_4^F\right)^2 + 2n_4^F n_0 + \frac{1}{3}n_2^l\left(-n_4^S+2n_2^h-3n_2^l\right)+3 \tag{6.91}$$

The boundary states corresponding to D4-branes wrapped on S are displayed in table 6.13. Here we have for the sheaves above the double line

$$\Delta^{(S)} = m^{(\mathrm{CFT},S)} - m^{(\mathrm{geom},S)} = 5(\nu-1)+2 \tag{6.92}$$

where we have chosen the factor in front of $(\nu-1)$ such that it agrees with the previous families $\mathbb{P}^4_{1,1,1,6,9}[18]$ and $\mathbb{P}^4_{1,1,1,3,6}[12]$ where the elliptic surface was the same and the states with the corresponding charges satisfied the same relationship.

- **The divisor F, a collection of 3 $\mathbb{P}^2$'s**
  For this divisor we have $n=(0,n_4^F,-3n_4^F,n_0,-n_4^F,n_2^l)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^F \tag{6.93}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^l-5n_4^F\right)l \tag{6.94}$$

$$\mathrm{ch}_2(\mathcal{F}) = \frac{9}{2}n_4^F - \frac{3}{2}n_2^l - n_0 \tag{6.95}$$

L-orbit	$n=(n_4^S,n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT},S)}$	ν	$m^{(\mathrm{geom},S)}$	$\Delta^{(S)}$
$\mid 2,0,0,0,0\,\rangle\!\rangle_B$	(0, 3, 0, 3) (0, 3, 0, 3) (0, 3, 0, 0)	3	1	3	0
$\mid 2,1,0,0,0\,\rangle\!\rangle_B$	(3, 0, −3, −3) (3, −6, −6, −3) (0, 6, 3, 0)	8	1	6	2
$\mid 2,2,0,0,0\,\rangle\!\rangle_B$	(3, −12, −6, −6) (3, −3, −6, 0) (3, 6, −3, 0) (0, 9, 3, 3) (0, 9, 3, 0) (3, −3, −3, −3)	14	1	12	2
$\mid 2,3,0,0,0\,\rangle\!\rangle_B$	(3, −9, −6, −3) (3, 3, −3, 0) (0, 12, 3, 3)	20	1	18	2
$\mid 2,2,1,0,0\,\rangle\!\rangle_B$	(6, 3, −6, −3) (6, −15, −12, −6) (0, 18, 6, 3)	29	1	39	−10

Table 6.13.: The boundary states corresponding to D4-branes wrapped on the elliptic surface S

as well as (5.99) yields for the dimension of the moduli space of the sheaves on F

$$m^{(\mathrm{geom},F)} = -\frac{11}{3}\left(n_4^F\right)^2 + 2n_0n_4^F + \frac{1}{3}n_2^l\left(n_2^l - n_4^F\right) + 1 \tag{6.96}$$

We don't find any boundary states corresponding to sheaves on the exceptional divisor F.

- **The divisor H**
  For this divisor we have $n = \left(0, n_4^H, 0, n_0, 3n_2^l, \frac{1}{3}\left(n_2^l + n_4^H\right)\right)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^F \tag{6.97}$$
$$\mathrm{c}_1(\mathcal{F}) = \frac{1}{3}\left(n_2^l + 13n_4^H\right)(3h+l) \tag{6.98}$$
$$\mathrm{ch}_2(\mathcal{F}) = -\frac{11}{2}n_4^H + \frac{1}{2}n_2^h - n_0 \tag{6.99}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = \frac{142}{27}\left(n_4^H\right)^2 + 2n_4^Hn_0 + \frac{1}{27}\left(-n_4^H + n_2^l\right) + 12 \tag{6.100}$$

The only boundary states we found are $\mid 2,2,0,0,0\,\rangle\!\rangle_B$ and $\mid 2,2,1,0,0\,\rangle\!\rangle_B$ which contain three sheaves supported on the curve $9h+3l$ which already appeared among the boundary states on the divisor S. This curve has genus 7.

6.3.8. D-branes on $\mathbb{P}^4_{1,1,2,8,12}[24]$

D6-branes

Here $n = (n_6, n_4^F, n_4^E - 2n_4^F, n_4^L - 2n_4^E, n_0, n_2^h, n_2^d, n_2^l)$ and we need the intersections numbers in (C.75), (C.77) and (C.78).

$$\mathrm{rk}(\mathcal{F}) = n_6 \tag{6.101}$$
$$\mathrm{c}_1(\mathcal{F}) = n_4^FF + n_4^EE + n_4^LL \tag{6.102}$$
$$\mathrm{ch}_2(\mathcal{F}) = n_2^hh + n_2^dd + n_2^ll \tag{6.103}$$
$$\mathrm{ch}_3(\mathcal{F}) = -n_0 + \frac{1}{3}n_4^F - 2n_4^L \tag{6.104}$$

The FMW construction

Here we follow closely the argument given in Section 6.3.5. We have seen in section C.3 that the base of the elliptic fibration $\pi : X \to B$ is a Hirzebruch surface $\mathbb{F}_2$. Furthermore the curve l can be identified with the section of $p : \mathbb{F}_2 \to \mathbb{P}_1$ and the curve d is a fiber of $\mathbb{F}_2$. Recall that the canonical bundle of the Hirzebruch surface is $K_{\mathbb{F}_2} = -2D|_F$. A general bundle V will have topological invariants (6.101) to (6.104). Setting $n_4^F = 0$ and using (C.77), (6.102), (6.103) and (5.111) yields for the dimension of the moduli space

$$n_6 + 4(n_4^E)^2 - n_4^E n_4^L + 4n_2^h - 2n_2^d \tag{6.105}$$

On the other hand, the bundles constructed in [271] satisfy $c_1(V) = c_3(V) = 0$, $\eta = a\, c_1(\mathbb{F}_2)$ for a odd and n has to be even. In this case (5.112) gives

$$c_2(V) = 2a(2d + l) - \frac{1}{3}(n^3 - n + 3a(a-n)n)h \tag{6.106}$$

and comparison with (6.103) finally gives

$$\begin{aligned} n_6 &= n && (6.107)\\ n_2^h &= \frac{1}{3}(n^3 - n + 3a(a-n)n) && (6.108)\\ n_2^d &= -4a && (6.109)\\ n_2^l &= -2a && (6.110)\\ n_0 &= n_4^F = n_4^E = n_4^L = 0 && (6.111) \end{aligned}$$

The comparison with the bundles obtained from the boundary states that none of them satisfies all these conditions. On the other hand, it is interesting to note, that all bundles with $n_4^F = n_4^E = n_4^L = 0$ have the property that $n_2^h = 2n_2^d = 4n_2^l$. The conditions that are then not satisfied are either a is not integral and odd or, most notably, that n_2^h does not have the required value. The equation $n_2^h = 2n_2^d$ has no positive, real solutions for a if n is an even integer.

D4-branes

- **The divisor L, a K3 surface**
  For this divisor we have $n = (0, 0, 0, n_4^L, n_0, n_2^h, n_2^d, 0)$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^L && (6.112)\\ c_1(\mathcal{F}) &= n_2^h h + n_2^d d && (6.113)\\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^L - n_0 && (6.114) \end{aligned}$$

  From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\text{geom}, L)}(\mathcal{F}) = 2n_2^d\left(n_2^h - n_2^d\right) + 2n_4^L\left(n_4^L + n_0\right) + 2 \tag{6.115}$$

  The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.14. We observe that

$$\Delta^{(L)} = m^{(\text{CFT},L)} - m^{(\text{geom},L)} = 3\nu - 2 \tag{6.116}$$

- **The divisor F, a Hirzebruch surface $\mathbb{F}_2$**
  For this divisor we have $n = (0, n_4^F, -2n_4^F, 0, n_0, 0, n_2^d, n_2^l)$ since $H \cdot F = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^F && (6.117)\\ c_1(\mathcal{F}) &= \left(n_2^d - 2n_4^F\right) d + \left(n_2^l - n_4^F\right) l && (6.118)\\ \mathrm{ch}_2(\mathcal{F}) &= n_4^F - n_2^d - n_0 && (6.119) \end{aligned}$$

L-orbit	Mukai vector $v = (n_4^L, n_2^h, n_2^d, -n_4^L - n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\mid 1,0,0,0,0\,\rangle\rangle_B$	$(1,-1,0,1)\,(0,0,-1,-1)\,(0,0,1,0)$ $(1,0,-1,0)\,(1,0,0,1)\,(1,-1,-1,1)$	1	1	0	1
$\mid 3,0,0,0,0\,\rangle\rangle_B$	$(1,-1,-1,0)\,(0,0,0,-1)\,(1,0,0,0)$ $(0,1,0,-1)\,(0,1,1,0)\,(0,0,1,0)$	3	1	2	1
$\mid 5,0,0,0,0\,\rangle\rangle_B$	$(1,-1,0,0)\,(1,0,-1,-1)\,(0,1,1,-1)$ $(1,1,0,0)\,(1,-2,-1,1)\,(0,-1,-1,-1)$	3	1	2	1
$\mid 5,0,1,0,0\,\rangle\rangle_B$	$(1,-3,-2,0)\,(1,-2,-1,-1)\,(2,-1,-1,-1)$ $(1,1,0,-2)\,(1,2,1,-1)\,(0,3,1,-1)$	7	1	6	1
$\mid 7,0,0,0,0\,\rangle\rangle_B$	$(2,-1,-1,0)\,(0,1,0,-2)\,(1,1,1,0)$ $(0,2,0,-1)\,(1,-2,-2,0)\,(0,-1,0,-1)$	3	1	2	1
$\mid 7,0,1,0,0\,\rangle\rangle_B$	$(1,-4,-2,1)\,(1,-3,-2,-1)\,(2,-2,-1,-1)$ $(2,0,-1,-2)\,(1,2,1,-2)\,(1,3,1,-1)$	9	1	8	1
$\mid 7,0,2,0,0\,\rangle\rangle_B$	$(3,1,0,-2)\,(1,4,1,-3)\,(0,5,3,-1)$ $(1,-5,-2,0)\,(3,-4,-3,-1)\,(2,-1,-1,-1)$	15	1	14	1
$\mid 7,0,3,0,0\,\rangle\rangle_B$	$(3,-6,-3,0)\,(3,-3,-3,-3)\,(3,0,0,-3)$ $(3,3,0,-3)\,(0,6,3,-3)\,(0,6,3,0)$	21	1	20	1
$\mid 9,0,0,0,0\,\rangle\rangle_B$	$(1,0,0,-1)\,(1,1,0,-1)\,(0,2,1,-1)$ $(0,2,1,0)\,(1,-2,-1,0)\,(1,-1,-1,-1)$	5	1	4	1
$\mid 9,0,1,0,0\,\rangle\rangle_B$	$(1,-4,-2,0)\,(2,-3,-2,-1)\,(2,-1,-1,-2)$ $(2,1,0,-2)\,(1,3,1,-2)\,(0,4,2,-1)$	11	1	10	1
$\mid 9,0,2,0,0\,\rangle\rangle_B$	$(2,3,1,-3)\,(1,5,2,-2)\,(1,-6,-3,1)$ $(2,-5,-3,-1)\,(3,-3,-2,-2)\,(3,0,-1,-3)$	19	1	18	1
$\mid 9,0,3,0,0\,\rangle\rangle_B$	$(2,-7,-4,0)\,(3,-5,-3,-2)\,(4,-2,-2,-3)$ $(3,2,0,-4)\,(2,5,2,-3)\,(0,7,3,-2)$	27	1	26	1
$\mid 9,0,4,0,0\,\rangle\rangle_B$	$(1,7,3,-3)\,(1,-8,-4,1)\,(3,-7,-4,-1)$ $(4,-4,-3,-3)\,(4,0,-1,-4)\,(3,4,1,-4)$	33	1	32	1
$\mid 11,0,0,0,0\,\rangle\rangle_B$	$(2,0,0,0)\,(0,2,0,-2)\,(0,2,2,0)$ $(0,2,0,0)\,(2,-2,-2,0)\,(0,0,0,-2)$	6	2	2	4
$\mid 11,0,1,0,0\,\rangle\rangle_B$	$(2,-4,-2,0)\,(2,-2,-2,-2)\,(2,0,0,-2)$ $(2,2,0,-2)\,(0,4,2,-2)\,(0,4,2,0)$	14	2	10	4
$\mid 11,0,2,0,0\,\rangle\rangle_B$	$(2,4,2,-2)\,(0,6,2,-2)\,(2,-6,-4,0)$ $(2,-4,-2,-2)\,(4,-2,-2,-2)\,(2,2,0,-4)$	22	2	18	4
$\mid 11,0,3,0,0\,\rangle\rangle_B$	$(2,-8,-4,2)\,(2,-6,-4,-2)\,(4,-4,-2,-2)$ $(4,0,-2,-4)\,(2,4,2,-4)\,(2,6,2,-2)$	30	2	26	4
$\mid 11,0,4,0,0\,\rangle\rangle_B$	$(0,8,4,-2)\,(2,-8,-4,0)\,(4,-6,-4,-2)$ $(4,-2,-2,-4)\,(4,2,0,-4)\,(2,6,2,-4)$	38	2	34	4
$\mid 11,0,5,0,0\,\rangle\rangle_B$	$(4,-8,-4,0)\,(4,-4,-4,-4)\,(4,0,0,-4)$ $(4,4,0,-4)\,(0,8,4,-4)\,(0,8,4,0)$	44	4	34	10

Table 6.14.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

as well as (5.99) yields for the dimension of the moduli space of the sheaves on F

$$m^{(\mathrm{geom},F)} = n_4^F\left(2n_0 - n_4^F\right) + 2n_2^l\left(n_2^d - n_2^l\right) + 1 \tag{6.120}$$

The boundary states with $n_4^F \neq 0$ corresponding to D4-branes wrapped on F are displayed in table 6.15. We note that

$$\Delta^{(F)} = m^{(\mathrm{CFT},F)} - m^{(\mathrm{geom},F)} = \nu - 1 \tag{6.121}$$

The sheaves corresponding to the boundary states in the first line are $\mathcal{O}_{\mathbb{F}_2}$, $\mathcal{O}_{\mathbb{F}_2}(-l-d)$, $\mathcal{O}_{\mathbb{F}_2}(-d)$ and $\mathcal{O}_{\mathbb{F}_2}(-l)$. Note that in [153] these bundles have been given an interpretation as pull-back bundles of twisted tangent bundles on d and l. The sheaves for $\mid 2,0,0,0,0\,\rangle\rangle_B$ are $\mathcal{I}_p$ and $\mathcal{I}_p(-l-d)$. The latter corresponds to a D4-D0 bound state and the dimensions being equal to 2 corresponds to the fact that the D0 is only allowed to move on $\mathbb{F}_2$. In the orbit of $\mid 1,1,0,0,0\,\rangle\rangle_B$ we find a sheaf with Chern classes $(2,-l-2d,2)$ and one with $(2,-l,1)$. Finally the last one corresponds to $(2,-l-d,2)$. For these sheaves we do not yet have a geometric interpretation.

L-orbit	$n = (n_4^F, n_2^d, n_2^l, n_0)$	$m^{(\mathrm{CFT},F)}$	ν	$m^{(\mathrm{geom},F)}$	$\Delta^{(F)}$
$\mid 0,0,0,0,0\,\rangle\!\rangle_B$	$(1,\ 2,\ 1,-1)\ (1,\ 1,\ 0,\ 0)$ $(1,\ 1,\ 1,\ 0)\ (1,\ 2,\ 0,\ 0)$	0	1	0	0
$\mid 1,0,0,0,0\,\rangle\!\rangle_B$	$(2,\ 3,\ 1,\ 0)\ (0,\ 1,\ 0,\ 0)$ $(0,\ 1,\ 0,-1)$	1	1	1	0
$\mid 2,0,0,0,0\,\rangle\!\rangle_B$	$(1,\ 1,\ 0,\ 1)\ (1,\ 2,\ 1,\ 0)$	2	1	2	0
$\mid 1,1,0,0,0\,\rangle\!\rangle_B$	$(2,\ 2,\ 1,\ 1)\ (2,\ 4,\ 1,\ 0)$	3	1	3	0
$\mid 2,1,0,0,0\,\rangle\!\rangle_B$	$(2,\ 3,\ 1,\ 1)$	5	1	5	0

Table 6.15.: The boundary states corresponding to D4-branes wrapped on the Hirzebruch surface $F = \mathbb{F}_2$

- **The divisor E, a ruled surface over an elliptic curve**
For this divisor we have $n = (0, 0, n_4^E, -2n_4^E, n_0, n_2^h, 0, n_2^l)$ since $D \cdot E = 0$ and (5.74) to (5.76) reduce to

$$
\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^E && (6.122)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - n_4^E\right) h + n_2^l l && (6.123)\\
\mathrm{ch}_2(\mathcal{F}) &= -n_2^l - n_0 && (6.124)
\end{aligned}
$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E

$$
m^{(\mathrm{geom},E)} = 2n_4^E n_0 + 2n_2^h n_2^l + 1 \tag{6.125}
$$

The boundary states with $n_4^E \neq 0$ corresponding to D4-branes wrapped on E are displayed in table 6.16. We note that

$$
\Delta^{(E)} = m^{(\mathrm{CFT},E)} - m^{(\mathrm{geom},E)} = \nu - 1 \tag{6.126}
$$

The bundles corresponding to the boundary state $\mid 0,0,1,0,0\,\rangle\!\rangle_B$ we find $\mathcal{O}_E$, $\mathcal{O}_E(-h)$ and $\mathcal{O}_E(-l)$. The remaining three are the ideal sheaf $\mathcal{I}_p(-h-l)$ and the torsion sheaves $j_*\mathcal{O}_l$ and $j_*\mathcal{I}_p(-1)$ where $j : l \to E$ is the embedding of the fiber of E. In the second line we find two bundles, $\mathcal{O}(l)$ and $\mathcal{O}(-2l)$. The sheaves in the last line can all be thought of as direct sums of twice a sheaf in the first line since $\widetilde{\nu} = 1$ and $N = 2$ indicates that they are $U(1) \times U(1)$ bundles.

L-orbit	$n = (n_4^E, n_2^h, n_2^l, n_0)$	$m^{(\mathrm{CFT},E)}$	ν	$m^{(\mathrm{geom},E)}$	$\Delta^{(E)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ 0,\ 0,\ 0)\ (0,\ 0,\ 1,\ 0)\ (1,\ 1,-1,\ 1)$ $(1,\ 1,\ 0,\ 0)\ (0,\ 0,\ 1,-1)\ (1,\ 0,-1,\ 0)$	1	1	1	0
$\mid 0,0,3,0,0\,\rangle\!\rangle_B$	$(1,\ 0,\ 1,\ 0)\ (1,\ 1,-2,\ 2)\ (2,\ 2,-1,\ 1)$ $(1,\ 1,\ 1,-1)\ (1,\ 0,-2,\ 0)\ (2,\ 0,-1,\ 0)$	1	1	1	0
$\mid 0,0,5,0,0\,\rangle\!\rangle_B$	$(0,\ 0,\ 2,\ 0)\ (2,\ 2,-2,\ 2)\ (2,\ 2,\ 0,\ 0)$ $(0,\ 0,\ 2,-2)\ (2,\ 0,-2,\ 0)\ (2,\ 0,\ 0,\ 0)$	2	2	1	1

Table 6.16.: The boundary states corresponding to D4-branes wrapped on the ruled surface E

- **The divisor D, an elliptic surface**
For this divisor we have $n = (0, 0, n_4^D, 0, n_0, n_2^h, 2n_2^l, n_2^l)$ and (5.74) to (5.76) reduce to

$$
\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^D && (6.127)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + n_4^D\right) h + n_2^l f && (6.128)\\
\mathrm{ch}_2(\mathcal{F}) &= -4n_4^D + n_2^l - n_0 && (6.129)
\end{aligned}
$$

where $f = 2d + l$ as introduced in Section C.3. (5.99) yields for the dimension of the moduli space of the sheaves on D

$$m^{(\mathrm{geom},D)} = 4\left(n_4^D\right)^2 + 2n_4^D n_0 + 2n_2^l\left(n_2^h - 2n_2^l\right) + 4 \tag{6.130}$$

The boundary states with $n_4^D \neq 0$ corresponding to D4-branes wrapped on D are displayed in table 6.17. Here we have for the sheaves above the double line

$$\Delta_1^{(D)} = m^{(\mathrm{CFT},D)} - m^{(\mathrm{geom},D)} = 7(\nu - 1) + 3 \tag{6.131}$$

while the three sheaves in the second part satisfy

$$\Delta_2^{(D)} = m^{(\mathrm{CFT},D)} - m^{(\mathrm{geom},D)} = 11(\nu - 1) + 7 \tag{6.132}$$

For the last three sheaves we do not see an obvious relation. Furthermore, one can check that there are no line bundles.

L-orbit	$n = (n_4^D, n_2^h, n_2^l, n_0)$	$m^{(\mathrm{CFT},D)}$	ν	$m^{(\mathrm{geom},D)}$	$\Delta^{(D)}$
$\vert\, 3,2,0,0,0\,\rangle\!\rangle_B$	$(1,-2,-1,-1)\ (1,1,0,-1)\ (0,3,1,0)$	9	1	6	3
$\vert\, 3,3,0,0,0\,\rangle\!\rangle_B$	$(1,-3,-1,-1)\ (1,-2,-1,0)\ (1,1,0,0)$ $(1,2,0,0)\ (0,4,1,1)\ (0,4,1,0)$	11	1	8	3
$\vert\, 3,4,0,0,0\,\rangle\!\rangle_B$	$(1,2,0,1)\ (0,5,1,1)\ (1,-3,-1,0)$	13	1	10	3
$\vert\, 3,6,0,0,0\,\rangle\!\rangle_B$	$(2,-1,-1,-1)\ (1,5,1,0)\ (1,-6,-2,-1)$	17	1	14	3
$\vert\, 3,8,0,0,0\,\rangle\!\rangle_B$	$(1,6,1,1)\ (1,-7,-2,-1)\ (2,-1,-1,0)$	21	1	18	3
$\vert\, 7,2,0,0,0\,\rangle\!\rangle_B$	$(2,-1,-1,-2)\ (1,4,1,-1)\ (1,-5,-2,-1)$	13	1	10	3
$\vert\, 7,3,0,0,0\,\rangle\!\rangle_B$	$(1,-7,-2,-2)\ (1,-6,-2,0)\ (2,-2,-1,-1)$ $(2,0,-1,-1)\ (1,5,1,1)\ (1,6,1,0)$	19	1	16	3
$\vert\, 5,5,0,0,0\,\rangle\!\rangle_B$	$(2,-6,-2,-2)\ (2,-4,-2,0)\ (2,2,0,0)$ $(2,4,0,0)\ (0,8,2,2)\ (0,8,2,0)$	23	1	20	3
$\vert\, 7,4,0,0,0\,\rangle\!\rangle_B$	$(1,7,1,2)\ (1,-8,-2,-1)\ (2,-1,-1,1)$	25	1	22	3
$\vert\, 7,6,0,0,0\,\rangle\!\rangle_B$	$(3,4,0,-1)\ (0,11,3,1)\ (3,-7,-3,-2)$	37	1	34	3
$\vert\, 7,7,0,0,0\,\rangle\!\rangle_B$	$(3,-9,-3,-3)\ (3,-6,-3,0)\ (3,3,0,0)$ $(3,6,0,0)\ (0,12,3,3)\ (0,12,3,0)$	43	1	40	3
$\vert\, 7,8,0,0,0\,\rangle\!\rangle_B$	$(0,13,3,2)\ (3,-8,-3,-1)\ (3,5,0,1)$	49	1	46	3
$\vert\, 11,2,0,0,0\,\rangle\!\rangle_B$	$(2,2,0,-2)\ (0,6,2,0)\ (2,-4,-2,-2)$	22	2	12	10
$\vert\, 11,3,0,0,0\,\rangle\!\rangle_B$	$(2,-6,-2,-2)\ (2,-4,-2,0)\ (2,2,0,0)$ $(2,4,0,0)\ (0,8,2,2)\ (0,8,2,0)$	30	2	20	10
$\vert\, 11,4,0,0,0\,\rangle\!\rangle_B$	$(0,10,2,2)\ (2,-6,-2,0)\ (2,4,0,2)$	38	2	28	10
$\vert\, 11,6,0,0,0\,\rangle\!\rangle_B$	$(2,10,2,0)\ (2,-12,-4,-2)\ (4,-2,-2,-2)$	54	2	44	10
$\vert\, 11,7,0,0,0\,\rangle\!\rangle_B$	$(2,-14,-4,-4)\ (2,-12,-4,0)\ (4,-4,-2,-2)$ $(4,0,-2,0)\ (2,10,2,2)\ (2,12,2,0)$	62	2	52	10
$\vert\, 11,8,0,0,0\,\rangle\!\rangle_B$	$(2,-14,-4,-2)\ (4,-2,-2,0)\ (2,12,2,2)$	70	2	60	10
$\vert\, 11,11,0,0,0\,\rangle\!\rangle_B$	$(4,-12,-4,-4)\ (4,-8,-4,0)\ (4,4,0,0)$ $(4,8,0,0)\ (0,16,4,4)\ (0,16,4,0)$	92	4	68	24
$\vert\, 10,3,0,0,0\,\rangle\!\rangle_B$	$(2,3,0,0)\ (0,8,2,1)\ (2,-5,-2,-1)$	27	1	20	7
$\vert\, 10,7,0,0,0\,\rangle\!\rangle_B$	$(2,11,2,1)\ (2,-13,-4,-2)\ (4,-2,-2,-1)$	59	1	52	7
$\vert\, 10,11,0,0,0\,\rangle\!\rangle_B$	$(0,16,4,2)\ (4,-10,-4,-2)\ (4,6,0,0)$	86	2	68	18
$\vert\, 10,0,1,0,0\,\rangle\!\rangle_B$	$(1,-3,-1,-1)\ (1,-2,-1,0)\ (1,1,0,0)$ $(1,2,0,0)\ (0,4,1,1)\ (0,4,1,0)$	13	1	8	5
$\vert\, 10,0,3,0,0\,\rangle\!\rangle_B$	$(1,-7,-2,-2)\ (1,-6,-2,0)\ (2,-2,-1,-1)$ $(2,0,-1,-1)\ (1,5,1,1)\ (1,6,1,0)$	29	1	16	13
$\vert\, 10,0,5,0,0\,\rangle\!\rangle_B$	$(2,-6,-2,-2)\ (2,-4,-2,0)\ (2,2,0,0)$ $(2,4,0,0)\ (0,8,2,2)\ (0,8,2,0)$	42	2	20	22

Table 6.17.: The boundary states corresponding to D4-branes wrapped on the elliptic surface D

- **The divisor H**
For this divisor we have $n = (0, n_4^H, 0, 0, n_0, 2n_2^d, n_2^d, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^H && (6.133) \\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^d + 2n_4^H\right)(2h+d) + \left(n_2^l + n_4^H\right) l && (6.134) \\ \mathrm{ch}_2(\mathcal{F}) &= -7n_4^H + n_2^d - n_0 && (6.135) \end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$m^{(\mathrm{geom},H)} = 7\left(n_4^H\right)^2 + 2n_4^H n_0 + 2n_2^l\left(n_2^d - n_2^l\right) + 9 \tag{6.136}$$

6.3.9. D-branes on $\mathbb{P}^4_{1,2,3,3,3}[12]$

- **The divisor L, a K3 surface**
For this divisor we have $n = (0, 0, 0, n_4^L, n_0, n_2^h, n_2^h, 0)$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^L && (6.137) \\ \mathrm{c}_1(\mathcal{F}) &= n_2^h(h+d) && (6.138) \\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^L - n_0 && (6.139) \end{aligned}$$

From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\mathrm{geom},\ L)}(\mathcal{F}) = \frac{1}{2}\left(n_2^h\right)^2 + 2n_4^L\left(n_4^L + n_0\right) + 2 \tag{6.140}$$

The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.18. We observe that

$$\Delta^{(L)} = m^{(\mathrm{CFT},L)} - m^{(\mathrm{geom},L)} = 3\nu - 2 \tag{6.141}$$

Note that we find precisely the same states as in the case of the $K3$ divisor in $\mathbb{P}^4_{1,1,2,2,2}[8]$, see table 6.1. Due to the fact that $A^6 \neq -1$ the states in the first line do not come in brane anti-brane pairs.

L-orbit	Mukai vector $v = (n_4^L, n_2^h, -n_4^L - n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\vert\, 2,0,0,0,0\, \rangle\!\rangle_B$	$(1,-4,\ 1)\ (3,-8,\ 3)$ $(1,-4,\ 3)\ (1,\ 0,\ 1)$	1	1	0	1
$\vert\, 5,0,0,0,0\, \rangle\!\rangle_B$	$(0,\ 4,-2)\ (2,-4,\ 0)$	7	1	6	1
$\vert\, 5,0,1,0,0\, \rangle\!\rangle_B$	$(2,\ 0,-2)\ (2,-8,\ 2)$	14	2	10	4
$\vert\, 5,0,1,1,0\, \rangle\!\rangle_B$	$(0,\ 8,-4)\ (4,-8,\ 0)$	28	4	18	10
$\vert\, 5,0,1,0,0\, \rangle\!\rangle_B$	$(4,\ 0,-4)\ (4,-16,\ 4)$	44	4	34	10

Table 6.18.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

- **The divisor E_1, a ruled surface over a genus $g = 3$ curve**
For this divisor we have $n = (0, n_4^{E_1}, 0, 0, n_0, n_2^h, n_2^d, 0)$ since $L \cdot E_1 = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_1} && (6.142) \\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - 4n_4^{E_1}\right) h + n_2^d d && (6.143) \\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^{E_1} + n_2^h - n_2^d - n_0 && (6.144) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_1

$$m^{(\mathrm{geom},E_1)} = 2\left(n_4^{E_1}\right)^2 + 2n_4^{E_1}n_0 + \frac{1}{4}n_2^h\left(2n_2^d - n_2^h\right) + 1 \tag{6.145}$$

The boundary states with $n_4^{E_1} \neq 0$ corresponding to D4-branes wrapped on E_1 are displayed in table 6.19. We note that

$$\Delta^{(E_1)} = m^{(\mathrm{CFT},E_1)} - m^{(\mathrm{geom},E_1)} = \nu - 1 \tag{6.146}$$

The rank one bundles in the first and second line are $\mathcal{O}_{E_1}$ and $\mathcal{O}_{E_1}(-4h-4d)$. The rank two bundles in the third and fourth line are topologically equivalent to $\mathcal{O}_{E_1}^{\oplus 2}$ and $\mathcal{O}_{E_1}^{\oplus 2}(-4h-4d)$. They have $\widetilde{\nu} = 1$ and $N = 2$ and are therefore $U(2)$ bundles and not holomorphic direct sums.

L-orbit	$n = (n_4^{E_1}, n_2^h, n_2^d, n_0)$	$m^{(\mathrm{CFT},E_1)}$	ν	$m^{(\mathrm{geom},E_1)}$	$\Delta^{(E_1)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	(1, 0, −4, 0)	3	1	3	0
$\mid 0,1,1,0,0\,\rangle\!\rangle_B$	(1, 4, 0, 2)	3	1	3	0
$\mid 0,0,1,1,1\,\rangle\!\rangle_B$	(2, 0, −8, 0)	12	4	9	3
$\mid 0,1,1,1,1\,\rangle\!\rangle_B$	(2, 8, 0, 4)	12	4	9	3

Table 6.19.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_1

- **The divisor E_2, a ruled surface over a genus $g = 3$ curve**
  For this divisor we have $n = (0, 0, n_4^{E_2}, 0, n_0, n_2^h, n_2^h, n_2^l)$ since $L \cdot E_2 = 2h + 2d$ and $H \cdot E_2 = D \cdot E_2$ and (5.74) to (5.76) reduce to

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^{E_2} && (6.147)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - 4n_4^{E_2}\right)(h+d) + \left(n_2^l + 2n_4^{E_2}\right) l && (6.148)\\
\mathrm{ch}_2(\mathcal{F}) &= -n_4^{E_2} + \frac{1}{2}n_2^h - n_2^l - n_0 && (6.149)
\end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_2

$$m^{(\mathrm{geom},E_2)} = 2\left(n_4^{E_2}\right)^2 + 2n_4^E n_0 + \frac{1}{2}n_2^h n_2^l + 1 \tag{6.150}$$

The boundary states with $n_4^{E_2} \neq 0$ corresponding to D4-branes wrapped on E_2 are displayed in table 6.20. We note that

$$\Delta^{(E_2)} = m^{(\mathrm{CFT},E_2)} - m^{(\mathrm{geom},E_2)} = \nu - 1 \tag{6.151}$$

The rank one bundles in the first and second line are $\mathcal{O}_{E_2}$ and $\mathcal{O}_{E_2}(-4h-4d)$. The rank two bundles in the third and fourth line are topologically equivalent to $\mathcal{O}_{E_2}^{\oplus 2}$ and $\mathcal{O}_{E_2}^{\oplus 2}(-4h-4d)$. These are exactly the same bundles as those on the divisor E_1. Since $4h+4d$ is the class of the section of both rulings $\pi_1 : E_1 \to C$ and $\pi_2 : E_2 \to C$ we could try to interpret these bundles as bundles on the whole Hirzebruch-Jung A_2 sphere-tree fibration over C. If this interpretation were correct, then we had an interesting phenomenon, namely that a stable D-brane configuration in the large volume limit splits into two different stable configurations at the Gepner point. They are different because e.g. the bundles $\mathcal{O}_{E_1}$ and $\mathcal{O}_{E_2}$ come from different boundary states, $\mid 0,1,1,0,0\,\rangle\!\rangle_B$ and $\mid 1,0,1,0,0\,\rangle\!\rangle_B$, respectively. Note also that all these bundles are the same as those obtained in table 6.2 for the ruled surface E in the family $\mathbb{P}^4_{1,1,2,2,2}[8]$.

L-orbit	$n = (n_4^{E_2}, n_2^h, n_2^l, n_0)$	$m^{(\mathrm{CFT},E_2)}$	ν	$m^{(\mathrm{geom},E_2)}$	$\Delta^{(E_2)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-2,\ \ 0)$	3	1	3	0
$\mid 1,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 4,-2,\ \ 2)$	3	1	3	0
$\mid 0,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 0,-4,\ \ 0)$	12	4	9	3
$\mid 1,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 8,-4,\ \ 4)$	12	4	9	3

Table 6.20.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_2

- **The divisor D_1, a $\mathbb{P}^2$ blown up in seven points**
For this divisor we have $n = (0, -n_4^{E_1}, 0, n_4^{E_1}, n_0, n_2^h, 0, 0)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = 2n_4^L \tag{6.152}$$
$$\mathrm{c}_1(\mathcal{F}) = (n_2^h - 2n_4^L)h \tag{6.153}$$
$$\mathrm{ch}_2(\mathcal{F}) = -n_4^L - \frac{1}{2}n_2^h - n_0 \tag{6.154}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on D_1

$$m^{(\mathrm{geom},D1)} = 2n_4^L\left(n_4^L + 2n_0\right) + \frac{1}{2}\left(n_2^h\right)^2 + 1 \tag{6.155}$$

The boundary states with corresponding to D4-branes wrapped on D_1 are displayed in table 6.21. We note that

$$\Delta^{(D_1)} = m^{(\mathrm{CFT},D_1)} - m^{(\mathrm{geom},D_1)} = \nu - 1 \tag{6.156}$$

The rank two sheaves have the following Chern classes: $(-6h,-12)$, $(-8h,-22)$, and $(-4h,-4)$. The first of them has gauge group $U(1)$ and the remaining two have gauge group $U(1)\times U(1)$ since $\widetilde{\nu} = 2$ and $N = 1$. The rank four sheaves are characterized by $(12h,-42)$, $(-16h,-76)$, and $(-8h,-16)$. Although their charges are twice the charges of the rank two sheaves they do not correspond to direct sums. The first one has gauge group $U(2)$ because $\widetilde{\nu} = 1$ and $N = 2$. The other two have gauge group $U(2)\times U(2)$ since $\widetilde{\nu} = 2$ and $N = 2$. By (3.44) the curve $2h$ has genus 1 and by (5.83) the line bundle supported on this curve in the first line is $\mathcal{O}_{2h}(-2)$. The rank two bundle supported on this curve is topologically equivalent to $\mathcal{O}_{2h}(-2)^{\oplus 2}$ but not holomorphically split since $\widetilde{\nu} = 1$ and $N = 2$.

L-orbit	$n = (2n_4^L, n_2^h, n_0)$	$m^{(\mathrm{CFT},D_1)}$	ν	$m^{(\mathrm{geom},D_1)}$	$\Delta^{(D_1)}$
$\mid 0,2,0,0,0\,\rangle\!\rangle_B$	$(0,\ \ 2,\ \ 2)$ $(2,-4,-2)$	3	1	3	0
$\mid 0,2,1,0,0\,\rangle\!\rangle_B$	$(2,-6,-4)$ $(2,-2,\ \ 0)$	6	2	5	1
$\mid 0,2,1,1,0\,\rangle\!\rangle_B$	$(0,\ \ 4,\ \ 4)$ $(4,-8,-4)$	12	4	9	3
$\mid 0,2,1,1,1\,\rangle\!\rangle_B$	$(4,-12,-8)$ $(4,-4,\ \ 0)$	24	8	17	7

Table 6.21.: The boundary states corresponding to D4-branes wrapped on D_1

- **The divisor J_2**
For this divisor we have $n = (0, 0, n_4^{E_2}, 2n_4^{E_2}, n_0, n_2^h, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^{E_2} \tag{6.157}$$
$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^h + 4n_4^{E_2}\right)(h+d) + \left(n_2^l + 2n_4^{E_2}\right)l \tag{6.158}$$
$$\mathrm{ch}_2(\mathcal{F}) = -4n_4^E + \frac{1}{2}n_2^h - n_0 \tag{6.159}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on J_2

$$m^{(\mathrm{geom},J_2)} = 4\left(n_4^{E_2}\right)^2 + 2n_4^E n_0 + \frac{1}{2}n_2^l\left(n_2^h - n_2^l\right) + 6 \tag{6.160}$$

The boundary states with $n_4^{E_2} \neq 0$ corresponding to D4-branes wrapped on J_2 are displayed in table 6.22. We have noted in Section C.1.2 that the topology of this divisor is the same as the topology of the divisor H in $\mathbb{P}^4_{1,1,2,2,2}[8]$. We note here that the Chern classes (6.157) to (6.159) and the dimension of the moduli space (6.160) of the sheaves $\mathcal{F}$ on J_2 agree with those of the sheaves on the divisor H of $\mathbb{P}^4_{1,1,2,2,2}[8]$, (6.29) to (6.31) and (6.32), respectively. All the sheaves obtained this way are contained in the sheaves in table 6.3. In particular, the sheaves in the orbits $|\,3,0,1,0,0\,\rangle\!\rangle_B$ and $|\,4,0,1,0,0\,\rangle\!\rangle_B$ have the same dimensions although they are in different orbits which agrees with the same sheaves being in the single orbit $|\,2,0,1,0,0\,\rangle\!\rangle_B$ for $\mathbb{P}^4_{1,1,2,2,2}[8]$. All the dimensions agree.

L-orbit	$n=(n_4^{E_2},n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT},J_2)}$	ν	$m^{(\mathrm{geom},J_2)}$	$\Delta^{(J_2)}$
$\|\,3,0,1,0,0\,\rangle\!\rangle_B$	$(1,-4,-2,-2)$	11	1	8	3
$\|\,4,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-2,\ \ 0)$	11	1	8	3
$\|\,5,1,1,0,0\,\rangle\!\rangle_B$	$(2,-8,-4,-4)$	30	2	14	16
$\|\,3,0,1,1,1\,\rangle\!\rangle_B$	$(2,-8,-4,-4)$	44	4	14	30
$\|\,4,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 0,-4,\ \ 0)$	44	4	14	30
$\|\,5,1,1,1,1\,\rangle\!\rangle_B$	$(4,-16,-8,-8)$	120	8	38	82

Table 6.22.: The boundary states corresponding to D4-branes wrapped on J_2

- **The divisor H**
  For this divisor we have $n = \left(0, \frac{1}{2}n_4^{E_2}, n_4^{E_2}, \frac{3}{2}n_4^{E_2}, n_0, n_2^h, n_2^d, n_2^l\right)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^{E_2} \tag{6.161}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^h + n_4^{E_2}\right)h + \left(n_2^d + 2n_4^{E_2}\right)d + \left(n_2^l + n_4^{E_2}\right)l \tag{6.162}$$

$$\mathrm{ch}_2(\mathcal{F}) = -\frac{7}{2}n_4^{E_2} + \frac{1}{2}n_2^h - n_0 \tag{6.163}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$\begin{aligned} m^{(\mathrm{geom},H)} = \ & 3\left(n_4^{E_2}\right)^2 + n_4^{E_2}\left(n_2^h - n_2^d + 2n_0\right) - \left(n_2^h\right)^2 \\ & - \left(n_2^d\right)^2 - \left(n_2^l\right)^2 + n_2^d\left(2n_2^h + n_2^l\right) + 4 \end{aligned} \tag{6.164}$$

6.3.10. D-branes on $\mathbb{P}^4_{1,2,3,3,9}[18]$

- **The divisor L, a K3 surface**
  For this divisor we have $n = (0,0,0,n_4^L,n_0,n_2^h,n_2^h,0)$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\mathrm{rk}(\mathcal{F}) = n_4^L \tag{6.165}$$

$$\mathrm{c}_1(\mathcal{F}) = n_2^h(h+d) \tag{6.166}$$

$$\mathrm{ch}_2(\mathcal{F}) = -2n_4^L - n_0 \tag{6.167}$$

From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\mathrm{geom},\ L)}(\mathcal{F}) = \frac{1}{2}\left(n_2^h\right)^2 + 2n_4^L\left(n_4^L + n_0\right) + 2 \tag{6.168}$$

The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.23. We observe that for the boundary states in the first part of the table

$$\Delta^{(L)} = m^{(\mathrm{CFT},L)} - m^{(\mathrm{geom},L)} = 3\nu - 2 \tag{6.169}$$

However, we observe a new phenomenon here. There are boundary states on the $K3$ fiber which do not appear in the Gepner model of the corresponding $K3$ surface and which, moreover, do not satisfy (6.169). In particular, we observe that the boundary states $|\,0,3,1,0,0\,\rangle\!\rangle_B$, $|\,0,3,1,1,0\,\rangle\!\rangle_B$ and $|\,0,3,2,1,0\,\rangle\!\rangle_B$ have the same charges as the states $|\,5,0,0,0,0\,\rangle\!\rangle_B$, $|\,5,0,1,0,0\,\rangle\!\rangle_B$ and $|\,8,0,1,0,0\,\rangle\!\rangle_B$, respectively, but the dimension of their CFT moduli space differs. Another observation is that the charges of the states with $L_1 = 8$, $L_2 = 0$ are twice the charges of the states with $L_1 = 0$, $L_2 = 3$. However, while the number of marginal operators of the former are also twice the number of marginal operators of the latter, the geometric dimension of the moduli differs by a factor of four (up to an additive constant of 2).

L-orbit	Mukai vector $v = (n_4^L, n_2^h, -n_4^L - n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\|\,2,0,0,0,0\,\rangle\!\rangle_B$	$(2,-2,\ 1)\ (1,\ 0,\ 1)\ (1,-2,\ 2)$	1	1	0	1
$\|\,5,0,0,0,0\,\rangle\!\rangle_B$	$(1,-2,\ 0)\ (1,\ 0,-1)\ (0,\ 2,-1)$	5	1	4	1
$\|\,5,0,1,0,0\,\rangle\!\rangle_B$	$(1,-4,\ 1)\ (2,-2,-1)\ (1,\ 2,-2)$	9	1	8	1
$\|\,5,0,1,1,0\,\rangle\!\rangle_B$	$(3,-6,\ 0)\ (3,\ 0,-3)\ (0,\ 6,-3)$	21	1	20	1
$\|\,8,0,0,0,0\,\rangle\!\rangle_B$	$(2,-4,\ 2)\ (2,\ 0,\ 0)\ (0,\ 0,-2)$	6	2	2	4
$\|\,8,0,1,0,0\,\rangle\!\rangle_B$	$(2,-4,\ 0)\ (2,\ 0,-2)\ (0,\ 4,-2)$	14	2	10	4
$\|\,8,0,1,1,0\,\rangle\!\rangle_B$	$(4,-4,-2)\ (2,\ 4,-4)\ (2,-8,\ 2)$	30	2	26	4
$\|\,8,0,2,0,0\,\rangle\!\rangle_B$	$(4,-4,\ 0)\ (0,\ 4,-4)\ (0,\ 4,\ 0)$	20	4	10	10
$\|\,8,0,2,1,0\,\rangle\!\rangle_B$	$(4,-8,\ 0)\ (4,\ 0,-4)\ (0,\ 8,-4)$	44	4	34	10
$\|\,8,0,2,2,0\,\rangle\!\rangle_B$	$(4,-12,\ 4)\ (4,-4,-4)\ (4,\ 4,-4)$	64	8	42	22
$\|\,0,3,0,0,0\,\rangle\!\rangle_B$	$(1,-2,\ 1)\ (0,\ 0,-1)\ (1,\ 0,\ 0)$	3	1	2	1
$\|\,0,3,1,0,0\,\rangle\!\rangle_B$	$(1,-2,\ 0)\ (1,\ 0,-1)\ (0,\ 2,-1)$	7	1	4	3
$\|\,0,3,1,1,0\,\rangle\!\rangle_B$	$(1,-4,\ 1)\ (2,-2,-1)\ (1,\ 2,-2)$	15	1	8	7
$\|\,0,3,2,0,0\,\rangle\!\rangle_B$	$(2,-2,\ 0)\ (0,\ 2,-2)\ (0,\ 2,\ 0)$	10	2	4	6
$\|\,0,3,2,1,0\,\rangle\!\rangle_B$	$(2,-4,\ 0)\ (2,\ 0,-2)\ (0,\ 4,-2)$	22	2	10	12
$\|\,0,3,2,2,0\,\rangle\!\rangle_B$	$(2,-6,\ 2)\ (2,-2,-2)\ (2,\ 2,-2)$	32	4	12	20

Table 6.23.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

- **The divisor E_1, a ruled surface over a genus $g = 2$ curve**
  For this divisor we have $n = (0, n_4^{E_1}, 0, 0, n_0, n_2^h, n_2^d, 0)$ since $L \cdot E_1 = 0$ and (5.74) to (5.76) reduce to

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^{E_1} && (6.170)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^h - 3n_4^{E_1}\right) h + n_2^d d && (6.171)\\
\mathrm{ch}_2(\mathcal{F}) &= -2n_4^{E_1} + n_2^h - n_2^d - n_0 && (6.172)
\end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_1

$$m^{(\mathrm{geom},E_1)} = \frac{1}{2}\left(n_4^{E_1}\right)^2 + n_4^{E_1}\left(n_2^h - n_2^d + 2n_0\right) + \frac{1}{2} n_2^h \left(2n_2^d - n_2^h\right) + 1 \tag{6.173}$$

The boundary states with $n_4^{E_1} \neq 0$ corresponding to D4-branes wrapped on E_1 are displayed in table 6.24. We note that

$$\Delta^{(E_1)} = m^{(\mathrm{CFT},E_1)} - m^{(\mathrm{geom},E_1)} = \nu - 1 \tag{6.174}$$

The first two line bundles are $\mathcal{O}_{E_1}(-2h-2d)$ and $\mathcal{O}_{E_1}$. The next two rank two bundles are topologically equivalent to $\mathcal{O}_{E_1}^{\oplus 2}(-2h-2d)$ and $\mathcal{O}_{E}^{\oplus 2}$. They have $\widetilde{\nu}=1$ and $N=2$ and are therefore $U(2)$ bundles and not holomorphic direct sums. The remaining rank two sheaves have Chern classes $(-4h-6d,4)$, $(-4h-2d,0)$, and $(\pm 2d,0)$.

L-orbit	$n=(n_4^{E_1},n_2^h,n_2^d,n_0)$	$m^{(\mathrm{CFT},E_1)}$	ν	$m^{(\mathrm{geom},E_1)}$	$\Delta^{(E_1)}$
$\vert\,0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ 1,-2,\ 0)$	2	1	2	0
$\vert\,0,1,1,0,0\,\rangle\!\rangle_B$	$(1,\ 3,\ 0,\ 1)$	2	1	2	0
$\vert\,0,0,2,1,0\,\rangle\!\rangle_B$	$(2,\ 2,-4,\ 0)$	6	2	5	1
$\vert\,0,1,2,1,0\,\rangle\!\rangle_B$	$(2,\ 6,\ 0,\ 2)$	6	2	5	1
$\vert\,0,0,2,2,0\,\rangle\!\rangle_B$	$(2,\ 2,-6,\ 0)\ (2,\ 2,-2,\ 0)$	8	4	5	3
$\vert\,0,1,2,2,0\,\rangle\!\rangle_B$	$(2,\ 6,\ 2,\ 0)\ (2,\ 6,-2,\ 4)$	8	4	5	3

Table 6.24.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_1

- **The divisor E_2, a ruled surface over a genus $g=2$ curve**
For this divisor we have $n=(0,0,n_4^{E_2},0,n_0,n_2^h,n_2^h,n_2^l)$ since $L\cdot E_2=2h+2d$ and $H\cdot E_2=D\cdot E_2$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^{E_2} \tag{6.175}$$
$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^h-2n_4^{E_2}\right)(h+d)+\left(n_2^l+n_4^{E_2}\right)l \tag{6.176}$$
$$\mathrm{ch}_2(\mathcal{F}) = -n_4^{E_2}+\frac{1}{2}n_2^h-n_2^l-n_0 \tag{6.177}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_2

$$m^{(\mathrm{geom},E_2)}=\left(n_4^{E_2}\right)^2+2n_4^E n_0+n_2^h n_2^l+1 \tag{6.178}$$

The boundary states with $n_4^{E_2}\neq 0$ corresponding to D4-branes wrapped on E_2 are displayed in table 6.25. We note that except for the last two boundary states

$$\Delta^{(E_2)}=m^{(\mathrm{CFT},E_2)}-m^{(\mathrm{geom},E_2)}=\nu-1 \tag{6.179}$$

The rank one bundles are $\mathcal{O}_{E_2}(-2h-2d)$ and $\mathcal{O}_{E_2}$. The rank two bundles in the third and fourth line are topologically equivalent to $\mathcal{O}_{E_2}^{\oplus 2}(-2h-2d)$ and $\mathcal{O}_{E_2}^{\oplus 2}$. The remaining rank two sheaves have Chern classes $(-4d-4h\mp 2l,\pm 2)$, $(\pm 2l,0)$, $(-8h-8d-2l,-8)$ and $(4h+4d+2l,6)$. The first four bundles are precisely the same as the first four in table 6.24 on the divisor E_1. Since $2h+2d$ is the class of the section of both rulings $\pi_1: E_1\to C$ and $\pi_2: E_2\to C$ we can interpret them as bundles on the Hirzebruch-Jung A_2 sphere-tree fibration over C in the same way as in the family $\mathbb{P}^4_{1,2,3,3,3}[12]$. Note also that all the configurations found here coincide with those in table 6.5 on the ruled surface E in the family $\mathbb{P}^4_{1,1,2,2,6}[12]$.

- **The divisor D_1, a $\mathbb{P}^2$ blown up at eight points**
For this divisor we have $n=(0,-n_4^{E_1},0,n_4^{E_1},n_0,n_2^h,0,0)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = 2n_4^L \tag{6.180}$$
$$\mathrm{c}_1(\mathcal{F}) = n_2^h h \tag{6.181}$$
$$\mathrm{ch}_2(\mathcal{F}) = -2n_4^L-\frac{1}{2}n_2^h-n_0 \tag{6.182}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on D_1

$$m^{(\mathrm{geom},D_1)}=4\left(n_4^L\right)^2+2n_4^L\left(n_2^h+2n_0\right)+\left(n_2^h\right)^2+1 \tag{6.183}$$

L-orbit	$n=(n_4^{E_2},n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT},E_2)}$	ν	$m^{(\mathrm{geom},E_2)}$	$\Delta^{(E_2)}$
$\mid 0,0,1,0,0\rangle\rangle_B$	$(1,\ 0,-1,\ 0)$	2	1	2	0
$\mid 1,0,1,0,0\rangle\rangle_B$	$(1,\ 2,-1,\ 1)$	2	1	2	0
$\mid 0,0,2,1,0\rangle\rangle_B$	$(2,\ 0,-2,\ 0)$	6	2	5	1
$\mid 1,0,2,1,0\rangle\rangle_B$	$(2,\ 4,-2,\ 2)$	6	2	5	1
$\mid 0,0,2,2,0\rangle\rangle_B$	$(2,\ 0,-4,\ 0)\ (2,\ 0,\ 0,\ 0)$	8	4	5	3
$\mid 1,0,2,2,0\rangle\rangle_B$	$(2,\ 4,\ 0,\ 0)\ (2,\ 4,-4,\ 4)$	8	4	5	3
$\mid 3,0,2,2,0\rangle\rangle_B$	$(2,-4,-4,\ 0)$	32	4	21	11
$\mid 4,0,2,2,0\rangle\rangle_B$	$(2,\ 8,\ 0,\ 4)$	32	4	21	11

Table 6.25.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_2

The boundary states with $n_4^L \neq 0$ corresponding to D4-branes wrapped on D_1 are displayed in table 6.26. We note that

$$\Delta^{(D_1)} = m^{(\mathrm{CFT},D_1)} - m^{(\mathrm{geom},D_1)} = \nu - 1 \tag{6.184}$$

We are not discussing all the bundles that appear in the table. Among all the rank one sheaves there is only bundle, namely $\mathcal{O}_{D_1}(-h)$ in the third line. There are also bundles topologically equivalent to $\mathcal{O}_{D_1}^{\oplus m}(-h)$ for $m = 2, 3$ and 4 in the orbits $\mid 0,2,2,0,0\rangle\rangle_B$, $\mid 0,2,1,1,0\rangle\rangle_B$ and $\mid 0,2,2,2,0\rangle\rangle_B$, respectively. The gauge groups are $U(1)\times U(1)$, $U(1)$ and $U(2)$, respectively. An interesting sheaf appears in the fourth line, the ideal sheaf of two points $\mathcal{I}_{p+q}$. Interpreting the two points as D0-branes, they are restricted to move in D_1 giving each of them two moduli for a total of four. We also find a sheaf topologically equivalent to $\mathcal{I}_{p+q}^{\oplus 2}$ with gauge group $U(1)\times U(1)$.

L-orbit	$n=(2n_4^L,n_2^h,n_0)$	$m^{(\mathrm{CFT},D_1)}$	ν	$m^{(\mathrm{geom},D_1)}$	$\Delta^{(D_1)}$
$\mid 0,0,0,0,0\rangle\rangle_B$	$(1,-1,-1)$	0	1	0	0
$\mid 0,1,0,0,0\rangle\rangle_B$	$(1,-2,-2)$	0	1	0	0
$\mid 0,2,0,0,0\rangle\rangle_B$	$(1,-1,\ 0)\ (0,\ 1,\ 1)\ (1,-2,-1)$	2	1	2	0
$\mid 0,2,1,0,0\rangle\rangle_B$	$(1,\ 0,\ 1)\ (1,-3,\ 2)\ (2,-3,-1)$	4	1	4	0
$\mid 0,2,1,1,0\rangle\rangle_B$	$(3,-3,\ 0)\ (0,\ 3,\ 3)\ (3,-6,-3)$	10	1	10	0
$\mid 0,2,2,0,0\rangle\rangle_B$	$(2,-2,\ 0)\ (0,\ 2,\ 2)\ (2,-4,-2)$	6	2	5	1
$\mid 0,2,2,1,0\rangle\rangle_B$	$(2,\ 0,\ 2)\ (2,-6,\ 4)\ (4,-6,-2)$	14	2	13	1
$\mid 0,2,2,2,0\rangle\rangle_B$	$(4,-4,\ 0)\ (0,\ 4,\ 4)\ (4,-8,-4)$	20	4	17	3

Table 6.26.: The boundary states corresponding to D4-branes wrapped on D_1

- **The divisor J_2**
  For this divisor we have $n = (0, 0, n_4^{E_2}, 2n_4^{E_2}, n_0, n_2^h, n_2^h, n_2^l)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_2} && (6.185)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^h + 2n_4^{E_2}\right)(h+d) + \left(n_2^l + n_4^{E_2}\right) l && (6.186)\\ \mathrm{ch}_2(\mathcal{F}) &= -4n_4^E + \frac{1}{2}n_2^h - n_0 && (6.187) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on J_2

$$m^{(\mathrm{geom},J_2)} = 4\left(n_4^{E_2}\right)^2 + 2n_4^{E_2}n_0 + n_2^l\left(n_2^h - n_2^l\right) + 5 \tag{6.188}$$

The boundary states with $n_4^{E_2} \neq 0$ corresponding to D4-branes wrapped on J_2 are displayed in table 6.27. We have noted in Section C.1.3 that the geometry of this divisor is the same as the geometry of the divisor H in $\mathbb{P}^4_{1,1,2,2,6}[12]$. We note here that the Chern classes (6.185) to (6.187) and the dimension of the moduli space (6.188) of the sheaves $\mathcal{F}$ on J_2 agree with those of the sheaves on the divisor H of $\mathbb{P}^4_{1,1,2,2,6}[12]$, (6.43) to (6.45) and (6.46), respectively. All the sheaves obtained this way are contained in the sheaves in table 6.6. In particular, the sheaves in the orbits $|\,3,0,2,2,0\,\rangle\!\rangle_B$ and $|\,4,0,2,2,0\,\rangle\!\rangle_B$ have the same dimensions although they are in different orbits which agrees with the same sheaves being in the single orbit $|\,2,0,2,2,0\,\rangle\!\rangle_B$ for $\mathbb{P}^4_{1,1,2,2,6}[12]$. All the dimensions agree.

L-orbit	$n=(n_4^{E_2},n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT},J_2)}$	ν	$m^{(\mathrm{geom},J_2)}$	$\Delta^{(J_2)}$
$\|\,5,1,0,0,0\,\rangle\!\rangle_B$	$(1,-2,-1,-1)$	10	1	8	2
$\|\,6,0,1,0,0\,\rangle\!\rangle_B$	$(1,-2,-1,-1)$	12	1	8	4
$\|\,7,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-1,\ \ 0)$	12	1	8	4
$\|\,5,1,1,1,0\,\rangle\!\rangle_B$	$(3,-6,-3,-3)$	42	1	32	10
$\|\,8,1,1,0,0\,\rangle\!\rangle_B$	$(2,-4,-2,-2)$	30	2	17	13
$\|\,6,0,2,1,0\,\rangle\!\rangle_B$	$(2,-4,-2,-2)$	38	2	17	21
$\|\,7,0,2,1,0\,\rangle\!\rangle_B$	$(2,\ \ 0,-2,\ \ 0)$	38	2	17	21
$\|\,3,0,2,2,0\,\rangle\!\rangle_B$	$(2,-4,\ \ 0,-4)$	32	4	5	27
$\|\,4,0,2,2,0\,\rangle\!\rangle_B$	$(2,\ \ 0,-4,\ \ 0)$	32	4	5	27
$\|\,6,0,2,2,0\,\rangle\!\rangle_B$	$(2,-8,-4,-4)\ (2,\ \ 0,\ \ 0,\ \ 0)$	56	4	21	35
$\|\,7,0,2,2,0\,\rangle\!\rangle_B$	$(2,-4,-4,\ \ 0)\ (2,\ \ 4,\ \ 0,\ \ 0)$	56	4	21	35
$\|\,8,1,2,1,0\,\rangle\!\rangle_B$	$(4,-8,-4,-4)$	92	4	53	39
$\|\,8,1,2,2,0\,\rangle\!\rangle_B$	$(4,-16,-8,-8)\ (4,\ \ 0,\ \ 0,\ \ 0)$	136	8	69	67

Table 6.27.: The boundary states corresponding to D4-branes wrapped on J_2

- **The divisor** H
For this divisor we have $n=\left(0,\frac{1}{2}n_4^{E_2},n_4^{E_2},\frac{3}{2}n_4^{E_2},n_0,n_2^h,n_2^d,n_2^l\right)$ and (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^{E_2} \tag{6.189}$$
$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^h+n_4^{E_2}\right)h+\left(n_2^d+2n_4^{E_2}\right)d+\left(n_2^l+n_4^{E_2}\right)l \tag{6.190}$$
$$\mathrm{ch}_2(\mathcal{F}) = -\frac{7}{2}n_4^{E_2}+\frac{1}{2}n_2^h-n_0 \tag{6.191}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$\begin{aligned} m^{(\mathrm{geom},H)} &= 3\left(n_4^{E_2}\right)^2+n_4^{E_2}\left(n_2^h-n_2^d+2n_0\right)-\left(n_2^h\right)^2 \\ &\quad -\left(n_2^d\right)^2-\left(n_2^l\right)^2+n_2^d\left(2n_2^h+n_2^l\right)+4 \end{aligned} \tag{6.192}$$

6.3.11. D-branes on $\mathbb{P}^4_{1,4,5,5,5}[20]$

- **The divisor L, a K3 surface**
Here we have $n=(0,0,0,0,0,n_4^L,n_0,n_2^{C_5},3n_2^{C_5},0,2n_2^{C_5},n_2^{C_5})$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\mathrm{rk}(\mathcal{F}) = n_4^L \tag{6.193}$$
$$\mathrm{c}_1(\mathcal{F}) = n_2^{C_5}(C_1+3C_2+2C_4+C_5) \tag{6.194}$$
$$\mathrm{ch}_2(\mathcal{F}) = -2n_4^L-n_0 \tag{6.195}$$

From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\mathrm{geom},\ L)}(\mathcal{F})=\frac{1}{4}\left(n_2^{C_5}\right)^2+2n_4^L\left(n_4^L+n_0\right)+2 \tag{6.196}$$

The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.28. We make the same observation as in [291] that

$$\Delta^{(L)} = m^{(\mathrm{CFT},L)} - m^{(\mathrm{geom},L)} = 3\nu - 2 \tag{6.197}$$

However, we observe here the same phenomenon as in Section 6.3.10. There are boundary states on the $K3$ fiber which do not appear in the Gepner model of the corresponding $K3$ surface and which, moreover, do not satisfy (6.197). In particular, we observe that the boundary states $|\ 1,1,1,1,0\ \rangle\rangle_B$ and $|\ 1,1,1,1,1\ \rangle\rangle_B$ have the same charges as the states $|\ 9,0,0,0,0\ \rangle\rangle_B$ and $|\ 9,0,1,0,0\ \rangle\rangle_B$, respectively, but the dimension of their CFT moduli space differs. Another observation is again that the charges of the states with $L_1 = 9$, $L_2 = 0$, $L_3 = 1$ are twice the charges of the states with $L_1 = 0$, $L_2 = 1$, $L_3 = 1$. However, while the number of marginal operators of the former are also twice the number of marginal operators of the latter, the geometric dimension of the moduli differs by a factor of four (up to an additive constant of 2).

L-orbit	Mukai vector $v = (n_4^L, n_2^{C_5}, -n_4^L - n_0)$	$m^{(\mathrm{CFT},L)}$	ν	$m^{(\mathrm{geom},L)}$	$\Delta^{(L)}$
$\|\ 4,0,0,0,0\ \rangle\rangle_B$	$(1,\ 0,\ 1)\ (3,-8,\ 3)$	1	1	0	1
$\|\ 9,0,0,0,0\ \rangle\rangle_B$	$(0,\ 4,-2)\ (2,-4,\ 0)$	7	1	6	1
$\|\ 9,0,1,0,0\ \rangle\rangle_B$	$(2,\ 0,-2)\ (2,-8,\ 2)$	14	2	10	4
$\|\ 9,0,1,1,0\ \rangle\rangle_B$	$(0,\ 8,-4)\ (4,-8,\ 0)$	28	4	18	10
$\|\ 9,0,1,1,1\ \rangle\rangle_B$	$(4,\ 0,-4)\ (4,-16,\ 4)$	56	8	34	22
$\|\ 1,1,1,0,0\ \rangle\rangle_B$	$(1,\ 0,-1)\ (1,-4,\ 1)$	7	1	4	3
$\|\ 1,1,1,1,0\ \rangle\rangle_B$	$(2,-4,\ 0)\ (0,\ 4,-2)$	14	2	6	8
$\|\ 1,1,1,1,1\ \rangle\rangle_B$	$(2,-8,\ 2)\ (2,\ 0,-2)$	28	4	10	18

Table 6.28.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

- **The divisor E_1, a ruled surface over a genus $g = 3$ curve**
For this divisor we have $n = (0, n_4^{E_1}, 0, 0, 0, 0, n_0, n_2^{C_1}, n_2^{C_2}, 0, 0, 0)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_1} && (6.198)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1} - 6n_4^{E_1}\right) C_1 + \left(n_2^{C_2} - 4n_4^{E_1}\right) C_2 && (6.199)\\ \mathrm{ch}_2(\mathcal{F}) &= -6n_4^{E_1} + 2n_2^{C_1} - n_2^{C_2} - n_0 && (6.200) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_1

$$m^{(\mathrm{geom},E_1)} = n_4^{E_1}\left(-n_4^{E_1} + 3n_2^{C_1} - n_2^{C_2} + 2n_0\right) + \frac{1}{4}n_2^{C_1}\left(2n_2^{C_2} - 3n_2^{C_1}\right) + 1 \tag{6.201}$$

The boundary states corresponding to D4-branes wrapped on E_1 are displayed in table 6.29. We note that

$$\Delta^{(E_1)} = m^{(\mathrm{CFT},E_1)} - m^{(\mathrm{geom},E_1)} = \nu - 1 \tag{6.202}$$

The two states correspond to $\mathcal{O}_{E_1}(-4C_1 - 12C_2)$ and a bundle which is topologically equivalent to $\mathcal{O}_{E_1}^{\oplus 2}(-4C_1 - 12C_2)$. It has $\widetilde{\nu} = 1$ and $N = 2$ and hence gauge group $U(2)$.

- **The divisor E_2, a ruled surface over a genus $g = 3$ curve**
For this divisor we have $n = (0, 0, n_4^{E_2}, 0, 0, 0, n_0, n_2^{C_1}, 3n_2^{C_1}, 0, n_2^{C_4}, 0)$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_2} && (6.203)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1} - 4n_4^{E_2}\right)(C_1 + 3C_2) + \left(n_2^{C_4} - 2n_4^{E_2}\right) C_4 && (6.204)\\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^{E_2} + \frac{3}{2}n_2^{C_1} - n_2^{C_4} - n_0 && (6.205) \end{aligned}$$

L-orbit	$n = (n_4^{E_1}, n_2^{C_1}, n_2^{C_2}, n_0)$	$m^{(\mathrm{CFT},E_1)}$	ν	$m^{(\mathrm{geom},E_1)}$	$\Delta^{(E_1)}$
$\lvert\, 0,0,1,0,0\, \rangle\!\rangle_B$	$(1, \;\; 2, -8, \;\; 0)$	3	1	3	0
$\lvert\, 0,0,1,1,1\, \rangle\!\rangle_B$	$(2, \;\; 4, -16, \;\; 0)$	12	4	9	3

Table 6.29.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_1

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_2

$$m^{(\mathrm{geom},E_2)} = 2n_4^{E_2} n_0 + \frac{1}{2} n_2^{C_1} \left(n_2^{C_4} - n_2^{C_1} \right) + 1 \tag{6.206}$$

The boundary states corresponding to D4-branes wrapped on E_2 are displayed in table 6.30. We note that

$$\Delta^{(E_2)} = m^{(\mathrm{CFT},E_2)} - m^{(\mathrm{geom},E_2)} = \nu - 1 \tag{6.207}$$

The two states correspond to $\mathcal{O}_{E_2}(-4C_1 - 12C_2 - 8C_4)$ and a bundle which is topologically equivalent to $\mathcal{O}_{E_2}^{\oplus 2}(-4C_1 - 12C_2 - 8C_4)$. It has $\widetilde{\nu} = 1$ and $N = 2$ and hence gauge group $U(2)$.

L-orbit	$n = (n_4^{E_2}, n_2^{C_1}, n_2^{C_4}, n_0)$	$m^{(\mathrm{CFT},E_2)}$	ν	$m^{(\mathrm{geom},E_2)}$	$\Delta^{(E_2)}$
$\lvert\, 0,0,1,0,0\, \rangle\!\rangle_B$	$(1, \;\; 0, -6, \;\; 0)$	3	1	3	0
$\lvert\, 0,0,1,1,1\, \rangle\!\rangle_B$	$(2, \;\; 0, -12, \;\; 0)$	12	4	9	3

Table 6.30.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_2

- **The divisor E_3, a ruled surface over a genus $g = 3$ curve**
  For this divisor we have $n = (0,0,0,n_4^{E_3},0,0,n_0,n_2^{C_1},3n_2^{C_1},0,2n_2^{C_1},n_2^{C_5})$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_3} && (6.208) \\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1} - 4n_4^{E_3} \right)(C_1 + 3C_2 + 2C_4) + n_2^{C_5} C_5 && (6.209) \\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^{E_3} + n_2^{C_1} - n_2^{C_5} - n_0 && (6.210) \end{aligned}$$

  as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_3

$$m^{(\mathrm{geom},E_3)} = 2n_4^{E_3} \left(n_4^{E_3} + n_0 \right) + \frac{1}{4} n_2^{C_1} \left(2n_2^{C_5} - n_2^{C_1} \right) + 1 \tag{6.211}$$

  The boundary states corresponding to D4-branes wrapped on E_3 are displayed in table 6.31. We note that

$$\Delta^{(E_3)} = m^{(\mathrm{CFT},E_3)} - m^{(\mathrm{geom},E_3)} = \nu - 1 \tag{6.212}$$

  The two states correspond to $\mathcal{O}_{E_3}(-4C_1 - 12C_2 - 8C_4 - 4C_5)$ and a bundle which is topologically equivalent to $\mathcal{O}_{E_3}^{\oplus 2}(-4C_1 - 12C_2 - 8C_4 - 4C_5)$. It has $\widetilde{\nu} = 1$ and $N = 2$ and hence gauge group $U(2)$.

- **The divisor E_4, a ruled surface over a genus $g = 3$ curve**
  For this divisor we have $n = (0,0,0,0,n_4^{E_4},0,n_0,n_2^{C_1},3n_2^{C_1},n_2^{C_3},2n_2^{C_1},n_2^{C_1})$ and (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_4} && (6.213) \\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1} - 4n_4^{E_4} \right)(C_1 + 3C_2 + 2C_4 + C_5) + \left(n_2^{C_3} + 2n_4^{E_4} \right) C_3 && (6.214) \\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^{E_4} + \frac{1}{2} n_2^{C_1} - n_2^{C_3} - n_0 && (6.215) \end{aligned}$$

L-orbit	$n=(n_4^{E_3},n_2^{C_1},n_2^{C_5},n_0)$	$m^{(\mathrm{CFT},E_3)}$	ν	$m^{(\mathrm{geom},E_3)}$	$\Delta^{(E_3)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-4,\ \ 0)$	3	1	3	0
$\mid 0,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 0,-8,\ \ 0)$	12	4	9	3

Table 6.31.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_3

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_4

$$m^{(\mathrm{geom},E_4)}=2n_4^{E_5}\left(n_4^{E_4}+n_0\right)+\frac{1}{2}n_2^{C_1}n_2^{C_3}+1 \tag{6.216}$$

The boundary states corresponding to D4-branes wrapped on E_4 are displayed in table 6.32. We note that

$$\Delta^{(E_4)}=m^{(\mathrm{CFT},E_4)}-m^{(\mathrm{geom},E_4)}=\nu-1 \tag{6.217}$$

The two states correspond to $\mathcal{O}_{E_4}(-4C_1-12C_2-8C_4-4C_5)$ and $\mathcal{O}_{E_4}$ and bundles which are topologically equivalent to $\mathcal{O}_{E_4}^{\oplus 2}(-4C_1-12C_2-8C_4-4C_5)$ and $\mathcal{O}_{E_4}^{\oplus 2}$. Both of them have $\widetilde{\nu}=1$ and $N=2$ and hence gauge group $U(2)$.

L-orbit	$n=(n_4^{E_4},n_2^{C_1},n_2^{C_3},n_0)$	$m^{(\mathrm{CFT},E_4)}$	ν	$m^{(\mathrm{geom},E_4)}$	$\Delta^{(E_4)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 0,-2,\ \ 0)$	3	1	3	0
$\mid 3,0,1,0,0\,\rangle\!\rangle_B$	$(1,\ \ 4,-2,\ \ 2)$	3	1	3	0
$\mid 0,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 0,-4,\ \ 0)$	12	4	9	3
$\mid 3,0,1,1,1\,\rangle\!\rangle_B$	$(2,\ \ 8,-4,\ \ 4)$	12	4	9	3

Table 6.32.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_4

- **The divisor D_1, a $\mathbb{P}^2$**
  For this divisor we have $n=(0,-3n_4^L,-2n_4^L,-n_4^L,0,n_4^L,n_0,n_2^{C_1},0,0,0,0)$ and (5.74) to (5.76) reduce to

$$\begin{aligned}\mathrm{rk}(\mathcal{F}) &= 4n_4^L &(6.218)\\ \mathrm{c}_1(\mathcal{F}) &= n_2^{C_1}C_1 &(6.219)\\ \mathrm{ch}_2(\mathcal{F}) &= -4n_4^L-\frac{3}{2}n_2^{C_1}-n_0 &(6.220)\end{aligned}$$

  as well as (5.99) yields for the dimension of the moduli space of the sheaves on D_1

$$m^{(\mathrm{geom},D1)}=4n_4^L\left(4n_4^L+3n_2^{C_1}+2n_0\right)-\frac{3}{2}\left(n_2^{C_1}\right)^2+1 \tag{6.221}$$

  We have not found any boundary states with these charges.

- **The divisor H**
  For this divisor we have $n=\left(0,\frac{1}{4}n_4^{E_4},\frac{1}{2}n_4^{E_4},\frac{3}{4}n_4^{E_4},n_4^{E_4},\frac{5}{4}n_4^L,n_0,n_2^{C_1},n_2^{C_2},n_2^{C_3},n_2^{C_4},n_2^{C_5}\right)$ and (5.74) to (5.76) reduce to

$$\begin{aligned}\mathrm{rk}(\mathcal{F}) &= n_4^{E_4} &(6.222)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1}+2n_4^{E_4}\right)C_1+\left(n_2^{C_2}+8n_4^{E_4}\right)C_2 &(6.223)\\ &\quad+\left(n_2^{C_3}+2n_4^{E_4}\right)C_3+\left(n_2^{C_4}+6n_4^{E_4}\right)C_4+\left(n_2^{C_5}+4n_4^{E_4}\right)C_5 &\\ \mathrm{ch}_2(\mathcal{F}) &= -4n_4^{E_4}+\frac{1}{2}n_2^{C_1}-n_0 &(6.224)\end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$\begin{aligned} m^{(\text{geom},H)} &= 3\left(n_4^{E_4}\right)^2 + n_4^{E_4}\left(3n_2^{C_1} - n_2^{C_2} + 2n_0\right) - 3\left(n_2^{C_1}\right)^2 \\ &\quad -\frac{1}{2}\left(\left(n_2^{C_2}\right)^2 + \left(n_2^{C_3}\right)^2 + \left(n_2^{C_4}\right)^2 + \left(n_2^{C_5}\right)^2\right) \\ &\quad -\frac{1}{2}\left(-n_2^{C_2}\left(4n_2^{C_1} + n_2^{C_4}\right) - n_2^{C_5}\left(n_2^{C_3} + n_2^{C_4}\right)\right) + 5 \end{aligned} \tag{6.225}$$

6.3.12. D-branes on $\mathbb{P}^4_{1,3,4,4,12}[24]$

- **The divisor L, a K3 surface**
Here we have $n = (0, 0, 0, 0, 0, n_4^L, n_0, 3n_2^{C_4}, 3n_2^{C_4}, 3n_2^{C_4}, n_2^{C_4}, 3n_2^{C_5})$. We have already computed the Chern classes for a sheaf $\mathcal{F}$ on L in (5.77) to (5.79) which for this model become

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^L && (6.226) \\ \mathrm{c}_1(\mathcal{F}) &= n_2^{C_4}(3C_1 + 3C_2 + 3C_3 + C_4 + 3C_5) && (6.227) \\ \mathrm{ch}_2(\mathcal{F}) &= -2n_4^L - n_0 && (6.228) \end{aligned}$$

From (5.102) the dimension of the moduli space of the sheaves $\mathcal{F}$ on L is

$$m^{(\text{geom, }L)}(\mathcal{F}) = \frac{1}{2}\left(n_2^{C_4}\right)^2 + 2n_4^L\left(n_4^L + n_0\right) + 2 \tag{6.229}$$

The boundary states corresponding to D4-branes wrapped on L are displayed in table 6.33. We note that

$$\Delta^{(L)} = m^{(\text{CFT},L)} - m^{(\text{geom},L)} = 3\nu - 2 \tag{6.230}$$

L-orbit	Mukai vector $v = (n_4^L, n_2^{C_5}, -n_4^L - n_0)$	$m^{(\text{CFT},L)}$	ν	$m^{(\text{geom},L)}$	$\Delta^{(L)}$
$\vert 3,0,0,0,0 \rangle\rangle_B$	(2, −2, 1) (1, 0, 1) (1, −6, 2)	1	1	0	1
$\vert 7,0,0,0,0 \rangle\rangle_B$	(1, 0, −1) (0, 2, −1) (1, −2, 0)	5	1	4	1
$\vert 7,0,1,0,0 \rangle\rangle_B$	(1, −4, 1) (2, 2, −1) (1, 2, −2)	9	1	8	1
$\vert 7,0,1,1,0 \rangle\rangle_B$	(0, 6, −3) (3, −6, 0) (3, 0, −3)	21	1	20	1
$\vert 11,0,0,0,0 \rangle\rangle_B$	(2, 0, 0) (2, −4, 2) (0, 0, −2)	6	2	2	4
$\vert 11,0,1,0,0 \rangle\rangle_B$	(2, −4, 0) (2, 0, −2) (0, 4, −2)	14	2	10	4
$\vert 11,0,1,1,0 \rangle\rangle_B$	(2, −8, 2) (4, −4, −2) (2, 4, −4)	30	2	26	4
$\vert 11,0,2,0,0 \rangle\rangle_B$	(0, 4, 0) (4, −4, 0) (0, 4, −4)	20	4	10	10
$\vert 11,0,2,1,0 \rangle\rangle_B$	(4, −8, 0) (4, 0, −4) (0, 8, −4)	44	4	34	10
$\vert 11,0,2,2,0 \rangle\rangle_B$	(4, −12, 4) (4, −4, −4) (4, 4, −4)	64	8	42	22

Table 6.33.: The boundary states corresponding to D4-branes wrapped on the K3 surface L

- **The divisor E_1, a ruled surface over a genus $g = 2$ curve**
For this divisor we have $n = (0, n_4^{E_1}, 0, 0, 0, 0, n_0, 3n_2^{C_3}, n_2^{C_2}, n_2^{C_3}, 3n_2^{C_3}, 2n_2^{C_3})$ and we set $C = 3C_1 + C_3 + 3C_4 + 2C_5$. Then (5.74) to (5.76) reduce to

$$\begin{aligned} \mathrm{rk}(\mathcal{F}) &= n_4^{E_1} && (6.231) \\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_3} - 2n_4^{E_1}\right)C + \left(n_2^{C_2} - 3n_4^{E_1}\right)C_2 && (6.232) \\ \mathrm{ch}_2(\mathcal{F}) &= -n_4^{E_1} + \frac{5}{2}n_2^{C_3} - n_2^{C_2} - n_0 && (6.233) \end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_1

$$m^{(\mathrm{geom},E_1)} = n_4^{E_1}\left(n_4^{E_1} + 2n_0\right) + n_2^{C_3}\left(n_2^{C_2} - 2n_2^{C_3}\right) + 1 \tag{6.234}$$

The boundary states with $n_4^{E_1} \neq 0$ corresponding to D4-branes wrapped on E_1 are displayed in table 6.34. We note that

$$\Delta^{(E_1)} = m^{(\mathrm{CFT},E_1)} - m^{(\mathrm{geom},E_1)} = \nu - 1 \tag{6.235}$$

The first state corresponds to the sheaf $\mathcal{O}_{E_1}(-2C - 6C_2) \otimes \mathcal{I}_{p_1,p_2}$. Since $\widetilde{\nu} = 2$ the second state is the direct sum of twice the first one. The remaining three rank two sheaves have gauge group $U(2)$ and Chern classes $(-4C - 14C_2, 6)$, $(-4C - 10C_2, 2)$, and $(-2C_2, 0)$.

L-orbit	$n = (n_4^{E_1}, n_2^{C_2}, n_2^{C_3}, n_0)$	$m^{(\mathrm{CFT},E_1)}$	ν	$m^{(\mathrm{geom},E_1)}$	$\Delta^{(E_1)}$
$\vert\, 0,0,1,0,0 \,\rangle\!\rangle_B$	$(1,-3,\ 0,\ 0)$	2	1	2	0
$\vert\, 0,0,2,1,0 \,\rangle\!\rangle_B$	$(2,-6,\ 0,\ 0)$	6	2	5	1
$\vert\, 0,0,2,2,0 \,\rangle\!\rangle_B$	$(2,-8,\ 0,\ 0)$ $(2,-4,\ 0,\ 0)$	8	4	5	3
$\vert\, 0,2,2,2,0 \,\rangle\!\rangle_B$	$(2,\ 4,\ 4,\ 4)$	8	4	5	3

Table 6.34.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_1

- **The divisor E_2, a ruled surface over a genus $g = 2$ curve**
  For this divisor we have $n = (0, 0, n_4^{E_2}, 0, 0, 0, n_0, 3n_2^{C_3}, 3n_2^{C_3}, n_2^{C_3}, 3n_2^{C_3}, n_2^{C_5})$ and we set $C = 3C_1 + 3C_2 + C_3 + 3C_4$. Then (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = n_4^{E_2} \tag{6.236}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^{C_3} - 2n_4^{E_2}\right) C + \left(n_2^{C_5} - 4n_4^{E_2}\right) C_5 \tag{6.237}$$

$$\mathrm{ch}_2(\mathcal{F}) = -n_4^{E_2} + 3n_2^{C_3} - n_2^{C_5} - n_0 \tag{6.238}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_2

$$m^{(\mathrm{geom},E_2)} = n_4^{E_2}\left(n_4^{E_2} + 2n_0\right) + \frac{1}{2}n_2^{C_3}\left(2n_2^{C_5} - 5n_2^{C_3}\right) + 1 \tag{6.239}$$

The boundary states with $n_4^{E_2} \neq 0$ corresponding to D4-branes wrapped on E_2 are displayed in table 6.3.12. We note that

$$\Delta^{(E_2)} = m^{(\mathrm{CFT},E_2)} - m^{(\mathrm{geom},E_2)} = \nu - 1 \tag{6.240}$$

The first state corresponds to the bundle $\mathcal{O}_{E_2}(-2C - 6C_5)$. Since $\widetilde{\nu} = 2$ the second state is the direct sum of twice the first one. The remaining rank two sheaves have gauge group $U(2)$ and Chern classes $(-4C - 14C_5, 4)$, and $(-4C - 10C_5, 0)$.

L-orbit	$n = (n_4^{E_2}, n_2^{C_3}, n_2^{C_5}, n_0)$	$m^{(\mathrm{CFT},E_2)}$	ν	$m^{(\mathrm{geom},E_2)}$	$\Delta^{(E_2)}$
$\vert\, 0,0,1,0,0 \,\rangle\!\rangle_B$	$(1,\ 0,-2,\ 0)$	2	1	2	0
$\vert\, 0,0,2,1,0 \,\rangle\!\rangle_B$	$(2,\ 0,-4,\ 0)$	6	2	5	1
$\vert\, 0,0,2,2,0 \,\rangle\!\rangle_B$	$(2,\ 0,-2,\ 0)$ $(2,\ 0,-6,\ 0)$	8	4	5	3

Table 6.35.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_2

- **The divisor E_3, a ruled surface over a genus $g = 2$ curve**
  For this divisor we have $n = (0,0,0,n_4^{E_3},0,0,n_0,3n_2^{C_3},3n_2^{C_3},n_2^{C_3},n_2^{C_4},3n_2^{C_3})$ and we ste $C = C_1 + 3C_2 + C_3 + 3C_5$. Then (5.74) to (5.76) reduce to

$$\begin{aligned}\mathrm{rk}(\mathcal{F}) &= n_4^{E_3} && (6.241)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_3} - 2n_4^{E_3}\right) C + \left(n_2^{C_4} - 5n_4^{E_3}\right) C_4 && (6.242)\\ \mathrm{ch}_2(\mathcal{F}) &= -n_4^{E_3} + \frac{7}{2}n_2^{C_3} - n_2^{C_4} - n_0 && (6.243)\end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on E_3

$$m^{(\mathrm{geom},E_3)} = n_4^{E_3}\left(n_4^{E_3} + 2n_0\right) + n_2^{C_3}\left(n_2^{C_4} - 3n_2^{C_3}\right) + 1 \qquad (6.244)$$

The boundary states with $n_4^{E_3} \neq 0$ corresponding to D4-branes wrapped on E_3 are displayed in table 6.3.12. We note that except for the last two boundary states we have

$$\Delta^{(E_3)} = m^{(\mathrm{CFT},E_3)} - m^{(\mathrm{geom},E_3)} = \nu - 1 \qquad (6.245)$$

The first two states correspond to the bundles $\mathcal{O}_{E_3}(-2C - 6C_4)$ and $\mathcal{O}_{E_3}$, respectively. Since $\widetilde{\nu} = 2$ the next two states are the direct sums of twice the first two states each, respectively. The remaining rank two sheaves have gauge group $U(2)$ and Chern classes $(-4C - 14C_4, 2)$, $(-4C - 10C_4, -2)$, $(-2C_4, 0)$, $(2C_4, 0)$, $(-8C - 26C_4, 8)$, and $(4C + 14C_4, 6)$.

L-orbit	$n = (n_4^{E_3}, n_2^{C_3}, n_2^{C_4}, n_0)$	$m^{(\mathrm{CFT},E_3)}$	ν	$m^{(\mathrm{geom},E_3)}$	$\Delta^{(E_3)}$
$\mid 0,0,1,0,0\,\rangle\!\rangle_B$	(1, 0, −1, 0)	2	1	2	0
$\mid 2,0,1,0,0\,\rangle\!\rangle_B$	(1, 2, 5, 1)	2	1	2	0
$\mid 0,0,2,1,0\,\rangle\!\rangle_B$	(2, 0, −2, 0)	6	2	5	1
$\mid 2,0,2,1,0\,\rangle\!\rangle_B$	(2, 4, 10, 2)	6	2	5	1
$\mid 0,0,2,2,0\,\rangle\!\rangle_B$	(2, 0, −4, 0) (2, 0, 0, 0)	8	4	5	3
$\mid 2,0,2,2,0\,\rangle\!\rangle_B$	(2, 4, 8, 4) (2, 4, 12, 0)	8	4	5	3
$\mid 4,0,2,2,0\,\rangle\!\rangle_B$	(2, −4, −16, 0)	32	4	21	11
$\mid 6,0,2,2,0\,\rangle\!\rangle_B$	(2, 8, 24, 4)	32	4	21	11

Table 6.36.: The boundary states corresponding to D4-branes wrapped on the ruled surface E_3

- **The divisor F, a collection of 2 $\mathbb{P}^2$'s**
  For this divisor we have $n = (0,0,0,0,n_4^F,0,n_0,0,n_2^{C_2},3n_2^{C_2},n_2^{C_2},2n_2^{C_2})$ and we set $h = C_2 + 3C_3 + C_4 + 2C_5$. Then (5.74) to (5.76) reduce to

$$\begin{aligned}\mathrm{rk}(\mathcal{F}) &= n_4^F && (6.246)\\ \mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_2} - 3n_4^F\right) h && (6.247)\\ \mathrm{ch}_2(\mathcal{F}) &= \frac{5}{2}n_4^F - \frac{3}{2}n_2^d C_2 - n_0 && (6.248)\end{aligned}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on F

$$m^{(\mathrm{geom},F)} = \frac{1}{2}n_4^F\left(4n_0 - 5n_4^F\right) + \frac{1}{2}\left(n_2^{C_2}\right)^2 + 1 \qquad (6.249)$$

The boundary states with $n_4^F \neq 0$ corresponding to D4-branes wrapped on F are displayed in table 6.37. We note that

$$\Delta^{(F)} = m^{(\mathrm{CFT},F)} - m^{(\mathrm{geom},F)} = \nu - 1 \qquad (6.250)$$

Recall from Section C.1.5 that F is not irreducible but consists of two $\mathbb{P}^2$'s. Denote the degree 1 curves in each $\mathbb{P}^2$ by h_1 and h_2. By analogy to the family $\mathbb{P}^4_{1,1,1,3,6}[12]$ we interpret the sheaves in the first row as $\mathcal{O}_F$ and $\mathcal{O}_F(-h_1-h_2)$ and $\Omega_F(-h_1-h_2)$ where Ω_F is the cotangent bundle of F. In the second line we then have $\mathcal{I}_{p_1+p_2}$ and $\mathcal{I}_{p_1+p_2}(-h_1-h_2)$ where p_1, p_2 are points on each of the $\mathbb{P}^2$'s. As in the case of the family $\mathbb{P}^4_{1,1,1,3,6}[12]$ we see that ν must be interpreted as counting the number of components in F. Therefore, we need to modify table 6.37. $m^{(\mathrm{geom},F)}$ should really be 0 and 4 and hence $\Delta^{(F)} = \nu - 2 = 0$.

L-orbit	$n = (n_4^F, n_2^{C_2}, n_0)$	$m^{(\mathrm{CFT},F)}$	ν	$m^{(\mathrm{geom},F)}$	$\Delta^{(F)}$
$\vert\, 0,3,0,0,0\,\rangle\!\rangle_B$	$(2,\ 4,\ 0)\ (1,\ 3,-2)\ (1,\ 1,\ 0)$	0	2	-1	1
$\vert\, 1,3,0,0,0\,\rangle\!\rangle_B$	$(1,\ 1,\ 2)\ (1,\ 3,\ 0)$	4	2	3	1

Table 6.37.: The boundary states corresponding to D4-branes wrapped on F

- **The divisor D_1, a $\mathbb{P}^2$ blown up at seven points**
For this divisor we have $n = (0, -2n_4^L, -n_4^L, 0, -n_4^L, n_4^L, n_0, n_2^{C_1}, n_2^{C_2}, n_2^{C_2}, 3n_2^{C_2}, 2n_2^{C_2})$ and we set $C = C_2 + C_3 + 3C_4 + 2C_5$. Then (5.74) to (5.76) reduce to

$$\mathrm{rk}(\mathcal{F}) = 3n_4^L \tag{6.251}$$

$$\mathrm{c}_1(\mathcal{F}) = \left(n_2^{C_1} - 6n_4^L\right) C_1 + \left(n_2^{C_2} - 3n_4^L\right) C \tag{6.252}$$

$$\mathrm{ch}_2(\mathcal{F}) = -\frac{3}{2}n_4^L - \frac{1}{2}n_2^{C_1} + \frac{1}{2}n_2^{C_2} - n_0 \tag{6.253}$$

as well as (5.99) yields for the dimension of the moduli space of the sheaves on D_1

$$m^{(\mathrm{geom},D1)} = \frac{3}{2}n_4^L\left(3n_4^L + 4n_0\right) + \frac{1}{2}\left(2n_2^{C_1} - 3n_2^{C_2}\right) + 1 \tag{6.254}$$

The boundary states with $n_4^L \neq 0$ corresponding to D4-branes wrapped on D_1 are displayed in table 6.38. We note that

$$\Delta^{(D_1)} = m^{(\mathrm{CFT},D_1)} - m^{(\mathrm{geom},D_1)} = \nu - 1 \tag{6.255}$$

These are two rank one sheaves with Chern classes $(-4C_1 - 2C, 2)$ and $(-6C_1 - 2C, 2)$, a rank two sheaf with gauge group $U(1)\times U(1)$ and Chern classes $(-12C_1 - 6C, 12)$, a rank four sheaf with gauge group $U(2)$ and Chern classes $(-24C_1 - 12C, 42)$ and two rank four sheaves with gauge group $U(2)\times U(2)$ and Chern classes $(-20C_1 - 12C, 12)$ and $(-28C_1 - 12C, 72)$.

L-orbit	$n = (3n_4^L, n_2^{C_1}, n_2^{C_2}, n_0)$	$m^{(\mathrm{CFT},D1)}$	ν	$m^{(\mathrm{geom},D1)}$	$\Delta^{(D_1)}$
$\vert\, 0,0,0,0,0\,\rangle\!\rangle_B$	$(1,-2,-1,-1)$	0	1	0	0
$\vert\, 0,1,0,0,0\,\rangle\!\rangle_B$	$(1,-4,-1,-2)$	0	1	0	0
$\vert\, 0,3,1,0,0\,\rangle\!\rangle_B$	$(2,-8,-4,-2)$	4	2	3	1
$\vert\, 0,3,2,0,0\,\rangle\!\rangle_B$	$(0,-4,-4,\ 0)$	4	4	1	3
$\vert\, 0,3,2,1,0\,\rangle\!\rangle_B$	$(4,-16,-8,-4)$	12	4	9	3
$\vert\, 0,3,2,2,0\,\rangle\!\rangle_B$	$(4,-12,-8,\ 0)\ (4,-20,-8-8)$	16	8	9	7

Table 6.38.: The boundary states corresponding to D4-branes wrapped on D_1

- **The divisor D_3**
For this divisor we have $n = \left(0, \frac{1}{3}n_4^{E_3}, \frac{2}{3}n_4^{E_3}, n_4^{E_3}, -\frac{1}{3}n_4^{E_3}, -\frac{4}{3}n_4^{E_3}, n_0, n_2^{C_1}, n_2^{C_2}, n_2^{C_3}, n_2^{C_4}, n_2^{C_5}\right)$

and (5.74) to (5.76) reduce to

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^{E_3} && (6.256)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1}+4n_4^{E_3}\right)C_1+\left(n_2^{C_2}+4n_4^{E_3}\right)C_2 && (6.257)\\
&\quad+\left(n_2^{C_3}+n_4^{E_3}\right)C_3+\left(n_2^{C_4}+4n_4^{E_3}\right)C_4+\left(n_2^{C_5}+4n_4^{E_3}\right)C_5\\
\mathrm{ch}_2(\mathcal{F}) &= -\frac{7}{2}n_4^{E_3}+\frac{1}{2}n_2^{C_3}-n_0 && (6.258)
\end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on D_3 is according to (5.99)

$$\begin{aligned}
m^{(\mathrm{geom},D_3)} &= \frac{7}{2}\left(n_4^{E_3}\right)^2+2n_4^{E_3}n_0-\frac{1}{2}\left(n_2^{C_1}\right)^2 && (6.259)\\
&\quad-\left(n_2^{C_2}\right)^2-\left(n_2^{C_4}\right)^2-\frac{15}{2}\left(n_2^{C_3}\right)^2-\left(n_2^{C_5}\right)^2\\
&\quad+n_2^{C_2}\left(n_2^{C_1}+n_2^{C_5}\right)+n_2^{C_4}\left(4n_2^{C_3}+n_2^{C_5}\right)+4
\end{aligned}$$

- **The divisor H**
  For this divisor we have $n=\left(0,\frac{1}{4}n_4^{E_4},\frac{1}{2}n_4^{E_4},\frac{3}{4}n_4^{E_4},n_4^{E_4},\frac{5}{4}n_4^{L},n_0,n_2^{C_1},n_2^{C_2},n_2^{C_3},n_2^{C_4},n_2^{C_5}\right)$ and (5.74) to (5.76) reduce to

$$\begin{aligned}
\mathrm{rk}(\mathcal{F}) &= n_4^{E_4} && (6.260)\\
\mathrm{c}_1(\mathcal{F}) &= \left(n_2^{C_1}+2n_4^{E_4}\right)C_1+\left(n_2^{C_2}+8n_4^{E_4}\right)C_2 && (6.261)\\
&\quad+\left(n_2^{C_3}+2n_4^{E_4}\right)C_3+\left(n_2^{C_4}+6n_4^{E_4}\right)C_4+\left(n_2^{C_5}+4n_4^{E_4}\right)C_5\\
\mathrm{ch}_2(\mathcal{F}) &= -4n_4^{E_4}+\frac{1}{2}n_2^{C_1}-n_0 && (6.262)
\end{aligned}$$

The dimension of the moduli space of the sheaves $\mathcal{F}$ on H is according to (5.99)

$$\begin{aligned}
m^{(\mathrm{geom},H)} &= 3\left(n_4^{E_4}\right)^2+n_4^{E_4}\left(3n_2^{C_1}-n_2^{C_2}+2n_0\right)-3\left(n_2^{C_1}\right)^2 && (6.263)\\
&\quad-\frac{1}{2}\left(\left(n_2^{C_2}\right)^2\left(n_2^{C_3}\right)^2+\left(n_2^{C_4}\right)^2+\left(n_2^{C_5}\right)^2\right)\\
&\quad-\frac{1}{2}\left(-n_2^{C_2}\left(4n_2^{C_1}+n_2^{C_4}\right)-n_2^{C_5}\left(n_2^{C_3}+n_2^{C_4}\right)\right)+5
\end{aligned}$$

6.4. Results on D-branes on toric Calabi-Yau hypersurfaces

We have seen in the previous section that in many cases of D4-branes wrapping divisors we were able to express the discrepancy Δ between the number of moduli of the boundary states in the boundary conformal field theory and the dimension of the moduli space of the corresponding sheaf in a simple formula involving the number of vacua ν. In the following sections we will collect these results and comment on them.

6.4.1. D4-branes on $K3$ fibers

The most important observation from (6.23), (6.37), (6.141), (6.169), (6.116), (6.197) and (6.230) is that

$$\Delta^{(L)} = 3\nu+2 \qquad (6.264)$$

for all $K3$ fibrations. We like to point out, however, that we also found two collections of boundary states in the families $\mathbb{P}^4_{1,2,3,3,9}[18]$ and $\mathbb{P}^4_{1,4,5,5,5}[20]$ which do not satisfy (6.264). We have thought of several possibilities to explain this, however, without success. The simplest one is to assume that these states lie on a branch of the moduli space of different dimension. However, since for a $K3$ surface L we have by (5.52) $\mathrm{ext}^2_{\mathcal{O}_L}(\mathcal{E},\mathcal{E}) = \mathrm{ext}^0_{\mathcal{O}_L}(\mathcal{E},\mathcal{E})$ and hence the moduli space is everywhere smooth of dimension $m^{(\mathrm{geom},L)}$. A further possibility is the following. We notice that the L_i characterizing the different boundary states have a structure. This structure can be understood from the boundary states on the $K3$ itself and the relation between the weights of the ambient space of the $K3$ and the weights of the ambient space of the Calabi–Yau threefold as explained in Section 3.3.4. For example, in the case where $l = 2$ in (3.51) we can write the charge vector using (4.38)

$$\begin{aligned} |n\rangle &= \sum_{l'=-\frac{L_j}{2}}^{\frac{L_j}{2}} g^{l'}(1-g)|\,0\,\rangle\!\rangle_B \\ &= g^{-\frac{1}{2}} \sum_{l'=-\frac{L_j-1}{4}}^{\frac{L_j-1}{4}} g^{2l'}(1-g^2)|\,0\,\rangle\!\rangle_B \\ &= g^{-\frac{1}{2}} \sum_{\tilde{l}'=-\frac{\tilde{L}_j}{2}}^{\frac{\tilde{L}_j}{2}} \tilde{g}^{\tilde{l}'}(1-\tilde{g})|\,0\,\rangle\!\rangle_B \end{aligned} \tag{6.265}$$

where $\tilde{g} = g^l$ and the tilde refers to the boundary conformal field theory of the Gepner model of the $K3$ surface. The Gepner models for the $K3$ surfaces appearing as fibers in our examples are $(2,2,2,2)$ for $\mathbb{P}^3_{1,1,1,1}[4]$, $(4,4,4,0)$ for $\mathbb{P}^3_{1,1,1,3}[6]$ and $(10,10,1,0)$ for $\mathbb{P}^3_{1,1,4,6}[12]$. We can repeat the comparison outlined in Section 6.3.2 in precisely the same way as for Calabi–Yau threefolds. The only important change is that $2\tilde{h}^{1,1}+2$ has to be replaced by $\rho+2$ where ρ is the rank of the Picard lattice, see Section 3.3.2. It turns out that the boundary states are precisely the ones given in the tables 6.1, 6.4 and 6.14, respectively. From this result and (6.265) we find

$$2\tilde{L}_1 = L_1 - 1 \tag{6.266}$$

which explains the labels of the boundary states in the $K3$ fibers in these tables. Using the relation

$$(1-g)\sum_{l'=0}^{n} g^{l'} = (1-g)\left(\sum_{i=0}^{l-1} g^i\right)\left(\sum_{j=0}^{\frac{n+1}{l}-1} g^{lj}\right) = (1-g^l)\sum_{j=0}^{\frac{n+1}{l}-1} g^{lj} \tag{6.267}$$

for $l|n+1$ this argument can be generalized to any $l \geq 2$. A similar argument applied to both L_1 and L_2 might also explain the additional boundary states for the families $\mathbb{P}^4_{1,2,3,3,9}[18]$ and $\mathbb{P}^4_{1,4,5,5,5}[20]$.

We observe that in all $K3$ fibrations considered there are configurations with charges $v(\mathcal{F}) = (2,0,2-2n)$, $n \geq 2$ which have $\nu = 2$. According to [222] such configurations can have a decomposition of the form $\mathcal{F} = \mathcal{L} \oplus \mathcal{L}^\vee$ into two branes with charges $v(\mathcal{L}) = (1, \pm\mathrm{c}_1(\mathcal{L}), \frac{1}{2}\mathrm{c}_1(\mathcal{L})^2)$ where $\mathcal{L}$ is a line bundle with $\mathrm{c}_1(\mathcal{L})^2 = 1-n$. The Higgs branches of these $U(1)\times U(1)$ theories happen to coincide with the moduli spaces of $SU(2)$ bundles with Chern classes $(2,0,2n)$. In the present case, $\mathcal{L}$ can, however, not be a line bundle because $\mathrm{c}_1(\mathcal{L}) = 0$. It is the ideal sheaf at n points on L, $\mathcal{I}_{p_1,\ldots,p_n}$ which appeared in our introduction to sheaf theory in Section 5.3.1. Since this sheaf is stable the configuration with charges $v(\mathcal{F}) = (2,0,2-2n)$ corresponds therefore to a polystable sheaf. (For both statements see Section 5.3.2.) This is a concrete example of the relationship between enhanced gauge symmetry and strictly semi-stable bundles [291], [184] mentioned in Sections 5.3.2, 6.3.1 and 6.3.2.

6.4.2. D4-branes on rational or ruled surfaces

Collecting (6.28), (6.42), (6.126), (6.146), (6.151), (6.174), (6.179), (6.202), (6.207), (6.212), (6.217), (6.235), (6.240) and (6.245) yields

$$\Delta^{(E)} = \nu - 1 \tag{6.268}$$

for all ruled surfaces over curves of genus $g > 0$. We note, however, that we find several states which do not satisfy (6.268). These are $|\, 2, 0, 2, 2, 0 \,\rangle\!\rangle_B$ in table 6.5, $|\, 3, 0, 2, 2, 0 \,\rangle\!\rangle_B$ and $|\, 4, 0, 2, 2, 0 \,\rangle\!\rangle_B$ in table 6.25, and $|\, 4, 0, 2, 2, 0 \,\rangle\!\rangle_B$ and $|\, 6, 0, 2, 2, 0 \,\rangle\!\rangle_B$ in table 6.3.12. All of them appear in $K3$ fibrations with the fiber being $\mathbb{P}^3_{1,1,1,3}[6]$ and obviously have a similar structure. We have not been able to either explain them or rule them out. One possibility is that since $\mathrm{ext}^2_{\mathcal{O}_E}(\mathcal{E}, \mathcal{E}) \neq \mathrm{ext}^0_{\mathcal{O}_E}(\mathcal{E}, \mathcal{E} \otimes \mathcal{K}_E)$ is not equal to $\mathrm{ext}^0_{\mathcal{O}_E}(\mathcal{E}, \mathcal{E})$ they might be in a component of different dimension.

Similarly for the rational surfaces we find from (6.67), (6.83), (6.121), (6.156), (6.184), (6.250) and (6.255) that

$$\Delta^{(F)} = \nu - 1 \tag{6.269}$$

Furthermore, we observe from the tables 6.8, 6.11, 6.15 and 6.37 that for D-branes wrapping rational surfaces arising from blow-ups of singular points (see Section 3.3.3) $\Delta^{(F)} = 0$ always. We found that the number of irreducible components of F must be taken into account as explained after (6.83) and (6.250). Hence we conclude that these D-branes do not gain any moduli when moving from the Gepner point to the large volume limit. We will return to them in Section 6.5.

The sheaves appearing on these rational surfaces share many common properties. First, among the exceptional sheaves, i.e. those whose moduli space is a point, there is always the trivial line bundle corresponding to a single D4-brane as well as the line bundles with first Chern classes being the generators of the Picard group corresponding to D4-D2 bound states with the D2-brane wrapping either the hyperplane in $\mathbb{P}^2$ or the fiber and the section of the Hirzebruch surfaces, respectively. This is because the moduli space of the line bundle is $b_1(F) = 0$, see Sections 5.5.1 and 5.5.3. Exceptional sheaves on toric rational surfaces have been studied in detail in [151]. Second, we find states which correspond to adding a D0-brane to the previous configurations. The D0-brane is allowed to move on the surface and hence these states have dimension two. Finally, the remaining configurations are all of rank two and can be written as extensions of two rank one sheaves corresponding to D-brane configurations that appeared in the first two cases. Hence, starting from the exceptional sheaves and the D0-brane, all the remaining states can be obtained as bound states of these. This supports the suggestion by Douglas (scattered in [236], [152], [251], [292], [293], [254]) that all D-branes obtained by this boundary state construction are bound states of elementary D-branes. However, while these elementary D-branes should have no moduli, we need here the D0-brane which has two moduli.

We note that there are only rank one and two sheaves on rational and ruled surfaces corresponding to Gepner model boundary states. The classification of stable rank two bundles on these surfaces is reviewed in [115].

6.4.3. D4-branes on elliptically fibered surfaces

Elliptically fibered surfaces do not satisfy the smoothness condition (5.100). But this just means that we cannot expect to find a smooth moduli space. We can nevertheless use (5.99) (as we actually have done in the corresponding cases) to compute the dimension of the tangent space of the moduli space at a smooth point. We have to take into account that $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ is generally not zero. But for the moment, we take the naive point of view and try to compare to the smooth points. The results for the main part of the boundary states wrapping elliptic surfaces in (6.61), (6.76) and (6.92) yield

$$\Delta^{(S)} = 5(\nu - 1) + 2 \tag{6.270}$$

and from (6.131)

$$\Delta^{(S)} = 7(\nu-1)+3 \tag{6.271}$$

We have noted in Sections 6.3.5, 6.3.6, 6.3.7 and 6.3.8 that there are more boundary states satisfying different relationships. We expect that these remaining boundary states do not correspond to smooth points in the moduli space and we will not consider them in the following discussion.

6.4.4. D4-branes on surfaces of general type

The same comment given at the beginning of the previous section also applies here. From the previous cases we see that we have to look for a relation between $\Delta^{(D_3)} = m^{(\mathrm{CFT},D_3)} - m^{(\mathrm{geom},D_3)}$ and the number of vacua ν. For illustrative purposes, we only consider a particular example, namely the divisor D_3 in $X = \mathbb{P}_{1,3,4,4,12}[24]$. This is shown in figure 6.1. In this example the linear relations emerge more clearly than in other examples we have computed. We find similar to the case of the elliptic fibrations

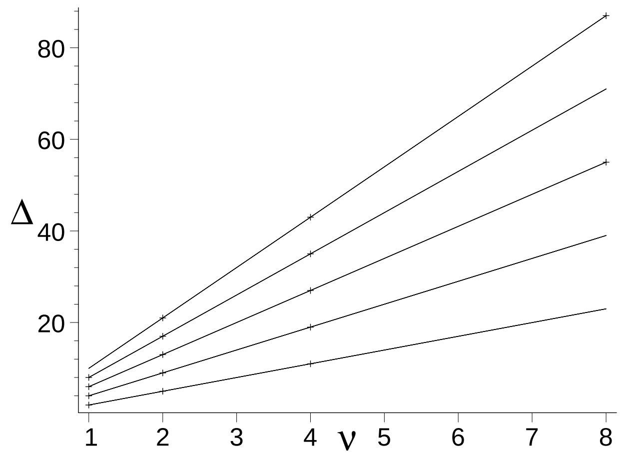

Figure 6.1.: The relation between the excess moduli Δ and the number of vacua ν for the divisor D_3 in $\mathbb{P}^4_{1,3,4,4,12}[24]$.

in Section 6.4.3 that there are several of these relations:

$$\Delta^{(D_3)} = (2k+1)\nu - 1 - 3 \tag{6.272}$$
$$= (2k+1)(\nu-1) + 2k - 3 \tag{6.273}$$

for $k = 1, \ldots, 5$. We do not yet have an understanding of (6.272).

6.4.5. General results for D4-branes

If we collect the results for Δ for the different kinds of divisors, (6.264), (6.268), (6.269), (6.270), (6.271) and (6.272) then we obtain the following interesting table

Surface	p_g	Δ
$\mathbb{P}^2, \mathbb{F}_2$, ruled surfaces	0	$1 \cdot (\nu-1)+0$
$K3$	1	$3(\nu-1)+1$
elliptic fibration	2	$5(\nu-1)+2$
elliptic fibration	3	$7(\nu-1)+3$
surface of general type	3	$(2k+1)(\nu-1)+2k-3$

(6.274)

Looking at the first four lines of this table we conjecture the general relation

$$\Delta = m^{(\mathrm{CFT},D)} - m^{(\mathrm{geom},D)} = c_D(\nu - 1) + p_g(D) \tag{6.275}$$

where c_D is a constant depending on the surface D. A natural guess for c_D would be

$$c_D = 2p_g(D) + 1 \tag{6.276}$$

Now note that in computing Δ we have used (5.99) and not (5.107), i.e. we have not yet allowed for the motions of the divisor D inside the Calabi-Yau manifold X. Therefore, if we take into account that we are not really dealing with sheaves $\mathcal{F}$ on the surface D but with torsion sheaves $i_*\mathcal{F}$ on X supported on the divisor D and hence have to work with the dimension $m_X^{(\mathrm{geom},D)}$ in eq. (5.107) then we find a high agreement between conformal field theory results and geometric expectations, namely

$$m^{(\mathrm{CFT},D)} - c_D(\nu - 1) = m_X^{(\mathrm{geom},D)} \tag{6.277}$$

and, in particular, for the $\nu = 1$ boundary states

$$m^{(\mathrm{CFT},D)} = m_X^{(\mathrm{geom},D)} \tag{6.278}$$

Let us comment on this result. Independently of the meaning of c_D this result gives a very strong confirmation of both the decoupling statement and the mirror geometric hypothesis stated in Section 6.3.1. Note that we have arbitrarily chosen subsets of the boundary states in the cases of elliptic surfaces and surfaces of general type. We can now give an *a posteriori* justification which still has to supported by more evidence. Since these subsets of boundary states satisfies the same relation as for the rational, ruled and $K3$ surfaces for which the moduli space is smooth, they should correspond to smooth points in the moduli space. Of course, we need much more evidence, or even an *a priori* argument to justify this claim. We recall here that even in the cases of ruled and $K3$ surfaces there are few boundary states which do net yet fit into this formula. In addition, it would be interesting to understand both the geometric and the conformal field theoretic meaning of the constant c_D.

6.4.6. D6-branes

In Section 5.5.3 we essentially discussed three classes of sheaves which can correspond to configurations involving D6-branes: the tangent bundle, the FMW bundles and the exceptional sheaves. Our analysis shows that there is no boundary state in any of the examples of Section 6.3 which corresponds to a tangent bundle of a Calabi–Yau threefold. Similarly, in all the examples we investigated which admit elliptic fibrations, there are no FMW bundles. In both cases it would be interesting to understand the reason for their absence in the spectrum of rational boundary states.

The exceptional sheaves, however, play an important role for the $\sum_j L_j = 0$ boundary states. We pointed out in Section 5.5.3 that there are exceptional rational boundary states. The reason for calling them in this way is that they correspond to the restriction of a foundation of a helix on the ambient space of the Calabi–Yau threefold [152], [294], [295] and [153]. This foundation contains the trivial bundle which by restriction to the Calabi–Yau hypersurface gives again the trivial bundle describing the pure D6-brane used in Section 6.3.2 to relate the charge vectors at the different points in moduli space. Interpreting the intersection form (6.9) as an inner product, the dual foundation of the dual helix is generated by the line bundles $\mathcal{O}(J_i)$ where the J_i are the generators of the Kähler cone [153]. This is related to the conjectured generalization of the McKay correspondence. In the above references it has been checked in several examples that this conjecture works and this has then be used in Appendix B to determine the analytic continuation matrix A.

For later reference we display the $\sum_j L_j = 0$ boundary states and the corresponding sheaves $\mathcal{E}$, as well as their duals $\mathcal{F}$ with respect to (6.9) in table 6.39. Here $\mathcal{S}$ is a sheaf with Chern character $\mathrm{ch}(\mathcal{S}) = 3 - (2H - L) + \frac{1}{12}(2H - L)^3$. The last column will be explained in Section 6.5.2.

M	$n=(n_6,n_4^E,n_4^L,n_0,n_2^h,n_2^l)$	$(r,\mathrm{c}_1,\mathrm{c}_2,\mathrm{c}_3)$	$\mathcal{E}$	$\mathcal{F}$	$\Delta\cdot J$
0	$(\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0)$	$(1,0,0,0)$	$\mathcal{O}_X$	$\mathcal{O}_X$	0
1	$(-1,\ \ 0,\ \ 1,-2,\ \ 0,\ \ 0)$	$(-1,L,0,0)$	$-\mathcal{O}_X(-L)$	$\mathcal{O}_X(-L)$	0
2	$(-3,\ \ 1,-2,-2,\ \ 4,\ \ 2)$	$(-3,E,-4l,0)$	$-\mathcal{S}^\vee(-E-L)$	$\mathcal{O}_X(-H)$	$16(t_1+t_2)$
3	$(\ \ 3,-1,-1,\ \ 6,\ \ 0,-2)$	$(3,-H-L,8h+4l,-2)$	$\mathcal{S}^\vee(-H)$	$\mathcal{O}_X(-H-L)$	$16(t_1+t_2)$
4	$(\ \ 3,-2,\ \ 4,\ \ 0,-8,\ \ 0)$	$(3,-2E,8l-8h,0)$	$\mathcal{S}(L)$	$\mathcal{O}_X(-2H)$	$16(t_1+t_2)$
5	$(-3,\ \ 2,-1,-6,\ \ 0,\ \ 0)$	$(-3,2H-L,-8h-8l,0)$	$-\mathcal{S}$	$\mathcal{O}_X(-2H-L)$	$16(t_1+t_2)$
6	$(-1,\ \ 1,-2,\ \ 2,\ \ 4,-2)$	$(1,E,0,0)$	$\mathcal{O}_X(-E)$	$\mathcal{O}_X(-3H)$	0
7	$(\ \ 1,-1,\ \ 1,\ \ 2,\ \ 0,\ \ 2)$	$(-1,H-L,0,0)$	$-\mathcal{O}_X(-H+L)$	$\mathcal{O}_X(-3H-L)$	0

Table 6.39.: Sheaves corresponding to the $\sum_j L_j=0$ boundary states

6.4.7. D2-branes

We have explained in Section 5.5.3 that it is difficult in general to make precise statements about sheaves supported on curves C which correspond to D2-branes wrapping C. Let us discuss a few examples.

For the family $\mathbb{P}^4_{1,1,2,2,2}[8]$ we find two boundary states representing D2-branes with support on the curve $C=4h$. This curve lies in the $K3$ fiber L and has, by (3.44), genus $g(C)=3$. The charges of the boundary state in the L-orbit $\mid 3,0,0,0,0\,\rangle\!\rangle_B$ are $n_2=4$ and $n_0=2$. By (5.82) and (5.83) this corresponds to the trivial line bundle on C. We find $m^{(\mathrm{CFT})}=7$ and by (5.113) $m^{(\mathrm{geom})}=3$. If we naively assume that the curve C can move inside L it will get $g=3$ additional moduli which together with the one modulus from the motion of L inside X makes up the difference of four between $m^{(\mathrm{CFT})}$ and $m^{(\mathrm{geom})}$. The bundle corresponding to the boundary state in the L-orbit $\mid 3,0,1,1,0\,\rangle\!\rangle_B$ has twice the charges, hence is the rank 2 trivial bundle. Since $\nu=4$ and $\widetilde{\nu}=1$ it has gauge group $U(2)$. We find $m^{(\mathrm{CFT})}=28$ and $m^{(\mathrm{geom})}=9$. We are not able to explain the difference in moduli.

For the family $\mathbb{P}^4_{1,1,2,2,6}[12]$ the set of boundary states corresponding to D2-brane configurations is much richer and is displayed in table 6.40. For the boundary states in the second part of the table we have not been able to determine the corresponding geometry in a unique way and we list them only for completeness. The last three columns have been computed using (3.44), (5.82), (5.83) and (5.113). It

L-orbit	$n=(n_2^h,n_2^l,n_0)$	$m^{(\mathrm{CFT})}$	ν	C	$p_a(C)$	(r,d)	$m^{(\mathrm{geom})}$
$\mid 3,0,0,0,0\,\rangle\!\rangle_B$	$(2,\ \ 0,\ \ 1)$	5	1	$2h$	2	$(1,\ \ 0)$	2
$\mid 5,0,1,0,0\,\rangle\!\rangle_B$	$(4,\ \ 0,\ \ 2)$	14	2	$2h$	2	$(2,\ \ 0)$	5
$\mid 5,0,2,0,0\,\rangle\!\rangle_B$	$(4,\ \ 0,\ \ 0)$	20	4	$2h$	2	$(2,\ \ 2)$	5
	$(4,\ \ 0,\ \ 4)$			$2h$	2	$(2,-2)$	5
$\mid 5,0,2,1,0\,\rangle\!\rangle_B$	$(8,\ \ 0,\ \ 4)$	44	4	$4h$	5	$(2,\ \ 0)$	17
$\mid 3,0,1,1,0\,\rangle\!\rangle_B$	$(6,\ \ 0,\ \ 3)$	21	1	$6h$	10	$(1,\ \ 9)$	10
$\mid 0,0,2,0,0\,\rangle\!\rangle_B$	$(2,\ \ 0,-2)$	2	2	l	0	$(2,\ \ 0)$	0
	$(2,\ \ 0,\ \ 0)$			l	0	$(2,-2)$	0
$\mid 2,2,0,0,0\,\rangle\!\rangle_B$	$(4,\ \ 2,\ \ 2)$	11	1	$4h+2l$	5	$(1,\ \ 2)$	5
	$(4,\ \ 2,\ \ 0)$			$4h+2l$	5	$(1,\ \ 4)$	5
$\mid 5,4,0,0,0\,\rangle\!\rangle_B$	$(8,\ \ 4,\ \ 2)$	38	2	$4h+2l$	5	$(2,\ \ 6)$	17
$\mid 5,5,0,0,0\,\rangle\!\rangle_B$	$(8,\ \ 4,\ \ 4)$	44	4	$4h+2l$	5	$(2,\ \ 4)$	17
	$(8,\ \ 4,\ \ 0)$			$4h+2l$	5	$(2,\ \ 8)$	17
$\mid 4,0,2,0,0\,\rangle\!\rangle_B$	$(4,\ \ 2,\ \ 2)$	18	2	$2h+l$			
	$(4,\ \ 2,\ \ 0)$			$2h+l$			
$\mid 5,0,2,0,0\,\rangle\!\rangle_B$	$(4,\ \ 4,\ \ 0)$	20	4	$2h+2l$			
$\mid 5,2,2,0,0\,\rangle\!\rangle_B$	$(12,\ \ 4,\ \ 4)$	68	4	$6h+2l$			

Table 6.40.: D2-brane configurations in $\mathbb{P}^4_{1,1,2,2,6}[12]$

seems as if the curves in the last three lines of the table do not lie in any of the divisors H, L and E and hence (3.44) can not be applied. Of course, it is again interesting to compare $m^{(\mathrm{CFT})}$ and $m^{(\mathrm{geom})}$. However, we can only repeat the argument given above for the line bundles on the curves $2h$ and $6h$.

Both of them lie in the $K3$ fiber L, so they can get additional 2 and 10 moduli, respectively, from the motion of these curves inside L. Adding the modulus from the motion of L inside X yields the 5 and 21 moduli expected from conformal field theory. Similarly, since l is the fiber of the ruled surface E (see Section 3.5.2) there is an extra modulus from the motion of l inside E. The exceptional divisor E has no moduli. However since $\widetilde{\nu} = 2$ the bundle is holomorphically split and this accounts for the two conformal field theory moduli.

6.4.8. D0-branes

In the development of the understanding of D-branes on Calabi-Yau spaces, the D0-brane has always been of most interest. It is the simplest D-brane and we expect it to be in the spectrum and to have a moduli space of dimension 3, since it can move everywhere on the Calabi–Yau manifold which is therefore its moduli space. In [5] it has been observed that the D0-brane was not in the D-brane spectrum of the rational boundary states in the quintic. This absence can be argued to be consistent with the geometric hypothesis as follows [2]. Any location we might pick for the D0-brane would break some of the symmetry group $\mathbb{Z}_5^4$, but all of the rational B-type boundary states are singlets under this group (see Section 4.3) and hence we should not find the D0-brane in this analysis. However, in [180] and [154] the D0-brane was found in the spectrum of other Gepner models. The existence of the D0-brane in many of the families listed in table 3.2 is documented in table 6.41. As we have seen in Section 6.3.2

X	L-orbit	n_0	$m^{(\mathrm{CFT})}$	ν
$\mathbb{P}^4_{1,1,1,1,1}[5]$	–	–	–	–
$\mathbb{P}^4_{1,1,1,1,2}[6]$	–	–	–	–
$\mathbb{P}^4_{1,1,1,1,4}[8]$	$\mid 3,0,0,0,0\,\rangle\!\rangle_B$	2	6	2
$\mathbb{P}^4_{1,1,1,2,5}[10]$	$\mid 0,0,0,1,0\,\rangle\!\rangle_B$ $\mid 3,0,0,0,0\,\rangle\!\rangle_B$	1 2	3 6	1 2
$\mathbb{P}^4_{1,1,2,2,2}[8]$	–	–	–	–
$\mathbb{P}^4_{1,1,2,2,6}[12]$	$\mid 5,0,0,0,0\,\rangle\!\rangle_B$	2	6	2
$\mathbb{P}^4_{1,2,2,3,4}[12]$	–	–	–	–
$\mathbb{P}^4_{1,2,2,2,7}[14]$	$\mid 0,2,0,0,0\,\rangle\!\rangle_B$ $\mid 6,0,0,0,0\,\rangle\!\rangle_B$	1 2	3 6	1 2
$\mathbb{P}^4_{1,1,1,6,9}[18]$	$\mid 2,0,0,0,0\,\rangle\!\rangle_B$ $\mid 8,0,0,0,0\,\rangle\!\rangle_B$	1 2	3 6	1 2
$\mathbb{P}^4_{1,1,1,3,6}[12]$	$\mid 5,0,0,0,0\,\rangle\!\rangle_B$	2	6	2
$\mathbb{P}^4_{1,1,1,3,3}[9]$	–	–	–	–
$\mathbb{P}^4_{1,2,3,3,3}[12]$	–	–	–	–
$\mathbb{P}^4_{1,2,3,3,9}[18]$	$\mid 0,3,0,0,0\,\rangle\!\rangle_B$ $\mid 8,0,0,0,0\,\rangle\!\rangle_B$	1 2	3 6	1 2
$\mathbb{P}^4_{1,1,2,8,12}[24]$	$\mid 3,0,0,0,0\,\rangle\!\rangle_B$ $\mid 11,0,0,0,0\,\rangle\!\rangle_B$	1 2	3 6	1 2
$\mathbb{P}^4_{1,1,2,4,8}[16]$	$\mid 7,0,0,0,0\,\rangle\!\rangle_B$	2	6	2
$\mathbb{P}^4_{1,1,2,4,4}[12]$	–	–	–	–

Table 6.41.: The boundary states corresponding to D0-branes for different families X

a central ingredient in computing this table is the monodromy matrix around the Gepner point in Kähler moduli space A in (3.76). We find that there is a rational D0-brane in the spectrum precisely if this matrix satisfies $A^{\frac{d}{2}} = -\mathbb{1}$ where d is the degree of the Calabi–Yau hypersurface. Another way to state this condition is to look at the symmetries of the boundary states. Looking at the $\sum_j L_j = 0$ boundary states we see that if the condition is satisfied, then these states form a representation of the $\mathbb{Z}_2$ symmetry acting as $S \to S + 2$ on the boundary states, see Section 4.3. In other words, $S \to S + 2$ and $M \to M + K'$ have the same effect. Therefore we have to look for a $\mathbb{Z}_2$ subgroup in the Gepner model group G in (2.89). There is such a $\mathbb{Z}_2$ subgroup if and only if the Gepner model contains a $k = 0$

factor theory whose symmetry group is this $\mathbb{Z}_2$. Looking at the table 6.41 confirms this statement. The D0-brane appears in the spectrum of Gepner model boundary state if and only if the Gepner model has a trivial subtheory [296].

6.4.9. D-branes and nested moduli spaces

In this section we point out an interesting fact about D-branes in different Calabi–Yau manifolds whose moduli spaces are nested as explained in Section 3.6. Note that in this particular case we have more than only two points in the moduli space to compare D-brane spectra. This therefore allows us to extend the comparison to more complicated situations. Due to the fact that we can get more information from D4-branes than from any other kind of D-brane we will mainly argue by referring to the results obtained for them. We focus on the example explained in detail in Section 3.6: The Kähler moduli space of $X' = \mathbb{P}^4_{1,1,2,2,6}[12]$ is embedded in the Kähler moduli space of $X = \mathbb{P}^4_{1,2,3,3,9}[18]$. In Section 6.3.10 we found that the sets of all rational D4-branes wrapping the divisors H, L and E of X' are contained in the sets of rational D4-branes wrapping the divisors J_2, L and E_2 of X'. This means that this part of the spectra at the four different points in Kähler moduli space is the same. As seen in Section 6.3.9 we can make the same statement for $X' = \mathbb{P}^4_{1,1,2,2,2}[8]$ and $X = \mathbb{P}^4_{1,2,3,3,3}[12]$ due to the great similarity of these families. Note that in all these cases note only the spectra agree but also the formulae for the Chern classes and, in particular, for the dimension of the moduli space. This is further strong evidence for the decoupling statement and the mirror geometric hypothesis stated in Section 6.3.1.

This extension from lower- to higher-dimensional moduli spaces does not always work as easy. For example, the reflexive polyhedron for $\mathbb{P}^4_{1,1,1,1,4}$ is contained in the reflexive polyhedron for $\mathbb{P}^4_{1,2,2,2,7}$. However, the vertex $\nu^* = (-1,-1,-1,-4)$ corresponds to a codimension one face in $\Delta(1,2,2,2,7)$ and hence does generically not meet the degree 14 hypersurface in $\mathbb{P}^4_{1,2,2,2,7}$.

6.5. Lines of marginal stability

6.5.1. Non-supersymmetric configurations

Next, we are going to consider bound states of D0- and D6-branes. In the large volume limit, when the Calabi–Yau manifold can be approximated by a flat space, unbroken supersymmetry requires that a Dp-Dq-brane bound state can only exist if $p = q \mod 4$ [192]. Hence, a bound state of a D0- and a D6-brane is unstable and cannot exist. This can actually be shown by computing the static force between them which turns out to be repulsive [297]. The configuration can be made supersymmetric if a very large B-field is turned on [298].

At the Gepner point, however, we find supersymmetric D-brane configurations which correspond to D0-D6-brane bound states in the large volume limit. They are described by the boundary states in table 6.42. The first of these has first been observed in [180] in the case of the family $\mathbb{P}^4_{1,1,1,6,9}[18]$. It

$n = (n_6, n_0)$	m^{CFT}	ν
$(1,1)$	3	1
$(1,2)$	6	1
$(2,2)$	6	2
$(2,4)$	14	2

Table 6.42.: D0-D6-brane bound states

occurs also in $\mathbb{P}^4_{1,1,1,2,5}[10]$, $\mathbb{P}^4_{1,2,2,2,7}[14]$, $\mathbb{P}^4_{1,2,3,3,9}[18]$ and in $\mathbb{P}^4_{1,1,2,8,12}[24]$. All the other bound states appear in the spectra of all Calabi–Yau families in table 6.41 except those for which the D0-brane did not appear in the spectrum. Some of them were noted in [154]. We make two very interesting observations here [154], [296].

The supersymmetric boundary state in the Gepner model corresponding to a D0-D6 brane bound state decays into a non-supersymmetric configuration of D-branes at the large volume limit. The authors of [180] have given an interesting interpretation from the point of view of the mirror X^* which we briefly repeat here. Recall the argument of Strominger, Yau and Zaslow given in Section 5.2. Under mirror symmetry the D0-brane and the D6-brane are mapped to the base B^* and the T^3 fiber of X^*, respectively. The above decay process now tells us that the corresponding homology class B^*+T^3 should not support a special Lagrangian cycle in a neighborhood of the large complex structure limit of X. It should support it instead in a region of the moduli space of X^* which is mapped to a neighborhood of the Gepner point of X by mirror symmetry. This yields a prediction of the Joyce transitions discussed in Section 6.2 in several concrete families. Returning to the original Calabi–Yau manifold X, this decay process indicates the presence of a line of marginal of marginal stability in Kähler moduli space. On one side, near the Gepner point, the D0-D6 bound state is supersymmetric and therefore stable, on the other side near the large volume limit its constituents are supersymmetric and stable.

A second interesting observation is that those families admitting a D0-brane also admit a D0-D6 bound state. The latter is nothing but the state obtained by a Fourier-Mukai monodromy transform about the conifold locus (5.120) from the former. The same happens with the 2 D0-brane states and the 2 D0-D6 brane states in all the families admitting them. One can furthermore check that these pairs of states always appear in the same L-orbit of the Gepner model. However, there are no states consisting of n D0-branes or of n D0-D6 brane bound states with $n > 2$. Following an argument in [236] this suggests the presence of another line of marginal stability along which these states decay into simpler configurations. A mathematical description of these bound states and the monodromy transform in terms of complexes of sheaves has been given in [252].

Finally, we discuss the number of moduli. The D0-D6 bound state has three moduli corresponding to the fact that the D6 brane has no moduli and the D0-brane has 3 moduli. We note that the monodromy transformation does not change the number of moduli in this case. The bound state of the D6-brane with 2 D0-brane branes has 6 moduli from the two independent motions of the D0-branes. The configuration with 2 D0-D6-brane bound states has 6 moduli and 2 vacua suggesting that it actually is a bound state of two D0-D6-bound states [296].

6.5.2. Unstable configurations

Since the prepotential $\mathcal{F}_K$ determining the central charges of the B-type D-branes receives world-sheet instanton corrections, we expect that the mathematical stability condition (5.20) is modified in the stringy regime in accordance with the discussion in Section 6.3.1. One way to check this is to compare the stability condition in the stringy regime (4.45) with the Bogomolov condition (5.20) for stable sheaves. It turns out that there are D-brane configurations corresponding to unstable sheaves in the large volume limit and to a supersymmetric boundary state at the Gepner point. Hence, this gives a further indication that there must be lines of marginal stability which have been crossed in the comparison.

We first consider the family $\mathbb{P}^4_{1,1,2,2,2}[8]$. Substituting (C.2) and (C.1) in (5.71) yields

$$\Delta(\mathcal{F}) \cdot J = -2n_6(n_2^h t_1 + n_2^l t_2) + 8n_4^H(n_4^H + n_4^L)t_1 + 4\left(n_4^H\right)^2 t_2 \tag{6.279}$$

The unstable sheaves that we find in the spectrum of rational boundary states are displayed in table 6.43. For the family $\mathbb{P}^4_{1,1,2,2,6}[12]$ we can analogously proceed. Substituting (3.110) and (3.107) in (5.71) yields

$$\Delta(\mathcal{F}) \cdot J = -2n_6(n_2^h t_1 + n_2^l t_2) + 4n_4^H(n_4^H + n_4^L)t_1 + 2\left(n_4^H\right)^2 t_2 \tag{6.280}$$

The unstable sheaves that we find in the spectrum of rational boundary states are displayed in table 6.44. Since these two these two tables look very similar, and indeed the processes we are going to discuss are the same, we will restrict ourselves to the first one. Note that due to (4.38) two consecutive boundary

L-orbit	M	$n = (n_6, n_4^E, n_4^L, n_0, n_2^h, n_2^l)$	(r, c_1, c_2, c_3)	$\Delta \cdot J$
$\mid 1,0,0,0,0\,\rangle\!\rangle_B$	7	$(2,-1,\ \ 1,\ \ 2,\ \ 0,\ \ 2)$	$(2,-E-L,0,0)$	$-4t_2$
$\mid 2,0,0,0,0\,\rangle\!\rangle_B$	6	$(1,\ \ 0,-1,\ \ 4,\ \ 4,\ \ 2)$	$(1,-L,-4h,-4)$	$-8t_1$
$\mid 2,0,0,0,0\,\rangle\!\rangle_B$	7	$(1,-1,\ \ 2,\ \ 0,\ \ 0,\ \ 2)$	$(1,-E,-4h,0)$	$-8t_1$
$\mid 1,1,0,0,0\,\rangle\!\rangle_B$	6	$(2,-1,\ \ 0,\ \ 6,\ \ 4,\ \ 2)$	$(2,-E-2L,0,0)$	$-8t_1-4t_2$
$\mid 1,1,0,0,0\,\rangle\!\rangle_B$	7	$(2,-1,\ \ 2,\ \ 0,\ \ 0,\ \ 2)$	$(2,-E,-4h,0)$	$-8t_1-4t_2$
$\mid 2,1,0,0,0\,\rangle\!\rangle_B$	6	$(2,-1,\ \ 1,\ \ 4,\ \ 4,\ \ 2)$	$(2,-L-E,-4h,0)$	$-16t_1-4t_2$

Table 6.43.: Unstable configurations in $\mathbb{P}^4_{1,1,2,2,2}[8]$

states with $\sum_j L_j = 0$, B^0_M and B^0_{M+1} will form a bound state $B^1_M = B^0_M + B^0_{M+1}$ which appears in the L-orbit with $\sum_j L_j = 1$. Hence, the first boundary state in table 6.43 is such a bound state of the two states with $M = 7$ and $M = 0$ in table 6.39 corresponding to $\mathcal{O}_X$ and $\mathcal{O}_X(-E-L)$. The latter two have $\Delta \cdot J = 0$ in (5.20). Recall from Section 5.3.2 that $\Delta \cdot J = 0$ defines a wall in the Kähler cone on which stability degenerates to semi-stability. After crossing such a wall a stable configuration will be unstable, and vice versa. This means that we try to form a bound state out of two bundles that sit on a wall of the Kähler cone. Moving away from this wall towards the Gepner point it will be a supersymmetric and stable boundary state. However, moving away from this wall towards the large volume limit, we will obtain the unstable sheaf with Chern classes $(2, -E-L, 0, 0)$.

Now, we can repeat this argument with the state $B^1_{M'-1}$ precedent to $B^1_{M'}$, and the state $B^1_{M'+1}$ subsequent to $B^1_{M'}$. They form bound states $B^{11}_{M'} = B^1_{M'} + B^1_{M'+1}$ in the L-orbit $\mid 1,1,0,0,0\,\rangle\!\rangle_B$. We have $M' = 7$, so B^1_6 and B^1_0 correspond to the bundles $\mathcal{O}_L(-4h)$ and $\mathcal{O}_L$, respectively, on the $K3$ fiber L and have $\Delta = 0$. (These bundles are listed in the first line of table 6.1.) The bound state formation of an unstable state and one on a wall in the Kähler cone will again yield an unstable state. The bound state of B^{11}_6 and B^1_7 yields the state B^{11}_6 in the fourth line of table 6.43, the bound state of B^1_7 and B^1_0 yields the state B^{11}_7 in the fifth line.

We can also consider three consecutive boundary states with $\Delta \cdot J = 0$ in table 6.43 which again by (4.38) give rise to bound states $B^2_{M''} = B^0_{M''} + B^0_{M''+1} + B^0_{M''+2}$ in the L-orbit $\mid 2,0,0,0,0\,\rangle\!\rangle_B$. Taking $M'' = 6$ yields the state in the second line of table 6.43, $M'' = 7$ yields the state in the third line. Two consecutive boundary states $B^2_{M''}$ and $B^2_{M''+1}$ form a bound state $B^{21}_{M''} = B^2_{M''} + B^2_{M''+1}$ in the L-orbit $\mid 2,1,0,0,0\,\rangle\!\rangle_B$. The two unstable states with $M'' = 6$ and $M'' = 7$ give rise to the boundary state in the last line of table 6.43 which is again unstable.

If we consider the bound state formed from the four consecutive boundary states with $\Delta \cdot J = 0$ in table 6.43 with $M = 6, 7, 0, 1$, it appears in the L-orbit $\mid 3,0,0,0,0\,\rangle\!\rangle_B$ and has charges $n = (0,0,0,-2,-4,0)$ with $\Delta \cdot J = 0$, too. This is the D2-brane configuration discussed in Section 6.4.7 corresponding to the trivial line bundle on the curve $4h$. The Bogomolov criterion does not imply that it is unstable. But since it is a line bundle the configuration is expected to be stable.

It seems as if bound states of configurations with positive $\Delta \cdot J$ and configurations with negative $\Delta \cdot J$ or zero have positive $\Delta \cdot J$ and therefore no other unstable states can be found. As mentioned above this does not necessarily mean that there are no further unstable configurations. At any rate, this example indicates again the existence of lines of marginal stability. We are however only able to give a glimpse of the full picture and this question definitely has to be investigated in much more detail.

Important questions that we are presently not able to answer concern the number of these lines of marginal stability and their location in the Kähler moduli space. Crossing a given a line of marginal stability not all the states will decay. So one might ask which ones do decay. We have seen in Section 6.4.2 that D-branes wrapping rational surfaces arising from blow-ups of singular points do not gain any moduli when moving from the Gepner point to the large volume limit and are therefore expected to be stable everywhere. Naively applying (5.20) to these sheaves is not allowed as they are torsion sheaves.

L-orbit	M	$n = (n_6, n_4^E, n_4^L, n_0, n_2^h, n_2^l)$	(r, c_1, c_2, c_3)	$\Delta \cdot J$
$\mid 1,0,0,0,0 \rangle\rangle_B$	5	$(2, -1, 1, 2, 0, 1)$	$(2, -E-L, 0, 0)$	$-2t_2$
$\mid 2,0,0,0,0 \rangle\rangle_B$	4	$(1, 0, -1, 3, 2, 0)$	$(1, -L, -2h, -2)$	$-4t_1$
$\mid 2,0,0,0,0 \rangle\rangle_B$	5	$(1, -1, 2, 0, 0, 1)$	$(1, -E, -2h, 0)$	$-4t_1$
$\mid 1,1,0,0,0 \rangle\rangle_B$	4	$(2, -1, 0, 5, 2, 1)$	$(2, -E-2L, 0, 0)$	$-4t_1 - 2t_2$
$\mid 1,1,0,0,0 \rangle\rangle_B$	5	$(2, -1, 2, 0, 0, 1)$	$(2, -E, -2h, 0)$	$-4t_1 - 2t_2$
$\mid 1,2,0,0,0 \rangle\rangle_B$	4	$(2, -1, 1, 3, 2, 1)$	$(2, -L-E, -2h, 0)$	$-8t_1 - 2t_2$

Table 6.44.: Unstable configurations in $\mathbb{P}^4_{1,1,2,2,6}[12]$

7. Outlook

D-branes on Calabi–Yau spaces are a very exciting though difficult to understand topic in string theory. It is only in the past two years that a lot of progress has been made. Their role in string theory has to be further elucidated. We have mentioned in the introduction that the ultimate goal is a classification of all supersymmetry preserving D-branes at each point in the Calabi–Yau moduli space. This necessitates a good mathematical framework that incorporates all their physical properties. We have seen that in the large volume limit this is provided by the coherent sheaves. On the other hand, at the Gepner point, the D-brane boundary states can be translated into quiver representations of a Landau-Ginzburg orbifold theory [152]. Both of these frameworks are known to form abelian categories [236]. At any other point in the moduli space such an explicit description of the D-branes is not yet known, and therefore one has to try and formulate it in terms of the known descriptions at these two special points. At present, this is claimed to be achieved by the construction of a larger, in particular non-abelian category of so-called topological D-branes [251], [252], [253], [254], [299]. This category contains a list of all the possible D-branes at all the points in Calabi–Yau moduli space. The physical D-brane spectrum at a given point in this moduli space is then believed to be a certain abelian subcategory of this large category. The reason for this is that abelian categories contain among their objects kernels and cokernels which have been recognized to be necessary for the notion of stability and bound state formation or tachyon condensation, respectively. It is not known how to explicitly determine this subcategory. For this purpose more physical information is required that can be provided by e.g. a more detailed study of the lines of marginal stability in other threefolds than the quintic. The present work provides a basis for such investigations.

On a more concrete level, there are many questions left open in this work. Just to recall a few we mention those boundary states for D4-branes that have not been understood, and the formula relating the conformal field theory dimension and the geometric dimension of a D-brane state which deserves a deeper understanding. In particular, one should be able to derive it from first principles and give a proof for it. This might involve a classification of the divisors in toric Calabi–Yau hypersurfaces. This formula together with the decoupling statement could allow for the computation of the dimension of the moduli space of coherent sheaves in terms of conformal field theory. Another method to proceed is to give a more direct geometric interpretation of the number of marginal boundary operators in the Gepner model. It is well-known that the number of marginal operators in the bulk theory can be determined by the Poincaré polynomial for the chiral ring which coincides with the Poincaré polynomial of the Calabi–Yau space. Similarly, the number of marginal operators in the boundary Gepner model should be encoded in the Poincaré polynomial for the boundary chiral ring. This should then be related to the Poincaré polynomial for the endomorphism bundle of the bundle describing the D-brane state. One important point that has to be taken into account in this argument is that the moduli space of the bundle will generally have several components of different dimensions. While the Gepner model often provides only one dimension, the quiver gauge theory obtained from the corresponding Landau-Ginzburg orbifold theory can provide more than one [236].

One of the most interesting phenomena in the moduli space of Calabi–Yau manifolds are the topology changing phase transitions [53], [55]. There are basically two types of such transitions. The first type involves transitions from a smooth Calabi–Yau phase to another smooth Calabi–Yau phase in which only the intersection numbers are altered. The second type in which the moduli of the Calabi–Yau space are tuned in such a way as to approach a phase boundary is more drastic since also the Hodge numbers are changed. We have discussed this type in detail in an example and found that the D4-branes are preserved in this process. The methods provided in this thesis also allow for the study of the first type of transitions which appear only in Fermat hypersurfaces with large $h^{1,1}$. Furthermore, they can be directly generalized to complete intersection Calabi–Yau spaces which are often the resulting manifolds after a phase transition of the second type.

Once the knowledge of the D-brane spectrum is sufficiently big, the most interesting physical application of these results might be the construction of type I compactifications on Calabi–Yau manifolds [300]. The supersymmetric vacua might yield to realistic world-volume theories. However, there is an important issue that has to be taken care of. In order to satisfy Gauss' law for the various RR charges in the non-compact four-dimensional space-time, one should either consider branes wrapping cycles in non-compact Calabi–Yau spaces, or consider configurations containing both branes and orientifolds for tadpole cancellation [301]. It is known that the superpotential is essentially a topological quantity and can be computed in an appropriately twisted theory. Since $\mathcal{N} = 2$ world-sheet supersymmetry is a consequence of $\mathcal{N} = 1$ $D = 4$ space-time supersymmetry, the twisted theories will still make sense in the presence of orientifolds. Much work in this direction still has to be done.

A. Analytic continuation of the periods

Analysis of the periods of $\mathbb{P}^4_{1,1,2,2,6}[12]$

Definitions and general properties

As we have pointed out in Section 3.4.2, there are two distinguished bases of $H^3(X^*, \mathbb{Z})$ for the periods of X^*. In this Appendix we explicitly show in the example of the family $X = \mathbb{P}^4_{1,1,2,2,6}[12]$ and its mirror X^* how the periods can be analytically continued from the Gepner point to the large volume limit. We use here the method developed in [105] but modified in such a way that the calculation simplifies considerably. In particular, due to the fact that (3.77) is an integral symplectic basis for $H^3(X^*, \mathbb{Z})$ it is not necessary any more to first analyze the divisors in the secondary fan $A(\Xi)$ and determine the corresponding monodromy matrices. Instead one can perform the analytic continuation directly.

We have $h^{1,1} = 2$ and $h^3 = 6$. Let us introduce the variables corresponding to (3.57) and (3.19)

$$x = x_1 \qquad\qquad y = x_2 \tag{A.1}$$

$$\psi = \psi_1 \qquad\qquad \phi = \psi_2 \tag{A.2}$$

The two sets of variables are related by

$$x = -\frac{2\phi}{(12\psi)^6}, \qquad y = \frac{1}{4\phi^2} \tag{A.3}$$

The fundamental period (3.58) then reads

$$w_0(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(6n)!}{(3n)!(n!)^2(m!)^2(n-2m)!} x^n y^m \tag{A.4}$$

or

$$\varpi_0(\psi,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(6n)!(-2\phi)^{n-2m}}{(3n)!(n!)^2(m!)^2(n-2m)!(12\psi)^{6n}} \tag{A.5}$$

$$= \sum_{n=0}^{\infty} \frac{(6n)!(-1)^n}{(3n)!(n!)^3(12\psi)^{6n}} u_n(\phi) \tag{A.6}$$

where

$$u_n(\phi) = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{(m!)^2(n-2m)!(2\phi)^{2m}} \tag{A.7}$$

and is valid in the range

$$\left| \frac{\phi \pm 1}{864\psi^6} \right| < 1 \tag{A.8}$$

The periods in the large volume limit

For the periods at the large volume point we set [95]

$$c(n,m)=\frac{(6n)!}{(3n)!(n!)^2(m!)^2(n-2m)!} \tag{A.9}$$

then we define

$$w_0(x,y;\rho,\sigma)=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}c(n+\rho,m+\sigma)x^{n+\rho}y^{m+\sigma} \tag{A.10}$$

Furthermore, using differential operators (3.67) the periods at the large volume limit are defined by [95]

$$\begin{aligned}
w_a^{(1)}(x,y) &= \lim_{\rho,\sigma\to 0} D_a^{(1)} w_0(x,y,\rho,\sigma) && \text{(A.11)}\\
w_a^{(2)}(x,y) &= \lim_{\rho,\sigma\to 0} D_a^{(2)} w_0(x,y,\rho,\sigma) && \text{(A.12)}\\
w^{(3)}(x,y) &= \lim_{\rho,\sigma\to 0} D^{(3)} w_0(x,y,\rho,\sigma) && \text{(A.13)}
\end{aligned}$$

where $\rho_1=\rho$, $\rho_2=\sigma$. The intersection numbers are given in (3.110) from which it follows that $A_{ab}=0,\forall a,b$. Similarly, the linear forms can be read off from (3.103). Hence we see that we have to compute only the following derivatives

$$\partial_\rho,\quad \partial_\sigma,\quad \partial_\rho^2,\quad \partial_\rho\partial_\sigma,\quad \partial_\rho^3,\quad \partial_\rho^2\partial_\sigma \tag{A.14}$$

If we define

$$\begin{aligned}
\Phi(n,m) &= 6\psi(6n+1)-\psi(3n+1)-2\psi(n+1)-\psi(n-2m+1) && \text{(A.15)}\\
\Psi(n,m) &= -2\psi(m+1)+2\psi(n-2m+1) && \text{(A.16)}
\end{aligned}$$

this yields for the periods

$$\begin{aligned}
w_1^{(1)}(x,y) &= \frac{\log x}{2\pi i}w_0(x,y)+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\Phi(n,m)c(n,m)x^n y^m && \text{(A.17)}\\
w_2^{(1)}(x,y) &= \frac{\log y}{2\pi i}w_0(x,y)+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\Psi(n,m)c(n,m)x^n y^m && \text{(A.18)}\\
w_1^{(2)}(x,y) &= \frac{2}{(2\pi i)^2}\Bigg(\left(\log^2 x+\log x\log y\right)w_0(x,y) && \text{(A.19)}\\
&\quad +\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\Big(\log x\left(2\Phi(n,m)+\Psi(n,m)\right)+\log y\,\Phi(n,m)\\
&\quad +\Phi(n,m)^2+\Phi(n,m)\Psi(n,m)+\Phi'(n,m)+\Psi'(n,m)\Big)c(n,m)x^n y^m\Bigg)\\
w_2^{(2)}(x,y) &= \frac{1}{(2\pi i)^2}\log^2 x\,w_0(x,y) && \text{(A.20)}\\
&\quad +\frac{1}{(2\pi i)^2}\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\left(2\log x\,\Phi(n,m)+\Phi(n,m)^2+\Phi'(n,m)\right)c(n,m)x^n y^m
\end{aligned}$$

$$
\begin{aligned}
w^{(3)}(x,y) \quad = \quad & -\frac{2}{3(2\pi i)^3}\Bigg(\log^3 x\, w_0(x,y) + \\
& + \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \Big(3\log^2 x\,\Phi(n,m) 3\log x \left(\Phi'(n,m)+\Phi(n,m)^2\right) \\
& \qquad \Phi''(n,m)+\Phi'(n,m)\Phi(n,m)+\Phi(n,m)^3\Big) c(n,m)x^n y^m \Bigg) \\
& -\frac{1}{(2\pi i)^3}\Bigg(\log x \log y\, w_0(x,y) + \\
& + \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \Big(\log^2 x\,\Psi(n,m) + 2\log x\log y\,\Phi(n,m) \\
& \qquad + 2\log x\left(\Phi(n,m)\Psi(n,m)+\Psi'(n,m)\right) + \log y\,\Phi'(n,m) \\
& \qquad + \Psi''(n,m) + 2\Phi(n,m)\Psi'(n,m) \\
& \qquad + \Psi(n,m)\left(\Phi'(n,m)+\Phi^2(n,m)\right)\Big) c(n,m)x^n y^m \Bigg) \\
& -\frac{1}{(2\pi i)}\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} \left(\frac{13}{3}\left(\log x + \Phi(n,m)\right) + 2\left(\log y + \Psi(n,m)\right)\right) c(n,m)x^n y^m
\end{aligned}
\tag{A.21}
$$

The periods at the Gepner point

We can relate $u_\nu(\phi)$ to the hypergeometric function [105]

$$
u_\nu(\phi) \quad = \quad (2\phi)^\nu {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}+\frac{1}{2}; 1; \frac{1}{\phi^2}\right) \tag{A.22}
$$

We will need the analytic continuation of (A.5) to small values of ψ by means of Barnes' integral representation [302, §1.19, §2.1.3].

$$
\varpi_0(\psi,\phi) \quad = \quad \sum_{n=0}^{\infty} \frac{\Gamma(6n+1)}{\Gamma(3n+1)\Gamma(n+1)^2}\frac{(-1)^n}{\Gamma(n+1)}\frac{u_n(\phi)}{(12\psi)^{6n}} \tag{A.23}
$$

$$
\quad = \quad -\frac{1}{2\pi i}\int_C \frac{\Gamma(6\nu+1)\Gamma(-\nu)}{\Gamma(3\nu+1)\Gamma(\nu+1)^2}\frac{u_\nu(\phi)}{(12\psi)^{6\nu}}\mathrm{d}\nu \tag{A.24}
$$

$$
\quad = \quad -\frac{1}{6}\sum_{m=0}^{\infty} \frac{\Gamma(\frac{m}{6})}{\Gamma(1-\frac{m}{2})\Gamma(1-\frac{m}{6})^2}\frac{(-1)^m}{\Gamma(m)}(12\psi)^m u_{-\frac{m}{6}}(\phi) \tag{A.25}
$$

Since $\Gamma(1-\frac{m}{6})$ has a pole for $m=6k$ and $\Gamma(1-\frac{m}{2})$ has a pole for $m=2k$ the sum runs only over $m=6k+r$, $r=1,3,5$. By the reflection formula for the Γ-function we have

$$
\begin{aligned}
\varpi_0(\psi,\phi) \quad = \quad & -\frac{1}{6\pi^3}\sum_{r=1,3,5}(-1)^{\frac{r+1}{2}}\sin^2\left(\frac{\pi r}{6}\right) \\
& \qquad \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+\frac{r}{6})^3\Gamma(3k+\frac{r}{2})}{\Gamma(6k+r)}(12^6\psi^6)^{k+\frac{r}{6}}u_{-k-\frac{r}{6}}(\phi)
\end{aligned}
\tag{A.26}
$$

The other periods are defined through(3.75b)

$$
\varpi_j(\psi,\phi) = \varpi_0(\alpha^j\psi, \beta^j\phi) \tag{A.27}
$$

where $\alpha^{12} = 1$ and $\beta^2 = 1$. Due to the pole of $\Gamma(1 - \frac{m}{6})$ in the denominator of (A.25) the periods satisfy the relations

$$\varpi_j + \varpi_{j+6} = 0 \qquad j = 0, 1, \ldots, 5 \tag{A.28}$$

hence we set $j = 2a + \sigma$, $a = 0, 1, 2(, 3, 4, 5)$ and $\sigma = 0, 1$. The periods then read

$$\begin{aligned}\varpi_{2a+\sigma}(\psi, \phi) &= -\frac{1}{6\pi^3} \sum_{r=1,3,5} (-1)^{\frac{r+1}{2}} \alpha^{2ar} \sin^2\left(\frac{\pi r}{6}\right) \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + \frac{r}{6})^3 \Gamma(3k + \frac{r}{2})}{\Gamma(6k + r)} (12^6 \psi^6)^{k+\frac{r}{6}} \beta^{\sigma(k+\frac{r}{6})} u_{-k-\frac{r}{6}}(\beta^\sigma \phi)\end{aligned} \tag{A.29}$$

From the last expression we see that it is useful to introduce the following functions

$$u_\nu^\sigma(\phi) = \beta^{-\nu\sigma} u_\nu(\beta^\sigma \phi), \qquad \sigma = 0, 1 \tag{A.30}$$

or, in other words,

$$u_\nu^0(\phi) = u_\nu(\phi), \qquad u_\nu^1(\phi) = e^{-i\pi\nu} u_\nu(-\phi) \tag{A.31}$$

We can write the periods then as follows

$$\varpi_{2a+\sigma}(\psi, \phi) = -\frac{1}{6\pi^3} \sum_{r=1,3,5} (-1)^{\frac{r+1}{2}} \alpha^{2ar} \sin^2\left(\frac{\pi r}{6}\right) \xi_r^\sigma(\psi, \phi) \tag{A.32}$$

where

$$\xi_r^\sigma(\psi, \phi) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + \frac{r}{6})^3 \Gamma(3k + \frac{r}{2})}{\Gamma(6k + r)} (12^6 \psi^6)^{k+\frac{r}{6}} u_{-k-\frac{r}{6}}^\sigma(\phi) \tag{A.33}$$

Analytic continuation of $u_\nu^\sigma(\phi)$

We also need a description of the analytic continuation of (A.7) to small values of ϕ by means of Barnes' integral representation which gives certain linear transformation formulae for hypergeometric functions. I.e. applying the linear transformation formula [303, (15.3.7)]

$$\begin{aligned}{}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2} + \frac{1}{2}; 1; \frac{1}{\phi^2}\right) &= \frac{\pi^{\frac{1}{2}} e^{\frac{i\pi}{2}\nu}}{\Gamma(-\frac{\nu}{2} + \frac{1}{2})\Gamma(1 + \frac{\nu}{2})} \frac{1}{\phi^\nu} {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; \phi^2\right) \\ &\quad - \frac{2\pi^{\frac{1}{2}} e^{\frac{i\pi}{2}(\nu-1)}}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \frac{1}{\phi^{\nu-1}} {}_2F_1\left(-\frac{\nu}{2} + \frac{1}{2}, -\frac{\nu}{2} + \frac{1}{2}; \frac{3}{2}; \phi^2\right)\end{aligned} \tag{A.34}$$

to (A.22) yields

$$\begin{aligned}u_\nu(\phi) &= \frac{2^\nu \pi^{\frac{1}{2}} e^{\frac{i\pi}{2}\nu}}{\Gamma(-\frac{\nu}{2} + \frac{1}{2})\Gamma(1 + \frac{\nu}{2})} {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; \phi^2\right) \\ &\quad - \frac{2^{\nu+1} \pi^{\frac{1}{2}} e^{\frac{i\pi}{2}(\nu-1)} \phi}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} {}_2F_1\left(-\frac{\nu}{2} + \frac{1}{2}, -\frac{\nu}{2} + \frac{1}{2}; \frac{3}{2}; \phi^2\right)\end{aligned} \tag{A.35}$$

We will have to analytically continue the functions $u_\nu^\sigma(\phi)$ to large values of ϕ. Let us first consider only $u_\nu^0(\phi) = u_\nu(\phi)$. We could use the same formula [303, (15.3.7)] as before, however, we have to be careful, since the parameters a and b in (A.35) are equal and hence will produce logarithmic terms. In this case we can apply the linear transformation formula [303, (15.3.13)] to each of the hypergeometric functions in eq. (A.35). Note that by the reflection formula for the ψ-function [303, (6.3.7)] and by the duplication formula for the ψ-function [303, (6.3.8)] we have

$$\psi\left(-\tfrac{\nu}{2}+r\right)+\psi\left(\tfrac{\nu}{2}+\tfrac{1}{2}-r\right) \;=\; 2\psi(1-2r+\nu)-\ln\left(2^2\right)+\pi\cot\left(\frac{\pi\nu}{2}\right) \tag{A.36}$$

Using this as well as [303, (15.3.13)] we obtain

$$\begin{aligned}{}_2F_1\left(-\frac{\nu}{2},-\frac{\nu}{2};\frac{1}{2};\phi^2\right) \;=\;& \frac{\sqrt{\pi}e^{-\frac{i\pi}{2}\nu}}{2^\nu\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}(2\phi)^\nu\sum_{r=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-2r+1)(r!)^2}\frac{1}{(2\phi)^{2r}} \\ &\cdot\left(\ln\left(-(2\phi)^2\right)+2\psi(r+1)-2\psi(1-2r+\nu)-\pi\cot\left(\frac{\pi\nu}{2}\right)\right)\end{aligned} \tag{A.37}$$

and

$$\begin{aligned}{}_2F_1\left(-\frac{\nu}{2}+\frac{1}{2},-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2};\phi^2\right) \;=\;& \frac{\sqrt{\pi}e^{-\frac{i\pi}{2}(\nu-1)}}{2^\nu\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}(2\phi)^{(\nu-1)}\sum_{r=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-2r+1)(r!)^2}\frac{1}{(2\phi)^{2r}} \\ &\cdot\left(\ln\left(-(2\phi)^2\right)+2\psi(r+1)-2\psi(1-2r+\nu)-\pi\cot\left(\frac{\pi}{2}(\nu-1)\right)\right)\end{aligned} \tag{A.38}$$

Substituting (A.37) and (A.38) into (A.35) returns the original definition of $u_\nu(\phi)$ in (A.7). Next, we turn to the analytic continuation of $u_\nu^1(\phi)$ to large ϕ. It is here where we choose a different method than [105]. Plugging (A.35) into the second equation of (A.31) results in a change of the relative sign of the two summands of $u_\nu^1(\phi)$ as compared to $u_\nu^0(\phi)$

$$\begin{aligned}u_\nu^1(\phi) \;=\;& \frac{2^\nu\pi^{\frac{1}{2}}e^{\frac{i\pi}{2}(\nu-2)}}{\Gamma(-\frac{\nu}{2}+\frac{1}{2})\Gamma(1+\frac{\nu}{2})}\,{}_2F_1\left(-\frac{\nu}{2},-\frac{\nu}{2};\frac{1}{2};\phi^2\right) \\ &+\frac{2^{\nu+1}\pi^{\frac{1}{2}}e^{\frac{i\pi}{2}(\nu-3)}\phi}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})}\,{}_2F_1\left(-\frac{\nu}{2}+\frac{1}{2},-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2};\phi^2\right)\end{aligned} \tag{A.39}$$

To perform the analytic continuation, we again substitute (A.37) and (A.38) into (A.39) and find

$$\begin{aligned}u_\nu^1(\phi) \;=\;& e^{-i\pi\nu}\left(\cos(\pi\nu)u_\nu(\phi)-\frac{\sin(\pi\nu)}{\pi}(2\phi)^\nu\sum_{r=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-2r+1)(r!)^2}\frac{1}{(2\phi)^{2r}}\right. \\ &\left.\left(\ln\left(-(2\phi)^2\right)+2\psi(r+1)-2\psi(1-2r+\nu)\right)\right)\end{aligned} \tag{A.40}$$

The two solutions $u_\nu(\phi)$ and $u_\nu(-\phi)$ are linearly independent except at the integers [105] and we therefore define

$$v_\nu(\phi)=\frac{\pi}{\sin\pi\nu}\left(u_\nu(\phi)\cos(\pi\nu)-u_\nu(-\phi)\right) \tag{A.41}$$

Using (A.40), recalling that the prefactor $e^{-i\pi\nu}$ stems from the definition of $u_\nu^1(\phi)$ in (A.31) we see that $v_\nu(\phi)$ can explicitly be written as

$$\begin{aligned}v_\nu(\phi) \;=\;& (2\phi)^\nu\sum_{r=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-2r+1)(r!)^2}\frac{1}{(2\phi)^{2r}} \\ &\cdot\left(\ln\left(-(2\phi)^2\right)+2\psi(r+1)-2\psi(1-2r+\nu)\right)\end{aligned} \tag{A.42}$$

Analytic continuation of $\varpi_j(\psi,\phi)$

Let us turn to the periods written in terms valid for small ψ and ϕ, (A.32)) and (A.33). Their analytic continuation to large values of ψ and ϕ is again performed by using Barnes' integral representation for $\xi_r^\sigma(\psi,\phi)$. This yields

$$\begin{aligned}\xi_r^\sigma(\psi,\phi) &= \sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{r}{6})^3\Gamma(k+\frac{3r}{2})\Gamma(k+1)}{\Gamma(6k+r)}\frac{(-1)^k}{\Gamma(k+1)}(12^6\psi^6)^{k+\frac{r}{6}}u^\sigma_{-k-\frac{r}{6}}(\phi) &\text{(A.43)}\\ &= \int_{C'}\frac{\Gamma(-\nu)^3\Gamma(-3\nu)}{\Gamma(-6\nu)}\frac{1}{2i\sin\left(\pi\left(\nu+\frac{r}{6}\right)\right)}(12^6\psi^6)^{-\nu}u_\nu^\sigma(\phi)\mathrm{d}\nu &\text{(A.44)}\end{aligned}$$

where the contour C' encloses the poles on the negative ν-axis. In order to obtain an integral representation valid for large ψ we wish to rotate the contour so as to run parallel to the imaginary axis, as usual in Barnes type arguments. It turns out that we have to treat ξ_r^0 and ξ_r^1 separately. For $\sigma=0$ the arcs at infinity give a vanishing contribution so for this case we have

$$\xi_r^0(\psi,\phi)=\int_C\frac{\Gamma(-\nu)^3\Gamma(-3\nu)}{\Gamma(-6\nu)}\frac{1}{2i\sin\left(\pi\left(\nu+\frac{r}{6}\right)\right)}(12^6\psi^6)^{-\nu}u_\nu(\phi)\mathrm{d}\nu \tag{A.45}$$

Closing the contour to the right, noting that the poles are of third order, and summing over the residues yields

$$\begin{aligned}\xi_r^0(x,y) &= \pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Bigg(\Phi'(n)+\pi^2\cot^2\left(\frac{\pi r}{6}\right)-7\pi^2 &\text{(A.46)}\\ &\qquad+\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)^2\Bigg)c(n,m)\frac{x^ny^m}{\sin\left(\frac{\pi r}{6}\right)}\end{aligned}$$

The case $\sigma=1$ requires a consideration of the convergence of the integrals and the contributions of the arcs at infinity [105]. Due to the exponential factor in $u_\nu^1(\phi)$ the integrand of $\xi_r^1(\psi,\phi)$ will not converge for large ν. Note however that the value of the integral is unchanged if we replace $u_\nu^1(\phi)$ by an appropriate linear combination of the $u_\nu^\sigma(\phi)$. There are several possible choices. We follow [105] and set

$$\widetilde{u}_\nu^1(\phi)=-u_\nu^0(\phi)\frac{\sin(\pi(\nu+\frac{r}{6}))}{\sin(\pi\nu)}-e^{i\pi\nu}u_\nu^1(\phi)\frac{\sin(\frac{\pi r}{6})}{\sin(\pi\nu)} \tag{A.47}$$

so that the integral for $\xi_r^1(\psi,\phi)$ converges. When ψ is large the contours can be closed to the right so as to encompass the poles where ν is an integer. The poles are of fourth order, however, using (A.41) the integrand can be transformed so as to make the evaluation of the residues slightly easier

$$\begin{aligned}\xi_r^1(\psi,\phi) &= -\frac{1}{2i}\int_C\frac{\Gamma(-\nu)^3\Gamma(-3\nu)}{\Gamma(-6\nu)}(12^6\psi^6)^{-\nu}\frac{u_\nu(\phi)}{\sin(\pi\nu)}\left(1+\frac{\cos(\pi\nu)\sin\left(\frac{\pi r}{6}\right)}{\sin\left(\pi\left(\nu+\frac{r}{6}\right)\right)}\right)\mathrm{d}\nu\\ &\quad+\frac{1}{2\pi i}\int_C\frac{\Gamma(-\nu)^3\Gamma(-3\nu)}{\Gamma(-6\nu)}(12^6\psi^6)^{-\nu}v_\nu(\phi)\frac{\sin\left(\frac{\pi r}{6}\right)}{\sin\left(\pi\left(\nu+\frac{r}{6}\right)\right)}\mathrm{d}\nu &\text{(A.48)}\end{aligned}$$

Note that the second integrand involving $v_\nu(\phi)$ has now only third order poles. A straightforward computation then gives

$$
\begin{aligned}
\xi_r^1(x,y) \quad = \quad & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{3}\left(-23\pi^2\left(-\pi\cot\left(\frac{\pi r}{6}\right)+2\left(\Phi(n)+\ln x+i\pi\right)\right)\right.\right. \\
& \qquad -2\pi^3\cot\left(\frac{\pi r}{6}\right)\left(1+3\cot^2\left(\frac{\pi r}{6}\right)\right) \\
& \qquad +6\left(\Phi(n)+\ln x+i\pi\right)\pi^2\cot^2\left(\frac{\pi r}{6}\right) \\
& \qquad -3\left(\left(\Phi(n)+\ln x+i\pi\right)^2+\Phi'(n)\right)\pi\cot\left(\frac{\pi r}{6}\right) \\
& \qquad +2\left(\Phi''(n)+3\Phi'(n)\left(\Phi(n)+\ln x+i\pi\right)\right. \\
& \qquad \left.\left.+\left(\Phi(n)+\ln x+i\pi\right)^3\right)\right) \\
& +\left(\Phi'(n)+\pi^2\cot^2\left(\frac{\pi r}{6}\right)-7\pi^2\right. \\
& \quad \left.+\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)^2\right)\left(\ln y-i\pi+\Psi(n)\right) \\
& \quad \left.+\left(2\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)\Psi'(n)+\Psi''(n)\right)\right) \\
& c(n,m)x^n y^m
\end{aligned}
\tag{A.49}
$$

The basis transformation

Now we can finally come back to (A.32). Let us first consider the case $\sigma=0$. By (A.46) we have

$$
\begin{aligned}
\varpi_{2a}(x,y) \quad = \quad & \frac{1}{(2\pi i)^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c(n,m)x^n y^m \\
& \cdot\frac{2}{3}\sum_{r=1,3,5}(-1)^{\frac{r+1}{2}}\alpha^{2ar}\sin\left(\frac{\pi r}{6}\right)\left(\Phi'(n)+\pi^2\cot^2\left(\frac{\pi r}{4}\right)-7\pi^2\right. \\
& \quad \left.+\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)^2\right)
\end{aligned}
\tag{A.50}
$$

The different r-dependent terms $S_1(n,a)=\frac{2}{3}\sum_{r=1,3,5}(-1)^{\frac{r+1}{2}}\alpha^{2ar}\sin\left(\frac{\pi r}{6}\right)\cot^n\left(\frac{\pi r}{6}\right)$ yield

$$S_1(0,a) \quad = \quad 0,\ -1,\ 1,\ 0,\ 1,\ -1 \tag{A.51}$$
$$S_1(1,a) \quad = \quad 0,\ -i,\ -i,\ 0,\ i,\ i \tag{A.52}$$
$$S_1(2,a) \quad = \quad -2,\ -1,\ 1,\ 2,\ 1,\ -1 \tag{A.53}$$

for $a = 0, \dots, 5$. Hence we get $\varpi_0(x,y) = w^{(0)}$ and

$$\begin{aligned}
\varpi_2(x,y) &= \frac{1}{(2\pi i)^2}\frac{2}{3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+n)^2\Gamma(m+1)^2\Gamma(n-2m+1)}x^n y^m \\
&\qquad \left(-\Phi'(n)+4\pi^2-(\ln x+\Phi(n))^2\right) \\
&= -w_2^{(2)} - w^{(0)} \qquad \text{(A.54)}
\end{aligned}$$

$$\begin{aligned}
\varpi_4(x,y) &= \frac{1}{(2\pi i)^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+n)^2\Gamma(m+1)^2\Gamma(n-2m+1)}x^n y^m \\
&\qquad \left(2\Phi'(n)-8\pi^2+2(\ln x+\Phi(n))^2+2(2\pi i)(\ln x+\Phi(n))\right) \\
&= w_2^{(2)} + 2w_1^{(1)} - 2w^{(0)} \qquad \text{(A.55)}
\end{aligned}$$

and similarly for ϖ_6, ϖ_8, ϖ_{10}. For the case $\sigma = 1$ we have by (A.49)

$$\begin{aligned}
\varpi_{2a+1}(x,y) &= \frac{1}{(2\pi i)^3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c(n,m)x^n y^m \qquad \text{(A.56)} \\
&\cdot\frac{2i}{3}\sum_{r=1,3,5}(-1)^{\frac{r+1}{2}}\alpha^{(2a+1)r}\sin^2\left(\frac{\pi r}{6}\right) \\
&\qquad\cdot\left(\frac{-1}{3}\left(-23\pi^2\left(-\pi\cot\left(\frac{\pi r}{6}\right)+2\left(\Phi(n)+\ln x+i\pi\right)\right)\right.\right. \\
&\qquad\qquad -2\pi^3\cot\left(\frac{\pi r}{6}\right)\left(1+3\cot^2\left(\frac{\pi r}{6}\right)\right) \\
&\qquad\qquad +6\left(\Phi(n)+\ln x+i\pi\right)\pi^2\cot^2\left(\frac{\pi r}{6}\right) \\
&\qquad\qquad -3\left(\left(\Phi(n)+\ln x+i\pi\right)^2+\Phi'(n)\right)\pi\cot\left(\frac{\pi r}{6}\right) \\
&\qquad\qquad +2\left(\Phi''(n)+3\Phi'(n)\left(\Phi(n)+\ln x+i\pi\right)\right. \\
&\qquad\qquad \left.\left.+\left(\Phi(n)+\ln x+i\pi\right)^3\right)\right) \\
&\qquad -\left(\left(\Phi'(n)+\pi^2\cot^2\left(\frac{\pi r}{6}\right)-7\pi^2\right.\right. \\
&\qquad\quad +\left.\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)^2\right)\left(\ln y-i\pi+\Psi(n)\right) \\
&\qquad\quad +\left.\left.\left(2\left(\Phi(n)-\pi\cot\left(\frac{\pi r}{6}\right)+\ln x+i\pi\right)\Psi'(n)+\Psi''(n)\right)\right)\right)
\end{aligned}$$

The different r-dependent terms $S_2(n,a)=\frac{2i}{3}\sum_{r=1,3,5}(-1)^{\frac{r+1}{2}}\alpha^{(2a+1)r}\sin^2\left(\frac{\pi r}{6}\right)\cot^n\left(\frac{\pi r}{6}\right)$ yield

$$\begin{aligned} S_2(0,a) &= -\frac{1}{2},\, 1,\, -\frac{1}{2},\, \frac{1}{2},\, -1,\, \frac{1}{2} && \text{(A.57)}\\ S_2(1,a) &= -\frac{i}{2},\, 0,\, \frac{i}{2},\, \frac{i}{2},\, 0,\, -\frac{1}{2} && \text{(A.58)}\\ S_2(2,a) &= \frac{1}{2},\, 1,\, \frac{1}{2},\, -\frac{1}{2},\, -1,\, -\frac{1}{2} && \text{(A.59)}\\ S_2(3,a) &= -\frac{3i}{2},\, 0,\, \frac{3i}{2},\, \frac{3i}{2},\, 0,\, -\frac{3i}{2} && \text{(A.60)} \end{aligned}$$

where $a=0,\ldots,5$. Hence

$$\begin{aligned} \varpi_1(x,y) &= \frac{1}{(2\pi i)^3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+n)^2\Gamma(m+1)^2\Gamma(n-2m+1)}x^n y^m \\ &\quad \left(\frac{2}{3}\left((\ln x+\Phi(n))^3+3(\ln x+\Phi(n))(\Phi'(n)+\Psi'(n))+\Phi''(n)\right)\right. \\ &\quad +\left((\ln y+\Psi(n))\left((\ln x+\Phi(n))^2+\Phi'(n)\right)+\Psi''(n)\right) \\ &\quad \left.-\frac{13}{3}(2\pi i)^2(\ln x+\Phi(N))-2(2\pi i)^2(\ln y+\Psi(N))+(2\pi i)^3\right) \\ &= -w^{(3)}+w^{(0)} && \text{(A.61)} \end{aligned}$$

$$\begin{aligned} \varpi_3(x,y) &= \frac{1}{(2\pi i)^3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+n)^2\Gamma(m+1)^2\Gamma(n-2m+1)}x^n y^m \\ &\quad \left(-\frac{4}{3}\left((\ln x+\Phi(n))^3+3(\ln x+\Phi(n))(\Phi'(n)+\Psi'(n))+\Phi''(n)\right)\right. \\ &\quad -2\left((\ln y+\Psi(n))\left((\ln x+\Phi(n))^2+\Phi'(n)\right)+\Psi''(n)\right) \\ &\quad -2(2\pi i)\left((\ln x+\Phi(n))(\ln y+\Psi(n))+\Psi'(n)\right) \\ &\quad -(2\pi i)\left((\ln x+\Phi(n))^2+\Phi'(n)\right) \\ &\quad \left.-\frac{20}{3}(2\pi i)^2(\ln x+\Phi(n))-3(2\pi i)^2(\ln y+\Psi(n))-2(2\pi i)^3\right) \\ &= 2w^{(3)}-w_1^{(2)}+w_2^{(2)}+2w_1^{(1)}+w_2^{(1)}-2w^{(0)} && \text{(A.62)} \end{aligned}$$

$$\begin{aligned} \varpi_5(x,y) &= \frac{1}{(2\pi i)^3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+n)^2\Gamma(m+1)^2\Gamma(n-2m+1)}x^n y^m \\ &\quad \left(\frac{2}{3}\left((\ln x+\Phi(n))^3+3(\ln x+\Phi(n))(\Phi'(n)+\Psi'(n))+\Phi''(n)\right)\right. \\ &\quad +3\left((\ln y+\Psi(n))\left((\ln x+\Phi(n))^2+\Phi'(n)\right)+\Psi''(n)\right) \\ &\quad +2(2\pi i)\left((\ln x+\Phi(n))(\ln y+\Psi(n))+\Psi'(n)\right) \\ &\quad +1(2\pi i)\left((\ln x+\Phi(n))+\Phi'(n)\right) \\ &\quad \left.-\frac{13}{3}(2\pi i)^2(\ln x+\Phi(n))-(2\pi i)^2(\ln y+\Psi(n))+(2\pi i)^3\right) \\ &= -w^{(3)}+w_1^{(2)}-w_2^{(2)}+w_2^{(1)}+w^{(0)} && \text{(A.63)} \end{aligned}$$

and similarly for the remaining periods. Now we are finally able to write down the matrix of the basis transformation (3.78). We read off from (A.54) to (A.55) and (A.61) to (A.63) that

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ -2 & 2 & 1 & 2 & -1 & 1 \\ 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} \tag{A.64}$$

The monodromy matrix in the Gepner basis $A^{(G)}$ can be read off from (3.76) and (A.28)

$$A^{(G)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.65}$$

Note that this matrix satisfies $A^6 = -\mathbb{1}$. Then

$$A^{(L)} = MA^{(G)}M^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 1 & -2 & -1 & -1 & 1 & -1 \end{pmatrix} \tag{A.66}$$

One can also explicitly check that the periods $\varpi_j(x,y)$ satisfy the relations (A.28).

B. McKay correspondence

Here we review the method developed by Mayr [153] known as the generalized McKay correspondence to compute the analytic continuation of the periods. The idea is to consider an orbifold $\mathbb{C}^5/\Gamma$ for a discrete subgroup $\Gamma \subset SU(5)$ and to blow up the singularity at the origin. The exceptional divisor that is created in this process is a weighted projective space $Y = \mathbb{P}^4_w$. The Calabi-Yau manifold X is then defined as usual as the zero set of a generic section of the hyperplane bundle of Y (see Section 3.2.2). The D-branes that can be constructed in the orbifold theory give rise to the so-called fractional D-branes on Y which are D-branes wrapped on the compact homology of Y [176]. They are the basic objects in the small volume limit and correspond to the generators S^a of the basis of the Grothendieck group $K(Y)$. Diaconescu and Douglas [152] have conjectured that the restrictions of these fractional brane states to the Calabi-Yau X represent the rational B-type boundary states of the Gepner model describing the small volume limit of X (see Sections 4.3.2 and 4.3.3). Let us denote the restrictions of the fractional branes to the Calabi-Yau by $V^a = S^a|_X$. Then we have a relation between the intersection form on $K(Y)$ given by (cf. (3.62) and (5.54)) [153]

$$\langle E, F\rangle_Y = \int_Y \mathrm{c}_1(Y)\,\mathrm{ch}(E)\,\mathrm{ch}(F^*)\,\mathrm{td}(Y) \tag{B.1}$$

and the intersection form on the $L = 0$ boundary states (4.42)

$$I^{B,ab}_{0,0} = \langle V^a, V^b\rangle_Y = \chi^{ab} - \chi^{ba} \tag{B.2}$$

where χ^{ab} is the inverse of

$$\chi_{ab} = \prod_{j=1}^{r}(1 - h^{w_j}) \tag{B.3}$$

and h is the $d \times d$ shift matrix satisfying $h^d = 0$ where $d = \sum_j w_j$. Let us take $Y = \mathbb{P}^4_{1,2,3,3,9}$ and $X = \mathbb{P}^4_{1,2,3,3,9}[18]$ as a concrete example. Then we have

$$\chi = \begin{pmatrix}
1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 & 53 & 64 & 80 & 91 & 107 \\
0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 & 53 & 64 & 80 & 91 \\
0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 & 53 & 64 & 80 \\
0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 & 53 & 64 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 & 53 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 & 46 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 & 35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 & 28 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 & 24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 & 17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{B.4}$$

and for the intersection matrix $I_{B,00}$ at the Gepner point

$$I_{B,00} = \begin{pmatrix}
0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 \\
1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 \\
-2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 \\
-2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\
1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 \\
1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 \\
1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 \\
0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 \\
-1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\
-1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 \\
2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 \\
2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 \\
-1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 \\
-1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0
\end{pmatrix} \tag{B.5}$$

Now recall the definition of helices in Section 5.5.3. From the foundation $\{R_a\}$ of the helix $\mathcal{H}_R$, $\{R_a\} = \{\mathcal{O}(-(d-1)K_Y), \ldots, \mathcal{O}\}$, in our example

$$\{R_a\} = \{\mathcal{O}(-17), \mathcal{O}(-16), \ldots, \mathcal{O}(-1), \mathcal{O}\} \tag{B.6}$$

one can construct the basis $\{\widetilde{R}_a\}$ for the geometric bundles on the (partial) resolution $\widetilde{X}$ of X

$$\begin{aligned}
\{\widetilde{R}_a\} \quad = \quad & \{\mathcal{O}(-5,0,-1), \mathcal{O}(-4,-1,0), \mathcal{O}(-5,0,0), \mathcal{O}(-4,0,-1), \mathcal{O}(-3,-1,0), \\
& \mathcal{O}(-4,0,0), \mathcal{O}(-3,0,-1), \mathcal{O}(-2,-1,0), \mathcal{O}(-3,0,0), \mathcal{O}(-2,0,-1), \\
& \mathcal{O}(-1,-1,0), \mathcal{O}(-2,0,0), \mathcal{O}(-1,0,-1), \mathcal{O}(0,-1,0), \mathcal{O}(-1,0,0), \\
& \mathcal{O}(0,0,-1), \mathcal{O}(1,-1,0), \mathcal{O}(0,0,0)\}
\end{aligned} \tag{B.7}$$

Here we use the standard notation $[\mathcal{O}(a)] = aK$ and $[\mathcal{O}(a,b,c)] = aJ_1 + bJ_2 + cJ_3$ where J_i are the $(1,1)$-forms on $\widetilde{X}$ related to the Mori generators $l^{(a)}$ as explained in 3.3.1 and given for this model in (C.24). We note that the single Mori generator $l^{(LG)}$ of the Landau-Ginzburg phase is related to the Mori generators of the large volume phase by $l^{(LG)} = 3l^{(1)} + 4l^{(2)} + 2l^{(3)}$. Then the dual foundation $\{S_a^*\}^*$ of the dual helix $\mathcal{H}_S$ restricted to X consists of the fractional branes $V^a = \chi^{ab} R_b^*|_X$. We won't give the explicit expressions here for our example but only note that taking the dual includes also reversing the order of the foundation. Given the sheaves V^a, we can repeat the process of comparison explained in Section 5.4.2 of the central charges $Z^a = Z(V^a)$ given by (5.66). Using (5.67) to (5.70) we can determine the charge matrix $N = (n_i^a)$ for the $L = 0$ boundary states. In our example this is

$$N = \begin{pmatrix} -1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & -2 & 0 \\ 1 & -2 & 1 & 1 & -1 & -3 & 0 & 1 \\ 2 & -2 & 1 & 0 & 2 & -2 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & 1 & 0 & 0 \\ -2 & 2 & -1 & -1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 & 2 & 0 \\ -1 & 2 & -1 & -1 & 1 & 3 & 0 & -1 \\ -2 & 2 & -1 & 0 & -2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 1 & -3 & -2 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.8}$$

This provides us with the starting point for (4.40).

The last ingredient we need is the representation matrix $A^{(L)}$ of the generator g of the $\mathbb{Z}_d$ action on $H^3(X^*, \mathbb{Z})$ in the large volume basis. This is given by the solution to the equation [153]

$$g^T \cdot N = N \cdot A^{(L)} \tag{B.9}$$

where $g = h + E_{d,1}$ is the $d \times d$ shift matrix satisfying $g^d = \mathbb{1}$. This yields

$$A^{(L)} = \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 3 & 1 & 0 \\ -1 & 2 & -1 & 0 & -1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \tag{B.10}$$

This is then used in (4.40) to compute the matrix $t_{LL'}$ in order to get the charges of the boundary states with $L > 0$. At the Gepner point we have d periods as in (3.75a) corresponding to the d central charges of the d fractional D-brane states. One may use $2h^{1,1} + 2$ of them as basis for the period vector at the

Gepner point and express the remaining ones through linear combinations of this basis as in (3.76). This gives the intersection matrix $I^{(G)}$ on $H^3(X^*,\mathbb{Z})$ by restriction of $I_{B,00}$ in (B.5) to the first $2h^{1,1}+2$ basis vectors

$$I^{(G)} = \begin{pmatrix} 0 & -1 & -1 & -1 & 2 & 2 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & 2 & 2 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 & 2 & 2 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 \\ -2 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ -2 & -2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -2 & -2 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -2 & -2 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{B.11}$$

and a matrix of relations R which consists of the zero eigenvectors of the matrix $I_{B,00}$ in (B.5)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \tag{B.12}$$

The transformation matrix M representing the analytic continuation is then the solution the equation [153]

$$N \cdot M = (1-g) \cdot R \tag{B.13}$$

yielding

$$M = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1/2 & 1 & -1 & 3/2 & -1 & 0 & -1/2 \\ 2 & -1/2 & 2 & 0 & 1/2 & 0 & 0 & -1/2 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & -1/2 & 1 & -1 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & -1/2 & 1 & -1 & 1/2 \end{pmatrix} \tag{B.14}$$

so that we finally can check that the representation $A^{(L)}$ of the $\mathbb{Z}_d$ generator g in the Gepner basis is

$$A^{(G)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \tag{B.15}$$

which is of the form given in (3.76).

C. Data of toric Calabi-Yau manifolds

C.1. The families $\mathbb{P}^4_{1,1,2,2,2}[8]$, $\mathbb{P}^4_{1,2,3,3,3}[12]$, $\mathbb{P}^4_{1,2,3,3,9}[18]$, $\mathbb{P}^4_{1,4,5,5,5}[20]$ and $\mathbb{P}^4_{1,3,4,4,12}[24]$

C.1.1. Toric description of $\mathbb{P}^4_{1,1,2,2,2}[8]$

The family $X = \mathbb{P}^4_{1,1,2,2,2}[8]$ has a singular $\mathbb{Z}_2$-curve $C = \mathbb{P}^2[4]$ with genus 3. A similar analysis yields for the toric data

$$
\begin{array}{lrrrrr|rrl}
 & & & & & & C_1 & C_2 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & K = -4H \\
D_1 & -1 & -2 & -2 & -2 & 1 & 0 & 1 & L \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & H \\
D_4 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & H \\
D_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & H \\
D_6 & 0 & -1 & -1 & -1 & 1 & 1 & -2 & E = H - 2L \\
 & & & & & & h & l &
\end{array}
\tag{C.1}
$$

for the intersection numbers and linear forms

$$L^3 = H \cdot L^2 = 0, \quad H^2 \cdot L = 4, \quad H^3 = 8 \tag{C.2}$$

$$\mathrm{c}_2 \cdot H = 56 \quad \mathrm{c}_2 \cdot L = 24 \tag{C.3}$$

The divisors H, L and E are the restriction of the hyperplane class of $\mathbb{P}^4_{1,1,2,2,2}$, a $K3$ fiber described as $\mathbb{P}^3_{1,1,1,1}[4]$ and a ruled surface over the curve C, respectively. They are characterized by

$$\chi(H) = 64 \qquad \chi(\mathcal{O}_H) = 6 \qquad q(H) = 0 \qquad p_g(H) = 5 \tag{C.4}$$

$$\chi(L) = 24 \qquad \chi(\mathcal{O}_L) = 2 \qquad q(L) = 0 \qquad p_g(L) = 1 \tag{C.5}$$

$$\chi(E) = -16 \qquad \chi(\mathcal{O}_E) = -2 \qquad q(E) = 3 \qquad p_g(E) = 0 \tag{C.6}$$

The Picard lattice of L is

$$\mathrm{Pic}(L) = \langle 4 \rangle \tag{C.7}$$

A basis for $H_2(X, \mathbb{Z})$ dual to H and L is

$$h = \frac{1}{4} H \cdot L \qquad\qquad l = \frac{1}{4} H \cdot E \tag{C.8}$$

C.1.2. Toric description of $\mathbb{P}^4_{1,2,3,3,3}[12]$

The family $X = \mathbb{P}^4_{1,2,3,3,3}[12]$ has a singular $\mathbb{Z}_3$-curve $C = \mathbb{P}^2[4]$ with genus 3. The toric data are

$$\begin{array}{cccccc|ccc|c}
 & & & & & & C_1 & C_2 & C_3 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & K=-4H \\
D_1 & -2 & -3 & -3 & -3 & 1 & -1 & 1 & 0 & \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & H \\
D_4 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & H \\
D_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & H \\
D_6 & -1 & -2 & -2 & -2 & 1 & 2 & -2 & 1 & E_1 \\
D_7 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & -2 & E_2 \\
 & & & & & & h & d & l &
\end{array} \tag{C.9}$$

while the intersection numbers and linear forms read

$$\begin{aligned}
E_1^3 &= -16 & E_1^2 \cdot E_2 &= 0 & E_1^2 \cdot L &= 0 \\
E_1 \cdot E_2^2 &= 4 & E_1 \cdot E_2 \cdot L &= 0 & E_1 \cdot L^2 &= 0 \\
E_2^3 &= -16 & E_2^2 \cdot L &= 4 & E_2 \cdot L^2 &= 0 \\
L^3 &= 0 & & & &
\end{aligned} \tag{C.10}$$

$$\mathrm{c}_2 \cdot E_1 = 8 \qquad \mathrm{c}_2 \cdot E_2 = 8 \qquad \mathrm{c}_2 \cdot L = 24 \tag{C.11}$$

The divisors H, L, E_1, E_2 and D_1 are characterized by

$$\chi(H) = 54 \qquad \chi(\mathcal{O}_H) = 5 \qquad q(H) = 0 \qquad p_g(H) = 4 \tag{C.12}$$
$$\chi(L) = 24 \qquad \chi(\mathcal{O}_L) = 2 \qquad q(L) = 0 \qquad p_g(L) = 1 \tag{C.13}$$
$$\chi(E_i) = -8 \qquad \chi(\mathcal{O}_{E_i}) = -2 \qquad q(E_i) = 3 \qquad p_g(E_i) = 0 \tag{C.14}$$
$$\chi(D_1) = 10 \qquad \chi(\mathcal{O}_{D_1}) = 1 \qquad q(D_1) = 0 \qquad p_g(D_1) = 0 \tag{C.15}$$

The divisor H is again the restriction of the hyperplane class of the ambient space. The divisor L is a $K3$ fiber given as $\mathbb{P}^3_{1,1,1,1}[4]$. The divisors E_1 and E_2 are each ruled surfaces over the curve C whose fibers form together the A_2 Hirzebruch-Jung sphere tree. Finally, D_1 is a blown-up rational surface. The Picard lattice of L is $\mathrm{Pic}(L) = \langle 4 \rangle$. The generators of the Kähler cone are

$$J_1 = H \qquad J_2 = D_1 + H \qquad J_3 = L \tag{C.16}$$

We note that the divisor J_2 has the same topological properties as the divisor H in $\mathbb{P}^4_{1,1,2,2,2}[8]$

$$\chi(J_2) = 64 \qquad \chi(\mathcal{O}_{J_2}) = 6 \qquad q(J_2) = 0 \qquad p_g(J_2) = 5 \tag{C.17}$$

A basis for $H_2(X, \mathbb{Z})$ dual to H, J_2 and L is

$$h = \frac{1}{4} D_1 \cdot E_1 \qquad d = \frac{1}{4} H \cdot E_1 \qquad l = \frac{1}{4} H \cdot E_2 \tag{C.18}$$

We further note that

$$L^2 = 0 \tag{C.19}$$
$$D_1 \cdot L = 0 \tag{C.20}$$
$$E_1 \cdot L = 0 \tag{C.21}$$
$$D_1 \cdot E_2 = 0 \tag{C.22}$$
$$H \cdot D_1 = -D_1^2 = 2h \tag{C.23}$$

C.1.3. Toric description of $\mathbb{P}^4_{1,2,3,3,9}[18]$

For $X = \mathbb{P}^4_{1,2,3,3,9}[18]$ the toric data are

$$\begin{array}{c|ccccc|ccc|c}
 & & & & & & C_1 & C_2 & C_3 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 & 0 & K = -6H \\
D_1 & -2 & -3 & -3 & -9 & 1 & -1 & 1 & 0 & \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & H \\
D_4 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & H \\
D_5 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & 0 & H \\
D_6 & -1 & -2 & -2 & -6 & 1 & 2 & -2 & 1 & E_1 \\
D_7 & 0 & -1 & -1 & -3 & 1 & 0 & 1 & -2 & E_2 \\
 & & & & & & h & d & l &
\end{array} \tag{C.24}$$

while the intersection numbers and linear forms read

$$\begin{array}{ccc}
E_1^3 = -8 & E_1^2 \cdot E_2 = 0 & E_1^2 \cdot L = 0 \\
E_1 \cdot E_2^2 = 2 & E_1 \cdot E_2 \cdot L = 0 & E_1 \cdot L^2 = 0 \\
E_2^3 = -8 & E_2^2 \cdot L = 2 & E_2 \cdot L^2 = 0 \\
L^3 = 0 & &
\end{array} \tag{C.25}$$

$$\mathrm{c}_2 \cdot E_1 = 4 \qquad \mathrm{c}_2 \cdot E_2 = 4 \qquad \mathrm{c}_2 \cdot L = 24 \tag{C.26}$$

The divisors H, L, E_1, E_2 and D_1 are characterized by

$$\chi(H) = 45 \qquad \chi(\mathcal{O}_H) = 4 \qquad q(H) = 0 \qquad p_g(H) = 3 \tag{C.27}$$

$$\chi(L) = 24 \qquad \chi(\mathcal{O}_L) = 2 \qquad q(L) = 0 \qquad p_g(L) = 1 \tag{C.28}$$

$$\chi(E_i) = -4 \qquad \chi(\mathcal{O}_{E_i}) = -1 \qquad q(E_i) = 2 \qquad p_g(E_i) = 0 \tag{C.29}$$

$$\chi(D_1) = 11 \qquad \chi(\mathcal{O}_{D_1}) = 1 \qquad q(D_1) = 0 \qquad p_g(D_1) = 0 \tag{C.30}$$

The divisor H is the restriction of the hyperplane class of the ambient space. The divisor L is a $K3$ fiber given as $\mathbb{P}^3_{1,1,1,3}[6]$. The divisors E_1 and E_2 are each ruled surfaces over the curve C whose fibers form together the A_2 Hirzebruch-Jung sphere tree. Finally, D_1 is a blown-up rational surface. The Picard lattice of L is $\mathrm{Pic}(L) = \langle 2 \rangle$. The generators of the Kähler cone are

$$J_1 = H \qquad J_2 = D_1 + H \qquad J_3 = L \tag{C.31}$$

We note that the divisor J_2 has the same topological properties as the divisor H in $\mathbb{P}^4_{1,1,2,2,6}[12]$

$$\chi(J_2) = 56 \qquad \chi(\mathcal{O}_{J_2}) = 5 \qquad q(J_2) = 0 \qquad p_g(J_2) = 4 \tag{C.32}$$

A basis for $H_2(X, \mathbb{Z})$ dual to H, J_2 and L is

$$h = \frac{1}{2} D_1 \cdot E_1 \qquad d = \frac{1}{2} H \cdot E_1 \qquad l = \frac{1}{2} H \cdot E_2 \tag{C.33}$$

We further note that

$$L^2 = 0 \tag{C.34}$$

$$D_1 \cdot L = 0 \tag{C.35}$$

$$E_1 \cdot L = 0 \tag{C.36}$$

$$D_1 \cdot E_2 = 0 \tag{C.37}$$

$$H \cdot D_1 = -D_1^2 = h \tag{C.38}$$

C.1.4. Toric description of $\mathbb{P}^4_{1,4,5,5,5}[20]$

For $X = \mathbb{P}^4_{1,4,5,5,5}[20]$ the toric data are

$$
\begin{array}{lrrrrr|rrrrrl}
 & & & & & & C_1 & C_2 & C_3 & C_4 & C_5 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 & K = -4H \\
D_1 & -4 & -5 & -5 & -5 & 1 & -3 & 1 & 0 & 0 & 0 & \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & H \\
D_4 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & H \\
D_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & H \\
D_6 & -3 & -4 & -4 & -4 & 1 & 4 & -2 & 0 & 1 & 0 & E_1 \\
D_7 & -2 & -3 & -3 & -3 & 1 & 0 & 1 & 0 & -2 & 1 & E_2 \\
D_8 & -1 & -2 & -2 & -2 & 1 & 0 & 0 & 1 & 1 & -2 & E_3 \\
D_9 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & -2 & 0 & 1 & E_4
\end{array}
\tag{C.39}
$$

while the non-zero intersection numbers and linear forms in the basis $\{E_1, E_2, E_3, E_4, L\}$ read

$$
\begin{array}{lll}
E_i^3 = -16 & i = 1, \dots, 4 & \\
E_1 \cdot E_2^2 = 12 & E_1^2 \cdot E_2 = -8 & E_2 \cdot E_3^2 = 8 \\
E_3 \cdot E_2^2 = -4 & E_3 \cdot E_4^2 = 4 & E_4^2 \cdot L = 4
\end{array}
\tag{C.40}
$$

$$
\mathrm{c}_2 \cdot E_i = 8 \qquad i = 1, \dots, 4 \qquad \mathrm{c}_2 \cdot L = 24 \tag{C.41}
$$

The divisors H, L, $E_1, \dots, E_4$, and D_1 are characterized by

$$\chi(H) = 55 \qquad \chi(\mathcal{O}_H) = 5 \qquad q(H) = 0 \qquad p_g(H) = 4 \tag{C.42}$$

$$\chi(L) = 24 \qquad \chi(\mathcal{O}_L) = 2 \qquad q(L) = 0 \qquad p_g(L) = 1 \tag{C.43}$$

$$\chi(E_i) = -8 \qquad \chi(\mathcal{O}_{E_i}) = -2 \qquad q(E_i) = 3 \qquad p_g(E_i) = 0 \tag{C.44}$$

$$\chi(D_1) = 3 \qquad \chi(\mathcal{O}_{D_1}) = 1 \qquad q(D_1) = 0 \qquad p_g(D_1) = 0 \tag{C.45}$$

The divisor H is the restriction of the hyperplane class of the ambient space. The divisor L is a $K3$ fiber given as $\mathbb{P}^3_{1,1,1,1}[4]$. The divisors $E_1 \dots E_4$ are each ruled surfaces over the curve C whose fibers form together the A_4 Hirzebruch-Jung sphere tree. Finally, D_1 is a $\mathbb{P}^2$. The Picard lattice of L is $\mathrm{Pic}(L) = \langle 4 \rangle$. The generators of the Kähler cone are

$$J_1 = H \qquad J_2 = D_1 + 3H \tag{C.46}$$

$$J_3 = L \qquad J_4 = E_3 + 2E_4 + 3L \qquad J_5 = E_4 + 2L \tag{C.47}$$

A basis for $H_2(X, \mathbb{Z})$ dual to $J_1, \dots, J_5$ is

$$C_1 = \frac{1}{4} D_1 \cdot E_1 \qquad C_2 = \frac{1}{4} H \cdot E_1 \qquad C_3 = \frac{1}{4} H \cdot E_4 \tag{C.48}$$

$$C_4 = \frac{1}{4} H \cdot E_2 \qquad C_5 = \frac{1}{4} H \cdot E_3 \tag{C.49}$$

C.1.5. Toric description of $\mathbb{P}^4_{1,3,4,4,12}[24]$

For $X = \mathbb{P}^4_{1,3,4,4,12}[24]$ the toric data are

$$\begin{array}{cccccc|ccccccl} & & & & & & C_1 & C_2 & C_3 & C_4 & C_5 & \\ D_0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & K=-2H \\ D_1 & -3 & -4 & -4 & -12 & 1 & -1 & 1 & 0 & 0 & 0 & \\ D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & 1 & 0 & L \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \\ D_4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \\ D_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & H \\ D_6 & -2 & -3 & -3 & -9 & 1 & 1 & -2 & 0 & 0 & 1 & E_1 \\ D_7 & -1 & -2 & -2 & -6 & 1 & 0 & 1 & 0 & 1 & -2 & E_2 \\ D_8 & 0 & -1 & -1 & -3 & 1 & 0 & 0 & 4 & -2 & 1 & E_3 \\ D_9 & -1 & -1 & -1 & -4 & 1 & 1 & 0 & -3 & 0 & 0 & F \end{array} \tag{C.50}$$

while the non-zero intersection numbers and linear forms in the basis $\{E_1, E_2, E_3, F, L\}$ read

$$E_i^3 = -8 \qquad E_3^2 \cdot L = 2 \qquad F^3 = 18 \tag{C.51}$$
$$E_1 \cdot E_2^2 = 4 \qquad E_1^2 \cdot E_2 = -2 \qquad E_2 \cdot E_3^2 = 2 \tag{C.52}$$
$$\mathrm{c}_2 \cdot E_i = 4 \qquad \mathrm{c}_2 \cdot F = -12 \qquad \mathrm{c}_2 \cdot L = 24 \tag{C.53}$$

where $i = 1, \ldots, 3$. The divisors H, L, $E_1, \ldots, E_3$, F, D_1 and D_3 are characterized by

$$\chi(H) = 192 \qquad \chi(\mathcal{O}_H) = 22 \qquad q(H) = 0 \qquad p_g(H) = 21 \tag{C.54}$$
$$\chi(L) = 24 \qquad \chi(\mathcal{O}_L) = 2 \qquad q(L) = 0 \qquad p_g(L) = 1 \tag{C.55}$$
$$\chi(E_i) = -4 \qquad \chi(\mathcal{O}_{E_i}) = -1 \qquad q(E_i) = 2 \qquad p_g(E_i) = 0 \tag{C.56}$$
$$\chi(F) = 6 \qquad \chi(\mathcal{O}_F) = 2 \qquad q(F) = 0 \qquad p_g(F) = 0 \tag{C.57}$$
$$\chi(D_1) = 10 \qquad \chi(\mathcal{O}_{D_1}) = 1 \qquad q(D_1) = 0 \qquad p_g(D_1) = 0 \tag{C.58}$$
$$\chi(D_3) = 46 \qquad \chi(\mathcal{O}_{D_3}) = 4 \qquad q(D_3) = 0 \qquad p_g(D_3) = 3 \tag{C.59}$$

The divisor H is the restriction of the hyperplane class of the ambient space. The divisor L is a $K3$ fiber given as $\mathbb{P}^3_{1,1,1,3}[6]$. The divisors $E_1 \ldots E_3$ are each ruled surfaces over the curve C whose fibers form together the A_3 Hirzebruch-Jung sphere tree. The divisor F is a collection of two $\mathbb{P}^2$'s. The divisor D_1 is a blown-up rational surface. Finally, the divisor D_3 is a surface of general type, and is the same as the divisor H in the family $\mathbb{P}^4_{1,1,1,1,4}[8]$ in (3.88). The Picard lattice of L is $\mathrm{Pic}(L) = \langle 2 \rangle$. The generators of the Kähler cone are

$$J_1 = H \qquad J_2 = D_1 + H \qquad J_3 = L + 3D_3 \tag{C.60}$$
$$J_4 = D_3 \qquad J_5 = E_1 + H + 2D_1 \tag{C.61}$$

A basis for $H_2(X, \mathbb{Z})$ dual to $J_1, \ldots, J_5$ is

$$C_1 = \frac{1}{2} D_1 \cdot E_3 \qquad C_2 = \frac{1}{6} H \cdot E_1 \qquad C_3 = \frac{1}{6} H \cdot E_3 \tag{C.62}$$
$$C_5 = \frac{1}{6} H \cdot E_2 \tag{C.63}$$

The curve C_4 cannot be written as the intersection of two of the listed divisors.

Note that this family has one non-toric divisor (see Section 3.2.1 and table 3.2). This accounts for having a reducible divisor F consisting of two $\mathbb{P}^2$'s as will be explained in more detail in Section C.2.

C.2. The families $\mathbb{P}^4_{1,1,1,6,9}[18]$, $\mathbb{P}^4_{1,1,1,3,6}[12]$ and $\mathbb{P}^4_{1,1,1,3,3}[9]$

The geometry of the first of these families has been studied in great detail in [142], [161] and [304]. These three families can be treated uniformly by writing $\mathbb{P}^4_{1,1,1,3a,3b}[3c]$ with $(a,b) = (2,3)$, $(1,2)$ and $(1,1)$ for $\mathbb{P}^4_{1,1,1,6,9}[18]$, $\mathbb{P}^4_{1,1,1,3,6}[12]$ and $\mathbb{P}^4_{1,1,1,3,3}[9]$, respectively, and $c = 1+a+b$. The toric data of these families can be summarized in the following table

$$\begin{array}{lrrrrr|rrl} & & & & & & C_1 & C_2 & \\ D_0 & 0 & 0 & 0 & 0 & 1 & -c & 0 & K=-cH \\ D_1 & -1 & -1 & -3a & -3b & 1 & 0 & 1 & S \\ D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & S \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & S \\ D_4 & 0 & 0 & 1 & 0 & 1 & a & 0 & aH \\ D_5 & 0 & 0 & 0 & 1 & 1 & b & 0 & bH \\ D_6 & 0 & 0 & -a & -b & 1 & 1 & -3 & F=H-3S \\ & & & & & & h & l & \end{array} \tag{C.64}$$

which is built in the same way as in section 3.5.2 and in the Stanley-Reisner ideal

$$\mathcal{I}_{SR} = \{D_1\cdot D_2\cdot D_3 = S^3 = 0, D_4\cdot D_5\cdot D_6 = \frac{c}{k}H^4 - \frac{3c}{k}H^3\cdot S = 0\} \tag{C.65}$$

which follows from the primitive collections $\{\nu_1^*,\nu_2^*,\nu_3^*\}$ and $\{\nu_4^*,\nu_5^*,\nu_6^*\}$. Here $k=1,2,3$ are associated to $(a,b) = (2,3),(1,2),(1,1)$, respectively. The intersection ring of X is generated by H and S satisfying

$$S^3 = 0, \quad H\cdot S^2 = k, \quad H^2\cdot S = 3k, \quad H^3 = 9k \tag{C.66}$$

From these intersections we see that no linear combination af H and S satisfies condition eq. (3.49b) for a $K3$ fibration. However, X is elliptically fibered as can be seen by taking $D=S, D'=H$ which satisfy conditions (3.46a) to (3.46c). The fiber is $\mathbb{P}^2_{1,a,b}[c]$, see Section 3.3.4. Although the Kähler cone is generated by $J_1 = H$ and $J_2 = S$ we will use F and S as basis for $H_4(X,\mathbb{Z})$. Hence the intersection numbers (3.47) and linear forms (3.48) read

$$S^3 = 0 \qquad S^2\cdot F = k \tag{C.67}$$
$$F^2\cdot S = -3k \qquad F^3 = -9k \tag{C.68}$$
$$\mathrm{c}_2\cdot F = -6k \qquad \mathrm{c}_2\cdot S = 36 \tag{C.69}$$

The divisors H, S and F are characterized by

$$\chi(H) = 108+3k \qquad \chi(\mathcal{O}_H) = 9+k \qquad q(H) = 0 \qquad p_g(H) = 8+k \tag{C.70}$$
$$\chi(S) = 36 \qquad \chi(\mathcal{O}_S) = 3 \qquad q(S) = 0 \qquad p_g(S) = 2 \tag{C.71}$$
$$\chi(F) = 3k \qquad \chi(\mathcal{O}_F) = 1 \qquad q(F) = 0 \qquad p_g(F) = 0 \tag{C.72}$$

A basis for $H_2(X,\mathbb{Z})$ dual to H and S is

$$h = S^2 \qquad l = F\cdot S \tag{C.73}$$

From (C.64) and (C.72) we see that F is a collection of k $\mathbb{P}^2$'s, the section of the fibration. As $l\cdot F = -3$, l must be contained in this $\mathbb{P}^2$, and from $l\cdot D_i = 1$, $i = 1, 2, 3$ it follows that l lies in this $\mathbb{P}^2$ with degree 1. $h\cdot F = 1$, hence h meets the section once and must be a curve in the fiber direction.

Note that $\mathbb{P}^4_{1,1,1,3,6}[12]$ and $\mathbb{P}^4_{1,1,1,3,3}[9]$ have one and two non-toric divisors, respectively (see Section 3.2.1 and table 3.2). We are not able to treat them with our toric methods without loosing the

Landau-Ginzburg orbifold phase and therefore the Gepner point. It is possible to modify the polyhedron Δ^* in such a way that $h^{1,1}(X)$ and $h^{2,1}(X)$ remain unchanged, but $\tilde{h}^{1,1}$ becomes equal to $h^{1,1}$ [108]. This is at the expense of introducing additional non-toric complex structure moduli, i.e. of making $\tilde{h}^{2,1}(X)$ smaller, and more importantly of changing the phase structure such that the Landau-Ginzburg orbifold phase disappears. We are however interested precisely in this phase and want to keep it. Therefore we can not treat these non-toric divisors as single divisors. Instead the divisor F contains besides the toric blow-up of the singularity one or two more $\mathbb{P}^2$'s and is therefore reducible in these cases. This will play a role when we wrap D4-branes around it in Section 6.3.

C.3. The families $\mathbb{P}^4_{1,1,2,8,12}[24]$, $\mathbb{P}^4_{1,1,2,4,8}[16]$ and $\mathbb{P}^4_{1,1,2,4,4}[12]$

In this section we will study in detail [305] the families $X = \mathbb{P}_{1,1,2,8,12}[24]$, $\mathbb{P}^4_{1,1,2,4,8}[16]$ and $\mathbb{P}^4_{1,1,2,4,4}[12]$ which have $h^{1,1}(X) = 3$. We first collect some results already obtained in [142] and [306] and extend these to get the geometric description from the toric data. Typically, a member of these families has the following form

$$x_1^{4c} + x_2^{4c} + x_3^{2c} + x_4^{\frac{c}{a}} + x_5^{\frac{c}{b}} = 0 \tag{C.74}$$

where $(a,b) = (2,3)$ for $\mathbb{P}^4_{1,1,2,8,12}[24]$, $(a,b) = (1,2)$ for $\mathbb{P}^4_{1,1,2,4,8}[16]$ and $(a,b) = (1,1)$ for $\mathbb{P}^4_{1,1,2,4,4}[12]$ and $c = 1 + a + b$. The space $\mathbb{P}^4_{1,1,2,4a,4b}$ has singularities which intersect the degree $4c$ hypersurface in a curve C of $\mathbb{Z}_2$ singularities given by $x_3^{2c} + x_4^{\frac{c}{a}} + x_5^{\frac{c}{b}} = 0$ with an additional $\mathbb{Z}_4$ singular point. The curve C is isomorphic to $\mathbb{P}^2_{1,a,b}[c]$ which is an elliptic curve, see Section 3.3.4. Our Calabi–Yau X is then obtained as the proper transform of the blow-up of these singularities which introduces the exceptional divisors E which is a ruled surface over the curve C and F which is a Hirzebruch surface $\mathbb{F}_2$. The (toric) data of X are most conveniently summarized in table C.75.

$$\begin{array}{cccccc|ccc l}
 & & & & & & C_1 & C_2 & C_3 & \\
D_0 & 0 & 0 & 0 & 0 & 1 & -c & 0 & 0 & K = -cH \\
D_1 & -1 & -2 & -4a & -4b & 1 & 0 & 0 & 1 & L \\
D_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & L \\
D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & D \\
D_4 & 0 & 0 & 1 & 0 & 1 & a & 0 & 0 & aH \\
D_5 & 0 & 0 & 0 & 1 & 1 & b & 0 & 0 & bH \\
D_6 & 0 & -1 & -2a & -2b & 1 & 0 & 1 & -2 & E = D - 2L \\
D_7 & 0 & 0 & -a & -b & 1 & 1 & -2 & 0 & F = H - 2D \\
 & & & & & & h & d & l &
\end{array} \tag{C.75}$$

The generators of the Kähler cone are identified as $J_1 = H$, $J_2 = D$ and $J_3 = L$. They form a basis of $H^2(X, \mathbb{Z})$ and the dual basis in $H_2(X, \mathbb{Z})$ is h, d and l. The Stanley-Reisner ideal is

$$\begin{aligned} \mathcal{I}_{SR} = \{ & D_1 \cdot D_2 = L^2 = 0, D_3 \cdot D_6 = D^2 - 2D \cdot L = 0, \\ & D_4 \cdot D_5 \cdot D_7 = \frac{ac}{k} H^3 - \frac{2ac}{k} H^2 \cdot D = 0 \} \end{aligned} \tag{C.76}$$

which follows from the primitive collections $\{\nu_1^*, \nu_2^*\}$, $\{\nu_3^*, \nu_6^*\}$ and $\{\nu_4^*, \nu_5^*, \nu_7^*\}$. Here $k = 1, 2, 3$ are associated to $(a,b) = (2,3), (1,2), (1,1)$, respectively. The intersection ring of X is generated by E, F and L satisfying

$$\begin{array}{rclrclrcl}
L^3 & = & 0 & F \cdot L^2 & = & 0 & F^2 \cdot L & = & -2k \\
F^3 & = & 8k & E \cdot L^2 & = & 0 & E \cdot F \cdot L & = & k \\
E \cdot F^2 & = & 0 & E^2 \cdot L & = & 0 & E^2 \cdot F & = & -2k \\
E^3 & = & 0 & & & & & &
\end{array} \tag{C.77}$$

From (3.29) we find for the second Chern class of X

$$\mathrm{c}_2 \cdot E = 0 \qquad \mathrm{c}_2 \cdot F = -4k \qquad \mathrm{c}_2 \cdot L = 24 \tag{C.78}$$

We can also give a geometric picture to these divisors and curves. First, note that the divisor L satisfies (3.46a) to (3.46c), hence X is a K3 fibration with L being a fiber. From (C.77) we see that its Picard lattice is $\mathrm{Pic}(L) = U$ (see Section 3.3.2). Furthermore, the divisor D satisfies (3.49a) and (3.49b) for $D' = H$. Therefore X is also an elliptic fibration with h being a curve in the elliptic fiber. Finally, since $D^2 \cdot L = 0$, the two fibrations are compatible according to (3.50) and we have therefore the well-known fact that X admits both an elliptic and a K3 fibration. Since $h \cdot F = 1$, we can identify F with the section of the elliptic fibration. From the intersection relations we see that h is a class of the intersection $E \cap L$, d is a class of the intersection $F \cap L$ and l is a class of the intersection $E \cap F$. Since $d^2|_F = 0, d \cdot l|_F = 1$ and $l^2|_F = -2$, d and l are the fiber and the section of the Hirzebruch surface $\mathbb{F}_2$, respectively[1]. Its canonical divisor is then $K_{\mathbb{F}_2} = -2l - 4d = -2D|_F$. Since $h^2|_E = 0, h \cdot l|_E = 1$ and $l^2|_E = 0$, l and h are the fiber and the section of the ruled surface E, respectively, whose canonical divisor is $K_E = -2h$. Furthermore, also the divisor D has a fibration structure. Its canonical divisor is $K_D = 2h$ which satisfies $K_D^2 = 0$. Hence it is an elliptic fibration. Setting $f = 2d + l$, we have $f^2|_D = -4$, $f \cdot h|_D = 1$ and $h^2|_D = 0$ and hence D is an elliptic fibration with section f. Note, that this agrees also with the general relation for elliptically fibered surfaces, $\chi(D) = -12f^2$. Finally, we also need the holomorphic Euler characteristics, given by (3.39)

$$\chi(\mathcal{O}_D) = 4, \quad \chi(\mathcal{O}_E) = 0, \quad \chi(\mathcal{O}_F) = 1, \quad \chi(\mathcal{O}_H) = 9 + k, \quad \chi(\mathcal{O}_L) = 2 \tag{C.79}$$

and the geometric genera

$$p_g(D) = 3, \quad p_g(E) = 0, \quad p_g(F) = 0, \quad p_g(L) = 1, \quad p_g(H) = 8 + k \tag{C.80}$$

Note that $\mathbb{P}^4_{1,1,2,4,8}[16]$ and $\mathbb{P}^4_{1,1,2,4,4}[12]$ have one and two non-toric divisors, respectively (see Section 3.2.1 and table 3.2). Here the same comment as in Section C.2 applies.

[1] Note that we are at a special point of the moduli space. A generic point is actually an elliptic fibration over $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

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