Derived Secondary Classes for Flags of Foliations

Dissertation zur Erlangung des Doktorgrades
an der Fakultät für Mathematik und Informatik
der Ludwig-Maximilians-Universität München

Vorgelegt am 10. September 2001 von

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Tag der mündlichen Prüfung:	5. Oktober 2001

Contents

Introduction

1. Introduction

A foliation $\mathcal{F}$ is a decomposition of a manifold M into immersed submanifolds which locally looks like a product. The vectors tangent to these immersed submanifolds span a distribution $T\mathcal{F} \subset TM$. Bott [**2**] noticed that there is an obstruction against the integrability of an arbitrary distribution D to a foliation $\mathcal{F}$. Bott's Vanishing Theorem (Theorem III.2.2) states that if $D = T\mathcal{F}$ is the tangent bundle of a foliation with rank q normal bundle $Q = TM/D$, then the ring of Pontrjagin classes $\mathrm{Pont}^*(Q) \subset H^*(M;\mathbb{R})$ of Q has to vanish in degrees greater than $2q$.

Reflecting a general phenomenon, the vanishing of these primary characteristic classes gives rise to so-called secondary characteristic classes of the foliation (cf. [**19**]). The oldest and best studied of all secondary characteristic classes is the Godbillon-Vey class constructed as follows. Consider a defining form α, i. e. a locally decomposable form of maximal rank with $\ker\alpha = T\mathcal{F}$. The Frobenius Theorem (Theorem A.5) says that there is a one-form β such that $d\alpha = \beta \wedge \alpha$. The Godbillon-Vey class $gv(\mathcal{F}) \in H^{2q+1}(M;\mathbb{R})$ is defined by $gv(\mathcal{F}) = [\beta \wedge (d\beta)^q]$. This class is a natural cobordism invariant of the foliation. If $\mathcal{F}$ is a codimension one foliation, then in some sense $gv(\mathcal{F})$ measures the "helical wobble" of the leaves of $\mathcal{F}$ (cf. [**30**]). For arbitrary codimension, if there is a holonomy invariant transverse volume form for $\mathcal{F}$, then the Godbillon-Vey class $gv(\mathcal{F})$ vanishes (Corollary II.1.6) – but this condition is not necessary for the vanishing of the Godbillon-Vey class.

Now, Gel'fand-Feigin-Fuks [**11**] and later Fuks [**10**] considered one-parameter families $\mathcal{F}_t$ of foliations and defined characteristic classes of these families in the context of the cohomology of Lie algebras of formal vector fields. For such one-parameter families of codimension q foliations the defining form α – and therefore the form β as well – depends on the parameter t. Kotschick [**21**] introduced a class $TGV(\mathcal{F}_t) = [\dot{\beta}_t \wedge \beta_t \wedge (d\beta_t)^q] \in H^{2q+2}(M;\mathbb{R})$, where the dot denotes the derivative with respect to t. In codimension one this is a characteristic class already considered by Fuks (and others, cf. [**22**]). In the forthcoming paper [**17**] Kamber, Kotschick and the author will construct more classes of this type for one-parameter families of foliations and explain the connection to Gel'fand-Fuks cohomology. Here, we will not use cohomology of infinite dimensional Lie algebras but rather the universal formalism based on finite dimensional Weil algebras.

By definition, the class $TGV(\mathcal{F}_t)$ looks very much like the Godbillon-Vey class itself or some kind of derivative of it. This was the initial point of this thesis: to clarify the connection between the Godbillon-Vey class, the time derivative of the family of Godbillon-Vey classes $gv(\mathcal{F}_t)$ and the class $TGV(\mathcal{F}_t)$. The second question arising immediately was: are there other characteristic classes for families of foliations of the same kind or – what would be even better – is there a general (maybe universal) way of constructing such characteristic classes. The constructions of Gel'fand-Feigin-Fuks, Fuks and in [**17**] as well are limited to one-parameter families of foliations. So finally, we have to ask wether it is possible to define characteristic classes for multi-parameter families of foliations. In the context of Gel'fand-Fuks cohomology Tsujishita [**31**] already worked in this direction.

These questions are answered in full detail in this thesis. The universal construction of derived secondary characteristic classes presented here (Theorem III.3.1) gives a variety of natural concordance invariants for multi-parameter families of foliations. A basis for the space of universal derived secondary characteristic classes will be computed explicitly (Theorem III.5.4). In particular, we will see that all the derived characteristic classes for one-parameter families of codimension one foliations are linear combinations of $TGV(\mathcal{F}_t)$ and the time derivative $\frac{\partial}{\partial t}gv(\mathcal{F}_t)$ of the Godbillon-Vey classes (Theorem III.3.7). Moreover, our construction yields characteristic classes for families parameterized by an arbitrary manifold and even more generally for flags of foliations.

The basic idea is the following. Consider a one-parameter family $\mathcal{F}_t$ of codimension q foliations on M. The union of all these foliations gives rise to a codimension $q+1$ foliation $\mathcal{G}_1$ on the cylinder $\mathbb{R} \times M$. The foliation $\mathcal{F}_t$ at time t appears as the intersection of $\mathcal{G}_1$ with the time slice $M = \{t\} \times M \subset \mathbb{R} \times M$. These time slices themselves constitute a (rather simple) codimension one foliation $\mathcal{G}_2$ on the cylinder $\mathbb{R} \times M$. Since the leaves of $\mathcal{G}_1$ are contained in the leaves of $\mathcal{G}_2$, we get a two-flag $(\mathcal{G}_2, \mathcal{G}_1)$ of foliations – a so-called subfoliation – on the cylinder $\mathbb{R} \times M$. The same form $\beta \wedge (d\beta)^{q+1}$ representing the Godbillon-Vey class $gv(\mathcal{G}_1)$ now represents a class $GV(\mathcal{G}_2, \mathcal{G}_1)$ in a cohomology module $H^*(M, \mathcal{G}_2)$ adapted to the top foliation $\mathcal{G}_2$ (Lemma II.1.4) which we call the Godbillon-Vey class of the subfoliation. Since the leaves of $\mathcal{G}_2$ are the fibres of the trivial bundle $\mathbb{R} \times M \to \mathbb{R}$, we shall compute that cohomology module $H^*(M, \mathcal{G}_2)$ to be the module of smooth maps from $\mathbb{R}$ to $H^*(M; \mathbb{R})$ (Corollary II.4.2). We will see that if we therefore interpret $GV(\mathcal{G}_2, \mathcal{G}_1)$ as a time-dependent class in $H^*(M; \mathbb{R})$, we just recover the class $TGV(\mathcal{F}_t)$ up to a constant (Theorem II.4.3).

This point of view will allow us to define derived classes not only for multi-parameter families of foliations but for flags of foliations in general. This will prove useful in the last chapter, where the derived secondary characteristic classes of a family of foliations are computed in terms of residues defined at the singularities of certain singular three-flags (Theorem IV.10). The greatest advantage of generalizing to flags of foliations is that we can formulate the definition of derived classes in a universal way. This universal construction is based on the ideas of Kamber-Tondeur [**19**] who used a filtration on the Weil algebra of a Lie group to define a characteristic homomorphism for foliated principal bundles giving rise to secondary characteristic classes of foliations. This was generalized to subfoliations by Carballés [**8**]. In view of the last chapter, we will give a construction for arbitrary k-flags of foliation (Theorem III.2.4). This will yield a space of universal classes which are mapped by a derived characteristic homomorphism to families of de Rham classes if the leaves of the top foliation are the fibres of a fibre bundle. These are the natural concordance invariants of families of foliations mentioned above.

Before beginning, I would like to thank my advisor Dieter Kotschick for his patience, the opportunity of fruitful collaboration and the friendly support during the preperation of this thesis. I have also the pleasure to thank Franz W. Kamber for many enlightening discussions and valuable help in understanding the Koszul spectral sequence. Let me finally express my gratitude to the Graduiertenkolleg "Mathematik im Bereich ihrer Wechselwirkung mit der Physik" at the University of Munich for financial support.

2. Outline of the contents

We will now give a detailed overview of the contents of this thesis.

Chapter I. The first chapter collects the basic properties of foliated cohomology for future reference. Parts of it are well-known – especially for the classical leafwise cohomology $H^*(\mathcal{F})$ – but as far as I know have not yet been collected in that general form keeping the whole Koszul spectral sequence in view. So, maybe this will be helpful to others too. The reader familiar with the Koszul spectral sequence should feel comfortable with proceeding to Chapter II and returning whenever needed.

The first section recalls the fundamental definitions, the Frobenius Theorem mentioned above and the Reeb class $[\beta]$ which is defined in the classical leafwise cohomology $H^1(\mathcal{F})$. This is the first cohomology group of the quotient $\Omega^*(M)/I^*(\mathcal{F})$ of the de Rham complex by the foliation ideal $I^*(\mathcal{F})$ of all forms vanishing along $\mathcal{F}$. The section goes on giving the construction of the holonomy groupoid of a foliation and showing that the Reeb class is an obstruction against the existence of a holonomy invariant transverse volume form. Of course, all of this is well-known and serves here as a motivation for the following constructions. The essence is that there are naturally defined classes in the cohomology of certain (sub)quotients of the de Rham complex which contain geometric information about the foliation. The rest of the chapter will be devoted to the study of such cohomology groups.

In the second section the Koszul spectral sequence $E_j^{r,s}(M, \mathcal{F})$ is defined which can be viewed as a generalization of the leafwise cohomology algebra. Actually, this spectral sequence arising from a natural filtration of the de Rham complex is concentrated in a rectangle in the first quadrant, and the left most column $E_1^{0,*}(M,\mathcal{F})$ of the E_1-term is equal to $H^*(\mathcal{F})$. In this thesis we will be especially interested in the right most column $E_1^{q,*}(M,\mathcal{F})$, where q is the codimension of $\mathcal{F}$, which we denote by $H^*(M,\mathcal{F})$. Our first result is that we can compute this $H^*(\mathcal{F})$-module $H^*(M,\mathcal{F})$ using the same graded algebra $\Omega^*(M)/I^*(\mathcal{F})$ as for the computation of $H^*(\mathcal{F})$ but with a perturbed differential d_α (Theorem I.2.6). In particular, if $\mathcal{F}$ admits a holonomy invariant transverse volume form, then $H^*(M,\mathcal{F}) \cong H^*(\mathcal{F})$ (Corollary I.2.7).

The next section deduces the functorial properties of the Koszul spectral sequence which are of course essential for constructing characteristic classes. The spectral sequence is natural with respect to pulling back forms via submersions. Moreover, it is invariant under homotopies which move leaves within leaves (Theorem I.3.2). As an application of this leafwise homotopy invariance we prove a Poincaré Lemma, i. e. we compute the spectral sequence of the standard p-dimensional foliation $\mathcal{T}^{q,p}$ on $\mathbb{R}^{q+p}$ given by the fibres of the projection $\mathbb{R}^{q+p} \to \mathbb{R}^q$ (Theorem I.3.3). The E_1-term equals

$$E_1^{r,s}(\mathbb{R}^{q+p},\mathcal{T}^{q,p}) \cong \begin{cases} \Omega^r(\mathbb{R}^q) & , \text{ for } 0 \le r \le q \text{ and } s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

At the end of the section we will explain what has to be changed if we consider the Koszul spectral sequences given by forms with compact or at least transversally compact support which will be useful for integration theorems as in Section I.6 or in Chapter IV.

Section I.4 proves the two most fundamental theorems for explicitly computing cohomology groups: the Mayer-Vietoris Sequence and the Künneth Formula. Actually, the Mayer-Vietoris Sequence exists for every foliation $\mathcal{F}$ on a manifold $M = U \cup V$, where U, V are open subsets of M (Theorem I.4.1). The Künneth Formula $E_1^{*,*}(M \times N, \mathcal{F} \times \mathcal{G}) \cong E_1^{*,*}(M,\mathcal{F}) \otimes E_1^{*,*}(N,\mathcal{G})$ on the other hand does *not* hold in general (Proposition I.4.7) – which may not be too surprising, since the E_1-terms tend to be far from having finite type. But if one of the foliations is trivial, say $\mathcal{G} = \{N\}$, then the Künneth Formula is valid (Theorem I.4.5). This we will use for constructing non-vanishing examples in Chapter II. For the classical leafwise cohomology algebras Theorem I.4.5 already appeared in [**1**].

In the following section we compute the spectral sequence for the crucial example of a simple foliation $\mathcal{F}$ whose leaves are the fibres of a fibre bundle $X \to B$ with fibre M. In this case the E_1-term is given by

$$E_1^{r,s}(M,\mathcal{F}) = \Omega^r(B;\mathcal{H}^s(M;\mathbb{R})) \ ,$$

where $\mathcal{H}^*(M;\mathbb{R})$ is the flat vector bundle associated to the fibre bundle $X \to B$ (Theorem I.5.2). This implies the Leray-Serre Theorem (Corollary I.5.3).

The last section of Chapter I will be relevant for the residue theorem in Chapter IV. It contains the construction of the integration along the fibre

$$\int_\pi : E_{0,vc}^{r,k+s}(M, \pi^*\mathcal{F}) \to E_0^{r,s}(B, \mathcal{F})$$

of an oriented fibre bundle $\pi : M \to B$ with fibre F, an oriented k-manifold with boundary ∂F (Theorem I.6.5 – the suffix vc denotes cohomology with compact supports along the fibres). If ∂F is empty, then this map induces a homomorphism

$$\int_\pi : E_{1,vc}^{r,k+s}(M, \pi^*\mathcal{F}) \to E_1^{r,s}(B, \mathcal{F})$$

between the E_1-terms. If M is actually a vector bundle over B, then this homomorphism is isomorphic (Theorem I.6.3). Its inverse is the Thom isomorphism of the vector bundle which is given by right multiplication by a Thom class $u \in H_{vc}^k(\pi^*\mathcal{F})$.

Chapter II. The second chapter focuses on the Godbillon-Vey class of families of foliations. The aim is to identify the class $TGV(\mathcal{F}_t)$ mentioned above as the Godbillon-Vey class of a subfoliation. In doing so we get the idea how to construct more invariants for families of foliations of this type.

In the first section k-flags of foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ are considered. These are sequences of foliations such that the leaves of each foliation are contained in the leaves of the next one, $\mathcal{F}_i \subset \mathcal{F}_{i+1}$. It is shown that there are canonical maps

$$E_j^{r,s}(M, \mathcal{F}_i) \to E_j^{r-q_i, s+q_i}(M, \mathcal{F}_{i+1}) \ ,$$

where q_i is the codimension of $\mathcal{F}_i$ in $\mathcal{F}_{i+1}$ (Proposition II.1.2). The classes which are in the image of one of the maps

$$H^*(M, \mathcal{F}_i) \to H^{*+q_i}(M, \mathcal{F}_{i+1})$$

we call derived classes. We are especially interested in the classes lying in the image of

$$H^*(M, \mathcal{F}_i) \to H^{*+q_i+\cdot\cdot+q_{k-1}}(M, \mathcal{F}_k) \ , \tag{2.1}$$

since we think of the top foliation $\mathcal{F}_k$ as a parameterization for the subordinate ones (as we indicated in the introduction). As a first example, we see that the same form $\beta_i \wedge (d\beta_i)^{q_i+\dots+q_k}$ which represents the classical Godbillon-Vey class $gv(\mathcal{F}_i)$ in de Rham cohomology gives rise to a class $GV(\mathcal{F}_k, \mathcal{F}_i)$ in the image of the canonical map (2.1) above (Lemma II.1.4). We call the sum of all these derived classes

$$GV(\mathcal{F}_k, \dots, \mathcal{F}_1) = \sum_{i=1}^{k-1} GV(\mathcal{F}_k, \mathcal{F}_i) \ \in H^*(M, \mathcal{F}_k)$$

the Godbillon-Vey class of the k-flag.

In Section II.2 we prove that there are k-flags of arbitrary codimension with non-trivial Godbillon-Vey class. First we notice that if the Godbillon-Vey class $GV(\mathcal{F}_k, \dots, \mathcal{F}_1)$ of a k-flag vanishes, then all the classical Godbillon-Vey classes $gv(\mathcal{F}_i)$ have to be zero for $1 \leq i < k$ (Proposition II.2.1). Making use of Roussarie's foliation $\mathcal{F}$ with $gv(\mathcal{F}) \neq 0$ (Theorem II.2.2) we deduce the general non-vanishing result: for arbitrary codimension $(q_k, \dots, q_1)$ there is a k-flag of foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ such that $GV(\mathcal{F}_i, \dots, \mathcal{F}_1)$ vanishes for $1 \leq i < k$ but the Godbillon-Vey class $GV(\mathcal{F}_k, \dots, \mathcal{F}_1)$ does not (Theorem II.2.5).

The short Section II.3 shows that the Godbillon-Vey class of a k-flag is a characteristic class, i. e. it is natural under pull-backs (Proposition II.3.1), and furthermore that it is invariant under leafwise concordance (Theorem II.3.5). Two k-flags are leafwise concordant, if there is a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on the cylinder $[0,1] \times M$ which is transverse to the boundary $M \sqcup M$ and restricts there to the two given k-flags and if moreover the top foliation $\mathcal{F}_k$ is a product foliation $\{[0,1]\} \times \mathcal{F}$. This implies the concordance invariance of the classical Godbillon-Vey class $gv(\mathcal{F})$ (Corollary II.3.6).

The central section of this chapter is Section II.4. A family of foliations on M parameterized by B is a subfoliation $(\mathcal{G}_2, \mathcal{G}_1)$, where the leaves of the top foliation $\mathcal{G}_2$ are the fibres of a fibre bundle $X \to B$ with fibre M (this makes sense as we saw in the introduction). Every choice of volume form on the parameter space B gives rise to an identification of $H^*(X, \mathcal{G}_2)$ with the smooth sections of the flat vector bundle $\mathcal{H}^*(M;\mathbb{R})$ (Corollary II.4.2). In particular, the derived classes of a q_2-parameter family of foliations $\mathcal{F}_t$ are smooth mappings from $\mathbb{R}^{q_2}$ to the de Rham cohomology $H^*(M;\mathbb{R})$. In Theorem II.4.3 we explicitly compute the Godbillon-Vey class $GV(\mathcal{F}_t) = GV(\mathcal{G}_2, \mathcal{G}_1)$ to be the map sending $t \in \mathbb{R}^{q_2}$ to the class

$$GV(\mathcal{F}_t) = \tfrac{(q_1+q_2)!}{q_1!} \left[(\tfrac{\partial}{\partial t_{q_2}}\beta)_t \wedge \cdots \wedge (\tfrac{\partial}{\partial t_1}\beta)_t \wedge \beta_t \wedge (d\beta_t)^{q_1} \right] \in H^{2q_1+q_2+1}(M;\mathbb{R}) \ .$$

So indeed, the class $TGV(\mathcal{F}_t)$ of a one-parameter family of foliations is just the Godbillon-Vey class of the associated subfoliation. Furthermore, by Section II.3 we know that the Godbillon-Vey class $GV(\mathcal{G}_2, \mathcal{G}_1)$ is a natural concordance invariant of the family of foliations (Theorem II.4.4 and Theorem II.4.5).

The chapter closes with Section II.5. Here, the dual of the Godbillon-Vey class of a family of foliations

$$GV(\mathcal{G}_2, \mathcal{G}_1)^* : \Gamma(\mathcal{H}^{n-2q_1-q_2-1}(M;\mathbb{R})) \to \Omega^{q_2}(B)$$

with respect to the pairing $\langle \cdot, \cdot \rangle$ given by integration along the fibre is decomposed into the composition of two maps, the Vey invariant

$$V : \Gamma(\mathcal{H}^{n-2q_1-q_2-1}(M;\mathbb{R})) \to H^{n-q_1-1}(X, \mathcal{G}_1)$$

and the Godbillon operator

$$G : H^{n-q_1-1}(X, \mathcal{G}_1) \to \Omega^{q_2}(B)$$

which is given by $G[\omega] = \int_\pi \omega \wedge \beta$ (Theorem II.5.2). If B equals a point, then this is the classical decomposition of $gv(\mathcal{F})^*$ due to Duminy (cf. [**12**]). Since the pairing $\langle\cdot,\cdot\rangle$ is non-degenerate (Lemma II.5.1), we get that the Godbillon-Vey class $GV(\mathcal{G}_2,\mathcal{G}_1)$ of a family of foliations vanishes whenever the Godbillon operator G vanishes. For example, if there is a sequence of representatives β_k for the Reeb class of $\mathcal{G}_1$ which tends to zero, then $GV(\mathcal{G}_2,\mathcal{G}_1)$ vanishes (Corollary II.5.3) – even though the Reeb class may be non-trivial.

Chapter III. This chapter is the heart of this thesis. We give two universal constructions for derived characteristic classes of k-flags of foliations: the first, more general one, following the construction of Kamber-Tondeur [**19**] of the characteristic homomorphism of a foliated principal bundle, and the second one in the fashion of Bott [**3**] which makes computations easier but is more or less restricted to the normal bundle of a k-flag.

In the first section we define the notion of a k-foliated principle bundle. In particular, the transverse frame bundle of a k-flag of foliations is canonically k-foliated in that sense (Proposition III.1.5). If we chose a connection ω on P which is adapted to the foliated structure, then the curvature of ω shows a certain vanishing phenomenon which is crucial for the construction (Proposition III.1.12).

In Section III.2 the characteristic homomorphism of a k-foliated principle bundle is constructed (for $k = 2$ this has already been done by Carballés [**8**]) generalizing Kamber-Tondeur's homomorphism (Theorem III.2.1). We get a homomorphism from the cohomology of a truncated relative Weil algebra to the de Rham cohomology of M,

$$\Delta(\tilde{\mathcal{F}}_k,\dots,\tilde{\mathcal{F}}_1,H)_* : H^*(W(\mathfrak{g},H)_{(q_k,\dots,q_1)}) \to H^*(M;\mathbb{R})$$

which is natural and concordance invariant (Theorem III.2.4).

Section III.3 begins by noticing that the truncated Weil algebra carries k filtrations $F_iW(\mathfrak{g},H)_{(q_k,\dots,q_1)}$ which are mapped by the Weil homomorphism into the k filtrations $F_{\mathcal{F}_i}\Omega^*(M)$, thanks to the vanishing phenomenon mentioned above. Thus, the characteristic homomorphism of the k-foliated principal bundle is already defined at the level of spectral sequences

$$E_j^{*,*}(W(\mathfrak{g},H)_{(q_k,\dots,q_1)},F_i) \to E_j^{*,*}(M,\mathcal{F}_i)$$

for $j \geq 1$. Considering the sum $DCh^*_{(q_k,\dots,q_1)}(\mathfrak{g},H)$ of the appropriate parts of the E_1-terms on the left hand side we get the derived characteristic homomorphism

$$D\Delta(\tilde{\mathcal{F}}_k,\dots,\tilde{\mathcal{F}}_1,H)_* : DCh^*_{(q_k,\dots,q_1)}(\mathfrak{g},H) \to H^{*-q_k}(M,\mathcal{F}_k)$$

of a k-foliated principal bundle which is natural under pull-backs and invariant under leafwise concordance (Theorem III.3.1). In Theorem III.3.5 we identify the universal class $GV \in DCh^*_{(q_k,\dots,q_1)}(\mathfrak{g},H)$ which is mapped to $GV(\mathcal{F}_k,\dots,\mathcal{F}_1)$ under the derived characteristic homomorphism, and thereby see that this homomorphism is not always trivial. The section closes by computing the space of universal derived characteristic classes of q_2-parameter families of foliations to be

$$H^*((\Lambda^*\mathfrak{gl}_{q_1}(\mathbb{R})^* \otimes S^{q_2+q_1}\mathfrak{gl}_{q_1}(\mathbb{R})^*)_{\mathrm{O}(q_1)}) \ .$$

In particular, the space of universal derived characteristic classes of one-parameter families of codimension one foliations $\mathcal{F}_t$ is two-dimensional, and a basis is given by the universal Godbillon-Vey class GV – which is mapped to $TGV(\mathcal{F}_t)$ – and the universal class which is mapped to the time derivative $\frac{\partial}{\partial t}gv(\mathcal{F}_t)$ of the Godbillon-Vey classes (Theorem III.3.7).

Section III.4 starts the second construction of the derived characteristic homomorphism using Bott's comparison technique. This works for foliated vector bundles rather than for foliated principal bundles. It is shown that the normal bundle of a k-flag is canonically k-foliated by any adapted linear connection (Proposition III.4.4). Of course, this can also be deduced from the fact that the transverse frame bundle is k-foliated (Proposition III.4.5).

The last section of this chapter is devoted to the comparison construction of the characteristic homomorphism of a k-foliated vector bundle,

$$\lambda(E, \nabla)_* : H^*(WO^*_{(q_k,\ldots,q_1)}) \to H^*(M; \mathbb{R}) \ ,$$

which is a natural concordance invariant (Theorem III.5.1). This homomorphism is also defined at the level of spectral sequences

$$E_j^{2r,s-r}(WO^*_{(q_k,\ldots,q_1)}, F_i) \to E_j^{r,s}(M, \mathcal{F}_i)$$

for $j \geq 1$. In the same way as before, we obtain a derived characteristic homomorphism for k-flags of foliations

$$D\lambda(\mathcal{F}_k, \ldots, \mathcal{F}_1)_* : DWO^*_{(q_k,\ldots,q_1)} \to H^{*-q_k}(M, \mathcal{F}_k)$$

which is natural under pull-backs and invariant under leafwise concordance (Theorem III.5.2). We identify the universal Godbillon-Vey class in the new setting (Theorem III.5.3), and we compute explicitly the space of universal derived characteristic classes for q_2-parameter families of foliations (Theorem III.5.4). A basis for this space is given by elements

$$(h_{i_1} \wedge \cdots \wedge h_{i_s}) \otimes (c_1^{j_1} \cdots c_{q_1}^{j_{q_1}})$$

with $1 \leq i_1 < \cdots < i_s \leq l_1 = 2\left[\frac{q_1+1}{2}\right] - 1$ odd and $j_1 + 2j_2 + \cdots + q_1 j_{q_1} = q_2 + q_1$.

Chapter IV. The last chapter introduces the notion of a singular three-flag of foliations, i. e. a three-flag $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ which is defined outside a (nice) singular set S and which can be extended to a subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ on the whole manifold. Such singular three-flags arise for example when we consider infinitesimal automorphisms of a subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$. Under certain conditions on the singular three-flag we can compute all derived characteristic classes y of the subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ out of residues

$$\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S) \in H^{j-q_2-q_1}(S, \mathcal{F}_1)$$

which only depend on the local behaviour of the three-flag around the singularity S (Theorem IV.10). This generalizes Heitsch's Residue Theorem [**16**] for infinitesimal automorphisms of foliations.

CHAPTER I

Foliated cohomology

I.1. Foliations and holonomy

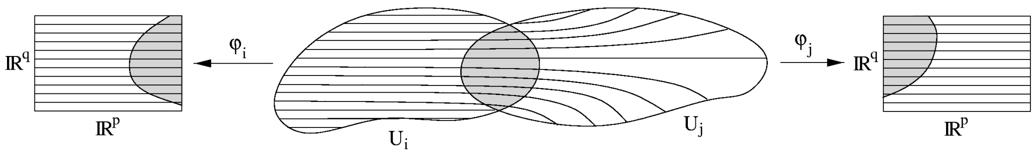
First let us recall some basic facts about foliations and the dynamics of their leaves. Let M be a smooth manifold of dimension $n = p + q$ (unless otherwise stated our manifolds are supposed to be without boundary). A *foliation* $\mathcal{F}$ of dimension p and codimension q on M is an atlas for the manifold M such that the charts

$$\varphi_i : U_i \subset M \to V_i \subset \mathbb{R}^q \times \mathbb{R}^p$$

satisfy the condition that all the coordinate transformations

$$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

respect the decomposition of $\mathbb{R}^q \times \mathbb{R}^p$ into the sets $\{x\} \times \mathbb{R}^p$, $x \in \mathbb{R}^q$.



Every foliated chart determines a submersion

$$f_i = pr_1 \circ \varphi_i : U_i \to \mathbb{R}^q \ .$$

The fibres $f_i^{-1}(x) = \varphi_i^{-1}(\{x\} \times \mathbb{R}^p)$ are called the *plaques* of $\mathcal{F}$. The transitive hull of the symmetric and reflexive relation "$x, y \in M$ lie in the same plaque of $\mathcal{F}$" is an equivalence relation on M. The equivalence classes are called the *leaves* of $\mathcal{F}$. The *leaf space* $M/\mathcal{F}$ of $\mathcal{F}$ is the quotient space by this equivalence relation. In general, this topological space is far from being a manifold. Since the leaves can accumulate, it need not even be Hausdorff.

By definition every leaf of $\mathcal{F}$ is an immersed p-dimensional submanifold of M. Denote by $T\mathcal{F}$ the set of all tangent vectors of M which are tangent to the leaves of $\mathcal{F}$. Again by definition, this set is a locally trivial vector bundle, i. e. a p-dimensional distribution $T\mathcal{F} \subset TM$, called the *tangent bundle* of $\mathcal{F}$. The vector bundle $Q = TM/T\mathcal{F}$ is the *normal bundle* of $\mathcal{F}$. Let $D \subset TM$ be some distribution. It is called *integrable* if there is a foliation $\mathcal{F}$ on M such that $D = T\mathcal{F}$. Not every distribution is integrable, but there is an easy criterion for integrability. A distribution D is called *involutive* if $\Gamma(D)$ is a Lie subalgebra of the Lie algebra $\Gamma(TM)$ of smooth vector fields on M, i. e. if for all vector fields $X, Y \in \Gamma(D)$ the Lie bracket $[X, Y]$ is in $\Gamma(D)$ as well.

Theorem I.1.1 (Frobenius)**.** *A distribution is integrable if and only if it is involutive.*

For a proof of this theorem see Appendix A. Now, suppose that D is a *coorientable* distribution, i. e. the normal bundle $Q = TM/D$ is orientable. Thus, if q is the codimension of D, then $\Lambda^q Q^*$ is a trivial line bundle. The projection $\pi : TM \twoheadrightarrow Q$ induces a monomorphism

$$\pi^* : \Gamma(\Lambda^q Q^*) \rightarrowtail \Omega^q(M) .$$

Every nowhere-vanishing form α in the image of this monomorphism is called a *defining form* for D. Since $\Lambda^q Q^*$ is a trivial line bundle such a form exists and is unique up to multiplication by a nowhere vanishing function on M. A defining form for the distribution D is characterized by the property $D = \ker \alpha$. Note, that this implies that α is a *transverse volume form* for D. This means whenever $T \subset M$ is an immersed q-dimensional submanifold transverse to D, then $\alpha|_T$ is a volume form on T. On the other hand, the kernel of a q-form $\alpha \in \Omega^q(M)$ is a coorientable distribution D if and only if α is of constant rank. Now, the Frobenius Theorem can be dualised to yield the following equivalent statement (cf. Appendix A).

Corollary I.1.2. *Let $\alpha \in \Omega^q(M)$ be a locally decomposable form of maximal rank. Then the coorientable distribution* $\ker \alpha$ *is integrable if and only if there is a one-form $\beta \in \Omega^1(M)$ such that*

$$d\alpha = \beta \wedge \alpha .$$

Let $\mathcal{F}$ be the foliation integrating $\ker \alpha$. Since α is a transverse volume form, the one-form β above is unique up to addition of one-forms vanishing along $\mathcal{F}$. The space $I^*(\mathcal{F})$ of all forms vanishing along $\mathcal{F}$,

$$I^k(\mathcal{F}) = \{\omega \in \Omega^k(M) |\ \omega(X_1, \ldots, X_k) = 0 \text{ for all } X_i \in T\mathcal{F}\} ,$$

is obviously a differential ideal called the *foliation ideal* associated to $\mathcal{F}$. It gives rise to a short exact sequence of differential graded algebras

$$0 \to I^*(\mathcal{F}) \to \Omega^*(M) \to \Omega^*(M)/I^*(\mathcal{F}) \to 0 . \tag{I.1.1}$$

The well-known *leafwise cohomology algebra* associated to the foliation $\mathcal{F}$ is the cohomology of the quotient algebra

$$H^*(\mathcal{F}) = H(\Omega^*(M)/I^*(\mathcal{F}), d) .$$

That α is a transverse volume form can be used again to see that, since

$$\begin{aligned} 0 = d(d\alpha) = d(\beta \wedge \alpha) &= d\beta \wedge \alpha - \beta \wedge d\alpha = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha \\ &= d\beta \wedge \alpha , \end{aligned}$$

the differential $d\beta \in I^2(\mathcal{F})$ is an element of the foliation ideal. Therefore α determines an element $[\beta] \in H^1(\mathcal{F})$ called the *Reeb class* of $\mathcal{F}$. An easy calculation shows that the Reeb class does not depend on the choice of defining form. Explicitly, if α' is another defining form for $\mathcal{F}$ giving the same coorientation (say), then $\alpha' = f \cdot \alpha$ with a positive function $f : M \to \mathbb{R}$. Hence,

$$d\alpha' = df \wedge \alpha + f d\alpha = f(d\log f) \wedge \alpha + f\beta \wedge \alpha = (d\log f + \beta) \wedge \alpha' \ .$$

Thus, β and $\beta' = \beta + d\log f$ represent the same class in $H^1(\mathcal{F})$.

So, the Reeb class is a well-defined invariant of the coorientable foliation $\mathcal{F}$. It is closely related to the dynamics of the leaves of the foliation as we will see immediately. The central object in the study of the dynamics of the leaves of $\mathcal{F}$ is the holonomy groupoid. Let us briefly recall its definition (cf. [**26**]).

A *transversal* $\mathcal{T}$ of $\mathcal{F}$ is an immersed submanifold $\mathcal{T} \subset M$ giving a decomposition $T\mathcal{T} \oplus T\mathcal{F}|_{\mathcal{T}} = TM|_{\mathcal{T}}$. Consider two points $x, y \in M$ lying in the same leaf L of $\mathcal{F}$ and two transversals $\mathcal{T}_x, \mathcal{T}_y$ through x, resp. y. We want to move the transversal $\mathcal{T}_x$ along the leaves of $\mathcal{F}$ into the transversal $\mathcal{T}_y$ in a unique way. This demands some considerations, since a leaf of $\mathcal{F}$ can meet the transversals more than once (– or never). First suppose that x and y lie in the same plaque and therefore in the domain of the same foliated chart $\varphi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^p$. The restriction of the submersion $f_i : U_i \to \mathbb{R}^q$ to the transversal $\mathcal{T}_x$ has bijective differential in x. Hence, f_i restricted to a small neighbourhood $N_x \subset \mathcal{T}_x$ of x in $\mathcal{T}_x$ is a diffeomorphism onto a small neighbourhood of $f_i(x)$ in $\mathbb{R}^q$. The same is true for f_i restricted to $\mathcal{T}_y$. Since $f_i(x) = f_i(y)$, we can assume that f_i maps $N_x \subset \mathcal{T}_x$ and $N_y \subset \mathcal{T}_y$ diffeomorphically onto the same open neighbourhood of $f_i(x)$ in $\mathbb{R}^q$. In other words, the prescription "map a point $x' \in N_x \subset \mathcal{T}_x$ to the point $y' \in N_y \subset \mathcal{T}_y$ which lies in the same plaque of $\mathcal{F}$ as x'" determines a diffeomorphism $h_{x,y} : N_x \to N_y$ mapping x to y. The germ at x of this diffeomorphism does not depend on the choice of the neighbourhoods N_x, N_y and of the chart φ_i. It only depends on the germ of $\mathcal{T}_x$ at x and the germ of $\mathcal{T}_y$ at y. For simplicity we will denote the germs by the same symbols as the representing objects. So, we have constructed a unique germ

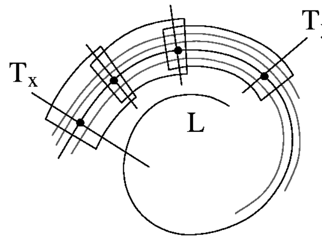
$$h_{y,x} : \mathcal{T}_x \to \mathcal{T}_y$$

for x and y lying in the same plaque. In the general case, let w be a path in the leaf L starting in x and ending in y. The *holonomy transformation* along w is the germ at x of a locally defined diffeomorphism

$$hol(w) : \mathcal{T}_x \to \mathcal{T}_y$$

mapping x to y, constructed in the following way. Cover the path w by foliated charts $U_1, \ldots, U_k$ such that $x = w(0) \in U_1$, $y = w(1) \in U_k$ and such that there are points $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k = 1$ with $w(t_i) \in U_i \cap U_{i+1}$ for $i = 1, \ldots, k-1$. Choose arbitrary transversals $\mathcal{T}_{w(t_i)}$ through $w(t_i)$ and set

$$hol(w) = h_{y,w(t_{k-1})} \circ h_{w(t_{k-1}),w(t_{k-2})} \circ \cdots \circ h_{w(t_2),w(t_1)} \circ h_{w(t_1),x} \ .$$



If we replace the i'th intermediate transversal $\mathcal{T}_{w(t_i)}$ by another transversal $\mathcal{T}'_{w(t_i)}$ through $w(t_i)$, then $h_{w(t_i),w(t_{i-1})}$ is replaced by $h \circ h_{w(t_i),w(t_{i-1})}$ and $h_{w(t_{i+1}),w(t_i)}$ is replaced by $h_{w(t_{i+1}),w(t_i)} \circ h^{-1}$, where h is the unique germ $h = h_{w(t_i),w(t_i)} : \mathcal{T}_{w(t_i)} \to \mathcal{T}'_{w(t_i)}$. Hence, the germ $hol(w)$ does not depend on the choice of intermediate transversals. The same argument shows that $hol(w)$ is independent of the choice of the points t_i. So, the holonomy transformation along w only depends on the germ of $\mathcal{T}_x$ at x, the germ of $\mathcal{T}_y$ at y and the path w. Furthermore, if w_0, w_1 are two paths in the leaf L starting in x and ending in y which are homotopic in L by a homotopy fixing the endpoints, then $hol(w_1) = hol(w_2)$.

Now, fix for every point $x \in M$ the germ at x of a transversal $\mathcal{T}_x$ through this point. The *holonomy groupoid* $\mathrm{Hol}(\mathcal{F})$ is the small category having as objects the points of M, and for two points x and y lying in the same leaf L of $\mathcal{F}$ the morphisms from x to y are the holonomy transformations from $\mathcal{T}_x$ to $\mathcal{T}_y$ along paths w in L joining x and y. Obviously, if we replace our choice of transversals by another one, then the new holonomy groupoid is canonically isomorphic to the old one. Instead of fixing germs of transversals, we could fix a foliated chart $\varphi_x : U_x \to \mathbb{R}^q \times \mathbb{R}^p$ for every $x \in M$ which maps x to 0. This not only gives rise to a unique transversal through x, namely $\varphi_x^{-1}(\mathbb{R}^q \times \{0\})$, but yields furthermore an identification of this transversal with $\mathbb{R}^q$. This choice identifies the morphisms of $\mathrm{Hol}(\mathcal{F})$ with elements of $\overline{\mathrm{Diff}}(\mathbb{R}^q, 0)$, the group of germs at 0 of diffeomorphisms between open neighbourhoods of 0 in $\mathbb{R}^q$ fixing 0. If L is a leaf of $\mathcal{F}$ and $\Pi(L)$ denotes the fundamental groupoid of L, then we get a representation of groupoids

$$\begin{aligned} hol : \Pi(L) &\to \mathrm{Hol}(\mathcal{F}) \\ [w] &\mapsto hol(w)\ . \end{aligned}$$

In particular, for $x \in L$ it yields a representation of groups

$$hol : \pi_1(L, x) \to \overline{\mathrm{Diff}}(\mathbb{R}^q, 0)$$

called the *holonomy representation* of the leaf L. The conjugacy class of this homomorphism is an invariant of the leaf L. To get a linear representation we can compose this homomorphism with the map taking the differential in 0 to get the *infinitesimal holonomy representation*

$$\mathit{infhol} : \pi_1(L, x) \to \mathrm{GL}_q(\mathbb{R})$$

of L. These homomorphisms reflect the (transverse) geometric properties of the foliation. For example, $\mathcal{F}$ is coorientable if and only if *infhol* takes values in $\mathrm{GL}_q^+(\mathbb{R})$, the linear group of matrices with positive determinant.

Let us call a form $\alpha \in \Omega^*(M)$ *holonomy invariant* if it satisfies $h^*(\alpha|_{\mathcal{T}_y}) = \alpha|_{\mathcal{T}_x}$ in x for every $h : \mathcal{T}_x \to \mathcal{T}_y$ in $\mathrm{Hol}(\mathcal{F})$.

Lemma I.1.3. *Consider a form $\alpha \in \Omega^*(M)$ with $T\mathcal{F} \subset \ker \alpha$. Then α is holonomy invariant if and only if $L_X\alpha = 0$ for every $X \in \Gamma(T\mathcal{F})$.*

PROOF. Let $\varphi : U \to \mathbb{R}^q \times \mathbb{R}^p$ be a foliated chart. Since $i_X\alpha = 0$ for every $X \in T\mathcal{F}$, we have

$$(\varphi^{-1})^*(\alpha|_U) = \sum_{1 \le k_1 < \cdots < k_r \le q} a_{k_1,\dots,k_r}(y, x)\, dy_{k_1} \wedge \cdots \wedge dy_{k_r}$$

with functions $a_{k_1,\dots,k_r} : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}$. Moreover, $L_X(\alpha|_U) = 0$ for every $X \in \Gamma(T\mathcal{F}|_U)$ if and only if for fixed $y \in \mathbb{R}^q$ none of the coefficients $a_{k_1,\dots,k_r}(y, x)$ depends on $x \in \mathbb{R}^p$. Suppose that $h^*(\alpha|_{\mathcal{T}_v}) = \alpha|_{\mathcal{T}_u}$ for every $h \in \mathrm{Hol}(\mathcal{F})$. In particular, if $u, v \in U$ are lying in the same plaque, then $h^*_{v,u}(\alpha|_{\mathcal{T}_v}) = \alpha|_{\mathcal{T}_u}$. Since $h^*_{v,u}(\varphi^* dy_i) = \varphi^* dy_i$, this implies that the $a_{k_1,\dots,k_r}(y, x)$ are independent of x. Hence, $L_X\alpha = 0$ for every $X \in \Gamma(T\mathcal{F})$. On the other hand, if α satisfies $L_X\alpha = 0$ for all $X \in \Gamma(T\mathcal{F})$, then the $a_{k_1,\dots,k_r}(y, x)$ are independent of x and therefore satisfy $h^*_{v,u}(\alpha|_{\mathcal{T}_v}) = \alpha|_{\mathcal{T}_u}$ for every $u, v \in U$ lying in the same plaque. Since $\mathrm{Hol}(\mathcal{F})$ is generated by such transformations $h_{v,u}$, we have $h^*(\alpha|_{\mathcal{T}_v}) = \alpha|_{\mathcal{T}_u}$ for all $h \in \mathrm{Hol}(\mathcal{F})$. □

If there is a holonomy invariant transverse volume form for $\mathcal{F}$, then *infhol* takes values in the special linear group $\mathrm{SL}_q(\mathbb{R})$ of matrices with determinant equal to one. Lemma I.1.3 shows that the Reeb class is an obstruction against the existence of such a holonomy invariant transverse volume form.

Theorem I.1.4. *A defining form α for $\mathcal{F}$ is a holonomy invariant transverse volume form if and only if it is closed. In particular, if $\mathcal{F}$ admits a holonomy invariant transverse volume form, then the Reeb class of $\mathcal{F}$ vanishes.*

PROOF. If α is a holonomy invariant defining form and $X \in \Gamma(T\mathcal{F})$ is tangential to the foliation, then

$$i_X d\alpha = L_X\alpha - d i_X\alpha = 0$$

by Lemma I.1.3. Since the normal bundle Q of $\mathcal{F}$ is q-dimensional and $d\alpha$ is a $(q+1)$-form, this implies that $d\alpha = 0$. On the other hand, if α is a closed form defining $\mathcal{F}$, then

$$L_X\alpha = i_X d\alpha + d i_X\alpha = 0$$

for every $X \in \Gamma(T\mathcal{F})$. Hence, again by Lemma I.1.3, α is holonomy invariant. □

The rest of this chapter is devoted to the study of cohomology modules related to the leafwise cohomology algebra which are the natural habitat of classes like the Reeb class. But actually, it would be more convenient to have such classes in the usual de Rham cohomology. That is the reason why one considers the so-called Godbillon-Vey class rather than the Reeb class. This class will be in the focus of Chapter II. The fact that the Reeb class can be interpreted as an obstruction against the reduction of the infinitesimal holonomy from $\mathrm{GL}_q^+(\mathbb{R})$ to the subgroup $\mathrm{SL}_q(\mathbb{R})$ motivates the universal constructions which are the content of Chapter III.

I.2. The spectral sequence of a foliation

Let M be a smooth n-manifold carrying a smooth foliation $\mathcal{F}$ of codimension q. As we saw in the last section there is a class, namely the Reeb class, which is defined in the cohomology of a certain (sub)quotient of the de Rham complex, containing geometric information about the foliation $\mathcal{F}$. There is a whole sequence of subquotients of $\Omega^*(M)$ reflecting geometric properties of the foliation. The *Koszul filtration* $F^*_{\mathcal{F}}\Omega^*(M)$ (for convenience let us drop the index $\mathcal{F}$ as long as there is no danger of confusion) of the space of differential forms on M (cf. [**19**]) is defined by

$$F^r\Omega^k(M) = \{\omega \in \Omega^k(M) |\ i_{X_{k-r+1}} \cdots i_{X_1}\omega = 0 \text{ for all } X_i \in T\mathcal{F}\}$$

for $0 < r \leq k$, $F^r\Omega^k(M) = 0$ for $r > k$ and $F^r\Omega^k(M) = \Omega^k(M)$ for $r \leq 0$. This filtration is compatible with the differential graded algebra structure on $\Omega^*(M)$.

Proposition I.2.1. *The Koszul filtration is a bounded decreasing filtration by differential ideals,*

$$\Omega^*(M) = F^0\Omega^*(M) \supset F^1\Omega^*(M) \supset \cdots \supset F^q\Omega^*(M) \supset F^{q+1}\Omega^*(M) = 0 \ ,$$

and it is multiplicative,

$$F^r\Omega^k(M) \wedge F^s\Omega^l(M) \subset F^{r+s}\Omega^{k+l}(M) \ .$$

This proposition will follow immediately from the dual characterization of the filtration terms $F^r\Omega^*(M)$ which we will state in the following lemma. With a multiindex $I = (i_1, \ldots, i_k)$ we write dx_I for the form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Denote by $l(I) = k$ the length of the multiindex and write $\mathfrak{I}_q$ for the set of all multiindices $\mathfrak{I}_q = \{I = (i_1, \ldots, i_k) |\ 1 \leq i_1 < \cdots < i_k \leq q\}$.

Lemma I.2.2. *Denote by $y = (y_1, \ldots, y_q)$ the standard coordinates of $\mathbb{R}^q$ and by $x = (x_1, \ldots, x_p)$ the standard coordinates of $\mathbb{R}^p$. Then $F^r\Omega^*(M)$ consists of all the k-forms ω on M such that for every foliated chart $\varphi : U \to \mathbb{R}^q \times \mathbb{R}^p$ the form $(\varphi^{-1})^*\omega$ can be written as*

$$\sum_{\substack{I\in\mathfrak{I}_q,\ l(I)\geq r \\ J\in\mathfrak{I}_p}} a_{I,J}(y,x)\, dy_I \wedge dx_J \ .$$

Loosely speaking this means that ω contains at least r covectors of the transverse coframe.

PROOF. This is obvious. □

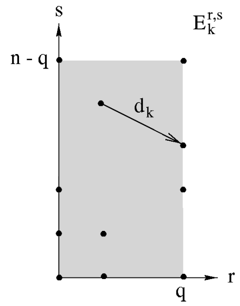
This filtration gives rise to a spectral sequence $E_k^{*,*}(M,\mathcal{F})$ of differential bigraded algebras (for the basics in spectral sequences see [**23**]) with initial term

$$E_0^{r,s}(M,\mathcal{F}) = F^r\Omega^{r+s}(M)/F^{r+1}\Omega^{r+s}(M)$$

converging to the de Rham cohomology $H^*(M;\mathbb{R})$ of M. As usual, the product in $E_k^{*,*}(M,\mathcal{F})$ induced by $\wedge$ is denoted by

$$E_k^{r,s}(M,\mathcal{F}) \otimes E_k^{i,j}(M,\mathcal{F}) \xrightarrow{\cup} E_k^{r+i,s+j}(M,\mathcal{F}) \ .$$

Since $E_k^{*,*}(M,\mathcal{F})$ is concentrated in the rectangle $0 \le r \le q$, $0 \le s \le n-q$ by Lemma I.2.2 and the differential d_k has bidegree $(k, 1-k)$, the spectral sequence collapses at stage $q+1$, i. e. $E_{q+1}^{r,s}(M,\mathcal{F}) = E_{q+2}^{r,s}(M,\mathcal{F}) = \cdots = E_\infty^{r,s}(M,\mathcal{F})$.



In particular, $H^s(M;\mathbb{R}) \cong \bigoplus_{r=0}^{s} E_{q+1}^{r,s-r}(M,\mathcal{F})$.

Let us see how the considerations of the last section fit into this scheme. By definition we have $I^*(\mathcal{F}) = F^1\Omega^*(M)$. Thus,

$$\Omega^*(M)/I^*(\mathcal{F}) = F^0\Omega^*(M)/F^1\Omega^*(M) = E_0^{0,*}(M,\mathcal{F})$$

and the first column of the E_1-term is just the leafwise cohomology algebra,

$$H^*(\mathcal{F}) = E_1^{0,*}(M,\mathcal{F}) \ .$$

Now, consider the other extreme. Since the filtration is bounded, we have

$$E_0^{q,s}(M,\mathcal{F}) = F^q\Omega^{q+s}(M) = \{\omega \in \Omega^{q+s}(M) |\ i_{X_{s+1}} \cdots i_{X_1}\omega = 0 \ \forall\ X_i \in T\mathcal{F}\} \ .$$

Denote by

$$H^*(M,\mathcal{F}) = E_1^{q,*}(M,\mathcal{F})$$

the cohomology module given by that subcomplex. Of course, this module is no longer closed under multiplication in $E_1^{*,*}(M,\mathcal{F})$. But just like every column of the E_1-term it is a module over the leafwise cohomology algebra $H^*(\mathcal{F}) = E_1^{0,*}(M,\mathcal{F})$. In Corollary I.2.7 below we will determine this $H^*(\mathcal{F})$-module structure for a certain type of foliation. Note finally that the inclusions $E_0^{q,s}(M,\mathcal{F}) = F^q\Omega^{q+s}(M) \subset \Omega^{q+s}(M)$ induce canonical homomorphisms

$$\epsilon : H^s(M,\mathcal{F}) \to H^{q+s}(M;\mathbb{R})$$

relating the foliated cohomology module $H^*(M,\mathcal{F})$ to the usual de Rham cohomology.

Let us look at some naive examples (a more interesting example, namely the foliation given by the fibres of a fibre bundle, will be examined in Section I.5).

EXAMPLE I.2.1. If $q=0$, i. e. $\mathcal{F}=\{M\}$ is the trivial foliation consisting of a single leaf, then we get

$$E_0^{r,s}(M,\mathcal{F})=\begin{cases}\Omega^s(M) & \text{, for } r=0\\ 0 & \text{, otherwise .}\end{cases}$$

Hence,

$$E_\infty^{r,s}(M,\mathcal{F})=\cdots=E_1^{r,s}(M,\mathcal{F})=\begin{cases}H^s(M;\mathbb{R}) & \text{, for } r=0\\ 0 & \text{, otherwise ,}\end{cases}$$

and the cohomology modules defined above are just the usual de Rham cohomology modules,

$$H^*(M,\mathcal{F})=H^*(\mathcal{F})=H^*(M;\mathbb{R})\ .$$

⊠

EXAMPLE I.2.2. If $q=1$, then $E_0^{1,s}(M,\mathcal{F})=F^1\Omega^{1+s}(M)=I^{s+1}(\mathcal{F})$. Hence, the cohomology module $H^*(M,\mathcal{F})$ is isomorphic to the cohomology of the foliation ideal,

$$H^s(M,\mathcal{F})=H^{s+1}(I^*(\mathcal{F}))\ ,$$

and we get a long exact sequence

$$\cdots\to H^{s-1}(M,\mathcal{F})\xrightarrow{\epsilon}H^s(M;\mathbb{R})\to H^s(\mathcal{F})\xrightarrow{\delta}H^s(M,\mathcal{F})\to\cdots$$

induced by the short exact sequence (I.1.1). The connecting homomorphism δ equals the differential $d_1:E_1^{0,s}(M,\mathcal{F})\to E_1^{1,s}(M,\mathcal{F})$. ⊠

EXAMPLE I.2.3. If finally q equals the dimension of M, i. e. $\mathcal{F}$ is the foliation by the points of M, then

$$E_1^{r,s}(M,\mathcal{F})=E_0^{r,s}(M,\mathcal{F})=\begin{cases}\Omega^r(M) & \text{, for } s=0\\ 0 & \text{, otherwise}\end{cases}$$

and

$$E_\infty^{r,s}(M,\mathcal{F})=\cdots=E_2^{r,s}(M,\mathcal{F})=\begin{cases}H^r(M;\mathbb{R}) & \text{, for } s=0\\ 0 & \text{, otherwise .}\end{cases}$$

⊠

From time to time it is easier to work with forms having compact support (for example, if we want to integrate foliated cohomology classes, as will be done in Section I.6). So, denote by $\Omega_c^*(M)$ the complex of compactly supported forms on M. We get an analogous spectral sequence $E_{k,c}^{r,s}(M,\mathcal{F})$ converging to the de Rham cohomology with compact supports $H_c^*(M;\mathbb{R})$ and foliated cohomology rings, resp. modules $H_c^*(\mathcal{F}),H_c^*(M,\mathcal{F})$. Obviously, if M is compact, then $\Omega_c^*(M)=\Omega^*(M)$ and we get nothing new. There is another compactness condition which is sometimes more adequate. Let $\Omega_{tr}^r(M)$ denote the space of forms with *transversally compact support*, i. e. the space of forms ω, such that the closure of the image of $\operatorname{supp}\omega$ in the leaf space $M/\mathcal{F}$ is compact. This yields a spectral sequence $E_{k,tr}^{r,s}(M,\mathcal{F})$ and cohomology rings, resp. modules $H_{tr}^*(\mathcal{F}), H_{tr}^*(M,\mathcal{F})$. If the leaf space $M/\mathcal{F}$ is compact, then $\Omega_{tr}^*(M) = \Omega^*(M)$ and we get the usual Koszul spectral sequence again. If the quotient map $\pi : M \to M/\mathcal{F}$ is proper, i. e. if the preimages of compact sets are compact, then $\Omega_{tr}^*(M) = \Omega_c^*(M)$ and the Koszul spectral sequence with transversally compact supports equals the Koszul spectral sequence with compact supports (in that case all the leaves have to be compact, of course).

EXAMPLE I.2.4. For example, if $\mathcal{F} = \{M\}$, then $E_{k,tr}^{r,s}(M,\mathcal{F}) = E_k^{r,s}(M,\mathcal{F})$,

$$E_{0,c}^{r,s}(M,\mathcal{F}) = \begin{cases} \Omega_c^s(M) & , \text{ for } r = 0 \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$E_{\infty,c}^{r,s}(M,\mathcal{F}) = \cdots = E_{1,c}^{r,s}(M,\mathcal{F}) = \begin{cases} H_c^s(M;\mathbb{R}) & , \text{ for } r = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

In particular, $H_c^*(\mathcal{F})$ and $H_c^*(M,\mathcal{F})$ are equal to the de Rham cohomology with compact supports $H_c^*(M;\mathbb{R})$. ⊠

EXAMPLE I.2.5. If $\mathcal{F}$ is the foliation by points, then the spectral sequences $E_{i,c}^{r,s}(M,\mathcal{F}) = E_{i,tr}^{r,s}(M,\mathcal{F})$ coincide, and

$$E_{1,tr}^{r,s}(M,\mathcal{F}) = E_{0,tr}^{r,s}(M,\mathcal{F}) = \begin{cases} \Omega_c^r(M) & , \text{ for } s = 0 \\ 0 & , \text{ otherwise} \end{cases}$$

as well as

$$E_{\infty,tr}^{r,s}(M,\mathcal{F}) = \cdots = E_{2,tr}^{r,s}(M,\mathcal{F}) = \begin{cases} H_c^r(M;\mathbb{R}) & , \text{ for } s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

⊠

Even though we will focus on the first and the last column, the other parts of the spectral sequence contain geometric information too. For example it would be helpful to have some kind of de Rham theory on $M/\mathcal{F}$, although the leaf space $M/\mathcal{F}$ is not a manifold in general. For that reason Reinhard [**29**] introduced the space of *basic forms* on M, namely

$$\Omega^*(M/\mathcal{F}) = \{\omega \in \Omega^*(M) |\ i_X\omega = 0,\ i_X d\omega = 0 \text{ for all } X \in T\mathcal{F}\} \ .$$

Obviously, this space is a differential graded algebra and indeed, if the foliation $\mathcal{F}$ is *simple*, i. e. the leaves of $\mathcal{F}$ are the fibres of a submersion $p : M \to B$, then the basic forms on M are just the pull-backs of forms on the base space B (for more details about the complex of basic forms, see Section I.5).

Proposition I.2.3. *If $\mathcal{F}$ is a simple foliation given by the fibres of a surjective submersion $p : M \twoheadrightarrow B$, then there is an isomorphism of differential graded algebras*

$$\Omega^*(M/\mathcal{F}) \cong \Omega^*(B) \ .$$

Proof. It is clear that $p^* : \Omega^*(B) \to \Omega^*(M)$ gives a monomorphism $p^* : \Omega^*(B) \rightarrowtail \Omega^*(M/\mathcal{F})$. If $\varphi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^p$ is a foliated chart, then the prescription $\psi_i \circ (p|_{U_i}) = pr_1 \circ \varphi_i = f_i$ defines a chart $\psi_i : p(U_i) \to \mathbb{R}^q$ of B. So, the foliation $\mathcal{F}$ induces an atlas of B. Since $i_X\omega = 0$ and $i_X d\omega = 0$ implies $L_X\omega = 0$, we have that every form $\omega \in \Omega^*(M/\mathcal{F})$ is holonomy invariant by Lemma I.1.3. Like in the proof of Lemma I.1.3 we see that on every foliated chart the form ω is the pull-back of a form on the corresponding chart of B. Glueing these forms together with a partition of unity yields that ω can be pushed down to a form on B. So, $p^* : \Omega^*(B) \to \Omega^*(M/\mathcal{F})$ is an isomorphism. □

The cohomology algebra of the complex $\Omega^*(M/\mathcal{F})$ is called the *basic cohomology* $H^*(M/\mathcal{F})$ of $\mathcal{F}$. In fact there are other approaches to the cohomology of the leaf space, like the transverse cohomology of Haefliger [**15**]. These different attempts have been related by Moerdijk [**9**]. Note that the bottom row $E_1^{*,0}(M,\mathcal{F})$ of the E_1-term of the Koszul spectral sequence is a complex with respect to the differential d_1.

Theorem I.2.4. *For every codimension-q foliation $\mathcal{F}$ on M we have an isomorphism of complexes*

$$E_1^{r,0}(M,\mathcal{F}) \cong \Omega^r(M/\mathcal{F}) \ .$$

Hence, the bottom row of the E_2-term of the Koszul spectral sequence is the basic cohomology of $\mathcal{F}$,

$$E_2^{r,0}(M,\mathcal{F}) \cong H^r(M/\mathcal{F}) \ .$$

Proof. Obviously, there are no coboundaries in $E_0^{r,0}(M,\mathcal{F})$. Hence, we have $E_1^{r,0}(M,\mathcal{F}) = \ker d_0 = \{\omega \in F^r\Omega^r(M) |\ d\omega \in F^{r+1}\Omega^{r+1}(M)\} = \Omega^r(M/\mathcal{F})$. □

The same can be done for the spectral sequence with transversally compact support. For example, if $\mathcal{F}$ is a simple foliation given by the fibres of a submersion $p : M \to B$, then

$$E_{1,tr}^{r,0}(M,\mathcal{F}) \cong \Omega_c^r(B) \quad \text{and} \quad E_{2,tr}^{r,0}(M,\mathcal{F}) \cong H_c^r(B;\mathbb{R}) \ .$$

This makes it natural to call the elements of $E_{1,tr}^{*,0}(M,\mathcal{F})$ (in slight abuse of notation) the *basic forms with compact support* $\Omega_c^*(M/\mathcal{F})$ and $E_{2,tr}^{*,0}(M,\mathcal{F})$ the *basic cohomology with compact supports* $H_c^*(M/\mathcal{F})$.

Let us return to the foliated cohomology modules $H^*(\mathcal{F})$ and $H^*(M,\mathcal{F})$ again. In particular, $H^0(\mathcal{F}) \cong \Omega^0(M/\mathcal{F})$ is the space of all smooth functions on M which are constant along the leaves (showing by the way that in general the foliated cohomology modules are far from being finitely generated). On the other hand, $H^0(M,\mathcal{F}) \cong \Omega^q(M/\mathcal{F})$. It is easy to see that if $\mathcal{F}$ can be defined by a holonomy invariant transverse volume form, then multiplication by this form defines an isomorphism $\Omega^0(M/\mathcal{F}) \cong \Omega^q(M/\mathcal{F})$ and hence an isomorphism $H^0(\mathcal{F}) \cong H^0(M,\mathcal{F})$. We will generalize this statement and will compute $H^*(M,\mathcal{F})$ from $H^*(\mathcal{F})$. For the remainder of this section let $\mathcal{F}$ be coorientable.

Then it can be defined by a q-form α satisfying $d\alpha = \beta \wedge \alpha$ for some 1-form β. We want to introduce an additive homomorphism

$$\begin{aligned} d_\alpha : \Omega^*(M)/I^*(\mathcal{F}) &\to \Omega^{*+1}(M)/I^{*+1}(\mathcal{F}) \\ \omega &\mapsto \beta \wedge \omega + d\omega \ . \end{aligned}$$

Lemma I.2.5. *The map d_α is a well-defined differential on $\Omega^*(M)/I^*(\mathcal{F})$ depending only on α. Moreover,*

$$d_\alpha(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d_\alpha \eta) \tag{I.2.1}$$

for each $\omega \in \Omega^k(M)/I^k(\mathcal{F})$ and every $\eta \in \Omega^*(M)/I^*(\mathcal{F})$.

Proof. Since $I^*(\mathcal{F})$ is a differential ideal, d_α is well-defined. For fixed α the form β is unique up to the addition of elements $\tau \in I^1(\mathcal{F})$ as we noticed in the last section. So, d_α only depends on α. Since α is a transverse volume form, we have $d\beta \in I^2(\mathcal{F})$. Hence, for $\omega \in \Omega^*(M)/I^*(M)$ the equation

$$d_\alpha d_\alpha \omega = \beta \wedge \beta \wedge \omega + d(\beta \wedge \omega) + \beta \wedge d\omega + dd\omega = d\beta \wedge \omega = 0$$

holds in $\Omega^*(M)/I^*(\mathcal{F})$ and d_α is a differential. Obviously, for $\omega \in \Omega^k(M)/I^k(\mathcal{F})$ and every $\eta \in \Omega^*(M)/I^*(\mathcal{F})$ we have

$$\begin{aligned} d_\alpha(\omega \wedge \eta) &= \beta \wedge \omega \wedge \eta + d(\omega \wedge \eta) \\ &= (d\omega) \wedge \eta + (-1)^k \omega \wedge (\beta \wedge \eta + d\eta) \\ &= (d\omega) \wedge \eta + (-1)^k \omega \wedge (d_\alpha \eta) \ . \end{aligned}$$

□

If we compute the cohomology $H(\Omega^*(M)/I^*(\mathcal{F}), d_\alpha)$ with respect to this perturbed differential, then Formula (I.2.1) implies that this cohomology group is a left $H^*(\mathcal{F})$-module. Since α is an element of $F^q\Omega^q(M) = E_0^{q,0}(M, \mathcal{F})$, right multiplication by α defines a homomorphism

$$\begin{aligned} R_\alpha : E_0^{0,*}(M, \mathcal{F}) &\to E_0^{q,*}(M, \mathcal{F}) \\ \omega &\mapsto \omega \wedge \alpha \end{aligned}$$

yielding the following identification.

Theorem I.2.6. *Let $\mathcal{F}$ be a coorientable foliation on M with defining form α. Then $H^*(M, \mathcal{F})$ is canonically isomorphic to $H(\Omega^*(M)/I^*(\mathcal{F}), d_\alpha)$ as a left $H^*(\mathcal{F})$-module.*

Proof. For $\omega \in \Omega^k(M)/I^k(\mathcal{F}) = E_0^{0,k}(M, \mathcal{F})$ we compute

$$\begin{aligned} dR_\alpha \omega = d(\omega \wedge \alpha) &= d\omega \wedge \alpha + (-1)^k \omega \wedge \beta \wedge \alpha = d\omega \wedge \alpha + \beta \wedge \omega \wedge \alpha \\ &= R_\alpha(d_\alpha \omega) \ . \end{aligned}$$

Hence, R_α is a cochain map from $(\Omega^*(M)/I^*(\mathcal{F}), d_\alpha)$ to $(E_0^{q,*}(M,\mathcal{F}), d_0)$, and it suffices to prove that

$$R_\alpha : \Omega^s(M)/I^s(\mathcal{F}) \to \{\omega \in \Omega^{q+s}(M) |\ i_{X_{s+1}} \cdots i_{X_1}\omega = 0 \text{ for all } X_i \in T\mathcal{F}\}$$

is an isomorphism. Let $\omega \in \Omega^s(M)$ with $R_\alpha(\omega) = 0$. Denote by Q some complement of $T\mathcal{F}$, i. e. $TM = T\mathcal{F} \oplus Q$. Because α is a transverse volume form and Q is orientable, there is a section $Y \in \Gamma(\Lambda^q Q)$ such that $i_Y\alpha \equiv 1$. For every $X \in \Gamma(\Lambda^s T\mathcal{F})$ we have

$$i_X\omega = i_X\omega \cdot i_Y\alpha = i_Y i_X(\omega \wedge \alpha) = i_Y i_X R_\alpha(\omega) = 0$$

because $\alpha \in F^q\Omega^q(M)$. Thus, R_α is injective. On the other hand, if $\eta \in E_0^{q,s}(M,\mathcal{F})$ then define ω to equal $\omega = (-1)^{qs} i_Y\eta$. This gives

$$i_Y i_X(\omega \wedge \alpha) = i_X\omega \cdot i_Y\alpha = (-1)^{qs} i_X(i_Y\eta) = i_Y i_X\eta\ .$$

Thus, $R_\alpha(\omega) = \eta$ because both $\omega \wedge \alpha$ and η are elements of $E_0^{q,s}(M,\mathcal{F})$ and $Q \cong TM/T\mathcal{F}$ is q-dimensional. This proves that R_α is isomorphic. □

If $\mathcal{F}$ can be defined by a closed form α, then this closed form represents a class $[\alpha] \in H^0(M,\mathcal{F})$. In this case Theorem I.2.6 can be interpreted in the following way.

Corollary I.2.7. *If $\mathcal{F}$ admits a holonomy invariant transverse volume form α, then $H^*(M,\mathcal{F})$ is a one-dimensional free left $H^*(\mathcal{F})$-module generated by $[\alpha] \in H^0(M,\mathcal{F})$.*

Note that Theorem I.2.6 remains true if we restrict ourselves to transversally compact supported forms or even to compactly supported forms.

I.3. Naturality, homotopy invariance and Poincaré Lemmas

The spectral sequence defined in the last section is functorial and invariant under homotopies (if the notion of homotopy is suitably defined) as will be made precise in this section. Let $(M,\mathcal{F}), (N,\mathcal{G})$ be foliated manifolds. A *morphism* from $(N,\mathcal{G})$ to $(M,\mathcal{F})$ is a smooth map $f : N \to M$ which maps every leaf of $\mathcal{G}$ into a leaf of $\mathcal{F}$. We denote that by $f(\mathcal{G}) \subset \mathcal{F}$. This defines the category $\mathcal{F}ol$ of foliated manifolds. Pulling back forms via a morphism f gives homomorphisms

$$f^* : F^s_{\mathcal{F}}\Omega^*(M) \to F^s_{\mathcal{G}}\Omega^*(N)\ .$$

Thus, we have induced homorphisms

$$f^* : E_k^{r,s}(M,\mathcal{F}) \to E_k^{r,s}(N,\mathcal{G})$$

for every $k \geq 0$. By this the Koszul spectral sequence becomes a contravariant functor $E : \mathcal{F}ol \to \mathcal{S}pec^{*,*}_{\mathbb{R}}$ from the category $\mathcal{F}ol$ into the category of multiplicative spectral sequences of bigraded $\mathbb{R}$-modules. In particular, a morphism induces a homomorphism

$$f^* : H^*(\mathcal{F}) \to H^*(\mathcal{G})\ .$$

turning $(M, \mathcal{F}) \mapsto H^*(\mathcal{F})$ into a functor $H : \mathcal{F}ol \to \mathcal{A}lg^*_{\mathbb{R}}$ from $\mathcal{F}ol$ into the category of graded $\mathbb{R}$-algebras. If furthermore $\operatorname{codim} \mathcal{F} = \operatorname{codim} \mathcal{G}$, then there is a homomorphism

$$f^* : H^*(M, \mathcal{F}) \to H^*(N, \mathcal{G}) \ .$$

Hence, if $\mathcal{F}ol^q$ is the subcategory of $\mathcal{F}ol$ consisting of all foliations of codimension q, then $(M, \mathcal{F}) \mapsto H^*(M, \mathcal{F})$ is a functor $H : \mathcal{F}ol^q \to \mathcal{M}od^*_{\mathbb{R}}$ from $\mathcal{F}ol^q$ into the category of graded $\mathbb{R}$-modules.

For example, if $f : N \to M$ is any map transverse to $\mathcal{F}$ (i. e. $f_*TN + T\mathcal{F}|_{\operatorname{im} f} = TM|_{\operatorname{im} f}$), then there is the pull-back foliation $f^*\mathcal{F}$ on N. The leaves of $f^*\mathcal{F}$ are just the preimages under f of the leaves of $\mathcal{F}$. This pull-back has the same codimension as the original foliation $\mathcal{F}$. So, there is a canonical homomorphism

$$f^* : H^*(M, \mathcal{F}) \to H^*(N, f^*\mathcal{F})$$

for every map f transverse to $\mathcal{F}$. Therefore, the functor $H : \mathcal{F}ol^q \to \mathcal{M}od^*_{\mathbb{R}}$ is natural with respect to such maps that are transverse to every foliation, i. e. it is natural with respect to submersions. Of course, the same is true for the functors $E : \mathcal{F}ol \to \mathcal{S}pec^{*,*}_{\mathbb{R}}$ and $H : \mathcal{F}ol \to \mathcal{A}lg^*_{\mathbb{R}}$.

If we want to consider homotopies between morphisms, then these homotopies should respect the foliations as well. Let us make this precise.

Definition I.3.1. Two morphisms $f_0, f_1 : (N, \mathcal{G}) \to (M, \mathcal{F})$ between two foliated manifolds are *leafwise homotopic* if there is a map $F : N \times [0, 1] \to M$ satisfying the following conditions. If F_t denotes the restriction

$$F_t = F|_{N \times \{t\}} : N \to M$$

for $t \in [0, 1]$, then

(1) $F_0 = f_0, F_1 = f_1$,
(2) $F_t(\mathcal{G}) \subset \mathcal{F}$ for all $t \in [0, 1]$,
(3) for every $x \in N$ the points $F_s(x), F_t(x)$ lie in the same leaf of $\mathcal{F}$ for all $s, t \in [0, 1]$.

Thus, a leafwise homotopy consists of morphisms and moves points along leaves. It is easy to see that the functors $E : \mathcal{F}ol \to \mathcal{S}pec^{*,*}_{\mathbb{R}}$, $H : \mathcal{F}ol \to \mathcal{A}lg^*_{\mathbb{R}}$ and $H : \mathcal{F}ol^q \to \mathcal{M}od^*_{\mathbb{R}}$ are invariant under leafwise homotopy.

Theorem I.3.2. *Let $(M, \mathcal{F})$, $(N, \mathcal{G})$ be two foliated manifolds and suppose that $f_0, f_1 : (N, \mathcal{G}) \to (M, \mathcal{F})$ are leafwise homotopic morphisms. Then we have*

$$f_0^* = f_1^* : E_k^{r,s}(M, \mathcal{F}) \to E_k^{r,s}(N, \mathcal{G})$$

for every $k \geq 1$. In particular,

$$f_0^* = f_1^* : H^*(\mathcal{F}) \to H^*(\mathcal{G}) \ .$$

If furthermore $\operatorname{codim} \mathcal{F} = \operatorname{codim} \mathcal{G}$*, then*

$$f_0^* = f_1^* : H^*(M, \mathcal{F}) \to H^*(N, \mathcal{G}) \ .$$

Proof. Consider the cylinder $N \times [0,1] \subset N \times \mathbb{R}$. Let $p : N \times [0,1] \to N$ be the canonical projection and foliate the cylinder by the foliation $p^*\mathcal{G}$. Then the canonical vector field $\frac{\partial}{\partial t}$ tangent to the fibres of p is a section of $T(p^*\mathcal{G})$ and we have the map

$$\begin{aligned} F^r_{p^*\mathcal{G}}\Omega^k(N \times [0,1]) &\to F^r_{\mathcal{G}}\Omega^{k-1}(N) \\ \omega &\mapsto \iota_t^*(i_{\frac{\partial}{\partial t}}\omega) \ , \end{aligned}$$

where $\iota_t : N = N \times \{t\} \subset N \times [0,1]$ is the inclusion map. This induces a map

$$\begin{aligned} I : E_0^{r,s}(N \times [0,1], p^*\mathcal{G}) &\to E_0^{r,s-1}(N, \mathcal{G}) \\ \omega &\mapsto \int_0^1 \iota_t^*(i_{\frac{\partial}{\partial t}}\omega)\ dt \end{aligned}$$

(for details about the integration along the fibre, see Section I.6). An easy computation shows that

$$\iota_1^* - \iota_0^* = Id + dI$$

on $\Omega^*(N \times [0,1])$, and the same equality holds if both sides are considered as maps

$$E_0^{r,s}(N \times [0,1], p^*\mathcal{G}) \to E_0^{r,s}(N, \mathcal{G}) \ .$$

Indeed, if we write $\omega = \alpha_t + dt \wedge \beta_t$ with $\alpha_t \in \Omega^k(N), \beta_t \in \Omega^{k-1}(N)$, then

$$\iota_t^*(L_{\frac{\partial}{\partial t}}\omega) = \iota_t^*(di_{\frac{\partial}{\partial t}}\omega + i_{\frac{\partial}{\partial t}}d\omega) = \iota_t^*(d\beta_t + i_{\frac{\partial}{\partial t}}(dt \wedge \dot{\alpha}_t + d\alpha_t - dt \wedge d\beta_t)) = \dot{\alpha}_t \ .$$

and

$$\begin{aligned} \iota_1^*\omega - \iota_0^*\omega = \alpha_1 - \alpha_0 &= \int_0^1 \dot{\alpha}_t\ dt = \int_0^1 \iota_t^*(L_{\frac{\partial}{\partial t}}\omega)\ dt \\ &= \int_0^1 \iota_t^*(i_{\frac{\partial}{\partial t}}d\omega)\ dt + \int_0^1 \iota_t^*(di_{\frac{\partial}{\partial t}}\omega)\ dt \\ &= Id(\omega) + dI(\omega) \ . \end{aligned}$$

Hence, we get

$$\iota_0^* = \iota_1^* : E_1^{r,s}(N \times [0,1], p^*\mathcal{G}) \to E_1^{r,s}(N, \mathcal{G}) \ .$$

Let $F : N \times [0,1] \to M$ be the leafwise homotopy from f_0 to f_1. Since $F(p^*\mathcal{G}) \subset \mathcal{F}$ we have

$$f_0^* = \iota_0^* F^* = \iota_1^* F^* = f_1^* : E_1^{r,s}(M, \mathcal{F}) \to E_1^{r,s}(N, \mathcal{G}) \ .$$

□

This gives the following foliated version of the Poincaré Lemma. Denote by $\mathcal{T}^{q,p}$ the foliation on $\mathbb{R}^q \times \mathbb{R}^p$ given by the fibres of the canonical projection $pr_1 : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q$ which is the local model for every codimension q foliation on a $(q+p)$-manifold.

Theorem I.3.3. *We have*

$$E_1^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega^r(\mathbb{R}^q) & , \text{ for } 0 \le r \le q \text{ and } s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

In particular,

$$H^s(\mathcal{T}^{q,p}) \cong H^s(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega^0(\mathbb{R}^q) & , \text{ for } s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

Proof. Foliate $\mathbb{R}^q \times \mathbb{R}^p$ by $\mathcal{T}^{q,p}$ and consider the homotopy

$$F : \mathbb{R}^q \times \mathbb{R}^p \times [0,1] \to \mathbb{R}^q \times \mathbb{R}^p$$
$$(x, y, t) \mapsto (x, (1-t)y) \ .$$

F is a leafwise homotopy from the identity $f_0 = id_{\mathbb{R}^{q+p}}$ to the map

$$f_1 : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q \times \mathbb{R}^p$$
$$(x, y) \mapsto (x, 0) \ .$$

Foliate $\mathbb{R}^q$ by its points and let ι_0 denote the inclusion

$$\iota_0 : \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^p$$
$$x \mapsto (x, 0) \ .$$

Then ι_0 and the projection $pr_1 : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q$ induce homomorphisms between the spectral sequences of the foliated manifolds $(\mathbb{R}^{q+p}, \mathcal{T}^{q,p})$ and $(\mathbb{R}^q, \mathcal{T}^{q,0})$ which satisfy

$$\iota_0^* pr_1^* = (pr_1 \iota_0)^* = id \ , \quad pr_1^* \iota_0^* = (\iota_0 pr_1)^* = f_1^* = id$$

by the homotopy invariance of $E : \mathcal{F}ol \to \mathcal{S}pec_{\mathbb{R}}^{*,*}$. Hence, pr_1 induces an isomorphism between the spectral sequences of $(\mathbb{R}^{q+p}, \mathcal{T}^{q,p})$ and $(\mathbb{R}^q, \mathcal{T}^{q,0})$. But by Example I.2.3

$$E_0^{r,s}(\mathbb{R}^q, \mathcal{T}^{q,0}) = E_1^{r,s}(\mathbb{R}^q, \mathcal{T}^{q,0}) = \begin{cases} \Omega^r(\mathbb{R}^q) & , \text{ for } s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

This proves the theorem. □

For $q = 0$ this is indeed the classical Poincaré Lemma. Note that the isomophism $E_1^*(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \Omega^*(\mathbb{R}^q)$ is an isomorphism of complexes. Hence,

$$E_\infty^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) = \cdots = E_2^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \mathbb{R} & , \text{ for } r = s = 0 \\ 0 & , \text{ otherwise .} \end{cases}$$

If we want to apply the above considerations to the spectral sequences with transversally compact supports, we have to restrict the morphism sets. We call a smooth map $f : N \to M$ which maps the leaves of $\mathcal{G}$ into the leaves of $\mathcal{F}$ a *transversally proper morphism* from the foliated manifold $(N, \mathcal{G})$ to the foliated manifold $(M, \mathcal{F})$ if the induced continous map $f_* : N/\mathcal{G} \to M/\mathcal{F}$ on the leaf spaces is proper. The category $\mathcal{F}ol_{tr}$ is the category consisting of the same objects as $\mathcal{F}ol$ and the transversally proper morphisms. Then the Koszul spectral sequence with transversally compact supports is a contravariant functor $E : \mathcal{F}ol_{tr} \to \mathcal{S}pec_{\mathbb{R}}^{*,*}$, and we get functors $H : \mathcal{F}ol_{tr} \to \mathcal{A}lg_{\mathbb{R}}^{*}, (M, \mathcal{F}) \mapsto H_{tr}^{*}(\mathcal{F})$ and $H : \mathcal{F}ol_{tr}^{q} \to \mathcal{M}od_{\mathbb{R}}^{*}, (M, \mathcal{F}) \mapsto H_{tr}^{*}(M, \mathcal{F})$. To formulate the naturality property of the Koszul spectral sequence with transversally compact supports recall that the *saturation* of a subset $U \subset M$ is the union of all the leaves of $\mathcal{F}$ passing through U.

Lemma I.3.4. *If $f : N \to M$ is any open map transverse to a foliation $\mathcal{F}$ on M such that the saturation of the image of f is closed in M, then f is a transversally proper morphism from $(N, f^*\mathcal{F})$ to $(M, \mathcal{F})$.*

PROOF. By definition the induced map $f_* : N/f^*\mathcal{F} \rightarrowtail M/\mathcal{F}$ is injective. Since the saturation of an open set is again open, the induced map f_* is open because f is. Hence, every open cover of a subset A of $N/f^*\mathcal{F}$ gives rise to an open cover of f_*A. This together with the injectivity of f_* yields that A is compact if and only if f_*A is. Let $K \subset M/\mathcal{F}$ be compact. That the saturation of the image of f is closed implies that the image of f_* is closed in $M/\mathcal{F}$. Hence, $f_*(f_*^{-1}(K)) = \operatorname{im} f_* \cap K$ is also compact and so is $f_*^{-1}(K)$. □

Note that under the assumptions of the lemma, the saturation of the image of f is a union of connected components of M. In particular, all the functors $E : \mathcal{F}ol_{tr} \to \mathcal{S}pec_{\mathbb{R}}^{*,*}$, $H : \mathcal{F}ol_{tr} \to \mathcal{A}lg_{\mathbb{R}}^{*}$ and $H : \mathcal{F}ol_{tr}^{q} \to \mathcal{M}od_{\mathbb{R}}^{*}$ are natural with respect to surjective submersions. Two transversally proper morphisms f_0, f_1 are *transversally proper leafwise homotopic* if there is a leafwise homotopy F from f_0 to f_1 such that every F_t is a transversally proper morphism. Then the proof of Theorem I.3.2 goes through in the new context to yield that the functors $E : \mathcal{F}ol_{tr} \to \mathcal{S}pec_{\mathbb{R}}^{*,*}$, $H : \mathcal{F}ol_{tr} \to \mathcal{A}lg_{\mathbb{R}}^{*}$ and $H : \mathcal{F}ol_{tr}^{q} \to \mathcal{M}od_{\mathbb{R}}^{*}$ are invariant under transversally proper homotopies. The proof of Theorem I.3.3 holds as well.

Theorem I.3.5. *We have*

$$E_{1,tr}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega_c^r(\mathbb{R}^q) & , \textit{for } 0 \le r \le q \textit{ and } s = 0 \\ 0 & , \textit{otherwise} \end{cases}$$

In particular,

$$H_{tr}^{s}(\mathcal{T}^{q,p}) \cong H_{tr}^{s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega_c^0(\mathbb{R}^q) & , \textit{for } s = 0 \\ 0 & , \textit{otherwise}\ . \end{cases}$$

By the classical Poincaré Lemma for cohomology with compact supports (which is a special instance of Theorem I.3.6 below) we get

$$E_{\infty,tr}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) = \cdots = E_{2,tr}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \mathbb{R} & , \text{for } r = q \text{ and } s = 0 \\ 0 & , \text{otherwise}\ . \end{cases}$$

Let $\mathcal{F}ol_c$ denote the category of foliated manifolds and proper morphisms. Then the Koszul spectral sequence with compact supports gives contravariant functors $E : \mathcal{F}ol_c \to \mathcal{S}pec_{\mathbb{R}}^{*,*}$, $H : \mathcal{F}ol_c \to \mathcal{A}lg_{\mathbb{R}}^{*}, (M, \mathcal{F}) \mapsto H_c^{*}(\mathcal{F})$ and $H : \mathcal{F}ol_c^{q} \to \mathcal{M}od_{\mathbb{R}}^{*}, (M, \mathcal{F}) \mapsto H_c^{*}(M, \mathcal{F})$ which are natural with respect to proper submersions (such as fibre bundles with compact fibres). A *proper leafwise homotopy* is a leafwise homotopy which is proper as a map $F : N \times [0,1] \to M$. Again the proof of Theorem II.3.3 shows that those functors are invariant under proper leafwise homotopies. For example, every *leafwise diffeotopy*, i. e. every leafwise homotopy F such that each F_t is a diffeomorphism, is a proper leafwise homotopy. Alas, the leafwise homotopy used in the proof of Theorem I.3.3 is obviously not proper. But we can adopt the proof of Theorem I.3.2 to get a Poincaré Lemma for the spectral sequence with compact supports.

Theorem I.3.6. *We have*

$$E_{1,c}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega_c^r(\mathbb{R}^q) & , \text{ for } 0 \le r \le q \text{ and } s = p \\ 0 & , \text{ otherwise .} \end{cases}$$

In particular,

$$H_c^s(\mathcal{T}^{q,p}) \cong H_c^s(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega_c^0(\mathbb{R}^q) & , \text{ for } s = p \\ 0 & , \text{ otherwise .} \end{cases}$$

PROOF. Denote by $pr_1 : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}^p$, $pr_2 : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ the canonical projections. Thus, $\mathcal{T}^{q,p+1} = pr_1^* \mathcal{T}^{q,p}$. Like in the proof of Theorem I.3.2 we get a map

$$\begin{aligned} I : E_{0,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^{p+1}, \mathcal{T}^{q,p+1}) &\to E_{0,c}^{r,s-1}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p}) \\ \omega &\mapsto \int_{-\infty}^{+\infty} \iota_t^*(i_{\frac{\partial}{\partial t}} \omega)\ dt \end{aligned}$$

which is well-defined, since we are considering forms with compact support. For the same reason we have

$$Id - dI = 0\ ,$$

i. e. I is a cochain map. Let $\gamma \in \Omega_c^1(\mathbb{R})$ be a compactly supported form with $\int_{\mathbb{R}} \gamma = 1$ and set

$$\begin{aligned} D : F_{\mathcal{T}^{q,p}}^r \Omega_c^k(\mathbb{R}^q \times \mathbb{R}^p) &\to F_{\mathcal{T}^{q,p+1}}^r \Omega_c^{k+1}(\mathbb{R}^q \times \mathbb{R}^{p+1}) \\ \omega &\mapsto pr_2^* \gamma \wedge pr_1^* \omega\ . \end{aligned}$$

This induces a map

$$D : E_{0,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p}) \to E_{0,c}^{r,s+1}(\mathbb{R}^q \times \mathbb{R}^{p+1}, \mathcal{T}^{q,p+1})$$

satisfying $ID = id$. Next we show that DI is homotopic to id. Set $g(t) = \int_{-\infty}^t \gamma$ and define another operator by

$$\begin{aligned} K : F_{\mathcal{T}^{q,p+1}}^r \Omega_c^k(\mathbb{R}^q \times \mathbb{R}^{p+1}) &\to F_{\mathcal{T}^{q,p+1}}^r \Omega_c^{k-1}(\mathbb{R}^q \times \mathbb{R}^{p+1}) \\ \omega &\mapsto \int_{-\infty}^t \iota_t^*(i_{\frac{\partial}{\partial t}} \omega)\ dt - pr_2^* g \cdot \int_{-\infty}^{+\infty} \iota_t^*(i_{\frac{\partial}{\partial t}} \omega)\ dt\ . \end{aligned}$$

For $\omega = \alpha_t + dt \wedge \beta_t$ we compute

$$\begin{aligned}
K(\omega) &= \int_{-\infty}^{t} \beta_t \; dt - pr_2^* g \cdot \int_{-\infty}^{+\infty} \beta_t \; dt \; , \\
K(d\omega) &= K(dt \wedge (\dot\alpha_t - d\beta_t) + d\alpha_t) \\
&= \int_{-\infty}^{t} (\dot\alpha_t - d\beta_t) \; dt - pr_2^* g \cdot \int_{-\infty}^{+\infty} (\dot\alpha_t - d\beta_t) \; dt \\
&= \alpha_t - \int_{-\infty}^{t} (d\beta_t) \; dt + pr_2^* g \cdot \int_{-\infty}^{+\infty} (d\beta_t) \; dt \; , \\
dK(\omega) &= dt \wedge \beta_t + \int_{-\infty}^{t} (d\beta_t) \; dt - pr_2^*\gamma \wedge \int_{-\infty}^{+\infty} \beta_t \; dt - pr_2^* g \cdot \int_{-\infty}^{+\infty} (d\beta_t) \; dt \; .
\end{aligned}$$

This gives

$$(dK + Kd)(\omega) = dt \wedge \beta_t + \alpha_t - pr_2^*\gamma \wedge \int_{-\infty}^{+\infty} \beta_t \; dt \; .$$

Thus,

$$dK + Kd = id - DI \; .$$

In particular, $D_* I_* = id : E_{1,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p}) \to E_{1,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p})$. So, I induces an isomorphism

$$E_{1,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^{p+1}, \mathcal{T}^{q,p+1}) \cong E_{1,c}^{r,s-1}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p}) \; .$$

By induction we see that

$$E_{1,c}^{r,s}(\mathbb{R}^q \times \mathbb{R}^p, \mathcal{T}^{q,p}) \cong E_{1,c}^{r,s-p}(\mathbb{R}^q, \mathcal{T}^{q,0}) \cong \begin{cases} \Omega_c^r(\mathbb{R}^q) & , \text{ for } s = p \\ 0 & , \text{ otherwise .} \end{cases}$$

□

For $q = 0$ we get the classical Poincaré Lemma for de Rham cohomology with compact supports which applies to give

$$E_{\infty,c}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) = \cdots = E_{2,c}^{r,s}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \mathbb{R} & , \text{ for } r = q \text{ and } s = p \\ 0 & , \text{ otherwise .} \end{cases}$$

I.4. Mayer-Vietoris sequence and Künneth formula

To compute the cohomology of a manifold out of local information the two most important tools (besides the Poincaré Lemma) are the Mayer-Vietoris sequence and the Künneth formula. While the first one holds in full generality in the foliated case, the second one is only valid under special assumptions for foliated manifolds. Suppose that $(M, \mathcal{F})$ is a foliated manifold, $U, V \subset M$ are open and M is the union of these two sets, $M = U \cup V$. There is a short exact sequence

$$0 \to \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{p} \Omega^*(U \cap V) \to 0 \; ,$$

where $i(\omega) = (\omega|_U, \omega|_V)$ and $p(\omega,\eta) = \eta|_{U\cap V} - \omega|_{U\cap V}$. Let $\mathcal{F}_U, \mathcal{F}_V$ and $\mathcal{F}_{U\cap V}$ denote the restrictions of $\mathcal{F}$ to U, V, resp. $U \cap V$. Then i and p induce a short sequence

$$\text{(I.4.1)}\quad 0 \to F^r_{\mathcal{F}}\Omega^k(M) \xrightarrow{i} F^r_{\mathcal{F}_U}\Omega^k(U) \oplus F^r_{\mathcal{F}_V}\Omega^k(V) \xrightarrow{p} F^r_{\mathcal{F}_{U\cap V}}\Omega^k(U\cap V) \to 0\ .$$

It gives rise to a long exact Mayer-Vietoris sequence.

Theorem I.4.1. *For open subsets $U, V \subset M$ with $M = U \cup V$ there is for every r a natural long exact sequence*

$$\cdots \to E_1^{r,s}(M,\mathcal{F}) \xrightarrow{i_*} E_1^{r,s}(U,\mathcal{F}_U) \oplus E_1^{r,s}(V,\mathcal{F}_V) \xrightarrow{p_*} E_1^{r,s}(U\cap V,\mathcal{F}_{U\cap V}) \xrightarrow{\delta} \cdots$$

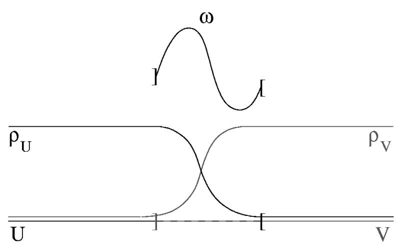
increasing in s. In particular, there are long exact sequences

$$\cdots \to H^s(\mathcal{F}) \xrightarrow{i_*} H^s(\mathcal{F}_U) \oplus H^s(\mathcal{F}_V) \xrightarrow{p_*} H^s(\mathcal{F}_{U\cap V}) \xrightarrow{\delta} \cdots$$

and

$$\cdots \to H^s(M,\mathcal{F}) \xrightarrow{i_*} H^s(U,\mathcal{F}_U) \oplus H^s(V,\mathcal{F}_V) \xrightarrow{p_*} H^s(U\cap V,\mathcal{F}_{U\cap V}) \xrightarrow{\delta} \cdots\ .$$

Proof. First we have to show is that the short sequence (I.4.1) is exact. This is obvious at the first two steps. Let $\omega \in F^r_{\mathcal{F}_{U\cap V}}\Omega^k(U\cap V)$. Choose a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the open cover $\{U, V\}$ of M.



Then $-(\rho_V)|_U \cdot \omega \in F^r_{\mathcal{F}_U}\Omega^k(U)$, $(\rho_U)|_V \cdot \omega \in F^r_{\mathcal{F}_V}\Omega^k(V)$ and

$$\omega = (\rho_U)|_{U\cap V}\cdot\omega + (\rho_V)|_{U\cap V}\cdot\omega = p(-(\rho_V)|_U\cdot\omega, (\rho_U)|_V\cdot\omega)\ .$$

So, Sequence (I.4.1) is exact. This gives a short sequence

$$\text{(I.4.2)}\quad 0 \to E_0^{r,s}(M,\mathcal{F}) \xrightarrow{i} E_0^{r,s}(U,\mathcal{F}_U) \oplus E_0^{r,s}(V,\mathcal{F}_V) \xrightarrow{p} E_0^{r,s}(U\cap V,\mathcal{F}_{U\cap V}) \to 0$$

which is obviously exact at the first and the last step. Suppose that $\omega \in F^r_{\mathcal{F}_U}\Omega^k(U)$ and $\eta \in F^r_{\mathcal{F}_V}\Omega^k(V)$ are elements with $p(\omega,\eta) \in F^{r+1}_{\mathcal{F}_{U\cap V}}\Omega^k(U\cap V)$. Set $\tau = p(\omega,\eta)$. Then the element $\bar\omega = \omega + (\rho_V)|_U\cdot\tau \in F^r_{\mathcal{F}_U}\Omega^k(U)$ represents the same element in $E_0^{r,k-r}(U,\mathcal{F}_U)$ as ω and the element $\bar\eta = \eta - (\rho_U)|_V\cdot\tau \in F^r_{\mathcal{F}_V}\Omega^k(V)$ represents the same element in $E_0^{r,k-r}(V,\mathcal{F}_V)$ as η. But

$$p(\bar\omega,\bar\eta) = \eta|_{U\cap V} - (\rho_U)|_{U\cap V}\cdot\tau - \omega|_{U\cap V} - (\rho_V)|_{U\cap V}\cdot\tau = \tau - \tau = 0\ .$$

Hence, $(\omega, \eta) = (\bar{\omega}, \bar{\eta}) \in E_0^{r,s}(U, \mathcal{F}_U) \oplus E_0^{r,s}(V, \mathcal{F}_V)$ is in the image of i. So, Sequence (I.4.2) is exact and induces a long exact sequence as stated. Note for future reference that the connecting homomorphism δ is given explicitly by

$$\delta : E_1^{r,s}(U \cap V, \mathcal{F}_{U\cap V}) \to E_1^{r,s+1}(M, \mathcal{F})$$

$$[\omega] \mapsto [\delta(\omega)],\ \delta(\omega) = \begin{cases} -(d\rho_V) \wedge \omega & , \text{ on } U \\ (d\rho_U) \wedge \omega & , \text{ on } V \ . \end{cases}$$

□

For $\mathcal{F} = \{M\}$ this becomes the classical Mayer-Vietoris sequence.

A subset of a foliated manifold is called *saturated* if it is a union of leaves. If we want to construct a Mayer-Vietoris sequence for the spectral sequence with transversally compact supports, we have to require that the subsets U and V are saturated, since we want to extend forms given on U, V in a canonical way.

Theorem I.4.2. *For saturated open sets $U, V \subset M$ with $M = U \cup V$ there is for every r a natural long exact sequence*

$$\cdots \to E_{1,tr}^{r,s}(U \cap V, \mathcal{F}_{U\cap V}) \xrightarrow{j_*} E_{1,tr}^{r,s}(U, \mathcal{F}_U) \oplus E_{1,tr}^{r,s}(V, \mathcal{F}_V) \xrightarrow{k_*} E_{1,tr}^{r,s}(M, \mathcal{F}) \xrightarrow{\delta} \dots\ .$$

Proof. Since $U \cap V$ is saturated, we can extend every form $\omega \in \Omega^*_{tr}(U \cap V)$ with transversally compact support trivially to the whole of M. Let $j : \Omega^*_{tr}(U \cap V) \to \Omega^*_{tr}(U) \oplus \Omega^*_{tr}(V)$ be this extension map followed by the restriction map i above. The map $k : \Omega^*_{tr}(U) \oplus \Omega^*_{tr}(V) \to \Omega^*_{tr}(M)$ is constructed similarily as the composition of the extension map with the difference homomorphism $(\omega, \eta) \mapsto \eta - \omega$. This gives a short exact sequence

$$0 \to F^r_{\mathcal{F}_{U\cap V}} \Omega^{r+s}_{tr}(U \cap V) \xrightarrow{j} F^r_{\mathcal{F}_U} \Omega^{r+s}_{tr}(U) \oplus F^r_{\mathcal{F}_V} \Omega^{r+s}_{tr}(V) \xrightarrow{k} F^r_{\mathcal{F}} \Omega^{r+s}_{tr}(M) \to 0$$

by a partition of unity argument analogous to the one in the last proof. Hence, we get a short sequence

$$0 \to E_{0,tr}^{r,s}(U \cap V, \mathcal{F}_{U\cap V}) \xrightarrow{j_*} E_{0,tr}^{r,s}(U, \mathcal{F}_U) \oplus E_{0,tr}^{r,s}(V, \mathcal{F}_V) \xrightarrow{k_*} E_{0,tr}^{r,s}(M, \mathcal{F}) \to 0$$

which is exact just like in the proof above. This short exact sequence induces the desired long exact sequence in the E_1-terms. □

The same proof goes through for the spectral sequence with compact supports and arbitrary open sets U, V.

Theorem I.4.3. *For open sets $U, V \subset M$ with $M = U \cup V$ there is for every r a natural long exact sequence*

$$\cdots \to E_{1,c}^{r,s}(U \cap V, \mathcal{F}_{U\cap V}) \xrightarrow{j_*} E_{1,c}^{r,s}(U, \mathcal{F}_U) \oplus E_{1,c}^{r,s}(V, \mathcal{F}_V) \xrightarrow{k_*} E_{1,c}^{r,s}(M, \mathcal{F}) \xrightarrow{\delta} \dots\ .$$

If $\mathcal{F} = \{M\}$ is the trivial foliation of codimension 0, then we get the usual Mayer-Vietoris sequence for de Rham cohomology with compact supports.

Let us now turn to the Künneth formula. Consider two foliated manifolds $(M, \mathcal{F})$, $(N, \mathcal{G})$. There is the product foliation $\mathcal{F} \times \mathcal{G}$ defined on $M \times N$ in the obvious way. The leaves of $\mathcal{F} \times \mathcal{G}$ are the Cartesian products of a leaf of $\mathcal{F}$ with a leaf of $\mathcal{G}$. This product foliation gives rise to an outer product on the Koszul spectral sequences.

Definition I.4.4. The *cross product* is the homomorphism

$$\times : E_k^{r,s}(M, \mathcal{F}) \otimes E_k^{i,j}(N, \mathcal{G}) \to E_k^{r+i,s+j}(M \times N, \mathcal{F} \times \mathcal{G})$$
$$[\omega] \otimes [\eta] \mapsto [pr_M^* \omega \wedge pr_N^* \eta] \ .$$

This is obviously well-defined, since $pr_M(\mathcal{F} \times \mathcal{G}) \subset \mathcal{F}$ and $pr_N(\mathcal{F} \times \mathcal{G}) \subset \mathcal{G}$. In particular, we get a cross product

$$\times : H^i(\mathcal{F}) \otimes H^j(\mathcal{G}) \to H^{i+j}(\mathcal{F} \times \mathcal{G}) \ ,$$

and because the codimension of $\mathcal{F} \times \mathcal{G}$ is the sum of the codimension of $\mathcal{F}$ and the codimension of $\mathcal{G}$ there is a cross product

$$\times : H^i(M, \mathcal{F}) \otimes H^j(N, \mathcal{G}) \to H^{i+j}(M \times N, \mathcal{F} \times \mathcal{G})$$

as well. Even though pr_M and pr_N are not (transversally) proper in general, it is obvious that $pr_M^* \omega \wedge pr_N^* \eta \in \Omega^*(M \times N)$ has (transversally) compact support if ω and η have, since $(M \times N)/(\mathcal{F} \times \mathcal{G}) = M/\mathcal{F} \times N/\mathcal{G}$ (as we already used in the proof of Theorem I.3.5). Hence, the cross product is also well-defined on the Koszul spectral sequence with (transversally) compact supports.

If we suppose that $\mathcal{G} = \{N\}$ is the trivial foliation of codimension 0 on N and thus $H^*(N, \mathcal{G}) = H^*(N; \mathbb{R})$, then there is the following foliated version of the Künneth formula. Recall that an open cover $\mathcal{U} = \{U_i\}$ of M is called a *good cover* if for every $U_i, U_j \in \mathcal{U}$ the intersection $U_i \cap U_j$ is diffeomorphic to $\mathbb{R}^n$. Every cover of M has a refinement which is a good cover (cf. [**4**]). A manifold M is of *finite type* if it posesses a finite good cover. For example, every compact manifold is of finite type. In the same fashion, we will say that a foliated manifold $(M, \mathcal{F})$ is of finite type, if it admits a finite good cover by the domains of foliated charts.

Theorem I.4.5. *Consider a manifold N of finite type and a foliated manifold $(M, \mathcal{F})$ of finite type. Then the cross product*

$$\times : \bigoplus_{i+j=s} E_1^{r,i}(M, \mathcal{F}) \otimes H^j(N; \mathbb{R}) \to E_1^{r,s}(M \times N, \mathcal{F} \times \{N\})$$

is an isomorphism. In particular, we have isomorphisms

$$H^*(\mathcal{F}) \otimes H^*(N; \mathbb{R}) \cong H^*(\mathcal{F} \times \{N\})$$

and

$$H^*(M, \mathcal{F}) \otimes H^*(N; \mathbb{R}) \cong H^*(M \times N, \mathcal{F} \times \{N\}) \ .$$

Proof. 1) Suppose that $M = \mathbb{R}^q \times \mathbb{R}^p$ and $\mathcal{F} = \mathcal{T}^{q,p}$ is the simple codimension q foliation on $\mathbb{R}^{q+p}$ like in Section I.3 and that $U_1, \ldots, U_l$ is a finite good cover of N. We will do an induction over l to prove the statement in this special case.

If $l = 1$, then $N = \mathbb{R}^n$. By Theorem I.3.3 we have to show that

$$\times : \Omega^r(\mathbb{R}^q) \otimes H^s(\mathbb{R}^n; \mathbb{R}) \to \begin{cases} \Omega^r(\mathbb{R}^q) & , \text{ for } s = 0 \\ 0 & , \text{ otherwise} \end{cases}$$

is isomorphic. This is obvious. Suppose that we know that the theorem is true for $M = \mathbb{R}^q \times \mathbb{R}^p$ and all manifolds N' that have a good cover of length l. If $U_1, \dots, U_{l+1}$ is a good cover of N, then let $V = U_1 \cup \dots \cup U_l$. Now, $V \cap U_{l+1}$ has a good cover of length l, namely $U_1 \cap U_{l+1}, \dots, U_l \cap U_{l+1}$. Hence, the hypothesis holds for the pairs $\{M, U_{l+1}\}$, $\{M, V\}$ and $\{M, V \cap U_{l+1}\}$. Consider the Mayer-Vietoris sequence for $N = U_{l+1} \cup V$. Then tensoring with the vector space $E_1^{r,0}(M, \mathcal{F}) = \Omega^r(\mathbb{R}^q)$ leaves the sequence exact. Hence, we get a long exact sequence

$$\begin{aligned} \dots \to\ & E_1^{r,0}(M, \mathcal{F}) \otimes H^s(N; \mathbb{R}) \\ \to\ & E_1^{r,0}(M, \mathcal{F}) \otimes H^s(U_{l+1}; \mathbb{R}) \oplus E_1^{r,0}(M, \mathcal{F}) \otimes H^s(V; \mathbb{R}) \\ \to\ & E_1^{r,0}(M, \mathcal{F}) \otimes H^s(U_{l+1} \cap V; \mathbb{R}) \to \dots \end{aligned}$$

which is mapped naturally into the Mayer-Vietoris sequence of the decomposition $M \times N = M \times U_{l+1} \cup M \times V$ by the cross product. So, by the induction hypothesis and the Five Lemma the theorem is also true for $\{M, N\}$. This proves the theorem for $M = \mathbb{R}^q \times \mathbb{R}^p$, $\mathcal{F} = \mathcal{T}^{q,p}$ and arbitrary N posessing a finite good cover.

2) Now, let $(M, \mathcal{F})$ be an arbitary foliated manifold with a finite good cover $U_1, \dots, U_l$ by the domains of foliated charts $\varphi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^p$. We do an induction over l again to prove the statement in the general case.

The case $l = 1$ has been treated in 1). The induction step goes through in the same way as in the first part, tensoring the Mayer-Vietoris sequence with the total vector space $H^*(N; \mathbb{R})$. To be more precise, we consider the long exact sequence

$$\begin{aligned} \dots \to\ & \bigoplus_{i=0}^{s} E_1^{r,i}(M, \mathcal{F}) \otimes H^{s-i}(N; \mathbb{R}) \\ \to\ & \bigoplus_{i=0}^{s} E_1^{r,i}(U_{l+1}, \mathcal{F}_{U_{l+1}}) \otimes H^{s-i}(N; \mathbb{R}) \oplus E_1^{r,i}(V, \mathcal{F}_V) \otimes H^{s-i}(N; \mathbb{R}) \\ \to\ & \bigoplus_{i=0}^{s} E_1^{r,i}(U_{l+1} \cap V, \mathcal{F}_{U_{l+1} \cap V}) \otimes H^{s-i}(N; \mathbb{R}) \to \dots \end{aligned}$$

Again, the Five Lemma gives the desired result. □

For the spectral sequence with compact supports the proof of Theorem I.4.5 goes through literally (for transversally compact supports the second part of the proof fails).

Theorem I.4.6. *Consider a manifold N of finite type and a foliated manifold $(M,\mathcal{F})$ of finite type. Then the cross product*

$$\times : \bigoplus_{i+j=s} E_{1,c}^{r,i}(M,\mathcal{F}) \otimes H_c^j(N;\mathbb{R}) \to E_{1,c}^{r,s}(M \times N, \mathcal{F} \times \{N\})$$

is an isomorphism.

One could ask wether the Künneth formula remains valid when the foliation $\mathcal{G}$ on N has positive codimension. This can not be true in general. Just consider the following example.

Proposition I.4.7. *The cross product*

$$\times : H^*(\mathcal{T}^{1,0}) \otimes H^*(\mathcal{T}^{1,0}) \to H^*(\mathcal{T}^{2,0})$$

is injective but not surjective.

PROOF. Denote by m the canonical homomorphism

$$m : \Omega^0(\mathbb{R}) \otimes \Omega^0(\mathbb{R}) \to \Omega^0(\mathbb{R} \times \mathbb{R}) \ ,$$

defined by $m(f \otimes g)(x,y) = f(x) \cdot g(y)$. By Example I.2.3 we have to show that m is injective but not surjective. Let $\{e_i\}$ be a basis for $\Omega^0(\mathbb{R})$. If

$$m\left(\sum_{i,j\in I} a_{ij} e_i \otimes e_j\right) = 0$$

for some finite index subset I, then for every $x \in \mathbb{R}$

$$\sum_{i,j\in I} a_{ij} e_i(x) \cdot e_j = 0 \ .$$

Since the $\{e_j\}$ are linearily independent we get

$$\sum_{i\in I} a_{ij} e_i(x) = 0$$

for all $j \in I$ and all $x \in \mathbb{R}$. Thus, $a_{ij} = 0$ for all $i,j \in I$ by the linear independence of the $\{e_i\}$. So, m is injective. Now, let $p_n(y) = y^n$. Extend the linearly independent family $\{p_n\}$ to a basis $\{e_i\}$ of $\Omega^0(\mathbb{R})$. Denote by $\{e_i^*\}$ the dual linear forms defined by $e_i^* e_j = \delta_{ij}$. For every $f \in \Omega^0(\mathbb{R}^2)$ and every $x \in \mathbb{R}$ define the support of f at x to be

$$\operatorname{supp}_x f = \{e_i |\ e_i^*(f(x,-)) \neq 0\} \ .$$

If f is in the image of m, then there is a constant N, namely the cardinality of the finite index subset, such that for every $x \in \mathbb{R}$ the support of f at x satisfies $|\operatorname{supp}_x f| \leq N$. Choose a function $\lambda \in \Omega^0(\mathbb{R})$ such that λ is equal to one in a neighbourhood of $\mathbb{Z}$ and vanishes on a neighbourhood of $\frac{1}{2}\mathbb{Z}$. Then

$$f(x,y) = \lambda(x)\left(1 + y + y^2 + \cdots + y^{[x+\frac{1}{2}]}\right)$$

is a smooth function on $\mathbb{R}^2$. But

$$p_n^* f(i,-) = \lambda(i) = 1$$

for $n = 0, \ldots, i$. In particular, $|\operatorname{supp}_i F| \geq i$ for all i. This means that f can not be in the image of m. Hence, m is not surjective. ☐

Even though the Künneth formula does not hold in general, there is the following injectivity statement which we will need in Chapter II. Recall from Section I.2 that there is a canonical homomorphism

$$\epsilon : H^j(N, \mathcal{G}) \to H^{q+j}(N; \mathbb{R}) \ .$$

for every codimension q foliation $\mathcal{G}$ on a manifold N.

Corollary I.4.8. *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be foliated manifolds of finite type. Suppose that $b \in H^j(N, \mathcal{G})$ with $\epsilon(b) \neq 0$. Then the homomorphism*

$$\begin{aligned} R_b : H^i(M, \mathcal{F}) &\to H^{i+j}(M \times N, \mathcal{F} \times \mathcal{G}) \\ a &\mapsto a \times b \end{aligned}$$

is injective.

Proof. Suppose that $\epsilon(b) \neq 0$ and $R_b(a) = 0$. There is a commuting diagram

$$\begin{array}{ccc} H^i(M, \mathcal{F}) \otimes H^j(N, \mathcal{G}) & \xrightarrow{\times} & H^{i+j}(M \times N, \mathcal{F} \times \mathcal{G}) \\ {\scriptstyle id \otimes \epsilon} \downarrow & & \downarrow \\ H^i(M, \mathcal{F}) \otimes H^{q+j}(N; \mathbb{R}) & \xrightarrow{\times} & H^{i+q+j}(M \times N, \mathcal{F} \times \{N\}) \end{array}$$

where the map on the right hand side is induced by the identity (as will be explained in Chapter II). Hence, the assumption $a \times b = 0$ and Theorem I.4.5 imply $a \otimes \epsilon(b) = 0$. If $a \neq 0$, then there is a linear map $a^* : H^i(M, \mathcal{F}) \to \mathbb{R}$ with $a^* a = 1$. This gives

$$\epsilon(b) = a^* a \cdot \epsilon(b) = (a^* \otimes id)(a \otimes \epsilon(b)) = 0 \ .$$

This is a contradiction. So, $a = 0$. ☐

I.5. The sheaf of basic forms and the cohomology of fibre bundles

The Koszul spectral sequence can be interpreted as the space of global sections in a spectral sequence of sheaves. We will use this to compute the Koszul spectral sequence for simple foliations given by the fibres of a fibre bundle which will be essential in the construction of derived classes for families of foliations (for the basic facts about sheaves and sheaf cohomology see [**5**]).

Let us write

$$\Omega^s(\mathcal{F}; \Lambda^r Q^*) = \Gamma(\Lambda^s T^* \mathcal{F} \otimes \Lambda^r Q^*) \ ,$$

where Q is the normal bundle of $\mathcal{F}$. By identifying Q with a complement of $T\mathcal{F}$ in TM, we get

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*\mathcal{F} \oplus Q^*)) = \Gamma\left(\bigoplus_{r+s=k} \Lambda^s T^*\mathcal{F} \otimes \Lambda^r Q^*\right) = \bigoplus_{r+s=k} \Omega^s(\mathcal{F}; \Lambda^r Q^*) \ .$$

Moreover, by Lemma I.2.2 we know that

$$F^r\Omega^{r+s}(M) = \bigoplus_{j=r}^{r+s} \Omega^{r+s-j}(\mathcal{F}; \Lambda^j Q^*) \ .$$

Hence,

$$E_0^{r,s}(M, \mathcal{F}) = \Omega^s(\mathcal{F}; \Lambda^r Q^*) \ .$$

So, d_0 induces a differential on $\Omega^*(\mathcal{F}; \Lambda^r Q^*)$. Therefore, the E_1-term of the Koszul spectral sequence can be interpreted as the leafwise cohomology with coefficients in the exterior powers of the normal bundle,

$$H^s(\mathcal{F}; \Lambda^r Q^*) = E_1^{r,s}(M, \mathcal{F}) \ .$$

In particular, we recover the basic forms on M as $H^0(\mathcal{F}; \Lambda^r Q^*) = \Omega^r(M/\mathcal{F})$.

Denote by A_M^* the de Rham presheaf which assigns to every open subset $U \subset M$ the complex of differential forms on U,

$$A_M^*(U) = \Omega^*(U) \ .$$

It generates the differential graded sheaf $\mathcal{A}_M^*$ of germs of differential forms on M. To abbreviate the notations we will write A_M for the ring of smooth functions on M and we will denote by $\mathcal{A}_M$ the sheaf of rings $\mathcal{A}_M^0$. It is natural to define the *presheaf* $B_\mathcal{F}^*$ *of basic forms* on M by

$$B_\mathcal{F}^*(U) = \begin{cases} f_i^*\Omega^*(\mathbb{R}^q) & , \text{ if } \varphi_i : U \to \mathbb{R}^q \times \mathbb{R}^p \text{ is a foliated chart} \\ \Omega^*(U) & , \text{ otherwise ,} \end{cases}$$

where $f_i = pr_1 \circ \varphi$ is the local submersion defined by φ_i. This definition is obviously independent of the choice of the chart φ_i. The sheaf generated by the presheaf $B_\mathcal{F}^*$ is the *sheaf* $\mathcal{B}_\mathcal{F}^*$ *of germs of basic forms* on M. Finally, define a presheaf $A_\mathcal{F}^{r,s}$ by

$$A_\mathcal{F}^{r,s}(U) = \Omega^s(\mathcal{F}_U; \Lambda^r(Q|_U)^*) = E_0^{r,s}(U, \mathcal{F}_U) \ .$$

It generates a sheaf $\mathcal{A}_\mathcal{F}^{r,s}$. Since $A_\mathcal{F}^{r,s}$ is a presheaf of sections of a vector bundle, the sheaf $\mathcal{A}_\mathcal{F}^{r,s}$ is fine. Furthermore, Theorem I.3.3 proves that the natural injection $j : \mathcal{B}_\mathcal{F}^r \rightarrowtail \mathcal{A}^{r,0}$ gives rise to a fine resolution

$$0 \to \mathcal{B}_\mathcal{F}^r \xrightarrow{j} \mathcal{A}_\mathcal{F}^{r,0} \xrightarrow{d_0} \mathcal{A}_\mathcal{F}^{r,1} \xrightarrow{d_0} \dots$$

of the sheaf of germs of basic forms. Hence, the E_1-term of the Koszul spectral sequence is canonically isomorphic to the cohomology of M with coefficients in the sheaf of germs of basic forms,

$$H^s(M; \mathcal{B}_\mathcal{F}^r) = H^s(\Gamma\mathcal{A}_\mathcal{F}^{r,*}) = H^s(\mathcal{F}; \Lambda^r Q^*) = E_1^{r,s}(M, \mathcal{F}) \ .$$

In particular, $\Omega^r(M/\mathcal{F}) = H^0(\mathcal{F};\Lambda^r Q^*) = H^0(M;\mathcal{B}^r_{\mathcal{F}}) = \Gamma\mathcal{B}^r_{\mathcal{F}}$, further justifying the notion of basic forms given in Section I.2. Let us collect these facts in the following theorem.

Theorem I.5.1. *For every foliation $\mathcal{F}$ on M we have isomorphisms*

$$E_0^{r,s}(M,\mathcal{F}) \cong \Omega^s(\mathcal{F};\Lambda^r Q^*)$$

and

$$E_1^{r,s}(M,\mathcal{F}) \cong H^s(\mathcal{F};\Lambda^r Q^*) \cong H^s(M;\mathcal{B}^r_{\mathcal{F}}) \ .$$

As an example, we will compute the spectral sequence for fibre bundles. Let $\pi : M \to B$ be a smooth fibre bundle with fibre F and consider the simple foliation $\mathcal{F}$ on M given by the fibres of π. The presheaf $U \mapsto H^*(\pi^{-1}(U);\mathbb{R})$ generates a locally constant sheaf, i. e. a local coefficient system. It can be retopologized as the flat vector bundle $\mathcal{H}^*(F;\mathbb{R})$ with fibre $H^*(F;\mathbb{R})$ associated to the monodromy homomorphism $\rho : \pi_1(B) \to \mathrm{Diff}(F)/\mathrm{Diff}_0(F)$. Let us write $\Omega^r(B;\mathcal{H}^s(F;\mathbb{R})) = \Gamma(\Lambda^r T^*B \otimes \mathcal{H}^s(F;\mathbb{R}))$.

Theorem I.5.2. *Suppose that B and F are manifolds of finite type. Let $\mathcal{F}$ be the simple foliation on M given by the fibres of a smooth fibre bundle $\pi : M \to B$ with fibre F. Then*

$$E_1^{r,s}(M,\mathcal{F}) \cong \Omega^r(B;\mathcal{H}^s(F;\mathbb{R})) \ .$$

In particular,

$$H^s(\mathcal{F}) \cong \Gamma\mathcal{H}^s(F;\mathbb{R}) \ .$$

Proof. Since $Q \cong \pi^*TB$, we have an isomorphism of $\mathcal{A}_M$-modules $\mathcal{A}^{r,0}_{\mathcal{F}} \cong (\pi^*\mathcal{A}^r_B) \otimes_{\pi^*\mathcal{A}_B} \mathcal{A}_M$. Hence,

$$\mathcal{A}^{r,s}_{\mathcal{F}} \cong \mathcal{A}^{r,0}_{\mathcal{F}} \otimes_{\mathcal{A}_M} \mathcal{A}^{0,s}_{\mathcal{F}} \cong (\pi^*\mathcal{A}^r_B) \otimes_{\pi^*\mathcal{A}_B} \mathcal{A}^{0,s}_{\mathcal{F}}$$

and

$$\pi_*\mathcal{A}^{r,s}_{\mathcal{F}} \cong \mathcal{A}^r_B \otimes_{\mathcal{A}_B} \pi_*\mathcal{A}^{0,s}_{\mathcal{F}}$$

because there is an isomorphism $(\pi_*\pi^*\mathcal{A}^r_B) \otimes_{\pi_*\pi^*\mathcal{A}_B} \mathcal{A}_B \cong \mathcal{A}^r_B$ of $\mathcal{A}_B$-modules. This gives

$$\begin{aligned} E_0^{r,s}(M,\mathcal{F}) &\cong \Gamma\mathcal{A}^{r,s}_F \cong \Gamma(\pi_*\mathcal{A}^{r,s}_{\mathcal{F}}) \cong \Gamma(\mathcal{A}^r_B \otimes_{\mathcal{A}_B} \pi_*\mathcal{A}^{0,s}_{\mathcal{F}}) \\ &= \Omega^r(B) \otimes_{A_B} \Gamma(\pi_*\mathcal{A}^{0,s}_{\mathcal{F}}) \\ &= \Omega^r(B) \otimes_{A_B} E_0^{0,s}(M;\mathcal{F}) \ . \end{aligned}$$

Since the differential df of a basic function f is in $F^1\Omega^1(M)$ we get

$$E_1^{r,s}(M,\mathcal{F}) \cong \Omega^r(B) \otimes_{A_B} E_1^{0,s}(M,\mathcal{F}) \ .$$

So, the general fact will follow from the special instance $E_1^{0,s}(M,\mathcal{F}) \cong \Gamma\mathcal{H}^s(F;\mathbb{R})$. Denote by $\mathcal{H}^s_F$ the fine sheaf of germs of sections of the flat vector bundle $\mathcal{H}^s(F;\mathbb{R})$, and let $\mathcal{H}^s_{\mathcal{F}}$ be the sheaf on B generated by the presheaf $H^s_{\mathcal{F}}$ given by $U \mapsto E_1^{0,s}(\pi^{-1}U, \mathcal{F}_{\pi^{-1}U})$. If $U \subset B$ is a trivializing open neighbourhood of the bundle $\pi : M \to B$ then

$$\begin{aligned}\mathcal{H}_F^s(U) = \Gamma(\mathcal{H}^s(F;\mathbb{R})|_U) \cong \Gamma(U \times H^s(F;\mathbb{R})) = A_U \otimes H^s(F;\mathbb{R}) \\ \cong E_1^{0,s}(\pi^{-1}U, \mathcal{F}_{\pi^{-1}U}) = H_{\mathcal{F}}^s(U)\end{aligned}$$

by Theorem I.4.5. So, $\mathcal{H}_F^s \cong \mathcal{H}_{\mathcal{F}}^s$. Since B is of finite type, a Mayer-Vietoris argument similar to the one used in the proof of Theorem I.4.5 shows that the presheaf $H_{\mathcal{F}}^s$ is already a sheaf. Hence,

$$\Gamma\mathcal{H}^s(F;\mathbb{R}) = \Gamma\mathcal{H}_F^s \cong \Gamma\mathcal{H}_{\mathcal{F}}^s \cong E_1^{0,s}(M, \mathcal{F}) \ .$$

□

As an instant corollary we get the Leray-Serre Theorem.

Corollary I.5.3 (Leray-Serre)**.** *Suppose that B and F are manifolds of finite type. For every smooth fibre bundle $\pi : M \to B$ with fibre F there is a spectral sequence with E_2-term*

$$E_2^{r,s} = H^r(B; \mathcal{H}^s(F;\mathbb{R}))$$

converging to the de Rham cohomology $H^*(M;\mathbb{R})$ of the total space.

I.6. Integration along the fibre and Thom isomorphism

The integration along the fibre which was already used (in its simplest form) in Section I.3 will now be discussed in greater detail. This is necessary for the residue theorem in Chapter IV. Suppose that $\pi : M \to B$ is a fibre bundle with fibre F and that the base space B carries a foliation $\mathcal{F}$. We will need another compactness condition. Let $\Omega_{vc}^*(M)$ denote the differential forms with *vertically compact supports*, i. e. the forms $\omega \in \Omega^*(M)$ such that the intersection of the support of ω with each fibre is compact. We get a Koszul spectral sequence $E_{k,vc}^{r,s}(M, \pi^*\mathcal{F})$ and cohomology rings, resp. modules $H_{vc}^*(\pi^*\mathcal{F}), H_{vc}^*(M, \pi^*\mathcal{F})$ as before. The proof of Theorem I.4.1 goes through to yield the following Mayer-Vietoris sequence.

Theorem I.6.1. *Let $\pi : M \to B$ be a smooth fibre bundle over a foliated manifold $(B, \mathcal{F})$. For open subsets $U, V \subset B$ with $B = U \cup V$ there is for every r a natural long exact sequence*

$$\begin{aligned}\cdots \to E_{1,vc}^{r,s}(M, \pi^*\mathcal{F}) \xrightarrow{i_*} E_{1,vc}^{r,s}(M|_U, \pi^*\mathcal{F}_U) \oplus E_{1,vc}^{r,s}(M|_V, \pi^*\mathcal{F}_V) \to \\ \xrightarrow{p_*} E_{1,vc}^{r,s}(M|_{U\cap V}, \pi^*\mathcal{F}_{U\cap V}) \xrightarrow{\delta} \cdots\end{aligned}$$

increasing in s.

Recall that the connecting homomorphism δ is defined by

$$\delta[\omega] = [\pi^*(\rho_V d\rho_U) \wedge \omega - \pi^*(\rho_U d\rho_V) \wedge \omega] \qquad \text{(I.6.1)}$$

where $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to $\{U, V\}$. The proof of Theorem I.3.6 can be carried over literally to get the Poincaré Lemma for cohomology with vertically compact supports.

Theorem I.6.2. *Let $\pi : \mathbb{R}^{q+p} \times \mathbb{R}^k \to \mathbb{R}^{q+p}$ be the canonical projection. Then $\pi^*\mathcal{T}^{q,p} = \mathcal{T}^{q,p+k}$. We have*

$$E_{1,vc}^{r,s}(\mathbb{R}^{q+p+k}, \mathcal{T}^{q,p+k}) \cong E_1^{r,s-k}(\mathbb{R}^{q+p}, \mathcal{T}^{q,p}) \cong \begin{cases} \Omega^r(\mathbb{R}^q) & , \textit{ for } s = k \\ 0 & , \textit{ otherwise }. \end{cases}$$

This theorem gives the local model for the following isomorphism called the *integration along the fibre* of a vector bundle.

Theorem I.6.3. *For every oriented k-dimensional vector bundle $\pi : E \to B$ over a foliated manifold $(B, \mathcal{F})$ of finite type there is an isomorphism*

$$\int_\pi : E_{1,vc}^{r,k+s}(E, \pi^*\mathcal{F}) \xrightarrow{\cong} E_1^{r,s}(B, \mathcal{F}) \ .$$

In particular,

$$H_{vc}^{k+s}(\pi^*\mathcal{F}) \cong H^s(\mathcal{F}) \ ,$$
$$H_{vc}^{k+s}(E, \pi^*\mathcal{F}) \cong H^s(B, \mathcal{F}) \ .$$

Proof. We want to define a homomorphism

$$\int_\pi : F_{\pi^*\mathcal{F}}^r \Omega_{vc}^{k+r+s}(E) \to F_{\mathcal{F}}^r \Omega^{r+s}(B) \ . \tag{I.6.2}$$

1) First, we will do this locally. So, suppose that $\pi : E = \mathbb{R}^{q+p} \times \mathbb{R}^k \to B = \mathbb{R}^{q+p}$ is trivial and $\mathcal{F} = \mathcal{T}^{q,p}$. Then $\pi^*\mathcal{F}$ is given by the fibres of the projection

$$\mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^q$$
$$(y_1, \dots, y_q, x_1, \dots, x_p, t_1, \dots, t_k) \mapsto (y_1, \dots, y_q) \ .$$

Every element of $\Omega_{vc}^{k+r}(E)$ can be written uniquely as

$$\omega = \sum_{I,J} a_{I,J}(y,x,t) dy_I \wedge dx_J \wedge dt_1 \wedge \dots \wedge dt_k + \omega' \ , \tag{I.6.3}$$

where the $a_{I,J}$ are functions on E and ω' is an element of $F_{\pi^*\mathcal{F}}^{r+1}\Omega_{vc}^{k+r}(E)$ containing less than k of the dt_i's. Define

$$\int_{\mathbb{R}^k} \omega = \sum_{I,J} \left(\int_{\mathbb{R}^k} a_{I,J}(y,x,t) dt_1 \wedge \dots \wedge dt_k \right) dy_I \wedge dx_J \ .$$

Note, that if ω is in $F_{\pi^*\mathcal{F}}^r \Omega_{vc}^{k+r+s}(E)$, then it contains at least r of the dy_i's and so does $\int_{\mathbb{R}^k} \omega$. Hence $\int_{\mathbb{R}^k} \omega$ is in $F_{\mathcal{F}}^r \Omega^{r+s}(B)$.

2) Now, back to the general case, choose a cover $\{U_i\}$ of B, such that the U_i's are the domains of foliated charts $\varphi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^p$. Choose orientation

preserving trivialisations $\vartheta_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ and a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$. Set

$$\psi_i = (\varphi_i \times id)\vartheta_i : \pi^{-1}(U_i) \to \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^k \ .$$

For $\omega \in F^r_{\pi^*\mathcal{F}}\Omega^{k+r+s}_{vc}(E)$ the forms $\omega_i = (\pi^*\rho_i)\omega$ have support in $\pi^{-1}(U_i)$ and $\omega = \sum_i \omega_i$. Hence, we can use 1) to define

$$\int_\pi \omega = \sum_i \varphi_i^* \left(\int_{\mathbb{R}^k} (\psi_i^{-1})^* \omega_i \right)$$

This gives the homomorphism (I.6.2).

3) Next, we have to show that this homomorphism does not depend on the choices made during the construction. Replace the foliated charts φ_i by foliated charts φ_i' defined on the same domains U_i. With $\psi_i' = (\varphi_i' \times id)\vartheta_i$ and $T = \varphi_i'\varphi_i^{-1}$, we get $\varphi_i' = T\varphi_i$ and $\psi_i' = (T \times id)\psi_i$. Hence,

$$\begin{aligned}
\sum_i (\varphi_i')^* \left(\int_{\mathbb{R}^k} (\psi_i'^{-1})^* \omega_i \right) &= \sum_i (T\varphi_i)^* \left(\int_{\mathbb{R}^k} (T^{-1} \times id)^* (\psi_i^{-1})^* \omega_i \right) \\
&= \sum_i (T\varphi_i)^* (T^{-1})^* \left(\int_{\mathbb{R}^k} (\psi_i^{-1})^* \omega_i \right) \\
&= \sum_i \varphi_i^* \left(\int_{\mathbb{R}^k} (\psi_i^{-1})^* \omega_i \right) \ .
\end{aligned}$$

So, $\int_\pi$ does not depend on the choice of foliated charts. If we replace the trivialisations ϑ_i by different orientation preserving trivialisations $\vartheta_i' : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$, then $T = \vartheta_i'\vartheta_i^{-1}$ preserves the orientation on each fibre. Thus, the integral is left invariant by T and $\int_\pi$ does not depend on the ϑ_i's. If ρ_i' is another partition of unity subordinate to $\{U_i\}$, then the change from $(\pi^*\rho_i)\omega$ to $(\pi^*\rho_i')\omega$ obviously does not inflict the integral. By passing from another cover $\{U_i'\}$ to the common refinement $\{U_i' \cap U_j\}$ we see that $\int_\pi$ does not depend on the choice of open cover. Hence, $\int_\pi$ is independent of all choices made in the construction.

4) The homomorphism (I.6.2) commutes with d. If the term ω' in the decomposition (I.6.3) is zero, then this is obvious. Moreover, partial integration shows that $\int_\pi d\omega' = 0 = d\int_\pi \omega'$ as well. Hence, there is an induced homomorphism

$$\text{(I.6.4)} \qquad \int_\pi : E^{r,k+s}_{1,vc}(E, \pi^*\mathcal{F}) \to E^{r,s}_1(B, \mathcal{F}) \ .$$

5) Finally, we see that (I.6.4) is indeed an isomorphism by a Mayer-Vietoris argument. Let U and V be trivialising foliated neighbourhoods. Then $\int_\pi$ induces

homomorphisms between the associated Mayer-Vietrois sequences,

$$\begin{array}{ccccccc} \cdots \to & E_{1,vc}^{r,k+s}(E|_{U\cap V},\pi^*\mathcal{F}_{U\cap V}) & \xrightarrow{\delta} & E_{1,vc}^{r,k+s+1}(E|_{U\cup V},\pi^*\mathcal{F}_{U\cup V}) & \to \cdots \\ & \Big\downarrow{\scriptstyle\int_\pi} & & \Big\downarrow{\scriptstyle\int_\pi} & \\ \cdots \to & E_1^{r,s}(U\cap V,\mathcal{F}_{U\cap V}) & \xrightarrow{\delta} & E_1^{r,s+1}(U\cup V,\mathcal{F}_{U\cup V}) & \to \cdots \ . \end{array}$$

For this we only have to check that the square above commutes. But the explicit formula (I.6.1) for the connecting homomorphism and Lemma I.6.4 below give

$$\begin{aligned} \int_\pi \delta\omega &= \int_\pi \pi^*(\rho_U d\rho_U)\wedge\omega - \int_\pi \pi^*(\rho_V d\rho_V)\wedge\omega \\ &= \rho_U d\rho_U \wedge \int_\pi \omega - \rho_V d\rho_V \wedge \int_\pi \omega \\ &= \delta\int_\pi \omega \ . \end{aligned}$$

So, by induction over the length of a finite good cover of B we can reduce ourselves to the case where E is a trivial bundle over $\mathbb{R}^q\times\mathbb{R}^p$, and the theorem follows by Theorem I.6.2. □

The vector space $H_{vc}^*(E,\pi^*\mathcal{F})$ is a left module over the algebra $H^*(\mathcal{F})$ thanks to the homomorphism $\pi^* : H^*(\mathcal{F}) \to H^*(\pi^*\mathcal{F})$ and the fact that the product of a form with vertically compact support with any other form has again vertically compact support. The integration along the fibre is a $H^*(\mathcal{F})$-module homomorphism. More generally, the following statement is true.

Lemma I.6.4. *For $b \in E_1^{i,j}(B,\mathcal{F})$ and $a \in E_{1,vc}^{r,s}(E,\pi^*\mathcal{F})$ we have*

$$\int_\pi \big((\pi^*b)\cup a\big) = b\cup\int_\pi a \ .$$

Proof. To prove the lemma it suffices to check this equality on the cochain level. Moreover, it is enough to do this locally, i. e. we may assume that E is the trivial bundle over $\mathbb{R}^q\times\mathbb{R}^p$. In that case the statement is obvious from the construction of $\int_\pi$. □

The *Thom isomorphism* of π is defined to be the inverse of $\int_\pi$,

$$\Phi : E_1^{r,s}(B,\mathcal{F}) \to E_{1,vc}^{r,k+s}(E,\pi^*\mathcal{F}) \ .$$

The function on B taking constantly value 1 defines a class $1\in H^0(\mathcal{F})$. Applying Φ gives a class $u=\Phi(1)\in H_{vc}^k(\pi^*\mathcal{F})$. Lemma I.6.4 shows that

$$\int_\pi \Phi(a) = a = a\cup\int_\pi u = \int_\pi((\pi^*a)\cup u) \ .$$

Hence, Φ is just right multiplication by the *Thom class* u,

$$\Phi(a) = (\pi^*a)\cup u \ .$$

Of course, integration along the fibre is possible not only on vector bundles. Consider an arbitrary oriented manifold F with boundary ∂F and an oriented fibre bundle $\pi : M \to B$ with fibre F. The same construction as above yields the following theorem which we will only use in the special case that M is the disc bundle of an oriented vector bundle $E \to B$ with respect to some metric and ∂M is the associated sphere bundle.

Theorem I.6.5. *Consider an oriented fibre bundle $\pi : M \to B$ with fibre F, an oriented k-manifold with boundary ∂F, over a manifold B without boundary. For every foliation $\mathcal{F}$ on B there are homomorphisms*

$$\int_\pi : E_{0,vc}^{r,k+s}(M, \pi^*\mathcal{F}) \to E_0^{r,s}(B, \mathcal{F}) \ ,$$
$$\oint_\pi : E_{0,vc}^{r,k-1+s}(\partial M, \pi^*\mathcal{F}) \to E_0^{r,s}(B, \mathcal{F}) \ .$$

defined by integration along the fibres. Furthermore, we have the equation

$$\int_\pi d_0 - d_0 \int_\pi = \oint_\pi i^* \ , \tag{I.6.5}$$

where $i : \partial M \hookrightarrow M$ is the inclusion map. In particular, if $\partial F = \varnothing$, then there is an induced homomorphism

$$\int_\pi : E_{1,vc}^{r,k+s}(M, \pi^*\mathcal{F}) \to E_1^{r,s}(B, \mathcal{F}) \ .$$

Proof. Let us review the proof of Theorem I.6.3 to see how it has to be adapted to the case with boundary.

1) The space $\mathbb{R}^k$ has to be replaced by the upper half space

$$\mathbb{H}^k = \{(t_1, \dots, t_k)|\ t_1 \geq 0\} \subset \mathbb{R}^k \ ,$$

and we define

$$\int_{\mathbb{H}^k} \omega = \sum_{I,J} \left(\int_{\mathbb{H}^k} a_{I,J}(y, x, t) dt_1 \wedge \cdots \wedge dt_k \right) dy_I \wedge dx_J \ .$$

2) In addition to the trivializations

$$\psi_i : \pi^{-1}(U_i) \to \mathbb{R}^q \times \mathbb{R}^p \times F$$

we have to choose orientation preserving local charts $\{V_j\}$ for F,

$$\Phi_j : V_j \subset F \xrightarrow{\cong} \begin{cases} \mathbb{H}^k & , \text{ if } V_j \text{ meets the boundary} \\ \mathbb{R}^k & , \text{ otherwise} \ , \end{cases}$$

to get diffeomorphisms

$$\psi_{ij} = (id \times id \times \Phi_j)\psi_i : W_{ij} \to \begin{cases} \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{H}^k & , \text{ if } W_{ij} \text{ meets the boundary} \\ \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^k & , \text{ otherwise} \ , \end{cases}$$

where $W_{ij} = \psi_i^{-1}(\mathbb{R}^q \times \mathbb{R}^p \times V_j) \subset M$. By choosing a partition of unity subordinate to $\{V_j\}$ we can decompose every $\omega \in F^r_{\pi^*\mathcal{F}}\Omega^{k+r+s}(M)$ into a sum $\omega = \sum \omega_{ij}$ of forms ω_{ij} having support in W_{ij}. Setting

$$\int_\pi \omega = \sum_{i,j} \varphi_i^* \left(\int_{\mathbb{H}^k, \text{resp.} \mathbb{R}^k} (\psi_{ij}^{-1})^* \omega_{ij} \right)$$

yields a homomorphism

$$\int_\pi : F^r_{\pi^*\mathcal{F}}\Omega^{k+r+s}_{vc}(M) \to F^r_{\mathcal{F}}\Omega^{r+s}(B) \ .$$

3) We have to check what happens if we change the partition of unity subordinate to $\{V_j\}$ or the charts Φ_j. This is just the well-definedness of the integral over F of k-forms on the fibre. Hence, $\int_\pi$ does not depend on any choices.

4') What remains to be shown is Formula (I.6.5). We only have to check it locally. If $\operatorname{supp} \omega_{ij} \subset W_{ij}$ does not meet the boundary, then $\int_\pi d\omega_{ij} = d \int_\pi \omega_{ij}$ just like in 4). If $W_{ij} \cong \mathbb{H}^k$ and ω_{ij} has the form $\sum_{I,J} a_{IJ} dt_2 \wedge \cdots \wedge dt_k \wedge dy_I \wedge dx_J$, then Fubini's Theorem gives $\int_\pi d\omega_{ij} = \oint_\pi i^* \omega_{ij}$. In the other cases both sides of the equation vanish. Hence, Formula (I.6.5) is satisfied. □

If B equals a single point, then this is just the classical Stokes Theorem. On the other hand, if $\partial F = \varnothing$ and $\operatorname{codim} \mathcal{F} = 0$, then we get the usual integration along the fibre. Note that the same proof goes through for compact supports to yield a homomorphism

$$\int_\pi : E_{0,c}^{r,k+s}(M, \pi^*\mathcal{F}) \to E_{0,c}^{r,s}(B, \mathcal{F})$$

which induces an isomorphism

$$\int_\pi : E_{1,c}^{r,k+s}(E, \pi^*\mathcal{F}) \xrightarrow{\cong} E_{1,c}^{r,s}(B, \mathcal{F})$$

for every oriented vector bundle $\pi : E \to B$ of rank k.

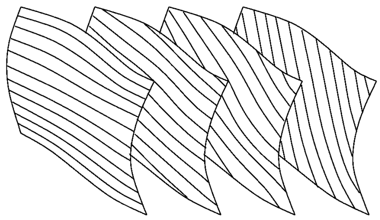
CHAPTER II

Derived classes for flags of foliations

II.1. Construction of the derived classes

In this chapter we will turn our attention to the interplay between the spectral sequences of two (or more) foliations on M, where the leaves of one foliation are contained in the leaves of the other. This originates in the examination of one-parameter families of foliations $\mathcal{F}_t$ on M, where every leaf of the bigger foliation is a time slice $M = \{t\} \times M$ of the cylinder $\mathbb{R} \times M$ which contains the leaves of the foliation $\mathcal{F}_t$ at time t. This crucial example will be discussed in detail in Section II.4.

Definition II.1.1. A *subfoliation* $(\mathcal{F}_2, \mathcal{F}_1)$ is a pair of smooth foliations on M such that every leaf of the foliation $\mathcal{F}_1$ is contained in a leaf of the foliation $\mathcal{F}_2$. A *k-flag of foliations* on M is a sequence $(\mathcal{F}_k, \mathcal{F}_{k-1}, \dots, \mathcal{F}_1)$ of smooth foliations on M such that every leaf of $\mathcal{F}_{i+1}$ is saturated with respect to $\mathcal{F}_i$ for $i = 1, \dots, k-1$.



Hence, a subfoliation is just a two-flag of foliations and a k-flag of foliations is a sequence in which each pair of succeeding foliations constitutes a subfoliation. We think of the top foliation as a parameterization for the subordinate ones. The codimension of $\mathcal{F}_i$ in $\mathcal{F}_{i+1}$ will always be denoted by q_i. Thus, the codimension of $\mathcal{F}_i$ in M is equal to $q_i + q_{i+1} + \dots + q_k$. Let $\mathcal{F}ol(k)$ denote the category of k-flags of foliations. The objects of $\mathcal{F}ol(k)$ are manifolds M together with a k-flag of foliations on M and the morphisms from a k-flag $(\mathcal{G}_k, \dots, \mathcal{G}_1)$ on N to a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on M are the smooth maps $f : N \to M$ mapping the leaves of $\mathcal{G}_i$ into the leaves of $\mathcal{F}_i$ for every $1 \le i \le k$. So, $\mathcal{F}ol(0)$ is the category of smooth manifolds, $\mathcal{F}ol(1) = \mathcal{F}ol$ is the category of foliations defined in Section I.3 of the previous chapter, $\mathcal{F}ol(2)$ is the category of subfoliations and so on. Obviously, we have a functorial inclusion map $\mathcal{F}ol(k) \to \mathcal{F}ol(k+1)$ sending a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on M to the $(k+1)$-flag $(\{M\}, \mathcal{F}_k, \dots, \mathcal{F}_1)$. Note, that there is the "forgetful operator" $\mathcal{F}ol(k+1) \to \mathcal{F}ol(k)$ sending $(\mathcal{F}_{k+1}, \dots, \mathcal{F}_1)$ to $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ as well.

But this operator is not natural with respect to the structures we are interested in (cf. Chapter III).

If $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is a k-flag of foliations on M, then each $\mathcal{F}_i$ gives rise to a Koszul filtration $F^r_{\mathcal{F}_i}\Omega^*(M)$ on $\Omega^*(M)$. Obviously, the inclusion map $F^r_{\mathcal{F}_{i+1}}\Omega^*(M) \subset F^r_{\mathcal{F}_i}\Omega^*(M)$ induces homomorphisms

$$E_j^{r,s}(M, \mathcal{F}_{i+1}) \to E_j^{r,s}(M, \mathcal{F}_i)$$

commuting with the differential d_j. But there is a more subtle connection between the two spectral sequences.

Proposition II.1.2. *Let $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ be a k-flag of foliations on M. Then there are canonical cochain maps*

$$E_j^{r,s}(M, \mathcal{F}_i) \to E_j^{r-q_i, s+q_i}(M, \mathcal{F}_{i+1})$$

for every $r, s \in \mathbb{Z}$ and $j \geq 0$.

Proof. All we have to show is that

$$F^r_{\mathcal{F}_i}\Omega^{r+s}(M) \subset F^{r-q_i}_{\mathcal{F}_{i+1}}\Omega^{r+s}(M) \ .$$

So, let $\omega \in F^r_{\mathcal{F}_i}\Omega^{r+s}(M)$. Then we have to prove

$$i_{X_{s+q_i+1}} \cdots i_{X_1}\omega = 0 \qquad \text{(II.1.1)}$$

for all $X_1, \dots, X_{s+q_i+1} \in T\mathcal{F}_{i+1}$. To do this, we may assume that every X_j is either in $T\mathcal{F}_i \subset T\mathcal{F}_{i+1}$ or in some fixed complement $Q_i \subset T\mathcal{F}_{i+1}$ with $T\mathcal{F}_{i+1} = T\mathcal{F}_i \oplus Q_i$. Because Q_i is q_i-dimensional we may assume further that at least $(s + q_i + 1) - q_i = s + 1$ of the vectors are in $T\mathcal{F}_i$. But then Equation (II.1.1) follows from $\omega \in F^r_{\mathcal{F}_i}\Omega^{r+s}(M)$. □

Setting $r = q_i + \cdots + q_k$ in Proposition II.1.2 leads to the following definition.

Definition II.1.3. We call the foliated classes in the image of the canonical homomorphism

$$\epsilon_i : H^*(M, \mathcal{F}_i) \to H^{*+q_i}(M, \mathcal{F}_{i+1})$$

the *derived secondary classes (at stage $i + 1$)* of the k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$.

If we regard $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ as a $(k+1)$-flag $(\{M\}, \mathcal{F}_k, \dots, \mathcal{F}_1)$, then the map

$$\epsilon_k \circ \cdots \circ \epsilon_i : H^*(M, \mathcal{F}_i) \to H^{*+q_i+\cdots+q_k}(M; \mathbb{R})$$

is just the canonical map ϵ mentioned in Section I.2. The classes in the image of this map can be viewed as derived classes at stage infinity. Let us begin our discussion with an explicit example for a derived class.

Lemma II.1.4. *For every k-flag of foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ there is a well-defined $(q_i + \cdots + q_k + 1)$-linear map*

$$\begin{aligned} \chi_i : H^1(\mathcal{F}_i) \times \cdots \times H^1(\mathcal{F}_i) &\to H^{q_i+\cdots+q_k+1}(M, \mathcal{F}_i) \\ ([\beta_0], [\beta_1], \dots, [\beta_{q_i+\cdots+q_k}]) &\mapsto [\beta_0 \wedge d\beta_1 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k}] \ . \end{aligned}$$

Furthermore, the map

$$\epsilon_k \circ \cdots \circ \epsilon_i \circ \chi_i : H^1(\mathcal{F}_i) \times \cdots \times H^1(\mathcal{F}_i) \to H^{2(q_i+\cdots+q_k)+1}(M;\mathbb{R})$$

is symmetric.

PROOF. Let $[\beta_0], \dots, [\beta_{q_i+\cdots+q_k}] \in H^1(\mathcal{F}_i) = E_1^{0,1}(M, \mathcal{F}_i)$. Since

$$[d\beta_i] = d_1[\beta_i] \in E_1^{1,1}(M, \mathcal{F}_i) \ ,$$

we get

$$[d\beta_1 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k}] = [d\beta_1] \cup \cdots \cup [d\beta_{q_i+\cdots+q_k}] \in E_1^{q_i+\cdots+q_k, q_i+\cdots+q_k}(M, \mathcal{F}_i) \ .$$

So, the class

$$\begin{aligned} \chi_i([\beta_0], \dots, [\beta_{q_i+\cdots+q_k}]) &= [\beta_0 \wedge d\beta_1 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k}] \\ &= [\beta_0] \cup [d\beta_1 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k}] \in H^{q_i+\cdots+q_k+1}(M, \mathcal{F}_i) \end{aligned}$$

is well-defined.

That χ_i is $(q_i + \cdots + q_k + 1)$-linear and symmetric in the last $q_i + \cdots + q_k$ variables is obvious. Moreover, we have

$$\begin{aligned} \beta_0 \wedge & d\beta_1 \wedge d\beta_2 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k} - \beta_1 \wedge d\beta_0 \wedge d\beta_2 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k} \\ &= -d(\beta_0 \wedge \beta_1) \wedge d\beta_2 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k} \\ &= d(-(\beta_0 \wedge \beta_1) \wedge d\beta_2 \wedge \cdots \wedge d\beta_{q_i+\cdots+q_k}) \ . \end{aligned}$$

Hence, $\epsilon_k \circ \cdots \circ \epsilon_i \circ \chi_i$ is indeed symmetric. □

Now, let $\mathcal{F}_i$ be coorientable with defining form α_i and $d\alpha_i = \beta_i \wedge \alpha_i$. The one-form β_i represents the Reeb class $[\beta_i] \in H^1(\mathcal{F}_i)$ which is an invariant of the coorientable foliation $\mathcal{F}_i$ as we saw in Section I.1.

Definition II.1.5. The *Godbillon-Vey class* of a k-flag of coorientable foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is the derived class

$$GV(\mathcal{F}_k, \dots, \mathcal{F}_1) = \sum_{i=1}^{k-1} \epsilon_{k-1} \circ \cdots \circ \epsilon_i \circ \chi_i([\beta_i], \dots, [\beta_i]) \ \in H^*(M, \mathcal{F}_k) \ ,$$

where $[\beta_i] \in H^1(\mathcal{F}_i)$ is the Reeb class of $\mathcal{F}_i$.

Obviously, we have

$$GV(\mathcal{F}_k, \dots, \mathcal{F}_1) = \sum_{i=1}^{k-1} GV(\mathcal{F}_k, \mathcal{F}_i) \ .$$

The de Rham cohomology class

$$gv(\mathcal{F}_i) = GV(\{M\}, \mathcal{F}_i) = [\beta_i \wedge (d\beta_i)^{q_i+\cdots+q_k}] \in H^{2(q_i+\cdots+q_k)+1}(M;\mathbb{R})$$

is the classical Godbillon-Vey invariant of the coorientable foliation $\mathcal{F}_i$ (cf. [**13**], [**12**]). As an immediate corollary from Theorem I.1.4 we get the following statement which explains the geometric significance of the Godbillon-Vey class.

Corollary II.1.6. *Let $\mathcal{F}$ be a coorientable foliation on a manifold M. If $\mathcal{F}$ admits a holonomy invariant transverse volume form, then $gv(\mathcal{F}) = 0$.*

Compared to the Reeb class this obstruction $gv(\mathcal{F})$ enjoys the advantage to be defined in the usual de Rham cohomology rather than in the leafwise cohomology. The derived class $GV(\mathcal{F}_k, \mathcal{F}_i)$ is a lift of $gv(\mathcal{F}_i)$ under the canonical map

$$\epsilon : H^{2(q_i+\cdots+q_{k-1})+q_k+1}(M, \mathcal{F}_k) \to H^{2(q_i+\cdots+q_k)+1}(M; \mathbb{R}) \ .$$

Corollary II.1.6 remains true, saying that if $\mathcal{F}_i$ admits a holonomy invariant transverse volume form, then $GV(\mathcal{F}_k, \mathcal{F}_i) = 0$. Even though this is a class in the foliated cohomology $H^*(M, \mathcal{F}_k)$, there are other cases of interest (apart from $\mathcal{F}_k = \{M\}$) where the Godbillon-Vey class of a k-flag of foliations can again be interpreted as a de Rham cohomology class. We will describe them in Section II.4. First we will be concerned with the non-triviality of the Godbillon-Vey class of a flag of foliations.

II.2. Non-vanishing examples

Immediate from the definition there is the following vanishing statement.

Proposition II.2.1. *Let $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ be a k-flag of coorientable foliations on M. If the derived class $GV(\mathcal{F}_k, \dots, \mathcal{F}_1)$ vanishes, so do the classical Godbillon-Vey invariants $gv(\mathcal{F}_i)$ for all $1 \leq i \leq k$.*

Proof. We have

$$\begin{aligned}\sum_{i=1}^{k} gv(\mathcal{F}_i) = \sum_{i=1}^{k} GV(\{M\}, \mathcal{F}_i) &= GV(\{M\}, \mathcal{F}_k, \dots, \mathcal{F}_1) \\ &= \epsilon(GV(\mathcal{F}_k, \dots, \mathcal{F}_1)) \ .\end{aligned}$$

Suppose that $GV(\mathcal{F}_k, \dots, \mathcal{F}_1) = 0$. If $\mathfrak{I}$ is a maximal set of indices $1 \leq i \leq k$ such that the $\mathcal{F}_i$ with $i \in \mathfrak{I}$ are pairwise different, then there are positive numbers n_i such that

$$0 = \epsilon(GV(\mathcal{F}_k, \dots, \mathcal{F}_1)) = \sum_{i \in \mathfrak{I}} n_i \cdot gv(\mathcal{F}_i)$$

For $\mathcal{F}_i \neq \mathcal{F}_j$ the degree of $gv(\mathcal{F}_i)$ is different from the degree of $gv(\mathcal{F}_j)$. So, $n_i \cdot gv(\mathcal{F}_i) = 0$ for every $i \in \mathfrak{I}$. The proposition follows, since $H^*(M; \mathbb{R})$ is a real vector space. □

Proposition II.2.1 makes it natural to ask if there are any examples for k-flags of coorientable foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ such that the classical Godbillon-Vey classes $gv(\mathcal{F}_i)$ vanish for every $1 \leq i \leq k$ but $GV(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is non-vanishing. We will give such examples using a non-vanishing result due to Roussarie (cf. [**13**]) for the classical Godbillon-Vey class.

Theorem II.2.2 (Roussarie)**.** *Let $T_1\Sigma$ be the sphere bundle of a closed hyperbolic Riemannian surface. Then there is a coorientable foliation $\mathcal{F}$ of codimension one on $T_1\Sigma$ such that $gv(\mathcal{F}) \neq 0$.*

PROOF. The Uniformization Theorem says $\Sigma \cong \mathbb{H}^2/\pi_1(\Sigma)$, where $\pi_1(\Sigma)$ acts as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^2$. Furthermore, $T_1\mathbb{H}^2 \cong \mathrm{PSL}_2(\mathbb{R})$ gives $T_1\Sigma \cong \mathrm{PSL}_2(\mathbb{R})/\pi_1(\Sigma)$. Hence, if we find an invariant codimension one foliation $\tilde{\mathcal{F}}$ on $\mathrm{PSL}_2(\mathbb{R})$, then $\tilde{\mathcal{F}}$ projects to a foliation $\mathcal{F}$ on $T_1\Sigma$ under the covering map $\mathrm{PSL}_2(\mathbb{R}) \twoheadrightarrow T_1\Sigma$. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$. There is a basis α, β, γ of the dual algebra $\mathfrak{sl}_2(\mathbb{R})^*$ satisfying

$$d\alpha = \alpha \wedge \beta,\ d\beta = -2\alpha \wedge \gamma,\ d\gamma = \beta \wedge \gamma\ .$$

By the Frobenius Theorem γ is a defining form for an invariant foliation $\tilde{\mathcal{F}}$ on $\mathrm{PSL}_2(\mathbb{R})$ (and so is α). Moreover,

$$\beta \wedge d\beta = -2\beta \wedge \alpha \wedge \gamma = 2\alpha \wedge \beta \wedge \gamma$$

is an invariant volume form on $\mathrm{PSL}_2(\mathbb{R})$ which projects down to $gv(\mathcal{F})$. Hence, $gv(\mathcal{F}) \neq 0$. □

We would like to take products of this example to create non-vanishing examples for the Godbillon-Vey classes of flags of foliations. To do this we need the following lemma.

Lemma II.2.3. *Consider a coorientable subfoliation $(\mathcal{F}_2, \mathcal{F}_1)$ on a closed manifold M and a coorientable subfoliation $(\mathcal{G}_2, \mathcal{G}_1)$ on a closed manifold N. If the derived class $GV(\mathcal{F}_2 \times \mathcal{G}_2, \mathcal{F}_1 \times \mathcal{G}_1)$ vanishes, so does either $GV(\mathcal{F}_2, \mathcal{F}_1)$ or the classical Godbillon-Vey invariant $gv(\mathcal{G}_1)$. Furthermore, $gv(\mathcal{F}_1 \times \mathcal{G}_1)$ always vanishes if $\mathcal{F}_1$ and $\mathcal{G}_1$ have positive codimension.*

PROOF. Denote by q the codimension of $\mathcal{F}_1$ in M and by r the codimension of $\mathcal{G}_1$ in N. Let α (resp. ω) be a defining form for $\mathcal{F}_1$ (resp. $\mathcal{G}_1$). Then there are one-forms β and η such that $d\alpha = \beta \wedge \alpha$ and $d\omega = \eta \wedge \omega$. Now, $\theta = pr_1^*\alpha \wedge pr_2^*\omega$ is a defining $(q+r)$-form for $\mathcal{F}_1 \times \mathcal{G}_1$. Here pr_1 (resp. pr_2) denotes the projection from $M \times N$ to M (resp. N). Because of

$$\begin{aligned} d\theta &= pr_1^* d\alpha \wedge pr_2^*\omega + (-1)^q pr_1^*\alpha \wedge pr_2^* d\omega \\ &= pr_1^*\beta \wedge pr_1^*\alpha \wedge pr_2^*\omega + (-1)^q pr_1^*\alpha \wedge pr_2^*\eta \wedge pr_2^*\omega \\ &= (pr_1^*\beta + pr_2^*\eta) \wedge (pr_1^*\alpha \wedge pr_2^*\omega) \end{aligned}$$

the Reeb class of $\mathcal{F}_1 \times \mathcal{G}_1$ is represented by $\nu = pr_1^*\beta + pr_2^*\eta$, i. e. $d\theta = \nu \wedge \theta$. Since $I^*(\mathcal{F}_1)^{q+1} = 0$ and $I^*(\mathcal{G}_1)^{r+1} = 0$ we have

$$\begin{aligned} (d\nu)^{q+r} &= \sum_{i=0}^{q+r} \binom{q+r}{i} pr_1^*(d\beta)^i \wedge pr_2^*(d\eta)^{q+r-i} \\ &= \binom{q+r}{q} pr_1^*(d\beta)^q \wedge pr_2^*(d\eta)^r\ . \end{aligned}$$

This gives

$$\begin{aligned} pr_1^*\beta \wedge \nu \wedge (d\nu)^{q+r} &= pr_1^*\beta \wedge pr_2^*\eta \wedge (d\nu)^{q+r} \\ &= \binom{q+r}{q} pr_1^*\left(\beta \wedge (d\beta)^q\right) \wedge pr_2^*\left(\eta \wedge (d\eta)^r\right) \end{aligned}$$

and hence, with $pr_1^*[\beta] \in E_1^{0,1}(M \times N, \mathcal{F}_1 \times \mathcal{G}_1)$,

$$pr_1^*[\beta] \cup GV(\mathcal{F}_2 \times \mathcal{G}_2, \mathcal{F}_1 \times \mathcal{G}_1) = \binom{q+r}{q} GV(\mathcal{F}_2, \mathcal{F}_1) \times GV(\mathcal{G}_2, \mathcal{G}_1) \ .$$

Suppose that $GV(\mathcal{F}_2 \times \mathcal{G}_2, \mathcal{F}_1 \times \mathcal{G}_1)$ vanishes, then by Corollary I.4.8 either $GV(\mathcal{F}_2, \mathcal{F}_1) = 0$ or $gv(\mathcal{G}_1) = 0$, since $\epsilon(GV(\mathcal{G}_2, \mathcal{G}_1)) = gv(\mathcal{G}_1)$. This proves the first part of the lemma.

If $q, r > 0$, then $gv(\mathcal{F}_1 \times \mathcal{G}_1) = 0$, since

$$\begin{aligned} \nu \wedge (d\nu)^{q+r} &= \binom{q+r}{q} \big(pr_1^*\beta \wedge pr_1^*(d\beta)^q \wedge pr_2^*(d\eta)^r \\ &\qquad + pr_2^*\eta \wedge pr_1^*(d\beta)^q \wedge pr_2^*(d\eta)^r \big) \\ &= -\binom{q+r}{q} d\big(pr_1^*\beta \wedge pr_1^*(d\beta)^q \wedge pr_2^*\eta \wedge pr_2^*(d\eta)^{r-1} \\ &\qquad - pr_1^*\beta \wedge pr_1^*(d\beta)^{q-1} \wedge pr_2^*\eta \wedge pr_2^*(d\eta)^r \big) \end{aligned}$$

is exact. □

Now, we are able to show the non-triviality of the new Godbillon-Vey classes. Let us start with the case of a two-flag.

Theorem II.2.4. *For every $q_1, q_2 > 0$ there is a closed oriented manifold M of dimension $3(q_1 + q_2)$ and a cooriented subfoliation $(\mathcal{F}_2, \mathcal{F}_1)$ of codimension (q_2, q_1) on M such that $gv(\mathcal{F}_1) = gv(\mathcal{F}_2) = 0$ but*

$$GV(\mathcal{F}_2, \mathcal{F}_1) \neq 0 \ .$$

Proof. Consider the foliation $\mathcal{F}$ on $X = T_1\Sigma$ with $gv(\mathcal{F}) \neq 0$ constructed in Theorem II.2.2. Proposition II.2.1 gives that $GV(\mathcal{F}, \mathcal{F}) \neq 0$. Thus, by Lemma II.2.3 we know that $GV(\mathcal{F} \times \mathcal{F}, \mathcal{F} \times \mathcal{F}) \neq 0$. The same argument shows that $GV(\{X\} \times \mathcal{F}, \mathcal{F} \times \mathcal{F}) \neq 0$, since $GV(\{X\}, \mathcal{F}) = gv(\mathcal{F}) \neq 0$. By induction via Lemma II.2.3 we see that on $M = X \times \cdots \times X = X^{q_1+q_2}$ the derived class

$$GV(\{X\}^{q_1} \times \mathcal{F}^{q_2}, \mathcal{F}^{q_1} \times \mathcal{F}^{q_2})$$

does not vanish. On the other hand, the second part of Lemma II.2.3 implies

$$gv(\{X\}^{q_1} \times \mathcal{F}^{q_2}) = gv(\mathcal{F}^{q_1} \times \mathcal{F}^{q_2}) = 0 \ .$$

□

This theorem can be generalized to the following non-vanishing statement about k-flags of coorientable foliations.

Theorem II.2.5. *Let $k > 1$. For each sequence $q_1, \dots, q_k > 0$ of positive numbers there is a closed oriented manifold M of dimension $3(q_k + q_{k-1}) + q_{k-2} + \cdots + q_1$ and a k-flag of cooriented foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ of codimension $(q_k, \dots, q_1)$ on M such that*

$$GV(\mathcal{F}_i, \dots, \mathcal{F}_1) = 0$$

for every $1 \le i < k$ and $gv(\mathcal{F}_i) = 0$ for all $1 \le i \le k$, but

$$GV(\mathcal{F}_k, \dots, \mathcal{F}_1) \neq 0 \ .$$

Proof. By Theorem II.2.4 we may assume that $k > 2$. Let $\mathcal{F}$ be Roussarie's foliation on $X = T_1\Sigma$ and $\mathcal{G}$ the foliation by points on S^1. Both foliations have codimension one. Define a k-flag of foliations on $M = X^{q_k+q_{k-1}} \times (S^1)^{q_{k-2}+\cdots+q_1}$ by

$$\begin{array}{lllllll}
\mathcal{F}_1 & = \mathcal{F}^{q_k} \times \mathcal{F}^{q_{k-1}} & \times \mathcal{G}^{q_{k-2}} & \times \cdots \times \mathcal{G}^{q_2} & \times \mathcal{G}^{q_1} \\
\mathcal{F}_2 & = \mathcal{F}^{q_k} \times \mathcal{F}^{q_{k-1}} & \times \mathcal{G}^{q_{k-2}} & \times \cdots \times \mathcal{G}^{q_2} & \times \{S^1\}^{q_1} \\
\vdots & & & & \\
\mathcal{F}_{k-1} & = \mathcal{F}^{q_k} \times \mathcal{F}^{q_{k-1}} & \times \{S^1\}^{q_{k-2}} & \times \cdots \times \{S^1\}^{q_2} & \times \{S^1\}^{q_1} \\
\mathcal{F}_k & = \mathcal{F}^{q_k} \times \{X\}^{q_{k-1}} & \times \{S^1\}^{q_{k-2}} & \times \cdots \times \{S^1\}^{q_2} & \times \{S^1\}^{q_1} \ .
\end{array}$$

Then $\mathcal{F}_i$ has codimension q_i in $\mathcal{F}_{i+1}$. Denote by α a defining form for $\mathcal{F}$ with $d\alpha = \beta \wedge \alpha$. The foliation $\mathcal{G}$ is defined by a volume form θ on S^1. The form

$$\omega_j = \left(\bigwedge_{i=1}^{q_k+q_{k-1}} pr_i^*\alpha \right) \wedge \left(\bigwedge_{i=q_k+q_{k-1}+1}^{q_k+\cdots+q_j} pr_i^*\theta \right)$$

is a defining form for $\mathcal{F}_j$, $j = 1, \dots, k-1$, with

$$d\omega_j = \left(\sum_{i=1}^{q_k+q_{k-1}} pr_i^*\beta \right) \wedge \omega_j \ .$$

Hence, the Godbillon-Vey class of $\mathcal{F}_j$ is represented by

$$\left(\sum_{i=1}^{q_k+q_{k-1}} pr_i^*\beta \right) \wedge \left(\sum_{i=1}^{q_k+q_{k-1}} pr_i^* d\beta \right)^{q_k+\cdots+q_j} \in F_{\mathcal{F}_{k-1}}^{q_k+q_{k-1}+\cdots+q_j}\Omega^*(M) \ ,$$

since $pr_i^* d\beta \in F^1_{\mathcal{F}_{k-1}}\Omega^*(M)$ for $1 \le i \le q_k + q_{k-1}$. Hence, the Godbillon-Vey class of each $\mathcal{F}_j$ with $1 \le j \le k-2$ can be represented by zero. Thus,

$$GV(\mathcal{F}_i, \dots, \mathcal{F}_1) = \sum_{j=1}^{i-1} GV(\mathcal{F}_i, \mathcal{F}_j) = 0$$

for $1 \leq i \leq k-1$. But

$$\begin{aligned} GV(\mathcal{F}_k, \dots, \mathcal{F}_1) &= \sum_{j=1}^{k-1} GV(\mathcal{F}_k, \mathcal{F}_j) = GV(\mathcal{F}_k, \mathcal{F}_{k-1}) \\ &= GV(\mathcal{F}^{q_k} \times \{X\}^{q_{k-1}}, \mathcal{F}^{q_k} \times \mathcal{F}^{q_{k-1}}) \times 1 \\ &\neq 0 \end{aligned}$$

by Theorem II.2.4 above and Theorem I.4.5 of the previous chapter. Here the class $1 \in H^0(\{S^1\}^{q_{k-2}+\dots+q_1})$ is represented by the constant function of value one. □

II.3. Naturality and leafwise concordance

The Godbillon-Vey class of a coorientable flag of foliations is actually a characteristic class, i. e. its natural with respect to pull-backs. Consider a morphism f from a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on M to a k-flag $(\mathcal{G}_k, \dots, \mathcal{G}_1)$ on N having the same codimension $(q_k, \dots, q_1)$. Then obviously, the maps defined in Lemma II.1.4 satisfy

(II.3.1) $$f^*\chi_i([\beta_0], \dots, [\beta_{q_2}]) = \chi_i(f^*[\beta_0], \dots, f^*[\beta_{q_2}]) \ .$$

for all $[\beta_j] \in H^1(\mathcal{F}_i)$. In particular, if $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is a k-flag of foliations on M and $f : N \to M$ is a smooth map transversal to the foliation $\mathcal{F}_1$ (and therefore transversal to all of the $\mathcal{F}_i$), then there is the pull-back k-flag $f^*(\mathcal{F}_k, \dots, \mathcal{F}_1) = (f^*\mathcal{F}_k, \dots, f^*\mathcal{F}_1)$ on N and we get the following naturality property.

Proposition II.3.1. *Let $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ be a k-flag of coorientable foliations on M and let $f : N \to M$ be a map transversal to $\mathcal{F}_1$. Then*

$$GV(f^*\mathcal{F}_k, \dots, f^*\mathcal{F}_1) = f^*GV(\mathcal{F}_k, \dots, \mathcal{F}_1) \ .$$

Proof. If $\alpha_i \in \Omega^{q_k+\dots+q_i}(M)$ is a defining form for $\mathcal{F}_i$, then $f^*\mathcal{F}_i$ is defined by $f^*\alpha_i \in \Omega^{q_k+\dots+q_i}(N)$. For $d\alpha_i = \beta_i \wedge \alpha_i$, we have

$$df^*\alpha_i = f^*d\alpha_i = f^*\beta_i \wedge f^*\alpha_i \ .$$

Hence, if $[\beta_i] \in H^1(\mathcal{F}_i)$ is the Reeb class of $\mathcal{F}_i$, then $f^*[\beta_i] = [f^*\beta_i] \in H^1(f^*\mathcal{F}_i)$ is the Reeb class of $f^*\mathcal{F}_i$. The assertion follows by Equation (II.3.1). □

The Godbillon-Vey class is not only natural but moreover it is a homotopy invariant in the following sense (cf. Definition I.3.1).

Definition II.3.2. Consider two k-flags $(\mathcal{F}_k^0, \dots, \mathcal{F}_1^0)$, $(\mathcal{F}_k^1, \dots, \mathcal{F}_1^1)$ and an adittional foliation $\mathcal{F}$ on M. A smooth map $F : M \times [0,1] \to N$ is a *leafwise homotopy along* $\mathcal{F}$ joining the two k-flags, if $\mathcal{F}_k^0 \subset \mathcal{F}$ and $\mathcal{F}_k^1 \subset \mathcal{F}$ and if there is a $(k+1)$-flag $(\mathcal{G}, \mathcal{G}_k, \dots, \mathcal{G}_1)$ on N such that the maps $F_t = F|_{M \times \{t\}} : M \to N$ satisfy the following conditions.

(1) F_0 and F_1 are transverse to $\mathcal{G}_1$ and $F_0^*(\mathcal{G}, \mathcal{G}_k, \dots, \mathcal{G}_1) = (\mathcal{F}, \mathcal{F}_k^0, \dots, \mathcal{F}_1^0)$, $F_1^*(\mathcal{G}, \mathcal{G}_k, \dots, \mathcal{G}_1) = (\mathcal{F}, \mathcal{F}_k^1, \dots, \mathcal{F}_1^1)$,

(2) $F_t(\mathcal{F}) \subset \mathcal{G}$ for all $t \in [0,1]$,
(3) for every $x \in M$ the points $F_s(x), F_t(x)$ lie in the same leaf of $\mathcal{G}$ for all $s, t \in [0,1]$.

Note that Condition (1) implies that the flags $(\mathcal{F}, \mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$, $(\mathcal{F}, \mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$ and $(\mathcal{G}, \mathcal{G}_k, \ldots, \mathcal{G}_1)$ have the same codimension.

Theorem II.3.3. *Let $(\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$ and $(\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$ be two k-flags of coorientable foliations on M. If there is a leafwise homotopy along a foliation $\mathcal{F}$ on M joining the two k-flags, then*

$$GV(\mathcal{F}, \mathcal{F}^0_k, \ldots, \mathcal{F}^0_1) = GV(\mathcal{F}, \mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$$

in $H^*(M, \mathcal{F})$.

Proof. This follows directly from Proposition II.3.1 above and Theorem I.3.2. □

The existence of a leafwise homotopy along $\mathcal{F}$ does not mean that there is a path in the "space of k-flags" joining $(\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$ and $(\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$. Actually, this condition establishes rather a concordance than a homotopy of k-flags. Let us put this notion right.

Definition II.3.4. Two k-flags $(\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$ and $(\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$ on M are *leafwise concordant* if there is a k-flag of foliations $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$ on the cylinder $\mathbb{R} \times M$ such that the inclusion maps $j_t : M \to \mathbb{R} \times M$ defined by $j_t(x) = (t, x)$ satisfy the following conditions.

(1) j_0 and j_1 are transverse to $\mathcal{F}_1$ and the pull-backs equal $j_0^*(\mathcal{F}_k, \ldots, \mathcal{F}_1) = (\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$ and $j_1^*(\mathcal{F}_k, \ldots, \mathcal{F}_1) = (\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$,
(2) $\mathcal{F}^0_k$ equals $\mathcal{F}^1_k$, and $\mathcal{F}_k$ is the product foliation $\{\mathbb{R}\} \times \mathcal{F}^0_k$.

Two k-flags are *concordant*, if they are leafwise concordant as $(k+1)$-flags (i. e. we do not require (2) any more).

By Theorem II.3.3 the Godbillon-Vey class is a leafwise concordance invariant.

Theorem II.3.5. *Let $(\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1)$ and $(\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$ be two leafwise concordant k-flags of coorientable foliations on a manifold M. Then*

$$GV(\mathcal{F}^0_k, \ldots, \mathcal{F}^0_1) = GV(\mathcal{F}^1_k, \ldots, \mathcal{F}^1_1)$$

in $H^*(M, \mathcal{F}^0_k) = H^*(M, \mathcal{F}^1_k)$.

Proof. Let $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$ the leafwise concordance foliation on the cylinder $N = \mathbb{R} \times M$. Then $F : M \times [0,1] \to N$ defined by $F(x,t) = (t,x)$ is a leafwise homotopy along $\mathcal{F}^0_k = \mathcal{F}^1_k$ joining $(\mathcal{F}^0_{k-1}, \ldots, \mathcal{F}^0_1)$ and $(\mathcal{F}^1_{k-1}, \ldots, \mathcal{F}^1_1)$. The assertion follows by Theorem II.3.3. □

Theorem II.3.5 implies the concordance invariance of the classical Godbillon-Vey class.

Corollary II.3.6. *If $(\mathcal{F}_k^0, \dots, \mathcal{F}_1^0)$ and $(\mathcal{F}_k^1, \dots, \mathcal{F}_1^1)$ are two concordant k-flags of coorientable foliations on M, then*

$$gv(\mathcal{F}_j^0) = gv(\mathcal{F}_j^1)$$

for all $1 \leq j \leq k$.

PROOF. This follows immediately from the equation $GV(\{M\}, \mathcal{F}_k^i, \dots, \mathcal{F}_1^i) = \sum_{j=1}^k gv(\mathcal{F}_j^i)$. □

II.4. Families of foliations

For the sake of simplicity, we will restrict ourselves to subfoliations for the rest of the chapter. In this section we will discuss the crucial example of subfoliations arising from multi-parameter families of foliations. The Godbillon-Vey class of such a subfoliation can be identified with a family of de Rham cohomology classes which depends naturally on the family of foliations. First, consider a one-parameter family $\mathcal{F}_t$ of codimension q foliations on a manifold M. Then there is a smooth family α_t of defining forms for $\mathcal{F}_t$ with $d\alpha_t = \beta_t \wedge \alpha_t$ for a family of one-forms β_t. The classical Godbillon-Vey class of $\mathcal{F}_t$ is given by $gv(\mathcal{F}_t) = [\beta_t \wedge (d\beta_t)^q] \in H^{2q+1}(M; \mathbb{R})$. But as noticed in [**21**], there is another class (actually a family of classes)

$$TGV(\mathcal{F}_t) = [\dot{\beta}_t \wedge \beta_t \wedge (d\beta_t)^q] \in H^{2q+2}(M; \mathbb{R})$$

associated to this family of foliations (here the dot denotes the derivative with respect to t). It can be proven by explicit calculations that this class is well-defined, independent of the choice of the family of defining forms. But these calculations are rather lengthy and not very enlightening. We will see in a moment that this class can be interpreted as the Godbillon-Vey class of a certain subfoliation.

Indeed we can think of the union of all the $\mathcal{F}_t$'s as a smooth codimension $q+1$ foliation $\mathcal{G}_1$ on the cylinder $X = \mathbb{R} \times M$. Then the foliation $\mathcal{F}_t$ is contained in the time slice $\{t\} \times M$. In other words, if $\mathcal{G}_2$ is the codimension one foliation on $\mathbb{R} \times M$ with leaves $\{t\} \times M$, then $(\mathcal{G}_2, \mathcal{G}_1)$ is a subfoliation. Let us use this point of view to generalize the notion of a family of foliations.

Definition II.4.1. A *family of foliations on M parametrized by B* is a subfoliation $(\mathcal{G}_2, \mathcal{G}_1)$ on the total space X of a fibre bundle $\pi : X \to B$ with fibre M, such that the leaves of $\mathcal{G}_2$ are the fibres of the fibre bundle. We will always assume that B is orientable and of finite type. If $B = \mathbb{R}^{q_2}$, $X = \mathbb{R}^{q_2} \times M$ and $\pi = pr_1$ is the projection onto $\mathbb{R}^{q_2}$, let us call $(\mathcal{G}_2, \mathcal{G}_1)$ a *q_2-parameter family of foliations on M*. For such a q_2-parameter family of foliations we will also use the more suggestive notation $\mathcal{F}_t$ again, where t now varies in $\mathbb{R}^{q_2}$.

If $(\mathcal{G}_2, \mathcal{G}_1)$ is a family of foliations, then every volume form ν on B gives rise to a closed defining form $\pi^*\nu$ for the simple foliation $\mathcal{G}_2$. By Corollary I.2.7, $H^*(X, \mathcal{G}_2)$ is a one-dimensional free left $H^*(\mathcal{G}_2)$-module with generator $[\pi^*\nu] \in H^0(X, \mathcal{G}_2)$.

On the other hand, we computed in Theorem I.5.2 the leafwise cohomology algebra $H^*(\mathcal{G}_2)$.

Corollary II.4.2. *Let $(\mathcal{G}_2, \mathcal{G}_1)$ be a family of foliations on M parameterized by B. Then every volume form on B gives rise to an identification*

$$H^*(X, \mathcal{G}_2) \cong \Gamma(\mathcal{H}^*(M; \mathbb{R})) \ .$$

Hence, the derived classes of $(\mathcal{G}_2, \mathcal{G}_1)$ can be interpreted as sections of the flat vector bundle $\mathcal{H}^*(M, \mathbb{R})$.

Now, $\mathbb{R}^{q_2}$ has a canonical volume form, namely $dt_1 \wedge \cdots \wedge dt_{q_2}$. Thus, the derived classes of a q_2-parameter family $\mathcal{F}_t$ on M are canonically identified with sections of the trivial bundle $\mathcal{H}^*(M; \mathbb{R})$, i. e. for q_2-parameter families we have a canonical identification

$$H^*(\mathbb{R}^{q_2} \times M, \mathcal{G}_2) = \mathcal{C}^\infty(\mathbb{R}^{q_2}, H^*(M; \mathbb{R})) \ .$$

In particular, the Godbillon-Vey class of a q_2-parameter family of foliations is a smooth map from $\mathbb{R}^{q_2}$ to $H^{2q_1+q_2+1}(M; \mathbb{R})$. If $\omega \in \Omega^*(\mathbb{R}^{q_2} \times M)$, then write $\omega_t \in \Omega^*(M)$ for the restriction of ω to the fibre over $t \in \mathbb{R}^{q_2}$.

Theorem II.4.3. *Let $\mathcal{F}_t$ be a q_2-parameter family of coorientable codimension q_1 foliations on M. Then*

$$GV(\mathcal{F}_t) = \tfrac{(q_1+q_2)!}{q_1!} \left[(\tfrac{\partial}{\partial t_{q_2}} \beta)_t \wedge \cdots \wedge (\tfrac{\partial}{\partial t_1} \beta)_t \wedge \beta_t \wedge (d\beta_t)^{q_1} \right] \ ,$$

where $\beta \in \Omega^1(\mathbb{R}^{q_2} \times M)$ represents the Reeb class of $\mathcal{F}_1$.

Proof. Decompose the differential d on $\Omega^*(\mathbb{R}^{q_2} \times M)$ into $d = d_B + d_M$, where $d_B \omega = \sum_{i=1}^{q_2} dt_i \wedge (\frac{\partial}{\partial t_i} \omega)$. Obviously $d_B \beta \in F^1_{\mathcal{G}_1} \Omega^2(\mathbb{R}^{q_2} \times M)$. Since we know that $d\beta \in F^1_{\mathcal{G}_1} \Omega^2(\mathbb{R}^{q_2} \times M)$, we have $d_M \beta \in F^1_{\mathcal{G}_1} \Omega^2(\mathbb{R}^{q_2} \times M)$ as well. This gives $(d_M \beta)^{q_1+1} \in F^{q_1+1}_{\mathcal{G}_1} \Omega^{2q_1+2}(\mathbb{R}^{q_2} \times M) \subset F^1_{\mathcal{G}_2} \Omega^{2q_1+2}(\mathbb{R}^{q_2} \times M)$. Thus, $(d_M \beta)^{q_1+1} = 0$ in $E_0^{0,*}(M, \mathcal{G}_2)$. On the other hand, $(d_B \beta)^{q_2+1} = 0$. Hence, we compute in $E_0^{0,*}(M, \mathcal{G}_2)$ that

$$\begin{aligned} (d\beta)^{q_1+q_2} &= \sum_{i=0}^{q_1+q_2} \binom{q_1+q_2}{i} (d_B\beta)^i \wedge (d_M \beta)^{q_1+q_2-i} \\ &= \binom{q_1+q_2}{q_2} (d_B \beta)^{q_2} \wedge (d_M \beta)^{q_1} \ . \end{aligned}$$

The left factor can be expanded to

$$\begin{aligned} (d_B \beta)^{q_2} &= q_2!\, dt_1 \wedge (\tfrac{\partial}{\partial t_1}\beta) \wedge \cdots \wedge dt_{q_2} \wedge (\tfrac{\partial}{\partial t_{q_2}} \beta) \\ &= (-1)^{\frac{q_2(q_2+1)}{2}} q_2!\, (\tfrac{\partial}{\partial t_1}\beta) \wedge \cdots \wedge (\tfrac{\partial}{\partial t_{q_2}}\beta) \wedge dt_1 \wedge \cdots \wedge dt_{q_2} \\ &= (-1)^{q_2} q_2!\, (\tfrac{\partial}{\partial t_{q_2}}\beta) \wedge \cdots \wedge (\tfrac{\partial}{\partial t_1}\beta) \wedge dt_1 \wedge \cdots \wedge dt_{q_2} \ . \end{aligned}$$

This proves the theorem. □

We recover the class $TGV(\mathcal{F}_t) = [\dot{\beta}_t \wedge \beta_t \wedge (d\beta_t)^q] \in H^{2q+4}(F;\mathbb{R})$ discussed in [**21**] as the Godbillon-Vey class of a 1-parameter family of foliations on M.

By what was said in the previous section, these classes are natural concordance invariants of the family of foliations as we will explain now. Let $(\mathcal{G}_2, \mathcal{G}_1)$ be a family of foliations on M parameterized by a fibre bundle $\pi : X \to B$. Suppose that $(\mathcal{G}_2', \mathcal{G}_1')$ is a family of foliations on a manifold N parameterized by a fibre bundle $\pi' : X' \to B$ over the same base space B. We will call a morphism $f : X' \to X$ from the subfoliation $(\mathcal{G}_2', \mathcal{G}_1')$ to the subfoliation $(\mathcal{G}_2, \mathcal{G}_1)$ a *morphism of families of foliations parameterized by* B if f is a fibre bundle map over the identity. In that case $\mathcal{G}_2' = f^*\mathcal{G}_2$ and any choice of volume form on B gives rise to a commuting diagram

$$\begin{array}{ccc} H^*(X,\mathcal{G}_2) & \xrightarrow{\cong} & \Gamma(\mathcal{H}^*(M;\mathbb{R})) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^*(X',\mathcal{G}_2') & \xrightarrow{\cong} & \Gamma(\mathcal{H}^*(N;\mathbb{R})) \ . \end{array}$$

Proposition II.3.1 gives the following theorem.

Theorem II.4.4. *Let $(\mathcal{G}_2, \mathcal{G}_1)$ be a family of foliations parameterized by a fibre bundle $\pi : X \to B$ with fibre M. If $\pi' : X' \to B$ is a fibre bundle with fibre N and $f : X' \to X$ a fibre bundle map over the identity, transverse to $\mathcal{F}_1$, then $(f^*\mathcal{G}_2, f^*\mathcal{G}_1)$ is a family of foliations parameterized by π' such that for every volume form ν on B we have*

$$GV(f^*\mathcal{G}_2, f^*\mathcal{G}_1) = f^*GV(\mathcal{G}_2, \mathcal{G}_1)$$

as sections of the flat vector bundle $\mathcal{H}^*(N;\mathbb{R})$ associated to the bundle π'.

In the same way we can carry over the concordance invariance to families of foliations. We call two families of foliations *concordant* if they are leafwise concordant as subfoliations. This means, two families $(\mathcal{G}_2^0, \mathcal{G}_1^0)$ and $(\mathcal{G}_2^1, \mathcal{G}_1^1)$ of foliations on M are concordant if they are parameterized by the same bundle $\pi : X \to B$ and if there is a family of foliations $(\mathcal{G}_2, \mathcal{G}_1)$ parameterized by $\pi \circ pr_1 : X \times [0,1] \to B$ such that $\mathcal{G}_1$ is transverse to $X^0 = X \times \{0\}$ and $X^1 = X \times \{1\}$ with $\mathcal{G}_1|_{X^0} = \mathcal{G}_1^0$ and $\mathcal{G}_1|_{X^1} = \mathcal{G}_1^1$. Theorem II.3.3 yields the concordance invariance of the Godbillon-Vey class of families of foliations.

Theorem II.4.5. *If two families of foliations $(\mathcal{G}_2^0, \mathcal{G}_1^0)$ and $(\mathcal{G}_2^1, \mathcal{G}_1^1)$ on M parameterized by the same bundle over B are concordant, then for every choice of volume form on B we have*

$$GV(\mathcal{G}_2^0, \mathcal{G}_1^0) = GV(\mathcal{G}_2^1, \mathcal{G}_1^1)$$

as sections of the flat bundle $\mathcal{H}^{2q_1+q_2+1}(M;\mathbb{R})$.

In particular, the classes $GV(\mathcal{F}_t)$ computed in Theorem II.4.3 are natural concordance invariants of the multi-parameter family of foliations $\mathcal{F}_t$. Of course, this can be computed directly, but with the argument above we will see in the next chapter that there is a huge variety of such natural concordance invariants for multi-parameter families of foliations.

II.5. Decomposition of GV into Godbillon and Vey invariant

There is a decomposition of the dual of the classical Godbillon-Vey invariant due to Duminy (cf. [**12**]) that can be worked out for the derived one as well. It leads to a non-trivial vanishing result for the derived classes discussed in the previous section. Let $(\mathcal{G}_2, \mathcal{G}_1)$ be a family of foliations on a closed and oriented n-manifold M parameterized by an oriented fibre bundle $\pi : X \to B$. If $\mathcal{B}$ denotes the trivial foliation by points on B, then $\mathcal{G}_2 = \pi^*\mathcal{B}$. Hence, the integration along the fibre constructed in Theorem I.6.3 gives a homomorphism

$$\int_\pi : H^n(X, \mathcal{G}_2) \to H^0(B, \mathcal{B}) = \Omega^{q_2}(B) \ .$$

Consider the pairing

$$\langle \cdot, \cdot \rangle : H^{n-k}(\mathcal{G}_2) \otimes H^k(X, \mathcal{G}_2) \xrightarrow{\cup} H^n(X, \mathcal{G}_2) \xrightarrow{\int_\pi} \Omega^{q_2}(B)$$

Lemma II.5.1. *The pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, i. e. $\langle a, b \rangle = 0$ for all $a \in H^{n-k}(\mathcal{G}_2)$ implies that $b = 0$ in $H^k(X, \mathcal{G}_2)$.*

PROOF. If we choose a volume form ν on B, then we can identify the pairing with a map

$$\langle \cdot, \cdot \rangle : \Gamma(\mathcal{H}^{n-k}(M; \mathbb{R})) \otimes \Gamma(\mathcal{H}^k(M; \mathbb{R})) \to \Gamma(\mathcal{H}^n(M; \mathbb{R})) \to \mathcal{C}^\infty(B, \mathbb{R}) \ .$$

The value of $\langle a, b \rangle$ at $t \in B$ only depends on the values of a and b in t. Therefore, it suffices to show that the pairing induced over every point $t \in B$ is non-degenerate. But this follows from Poincaré duality. □

Let $[\beta]$ be the Reeb class of $\mathcal{G}_1$. The dual of $GV(\mathcal{G}_2, \mathcal{G}_1)$ with respect to the pairing $\langle \cdot, \cdot \rangle$ is the map

$$\begin{aligned} GV(\mathcal{G}_2, \mathcal{G}_1)^* : \Gamma(\mathcal{H}^{n-2q_1-q_2-1}(M; \mathbb{R})) &\to \Omega^{q_2}(B) \\ a &\mapsto \langle a, GV(\mathcal{G}_2, \mathcal{G}_1) \rangle \ . \end{aligned}$$

The decomposition now takes the following form.

Theorem II.5.2. *Let $(\mathcal{G}_2, \mathcal{G}_1)$ be a family of foliations on a closed oriented manifold M parametrized by an orientable fibre bundle $X \to B$. There are two well-defined homomorphisms,*

$$\begin{aligned} V : \Gamma(\mathcal{H}^{n-2q_1-q_2-1}(M; \mathbb{R})) &\to H^{n-q_1-1}(X, \mathcal{G}_1) \\ [\omega] &\mapsto [\omega \wedge (d\beta)^{q_1+q_2}] \end{aligned}$$

and

$$G : H^{n-q_1-1}(X, \mathcal{G}_1) \to \Omega^{q_2}(B)$$
$$[\omega] \mapsto \int_\pi \omega \wedge \beta \ ,$$

only depending on $(\mathcal{G}_2, \mathcal{G}_1)$ *such that*

$$GV(\mathcal{G}_1, \mathcal{G}_2)^* = G \circ V \ .$$

G *is called the* Godbillon invariant *and* V *is the* Vey invariant.

Proof. Let $j : H^*(\mathcal{G}_2) \to H^*(\mathcal{G}_1)$ be the homomorphism induced by the identity and let $[\omega] \in H^*(\mathcal{G}_1)$. Then

$$V[\omega] = j[\omega] \cup [(d\beta)^{q_1+q_2}] \in H^{*+q_1+q_2}(X, \mathcal{G}_1)$$

is well-defined because the class $[(d\beta)^{q_1+q_2}] = (d_1[\beta])^{q_1+q_2} \in H^{q_1+q_2}(X, \mathcal{G}_1)$ does not depend on any choices, since the Reeb class $[\beta] \in H^1(\mathcal{G}_1)$ does not. Moreover, we have $[\omega \wedge \beta] = [\omega] \cup [\beta] \in H^{n-q_1}(X, \mathcal{G}_1)$ and $\epsilon_1([\omega] \cup [\beta]) \in H^n(X, \mathcal{G}_2)$ for every class $[\omega] \in H^{n-q_1-1}(X, \mathcal{G}_1)$. Thus,

$$G[\omega] = \int_\pi \epsilon_1[\omega \wedge \beta]$$

is well-defined. □

For example, if B is just a point, then this gives the decomposition of the dual $gv(\mathcal{F})^* : H^{n-2q-1}(M; \mathbb{R}) \to \mathbb{R}$ of the classical Godbillon-Vey class of a coorientable foliation $\mathcal{F}$ of codimension q. Then the Vey operator is a homomorphism $V : H^{n-2q-1}(M; \mathbb{R}) \to H^{n-q-1}(M, \mathcal{F})$ and the Godbillon operator is a functional $G : H^{n-q-1}(M, \mathcal{F}) \to \mathbb{R}$. In [**12**] the special case of a codimension one foliation was discussed. In this case $H^{n-2}(M, \mathcal{F})$ is equal to $H^{n-1}(I^*(\mathcal{F}))$ as we noticed in Example I.2.2.

By Lemma II.5.1 we in particular know that if the Godbillon operator of a family $(\mathcal{G}_2, \mathcal{G}_1)$ vanishes, so does the Godbillon-Vey class $GV(\mathcal{G}_2, \mathcal{G}_1)$. This gives the following vanishing statement.

Corollary II.5.3. *Let* $(\mathcal{G}_2, \mathcal{G}_1)$ *be a family of foliations on a closed oriented manifold* M *parameterized by an oriented fibre bundle* $X \to B$. *Suppose that there is a sequence* α_k *of defining forms for* $\mathcal{G}_1$ *with* $d\alpha_k = \beta_k \wedge \alpha_k$ *such that on each fibre* M_t *the class* $\beta_k|_{M_t}$ *tends to zero in the* $\mathcal{C}^0$*-topology as* k *tends to infinity. Then the Godbillon-Vey class* $GV(\mathcal{G}_2, \mathcal{G}_1)$ *vanishes.*

Note that the condition $\beta_k \to 0$ does *not* imply that the Reeb class $[\beta_k] \in H^1(\mathcal{G}_1)$ vanishes, since the space of boundaries need not be closed in the quotient $\Omega^1(M)/I^1(\mathcal{G}_1)$.

CHAPTER III

Universal constructions

III.1. k-Foliated principal bundles

In this chapter we will show that the Godbillon-Vey class discussed in the previous chapter is only one of a whole variety of derived classes of flags of foliations which all lead to concordance invariants of families of foliations in the way the Godbillon-Vey class did. We will explicitly compute all these derived characteristic classes for multi-parameter families of foliations. In particular, we will see that the module of derived characteristic classes for one-parameter families of foliations $\mathcal{F}_t$ is generated by the Godbillon-Vey class $GV(\mathcal{F}_t)$ of the family and the time derivative of the classical Godbillon-Vey class $\frac{\partial}{\partial t}gv(\mathcal{F}_t)$. There will be two universal constructions presented in this chapter: a more general one using k-foliated principal bundles following the construction of secondary characteristic classes of foliated principal bundles due to Kamber-Tondeur and another approach for foliated vector bundles in the fashion of Bott's construction of secondary characteristic classes of foliations. First let us describe the Chern-Weil theory for flags of foliations.

Let G be a Lie group and suppose that $p : P \to M$ is a principal G-bundle. Recall that a *connection* on P is a 1-form on P taking values in the Lie algebra $\mathfrak{g}$ which is vertical and equivariant, i. e.

(1) $\omega(X^*) = X$ for every $X \in \mathfrak{g}$,
(2) $R_g^*\omega = \mathrm{Ad}(g^{-1})(\omega)$ for every $g \in G$.

Here X^* is the fundamental vector field on P associated to $X \in \mathfrak{g}$,

$$X^*(x) = \frac{d}{dt}R_{\exp(tX)}(x)\Big|_{t=0} .$$

A connection is completely determined by its *horizontal space*, $H = \ker\omega$. Since ω is vertical, H is a distribution intersecting the tangent space of each fibre of p trivially. Because ω is equivariant, the distribution H is G-invariant. This makes it natural to consider the following generalization.

Definition III.1.1. A distribution $H \subset TP$ is a *partial connection* on P if

(1) $H_x \cap T_x p = \{0\}$ for every $x \in P$,
(2) $H_{R_g(x)} = R_{g*}H_x$ for every $x \in P$ and $g \in G$.

Since the partial connection H is G-invariant and intersects the tangent bundle to the fibres trivially, it induces a distribution $p_*(H)$ of the same dimension on M.

Definition III.1.2. A connection ω is called *adapted* to a partial connection H if the distribution H is horizontal, $\omega(H) = 0$.

By a partition of unity argument one proves that every partial connection H admits some adapted connection ω (cf. the proof of Lemma III.3.4 below). Moreover, a partial connection H is completely determined by an adapted connection and the induced distribution $p_*(H)$ on M. We are particularly interested in the case where H is the tangent bundle of a foliation on P.

Definition III.1.3. A partial connection H is called *flat* if H is involutive.

The reason for this notation is given by Lemma III.1.7 below.

Definition III.1.4. A principal bundle P is called a *foliated principal bundle* if it is equipped with a flat partial connection H.

In this case the distribution $p_*(H)$ is integrable as well. Hence, $p : P \to M$ is foliated if and only if there is a foliation $\mathcal{F}$ on M and a lift $\tilde{\mathcal{F}}$ of this foliation to P which is a foliation of the same dimension and is invariant under the principal G-action.

Example III.1.1. If $\mathcal{F}$ is the trivial foliation of M by its points, then every principal G-bundle over M is foliated. In that case, the constructions described below will yield the classical Chern-Weil construction. On the other hand, if $\mathcal{F} = \{M\}$ is the codimension 0 foliation on M, then a principal bundle over $(M, \mathcal{F})$ is foliated if and only if it is flat (cf. Proposition III.1.8). ⊠

Example III.1.2. The natural example of a foliated principal bundle is the transverse frame bundle of a foliation. Let $\mathcal{F}$ be some foliation of codimension q on M. Denote by $Q = TM/T\mathcal{F}$ the normal bundle of the foliation and let $p : P \to M$ be the transverse frame bundle, i. e. the associated principal $G = \mathrm{GL}_q(\mathbb{R})$-bundle. Let $\pi : TM \twoheadrightarrow Q$ denote the projection. The *fundamental one-form* θ on P taking values in $\mathbb{R}^q$ is defined as follows. Every point $y \in P$ with $p(y) = x$ defines a parametrization

$$y : \mathbb{R}^q \xrightarrow{\cong} Q_x \ .$$

For $Y \in T_y P$ set

$$\theta(Y) = y^{-1} \pi p_*(Y) \ .$$

Define a distribution H on P by

$$H = \{Y \in TP | \ \theta(Y) = 0, \ i_Y d\theta = 0\} \ .$$

This distribution is integrable and $\mathrm{GL}_q(\mathbb{R})$-invariant (cf. [**26**]).

Proposition III.1.5. *H is a flat partial connection and the integrating foliation $\tilde{\mathcal{F}}$ projects to $\mathcal{F}$ under p.*

Proof. It is sufficient to check this on the preimage of a foliated chart of $(M, \mathcal{F})$, i. e. we can assume that $P = \mathbb{R}^q \times \mathbb{R}^p \times \mathrm{GL}_q(\mathbb{R})$ and p is the projection onto $\mathbb{R}^q \times \mathbb{R}^p$. In that case

$$\theta_{(y,x,U)}((Y, X, A)) = U^{-1} Y$$

for $(Y,X,A) \in T_{(y,x,U)}P = \mathbb{R}^q \times \mathbb{R}^p \times \mathrm{Mat}_{q,q}(\mathbb{R})$. Hence,

$$(Y,X,A) \in \ker\theta \iff Y = 0 \ ,$$
$$(0,X,A) \in \ker d\theta \iff A = 0 \ .$$

So, $H = \{(0,X,0) \in TP\}$ and the proposition follows. □

By this proposition the transverse frame bundle of a foliation is canonically foliated. ⊠

Sometimes there are connections on a foliated bundle which are not only adapted, but also constant along the leaves of the lifted foliation. These are called basic connections.

Definition III.1.6. Let ω be a connection adapted to the partial connection H. It is called *basic* if $L_X\omega = i_X d\omega = 0$ for every $X \in H$.

The existence of a basic connection is indeed a restrictive condition on the foliated bundle P (cf. Theorem III.3.3). Be aware that some authors, e. g. Heitsch [**16**], use the notion "basic" in the sense of "adapted".

For any connection ω let Ω be the curvature of ω, i. e. the 2-form on P taking values in $\mathfrak{g}$ defined by

$$\Omega(X,Y) = (d\omega)(X,Y) + [\omega(X), \omega(Y)] \ .$$

The next lemma is immediate from the definitions.

Lemma III.1.7. *Let H be a flat partial connection on P. If ω is adapted to H, then $\Omega(X,Y) = 0$ for every $X,Y \in H$. If ω is basic, then $i_X\Omega = 0$ for every $X \in H$.*

Now, the one-form ω taking values in $\mathfrak{g}$ defines a homomorphism

$$\omega : \mathfrak{g}^* \to \Omega^1(P)$$
$$\alpha \mapsto \alpha(\omega) \ .$$

In the same way, Ω defines a homomorphism $\Omega : \mathfrak{g}^* \to \Omega^2(P)$. Denote by $\tilde{\mathcal{F}}$ the foliation integrating the flat partial connection H, then the lemma above can be restated as follows. If ω is adapted, then

$$\Omega(\beta) \in I^2(\tilde{\mathcal{F}}) = F^1_{\tilde{\mathcal{F}}}\Omega^2(P)$$

for every $\beta \in \mathfrak{g}^*$. If moreover ω is basic, then

$$\Omega(\beta) \in F^2_{\tilde{\mathcal{F}}}\Omega^2(P)$$

for every $\beta \in \mathfrak{g}^*$. We need an even stronger vanishing statement. Consider the filtration defined on $\Omega^*(P)$ by

$$F^r\Omega^*(P) = \{\eta \in \Omega^*(P) | \ \eta \text{ is locally in the ideal spanned by } p^*F^r_{\mathcal{F}}\Omega^*(M)\} \ .$$

Obviously, we have $F^r\Omega^*(P) \subset F^r_{p^*\mathcal{F}}\Omega^*(P) \subset F^r_{\tilde{\mathcal{F}}}\Omega^*(P)$. The following proposition is essential for the construction of the characteristic homomorphism of the foliation $\mathcal{F}$ due to Kamber-Tondeur [**19**].

Proposition III.1.8. *If ω is an adapted connection with curvature Ω on a foliated principal G-bundle $P \to M$, then $\Omega(\beta) \in F^1\Omega^2(P)$ for every $\beta \in \mathfrak{g}^*$. If ω is basic, then $\Omega(\beta) \in F^2\Omega^2(P)$ for every $\beta \in \mathfrak{g}^*$.*

PROOF. Let ω be an adapted connection. Since ω is vertical, the curvature Ω is horizontal, i. e. $i_{X^*}\Omega = 0$ for every fundamental vector field X^*. Again it is sufficient to check the proposition on the preimage of foliated chart, i. e. we may assume that $P = \mathbb{R}^q \times \mathbb{R}^p \times G$ is the trivial bundle. Let $dy_1, \dots, dy_q, dx_1, \dots, dx_p$ be the elements of the dual local framing of T^*P corresponding to the factor $\mathbb{R}^q \times \mathbb{R}^p$. So, the dy_i's vanish on the lifted distribution $T\tilde{\mathcal{F}}$. Since Ω is horizontal, there are functions a_{ij}, b_{ij}, c_{ij} such that $\Omega(\beta)$ has the form

$$\begin{aligned}\Omega(\beta) &= \sum_{i<j} a_{ij} dy_i \wedge dy_j + \sum_{i,j} b_{ij} dy_i \wedge dx_j + \sum_{i<j} c_{ij} dx_i \wedge dx_j \\ &= \sum_{i<j} a_{ij} dy_i \wedge dy_j + \sum_{i,j} b_{ij} dy_i \wedge dx_j\end{aligned}$$

by Lemma III.1.7. Indeed, this is an element of $F^1\Omega^2(P)$ by Lemma I.2.2. If ω is basic, then the second summand vanishes and $\Omega(\beta)$ is an element of $F^2\Omega^2(P)$. □

We are interested in flags of foliations. Hence, we have to adapt the above notions to the case where M carries a k-flag of foliations. For $k = 2$ this was done by Carballés [**8**].

Definition III.1.9. Suppose that $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is a k-flag of foliations on M. A principal G-bundle $P \to M$ is called *k-foliated*, if $G = G_k \times \cdots \times G_1$ is a product and $P = (P_k, \dots, P_1)$ is the fibre product of foliated principal G_i-bundles P_i over $(M, \mathcal{F}_i)$, where $1 \leq i \leq k$.

EXAMPLE III.1.3. Of course, every foliated principal G-bundle over $(M, \mathcal{F})$ can be viewed as a k-foliated bundle by setting $G = 1 \times \cdots \times 1 \times G$ and $(\mathcal{F}_k, \dots, \mathcal{F}_2, \mathcal{F}_1) = (\{M\}, \dots, \{M\}, \mathcal{F})$. ⊠

EXAMPLE III.1.4. The fundamental example is the following. Consider a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ of foliations on a Riemannian manifold (M, g). The *normal bundle of the k-flag* is

$$\nu(\mathcal{F}_k, \dots, \mathcal{F}_1) = Q_k \oplus \cdots \oplus Q_1 \ ,$$

where $Q_k = TM/T\mathcal{F}_k$ and $Q_i = T\mathcal{F}_{i+1}/T\mathcal{F}_i$ for $1 \leq i \leq k-1$. Via the given Riemannian metric g on M we can identify $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$ with a subbundle of TM isomorphic to the normal bundle $Q = TM/T\mathcal{F}_1$ of $\mathcal{F}_1$. In particular, we have orthogonal projections $\pi_i : TM \twoheadrightarrow Q_i$. Denote by $P_1, \dots, P_k$ the frame bundles of $Q_1, \dots, Q_k$ and set $G = \mathrm{GL}_{q_k}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{q_1}(\mathbb{R})$. The principal G-bundle $P = (P_k, \dots, P_1)$ is the *transverse frame bundle of the k-flag* $(\mathcal{F}_k, \dots, \mathcal{F}_1)$.

Proposition III.1.10. *The transverse frame bundle of a k-flag of foliations on a Riemannian manifold is canonically k-foliated.*

PROOF. The same construction as in Example III.1.2 yields fundamental forms θ_i with $\theta_i(Y) = y^{-1}\pi_i p_{i*}Y$ for $Y \in T_yP_i$. Set

$$H_i = \{Y \in TP_i|\ p_{i*}Y \in T\mathcal{F}_{i+1},\ \theta_i(Y) = 0,\ i_Y d\theta_i = 0\}\ .$$

The same argument as in the proof of Proposition III.1.5 shows that H_i is a flat partial connection on P_i which projects to $T\mathcal{F}_i$. □

Note, that $(P_k, \ldots, P_i)$ is a reduction of the transverse frame bundle of $\mathcal{F}_i$. The fundamental form θ of $\mathcal{F}_i$ then reduces to the sum $\theta_k \oplus \cdots \oplus \theta_i$ and H_i gets identified with the canonical flat partial connection of $\mathcal{F}_i$. ⊠

Definition III.1.11. A connection ω on a k-foliated bundle $P = (P_k, \ldots, P_1)$ is called *adapted* if it is a sum $\omega = \omega_k \oplus \cdots \oplus \omega_1$ of adapted connections ω_i on P_i. It is called *basic* if all ω_i are basic.

In complete analogy with the case considered before we get the following vanishing statement. Consider the filtrations

$$F_i^r\Omega^*(P) = \{\eta \in \Omega^*(P)|\ \eta \text{ is locally in the ideal spanned by } p^*F^r_{\mathcal{F}_i}\Omega^*(M)\}\ .$$

Proposition III.1.12. *Let $P = (P_k, \ldots, P_1) \to M$ be a k-foliated bundle over a manifold M equipped with a k-flag of foliations $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$. If $\omega = \omega_k \oplus \cdots \oplus \omega_1$ is an adapted connection, then $\Omega(\beta) \in F_i^1\Omega^2(P)$ for every $\beta \in \mathfrak{g}_i^*$. If ω is basic, then moreover $\Omega(\beta) \in F_i^2\Omega^2(P)$ for every $\beta \in \mathfrak{g}_i^*$.*

III.2. The characteristic homomorphism of a k-foliated bundle

Let us recall the definition of the characteristic homomorphism of a foliated principal bundle due to Kamber-Tondeur [**19**]. The *Weil algebra* associated to the Lie algebra $\mathfrak{g}$ of a Lie group G is the graded commutative algebra

$$W^*(\mathfrak{g}) = \bigoplus_{i,j\geq 0} \Lambda^i\mathfrak{g}^* \otimes S^j\mathfrak{g}^*\ .$$

The grading is defined by assigning degree i to the antisymmetric elements in $\Lambda^i\mathfrak{g}^*$ and degree $2j$ to the symmetric elements in $S^j\mathfrak{g}^*$. There is a differential d on $W^*(\mathfrak{g})$ defined as follows. If $\alpha \in \Lambda^1\mathfrak{g}^*$ and $\beta \in S^1\mathfrak{g}^*$, then

$$\begin{aligned} d(1 \otimes 1) &= 0\ , \\ d(\alpha \otimes 1) &= (d_\Lambda\alpha) \otimes 1 + 1 \otimes \alpha\ , \\ d(1 \otimes \beta) &= \sum X_j^* \otimes (-\operatorname{ad}(X_j)^*\beta)\ . \end{aligned}$$

This determines the derivation d on the whole of $W^*(\mathfrak{g})$. Here X_j is a basis of $\mathfrak{g}$ and X_j^* is the dual basis of $\mathfrak{g}^*$. The derivation $d_\Lambda : \Lambda^i\mathfrak{g}^* \to \Lambda^{i+1}\mathfrak{g}^*$ is the Chevalley-Eilenberg differential determined by

$$(d_\Lambda\alpha)(X, Y) = -\operatorname{ad}(X)^*\alpha(Y) = -\alpha([X, Y])\ .$$

The Weil algebra is a G-DG-algebra in the sense of Kamber-Tondeur [**19**] (for a quick overview see Appendix B). The G-DG-structure is given by

$$\begin{aligned} \rho(g)(\alpha \otimes \beta) &= (\mathrm{Ad}(g^{-1})^*\alpha) \otimes (\mathrm{Ad}(g^{-1})^*\beta) \ , \\ i(X)(\alpha) &= (i_X\alpha) \otimes \beta \ , \\ \theta(X)(\alpha \otimes \beta) &= (-\operatorname{ad}(X)^*\alpha) \otimes \beta + \alpha \otimes (-\operatorname{ad}(X)^*\beta) \ . \end{aligned}$$

The homomorphism defined by a connection ω on P extends to a multiplicative homomorphism

$$\omega : \bigoplus_i \Lambda^i \mathfrak{g}^* \to \Omega^*(P) \ ,$$

and the curvature of ω gives rise to a multiplicative homomorphism

$$\Omega : \bigoplus_j S^j \mathfrak{g}^* \to \Omega^{2*}(P) \ .$$

Together, they give a G-DG-algebra homomorphism (as is easily checked)

$$\begin{aligned} k(\omega) : W^*(\mathfrak{g}) &\to \Omega^*(P) \\ \alpha \otimes \beta &\mapsto \omega(\alpha) \wedge \Omega(\beta) \ , \end{aligned}$$

called the *Weil homomorphism* of ω. If $H \subset G$ is a closed subgroup, then denote by

$$W^*(\mathfrak{g}, H) = W^*(\mathfrak{g})_H = \{a \in W^*(\mathfrak{g}) |\ \rho(h)a = a,\ i_X a = 0 \text{ for all } h \in H,\ X \in \mathfrak{h}\}$$

the subalgebra of H-basic elements of $W^*(\mathfrak{g})$ (cf. Appendix B). Since $k(\omega)$ is a G-DG-algebra homomorphism, it induces a homomorphism

$$k(\omega)_H : W^*(\mathfrak{g}, H) = W^*(\mathfrak{g})_H \to \Omega^*(P)_H = \Omega^*(P/H)$$

which commutes with the differentials.

To make use of the vanishing phenomenon observed in Proposition III.1.8, define an even filtration on $W^*(\mathfrak{g})$ by

$$F^{2r-1}W^*(\mathfrak{g}) = F^{2r}W^*(\mathfrak{g}) = \bigoplus_i \bigoplus_{j \geq r} \Lambda^i \mathfrak{g}^* \otimes S^j \mathfrak{g}^*$$

for $r \geq 0$. Hence, $F^{2r}W^*(\mathfrak{g})$ is the G-DG-ideal generated by $S^r\mathfrak{g}^*$. For a foliated principal bundle P the filtration ideal $F^r\Omega^*(P)$ defined in the last section is also a G-DG ideal, and obviously $F^r\Omega^*(P) = 0$ for $r > q$. If the connection ω is adapted to $\mathcal{F}$, then $k(\omega)$ respects these filtrations by Proposition III.1.8,

$$k(\omega)\left(F^{2r}W^*(\mathfrak{g})\right) \subset F^r\Omega^*(P) \ .$$

In particular, $k(\omega)(F^{2(q+1)}W^*(\mathfrak{g})) \subset F^{q+1}\Omega^*(P) = 0$. Thus, if we denote by

$$W^*(\mathfrak{g}, H)_q = W^*(\mathfrak{g}, H)/F^{2(q+1)}W^*(\mathfrak{g}, H)$$

the truncated relative Weil algebra, then $k(\omega)_H$ gives a well-defined map

$$k(\omega)_H : W^*(\mathfrak{g}, H)_q \to \Omega^*(P/H)$$

commuting with the differentials. By standard techniques it can be seen that the induced map in cohomology does not depend on the adapted connection ω. Now, suppose that H contains a maximal compact subgroup of G. Then by a result of Mostow [**27**] the quotient G/H is contractible. Let $s : M \to P/H$ be any section which defines an H-reduction of P. Since G/H is contractible such a section exists and any two such sections are homotopic. Hence, the map

$$\Delta(\tilde{\mathcal{F}}, H) = s^* \circ k(\omega)_H : W^*(\mathfrak{g}, H)_q \to \Omega^*(M)$$

induces a homomorphism

$$\Delta(\tilde{\mathcal{F}}, H)_* : H^*(W^*(\mathfrak{g}, H)_q) \to H^*(M; \mathbb{R})$$

independent of any choices.

Theorem III.2.1 (Kamber-Tondeur)**.** *Let $P \to M$ be a foliated principal G-bundle over a foliated manifold $(M, \mathcal{F})$ and suppose that $H \subset G$ is a closed subgroup containing a maximal compact subgroup of G. Then there is a well-defined homomorphism*

$$\Delta(\tilde{\mathcal{F}}, H)_* : H^*(W^*(\mathfrak{g}, H)_q) \to H^*(M; \mathbb{R})$$

called the characteristic homomorphism of the foliated G-bundle P with respect to the subgroup H. *This homomorphism is natural with respect to pull-backs and invariant under concordance.*

Proof. It remains to show the last two statements. If $f : N \to M$ is transverse to $\mathcal{F}$, then f^*P is a foliated G-bundle over $(N, f^*\mathcal{F})$ with foliated structure $f^*\tilde{\mathcal{F}}$. If $s : M \to P/H$ is a section, then $f^*s(x) = (x, s(f(x)))$ defines a section $f^*s : M \to f^*P/H$. Using these data in the construction, it is obvious that

$$\Delta(f^*\tilde{\mathcal{F}}, H)_* = f^*\Delta(\tilde{\mathcal{F}}, H)_* \ .$$

Thus the characteristic homomorphism is natural with respect to pull-backs (justifying its name). Let us turn to the last statement. We call two foliated principal bundles P_0, P_1 over $(M, \mathcal{F}_0)$, resp. over $(M, \mathcal{F}_1)$ *concordant*, if there is a foliated principal G-bundle $P \to M \times [0,1]$ over the cylinder and a codimension q foliation $\mathcal{F}$ on $M \times [0,1]$ such that the inclusion maps $j_i : M \to M \times [0,1]$ defined by $j_i(x) = (x, i)$ are transverse to $\mathcal{F}$ with $j_i^*\mathcal{F} = \mathcal{F}_i$ and $j_i^*P \cong P_i$ as foliated bundles for $i = 0, 1$. By the naturality property we have

$$\begin{aligned}\Delta(\tilde{\mathcal{F}}_0, H)_* &= \Delta(j_0^*\tilde{\mathcal{F}}, H)_* = j_0^*\Delta(\tilde{\mathcal{F}}, H)_* = j_1^*\Delta(\tilde{\mathcal{F}}, H)_* = \Delta(j_1^*\tilde{\mathcal{F}}, H)_* \\ &= \Delta(\tilde{\mathcal{F}}_1, H)_* \ ,\end{aligned}$$

since j_0 and j_1 are homotopic. This proves the concordance invariance. □

If P is the canonically foliated transverse frame bundle of a foliation $\mathcal{F}$ on M as in Example III.1.2 with structure group $G = \mathrm{GL}_q(\mathbb{R})$ and maximal compact subgroup $H = \mathrm{O}(q)$, then we will write

$$\Delta(\mathcal{F}) : H^*(W^*(\mathfrak{gl}_q(\mathbb{R}), \mathrm{O}(q))_q) \to H^*(M; \mathbb{R})$$

instead of $\Delta(\tilde{\mathcal{F}}, H)$. This is the characteristic homomorphism of the foliation $\mathcal{F}$. Hence, all the classes in the image of $\Delta(\mathcal{F})_*$ are natural concordance invariants of the foliation $\mathcal{F}$.

EXAMPLE III.2.1. Consider $H = G$. In this case the relative Weil algebra $W^*(\mathfrak{g}, G) = (S^*\mathfrak{g}^*)^G = I^{2*}(G)$ is just the algebra of invariant polynomials on G (cf. Section III.4 below). So, $H^*(W^*(\mathfrak{g}, G)_q) = I^{2*}(G)_q = I^{2*}(G)/I^{2(*+q+1)}(G)$. For example, if $P \to M$ is any principal G-bundle foliated by points, then

$$\Delta(P)_* : I^{2*}(G) \twoheadrightarrow I^{2*}(G)/I^{2(*+n+1)}(G) \to H^{2*}(M;\mathbb{R})$$

is the classical Chern-Weil homomorphism of the principal G-bundle P. So, the image of $\Delta(P)_*$ is the ring of Pontrjagin classes $\mathrm{Pont}^*(P) \subset H^*(M;\mathbb{R})$ – the so-called primary characteristic classes of P. If the principal bundle is truely foliated, then a vanishing phenomenon for these primary classes comes into view: since the Chern-Weil homomorphism of P factors through $I^{2*}(G)/I^{2(*+q+1)}(G)$, the ring of Pontrjagin classes has to vanish in large degrees. This is the Bott Vanishing Theorem.

Theorem III.2.2 (Bott)**.** *Let $D \subset TM$ be a codimension q distribution on M. If D is integrabel, then the Pontrjagin classes of the normal bundle $Q = TM/D$ have to vanish in degrees greater than $2q$.*

⊠

Note that if the foliated bundle $P \to M$ admits a basic connection ω, then by Proposition III.1.8 the Weil homomorphism has the stronger filtration property

$$k(\omega)\left(F^{2r}W^*(\mathfrak{g})\right) \subset F^{2r}\Omega^*(P)\ . \tag{III.2.1}$$

This gives a refinement of the Bott Vanishing Theorem due to Molino [**25**].

Theorem III.2.3 (Bott-Molino)**.** *Let $(M, \mathcal{F})$ be a foliated manifold of codimension q. If a foliated principal bundle $P \to M$ over $(M, \mathcal{F})$ admits a basic connection, then* $\mathrm{Pont}^i(P) = 0$ *for $i > q$.*

The construction of the homomorphism in Theorem III.2.1 can be generalized to arbitrary k-foliated bundles in the same way that Carballés [**8**] did for $k = 2$. Suppose that $G = G_k \times \cdots \times G_1$ and that $P \to M$ is a k-foliated principal G-bundle. Then the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1$ and the Weil algebra of $\mathfrak{g}$ can be written as

$$W^*(\mathfrak{g}) = \bigoplus_{j_k,\dots,j_1 \geq 0} \Lambda^*\mathfrak{g}^* \otimes S^{j_k}\mathfrak{g}_k^* \otimes \cdots \otimes S^{j_1}\mathfrak{g}_1^*\ .$$

Now, there are k even filtrations defined on $W^*(\mathfrak{g})$. For $r \geq 0$ and $1 \leq i \leq k$ we set

$$F_i^{2r-1}W^*(\mathfrak{g}) = F_i^{2r}W^*(\mathfrak{g}) = \bigoplus_{j_k+\cdots+j_i \geq r}\ \bigoplus_{j_{i-1},\dots,j_1 \geq 0} \Lambda^*\mathfrak{g}^* \otimes S^{j_k}\mathfrak{g}_k^* \otimes \cdots \otimes S^{j_1}\mathfrak{g}_1^*\ .$$

Thus, $F_i^{2r}W^*(\mathfrak{g})$ is the G-DG-ideal generated by $(S^*\mathfrak{g}_k \otimes \cdots \otimes S^*\mathfrak{g}_i)^r$. On the other hand, there are the k filtrations $F_i^r\Omega^*(P)$ on $\Omega^*(P)$ defined in the last section. If ω is an adapted connection on P, then by Proposition III.1.12 the Weil homomorphism $k(\omega)$ respects all of these filtrations,

$$k(\omega)\left(F_i^{2r}W^*(\mathfrak{g})\right) \subset F_i^r\Omega^*(P) \ .$$

In particular, if we consider the G-DG-ideal

$$I = F_k^{2(q_k+1)}W^*(\mathfrak{g}) + \cdots + F_1^{2(q_k+\cdots+q_1+1)}W^*(\mathfrak{g}) \ ,$$

then we have $k(\omega)(I) \subset F_k^{q_k+1}\Omega^*(P) + \cdots + F_1^{q_k+\cdots+q_1+1}\Omega^*(P) = 0$. For $1 \le i \le k$ let $H_i \subset G_i$ be a closed subgroup containing a maximal compact subgroup of G_i and set $H = H_k \times \cdots \times H_1 \subset G$. Then we can define the k-truncated relative Weil algebra

$$W^*(\mathfrak{g}, H)_{(q_k,\dots,q_1)} = (W^*(\mathfrak{g})/I)_H \ .$$

In the same way as before we get a cochain map

$$\Delta(\tilde{\mathcal{F}}_k, \dots, \tilde{\mathcal{F}}_1, H) = s^* \circ k(\omega)_H : W^*(\mathfrak{g}, H)_{(q_k,\dots,q_1)} \to \Omega^*(M)$$

inducing a homomorphism

$$\Delta(\tilde{\mathcal{F}}_k, \dots, \tilde{\mathcal{F}}_1, H)_* : H^*(W(\mathfrak{g}, H)_{(q_k,\dots,q_1)}) \to H^*(M; \mathbb{R})$$

independent of any choices.

Theorem III.2.4. *Let $P \to M$ be a k-foliated principal G-bundle over a manifold M carrying a k-flag of foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$. Suppose that $H = H_k \times \cdots \times H_1 \subset G = G_k \times \cdots \times G_1$ is a product of closed subgroups $H_i \subset G_i$ containing a maximal compact subgroup of G_i. Then there is a well-defined homomorphism*

$$\Delta(\tilde{\mathcal{F}}_k, \dots, \tilde{\mathcal{F}}_1, H)_* : H^*(W(\mathfrak{g}, H)_{(q_k,\dots,q_1)}) \to H^*(M; \mathbb{R})$$

called the characteristic homomorphism of the k-foliated G-bundle P with respect to the subgroup H. *This homomorphism is natural with respect to pull-backs and invariant under concordance.*

For the definition of concordance of k-foliated principal bundles we refer to Definition III.3.2 below.

Again, if P is the canonically k-foliated transverse frame bundle of the k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ of foliations as in Example III.1.4 with structure group $G = \mathrm{GL}_{q_k}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{q_1}(\mathbb{R})$ and maximal compact subgroup $H = \mathrm{O}(q_k) \times \cdots \times \mathrm{O}(q_1)$, then we will write

$$\Delta(\mathcal{F}_k, \dots, \mathcal{F}_1)_* : H^*(W(\mathfrak{g}, H)_{(q_k,\dots,q_1)}) \to H^*(M; \mathbb{R})$$

for the characteristic homomorphism of the k-flag of foliations. The image of $\Delta(\mathcal{F}_k, \dots, \mathcal{F}_1)_*$ consists of classes which are natural concordance invariance of the k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$.

Obviously, for $k = 1$ this is the same construction as before, and moreover the characteristic homomorphism $\Delta(\{M\}, \mathcal{F}_k, \dots, \mathcal{F}_1)_*$ is equal to the characteristic homomorphism $\Delta(\mathcal{F}_k, \dots, \mathcal{F}_1)_*$. So, the characteristic homomorphism is natural with respect to the inclusion map $\mathcal{F}ol(k) \to \mathcal{F}ol(k+1)$.

III.3. Universal derived characteristic classes

The characteristic homomorphisms are already defined at the level of the spectral sequences. Because if we choose the reduction $s : M \to P/H$ to be a product of reductions $s_i : M \to P_i/H_i$, then the induced map respects the filtrations,

$$s^* \left(F_i^r \Omega^*(P/H)\right) \subset F_{\mathcal{F}_i}^r \Omega^*(M) \ .$$

Thus, the construction above gives rise to induced homomorphisms

$$\Delta(\tilde{\mathcal{F}}_k, \ldots, \tilde{\mathcal{F}}_1, H)_* : E_j^{2r,s-r}(W(\mathfrak{g}, H)_{(q_k,\ldots,q_1)}, F_i) \to E_j^{r,s}(M, \mathcal{F}_i)$$

which are independent of choices for $j \geq 1$. Note, that the image of the homomorphism $\Delta(\tilde{\mathcal{F}}, H)_* : E_j^{*,*}(W(\mathfrak{g}, H)_q, F) \to E_j^{*,*}(M, \mathcal{F})$ is exactly what Kamber-Tondeur [**18**] called the derived characteristic classes of the foliation $\mathcal{F}$.

Since we are interested in derived characteristic classes at stage k, we have to consider the homomorphisms $\epsilon_{k-1} \cdots \epsilon_i \circ \Delta(\tilde{\mathcal{F}}_k, \ldots, \tilde{\mathcal{F}}_1, H)_*$ which are maps

$$E_1^{2(q_i+\cdots+q_k),s-2(q_i+\cdots+q_k)}(W(\mathfrak{g}, H)_{(q_k,\ldots,q_1)}, F_i) \to H^{s-q_k}(M, \mathcal{F}_k) \ .$$

So, let

$$D_iCh_{(q_k,\ldots,q_1)}^{2(q_i+\cdots+q_k)+s}(\mathfrak{g}, H) = E_1^{2(q_i+\cdots+q_k),s}(W(\mathfrak{g}, H)_{(q_k,\ldots,q_1)}, F_i) \ .$$

We define the module of *universal derived characteristic classes* to be

$$DCh_{(q_k,\ldots,q_1)}^*(\mathfrak{g}, H) = \bigoplus_{i=1}^{k-1} D_iCh_{(q_k,\ldots,q_1)}^*(\mathfrak{g}, H) \ .$$

The direct sum of the homomorphisms $\epsilon_{k-1} \cdots \epsilon_i \circ \Delta(\tilde{\mathcal{F}}_k, \ldots, \tilde{\mathcal{F}}_1, H)_*$ defines the derived characteristic homomorphism.

Theorem III.3.1. *Let $P \to M$ be a k-foliated bundle over a manifold M carrying a k-flag $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$. There is a well-defined homomorphism*

$$D\Delta(\tilde{\mathcal{F}}_k, \ldots, \tilde{\mathcal{F}}_1, H)_* : DCh_{(q_k,\ldots,q_1)}^*(\mathfrak{g}, H) \to H^{*-q_k}(M, \mathcal{F}_k) \ ,$$

called the derived characteristic homomorphism of the k-foliated principal bundle*, which is natural under pull-backs and invariant under leafwise concordance.*

We have to explain what we mean by leafwise concordance of k-foliated principal bundles.

Definition III.3.2. Let P_0 and P_1 be two k-foliated principal G-bundles over M, k-foliated with respect to k-flags of foliations $(\mathcal{F}_k^0, \ldots, \mathcal{F}_1^0)$, resp. $(\mathcal{F}_k^1, \ldots, \mathcal{F}_1^1)$. Suppose that there is a k-foliated principal G-bundle P over the cylinder $M \times [0,1]$ carrying a k-flag of foliations $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$. Let $j_t : M \to M \times [0,1]$ denote the inclusion maps defined by $j_t(x) = (x,t)$. Then P_0 and P_1 are *leafwise concordant* if the following conditions are satisfied.

(1) j_0 and j_1 are transverse to $\mathcal{F}_1$, such that j_0^*P and j_1^*P are k-foliated bundles over M foliated by $(j_0^*\mathcal{F}_k, \ldots, j_0^*\mathcal{F}_1)$, resp. by $(j_1^*\mathcal{F}_k, \ldots, j_1^*\mathcal{F}_1)$.
(2) P_i are isomorphic to j_i^*P as foliated bundles for $i = 0, 1$.

(3) $\mathcal{F}_k^0$ equals $\mathcal{F}_k^1$, and $\mathcal{F}_k$ is the product foliation $\mathcal{F}_k^0 \times [0,1]$.

We call two k-foliated principal bundels P_0 and P_1 *concordant* if they are leafwise concordant except that we do not require (3) any more.

Then the same argument as in the proof of Theorem III.2.1 shows that the derived characteristic homomorphism is indeed invariant under leafwise concordance, since j_0 and j_1 are leafwise homotopic with respect to the top foliation.

Note that the filtration property (III.2.1) gives a general vanishing theorem for derived classes.

Theorem III.3.3. *Suppose that $P \to M$ is a k-foliated principal bundle over a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ of foliations admitting a basic connection. If $\mathcal{F}_k$ has positive codimension, then the derived characteristic homomorphism is trivial,*

$$D\Delta(\tilde{\mathcal{F}}_k, \dots, \tilde{\mathcal{F}}_1, H)_* = 0 \ .$$

Let us see if we can find a non-trivial derived characteristic homomorphism.

Example III.3.1. Reconsider Example III.1.4 again. We want to identify the Godbillon-Vey class of a k-flag of foliations as an element in the image of the derived characteristic homomorphism. If we denote by

$$tr_i : \mathfrak{g}_i = \mathrm{Mat}_{q_i,q_i}(\mathbb{R}) \to \mathbb{R}$$

the trace function, then tr_i is obviously G_i-invariant and $d_\Lambda tr_i = 0$. Since $\mathfrak{h}_i$ consists of trace-free matrices, $i_X tr_i$ vanishes for all $X \in \mathfrak{h}_i$ and the differential of

$$x_i = \underbrace{(tr_k + \cdots + tr_i)}_{\in \Lambda \mathfrak{g}^*} \otimes \underbrace{(tr_k + \cdots + tr_i)^{q_k + \cdots + q_i}}_{\in S^* \mathfrak{g}_k^* \otimes \cdots \otimes S^* \mathfrak{g}_i^*} \in W^{2(q_i + \cdots + q_k)+1}(\mathfrak{g}, H)_{(q_k, \dots, q_1)}$$

satisfies

$$d(x_i) = 1 \otimes (tr_k + \cdots + tr_i)^{q_k + \cdots + q_i + 1} = 0$$

in the k-truncated relative Weil algebra. Thus, x_i represents a class

$$[x_i] \in D_i Ch^{2(q_i + \cdots + q_k)+1}_{(q_k, \dots, q_1)}(\mathfrak{g}, H) \ .$$

Lemma III.3.4. *If $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is a k-flag of cooriented foliations on a Riemannian manifold M, then there is an adapted connection ω on the transverse frame bundle P of the k-flag, a section $s : M \to P/H$ and a representative $\beta_i \in \Omega^1(M)$ of the Reeb class of $\mathcal{F}_i$ such that*

$$s^* k(\omega)_H((tr_k + \cdots + tr_i) \otimes 1) = \beta_i \ .$$

Proof. Choose sections $s_i : M \to P_i/\,\mathrm{O}(q_i)$ compatible with the fixed coorientations. Locally these sections can be lifted to local framings $s_i : M \to P_i$. For simplicity let us assume $k = 1$, i. e. let $P \to M$ be the transverse frame bundle of a foliated manifold $(M,\mathcal{F})$, and $s=(g_1,\dots,g_q)$ with linearly independent sections g_j of $Q=TM/T\mathcal{F}$. Denote by $\gamma_1,\dots,\gamma_q$ the dual local framing of $Q^*\subset\Omega^1(M)$. Then locally

$$\alpha=\gamma_1\wedge\cdots\wedge\gamma_q$$

is a defining form for $\mathcal{F}$. The proof of Theorem A.5 in Appendix A gives local one-forms η_{ij} such that

$$d\gamma_i=\sum_{j=1}^{q}\eta_{ij}\wedge\gamma_j\ ,$$

and such that $\beta=\sum_{j=1}^{q}\eta_{jj}$ satisfies $d\alpha=\beta\wedge\alpha$. Denote by $\eta=(\eta_{ij})$ the matrix of one-forms. Extending the prescription $\omega(s_*X)=\eta(X)$ for $X\in TM$ equivariantly to the whole of TP defines locally an adapted connection $\omega: TP\to\mathfrak{g}=\mathrm{Mat}_{q,q}(\mathbb{R})$. Hence,

$$s^*k(\omega)(tr\otimes 1)(X)=(tr\circ\omega)(s_*X)=tr(\eta(X))=\sum_{j=1}^{q}\eta_{jj}(X)=\beta(X)\ .$$

Glueing with a partition of unity gives an adapted connection ω and a representative β of the Reeb class of $\mathcal{F}$ such that

$$s^*k(\omega)_{\mathrm{O}(q)}(tr\otimes 1)=\beta\ .$$

The same argument goes through for $k>1$. □

Call $GV=\sum_{i=1}^{k-1}[x_i]$ the *universal Godbillon-Vey class.* Then the following theorem is immediate.

Theorem III.3.5. *Let $(\mathcal{F}_k,\dots,\mathcal{F}_1)$ be a k-flag of cooriented foliations on M. If $GV\in DCh^*_{(q_k,\dots,q_1)}(\mathfrak{g},H)$ is the universal Godbillon-Vey class, then*

$$D\Delta(\mathcal{F}_k,\dots,\mathcal{F}_1)_*(GV)=GV(\mathcal{F}_k,\dots,\mathcal{F}_1)\ .$$

Proof. The previous lemma yields an adapted connection on the transverse frame bundle P, a section $s: M\to P$ and a representative β_i of the Reeb class of $\mathcal{F}_i$ such that

$$s^*k(\omega)_H((tr_k+\cdots+tr_i)\otimes 1)=\beta_i\ .$$

This implies

$$\begin{aligned}s^*k(\omega)_H(1\otimes(tr_k+\cdots+tr_i)) &= s^*k(\omega)_{\mathrm{O}(q)}(d((tr_k+\cdots+tr_i)\otimes 1))\\ &= d(s^*k(\omega)_{\mathrm{O}(q)}((tr_k+\cdots+tr_i)\otimes 1))\\ &= d\beta_i\ .\end{aligned}$$

So, $s^*k(\omega)_H(x_i)=\beta_i\wedge(d\beta_i)^{q_k+\cdots+q_i}$ which is a representative for $GV(\mathcal{F}_k,\mathcal{F}_i)$ in $H^{2(q_i+\cdots+q_{k-1})+q_k+1}(M,\mathcal{F}_k)$. This proves the theorem. □

By Theorem II.2.5 and Theorem III.3.5 we know that the derived characteristic homomorphism is not always trivial. ⊠

Of course, we can apply the above construction to families of foliations.

Example III.3.2. Consider a family of foliations $(\mathcal{G}_2,\mathcal{G}_1)$ parameterized by a fibre bundle $p : X \to B$ with fibre M. If we fix a volume form on B, then the derived characteristic homomorphism becomes a map

$$D\Delta(\mathcal{G}_2,\mathcal{G}_1)_* : DCh^*_{(q_2,q_1)}(\mathfrak{g},H) \to \Gamma(\mathcal{H}^{*-q_2}(M;\mathbb{R})) \ .$$

Assume that B is an affine flat manifold, i. e. TB carries a flat linear connection. Since P_2 is the pull-back of the frame bundle of TB it admits a flat basic connection. Hence, all classes with representatives containing a non-trivial $S^*\mathfrak{g}_2$-part are mapped to zero. This means we can replace $DCh^*_{(q_2,q_1)}(\mathfrak{g},H)$ by

$$\begin{aligned} C^*_{(q_2,q_1)} &= H^*((\Lambda^*(\mathfrak{gl}_{q_2}(\mathbb{R})^* \oplus \mathfrak{gl}_{q_1}(\mathbb{R})^*) \otimes S^{q_1+q_2}\mathfrak{gl}_{q_1}(\mathbb{R})^*)_{\mathrm{O}(q_2)\times\mathrm{O}(q_1)}) \\ &= H^*(\mathfrak{gl}_{q_2}(\mathbb{R}),\mathrm{O}(q_2)) \otimes H^*((\Lambda^*\mathfrak{gl}_{q_1}(\mathbb{R})^* \otimes S^{q_2+q_1}\mathfrak{gl}_{q_1}(\mathbb{R})^*)_{\mathrm{O}(q_1)}) \ . \end{aligned}$$

For example, let $q_2 = q_1 = 1$. If dt denotes the generator of $\mathfrak{gl}_{q_2}(\mathbb{R})^* = \mathfrak{gl}_1(\mathbb{R})^*$ and dy the generator of $\mathfrak{gl}_{q_1}(\mathbb{R})^* = \mathfrak{gl}_1(\mathbb{R})^*$, then

$$C^i_{(1,1)} = \begin{cases} \mathbb{R}\cdot 1 \otimes (dy)^2 & , \text{ for } i=4 \\ \mathbb{R}\cdot dt \otimes (dy)^2 \oplus \mathbb{R}\cdot dy \otimes (dy)^2 & , \text{ for } i=5 \\ \mathbb{R}\cdot (dt\wedge dy) \otimes (dy)^2 & , \text{ for } i=6 \\ 0 & , \text{ otherwise} \ . \end{cases}$$

If $(\mathcal{G}_2,\mathcal{G}_1)$ is a q_2-parameter family of foliations, then the section $s_2 : X \to P_2$ used in the construction of the derived characteristic homomorphism can be chosen to be horizontal. In this case all classes in $C^*_{(q_2,q_1)}$ with representatives containing a $H^i(\mathfrak{gl}_{q_2}(\mathbb{R}),\mathrm{O}(q_2))$-part with $i > 0$ are mapped to zero. Hence, we can replace $C^*_{(q_2,q_1)}$ by

$$D^*_{(q_2,q_1)} = H^*((\Lambda^*\mathfrak{gl}_{q_1}(\mathbb{R})^* \otimes S^{q_2+q_1}\mathfrak{gl}_{q_1}(\mathbb{R})^*)_{\mathrm{O}(q_1)})$$

to get the following theorem.

Theorem III.3.6. *Let $\mathcal{F}_t$ be a q_2-parameter family of codimension q_1 foliations on M. There is a well-defined homomorphism*

$$D\Delta(\mathcal{F}_t)_* : H^*((\Lambda^*\mathfrak{gl}_{q_1}(\mathbb{R})^* \otimes S^{q_2+q_1}\mathfrak{gl}_{q_1}(\mathbb{R})^*)_{\mathrm{O}(q_1)}) \to \mathcal{C}^\infty(\mathbb{R}^{q_2}, H^{*-q_2}(M;\mathbb{R})) \ ,$$

called the derived characteristic homomorphism of the q_2-parameter family of foliations*, which is natural with respect to pull-backs and invariant under concordance of families of foliations.*

Let us compute the derived characteristic classes for one-parameter families of codimension one foliations.

Theorem III.3.7. *Let $\mathcal{F}_t$ be a one-parameter family of cooriented codimension one foliations on M and β_t a smooth family of representatives for the Reeb classes of the foliations. Then the subspace of derived characteristic classes in the space of smooth functions from $\mathbb{R}$ to $H^*(M;\mathbb{R})$ is at most two-dimensional, generated by the Godbillon-Vey class of the family,*

$$GV(\mathcal{F}_t) = 2\cdot[\dot{\beta}_t \wedge \beta_t \wedge d\beta_t] \in H^4(M;\mathbb{R}) \ ,$$

and by the time derivative of the family of Godbillon-Vey classes of the foliations,

$$\tfrac{\partial}{\partial t} gv(\mathcal{F}_t) = 2 \cdot [\dot{\beta}_t \wedge d\beta_t] \in H^3(M;\mathbb{R}) \ .$$

Proof. The space of universal derived classes is

$$D^i_{(1,1)} = \begin{cases} \mathbb{R} \cdot 1 \otimes (dy)^2 & , \text{ for } i = 4 \\ \mathbb{R} \cdot dy \otimes (dy)^2 & , \text{ for } i = 5 \\ 0 & , \text{ otherwise .} \end{cases}$$

The universal Godbillon-Vey class is

$$GV = (dt + dy) \otimes (dt + dy)^2 = (dt + dy) \otimes (2dt \otimes dy + (dy)^2)$$

in $W^5(\mathfrak{g}, H)_{(1,1)}$. So, in $D^5_{(1,1)}$ the Godbillon-Vey class is represented by $dy \otimes (dy)^2$. By Theorem III.3.5 and Theorem II.4.3 we get

$$D\Delta(\mathcal{F}_t)_*(dy \otimes (dy)^2) = GV(\mathcal{F}_t) \ .$$

By Lemma III.3.4 the class $1 \otimes (dy)^2$ in $D^4_{(1,1)}$ which is equivalent to the universal class $1 \otimes (dt + dy)^2$ in $W^4(\mathfrak{g}, H)_{(1,1)}$ is mapped to $[(d\beta)^2] \in H^3(\mathbb{R} \times M, \mathcal{G}_2)$. By the proof of Theorem II.4.3 this is the class evaluating as

$$D\Delta(\mathcal{F}_t)_*(1 \otimes (dy)^2) = -[\dot{\beta}_t \wedge d\beta_t] \ \in H^3(M;\mathbb{R}) \ .$$

On the other hand,

$$\tfrac{\partial}{\partial t}(\beta_t \wedge d\beta_t) = \dot{\beta}_t \wedge d\beta_t + \beta_t \wedge d\dot{\beta}_t \sim 2 \cdot \dot{\beta}_t \wedge d\beta_t \ ,$$

since $d(\dot{\beta}_t \wedge \beta_t) = d\dot{\beta}_t \wedge \beta_t - \dot{\beta}_t \wedge d\beta_t$. This proves the second statement. □

This theorem justifies again the term "derived classes". ⊠

III.4. Linear connections and flags of foliations

If we are just interested in the normal bundle of a k-flag of foliations and not in k-foliated principal bundles in general, then there is a construction analogous to the classical construction by Bott [**3**] which gives another description of the characteristic homomorphism of a k-flag of foliations. For $k = 2$ this is due to Cordero and Masa [**7**]. With regard to the computations done in the next chapter this discription is more convenient.

Let $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ be a k-flag of foliations on a Riemannian manifold M and denote by $Q_k = TM/T\mathcal{F}_k$, $Q_{k-1} = T\mathcal{F}_k/T\mathcal{F}_{k-1}, \dots$, $Q_1 = T\mathcal{F}_2/T\mathcal{F}_1$ the relative normal bundles of the foliations which we can identify via the given Riemannian metric with mutually orthogonal subbundles of TM. Denote the orthogonal projections by $\pi_i : TM \twoheadrightarrow Q_i$. We have an orthogonal decomposition

$$TM = \nu(\mathcal{F}_k, \dots, \mathcal{F}_1) \oplus T\mathcal{F}_1 \ .$$

Use the notation $\pi_0 : TM \twoheadrightarrow T\mathcal{F}_1$ for the orthogonal projection onto the remaining summand $T\mathcal{F}_1 \subset TM$. To shorten the notation we will often write Y_i for $\pi_i(Y)$ in the following.

Definition III.4.1. A linear connection $\nabla^i : \Gamma(TM) \otimes \Gamma(Q_i) \to \Gamma(Q_i)$ on Q_i is *adapted* to $\mathcal{F}_i$ if for every $X \in \Gamma(T\mathcal{F}_i)$ and every $Y_i \in \Gamma(Q_i)$ we have

$$\nabla^i_X Y_i = \pi_i[X, Y_i] \ .$$

An adapted linear connection on $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is a linear connection which decomposes as a direct sum $\nabla = \nabla^k \oplus \cdots \oplus \nabla^1$ such that each ∇^i is a linear connection on Q_i adapted to $\mathcal{F}_i$.

Lemma III.4.2. *Suppose that $\tilde{\nabla}^i$ is any connection on Q_i. Then*

$$\nabla^i_X Y_i = \tilde{\nabla}^i_{X_k + \cdots + X_i} Y_i + \pi_i[X_{i-1} + \cdots + X_0, Y_i]$$

defines a linear connection on Q_i which is adapted to $\mathcal{F}_i$.

Proof. First we have to check that the prescription above defines a linear connection on Q_i. Consider $X \in \Gamma(TM), Y_i \in \Gamma(Q_i)$ and a function f on M, then

$$\begin{aligned}
\nabla^i_{fX} Y_i &= \tilde{\nabla}^i_{f(X_k + \cdots + X_i)} Y_i + \pi_i[f(X_{i-1} + \cdots + X_0), Y_i] \\
&= f\tilde{\nabla}^i_{X_k + \cdots + X_i} Y_i + f\pi_i[X_{i-1} + \cdots + X_0, Y_i] + (Y_i.f)\pi_i(X_{i-1} + \cdots + X_0) \\
&= f\nabla^i_X Y_i
\end{aligned}$$

and

$$\begin{aligned}
\nabla^i_X (fY_i) &= \tilde{\nabla}^i_{X_k + \cdots + X_i} (fY_i) + \pi_i[X_{i-1} + \cdots + X_0, fY_i] \\
&= ((X_k + \cdots + X_i).f)Y_i + f\tilde{\nabla}^i_{X_k + \cdots + X_i} Y_i \\
&\quad + ((X_{i-1} + \cdots + X_0).f)Y_i + f\pi_i[X_{i-1} + \cdots + X_0, Y_i] \\
&= (X.f)Y_i + f\nabla^i_X Y_i \ .
\end{aligned}$$

Moreover, if $X \in T\mathcal{F}_i = Q_{i-1} \oplus \cdots \oplus Q_1 \oplus T\mathcal{F}_1$, then $\nabla^i_X Y_i = \pi_i[X, Y_i]$. Thus, ∇^i is a linear connection adapted to $\mathcal{F}_i$. □

This lemma yields that for every k-flag of foliations $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ there is an adapted linear connection on $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$. Note that such a linear connection ∇ defines a linear connection on $\nu(\mathcal{F}_1) = TM/T\mathcal{F}_1$ as well. But in general ∇ is not adapted to $\mathcal{F}_1$. Nevertheless it induces an adapted linear connection ∇' on $\nu(\mathcal{F}_1)$ via

$$\nabla'_X(Y_k + \cdots + Y_1) = \nabla_X(Y_k + \cdots + Y_1) + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \pi_i[X_0, Y_j] \ .$$

Definition III.4.3. A vector bundle $E_i \to M$ is called *foliated* with respect to $\mathcal{F}_i$ if there is a linear connection ∇^i on E_i such that the curvature R^i of ∇^i is zero along $\mathcal{F}_i$, i. e.

$$R^i(X, Y) = \nabla^i_X \nabla^i_Y - \nabla^i_Y \nabla^i_X - \nabla^i_{[X,Y]} = 0$$

for all $X, Y \in T\mathcal{F}_i$. A vector bundle $E \to M$ is called *k-foliated* with respect to a k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on M if there is a linear connection ∇ on E, and E

decomposes into a direct sum $E = E_k \oplus \cdots \oplus E_1$ of vector bundles over M such that ∇ decomposes as a direct sum $\nabla = \nabla^k \oplus \cdots \oplus \nabla^1$ of linear connections ∇^i on E_i turning each E_i into a vector bundle foliated with respect to $\mathcal{F}_i$.

Proposition III.4.4. *Let $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ be a k-flag of foliations on a Riemannian manifold M. Any linear connection on the normal bundle $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$ adapted to the k-flag turns $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$ into a k-foliated vector bundle.*

Proof. We only have to show that the bundle $Q_i \to M$ together with any linear connection ∇^i adapted to $\mathcal{F}_i$ is foliated with respect to $\mathcal{F}_i$. Consider $X, Y \in T\mathcal{F}_i$ and $Z_i \in Q_i$. Since ∇^i is adapted,

$$\begin{aligned} R^i(X,Y)Z_i &= \pi_i[X, \pi_i[Y, Z_i]] - \pi_i[Y, \pi_i[X, Z_i]] - \pi_i[[X,Y], Z_i] \\ &= \pi_i[X, \pi_i[Y, Z_i]] + \pi_i[Y, \pi_i[Z_i, X]] + \pi_i[Z_i, [X,Y]] \\ &= \pi_i[X, [Y, Z_i]] - \sum_{j=0}^{i-1} \pi_i[X, \pi_j[Y, Z_i]] \\ &\quad + \pi_i[Y, [Z_i, X]] - \sum_{j=0}^{i-1} \pi_i[Y, \pi_j[X, Z_i]] \\ &\quad + \pi_i[Z_i, [X,Y]] \ , \end{aligned}$$

for $X, Y, Z_i \in T\mathcal{F}_{i+1}$ implies $[X, Z_i], [Y, Z_i] \in T\mathcal{F}_{i+1}$,

$$\begin{aligned} &= \pi_i([X, [Y, Z_i]] + [Y, [Z_i, X]] + [Z_i, [X,Y]]) + \\ &\quad - \sum_{j=0}^{i-1} (\pi_i[X, \pi_j[Y, Z_i]] + \pi_i[Y, \pi_j[X, Z_i]]) \\ &= 0 \end{aligned}$$

by the Jacobian identity and the fact that $X, Y \in T\mathcal{F}_i$ implies $[X, \pi_j W] \in T\mathcal{F}_i$ and $[Y, \pi_j W] \in T\mathcal{F}_i$ for every $j < i$ and all $W \in TM$. □

This proposition is the linear analogon to Proposition III.1.10. More generally we have the following.

Proposition III.4.5. *Let $P \to M$ be a k-foliated principal G-bundle and suppose that $\rho : G = G_k \times \cdots \times G_1 \to \mathrm{GL}_{q_k}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{q_1}(\mathbb{R})$ is a product of k linear representations $\rho_i : G_i \to \mathrm{GL}_{q_i}(\mathbb{R})$. Denote by $E = P \times_G \mathbb{R}^{q_k + \cdots + q_1} \to M$ the associated vector bundle. If ω is an adapted connection on P, then the associated linear connection ∇ turns E into a k-foliated vector bundle.*

Proof. A section $s \in \Gamma(E)$ is the same as a G-equivariant map $s : P \to \mathbb{R}^{q_k + \cdots + q_1}$. Let H be the horizontal space of ω. If $X \in T_x M$ and $y \in P_x$, then denote by X^h the unique horizontal lift of X to H_y. The linear connection associated to ω is defined by

$$(\nabla_X s)(y) = s_*(X^h) \ .$$

This map is G-equivariant in y, thus is indeed a section $\nabla_X s$ of E. Obviously, ∇ decomposes into linear connections ∇^i under the decomposition of the bundle $E = E_k \oplus \cdots \oplus E_1$ into vector bundles $E_i = P_i \times_G \mathbb{R}^{q_i}$ associated to the representations ρ_i. To be precise $(\nabla^i_X s_i)(y) = s_{i*}(X^h)$ for a section $s_i : P_i \to \mathrm{GL}_{q_i}(\mathbb{R})$ of E_i. The curvatures of these linear connections are

$$(R^i(X,Y)s_i)(y) = -(\rho_{i*}\Omega_i(X^h,Y^h))(s_i(y)) \ ,$$

where Ω_i is the i'th summand in the curvature of the adapted connection $\omega = \omega_k \oplus \cdots \oplus \omega_1$ on P, as is easily computed (cf. [**28**], where another sign convention is used in the definition of the curvature R^i). In particular, if X, Y are tangent to $\mathcal{F}_i$, then X^h, Y^h are tangent to the lifted foliation $\tilde{\mathcal{F}}$ on P because ω_i is adapted to $\mathcal{F}_i$. So, by Proposition III.1.12 we have $R^i(X,Y) = 0$ for $X, Y \in T\mathcal{F}_i$. Hence, $E = E_k \oplus \cdots \oplus E_1$ is k-foliated. □

Let $E_i \to M$ be a vector bundle foliated with respect to $\mathcal{F}_i$ by the linear connection ∇^i. Choose a local frame on E_i. Then ∇^i with respect to this frame is described by a matrix (ω^i_{rs}) of one-forms on M and the curvature is described by a matrix (Ω^i_{rs}) of two-forms on M. By definition, we have the following linear version of Proposition III.1.12.

Proposition III.4.6. *Let $E = E_k \oplus \cdots \oplus E_1$ be a vector bundle, k-foliated by a linear connection ∇. For $i = 1, \ldots, k$ choose a local frame s_i of E_i. Then $s_k \wedge \cdots \wedge s_1$ is a local frame of E. With respect to this frame, the curvature of ∇ is described by a matrix (Ω_{rs}) of two-forms decomposing into k blocks (Ω^i_{rs}) such that each two-form Ω^i_{rs} is in $F^1_{\mathcal{F}_i}\Omega^2(M)$.*

III.5. Universal derived characteristic classes in the spirit of Bott

Let us use the above to construct the characteristic homomorphism in the linear setting. Recall (e. g. from [**3**]) that the ring $I^*(\mathrm{GL}_{q_i}(\mathbb{R})) = (S^*\mathfrak{gl}_{q_i}(\mathbb{R}))^{\mathrm{GL}_{q_i}(\mathbb{R})}$ of invariant polynomials on $\mathrm{GL}_{q_i}(\mathbb{R})$ is a free commutative algebra generated by the *Chern polynomials* $c_1, \ldots, c_{q_i}$,

$$I^*(\mathrm{GL}_{q_i}(\mathbb{R})) = \mathbb{R}[c_1, \ldots, c_{q_i}]$$

which are given by the equation

$$\det\left(I + \frac{t}{2\pi}A\right) = \sum_{j=1}^{q_i} c_j(A)t^j$$

for all $A \in \mathfrak{gl}_{q_i}(\mathbb{R})$. The Chern polynomial c_j is considered to have degree $2j$. If $E_i \to M$ is a vector bundle and ∇^i is an arbitrary linear connection on E_i, then we can choose a local frame of E_i and apply c_j to the matrix (Ω^i_{rs}) of local curvature forms of ∇^i to get a locally defined $2j$-form on M. Using a partition of unity we can paste these local forms together to get a global form $c_j(R^i) \in \Omega^{2j}(M)$. This form only depends on ∇^i and is known to be closed (cf. [**24**]). So, the algebra homomorphism

$$\begin{aligned} I^*(\mathrm{GL}_{q_i}) &\to \Omega^*(M) \\ c_j &\mapsto c_j(R^i) \end{aligned}$$

induces an algebra homomorphism

$$I^*(\mathrm{GL}_{q_i}) \to H^*(M;\mathbb{R}) \ , \tag{III.5.1}$$

known as the *Chern-Weil homomorphism* of the vector bundle $E_i \to M$.

If $\tilde\nabla^i$ is another linear connection on E_i then we can define "comparison forms" $c_j(\nabla^i, \tilde\nabla^i)$ in the following way. Let $\hat\nabla^i$ be the linear connection on the vector bundle $E^i \times \mathbb{R} \to M \times \mathbb{R}$ defined by convex combination,

$$\hat\nabla^i = (1-t)\nabla^i + t\tilde\nabla^i \ .$$

Set

$$c_j(\nabla^i, \tilde\nabla^i) = \int_0^1 c_j(\hat R^i)|_{M\times[0,1]} \ \in \Omega^{2j-1}(M) \ .$$

These forms satisfy

$$dc_j(\nabla^i, \tilde\nabla^i) = c_j(R^i) - c_j(\tilde R^i) \ . \tag{III.5.2}$$

This shows that the Chern-Weil homomorphism (III.5.1) is independent of the choice of linear connection. But actually it gives more than that. Denote by $\Lambda(h_1, \dots, h_{q_i})$ the exterior algebra generated by elements h_j of degree $2j-1$. Set

$$W_i^* = \Lambda(h_1, \dots, h_{q_i}) \otimes \mathbb{R}[c_1, \dots, c_{q_i}]$$

and define a graded algebra map by

$$\begin{aligned} \lambda_i : W_i^* &\to \Omega^*(M) \\ c_j &\mapsto c_j(R^i) \\ h_j &\mapsto c_j(\nabla^i, \tilde\nabla^i) \ . \end{aligned}$$

If we have a vector bundle $E = E_k \oplus \cdots \oplus E_1$ with linear connections $\nabla, \tilde\nabla$ which decompose into direct sums $\nabla = \nabla_k \oplus \cdots \oplus \nabla_1$ (and similarly for $\tilde\nabla$), then we consider the tensor product

$$W^* = W_k \otimes \cdots \otimes W_1$$

and set

$$\lambda(E, \nabla, \tilde\nabla) = \lambda_k \otimes \cdots \otimes \lambda_1 : W^* \to \Omega^*(M) \ .$$

The algebra W^* carries even filtrations $F_i W^*$. Namely $F_i^{2r} W^* = F_i^{2r-1} W^*$ is the ideal generated by monomials $(c_{J_k})_k \otimes \cdots \otimes (c_{J_i})_i$ of degree $2(|J_k| + \cdots + |J_i|) \geq 2r$, where $|J_s| = |(j_1, \dots, j_{q_s})| = \sum_{k=1}^{q_s} k j_k$. Denote by I^* the ideal

$$I^* = F_k^{2(q_k+1)} W^* + \cdots + F_1^{2(q_k+\cdots+q_1+1)} W^* \ .$$

If we suppose that $E = E_k \oplus \cdots \oplus E_1 \to M$ is a vector bundle k-foliated with respect to $(\mathcal{F}_k, \ldots, \mathcal{F}_1)$ by the partially flat connections ∇^i, then by Proposition III.4.6 the forms $c_j(R^i)$ are in $F^j_{\mathcal{F}_i}\Omega^{2j}(M)$ and the homomorphism λ vanishes on I^*. So we get a well-defined homomorphism

$$\lambda(E, \nabla, \tilde{\nabla}) : W^*_{(q_k,\ldots,q_1)} = W^*/I^* \to \Omega^*(M) \ .$$

On W_i^* we can define a differential d by

$$dc_j = 0,\ dh_j = c_j \ .$$

This turns W^* into a differential graded algebra. Since I^* is a differential ideal, $W^*_{(q_k,\ldots,q_1)}$ inherits a differential graded structure. Note, that although W^* is acyclic, $W^*_{(q_k,\ldots,q_1)}$ is not. Suppose that all the $\tilde{\nabla}^i$ are flat linear connections. Then Equation (III.5.2) implies that λ commutes with the differentials. Hence, in this case we get a homomorphism

$$\lambda(E, \nabla)_* : H^*(W^*_{(q_k,\ldots,q_1)}) \to H^*(M; \mathbb{R}) \ .$$

This homomorphism does not depend on the choice of the flat connections $\tilde{\nabla}^i$ any more (cf. the proof of Theorem III.5.1 below) and depends only on the restrictions of ∇^i to the tangent bundle of the foliation $T\mathcal{F}_i$, i. e. of the foliated structure on the foliated vector bundles E_i. Applying this to the normal bundle $\nu(\mathcal{F}_k, \ldots, \mathcal{F}_1)$ of the k-flag, we get a canonical homomorphism

$$\lambda(\mathcal{F}_k, \ldots, \mathcal{F}_1)_* : H^*(W^*_{(q_k,\ldots,q_1)}) \to H^*(M; \mathbb{R})$$

under the assumption that for every i the normal bundle of $\mathcal{F}_i$ in $\mathcal{F}_{i+1}$ carries a flat linear connection.

If the vector bundles E_i do not admit a flat linear connection, we have to consider a subcomplex of W^*. Let $l_i = 2\left[\frac{q_i+1}{2}\right] - 1$ be the largest odd number less than or equal to q_i and set

$$WO_i^* = \Lambda(h_1, h_3, \ldots, h_{l_i}) \otimes \mathbb{R}[c_1, \ldots, c_{q_i}] \ .$$

Then we get a homomorphism

$$\lambda(E, \nabla, \tilde{\nabla}) : WO^*_{(q_k,\ldots,q_1)} \to \Omega^*(M)$$

in the same way as above. We can not choose $\tilde{\nabla}^i$ to be a flat linear connection any more, but we can suppose that $\tilde{\nabla}^i$ is a orthogonal connection on E_i, i. e. for some metric g on E_i we have $X.g(s_1, s_2) = g(\tilde{\nabla}^i_X s_1, s_2) + g(s_1, \tilde{\nabla}^i_X s_2)$ for all $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E_i)$. This implies that the local connection matrices ω (with respect to an orthonormal local frame) are skew-symmetric. Then the local curvature matrices (Ω_{rs}) are skew-symmetric as well and the odd classes $c_{2j-1}(\tilde{R}^i)$ have to vanish, since $c_j(A^t) = c_j(A)$ and $c_j(-A) = (-1)^j c_j(A)$. So, by Equation (III.5.2) again, $\lambda(E, \nabla, \tilde{\nabla})$ is a cochain map inducing a homomorphism

$$\lambda(E, \nabla)_* : H^*(WO^*_{(q_k,\ldots,q_1)}) \to H^*(M; \mathbb{R}) \ .$$

Let us summarize this.

Theorem III.5.1. *Suppose that $E \to M$ is a vector bundle which is k-foliated by a linear connection $\nabla = \nabla^k \oplus \cdots \oplus \nabla^1$. Then there is a well-defined homomorphism*

$$\lambda(E, \nabla)_* : H^*(WO^*_{(q_k,\dots,q_1)}) \to H^*(M; \mathbb{R}) \ .$$

called the characteristic homomorphism of the k-foliated vector bundle E, *only depending on the foliated structure of the vector bundle E. In particular, for any k-flag $(\mathcal{F}_k, \dots, \mathcal{F}_1)$ on M there is a canonical characteristic homomorphism*

$$\lambda(\mathcal{F}_k, \dots, \mathcal{F}_1)_* : H^*(WO^*_{(q_k,\dots,q_1)}) \to H^*(M; \mathbb{R})$$

which is natural with respect to pull-backs and is invariant under concordance.

Proof. To simplify the notation assume $k = 1$. Suppose that $\tilde{\nabla}^0$ and $\tilde{\nabla}^1$ are two orthogonal connections (with respect to the same metric) on E. Then the convex combination $\tilde{\nabla} = (1-s)\tilde{\nabla}^0 + s\tilde{\nabla}^1$ is an orthogonal connection on the product bundle $E \times \mathbb{R} \to M \times \mathbb{R}$ with curvature forms

$$\tilde{\Omega}_{ij} = (\tilde{\omega}^1_{ij} - \tilde{\omega}^0_{ij}) \times ds + (1-s)\tilde{\Omega}^0_{ij} \times 1 + s\tilde{\Omega}^1_{ij} \times 1 \ . \tag{III.5.3}$$

Consider the convex combination

$$\begin{aligned} \hat{\nabla} &= (1-s)\big((1-t)\nabla + t\tilde{\nabla}^0\big) + s\big((1-t)\nabla + t\tilde{\nabla}^1\big) \\ &= (1-t)\nabla + t\big((1-s)\tilde{\nabla}^0 + s\tilde{\nabla}^1\big) \end{aligned}$$

on $E \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R} \times \mathbb{R}$. If $j_s : M \to M \times \mathbb{R}$ denotes the inclusion map $j_s(x) = (x, s)$, then by Equation (III.5.3)

$$c_j(\nabla, \tilde{\nabla}^i) = \int_0^1 c_j(\tilde{R}^i) = \int_0^1 j_i^* c_j(\hat{R}) = j_i^* c_j(\nabla, \tilde{\nabla})$$

for $i = 0, 1$. By the homotopy invariance of the de Rham cohomology we get that the homomorphism in cohomology induced by $\lambda(E, \nabla, \tilde{\nabla}^0)$ is the same as the one induced by $\lambda(E, \nabla, \tilde{\nabla}^1)$. Formula (III.5.3) shows that if we have two linear connections which agree and are both flat along $\mathcal{F}$, then their convex combination is flat along $\mathcal{F} \times \{\mathbb{R}\}$. The same argument as before shows that the induced homomorphisms in cohomology agree. So, the characteristic homomorphism $\lambda(E, \nabla)_*$ depends only on the foliated structure of E.

If $f : N \to M$ is transverse to $\mathcal{F}_1$, then $f^*\nu(\mathcal{F}_k, \dots, \mathcal{F}_1) \cong \nu(f^*\mathcal{F}_k, \dots, f^*\mathcal{F}_1)$. By definition of the adapted linear connection the characteristic homomorphisms satisfy $\lambda(f^*\mathcal{F}_k, \dots, f^*\mathcal{F}_1)_* = f^*\lambda(\mathcal{F}_k, \dots, \mathcal{F}_1)_*$. The concordance invariance follows by this naturality property and the homotopy invariance of the de Rham cohomology. □

Since the foliated structure on $\nu(\mathcal{F}_k, \dots, \mathcal{F}_1)$ is canonically given by the Bott connection $\pi_i[X, Y_i]$, the characteristic homomorphism $\lambda(\mathcal{F}_k, \dots, \mathcal{F}_1)_*$ of a k-flag of foliations on a Riemannian manifold is canonical.

By Proposition III.4.6 the homomorphism $\lambda(E,\nabla,\tilde{\nabla})$ respects the filtrations in the sense that

$$\lambda(E,\nabla,\tilde{\nabla})(F_i^{2r}WO^*_{(q_k,\dots,q_1)}) \subset F^r_{\mathcal{F}_i}\Omega^*(M) \ .$$

Hence, it induces spectral sequence maps

$$\lambda(E,\nabla)_* : E_j^{2r,s-r}(WO^*_{(q_k,\dots,q_1)},F_i) \to E_j^{r,s}(M,\mathcal{F}_i)$$

only depending on the foliated structure on E for $j \geq 1$. We can use this construction to yield derived characteristic classes for the k-flag of foliations. Set

$$D_iWO^{2(q_i+\dots+q_k)+s}_{(q_k,\dots,q_1)} = E_1^{2(q_i+\dots+q_k),s}(WO^*_{(q_k,\dots,q_1)},F_i)$$

and

$$DWO^*_{(q_k,\dots,q_1)} = \bigoplus_{i=1}^{k} D_iWO^*_{(q_k,\dots,q_1)} \ .$$

Then the direct sum of the homomorphisms $\epsilon_{k-1}\cdots\epsilon_i \circ \lambda(\mathcal{F}_k,\dots,\mathcal{F}_1)_*$ defines the derived characteristic homomorphism in the manner of Bott.

Theorem III.5.2. *Let $(\mathcal{F}_k,\dots,\mathcal{F}_1)$ be a k-flag of foliations on a Riemannian manifold M. There is a well-defined* derived characteristic homomorphism

$$D\lambda(\mathcal{F}_k,\dots,\mathcal{F}_1)_* : DWO^*_{(q_k,\dots,q_1)} \to H^{*-q_k}(M,\mathcal{F}_k)$$

which is natural under pull-backs and invariant under leafwise concordance.

Example III.5.1. Since c_1 is equal to $\frac{1}{2\pi}\, tr$ we can easily identify the universal Godbillon-Vey class in our new model. By definition $dc_1 = 0$ and $dh_1 = c_1$. Therefore the element

$$x_i = ((h_1)_k + \dots + (h_1)_i) \otimes ((c_1)_k + \dots + (c_1)_i)^{q_k+\dots+q_i} \ \in WO^{2(q_k+\dots+q_i)+1}_{(q_k,\dots,q_1)} \ .$$

defines a class $[x_i]$ in $D_iWO^{2(q_k+\dots+q_i)+1}_{(q_k,\dots,q_1)}$. Call $GV = \sum_{i=1}^{k-1}(2\pi)^{q_k+\dots+q_i+1}\cdot[x_i]$ the universal Godbillon-Vey class in $DWO^*_{(q_k,\dots,q_1)}$. Then we get the following analogue of Theorem III.3.5.

Theorem III.5.3. *Let $(\mathcal{F}_k,\dots,\mathcal{F}_1)$ be a k-flag of foliations. For the universal Godbillon-Vey class $GV \in DWO^*_{(q_k,\dots,q_1)}$ we have*

$$D\lambda(\mathcal{F}_k,\dots,\mathcal{F}_1)_*(GV) = GV(\mathcal{F}_k,\dots,\mathcal{F}_1) \ .$$

Proof. Just like in the proof of Theorem III.3.5 we only have to show that there are an adapted linear connection ∇ and an orthogonal connection $\tilde{\nabla}$ on $\nu(\mathcal{F}_k,\dots,\mathcal{F}_1)$ such that

$$\lambda(\nu(\mathcal{F}_k,\dots,\mathcal{F}_1),\nabla,\tilde{\nabla})(((h_1)_k + \dots + (h_1)_i) \otimes 1) = \tfrac{1}{2\pi}\,\beta_i$$

for some representative β_i of the Reeb class of $\mathcal{F}_i$. For simplicity we may assume that $k=1$. Again we can construct the data locally and get the general statement by a partition of unity argument. Let $s_1, \ldots, s_q$ be a an orthonormal frame of $\nu(\mathcal{F})$ and denote by $\gamma_1, \ldots, \gamma_q$ the dual forms in $\Omega^1(M)$. If $d\gamma_i = \sum_{j=1}^q \eta_{ij} \wedge \gamma_j$, then a representative of the Reeb class of $\mathcal{F}$ is given by $\beta = \sum_{i=1}^q \eta_{ii}$. An adapted linear connection can be defined by

$$\nabla_{s_k} Y = \sum_{i=1}^q \left(s_k.\gamma_i(Y) - \sum_{j=1}^q \gamma_j(Y)\eta_{ij}(s_k) \right) s_i$$

and by the Bott connection

$$\nabla_X Y = \sum_{i=1}^q \gamma_i([X,Y]) s_i$$

for $X \in T\mathcal{F}$. This linear connection is described by the connection matrix (ω_{ij}) with

$$\omega_{ij}(X) = \gamma_i([X, s_j]) = -d\gamma_i(X, s_j)$$

for $X \in T\mathcal{F}$ and $\omega_{ij}(s_k) = -\eta_{ij}(s_k)$. Let $\tilde{\nabla}$ be any orthogonal connection. By Equation (III.5.3) we get

$$c_1(\nabla, \tilde{\nabla}) = \frac{1}{2\pi} tr(\tilde{\omega}_{ij} - \omega_{ij}) = -\frac{1}{2\pi} \sum_{i=1}^q \omega_{ii} \ ,$$

since $(\tilde{\omega}_{ij})$ is skew-symmetric. If $X \in T\mathcal{F}$, then

$$\begin{aligned} c_1(\nabla, \tilde{\nabla})(X) &= \frac{1}{2\pi} \sum_{i=1}^q d\gamma_i(X, s_i) = \frac{1}{2\pi} \sum_{i,j=1}^q (\eta_{ij} \wedge \gamma_j)(X, s_i) \\ &= \frac{1}{2\pi} \sum_{i=1}^q \eta_{ii}(X) = \frac{1}{2\pi} \beta(X) \end{aligned}$$

and of course $c_1(\nabla, \tilde{\nabla})(s_k) = \frac{1}{2\pi} \sum_{i=1}^q \eta_{ii}(s_k) = \frac{1}{2\pi} \beta(s_k)$. Hence, $c_1(\nabla, \tilde{\nabla}) = \frac{1}{2\pi} \beta$ as it should be. □

By Theorem III.5.3 and Theorem II.2.5 the derived characteristic homomorphism $D\lambda(\mathcal{F}_k, \ldots, \mathcal{F}_1)_*$ is not trivial in general. ⊠

Again we can consider the derived characteristic homomorphism of families of foliations.

Example III.5.2. Suppose that $(\mathcal{G}_2, \mathcal{G}_1)$ is a family of foliations parameterized by a fibre bundle $p : X \to B$ with fibre M. Assume that B is parallelizable. Then $\nu(\mathcal{F}_2) = p^*TB$ carries a flat orthogonal connection ∇^2 adapted to $\mathcal{F}_2$ (see Lemma IV.9 in Chapter IV). Hence, $\lambda_2 : W_2^* \to \Omega^*(M)$ is trivial. Thus, we can replace $WO^*_{(q_2,q_1)}$ by

$$\Lambda(h_1, \ldots, h_{l_1}) \otimes \mathbb{R}[c_1, \ldots, c_{q_1}]/J^* \ ,$$

where J^* is the ideal generated by the monomials $c_1^{j_1} \cdots c_{q_1}^{j_{q_1}}$ with $j_1 + 2j_2 + \cdots + q_1 j_{q_1} \geq q_2 + q_1 + 1$. Then $DWO^*_{(q_2,q_1)}$ is replaced by the cohomology module $D^*_{(q_2,q_1)}$ of the subcomplex generated by the monomials $h_I c_1^{j_1} \cdots c_{q_1}^{j_{q_1}}$ with exponents satisfying $j_1 + 2j_2 + \cdots + q_1 j_{q_1} = q_2 + q_1$.

Theorem III.5.4. *For every q_2-parameter family $\mathcal{F}_t$ of codimension q_1 foliations on a manifold M there is a well-defined derived characteristic homomorphism*

$$D\lambda(\mathcal{F}_t)_* : D^*_{(q_2,q_1)} \to \mathcal{C}^\infty(\mathbb{R}^{q_2}, H^{*-q_2}(M;\mathbb{R}))$$

which is natural under pull-backs and invariant under concordance. A basis for the real vector space $D^*_{(q_2,q_1)}$ is given by

$$(h_{i_1} \wedge \cdots \wedge h_{i_s}) \otimes (c_1^{j_1} \cdots c_{q_1}^{j_{q_1}})$$

with $1 \leq i_1 < \cdots < i_s \leq l_1 = 2\left[\frac{q_1+1}{2}\right] - 1$ odd and $j_1 + 2j_2 + \cdots + q_1 j_{q_1} = q_2 + q_1$.

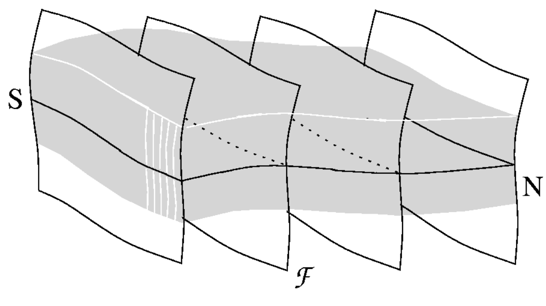
We see again that the space of derived characteristic classes for one-parameter families of codimension one foliations has the two generators $GV(\mathcal{F}_t)$ and $\frac{\partial}{\partial t} gv(\mathcal{F}_t)$ which correspond to the universal derived classes $8\pi^3\, h_1 \otimes (c_1)^2$ and $-8\pi^2\, 1 \otimes (c_1)^2$ respectively. ⊠

CHAPTER IV

The residue theorem

In this chapter we shall consider singular three-flags of foliations, i. e. subfoliations $(\mathcal{F}_3, \mathcal{F}_1)$ on M such that there is a third foliation $\mathcal{F}_2$ defined outside a closed subset $S \subset M$ such that $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is a three-flag of foliations on $M \setminus S$. We will see that if the foliation $\mathcal{F}_2$ is nice enough, then the derived characteristic classes of the subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ can be computed out of the local behaviour of the three-flag $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ near the singular set S.

For this we need a localizing tool which we will develop first: the Thom homomorphism of the singular set. Let M be an n-manifold carrying a foliation $\mathcal{F}$ of codimension q. Suppose that there is a closed cooriented submanifold $S \subset M$ of dimension $p \geq q$ which intersects $\mathcal{F}$ transversally. Then the intersections of the leaves of $\mathcal{F}$ with S define a foliation $\mathcal{F}_S$ on S. Recall that a *tubular neighbourhood* of S is an open neighbourhood N of S in M which carries a vector bundle structure $\pi : N \to S$, such that $S \subset N$ is the zero section of π. Moreover, our tubular neighbourhoods are supposed to be *regular*, i. e. the closure $\bar{N}$ of N should carry the structure of a closed disc bundle $D^{n-p} \rightarrowtail \bar{N} \xrightarrow{\bar{\pi}} S$, such that the open disc bundle is isomorphic to $\pi : N \to S$ as a smooth fibre bundle. The fibres of π are called the *slices* of N.



Lemma IV.1. *There is a tubular neighbourhood $N \subset M$ of S, such that the slices of N are tangent to the leaves of $\mathcal{F}$.*

Proof. Choose a Riemannian metric on M and consider the Riemannian exponential map $\exp : \Omega \subset TM \to M$ which assigns $c_X(1)$ to every $X \in \Omega_x \subset T_xM$, where c_X is the geodesic with $c_X(0) = x$ and $\dot{c}_X(0) = X$. This is a well-defined smooth map on some neighbourhood Ω of the zero section M of TM.

Moreover, the differential of exp on the zero section of TM is the homomorphism

$$\exp_*|_M : T\Omega|_M = TM \oplus TM \to TM \ .$$
$$(v,w) \mapsto v+w \ .$$

Denote by $Q \to S$ the normal bundle of the submanifold $S \subset M$. Then we can restrict the exponential map to $Q^\Omega = \Omega|_S \cap Q$ to get a map $\exp_Q : Q^\Omega \to M$ whose differential restricted to the zero section S of Q^Ω,

$$\exp_{Q_*}|_S : T(TQ^\Omega)|_S = TS \oplus Q \to TS \oplus Q = TM|_S \ ,$$

is the identity. Since S is compact, there is an $\epsilon > 0$ such that $Q^\epsilon = \{X \in Q| \ ||X|| < \epsilon\}$ is a neighbourhood of the zero section in Q^Ω and $\exp_Q : Q^\epsilon \to M$ is a diffeomorphism onto a neighbourhood N of S. Because S is transverse to $\mathcal{F}$ we can choose the Riemannian metric on M in such a way that Q is tangent to $\mathcal{F}$. Then $N = \exp_Q(Q^\epsilon)$ has the desired property. □

Let us call such an N a *combed* tubular neighbourhood of S. On any tubular neighbourhood N of S there are two induced foliations of codimension q. The one given by pulling back $\mathcal{F}_S$ to N via the projection $\pi : N \to S$ (containing the foliation by the slices) and the one obtained by restriction of $\mathcal{F}$ to the open set N. If N is a combed tubular neighbourhood, then these two foliations coincide, $\mathcal{F}_N = \pi^*\mathcal{F}_S$. Because S is cooriented, the vector bundle $\pi : N \to S$ carries a natural orientation. As shown in Chapter I we get a Thom isomorphism $\Phi : H^*(S, \mathcal{F}_S) \to H^{*+n-p}_{vc}(N, \pi^*\mathcal{F}_S)$. Denote by $t : H^*(S, \mathcal{F}_S) \to H^{*+n-p}(M, \mathcal{F})$ the composition

$$H^*(S, \mathcal{F}_S) \xrightarrow{\Phi} H^{*+n-p}_{vc}(N, \pi^*\mathcal{F}_S) = H^{*+n-p}_{vc}(N, \mathcal{F}_N) \to H^{*+n-p}(M, \mathcal{F})$$

of the Thom isomorphism with the map given by extending forms with fibrewise compact support in N trivially to the rest of M.

Lemma IV.2. *The homomorphism t does not depend on the combed tubular neighbourhood N.*

Proof. If we have two different vector bundle structures on the same N, then they are linked by an orientation preserving isomorphism, since N can always be identified with the oriented normal bundle of S. Hence $\int_\pi$ does not depend on the vector bundle structure on N and neither does Φ. Suppose that $N' \subset N$ is a neighbourhood of S contained in N. Then it contains an open disc bundle N_0 inheriting the bundle structure from N turning it into a combed tubular neighbourhood. Let a be in $H^*(S, \mathcal{F}_S)$. Since we can choose the support of the form ϵ in the proof of Theorem I.6.2 arbitrarily small, we can construct a representative for the class $\Phi(a) \in H^{*+p-q}(N, \mathcal{F}_N)$ with fiberwise compact support contained in N_0. Hence, it does not matter if we use N or N_0 to define t. Finally, if we start with two different combed tubular neighbourhoods N_1 and N_2, then choosing a combed tubular neighbourhood $N_0 \subset N_1 \cap N_2$ yields that both give the same t. □

This homomorphism $t : H^*(S, \mathcal{F}_S) \to H^{*+n-p}(M, \mathcal{F})$ is the *Thom homomorphism of* S.

Definition IV.3. A *singular three-flag of foliations* $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ on a manifold M is a subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ on M together with a foliation $\mathcal{F}_2$ defined outside a closed subset $S \subset M$, the *singular set*, such that S is saturated with respect to $\mathcal{F}_1$ and $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is a three-flag of foliations on $M \setminus S$. Let us call the singular set S *tame* if S is a closed submanifold with finitely many connected components transverse to $\mathcal{F}_3$ such that $\mathcal{F}_3$ intersects S along $\mathcal{F}_1$, i. e. $\mathcal{F}_3|_S = \mathcal{F}_1|_S$. In that case S has codimension $q_1 + q_2$.

Such singular three-flags arise for example by varying the foliation $\mathcal{F}_1$ in a direction tangent to $\mathcal{F}_3$ as will be explained in Example IV.3 below.

Definition IV.4. An *infinitesimal automorphism* X of a two-flag $(\mathcal{F}_3, \mathcal{F}_1)$ of foliations is a vector field $X \in \Gamma(T\mathcal{F}_3)$ such that for every vector field $Y \in \Gamma(T\mathcal{F}_1)$ the Lie bracket $[X, Y]$ is in $\Gamma(T\mathcal{F}_1)$ as well.

Example IV.3. Suppose X is an infinitesimal automorphism of a subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ of codimension $(q_3, q_2 + 1)$. If X is tangent to $\mathcal{F}_1$ in some point $x \in M$, then X is tangent to $\mathcal{F}_1$ on the whole leaf of $\mathcal{F}_1$ containing x. So, the space

$$S = \{x \in M |\ X(x) \text{ is tangent to } \mathcal{F}_1\}$$

is a union of leaves of $\mathcal{F}_1$. By the Frobenius Theorem the distribution on $M \setminus S$ spanned by $T\mathcal{F}_1$ and X is integrable to some foliation $\mathcal{F}_2$. So, we get a singular three-flag of foliations $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ of codimension $(q_3, q_2, 1)$ on M with singular set S. ⊠

Suppose that $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is a singular three-flag of foliations with oriented tame singular set S. Denote by $S_1, \ldots, S_k$ the connected components of S. For every component S_i choose a tubular neighbourhood $N_i \subset M$ which is combed with respect to $\mathcal{F}_3$ such that $\bar{N}_i \cap \bar{N}_j = \varnothing$ for $i \neq j$. Set $N = \bigcup_i N_i$. There is a Thom homomorphism

$$t_i : H^*(S_i, \mathcal{F}_1) \to H^{*+q_1+q_2}(M, \mathcal{F}_3)$$

for each singularity S_i.

Definition IV.5. Consider a singular three-flag of foliations $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ and a combed tubular neighbourhood N of the singular set S. A linear connection ∇ on $\nu(\mathcal{F}_3, \mathcal{F}_1)$ is *adapted to* $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ *with respect to* U, if it is adapted to the subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$ and on a small neighbourhood U of $M \setminus N$ it decomposes under the isomorphism $\nu(\mathcal{F}_3, \mathcal{F}_1)|_U \cong \nu(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)|_U$ into a sum of linear connections such that it is adapted to $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$.

Lemma IV.6. *There is an adapted linear connection* ∇ *for every singular three-flag of foliations* $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ *with respect to every neighbourhood* U *of* $M \setminus N$ *with* $\bar{U} \cap S = \varnothing$.

Proof. Consider any adapted linear connection $\nabla^3 \oplus \nabla^2 \oplus \nabla^1$ on the normal bundle $\nu(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)|_{M \setminus S} = Q_3 \oplus Q_2 \oplus Q_1$ of the three-flag $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$. Define a linear connection $\tilde{\nabla}^1$ on $\tilde{Q}_1 = Q_2 \oplus Q_1$ by

$$\tilde{\nabla}^1_X(Y_2 + Y_1) = \nabla^2_X Y_2 + \nabla^1_X Y_1 + \pi_1[X_0, Y_2]$$

(cf. Section III.4). Then $\tilde{\nabla} = \nabla^3 \oplus \tilde{\nabla}^1$ is an adapted linear connection on the normal bundle $\nu(\mathcal{F}_3, \mathcal{F}_1)|_{M \setminus S} = Q_3 \oplus \tilde{Q}_1$. Choose an open neighbourhood $V \subset N$ of S with $V \cap U = \varnothing$. If $\tilde{\nabla}'$ is any adapted linear connection on $\nu(\mathcal{F}_3, \mathcal{F}_1)|_V$, then we can paste $\tilde{\nabla}$ and $\tilde{\nabla}'$ together by a partition of unity subordinate to the cover $\{M \setminus S, V\}$ of M to get an adapted linear connection ∇ on $\nu(\mathcal{F}_3, \mathcal{F}_1)$ which decomposes over U into summands

$$\pi_i \nabla_X Y_i = \pi_i \tilde{\nabla}_X Y_i = \nabla^i_X Y_i \ .$$

Hence, ∇ is adapted to the singular three-flag $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$. □

For our residue theorem we need an adapted linear connection whose first summand does not contribute to the characteristic homomorphism.

Definition IV.7. A linear connection ∇ adapted to a singular three-flag of foliations $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ with respect to U is called *reducible* if it decomposes over U in such a way that the summand ∇^1 on Q_1 is a flat orthogonal connection adapted to $\mathcal{F}_1$. We say that $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is *reducible* if it admits a reducible adapted linear connection for every neighbourhood U of $M \setminus N$ with $\bar{U} \cap S = \varnothing$.

Lemma IV.8. *Suppose that $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is a singular three-flag of foliations such that the bundle $Q_1 = T\mathcal{F}_2/T\mathcal{F}_1$ is trivial and there is an orthonormal frame $s_1, \dots, s_{q_1}$ spanning Q_1 which consists of infinitesimal automorphisms of the subfoliation $(\mathcal{F}_2, \mathcal{F}_1)$. Then $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is reducible.*

Proof. By the proof of Lemma IV.6 we just have to construct a flat orthogonal connection ∇^1 on the trivial bundle $Q_1 \to M \setminus S$ adapted to $\mathcal{F}_1$. Set

$$\nabla^1_X \left(\sum_{j=1}^{q_1} a_j s_j \right) = \sum_{j=1}^{q_1} (X.a_j) s_j \ .$$

Then the connection matrix vanishes. Hence, ∇^1 is a flat orthogonal connection on Q_1. Moreover, if $X_0 \in \Gamma(T\mathcal{F}_1)$, then

$$\begin{aligned} \pi_1 \left[X_0, \left(\sum_{j=1}^{q_1} a_j s_j \right) \right] &= \sum_{j=1}^{q_1} (X_0.a_j) s_j + \sum_{j=1}^{q_1} a_j \pi_1 [X_0, s_j] \\ &= \nabla^1_{X_0} \left(\sum_{j=1}^{q_1} a_j s_j \right) \end{aligned}$$

since $[X_0, s_j] \in T\mathcal{F}_1$. Thus ∇^1 is an adapted linear connection. □

EXAMPLE IV.4. Of course, if $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ is the singular three-flag generated by an infinitesimal automorphism X of the subfoliation $(\mathcal{F}_3, \mathcal{F}_1)$, then the conditions of the lemma are satisfied with $s_1 = X$. ⊠

The same argument as in the proof of Lemma IV.8 shows that for a family of foliations parameterized by a parallelizable manifold the normal bundle of the ambient foliation does not contribute to the characteristic homomorphism.

Lemma IV.9. *If $(\mathcal{G}_2, \mathcal{G}_1)$ is a family of foliations parameterized by a fibre bundle $p : X \to B$ with parallelizable base space B, then $\nu(\mathcal{G}_2, \mathcal{G}_1)$ carries an adapted linear connection $\nabla = \nabla^2 \oplus \nabla^1$ such that ∇^2 is a flat orthogonal connection.*

PROOF. To use the argument above, we only have to show that $\nu(\mathcal{G}_2)$ has an orthonormal frame $s_1, \dots, s_{q_2}$ such that $[s_j, X] \in \Gamma(T\mathcal{G}_2)$ for every $X \in \Gamma(T\mathcal{G}_2)$. Since B is parallelizable, there is an orthonormal frame $s_1, \dots, s_{q_2}$ of TB. Since $p_* : Q_2 \to p^*(TX)$ is an isomorphism, we can pull back this frame to get an orthonormal frame $p^*s_1, \dots, p^*s_{q_2}$ of Q_2 with respect to the pull-back metric. But obviously the Lie bracket of a vector field tangent to the fibres with the pull-back of a vector field on the base space is again tangent to the fibres. □

This lemma was already used in Section III.5 of the previous chapter.

Let us return to singular three-flags of foliations. The main result of this chapter is the following theorem which is a generalization of Heitsch's Residue Theorem [**16**] to singular three-flags of foliations.

Theorem IV.10. *Let $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ be a reducible singular three-flag of foliations with cooriented tame singular set $S = S_1 \cup \dots \cup S_k$ and $q_1 > 0$. Then every class $y \in D^j_{(q_3, q_2+q_1)}$ determines a class*

$$\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S_i) \in H^{j-q_3-q_2-q_1}(S_i, \mathcal{F}_1)$$

such that

1. $\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S_i)$ *depends only on the behaviour of $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ in an arbitrarily small neighbourhood of S_i,*
2. $\sum_{i=1}^k t_i(\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S_i)) = D\lambda(\mathcal{F}_3, \mathcal{F}_1)_*(y)$.

PROOF. Let U be some neighbourhood of $M \smallsetminus N$ disjoint from S with a reducible linear connection ∇ adapted to $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$. Let

$$y = (h_{i_1} \wedge \dots \wedge h_{i_s}) \otimes (c_1^{j_1} \cdots c_{q_2+q_1}^{j_{q_2+q_1}})$$

with $1 \le i_1 < \dots < i_s \le l_1 = 2\left[\frac{q_1+q_2+1}{2}\right] - 1$ odd and $j_1 + 2j_2 + \dots + (q_1+q_2)j_{q_1+q_2} = q_3 + q_2 + q_1$ be some basis element of $D^*_{(q_3, q_2+q_1)}$. Then

$$\begin{aligned}(\lambda(\mathcal{F}_3, \mathcal{F}_1)(y))|_U &= (\lambda(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)(y))|_U \\ &= (\lambda(\mathcal{F}_3, \mathcal{F}_2)(y))|_U \;\in F^{q_3+q_2+q_1}_{\mathcal{F}_2}\Omega^*(M) = 0 \;,\end{aligned}$$

since $\mathcal{F}_2$ has codimension $q_3 + q_2 < q_3 + q_2 + q_1$. The first equality holds because ∇ is adapted to the singular three-flag and the second one because ∇ is reducible.

So, $(\lambda(\mathcal{F}_3, \mathcal{F}_1)[y])|_N$ has fibrewise compact support in N and we get a class

$$(D\lambda(\mathcal{F}_3, \mathcal{F}_1)_*[y])|_N \in H^{j-q_3}_{vc}(N, \mathcal{F}_3|_N) \ ,$$

where $j = i_1 + \cdots + i_s + 2(q_3 + q_2 + q_1)$ is the degree of y. Now, define

$$\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S) = \int_\pi (D\lambda(\mathcal{F}_3, \mathcal{F}_1)_*[y])|_N \in H^{j-q_3-q_2-q_1}(S, \mathcal{F}_1) \ ,$$

and $\mathrm{Res}_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S_i)$ to be the i'th summand in the decomposition of the residue $Res_y(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S) \in H^*(S, \mathcal{F}_1) = H^*(S_1, \mathcal{F}_1) \oplus \cdots \oplus H^*(S_k, \mathcal{F}_1)$. Here $\pi : N \to S$ denotes the projection of the tubular neighbourhood N onto S. Properties 1. and 2. are immediate. The only thing we have to check is that the class $(D\lambda(\mathcal{F}_3, \mathcal{F}_1)_*[y])|_N \in H^{j-q_3}_{vc}(N, \mathcal{F}_3|_N)$ does not depend on the choice of adapted linear connection (up to now we just know this for its image in $H^{j-q_3}(N, \mathcal{F}_3|_N)$). This follows by the standard convexity argument. □

This theorem has an immediate corollary.

Corollary IV.11. *If a q_2-parameter family of foliations $\mathcal{F}_t$ of positive codimension on M admits an infinitesimal automorphism without singularities, then the derived characteristic homomorphism is trivial, $D\lambda(\mathcal{F}_t)_* = 0$.*

Now, let us change our point of view to the local behaviour of a singular three-flag of foliations generated by an infinitesimal automorphism of a subfoliation. Let $\pi : N \to S$ be an oriented $(q_2 + 1)$-disc bundle carrying a reducible singular three-flag of foliations $(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1)$ with singular set equal to the zero section S, such that $\pi^*(\mathcal{F}_1|_S) = \mathcal{F}_3$ and $q_1 = 1$. Denote by ∂N the sphere bundle of N and let $i : \partial N \rightarrowtail N \smallsetminus S$ be the inclusion map. Since $(N \smallsetminus S, \pi^*\mathcal{F}_1)$ is leafwise homotopy equivalent to $(\partial N, \pi^*\mathcal{F}_1)$ we get an isomorphism $H^*(N \smallsetminus S, \mathcal{F}_3) \to H^*(\partial N, \mathcal{F}_3)$. Denote by

$$i^* : H^*(N, \mathcal{F}_3) \to H^*(\partial N, \mathcal{F}_3)$$

the restriction map and by

$$\delta : D^*_{(q_3,q_2)} \to D^{*+1}_{(q_3,q_2+1)}$$

the map induced by the composition of the inclusion $j : WO^*_{(q_3,q_2)} \to WO^*_{(q_3,q_2+1)}$ with the differential $d : WO^*_{(q_3,q_2+1)} \to WO^{*+1}_{(q_3,q_2+1)}$. The next theorem may prove helpful when constructing further examples.

Theorem IV.12. *In this situation the equality*

$$\oint_\pi i^* D\lambda(\mathcal{F}_3, \mathcal{F}_2)_*[y] = \mathrm{Res}_{\delta y}(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_1, S)$$

holds for every $y \in D^*_{(q_3,q_2)}$.

PROOF. By Theorem I.6.5 we have

$$\begin{aligned}\oint_\pi i^*\lambda(\mathcal{F}_3,\mathcal{F}_2)(y) &= \int_\pi d\lambda(\mathcal{F}_3,\mathcal{F}_2)(y) - d\int_\pi \lambda(\mathcal{F}_3,\mathcal{F}_2)(y)\\ &= \int_\pi \lambda(\mathcal{F}_3,\mathcal{F}_2)(dy) - d\int_\pi \lambda(\mathcal{F}_3,\mathcal{F}_2)(y)\ .\end{aligned}$$

Hence,

$$\oint_\pi i^* D\lambda(\mathcal{F}_3,\mathcal{F}_2)_*[y] = \mathrm{Res}_{\delta y}(\mathcal{F}_3,\mathcal{F}_2,\mathcal{F}_1,S)$$

□

Note that δy vanishes, unless y is a linear combination of basis elements of the form $h_1 \otimes (c_1^{j_1}\cdots c_{q_2}^{j_{q_2}})$ with $j_1 + 2j_2 + \cdots + (q_2)j_{q_2} = q_3 + q_2$. In particular, this theorem yields

$$\oint_\pi i^* GV(\mathcal{F}_3,\mathcal{F}_2) = \mathrm{Res}_{c_1^{q_3+q_2+1}}(\mathcal{F}_3,\mathcal{F}_2,\mathcal{F}_1,S)$$

in $H^{q_3+q_2+1}(S,\mathcal{F}_1)$. So, constructing non-trivial residues on the right hand side will give non-trivial Godbillon-Vey classes on the left hand side.

Appendices

A. The Frobenius Theorem

The Frobenius Theorem is the fundamental theorem in foliation theory, and hence is well-known. Nevertheless we will give a proof of it here, since parts of the proof of Theorem A.5 expressing it in the language of differential forms are used in the computation of examples at various places in the text. First we will review the beautiful proof of the classical theorem due to Karcher [**20**].

Let D be a p-dimensional distribution on a smooth manifold M, i. e. a smooth p-dimensional subbundle of TM. Choose a complementary distribution H and a torsionfree linear connection ∇ on TM. The decomposition $TM = D \oplus H$ defines two natural projections, $\mathcal{D} : TM \to D$ and $\mathcal{H} : TM \to H$. Since $\mathcal{D} + \mathcal{H} = id_{TM}$, we have

$$\nabla_Y \mathcal{D} + \nabla_Y \mathcal{H} = 0 \tag{A.1}$$

for $Y \in TM$. The connection ∇ on TM induces a linear connection ∇^D on D via

$$\nabla^D_Y X = \mathcal{D}(\nabla_Y X)$$

for $Y \in TM, X \in \Gamma(D)$. We compute

$$0 = \nabla_Y(\mathcal{H}X) = (\nabla_Y \mathcal{H})X + \mathcal{H}(\nabla_Y X) = (\nabla_Y \mathcal{H})X + \nabla_Y X - \mathcal{D}(\nabla_Y X) \ .$$

Hence, we can rewrite the induced connection ∇^D as

$$\nabla^D_Y X = \nabla_Y X + (\nabla_Y \mathcal{H})X \ .$$

Consider a smooth path $\gamma : \mathbb{R} \to M$ in M. The last formula makes it natural to call γ a *geodesic in D* if $\dot\gamma(0) \in D_{\gamma(0)}$ and

$$\nabla_{\dot\gamma}\dot\gamma + (\nabla_{\dot\gamma}\mathcal{H})\dot\gamma = 0 \ .$$

The following lemma shows that this notion is well-chosen.

Lemma A.1. *If γ is a geodesic in D then $\dot\gamma(t) \in D_{\gamma(t)}$ for all $t \in \mathbb{R}$.*

Proof. Equation (A.1) gives

$$0 = \nabla_Y(\mathcal{D}\mathcal{H}) = (\nabla_Y \mathcal{D})\mathcal{H} + \mathcal{D}(\nabla_Y \mathcal{H}) = -(\nabla_Y \mathcal{H})\mathcal{H} + \mathcal{D}(\nabla_Y \mathcal{H}) \ .$$

Thus, if γ is a geodesic in D, then

$$\begin{aligned}\nabla_{\dot\gamma}(\mathcal{H}\dot\gamma) &= (\nabla_{\dot\gamma}\mathcal{H})\dot\gamma + \mathcal{H}(\nabla_{\dot\gamma}\dot\gamma) = (\nabla_{\dot\gamma}\mathcal{H})\dot\gamma - \mathcal{H}((\nabla_{\dot\gamma}\mathcal{H})\dot\gamma) = \mathcal{D}((\nabla_{\dot\gamma}\mathcal{H})\dot\gamma)\\ &= (\nabla_{\dot\gamma}\mathcal{H})(\mathcal{H}\dot\gamma) \ .\end{aligned}$$

This is a linear differential equation for $\mathcal{H}\dot\gamma$ with initial value $\mathcal{H}\dot\gamma(0) = 0$. Hence, $\mathcal{H}\dot\gamma(t) = 0$ for all t. So indeed, a geodesic in D stays tangential to D. □

This proof shows furthermore that on a small neighbourhood $\Omega \subset D$ of the zero section in D there is the partial exponential map

$$\begin{aligned} \exp^D : \Omega &\to M \\ Y &\mapsto \gamma_X(1)\ , \end{aligned}$$

where γ_X is a geodesic in D with $\gamma_X(0) = x$ and $\dot\gamma_X(0) = X \in \Omega_x$. Note that $\frac{d}{dt}\gamma(\lambda t) = \lambda\dot\gamma(\lambda t)$. Hence, if M is compact, then $\exp^D$ is defined on the whole distribution D.

Lemma A.2. *If the distribution D is involutive, then*

$$(\nabla_{X_1}\mathcal{H})X_2 = (\nabla_{X_2}\mathcal{H})X_1$$

for all $X_1, X_2 \in \Gamma(D)$.

Proof. Let $X_1, X_2 \in \Gamma(D)$. If D is involutive, then $[X_1, X_2] \in \Gamma(D)$ as well. Thus, $\mathcal{H}[X_1, X_2] = 0$. Since ∇ is torsion-free, we deduce

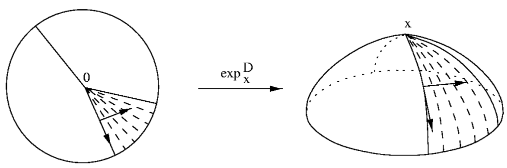
$$\begin{aligned} 0 = \mathcal{H}[X_1, X_2] &= \mathcal{H}(\nabla_{X_1}X_2) - \mathcal{H}(\nabla_{X_2}X_1) \\ &= \nabla_{X_1}(\mathcal{H}X_2) - (\nabla_{X_1}\mathcal{H})X_2 - \nabla_{X_2}(\mathcal{H}X_1) + (\nabla_{X_2}\mathcal{H})X_1 \\ &= -(\nabla_{X_1}\mathcal{H})X_2 + (\nabla_{X_2}\mathcal{H})X_1\ . \end{aligned}$$

□

We are now able to prove the classical theorem of Frobenius.

Theorem A.3 (Frobenius)**.** *A distribution D is integrable if and only if it is involutive.*

Proof. If D is the tangent bundle of a foliation, then obviously D is involutive, since this is a local property and the statement is true for the canonical p-dimensional foliation of $\mathbb{R}^q \times \mathbb{R}^p$. We only have to prove that any involutive distribution is integrable. So, let D be an involutive distribution. Then we propose that for every $x \in M$ the image $\exp_x^D(D_x)$ is an integral submanifold of D_x. Consider a family γ_s of one-parameter groups of geodesics in D with $\gamma_s(0) \in D_x$. We already know by Lemma A.1 that $\dot\gamma_s$ stays tangent to D. All we have to show is that $\frac{d}{ds}\gamma_s$ is tangent to D as well.



Since $\gamma_s(0) = x$, we have $\frac{d}{ds}\gamma_s(0) = 0$. Furthermore, the equation

$$\left(\nabla_{\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s + \mathcal{H}\frac{\nabla}{ds}\dot{\gamma}_s = \nabla_{\frac{d}{ds}\gamma_s}(\mathcal{H}\dot{\gamma}_s) = 0$$

implies that

$$\begin{aligned}\mathcal{H}\frac{\nabla}{ds}\dot{\gamma}_s &= -\left(\nabla_{\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s = -\left(\nabla_{\mathcal{H}\frac{d}{ds}\gamma_s+\mathcal{D}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s \\ &= -\left(\nabla_{\mathcal{H}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s - \left(\nabla_{\mathcal{D}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s \\ &= -\left(\nabla_{\mathcal{H}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s - (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\mathcal{D}\frac{d}{ds}\gamma_s\end{aligned}$$

by Lemma A.2. We compute

$$\begin{aligned}\nabla_{\dot{\gamma}_s}\left(\mathcal{H}\tfrac{d}{ds}\gamma_s\right) &= (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\tfrac{d}{ds}\gamma_s + \mathcal{H}\nabla_{\dot{\gamma}_s}\tfrac{d}{ds}\gamma_s \\ &= (\nabla_{\dot{\gamma}_s}\mathcal{H})\,(\mathcal{H}+\mathcal{D})\tfrac{d}{ds}\gamma_s + \mathcal{H}\tfrac{\nabla}{ds}\dot{\gamma}_s \\ &= (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\mathcal{H}\tfrac{d}{ds}\gamma_s + (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\mathcal{D}\tfrac{d}{ds}\gamma_s - \left(\nabla_{\mathcal{H}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s - (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\mathcal{D}\tfrac{d}{ds}\gamma_s \\ &= (\nabla_{\dot{\gamma}_s}\mathcal{H})\,\mathcal{H}\tfrac{d}{ds}\gamma_s - \left(\nabla_{\mathcal{H}\frac{d}{ds}\gamma_s}\mathcal{H}\right)\dot{\gamma}_s\ .\end{aligned}$$

So, $\mathcal{H}\frac{d}{ds}\gamma_s$ satisfies a linear differential equation with initial value $\mathcal{H}\frac{d}{ds}\gamma_s(0) = 0$. This proves that $\frac{d}{ds}\gamma_s(t) \in D_{\gamma_s(t)}$ for all $t \in \mathbb{R}$. □

Consider now a q-form $\alpha \in \Omega^q(M)$ on M and a point $x \in M$. Recall that the *rank* of α in x is defined to be the rank of the linear map

$$\begin{aligned}T_xM &\to \Lambda^{q-1}T_x^*M \\ Y &\mapsto i_Y\alpha\ .\end{aligned}$$

Hence, the subspace $\ker\alpha \subset TM$,

$$\ker\alpha = \{Y \in TM|\ i_Y\alpha = 0\}\ ,$$

is a distribution if and only if the rank of α is constant on M. Suppose that $\alpha \in \Omega^q(M)$ is a locally decomposable form, i. e. there is an open cover $\{U_i\}$ of M such that $\alpha|_{U_i} = \gamma_1 \wedge \cdots \wedge \gamma_q \in \Omega^q(U_i)$ is a product of one-forms. Then the rank of α is everywhere less than or equal to q. So, the locally decomposable q-form α defines a distribution of codimension q if and only if α is of maximal rank.

Lemma A.4. *A subspace $D \subset TM$ is a coorientable distribution of codimension q if and only if there is a locally decomposable q-form $\alpha \in \Omega^q(M)$ of maximal rank such that* $\ker\alpha = D$.

Proof. We just saw that if α is a locally decomposable q-form of maximal rank, then $\ker\alpha$ is a distribution of codimension q. Since α induces a nowhere vanishing section of $\Lambda^q(TM/\ker\alpha)^*$, this distribution is coorientable. It remains to be shown that every coorientable distribution can be defined by a locally decomposable form. Consider a distribution $D \subset TM$ and choose a complementary distibution Q such that $TM = D \oplus Q$. Fix an orientation of Q. Let $\{U_i\}$ be a open covering of M consisting of open neighbourhoods simultaniously trivialising D and Q. Choose a framing $X_1, \dots, X_p$ of $D|_{U_i}$ and a framing $Y_1, \dots, Y_q$ of $Q|_{U_i}$ giving the fixed orientation of $Q|_{U_i}$. Set $\alpha_i = Y_1^* \wedge \dots \wedge Y_q^* \in \Omega^q(U_i)$, where Y_j^* is the one-form dual to Y_j, i. e. $Y_j(Y_k) = \delta_{jk}$ and $Y_j^*(X_k) = 0$. Then $\ker \alpha_i = D|_{U_i}$. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define $\alpha = \sum_i \rho_i \alpha_i$. Obviously, $\ker \alpha = D$ since $\alpha|_{U_i}(Y_1, \dots, Y_q) > 0$. Moreover, $i_{X_j}\alpha = 0$ implies that $\alpha|_{U_i} = f Y_1^* \wedge \dots \wedge Y_q^*$ with a positive function f. Hence α is locally decomposable of maximal rank. Note that the same argument shows that every defining form for D is locally decomposable of maximal rank. □

This lemma makes it possible to reformulate the Frobenius Theorem in the language of de Rham theory. The fact that we reduce ourselves to coorientable foliations is no restriction at all, since locally every distribution is coorientable and the integrability of a distribution is a local property.

Theorem A.5 (Frobenius)**.** *Let $\alpha \in \Omega^q(M)$ be a locally decomposable form of maximal rank. Then the coorientable distribution* $\ker \alpha$ *is integrable if and only if there is a one-form $\beta \in \Omega^1(M)$ such that*

$$d\alpha = \beta \wedge \alpha \ .$$

Proof. Consider a q-form $\alpha \in \Omega^q(M)$ of constant rank q. If $d\alpha = \beta \wedge \alpha$ for some one-form β, then for $X_1, X_2 \in \ker \alpha$

$$0 = i_{X_2} i_{X_1}(\beta \wedge \alpha) = i_{X_2} i_{X_1} d\alpha = i_{[X_1, X_2]}\alpha$$

gives $[X_1, X_2] \in \ker \alpha$. So, $\ker \alpha$ is involutive, and by Theorem A.3 the coorientable distribution $\ker \alpha$ is integrable. Now, suppose that $\alpha \in \Omega^q(M)$ is a locally decomposable form of maximal rank. Let $\{U_i\}$ be an open covering of M such that $\alpha|_{U_i}$ is decomposable, $\alpha_i = \alpha|_{U_i} = \gamma_1 \wedge \dots \wedge \gamma_q$. Since α is nowhere-vanishing, the γ_i are of maximal rank 1 and $\ker \alpha_i = \bigcap_{j=1}^q \ker \gamma_j$. Therefore, if $X_1, X_2 \in \ker \alpha_i$, then

$$\begin{aligned} i_{[X_1,X_2]}\alpha_i = i_{X_2} i_{X_1} d\alpha_i &= i_{X_2} i_{X_1} \sum_{j=1}^q (-1)^j d\gamma_j \wedge \gamma_1 \wedge \dots \wedge \hat{\gamma}_j \wedge \dots \wedge \gamma_q \\ &= \sum_{j=1}^q (-1)^j d\gamma_j(X_1, X_2) \cdot \gamma_1 \wedge \dots \wedge \hat{\gamma}_j \wedge \dots \wedge \gamma_q \ . \end{aligned}$$

If $\ker \alpha$ is integrable and thus involutive, then we get

$$0 = \sum_{j=1}^q (-1)^j d\gamma_j(X_1, X_2) \cdot \gamma_1 \wedge \dots \wedge \hat{\gamma}_j \wedge \dots \wedge \gamma_q \ .$$

Multiplying from the left with γ_k gives the equation

$$d\gamma_k(X_1, X_2) \cdot \gamma_1 \wedge \dots \wedge \gamma_q = 0 \ .$$

Since $\ker \alpha$ has codimension q, this implies the so-called *integrability condition*

$$d\gamma_k \wedge \gamma_1 \wedge \cdots \wedge \gamma_q = 0 \tag{A.2}$$

for all $k = 1, \ldots, q$. The γ_j are linearly independent, so locally we can extend $\{\gamma_j\}$ to a basis of $\Omega^1(U_i)$ by the dual of a local framing of $\ker \alpha$. Then the integrability condition (A.2) gives that $d\gamma_k$ is in the ideal spanned by the γ_j,

$$d\gamma_k = \sum_{j=1}^{q} \eta_{kj} \wedge \gamma_j$$

with $\eta_{kj} \in \Omega^1(U_i)$. Set $\beta_i = \sum_{j=1}^{q} \eta_{jj}$. Then

$$\begin{aligned} d\alpha_i &= \sum_{k=1}^{q} (-1)^k d\gamma_k \wedge \gamma_1 \wedge \cdots \wedge \hat{\gamma}_k \wedge \cdots \wedge \gamma_q \\ &= \sum_{j,k=1}^{q} (-1)^k \eta_{kj} \wedge \gamma_j \wedge \gamma_1 \wedge \cdots \wedge \hat{\gamma}_k \wedge \cdots \wedge \gamma_q \\ &= \beta_i \wedge \alpha_i \ . \end{aligned}$$

Choosing a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$ and defining $\beta = \sum_i \rho_i \beta_i$ gives a one-form $\beta \in \Omega^1(M)$ with $d\alpha = \beta \wedge \alpha$. □

B. G-DG-algebras

Let us recall the notion of a G-DG-algebra introduced by Cartan, Koszul and Weil in [**6**]. This appendix will follow the exposition in [**19**]. A *graded algebra* A^* is an algebra over the reals with a decomposition

$$A^* = \bigoplus_{j=0}^{\infty} A^j$$

into subspaces A^j such that the multiplication satisfies

$$A^j \cdot A^k \subset A^{j+k} \ .$$

Moreover, we will demand a graded algebra to be graded commutative, i. e. for $a \in A^j$ and $b \in A^k$ we have

$$a \cdot b = (-1)^{jk} b \cdot a \ .$$

A *derivation* of degree p on A^* is a linear map $\delta : A^* \to A^{*+p}$ such that

$$\delta(a \cdot b) = \delta(a) \cdot b + (-1)^{jp} a \cdot \delta(b) \ .$$

for $a \in A^j$ and $b \in A^k$. Note, that the vector space of derivations carries the structure of a graded Lie algebra with respect to the Lie bracket

$$[\delta, \delta'] = \delta\delta' - (-1)^{pq} \delta'\delta \ ,$$

where δ a derivation of degree p and δ' is a derivation of degree q. A *differential graded algebra* (A^*, d) is a graded algebra A^* together with a derivation

$$d : A^* \to A^{*+1}$$

of degree 1 which satisfies $d^2 = 0$. Obviously, this implies that the cohomology spaces $H^*(A^*) = H(A^*, d) = \ker d \,/\, \operatorname{im} d$ form again a graded algebra. A differential graded algebra homomorphism is an algebra homomorphism preserving the graduation and commuting with the differentials.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider a differential graded algebra (A^*, d) equipped with an action ρ of G on (A^*, d) and an inner product i with elements of $\mathfrak{g}$. To be precise, for every $g \in G$ we have a differential graded automorphism

$$\rho(g) : A^* \to A^*$$

such that $\rho(gh) = \rho(g)\rho(h)$. And for every $X \in \mathfrak{g}$ there is a derivation

$$i(X) : A^* \to A^{*-1}$$

of degree -1 depending linearly on X. The action ρ induces an infinitesimal action of $\mathfrak{g}$ on A^*. Namely, for every $X \in \mathfrak{g}$ we get a homomorphism $\theta(X) : A^* \to A^*$,

$$\theta(X)(a) = \frac{d}{dt} \rho(\exp(tX)) a \Big|_{t=0} \ .$$

This is a Lie algebra homomorphism into the derivations of degree zero,

$$\theta([X,Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X)\ .$$

Now, suppose that these structures are consistent in the following sense.

Definition B.1. A *G-differential graded algebra* (short: a G-DG-algebra) is a differential graded algebra (A^*, d) equipped with a G-action ρ and an inner product i satisfying the following conditions for all $g \in G$ and $X \in \mathfrak{g}$.

(1) $i(X)^2 = 0$,
(2) $\rho(g)i(X)\rho(g^{-1}) = i(\mathrm{Ad}(g)X)$,
(3) $\theta(X) = i(X)d + di(X)$.

Differentiating Condition (2) gives moreover, that

$$i([X,Y]) = \theta(X)i(Y) - i(X)\theta(X)\ .$$

Example B.1. The natural example for such a G-DG-algebra is the de Rham complex $\Omega^*(P)$ of a principal G-bundle $P \to B$. Use the fundamental vector field X^* associated to $X \in \mathfrak{g}$ to define

$$i(X)\omega = i_{X^*}\omega\ ,$$
$$\rho(g)\omega = R_g^*\omega$$

for $\omega \in \Omega^*(P)$. With these definitions $\Omega^*(P)$ becomes a G-DG-algebra. This applies in particular if $P = G$ is the trivial bundle over a point. Restricting $\Omega^*(G)$ to the left invariant forms restricts $d : \Omega^n(G) \to \Omega^{n+1}(G)$ to the Chevalley-Eilenberg differential $d_\Lambda : \Lambda^n\mathfrak{g}^* \to \Lambda^{n+1}\mathfrak{g}^*$. So, the above implies that the exterior algebra

$$\Lambda^*\mathfrak{g}^* = \bigoplus_{n=0}^{\dim G} \Lambda^n\mathfrak{g}^*$$

is a G-DG-algebra with

$$i(X)\alpha = i_X\alpha\ ,$$
$$\rho(g)\alpha = \mathrm{Ad}(g^{-1})^*\alpha\ ,$$
$$\theta(X)\alpha = -\,\mathrm{ad}(X)^*\alpha\ .$$

⊠

Now, let $H \subset G$ be any Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Then there is the subspace $(A^*)^H \subset A^*$ of elements invariant under the H-action,

$$(A^*)^H = \{a \in A^* |\ \rho(h)a = a \text{ for all } h \in H\}\ .$$

Since $\rho(h)$ is a differential graded automorphism, $(A^*)^H$ is a differential graded subalgebra of A^*. The *H-basic elements* are the elements of the subspace $(A^*)_H \subset (A^*)^H$ defined by

$$(A^*)_H = \{a \in (A^*)^H |\ i(X)a = 0 \text{ for all } X \in \mathfrak{h}\}\ .$$

Because $i(X)$ is a derivation, $(A^*)_H$ is a graded subalgebra. Moreover, Condition 3. of Definition B.1 and $(A^*)_H \subset (A^*)^H$ imply that $(A^*)_H$ is closed with respect to the differential d. Hence, we can define the relative cohomology of A^* to equal

$$H^*(A^*, H) = H((A^*)_H, d) \ .$$

Example B.2. The elements of $(A^*)_H$ are called H-basic because for a principal G-bundle $P \to B$, we have that $\Omega^*(P)^H$ is the algebra of H-invariant forms on P and $\Omega^*(P)_H$ is the algebra of H-invariant and horizontal forms, i. e. the image of the injective map $\pi^* : \Omega^*(P/H) \to \Omega^*(P)$ where $\pi : P \to P/H$ denotes the principal H-bundle induced by P. Thus,

$$\Omega^*(P)_H \cong \Omega^*(P/H)$$

and in particular,

$$\Omega^*(P)_G \cong \Omega^*(B) \ .$$

Restricting again to the left invariant forms on G we deduce

$$(\Lambda^*\mathfrak{g}^*)_H \cong ({}^G\Omega^*(G))_H = {}^G(\Omega^*(G)_H) \cong {}^G\Omega^*(G/H) \cong (\Lambda^*(\mathfrak{g}/\mathfrak{h})^*)^H \ .$$

Recall that the cohomology of the Lie algebra $\mathfrak{g}$ is defined to equal

$$H^*(\mathfrak{g}) = H(\Lambda^*\mathfrak{g}^*, d_\Lambda) \ .$$

The relative cohomology then is

$$H^*(\mathfrak{g}, H) = H^*((\Lambda^*\mathfrak{g}^*)_H) = H^*((\Lambda^*(\mathfrak{g}/\mathfrak{h})^*)^H) \ .$$

⊠

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List of Notations

$\square$	End of proof,
$\boxtimes$	End of example,
$\mathcal{F}, \mathcal{G}, \ldots$	Foliations, 1
$(\mathcal{F}_k, \ldots, \mathcal{F}_1)$	k-Flag of foliations, 33
$M/\mathcal{F}$	Leaf space of a foliation, 1
$T\mathcal{F}$	Tangent bundle of a foliation, 1
$\nu(\mathcal{F}_k, \ldots, \mathcal{F}_1)$	Normal bundle of a k-flag of foliations, 50
$f(\mathcal{G}) \subset \mathcal{F}$	Morphism from $(N, \mathcal{G})$ to $(M, \mathcal{F})$, 12
$\mathcal{F}_U$, $\mathcal{F}\vert_U$	Restriction of $\mathcal{F}$ to the transverse subset U, 19
$\mathcal{F} \times \mathcal{G}$	Product foliation, 20
$a \times b$	Cross product, 21
$\tilde{\mathcal{F}}$	The foliated structure on a foliated bundle $P \to (M, \mathcal{F})$, 48
$\{M\}$	Trivial foliation of codimension zero, 8
$\mathcal{T}^{q,p}$	Canonical foliation of codimension q on $\mathbb{R}^q \times \mathbb{R}^p$, 14
$hol(w)$	Holonomy transformation along a path w, 3
$\mathrm{Hol}(\mathcal{F})$	Holonomy groupoid of a foliation, 4
$\Pi(L)$	Fundamental groupoid of a leaf L, 4
$\mathrm{Diff}(M)$	Diffeomorphism group of M, 26
$\mathrm{Diff}_0(M)$	Connected component of the identity in $\mathrm{Diff}(M)$, 26
$\overline{\mathrm{Diff}}(\mathbb{R}^q, 0)$	Group of germs at 0 of diffeomorphisms of $\mathbb{R}^q$, 4
hol	Holonomy representation, 4
$infhol$	Infinitesimal holonomy representation, 4
$\Omega^*(M)$	De Rham complex of a manifold M, 2
$\Omega^*_c(M)$	Complex of forms with compact supports, 8
$\Omega^*_{tr}(M)$	Complex of forms with transversally compact support, 8
$\Omega^*_{vc}(M)$	Complex of forms with vertically compact support, 27
$\Omega^*(M/\mathcal{F})$	Complex of basic forms, 9
$\Omega^*_c(M/\mathcal{F})$	Complex of basic forms with compact support, 10
$\Omega^*(\mathcal{F}; \Lambda^r Q^*)$	Complex of foliated forms, 24

Index

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