Emergence of Yukawa couplings in M-theory

Aleksandar Gligovic

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Emergenz von Yukawa-Kopplungen in der M-theorie

Aleksandar Gligovic

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Aleksandar Gligovic
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Abstract

Throughout history, theoretical physics has advanced by challenging deep-seated assumptions about how our universe should be described. Theories of quantum gravity, which aim to unify gravity with the principles of quantum mechanics, indicate that even spacetime itself may only arise in certain limits of the fundamental theory, where an effective geometric description becomes valid. This raises the question of what the basic ingredients of quantum gravity theories really are and how exactly they give rise to the physics observed at low energies. The Emergence Proposal offers a perspective to approach these questions: it suggests that low-energy interactions - including gravity - emerge from integrating out infinite towers of states, which are widely regarded as a hallmark of quantum gravity theories. This proposal originates from the swampland program, whose goal is to identify universal features of effective field theories that can be consistently UV-completed to quantum gravity. Its hypotheses are tested in explicit models, mostly within the framework of string theory.

In this thesis we argue why the Emergence Proposal is naturally realized in M-theory, an eleven-dimensional theory of quantum gravity that was encountered through string dualities, but whose microscopic formulation is still unknown. We propose a refined version of the Emergence Proposal, suggesting that the entire low-energy effective action of M-theory should arise as a pure quantum effect after integrating out those infinite towers of states whose mass scale is parametrically not larger than the species scale, the energy scale where quantum gravity effects become relevant. We gather evidence from 1/2-BPS protected couplings, for which the contributing BPS states can be reliably described in terms of their weakly coupled string theory realizations. Concretely, we show that for compactifications of type IIA string theory on Calabi-Yau manifolds, the classical Yukawa couplings - corresponding to the triple intersection numbers of the Calabi-Yau threefold - can be obtained from a one-loop Schwinger integral over bound states of $D0$- and $D2$-branes. For compact Calabi-Yau threefolds, we propose a novel regularization method for the infinite sum over Gopakumar-Vafa invariants by employing finite distance degeneration limits of the Calabi-Yau geometry. We test our proposal through the explicit determination of the periods near such degeneration points for threefolds with a small number of Kähler moduli.

Zusammenfassung

Im Laufe der Geschichte hat sich die theoretische Physik stets dadurch weiterentwickelt, dass tief verwurzelte Annahmen über die angemessene Beschreibung unseres Universums hinterfragt wurden. Theorien der Quantengravitation, deren Ziel die Vereinheitlichung der Gravitation mit den Prinzipien der Quantenmechanik ist, legen nahe, dass selbst die Raumzeit nur in bestimmten Grenzbereichen der fundamentalen Theorie entsteht, in denen eine effektive geometrische Beschreibung gültig ist. Dies wirft die Frage auf, was die grundlegenden Bausteine von Quantengravitationstheorien eigentlich sind und wie aus ihnen die bei niedrigen Energien beobachtete Physik hervorgeht. Die sogenannte Emergenzhypothese bietet eine Perspektive, um diese Fragen anzugehen: Sie besagt, dass Interaktionen bei niedrigen Energien - selbst die Gravitation - durch das Ausintegrieren unendlicher Reihen von Zuständen entstehen, die als charakteristisches Merkmal von Quantengravitationstheorien gelten. Diese Hypothese stammt vom Swampland Programm, dessen Ziel es ist, universelle Eigenschaften effektiver Feldtheorien zu bestimmen, die konsistent zu einer Theorie der Quantengravitation vervollständigt werden können. Die Vermutungen werden in konkreten Modellen getestet, die zumeist von der Stringtheorie hergeleitet sind.

In dieser Arbeit erläutern wir, warum die Emergenzhypothese auf natürliche Weise in der M-theorie realisiert ist. Diese ist eine elf-dimensionale Quantengravitationstheorie, die durch String-Dualitäten entdeckt wurde, deren mikroskopische Beschreibung jedoch bis heute unbekannt ist. Wir schlagen eine präzisierte Variante der Emergenzhypothese vor, laut der die gesamte effektive Wirkung als reiner Quanteneffekt durch das Ausintegrieren derjenigen Reihen von Zuständen entsteht, deren Massenskala parametrisch nicht größer als die Species-Skala ist, bei welcher die Effekte der Quantengravitation eine wesentliche Rolle spielen. Wir sammeln Belege anhand von 1/2-BPS-geschützten Kopplungen, für welche die beitragenden BPS-Zustände zuverlässig durch ihre Realisierungen in schwach gekoppelter Stringtheorie beschrieben werden können. Konkret zeigen wir, dass für Kompaktifizierungen der Typ-IIA-Stringtheorie auf Calabi-Yau Mannigfaltigkeiten die klassischen Yukawa Kopplungen - den Dreifach-Schnittzahlen von Calabi-Yau Mannigfaltigkeiten entsprechend - aus einem einschleifigem Schwinger Integral über gebundene Zustände von $D0$- und $D2$-Branen gewonnen werden können. Für kompakte Calabi-Yau Mannigfaltigkeiten schlagen wir eine neue Regularisierungsmethode für die unendliche Summe über Gopakumar-Vafa Invarianten vor, die von Degenerationen von Calabi-Yau Geometrien bei endlicher Distanz Gebrauch macht. Wir testen unsere Hy-

pothese durch die explizite Bestimmung der Perioden nahe dieser Degenerationspunkte für Calabi-Yau Räume mit wenigen Kähler-Moduli.

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Contents

Chapter 1

Introduction

Ever since humans began developing theories to describe the laws of nature, a central theme has been the identification of the most basic building blocks of our universe. As these theories evolved, so did our perception of what counts as "fundamental" and what turns out to be just coarse-grained modeling. Conceptually, the simplest step forward is to reveal the internal structure of objects that were once regarded as indivisible. This is nicely illustrated by the evolution of our understanding of the atom. From its origin in ancient Greek philosophy up until the more quantitative atomic theory by John Dalton, the atom was conceived as the fundamental unit of matter. This viewpoint was disproven by the experiments of J.J. Thomson and Ernest Rutherford in the late 19th and early 20th centuries. In the resulting nuclear model by Rutherford, the atom consisted of a dense, positively charged core and electrons orbiting around it. Scattering experiments in the mid-20-th-century then revealed that even protons and neutrons, the building blocks of the nucleus, are by themselves composite, consisting of quarks that are bound by the strong force. Whether this hierarchy continues at yet smaller scales is of course unknown and will ultimately be decided by the reach of future experiments.

However, the development of physical theories is not limited to the pure subdivision of objects into smaller components. Instead, the notion of fundamental objects tends to shift towards more abstract entities to accommodate new experimental observations. An early example of such a paradigm shift is the rise of classical field theory in the 19th century, most notably James Clerk Maxwell's theory of electromagnetism. Previously, it was assumed that not only matter but also light consists of localized objects, so called "corpuscles". The intrinsic properties and dynamics of those constituents were believed to explain all observable phenomena. If one assumes that corpuscles move freely and are governed by Newtonian mechanics, light should always travel in straight lines. But as experiments showed that light exhibits wave-like properties such as interference and diffraction and that electromagnetic waves carry energy independent of any material source, the corpuscular description of light was ultimately invalidated. Certainly, the issue could not be resolved by just sub-dividing corpuscles even further. Instead, electromagnetism required the introduction of a completely new physical object known as the *field* - a continuous, dynamical entity that permeates all of space and carries energy when being excited. In hindsight, the particle picture turned out to be only an effective description of light, valid when its wavelength is negligible compared to the relevant length scales. In that case the electromagnetic field oscillates so rapidly that the phase information averages out and becomes irrelevant.

With the advent of quantum mechanics (QM) in the early 20th century, initiated by Max Planck and further developed by many others, our understanding of the atom also had to fundamentally change. At its core, QM replaces the deterministic description of particles with a probabilistic framework. Electrons cannot be envisioned as orbiting point particles with a definite trajectory, but behave like waves at small enough scales. The regions where these waves have the highest amplitude are called orbitals, and they indicate the spatial regions where an electron is most likely to be found. In that sense, QM continued the historical trend of increasing the level of abstraction: It introduces fundamental mathematical objects, namely state vectors in Hilbert spaces, that do not directly describe definite properties of a system. Instead, they encode what outcomes are expected when a measurement is being performed and assign probabilities to those events. The framework was able to explain several important observations, such as the discrete atomic spectra and the stability of atoms. But this formulation turned out to be incompatible with Albert Einstein's theory of special relativity, which unifies space and time and requires that the laws of physics take the same form in all inertial frames. Since non-relativistic QM treats time as an absolute parameter, probability densities (computed from wavefunctions associated to particles) do not transform consistently between different inertial frames. This issue was resolved by *quantum field theory* (QFT), which embedded QM into a relativistic framework and made particles lose their status as fundamental objects altogether. In a QFT, particles are interpreted as excitations of quantum fields (fields exhibiting quantum fluctuations), whose dynamics are captured by equations of motion that respect the spacetime symmetries. These fields and their interactions are the key ingredients in the formulation of QFTs, and all measurable properties of matter (and radiation) result from the dynamics and correlations of those fields.

The framework of QFT made it possible to formulate the *Standard Model of Particle Physics* (SM), a theory that describes all fundamental forces except gravity and classifies all known elementary particles. It has been extensively tested in experiments, e.g. through the precise measurement of scattering events or the discovery of the Higgs boson through which SM particles acquire their mass. The complementary theory that describes gravity is General Relativity (GR), a completely classical theory in which the gravitational force arises due to the curvature of spacetime that is caused by mass/energy. Likewise, it is incredibly well tested, and correctly describes gravitational phenomena on large scales. Despite their empirical success, the SM and GR are hard to combine into one single framework. When GR is treated as QFT for the spacetime metric, quantum corrections arising from high-energy (short-distance) fluctuations generate infinitely many divergences. This requires an infinite number of parameters (and therefore an infinite number of measurements) to define the theory at arbitrarily small length scales. Such a breakdown may indicate that classical spacetime geometry cannot remain a fundamental concept at very short distances. A framework aiming to unify gravity with the principles of quantum mechanics at all scales is commonly called *quantum gravity* (QG). Such a theory should be able to describe situations in which strong gravitational effects coincide with quantum phenomena, such as the physics of black holes or the early universe. And from a purely theoretical viewpoint, it should resolve the incompatibility of the SM and GR, possibly by introducing a further conceptual shift[1]. While formulating a QG theory is highly non-trivial, at the very least it should be approximated by an effective field theory (EFT) at low energies, consisting of the SM coupled to GR at leading order.

Quantum Gravity and Emergence

Among the various approaches to QG, string theory is the most developed framework to this date, making it particularly well suited as a theoretical laboratory. The starting point of string theory is a very conservative modification of QFT and GR, the only dynamical input being that fundamental objects are one-dimensional, vibrating strings. Their dynamics are encoded in a *conformal field theory* (CFT), a highly constrained QFT defined on the so-called string *world-sheet*. The world-sheet is a two-dimensional surface that is swept out by the string as it moves in time (the generalization of the particle world-line). Remarkably, the consistency constraints of the CFT are so restrictive that they fix crucial features of the spacetime background, including the dimension of spacetime, as well as the spectrum coming from the string excitations. Among the string's massless excitations is a spin-two field corresponding to the graviton, showing that gravity is a prediction of the theory rather than an extra input.

The mathematical structure of string theory eliminates the ultraviolet divergences that make gravity non-renormalizable as an ordinary QFT, while reproducing the familiar equations of GR at low energies. However, string theory obeys a set of intricate symmetries, known as dualities, which reinforce the idea that classical spacetime geometry is not fundamental. A well-known example is T-duality, which in its simplest form establishes an equivalence between two string theories with one compact dimension (meaning that one spatial dimension is curled up into a circle). The claim is that a string propagating along a very small circle behaves exactly like a string moving along a large circle, indicating the existence of a minimum length scale that strings can probe. This idea is pushed even further by mirror symmetry, which relates completely different spacetime topologies to equivalent physics. More concrete evidence for the claim that spacetime is not a fundamental concept is provided by the famous *AdS/CFT correspondence* [8]. It states that a gravitational theory on an Anti-de Sitter space (AdS) is exactly equivalent to a non-gravitational QFT (namely, a CFT) defined on the boundary of that AdS space. In this duality, the bulk spacetime geometry can be reconstructed from the entanglement structure among the degrees of freedom of the boundary CFT. Roughly speaking, strongly entangled CFT degrees of freedom behave as if they are close together in the emergent

[1] We should stress that there are plenty QFT-based approaches to QG that keep the spacetime metric as fundamental. Examples are Asymptotic Safety [1, 2], quadratic gravity [3, 4], non-local gravity [5, 6] and Causal Dynamical Triangulation [7].

bulk spacetime, and vice versa. From this perspective, spacetime itself appears as an *emergent* phenomenon, given that its existence is not assumed a priori.

All these observations naturally raise an interesting question: If spacetime is not fundamental, what are the basic ingredients required by a theory of QG, and how does such a framework reproduce the familiar low-energy effective action governing gravity and matter? While we need a fully developed QG theory to ultimately settle these questions, there is an intriguing approach to them known as the *Emergence Proposal*. It suggests that terms in the low-energy effective action *emerge* by integrating out the fundamental degrees of freedom of an underlying UV-complete theory of QG. In its most radical formulation, all terms in the effective action - including those traditionally regarded as classical - are a *pure quantum effect*. Outside the geometric regime, these terms may be interpreted as an effective description of a more abstract structure that makes no reference to spacetime or a local action. While this is an interesting idea, several follow-up questions need to be answered to make this proposal quantitative and testable: (*i*) Is the Emergence Proposal realized in any regime of the parameter space of a QG theory? (*ii*) What are the fundamental degrees of freedom in a given regime? And (*iii*), how can one set up a computation where those degrees of freedom are integrated out, such that the result is finite and completely matches the classical expectation from the low-energy effective action? This thesis aims to answer these questions, with the goal of sharpening the formulation of the Emergence Proposal and providing explicit evidence in models constructed within the string theory framework, our chosen testing ground.

Without going into the details of our analysis, one of the central claims of this thesis is that the Emergence Proposal is *not* realized in any known regime of QG. We present evidence that it should be realized in *M-theory*, an eleven-dimensional theory of QG best known for unifying all known supersymmetric string theories. To this day, M-theory remains rather poorly understood. Extrapolations from string theory teach us that its spectrum includes membranes and higher-dimensional five-branes (but no strings), and that its low-energy dynamics are approximated by eleven-dimensional supergravity. Nonetheless, the technical challenges of consistently quantizing branes have so far prevented the construction of a complete formulation of M-theory. Interestingly to us, there is a proposal for such a formulation known as the BFSS matrix model (named after Banks, Fischler, Shenker and Susskind), which explicitly realizes the idea that gravitational interactions arise as a quantum effect. In this model, one studies the dynamics of large matrices that, in certain geometric regimes, can be interpreted as encoding the positions of gravitons in terms of their eigenvalues. In generic situations however these matrices represent non-geometric degrees of freedom. An important result is that the familiar long-range gravitational potential between gravitons is absent at tree-level and only generated at one-loop, after integrating out heavy degrees of freedom represented by off-diagonal matrix entries. We should note that the BFSS model is unlikely to provide a complete formulation of M-theory, given that it captures only a particular kinematic regime and so far cannot model the entire M-theory spectrum. But the emergence of gravity as a quantum effect is a highly distinct feature that does *not* arise in weakly coupled string theories and strongly resonates with the general philosophy of the Emergence Proposal, making M-theory a promising candidate.

While the BFSS matrix model provides encouraging hints for the viability of the Emergence Proposal, a central challenge is to obtain compelling evidence given that a full quantization of M-theory is still lacking. This issue can be circumvented by focusing on certain couplings in the effective action that are protected by supersymmetry and only receive contributions from states that preserve part of the supersymmetry (called BPS states). For such couplings, the relevant states can be reliably described in terms of their weakly coupled string theory counterparts, and supersymmetry ensures that their properties can be extrapolated to the strong-coupling limit yielding M-theory. The positive results obtained for these protected couplings currently provide our strongest indication in favor of the M-theoretic Emergence Proposal. The ultimate question is whether emergence is just a special aspect of this simplified setting or whether it represents a general property of M-theory, a question that demands extensive further study.

The thesis is structured as follows: in chapter 2 we review basic aspects of the string theory framework. This includes the quantization of weakly coupled strings, the derivation of low-energy effective descriptions and an overview of string dualities. These dualities form a web of relations among consistent string theories and point toward the existence of M-theory, whose basic properties and proposed formulation in terms of the BFSS matrix model will also be discussed. In chapter 3 we introduce the so-called *swampland program*, a research field that aims to find universal properties of EFTs that can be consistently UV-completed to QG. General lessons from the swampland program suggest a connection between the degrees of freedom of a QG theory and the terms appearing in its associated low-energy effective action. This relation is made concrete by the Emergence Proposal, whose original formulation and refined version for M-theory will be a central part of the discussion. A key ingredient of this proposal is the *species scale*, the energy scale at which QG effects become relevant. As we will argue, this scale distinguishes between fundamental and classical, solitonic objects and thus identifies the degrees of freedom that should be integrated out. In the next three chapters we challenge the M-theoretic Emergence Proposal in a concrete setup, namely a class of string theory models that are approximated by 4D $N = 2$ supergravity at low energies. By taking the appropriate strong-coupling limit towards M-theory, we demonstrate that certain couplings which encode the kinetic terms of vector multiplet fields can be exactly reproduced by integrating out the full set of fundamental BPS states at one-loop. The main technical challenge lies in regularizing the corresponding amplitude in such a way that the expected classical result is obtained. Chapter 4 focuses on a benchmark model in which the regularization can be performed using the analytic continuation of the Riemann ζ-function. In Chapters 5 and 6 we then tackle models with a much higher degeneracy of states and develop a regularization method for the according divergences. In chapter 7 we briefly summarize our findings and comment on potential directions for future research.

The thesis is based on the following publications:

- **Demystifying the Emergence Proposal**
  Ralph Blumenhagen, Niccolò Cribiori, Aleksandar Gligovic, Antonia Paraskevopoulou
  JHEP 04 (2024) 053, arXiv: 2309.11551 [hep-th]

- **Emergent M-theory limit**
  Ralph Blumenhagen, Niccolò Cribiori, Aleksandar Gligovic, Antonia Paraskevopoulou
  Phys.Rev.D 109 (2024) 2, L021901, arXiv: 2309.11554 [hep-th]

- **Reflections on an M-theoretic Emergence Proposal**
  Ralph Blumenhagen, Niccolò Cribiori, Aleksandar Gligovic, Antonia Paraskevopoulou
  PoS CORFU2023 (2024) 238, arXiv: 2404.05801 [hep-th]

- **Emergence of CY triple intersection numbers in M-theory**
  Ralph Blumenhagen, Aleksandar Gligovic
  JHEP 10 (2025) 048, arXiv: 2506.20725 [hep-th]

During his doctoral studies, the author has also contributed to the following papers:

- **The emergence proposal and the emergent string**
  Ralph Blumenhagen, Aleksandar Gligovic, Antonia Paraskevopoulou
  JHEP 10 (2023) 145, arXiv: 2305.10490 [hep-th]

- **Emergence of R^4-terms in M-theory**
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Chapter 2

Primer on string theory and M-theory

In the following chapter we review basic concepts of string theory, our chosen theoretical laboratory for testing the idea of emergence in QG. From a practical viewpoint string theory is a good laboratory because it is one of the few QG frameworks where detailed calculations can actually be performed, assuming one works in controlled regimes of the theory. But even outside these regimes it is possible to probe the theory at least partially by using powerful duality relations that were mentioned previously. The canonical starting point is to consider strings that are light and weakly interacting, known as the perturbative string limit. We will first consider the classical bosonic string and discuss one particular method for its quantization. The analysis will reveal a pathological property of bosonic string theory that can be removed by considering supersymmetric extensions, of which there exist five in ten dimensions. To engineer (semi-)realistic models with four macroscopic dimensions, one needs to assume that the residual six dimensions of those string theories form a compact space that is small enough to escape detection. We will describe the systematic procedure of deriving effective four-dimensional models with $N = 2$ supersymmetry and use these as a testing ground for the Emergence Proposal in later parts of this work. We finish the chapter with a discussion of dualities, which create relations among different string theories and point towards the existence of M-theory, an eleven-dimensional quantum gravity theory that only contains membranes and no strings. Most of the topics are standard textbook material. Some useful references[1] are [9–13].

2.1 Bosonic string

Following common introductions on string theory, we start our discussion with the bosonic string. The two-dimensional world-sheet swept out be the string will be denoted by Σ. There are two topologically distinct variants, namely closed and open strings. The former

[1]The author has also benefited from the string theory lecture notes by Michael Haack in drafting Sections 2.1 and 2.2.

are mediators of the gravitational force, while the latter describe gauge interactions. Our explicit discussion will deal with closed strings exclusively. In the following we will sketch important steps for arriving at a quantum theory describing those objects.

2.1.1 Classical analysis

We parametrize the worldsheet with two coordinates σ^α with $\alpha = 1, 2$, where $\sigma^1 \equiv \tau \in \mathbb{R}$ is a time-like parameter and $\sigma^2 \equiv \sigma$ is a compact parameter indicating the position along the string. Conventionally, $\sigma \in [0, 2\pi]$ for closed strings and $\sigma \in [0, \pi]$ for open strings. The embedding of the worldsheet into a D-dimensional spacetime $\mathcal{M}$ is formally defined by the fields $X^\mu(\tau, \sigma) : \Sigma \to \mathcal{M}$ where $\mu = 0, \ldots, D-1$. The action of a freely moving string can be written in terms of the area of the worldsheet, leading to the *Nambu-Goto action*

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{-\gamma}\,. \tag{2.1}$$

Here, $\gamma_{\alpha\beta}$ are components of the induced metric on Σ and $\gamma = -\det(\gamma_{\alpha\beta})$. T is the tension of the string and measures the string energy per unit length, with units $[T] = [\text{length}]^{-2}$. It is commonly denoted as $T = 1/(2\pi\alpha')$ and is the only dimensionful free parameter of the theory. The string length scale (l_s) and mass scale (M_s) are defined as $l_s = M_s^{-1} = 2\pi\sqrt{\alpha'}$. The Nambu-Goto action features manifest Poincarè symmetry and is invariant under local coordinate reparametrizations $\sigma^\alpha \to \tilde{\sigma}^\alpha$.

The Nambu-Goto action has some shortcomings: it leads to rather complicated equations of motion and is difficult to quantize in the path integral formalism due to the appearance of the square root. One can rewrite the action in a useful manner by introducing a dynamical worldsheet metric $g_{\alpha\beta}(\sigma)$ that couples to the worldsheet scalars X^μ, leading to the *Polyakov action*

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta}(\sigma) \partial_\alpha X \cdot \partial_\beta X\,. \tag{2.2}$$

This action is the true starting point for studying string theory. If $g_{\alpha\beta}(\sigma)$ is on-shell, S_P reduces to (2.1) again. A change of the metric leads to a response in the matter action, leading to the definition of the energy momentum tensor

$$T_{\alpha\beta} = -\frac{1}{\alpha'} \left(\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X \right)\,, \tag{2.3}$$

which is traceless ($T^\alpha_\alpha = 0$) and conserved ($\nabla^\alpha T_{\alpha\beta} = 0$). The action (2.2) is still invariant under Poincarè transformations and local diffeomorphisms $\sigma^\alpha \to \tilde{\sigma}^\alpha$, but is also invariant under local Weyl transformations which act as

$$X^\mu(\sigma) \to X^\mu(\sigma)\,, \quad g_{\alpha\beta}(\sigma) \to e^{2\omega(\sigma)} g_{\alpha\beta}(\sigma)\,. \tag{2.4}$$

These transformations locally deform the worldsheet while preserving angles among lines and are usually called conformal transformations. Due to Weyl invariance, the only other term that one can add to the Polyakov action is a (topological) 2D Einstein-Hilbert term. Moreover, in two dimensions diffeorphism and Weyl invariance can be used to gauge fix the metric locally to $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,1)$, while globally only $g_{\alpha\beta} = e^{2\omega(\sigma)}\eta_{\alpha\beta}$ is achievable. We are then led to the free action of D scalar fields given by

$$S_P = \frac{T}{2}\int d^2\sigma\Big(\dot{X}^2 - (X')^2\Big)\,. \tag{2.5}$$

However, even after gauge fixing the equations of motion of the metric, namely $T_{\alpha\beta} = 0$, must be implemented as a constraint. This is tied to the fact that there are still residual diffeomorphisms whose effect can be undone by Weyl rescalings. Infinitesimal coordinate shifts ϵ^α belonging to this group are called conformal Killing vectors and satisfy

$$\nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha - (\nabla_\gamma\epsilon^\gamma)g_{\alpha\beta} = 0\,. \tag{2.6}$$

Every conformal Killing vector yields a conserved current $J^\alpha_\epsilon = T^{\alpha\beta}\epsilon_\beta$.

The equations of motion from (2.5) are the free wave equations $\partial_\alpha\partial^\alpha X^\mu = 0$. It is best to solve them using the light-cone coordinates $\sigma^\pm = \sigma \pm \tau$, in which case $\partial_+\partial_- X^\mu = 0$. The most general solution is decomposed into left- and right-movers as

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)\,. \tag{2.7}$$

To find the classical closed string solution, we require periodicity $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$ and impose the constraints, which now read $\alpha' T_{\pm\pm} = \partial_\pm X\cdot\partial_\pm X = 0$. General periodic functions have the Fourier expansion

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-in\sigma^+}\,,\\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}\,, \end{aligned} \tag{2.8}$$

where x^μ, p^μ are the position and momentum of the center of mass of the string and the Fourier coefficients satisfy $\alpha^\mu_n = (\alpha^\mu_{-n})^*$, $\bar{\alpha}^\mu_n = (\bar{\alpha}^\mu_{-n})^*$ since X^μ must be real. The ansatz satisfies the constraints if

$$L_n = \sum_{m\in\mathbb{Z}}\alpha_m\cdot\alpha_{n-m} \overset{!}{=} 0\,, \quad \bar{L}_n = \sum_{m\in\mathbb{Z}}\bar{\alpha}_m\cdot\bar{\alpha}_{n-m} \overset{!}{=} 0 \quad \forall n\in\mathbb{Z}\,, \tag{2.9}$$

where $\alpha^\mu_0 = \bar{\alpha}^\mu_0 = \sqrt{\alpha'/2}\,p^\mu$. One can show that the infinite set of $L_n, \bar{L}_n$ form a closed algebra, namely the Witt algebra given by

$$\{\overset{(-)}{L}_m, \overset{(-)}{L}_n\}_{\mathrm{PB}} = -i(m-n)\overset{(-)}{L}_{m+m}\,, \quad \{\bar{L}_m, L_n\}_{\mathrm{PB}} = 0\,, \tag{2.10}$$

where $\{.,.\}_{\mathrm{PB}}$ denotes a Poisson bracket and for our convenience we introduced the compact accent notation $(-)$ above symbols indicating that an equation holds seperately for both the right- and leftmovers. The quantities L_n generate the residual conformal transformations that were mentioned previously and are typically called Virasoro generators, while the constraints (2.9) are known as the Virasoro constraints. For $n = 0$, they lead to the so-called level matching condition

$$M^2 = p^\mu p_\mu = \frac{4}{\alpha'} \sum_{m>0} \alpha_m \cdot \alpha_{-m} = \frac{4}{\alpha'} \sum_{m>0} \bar{\alpha}_m \cdot \bar{\alpha}_{-m}\,, \tag{2.11}$$

connecting the left- and right-moving sectors which are otherwise independent.

2.1.2 Quantization and spectrum

In order to canonically quantize the above theory one is guided by the standard rules from QFT: The canonical phase space variables $X^\mu(\tau,\sigma), \Pi^\mu(\tau,\sigma) = T\,\dot{X}^\mu(\tau,\sigma)$ are promoted to operators and Poisson brackets are replaced by commutators via $\{.,.\}_{\mathrm{PB}} \to \frac{1}{i}[.,.]$. If one analyzes the closed string expansion of X^μ one is guided to

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}\,, \quad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] \tag{2.12}$$

and zero otherwise. From (2.12) one infers that after defining the operators $a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m$ and $(a^\dagger)^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ with $m > 0$ one obtains an infinite set of algebras for harmonic oscillators. The Fock space can then be constructed by acting with creation operators on oscillator ground states with given momentum, yielding

$$\mathcal{H}_k \equiv \Big\{ |\{\lambda_{n,\mu}, \bar{\lambda}_{n,\mu}\rangle\,; k \Big\} = \prod_{\mu=1}^{D}\prod_{n=1}^{\infty} (a^{\dagger,\mu}_n)^{\lambda_{n,\mu}} (\bar{a}^{\dagger,\mu}_n)^{\lambda_{n,\mu}}\, |0;k\rangle\;, \tag{2.13}$$

where $|0;k\rangle$ denote momentum eigenstates. However, not all such states satisfy the mass-shell conditions and, even worse, there are states of negative norm as

$$\forall m > 0: \quad \langle 0;k|\,\alpha^0_m \alpha^0_{-m}\,|0;k'\rangle = -m\delta^{(D)}(k-k')\,. \tag{2.14}$$

We recall that the Virasoro constraints still need to be imposed, which in the classical theory corresponded to setting $L_n = \bar{L}_n = 0$. For the implementation of those constraints in a quantum framework there are two well-known procedures: there is *light-cone quantization* (LCQ), where one solves the Virasoro constraints already at the classical level. This leads to a description that is manifestly unitary, but only Lorentz invariant in the critical dimension $D = 26$. In the second method, known as *old-covariant quantization* (OCQ), the Virasoro generators impose constraints on states in the quantum theory, resulting in manifest Lorentz invariance whereas unitarity only holds in $D = 26$. We will focus on the light-cone approach since there the derivation of the physical spectrum is more efficient. If we were interested in the calculation of string interactions via the path integral, the covariant approach would be more illuminating.

After going to conformal gauge in the Polyakov action, the residual gauge symmetry in light-cone coordinates is $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$. Since every $\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-)$ is a solution to the free wave equation, one can identify this $\tilde{\tau}$ with one of the string coordinates. After introducing spacetime light-cone coordinates $X^{\pm} = (X^0 \pm X^{D-1})/\sqrt{2}$ and X^i with $i = 1, \ldots, D-2$, one sets $X^+ = \alpha' p^+ \tau$ for closed strings, defining the light-cone gauge. Center of mass momenta such as p^+ are defined in the same way as before. This choice implies $x^+ = 0$ and $\overset{(-)}{\alpha}{}^+_n = 0 \ \forall_{n\neq 0}$. The Virasoro generators become

$$\overset{(-)}{L}_n = -\sqrt{\frac{\alpha'}{2}} p^+ \overset{(-)}{\alpha}{}^-_0 + \overset{(-)}{L}{}^{\perp}_n \quad \text{with} \quad \overset{(-)}{L}{}^{\perp}_n = \frac{1}{2} \sum_{m\in\mathbb{Z}} \overset{(-)}{\alpha}{}^i_{n-m} \overset{(-)}{\alpha}{}^i_m \,, \tag{2.15}$$

where the sum over i is implicit. Solving the Virasoro constraints yields

$$\forall_{n\neq 0}: \quad \overset{(-)}{\alpha}{}^-_n = \sqrt{\frac{\alpha'}{2}} \frac{1}{p^+} \overset{(-)}{L}{}^{\perp}_n \,, \quad p^- = \frac{1}{\alpha' p^+}(L_0^{\perp} + \bar{L}_0^{\perp}) \,, \tag{2.16}$$

while the level-matching constraint $L_0 - \bar{L}_0$ still needs to be imposed in the quantum theory. One can see that physical states will be built only from the transverse oscillator excitations α^i_{-n}. Quantization proceeds by imposing the non-trivial commutators

$$[x^i, p^j] = i\delta^{ij} \,, \quad [x^-, p^+] = -i \,, \quad [\overset{(-)}{\alpha}{}^i_n, \overset{(-)}{\alpha}{}^j_m] = n\delta^{ij}\delta_{m+n,0} \tag{2.17}$$

and building the Fock space by acting with creation operators on the ground states $|0; p^+, p^i\rangle$. After implementing the remaining constraint no negative-norm states remain.

The mass formula (2.11) reveals a subtlety in the quantum theory: The commutation relations in (2.17) imply that for $L_0^{\perp}, \bar{L}_0^{\perp}$ there is an ordering ambiguity, which is taken care of by defining the normal-ordered operators

$$\overset{(-)}{L}{}^{\perp}_n = \frac{1}{2} \sum_{m\in\mathbb{Z}} : \overset{(-)}{\alpha}{}^i_{n-m} \overset{(-)}{\alpha}{}^j_m : \,, \tag{2.18}$$

where as usual normal ordering $: \ldots :$ puts annihilation operators to the right of creation operators. As a consequence, we obtain modified operators $L_0 \to L_0 + a$ and $\bar{L}_0 \to \bar{L}_0 + a$ in the quantum theory. The constant a shifts the ground state energy due to the finite length of the string and therefore encodes a Casimir effect. There is a nice heuristic derivation for the value for a: Say we were to spell out the naive classical formula for $L_0^{\perp}$ (denoted as H) in the quantum theory, then

$$H = \frac{\alpha'}{4} p^i p^i + \frac{1}{2} \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \frac{1}{2} \sum_{n=1}^{\infty} \alpha^i_n \alpha^i_{-n} = L_0^{\perp} + \frac{D-2}{2} \sum_{n=1}^{\infty} n \,, \tag{2.19}$$

where in the second step we brought the right-hand sum into normal-ordered form by using the relevant commutator from (2.17). To regularize the infinite sum over n we introduce a regulator $q = e^{-2\pi/\Lambda}$ and rewrite the expression as

$$\lim_{\Lambda\to\infty} \sum_{n=1}^{\infty} n\, q^n = \lim_{\Lambda\to\infty} \left[q\partial_q \left(\frac{1}{1-q} \right) \right] = \lim_{\Lambda\to\infty} \left[\frac{\Lambda^2}{4\pi^2} - \frac{1}{12} + \mathcal{O}(\Lambda^{-1}) \right] , \tag{2.20}$$

where in the last step we expanded for large Λ. The divergent term yields a constant contribution to the vacuum energy that needs to be cancelled by a counterterm, i.e. a bare contribution to the cosmological constant of the world-sheet. From the finite piece one reads of that the Casimir energy is $a = -(D-2)/24$. Keeping in mind that p^- in (2.16) acquires the Casimir shift, the mass formula for closed string states reads

$$M^2 = -2p^+p^- + p^ip^i = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2a)\,, \tag{2.21}$$

where we used that $a = \bar{a}$. The number operators $N^\perp$, defined via $L_0^\perp = \alpha'(p^ip^i) + N^\perp$, and $\bar{N}^\perp$, which is defined analogously, satisfy

$$N^\perp = \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i \stackrel{!}{=} \sum_{m=1}^{\infty} \bar{\alpha}_{-m}^i \bar{\alpha}_m^i = \bar{N}^\perp \tag{2.22}$$

due to level matching. As mentioned already, in LCQ Lorentz-invariance is not guaranteed but only holds in a specific spacetime dimension. By demanding that certain components of the angular momentum operator satisfy the Lorentz algebra, one can derive both $D_{\text{crit}} = 26$ and $a = -1$ (see e.g. [14] for details).

Concerning the spectrum, at the lowest level one finds a tachyonic ground state T with $M^2 = -4/\alpha'$, signaling that we are expanding the theory about an unstable maximum of the tachyon field potential. Since this tachyon is an omnipotent part of the bosonic closed string spectrum, the theory does not feature a stable vacuum. This flaw can only be fixed by introducing supersymmetry, to which we will come shortly. At the first excited level we find 24^2 massless states $\alpha_{-1}^i \bar{\alpha}_{-1}^j |0; p^+, p^i\rangle$ transforming in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$. The irreducible representations are the traceless symmetric tensor $G_{\mu\nu}$ (spacetime metric), the anti-symmetric tensor $B_{\mu\nu}$ (Kalb-Ramond field) and the singlet Φ (the dilaton), respectively. These modes appear in all string theories except $B_{\mu\nu}$ which is absent for unoriented strings. In accordance with Wigner's classification, the massless states fill out a representation of $SO(D-2)$, while all higher excitations (with fixed momentum) are massive and fall into representations of $SO(D-1)$. Table 2.1 provides an overview over the low-energy spectrum.

$N^\perp = \bar{N}^\perp$	$\alpha' M^2$	little group	representation	field content
0	-4	$SO(25)$	1	T
1	0	$SO(24)$	$1 + \ydiagram{1,1} + \ydiagram{2}$	(Φ, B_{MN}, G_{MN})

Table 2.1: Low-energy spectrum bosonic string.

Quantum consistency of string backgrounds

It is instructive to understand how the Casimir effect manifests in the alternative covariant quantization approach. There the classical Virasoro constraints are imposed as conditions on the states in the quantum theory. The generators L_m become (normal-ordered) operators that satisfy a quantum version of the Witt algebra from (2.10), known as the Virasoro algebra, given by

$$[\overset{(-)}{L}_m, \overset{(-)}{L}_n] = (m-n)\overset{(-)}{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \,. \tag{2.23}$$

The second term is the central extension and c is known as the central charge. c is part of the data defining a 2D *conformal field theory* (CFT) [15, 16], a highly constrained QFT that naturally implements local scale invariance on the string world-sheet. In the CFT framework, the Hamiltonian on the closed string worldsheet has the form $H = L_0 + \bar{L}_0 - (c+\bar{c})/24$, so by comparison with the LCQ approach we recognize that a single free boson contributes to the CFT with $c = \bar{c} = 1$. Using CFT techniques, one can show that in the quantum theory the trace of the energy momentum tensor has the expectation value

$$\langle T^\alpha_\alpha \rangle = -\frac{c}{12} R^{(2)} \tag{2.24}$$

for any state. It is therefore not traceless on a curved worldsheet for $c \neq 0$ and hence Weyl invariance no longer holds. This trace anomaly is also known as *Weyl-anomaly*. The canonical way of removing the Weyl-anomaly in bosonic string theory is couple the CFT of D free bosons with central charge c_X to a CFT of ghost fields with $c_{\text{ghost}} = -26$. In the critical dimension $D_{\text{crit}} = 26$ one then obtains $c_{\text{tot}} = c_{\text{X}} + c_{\text{ghost}} = 0$.

Our whole discussion so far was based on the Polyakov action describing a free bosonic string in flat spacetime. We could generalize this action by including the massless string excitations from table (2.1) as a classical background, giving rise to the action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-g} \Big[\Big(g^{\alpha\beta} G_{\mu\nu}(X) + i\epsilon^{\alpha\beta} B_{\mu\nu}(X) \Big) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)} \Phi(X) \Big] \,, \tag{2.25}$$

which describes a *non-linear sigma model*. The first term captures that the string is moving in a curved spacetime. This spacetime corresponds to a coherent state of gravitons, i.e. it consists of a high number of quanta whose expectation value behaves like a smooth geometry. The second term describes the minimal coupling of the string to the two-form gauge potential $B_{\mu\nu}$. The analogy with the point particle coupled to a $U(1)$ gauge field A_μ can be made apparent by introducing a source term such that

$$S_{B_2} = \int d^D x \, B_{\mu\nu}(x) j^{\mu\nu}(x) \,, \quad j^{\mu\nu} = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \delta^{(D)}(x - X(\tau,\sigma)) \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu . \tag{2.26}$$

The third term is the coupling to the dilaton, whose background value Φ_0 controls the strength of string interactions in terms of the string coupling $g_s = e^{\Phi_0}$. Scattering amplitudes in perturbative string theory are sums over worldsheets of different genera. Indeed, for a constant background value Φ_0 the dilaton action is purely topological since

$$S_{\Phi_0} = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-g} \, \Phi_0 R^{(2)} = \Phi_0 \chi(\Sigma) \tag{2.27}$$

according to the Gauss-Bonnet theorem. Here $\chi(\Sigma) = 2 - 2g$ is the Euler number and g the genus of Σ. In the path integral every closed string world-sheet diagram will thus be weighed by the prefactor $e^{-S_{\Phi_0}} = g_s^{2-2g}$.

As before, the model can only yield a consistent quantum theory if Weyl invariance is preserved. The trace of the energy momentum tensor now reads

$$2\alpha' T^\alpha_\alpha = \alpha' \beta^\Phi R^{(2)} + \beta^G_{\mu\nu} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \beta^B_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \,, \tag{2.28}$$

where $\beta^\Phi, \beta^G_{\mu\nu}, \beta^B_{\mu\nu}$ are the beta functions for the fields $\Phi, G_{\mu\nu}, B_{\mu\nu}$ (couplings on the worldsheet). A vanishing Weyl anomaly requires all beta functions to vanish, which puts constraints on said fields and defines a string vacuum. If the target space has curvature radius $R \gg \sqrt{\alpha'}$, the world-sheet sigma model is weakly coupled and the β-functions admit a perturbative α'-expansion. For example, the beta function for G given by

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} - \frac{\alpha'}{4} H^{\rho\sigma}_\mu H^{\nu\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi + \mathcal{O}(\alpha'^2) \tag{2.29}$$

imposes Ricci-flatness of the spacetime metric at leading order, assuming a pure metric background. While there the condition for Weyl invariance is only known to some leading order, there can be setups where the background is exact. One example are compactifications on CY spaces, to be introduced later. Finally, let us mention that the condition of vanishing Weyl anomaly can also be met by so-called no-critical string theories where $D \neq D_{\text{crit}}$. This can be for example achieved by picking a non-trivial dilaton profile or coupling the worldsheet scalars to an internal CFT.

One-loop partition function

We finish our discussion of bosonic string theory by computing the one-loop partition function, which nicely illustrates how string theory organizes infinite towers of states into a consistent quantum-gravitational amplitude. We consider a Euclidean string worldsheet with the topology of a torus. In the complex plane with coordinate z a torus is obtained by identifying $z \equiv z + 2\pi$ and $z \equiv z + 2\pi\tau$, where $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ parametrizes how "skewed" the torus is. We identify $z = \sigma + i\tau$, where (σ, τ) are the Euclidean worldsheet coordinates. Since the Euclidean time is compact by definition, the path integral corresponding to the vacuum energy is encoded in the thermal partition function

$$Z(\tau, \bar\tau) = \text{Tr}(e^{-2\pi\tau_2 H} e^{-2\pi i \tau_1 P}) \,. \tag{2.30}$$

H and P are the world-sheet Hamiltonian and momentum operators defined in terms of the Virasoro generators via $H = L_0 + \bar L_0 - (c + \bar c)/24$ and $P = L_0 - \bar L_0$. (2.30) shows that Euclidean time evolution by $2\pi\tau_2$ is accompanied by a shift of the spatial coordinate by $2\pi\tau_1$. The trace is taken over the full closed string Hilbert space. If we fix to light-cone gauge, we work with the Virasoro generators $L_0^\perp, \bar L_0^\perp$ and trace over the the spectrum of states defined in (2.21). The partition function is then given by

$$Z(\tau, \bar\tau) = \text{Tr}_{\mathcal{H}_{\text{phys}}}(q^{L_0^\perp - 1} \bar q^{\bar L_0^\perp - 1}) = \text{Tr}_{\mathcal{H}_{\text{phys}}}(e^{-\pi\tau_2 \alpha' p^i p^i} q^{N^\perp - 1} \bar q^{\bar N^\perp - 1}) \,, \tag{2.31}$$

where $q = e^{2\pi i\tau}$. The trace over the zero modes correspond to an ordinary Gaussian integral and yields

$$\int \frac{d^{24}p}{(2\pi)^{24}} e^{-\pi\tau_2\alpha' p^i p^i} = \frac{1}{(2\pi\sqrt{\alpha'})^{24}} \frac{1}{\tau_2^{12}} \,. \tag{2.32}$$

The contribution of each single oscillator is given by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \,. \tag{2.33}$$

The factors $q^{\frac{1}{24}}$ of all oscillators $i = 1, \ldots, 24$ cancel against the contribution from the Casimir energy. Equation (2.33) defines the Dedekind eta function $\eta(\tau)$, a modular form whose properties we have summarized in appendix A. These types of functions have definite behavior under modular transformations of the τ-parameter, which acts as

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,, \tag{2.34}$$

where $SL(2,\mathbb{Z})$ is the group of 2×2 matrices with determinant 1. In our case they are transformations of the complex torus parameter that leave the torus invariant (after using its coordinate identifications). In section 3.3 we will also briefly discuss the space of τ-parameters describing physically inequivalent tori. Coming back to the one-loop partition function, the final result reads

$$Z(\tau,\bar{\tau}) = \frac{1}{(2\pi\sqrt{\alpha'})^{24}} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} \,. \tag{2.35}$$

With the properties of $\eta(\tau)$ one can show that the whole function is modular invariant, as every well-defined function on the torus should be. Modular invariance is a key consistency condition in string theory. In the supersymmetric string theories, to which we will come next, it even serves as a guiding principle to construct a tachyon-free, spacetime supersymmetric spectrum.

2.2 Superstring

While the bosonic string has many more interesting features than we were able to cover, the unavoidable presence of the tachyon ultimately forces to adjust our starting point in order to obtain a theory with a classically stable vacuum. Moreover, the field content of the worldsheet theory does not lead to any fermionic matter. Supersymmetry can fix the spectrum in that regard. In this section we present the the RNS formulation (after Ramond-Neveu-Schwarz), in which supersymmetry is made manifest on the worldsheet, and spacetime supersymmetry arises after quantization and a properly chosen projection in the space of states. The spinor formalism introduces a couple of new technicalities, but we will try to keep the discussion light and focus on the low-energy spectra that arise in the closed string sector.

2.2.1 The RNS action and closed string solutions

Compared to the Polyakov action with D worldsheet scalars X^μ, the RNS action hosts additional D Majorana worldsheet spinors $\psi^\mu_A = (\psi^\mu_+, \psi^\mu_-)$, where $\psi^\mu_\pm$ are Weyl spinors, i.e. they have definite chirality. In flat gauge the action reads

$$S = -\frac{1}{8\pi}\int d^2\sigma \left(\frac{2}{\alpha'}\partial_\alpha X_\mu \partial^\alpha X^\mu + 2i\bar{\psi}^\mu_A \gamma^\alpha_{AB}\partial_\alpha \psi_{B,\mu}\right) . \tag{2.36}$$

Here γ^α_{AB} denote two-dimensional gamma matrices and and $\bar{\psi} = \psi^T\gamma^0$. The supersymmetry transformations $\delta X^\mu = \sqrt{\alpha'/2}i\bar{\epsilon}\psi^\mu$ and $\delta\psi^\mu = \sqrt{\alpha'/2}\gamma^\alpha\partial_\alpha X^\mu\epsilon$ with Majorana spinor ϵ relate bosons and fermions, and on-shell the number of bosonic and fermionic degrees of freedom matches. The RNS action in flat gauge arises from gauge-fixing the supersymmetric Polyakov action, which describes two-dimensional supergravity coupled to matter fields. Since spinors live in flat tangent spaces with Minkowskian metric η^{ab} $(a, b = 0, 1)$, the supergravity field content is described by a vielbein e^a_α defined as $e^a_\alpha e^b_\beta g^{\alpha\beta} = \eta^{ab}$ and its superpartner, the gravitino χ_α. The local symmetries of the respective action can be used to reach "superconformal gauge" and thereby reproduce (2.36). In worldsheet lightcone coordinates $\sigma^\pm$ the RNS action takes the form

$$S = \frac{1}{2\pi}\int d^2\sigma \left(\frac{2}{\alpha'}\partial_+ X\cdot\partial_- X + i\left(\psi_+\cdot\partial_-\psi_+ + \psi_-\cdot\partial_+\psi_-\right)\right) , \tag{2.37}$$

where according to the equations of motion the spinors $\psi_\pm$ are also chiral in the sense that $\psi_\pm = \psi_\pm(\sigma^\pm)$. Compared to the bosonic discussion, in superconformal gauge one has to impose the equations of motion of the two fields e^a_α and χ_α from the supersymmetric Polyakov action. While the vielbein yields the energy momentum tensor $T_{\alpha\beta}$, the variation with respect to the gravitino leads to a new fermionic tensor $T_{F\alpha}$. In light-cone coordinates their non-trivial components read

$$\begin{aligned} T_{\pm\pm} &= -\frac{1}{\alpha'}\partial_\pm X\cdot\partial_\pm X - \frac{i}{2}\psi_\pm\cdot\partial_\pm\psi_\pm \, , \\ T_{F\pm} &= -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_\pm\cdot\partial_\pm X \, . \end{aligned} \tag{2.38}$$

These quantities are conserved if $\partial_+ T_{--} = \partial_- T_{++} = 0$ and $\partial_- T_{F+} = \partial_+ T_{F-} = 0$, respectively. Conservation of T and T_F once again yields infinitely many conserved charges generating remnant conformal transformations ($\delta\sigma^\pm = \delta\sigma^\pm(\sigma^\pm)$) and supersymmetry transformations ($\epsilon^\pm = \epsilon^\pm(\sigma^\pm)$).

According to (2.37) the bosons and fermions are decoupled, hence the bosonic mode expansion and boundary conditions remain the same. If we vary the fermionic part of (2.37) for the closed string, we find that the most general solution that does not mix ψ_+ and ψ_- and that respects Poincarè symmetry falls in one of two categories:

$$\begin{aligned} \text{R-sector}: \ \psi^\mu_\pm(\sigma + 2\pi) &= \psi^\mu_\pm(\sigma) \, , \\ \text{NS-sector}: \ \psi^\mu_\pm(\sigma + 2\pi) &= -\psi^\mu_\pm(\sigma) \, . \end{aligned} \tag{2.39}$$

If one combines the left-and right-movers, one obtains the four independent sectors R-R, NS-NS, R-NS and NS-R, where the left (right) symbol refers to left-(right-)movers. As we shall see soon, the former two describe spacetime bosons while the latter two lead to spacetime fermions. Due to periodicity, the R-sector expansion reads

$$\psi^\mu_+(\sigma^-) = \sum_{n\in\mathbb{Z}} b^\mu_n e^{-in\sigma^-} \quad \text{and} \quad \psi^\mu_-(\sigma^+) = \sum_{n\in\mathbb{Z}} \bar{b}^\mu_n e^{-in\sigma^+} , \tag{2.40}$$

while the NS-sector is anti-periodic and therefore half-integer moded:

$$\psi^\mu_+(\sigma^-) = \sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-in\sigma^-} \quad \text{and} \quad \psi^\mu_-(\sigma^+) = \sum_{r\in\mathbb{Z}+1/2} \bar{b}^\mu_r e^{-in\sigma^+} . \tag{2.41}$$

2.2.2 Quantization, GSO projection and type II strings

To canonically quantize the fermion fields, one starts similarly by imposing canonical anti-commutation relations among the fermions $\psi^\mu_\pm$. After plugging in the mode expansion, one obtains the relations

$$\begin{aligned} \text{R-sector}: \ \{b^\mu_m\, b^\nu_n\} = \{\bar{b}^\mu_m\, \bar{b}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \, , \\ \text{NS-sector}: \ \{b^\mu_r\, b^\nu_s\} = \{\bar{b}^\mu_r\, \bar{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \, . \end{aligned} \tag{2.42}$$

The construction of the state space reveals a crucial difference between the ground states in the NS-sector and the R-sector. For ease of notation, we will write explicit formulas only for the right-moving sector of the closed string in the following. The NS ground state $|0\rangle_{\text{NS}}$ is a spacetime scalar, on top of which the state space can be built by acting with the operators α^μ_{-m} and b^μ_{-r} with $m>0, r>0$ as we are used to already. In the R-sector on the other hand, a ground state $|0\rangle_{\text{R}}$ that is annihilated by α^μ_m for $m \geq 0$ and b^μ_m for $m>0$ is not uniquely defined as each operator b^μ_0 acting on $|0\rangle_{\text{R}}$ leads to a state with identical excitation number. According to (2.42), the operators $\sqrt{2}b^\mu_0$ satisfy the spacetime Clifford algebra, suggesting that the ground states form a spinor representation of $SO(1, D-1)$. One can show that this Dirac representation decomposes into two irreducible Weyl representations of opposite chirality, denoted as $\mathbf{8}/\mathbf{8}'$ for positive/negative chirality.

The implementation of the Super-Virasoro constraints $T_{\pm\pm} = 0 = T_{F\pm}$ can be done again either at the classical or quantum level. While the analogous light-cone approach gives a transparent physical picture, it is technically more involved to solve the constraints on the Hilbert space, which is why the covariant approach is often preferred in this case. We will just summarize the main results in the following. The Super-Virasoro generators of T_{--} and T_{F-} have the mode expansion

$$\begin{aligned} L_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{-in\sigma} T_{--} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{m+n} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r+\frac{n}{2}\right) b_{-m}\cdot b_{r+n} \, , \\ G_r &= -\frac{1}{\pi}\int_0^{2\pi} d\sigma e^{-ir\sigma} T_{F-} = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_{m+r} \, , \end{aligned} \tag{2.43}$$

where in the first formula $\phi = 0$ in the R-sector and $\phi = 1/2$ in the NS sector, and in the second line $r \in \mathbb{Z}$ for R-sector and $r + 1/2 \in \mathbb{Z}$ for NS. The above operators are all normal-ordered in the quantum theory, leading to potential shifts $a_{\mathrm{R}}, a_{\mathrm{NS}}$ in L_0 in both sectors. Physical states are defined by $L_n |\phi\rangle = 0$ for $(n > 0)$ and

$$\begin{aligned} \text{R-sector}: \ G_n |\phi\rangle = 0 \ \ (n \geq 0) \ \ \& \ \ (L_0 - a_{\mathrm{R}}) |\phi\rangle = 0 \,, \\ \text{NS-sector}: \ G_r |\phi\rangle = 0 \ \ (r \geq 0) \ \ \& \ \ (L_0 - a_{\mathrm{NS}}) |\phi\rangle = 0 \,, \end{aligned} \tag{2.44}$$

and are subject to the level matching condition once again. To obtain the Hilbert space, one identifies all states differing by null states. Imposing absence of negative-norm states and unitarity at 1-loop leads to the critical dimension $D = 10$ and the normal ordering constants $a_{\mathrm{R}} = 0, a_{NS} = 1/2$. A detailed analysis of the spectrum shows that the massless states in the R-sector are the two Weyl spinors $\mathbf{8}/\mathbf{8}'$ comprising 16 physical states, while in the NS-sector there is still a tachyonic state at zeroth excitation and a massless vector $\mathbf{8}_V$ of the little group $SO(8)$ describing 8 on-shell states.

At this stage the physical spectrum is not spacetime supersymmetric even at the massless level and the tachyon is still present. Both issues can be fixed with the *GSO-projection* (after Gliozzi-Scherk-Olive), which performs a projection in the state space with respect to the worldsheet fermion number $(-1)^F$, defined as

$$\begin{aligned} \text{NS-sector}: \ (-1)^F = -(-1)^{\sum_{r>0} b_r \cdot b_r} \,, \\ \text{R-sector}: \ (-1)^F = b_0^0 \dots b_0^9 (-1)^{\sum_{n>0} b_n \cdot b_n} \,. \end{aligned} \tag{2.45}$$

Imposing $(-1)^F = +1 = (-1)^{\bar{F}}$ in the NS sector removes the tachyon. In the R-sector there are two inequivalent choices: one can either impose $(-1)^F = +1 = (-1)^{\bar{F}}$, leading to type IIB string theory, or $(-1)^F = +1 = -(-1)^{\bar{F}}$, yielding type IIA. In each case the number of bosonic and fermionic degrees of freedom is equal. In the bosonic spectrum one finds the fields from the bosonic string and novel RR p-form gauge fields, while on the fermionic side there are two gravitini and dilatini, albeit with different chirality structure. The tables 2.2 and 2.3 display the massless spectra of each theory, indicating the ten-dimensional fields and the irreducible representations of the little group $SO(8)$.

sector	$SO(8)$ representation	irred. representations	$10D$ fields
NS-NS	$\mathbf{8}_V \otimes \mathbf{8}_V$	$\mathbf{1} \oplus \mathbf{28}_V \oplus \mathbf{35}_V$	(Φ, B_{MN}, G_{MN})
R-R	$\mathbf{8}_C \otimes \mathbf{8}_C$	$\mathbf{1} \oplus \mathbf{28}_V \oplus \mathbf{35}_+$	$(C^{(0)}, C^{(2)}_{MN}, C^{(4)}_{MNPQ})$
NS-R	$\mathbf{8}_V \otimes \mathbf{8}_C$	$\mathbf{8}_S \oplus \mathbf{56}_C$	$(\tilde{\lambda}^1_A, \psi^{1\,M}_A)$
R-NS	$\mathbf{8}_C \otimes \mathbf{8}_V$	$\mathbf{8}_S \oplus \mathbf{56}_C$	$(\tilde{\lambda}^2_A, \psi^{2\,M}_A)$

Table 2.2: Massless spectrum of type IIB.

Let us stress that the GSO-projection is not only motivated by the previously mentioned problems but is in fact required to achieve modular invariance of the torus partition function. This can be seen as follows: each of the four closed string sectors has a different

sector	$SO(8)$ representation	irred. representations	$10D$ fields
NS-NS	$\mathbf{8}_V \otimes \mathbf{8}_V$	$\mathbf{1} \oplus \mathbf{28}_V \oplus \mathbf{35}_V$	(Φ, B_{MN}, G_{MN})
R-R	$\mathbf{8}_S \otimes \mathbf{8}_C$	$\mathbf{8}_V \oplus \mathbf{56}_V$	$(C^{(1)}_M, C^{(3)}_{MNP})$
NS-R	$\mathbf{8}_V \otimes \mathbf{8}_C$	$\mathbf{8}_S \oplus \mathbf{56}_C$	$(\tilde{\lambda}_A, \psi^M_A)$
R-NS	$\mathbf{8}_S \otimes \mathbf{8}_V$	$\mathbf{8}_C \oplus \mathbf{56}_S$	$(\lambda_A, \tilde{\psi}^M_A)$

Table 2.3: Massless spectrum of type IIA.

set of boundary conditions along the directions of the torus. Since the modular group mixes these sectors non-trivially, all of their contributions must be included. However, as modular transformations act with non-trivial phases on these terms, modular invariance is guaranteed only for certain relative signs, which determines the GSO-projection.

For both theories, the leading part of the low-energy effective action for the massless closed string states is given by the corresponding type IIB or type IIA supergravity theory. In practice this can be checked by computing string amplitudes for massless external states with the vertex operator formalism in the appropriate picture and comparing these results with field-theoretic diagrams from a candidate supergravity action. The bosonic action splits into the pieces $S^{\text{IIA/B}} = S_{NS} + S_R^{\text{IIA/B}}$ with the universal NS-sector action

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) , \tag{2.46}$$

where κ_{10} is the 10D gravitational coupling, related to string theory quantities via

$$2\kappa_{10}^2 = (2\pi)^7 g_s^2 (\alpha')^4 . \tag{2.47}$$

Moreover, $H_3 = dB_2$ and $|F_p|^2 = F_{\mu_1 \dots \mu_p} F^{\mu_1 \dots \mu_p}/p!$ holds for all p-forms. The theory-specific parts of the action read

$$\begin{aligned} S_R^{\text{IIA}} &= -\frac{1}{4\kappa_{10}^2} \left[\int d^{10}x \sqrt{-G} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) + \int B_2 \wedge F_4 \wedge F_4 \right] , \\ S_R^{\text{IIB}} &= -\frac{1}{4\kappa_{10}^2} \left[\int d^{10}x \sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) + \int C_4 \wedge H_3 \wedge F_3 \right] . \end{aligned} \tag{2.48}$$

The field strengths of the RR fields are defined as $F_p = dC_{p-1}$, $\tilde{F}_3 = F_3 - C_0 \wedge H_3$, $\tilde{F}_4 = F_4 - C_1 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. The left-hand terms are standard kinetic terms while the right-hand terms are topological Chern-Simons interactions, which are required for the closure of the $N = 2$ action under supersymmetry transformations. There is also a subtlety regarding the self-duality constraint of the field strength $\tilde{F}_5$, given by $\star \tilde{F}_5 = \tilde{F}_5$. Here $\star$ is the Hodge-$\star$ operator, a linear map with the defining property

$$\alpha \wedge \star \beta = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} \beta^{\mu_1 \dots \mu_p} \, v \equiv \langle \alpha, \beta \rangle \, v , \tag{2.49}$$

where α, β are p-forms, $\star$ maps a p-form to a $(D-p)$-form in D dimensions and v is the volume form defined as $v = \sqrt{\det g} \, dx^1 \wedge \ldots \wedge dx^D$. In particular, one has $\star v = 1$ and

$\star^2 = \xi(-1)^{p(D-p)}$, where $\xi = -1$ in Lorentzian signature. The condition on $\tilde{F}_5$ must be imposed after deriving the equations of motion, given that it is not possible to write a covariant action respecting the condition.

The type IIB action has a hidden $SL(2,\mathbb{R})$ symmetry which can be made manifest by switching to Einstein frame, where the Einstein-Hilbert term has a canonical form. For that one performs a Weyl transformation of the metric such that

$$\tilde{G}_{MN} = e^{-2\Phi(x)} G_{MN} \, . \tag{2.50}$$

Using the general transformation rule of the Ricci tensor in D dimensions and introducing the fields $\tau = C_0 + ie^{-\Phi}$ and $G_3 = F_3 - ie^{-\Phi} H_3$, one obtains

$$\begin{aligned} S_{NS}^{\mathrm{IIB}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\tilde{G}} \left[R - \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{1}{2} \frac{|G_3|^2}{\mathrm{Im}(\tau)} - \frac{1}{4} |F_5|^2 \right] , \\ S_R^{\mathrm{IIB}} &= \frac{1}{8i\kappa_{10}^2} \frac{1}{\mathrm{Im}(\tau)} C_4 \wedge G_3 \wedge \bar{G}_3 \, . \end{aligned} \tag{2.51}$$

This action is invariant under an $SL(2,\mathbb{R})$ transformation which leaves the Einstein frame metric and R-R four-form invariant and otherwise acts as

$$\tau \to \frac{a\tau + b}{c\tau + d} \, , \quad \begin{pmatrix} C_2' \\ B_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} , \tag{2.52}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In section (2.4.1) we will revisit this symmetry and discuss its physical relevance.

For later purposes we also introduce a more symmetric way of writing the type II actions, known as the democratic formulation. Note that for each $p \leq 4$ the R-R forms $C^{(p)}$ and $C^{(8-p)}$ have the same number of on-shell degrees of freedom and are related through the Hodge-duality of their corresponding field strengths. In the democratic formulation one includes these higher form fields such that the massless R-R sector becomes

$$\begin{aligned} &\text{Type IIA:} \; C^{(1)}, \; C^{(3)}, \; C^{(5)}, \; C^{(7)} \, , \\ &\text{Type IIB:} \; C^{(0)}, \; C^{(2)}, \; C^{(4)}, \; C^{(6)}, \; C^{(8)} \, . \end{aligned} \tag{2.53}$$

To have the same number of degrees of freedom as before one imposes the constraints $\star F_{2p} = (-1)^p F_{10-2p}$ and $\star F_{2p+1} = (-1)^p F_{9-2p}$ by hand at the level of equations of motion. The resulting actions only have kinetic terms of the form $|F_p|^2$ with altered prefactors and no Chern-Simons terms.

2.3 Effective Theories in lower dimensions

As anticipated already, supersymmetry removes several pathological properties of bosonic string theory and provides a better starting point for building (semi-)realistic models for particle physics and cosmology. Nevertheless, we still have to deal with the obvious discrepancy between the crititical dimension of the superstring and the observed four macroscopic dimensions for starters. In technical jargon, we need to *compactify* the ten-dimensional string theory to four dimensions by assuming that the extra six dimensions form a compact space. Moreover, there are many good reasons to consider lower-dimensional models that still preserve some degree of supersymmetry, even though to this day there is no evidence for supersymmetry from accelerators. On the phenomenological side, supersymmetry can explain the electroweak hierarchy problem since fermionic and bosonic loop corrections to the Higgs mass cancel, provide a dark matter candidate in the form of a light susy partner, enable gauge coupling unification and much more. Given that, we first need to understand for which compact spaces supersymmetry can be preserved and how these spaces can be mathematically described. As we will see, the proper candidates for that are special complex manifolds called Calabi-Yau manifolds. In the next step we will derive low-energy theories from type II superstrings obtained from Calabi-Yau compactifications and discuss their properties.

2.3.1 Supersymmetry in lower dimensions

Our first task is to find the conditions to have (partially) unbroken supersymmetry in four dimensions. Assuming that the 10D spacetime is given by the direct product $\mathcal{M}_{10} = \mathcal{M}_4 \times K_6$ with compact K_6, the full metric has a block-diagonal form given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \,, \tag{2.54}$$

where x^μ with $\mu = 0, \dots, 3$ and y^m with $m = 4, \dots, 9$ are the coordinates of $\mathcal{M}_4$ and K_6, respectively. The space K_6 and its metric $g_{mn}(y)$ are so far unspecified, while $\mathcal{M}_4$ is assumed to be a maximally symmetric space whose Riemann tensor is

$$R_{\mu\nu\rho\sigma} = \frac{k}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu}) \,, \tag{2.55}$$

with either zero (Minkowksi), negative (AdS) or positive (dS) constant curvature k everywhere. To maintain Lorentz-invariance we also need a vanishing background for all field components that are not scalars under the Lorentz group $SO(1,3)$. In other words, the only allowed non-zero backgrounds are scalars that may depend on internal coordinates, purely internal components of p-forms and spacetime-filling forms that are proportional to the volume form $\epsilon_{\mu\nu\rho\sigma}$ in four dimensions. In particular, fermionic fields must always vanish in a classical background. The most direct analysis one could make is to check whether an ansatz of the form (2.54) and (2.55) can be a solution of the 10D supergravity equations of motion. However, it is way simpler to demand (2.54) and (2.55) and require unbroken supersymmetry in four dimensions since the resulting differential equations are of lower order. One can then show that under very mild assumptions the solutions to the supersymmetry conditions solve the field equations as well.

The type II theories both contain two independent supersymmetry generators Q^A_α with $A = 1, 2$, which are Majorana-Weyl spinors with 16 real components. Each supercharge generates infinitesimal transformations according to $\delta_\epsilon \Phi = [\bar{\epsilon}Q, \Phi]$, where Φ is a generic field and ϵ_α a supersymmetry parameter. An unbroken supersymmetry that is defined by $\bar{\epsilon}Q\,|0\rangle = 0$ therefore satisfies

$$\langle \delta_\epsilon \Phi \rangle = \langle 0|\,[\bar{\epsilon}Q, \Phi]\,|0\rangle = 0\,, \tag{2.56}$$

which means that classically a given background is left invariant. The supersymmetry variations of bosonic fields are trivial as they are by definition given by fermionic fields whose background value is zero. Assuming that we only have a non-trivial background for the metric, also the variation of the dilatini gives no further constraints. For each gravitino ψ^A_M the variation is given by $\delta_\epsilon \psi^A_M = \nabla_M \epsilon^A + \ldots$, where ∇_M is the covariant derivative on spinors and ellipses denote terms containing other bosonic fields whose vacuum expectation values are zero. Vanishing of the gravitini vevs gives

$$\langle \nabla_M \epsilon^A \rangle = \bar{\nabla}_M \epsilon^A = 0 \;\longrightarrow\; \bar{\nabla}_\mu \epsilon^A = 0 \quad \text{and} \quad \bar{\nabla}_m \epsilon^A = 0\,, \tag{2.57}$$

where the bar indicates evaluation on the metric backgrounds $\bar{g}_{\mu\nu}$ and $\bar{g}_{mn}$. Unbroken supersymmetry therefore requires a spinor that is covariantly constant on both spaces. The first equation implies that the non-compact space has zero curvature (and therefore is Minkowksi) and ϵ^A is in fact constant along $\mathcal{M}_4$, while the second one imposes that the compact space is Ricci-flat, i.e. the Ricci tensor satisfies $R_{mn} = 0$. For a spinor to be covariantly constant on K_6, it must remain invariant under the action of the holonomy group H, the group of transformations that parallel transport vectors and spinors along an arbitrary closed loop of K_6. But under $SO(1,3) \times SO(6)$, each ϵ^A decomposes as

$$\epsilon^A(x,y) = \zeta^A_+ \otimes \eta^A_+(y) \oplus \zeta^A_- \otimes \eta^A_-(y)\,, \tag{2.58}$$

where (ζ^A_+, ζ^A_-) are 4D Weyl spinors and (η^A_+, η^A_-) are 6D Weyl spinors that transform as $(\mathbf{4}, \bar{\mathbf{4}})$ of $SU(4) \simeq SO(6)$. If the holonomy group is unrestricted, the spinors on the compact manifold are not invariant, but for $H = SU(3) \subset SU(4)$ the $\mathbf{4}$ decomposes as $\mathbf{4}_{SU(4)} = (\mathbf{1} + \mathbf{3})_{SU(3)}$ and yields a singlet under H. Since ϵ^A is Majorana, we have

$$\eta^A_- = (\eta^A_+)^*\,, \quad \zeta^A_- = (\zeta^A_+)^*\,, \tag{2.59}$$

so for each A we have exactly one covariantly constant spinor and under the dimensional reduction we obtain $N = 2$ supersymmetry in $4D$ when starting from $N = 2$ in $10D$. By further restricting the holonomy one can get more covariantly constant spinors, up to the point where the holonomy is trivial and K_6 is a six-dimensional torus, leading to four singlets per $10D$ Majorana-Weyl spinor and hence $N = 8$ in $D = 4$.

The physical arguments suggest that the compactification space should be a Ricci-flat manifold with holonomy group $SU(3) \subset SO(6)$ to arrive at a four-dimensional effective theory preserving $N = 2$ supersymmetry. These types of spaces are known as *Calabi-Yau* (CY) manifolds and are described in the language of complex geometry. In the following we will introduce some essential definitions and notions and refer to the physicist-friendly introductions [17–19] for a more in-depth view.

2.3.2 Calabi-Yau manifolds

We start by defining complex manifolds and some structures that can be defined on them. Most of these definitions are just complex analogs of familiar concepts from real geometry. Let us consider a real $2n$-dimensional manifold M covered by open sets $\{U_a\}_{a\in A}$ where on each set a map $\psi_a : U_a \to \mathbb{C}^n$ defines a set of local holomorphic coordinates $z_a \equiv (z_1^a, \ldots, z_n^a)$. We say that the data $(M, \{U_a, \psi_a\})$ defines a complex manifold if for every overlap $U_a \cap U_b$ the transition functions $\psi_{ab} \equiv \psi_b \circ \psi_a^{-1} : \psi_a(U_a \cap U_b) \to \psi_b(U_a \cap U_b)$ are holomorphic. The simplest instance is $\mathbb{C}^n$ itself, which can be covered by just one patch and whose complex coordinates are related to those of $\mathbb{R}^{2n}$ via

$$z^j = x^{2j-1} + ix^{2j}\,, \quad \bar{z}^j = x^{2j-1} - ix^{2j}\,, \quad j = 1, \ldots, n\,. \tag{2.60}$$

Complex derivatives $\partial_j \equiv \partial/\partial z_j$, $\partial_{\bar{j}} \equiv \partial/\partial \bar{z}_j$ and differentials $dz^j, d\bar{z}^j$ can be defined accordingly. For a general complex manifold, the complexified tangent bundle $T_{\mathbb{C}}(M) \equiv T(M) \otimes \mathbb{C}$ can be easily shown to split as $T_{\mathbb{C}}(M) = T^{(1,0)}(M) \oplus T^{(0,1)}(M)$, where the $\{\partial_j\}$ span the $(1,0)$-component and the $\{\partial_{\bar{j}}\}$ span the $(0,1)$-component. For the complexified cotangent bundle we have a completely analogous splitting into $T^{*(1,0)}(M)$ and $T^{*(0,1)}(M)$. These bundles are special instances of complex vector bundles, which we define briefly for later purposes. Given a complex manifold M, a complex vector bundle is a family of complex vector spaces $\{E_p\}_{p\in M}$, attached to each point $p \in M$, such that in each local patch $U \subset M$ the bundle locally looks like $U \times \mathbb{C}^n$ and on overlaps the transition functions are smooth. If the transition functions are holomorphic, the vector bundle is also called holomorphic. This applies to the components of the (co-)tangent bundles, since there the transition functions are just the Jacobians of the coordinate change on M.

By defining a map that picks one element of E_p at each point $p \in M$ such that the map varies smoothly over M, one defines a section of the vector bundle. An important example are differential (p, q)-forms

$$\omega = \frac{1}{p!q!}\omega_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz^{i_1} \wedge \ldots dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}\,, \tag{2.61}$$

which are sections of the exterior product $\wedge^p T^{*(1,0)}(M) \otimes \wedge^q T^{*(0,1)}(M)$. For forms of total degree $k = p+q$, we denote the space of sections as $\wedge^k T^*(M)$. One can define the exterior derivative d where $d\omega \in A^{p+1,q} \oplus A^{p,q+1}$. This suggests the decomposition $d = \partial + \bar{\partial}$, where $\partial\omega \in A^{p+1,q}$ and $\bar{\partial}\omega \in A^{p,q+1}$ have the explicit form

$$\begin{aligned}\partial\omega &= \frac{1}{p!q!}\partial_i\, \omega_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz^i \wedge dz^{i_1} \wedge \ldots \wedge d\bar{z}^{j_q}\,,\\ \bar{\partial}\omega &= \frac{(-1)^p}{p!q!}\partial_{\bar{j}}\, \omega_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz^{i_1} \wedge \ldots dz^{i_p} \wedge d\bar{z}^j \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}\,.\end{aligned} \tag{2.62}$$

Given that $d^2 = 0$, one has $\partial^2 = 0, \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$ since the resulting forms live in different spaces. ∂ and $\bar{\partial}$ are called Dolbeault operators. Holomorphic $(p, 0)$-forms satisfy $\bar{\partial}\omega = 0$, while anti-holomorphic $(0, q)$-forms obey $\partial\omega = 0$.

Every complex manifold admits a Hermitian metric g whose only non-vanishing components are $g_{i\bar{\jmath}} = g(\partial_i, \partial_{\bar{\jmath}})$, whereas $g_{ij} = g(\partial_i, \partial_j) = 0$ and $g_{\bar{\imath}\bar{\jmath}} = g(\partial_{\bar{\imath}}, \partial_{\bar{\jmath}}) = 0$. A Hermitian metric can be used to define a real and natural $(1,1)$-form

$$J = J_{i\bar{\jmath}} dz^i \wedge d\bar{z}^j = i g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^j \,. \tag{2.63}$$

The wedge product of n J-factors gives $J^n = J \wedge \ldots \wedge J = 2^n n!\, g(z)\, dx^1 \wedge \ldots \wedge dx^{2n}$, where $g(z) = \det(g_{i\bar{\jmath}}) > 0$. If we also use $2^n g(z) = \sqrt{\det(g)}$, where g is the original Riemannian metric, we see that J^n is a good volume form on M. The connection on manifolds with a Hermitian metric can be fixed by demanding that the decomposition of the tangent space into holomorphic and anti-holomorphic parts is respected by parallel transport, which is guaranteed if the only non-vanishing Christoffel symbols are Γ^i_{jk} and $\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}$. If the metric is also covariantly constant, the connection is unique and the coefficients are

$$\Gamma^i_{jk} = g^{\bar{l}i} \partial_j g_{k\bar{l}} \,, \quad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}l} \partial_{\bar{\jmath}} g_{l\bar{k}} \,. \tag{2.64}$$

This connection called the Hermitian connection. It leads to a simplified expression for the Riemann tensor, whose only non-vanishing components read

$$R^i_{jk\bar{l}} = -\partial_{\bar{l}} \Gamma^i_{kj} = -\partial_{\bar{l}}(g^{\bar{m}i} \partial_k g_{j\bar{m}}) \,. \tag{2.65}$$

All other components arise from complex conjugation or the symmetry properties. We can define an associated (1,1)-form called Ricci form, given by

$$\mathcal{R} = \mathcal{R}_{i\bar{\jmath}} dz^i \wedge d\bar{z}^j = i R^k_{ki\bar{\jmath}} dz^i \wedge d\bar{z}^j \,. \tag{2.66}$$

Note that, despite the name, the components of this tensor do not generally coincide with components of the Ricci tensor. The latter is obtained from the Levi-Civita connection which differs from the Hermitian connection by a torsion contribution.

It turns out that the two aforementioned connections do coincide for a special class of Hermitian manifolds whose natural two-form J is closed, i.e. $dJ = 0$. In this case J is called the Kähler form J and the Hermitian metric/manifold is called Kähler metric/manifold. Let us stress that unlike Hermiticity the Kähler condition is an actual restriction on the differential geometry that cannot be fulfilled by all complex manifolds. The fact that J is closed implies the relations $\partial_i g_{k\bar{\jmath}} = \partial_k g_{i\bar{\jmath}}$ and $\partial_{\bar{\imath}} g_{k\bar{\jmath}} = \partial_{\bar{\jmath}} g_{k\bar{\imath}}$. This in turn implies that on each coordinate neighborhood there exists a real function K, called Kähler potential, in terms of which the Kähler metric is given by

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K \,. \tag{2.67}$$

This function is not unique as K and $K + f(z) + \bar{f}(\bar{z})$ give rise to the same metric. Here $f(z), \bar{f}(\bar{z})$ are holomorphic/anti-holomorphic functions on the patch where K is defined. Moreover, there are additional symmetries for the Riemann tensor which make sure that the components of the form (2.66) are indeed given by the Ricci tensor and lead to the simplified expression

$$\mathcal{R} = -i\partial_i \partial_{\bar{\jmath}}(\log(\det(g))dz^i \wedge dz^{\bar{\jmath}} \,. \tag{2.68}$$

(Co)homology primer

Next, we review some important notions from differential geometry which identify equivalence classes of geometric objects that are globally indistinguishable in terms of integration or global calculus. We will first introduce these concepts for real manifolds and discuss their implications for complex (Kähler) manifolds afterwards.

We start by introducing homology groups, which identify submanifolds that close on themselves. If M is a smooth and orientable Riemannian manifold, we can define the space of p-chains $C_p(M) = \{a_p = \sum_k c_k N_k\}$, where $\{N_k\}$ are oriented p-dimensional submanifolds of M and the c_k are integer, real or complex. Special instances are p-cycles $Z_p(M) = \{a_p | \partial a_p = \varnothing\}$, which have no boundary, and p-dimensional boundaries $B_p(M) = \{a_p | a_p = \partial a_{p+1}\}$. Since the boundary operator ∂ squares to zero, $B_p(M) \subset Z_p(M) \subset C_p(M)$. We define the p-th integral/real/complex homology group of M as

$$H_p(M, R) = Z_p(M)/B_p(M)\,, \tag{2.69}$$

where $R \in \{\mathbb{Z}, \mathbb{R}, \mathbb{C}\}$ depending on the type on p-chains. Elements of $H_p(M, R)$ are thus equivalence classes $[z_p]$ with $z_p \sim z_p + \partial a_{p+1}$, i.e. p-cycles that differ by a boundary. In figure 2.1 we depict examples for the different kinds of submanifolds.

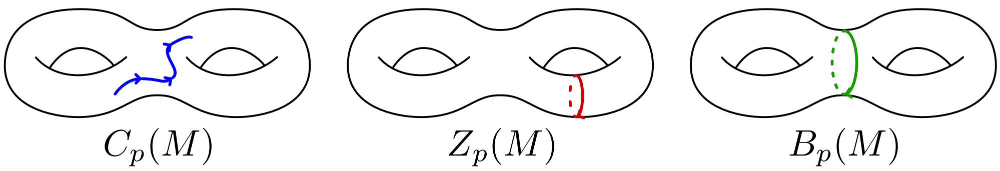


Figure 2.1: Examples for chains (blue), cycles (red) and boundaries (green).

A similar construction can be made for p-forms by noting that the exterior derivative obeys $d^2 = 0$. We define Z^p as the space of closed p-forms, $Z^p(M) = \{\omega_p | d\omega_p = 0\}$, and B^p as the space of exact p-forms $B^p(M) = \{\omega_p | \omega_p = d\omega_{p-1}\}$, which directly leads to definition of the p-th de Rham cohomology group

$$H^p_{dR}(M, \mathbb{R}) = Z^p(M)/B^p(M)\,. \tag{2.70}$$

Here the equivalence classes are closed forms that differ by exact forms. The dimensions of the de Rham cohomology groups are the Betti numbers $b^p(M) = \dim(H^p(M, \mathbb{R}))$, which lead to the definition of the Euler number

$$\chi(M) = \sum_{p=0}^{\dim(M)} (-1)^p\, b^p(M)\,. \tag{2.71}$$

On complex manifolds, one can use the complex structure to define cohomology groups in terms of the Dolbeault operators $\partial, \bar{\partial}$ since these also square to zero. This gives rise to the definition of the Dolbeault cohomology groups $H^{p,q}_{\bar\partial} = Z^{p,q}_{\bar\partial}(M)/\bar\partial A^{p,q-1}(M)$, where $Z^{p,q}_{\bar\partial}(M)$ is the the group of $\bar\partial$-closed (p,q)-forms and (p,q)-forms from $\bar\partial A^{p,q-1}(M)$ are $\bar\partial$-exact. The dimensions $h^{p,q}(M)$ of these groups are called Hodge numbers. According to the complex version of Hodge's theorem, there is an orthogonal decomposition of the space of (p,q)-forms according to which the space $Z^{p,q}_{\bar\partial}(M)$ is of the form

$$Z^{p,q}_{\bar\partial}(M) = \mathcal{H}^{p,q}(M) \oplus \bar\partial A^{p,q-1}(M)\,. \tag{2.72}$$

Here, $\mathcal{H}^{p,q}(M)$ is the space of $\bar\partial$-harmonic (p,q)-forms, i.e. forms with zero eigenvalue under the Laplacian operator

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial\,. \tag{2.73}$$

The adjoint operator $\bar\partial^\dagger$ is defined in terms of the Hogde-$\star$ operator and acts as $\bar\partial^\dagger w_{p,q} \in A^{p,q-1}(M)$. By the definition of the $\bar\partial$-cohomology group each cohomology class thus has a unique $\bar\partial$-harmonic representative from $\mathcal{H}^{p,q}(M)$. In addition, one can define two further Laplacians Δ_∂ and Δ_d and cohomology groups with respect to the appropriate differential operators, where d gives rise to the complex de Rham cohomology groups $H^p_{dR}(M,\mathbb{C})$. Generally the Laplacians are unrelated, but for Kähler manifolds one finds $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$. In that case the de Rham cohomology groups can be decomposed as

$$H^r_{dR}(M,\mathbb{C}) = \bigoplus_{r=p+q} H^{p,q}_{\bar\partial}(M) \tag{2.74}$$

and the Euler number is given by $\chi(M) = \sum_{p,q}(-1)^{p+q}h^{p,q}(M)$. Moreover, the Hodge numbers of Kähler manifolds obey $h^{p,q}(M) = h^{n-q,n-p}(M)$ and $h^{p,q}(M) = h^{q,p}(M)$.

Defining Calabi-Yau manifolds

We now come to the formal definition of CY manifolds. In section 2.3.1, supersymmetry arguments suggested that the compact space should be Ricci-flat. On Kähler manifolds, the Ricci-form given in (2.68) is closed, $d\mathcal{R} = 0$, which immediately follows from the relations among d, ∂ and $\bar\partial$. But since $\log(\det(g))$ is not globally defined, $\mathcal{R}$ is not exact and hence defines a non-trivial element of $H^2_{dR}(M,\mathbb{C})$ known as the first Chern class

$$c_1(M) = \left[\frac{\mathcal{R}}{2\pi}\right]. \tag{2.75}$$

In general, Chern classes are cohomology classes built from the curvature of a natural connection on a complex vector bundle. $c_1(M)$ is defined in terms of the Hermitian connection on the holomorphic tangent bundle $T^{(1,0)}(M)$ and is a crucial ingredient for defining CY spaces. Obviously, a Ricci-flat Kähler manifold has vanishing first Chern class. It was however conjectured by Calabi and later proven by Yau that $c_1(M) = 0$ is a sufficient condition for the existence of a unique Ricci-flat metric on M for given complex structure and Kähler class J, giving rise to the following definition:

Definition: A CY manifold M is a compact Kähler manifold with $c_1(M) = 0$.

Connecting to the arguments from section 2.3.1, $c_1(M) = 0$ automatically implies a restricted holonomy group contained in $SU(n)$[2]. The reduced holonomy can also be used to argue that there exists a nowhere vanishing and unique holomorphic $(n, 0)$-form

$$\Omega = \frac{1}{n!}\Omega_{i_1,\ldots,i_n} dz^1 \wedge \ldots \wedge dz^n \,, \tag{2.76}$$

where $\Omega_{i_1,\ldots,i_n}$ are holomorphic coefficients and Ω is both harmonic and covariantly constant with respect to the Ricci-flat metric. Note that this implies $h^{n,0} = 1$. The existence of such a form Ω is tied to the fact that the canonical bundle defined as $K_M = \wedge^n T^{*(0,1)}$ is topologically trivial. K_M is an example for a holomorphic line bundle, a complex vector bundle of rank one whose transition functions are holomorphic. Since the sections of this bundle are holomorphic $(n, 0)$-forms, triviality implies that there is no obstruction to globally define a nowehere vanishing, holomorphic $(n, 0)$-form.

The CY property leads to additional symmetries for the Hodge numbers, namely $h^{p,0} = h^{n-p,0}$ and $h^{p,0} = 0$ for $0 < p < n$. For a CY threefold this implies that the only independent Hodge numbers are $h^{1,1} \geq 1$ and $h^{2,1} \geq 0$ and that the Euler number has the simple form $\chi = 2(h^{1,1} - h^{2,1})$. The so-called *Hodge diamond* displaying all Hodge numbers is given by

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.77}$$

The element of the cohomology groups $H^{p,q}_{\bar{\partial}}(M) = H^{p,q}(M)$ will be expanded in terms of a basis, for which we choose the respective unique harmonic representative appearing in (2.72). The basis of harmonic $(1, 1)$-forms will be denoted by ω_A with $A = 1, \ldots, h^{1,1}$, while elements in $H^{2,2}(M)$ are expanded in terms of the dual basis $\tilde{w}^A$. Harmonic $(2, 1)$-forms will be labeled by χ_K where $K = 1, \ldots, h^{2,1}$. The cohomology group $H^3(M)$ is spanned by a real, symplectic basis of harmonic three-forms $(\alpha_{\hat{K}}, \beta^{\hat{K}})$, where $\hat{K}, \hat{L} = 0, ..., h^{2,1}$. The non-trivial intersections are

$$\int_M \omega_A \wedge \tilde{\omega}^B = \delta^B_A \,, \quad \int_M \alpha_{\hat{K}} \wedge \beta^{\hat{L}} = \delta^{\hat{L}}_{\hat{K}} \,. \tag{2.78}$$

[2]It is however common to speak of CY spaces if the holonomy group is precisely $SU(n)$, thereby excluding tori and direct product spaces.

2.3.3 Compactification

As we are now well acquainted with proper candidates for compactification spaces, we can discuss how to extract lower-dimensional effective theories from the ten-dimensional string theories. Note that since our starting point will be the supergravity approximation in $10D$, the massive string excitations are thereby discarded already. The idea of compactifications is already known since the field-theoretic considerations of Kaluza and Klein (KK) in the early twentieth century, in which they proposed that the electromagnetic force in four dimensions can arise from a five-dimensional spacetime if one assumes that one spatial dimension is tiny and compact. It is instructive to first review such a simple setup as it already exhibits many relevant features.

We consider the spacetime $\mathcal{M}_D = \mathcal{M}_d \times S^1_R$, where S^1_R is a circle of radius R with coordinate $x^d \simeq x^d + 2\pi R$. The coordinates of $\mathcal{M}_D$ are denoted as $x^M = (x^\mu, x^d)$ with $\mu = 0, \dots, d-1$. For simplicity we consider a massless scalar field Φ in D dimensions, which due to its required periodicity $\Phi(x^\mu, x^d) = \Phi(x^\mu, x^d + 2\pi R)$ has the Fourier expansion

$$\Phi(x^\mu, x^d) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{i \frac{n}{R} x^d} \,. \tag{2.79}$$

Plugging this ansatz into the D-dimensional equations of motion $\partial_M \partial^M \Phi = 0$ yields

$$\partial_\mu \partial^\mu \phi_n = \frac{n^2}{R^2} \phi_n(x^\mu) \quad \forall_{n \in \mathbb{Z}} \,. \tag{2.80}$$

This shows that in d dimensions there is a massless scalar ϕ_0 and an infinite collection of massive scalar fields ϕ_n with $m_n^2 = n^2/R^2$. When performing a dimensional reduction, one discards all massive ϕ_n since they become very heavy for $R \to 0$.

In the following we will work with a rescaled compact coordinate $\theta \simeq \theta + 2\pi$ instead of x^d. The general ansatz for the full metric can be parametrized as

$$ds^2 = G_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + r^2 (d\theta + A_\mu dx^\mu)^2 \,, \tag{2.81}$$

where we defined $A_\mu = G_{\mu d}, r = G_{dd}$ and we only consider the θ-independent zero modes of $G_{\mu\nu}, r$ and A_μ. The diffeomorphisms of this metric are $x^\mu \to x'^\mu(x^\mu)$ and $\theta \to \theta + \lambda(x^\mu)$. The latter leads to the transformation $A_\mu \to A_\mu - \partial_\mu \lambda$, showing that A_μ is a $U(1)$ gauge field in the lower-dimensional theory. To understand the role of r, one inserts the ansatz (2.81) into the Einstein-Hilbert term in D dimensions. Using that under the dimensional reduction the Ricci-scalars in D and d dimensions are related by

$$R_D = R_d - 2\frac{1}{r}\nabla^2 r - \frac{1}{4} r^2 F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \,, \tag{2.82}$$

one obtains

$$\begin{aligned} S_{EH}^{(D)} &= M_{\text{pl},D}^{D-2} \int d^D x \sqrt{-G^{(D)}} R_D \\ &= 2\pi M_{\text{pl},D}^{D-2} \int d^d x \sqrt{-G^{(d)}}\, r \left[R_d - \frac{1}{4} r^2 F_{\mu\nu} F^{\mu\nu} \right] . \end{aligned} \tag{2.83}$$

Here we have performed the integration over θ and dropped the total derivative term. To bring the action into Einstein frame in d dimensions, we need to perform a Weyl rescaling of the metric given by

$$G^{(d)}_{\mu\nu} = \left(\frac{r_0}{r}\right)^{\frac{2}{d-2}} \tilde{G}^{(d)}_{\mu\nu} \,. \tag{2.84}$$

Here $r_0 = \langle r \rangle$ is the vacuum expectation value of r. The action then becomes

$$S^{(D)}_{EH} = M_{\rm pl}^{d-2} \int d^d x \sqrt{\tilde{G}^{(d)}} \left(\tilde{R}_d - \frac{d-1}{d-2}\frac{1}{r^2}\partial_\mu r \tilde{\partial}^\mu r + \frac{1}{2g^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \right) . \tag{2.85}$$

We have further introduced the lower-dimensional Planck mass $M^{D-2}_{\rm pl,D}(2\pi r_0) = M^{d-2}_{\rm pl,d}$ and the gauge coupling g of A_μ is defined through the equation

$$\frac{1}{2} g^2 M^{d-2}_{\rm pl,d} = \frac{1}{r^2}\left(\frac{r_0}{r}\right)^{\frac{2}{d-2}} \,. \tag{2.86}$$

Equations (2.85) and (2.86) teach us two important lessons about the dynamical field r, whose vev parametrizes the size of the circle direction. The first is that r has no potential and can therefore be changed without energy expense, giving rise to a continuous family of vacuum solutions. We refer to such scalar fields as moduli, and the geometric space that is spanned by the set of possible vevs of the moduli fields is called the moduli space. Second, the vev of r also sets the value of the gauge coupling and the lower-dimensional Planck mass, showing that the geometric data influences physical couplings in the lower-dimensional theory. We will revisit this model in chapter 3 to highlight some additional features, but now we have gained enough intuition about compactifications to study the dimensional reduction of type II superstrings.

KK reduction of type II theories

Let us now consider a ten-dimensional spacetime given by $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times Y$, where Y is a six-dimensional CY threefold. Let x^μ with $\mu = 0, \dots, 3$ denote the coordinates of Minkowksi space and y^m with $m = 1, \dots, 6$ the local real coordinates of the CY. We also introduce the compact notation $z^M = (x^\mu, y^m)$. The generalization of the scalar field reduction from our toy model goes as follows: We consider generic ten-dimensional tensor fields $\Phi^{mn\dots}_{\mu\nu\dots}(x,y)$ with infinitesimal fluctuations $\varphi^{mn\dots}_{\mu\nu\dots}(x,y)$ around their vev. The linearized equations of motion in ten dimensions can then be brought to the form

$$\left(\mathcal{O}_4 + \mathcal{O}_6\right) \varphi^{mn\dots}_{\mu\nu\dots}(x,y) \,, \tag{2.87}$$

where $\mathcal{O}_4, \mathcal{O}_6$ are differential operators of order p ($p = 2$ for bosons, $p = 1$ for fermions), defined on the spaces $\mathbb{R}^{1,3}$ and Y respectively. A generic field can then be expanded in terms of the eigenfunctions $Y^{mn\dots}_a(y)$ of the operator $\mathcal{O}_6$, yielding

$$\varphi^{mn\dots}_{\mu\nu\dots}(x,y) = \sum_a \varphi_{a,\mu\nu}\, Y^{mn\dots}_a(y) \,. \tag{2.88}$$

The eigenvalues λ_a of the functions $Y_a^{mn\cdots}(y)$ will generically scale as $\lambda_a \sim 1/R^p$, where R is a typical length scale of Y and p is a non-negative power. Thus, only the zero modes of the operator $\mathcal{O}_6$ will give rise to massless fields in the four-dimensional theory, and for small R the other modes will become heavy. We have observed these patterns already in the circle compactification. Unlike for the circle however, the zero modes of the internal operator can depend on the internal coordinates in general.

Let us now consider a p-form $B_p = B_{M_1,\dots,M_p} dz^{M_1} \wedge \dots \wedge dz^{M_p}$ and analyze what the above recipe implies. With an appropriate gauge choice, the equations of motion are

$$\Box_{10} B_{M_1,\dots,M_p} = (\Box_{\mathbb{R}^{1,3}} + \Delta)\, B_{M_1,\dots,M_p} = 0\,, \tag{2.89}$$

where Δ is the CY Laplacian that was constructed in section 2.3.2. We can now expand this p-form in terms of eigenfunctions of the Laplacian to obtain

$$B_p = \sum_n b^{(n)}(x)\omega_p^{(n)}(y) + \sum_m b_\mu^{(m)}(x) dz^\mu \wedge \omega_{p-1}^{(m)}(y) + \dots\,, \tag{2.90}$$

where $\{\omega_r^{(m)}(y)\}$ is a complete set of Eigenforms of degree r. For each sum, the number of massless fields in four dimensions therefore corresponds to the number of Δ-harmonic forms of respective degree. But we have learned in section 2.3.2 that the number of harmonic r-forms is counted by the Betti number b^r, which for Kähler manifolds is in turn determined by the Hodge numbers $h^{p,q}$ with $p+q=r$. This way, the topological data of the CY directly determines the low-energy spectrum in four dimensions. Also, each scalar from the KK reduction again corresponds to a modulus that parametrizes the degeneracy of the vacuum solution.

The above technique can be directly applied to all bosonic fields from the NS-NS and R-R sectors from the tables 2.2 and 2.3. Among these, the metric tensor g_{MN} requires some special care. We will denote its different zero modes by $g_{\mu\nu}, g_{\mu m}$ and g_{mn}. Here $g_{\mu\nu}$ is the lower-dimensional graviton and the fields $g_{\mu m}$ are absent for CYs since $b^1 = 0$. The purely internal components g_{mn} correspond to deformations of the compact space that respect the Ricci-flatness condition, i.e. for a variation δg_{mn} one obtains

$$R_{mn}(g + \delta g) = 0\,. \tag{2.91}$$

After eliminating diffeomorphism invariance via the gauge choice $\Delta^m \delta g_{mn} = 0$, expanding to linear order and taking the trace, equation (2.91) becomes

$$\nabla^l \nabla_l \delta g_{mn} + 2R_{m\ \ n}^{\ \ l\ \ r} \delta g_{lr} = 0\,. \tag{2.92}$$

If one analyzes this equation for Kähler manifolds, the constraints on the components $\delta g_{i\bar{\jmath}}$ and δg_{ij} decouple. For the mixed components one finds the condition $(\Delta g)_{i\bar{\jmath}} = 0$, i.e. they are harmonic $(1,1)$-forms. As such, they yield metric variations that correspond to cohomologically non-trivial changes of the Kähler class J (recall definition (2.63)).

The $\delta g_{i\bar{\jmath}}$ are commonly combined with the harmonic $(1,1)$-forms arising from the B-field, denoted as $\delta B_{i\bar{\jmath}}$, in which case the expansion in the basis of $(1,1)$-forms reads

$$(\delta B_{i\bar{\jmath}} + i\delta g_{i\bar{\jmath}}) = \sum_{a=1}^{h^{1,1}} t^a (\omega^a)_{i\bar{\jmath}} \,. \tag{2.93}$$

The expansion coefficients $t^a = b^a + i\tau^a$ are known as the complexified Kähler moduli. To maintain the geometric picture, the real Kähler moduli t^a need to be chosen such that the metric g with Kähler class J is positive definite, which is ensured if

$$\int_C J > 0\,, \quad \int_S J \wedge J > 0\,, \quad \int_Y J \wedge J \wedge J > 0 \tag{2.94}$$

holds for all complex curves C and surfaces S in the CY Y. The resulting subspace of $\mathbb{R}^{h^{1,1}}$ is known as the Kähler cone. In the geometric regime, the imaginary parts of the Kähler moduli parametrize the size of two-cycles that are elements of $H_2(X,\mathbb{Z})$.

The pure metric deformations can be shown to satisfy $\Delta_{\bar{\partial}}\, \delta g^i = 0$. Here $\delta g^i = \delta g^i_{\bar{\jmath}} d\bar{z}^j$ is a $(0,1)$-form with values in $T^{(1,0)}(Y) \equiv T_Y$ due to its free holomorphic index and hence determines an element in the generalized cohomology group $H^{0,1}(Y, T_Y)$. A shift by δg_{ij} introduces pure components in the metric, which can only be undone by a non-holomorphic transformation $z_i \to z'_i = f(z_i, \bar{z}_i)$. Since the resulting metric is then Kähler with respect to a new complex structure, we conclude that these shifts alter the complex structure of the CY. Using the holomorphic 3-form Ω, we can map the metric deformations to the $(2,1)$-forms $\Omega_{ijk}\delta g^i_{\bar{\jmath}}\, dz^i \wedge dz^j \wedge d\bar{z}^l$, which are also harmonic and therefore represent elements of $H^{2,1}(Y)$. Their expansion in terms of basic $(2,1)$-forms

$$\Omega_{ijk}\, \delta g^k_{\bar{l}} = \sum_{K=1}^{h^{2,1}} z^K (\chi_K)_{ij\bar{l}} \tag{2.95}$$

introduces the complex structure moduli z^K. In summary, we saw that the geometric moduli of a CY are Kähler class deformations lying in $H^{1,1}(Y)$ and complex structure deformations lying in $H^{2,1}(Y)$. The corresponding parameters span generally intricate moduli spaces which are respectively called the Kähler moduli space and complex structure moduli space. We will analyze some of their properties in the next subsection.

For concreteness, let us determine the action for the effective theory of type IIA supergravity when compactifying on a CY. Expanding the relevant ten-dimensional p-form fields in terms of the harmonic forms yields

$$\begin{aligned} \Phi(x,y) &= \phi(x)\,, \quad C_1(x,y) = A^0(x)\,, \\ B_2(x,y) &= B_2(x) + b^A(x)\,\omega_A\,, \\ C_3(x,y) &= \xi^{\hat{K}}(x)\,\alpha_{\hat{K}} - \tilde{\xi}_{\hat{K}}(x)\,\beta^{\hat{K}} + A^A(x)\wedge\omega_A\,, \end{aligned} \tag{2.96}$$

where $\phi, b^A, \xi^{\hat{K}}$ and $\tilde{\xi}_{\hat{K}}$ are $4D$ scalars, A^0, A^A are 1-forms and B_2 is a two-form. On the fermionic side, we recall that for a CY compactification we obtain two four-dimensional

name	# of multiplets	field content
gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vector multiplets	$h^{1,1}$	(A^A, t^A)
hypermultiplets	$h^{2,1}$	$(z_K, \xi^K, \tilde{\xi}_K)$
	1	$(B_2, \phi, \xi^0, \tilde{\xi}_0)$

Table 2.4: $N = 2$ multiplets for type IIA supergravity compactified on a CY.

gravitinos, leading to an $N = 2$ supergravity theory in which the zero modes assemble into $N = 2$ multiplets. The structure of these multiplets is shown in table 2.4.
If one inserts the expansions of the $10D$ fields into the ten-dimensional type IIA action provided in (2.46) and (2.48) and performs a Weyl rescaling to obtain a standard Einstein-Hilbert term, the bosonic part of the action reads

$$\begin{aligned} S^{\text{IIA}}_{(4)} = \int d^4x\sqrt{-g^{(4)}}\Big(\frac{1}{2}R + \frac{1}{4}\,\text{Im}\mathcal{N}_{\hat{A}\hat{B}}F^{\hat{A}}_{\mu\nu}F^{\hat{B},\mu\nu} + \text{Re}\mathcal{N}_{\hat{A}\hat{B}}F^{\hat{A}}_{\mu\nu}\tilde{F}^{\hat{B},\mu\nu} \\ - g_{A\bar{B}}(t)\,\partial_\mu t^A\partial^\mu\bar{t}^A - h_{uv}(q)\partial_\mu q^u\partial^\mu q^v\Big)\,. \end{aligned} \tag{2.97}$$

We defined the field strengths $F^{\hat{A}}_{\mu\nu} = 2\partial_{[\mu}A^{\hat{A}}_{\nu]}$ and the dual fields $\tilde{F}^{\hat{A}}_{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F^{\hat{A}}_{\rho\sigma}$ with $\epsilon^{0123} = 1$. $\mathcal{N}_{\hat{A}\hat{B}}$ is the symmetric, complex gauge-kinetic matrix depending on the scalars t^A. The real scalars q^u with $u = 1,\ldots,4(h^{2,1}+1)$ collectively denote scalar fields from the hypermultiplets. g_{AB} is the metric on Kähler moduli space, while the metric of the complex structure space is part of h_{uv}. It is due to the restrictions of $N = 2$ supersymmetry that the kinetic terms of the scalars from vector multiplets and hypermultiplets do not mix. For further details on type II effective actions we refer to [20].

A closer look at the CY moduli space

Since supersymmetry imposes stringent conditions on the effective action, it naturally affects also the geometric properties of the CY moduli space $\mathcal{M}$. For type II theories on CYs, the moduli space is given by the direct product

$$\mathcal{M} = \mathcal{M}_{\text{SK}} \times \mathcal{Q}\,. \tag{2.98}$$

The first factor is a special Kähler manifold that is spanned by the scalars in the vector multiplets. The definition of $\mathcal{M}_{\text{SK}}$ will be provided shortly. $\mathcal{Q}$ is a quaternionic manifold, a complex manifold of real dimension m and holonomy $Sp(2)\times Sp(2m)$, which is spanned by the hypermultiplets. For both type IIA and IIB the universal hypermultiplet contains the dilaton and therefore $\mathcal{Q}$ receives both perturbative and non-perturbative corrections in $g_s = e^{\langle\Phi\rangle}$, while $\mathcal{M}_{\text{SK}}$ is exact at g_s tree-level.

Let us focus on $\mathcal{M}_{\text{SK}}$, starting with its general definition. A special Kähler manifold of complex dimension n is a Kähler manifold that allows to express the Kähler potential and metric in terms of a single holomorphic function $\mathcal{F}_0(X)$ of the homogeneous coordinates

X^I with $I = 0, \dots, n$, where $\mathcal{F}_0(X)$ called the prepotential. The prepotential is of homogeneous degree two, i.e. it satisfies $\mathcal{F}_0(\lambda X) = \lambda^2 \mathcal{F}_0(X)$ for $\lambda \in \mathbb{C}/\{0\}$. $\mathcal{M}_{\mathrm{SK}}$ admits a holomorphic vector bundle whose sections are encoded in the period vector $\Pi = (X^I, \mathcal{F}_I)^T$, where $\mathcal{F}_I = \partial \mathcal{F}_0 / \partial X^I$. The bundle is moreover symplectic and flat, which means that on each fiber one can define a symplectic form such that the transition functions between different frames are constant maps $M \in Sp(2n+2, \mathbb{R})$ that preserve the symplectic form. The Kähler potential is then defined in terms of the symplectic pairing of the period vector with itself, namely

$$K = \log\left(i \Pi^\dagger \Sigma \Pi\right) \quad \text{with} \quad \Sigma = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, \tag{2.99}$$

where the matrix Σ represents the symplectic form and $\mathbb{1}_n$ denotes the n-dimensional identity matrix. In general $4D$ $N = 2$ supergravity, the scalar fields in the vector multiplets are the local complex coordinates on the special Kähler manifold. If we denote these scalars as ξ^i with $i = 1, \dots, n$, they are related to the homogeneous coordinates via $\xi^i = X^i / X^0$ where $X^0 \neq 0$.

Let us have a brief look at the vector multiplet moduli space from the perspective of both type IIA and type IIB. In the type IIB case, it is spanned by $n = h^{2,1}$ complex structure moduli. To define the periods, we consider the basis of the homology group $H_3(Y, \mathbb{Z})$, denoted as $(A^{\hat{K}}, B_{\hat{L}})$ with $\hat{K}, \hat{L} = 0, \dots, h^{2,1}$, with intersections

$$A^{\hat{K}} \cap B_{\hat{L}} = -B_{\hat{L}} \cap A^{\hat{K}} = \delta^{\hat{K}}_{\hat{L}} \,. \tag{2.100}$$

The basis is dual to the $H^3(Y, \mathbb{Z})$-basis in the sense that

$$\int_{A^{\hat{L}}} \alpha_{\hat{K}} = \int \alpha_{\hat{K}} \wedge \beta^{\hat{L}} = \delta^{\hat{L}}_{\hat{K}} \,, \quad \int_{B_{\hat{L}}} \beta^{\hat{K}} = \int \beta^{\hat{K}} \wedge \alpha_{\hat{L}} = -\delta^{\hat{K}}_{\hat{L}} \,. \tag{2.101}$$

Integrating the holomorphic 3-form Ω over the basic 3-cycles yields the CY periods

$$X^{\hat{K}} = \int_{A^{\hat{K}}} \Omega \,, \quad \mathcal{F}_{\hat{K}} = \int_{B_{\hat{K}}} \Omega \,. \tag{2.102}$$

It is easy to check that Ω can then be expanded as $\Omega = X^{\hat{K}} \alpha_{\hat{K}} - \mathcal{F}_{\hat{K}} \beta^{\hat{K}}$. With the basic relation $\int \Omega \wedge \partial_{\hat{K}} \Omega = 0$ one can then derive the form of the prepotential $\mathcal{F}_0(X) = \frac{1}{2} X^{\hat{K}} \mathcal{F}_{\hat{K}}$. The Kähler potential consequently reads

$$K = \log\left(i \int \Omega \wedge \bar{\Omega}\right). \tag{2.103}$$

The remarkable feature of special Kähler geometry is that the entire effective action describing the vector multiplet sector is encoded in the prepotential $\mathcal{F}(X)$, at least locally in field space. Though in order to find the explicit dependence on the complex structure moduli, one needs to compute the periods of the CY. For that one needs to solve a set of differential equations, known as the Picard-Fuchs equations, that describe how Ω changes as the complex structure moduli are varied. This is a calculation in classical geometry and does not receive any corrections, neither in g_s nor α'. The methods for explicitly solving these equations will be presented in chapter 5.

If we consider type IIA string theory, the scalars of the $n = h^{1,1}$ vector multiplets parametrize a special Kähler manifold. Since the sizes of CY two-cycles depend on Kähler moduli, the Kähler moduli space does receive α' corrections, both at perturbative and non-perturbative level. At this point, a very powerful duality relation known as *mirror symmetry* comes into play (see [21–27] for a a selection of pioneering papers and [19] for a broad review): It has been conjectured that CY threefolds come in pairs $(Y, \tilde{Y})$ such that their Hodge numbers are related via

$$h^{1,1}(Y) = h^{2,1}(\hat{Y})\,, \quad h^{2,1}(Y) = h^{1,1}(\hat{Y})\,. \tag{2.104}$$

It has been further conjectured that type IIA string theory compactified on Y is completely equivalent to type IIB string theory on $\hat{Y}$. It proposes the equivalence

$$\mathcal{M}^{\text{IIA}}_{\text{Kähler}} = \mathcal{M}^{\text{IIB}}_{\text{CS}} \tag{2.105}$$

between the Kähler moduli space of type IIA and the complex structure moduli of type IIB (and vice versa), assuming they are compactified on mirror dual CYs. Then the completely classical geometric computation on the type IIB side can be leveraged to determine the quantum corrections to the prepotential (and further quantities) in type IIA (to be revisited in chapter 5). Assuming that we work in the large volume regime $\text{Im}(t^A) \gg 1$, such that the α' corrections are small, the type IIA prepotential reads

$$g_s^2 \mathcal{F}_0(X) = \frac{1}{6}\kappa_{ijk}\frac{X^i X^j X^k}{X^0} - \frac{i}{2}\zeta(3)\chi(X)(X^0)^2 + \mathcal{F}_{\text{inst}}(X). \tag{2.106}$$

The first term is the classical volume term, which is cubic in the Kähler moduli defined via $X^i/X^0 = 2\pi t^i$ and depends on the triple intersection numbers

$$\kappa_{ijk} = \int_Y \omega_i \wedge \omega_j \wedge \omega_k\,. \tag{2.107}$$

On the perturbative side, there is only an $(\alpha')^3$-correction represented by the second term in (2.106). The gauge redundancy of the NS-NS 2-form B_2 leads to a Peccei-Quinn shift symmetry of the Kähler moduli, which together with the fact that $\mathcal{F}_0$ is homogeneous of degree two only allows for a constant term in the t^i. Finally, the third term encodes non-perturbative corrections in α' and reads

$$\mathcal{F}_{\text{inst}}(X) = i(X^0)^2 \sum_{\beta\in H_2(Y,\mathbb{Z})} \alpha_0^\beta \, \text{Li}_3(e^{i\beta^i X^i/X^0})\,, \tag{2.108}$$

The sum is taken over world-sheet instantons. A world-sheet instanton is a Euclidean string world-sheet Σ_E with holomorphic embedding map $\phi : \Sigma_E \to Y$ that wraps a homologically non-trivial 2-cycle in the CY threefold. The classical action

$$S_{\text{inst}} = \int_{\Sigma_E} (B + iJ) \tag{2.109}$$

enters (2.108) via $e^{-S_{\rm inst}} = e^{2\pi i \vec{\beta}\cdot\vec{t}}$ as the argument of the polylogarithm if the string world-sheet wraps a cycle specified by $\vec{\beta} \in H_2(Y,\mathbb{Z})$. The polylogarithm enters for the following reason: Since a string worldsheet can wrap the same curve $k > 1$ times, each curve gives rise to an entire series of corrections with instanton action $S_k = kS_{\rm inst}$ for k-fold wrapping. The string path integral weighs the fluctuations of the world-sheet fields around a k-fold cover with a suppression factor $1/k^3$, and the sum over the weighted contributions produces the trilogarithm.

The constants $\alpha_0^{\vec{\beta}} \in \mathbb{Z}$ are known as the genus-zero Gopakumar-Vafa (GV) invariants. These integers count the number of distinct genus-zero curves in the primitive homology class $\vec{\beta}$ on a CY manifold. Physically, they correspond to a BPS index that counts the net number of BPS states coming from membranes wrapped on curves of class $\vec{\beta}$. The GV invariants are related to the so-called Gromov-Witten (GW) invariants $n_0^{\vec{\beta}} \in \mathbb{Q}$ via

$$n_0^{\vec{\beta}} = \sum_{k\,|\,\vec{\beta}} \frac{1}{k^3}\, \alpha_0^{\vec{\beta}/k}\,, \tag{2.110}$$

where the notation $k\,|\,\vec{\beta}$ means that k divides every component of $\vec{\beta}$. $n_0^{\vec{\beta}}$ measures how many ways a sphere can be embedded into a CY so that it wraps a curve of type $\vec{\beta}$, including the possibility of multi-wrappings (hence the fractional values)[3]. In section 5.3 we will show how one obtains the GV invariants from the computation of periods on the mirror dual CY. The corresponding instanton contributions affect various physical quantities, including the singularity structure of the moduli space and the couplings in the low-energy effective action.

2.4 Dualities and M-theory

Quantum theories are generally well-understood in regimes where interactions are weak and a perturbative expansion in terms of a small coupling parameter exists. At strong coupling however these expressions become ill-defined and cease to describe non-perturbative phenomena such as the confinement of quarks or the dynamics of solitons, to name a few examples. In such situations, dualities can prove to be useful: it may be possible to find a complementary formulation of the theory in terms of different variables, such that a non-perturbative problem in the original description becomes tractable. The framework of string theory offers a rich set of such dualities (reviewed e.g. in [28, 29]), which led Edward Witten to propose that the different consistent superstring theories are all limits of a single, more fundamental theory. It also led to the proposal of an eleven-dimensional quantum gravity called M-theory, whose low-energy effective description is given by 11D supergravity, but whose underlying microscopic degrees of freedom remain obscure to this day. It is however possible to probe M-theory to some extent by exploiting its duality to type IIA string theory and analyzing the supersymmetric spectrum. In this section we

[3]While the GV invariants have a direct physical interpretation, the GW invariants have a more precise mathematical definition in terms of the moduli space of stable maps.

give a selective and brief overview over the web of string dualities, followed by essential information on M-theory and a famous proposed description in terms of a matrix model.

2.4.1 Differents superstrings and their relations

As a reminder, the massless spectrum of both type IIA and type IIB string theory features two gravitini from the R-NS and NS-R sectors, which implies $N = 2$ supersymmetry in ten dimensions. There exist three additional $N = 1$ supersymmetric string theories in 10D, known as the two heterotic string theories with gauge groups $SO(32)$ or $E_8 \times E_8$ and type I string with gauge group $SO(32)$. At low energies, these are all described by $N = 1$ supergravity coupled to Super Yang-Mills (SYM) theory, which has a chiral spectrum with a left-handed dilatino and right-handed gaugino and gravitino (see table 2.5 for the massless spectrum). We will refrain from discussing the derivation of the massless spectra as for the type II theories, but we would like to mention the main ideas behind the construction of those models. The reader can find nice a discussion e.g. in [30].

name	irred. representations $SO(8)$	field content
supergravity multiplet	$\mathbf{1} \oplus \mathbf{28}_V \oplus \mathbf{35}_V \oplus \mathbf{8}_S \oplus \mathbf{8}_C$	$(\Phi, B_{MN}, G_{MN}, \tilde{\lambda}_A, \psi_A^M)$
vector multiplet	$\mathbf{8}_V \oplus \mathbf{8}_C$	(A_M^a, χ_A^a)

Table 2.5: Massless $N = 1$ multiplets for three superstring theories in $D = 10$. In type I, the Kalb-Ramond field is projected out.

Heterotic string theories are hybrid models that combine the right-moving spectrum of the type II superstring theories with a non-supersymmetric spectrum of left-movers. The two sectors are completely independent up to the level-matching condition. The simplest possibility to cancel the Weyl anomaly is to introduce 32 left-moving Majorana-Weyl fermions λ_α with $\alpha = 1, \ldots, 32$ such that the total action in light-cone gauge reads

$$S = \frac{1}{2\pi} \int d^2\sigma \left(\frac{2}{\alpha'} \partial_+ X \cdot \partial X + i\psi \cdot \partial_+ \psi + \sum_{\alpha=1}^{32} \lambda_\alpha \partial_- \lambda_\alpha \right) . \qquad (2.111)$$

From there one can build several string theories differing by the choice of boundary conditions for the λ_α. The simplest possibility is to treat all λ_α equally, leading to two sectors in which either all are periodic (P) or anti-periodic (A). The global $SO(32)$ symmetry, under which the λ_α transform in the fundamental representation, becomes a gauge symmetry in spacetime. While in the right-moving sector one implements the Super-Virasoro constraints and performs the same GSO-projection as in type II, the level-matching condition enforces a GSO-projection in the left-moving sector that removes the tachyon.

Another viable possibility is to split the 32 fermions into two groups $\lambda_\alpha^{(1)}, \lambda_\alpha^{(2)}$ with $\alpha = 1, \ldots, 16$, leading to four distinct sectors. The naively expected $SO(16)_1 \times SO(16)_2$ gauge group is enhanced to $E_8 \times E_8$. This can be seen as follows: one can show that the massless vector fields transform in the $(\mathbf{120}, \mathbf{1}) \oplus (\mathbf{128}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{120}) \oplus (\mathbf{1}, \mathbf{128})$

representations of $SO(16)_1 \times SO(16)_2$. To obtain vector fields that transform in the adjoint of the gauge group one needs to consider the group E_8, which has an $SO(16)$ subgroup with respect to which the adjoint decomposes as $\mathbf{248} = \mathbf{120} \oplus \mathbf{128}$. The $SO(32)$ and $E_8 \times E_8$ theories are the only anomaly- and tachyon-free supersymmetric heterotic theories that are known.

Finally, there is also the type I string. In type IIB string theory the GSO-projection for the left- and rightmoving sectors acts identically and hence the spectrum is invariant under the world-sheet parity transformation $\Omega : (\sigma, \tau) \to (2\pi - \tau, \sigma)$. The type I string arises if one gauges the symmetry Ω in type IIB, i.e. one keeps only states which are invariant under Ω. In the R-NS and NS-R sectors the projection only leaves linear combinations of the two gravitini and dilatini invariant, reducing the amount of spacetime supersymmetry to $N = 1$. The degrees of freedom of the SYM sector are described by open strings, unlike for the heterotic theories which only admit closed strings.

String dualities and D-branes

The five consistent superstring theories can all be related to each other through chain of dualities, of which there are two categories: there is T-duality, a kinematical symmetry that relates target space geometries of different sizes, and S-duality, a dynamical symmetry exchanging weak and strong coupling. We start with T-duality, which reveals itself already in the KK compactification of bosonic string theory (in $D = d+1$ dimensions) on a circle S^1_R with $x^d \simeq x^d + 2\pi R$. From the mode expansion (2.8) one can derive that the quantized momenta in the d-th direction are given my

$$p^d_{L/R} = \frac{n}{R} \pm \frac{mR}{\alpha'} \,. \tag{2.112}$$

The first component is a KK momentum that already appeared in the field theory example in section 2.3.3. The second effect is tied to the extended nature of strings and arises if the string closes on itself only after m circumferences along the circle, where m is the winding number. Imposing level matching, the effective mass in $D-1$ dimensions is found to be

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N + \bar{N} - 2) \,, \tag{2.113}$$

where level matching relates left- and right-movers via $N - \bar{N} = n \cdot m$. T-duality corresponds to the invariance of the spectrum under $R \to \alpha'/R$ and $n \leftrightarrow m$. It therefore suggests that spacetime geometry is not really fundamental since small and large compactification radii describe the same physics as long as the full string spectrum is included. Moreover, there is no physical radius smaller than the critical value $R_{\text{crit}} = \sqrt{\alpha'}$. Note that according to (2.112), T-duality maps $(p^d_L, p^d_R) \to (p^d_L, -p^d_R)$. But in order for T-duality to be a symmetry of the full interacting theory, such that all correlation functions and the full spectra match, the action needs to be extended to the full world-sheet fields, i.e. it acts as $(X^d_L, X^d_R) \to (X^d_L, -X^d_R)$.

T-duality relates superstring theories if one considers their compactifications. For the type II theories for instance, T-duality states that

$$\text{Type IIB on } S^1_R \;\cong\; \text{Type IIA on } S^1_{\alpha'/R}\,. \tag{2.114}$$

A hint for this duality can be found by inspecting the R-sector. Say we compactify the 9-direction, then superconformal invariance of the world-sheet theory imposes that the flip $X^9_R \to -X^9_R$ must be accompanied by $\psi^9_R \to -\psi^9_R$. Recalling the R-sector expansion (2.40) and the GSO-projection (2.45), one finds that due to $b^9_0 \to -b^9_0$ the parity operator flips as $(-1)^F \to -(-1)^F$. Thus, in the R-sector one performs the opposite GSO-projection and obtains the opposite type II theory. If both the heterotic $SO(32)$ and $E_8 \times E_8$ theories are compactified on a circle they can also be related via T-duality. For that one needs to choose an appropriate gauge background that breaks the symmetry to $SO(16)_1 \times SO(16)_2$ and relate the radii of the two circles via $R_1 R_2 = \alpha'/2$ (for details see [31]).

Under S-dualities, the strong coupling regime of one theory is mapped to the weakly coupled regime of the other one. This concept was first developed in the context of $N = 4$ SYM theories in 4D [32]. These gauge theories exhibit self-duality in the sense that the theory at coupling g is equivalent to the same theory at inverse coupling $g' \sim g^{-1}$ if the roles of electrically and magnetically charged objects are exchanged. In more general terminology, fundamental objects become solitonic and vice versa. This duality is part of the $SL(2,\mathbb{Z})$-invariance of the SYM action under which the (complexified) gauge coupling and fields are transformed. We encountered such a duality in the type IIB supergravity action in section (2.2.2). Indeed, assuming a vanishing background $C_0 = 0$, the $SL(2,\mathbb{Z})$ map S with $a = 0, b = -1, c = 1, d = 0$ yields

$$B_2' = C_2\,, \quad C_2' = -B_2\,, \quad e^{-\Phi'} = e^{\Phi}\,, \tag{2.115}$$

so $g_s' = 1/g_s$ for a constant dilaton field. Thus, if $SL(2,\mathbb{Z})$ is a duality symmetry of type IIB string theory in the sense that backgrounds differing by $SL(2,\mathbb{Z})$ mappings are physically indistinguishable configurations, type IIB is self-dual and should be described in terms of a different fundamental object for $g_s \gg 1$, charged under C_2.

To understand how S-duality affects the spectrum we need to consider an important class of dynamical objects that were not mentioned so far, namely D-branes. In the following we provide some essential information about them and refer to [31,33,34] for in-depth reviews. A Dp-brane is a $(p+1)$-dimensional hypersurface on which open strings can end. The low-energy effective action of the worldvolume theory is split as $S_{\text{eff}} = S_{\text{DBI}} + S_{\text{WZ}}$, whose bosonic parts are given by

$$\begin{aligned} S_{\text{DBI}} &= -T_{Dp} \int d^{p+1}\xi\, e^{\Phi} \Big[-\det\Big(g_{ab}(X) + 2\pi\alpha' F_{ab} + B_{ab}(X) \Big) \Big]^{1/2}\,, \\ S_{\text{WZ}} &= i\mu_{Dp} \int_{p+1} \Big(\sum_q C^{(q)} \Big) \wedge e^{2\pi\alpha' F + B} = i\mu_{Dp} \int_{p+1} C^{(p+1)} + \ldots\,. \end{aligned} \tag{2.116}$$

The Dirac-Born-Infeld action (S_{DBI}) is the analog of the Nambu-Goto action of the string, with worldvolume coordinates ξ^a $(a = 0, \ldots, p)$ and embedding functions $X^M(\xi)$, and describes the coupling of the brane to the bulk NS-NS fields. $F_{ab} = 2\partial_{[a}A_{b]}$ denotes a $U(1)$ worldvolume field strength and g_{ab}, B_{ab} denote the respective pullbacks of g_{MN}, B_{MN}. In static gauge, the embedding coordinates are given by

$$X^a = \xi^a\,, \quad X^i = x^i + 2\pi\alpha'\phi^i(\xi) + \ldots\,. \tag{2.117}$$

The x^i mark the position of the brane, while the massless bosonic fields (ϕ^i, A) describe the motion and fluctuations of the brane in the embedding space. Expanding the square root in S_{DBI} leads to the kinetic term of Yang-Mills theory plus higher-curvature corrections. At leading order, the Wess-Zumino action S_{WZ} captures the coupling of a Dp-brane to a R-R gauge field $C^{(p+1)}$. Since only those Dp-branes with matching R-R forms can be stable objects, each of the type II theories only supports a subset of possible branes:

$$\begin{aligned} &\text{Type IIA}: \ D0,\ D2,\ D4,\ D6,\ D8\,,\\ &\text{Type IIB}: \ D1,\ D3,\ D5,\ D7,\ D9\,. \end{aligned} \tag{2.118}$$

This is best seen in the democratic formulation of the supergravity theories. By Hodge duality, a Dp-brane couples electrically to a $C^{(p+1)}$ form and magnetically to a $C^{(7-p)}$-form. Note that in non-trivial backgrounds a Dp-brane can couple to q-forms with $q < p$, encoded by the higher expansion terms in S_{WZ} that we omitted to write. Dp-branes of the above type are BPS objects whose tension T_{Dp} and charge μ_{Dp} coincide. The physically measured brane tension is then given by

$$\tau_{Dp} = \frac{1}{g_s}T_{Dp} = \frac{1}{g_s}|\mu_{Dp}| = \frac{1}{g_s}\frac{2\pi}{l_s^{p+1}}\,. \tag{2.119}$$

Between two BPS-branes the two competing forces, namely attraction due to exchange of gravitons (closed strings) and dilatons and the repulsion from R-R form exchange, cancel at all distances. Both type IIA and type IIB string theory feature yet another extended solitonic BPS object, called the $NS5$-brane, with tension

$$\tau_{NS5} = \frac{1}{g_s^2}\frac{2\pi}{l_s^6}\,. \tag{2.120}$$

It is the magnetic dual object of the fundamental string $F1$, i.e. it couples magnetically to the NS-NS 2-form B_2. $NS5$-branes have no open strings ending on them and they are not charged under R-R-forms, hence they are lacking a low-energy description in terms of DBI- and WZ-like actions.

At $g_s \ll 1$, the picture of a static D-brane or $NS5$-brane as a heavy solitonic object in space-time is perfectly valid, but it fails for $g_s \gg 1$ in which case the branes substantially back-react to the geometry of the ambient spacetime. The underlying BPS object however persists, and some brane is expected to become a light fundamental object in a dual framework. In type IIB the BPS brane spectrum indeed closes on itself, as the S-duality map $(g_s, l_s) \to (1/g_s, \sqrt{g_s}\,l_s)$ maps the tensions as

$$T_{F1} \leftrightarrow T_{D1}\,, \quad T_{D3} \to T_{D3}\,, \quad T_{D5} \leftrightarrow T_{NS5}\,. \tag{2.121}$$

In particular, the $D1$-brane becomes the lightest object of the theory and replaces the fundamental string. Another example of S-duality is realized between Heterotic $SO(32)$ and type I theory, where $g_s^{\text{Het}} \sim 1/g_s^I$ and the type I $D1$-brane becomes the heterotic fundamental string. Once again, the duality can be explicitly shown at the supergravity level by transforming fields appropriately. Figure 2.2 summarizes the duality relations of the various superstring theories. It also depicts a so far neglected quantum gravity theory called M-theory, which is also a part of the string duality web and which will be introduced next.

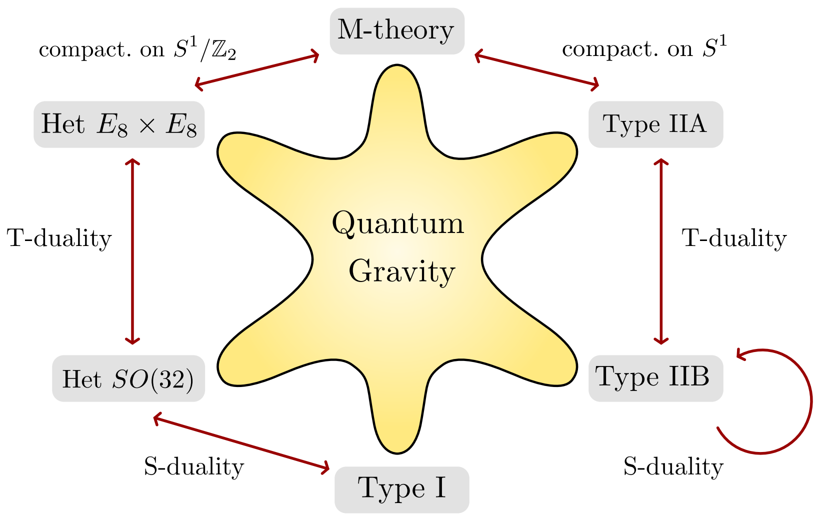


Figure 2.2: Dualities among different superstring theories.

2.4.2 M-theory

The results from the previous subsection beg the question whether one can find an S-dual framework for type IIA string theory by similar arguments. If we consider the BPS brane spectrum, the $D0$-brane with mass $\tau_{D0} = 2\pi M_s/g_s$ becomes the lightest state for $g_s \to \infty$. Furthermore, one can show that n $D0$-branes can have a single supersymmetric bound state with mass $n\tau_{D0}$, i.e. a state with no binding energy (see [31]). They thus form an evenly spaced tower of mass states charged under the R-R form $C^{(1)}$, which coincides with the KK spectrum for a compactification on a circle with radius $R_{11} = g_s/(2\pi M_s) = g_s\sqrt{\alpha'}$. This leads to the conjecture that type IIA string theory is in fact an eleven-dimensional quantum gravity theory, called M-theory, compactified on a circle of radius R_{11} [35]. In the low-energy limit, M-theory is described by $N = 1$ supergravity in eleven dimensions (32 supercharges, one Majorana spinor), with bosonic action

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{48}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int A_3 \wedge F_4 \wedge F_4\,, \tag{2.122}$$

where $F_4 = dA_3$. This is also the largest possible dimension for a supersymmetric theory describing particles with spin ≤ 2. The unique supergravity multiplet comprises 256 on-shell degrees of freedom (128 bosonic from the metric and three-form, 128 fermionic from the gravitino). Support for type IIA-M-theory relation comes from the fact that under dimensional reduction of (2.122) on a circle S^1_R one recovers the spectrum of type IIA supergravity in 10D. The gravitational couplings are related as

$$2\kappa_{11}^2 = 2\pi R_{11}(2\kappa_{10}^2) = (2\pi)^8 g_s^3 (\alpha')^{9/2} = \frac{1}{2\pi}\left(\frac{2\pi}{M_*}\right)^9 , \tag{2.123}$$

where in the last equality we introduced the eleven-dimensional Planck mass M_*. Solving for the corresponding Planck length $l_{11} = M_*^{-1}$ yields $l_{11} = g_s^{1/3}\sqrt{\alpha'}$, such that the dictionary between type IIA and M-theory quantities becomes

$$(2\pi M_s)^2 = R_{11} M_*^3 , \quad g_s = (R_{11} M_*)^{3/2} . \tag{2.124}$$

With these relations one can trace back from which M-theory objects the type IIA branes originate. The analysis shows that M-theory hosts two types of branes, an $M2$-brane and an $M5$-brane, whose tensions satisfy

$$\begin{aligned} T_{M2} &= \frac{M_*^3}{(2\pi)^2} = \tau_{D2} , \quad (2\pi R_{11})\, T_{M2} = T_{F1} , \\ T_{M5} &= \frac{M_*^6}{(2\pi)^5} = \tau_{NS5} , \quad (2\pi R_{11}) T_{M5} = \tau_{D4} . \end{aligned} \tag{2.125}$$

So depending on whether an M-theory brane wraps the circle with radius R_{11} or not one obtains a different type II brane or the fundamental string. $D6$-branes, which are the magnetic duals of the $D0$-branes, turn out to be the KK monopoles of compactified M-theory. The existence of $M2$-and $M5$-branes can also be inferred by studying the 11D supersymmetry algebra directly. One can show that the relevant part of the supersymmetry algebra can be decomposed into the irreducible components

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta} P^M &+ \frac{1}{2}(\Gamma_{MN} C^{-1})_{\alpha\beta} Z^{MN} \\ &+ \frac{1}{5!}(\Gamma^{M_1 \ldots M_5} C^{-1})_{\alpha\beta} Z^{M_1 \ldots M_5} . \end{aligned} \tag{2.126}$$

Here Γ^M denote Gamma matrices in 11D, $\Gamma^{M_1 \ldots M_p} = \Gamma^{[\cdots}\Gamma^{M_p]}$ and C is the charge conjugation matrix. The antisymmetric two- and five-forms Z^{MN} and $Z^{M_1 \ldots M_5}$ correspond to topological charges of extended objects, so they enforce the presence of $M2$- and $M5$-branes that are charged under the three-form A_3 ($M2$-branes electrically and $M5$-branes magnetically). Obtaining effective worldvolume theories for M-theory branes has proven far more difficult than for D-branes as the dynamics of $M2$- and $M5$-branes cannot be derived from open strings, making them intrinsically non-perturbative objects. While the classical supermembrane action was formulated very early [36, 37], it was found much later that stacks of M2-branes admit a description in terms interacting three-dimensional superconformal field theories [38–40]. A fully fledged worldvolume theory for multiple M5-branes has not been found yet.

Matrix theory and BFSS proposal

While the duality to perturbative string theory and the low-energy approximation by supergravity are powerful consistency checks, they do not by themselves furnish a microscopic definition of M-theory. In this section we will briefly review a famous proposal in this direction, known as the BFSS conjecture [41] (for reviews see [42–44]). Their claim was that asymptotically flat M-theory is dual to a large N limit of the matrix quantum mechanics describing N non-relativistic D0-branes[4]. This would allow to compute physical observables such as S-matrix elements between asymptotic states of M-theory in the matrix model instead.

Let us first be more precise on the duality itself. On the M-theory side, we introduce light-cone coordinates and momenta[5]

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{11}) , \quad P_{\pm} = \frac{1}{\sqrt{2}}(P_0 \pm P_{11}) . \tag{2.127}$$

In the light-cone frame (LCF), X^+ plays the role of time, while P^+ is the Hamiltonian. From the mass-shell condition for a general system with center of mass energy M one derives the Hamiltonian

$$H = P_+ = P_i^2/(2M) + M^2/(2P_-) , \tag{2.128}$$

where $i = 1, \ldots, 9$. The form of H shows that relativistic physics in the LCF is described by a Galilean invariant theory in the $(X^1, \ldots, X^9)$-plane. One then proceeds by compactifying M-theory on a light-like circle, such that $(X^+, X^-) \simeq (X^+, X^- + 2\pi R)$ and

$$P_- = N/R , \tag{2.129}$$

where N is a positive integer. As P_- is conserved, the Hilbert space splits into an infinite number of superselection sectors characterized by N (no physical observable connects states belonging to different sectors). Note that a light-like circle has zero proper length. To make the relation to type IIA more evident, one first compactifies M-theory on a small space-like circle of radius R_{11}, related to g_s via (2.124). After fixing the momentum along this circle, one sends $R_{11} \to 0$ and performs a Lorentz boost back to the LCF in which the compactification radius is R (see [51] for details). As a result, the spacelike circle is "rotated" into a light-like circle and the system has finite light-cone momentum N/R. By the dictionary between type IIA and M-theory, a system with momentum number N along the original spatial circle is described by a bound state of N $D0$-branes. BFSS thus proposed that the M-theory sector with light-cone momentum N/R is described by the effective action for a system of N $D0$-branes. This action can be obtained from

[4]Note that there are also several conjectured matrix model formulations of superstring theories, including the IKKT model for type IIB [45, 46], the Itoyama–Tsuchiya model for type I [47], and matrix string theory describing the type IIA string [48–50].

[5]To avoid a clash of notation, we will denote the tenth spatial coordinate by X^{11} instead of X^{10}.

the dimensional reduction of maximally supersymmetric 10D Yang-Mills theory to $0+1$ dimensions. In string units, the action reads [52]

$$S_{D0}^{(N)} = \int dt \operatorname{Tr}\left(\frac{(D_t X^i)^2}{2g_s\sqrt{\alpha'}} + \frac{[X^i, X^j]^2}{16\pi^2 g_s(\alpha')^{5/2}} - \frac{i}{2}\theta_\alpha D_t\theta_\alpha + \frac{1}{4\pi\alpha'}\theta_\alpha^T\gamma_{\alpha\beta}^i[X_i,\theta_\beta]\right). \quad (2.130)$$

Here $i, j = 1, \ldots, 9$ denote spatial directions in $\mathbb{R}^9$ and $\alpha, \beta = 1, \ldots, 16$ are spinor indices of $SO(9)$ (both implicitly summed). Each X^i is a $N\times N$ Hermitian matrix whose eigenvalues represent the positions of N $D0$-branes in the i-th direction. θ_α is an $N \times N$ Hermitian, matrix-valued Majorana spinor of $SO(9)$ and the γ^i denote (real) Gamma matrices. The action exhibits a $U(N)$ gauge symmetry, with respect to which both X^i and θ_α transform in the adjoint representation. The gauge-covariant derivative acts as

$$D_t\Phi = \partial_t\Phi - i[A_t, \Phi]\,. \quad (2.131)$$

for any adjoint-valued field Φ. As there are no remaining appearances of A_t, it can be thought of as a constraint field that enforces $U(N)$ gauge invariance. In the gauge $A_t = 0$, the corresponding Hamiltonian is found to be

$$H_{D0}^{(N)} = \frac{R_{11}}{2}\int dt \operatorname{Tr}\Big(P_i^2 - \frac{[X^i, X^j]^2}{8\pi^2 l_{11}^6} - \frac{1}{2\pi l_{11}^3}\theta_\alpha^T\gamma_{\alpha\beta}^i[X_i,\theta_\beta]\Big)\,, \quad (2.132)$$

expressed in terms of M-theory units by using (2.124). $P^i = \dot{X}^i/R_{11}$ (where dot stands for time derivative) denote the canonical momenta. When boosting this system to the LCF the Hamiltonian picks up a time dilation factor such that R_{11} is replaced by R.

According to the BFSS conjecture, a 1×1 matrix describing a single D0-brane in type IIA string theory corresponds to a graviton in M-theory with longitudinal momentum $P_- = 1/R$. Indeed, for $N = 1$ the Hamiltonian (2.132) reduces to

$$H_{D0}^{(1)} = \frac{R}{2}P_i^2 = \frac{P_i^2}{2P_-}\,, \quad (2.133)$$

yielding the correct expression for massless states in the LCF according to (2.128). Moreover, there are 16 fermionic fields that generate a $2^8 = 256$-dimensional supermultiplet, corresponding precisely to the field content of the massless supermultiplet in 11D supergravity. For $N > 1$, one can decompose the degrees of freedom into $U(1)$ (center of mass motion) and $SU(N)$ (relative motion) by separating off the trace. As a result, the two components decouple and the Hamiltonian splits as $H = H_{\text{cm}} + H_{\text{rel}}$. It has been shown [53, 54] that H_{rel} has zero energy bound states whose total energy is given by

$$H_{D0}^{(N)} = H_{\text{cm}} = \frac{(P_{\text{cm},i})^2}{2P_-}\,, \quad (2.134)$$

where $P_{\text{cm},i} = \operatorname{Tr}(P_i)$. These bound states again fill out a supergravity multiplet in 11D. The existence of a truly normalizable ground state in the quantum theory is subtle.

Classically, the bosonic potential $V \sim \mathrm{Tr}\,[X^i, X^j]^2$ in (2.132) has a minimum if $[X^i, X^j] = 0$ for all $i, j = 1, \dots, 9$. Then all matrices can be diagonalized simultaneously, such that

$$X^i_{(N)}(t) = \mathrm{diag}(x^i_1(t), \dots, x^i_N(t)) \tag{2.135}$$

describe N non-interacting $D0$-branes traveling along trajectories $x^i_I(t)$ with $I = 1, \dots, N$. These trajectories parametrize non-compact directions in the classical moduli space. If the wavefunction of the ground state spreads out to infinity along these flat directions, it may fail to be normalizable. Yet, it was possible to show that despite these flat directions there is a truly normalizable zero energy state in the BFSS matrix model, and there are results on how fast the wavefunction decays (see [55] and references therein).

The technique of splitting the degrees of freedom into center of mass/relative motion can also be applied to block-diagonal matrices of the form

$$X^i \;=\; \begin{pmatrix} X^i_{(N_1)} & 0 & \cdots & 0 \\ 0 & X^i_{(N_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X^i_{(N_k)} \end{pmatrix} \qquad \sum_{a=1}^{k} N_a = N\,, \tag{2.136}$$

where the a-th block is interpreted as a supergraviton state with longitudinal momentum $P_- = N_a/R$. Matrix theory is thus capable of describing multiple objects with finite-size matrices and should be understood as a second-quantized theory from the target space point of view.

Interactions at quantum level

The (block-)diagonal matrices discussed so far describe a set of non-interacting $D0$-branes. Such matrices provide an exact description of static brane configurations in the quantum theory as these preserve all supersymmetries and therefore the forces due to graviton and p-form exchange cancel. When the $D0$-branes move with non-zero relative velocity, supersymmetry is broken and the cancellation is no longer exact. The resulting interactions are described by open strings stretching between two $D0$-branes, giving rise to off-diagonal entries in the above matrices. If we consider two $D0$-branes A and B with separation r_{AB}, an open string stretched between the two will have energy $E_{\mathrm{str}} \sim r_{AB} R_{11} M_*^3$. By comparison, excitations of these strings will have additional energy $E_{\mathrm{exc}} \sim 1/l_s \sim (R_{11} M_*^3)^{1/2}$. In the limit $R_{11} \to 0$ the excitation energy becomes parametrically bigger than E_{str}, suggesting that the ground states of the stretched strings yield the dominant contribution. If the brane separation r_{AB} is very large, even the lightest strings have a high mass and the diagonal matrices provide a good leading approximation.

Studying the scattering of gravitons reveals one of the key properties of the BFSS model, namely that gravitational interactions arise as a *quantum effect* in matrix theory. To give a simple example, consider the scattering of two gravitons with (low) relative velocity v in the transverse 9-dimensional space and impact parameter b (the initial transverse separation). One first chooses a background of diagonal matrices describing two widely separated $D0$-branes and studies fluctuations around these, i.e.

$$X^1 = \frac{1}{2}\begin{pmatrix} vt & 0 \\ 0 & -vt \end{pmatrix} + \delta X^1 \,, \quad X^2 = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} + \delta X^2 \tag{2.137}$$

and $X^i = \delta X^i$ for $i > 2$. One then expands the BFSS action around this background to quadratic order. The off-diagonal modes connecting the two branes acquire masses proportional to the separation and therefore behave like time-dependent harmonic oscillators. Integrating out these massive bosonic and fermionic fluctuations at one-loop level (using the background-field method) yields an effective action for the diagonal degrees of freedom. The resulting effective long-range potential is a pure one-loop effect and has the leading order expansion

$$V_{\text{eff}}(r) = -\frac{15}{16}\frac{v^4}{r^7} + \mathcal{O}\left(\frac{v^6}{r^{11}}\right) \,, \tag{2.138}$$

where $r = \sqrt{(vt)^2 + b^2}$ is the instantaneous separation between the gravitons[6]. Remarkably, this potential exactly coincides with the corresponding tree-level result from 11D supergravity, including the numerical prefactor. Higher-loop corrections to the v^4-term would spoil the agreement. It has been shown that at two-loop level there are no corrections [57], but a full non-renormalization theorem to all orders is lacking so far.

The properties of the BFSS matrix model support the idea of an emergent spacetime and the correlated emergence of interactions. In matrix theory, the fundamental objects are non-commuting matrices, and the usual notion of spacetime is only recovered if $D0$-branes are widely separated and the diagonal entries of the corresponding matrices yield the classical spacetime coordinates. In the long-distance, low-velocity regime the dynamics of the matrix degrees of freedom yield the classically expected gravitational potential in the form of a quantum effect. This suggests that M-theory could provide a natural framework in which spacetime and dynamics of fields are not put in by hand, but instead arise as a low-energy imprint from the degrees of freedom of the full QG theory - a feature that closely aligns with the Emergence Proposal (to be discussed in section 3.3). However, we should stress an important caveat: The BFSS model is formulated in a light-cone frame and involves a weak coupling ($g_s \to 0$) and tensionless string ($M_s \to 0$) limit from the type IIA perspective. There is no obvious duality that directly relates this limit to the standard strong-coupling regime of type IIA string theory. So while the hints from the BFSS model are compelling, more work is needed to explicitly connect it to our arguments for the Emergence Proposal in M-theory.

[6]This result was previously derived from the annulus diagram [56] for an open string connecting two $D0$-branes with relative velocity v and distance r

Chapter 3

Lessons from the swampland

At high energies, the consistency conditions of string theory are remarkably constraining and almost uniquely fix the whole theory. Yet, this high degree of uniqueness does not immediately translate into comparably sharp, universal restrictions on low-energy effective field theories (EFTs), because the string theory framework admits an enormous variety of consistent vacua. That being said, not any EFT can arise from a string theory construction. There are necessary, model-independent criteria distinguishing between EFTs that are compatible or incompatible with QG. These criteria are studied within the swampland program, which will be introduced (selectively) in the following chapter. We will focus on conjectures that predict the appearance of infinite towers of light states in extreme regions of the moduli space of QG theories. These light towers dominate the low-energy physics and set the scale where QG effects become relevant. These concepts will guide us to the intriguing idea that terms in the low energy effective action in QG theories could be purely quantum effects, obtained by integrating out the appropriate towers of states. This is known as the Emergence Proposal, whose original formulation we will review briefly. We will then argue why this proposal should hold specifically in M-theory and formulate a refined version, preparing the discussion of the concrete evidence in the upcoming chapters.

3.1 General scope

In the previous chapter we saw that in perturbative string theory the conistency constraints of the world-sheet theory have an enormous impact on the spacetime background in which the string is allowed to propagate. The non-trivial backgrounds arising from CY compactifications solve these constraints even to all orders in α'. However, we should keep in mind that a compactification is merely a specific ansatz, motivated by some desired properties of the effective theory that we want to obtain. What we are lacking is a dynamical principle, inherent to the theory, which favors a particular background solution. One top of that, one faces the issue that the number of possible background solutions is astronomically large (although believed to be finite, see [58–61] for earlier and [62, 63] for more recent work). The space of distinct, consistent string vacua obtained by choosing background data such that all moduli are stabilized, is called the *string landscape* (reviewed e.g. in [64, 65]). For models derived e.g. from type IIB, an often quoted, rough estimate for the number of solutions is $\mathcal{O}(10^{100} - 10^{500})$. The dominant source for this huge figure is the vast number of possible flux quanta that one may choose, but also the choice of the compactification manifold and many more factors. With a pessimistic attitude one would start to believe that basically any low-energy theory may be constructed from string theory. But, as it turns out, this is far from being true.

The immensity of the string landscape has led to a major philosophical shift in the area of string phenomenology. In the early era of string model building, the dominant program was about finding the "right" string vacuum whose low-energy physics yields the Standard Model (or supersymmetric extensions of it). At some point however string theorists started to ask whether any conceivable QFT, weakly coupled to Einstein gravity, that describes some low-energy physics could be originating from string theory. To make this point a bit more precise, suppose that we are given a gravitational theory in d dimensions, described by a Wilsonian effective action $S_{\text{eff}} = S^{(\text{grav})} + S^{(m)}$, with components

$$\begin{aligned} S^{(\text{grav})} &= \frac{M_{\text{pl},d}^{d-2}}{2} \int d^d x \sqrt{-g} \left(R + \sum_k \alpha_k \frac{\mathcal{O}_k(g, \Phi)}{\Lambda_{\text{UV}}^{k-2}} \right) , \\ S^{(m)} &= \int d^d x \sqrt{-g} \left(\mathcal{L}^{(m)} + \sum_l \beta_l \frac{\mathcal{O}_l^{(m)}(g, \Phi)}{\Lambda_{\text{EFT}}^{l-d}} \right) . \end{aligned} \tag{3.1}$$

The action shows two different sectors: First, there is the gravitational sector, which is the part of the EFT controlled by the d-dimensional Planck scale $M_{\text{pl},d}$. Its leading part is the Einstein-Hilbert term, but it also has a series of corrections that are suppressed by Λ_{UV}, the energy scale where quantum gravity effects are no longer negligible. The corrections are given in terms of the irrelevant operators $\mathcal{O}_k$ of mass-dimension $k > d$ and dimensionless Wilson coefficients α_k. The operators may involve higher-curvature corrections (abbreviated by g) and generally depend on the low-energy matter and gauge fields, collectively denoted by Φ. Second, there is the matter sector, a completely model-dependent part of the action capturing the dynamics of scalars, gauge fields, fermions and so on. While $\mathcal{L}^{(m)}$ is the renormalizable part, there is also series of irrelevant terms $\mathcal{O}_l^{(m)}$ with Wilson coefficients β_l, but these are suppressed by an a priori different scale Λ_{EFT}, attributed to the mass of heavy states that were integrated out.

From the point of view of a low-energy observer, no particular choice of matter content, Wilson coefficients or operators seems more preferable, and most of them are perfectly fine in the low-energy regime. However, the remarkable feature of quantum gravity is that it creates a link between the gravitational sector and the matter sector, which is sometimes dubbed "UV-IR mixing", and teaches us that the matter sector cannot be arbitrary for the theory to admit a UV completion. This leads to the notion of the so-called *swampland* (proposed in [66, 67]), the complementary part of the landscape:

Definition: The swampland is the set of (apparently) consistent EFTs that do not admit a UV-completion to quantum gravity.

Above the quantum gravity cutoff Λ_{UV}, the description in terms of a weakly interacting QFT breaks down entirely. For the vast majority of gravitational effective theories this cutoff is given by the Planck scale $M_{\mathrm{pl},d}$. The reason is that interactions involving gravitons are suppressed by $1/M_{\mathrm{pl},d}$, which means that the according amplitude will be weighed by the effective coupling $E/M_{\mathrm{pl},d}$ at some invariant energy scale E. Gravity thus becomes strongly coupled at $E \simeq M_{\mathrm{pl,d}}$ at the latest. However, there can be situations in which Λ_{UV} is significantly lower than $M_{\mathrm{pl},d}$, which will be explored in future sections. If this UV cutoff is lowered to the extent that

$$\Lambda_{\mathrm{UV}} \ll \Lambda_{\mathrm{EFT}} \ll M_{\mathrm{pl},d} \tag{3.2}$$

holds, the constraints from quantum gravity become very severe and the domain of validity of the EFT is much smaller than initially thought. In section 3.2.3 we will learn how to obtain an upper bound for the cutoff Λ_{UV}.

The so-called *swampland program* is a branch of string phenomenology that aims to distinguish EFTs admitting a UV completion to quantum gravity from those that do not. The relevant criteria are formulated in terms of *swampland conjectures*, which identify model-independent properties of EFTs that make them compatible with quantum gravity, without any reference to the full microscopic theory. They cover various different aspects, like the relative strength of gravity and gauge interactions [68–70] (see [71,72] for reviews), constraints on scalar potentials [73–75] or the absence of global symmetries [76, 77], to name just a few examples. Some of them will be introduced in section 3.2, for a more complete picture we refer to the exhaustive reviews [78–80]. The conjectures are often not sharply defined and typically provide some bounds involving unknown constants of $\mathcal{O}(1)$. An EFT that parametrically violates a bound is expected to have an issue when being extrapolated to higher energies. As one climbs up the energy ladder, quantum gravitational effects become more dominant and as a result, swampland criteria become more constraining. As the sketch in figure 3.1 suggests, one could speculate that in the far UV the constraints are so severe that they point towards a unique quantum gravity theory. This hypothesis is also backed up by the idea of string universality, the conjecture that all consistent quantum gravity theories are somehow realized by string theory.

The scientific activity within the swampland program could be roughly divided into two complementary segments: On the one hand, there is the bottom-up approach, where physicists formulate conjectures based on some general physical consistency arguments, invoking for example black hole thought experiments, entropy bounds, positivity bounds etc. On the other hand, in the top-down approach the conjectures are tested by using explicit string theory constructions. These two disciplines constantly interact, since the gathering of evidence from the string landscape can reveal a limitation of the original argument and lead to a refined version of it. The ultimate goal however is to apply the conjectures to make a prediction that may be relevant to particle physics and cosmology. To give one example, an attractive recent proposal making such a quantitative prediction is the dark dimension scenario [81]. By combining swampland constraints with observational data, the authors proposed that our universe may feature extra mesoscopic dimensions of

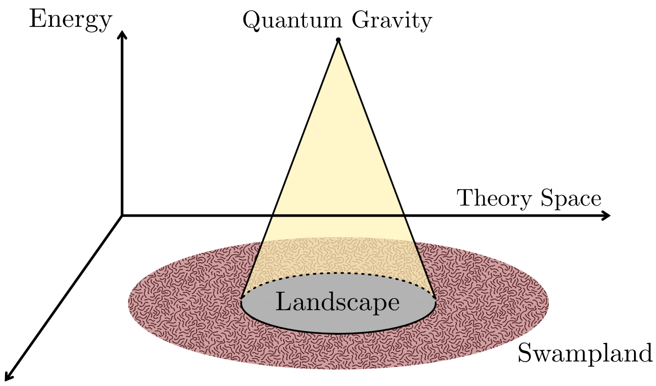


Figure 3.1: The Landscape and the Swampland.

micron size, whose size is tied to the smallness of the cosmological constant. This would lead to direct measurable deviations from the inverse-square law of Newtonian gravity at short distances and provide a dark matter candidate given by the KK excitations of the graviton. The scenario continues to be explored in recent research (see [82–86] for a tiny selection of works), while upcoming precision experiments and cosmological surveys could soon test those predictions.

3.2 Swampland conjectures and related concepts

After our general introduction we will concretize some swampland conjectures that are relevant in the context of the Emergence Proposal. Our review is by no means exhaustive since many of the most famous conjectures will not be covered at all. We will begin with the *Swampland Distance Conjecture* (SDC), which makes a generic claim about the breakdown of EFTs in cases when extreme regions of the moduli space are explored. We then continue with the *Emergent String conjecture* (ESC), a refinement of the SDC that classifies the previously mentioned limits in moduli space. After that we introduce the notion of the species scale, an upper bound for the quantum gravity cutoff Λ_{UV}. This scale is crucial for the determination of the fundamental degrees of freedom and is thus a core object in the definition of the Emergence Proposal, which will be introduced right afterwards. After briefly discussing its original version we will argue why it should be realized only in M-theory.

3.2.1 Swampland Distance Conjecture

The distance conjecture makes statements about the moduli spaces of gravitational EFTs. We recall that in a physical context, a moduli space is a geometric space spanned by the vevs of massless scalar fields with a flat potential[1]. The vevs can therefore be changed

[1]While moduli spaces can be defined without supersymmetry, exact flat directions and control over quantum corrections typically rely on it.

without energy cost and parametrize the vacuum degeneracy of a theory. But since physical parameters such as gauge or Yukawa couplings depend on the moduli, one could pose the question whether the theory could drastically change in certain regions of the moduli space. Suppose that we are dealing with a d-dimensional EFT coupled to n scalar fields $\{\phi^i\}_{i=1,\dots,n}$ whose vevs parametrize a moduli space $\mathcal{M}$. The action is

$$S = \frac{M_{\mathrm{pl},d}}{2} \int d^d x \sqrt{-g} \left(R - g_{ij}^{\mathcal{M}} \partial_\mu \phi^i \partial^\mu \phi^j \right) , \tag{3.3}$$

where $g_{ij}^{\mathcal{M}}$ is the metric on moduli space. Given the metric, the geodesic distance between two points P, Q on $\mathcal{M}$ can be defined as

$$d(P,Q) = \int_\gamma ds \sqrt{g_{ij}^{\mathcal{M}} \frac{\partial \phi^i}{\partial s} \frac{\partial \phi^j}{\partial s}} . \tag{3.4}$$

Here γ is the geodesic connecting the two points P and Q, and ds is the line element along the geodesic. We can now state the definition of the SDC, proposed in [67]:

Consider a gravitational EFT with non-trivial moduli space $\mathcal{M}$. For any point $P \in \mathcal{M}$ there is another point $Q \in \mathcal{M}$ such that the geodesic distance $\Delta\phi \equiv d(P,Q)$ is infinite. Moreover, as one traverses an infinite distance in $\mathcal{M}$, an infinite tower of states with mass scale M becomes asympotically massless at an exponential rate, i.e.

$$M(Q) = M(P) e^{-\alpha \Delta\phi} , \quad \Delta\phi \to \infty , \tag{3.5}$$

where α is a positive $\mathcal{O}(1)$ constant.

The first part of the conjecture is a more formal statement about the existence of infinite distance directions in moduli space. For instance, if we have a single massless scalar with canonical kinetic term, its moduli space $\mathcal{M} = \mathbb{R}$ is valid in that regard. The condition is however violated if the scalar is periodic $\phi \simeq \phi + 2\pi$, in which case $\mathcal{M} = S^1$. This doesn't exclude periodic scalars in quantum gravity, but imposes that their moduli space must be part of a bigger moduli space with non-compact directions.

The second part of the conjecture predicts a restricted domain of validity of EFTs. If we start from an EFT with cutoff Λ_{EFT} and we explore infinite distance regions in moduli space, there will be a tower of infinitely many states with a mass exponentially decreasing in $\Delta\phi$. If the lightest state of the tower has mass M_0, then $\Lambda_{\mathrm{EFT}} \lesssim M_0$ if we choose not to include the state in the EFT description. But then Λ_{EFT} drops off exponentially in $\Delta\phi$ as well, and eventually the EFT looses all of its predictive power. If we look at the SDC from the perspective of the full quantum gravity theory, it is closely related to the appearance of a weakly coupled dual description in infinite distance regions of moduli space. The new fundamental degrees of freedom in that limit are determined by the asymptotically massless, infinite towers of states. For this reason the SDC is often also called *Duality Conjecture*. The nature of the previously mentioned towers is specified by the ESC, to which we will come soon.

Let us check the validity of the SDC by looking at some simple examples. First, we revisit a model introducted in section 2.3.3, the theory of a scalar coupled to Einstein gravity, compactified on a circle from $d+1$ to d dimensions. Omitting the kinetic term of the graviphoton, the action reads

$$S \supset M_{\mathrm{pl},d}^{d-2} \int d^d x \sqrt{\tilde{G}^{(d)}} \left(\tilde{R}_d - \frac{d-1}{d-2} \frac{1}{r^2} \partial_\mu r \tilde{\partial}^\mu r \right) . \tag{3.6}$$

As a reminder, the mark "$\sim$" indicates that the quantities are defined in Einstein frame. The radion r is the only modulus and controls the radius of the S^1, hence $\mathcal{M} = \mathbb{R}_{\geq 0}$. The proper field distance is defined in terms of the canonically normalized field

$$\Delta(r) = \sqrt{\frac{d-1}{d-2}} \log(r) . \tag{3.7}$$

We see that $\Delta(r)$ diverges for the two infinite distance limits $r \to \infty$ and $r \to 0$. In the decompactification limit $r \to \infty$, an infinite tower of KK states satisfying (2.80) becomes light. If we transform the action of these states into Einstein frame and express the mass in terms of the field $\Delta(r)$, we obtain

$$m_n(r) = \left(\frac{n}{r}\right) \left(\frac{r_0}{r}\right)^{\frac{1}{d-2}} = n r_0^{\frac{1}{d-2}} \exp(-\alpha \Delta) , \tag{3.8}$$

confirming the prediction of the SDC. However, the SDC demands that *any* infinite distance limit is accompanied by an exponentially light tower. This is not fulfilled in the limit $r \to 0$, since there the only present tower that was just discussed becomes heavy.

At this point the distinctive features of string theory compactifications become vital. Recall that under a compactification on S^1_R string theory features two towers, namely KK modes and winding states, with masses

$$M_n^2 = \left(\frac{n}{R}\right)^2 , \quad M_w^2 = \left(\frac{wR}{\alpha'}\right)^2 . \tag{3.9}$$

If one performs the dimensional reduction, transforms to Einstein frame and extracts the moduli space metric, one finds that for $R \to \infty$ the KK modes are exponentially suppressed in proper field distance, while for $R \to 0$ this is true for the winding states. Thus, in the limit $R \to 0$ one can describe the theory in terms of new fundamental degrees of freedom, the topological winding states, corresponding to the dual frame under T-duality that was anticipated before.

Let us stress once again that the original SDC only applies to moduli spaces, i.e. scalar manifolds with no potential. But many phenomenologically interesting models, such as inflation, quintessence or dark energy, require scalar fields with a non-trivial potential. If one could extend the SDC to more general field spaces, swampland constraints could have a real impact on phenomenology, e.g. by bounding field ranges or restricting the allowed form of potentials. But for a prediction one needs to make a statement about *finite* excursions in field space. This originally motivated the *Refined Swampland Distance conjecture* (RSDC) [87, 88], according to which an infinite tower of states with exponentially decreasing mass appears already for trans-Planckian field excursions $\Delta\phi \gtrsim 1$, and not just at the pure boundary of moduli space. However, to apply this conjecture to general field spaces one needs to formulate a notion of distance therein, which is a delicate task. There has been much recent activity to provide an answer, see for example [89–92]

It is fair to say that the SDC for moduli spaces ranks among the most robust swampland conjectures, given that all investigated infinite distance limits in string- and M-theory agree with its predictions. Studying the SDC in compactification setups requires to compute the metric on moduli space, which can be quite challenging. For instance, if one is interested in CY compactifications of type IIA, one needs to invoke mirror symmetry to compute the metric of the Kähler moduli space. In the large volume regime of type IIA, one can obtain the metric from the periods of the mirror dual CY and the mirror map. But the full Kähler moduli space also features non-geometric regions, so for a full check of the SDC one needs to analytically continue the metric to these phases and carry out separate checks. We will discuss the phases of CYs in section 5.4.

3.2.2 Emergent String Conjecture

In its purest form, the SDC leaves several important questions unanswered. Most notably, there is no comment about the nature of the infinite tower that becomes exponentially light. By "nature" we mean the mass spacing in the tower, the degeneracy of states, the value of the constant α and the origin of these states in the full quantum gravity theory. Also, we do not know what the asymptotic theory at the boundary of moduli space actually is, but only that our initial EFT breaks down if left unmodified. An answer to all these questions in provided by the ESC, put forward in [93]:

In a theory of quantum gravity in $\mathbb{R}^{1,d-1}$ $(d \geq 3)$, all infinite distance limits are either

1. *decompactification limits, where the lightest tower has a (dual) interpretation as a KK tower.*

2. *emergent string limits, where the lightest tower are the excitations of a unique, critical, asymptotially tensionless (with respect to $M_{\mathrm{pl},d}$) string. Unless $d = 9$, the tower of string excitations is always accompanied by a KK tower at the same scale.*

In string theory both of these cases can be encountered in very straightfoward setups. For instance, if we consider critical superstring theory, the relation between the string scale and the 10D Planck scale is given by

$$M_s \simeq M_{\mathrm{pl},10}\, g_s^{1/4} \simeq M_{\mathrm{pl},10}\, e^{\Phi/4} \,. \tag{3.10}$$

Therefore, in the infinite distance of weak string coupling $\Phi \to \infty$ (with Planck mass fixed), the tower of string excitations with mass $M_n = \sqrt{n} M_s$ becomes exponentially light. Asymptotically this tower has an exponential degeneracy of states at mass level n given by $\deg_n \simeq \exp(\sqrt{n})$. Since we are already in ten dimensions, no equally light KK tower is expected to appear. On the other hand, compactifying string theory on a circle yields a KK tower with mass scaling (3.8) in the canonically normalized field, corresponding to a decompactification limit for $R \to \infty$. The limit $R \to 0$ leads to analogous conclusions for the tower of winding states, corresponding to a decompactification in the T-dual frame. A KK tower generally has a polynomial degeneracy factor $\deg_n$, which for the case of the circle is just 1. If in a certain limit a KK tower and a string tower are parametrically at same scale, the string excitations will dominate the spectrum since the corresponding tower is denser both in terms of spacing and degeneracy.

There is a number of conceivable asymptotic limits that could threaten the validity of the ESC, but explicit checks carried out so far indicate that these do not occur. In the following, we provide a list of forbidden scenarios and provide some references with explicit checks demonstrating their obstruction:

- the light spectrum cannot be dominated by objects other than KK states or string excitations, such as asymptotically tensionless membranes [94],
- there are no pathological string limits for which in $d < 9$ an asymptotically tensionless strings is *not* accompanied by a KK tower of the same scale [95, 96],
- limits with two or more critical strings that are weakly coupled at the same time are forbidden [93].

As a side remark, also non-critical strings can become asymptotically tensionless, but they are always found to be at finite distance in moduli space and hence do not fall under the ESC classification scheme. The ESC has been tested in a wide range of setups, see [97–110] for a selection of works.

The nature of the lightest tower in a given infinite distance limit is connected to the value of the exponent α in the SDC mass formula (3.5). In [111] it was proposed that, for any infinite distance limit of a d-dimensional quantum gravity theory, the coefficient α associated to the lightest tower will have the universal lower bound $\alpha_{\min} = 1/\sqrt{d-2}$. This hypothesis is known as the *Sharpened Distance Conjecture.* Support for this claim was gathered by showing that the bound is preserved under dimensional reduction and by checking string/M-theory compactifications. The bound can also be motivated with an argument that assumes the validity of the ESC [80, 112]. First one can show that in the two different ESC limits the dominant tower gives rise to the coefficient

$$\alpha_{\text{string}} = \frac{1}{\sqrt{d-2}} \quad \text{or} \quad \alpha_{\text{KK}} = \sqrt{\frac{D-2}{(D-d)(d-2)}} \geq \alpha_{\text{string}} \, . \tag{3.11}$$

Here α_{string} is associated with a weakly coupled string in d dimensions, while α_{KK} appears for a decompactification from d to D dimensions, where only the overall volume modulus of the internal space is varied, while other moduli stay fixed (often called *rigid* decompactification). As was argued in [80], among all decompactifications from d to D dimensions the rigid variant gives rise to the highest value of α_{KK}. The reason is that in the rigid case there are residual moduli that could either trigger further decompactification, leading to a lower value for α_{KK} according to (3.11), or an emergent string, in which case α_{string} imposes a lower bound. Repeated application of this argument leads to the conclusion that α_{string} must be the universal lower bound.

Some examples

To appreciate the meaning of the ESC, it is important to notice that the two limiting behaviors can come in various disguises. In the following we look at one example for each class of limit where a weakly coupled string or KK tower only becomes apparent in a dual frame. In each case we engineer an appropriate limit and analyze the resulting mass hierarchies.

(a) Emergent string limit: The most obvious such limit is the aforementioned co-scaled weak coupling limit where we rescale

$$g_s \to \lambda g_s\,, \quad M_s \to \lambda^{1/4} M_s \tag{3.12}$$

and consider $\lambda \ll 1$, in which case the 10D Planck mass $M_{\mathrm{pl},10} = M_s/g_s^{1/4}$ stays fixed. For the type IIB superstring, one can also consider the analogous strong coupling regime, $\lambda \gg 1$, leading to an emergent string limit where the string tower comes from $D1$-branes (the S-dual frame of type IIB). This can be generalized to compactifications on k-dimensional tori T^k down to $d = 10 - k$ dimensions. We can engineer a limit where the radii of the torus are rescaled isotropically and the d-dimensional Planck mass defined via

$$M_{\mathrm{pl},d}^{d-2} = M_s^{d-2} \frac{\mathcal{V}_k}{g_s^2} \tag{3.13}$$

remains constant. $\mathcal{V}_k$ denotes the volume of T^k in string units. In this case, the co-scaled emergent $D1$-string limit is given by

$$g_s \to \lambda g_s\,, \quad M_s \to \lambda^{\frac{d-6}{2(d-2)}} M_s\,, \quad \rho_1 \to \lambda^{1/2} \rho_1\,, \quad \rho_i \to \lambda^{1/2} \rho_i\,, \tag{3.14}$$

where ρ_1 and ρ_i with $i = 2, \ldots, k$ denote denote internal, dimensionless radii in string units. The radius ρ_1 has been singled out since we would like to T-dualize along this direction to translate the above limit to M-theory units. If one considers a circle compactification of 10D type IIA on R_1 and type IIB on α'/R_1 and compares the string-frame actions, one can show that the string couplings of both theories are related via

$$g_s^{\mathrm{IIB}} = g_s^{\mathrm{IIA}} \frac{\sqrt{\alpha'}}{R_1}\,. \tag{3.15}$$

We can now relate type IIB on T^k to M-theory on $T^2 \times T^{k-1}$, where in the latter case $T^2 = S^1_{R_1} \times S^1_{R_{11}}$. Using (2.124) and (3.15), we obtain

$$(2\pi M_s)^2 = M_*^2 r_{11}\,, \quad g_s = \frac{r_{11}}{r_1}\,, \quad \rho_1 = \frac{1}{2\pi} \frac{1}{r_1 r_{11}^{1/2}}\,, \quad \rho_i = \frac{1}{2\pi} r_i r_{11}^{1/2}\,, \tag{3.16}$$

where r_1, r_i and r_{11} denote internal radii in 11D Planck units. With the dictionary (3.16) one can translate (3.14) to the scaling of M-theory quantities, yielding

$$r_{11} \to \lambda^{1/3} r_{11}\,, \quad M_* \to \lambda^{\frac{d-8}{3(d-2)}} M_*\,, \quad r_1 \to \lambda^{-2/3} r_1\,, \quad r_i \to \lambda^{1/3} r_i\,. \tag{3.17}$$

In this limit, the lightest states are the $D1$-branes, whose *typical mass scale* is

$$M_{D1} = \tau_{D1}^{1/2} \simeq \frac{M_s}{g_s^{1/2}} \simeq \frac{M_{\text{pl},d}}{\lambda^{\frac{1}{d-2}}}\,. \tag{3.18}$$

Let us clarify what we mean by typical mass scales: In a weakly coupled regime where the quantization scheme of the theory is known, the typical mass scale is simply the mass gap between consecutive levels of the tower (like for the fundamental string). At strong (string) coupling, where the quantization is usually not known and the former weakly coupled states might migrate across the mass spectrum or even become unstable, we still assume that the naive mass scale, given for instance by the tension of a string or a p-brane, is a good measure for the typical mass scale of the tower. This is certainly true for BPS states, whose weakly coupled mass can be trusted also for strong coupling.

The emergent string limit should also yield a light KK tower at the same mass scale as the $D1$ string. Indeed, the SUGRA KK modes along the internal directions have mass

$$M_{\text{KK}} = \frac{1}{R_k} = \frac{M_s}{\rho_k} \simeq \frac{M_{\text{pl},d}}{\lambda^{\frac{1}{d-2}}}\,. \tag{3.19}$$

All remaining branes of type IIB give rise to states that are parametrically heavier. When expressed in terms of $\mathfrak{m}_{D1}$, the fundamental string has the typical mass scale

$$M_{F1} \simeq M_s \simeq M_{D1} \lambda^{1/2} \tag{3.20}$$

while the $NS5$-brane and the remaining Dp-branes (for $p \geq 3$) give

$$M_{NS5} \simeq \frac{M_s}{g_s^{1/3}} \simeq M_{D1}\lambda^{1/6}\,, \quad M_{Dp} \simeq \frac{M_s}{g_s^{1/(p+1)}} \simeq M_{D1}\lambda^{\beta(p)}\,, \tag{3.21}$$

where $\beta(p) = \frac{p-1}{2(p+1)} > 0$. Thus, we have also confirmed the uniqueness of the parametrically lightest string tower. The importance of the scale M_{D1} will be discussed soon.

In different setups, the appearance of a light string can be more subtle. In section 4.4.2 we will consider a compactification of type IIA on a CY which is $K3$-fibered over $\mathbb{P}^1$. In that case one can show that the lightest modes are given by the excitations of a string resulting from wrapping an $NS5$-brane on the $K3$ fiber, as well as other particle-like states from wrapped branes. In this limit there exists a weakly coupled dual heterotic model compactified on $K3 \times T^2$ such that the complexified Kähler modulus for the base, $t_B = b + i\tau_B$, is mapped to the dilaton of the heterotic string, $S = B + i\exp(-\Phi_{\text{het}})$. The light towers of states are mapped to the heterotic string excitations and to the KK and winding modes on $K3 \times T^2$. Hence this type IIA infinite distance limit is an emergent string limit, where a (dual) fundamental string is among the lightest modes.

(b) Decompactification limit: The type IIA Kähler moduli space of CY compactifications also admits a co-scaled decompactification limit, which is given by scaling all of its Kähler moduli isotropically as $t_I \to \lambda^{2/3} t_I$ and co-scaling the dilaton as $g_s \to \lambda g_s$ to keep the 4D Planck scale fixed. Let us discuss this strong coupling limit more generally, i.e. upon compactifying type IIA string theory on an internal manifold X of dimension $k = 10 - d$ and volume V_{10-d}. Since this limit will be related to M-theory, we recall the dictionary between strongly coupled type IIA and M-theory, given by

$$M_s^2 = M_*^2 r_{11}\,, \quad g_s = r_{11}^{3/2}\,. \tag{3.22}$$

We consider the strong coupling limit $g_s \to \lambda g_s$ where $\lambda \to \infty$, such that the d-dimensional Planck scale $M_{\rm pl,d}$ and the size of the internal space in units of M_* remain fixed, i.e.

$$(M_{{\rm pl},d})^{d-2} = V_{10-d}\, r_{11}\, M_* \stackrel{!}{=} \text{const.}\,, \quad M_*^{10-d}\, V_{10-d} \stackrel{!}{=} \text{const.} \tag{3.23}$$

in the infinite distance limit. In terms of M-theory quantities the limit is defined as

$$r_{11} \to \lambda^{2/3}\, r_{11}\,, \quad M_* \to \lambda^{-\frac{2}{3(d-2)}} M_*\,, \quad r_I \to r_I\,, \tag{3.24}$$

where indeed all internal directions scale isotropically. One can also verify that the $(d+1)$-dimensional Planck scale $M_{{\rm pl},d+1}$ scales in the same way as M_*. With the dictionary we can translate the limit to type IIA quantities and find

$$g_s \to \lambda g_s\,, \quad M_s \to \lambda^{\frac{d-4}{3(d-2)}} M_s\,, \quad \rho_I \to \lambda^{1/3} \rho_I\,. \tag{3.25}$$

Note that $d = 4$ is special in the sense that the string scale does not scale with λ.

From the M-theory perspective, this particular type IIA strong coupling limit corresponds to decompactification from d to $d+1$ dimensions. The lightest tower of states are particle-like $D0$-branes, or equivalently KK states of the eleventh direction of mass

$$M_{D0} \simeq \frac{M_s}{g_s} \simeq \frac{M_{{\rm pl},d}}{\lambda^{\frac{2(d-1)}{3(d-2)}}}\,. \tag{3.26}$$

The next lightest states arise from wrapped $D2$-branes and $NS5$-branes, with mass scale[2]

$$M_{D2,NS5} \simeq \frac{M_s}{g_s^{1/3}} \simeq \frac{M_{{\rm pl},d}}{\lambda^{\frac{2}{3(d-2)}}} \simeq M_*\,. \tag{3.27}$$

From the M-theory perspective, these are the membranes that are *not* wrapping the M-theory circle. The KK modes along the internal directions are parametrically of the same scale as $M_{\rm KK} = M_*/r_I$. Is is easy to check that other states, like wrapped $D4$-branes or the fundamental string (membranes wrapped along the M-theory circle), are parametrically heavier. Hence, this is a typical example of a decompactification limit where the lightest tower is (dual) to a KK tower of particle-like states.

[2]In our limit, the mass/tension of an object resulting from a brane wrapping cycles in X scales like the appropriate root of the tension of an unwrapped brane.

The evidence for the ESC comes mostly from string theory examples, although there are also attempts to give a bottom-up rationale, see e.g. [113–118]. It is nevertheless remarkable that this string lamppost approach already led to more than just emergent string limits, namely the existence of decompactification limits. As we have seen, one of them is closely related to the M-theory corner of the known string duality diagram. It would be interesting to know how generic this limit is. The suspicion is that all infinite distance decompactification limits are combinations of the M-theory limit and its conventional further decompactification limits of additional compact directions.

3.2.3 Species scale

Back in section 3.1 we argued that for a large class of possible EFTs coupled to gravity the Planck mass $M_{\text{pl},d}$ sets the scale at which a UV completion to quantum gravity is enforced. However, already the weakly coupled string theory shows the relevance of another mass scale, namely the string scale M_s, which in ten dimensions is related to the Planck scale via (3.10). For fixed Planck scale and in the infinite distance limit $g_s \to 0$, the string scale can become arbitrarily small. This means that quantum gravity effects will already occur at an energy scale far below the Planck scale. Since for the weakly coupled string the lowering of the UV cutoff is accompanied by a light tower of string excitations, one could ask whether the second class of limits, the decompactifications, also has such an intermediate mass scale associated with the respective light tower.

Even with no reference to the expected UV-completion, it is possible to estimate the scale at which quantum gravity effects become important. Indeed, the general appearance of an effective quantum gravity cutoff in the presence of a large number of light states was pointed out independently of any string theory reasoning in [119–121] (see also [122] for earlier work). An upper bound for this scale can be derived by studying the quantum corrections to the graviton propagator. In d dimensions, the relative one-loop correction in momentum space is proportional to $N(E)E^2/M_{\text{pl},d}^{d-2}$, where $N(E)$ is the number of species in the spectrum with masses up to E. If this combination is of order one, the one-loop contribution is comparable to the tree-level term and perturbation theory definitely breaks down. The energy scale at which this happens is known as the *species scale*, defined as

$$\tilde{\Lambda} = \frac{M_{\text{pl},d}}{N_{\text{sp}}^{\frac{1}{d-2}}} \,. \tag{3.28}$$

The precise dependence of this scale on physical parameters of the theory is tied to the spectrum of states, but the above expression shows that even the inclusion of an incomplete spectrum does provide an upper bound. If we consider towers of states predicted by the ESC, the number of light species N_{sp} with mass below the species scale can be effectively counted as

$$N_{\text{sp}} = \#(m \leq \tilde{\Lambda}) \,. \tag{3.29}$$

Then, the latter two equations can be solved for the two unknowns $\tilde{\Lambda}$ and N_{sp}. Since in quantum gravity the masses of states depend on the vevs of moduli fields, also $N_{\rm sp}$ and $\tilde{\Lambda}$ will be moduli-dependent functions.

One could alternatively think of the species scale as the scale where quantum corrections to the leading order Einstein-Hilbert term become relevant. One way of seeing this is to define the species scale as the radius $r_0 = 1/\tilde{\Lambda}$ of the minimal-sized black hole that can be described within the EFT. The mass and Bekenstein-Hawking entropy of such a black hole are

$$M_{\rm BH} = \frac{M_{{\rm pl},d}^{d-2}}{\tilde{\Lambda}^{d-3}}\,, \quad S_{\rm BH} = \frac{M_{{\rm pl},d}^{d-2}}{\tilde{\Lambda}^{d-2}}\,. \tag{3.30}$$

The number of species is defined via the statistical entropy as

$$S_{\rm BH} = \log \Omega(M_{\rm BH}) =: N_{\rm sp}\,, \tag{3.31}$$

where $\Omega(M_{\rm BH})$ is the number of ways the macroscopic black hole of mass $M_{\rm BH}$ can be realized by the microstates. Note that this definition of the species number satisfies the relation (3.28) and has also been developed into a more complete thermodynamic picture in [114, 116].

It turns out that for a tower of light string excitations the two definitions of the species scale do not agree. Let us consider a d-dimensional EFT and compute the species scale coming from a fundamental string that is becoming asymptotically tensionless in Planck units, i.e. $M_s/M_{{\rm pl},d} \to 0$. We use the asymptotic mass formula $M_n = \sqrt{n} M_s$ and an approximation for the degeneracy[3] given by $\deg_n = e^{\sqrt{n}}$. If one invokes the perturbative argument, one has $N_s = (\tilde{\Lambda}/M_s)^2$ states below $\tilde{\Lambda}$ and $N_{\rm sp}$ from (3.29) is given by

$$N_{\rm sp} = \sum_{n=1}^{N_s} \deg_n \simeq \frac{\tilde{\Lambda}}{M_s} e^{\tilde{\Lambda}/M_s}\,, \tag{3.32}$$

where the sum over n is approximated by an integral, and numerical prefactors as well as sub-leading contributions are neglected. One can then solve the two equations (3.28) and (3.32) for $\tilde{\Lambda}$ and $N_{\rm sp}$. For $\tilde{\Lambda}$, the leading order result is

$$\tilde{\Lambda} \simeq M_s \log\left(\frac{M_{{\rm pl},d}}{M_s}\right)\,. \tag{3.33}$$

This should be compared with the result from the black hole-based definition. Now, the excitation level required for the black hole mass is

$$\sqrt{N_{\rm BH}} \simeq \frac{M_{\rm BH}}{M_s} \simeq \frac{M_{{\rm pl},d}^{d-2}}{M_s\,\tilde{\Lambda}^{d-3}}\,, \tag{3.34}$$

which for small g_s is expected to be a very large number. Then the black hole entropy is

$$S_{\rm BH} \simeq \log\left(\deg_{N_{\rm BH}}\right) \simeq \sqrt{N_{\rm BH}}\,. \tag{3.35}$$

[3] A more precise estimation is $\deg_n \sim n^\alpha e^{\beta\sqrt{n}}$, where α, β depend on the actual string theory [123].

Setting this equal to the Bekenstein-Hawking entropy (3.30) and using (3.34) gives the leading order result

$$\tilde{\Lambda} \simeq M_s\,, \qquad N_{\rm sp} \simeq \left(\frac{M_{{\rm pl},d}}{M_s}\right)^{d-2}\,. \tag{3.36}$$

Hence, the so defined species scale is equal to the string scale. Note that here we have only exploited the presence of a string tower and the self-consistency of the relations (3.30) and (3.31). This does not exclude that there might be a lower scale where the black hole dynamically undergoes a phase transition. This was discussed in the context of transitions from towers of states to (minimal) black holes in [113] and was also employed for a conjecture on the characteristic energy scales appearing in an EFT of QG in [124].

The prediction from the black hole argument, $\tilde{\Lambda} = M_s$, is consistent with the known string corrections to the Einstein-Hilbert action, which include higher derivative terms generically suppressed by the string scale M_s. The additional multiplicative log-factor in (3.33) appears to be an artifact of approximating string excitations as point particles and describing their loop-corrections in terms of QFT diagrams. The result $\tilde{\Lambda} = M_s$ is also consistent with a fairly recent proposal for a definition of the species scale that is valid across all of moduli space. For the Kähler moduli space of type IIA compactifications on CY threefolds, it was argued in [125] that the one-loop topological free energy $\mathbb{F}_1(t,\bar{t})$ provides a good measure for the number of light species, so that

$$\tilde{\Lambda} \simeq \frac{M_{{\rm pl},4}}{\mathbb{F}_1^{1/2}} \tag{3.37}$$

holds *everywhere* in the vector multiplet moduli space[4]. $\mathbb{F}_1$ is related to a certain BPS-protected R^2-correction in the type IIA effective action and receives only additive, but not multiplicative corrections [126]. In [125] the formula has been applied to the large base limit of type IIA on a $K3$-fibered CY, which has a weakly coupled heterotic dual, as discussed before. Indeed, one finds $\mathbb{F}_1 \simeq t_B$ and hence $\tilde{\Lambda} \simeq M_s$ after invoking the duality map. A bottom-up motivation for the formula (3.37) in terms of black hole arguments was provided in [127]. In [112] it was argued that the coefficients of generic higher-curvature terms in the effective action encode the moduli dependence of the species scale, which was checked in various different setups in [128, 129].

We summarize that in an emergent string limit the species scale coincides with the string scale (of the emergent string) and that there are no towers of states with a parametrically lighter mass. It is now straightforward to also apply it to the type IIA decompactification limit discussed in the previous section. In this case, it is much easier to employ the definitions (3.28) and (3.29) for the computation of the species scale. The corresponding black hole computation was presented in [130] and gives the same result. We recall that the parametrically lightest states were BPS bound states of $D0$-branes

[4]The identification made it possible to study the behavior of the species scale in the interior of moduli space, where there are no parametrically light towers and the spectrum is much more complicated.

leading to towers with masses

$$M_{D0} \simeq \frac{M_s}{g_s} \simeq \frac{M_{\text{pl},d}}{\lambda^{\frac{2(d-1)}{3(d-2)}}} \,. \tag{3.38}$$

For this non-degenerate tower the number of light species is given by the maximal KK mode $n_{\max} = \lambda^{\frac{2(d-1)}{3(d-2)}} \tilde{\Lambda}/M_{\text{pl},d}$, so that we can solve for $\tilde{\Lambda}$ to find

$$\tilde{\Lambda} \simeq \frac{M_{\text{pl},d}}{\lambda^{\frac{2}{3(d-2)}}} \simeq M_{\text{pl},d+1} \simeq M_* \,. \tag{3.39}$$

As expected for decompactification limits, the species scale is given by the Planck scale of one dimension higher, which here scales in the same way as the 11D Planck scale. In the infinite distance limit $\lambda \to \infty$ the species scale goes to zero, signaling that the d-dimensional theory breaks down and one has to describe the theory in $(d+1)$ dimensions. However, since $\tilde{\Lambda}$ is parametrically larger than the mass scale of the $D0$-brane tower, there could still be towers of states with a mass scale below $\tilde{\Lambda}$. But as we have discussed, the next lightest states are $D2$- and $NS5$-branes as well as KK modes along internal directions, whose mass scale scales precisely like $\tilde{\Lambda}$. Therefore, they do not further lower the species scale, which is indeed given by the higher-dimensional Planck scale.

In [131] a similar analysis was also done for the co-scaled strong coupling limit of the 10D type IIB superstring. As expected, in this case the lightest towers are the string towers of the $D1$-branes, so that the species scale is nothing else than the $D1$-string mass scale $\tilde{\Lambda} \simeq T_{D1}^{1/2} \simeq M_{\text{pl},10}/g_s^{1/4}$ with all other mass scales parametrically larger than $\tilde{\Lambda}$. The analogous result holds for the co-scaled type IIB limit in lower dimensions with the resulting species scale $\tilde{\Lambda} \simeq T_{D1}^{1/2} \simeq M_{\text{pl},d}/\lambda^{2/(d-2)}$.

3.3 The Emergence Proposal

We have seen that a genuine feature of QG is the existence of infinite distance limits coming in two different types, namely emergent string and decompactification limits. These infinite distance boundaries of the moduli space are one aspect of the *kinematic* properties of a gravitational theory, which involve all structural data of the underlying effective description. Other examples are the field content, symmetries, charges and representations etc. A second important pillar are *dynamical* properties, describing how those fields interact and how states evolve by the equations of motion. In QFT the distinction between those two notions is sharp, since one first specifies the field content and symmetries, and then studies the resulting dynamics. QG on the other hand blurs the line between kinematic and dynamical aspects. For instance, we saw that couplings in the EFT depend on the vevs of moduli fields. The latter can only be fixed to a certain value by generating a potential with some dynamical ingredients, such as p-form fluxes along the compact manifold or non-perturbative effects. Hence, what looks like kinematic input in the EFT is in fact dynamically generated in QG. This is in line with the swampland philosophy, since the UV complete theory thereby imposes consistency conditions on EFTs. Then the question arises whether the EFT restrictions captured by swampland conjectures can be explained in terms of the dynamics in QG.

In this context, in [132–134] an interesting observation was made, namely that the metric on moduli space can be recovered by integrating out the tower of asymptotically massless states at one-loop. This observation in some sense reverses the logical arrow in the SDC, since the infinite distance limit occurs because towers of exponentially light states generate the appropriate divergent metric through loop effects. As a simple toy model, consider a light modulus ϕ and a tower of massive KK states h_n with mass $m_n(\phi) = n\Delta m(\phi)$ governed by a d-dimensional effective action

$$S = M_{\mathrm{pl},d}^{d-2} \int d^d x \left(\frac{1}{2} G_{\phi\phi} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \sum_n \partial_\mu h_n \partial^\mu h_n + \frac{1}{2} m_n^2(\phi) h_n^2 \right) , \tag{3.40}$$

where $G_{\phi\phi}$ denotes the metric on field space. At a particular value ϕ_0 one can expand $\phi = \phi_0 + \delta\phi$ and obtain the three-point coupling

$$y_n = \lambda_n h_n^2 \delta\phi \, , \quad \lambda_n = m_n \partial_\phi m_n |_{\phi=\phi_0} \tag{3.41}$$

after expanding the mass term of the h_n fields. The one-loop contribution to the ϕ propagator, with the massive modes h_n running in the loop, can be computed by using Feynman rules. The result is

$$\Pi_n(p^2) = \frac{\lambda_n^2}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m_n^2)} \frac{1}{((q-p)^2 + m_n^2)} \tag{3.42}$$

for external momentum p and internal momentum q. The correction to the field-space metric (corresponding to the wave-function renormalization) is given by the term linear in p^2 in (3.42). Integrating out these modes up to the UV-cutoff, taken to be the species scale $\tilde{\Lambda}$, leads to the leading order one-loop correction (see e.g. [135] for more details)

$$G_{\phi\phi}^{1-\mathrm{loop}} = \sum_{m_n \leq \tilde{\Lambda}} \frac{\partial \Pi_n(p^2)}{\partial p^2}\Big|_{p^2=0} \simeq \frac{\tilde{\Lambda}^{d-1}}{M_{\mathrm{pl},d}^{d-2}} \frac{(\partial_\phi \Delta m(\phi))^2}{(\Delta m(\phi))^3} + \dots \, . \tag{3.43}$$

To be consistent with the sum over states up to mass $\tilde{\Lambda}$, one also performs the momentum integral (3.42) only up to the cutoff scale $\tilde{\Lambda}$. For a KK tower with $\Delta m = M_{\mathrm{pl},d}/r$, the species scale is the $(d+1)$-dimensional Planck scale, i.e. $\tilde{\Lambda}^{d-1} \simeq M_{\mathrm{pl},d}^{d-1}/r$. Then we find $G_{rr}^{1-\mathrm{loop}} \simeq 1/r^2$, which has the same functional dependence on the modulus r as the tree level metric, $G_{rr}^{(0)}$, resulting from the dimensional reduction of the Einstein-Hilbert term[5]. Even though this is just a simple toy model, the above was considered quite a remarkable correlation leading to the formulation of the so-called *Emergence Proposal*:

Emergence Proposal (Strong): The dynamics (kinetic terms) for all fields are emergent in the infrared by integrating out towers of states down from an ultraviolet scale $\tilde{\Lambda}$, which is below the Planck scale.

[5] In [135] one can find similar discussions on the generation of gauge and fermionic kinetic terms. In [136] the results have been used to study implications for SM hierarchies.

There have been slightly different formulations and also a weak version [135], but for the purpose of this presentation let us stick to this version formulated in the review [78]. As it stands, this proposal is very generic and it is not clear what its realm of validity could be. Of course, it is certainly meant in the context of QG and the swampland program, where one usually identifies the UV cutoff scale Λ_{UV} with the species scale, i.e. $\Lambda_{\mathrm{UV}} \simeq \tilde{\Lambda}$. Moreover, the towers of states to be integrated out are those described in the previous sections and which become light in infinite distance limits of the moduli space.

In the previous toy example, a hard UV cutoff for the one-loop integral was introduced, i.e both internal momenta and the mass of the states running in the loop were cut off at the species scale. However, when extending this proposal beyond the pure EFT setting to theories of QG, the example of the fundamental string tells us that one should not cut off the loop-integrals at a finite energy scale and keep only the string modes with masses below that scale. In fact, such string loop amplitudes have nice UV properties precisely by including an infinite tower of states. The simplest example illustrating this feature is the one-loop vacuum energy of bosonic string theory. The amplitude is obtained by integrating the partition function (2.35), which traces over the full Hilbert space of string excitations, over the moduli space of inequivalent tori. The latter is given by the quotient $\mathcal{M} = \mathbb{C}/SL(2,\mathbb{Z})$, where $SL(2,\mathbb{Z})$ acts via (2.34). The final result for the amplitude is

$$\mathcal{A}_0^{1-\mathrm{loop}} = \frac{1}{l_s^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} \,. \tag{3.44}$$

Note that the integrand as well as the measure $d^2\tau/d\tau_2^2$ are separately invariant under $SL(2,\mathbb{Z})$. The integration is carried out over the fundamental domain $\mathcal{F}$, defined as

$$\mathcal{F} = \left\{ \tau \in \mathbb{H}^+ \middle| -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2} \,,\; |\tau| \geq 1 \right\} . \tag{3.45}$$

A proof showing that $\mathcal{F}$ covers all $SL(2,\mathbb{Z})$-inequivalent tori can be found in [137]. The integration over the fundamental domain (see figure 3.2) in string theory is analogous to loop integrals over momenta in QFT, with the $\mathrm{Im}(\tau) = \tau_2 \to 0$ region corresponding to the UV-limit of loop momenta in Feynman diagrams. In string theory this putative

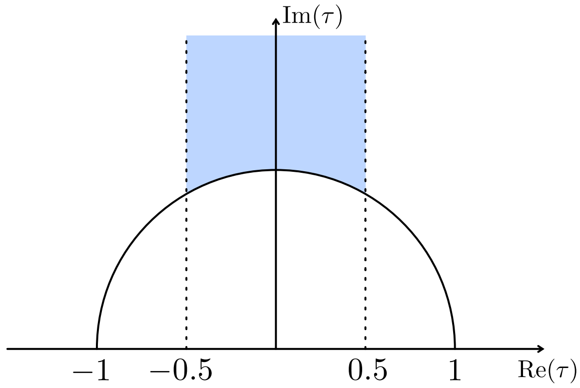


Figure 3.2: Fundamental domain $\mathcal{F}$ (blue) of the moduli space of a torus.

source of UV-divergence is absent, as modular transformations relate the limits $\tau_2 \to 0$ and $\tau_2 \to \infty$, the latter being identified with the IR regime. Thus, the theory never explores arbitrarily short distances and makes amplitudes UV-finite without imposing a hard cutoff.

Hence, despite the treatment of the simple toy example, in the Emergence Proposal it is implicitly meant that one really integrates out the full infinite tower of states. This is also compatible with the calculations of [78]. Then the question of which towers one has to integrate out arises: only the lightest one or even all conceivable towers? The latter option can be excluded, as in weakly coupled string theory, one only integrates out those towers with a mass scale M_s. A lesson we have drawn from the swampland conjectures is that in infinite distance limits there will generally be a hierarchy among the masses of states. More precisely, for a limit $\lambda \gg 1$ there is a naturally small parameter $g = \lambda^{-1} \ll 1$ such that the masses scale as

$$m_{\text{pert}} \simeq g^{\alpha}\Lambda\,, \quad m_{\text{class}} \simeq \frac{\Lambda}{g^{\beta}}\,, \tag{3.46}$$

where $\alpha \geq 0, \beta > 0$ and Λ is a characteristic mass scale in the limit. This categorization is well known from QFT, where one distinguishes classical states, corresponding to saddle points of the path integral, and perturbative states, given by the fluctuations around a saddle. The weakly coupled string limit $g_s = e^{\Phi} \ll 1$ suggests that the scale in (3.46) should be identified with the species scale, $\Lambda = \tilde{\Lambda}$. Then the oscillation modes (and potentially KK and winding modes) of the fundamental string are the perturbative degrees of freedom, while all p-branes with tensions $T_p \simeq M_s^{p+1}/g_s^{\beta}$ with $\beta = 1, 2$ are non-perturbative objects. Indeed, in the perturbative regime the p-branes correspond to classical saddles of the string path integral and act as a source for closed string states. This can be made very explicit for Dp-branes with $\beta = 1$, since those can be described as a coherent state of closed string states (boundary states in the CFT language, see [16]). Recall that a Dp-brane with open strings attached on it is interpreted as an excitation of the brane. As these vibration modes are of the scale of the brane tension itself, they are heavy and freeze out in the $g_s \to 0$ limit.

Given the classification of infinite distance limits by the Emergent String Conjecture, one can ask whether the Emergence Proposal should be realized in both types of limits after all. We think that for emergent string limits the answer is negative, for the following reasons. First, we notice that even the naive computation of (3.43) for a string tower does not give correct leading order results, as the previously mentioned multiplicative log-terms in the species scale are transferred to the one-loop corrections [130]. Second and on a more fundamental level, we know how to quantize the weakly coupled fundamental string such that, similar to QFT, one obtains a loop expansion in terms of higher genus Riemann surfaces. Here, e.g. the one-loop Schwinger integral over the tower of string states gives (tautologically) only the one-loop correction to certain terms in the low energy effective action. Hence, none of the tree-level terms in the g_s expansion are generated from quantum effects. This will also be true for other emerging string limits, like the one discussed in the type IIA Kähler moduli space for a $K3$-fibered Calabi-Yau, as these are conjectured to be dual to a fundamental string.

Emergence in decompactification limits

With the emergent string limit excluded, it is only the decompactification limit that remains, of which the M-theory limit is the typical example and perhaps the only non-trivial one. To see this, consider a d-dimensional EFT obtained by compactifying a 10D string theory on a k-dimensional space X with $k = 10 - d$ and volume V_k. Suppose the internal space factorizes as $X = S^1_R \times X_{9-d}$. If we decompactify the radius $R \to \infty$ while keeping $g_s \ll 1$ and the (string unit) volume $\mathcal{V}_\perp$ of the space X_{9-d} fixed, the lightest tower is certainly the KK tower with mass scale $m_{\text{KK}} = 1/R$ and the induced species scale is the $(d+1)$-dimensional Planck scale, as expected. However, the theory still has light strings, and their mass scale is related to the species scale via

$$\tilde{\Lambda} = R^{-\frac{1}{d-1}} M_{\text{pl},d}^{\frac{d-2}{d-1}} \simeq M_s \left(\frac{\mathcal{V}_\perp}{g_s^2}\right)^{\frac{1}{d-2}} , \tag{3.47}$$

where we have used the general relation (3.10) in the second step. For $g_s \ll 1$ and $\mathcal{V}_\perp > 1$, $\tilde{\Lambda}$ is larger than the string scale. Therefore, as expected, such a limit is just a higher dimensional perturbative string theory. Even though the lightest states are given by KK towers, the QG theory is described by quantized strings and as for the aforementioned emergent string limit, the Emergence Proposal is not realized[6].

As we have seen, the M-theoretic decompactification limit is of a different type as string towers are heavier than the species scale. The QG theory of M-theory is arguably one of the deepest mysteries and only partial results are available at present, like a formulation in terms of $D0$-branes by the BFSS matrix model, which be briefly discussed in section 2.4.2. Recall that in this model the interaction between gravitons was found to be absent classically and only generated via quantum (loop) effects. This can indicate that there is a good chance for the M-theory limit to be the natural home of the Emergence Proposal. In this spirit, from our discussion we extrapolate a lesson in the form of an M-theoretic refinement of the Emergence Proposal:

> *Emergence Proposal (M-theory): In the infinite distance M-theory limit $M_* R_{11} \gg 1$, with the Planck scale kept fixed, a perturbative QG theory arises whose low energy effective description emerges via quantum effects by integrating out the full infinite towers of states with a mass scale parametrically not larger than the 11D Planck scale. These are transverse $M2$-, $M5$-branes carrying momentum along the eleventh direction ($D0$-branes) and along any potentially present compact direction.*

Note that in this limit the longitudinally wrapped $M2$-brane, corresponding to the type IIA fundamental string, and the longitudinally wrapped $M5$-brane, corresponding

[6] There have been examples of the emergence proposal not being straightforwardly realized in such decompactification limits, like the partial emergence of certain quartic gauge couplings analyzed in [138].

to the type IIA $D4$-brane, have masses

$$M_{F1} = \frac{M_*}{g_M^{1/2}}, \quad M_{D4} = \frac{M_*}{g_M^{1/5}} \tag{3.48}$$

with the formal coupling constant $g_M = 1/(M_* R_{11}) \ll 1$. Hence, they are not among the light modes and are considered as non-perturbative classical objects. We note that the BFSS matrix model is indeed containing these two longitudinal branes as bound states of $D0$-branes, which one might speculate to be the analogue of the description of non-perturbative D-branes as coherent (boundary) states of weakly coupled closed string modes. However, the transverse $M5$-brane was lacking in the original BFSS matrix model which in view of the Emergence Proposal might indicate that it is not yet the complete description of quantum M-theory.

Evidence

The central challenge in providing evidence for the proposal is that we do not understand the full quantization of M-theory, yet. In addition, as M(matrix) theory teaches us, the leading order supergravity action at the second derivative level should emerge via loops, and hence also space-time itself should be somehow emergent. The loophole bypassing these difficulties is that there are certain couplings in the effective action that are protected by supersymmetry (non-renormalization theorems) and only receive contributions from 1/2 BPS states. These states are under good control and, to a certain extent, can already be reliably described by their weak string coupling counterparts, i.e. in the weakly coupled type IIA theory. Hence, this sector of M-theory is special and admits the usual geometric interpretation we are used to from string theory.

Current indications for the M-theoretic Emergence Proposal are mainly based on the evaluation of those 1/2 BPS saturated couplings. Explicit checks were performed by deriving such couplings from a one-loop Schwinger integral, where one integrates out solely the light, perturbative towers of particle-like states with masses not larger than the species scale, i.e. the 11D Planck scale. The following quantities were investigated:

- R^4-term in supergravity theories with 32 supercharges, arising from toroidal compactifications of type IIA/M-theory[7]. The analysis builds on the pioneering work of Green-Gutperle-Vanhove (GGV) [140], in which the couplings in 10D and 9D were obtained by integrating out the KK spectrum along the eleventh direction, i.e. bound states of $D0$-branes. This was generalized in [141] (see also [142–146] for closely related work). In [147] we showed that the transverse $M2$-brane, which can be particle-like only for $d \leq 8$, yields the constant 1-loop term in the small g_s expansion (in the work by GGV, this term was "added by hand" to preserve modular invariance). From the decompactification of the 8D result one deduces the presence of the same constant in 9D and 10D. For $d \geq 7$, it was shown explicitly that the full

[7] This term has also received attention in discussions of the emergence of species scale black hole horizons [139].

coupling in the small g_s expansion, including worldsheet and spacetime instanton contributions, is recovered from the M-theory Schwinger integral.

- F^4-term in 6d supergravity with 16 supercharges, obtained by compactifying type IIA on a $K3$ surface or the dual heterotic string on a T^4. At a special orbifold point $(T^4/\mathbb{Z}_2)$ in the $K3$ moduli space, one can apply a concrete duality map to relate the moduli and masses of BPS states in those models. The work [148] focuses on the F^4-coupling for a linear combination of 16 $U(1)$ gauge fields, each arising from the dimensional reduction of the RR 3-form C_3 along a two-sphere. With a careful case-by-case study of the worldsheet instanton corrections on the heterotic side, it was possible to show that $D4$-brane contributions are indeed redundant, in the sense that they lead to mutually canceling terms.

- the topological string couplings $\mathcal{F}_0$ and $\mathcal{F}_1$ in 4d $N = 2$ supergravity (8 supercharges). These are related to F-terms in the effective action of type IIA compactified on a CY threefold. Note that $\mathcal{F}_0$ contains information about the second order supergravity action, namely about the gauge couplings and kinetic terms of the vector multiplet fields. Also here we build on important seminal work [149, 150], where the authors expressed the topological string amplitudes $\mathcal{F}_g$ for $g \geq 0$ in terms of a Schwinger integral, integrating out particle-like bound states of $D0$- and $D2$-branes. Our checks of the M-theoretic Emergence proposal in this specific setup will be described in chapters 4 and 6.

Chapter 4

Emergence for the resolved conifold

In the following we present the first concrete evidence for the M-theoretic Emergence Proposal. The setup we work in is the Kähler moduli space of type IIA string theory compactified on a CY threefold. In this chapter we will focus on the resolved conifold, which is a non-compact CY. We will show that the string tree-level prepotential and the string one-loop correction can be obtained exactly by integrating out the relevant perturbative degrees of freedom in the isotropic M-theory limit. Here we make use of the seminal work by Gopakumar and Vafa, where it was shown that the above quantities can be expressed as one-loop Schwinger-like integrals with BPS particles running in the loop. Before we dive into the emergence computation, we will first review how effective actions can be obtained using Schwinger proper time by discussing a field-theoretic example. Then we will explain how this method was generalized to the supersymmetric string theory setup.

4.1 One-loop effective actions

We begin with a short review of the Schwinger proper time representation of effective actions in quantum field theory. The most famous example in this context is the Euler-Heisenberg Lagrangian, which describes the non-linear dynamics of a slowly varying electromagnetic field after the electron has been integrated out from the QED Lagrangian. With the imprint of the fermion the Euler-Heisenberg Lagrangian takes into account the polarization of the QED vacuum to one-loop, predicting several phenomena like the spontaneous production of $e^+ - e^-$ pairs ("Schwinger effect") or light-by-light scattering. We want to look at a closely related and slightly simpler model, called scalar QED, where instead of a fermion we consider a massive complex scalar field. The reason is that the Gopakumar-Vafa calculation very much resembles the one from scalar QED, which will become apparent at a later stage. In the following we will explain the key steps in the derivation of the effective action of scalar QED and show the explicit solution for a constant electromagnetic field. We will mostly follow [151], which provides a very pedagogical and thorough discussion of both scalar and fermionic case.

Starting from the scalar QED action, we want to derive an effective action $\Gamma[A]$ for the photon (A_μ) by integrating out the scalar (ϕ) with mass m in the path integral, i.e.

$$\int \mathcal{D}A \exp(i\Gamma[A]) = \int \mathcal{D}A\, \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\left[i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 - \phi^*(D^2 + m^2)\phi\right)\right], \tag{4.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$. Since the action is quadratic in ϕ, the path integral over the scalar is Gaussian and yields

$$\int \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\left(i \int d^4x\, \phi^*(-D^2 - m^2)\phi\right) = \mathcal{N}\, \frac{1}{\det(-D^2 - m^2)}\,, \tag{4.2}$$

where $\mathcal{N}$ is some infinite normalization constant which we will not keep track of in the following. The determinant can be easily converted into a sum by taking the logarithm on both sides of (4.1), yielding

$$i\Gamma[A] + i \int d^4x \left(\frac{1}{4}F_{\mu\nu}^2\right) = -\log[\det(-D^2 - m^2)] = -\mathrm{Tr}[\log\big(-D^2 - m^2\big)]\,. \tag{4.3}$$

Here, Tr stands for a trace of the operator $(-D^2 - m^2)$ over position or momentum eigenstates. We proceed by differentiating the effective action with respect to m^2, which gives us

$$\frac{d}{dm^2}\Gamma[A] = -i\, \mathrm{Tr}\left(\frac{1}{-D^2 - m^2}\right), \tag{4.4}$$

Now we can apply Schwinger's parametrization of an operator A given by

$$\frac{i}{A + i\epsilon} = \int_0^\infty ds\, e^{is(A+i\epsilon)}\,. \tag{4.5}$$

After applying this identity to (4.4), we integrate the latter over m^2 and (up to irrelevant constants) obtain

$$\Gamma[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2\right) - i \int_0^\infty \frac{ds}{s} e^{-s\epsilon} e^{-ism^2}\, \mathrm{Tr}\left(e^{-iD^2 s}\right)\,. \tag{4.6}$$

In terms of momentum variables, the operator inside the exponential has the form

$$D^2 \equiv \hat{H} = -(\hat{p}_\mu - eA_\mu(\hat{x}))^2\,. \tag{4.7}$$

We can interpret $\hat{H}$ as a Hamiltonian describing a quantum particle minimally coupled to the gauge field A_μ and evolving it in Schwinger proper time s. $\hat{H}$ should not be mistaken with the Hamiltonian of the actual field that we are integrating out, but the analogy is useful to re-interpret and rewrite the effective action (4.6). If we evaluate the trace in position space by writing

$$\mathrm{Tr}\left(e^{-iD^2 s}\right) = \int d^4x\, \langle x|\, e^{-i\hat{H}s}\, |x\rangle\,, \tag{4.8}$$

the term $\langle x; 0|x; s\rangle \equiv \langle x| e^{-i\hat{H}s} |x\rangle$ inside the integral corresponds to the amplitude for the particle to go around a closed loop in proper time s when it evolves according to $\hat{H}$. The operator (4.7) can be decomposed as $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is the free part and $\hat{V} = 2eA_\mu\partial_\mu - e^2A_\mu^2$. One can expand the exponential $e^{-i\hat{H}s}$ in the Dyson series

$$e^{-i\hat{H}s} = e^{-i\hat{H}_0 s}\left[1 - isV - \frac{s^2}{2}V^2 + \dots\right], \tag{4.9}$$

which is perturbative if as long as the electromagnetic field is weakly coupled and slowly varying. In the language of Feynman diagrams, we are summing over closed loops with an arbitrary number of external photons legs, as it's depicted in figure 4.1.

$$\Gamma[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2\right) +$$

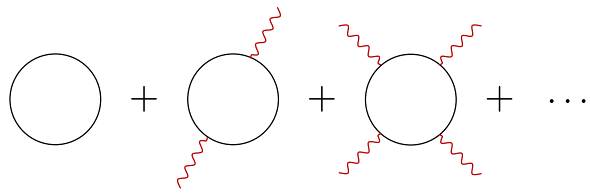


Figure 4.1: One-loop diagrams encoded in the effective action $\Gamma[A]$ of the photon after integrating out the massive scalar. Red lines represent external photon states.

Computing the trace for constant background

It is possible to derive an analytic expression for the effective action by assuming that the field strength $F_{\mu\nu}$ is constant, i.e. $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\mu$. In this case, one can always rotate to a Lorentz frame where the electric and magnetic field are parallel. For concreteness, we may choose $\vec{E} = E\hat{z}$ and $\vec{B} = B\hat{z}$, implying $F_{03} = -E$ and $F_{12} = B$. With this convenient choice we obtain two decoupled systems in orthogonal two-dimensional planes, namely the magnetic sector in the (x^1, x^2)-plane and the electric sector in the (x^0, x^3)-plane. As a consequence, the trace decomposes into

$$\mathrm{Tr}\left(e^{-i\hat{H}s}\right) = \mathrm{Tr}_{\mathrm{m}}\left(e^{-i\hat{H}_m s}\right)\,\mathrm{Tr}_{\mathrm{el}}\left(e^{-i\hat{H}_{\mathrm{el}}s}\right), \tag{4.10}$$

where $\mathrm{Tr}_{\mathrm{m/el}}$ denotes the trace over the Hilbert space of the respective subsector and the Hamiltonians are given by

$$\begin{aligned} \hat{H}_m &= \left(\hat{p}_x + \frac{eB}{2}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{eB}{2}\hat{x}\right)^2, \\ \hat{H}_{\mathrm{el}} &= \left(-\hat{p}_t + \frac{eE}{2}\hat{z}\right)^2 + \left(\hat{p}_z - \frac{eE}{2}\hat{t}\right)^2. \end{aligned} \tag{4.11}$$

Let us discuss the magnetic sector explicitly. A quick way to evaluate the trace is to work in the basis of orthonormal eigenstates $\{\psi_{\vec{n}}^{(m)}\}$ of $\hat{H}_m$. If we work in position space, the trace becomes

$$\int d^4x \,\langle x| e^{-i\hat{H}_m s} |x\rangle = \int d^4x \sum_{\vec{n}} \langle x\rangle\, \psi_{\vec{n}}^{(m)} \left\langle \psi_{\vec{n}}^{(m)} \middle| e^{-i\hat{H}_m s} \middle| x\right\rangle = \sum_{\vec{n}} e^{-iE_{\vec{n}}s}, \tag{4.12}$$

where $E_{\vec{n}}$ are the eigenenergies and $\vec{n}$ denotes a sum over all quantum numbers. The magnetic Hamiltonian describes a plain two-dimensional harmonic oscillator whose spectrum can be easily computed by introducing canonical momenta $\hat{\Pi}_i = \hat{p}_i - e\hat{A}_i$ and introducing the ladder operators

$$\hat{a} = \frac{1}{\sqrt{2eB}}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right)\,, \quad \hat{a}^\dagger = \frac{1}{\sqrt{2eB}}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right) \tag{4.13}$$

which satisfy $[\hat{a}, \hat{a}^\dagger] = 1$. The spectrum consists of the well-known Landau energies

$$E_n = 2eB\left(n + \frac{1}{2}\right) \quad \text{with} \quad n \in \mathbb{N}\,. \tag{4.14}$$

Note that each Landau level has an infinite degeneracy associated to its angular momentum. Assuming that the particle is confined to a box with area V_{xy}, one can show explicitly that the degeneracy per Landau level is $\deg_n = eB\, V_{xy}/(2\pi)$ [152]. To evaluate the trace, one then simply needs to sum over all Landau levels and include their degeneracies. For the magnetic sector we then obtain

$$\mathrm{Tr}_\mathrm{m}\left(e^{-i\hat{H}_m s}\right) = V_{xy}\frac{eB}{2\pi}\sum_{n=0}^{\infty} e^{-ieB(2n+1)s} = -iV_{xy}\frac{eB}{4\pi}\frac{1}{\sin(seB)}\,. \tag{4.15}$$

The electric sector can be treated in a completely analogous way by performing a Wick-rotation $t \to -i\tau$, where τ denotes Euclidean time. We can immediately obtain the result by making the substitutions $B \to iE$ (and $V_{xy} \to -iV_{\tau z}$) in (4.15). The combined trace can then be written as

$$\mathrm{Tr}\left(e^{-i\hat{H}s}\right) = V_4\frac{ie^2}{16\pi^2}EB\frac{1}{\sinh(seE)\sin(seB)}\,, \tag{4.16}$$

where $V_4 = V_{xy}V_{\tau z}$ is the total spacetime volume. One can express this result in terms of the Lorentz-invariant quantities $F_{\mu\nu}^2 = 2(B^2 - E^2)$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = 4EB$, where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. If we introduce the combination $X^2 \equiv \frac{1}{2}(F_{\mu\nu}^2 + iF_{\mu\nu}\tilde{F}^{\mu\nu})$, we can bring the effective Lagrangian from (4.6) into the form

$$\mathcal{L}_\text{eff} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{e^2}{64\pi^2}\int_0^\infty \frac{ds}{s}e^{-sm^2}\frac{1}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{4.17}$$

where we have also rotated $s \to -is$ to make the integral manifestly real. This action contains a UV-divergent term from the small proper time region, which can be made explicit by expanding

$$\frac{1}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{4}{e^2s^2} + \mathcal{O}(s)\,. \tag{4.18}$$

We see that the only UV-divergent term is independent of $F_{\mu\nu}$, hence it corresponds to the vacuum energy of the system (first diagram in figure 4.1).

Path integral interpretation

We have a second option to interpret and evaluate the operator trace (4.8) by using the path integral formalism. In quantum mechanics, the transition amplitude of a particle to go from x_i to x_f can be computed by summing over all possible paths, where each path is weighed by the appropriate action with the factor e^{iS}, i.e.

$$\langle x_f|\, e^{-i\hat{H}(t_f-t_i)}\, |x_i\rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{iS[x(t)]}\,. \tag{4.19}$$

In view of the upcoming discussion, we would like to consider the path integral in a space with Euclidean signature. For that we Wick-rotate $t \to -i\tau$, which transforms the action $S[x(t)] \to iS_E[x(\tau)]$. The trace imposes to identify the initial and final points and to integrate over all positions, which leads to the formula

$$\mathrm{Tr}(e^{-\hat{H}s}) = \int d^4x\, \langle x|\, e^{-\hat{H}s}\, |x\rangle = \int_{x(0)=x(s)} Dx(\tau) e^{-S_E[x(\tau)]}\,, \tag{4.20}$$

where we introduced the Schwinger proper time $s = \tau_f - \tau_i$ to connect to previous expressions. We can implement the boundary condition from the trace equivalently by working on an Euclidean circle with period $\tau \simeq \tau + s$. We then sum over all maps $x^\mu(\tau) : \tau \to \mathbb{R}^4$ with $x(\tau) = x(\tau + s)$ assuming that the particle loop winds once around the circle[1]. In order to reproduce the Euclidean version of (4.16), we would need to carry out the path integral using the action

$$S_E[x(\tau)] = \int_0^s d\tau \left[\frac{1}{2}\dot{x}^\mu \dot{x}_\mu - ie\, F_{\mu\nu}\dot{x}^\mu x^\nu\right]\,, \tag{4.21}$$

again with $F_{\mu\nu}$ constant. This is a standard textbook problem, see e.g. [153]

4.2 The Gopakumar-Vafa formula

In the previous section we saw how effective actions in quantum field theory can be reinterpreted as sums of one-loop diagrams and one can even obtain analytic expressions for suitably chosen backgrounds. Remarkably, these ideas resurface in the context of type IIA compactifications on CYs, where one can express special terms of the $N = 2$ supergravity action as sums over one-loop diagrams through Schwinger-like integrals. This idea was put forward in the pioneering work by Gopakumar and Vafa [149, 150]. The purpose of this section is to explain what exactly is computed with the Gopakumar-Vafa formula and which assumptions have to be made to make the calculation work.

[1] As is usual in the worldline formalism, we will always assume that the wordline metric has been gauge-fixed to a constant size. We then fix the worldline length and integrate over all possible loops parametrized by the maps we have just mentioned.

4.2.1 Setup and general idea

We want to study the effective action of a type IIA compactification on a CY threefold Y with Hodge numbers (h_{11}, h_{21}), which is described by four-dimensional $N = 2$ supergravity (see section 2.3.3). Recall that the four-dimensional theory has $(h_{11}+1)$ $U(1)$ gauge fields in total: the gauge fields A^i_μ with $i = 1, \ldots, h_{11}$ arising from the ten-dimensional RR 3-form with two legs along compact directions and one additional field A^0_μ coming from the RR 1-form. These fields are parts of supermultiplets with bosonic field content

$$\begin{aligned} \{g_{\mu\nu}, V_\mu\} \quad & \text{in supergravity multiplet}\,, \\ \{t^i, A^i_\mu\} \quad & \text{in vector multiplets}\,, \end{aligned} \tag{4.22}$$

where the $t^i = b^i + i\tau^i$ are the complexified Kähler moduli, $g_{\mu\nu}$ is the 4d metric tensor and V_μ is the graviphoton, which is a particular linear combination of the gauge fields (A^0_μ, A^i_μ) depending on the supersymmetric background. In the following the field strengths of the fields A^I_μ with $I = (0, i)$ will be denoted by $F^I_{\mu\nu}$ and the field strength of the graviphoton will be $W_{\mu\nu}$. As was reviewed already, the scalars in the vector multiplets are usually described in terms of the projective coordinates X^I defined via $2\pi t^i = X^i/X^0$. The redundant component is fixed by imposing the conformal gauge constraint

$$i(\bar{X}^I \mathcal{F}_I - \mathcal{F}_I X^I) = 1 \quad \text{with} \quad \mathcal{F}_I = \frac{\partial \mathcal{F}_0}{\partial X^I}\,, \tag{4.23}$$

where $\mathcal{F}_0$ is the supergravity prepotential. It is a holomorphic function of the X^I of homogeneous degree 2.

The core objects we are interested in are defined most conveniently by using superspace formalism, which is an extension of spacetime by fermionic coordinates. The superspace of a theory with $N = 2$ supersymmetry is spanned by four bosonic coordinates x^μ, four fermionic coordinates θ^{Ai} of negative chirality and four $\theta^{\dot{A}i}$ of positive chirality. While generic superfields can depend on all these coordinates, there can be fields depending only on a subset of fermionic variables. In our conventions these are

$$\Psi = \Psi(x, \theta) \quad \text{chiral superfields}\,, \quad \tilde{\Psi} = \tilde{\Psi}(x, \bar{\theta}) \quad \text{anti-chiral superfields}\,. \tag{4.24}$$

Each of the h_{11} vector multiplets are described by a chiral superfield of spin 0, denoted as $\mathcal{X}^I$. The lowest components are given by

$$\mathcal{X}^I = X^I + \ldots + \frac{1}{2}\epsilon_{ij}\bar{\theta}^i \sigma^{\mu\nu}\theta^j \mathcal{F}^I_{\mu\nu} + \ldots + \frac{1}{6}(\epsilon_{ij}\bar{\theta}^i\sigma^{\mu\nu}\theta^j)^2 D_\mu D^\mu \bar{X}^I\,, \tag{4.25}$$

where we have only written out the bosonic components. Here D_μ is a covariant derivative and the field strengths $\mathcal{F}^I_{\mu\nu}$ are defined as

$$\mathcal{F}^I_{\mu\nu} = F^I_{\mu\nu} - \frac{1}{2}\bar{X}^I W^-_{\mu\nu}\,. \tag{4.26}$$

where $W^-_{\mu\nu}$ is the antiself-dual part of the graviphoton field strength. The field content of the supergravity multiplet can be packed in a chiral multiplet of spin 1 (also known as a Weyl multiplet). The expansion in superspace coordinates reads

$$\mathcal{W}_{\mu\nu} = W^-_{\mu\nu} - R^-_{\mu\nu\rho\lambda}\epsilon_{ij}\bar{\theta}^i\sigma^{\lambda\rho}\theta^j + \ldots\,, \tag{4.27}$$

where $R^-_{\mu\nu\rho\lambda}$ is the anti-self dual part of the Riemann tensor. In order to have a supersymmetric background the supersymmetry variations of (4.25) and (4.27) have to vanish. This requires

$$F^I_{\mu\nu} = \frac{1}{2}\bar{X}^I W^-_{\mu\nu} \quad \text{and} \quad R^-_{\mu\nu\rho\lambda} = 0\,. \tag{4.28}$$

We have already seen that the kinetic terms and interactions of vector multiplet fields at the two-derivative level are encoded in the prepotential F_0. In the $N = 2$ superspace formalism the associated part of the effective action is a so-called F-term, which is a term that can be written has an integral over half of fermionic superspace coordinates, i.e.

$$S_0 = -\frac{M^2_{\text{pl},4}}{4\pi^2}\int d^4x\, d^4\theta \sqrt{-g^E}\, \mathcal{F}^{(S)}_0(\mathcal{X}^{\mathcal{I}})\,. \tag{4.29}$$

Here $\mathcal{F}^{(S)}_0(\mathcal{X}^{\mathcal{I}})$ is a holomorphic function of the chiral superfields $\mathcal{X}^{\mathcal{I}}$. If one performs the integral over the θ coordinates one recovers the prepotential from special Kähler geometry. In contrast to D-terms, which are defined as integrals over all of superspace, F-terms are holomorphic and protected from perturbative corrections. In the case at hand there can be more general F-terms depending on the superfields $\mathcal{X}^I, \mathcal{W}_{\mu\nu}$. For every integer $g \geq 0$ one can construct the general term

$$S_g = -\frac{M^2_{\text{pl},4}}{4\pi^2}\int d^4x\, d^4\theta \sqrt{-g^E}\, \mathcal{F}^{(S)}_g(\mathcal{X}^I)\left(\frac{1}{16}\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\right)^g\,, \tag{4.30}$$

where $\mathcal{F}^{(S)}_g(\mathcal{X}^I)$ is the genus-g (supergravity) prepotential, also holomorphic in the $\mathcal{X}^I$. To interpret these F-terms in type IIA string theory one can transform a generic S_g term to the string frame and find that the result is proportional to g_s^{2g-2}. This implies that a scattering amplitude in type IIA string theory with string worldsheets of genus g generates the term S_g in the low-energy effective theory. Since each of the terms appears exactly at order g_s^{2g-2}, there are no higher perturbative or non-perturbative corrections in g_s.

4.3 Integrating out $M2$-branes

Let us now discuss the details of the Gopakumar-Vafa calculation, following [154, 155]. The ultimate goal is to relate the full genus expansion of the topological string free energy to a one-loop effective action. The framework is M-theory on the background $\mathbb{R}^4 \times S^1 \times Y$, where we integrate out particle-like BPS states in a constant, anti-self dual graviphoton background[2]. First we will work out the contribution of an $M2$-brane wrapped on an holomorphic, rigid genus zero curve[3], which is nothing but an S^2 of fixed size. After that we will generalize this result to cover curves of arbitrary genus g. In the case of an S^2, the low-energy degrees of freedom of a wrapped $M2$ are encoded in a massive hypermultiplet

[2]Recently, the Gopakumar-Vafa formula was applied to the near-horizon geometry of BPS black holes in $N = 2$ supergravity to determine higher-derivative corrections [156] (see [157, 158] for related work).

[3]By rigidity we mean the absence of any deformations of the curve as a holomorphic submanifold.

in five dimensions. Assuming the M-theory circle to be large, we can ignore the internal structure of the CY and treat the object as a non-relativistic point particle.

From the low-energy perspective our starting point is a supersymmetric background of *five-dimensional supergravity*. From there we will Wick-rotate and then compactify the Euclidean time direction. For the so-called *particle-calculation* in [154] the authors view each BPS particle as an instanton in $\mathbb{R}^4$ that winds (multiple times) around the Euclidean time circle. However, we will take the viewpoint of a four-dimensional observer and describe those objects as BPS particles with different KK momenta along that circle direction. When integrating out the BPS states in the path integral approach, their worldlines will actually wind around a loop parametrized by Schwinger proper time. The resulting contribution to the four-dimensional superspace effective action will be the starting point for the emergence analysis in section 4.4.

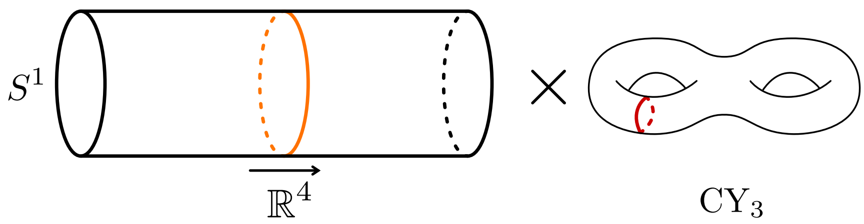


Figure 4.2: BPS particle arising from a wrapped $M2$-brane in $\mathbb{R}^4 \times S^1 \times Y$.

4.3.1 Contribution from hypermultiplets

To construct the worldline action, the supersymmetry algebra is the key ingredient. In five-dimensional Minkowski space, its most general form reads

$$\{Q_{\alpha i}, Q_{\beta j}\} = -i\Gamma^M_{\alpha\beta}\epsilon_{ij}P_M + C_{\alpha\beta}\epsilon_{ij}\zeta\,. \tag{4.31}$$

Here $Q_{\alpha i}$ with $\alpha = 1,\ldots,4$ and $i = 1,2$ are the supersymmetry generators and P_M with $M = 0,\ldots,4$ the momentum generators. $C_{\alpha\beta}$ is the charge conjugation matrix, used to raise or lower indices. $(\Gamma^M)_\beta^\alpha$ are the Dirac gamma matrices, ϵ_{ij} an antisymmetric 2-tensor and ζ the real central charge. For supergravity conventions see [154, 159].

An $M2$-brane wrapped on an isolated genus 0 curve $\Sigma \subset Y$ gives rise to a massive BPS hypermultiplet with mass $M = \zeta$. If the particle is at rest, (4.31) yields $\{Q_{Ai}, Q_{Bj}\} = 2M\epsilon_{ij}\epsilon_{AB}$, so the operators $S_i \equiv Q_i/\sqrt{M}$ generate a Clifford algebra. The $Q_{\dot{A}i}$ on the other hand annihilate the multiplet and correspond to the unbroken supercharges. With the Clifford algebra we can construct the Hilbert space $\mathcal{H}_H$ of the hypermultiplet, which is spanned by the following four states:

$$\mathcal{H}_H = \mathrm{span}\{|0\rangle\,, S_1^\dagger\,|0\rangle\,, S_2^\dagger\,|0\rangle\,, S_1^\dagger S_2^\dagger\,|0\rangle\}\,. \tag{4.32}$$

These degrees of freedom as well as the center-of-mass coordinates x^μ will now be collectively described by the worldline theory of a BPS particle in supersymmetric quantum mechanics. Assuming that the particle can be treated non-relativistically, we can approximate the Hamiltonian as $H_H^{\text{free}} = \frac{p^2}{2M} + M$, where $p^2 = p^\mu p_\mu$. The full supersymmetry algebra takes the form

$$\{Q_{Ai}, Q_{Bj}\} = 2M\epsilon_{AB}\epsilon_{ij}\,, \quad \{Q_{\dot{A}i}, Q_{\dot{B}j}\} = \frac{P^2}{2M}\epsilon_{\dot{A}\dot{B}}\epsilon_{ij}\,, \quad \{Q_{Ai}, Q_{\dot{A}j}\} = -i\Gamma^\mu_{A\dot{A}}P_\mu\epsilon_{ij}\,. \tag{4.33}$$

The Hamiltonian H_H^{free} commutes with $Q_{Ai}, Q_{\dot{A}i}$ and P_μ, so all these generators are conserved. For later convenience we define the rescaled fermion fields $\psi_{iA} = Q_{iA}/(\sqrt{2}M)$, which according to (4.33) and the commutation relation with H_H^{free} must satisfy

$$\{\psi_{Ai}, \psi_{Bj}\} = \frac{1}{M}\epsilon_{AB}\epsilon_{ij}\,, \qquad \frac{d}{dt}\psi_{Ai} = 0\,. \tag{4.34}$$

To be compatible with all these facts, the bosonic and fermionic parts of the action must contain only a free kinetic term, and in addition a constant mass term accounting for the rest energy. The minimal action therefore reads[4]

$$I_H^{\text{free}} = \int dt\, \mathcal{L}_H^{\text{free}} = M\int dt\left(\frac{1}{2}\dot{x}^\mu\dot{x}_\mu + \frac{i}{2}\epsilon^{AB}\epsilon^{ij}\psi_{Ai}\dot{\psi}_{Bj}\right) - \int dt\, M\,, \tag{4.35}$$

where the dot symbol above fields indicates derivation with respect to t. The action satisfies the algebra (4.33) if in addition we have

$$P^\mu = p^\mu \equiv \frac{\delta\mathcal{L}_H^{\text{free}}}{\delta\dot{x}_\mu} = M\dot{x}^\mu\,, \quad Q_{\dot{A}i} = -\frac{i}{\sqrt{2}}(M\dot{x}_\mu)\Gamma^\mu_{A\dot{A}}\psi_i^A\,. \tag{4.36}$$

Gödel background and modified worldline action

In the next step we want to couple the particle to an anti-self dual graviphoton background that preserves the same amount of supersymmetry. This can be achieved with the supersymmetric Gödel solution [160], a solution of pure $N = 1$ supergravity in five dimensions. In a spacetime with Lorentzian signature and coordinates (t, x^μ) with $\mu = 1, \ldots, 4$, the metric is given by

$$ds^2 = (-dt + V_\mu dx^\mu)^2 + \delta_{\mu\nu}dx^\mu dx^\nu \equiv G_{MN}dx^M dx^N\,, \tag{4.37}$$

where the graviphoton $V_\mu = -\frac{1}{2}T^-_{\mu\nu}x^\nu$ has constant field strength and no component along the time direction. We want to switch to Euclidean signature and eventually identify time with the direction of the M-theory circle[5]. For that purpose we introduce a periodic variable $y \sim y + 2\pi$ and Wick-rotate, so that $t = -iye^\sigma$, where e^σ is the radion parametrizing the size of the S^1. Moreover, in [154] the solution has been generalized to supergravities

[4]For future reference, the symbol I will denote worldline actions for particles in a 5-dimensional spacetime, while S will be used for 4 dimensions.

[5]The Gödel solution famously allows for closed time-like curves in spacetimes with Lorentzian signature. These enable massive particles to travel back to their own past and therefore violate causality. Of course, in the Euclidean signature the pathological property will play no role.

with vectormultiplets. To preserve the same amount of supersymmetry, the gauge fields A^i_μ with $i = 1, \ldots, h_{11}$ need to take the value $A^i_\mu = \tau^i V_\mu$. The metric is then given by

$$ds^2 = e^{2\sigma}(dy - \frac{i}{4}e^{-3\sigma/2}U_\mu dx^\mu)^2 + \delta_{\mu\nu}dx^\mu dx^\nu \,, \tag{4.38}$$

where we introduced $U_\mu = 4e^{\sigma/2}V_\mu$, where U_μ is the four-dimensional graviphoton with field strength $W^-_{\mu\nu}$. In Euclidean signature we are forced to assume that the graviphoton is purely imaginary to work with real metric, i.e. $W^-_{\mu\nu} \in i\mathbb{R}$. We have already seen that the Schwinger calculation in field theory works fine for both real and imaginary fields, and it poses no problem in this context either.

Let us for the moment work with the five-dimensional quantities and specify how the BPS multiplet couples to V_μ. Recall that each vectormultiplet gauge field A^i_μ couples to a conserved charge. We assume that the $M2$-brane state has specific eigenvalues

$$\beta_i = \int_\Sigma \omega_i \in \mathbb{Z} \,, \tag{4.39}$$

corresponding to the contribution of one particular homology class $\vec{\beta} = (\beta_1, \ldots, \beta_{h_{11}})$ in $H_2(Y, \mathbb{Z})$. A BPS particle with these charges couples to the effective gauge field $A^{(\vec{\beta})}_\mu = \sum_{i=1}^{h_{11}} \beta_i A^i_\mu$. Assuming the generalized Gödel background, this gauge field has field strength

$$F^{(\vec{\beta})}_{\mu\nu} = \sum_{i=1}^{h_{1,1}} \beta_i \tau^i T^-_{\mu\nu} = \zeta(\vec{\beta})T^-_{\mu\nu} = MT^-_{\mu\nu} \,, \tag{4.40}$$

which means the particle couples to the graviphoton through its mass. The new worldline action needs to take into account both the modified geometry (4.37) as well as the external coupling to the graviphoton V_μ. Hence we need to take a step back and start from the relativistic action of a point particle coupling to a curved geometry and a gauge field in five-dimensional Minkowski space. The bosonic (I_B) and fermionic (I_F) parts then read

$$\begin{aligned} I_B &= -M \int dt \sqrt{-G_{MN}\dot{x}^M\dot{x}^N} - M \int dt\, V_\mu \dot{x}^\mu \,, \\ I_F &= \frac{iM}{2} \int dt\, \epsilon^{ij}\epsilon^{AB}\psi_{A,i}\dot{\psi}_{B,j} \,, \end{aligned} \tag{4.41}$$

and $I_H = I_B + I_F$ is the total action. As before, we want to treat the point particle non-relativistically, so we take the leading term in the $\dot{x}^M \ll 1$ expansion. Using the five-dimensional Lorentzian background (4.37), the total action becomes

$$I_H = M \int dt \left(\frac{1}{2}\dot{x}^\mu \dot{x}_\mu + T^-_{\mu\nu}x^\mu\dot{x}^\nu + \frac{i}{2}\epsilon^{ij}\epsilon^{AB}\psi_{A,i}\dot{\psi}_{B,j} \right) - \int dt\, M \,. \tag{4.42}$$

This action is again completely fixed by the supersymmetry algebra, which is deformed under the presence of the graviphoton. The essential difference is that the translation generators do not coincide with the canonical momenta $\pi^\mu = \frac{\delta \mathcal{L}_H}{\delta \dot{x}_\mu}$, but are given by

$$P^\mu = \pi^\mu - M(T^-)^{\mu\nu}x_\nu \tag{4.43}$$

and no longer commute. If we replace $M\dot{x}^\mu$ by the new P^μ in (4.36), all supersymmetry charges remain conserved. The new Hamiltonian can be expressed as

$$H_H = \frac{P^2}{2M} + (T^-)^{\mu\nu} L_{\mu\nu}\,, \tag{4.44}$$

where $L_{\mu\nu} = x_\mu \pi_\nu - \pi_\mu x_\nu$. For all remaining details regarding the modified supersymmetry algebra we refer to [154].

Once we compactify the five-dimensional spacetime and Wick-rotate so that the metric is (4.38), the superparticle action becomes

$$I_H^{(E)} = M \int_0^{2\pi e^\sigma} d\tau \left(-1 + \frac{1}{2}\left(\frac{dx_\mu}{d\tau}\right)^2 - iT^-_{\mu\nu} x^\mu \frac{dx^\nu}{d\tau} + \frac{1}{2}\epsilon^{ij}\epsilon^{AB}\psi_{A,i}\frac{d}{d\tau}\psi_{B,j}\right)\,. \tag{4.45}$$

This would be the starting point of the *particle calculation* in [154] where the particle worldline winds around the M-theory circle. However, as initially announced, we will switch to the perspective of a four-dimensional observer for the remaining calculation. This involves a few conceptual differences: First, we view the wrapped $M2$-brane as a particle in four dimensions whose mass receives an additional contribution from KK momentum along the M-theory circle. Second, when we calculate the effective action from the path integral each particle worldline will wind around a circle whose length is given by the Schwinger proper time parameter s.

On the more formal side we should adjust (4.45) by introducing the relevant four-dimensional quantities. We will drop the constant term in (4.45) since it corresponds to the mass-dependent part of the Schwinger integral that we will separately add. Moreover, from the 4d perspective the particle of charge $\vec{q} = (q_0, ..., q_{h_{11}})$ with $q_0 = n$ and $q_i = \beta_i/2\pi$ for $i = 1, \ldots, h_{11}$ will couple to the gauge field $A^{\vec{q}}_\mu = q_\Lambda A^\Lambda_\mu$ with field strength

$$F^{(\vec{q})}_{\mu\nu} = \frac{1}{2} q_\Lambda \bar{X}^\Lambda W^-_{\mu\nu} = \frac{1}{4}\bar{Z}(\vec{q}) W^-_{\mu\nu}\,. \tag{4.46}$$

To summarize, the Euclidean worldline action in the four-dimensional setting is given by

$$S_H^{(E)} = M \int_0^{s} d\tau \left(\frac{1}{2}\left(\frac{dx_\mu}{d\tau}\right)^2 + \frac{i}{4}\bar{Z}(\vec{q}) W^-_{\mu\nu} x^\mu \frac{dx^\nu}{d\tau} + \frac{1}{2}\epsilon^{ij}\epsilon^{AB}\psi_{A,i}\frac{d}{d\tau}\psi_{B,j}\right)\,. \tag{4.47}$$

One-loop effective action from hypermultiplet

With all ingredients in place, we can proceed with the evaluation of the one-loop effective action from a single hypermultiplet. The action (4.47) indicates that the situation is very similar to the charged scalar in scalar QED, since the above action describes the dynamics of a spinless particle minimally coupling to a particular $U(1)$ gauge field. Since the fermions are free and uncharged under $W^-_{\mu\nu}$, their contribution factorizes and allows us to apply formula (4.6) in our setup immediately. The appropriate Schwinger integral then reads

$$\Delta\Gamma_H = -\int_\epsilon^\infty \frac{ds}{s} e^{-sM(\vec{q})^2}\, \mathrm{Tr}_{\mathcal{H}_H}(e^{-s\hat{H}_H})\,, \tag{4.48}$$

where $\hat{H}_H$ is the Hamiltonian associated with (4.47), $M(\vec{q}) = Z(\vec{q})\, M_{\mathrm{pl},4}$ and we have the trace of $e^{-s\hat{H}_H}$ over all states in the Hilbert space $\mathcal{H}_H$. We will now switch to the path integral language and integrate out the bosonic (x^μ) and fermionic ($\psi_{A,i}$) worldline fields. The dictionary for the two approaches is

$$\mathrm{Tr}_{\mathcal{H}_H}(e^{-s\hat{H}_H}) = \int_{\mathrm{PBC}} \mathcal{D}x^\mu(\tau) \int_{\mathrm{PBC}} \mathcal{D}\psi^i(\tau)\, e^{-S_{\mathrm{H}}^{(E)}[x,\psi]} \equiv \Xi\,, \tag{4.49}$$

where PBC stands for $x^\mu(0) = x^\mu(s)$ and $\psi^i_A(0) = \psi^i_A(s)$, respectively. To treat the path integral semiclassically, we study small fluctuations (x^μ, ψ^i_A) around a particular classical particle trajectory. Such an orbit breaks all translational and half of the supersymmetries, leading to modes with zero eigenvalue under the respective kinetic operator, called zero modes. Instead of regularizing their contributions, we project them out, integrate over the non-zero modes and perform an explicit integration over the so-called collective coordinates. With this in mind, we can perform the Gaussian path integrals and obtain

$$\Xi = \frac{1}{(2\pi)^2} \int d^4x\, d^4\psi^{(0)} \sqrt{\frac{\det' D_F}{\det' D_B}}\,. \tag{4.50}$$

A few comments about our conventions: First of all, we have included the integration over the zero modes $d^4x\, d^4\psi^{(0)}$ along with the standard normalization $1/\sqrt{2\pi}$ per mode. We have also defined the bosonic and fermionic kinetic operators from the Euclidean action (4.47) given by

$$D_B = \frac{d^2}{d\tau^2}\delta_{\mu\nu} - \frac{i}{2}\bar{Z}(\vec{q}) W^-_{\mu\nu}\frac{d}{d\tau} \quad \text{and} \quad D_F = \frac{d}{d\tau}\epsilon_{AB}\epsilon_{ij}\,. \tag{4.51}$$

and $\det'(\ldots)$ indicates that we take the determinant in the space orthogonal to the zero modes. The result for the determinant ratio, assuming $W^-_{03} = -W^-_{12} = W/2$, is

$$\sqrt{\frac{\det' D_F}{\det' D_B}} = \frac{1}{64}\frac{\bar{Z}(\vec{q})^2 W^2}{\sinh^2(s\bar{Z}(\vec{q})W/8)}\,. \tag{4.52}$$

The next step is to express this result in terms of the variables of the four-dimensional action in superspace language. For that we need the relation between the zero-mode measure and measure in the supergravity action given by

$$d^4x\, d^4\psi^{(0)} = -\frac{1}{\bar{Z}(\vec{q})^2} d^4x\, d^4\theta \sqrt{g^E}\,. \tag{4.53}$$

Now we can combine all intermediate results to determine the one-loop effective action. However, we should be careful about the fermionic integration variables: Since they are Grassmann variables, they have to appear linearly in the integrand for the result to be non-vanishing. Hence, in each superspace action we implicitly mean that the graviphoton and the central charge are replaced by their respective superfields $\mathcal{W}_{\mu\nu}$ and $\mathcal{Z}$. With our choice for the graviphoton we have $W^-_{\mu\nu}W^{-,\mu\nu} = W^2$, so we find

$$\begin{aligned}\Delta\Gamma_H &= \frac{1}{64\pi^2}\int d^4x\, d^4\theta\sqrt{-g^E}\int_\epsilon^\infty \frac{ds}{s}e^{-sM(\vec{q})^2}\frac{-W^2}{\big(2i\sinh\big(s\bar{Z}(\vec{q})W/8\big)\big)^2} \\ &\equiv \int d^4x\, d^4\theta\sqrt{-g^E}\,\mathcal{L}_H\,. \end{aligned} \tag{4.54}$$

To summarize, what we have just found is the one-loop effective action generated by a wrapped $M2$ brane on an S^2, which in four dimensions contributes as a BPS particle with mass $M(\vec{q})$, defined by the homology class $\vec{\beta} \in H_2(Y,\mathbb{Z})$ of the $M2$ as well as a KK momentum number along the M-theory circle. For the remainder of this section, we will refer to Lagrangians instead of the full superspace actions for ease of notation.

4.3.2 Generalizations

To capture the full contribution of BPS states from wrapped $M2$ branes, we should generalize the building block (4.54) by looking at $M2$ branes wrapped on genus r curves Σ_r, assuming again for the moment that they are rigid. In that case the curve has $b_1(\Sigma_r) = 2r$ independent 1-cycles that lead to new internal degrees of freedom related to the worldvolume topology. There is a precise notion [161] to map these massless spinors to differential forms on Σ_r. The universal fermions ψ_{Ai} with $i = 1, 2$ from the supersymmetry algebra are related to the bottom and top form of the curve, which has $b_0(\Sigma_r) = b_2(\Sigma_r) = 1$ for all r. To cover the additional harmonic forms we introduce r fermion pairs with correspondence

$$\eta_{A,\sigma} \leftrightarrow \text{(1,0)-forms} \quad \text{and} \quad \tilde{\eta}_{A,\sigma} \leftrightarrow \text{(0,1)-forms} \tag{4.55}$$

with $\sigma = 1, \ldots, r$ and $A = 1, 2$. From the five-dimensional low-energy perspective, we are then dealing with a more general supermultiplet with additional spin quantum numbers (j_L, j_R) under the rotation group $SO(4) \simeq SU(2)_L \times SU(2)_R$. Crucially, the zero-modes are singlets under $SU(2)_R$, so they transform as $(1/2, 0)$. We can now simply add these fermions to our effective quantum mechanical description by defining their corresponding worldline action. Since these fermions now also couple to the graviphoton field, we find the Euclidean action

$$S_\eta^{(E)} = \sum_{\sigma=1}^{g}\int_0^s d\tau\left(\tilde{\eta}_{A,\sigma}\frac{d}{d\tau}\eta_{A,\sigma} - \frac{i}{8}W^-_{\mu\nu}\gamma^{\mu\nu}_{AB}\tilde{\eta}_{A,\sigma}\eta_{B,\sigma}\right)\,, \tag{4.56}$$

while the action of a general BPS supermultiplet is $S^{(E)}_{\text{BPS}} = S^{(E)}_H + S^{(E)}_\eta$. The path integral for these fermions can be evaluated in a completely analogous way. The only obvious difference is that the new fermions are internal degrees of freedom of the worldvolume, so a classical solution of them does not break any symmetries of the Super-Poincarè algebra. Hence we do not need to introduce any additional zero mode integrations. For the total action S_{BPS} the result is

$$\mathcal{L}^{(r)}_{\text{rig}} = \frac{1}{64\pi^2}\int_\epsilon^\infty \frac{ds}{s}e^{-sM(\vec{q})^2}\frac{-W^2}{\big(2i\sinh\big(s\bar{Z}(\vec{q})W/8\big)\big)^{2-2r}}\,. \tag{4.57}$$

The last step is the generalization to genus r curves that are *not rigid*, in which case the $M2$ worldvolume can acquire extra deformations related to moduli of the embedding of Σ_r in Y. Because of supersymmetry we obtain sets of additional bosonic and fermionic zero modes that contribute in the same way as the set of fields $(x^\mu, \psi_{A,i}, \eta_{A\sigma}, \tilde{\eta}_{A,\sigma})$ before. Hence the only modification are overall degeneracy factors $\alpha_r^{\vec{\beta}}$, the genus r GV invariants, which count (in a BPS-sense) the number of zero modes per wrapping class and per genus.

Summary and remarks

Let us summarize: We have reviewed that F-terms in the type IIA effective action, generated at g-th loop order in perturbative string theory, can be alternatively obtained by summing over contributions from wrapped $M2$-brane states on genus r curves. Their contribution can be expressed through a Schwinger-like integral and involves an infinite sum over the homology lattice and KK momenta around the M-theory circle. We now transform the relevant part of the result into string units for the subsequent analysis. The central charge can be brought to string units via

$$Z(\vec{q})M_{\mathrm{pl},4} = -i\frac{M_{\mathrm{pl},4}}{\mathcal{V}_6^{1/2}}\left(q_i\frac{X^i}{X^0}+n\right) = -i\frac{M_s}{g_s}(\beta_i t^i + n) \equiv -Z_n^{\mathrm{str}}(\vec{\beta})M_s\,, \tag{4.58}$$

where we used $2X^0 = -i/\mathcal{V}_6^{1/2}$ with $\mathcal{V}_6$ denoting the CY volume in string units and the mass scale relation $M_s/g_s = M_{\mathrm{pl},4}/\mathcal{V}_6^{1/2}$. We also defined $Z_n^{\mathrm{str}}(\vec{\beta}) = i(\beta_i t^i + n)/g_s$. The total contribution can then be written as

$$\begin{aligned}
\mathcal{L}_{\mathrm{rig}}^{(r)} &= -\frac{M_{\mathrm{pl},4}^2}{4\pi^2}\int_\epsilon^\infty \frac{ds}{s} e^{-s\left|Z_n^{\mathrm{str}}(\vec{\beta})\right|^2 M_s^2} \frac{\lambda^2}{\left(2i\sinh\left(-s\bar{Z}_n^{\mathrm{str}}(\vec{\beta})M_s\lambda/2\right)\right)^{2-2r}}\,,\\
\mathcal{L}_{\mathrm{tot}} &= \sum_{\vec{\beta}\in H_2(Y,\mathbb{Z})}\sum_{n\in\mathbb{Z}}\sum_{r>0}\alpha_r^{\vec{\beta}}\,\mathcal{L}_{\mathrm{rig}}^{(r)}\,,
\end{aligned} \tag{4.59}$$

where we also introduced the coupling $\lambda = W/(4M_{\mathrm{pl},4})$. In $\mathcal{L}_{\mathrm{tot}}$ we sum over all KK momenta $n \in \mathbb{Z}$ and homology classes $\vec{\beta} \in H_2(Y,\mathbb{Z})$ such that the resulting BPS state is not massless. Moreover, the index r labels massive BPS states in five dimensions with $SO(4)$-spin content $I_1 \otimes \sum_r \alpha_r^{\vec{\beta}} I_r$, where $I_1 = 2[0] \oplus [1/2]$ and $I_r = (I_1)^r$. Note that in the seminal papers [149, 150], a slightly different formula was proposed. In our conventions, their result for $\mathcal{L}_{\mathrm{rig}}^{(r)}$ was

$$\mathcal{L}_{\mathrm{rig,GV}}^{(r)} = -\frac{M_{\mathrm{pl},4}^2}{4\pi^2}\int_\epsilon^\infty \frac{ds}{s} e^{sZ_n^{\mathrm{str}}(\vec{\beta})M_s} \frac{\lambda^2}{\left(2i\sinh\left(s\lambda/2\right)\right)^{2-2r}} \tag{4.60}$$

and $\mathcal{L}_{\mathrm{tot,GV}}$ given by the analogous sum over states. Naively, one would think that $\mathcal{L}_{\mathrm{rig}}^{(r)}$ in equation (4.59) should reduce to (4.60) through the change of variables $s' = -s\,\bar{Z}_n^{\mathrm{str}}(\vec{\beta})M_s$. However, this turns the Schwinger parameter into a complex variable and makes the path of integration depend on the concrete value of $\vec{\beta}$ and n. To correctly carry out the integral, one needs to integrate over a line that originates from the vicinity of the origin of the complex plane and stretches out to infinity at an angle depending on $Z_n(\vec{\beta})$. In [162] (see also [155]) it was shown that the proper integration of (4.59) over s also yields the non-perturbative corrections (in λ) to the corresponding amplitudes. In our analysis, we will work with the expression (4.60), which should still encode the classical contributions as well as the world-sheet instanton corrections from the weakly coupled string theory point of view. If we now compare the superspace action of $\mathcal{L}_{\text{tot,GV}}$ with (4.30), we find that the genus expansion of the prepotential reads[6]

$$\sum_{g=0}^{\infty} \mathcal{F}_g(t^i)\lambda^{2g} = \sum_{\vec{\beta},n,r} \alpha_r^{\vec{\beta}} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{sZ_n^{\text{str}}(\vec{\beta})M_s} \frac{\lambda^2}{\left(2i\sinh\left(s\lambda/2\right)\right)^{2-2r}} \,. \tag{4.61}$$

One can show that the right-hand side gives the expected scaling g_s^{2g-2} for the genus g prepotential $\mathcal{F}_g$. Moreover, the sum is such that each $\mathcal{F}_g$ on the left-hand side receives contributions from terms on the right-hand side for all $r \leq g$.

4.4 Emergence in strong coupling limits

In the previous sections we studied F-terms in the low-energy effective action of type IIA compactifications on CY threefolds and saw how certain particle-like BPS states in four dimensions contribute to these F-terms when being integrated out at one-loop and at strong coupling. The contributions are encoded in the GV formula, which will serve as our starting point for a first quantitative test of the Emergence Proposal beyond the effective field theory approach. The general idea, motivated in section 3.3, is to reproduce terms in the low-energy effective action from a one-loop diagram such as the GV formula - even those terms that are considered "classical" and are usually obtained from dimensional reduction. To go beyond the EFT paradigm, we need integrate out the *full towers* of states which are the fundamental degrees of freedom in a given limit. However, even at strong coupling theories with different fundamental states can arise. The aim of the present section is to explore these different limits and to underline that the Emergence Proposal is only realized in a very specific class.

Let us start with some general remarks about the asymptotic limits: We study the Kähler moduli space of type IIA on CY_3 and consider cases where the vacuum expectation value of a certain Kähler modulus, denoted as $t_0 = \langle t \rangle$, grows to infinity. The limit is always taken such that the four-dimensional dilaton stays fixed, so that the four-dimensional Planck mass does not diverge and hence gravity does not decouple. In each limit we obtain a perturbative quantum gravity theory with a characteristic set of fundamental degrees of freedom, given by those infinite towers with typical mass scale $\Delta m \lesssim \tilde{\Lambda}$, where $\tilde{\Lambda} = M_{\text{pl},4}/\sqrt{N_{\text{sp}}}$ is the species scale and N_{sp} the number of light species. This emergent theory then admits a perturbative expansion in terms of the naturally small parameter $g_E \simeq t_0^{-1} \simeq N_{\text{sp}}^{-1}$. The expansion in the coupling constant of the emergent

[6] To match the conventions in (4.30), we must identify $\mathcal{F}_g^{(S)}(X^i/X^0)(X^0)^{2-2g}M_{\text{pl},4}^{2g} = \mathcal{F}_g(t^i)$.

theory g_E will have perturbative polynomial corrections as well as non-perturbative exponentially suppressed corrections of the type $\exp(-1/g_E)$. In general we do not know how to quantize this emergent theory from first principles but, as we will see, in certain cases one can understand it via an existing dual fundamental string theory.

Infinite distance limits in this setup fall in one of two categories, namely decompactification limits and emergent string limits. In the following we will consider one particular limit from each of these two classes and pick specific models where explicit checks can be performed with very few technical tools. For concreteness, we will study the following two scenarios (following [163]):

- a *decompactification* to M-theory in which only the M-theory circle decompactifies, while the Calabi-Yau volume doesn't scale in M-theory units. We derive the full prepotential and the genus-one free energy for the (non)-compact resolved conifold, which features only a single rigid 2-cycle. Here a Schwinger integral is sufficient to recover the full answer and the Emergence Proposal is realized.

- an *emergent string limit*, which requires a $K3$ fibration over a base $\mathbb{P}^1$, whose volume is sent to infinity. Since the emergent string is given by an $NS5$-brane wrapped on the $K3$, we do not know how to quantize this theory but we can understand it in terms of the heterotic dual on $K3 \times T^2$. In that case the Schwinger integral only provides the one-loop correction in the heterotic dual string coupling, hence the prepotential does not fully emerge at one-loop. Instead, all terms in the prepotential will arise via the correct quantization of the perturbative quantum gravity theory that emerges in the considered asymptotic field limit.

In summary, as depicted in figure 4.3, the type IIA vector multiplet moduli space in four dimensions features a pattern very similar to the familiar duality star in ten/eleven dimensions. The figure also visualizes the difference between the mass spectrum in the asymptotic limits and the mass spectrum in the bulk region, which has been called the desert in [164]. For the latter, there are only a few light states, while the majority of the towers and the species scale are all close to the Planck scale.

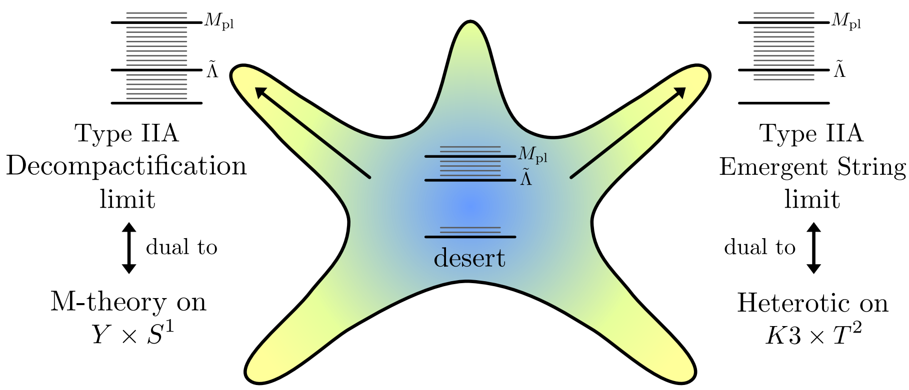


Figure 4.3: Cartoon of type IIA vector multiplet moduli space and asymptotic limits.

4.4.1 Decompactification limit

First we consider decompactifications to M-theory, starting from a four-dimensional effective theory from type IIA on CY_3. We recall that the limit is defined through two criteria: (*i*) the four-dimensional Planck mass $M_{\mathrm{pl},4}$ doesn't scale and (*ii*) the compact six-dimensional volume in units of the 11-dimensional Planck scale stays finite. In section 3.2.2 we have already determined the scaling of the relevant quantities if we send $g_s \to \lambda g_s$ and $\lambda \to \infty$. Specializing now to $d = 4$, formula (3.25) demands that in this limit the type IIA quantities transform as

$$g_s \to \lambda g_s\,, \quad M_s \to M_s\,, \quad t_I \to \lambda^{2/3} t_I\,, \tag{4.62}$$

implying that the volume in string units scales as $\mathcal{V}_6 \to \lambda^2 \mathcal{V}_6$. We emphasize once again that $d = 4$ is special in the sense that M_s doesn't scale with λ. We also recall that the mass scales of $D0$-branes and wrapped $D2$-branes on 2-cycles Σ_2 are given by

$$m_{D0} = \frac{M_{\mathrm{pl},4}}{\mathcal{V}_6^{1/2}} \simeq \frac{M_{\mathrm{pl},4}}{\lambda}\,, \quad m_{D2} = \frac{M_{\mathrm{pl},4}}{\mathcal{V}_6^{1/2}} \mathcal{V}(\Sigma_2) \simeq \frac{M_{\mathrm{pl},4}}{\lambda^{1/3}}\,, \tag{4.63}$$

with $\mathcal{V}(\Sigma_2)$ the volume of Σ_2 in string units. We have also applied the relation $M_s/g_s = M_{\mathrm{pl},4}/\mathcal{V}_6^{1/2}$. The tower of $D0$-branes determines the species scale to be

$$\tilde{\Lambda} = M_{\mathrm{KK}}^{1/3} M_{\mathrm{pl},4}^{1/3} \simeq \frac{M_{\mathrm{pl},4}}{\lambda^{1/3}}\,, \tag{4.64}$$

and the $D2$-brane tower scale is at the threshold. These are precisely the states being integrated out in Gopakumar-Vafa formula. However, according to the Emergence Proposal this should not be the complete set of relevant states. The pure analysis of mass scales suggests to also include contributions from wrapped $NS5$-branes and KK modes along all compact directions, since these scale as

$$m_{NS5} = \frac{M_{\mathrm{pl},4}}{\mathcal{V}_6} \sqrt{\mathcal{V}(\Sigma_4)} \simeq \frac{M_{\mathrm{pl},4}}{\lambda^{1/3}}\,, \quad m_{\mathrm{KK},I} = \frac{M_{\mathrm{pl},4}}{\mathcal{V}_6^{1/2}} \frac{g_s}{\rho_I} \simeq \frac{M_{\mathrm{pl},4}}{\lambda^{1/3}}\,. \tag{4.65}$$

To reconcile these facts, recall that only BPS states charged under the $U(1)$ gauge fields A^i_μ with $i = 1, \ldots, h_{11}$ contribute to the central charge in the Gopakumar-Vafa formula. Since these gauge fields originate from the RR one-form C_1 and three-form C_3 in type IIA, the central charge only receives contributions from $D0$- and $D2$-branes and their bound states. As branes carrying generic KK momentum along compact directions break the BPS condition, such states should not contribute to the Gopakumar-Vafa formula. Wrapped $NS5$-branes on the other hand would give rise to strings in four dimensions and upon quantization could result in contributions to the GV invariants. However, at present a complete quantization of $NS5$-branes is lacking, and there is likewise no framework for integrating out extended objects. Our analysis is therefore restricted to the BPS particle spectrum considered in the original works.

Let us now specify the concrete model for the analysis in the decompactification limit. We will study the resolved conifold, which has been analyzed already in [165] to develop a duality to Chern-Simons theory. The resolved conifold has only one two-cycle with the topology of an S^2 to which we assign the complex Kähler modulus $t = b + i\tau$. The holomorphic prepotential and its lowest-order correction are given by

$$\begin{aligned} \mathcal{F}_0 &= \frac{(2\pi i)^3}{g_s^2}\left(\frac{t^3}{12} + \frac{t}{24} + \frac{t^2}{8}(1+4m) - \frac{\zeta(3)}{(2\pi i)^3} + \frac{1}{(2\pi i)^3}\mathrm{Li}_3(e^{2\pi i t})\right), \\ \mathcal{F}_1 &= -\frac{2\pi i t}{24} - \frac{1}{12}\mathrm{Li}_1(e^{2\pi i t}). \end{aligned} \tag{4.66}$$

If we compare with the general formula for the prepotential (2.106), we see that the resolved conifold has the formal triple intersection number $C = 1/2$. For $g > 1$ one gets

$$\mathcal{F}_g = g_s^{2g-2}\left((-1)^g \chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} + \frac{\chi_g}{(2g-3)!}\mathrm{Li}_{3-2g}(e^{2\pi i t})\right), \tag{4.67}$$

where χ_g is the Euler characteristic of the moduli space of all Riemann surfaces with genus g and h punctures. While the integer m in $\mathcal{F}_0$ is naively related to h as $m = \frac{h}{2} - 1$, it is actually not uniquely fixed. Note that these higher genus $\mathcal{F}_g$ were derived in [149] by computing a Schwinger integral with a single-wrapped $D2$-brane bound to an arbitrary number of $D0$-branes in the loop.

Let us determine the Schwinger integral for the resolved conifold. With only one rigid S^2 we have $\alpha_0^\beta = 1$ and all other invariants are vanishing. As a result, the sum over r in (4.61) collapses to just one term and we obtain

$$\sum_{g=0}^{\infty} \mathcal{F}_g \lambda^{2g} = -\frac{1}{4}\sum_{n\in\mathbb{Z}} \int_0^\infty \frac{ds}{s}\frac{\lambda^2}{\sinh^2(s\lambda/2)} e^{sZ_n'}, \tag{4.68}$$

where now we work with the central charge $Z_n' = \frac{M_s i}{g_s}(t-n)$ and set $M_s = 2\pi$ henceforth. Now one can expand the sinh-function in the denominator using

$$\frac{1}{\sinh^2(x)} = -\frac{d}{dx}\coth(x) = \sum_{m=0}^{\infty} \frac{2^{2m}(2m-1)}{(2m)!} B_{2m}\, x^{2m-2}, \tag{4.69}$$

where B_{2g} are the Bernoulli numbers and the above expansion is convergent for $0 < |x| < \pi$. Inserting this expansion into (4.68) yields a formula for each $\mathcal{F}_g$, given by

$$\mathcal{F}_g = -\frac{2g-1}{(2g)!} B_{2g} \sum_{n\in\mathbb{Z}} \int_0^\infty ds\, s^{2g-3} e^{sZ_n'}. \tag{4.70}$$

As shown in [149], for $g > 1$ the integral and the sum over all $n \in \mathbb{Z}$ can be carried out explicitly and turn out to be finite. It gives the non-perturbative contributions in (4.67). For $g = 0, 1$ the objects we are interested in are at first sight divergent, so that one has to introduce an ultraviolet regulator, $\epsilon > 0$, in the integral (4.70). Using that $B_0 = 1/2$ and $B_2 = 1/6$, the resulting expressions are

$$\mathcal{F}_1 = -\frac{1}{12}\sum_{n\in\mathbb{Z}} \int_\epsilon^\infty \frac{ds}{s} e^{sZ_n'}, \qquad \mathcal{F}_0 = \sum_{n\in\mathbb{Z}} \int_\epsilon^\infty \frac{ds}{s^3} e^{sZ_n'}. \tag{4.71}$$

In the following we will demonstrate how the exact expressions for $\mathcal{F}_0$ and $\mathcal{F}_1$ in (4.66) can be obtained from (4.71) by regularizing the infinite sum over KK momenta with the Riemann-ζ function. We start with the simpler calculation, namely $\mathcal{F}_1$.

Evaluating $\mathcal{F}_1$

The first step is to perform the integral in s and expand the result for small ϵ. We get

$$\mathcal{F}_1 = \frac{1}{12}\sum_{n\in\mathbb{Z}}\left(\gamma_E + \log(-\epsilon\, Z_n') + \mathcal{O}(\epsilon)\right) , \tag{4.72}$$

where γ_E is the Euler-Mascheroni constant. We can then substitute the expression for the central charge and keep only those terms that are non-vanishing in the limit $\epsilon \to 0$. The result we find is

$$\mathcal{F}_1^{D0-D2} = \frac{1}{12}\sum_{n\in\mathbb{Z}}\log\left(\frac{i(n-t)}{\mu}\right) , \tag{4.73}$$

where $\mu = e^{-\gamma_E}\mathcal{V}_6^{1/2}/(2\pi\epsilon M_{\text{pl},4})$. Now we can decompose this sum as

$$\mathcal{F}_1^{D0-D2} = \frac{1}{12}\left(\log(-it) + \sum_{n\geq 1}\log\left(1-\frac{t^2}{n^2}\right) + 2\sum_{n\geq 1}\log(n) - \log(\mu)\sum_{n\in\mathbb{Z}}1\right) . \tag{4.74}$$

The second term on the right-hand side can be evaluated with a trigonometric identity which we show in appendix A. For the third and fourth expression we exploit ζ-function regularization. For the third term we apply the identity

$$\sum_{n\geq 1}\log(n) = -\zeta'(0) = \frac{1}{2}\log(2\pi) \tag{4.75}$$

and the fourth term depending on μ vanishes since

$$\sum_{n=1}^{\infty}1 = \zeta(0) = -\frac{1}{2} \quad \to \quad \sum_{n\in\mathbb{Z}}1 = 1 + 2\zeta(0) = 0\,. \tag{4.76}$$

Putting everything together, we arrive at the result

$$\mathcal{F}_1^{D0-D2} = -\frac{2\pi it}{24} + \frac{1}{12}\log\left(1-e^{2\pi it}\right) = -\frac{2\pi it}{24} - \frac{1}{12}\sum_{n\geq 1}\frac{1}{m}e^{2\pi int} , \tag{4.77}$$

where in the second step we used the series expansion of the logarithm. To sum up, we have shown that by summing over the entire infinite tower of $D0-D2$ BPS bound states and treating the divergences via ζ-function regularization one obtains the exact one-loop topological free energy. In particular, this includes the term linear in t, which is usually obtained from dimensional reduction.

To finish the discussion of $\mathcal{F}_1$ we should also consider the possibility of a pure $D0$-brane tower, whose contribution to the one-loop free energy is

$$\mathcal{F}_1^{D0} = \frac{1}{12}\sum_{n\neq 0}\log\left(\frac{in}{\mu}\right) . \tag{4.78}$$

The case $n = 0$ was excluded from the sum as some particle should really run in the loop. Proceeding with ζ-function regularization, we get

$$\mathcal{F}_1^{D0} = \frac{1}{12}\log(2\pi\mu)\,. \tag{4.79}$$

Since μ contains a factor of $\mathcal{V}_6^{1/2}$, it is tempting to speculate that this non-holomorphic logarithmic term is related to the holomorphic anomaly of $\mathcal{F}_1$. Furthermore, this time the result is not independent of the regulator ϵ, arising due to the $\log(\mu)$ factor in $\mathcal{F}_1^{D0}$, but we can avoid this by subtracting this term from the final result.

Evaluating $\mathcal{F}_0$

Next, we look at the prepotential $\mathcal{F}_0$. We start again by performing the integral over s and expanding in small ϵ, which gives

$$\mathcal{F}_0 = \frac{1}{g_s^2}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2(\epsilon')^2} + \frac{g_s Z_n'}{\epsilon'} - \frac{(g_s Z_n')^2}{4}\Big(-3 + 2\gamma_E + 2\log(-\epsilon' g_s Z_n')\Big)\right) + \mathcal{O}(\epsilon)\,, \tag{4.80}$$

where we defined $\epsilon' = \epsilon/g_s$ to display an overall multiplicative factor g_s^{-2}, matching with the resolved conifold. Using (4.76), we see that at fixed (small) ϵ' the first two terms in the above expression vanish. For the remainder we can again introduce a scale $\nu = e^{-3/2+\gamma_E}\frac{\mathcal{V}_6^{1/2}}{2\pi\epsilon M_{\mathrm{pl},4}}$ and rewrite (4.80) into

$$\mathcal{F}_0 = \frac{(2\pi)^2}{2g_s^2}\sum_{n\in\mathbb{Z}}(n-t)^2\log\left(\frac{i(n-t)}{\nu}\right)\,. \tag{4.81}$$

We already saw similar expressions arising also in the EFT approach to the Emergence Proposal, see for example the computation of gauge couplings in [130]. To calculate $\mathcal{F}_0$, first notice that the dependence on ν completely drops out because of (4.76) and $\zeta(-2) = 0$. The contribution from $D0 - D2$ bound states can then be decomposed as

$$\begin{aligned}\mathcal{F}_0 = \frac{(2\pi)^2}{2g_s^2}\Bigg[&t^2\log(-it) + \sum_{n\geq 1}(t^2+n^2)\log\left(1-\frac{t^2}{n^2}\right)\\ &+ 2\sum_{n\geq 1}(t^2+n^2)\log(n) + 4t\sum_{n\geq 1} n\,\mathrm{arccoth}\left(\frac{t}{n}\right)\Bigg]\,,\end{aligned} \tag{4.82}$$

where we use the principal domain for inverse trigonometric functions. To proceed, we first employ the identity (A.11) from appendix A and then regularize all infinite sums over n via ζ-function regularization. Using in particular

$$2\sum_{n\geq 1}\log\left(1-\frac{t^2}{n^2}\right) = t^2\log(2\pi) + \frac{\zeta(3)}{2\pi^2}\,, \tag{4.83}$$

we arrive at the simple expression

$$\mathcal{F}_0^{D0-D2} = \frac{(2\pi i)^3}{g_s^2}\left[\frac{t^3}{12} + \frac{1}{(2\pi i)^3}\mathrm{Li}_3(e^{2\pi i t})\right]\,. \tag{4.84}$$

Finally, we consider a bound state of only $D0$-branes, which contributes to $\mathcal{F}_0$ as

$$\mathcal{F}_0^{D0} = \frac{(2\pi)^2}{2g_s^2} \sum_{n\in\mathbb{Z}} n^2 \log\left(\frac{in}{\nu}\right) = \frac{1}{g_s^2}\zeta(3)\,. \tag{4.85}$$

In contrast to the analogous contribution in $\mathcal{F}_1$, this term does not have any dependence on ν upon using once more $\zeta(-2) = 0$. Hence, $\mathcal{F}_0^{D0}$ comes out holomorphic.

We have just shown that, similarly to $\mathcal{F}_1$, also the full prepotential $\mathcal{F}_0$ of the resolved conifold, including the polynomial terms, can be obtained by a loop computation in the emergent theory. In both instances it has been essential to sum over the full towers of BPS states and appropriately regularize the infinite sums with the ζ-function. This can be seen as a as a first proof of principle that exact emergence is possible via the (strong) Emergence Proposal when including all towers whose mass spacing is smaller or equal to the species scale. Had we neglected, say, the tower of $D0 - D2$ bound states, we would not have obtained the full result. Then, if these BPS states need to be included, it seems inconsistent to not include as well bound states involving $NS5$-branes.

Comparison with EFT approach

In order to appreciate the difference between the calculation above and the simplified EFT approach usually taken in the Emergence Proposal, let us sketch how the latter would work for the case of $\mathcal{F}_0$. The logic would be to sum only over those states whose masses lie below the species scale $\tilde{\Lambda}$. In fact, the application of this method to the sums that we are working with has been already proposed in [135]: one is instructed to choose the scale $\nu = \tilde{\Lambda}$ and to explicitly cut-off the sum in (4.82) at the level $n = N_{\rm sp} = \mathcal{V}_6^{1/3}$. This leads to a modified expression for the prepotential given by

$$\mathcal{F}_0 = \frac{(2\pi)^2}{2g_s^2} \sum_{n=0}^{\mathcal{V}_6^{1/3}} \left[(t^2+n^2)\log\left(\frac{n^2-t^2}{2\mathcal{V}_6^{2/3}}\right) + 4tn\,\mathrm{arccoth}\left(\frac{t}{n}\right)\right], \tag{4.86}$$

where a factor of 2 has been introduced inside the first log-factor for convenience (in any case, we have not been keeping track of several constant factors and normalizations). Proceeding then along with the logic same as in standard emergence computations, one considers the scaling $\mathcal{V}_6^{1/3} \to \lambda^{2/3}\mathcal{V}_6^{1/3}$ with $\lambda \to \infty$ and thus one is allowed to approximate the sum by an integral. Hence, one gets the leading order result

$$\begin{aligned}\mathcal{F}_0 \simeq \frac{(2\pi)^2}{18g_s^2}\Bigg[&3i\pi\left(t^3+3t\mathcal{V}_6^{2/3}\right) + 6\left(t^3+3t\mathcal{V}_6^{2/3}\right)\mathrm{arctanh}\left(\frac{t}{\mathcal{V}_6^{1/3}}\right)\\ &-\left(3t^2\mathcal{V}_6^{1/3}+\mathcal{V}_6\right)\left(2+3\log(2)-3\log\left(1-\frac{t^2}{\mathcal{V}_6^{2/3}}\right)\right)\Bigg].\end{aligned} \tag{4.87}$$

From this expression, it can be seen that only when $\mathcal{V}_6^{1/3} \sim it$, i.e. for the one-modulus case, one gets the simple leading order expression

$$\mathcal{F}_0 \simeq \frac{(2\pi i)^3}{g_s^2}\left(1+\frac{4}{3\pi}\right)\frac{t^3}{12}\,. \tag{4.88}$$

When comparing to the exactly emergent prepotential that we have been able to recover this result features some obvious shortcomings:

- The full prepotential is supposed to be holomorphic, but the explicit cut-off for the sum introduces a non-holomorphic dependence on the moduli.
- Even if a cubic polynomial part in t arises, the prefactor is not correct and the higher-order corrections are certainly not only those known for the prepotential of type IIA CY compactification. Indeed, the computation of the exact discrete sum instead of the continuous integral would produce these exponential terms, but also several non-holomorphic corrections.
- The higher-order corrections arising when insisting on this approach would also depend on whether one takes the sum up to $N_{\rm sp}$ or to, say, $N_{\rm sp}+1$. In this sense, such a cut-off procedure is not well defined if one wants to extract more information than the qualitative leading cubic behavior.

In view of the very simple and yet successful ζ-function regularization that we performed, we interpret all these issues as shortcomings of the naive EFT approach to the Emergence Proposal. Instead, our computation shows that exact quantities emerge in this asymptotic decompactification limit if all of the states in the towers with a typical mass scale m up to the species scale $\tilde{\Lambda}$ are running in the one-loop diagram.

4.4.2 Emergent String Limit

Given the initial concerns regarding the inclusion of string towers in the standard emergence calculations, one might expect that an emergent string limit is more involved. Instead, this limit is actually more familiar from the point of view of string dualities. There is an important difference between decompactification and emergent string limits: In the first case, as we have just seen, one can encounter multiple string-like objects which are actually emergent (wrapped) branes becoming asymptotically tensionless. In the second case, there is really only a single and fundamental string emerging. As it turns out, this requires the CY to admit a $K3$-fibration (see [166] for details).

In order to have a $K3$-fibration over a base $\mathbb{P}^1$, one needs to have at least two Kähler moduli, see e.g. [167]. A well-studied example is the CY $\mathbb{P}^4_{1,1,2,2,6}[12]$, which has Hodge numbers $(h_{21}, h_{11}) = (128, 2)$ and for which the prepotential has an expansion

$$\mathcal{F}_0^{\rm IIA} = t_1^2 t_2 + \underbrace{\left(\frac{2}{3}t_1^3 + \mathcal{O}(e^{2\pi i t_1})\right)}_{=g^{(1)}(t_1)} + \mathcal{O}(e^{2\pi i t_2})\,. \tag{4.89}$$

Here, $\tau_2 = {\rm Im}(t_2)$ measures the size of the $\mathbb{P}_1$ and $\tau_1 = {\rm Im}(t_1)$ controls the size of a homological 2-cycle sitting in the $K3$-fiber. The emergent string limit is defined via

$$t_1 \to t_1\,, \quad t_2 \to \lambda\, t_2\,, \quad g_s \to \lambda^{1/2} g_s\,, \tag{4.90}$$

following again from the requirement of a constant $M_{\text{pl},4}$. M_s also stays fixed in this case.

Once again we want to identify the species scale and the light towers of states in the strongly coupled region $\lambda \to \infty$. Wrapping an $NS5$-brane on the $K3$ fiber yields a four-dimensional string of tension

$$T_{\text{str}} = M_{\text{pl},4}^2 \frac{\mathcal{V}_{K3}}{\mathcal{V}_6} \simeq \frac{M_{\text{pl},4}^2}{\lambda}\,, \tag{4.91}$$

whose excitations will have the typical mass scale

$$m_{\text{str}} = \frac{M_{\text{pl},4}}{\sqrt{\tau_2}} \simeq \frac{M_{\text{pl},4}}{\sqrt{\lambda}}\,. \tag{4.92}$$

Here, $\mathcal{V}_6$ is the total volume of the Calabi-Yau, while $\mathcal{V}_{K3}$ is the volume of the $K3$ fiber. The latter is independent of the scaling of τ_2. In the presence of a string tower, the species scale is equal to the string mass scale itself, hence we identify

$$\tilde{\Lambda}_{\text{sp}} \simeq m_{\text{str}} \simeq \frac{M_{\text{pl},4}}{\sqrt{\lambda}}\,. \tag{4.93}$$

As opposed to the decompactification limit, in this case the scale given by the sixth root of the $NS5$-brane is parametrically above the species scale since

$$m_{NS5} = T_{NS5}^{1/6} \simeq M_{\text{pl},4} \frac{g_s^{2/3}}{\mathcal{V}_6^{1/2}} \simeq \frac{M_{\text{pl},4}}{\lambda^{1/6}}\,. \tag{4.94}$$

This provides yet another justification for talking about an emergent string limit in this case. While there are no other states which are parametrically lighter than the species scale, there are a couple of particle-like towers with a mass of the same order. First, due to the large size of $\mathbb{P}^1$, we have light KK modes with a typical mass scale

$$m_{\text{KK}} = \frac{M_s}{\sqrt{t_2}} \simeq \frac{M_{\text{pl},4}}{\sqrt{\lambda}}\,. \tag{4.95}$$

The remaining light states are once again given by D-branes, namely $D0$-branes as well as $D4$-branes wrapped around the $K3$ and $D2$-branes wrapped around the second homological 2-cycle inside the $K3$ fiber. Their typical masses read

$$m_{D0} = \frac{M_{\text{pl},4}}{\sqrt{\tau_2 \mathcal{V}_{K3}}}\,, \quad m_{D2} = \frac{M_{\text{pl},4}}{\sqrt{\tau_2}}\,, \quad m_{D4} = M_{\text{pl},4}\sqrt{\frac{\mathcal{V}_{K3}}{\tau_2}} \tag{4.96}$$

and all of them scale as $M_{\text{pl},4}/\sqrt{\lambda}$. Again, the wrapped $NS5$-branes and the KK modes are uncharged under the three RR vector fields. Furthermore, note that a $D2$-brane wrapping the large base $\mathbb{P}^1$ is now too heavy to be included in the perturbative states.

Our claim of exact emergence turns out to hold also in this new perturbative theory, where the full prepotential can be completely recovered again, even if not from a Schwinger integral exclusively. Recall that in the decompactification limit the one-loop integral was sufficient to recover the full $\mathcal{F}_0$. As we will show, this is not possible in the present limit. Instead, the complete prepotential will arise via the correct quantization of the perturbative quantum gravity theory that emerges asymptotically. For the present example we are in the convenient situation in which a known dual string description exists, namely the weakly coupled heterotic string, and we will exploit this duality in order to recover the full prepotential.

The dual heterotic model is a compactification on $K3 \times T^2$. Denoting the toroidal Kähler and complex structure moduli with T and U respectively, the dual model sits at the special locus $T = U$, where the Abelian gauge group $U(1)_U \times U(1)_T$ is enhanced to $SU(2) \times U(1)$. The former type IIA abelian gauge fields can be identified with the heterotic gauge fields coming from the T^2 factor. Moreover, the particle-like states arising from wrapped $D0$-, $D2$- and $D4$-branes correspond to heterotic KK and winding modes along the T^2. KK modes on the $\mathbb{P}^1$ correspond to KK modes along the $K3$ directions in the heterotic dual picture, hence they are uncharged under the gauge fields.

With respect to the type IIA frame, a crucial difference is that the heterotic dilaton S sits in a vector multiplet so that the prepotential can receive corrections at string one-loop and at the string non-perturbative level. Therefore, it takes the schematic form

$$\mathcal{F}_0^{\text{het}} = i\Big(- ST^2 + h^{(1)}(T) + h^{(\text{np})}(e^{-2\pi S})\Big) \,. \tag{4.97}$$

Here, the first term arises at string tree-level, while the second is the dilaton-independent one-loop correction. The function $h^{(1)}(T)$ is determined by its transformation properties under the exact quantum symmetry $SL(2,\mathbb{Z})_T$, acting on T as in (A.13), and by the singular structure at special points in moduli space. Modular invariance imposes that its second derivative reads

$$\partial_T^2 h^{(1)}(T) \sim \log(j(iT) - j(i)) + \text{finite terms}\,, \tag{4.98}$$

where $j(iT)$ is the modular invariant j-function (see appendix A for more details). Microscopically, $h^{(1)}(T)$ receives contributions from heterotic 1/2-BPS states which also contain contributions from left-moving string excitations. Via mapping the moduli according to

$$t_1 \to iT\,, \quad t_2 \to iS \tag{4.99}$$

it has been shown in [167–169] that the part $g^{(1)}(T)$ of the type IIA prepotential nicely matches the heterotic 1-loop correction $h^{(1)}$. The other terms in (4.89) are then trivially mapped to those in (4.97).

One might wonder what the role of the former type IIA Schwinger integral is in this setting. In the present case, when expressed in terms of M_s, all light towers listed in (4.96) are insensitive to the size of the large base, so that the sum over the Schwinger integrals will depend only on the type IIA modulus t_1. The KK modes on the large $\mathbb{P}^1$ do depend on t_2 but, just like the $NS5$-branes, they do not contribute to the Schwinger-integral, as we have already explained in section 4.4.1. Hence, the (regularized) Schwinger integral only gives the part $g^{(1)}(t_1)$ in the prepotential, which from the dual heterotic point of view is the string one-loop correction. This shows that the full prepotential is not emerging just from Schwinger integrals.

All this is supporting our claim that in the asymptotic limit, we can consider the type IIA prepotential (4.89) as a perturbative expansion in the small parameter $(\lambda\tau_2)^{-1}$, which is the inverse of the number of light species. Moreover, via duality we are also led to the conclusion that $g^{(1)}(t_1)$ does indeed receive contributions not only from $D2$-$D0$ bound states but also from excitations of $NS5$-$D2$-$D0$ bound states. The latter are dual to heterotic 1/2-BPS left-moving string excitations with KK momentum and winding.

Comments on recent developments

In a recent series of papers, starting with [170], an alternative interpretation of the emergence of classical contributions to $\mathcal{F}_0$ and $\mathcal{F}_1$ was provided. Studying the same setup, the authors argued that the original Schwinger integral by Gopakumar and Vafa, describing particles in the four-dimensional sense, does not correctly capture the contributions of the $D0$-$D2$ brane tower at the pole $s = 0$. Their proposal was to complexify the Schwinger parameter and to modify the integrand such that integrating along a contour encircling the pole at the origin reproduces the full tree-level contribution. The world-sheet instanton terms on the other hand originate from the integer poles on the positive real axis. In subsequent work [155, 162] (see also [171] for a different setup) it was shown that a careful contour integration of (4.59) and inclusion of additional poles also yields the non-perturbative corrections in the topological string coupling. These results are found to be in agreement with resurgence analyses (see [172, 173] for general reviews).

For the resolved conifold, the modified Schwinger integral encoding $\mathcal{F}_0$ yields a contour integral representation of the Hurwitz ζ-function (see appendix A). This function was also utilized in our work [163] to provide an alternative derivation of $\mathcal{F}_0$ and $\mathcal{F}_1$. For compact CYs, the refined Schwinger integral maps to integral representations of the CY periods. For these expressions, which already implicitly encode the sum over all BPS states, the analytic continuation to the pole at the origin is expected to yield the tree-level term. In chapter 6 we will propose a regularization technique for compact CYs that starts from the explicit infinite sums in the Gopakumar-Vafa formula and utilizes period data near finite distance degeneration points in the CY moduli space. The relation between the two approaches remains to be investigated.

Chapter 5

Computational methods from mirror symmetry

In our first explicit test of the M-theoretic Emergence Proposal in type IIA compactifications we have evaded a severe technical obstacle by working with the resolved conifold. Since the S^2 resolving the conifold can only be wrapped once by $D2$-branes, the infinite sum over the homology lattice $H_2(Y, \mathbb{Z})$ in the Gopakumar-Vafa formula collapsed to just a single term. While non-compact CYs are mathematically simple and indispensable for studying local models in particle physics, our ultimate goal is to compute emergent kinetic terms in fully fledged theories of quantum gravity, which is why we eventually need to consider compact CY spaces. The fact that multiple wrappings in curve classes of compact CYs lead to exponential degeneracies of $D2$-brane states makes the task of finding a sensible regularization even harder. Before we can embark on that mission, we need the right tools to study properties of CYs at special loci of their moduli spaces. The purpose of this (rather formal) chapter is to deliver these tools. Concretely, we want to study CYs defined as hypersurfaces in weighted projective spaces and find explicit expressions for their quantum-corrected Yukawa couplings at special loci. The latter will be needed in chapter 6 to propose a regularization that yields the triple intersection numbers in isotropic M-theory limits of CY threefolds.

5.1 Constructing CY manifolds

Our first goal is understand how compact CY manifolds can be explicitly constructed and conveniently described. In our discussion we will focus on the class of CYs given by hypersurfaces in weighted projective spaces. To develop the right mathematical tools, we need to consider these projective spaces as a particular example of the more general class of *toric varieties*. The latter in turn have a useful description in terms of *lattices, fans and reflexive polytopes*, which will all be introduced step by step. Introductions to these topics can be found in [19, 174–176], but we will assume that the reader had *no* exposure to these topics at all. Within this framework mirror symmetry is made manifest and one can proceed by setting up the differential equations that determine the CY periods.

5.1.1 Weighted projective spaces and toric varieties

Let us start by defining a weighted projective space of complex dimension n, which is obtained by taking a quotient of the space $\mathbb{C}^{n+1}$ with complex coordinates $(z_0, \ldots, z_n)$. A weighted projective space is defined as

$$\mathbb{P}^n_{\vec{w}} = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*}\,, \tag{5.1}$$

where $\vec{w} = (w_0, \ldots, w_n)$ is the weight vector with entries $w_i \in \mathbb{Z}_+$ and an element $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ acts as $\lambda(z_0, \ldots, z_n) = (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$. The coordinates $z_0, \ldots z_n$ are called homogeneous coordinates on the space $\mathbb{P}^n_{\vec{w}}$ and are denoted as $(z) \equiv (z_0 : \ldots : z_n)$. For the weight vector $\vec{w} = (1, \ldots, 1)$ one recovers the ordinary projective space $\mathbb{P}^n$, which is the set of complex lines going through the origin. Due to isomorphisms one only needs to consider "well-formed" weighted projective spaces, for which all weights w_i are relatively co-prime.

The space $\mathbb{P}^n_{\vec{w}}$ can be covered with $(n+1)$ coordinate patches $U_i = \{(z) : z_i \neq 0\}$ with $i = 0, \ldots, n$. Moreover, one can define local coordinates on each U_i via $\xi^k_{(i)} = z_k/z_i$. The set $\xi_{(i)} = (\xi^1_{(i)}, \ldots, \xi^n_{(i)})$ then gives the inhomogeneous coordinates on U_i. We can obtain transition functions on overlap regions $U_i \cap U_j$ by noting that

$$\xi^k_{(i)} = \frac{z_k}{z_j} = \frac{z_k/z_j}{z_i/z_j} = \frac{\xi^k_{(j)}}{\xi^j_{(i)}}\,, \tag{5.2}$$

where $\xi^j_{(i)} \neq 0$. The transition functions are therefore given by simple Laurent monomials and $\mathbb{P}^n_{\vec{w}}$ is indeed a complex manifold. However, the manifold doesn't have to be necessarily smooth. Let us for example consider the space $\mathbb{P}^2_{(2,3,1)}$, which has the equivalence relation $(z_0, z_1, z_2) \sim (\lambda^2 z_0, \lambda^3 z_1, \lambda z_2)$. It is easy to see that this space has orbifold singularities: If we look at a chart with $z_0 = 1$, we can act on the remaining coordinates (z_1, z_2) with the residual symmetry $(z_1, z_2) \to (\omega^3 z_1, \omega z_2)$ where $\omega \in \{\pm 1\}$. The fixed point set of this group action is $(z_1, z_2) = (0, 0)$ and the patch locally looks like $\mathbb{C}^2/\mathbb{Z}_2$. One can easily demonstrate that around the point $(0 : 1 : 0)$ one encounters a $\mathbb{C}^2/\mathbb{Z}_3$ structure.

One can now consider hypersurfaces in weighted projective spaces, which are submanifolds of codimension one. They are defined as the zero locus of a (quasi-)homogeneous polynomial P of degree $d \in \mathbb{Z}_+$. This means that under the equivalence relation of $\mathbb{P}^n_{\vec{w}}$ the polynomial satisfies $P(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d P(z_0, \ldots, z_n)$ and the hypersurface $X_{\vec{w}}$ is defined as

$$X_{\vec{w}} = \left\{ (z_0 : \ldots : z_n) \in \mathbb{P}^n_{\vec{w}} \;\middle|\; P(z) = 0 \right\}\,. \tag{5.3}$$

The vanishing of the first Chern class $c_1(X_{\vec{w}}) = (\sum_{i=0}^n w_i - d)J$ requires the weight vector components to add up to the degree d. For the embedding $X_{\vec{w}} \hookrightarrow \mathbb{P}^n_{\vec{w}}$ to be smooth, the polynomial must satisfy the transversality condition imposing that the equations

$$P(z) = 0 \quad \text{and} \quad dP(z) = 0 \tag{5.4}$$

have no simultaneous solutions. Concerning CY threefolds, there are 7555 $\mathbb{P}^4_{\vec{w}}$ spaces admitting transverse hypersurfaces. The corresponding polynomial is then said to be of Fermat type with $P(z) = \sum_{i=0}^{4} z_i^{q_i}$ and $q_i = d/w_i \in \mathbb{Z}$.

Toric varieties

Even though defining CYs in terms of weighted projective spaces is quite straightforward, a fully fledged analysis of the CY properties in that language would be cumbersome. String theorists therefore prefer to recast the projective spaces as toric varieties and to work with certain representative combinatorial objects (like polytopes) to study the geometry of the CY. Generally speaking, a toric variety $\mathcal{M}_\Sigma$ is obtained if we allow for a more general quotient than by $\mathbb{C}^*$, namely

$$\mathcal{M}_\Sigma = \frac{\mathbb{C}^n \; - \; Z_\Sigma}{\mathbb{C}^*_\Sigma}\,, \tag{5.5}$$

where we mod out the action of an algebraic torus $\mathbb{T} = \mathbb{C}^*_\Sigma$ and Z_Σ is a subset fixed by a continuous subgroup of $\mathbb{C}^*_\Sigma$ that we remove. Before we get any more specific about the detailed description, let us name a few reasons why this framework is so powerful: the toric description makes mirror symmetry almost trivial since a mirror pair of CYs is nothing but a pair of two polytopes obeying a simple relation. Moreover, it is easy to retrieve topological quantities like intersection or Hodge numbers from toric data and it is straightforward to resolve the CY singularities. Of course, all these arguments will be explained in the course of the chapter.

We proceed by providing a first handy definition of toric varieties in terms of lattices and auxiliary objects called fans, to be defined soon. We consider a pair of dual lattices (N, M) which are both isomorphic to $\mathbb{Z}^m$ and considered as parts of the vector spaces $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, respectively. There is a pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ extending to the $\mathbb{R}$ bilinear pairing $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. The first relevant object is a *(strongly convex rational polyhedral) cone* in N (or *cone* for short) defined as

$$\sigma(v_1, \ldots, v_n) = \left\{ a_1 v_1 + \ldots + a_n v_n \;\middle|\; a_i \geq 0 \right\}\,, \tag{5.6}$$

generated by finitely many $v_1, \ldots, v_n \in N$ and satisfying $\sigma \cap (-\sigma) = \{0\}$ (*strongly convex*). Let us go through all these adjectives: Strong convexity means that the cone contains no line going through the origin. The cone is polyhedral if it is generated by finitely many vectors and it is rational if those generating vectors all lie in the lattice N.

A *face* of a cone σ is either the cone σ itself or the intersection of σ with one of the hyperplanes bounding σ. With these ingredients one can finally define *fans*: A collection of cones in $N_\mathbb{R}$ denoted as Σ is called a *fan* if

1. Each face of a cone in Σ is also a cone in Σ and

2. the intersection of two cones in Σ is a face of each.

A fan contains cones of different dimensions. We denote the set of k-dimensional cones by $\Sigma(k)$, where $k = 1, \ldots, m$. Each fan contains the zero cone $\Sigma(0)$, which corresponds to the whole toric variety. One-dimensional cones $v_i \in \Sigma(1)$ correspond to toric divisors and higher-dimensional cones to submanifolds of respective codimension.

Let us consider a fan Σ in N and focus on the set of one-dimensional cones (or edges). Let $\{v_i\}$ with $i = 1, \ldots, n$ be the vectors generating $\Sigma(1)$ where $n = |\Sigma(1)|$ and $n \geq m$. To each v_i we associate a homogeneous coordinate $z_i \in \mathbb{C}$ giving one of the holomorphic coordinates of $\mathbb{C}^n$ in (5.5). Let us first define the set Z_Σ in (5.5). We consider subsets of $\Sigma(1)$ with vectors $\{v_{i_1}, \ldots, v_{i_l}\}$ that do *not* belong to one cone Σ and denote the respective index set as $I \subseteq \{1, \ldots, n\}$. We then consider algebraic sets defined as $\{z_i = 0\}$ for all $i \in I$ and take their union:

$$Z_\Sigma = \bigcup_I \left\{ (z_1, \ldots, z_n) \,\middle|\, z_i = 0 \ \ \forall\, i \in I \right\} . \tag{5.7}$$

Next, we specify the action of the group $\mathbb{C}^*_\Sigma$. The set of the n generators of $\Sigma(1)$ with m components forms an $(n \times m)$ matrix $(v_i^k) = (v_1^k, \ldots, v_n^k)$ with $k = 1, \ldots, m$ and induces a map $\phi : \mathbb{C}^n \to \mathbb{C}^m$ with the action

$$\phi : (z_1, \ldots, z_n) \to (\prod_{i=1}^n z_i^{v_i^1}, \ldots, \prod_{i=1}^n z_i^{v_i^m}) . \tag{5.8}$$

ϕ projects the homogeneous coordinates of the ambient $\mathbb{C}^n$ to the geometric coordinates of the toric variety. The group $\mathbb{C}^*_\Sigma$ is defined as the kernel of ϕ, i.e. $\mathbb{C}^*_\Sigma = \mathrm{Ker}(\phi)$, and tells us which rescalings of the $\mathbb{C}^n$ coordinates that are identified under the quotient. It consists of those vectors $Q^{(a)} = (Q_1^{(a)}, \ldots, Q_n^{(a)})$ with $a = 1, \ldots, n - m$ that satisfy

$$\sum_{i=1}^n Q_i^{(a)} v_i = 0 . \tag{5.9}$$

In other words, the vectors $Q^{(a)}$ encode the linear relations among the generators of $\Sigma(1)$. They are commonly referred to as *charge vectors* due to the relation of toric varieties to gauged linear sigma models, which we will explore in section 5.4. One can easily check that $Q^{(a)}$ imposes the identification

$$(z_1, \ldots, z_n) \sim (\lambda^{Q_1^{(a)}} z_1, \ldots, \lambda^{Q_n^{(a)}} z_n) . \tag{5.10}$$

Let us look at a concrete and simple example, the ordinary projective space $\mathbb{P}^2$. The fan Σ describing this space lives on a two-dimensional lattice and the generators of $\Sigma(1)$ are given by the three vectors

$$v_1 = (1, 0) , \quad v_2 = (0, 1) , \quad v_3 = (-1, -1) . \tag{5.11}$$

Apart from the trivial zero fan there also three two-dimensional cones as shown in figure 5.1. To write down the toric variety, we need Z_Σ and $\mathbb{C}^*_\Sigma$. First we notice that only the complete set $\Sigma(1) = \{v_1, v_2, v_3\}$ doesn't generate a cone in Σ, so we only need to remove the origin. Second, since the vectors obey the relation $1 \cdot v_1 + 1 \cdot v_2 + 1 \cdot v_3 = 0$, we learn that the only charge vector Q has components $Q = (1, 1, 1)$, implying the equivalence relation $(z_1, z_2, z_3) \sim \lambda(z_1, z_2, z_3)$. In the toric diagram 5.1 this is visually represented by the fact that adding $Q_1 v_1$, $Q_2 v_2$ and $Q_3 v_3$ brings us back to the same point. In summary, we consistently find

$$\mathcal{M}_\Sigma = \frac{\mathbb{C}^2 - \{(0,0,0)\}}{\mathbb{C}^*} = \mathbb{P}^2 . \tag{5.12}$$

This can be straightforwardly generalized to higher-dimensional $\mathbb{P}^n$: one simply chooses $v_i = e_i$ for $i = 1, \ldots, n$ and $v_{n+1} = -\sum_{i=1}^n v_i$.

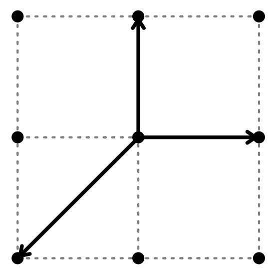


Figure 5.1: The fan for $\mathbb{P}^2$, generated by three one-dimensional cones.

5.1.2 Singularities and blow-ups

Now we demonstrate how the previously introduced combinatorial objects encode important properties of toric varieties in a geometric and intuitive way. In this subsection we focus on certain types of singularties of toric varieties and show how to resolve them by modifying the according fan. The resolution of singularities is critical for the analysis of periods: the moduli spaces of CY manifolds can also be described as toric varieties, and they need to be made entirely smooth to be able to set up the Picard-Fuchs equations at all points of interest.

The discussion of singularities and resolutions requires a few additional concepts. We first need to define *simplicial cones*. A k-dimensional cone σ is called simplicial if it is generated by k linearly independent vectors $v_1, \ldots, v_k$. We say that a fan Σ is simplicial if all its cones are simplicial (this was the case for $\mathbb{P}^2$, for example). This condition puts an enormous restriction on toric varieties: Given a simplicial fan Σ, one can show that the associated toric variety $\mathcal{M}_\Sigma$ can only have orbifold singularities.

Let us revisit the example $\mathbb{P}^2_{(2,3,1)}$ with homogeneous coordinates $(z_1 : z_2 : z_3)$. The fan Σ is depicted on the left-hand side in figure 5.2. The three generating vectors of $\Sigma(1)$ given by $v_1 = (1, 0), v_2 = (0, 1), v_3 = (-2, -3)$ obey the relation $2v_1 + 3v_2 + v_3 = 0$ and yield the charge vector $Q = (2, 3, 1)$. Each two-dimensional cone $\sigma(v_i, v_j)$ represents a patch of $\mathbb{P}^2_{(2,3,1)}$ with $z_k \neq 0$ where $k \notin \{i, j\}$. Since the fan is simplicial, we should only find orbifold singularities. And indeed, as discussed previously, there is only a $\mathbb{Z}_2$ singularity at $(z_1 : z_2 : z_3) = (1 : 0 : 0)$ and a $\mathbb{Z}_3$ singularity at $(0 : 1 : 0)$.

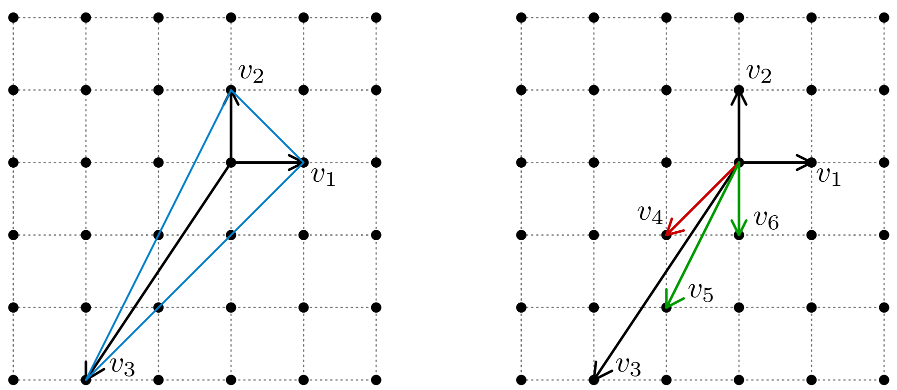


Figure 5.2: Left: Fan of the singular weighted projective space $\mathbb{P}^2_{(2,3,1)}$ with primitive vectors v_1, v_2, v_3. To make the toric variety smooth, one needs to introduce the three additional vectors v_4, v_5, v_6 as shown on the right-hand side. These new vectors end precisely at the lattice points through which the blue edges of the triangle-shaped polygon on the left go.

In the language of toric geometry the resolution of quotient singularities of toric varieties is very straightforward. To obtain a smooth toric variety, a given simplicial fan Σ must be converted into a so-called *regular* fan, for which each m-dimensional cone is generated by vectors that generate the whole lattice N. Individual cones whose generating vectors generate N are also called regular. Clearly, the left toric diagram in 5.2 shows two two-dimensional cones $\sigma(v_1, v_3)$ and $\sigma(v_2, v_3)$ whose vectors do *not* generate N. Given two vectors $\{v_i, v_j\}$, one can compute the order of the orbifold singularity from the determinant of the matrix with columns v_i, v_j. Here we find $|\det(v_2, v_3)| = 2$ and $|\det(v_1, v_3)| = 3$, indicating the orders of the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in the respective patches[1]. So to make $\mathbb{P}^2_{(2,3,1)}$ smooth, we must keep adding vectors to the fan until every two-dimensional cone is spanned by vectors generating N.

Let us show how to resolve the $\mathbb{Z}_2$ singularity as an example. If we add the vector $v_4 = (-1, -1)$ to the fan, the new cones $\sigma(v_2, v_4)$ and $\sigma(v_3, v_4)$ become regular. Since the vector sets $\{v_1, v_4\}$ and $\{v_2, v_3\}$ do not belong to one cone, our toric variety is

$$\mathcal{M}_{\Sigma'} = \frac{\mathbb{C}^4 - \{\{z_1 = z_4 = 0\} \cup \{z_2 = z_3 = 0\}\}}{\mathbb{C}^*_{\Sigma'}} . \tag{5.13}$$

Due to the additional relation $v_1 + v_2 + v_4 = 0$ we have two charge vectors $Q_1 = (2, 3, 1, 0)$ and $Q_2 = (1, 1, 0, 1)$ and the action of $\mathbb{C}^*_{\Sigma'}$ is $(z_1, z_2, z_3, z_4) \sim (\lambda^2 \mu z_1, \lambda^3 \mu z_2, \lambda z_3, \mu z_4)$ with $\lambda, \mu \in \mathbb{C}^*$. We can now look at different patches:

[1]This argument extends to higher-dimensional lattices: the determinant of the generators $v_1, \ldots, v_m$ gives the index of the sublattice, which is the order of the orbifold group. There the toric variety locally looks like $\mathbb{C}^m/G$, where G is a finite abelian group of order m.

- in the patch $z_4 \neq 0$, we can use μ to set $z_4 = 1$ and find a copy of the space $\mathbb{P}^2_{(2,3,1)}$ with the singularity $z_2 = z_3 = 0$ removed.

- If we take $z_4 = 0$, we must have $z_1 \neq 0$ according to (5.13). We can then define $\rho \equiv \lambda^2\mu$ and choose ρ such that $z_1 = 1$. As a result, we find $(1, z_2, z_3, 0) \sim (1, \lambda z_2, \lambda z_3, 0)$.

So in the patch with $z_4 = 0$ we find a $\mathbb{P}^1$ parametrizing a sphere in the (z_2, z_3)-directions. What we have done is to remove the singular point of the toric variety and replace it by a higher-dimensional smooth subvariety. This is a standard technique in algebraic geometry known as a *blow-up*.

Following the same logic, one can show that the resolution of the $\mathbb{Z}_3$ singularity requires two additional vectors v_5, v_6 shown on the right-hand side of 5.2. Geometrically, one can find all additional vectors by drawing line segments that connect the endpoints of $\{v_1, v_2, v_3\}$. The lattice points through which these lines go are the "missing" primitive vectors needed to construct a regular fan. Also this picture carries over to higher-dimensional lattices, where the additional lattice points lie to the inside of faces/interiors of higher-dimensional simplices.

5.1.3 CYs from toric varieties

We now want to focus on toric varieties that can describe CY spaces, in which case additional criteria must be imposed. The relevant condition for us is that for a CY variety Y the canonical bundle $K_Y = \wedge^m T^*Y$, defined as the m-th exterior product of the cotangent bundle T^*Y, must be trivial. A trivial canonical bundle implies the existence of a never vanishing global section, which for CY manifolds is the unique holomorphic top-form Ω. Recall that K_Y is an example of a line bundle, a family of one-dimensional vector spaces, one attached to each point in Y. On a general complex variety X, the holomorphic sections of a line bundle (locally, holomorphic functions) define a hypersurface inside X in terms of their zeros. To be more precise, an effective divisor $D = \sum_{i=1}^n n_i V_i$ with $n_i \in \mathbb{Z}^+$ and V_i irreducible hypersurfaces corresponds to a line bundle denoted as $\mathcal{O}_X(D)$. The sections of this bundle will vanish on each V_i with a zero of n_i-th order. We can extend the definition to arbitrary divisors if the $n_i < 0$ correspond to order n_i poles of the section.

For toric varieties $\mathcal{M}_\Sigma$ one can show that the canonical bundle is given

$$K_{\mathcal{M}_\Sigma} = \mathcal{O}_{\mathcal{M}_\Sigma}(-\sum_{i=1}^n D_i)\,, \tag{5.14}$$

where i runs over the one-dimensional cones in $\Sigma(1)$ and the D_i are toric divisors defined by the vanishing of the homogeneous coordinate z_i, so $D_i = \{z_i = 0\} \cap \mathcal{M}_\Sigma$. Hence we need to find the constraint that trivializes the bundle (5.14). Since toric divisors D_i are irreducible hypersurfaces defined by the vanishing of the monomial z_i, they naturally define a line bundle $\mathcal{O}_{\mathcal{M}_\Sigma}(D_i)$ whose section z_i has a zero of order one. Let us now consider the monomial $z_1^{a_1} \ldots z_n^{a_n}$ defining the line bundle $\mathcal{O}_{\mathcal{M}_\Sigma}(\sum_i a_i D_i)$ and assume that the coefficients are defined as $a_i = \langle v_i, m\rangle$, where $m \in M$ is a vector defined in the lattice dual to N. With the equivalence relations of the toric variety one finds

$$(\lambda^{Q_a^1} z_1)^{\langle v_1, m\rangle} \dots (\lambda^{Q_a^n} z_n)^{\langle v_n, m\rangle} = \lambda^{\langle \sum_i Q_a^i v_i, m\rangle}\, z_1^{\langle v_1, m\rangle} \dots z_n^{\langle v_n, m\rangle} \,. \tag{5.15}$$

Since $\sum_{i=1}^n Q_a^i v_i = 0$ for each a, the above monomial is globally defined and meromorphic and consequently the associated line bundle is trivial. So for the canonical bundle (5.14) to be trivial, there must be a vector $m_0 \in M$ satisfying $\langle v_i, m_0\rangle = 1$ for all i. Thus, if all primitive vectors v_i lie in the same affine hyperplane $H = \{v \in N_{\mathbb{R}} | \, \langle m_0, v\rangle = 1\}$, the associated toric variety satisfies the CY condition.

There is a second equivalent formulation of the CY condition: If we take the inner product of the generic relation (5.9) with an $m \in M$, we obtain $\sum_{i=1}^n Q_a^i \langle v_i, m\rangle = 0$. But if a toric variety is a CY, the existence of the special vector m_0 for which $\langle v_i, m_0\rangle = 1$ for all i implies $\sum_{i=1}^n Q_a^i = 0$ for all a.

Hypersurfaces inside toric varieties

If we restrict ourselves to CYs from toric varieties, we face a serious restriction: Toric varieties can be compact if and only if their fan Σ defined on a lattice $N_{\mathbb{R}}$ fills the entire lattice. But we have just argued that the generating vectors of the fan of a toric CY variety all lie in one affine hyperplane and thus such a variety must be non-compact. To construct a compact CY in toric geometry, we must consider hypersurfaces in compact toric varieties instead. Embeddings of this type also have a very elegant geometric description in terms of the so-called *Batyrev reflexive polytopes* [177]. In this language the information about the ambient toric variety and the defining polynomial of the CY hypersurface is encoded in a pair of dual polytopes. For a given fan Σ of a compact toric variety we consider the lattice polytope $\Delta^* \subset N_{\mathbb{R}}$ defined as the convex hull of the generators of $\Sigma(1)$, i.e. $\Delta^* = \mathrm{Conv}(v_1^*, \dots, v_n^*)$. This lattice polytope is called *reflexive* if

- the origin $v_0^* = (0, \dots, 0)$ lies strictly in the interior of Δ^*.
- the dual polytope defined as $\Delta = \{v \in M_{\mathbb{R}} \,|\, \langle v, v^*\rangle \geq -1 \;\forall\, v^* \in \Delta^*\}$ is again a lattice polytope, i.e. its vertices lie in M.

To construct the dual polytope Δ, we consider facets (co-dimension one faces) of Δ^*, which are in one-to-one correspondence with vertices of Δ. Suppose that such a facet is defined by a set of vertices $\{v_i^*\}$, where $i \in \{i_1, \dots, i_d\} \subset \{1, \dots, m\}$. We then need to solve the set of equations $\langle v, v_i^*\rangle = -1$ for the lattice vector $v \in M$ and check afterwards whether the solution satisfies the inequalities $\langle v, v_k^*\rangle = -1$ for the omitted k. The dual polytope Δ is the convex hull of all distinct vertices that we obtain. It is by construction also reflexive and $(\Delta^*)^* = \Delta$ holds[2]. A simple example illustrating this construction is the fan of $\mathbb{P}^1 \times \mathbb{P}^1$, whose generating vectors are $v_1^* = (1, 0)$, $v_2^* = (0, 1)$, $v_3^* = (-1, 0)$, $v_4^* = (0, -1)$. The dual polytope is just a rotated square, depicted in figure 5.3.

[2]Note that by definition each facet of Δ^* lies in some hyperplane at distance one from the origin. This is a weaker statement than the toric CY condition from before, which imposes that all primitive rays of the fan lie in a *single* hyperplane of height one. The toric variety is therefore not a CY by itself.

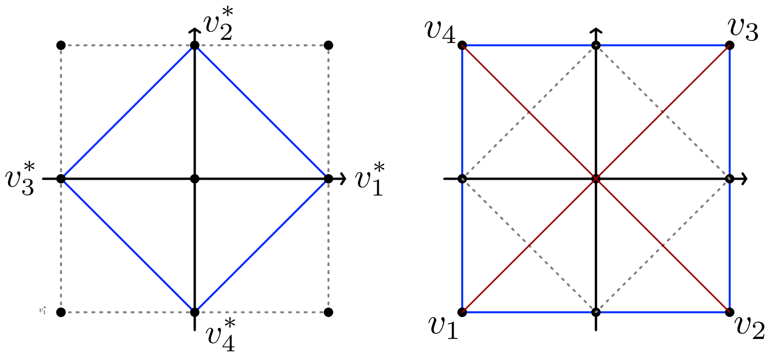


Figure 5.3: Reflexive lattice polytope from fan describing $\mathbb{P}^1 \times \mathbb{P}^1$ on the left and its dual on the right, each of them indicated in blue.

One might wonder how exactly the information about CYs is stored in the pair of dual polytopes. When we aim to construct CYs as hypersurfaces, the polytope Δ^* encodes the fan of the *ambient toric variety* $\mathcal{M}_\Sigma$ (in our use cases, weighted projective spaces). Lattice points on the boundary of Δ^* correspond to fan generators and are associated to homogeneous coordinates z_i of $\mathcal{M}_\Sigma$. A toric divisor D_i is defined by the vanishing of the coordinate z_i, which defines a section of the corresponding line bundle $\mathcal{O}_{\mathcal{M}_\Sigma}(D_i)$. We have also seen that the canonical divisor of the toric variety is defined by $K_{\mathcal{M}_\Sigma} = -\sum_i D_i$ where the sum runs over all toric prime divisors.

If we cut out a hypersurface from the toric variety, we need to make sure it meets the CY condition of a trivial canonical class. For a general complex variety X, a hypersurface will be defined by the section of the so-called *anti-canonical* bundle $K_X^{-1} \equiv \mathcal{O}_X(-K_X)$ of the variety, corresponding to the anti-canonical divisor $-K_X$[3]. This bundle is the dual of K_X and its sections are homogeneous polynomials $P(z_1, \dots, z_n)$ in the toric coordinates. According to the *adjunction formula*, a hypersurface $Y \subset X$ will have the canonical class

$$K_Y \simeq (K_X + [Y])|_Y, \tag{5.16}$$

where $[Y]$ denotes the divisor class of Y. We see that K_Y can be made trivial by imposing that Y is a divisor lying in the anti-canonical class $-K_X$, then Y is indeed a CY. In geometric terms, the bundle K_X measures the obstruction to having a globally defined holomorphic top-form and divisors defined through sections of K_X^{-1} fully compensate the "twist" of K_X.

Specializing again to toric varieties, we can now explain the importance of the dual polytope Δ. For a toric variety $\mathcal{M}_\Sigma$, the sections of the anticanonical bundle $K_{\mathcal{M}_\Sigma}^{-1}$ are parametrized by the entire set of lattice points contained by Δ. In other words, the lattice points in Δ are bijectively mapped to the allowed monomials that can be used to write a polynomial defining a CY hypersurface inside the toric variety. To each point $m \in M \cap \Delta$

[3]To avoid confusion, we write K_X^{-1} when talking about the anti-canonical line bundle and $-K_X$ when talking about the respective divisors.

we can associate the monomial

$$M(m) = \prod_{i=1}^{n} z_i^{\langle m, v_i \rangle + 1}, \tag{5.17}$$

where v_i are again the vectors in $\Sigma(1)$ defined in N. Equation (5.17) is famously known as the *monomial-divisor map*. Reflexivity of the polytopes ensures that $\langle m, v_i \rangle + 1 \geq 0$, so the monomial has no poles. It also ensures that $-K_{\mathcal{M}_\Sigma}$ is an honest integral divisor, so each term in (5.17) appears with integer powers (also known as *Gorenstein* condition). Therefore, a general polynomial defining a CY hypersurface in a toric variety is

$$P(z_1, \ldots, z_n) = \sum_{m \in M \cap \Delta} a_m \prod_{i=1}^{n} z_i^{\langle m, v_i \rangle + 1}, \tag{5.18}$$

where α_m are complex coefficients parametrizing the shape of the hypersurface. Thus the dual polytope Δ contains information about an entire *family* of CYs. To find the true number of complex structure moduli of the CY, one applies all scaling symmetries of the ambient toric variety to set as many a_i's to one as possible.

As an example we might consider the space $\mathbb{P}^2_{(2,3,1)}$ from section 5.1.2. Using the standard generators $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-2, -3)$, one can easily derive the dual polytope Δ, which in this case contains seven different lattice points. With formula (5.17) we can associate a monomial to each of these points (see figure 5.4).

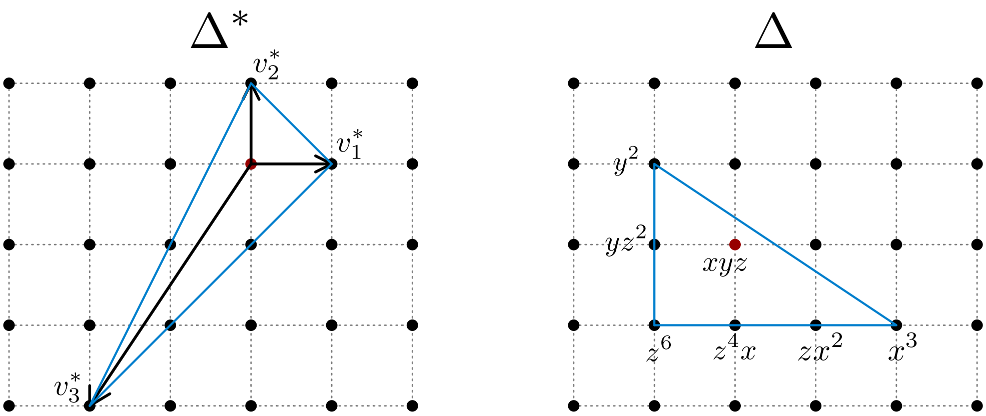


Figure 5.4: Dual polytopes for $\mathbb{P}^2_{(2,3,1)}$. Red dots indicate the origin of each lattice.

5.2 Computing CY periods

Section 5.1 taught us how defining properties of a CY manifold Y with (h_{11}, h_{21}) and its mirror dual $\hat{Y}$ with $(\hat{h}_{11} = h_{21}, \hat{h}_{21} = h_{11})$ are imprinted on a pair of reflexive polytopes (Δ^*, Δ). In this section we finally leverage this combinatorial data to set up the differential equations governing the periods of $\hat{Y}$, known as the *Picard-Fuchs equations*. For the most part of the section we will explicitly treat CY 3-folds from hypersurfaces in $\mathbb{P}^4_{\vec{w}}$, although these methods straightfowardly generalize to higher-dimensional d-folds (and more general toric varieties). The Picard-Fuchs equations of a 3-fold schematically read

$$\mathcal{L}_{(k)}\,\Pi_i(a_i) = 0 \quad \text{with} \quad \Pi_i(a_i) = \int_{\Gamma_i} \Omega(a_i)\,, \tag{5.19}$$

where a_i denote complex structure moduli, $\Gamma_i \in H_3(\hat{Y}, \mathbb{Z})$ constitute a basis of 3-cycles and $\mathcal{L}_{(k)}$ are differential operators. First we will show how to define a system of differential equations from the polytope data. This so-called GKZ hypergeometric system [178] (after Gelf'and, Kapranov and Zelevinsky) can then be directly related to the Picard-Fuchs system. We then describe an algorithm for determining a basis of local solutions around arbitrary singular points and show how to obtain an integral symplectic basis afterwards. Finally, we discuss how the set of periods can be continued to other singular points in moduli spaces.

5.2.1 Picard-Fuchs equations from toric data

We begin by setting up the differential equations in (5.19) for CY hypersurfaces, following the discussion in [179]. The initial input for a GKZ system are integral points $\{v_0, \ldots, v_p\}$ contained in the reflexive polytope Δ^* in $\mathbb{R}^n$. The origin v_0 is always contained in that set and $\{v_1, \ldots, v_p\}$ generate the fan of the toric variety. We start by embedding this polytope in a plane in $\mathbb{R}^{n+1}$ with distance one from the origin, choosing $\bar{v}_k = (1, v_k)$ where $k = 0, \ldots, p$. We then have $p+1$ integral points in the space $\mathbb{R}^{n+1}$, leading to a lattice of relations given by

$$L = \left\{ (l_0, \ldots, l_p) \in \mathbb{Z}^5 \;\middle|\; \sum_{i=0}^{p} l_i \bar{v}_i = 0 \right\}\,. \tag{5.20}$$

With our choice we have automatically implemented the condition $\sum_{i=0}^{p} l_i = 0$ for all vectors $l \in L$. With the height-one uplift to $\mathbb{R}^{n+1}$ we ensure that all monomials defining the hypersurface appear with a common degree and the total polynomial is homogeneous. In a suitable basis these l-vectors correspond to the generators of the Mori cone, so the number of distinct l-vectors gives the number of Kähler moduli on the A-side and number of complex structure deformations on the B-side. We then consider the affine complex space $\mathbb{C}^{p+1}$ with coordinates $(a_0, \ldots, a_p)$. To each vector $l \in L$ we assign the differential operator

$$\mathcal{D}_k = \prod_{l_i^{(k)}>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i^{(k)}} - \prod_{l_i^{(k)}<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i^{(k)}}\,. \tag{5.21}$$

We also introduce a second type of operators depending on the vertices $\bar{v}_i$, given by

$$\mathcal{Z}_j = \sum_{i=0}^{4} \bar{v}_{ij} a_i \frac{\partial}{\partial a_i} - \beta_j\,, \tag{5.22}$$

where $j = 1, \ldots, r$, so the number of $\mathcal{Z}$ is equal to the dimension of the fan. $\bar{v}_{ij}$ denotes the j-th component of the i-th vector $\bar{v}_i$. The set of differential equations defined by the operators (5.21) and (5.22) is known as a GKZ hypergeometric system with exponent vector β. For the special value $\beta = (-1, \ldots, 0)$, the periods Π_i satisfy such a GKZ system in the sense that they are annihilated by the respective operators [177]. Thus we have

$$\mathcal{D}_l \, \Pi_i = \mathcal{Z}_j \, \Pi_i = 0 \, , \quad i = 0, \ldots, 2\hat{h}_{21} + 1 \, . \tag{5.23}$$

The two types of operators have different meanings: the $\mathcal{Z}$-operators reduce the redundant coefficients space $\mathbb{C}^{p+1}$ to the true complex structure moduli space of the CY. Put differently, they enforce the homogeneity constraints defining the weighted projective space and thereby ensure that the periods are functions of coordinates invariant under these changes. At the LCS point the appropriate coordinates are readily defined in terms of the Mori cone generators and read

$$x_k = (-1)^{l_0^{(k)}} \prod_{i=0}^{p} a_i^{l_i^{(k)}} \, , \tag{5.24}$$

where $k = 1, \ldots, s$ with $s = |L|$. The determination of "good" coordinates around other singular points will be the topic of the next section. On the other hand, the $\mathcal{D}$-operators encode the relations among monomials defining the CY hypersurface. For the explicit CYs that we will analyze, they are directly related to Picard-Fuchs differential operators once we implement the constraints from the $\mathcal{Z}$-operators. In more general solution space of the GKZ system can be bigger in which case one needs to add further differential operators to obtain the Picard-Fuchs system (see type II and type III models in [179]). These additional operators represent relations of monomials corresponding to integral points of co-dimension one vertices in Δ^*.

With the coordinate choice (5.24) we can write down an explicit formula for the $\mathcal{D}$-operators in terms of the Mori cone generators. For each $l^{(k)} \in L$ we have

$$\mathcal{D}_k = \prod_{l_j^{(k)} > 0} \left(\prod_{i=0}^{l_j^{(k)} - 1} (\vartheta_j + i) \right) - x_k \prod_{l_j^{(k)} < 0} \left(\prod_{i=0}^{\left|l_j^{(k)}\right| - 1} (\vartheta_j - i) \right) . \tag{5.25}$$

where $\vartheta_j \equiv a_j \frac{\partial}{\partial a_j}$. We then introduce the logarithmic operators $\theta_{x_i} \equiv x_i \frac{\partial}{\partial x_i}$ which are related to the ϑ_j via

$$\vartheta_j = \sum_{k=1}^{s} l_j^{(k)} \theta_{x_k} \, . \tag{5.26}$$

The differential operators resulting from (5.25) oftentimes factorize into several irreducible pieces, where only one of these factors corresponds to the true LCS Picard-Fuchs operator, i.e. the minimal operator annihilating the periods. This reducibility also indicates that the GKZ system has a larger solution space, so the remaining operators eliminate the spurious solutions not corresponding to periods.

A concrete example

We want to supplement our general discussion about the computation of periods with a well-known example, namely the hypersurface $\mathbb{P}^4_{(1,1,2,2,6)}[12]$ with $(h_{11}, h_{21}) = (2, 128)$ defined by a degree 12 polynomial in the given weighted projective space. Several properties of this CY have been extensively studied in the seminal paper [180], including the geometry, the description of the mirror and its moduli space and many other aspects. In the following we will apply the general algorithm from toric geometry to set up the Picard-Fuchs system in the LCS regime of its mirror dual CY.

Step 1: Fetch polytope data. Given a weighted projective space $\mathbb{P}^4_{\vec{w}}$, the integral points of the reflexive polytope $\Delta^* \subset N_{\mathbb{R}}$ encoding its fan can be defined in terms of the weights $\vec{w}$ [179]. For $\mathbb{P}^4_{(1,1,2,2,6)}$, they are given by the origin $v_0^* = (0,0,0,0)$ and

$$\begin{aligned} &v_1^* = (-1,-1,-2,-6)\,,\ v_2^* = (1,0,0,0)\,,\ v_3^* = (0,1,0,0)\,,\\ &v_4^* = (0,0,1,0)\,,\ v_5^* = (0,0,0,1)\,. \end{aligned} \tag{5.27}$$

As we discussed before, weights $w_i > 1$ in weighted projective spaces produce orbifold singularities. Let $(z_1 : \ldots : z_5)$ denote the homogeneous coordinates of $\mathbb{P}^4_{(1,1,2,2,6)}$, then the $\mathbb{Z}_2$-type singularity is running along the curve $C = \{z_1 = z_2 = 0\}$. A hypersurface would inherit the singularity of the ambient space. To avoid that, we introduce an exceptional divisor that resolves C into $C \times \mathbb{P}^1$, i.e. each singular point in C is replaced by a $\mathbb{P}^1$. In toric geometry we resolve by refining the fan with the appropriate ray, which in this case adds the vertex $v_6^* = (0,-1,-1,-3)$.

Using the generators $\{v_1^*, \ldots, v_5^*\}$, we can derive the vertices of the dual polytope $\Delta \subset M_{\mathbb{R}}$. These vertices correspond to the monomials required to build a minimal Fermat-type hypersurface, while other integer lattice points contained by Δ give further allowed deformations. Each co-dimension one facet of Δ^* is given by the convex hull of four vertices. To determine a vertex of Δ we solve $\langle m, v_i \rangle = -1$ for $i \in \{i_1, \ldots, i_4\} \subset \{1, \ldots, 5\}$ and $m \in M$ and find [179]

$$\begin{aligned} &v_1 = (-1,-1,-1,-1)\,,\ v_2 = (11,-1,-1,-1)\,,\ v_3 = (-1,5,-1,-1)\,,\\ &v_4 = (-1,-1,5,-1)\,,\ v_5 = (-1,-1,-1,1)\,. \end{aligned} \tag{5.28}$$

Step 2: Determine polynomial of mirror. We first check the number of Kähler moduli on the A-side. To do so, we embed the generators $\{v_1^*, \ldots, v_6^*\}$ in a height-one plane in $\mathbb{R}^5$ via $\overline{v}_i^* = (1, v_i^*)$ and compute the distinct l-vectors defined via $\sum_i \overline{v}_i^* l_i^{(a)} = 0$. We find the two solutions

$$l^{(1)} = (-6,0,1,1,3,0,1)\,, \quad l^{(2)} = (0,1,0,0,0,1,-2) \tag{5.29}$$

and conclude that the CY has $h_{11} = 2$. One Kähler class is inherited from the ambient space while the other comes from the blow-up. The mirror dual should therefore have two complex structure moduli, limiting the number of lattice points in Δ that we should consider. To reproduce the choice of [180] we need the additional points $v_0 = (0,0,0,0)$ and

$v_6 = (5, -1, -1, -1)$. According to the monomial-divisor map (5.17), the corresponding polynomial reads

$$P = a_1 z_1^{12} + a_2 z_2^{12} + a_3 z_3^6 + a_4 z_4^6 + a_5 z_5^2 + a_0 z_1 z_2 z_3 z_4 z_5 + a_6 z_1^6 z_2^6 = 0\,, \tag{5.30}$$

where $a_1, \ldots, a_6 \in \mathbb{C}$ and the monomial multiplying a_i comes from the pairing of v_i with the vectors from Δ^*. Using coordinate reparametrizations $z_i \to c_i z_i$ with $c_i \in \mathbb{C}$ we can set $a_1 = \ldots = a_5 = 1$. For the residual parameters we choose $a_0 = -12\psi$ and $a_6 = -2\phi$ and arrive at the well-known form

$$P = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 = 0\,. \tag{5.31}$$

The parameters (ϕ,ψ) parametrize a whole family of hypersurfaces and span the complex structure moduli space of the mirror of $\mathbb{P}^4_{(1,1,2,2,6)}[12]$.

Step 3: Determine Picard-Fuchs equations. The Picard-Fuchs equations at the LCS are obtained from the respective GKZ system. From the linear relations (5.29) we construct the canonical LCS coordinates via (5.24) and find

$$x_1 \equiv x = \frac{a_2 a_3 a_4^3 a_6}{a_0^6} = -\frac{1}{2^{11}3^6}\frac{\phi}{\psi^6}\,, \quad x_2 \equiv y = \frac{a_1 a_5}{a_6^2} = \frac{1}{4\phi^2}\,. \tag{5.32}$$

Following [179], we also introduce rescaled variables $\bar{x} = 2^6 3^3 x$ and $\bar{y} = 4y$. Periods as functions of $(\bar{x}, \bar{y})$ automatically satisfy the constraints from the $\mathcal{Z}$-operators in (5.23). Using formula (5.25), we can derive the two Picard-Fuchs operators

$$\begin{aligned} \mathcal{L}_1 &= \theta_x^2(\theta_x - 2\theta_y) - 8x(6\theta_x + 1)(6\theta_x + 3)(6\theta_x + 5)\,, \\ \mathcal{L}_2 &= \theta_y^2 - y(2\theta_y - \theta_x)(2\theta_y - \theta_x + 1)\,, \end{aligned} \tag{5.33}$$

where $\theta_x = x\frac{d}{dx}$ and $\theta_y = y\frac{d}{dy}$. Note that in the first GKZ operator we have factored out $3\theta_x(3\theta_x+1)(3\theta_x+2)$ to obtain the Picard-Fuchs operator and eliminate spurious solutions.

5.2.2 Patches and local coordinates in moduli space

Picard-Fuchs systems are most often solved in the LCS regime of the mirror since the basis of periods at that locus encodes quantum corrections to the type IIA prepotential and higher-genus amplitudes in the weak coupling regime. However, the expansion of the LCS periods is only valid within a finite domain of the complex structure moduli space. If one is interested in the behavior of physical quantities away from that region, one needs to find a new set of local coordinates that covers the appropriate patch of moduli space and solve the Picard-Fuchs equations in terms of the new coordinates once again. In this section we outline how to construct these local coordinates systematically.

First we need to understand how moduli spaces can be described explicitly. In the first constructions by Greene and Plesser [22] a mirror dual manifold was obtained by taking the quotient of a hypersurface CY by a discrete symmetry group, which also induces an orbifold action in the moduli space of the mirror. To obtain a tractable moduli space one has resolve these singularities, compactify that space and make sure that all divisors at the boundary have normal crossings. While these steps are all rather case-specific in the historic approach, Batyrev's reflexive polytopes provide a universal framework that allows us to construct a moduli space as a toric variety that satisfies all above criteria. The construction goes as follows: We start from a pair of reflexive polytopes (Δ, Δ^*) and the complete lattice of relations $L = (l_i)^{(a)}$ with $a = 1, \ldots, r$. For $\mathbb{P}^4_{1,1,2,2,6}$, we found

$$L = \begin{pmatrix} -6 & 0 & 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} . \tag{5.34}$$

Let us now switch our perspective by viewing the columns in L as vectors in an r-dimensional lattice $N' \cong \mathbb{Z}^r$. We restrict ourselves to the subset of primitive generators, denoted as b_j $(j = 1, \ldots, n)$, which determine a subdivision of the space $\mathbb{R}^r$ into maximal cones. The moduli space of the toric variety is then given by another toric variety whose fan Σ' is built from the one-dimensional cones defined by the b_j [181, 182]. Σ' is usually referred to as the *secondary fan*[4]. In our example, the primitive generators are

$$b_1 = (1, 0)\,, \quad b_2 = (0, 1)\,, \quad b_3 = (1, -2)\,, \quad b_4 = (-1, 0)\,, \tag{5.35}$$

where each b_j is associated to a toric divisor $D_{b_j} = \{x_j = 0\}$ in the moduli space, as usual. The fan generated by these four vectors is not regular, hence we expect orbifold singularities according to our discussion in section 5.1.2. But in toric geometry we can easily resolve these singularities by subdividing the fan. In the concrete case, the vectors $b_5 = (1, -1)$ and $b_6 = (0, -1)$ make the fan regular and introduce two exceptional toric divisors. As a result we obtain smooth toric variety with honest normal crossings among the boundary divisors.

Toric geometry also provides a straightforward algorithm to construct local coordinates in the different patches of the moduli space. A patch is defined by a maximal cone in $\mathbb{R}^r$ that is spanned by the generators of the secondary fan. In general local coordinates need to be invariant combinations under toric rescalings, so if a fan has n generators obeying p linear relations Q_i^a $(a = 1, \ldots, p)$, a local coordinate x_{loc} satisfies

$$x_{loc} = x_1^{n_1} \cdot \ldots \cdot x_n^{n_n}\,, \quad \text{where} \quad x_{loc} \to \lambda^{\sum_j Q_j^a n_j} x_{loc} \overset{!}{=} x_{loc}\,. \tag{5.36}$$

Using the defining property $\sum_j Q_j^a b_j^k = 0$, we can set $n_j = \langle w, v_j \rangle$ for any N' lattice vector w to satisfy rescaling invariance. Given a maximal cone σ defining a patch U_σ, we can find the set of regular monomial functions defining valid local coordinates by constructing the dual cone $\sigma^\vee$ in the dual lattice M' as

$$\sigma^\vee = \{au \in \mathbb{R}^r \mid \langle u, b_j \rangle \geq 0 \text{ and } a \geq 0\}\,. \tag{5.37}$$

[4] The secondary fan divides the space of divisor classes into chambers, where each chamber corresponds to a region in which the same divisors remain effective. These chambers define different phases of the CY, which will be discussed in section 5.4.

The lattice points $\sigma \cap M'$ are in one-to-one correspondence to the monomials in the coordinate ring. To determine transition functions between two patches U_{σ_i} and U_{σ_j}, we need to find the generators of $\sigma_i \cap M'$ and $\sigma_j \cap M'$, denoted as $w_k^{(i)}$ and $w_{k'}^{(j)}$, and determine the linear relations

$$\sum_{k=1}^{r_i} q_k w_k^{(i)} = \sum_{k'=1}^{r_j} q_{k'} w_{k'}^{(j)} \,, \quad q_k, q_{k'} \in \mathbb{Z} \,. \tag{5.38}$$

Then the transition functions among the coordinates in the two patches are defined as

$$\prod_{k=1}^{r_i} (x_k^{(i)})^{q_k} = \prod_{k'=1}^{r_j} (x_{k'}^{(j)})^{q_{k'}} \,. \tag{5.39}$$

Let us apply these techniques to our example. With our choice of the Mori cone basis and the secondary fan we made sure that the LCS patch of the moduli space lies in the maximal cone $\sigma(b_1, b_2)$ by default. This patch is centered around the degeneration locus where the two toric divisors $D_{(1,0)} = \{x_1 = 0\}$ and $D_{(0,1)} = \{x_2 = 0\}$ intersect, with the local coordinates (x_1, x_2) given by (5.24). Coordinates in different patches can be easily expressed in terms of (x_1, x_2): Since $\sigma(b_1, b_2)$ is self-dual, formula (5.39) becomes

$$x_k^{(i)} = \prod_{k'=1}^{2} x_{k'}^{\langle w_{k'}^{(i)}, b_{k'} \rangle} \,, \tag{5.40}$$

implying that the powers of monomials are given by the components of the extremal rays of the respective dual cone. When expressed in terms of the rescaled variables $(\bar{x}, \bar{y})$, the local coordinates around the different intersections of boundary divisors are thus given by

$$\begin{gathered} D_{(1,0)} \cap D_{(0,1)} : (\bar{x}, \bar{y}) \,, \quad D_{(1,-1)} \cap D_{(1,0)} : \left(\bar{x}\bar{y}, \frac{1}{\bar{y}} \right) \,, \\ D_{(0,-1)} \cap D_{(1,-1)} : \left(\bar{x}, \frac{1}{\bar{x}\bar{y}} \right) \,, \quad D_{(-1,0)} \cap D_{(0,1)} : \left(\frac{1}{\bar{x}}, \bar{y} \right) \,. \end{gathered} \tag{5.41}$$

Discriminant loci and blow-ups

To fully assemble the moduli space we also need to incorporate the discriminant loci of the complex structure parameters, which are the set of values for which the CY becomes singular. If a family of CY hypersurfaces Y_{λ_a} with complex structure parameters λ_a is defined by the polynomials $P_{\lambda_a}(z_1, \ldots, z_n) = 0$, these loci are defined by the equations

$$P_{\lambda_a} = 0 \quad \text{and} \quad \frac{\partial}{\partial z_i} P_{\lambda_a} = 0 \quad \forall\, i = 1, \ldots, n \,. \tag{5.42}$$

In moduli space the corresponding parameter sets lead to divisors that can potentially lead to non-normal crossings or tangencies with toric divisors or among themselves. We must therefore find the explicit parametrization of the discriminants and introduce further blow-ups to be able to compute periods around these special points.

Let us focus on the moduli space of $\mathbb{P}^4_{1,1,2,2,6}[12]$ right away. The discriminants are found by solving the equations (5.42) for the polynomial (5.31). When expressed in terms of the rescaled coordinates $(\bar{x},\bar{y})$, the discriminants read

$$\begin{aligned} \Delta_1 &= (1-\bar{x})^2 - \bar{x}^2\bar{y} = 0\,, \\ \Delta_2 &= 1-\bar{y} = 0\,. \end{aligned} \tag{5.43}$$

From the perspective of the CY geometry, $\Delta_1 = 0$ gives rise to a conifold singularity at which one of the A-cycles shrinks to zero size, implying that the respective period also vanishes. $\Delta_2 = 0$ is rather difficult to interpret geometrically, but can be understood as the weak coupling limit of the heterotic dual string theory. To these equations we associate the moduli space divisors $D_1 = \{\Delta_1 = 0\}$ and $D_2 = \{\Delta_2 = 0\}$. Inspection of the defining algebraic equations reveals that the discriminants have the following intersections:

- $D_1 \cap D_2$ intersect vertically at $(\bar{x} = \frac{1}{2}, \bar{y} = 1)$.
- intersections with divisors partially lying at the LCS are $D_1 \cap D_{(0,1)}$ at $(\bar{x} = 1, \bar{y} = 0)$ and $D_2 \cap D_{(1,0)}$ at $(\bar{x} = 0, \bar{y} = 1)$. The former is a tangency of order 2, which can be seen by setting $\bar{y} = 0$ in Δ_1, while the latter is a vertical intersection.
- there is a triple intersection $D_1 \cap D_2 \cap D_{(-1,0)}$ at $(\bar{w} = \frac{1}{\bar{x}} = 0, \bar{y} = 1)$. To show that, one writes $\Delta_1 = 1 - 2\bar{x} + \bar{x}^2\Delta_2$ and introduces $\bar{x} = 1/\bar{w}$.

To achieve normal crossings among all boundary divisors, we study a model that can locally describe all appearing tangencies/intersections at once. This model is parametrized by $W(x,y) = x^2 - y^3 = 0$, which defines a divisor D in $\mathbb{C}^2$ in terms of complex coordinates (x,y). The singularity at $(0,0)$ can be resolved completely by performing three consecutive blow-ups. Generally speaking, a blow-up replaces a point by the set of all lines passing through it. This means that this point is resolved into a copy of $\mathbb{P}^1$, called the exceptional divisor E, while rest of the space remains unchanged. Formally, we write

$$\mathrm{Bl}_{(0,0)}(\mathbb{C}^2) = \left\{ \Big((x,y),(u:v) \Big) \in \mathbb{C}^2 \times \mathbb{P}^1 \,\middle|\, xu = yv \right\} \tag{5.44}$$

for the blow-up of $\mathbb{C}^2$ at the origin, where $(u:v)$ parametrize the $\mathbb{P}^1$ and encode the line passing through the origin and the point (x,y). The resulting $\mathbb{P}^1$ can be covered by two charts defined via $u = 1$ and $v = 1$, respectively. In each chart one then substitutes the blow-up coordinates into the curve equation and factors out the maximal power of the coordinate that parametrizes the exceptional divisor. The remaining equation defines the strict transform of the curve, which intersects with the exceptional divisor in only one of the two charts. The procedure is repeated until all remaining intersections are transverse. In the following we spell out this algorithm for our concrete model (note that we *very closely* follow the discussion in [183]):

1st blow-up: We introduce a $\mathbb{P}^1$ with homogeneous coordinates $(u_0 : v_0)$ satisfying $u_0 x - v_0 y = 0$ and denote this divisor by E_0. In the two different patches we get:

- patch $u_0 = 1$: $x = v_0 y$, then $W = y^2(v_0^2 - y)$. Divisors $E_0 = \{y = 0\}$ and $D = \{v_0^2 - y = 0\}$ do not intersect transversely in $(0, 0)$.

- patch $v_0 = 1$: $y = u_0 x$, then $W = x^2(1 - u_0^3 x)$. Divisors $E_0 = \{x = 0\}$ and $D = \{1 - u_0^3 x = 0\}$ do not intersect.

2nd blow-up: We introduce a $\mathbb{P}^1$ with homogeneous coordinates $(u_1 : v_1)$ satisfying $u_1 v_0 - v_1 y = 0$ and denote this divisor by E_1. In the two different patches we get:

- patch $u_1 = 1$: $v_0 = v_1 y$, then $W = y^3(v_1^2 y - 1)$. Divisors $E_1 = \{y = 0\}$ and $D = \{v_1^2 y - 1 = 0\}$ do not intersect.

- patch $v_1 = 1$: $y = u_1 v_0$, then $W = u_1^2 v_0^3(v_0 - u_1)$. Divisors $E_0 = \{u_1 = 0\}$, $E_1 = \{v_0 = 0\}$ and $D = \{v_0 - u_1 = 0\}$ do not intersect transversely in $(0, 0)$.

3rd blow-up: We introduce a $\mathbb{P}^1$ with homogeneous coordinates $(u_2 : v_2)$ satisfying $u_2 v_0 - v_2 u_1 = 0$ and denote this divisor by E_2. In the two different patches we get:

- patch $u_2 = 1$: $v_0 = v_2 u_1$, then $W = u_1^6 v_2^3(v_2 - 1)$. Divisors $E_2 = \{u_1 = 0\}$, $E_1 = \{v_2 = 0\}$ and $D = \{v_2 - 1 = 0\}$ do not intersect.

- patch $v_2 = 1$: $u_1 = u_2 v_0$, then $W = u_2^2 v_0^6(1 - u_2)$. Divisors $E_2 = \{v_0 = 0\}$, $E_0 = \{u_2 = 0\}$ and $D = \{1 - u_2 = 0\}$ do intersect transversely in $(0, 0)$.

These relations can be directly applied to model the crossings of the moduli space divisors. After the first blow-up, W locally describes the tangency of D_1 and $D_{(0,1)}$ in the chart with $u_0 = 1$ upon identifying $D \to D_1$ and $E_0 \to D_{(0,1)}$. At the level of coordinates we obtain $v_0 = 1 - \bar{x}$ and $y = \bar{x}^2\bar{y}$, and for the remaining ones we find

$$\begin{aligned} u_1 &= \frac{y}{v_0} = \frac{\bar{x}^2\bar{y}}{1-\bar{x}}\,, \quad & v_1 &= \frac{v_0}{y} = \frac{1-\bar{x}}{\bar{x}^2\bar{y}}\,, \\ u_2 &= \frac{u_1}{v_0} = \frac{\bar{x}^2\bar{y}}{(1-\bar{x})^2}\,, \quad & v_2 &= \frac{v_0}{u_1} = \frac{(1-\bar{x})^2}{\bar{x}^2\bar{y}}\,. \end{aligned} \tag{5.45}$$

The local coordinates around the different intersections are provided in table 5.1. The labeling P_1, P_1', P_1'' was introduced for later convenience.

Local coordinates #1			
Locus	$P_1 = D_{(0,1)} \cap E_2$	$P_1' = D_1 \cap E_2$	$P_1'' = E_1 \cap E_2$
(x_1, x_2)	$\left(\frac{\bar{x}^2\bar{y}}{(1-\bar{x})^2}, 1-\bar{x}\right)$	$\left(1 - \frac{\bar{x}^2\bar{y}}{(1-\bar{x})^2}, 1-\bar{x}\right)$	$\left(\frac{(1-\bar{x})^2}{\bar{x}^2\bar{y}}, \frac{\bar{x}^2\bar{y}}{1-\bar{x}}\right)$

Table 5.1: Coordinates in local charts after first blow-up.

After the second blow-up, the patch with $v_1 = 1$ locally describes the triple crossing $D_1 \cap D_2 \cap D_{(-1,0)}$. To see the correspondence, we describe the discriminants in terms of

$\bar{w} = 1/\bar{x}$ once again. We then identify $D \to D_1$, $E_0 \to D_2$ and $E_1 \to D_{(-1,0)}$, which implies $u_1 = 1 - \bar{y}$ and $v_0 = \alpha\bar{w}$ with $\alpha = 2 - \bar{w}$. For the residual coordinates we obtain

$$u_2 = \frac{u_1}{v_0} = \frac{\bar{x}}{\alpha}(1 - \bar{y}) \,, \quad v_2 = \frac{v_0}{u_1} = \frac{\alpha}{\bar{x}(1 - \bar{y})} \,. \tag{5.46}$$

To avoid a mix-up of the divisors, the exceptional divisor separating D_1 and D_2 will be relabeled $E_2 \to E_3$. The coordinates at the remaining intersections are shown in table 5.2. Again, we introduced P_{13}, P_{23}, P_{03} for later notational convenience. Note that overall

Local coordinates #2			
Locus	$P_{13} = D_1 \cap E_3$	$P_{23} = D_2 \cap E_3$	$P_{03} = D_{(-1,0)} \cap E_3$
(x_1, x_2)	$\left(\frac{1}{\bar{x}}\left(\bar{x}^2\bar{y} - (1-\bar{x})^2\right), \frac{1}{\bar{x}}\right)$	$\left(\bar{x}(1-\bar{y}), \frac{1}{\bar{x}}\right)$	$\left(\frac{1}{\bar{x}(1-\bar{y})}, 1 - \bar{y}\right)$

Table 5.2: Coordinates in local charts after second blow-up.

factors of α can be safely neglected since $\alpha \neq 0$ at $\bar{w} = 0$. A sketch of the resolved moduli space with all toric and non-toric exceptional divisors is shown in figure 5.5 (essentially the figure from [180]). Not all intersections will be needed for our purpose, but we wanted to nevertheless provide a complete description of the resolution procedure that can be easily generalized to other CYs.

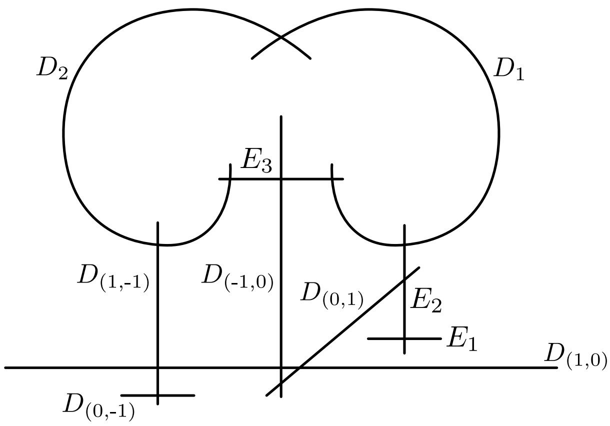


Figure 5.5: Sketch of completely resolved moduli space of $\mathbb{P}^4_{(1,1,2,2,6)}[12]$ with normal crossings among divisors only.

5.2.3 Frobenius solution

Now we can discuss how to obtain solutions of Picard-Fuchs equations in the vicinity of singular points of a CY moduli space. At each point one essentially follows two steps: first one needs to determine a complete basis of local periods analytically using the method of Frobenius, which applies to linear ODE's near regular singular points. We will denote such solutions as π_j, where $j = 1, \ldots, 2\hat{h}_{21} + 2$. In very broad terms these are power series in the complex structure moduli and logarithms thereof. In the second step one determines a transition matrix $\mathcal{T}$ that rotates this solution to an *integral symplectic basis* via $\Pi = \mathcal{T}\pi$. This section deals with the first step, while the computation of $\mathcal{T}$ will be discussed in the next part.

First we introduce some shorthand notation to keep expressions compact. Let $x = (x_1, \ldots, x_{\hat{h}_{21}})$ be the local coordinates near a singular point in moduli space. Let $\rho = (\rho_1, \ldots, \rho_{\hat{h}_{21}})$ be an index vector indicating the lowest polynomial powers of a power series. We also introduce the multi-index $n = (n_1, \ldots, n_{\hat{h}_{21}}) \in \mathbb{N}^{\hat{h}_{21}}$ that we use to sum over all orders. We start our analysis by making a power series ansatz of the form

$$w_j = \sum_{n=0}^{\infty} c_j(n) x^{n+\rho} \,, \tag{5.47}$$

where $x^{n+\rho} \equiv x_1^{n_1+\rho_1} \ldots x_{\hat{h}_{21}}^{n_{\hat{h}_{21}}+\rho_{\hat{h}_{21}}}$ and $\sum_{n=0}^{\infty} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_{\hat{h}_{21}}=0}^{\infty}$. After inserting this into the Picard-Fuchs equations we obtain a plethora of constraints on the coefficients $c_j(n)$ at different orders in the variables $x_1, \ldots, x_{\hat{h}_{21}}$. First we restrict ourselves to the lowest order equations from all Picard-Fuchs operators, which are known as the *index equations*. If we solve these we obtain the allowed index sets $\rho^{(a)}$, each of which is associated to one distinct pure power series solution. There is always one distinguished period known as the *fundamental period*, often denoted by ω_0, which has the index $\rho^{(0)} = (0, \ldots, 0)$. The full power series solutions can then be found by plugging the ansatz with the known $\rho^{(a)}$ back into the Picard-Fuchs equations and solving for the $c_j(n)$ order by order.

The relations among the different root vectors $\rho^{(a)}$ solving the index equations also affect the structure of the whole Frobenius solution. Typically one finds less than $2\hat{h}_{21} + 2$ distinct index sets, in which case the remaining periods must be constructed from descendants of the pure power series containing logarithmic factors. The more general ansatz for a period can then be written as

$$\pi_j = \text{const.} \sum_{0 \leq |t| \leq 3} \sum_{n=0}^{\infty} c_j(n) x^{n+\rho} \log^t(x) \,, \tag{5.48}$$

where we again introduced the compact notation $t = (t_1, \ldots, t_{\hat{h}_{21}})$. Here $|t| = \sum_{i=1}^{\hat{h}_{21}} t_i$, $\log^t(x) = \prod_{i=1}^{\hat{h}_{21}} \log(x_i)^t$ and the power to the multi-index t is understood as before. Regarding normalization, we choose to divide each solution by $1/(2\pi i)^\alpha$, where α is the highest logarithmic power appearing in the period. Note that highest possible power for the logarithm in (5.48) is determined by the maximum degree of nilpotency of the monodromy matrix .

To find the entire set of periods, one can proceed as follows: first, one determines the solutions of the index equations and associated pure power series. Then one makes an ansatz for a period containing terms with powers of logarithm factors up to one. This includes a yet undetermined pure power series and the power series from the first step multiplied by all possible linear log-factors with unknown prefactors. To give a simple example, suppose that we compute the local periods of a model with two moduli at the LCS with local coordinates (x, y). The only power series solution is the fundamental period ω_0 with index $(0, 0)$. Solutions with linear logarithms then have the general form

$$(2\pi i)\,\pi_j^{(1)} = \omega_j^{(0)} + \alpha_j^{(1,0)}\omega_0 \log(x) + \alpha_j^{(0,1)}\omega_0 \log(y)\,, \tag{5.49}$$

where $\alpha_j^{(1,0)}, \alpha_j^{(0,1)} \in \mathbb{C}$ and the pure power series $\omega_j^{(0)}$ are determined from the constraints of the Picard-Fuchs operators. Once these periods are determined one can continue by making an ansatz for periods with logarithms up to quadratic order and higher, following the same recipe. This procedure needs to be repeated until a set of $2\hat{h}_{21} + 2$ linearly independent period solutions is obtained.

When solving for the coefficients of the periods, one needs to pay special attention to the order to which the constraints from the Picard-Fuchs equations are valid. If one inserts an ansatz for periods with some maximal order $|n| = \sum_i n_i$, the order up to which the constraints are valid is typically lower than $|n|$, depending on the individual case. If too high orders are taken into account, solutions become over-constrained and one may not be able to find the higher logarithmic descendants in the following steps.

Solution at the LCS

At the LCS we can write down the solutions for all periods without going through the general procedure. The full basis of periods is built upon the fundamental period $\omega_0(x)$ given by

$$\omega_0 = \sum_{n=0}^{\infty} c(n) x^n = \sum_{n=0}^{\infty} \frac{(\sum_j |l_j^{(0)}| n_j)!}{\prod_i \left(\sum_j l_j^{(i)} n_j\right)!} x^n\,. \tag{5.50}$$

To construct the remaining periods we take derivatives of the function

$$\omega_0(x, \rho) = \sum_{n=0}^{\infty} c(n+\rho) x^{n+\rho} \tag{5.51}$$

with respect to indices in ρ and eventually evaluate at $\rho = (0, \ldots, 0)$. The structure at the LCS is always the same: we have a unique power series given by the fundamental period, $\hat{h}_{21}$ solutions with linear logarithms, the same number of solutions with quadratic logarithms and finally one solution with cubic terms. We can immediately determine the periods in the symplectic basis by acting with the differential operators

$$\begin{aligned} D_i^{(1)} &= \frac{1}{2\pi i}\frac{\partial}{\partial \rho_i}\,, \\ D_i^{(2)} &= \frac{1}{2}\frac{1}{(2\pi i)^2}\kappa_{ijk}\frac{\partial}{\partial \rho_j}\frac{\partial}{\partial \rho_k}\,, \\ D^{(3)} &= -\frac{1}{6}\frac{1}{(2\pi i)^3}\kappa_{ijk}\frac{\partial}{\partial \rho_i}\frac{\partial}{\partial \rho_j}\frac{\partial}{\partial \rho_k}\,, \end{aligned} \tag{5.52}$$

where $i = 1, \ldots, \hat{h}_{21}$ and κ_{ijk} are the topological triple intersection numbers. The period vector at the LCS is then given by

$$\pi^{\mathrm{LCS}}(x) = \begin{pmatrix} \omega_0(x) \\ D_i^{(1)}\omega_0(x,\rho)|_{\rho=0} \\ D_i^{(2)}\omega_0(x,\rho)|_{\rho=0} \\ D^{(3)}\omega_0(x,\rho)|_{\rho=0} \end{pmatrix} . \tag{5.53}$$

5.2.4 Integral symplectic basis

The Frobenius method gives us a complete basis of local analytic periods, but it does not tell us how these are related to the periods (2.102), which are defined as integrals over basic three-cycles in $H_3(\hat{Y}, \mathbb{Z})$. First we need to determine the integral symplectic basis in the LCS regime using topological data of the CY as an input.

Obtaining the integral symplectic basis at the LCS is quite straightfoward. The LCS regime of $\hat{Y}$ corresponds to the large volume regime of the dual CY Y, i.e. the Kähler moduli t^i of Y satisfy $\mathrm{Im}(t^i) \gg 1$ and the prepotential is hence dominated by its classical part. These classical terms depend on on the topological data of the CY and contain enough information to fix the transition matrix. For a general CY threefold Y the prepotential in the large volume regime is given by[5]

$$\mathcal{F}_0(t) = \frac{1}{6}\kappa_{ijk}t^i t^j t^k + \frac{1}{2}a_{ij}t^i t^j + b_i t^i + \frac{c}{2} + \frac{1}{(2\pi i)^3}\sum_{\beta \in H_2(Y,\mathbb{Z})} \mathrm{Li}_3(e^{2\pi i \beta\cdot t}) , \tag{5.54}$$

where $i = 1, \ldots, h_{11} = \hat{h}_{21}$. The constants b_i and c are defined in terms of topological invariants and read

$$b_i = -\frac{1}{24}\int_Y c_2 \wedge J_i , \quad c = -\frac{\chi}{(2\pi i)^3}\zeta(3) , \tag{5.55}$$

where J_i is the Kähler form associated to the Kähler parameter t^i. The constants a_{ij} are not related to topological quantities. The continuous Peccei-Quinn shift symmetry of the B-field axion fields is broken down to a discrete subgroup $t^i \to t^i + 1$ by quantum effects and enforces $\mathrm{Im}(a_{ij}) = \mathrm{Im}(b_i) = 0$. One can fix the values of the a_{ij} by acting with this symmetry on the period vector and imposing that the action corresponds to a symplectic transformation. The corresponding period vector will be normalized as[6]

$$\hat{\Pi}(t) = (1, t^i, \partial_{t^i}\mathcal{F}_0, 2\mathcal{F}_0 - t^i\partial_{t^i}\mathcal{F}_0) , \tag{5.56}$$

[5] Here the overall prefactor $(2\pi i)^3/g_s^2$ has been set to 1. The relation to the prepotential in terms of homogeneous coordinates is given by $\mathcal{F}_0(t) = (X^0)^{-2}\mathcal{F}_0(X)|_{X^i=2\pi t^i X^0}$. We will denote the prepotentials with the same symbol; the distinction should be clear from the variables on which $\mathcal{F}_0$ depends.

[6] To preserve the standard canonical symplectic pairing, we order the period vector such that the t^i appear in the usual order, while the derivatives $\partial_{t^i}\mathcal{F}_0(t)$ are listed in the reverse order. Matching this to the standard supergravity normalization requires a constant symplectic transformation.

where $\hat{\Pi}(t)$ is written in terms of the coordinates t^i defined via $2\pi\, t^i = X^i/X^0$. At this point, we can apply mirror symmetry: the flat Kähler coordinates of Y in the $\mathrm{Im}(t^i) \gg 1$ regime can be expressed through the LCS periods of $\hat{Y}$ in the Frobenius basis. Each Kähler parameter of Y is given by the ratio of a period with (at most) linear logarithms of the complex structure moduli and the fundamental period, i.e.

$$t^i = \frac{\pi_{i+1}^{\mathrm{LCS}}(x)}{\pi_1^{\mathrm{LCS}}(x)} = \frac{1}{2\pi i}\log\big(x^i\big) + \dots\,, \tag{5.57}$$

where we have only displayed the leading order term in the expansion. To obtain the transition matrix, we compare the leading terms of the period vectors on the A-side and the B-side, both expressed in terms of complex structure moduli. So on the one hand we take the Frobenius solution (5.53), divide out the fundamental period and only keep the lowest order in the expansion around $x = (0, \dots, 0)$, while on the other hand we insert the dominant term from (5.57) into (5.56). There exists a complementary approach for determining this transformation matrix, which utilizes the transformation behaviour of the periods under the monodromies [184]. In fact, the LCS is the only point where the monodromies alone are sufficient to completely fix the integral symplectic basis.

Moving away from the LCS regime

With the integral symplectic basis at the LCS at hand, the missing piece is to compute transition matrices at arbitrary singular points that bring us from a local Frobenius basis to an integral symplectic one. The local solutions from the Picard-Fuchs equations only have a finite convergence radius, typically set by the location of the nearest singularity. Suppose we are given two (general) neighboring moduli space patches A and B centered around singular points, with local Frobenius solutions π_A and π_B. Let us first describe the most general way of finding a numerical transition matrix between those two patches. First we need to select a certain basepoint $x^{(0)} = (x_1^{(0)}, \dots, x_{\hat{h}_{21}}^{(0)})$ at which both period sets (truncated at a sufficiently high order) nicely converge. We can then Taylor expand both of these periods around that basepoint to obtain

$$\pi_{A,B} = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \partial^\alpha \pi_{A,B}\Big|_{x^{(0)}} (x - x^{(0)})^\alpha\,, \tag{5.58}$$

where once again we have introduced a multi-index $\alpha = (\alpha_1, \dots, \alpha_{\hat{h}_{21}})$ and $\alpha! = \alpha_1! \dots \alpha_{\hat{h}_{21}}!$. We have also introduced the abbreviated derivative notation

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial\alpha_1 \dots \partial\alpha_{\hat{h}_{21}}} \tag{5.59}$$

with $|\alpha| = \alpha_1 + \dots + \alpha_{\hat{h}_{21}}$. In order to compare the two period sets we must express both of them in terms of one common set of complex structure coordinates. For a transition matrix defined as $\pi_A = \mathcal{T}_A^B \pi_B$ the comparison leads to a collection of linear equations for the matrix entries, valid at different powers of all possible monomials. In multi-modulus scenarios these constraints must be solved order by order in total degree $|\alpha|$.

In general multi-dimensional setups it can be quite tricky to find a convenient basepoint in which the convergence regions intersect. We can circumvent this issue by simply setting some of the moduli to a fixed value, typically the midpoint in a certain direction, and work with expansions in fewer variables instead. We will apply this technique to the concrete CYs with $h_{11} = 2$, fixing one of the two complex structure moduli and matching the expansions in the remaining free coordinate.

If we are after the transition matrix at a singular point whose patch is not overlapping with our initial starting point, several transition matrices need to be computed and chained together. For instance, one might have three patches A, B, C around singular points. The transition matrix for $A \to C$ can then be found by first analytically continuing $A \to B$ and then $B \to C$, so that

$$\mathcal{T}_A^C = \mathcal{T}_A^B \mathcal{T}_B^C \,. \tag{5.60}$$

In practice, we use the integral symplectic basis at the LCS as our initial reference point. We then match this basis with periods defined in charts that are "off" by only one complex structure coordinate relative to the LCS, using the above technique of partially fixing coordinates. With the newly obtained analytic continuation we can then match with periods at other singular points that are further away from the LCS.

The results for the transition matrices can be double-checked with another method: One may simply choose $\hat{h}_{21}$ points $x_i^{(p)}$ with $i = 1, \ldots, \hat{h}_{21}$ in the overlap region between two patches A and B, evaluate the periods at these points at a sufficiently high order and then directly solve the equations

$$\pi_A(x_i^{(p)}) = \mathcal{T}_A^B \pi_B(x_i^{(p)}) \tag{5.61}$$

numerically. This produces enough equations to solve for the entries of the matrix. The result for the transition matrix from both methods should agree, but the latter method of choosing points is much more unstable numerically. In practice, it is hard to decide whether a given matrix entry is really zero or just a numerically small number, unless a very high expansion order of the periods is taken into account.

5.3 Mirror map and Yukawa couplings

By now the main part of our technical preparations is finished and we can finally zoom in on the physical quantities of interest. As a reminder, our goal is to test whether the classical Yukawa couplings in the effective action of type IIA CY compactifications can arise as a quantum effect in M-theory limits. Recall that Yukawa couplings are three-point functions of scalar fields from vector multiplets, which in the type IIA case are the Kähler moduli t^i. Taking the weak coupling expansion of the prepotential in (5.54) as given, the full Yukawa couplings take the form

$$Y_{ijk}(t) = \partial_{t^i}\partial_{t^j}\partial_{t^k}\mathcal{F}_0(t) = \kappa_{ijk} + \sum_{\beta \in H_2(Y,\mathbb{Z})} \alpha_0^\beta \beta^i \beta^j \beta^k \frac{q^\beta}{1-q^\beta} \,, \tag{5.62}$$

where $q^\beta = \prod_i q_i^{\beta_i}$ and $q_i = e^{2\pi i t^i}$. We will refer to these as *A-side Yukawa couplings.* On top of the classical triple intersection numbers κ_{ijk} there is a sum over instanton corrections coming from string worldsheets wrapping genus zero curves of a CY Y. The sum is over effective curve classes $\beta \in H_2(Y,\mathbb{Z})$ and each such contribution is weighed by α_0^β. Our proposal for how to obtain the classical Yukawa couplings in (5.62) from a one-loop Schwinger integral will not be presented before section 6. However, we want to mention already here that our regularization will require the knowledge of the full functions (5.62) also away from the LCS regime. The purpose of this section is to discuss how the period data of the mirror manifold will be helpful in that regard.

5.3.1 B-side and A-side Yukawa couplings

Computing the enumerative invariants α_0^β of compact CYs directly by counting stable maps is extremely difficult. Here, mirror symmetry comes to the rescue: it establishes a relation between (5.62) and the Yukawa couplings of the complex structure moduli x^i of the mirror $\hat{Y}$, denoted as $C_{ijk}(x)$ and usually called *B-side Yukawa couplings.* Crucially, the latter can be determined from purely classical geometric quantities and do not receive worldsheet instanton corrections. When expressed in terms of the holomorphic 3-form Ω, they are defined as

$$C_{ijk}(x) = \int_{\hat{Y}} \Omega \wedge \partial_i \partial_j \partial_k \Omega \,. \tag{5.63}$$

Recall that Ω can be expanded as $\Omega = \alpha_{\hat{K}} X^{\hat{K}} + \beta^{\hat{K}} \mathcal{F}_{\hat{K}}$, where $(\alpha_{\hat{K}}, \beta^{\hat{L}})$ is a basis of $H^3(\hat{Y},\mathbb{Z})$ and $\Pi^T(X) = (X^{\hat{K}}, \mathcal{F}_{\hat{K}})^T$ are the integral periods. Then, using the defining properties of the symplectic basis, the Yukawa couplings can be written as

$$C_{ijk}(x) = \Pi^T(X)\Sigma\, \partial_i\partial_j\partial_k \Pi(X) = \sum_{\hat{K}=0}^{\hat{h}_{21}} \left(X^{\hat{K}} \partial_i\partial_j\partial_k \mathcal{F}_{\hat{K}} - \mathcal{F}_{\hat{K}} \partial_i\partial_j\partial_k X^{\hat{K}} \right), \tag{5.64}$$

where the symplectic pairing Σ was introduced in (2.99). The B-side Yukawa couplings can thus be calculated directly from the periods in the integral symplectic basis. Alternatively, one can even find closed rational expressions for the B-side Yukawa couplings from the Picard-Fuchs system (for more detailed instructions on that see [179]). The basic idea is to obtain a linear system relating the different Yukawa couplings with their derivatives. One can show that the Yukawa couplings close under differentiation with respect to moduli, i.e. derivatives are given by

$$\partial_a C_{ijk}(x) = \sum_{m,n,p} R^{m,n,p}_{a;i,j,k}(x)\, C_{mnp}(x)\,, \tag{5.65}$$

where $R^{m,n,p}_{l,i,k,l}$ only contain rational functions determined by the Picard-Fuchs operators. These relations are found by differentiating (5.63) and reducing the fourth-rank derivatives to rank three and lower by means of the PF operators, which by definition annihilate Ω.

The procedure is iterated until every Yukawa coupling appears in the set of equations (5.65). Solutions of the resulting differential equation have the general form

$$C_{ijk}(x) = \mathcal{C}^0_{ijk}\,\frac{p_{ijk}(x)}{q_{ijk}(x)}\frac{1}{\mathrm{dis}_1(\hat{Y})}\,, \tag{5.66}$$

with p_{ijk} and q_{ijk} polynomials, $\mathcal{C}^0_{ijk}$ normalization constants and $\mathrm{dis}_1(\hat{Y})$ the discriminant locus of the hypersurface CY, parametrizing co-dimension one loci in the CY moduli space along which the CY degenerates. The constants $\mathcal{C}^0_{ijk}$ are fixed by matching with the normalization of the triple intersection numbers in the large volume regime of the mirror CY. Appendix A of [179] provides the discriminant loci and exact B-side Yukawa couplings for a couple of CY hypersurfaces with few Kähler moduli (without normalization constants). Note that (5.66) are globally defined functions across all of moduli space.

In order to relate B-side Yukawa couplings on $\hat{Y}$ to A-side Yukawa couplings on Y we invoke the mirror map. According to mirror symmetry, the complex Kähler moduli on Y are identified with a period ratio on the mirror dual, namely $t^i(x) = \Pi_{i+1}(x)/\Pi_1(x)$ with $i = 1, \ldots, h_{11}(Y)$. Since the coordinates x^i on the B-side are globally defined, this map is also valid away from the LCS regime. At the LCS, the map takes the familiar form

$$q_i = e^{2\pi i t^i} = x^i + \ldots\,, \tag{5.67}$$

where each q_i is a multivariate power series in all variables, starting with x^i as indicated above. The equations (5.67) can be inverted order by order to obtain the inverse relations $x^i = x^i(q_j)$. We can now perform an easy consistency check for our LCS periods in the integral symplectic basis. For that we divide out the fundamental period, expand all components and insert the inverse mirror relations. Assuming our convention for the symplectic pairing, we can then compute the A-side prepotential via

$$\mathcal{F}_0(t) = \frac{1}{2}\left(\hat{\Pi}_1^{\mathrm{LCS}}\hat{\Pi}_6^{\mathrm{LCS}} + \hat{\Pi}_2^{\mathrm{LCS}}\hat{\Pi}_4^{\mathrm{LCS}} + \hat{\Pi}_3^{\mathrm{LCS}}\hat{\Pi}_5^{\mathrm{LCS}}\right)\,, \tag{5.68}$$

where hats again indicate a period vector expressed in inhomogeneous coordinates. The results for both the classical terms as well the GV invariants from the non-perturbative terms can be compared with known results in the literature or the output from the software package `CY-Tools`.

To compute the A-side Yukawas at general points in moduli space is also straightforward: their definition in terms of derivatives in (5.63) indicates a tensor transformation behavior under coordinate changes. We should also bear in mind that the holomorphic three-form can be altered by holomorphic rescalings $\Omega \to f(x)\,\Omega$, which leads to $C_{ijk} \to f^2(x)C_{ijk}$. To have scale-invariant Yukawas, we choose a gauge for Ω by dividing out the fundamental period X^0. In this normalization, the A-side Yukawas read

$$Y_{ijk}(t) = \frac{1}{(X^0)^2}\sum_{\alpha,\beta,\gamma}\frac{\partial x_\alpha}{\partial t^i}\frac{\partial x_\beta}{\partial t^j}\frac{\partial x_\gamma}{\partial t^k}C_{\alpha\beta\gamma}(x(t))\,. \tag{5.69}$$

Thus, for the calculation of the A-side Yukawas we do not actually need to know the inverse mirror relations $x_i = x_i(q_j)$ but only its derivatives. It is therefore sufficient to compute the Jacobian $J_{ij} = \frac{\partial t_i}{\partial x_j}$ and to invert the matrix.

5.3.2 Caveat: Inverting the mirror map

To calculate the Yukawa couplings in moduli space regions away from the LCS, we need to newly assemble the mirror map using the period data in the respective patch. The inversion of that map, however, cannot be achieved in arbitrary patches of a CY moduli space. There are only few exceptions to this general rule, one of them being the moduli space of the torus T^2. The fundamental domain (see figure 3.2) for both the complex structure and Kähler modulus is obtained by dividing the upper half-plane by the action of the modular group $SL(2,\mathbb{Z})$. And importantly, modular transformations neatly map one fundamental domain to its other representations without creating any overlaps between those images. As a consequence, mirror symmetry assigns each value of the complex structure modulus to a unique value of the Kähler parameter of the dual torus, so the mirror map has a global inverse.

If we consider generic CY moduli spaces, the situation changes. At discriminant loci such as conifold points the Jacobian of the mirror map degenerates and the mirror map fails to be invertible. Moreover, in the vicinity of certain singularities it may happen that no single-valued analytic inverse function exists. This happens precisely because monodromies of the periods lead to multi-valued functions $t_i(z)$ (with complex structure parameter z), meaning a putative inverse has different branches, and it is unclear which one to choose. Let us look at the quintic once again to see an explicit example. Suppose that we have determined the local periods around the LCS, the conifold and the Landau-Ginzburg point and glued them to the integral symplectic basis. The mirror map is

$$t(z) = \frac{\Pi_2(z)}{\Pi_1(z)}\,, \quad \Pi(z) \in \{\Pi^{\mathrm{LCS}}, \Pi^{\mathrm{C}}, \Pi^{\mathrm{LG}}\}\,. \tag{5.70}$$

To display the fundamental domain, we parametrize the complex structure parameter as $z = re^{i\phi}$ and plot $(\mathrm{Re}(t), \mathrm{Im}(t))$ for $0 \leq \phi \leq 2\pi$ and different values of r, always choosing the right set of periods for which r lies within the convergence radius. For our calculations we choose the branch cut of the logarithm (and fractional powers) to lie on the positive real axis. We can then check how these regions get mapped under the action of monodromies. The coefficients of the monodromy matrices can be obtained by solving

$$\Pi(e^{2\pi i}z) = M\Pi(z)\,, \quad M \in Sp(4,\mathbb{Z}) \tag{5.71}$$

around each respective point in moduli space. At the LCS and conifold they are given by

$$M_{\mathrm{LCS}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & -5 & 1 & 0 \\ 5 & 2 & -1 & -1 \end{pmatrix}\,, \quad M_{\mathrm{C}} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{5.72}$$

The monodromy at the Landau-Ginzburg point is fixed by the inverse product of the former two[7] such that $M_{\mathrm{LG}} = (M_{\mathrm{LCS}} \cdot M_C)^{-1}$.

[7]The reason is that the moduli space of the quintic has the topology of a sphere with three punctures,

In figure 5.6 we display the fundamental domain of the quintic moduli space and its images under the action of the monodromies M_{LCS} and M_{LG}. The LCS monodromy just results in a shift of the Kähler parameter given by $t \to t+1$. Around the LG point, the associated monodromy multiplies the complex structure parameter by a phase factor, inducing a rotation in the complex t-plane that satisfies $M_{\mathrm{LG}}^5 = 1$. If the LG monodromy is applied in the patch around the conifold, it leads to the problematic behaviour mentioned above: the monodromy acts such that period vectors with a different number of M_{LG} actions are still mapped to the same t-value by means of the mirror map. Put differently, the monodromy does not generate clearly separated fundamental domains (as it does for the LCS and LG regions) but rather creates overlaps, as can be seen in the plot. This multivaluedness explains the absence of a global inversion of the mirror map.

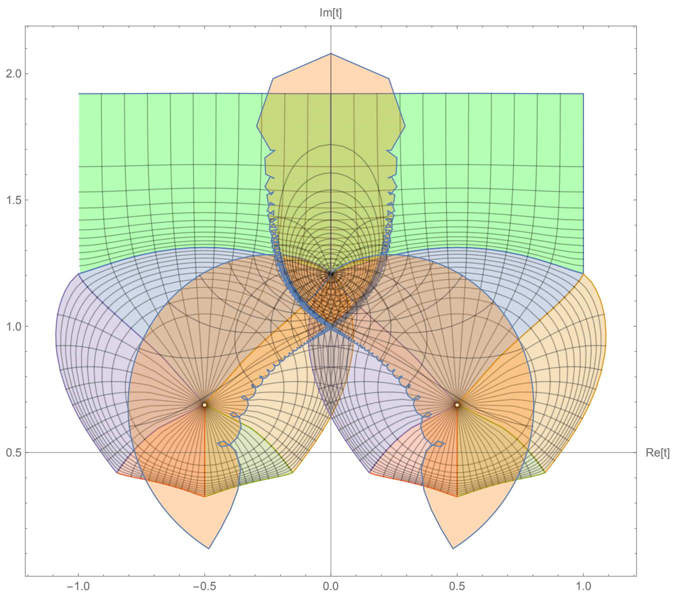


Figure 5.6: Moduli space of the quintic. The light green area at the top corresponds to the LCS domain, while each of the circular arcs below is a representative of the LG region, with the different colors being related by the LG monodromy. For these areas we show the fundamental regions with respect to the LCS monodromy (appearing on the left-hand side) as well as their images under one shift induced by M_{LCS}. The orange wedges represent the conifold patch and its transformations under the LG monodromy.

Another indication for the absence of a global inverse can be found by studying the explicit form of the mirror map at the conifold. In terms of the local coordinate $u = 1 - 5^{-5}$, the expansion of the mirror map is given by

$$t(u) = \frac{\Pi_2^C}{\Pi_1^C} = \sum_{n=0}^{\infty} (u \log(u))^n c_n(u)\,, \tag{5.73}$$

corresponding to the three singular points. If one considers a closed loop surrounding all three punctures, it can thus be contracted to a single point. As the monodromy matrices represent the effect on the period induced by these loops, the product of all three of them must be one.

where each c_n denotes a power series in z starting with a constant. Unlike for the LCS case, (5.73) contains logarithmic terms at each power. Therefore one cannot just exponentiate both sides to obtain a power series that can then be inverted order by order.

Even though the presence of monodromies prevents us from constructing a global inverse function, we can still make a *local* statement. The inverse function theorem guarantees the existence of a holomorphic inverse of the mirror map as long as the determinant of the Jacobian satisfies $\det J \neq 0$, i.e. away from a singularity. So if we look at a small subset not containing a singularity and choose a particular branch, a holomorphic inverse does indeed exist and the inverse derivatives dz^i/dt^j are well-defined accordingly.

5.3.3 Divergence of Yukawa couplings

We want to review a crucial observation about the divergence of Yukawa couplings which was already made by Candelas, de la Ossa, Green and Parkes in their seminal paper about the quintic [23]. They advertised the idea that singularities in the moduli space of the B-model carry information about the plethora of instanton corrections on the A-side. In the case of the Yukawa couplings these corrections are counted by the genus zero GV invariants. To provide evidence, the authors made an ansatz for the asymptotic growth of these invariants and fixed the unknown free parameters by demanding that this growth formula reproduces the expected singularity of the Yukawa coupling, as predicted by mirror symmetry. Since the general idea of a correlation between singularities and GV invariants will also be relevant for our regularization in the M-theory limit, we will sketch the original calculation by Candelas et al. in the following.

If we consider the mirror map at large complex structure and approach smaller values of the Kähler parameter, the conifold at $t = t_c$ is the first (and only) singularity in moduli space that we encounter for the quintic. The value of $\mathrm{Im}(t_c)$ also marks the smallest two-cycle size for which the CY is defined. The leading-order term of the mirror map expansion (5.73) is given by

$$t - t_c \sim u \log(u) + \dots \,. \tag{5.74}$$

We can now apply formula (5.69) after having expressed all factors in terms of the local coordinate u. We then find the leading divergence

$$Y_{ttt} \sim \frac{1}{u^3 \log(u)} + \dots \sim \frac{1}{(t - t_c)^2 \log(t - t_c)} + \dots \,. \tag{5.75}$$

Candelas et al. now raised the question how the GV invariants α_0^n need to scale to reproduce this singularity from the behavior of the sum

$$\kappa_{ttt} \sim \sum_{n=1}^{\infty} \alpha_0^n n^3 e^{2\pi i n t} \,. \tag{5.76}$$

Since the LCS expansion (5.76) is expected to converge only up to the conifold locus, i.e. for Kähler parameters with $\mathrm{Im}(t) > \mathrm{Im}(t_c)$, the GV invariants α_0^n cannot grow faster than

$e^{2\pi n \mathrm{Im}(t_c)}$ (if they would grow faster, the singularity would appear "earlier"). This portrays the rapid growth in the asymptotics, but to correctly account for the GV invariants for lower n they made the more general ansatz

$$\alpha_0^n \sim n^{\rho-3} \log(n)^\sigma e^{2\pi n \mathrm{Im}(t_c)} . \tag{5.77}$$

One can then plug this ansatz into (5.76) and approximate the sum by an integral to obtain

$$\kappa_{ttt} \sim \sum_{n=1}^{\infty} n^\rho \log(n)^\sigma e^{-2\pi n \mathrm{Im}(t-t_c)} \sim \frac{\Gamma(1+\rho)}{(\Delta t)^{1+\rho}} (-\log \Delta t)^\sigma , \tag{5.78}$$

with $\Delta t = 2\pi \mathrm{Im}(t - t_c)$. Comparing this to (5.75) one concludes that the ansatz works for $\rho = 0$ and $\sigma = -2$.

As a complementary approach, one could try to fix the values of the free parameters in the asymptotic growth formula by performing a fit to the known GV invariants from the literature. Starting with the even more general ansatz

$$\alpha_0^n \sim c_0 \, n^{\rho-3} \log(n)^\sigma e^{2\pi\lambda n} , \tag{5.79}$$

the strategy is to fix one parameter at a time by creating a sequence with GV invariants that converges to the exact value of this parameter. In [185] such an analysis was applied to genus g Gromov-Witten invariants. One starts by determining the exponent in the exponential and continues with the parameters of the functions that are less and less dominant. However, to fix the coefficients with the raw data one needs to know the GV invariants to a very high degree. The standard technique to speed up convergence is to apply Richardson transformations to the data sequences. If we assume that a general sequence $\{S(n)\}_{n\in\mathbb{N}}$ converges to a limit s and that the asymptotic expansion is of the form $S(n) = s + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots$, we can define the k-th Richardson transform as

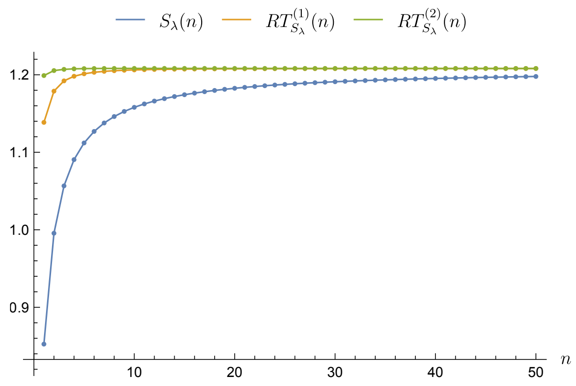
$$\mathrm{RT}_S(n,k) = \sum_{m=0}^{k} (-1)^{m+k} \frac{(n+m)^k}{m!(k-m)!} S(n+m) . \tag{5.80}$$

One can check explicitly that the k-th Richardson transform cancels the first k correction terms, so that $\mathrm{RT}_S(n,k) = s + \mathcal{O}\left(\frac{1}{n^{k+1}}\right)$ and thus the convergence is faster for larger k.

We have applied these techniques to fix the parameters in (5.79) for the quintic using GV invariants up to order 1000, but only managed to reproduce λ and ρ with an acceptable low numerical error. The coefficient in the exponential was determined from the sequence

$$S_\lambda(n) = \frac{1}{2\pi} \log\left(\frac{\alpha_0^{n+1}}{\alpha_0^n}\right) , \tag{5.81}$$

where α_0^n are the exact GV invariants. The result for the third Richardson transform (see figure 5.7, left) deviates from the literature value by only 0.02% for $n = 50$.



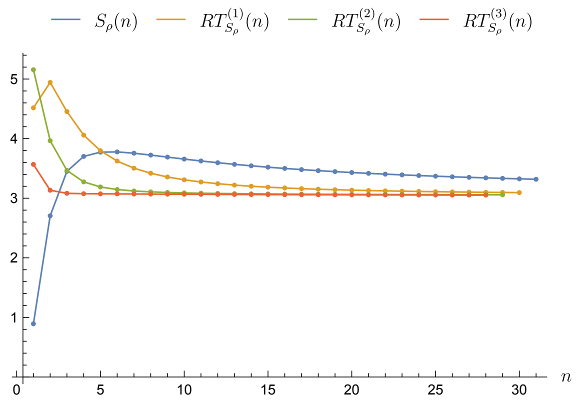


Figure 5.7: Sequences $S_\lambda(n)$ (left) and $S_\rho(n)$ (right) together with their first few Richardson transforms.

The parameter of the monomial term is fixed through

$$S_\rho(n) = f_\rho(n) - 2f_\rho(n^2)\,, \quad \text{with} \quad f_\rho(n) = n\left(e^{-2\pi\lambda}\frac{\alpha_0^{n+1}}{\alpha_0^n} - 1\right)\,. \tag{5.82}$$

The sequence as well as the Richardson transforms are shown on the right-hand side of figure 5.7. Here the relative numerical error is approximately 1.7% for $n = 28$. The parameters σ and c_0 could not be determined with our data. For enumerative invariants from higher genus amplitudes one can also usually retrieve the power of the logarithmic term with these techniques, as was mentioned for example in [186]. This is expected given that for higher genus g GV invariants the logarithm appears with a higher power, namely [187]

$$\alpha_g^n \sim c_g n^{2g-3}\log^{2g-2}(n)\, e^{2\pi n \mathrm{Im}(t_c)}\,. \tag{5.83}$$

5.4 Gauged Linear Sigma Models

In the final section of this chapter we review an alternative and complementary approach to the construction of toric varieties in which one directly takes a quotient of $\mathbb{C}^n$ such that the resulting variety is a Kähler manifold (known as *Kähler quotient*). In this language, one can understand toric varieties as the set of gauge-inequivalent vacuum states of a class of supersymmetric gauge theories known as gauged linear sigma models (GLSMs), to be defined soon. Depending on the values of certain couplings in the GLSM one typically finds distinct "phases" of the theory where the low-energy physics can vastly differ. Since these couplings are directly related to the Kähler parameters of the variety, GLSMs are a convenient tool to understand regions in the moduli space where the Kähler parameters lie outside of the Kähler cone, i.e. what happens under so-called *flop-transitions*.

5.4.1 From toric varieties to Gauged Linear Sigma Models

We start by introducing the Kähler quotient. Instead of using representatives such as fans and polytopes like in section 5.1, we implement the quotient in (5.5) directly. This construction in principle holds for arbitrary symplectic manifolds, but we will focus on Kähler manifolds from the start. The space $\mathbb{C}^n$ with complex coordinates $z = (z_1, \ldots, z_n)$ has the canonical Kähler form $\omega = -i \sum_{i=1}^n dz_i \wedge d\bar{z}_i$. The group $G = U(1)^r \subset \mathbb{T}$ with $r = n - m$ acts on the coordinates as $z_i \to e^{iQ_i^a \xi_a} z_i$, where $\xi = (\xi_1, \ldots, \xi_r) \in u(1)^r$ are the generators and $Q_i = (Q_i^1, \ldots, Q_i^r)$ the weights or "charges" of z_i. Obviously, G leaves ω invariant. The infinitesimal action of ξ is captured by a vector field v_ξ on $\mathbb{C}^n$, which acts on a test function as

$$v_\xi f(z, \bar{z}) = \frac{d}{dt}\Big|_{t=0} f(z'(t), \bar{z}'(t)) = i\xi_a \sum_{i=1}^n Q_i^a (z_i \partial_{z_i} - \bar{z}_i \partial_{\bar{z}_i}) f(z, \bar{z}) \,, \tag{5.84}$$

where we used the compact notation $z'(t) = (z_1'(t), \ldots, z_n'(t))$ with $z_i'(t) = e^{iQ_i^a \xi_a t} z_i(t)$ and likewise for the complex conjugate. We want to quotient by G such that the reduced space inherits a non-degenerate Kähler form. This can be done by means of the momentum map $\mu : \mathbb{C}^n \to \mathbb{R}^r$ defined via

$$d\langle \mu, \xi \rangle \equiv d(\mu_a \xi^a) = \iota_{v_\xi} \omega \,, \tag{5.85}$$

where $\iota_{v_\xi}\omega$ is the contraction of ω with the vector field appearing in (5.84). In the case at hand we find that the momentum map has components

$$\mu_a = \sum_{i=1}^n Q_i^{(a)} |z_i|^2 - \tau_a \,, \tag{5.86}$$

where τ_a are real integration constants that define a so-called *level-set*. If we now restrict ourselves to those points for which all μ_a at a given level vanish, (5.85) indicates that ω will be invariant along the G-orbits. We therefore define the variety as

$$\mathcal{M}(\tau) = \left\{ z \in \mathbb{C}^n \Big| \sum_{i=1}^n Q_i^{(a)} |z_i|^2 = \tau_a \right\} \Big/ U(1)^r \,, \tag{5.87}$$

leading to a space with well-defined Kähler form and complex structure. As the τ_a vary, the topology and even the dimension of the resulting space may change.

It is instructive to relate the level-set restriction in (5.87) to the vacuum constraints of a class of supersymmetric and renormalizable gauge theories in two dimensions, namely the GLSMs. We will just briefly introduce their key ingredients and refer to the seminal papers [188, 189] for a more detailed account. We consider a theory with

1. n chiral superfields Φ_i with charges $Q_i^{(a)}$ under $G = U(1)^r$
2. r abelian vector multiplets V_a with field strengths Σ_a (common gauge coupling e)
3. r Fayet-Iliopoulos (FI) terms with parameter r_a for each vector multiplet

4. a superpotential $W(\Phi_i)$, depending holomorphically on the Φ_i

We are interested in the vacuum solutions of this theory. The potential energy for the scalars ϕ_i (bottom components of Φ_i) and σ_a (bottom components of Σ_a) reads

$$U(\phi_i, \sigma_a) = \frac{1}{2e^2} \sum_{a=1}^{r} D_a^2 + \sum_{i=1}^{n} |F_i|^2 + 2 \sum_{a,b=1}^{r} \bar{\sigma}_a \sigma_b \sum_{i=1}^{n} Q_i^a Q_i^b |\phi_i|^2 \,, \tag{5.88}$$

where D_a and F_i are auxiliary fields which are fixed by the equations of motion, yielding the D-term and F-term constraints

$$D_a = -e^2 \left(\sum_{i=1}^{n} Q_i^{(a)} |\phi_i|^2 - r_a \right) \,, \quad F_i = \frac{\partial W}{\partial \phi_i} \,. \tag{5.89}$$

For the moment we assume that no superpotential is present. Classical vacua satisfy $U = 0$, leading to the constraints $D_a = 0$ for $a = 1, \ldots, r$. These constraints precisely match with those from the Kähler quotient (5.87) if we identify

$$z_i \longleftrightarrow \phi_i \quad \text{and} \quad \tau_a \longleftrightarrow r_a \,. \tag{5.90}$$

In addition, the weights of the coordinate z_i under the a-th toric quotient correspond to the $U(1)$ charges of the field ϕ_i (justifying the identical labeling used already). Then the manifold of classical, gauge inequivalent vacua is indeed $D^{-1}(0)/G = \mathcal{M}(r)$.

As in the previous sections we want to study compact CY manifolds which are hypersurfaces inside toric varieties that lie in the anticanonical class. To implement the CY condition at GLSM level we introduce an additional chiral superfield Φ_0 with $U(1)$ charges $Q_0^{(a)} = -\sum_{i=1}^{n} Q_i^{(a)}$. In the GLSM this has the effect of canceling the gauge anomalies in the R-symmetries [188]. We take the superpotential to be the gauge invariant function

$$W(\Phi) = \Phi_0 P(\Phi_1, \ldots, \Phi_n) \,, \tag{5.91}$$

where G is a homogeneous polynomial of degree n, which keeps the R-symmetry intact. We also want to choose P to be transverse so that the equations

$$0 = \frac{\partial P}{\partial \Phi_1} = \ldots = \frac{\partial P}{\partial \Phi_n} \tag{5.92}$$

have no common root except $\Phi_1 = \ldots \Phi_n = 0$. The potential energy of the modified system just changes by the inclusion of the additional chiral superfield so that the sums over i in (5.88) and (5.89) run from 0 to n. Concretely, the D- and F-terms become

$$D_a = -e^2 \left(\sum_{i=0}^{n} Q_i^{(a)} |\phi_i|^2 - r_a \right) \,, \quad F_0 = P \,, \quad F_i = \phi_0 \frac{\partial P}{\partial \phi_i} \,. \tag{5.93}$$

To understand which kinds of phases can occur we need to analyze the vacuum equations $D_a = 0, F_i = 0$ for different values of the FI parameters. For concreteness, we

	ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
$Q^{(1)}$	-6	0	1	1	3	0	1
$Q^{(2)}$	0	1	0	0	0	1	-2

Table 5.3: GLSM data for $\mathbb{P}^4_{1,1,2,2,6}[12]$

revisit the crepantly resolved hypersurface $\mathbb{P}^4_{1,1,2,2,6}[12]$, whose Mori cone vectors in (5.29) provide the two GLSM charge vectors shown in table 5.3. Compared to the singular ambient weighted projective space $\mathbb{P}^4_{1,1,2,2,6}$, we have the additional chiral multiplet Φ_6 associated to the exceptional divisor that resolves the $\mathbb{Z}_2$ singularity and Φ_0 which is used to implement the CY condition. The D-terms read

$$\begin{aligned} D_1 &= -e^2(|\phi_2|^2 + |\phi_3|^2 + 3|\phi_4|^2 + |\phi_6|^2 - 6|\phi_0|^2 - r_1)\,, \\ D_2 &= -e^2(|\phi_1|^2 + |\phi_5|^2 - 2|\phi_6|^2 - r_2)\,. \end{aligned} \tag{5.94}$$

The classical boundaries of the phase diagram lie along co-dimension one cones for which solutions to the D-term equations $D_a = 0$ leave a continuous subgroup of $G = U(1)_1 \times U(1)_2$ unbroken. The sub-cones mark singularities of the theory because there additional degrees of freedom become light, invalidating the effective description used away from that locus. For the generators $g_a : \phi_i \to \lambda^{Q_i^{(a)}} \phi_i$ with $a = 1, 2$ we find

- g_1 unbroken if $\phi_0 = \phi_2 = \phi_3 = \phi_4 = \phi_6 = 0$, then $r_1 = 0, r_2 \geq 0$.
- g_2 unbroken if $\phi_1 = \phi_5 = \phi_6 = 0$, then $r_2 \geq 0$.
- $g_1^2 g_2$ unbroken if all fields but ϕ_6 equal 0, then $r_1 \geq 0, 2r_1 + r_2 = 0$.

These cones divide the parameter space into four chambers, which we will analyze one by one as was done in [189]. Generally speaking, in each chamber different parts of the gauge group remain unbroken, meaning that different sets of fields acquire a mass via the Higgs mechanism. Our goal is to indicate which kinds of effective theories arise in those cases.

Phase I: Geometric phase ($r_1 > 0, r_2 > 0$). We need to exclude $\{\phi_1 = \phi_5 = 0\} \cup \{\phi_2 = \phi_3 = \phi_4 = \phi_6 = 0\}$. This exactly matches the excluded set Z_Σ from (5.5) if one constructs the resolved $\mathbb{P}^4_{1,1,2,2,6}$ torically. According to the F-terms in (5.93), $\phi_0 = 0$ since not all coordinates are allowed to vanish simultaneously and we imposed (5.92). Moreover $P = 0$, imposing the hypersurface constraint. In this phase the gauge group is Higgsed completely since all ϕ_i acquire a non-zero vev. The only light fields are gauge-invariant combinations of the ϕ_i, which correspond to the local coordinates in the quotient space. As a result, the theory flows in the IR to a non-linear sigma model (NLSM) whose target space is the CY specified by $P = 0$, so as expected the GLSM describes a smooth CY geometry when the FI parameters lie in the Kähler cone.

Phase II: Orbifold phase ($r_1 > 0, r_2 < 0, 2r_1 + r_2 > 0$). Here one excludes $\{\phi_6 = 0\} \cup \{\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0\}$. The F-terms again impose $\phi_0 = 0$ and $P = 0$. However, since different parameter values have been omitted, the ambient space changes. Suppose we do not fix (r_1, r_2) to a level but still keep them within the specified chamber, allowing us to quotient by rescalings in addition to the $U(1)$ factors. Then one can fix $\phi_6 = 1$ by the g_2 action and finds that the remaining $g_1^2 g_2$ precisely implements the $\mathbb{C}^*$ action in the toric construction (5.5) of the unresolved weighted projective space $\mathbb{P}^4_{1,1,2,2,6}$ (after relabeling some coordinates). Thus, the low-energy theory is again a NLSM but the target space is a hypersurface in the unresolved space. Due to the presence of orbifold singularities, the phase is usually dubbed "orbifold phase".

Phase III: Landau-Ginzburg phase ($r_2 < 0, 2r_1 + r_2 < 0$). The omitted values are $\{\phi_0 = 0\} \cup \{\phi_6 = 0\}$. Here the situation drastically changes: The F-terms impose $\phi_1 = \ldots = \phi_5 = 0$, which leads to $|\phi_0| = \sqrt{-r_1/6}$ and $|\phi_6| = \sqrt{-r_2/2}$ according to the D-terms. The gauge group $U(1)_1 \times U(1)_2$ is broken to $\mathbb{Z}_6 \times \mathbb{Z}_2$ accordingly, with $\mathbb{Z}_n$ acting via $\phi_i \to \exp(2\pi i/n)\phi_i$ for all i. As a result, we obtain a unique vacuum configuration (modulo gauge transformation), with the original toric variety collapsing to a point. The fields ϕ_i with $i = 1, \ldots, 5$ remain massless and interact via the effective superpotential

$$W_{\text{eff}} = \sqrt{-r_1/6}\, W(\phi_1, \ldots, \phi_5)\,, \tag{5.95}$$

which features a degenerate critical point and is obtained after intergrating out the fields ϕ_0, ϕ_6. These types of theories are known as Landau-Ginzburg theories.

Phase IV: Hybrid phase ($r_1 < 0, r_2 > 0$). The omitted values are $\{\phi_0 = 0\} \cup \{\phi_1 = \phi_5 = 0\}$. Satisfying the F-terms requires $\phi_2 = \phi_3 = \phi_4 = \phi_6 = 0$ and according to D_1 we have $|\phi_0| = \sqrt{-r_1/6}$. Now the gauge group is broken to $\mathbb{Z}_6$. The D_2 constraint reads

$$r_2 = |\phi_1|^2 + |\phi_5|^2\,, \tag{5.96}$$

i.e. (ϕ_1, ϕ_5) parametrize a 3-sphere in $\mathbb{C}^2$. After implementing the $U(1)$ quotient one finds that the moduli space is $\mathbb{P}^1$. The fields (ϕ_2, ϕ_3, ϕ_4) remain massless and interact via an effective superpotential that depends on the location on $\mathbb{P}^1$. Since the result is a NLSM with target space $\mathbb{P}^1$ paired with a Landau-Ginzburg model, the phase is hybrid.

5.4.2 Quantum-corrections to FI parameters

Let us finally also address the quantum corrections for GLSMs and how they affect the phase space structure discussed so far. When we introduced the constituents of GLSMs we have neglected the possibility of including θ-terms to the action. If we do so, the tree-level superpotential for the chiral fields Σ_a is given by

$$\widetilde{W}_{\text{tree}}(\Sigma) = \frac{i}{2\sqrt{2}} \sum_{a=1}^{r} \xi_a \Sigma_a\,, \quad \text{with} \quad \xi_a = \frac{\theta_a}{2\pi} + i r_a\,. \tag{5.97}$$

Classically, the correspondence with the complexified Kähler parameters is $\xi_a = t_a = b_a + i\tau_a$, i.e. the θ-terms correspond to B-fields on the Kähler manifold. Due to $N = 2$ non-renormalization theorems, quantum corrections to these relations are severely restricted: since (5.97) is holomorphic, the perturbative series $t_a = \xi_a + c_a^{(1)} + \mathcal{O}(r_a^{-1})$ already terminates at one-loop level. In principle there can be non-perturbative corrections as well, but for these models they are found to be absent [189]. To compute the one-loop correction, one assumes that the fields σ_a take large expectation values, giving the charged matter fields ϕ_i a heavy mass according to (5.88). In other words, the theory lies entirely in the Coulomb phase. Integrating out the ϕ_i leads to the corrected superpotential

$$\widetilde{W}(\Sigma) = \frac{1}{\sqrt{2}} \sum_{a=1}^{r} \Sigma_a \left(i\xi_a - \frac{1}{2\pi} \sum_{i=0}^{n} Q_i^{(a)} \log \left(\sqrt{2} \sum_{b=1}^{r} \frac{Q_i^{(b)} \Sigma_b}{\Lambda} \right) \right) , \tag{5.98}$$

where Λ is a UV-cutoff introduced to regularize the divergent graphs. We find the vacua of the theory by computing the stationary points $\partial \widetilde{W} / \partial \sigma_a = 0$, yielding

$$q_a \equiv e^{2\pi i \xi_a} = \prod_{i=0}^{n} \left(\frac{\sqrt{2} e}{\Lambda} \delta_i \right)^{Q_i^{(a)}} = \prod_{i=0}^{n} \delta_i^{Q_i^{(a)}} , \quad a = 1, \ldots, r , \tag{5.99}$$

where $\delta_i = \sum_{b=1}^{r} Q_i^{(b)} s_b$ with expectation values $s_b = \langle \sigma_b \rangle$ and in the second equality we used that $\sum_{i=0}^{n} Q_i^{(a)} = 0$ for all a due to the CY condition. The fact that the charges for fixed a sum to zero makes a crucial difference in the quantum picture. To see why, suppose we were looking at a GLSM for which one set of charges, say $Q_i^{(1)}$, does *not* sum up to zero. For instance, this is the case for the ambient $\mathbb{P}^4_{1,1,2,2,6}$ on its own, with ϕ_0 from table 5.3 removed. If for this model we approach the classical singularity $r_1 \to 0$ while staying far away from the origin in r-space, the generator g_2 gets Higgsed, so σ_2 becomes massive and needs to be integrated out as well (by setting it to zero in the superpotential). One can then analyze the effective potential $U(\sigma_1)$ and finds that in this case no minimum can occur, meaning that there is no phase change and hence in the quantum theory there is no singularity. On the other hand, the charges $Q_i^{(2)}$ of $\mathbb{P}^4_{1,1,2,2,6}$ do sum up to zero. If one repeats the analysis close to $r_2 = 0$, one finds the constant correction

$$\xi_{2,\text{eff}} = \xi_2 + \frac{i}{2\pi} \sum_{i=1}^{5} Q_i \log(Q_i) \tag{5.100}$$

at which the minimum occurs, with the singularity remaining intact. Crucially, it occurs only for specific values of the θ parameters, which for this case is $\theta_2 \in \{0, \pi\}$. A sketch of the classical and quantum-corrected moduli spaces is shown in figure 5.8.

The singular loci (5.99) are derived under the assumption that all σ_a are in the Coulomb phase, but as mentioned before there can be situations where partial Higgsing occurs, so that some ϕ_i remain massless. To construct associated singular loci, one focuses on k-dimensional subgroups $H \subset G$ with a special property: the charges of the *complement* of H must generate all of $\mathbb{R}^{r-k}$ by positive linear combinations. In practice

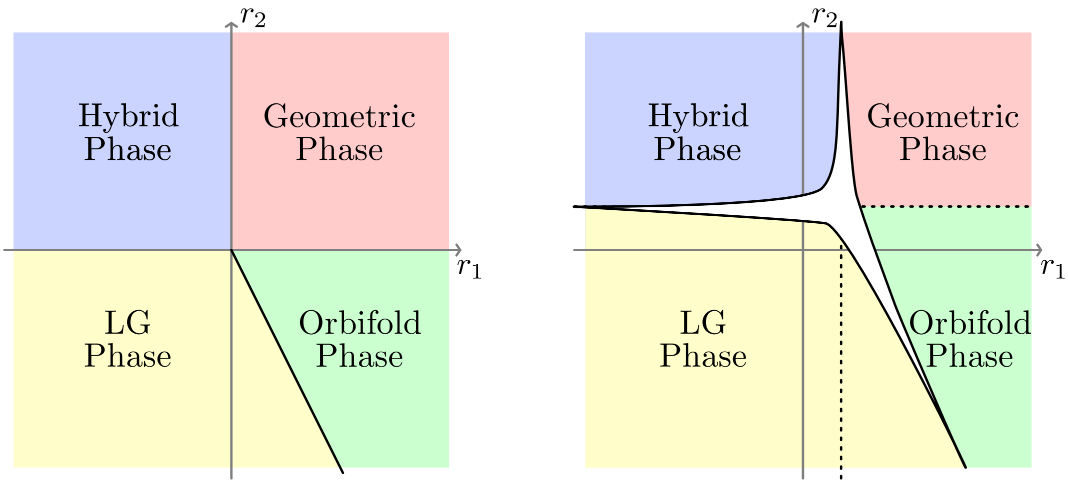


Figure 5.8: Phases of the classical (left) and quantum-corrected (right) Kähler moduli space of $\mathbb{P}^4_{1,1,2,2,6}[12]$.

this just means that the complement must have both positive and negative charges in each component. H divides the fields ϕ_i into those which are (partially) charged ($\{\phi_i\}_{i\in I}$) and those which are completely neutral ($\{\phi_i\}_{i\in I^c}$) with respect to H. Then the $U(1)$'s different from H get Higgsed and only the fields ϕ_i for $i \in I^c$ remain massless, eventually yielding the following equations for the minima:

$$q_p = \prod_{i\in I} \langle \delta_i|_H \rangle^{Q_i^{(p)}}\,, \quad p = 1, \dots, k-r\,. \tag{5.101}$$

The product is over all chiral fields charged under H and the $\delta_i|_H$ are obtained from δ_i by setting all fields σ related to the complement of H to zero.

Let us finally evaluate the implications for the CY $\mathbb{P}^4_{1,1,2,2,6}[12]$. For GLSMs describing CYs in their geometric phase, we have learned that all equations in (5.99) give honest singular regions in the parameter space since $\sum_{i=0}^n Q_i^{(a)} = 0$ is guaranteed for all a. In our concrete case they require

$$q_1 = \frac{1}{2^6 3^3}\left(1 - \frac{s_2}{s_1}\right)\,, \quad q_2 = s_2^2(1-s_2)^2\,. \tag{5.102}$$

After noticing the relation $2^{12}3^6 q_1^2 q_2 = (s_2/s_1)^2$ and using the implicit constraint on s_2/s_1 from the first equation, it is easy to see that

$$(1 - 2^6 3^3 q_1)^2 - 2^{14}3^6 q_1^2 q_2 = 0\,. \tag{5.103}$$

Moreover, the only subgroup of G whose charges span $\mathbb{R}$ is the first $U(1)$ so that H is equal to the second $U(1)$. Therefore, (5.101) produces the additional relation

$$1 - 2^2 q_2 = 0\,. \tag{5.104}$$

Hence, the two relations (5.103) and (5.104) specify the quantum shift of the classical singularity at $\xi_1 = \xi_2 = 0$. Comparing this with the discriminant loci (5.43) in the complex structure moduli of the mirror dual, we can read off the identification

$$\bar{x} = 2^6 3^3 q_1 \,, \quad \bar{y} = 4q_2 \,. \tag{5.105}$$

If we now also identify the Kähler parameters with the complexified FI parameters, this clearly shows that the intersection of the two singular divisors D_1 and D_2 corresponds to the quantum shifted singular point of the GLSM.

Chapter 6

Emergent Yukawa couplings

With all the technical preparations being complete, we can now generalize our computation from chapter 4 to obtain the triple intersection numbers (TINs) $\kappa_{t_it_jt_k}$ of *compact* CY threefolds from the one-loop Schwinger integral by Gopakumar and Vafa. As a reminder, this task is quite challenging since we must integrate out all particle-like 1/2-BPS bound states of $D0$- and $D2$-branes that wrap 2-cycles. In other words, we must sum over the full homology lattice $H_2(X,\mathbb{Z})$ of a threefold X and weigh those contributions with the exponentially growing GV invariants α_0^β, leading to expressions of the general form

$$Y_{ijk}^{(0)} = \frac{1}{2} \sum_{\beta \in H_2(X,\mathbb{Z})} \alpha_0^\beta \beta^i \beta^j \beta^k \,, \tag{6.1}$$

which after successful regularization are supposed to yield the TINs. Let us briefly outline how we will proceed: First, we want to explain the origin of (6.1) and provide a mathematical toy example, which already contains an essential hint of how one could approach our main problem. We will then consider a couple of special CYs and take infinite distance limits where only a subset of GV invariants "count" the light states that need to be integrated out. These cases do not yet require any input from mirror symmetry and serve as a first hint that the regularization of infinite sums of the type (6.1) leads to sensible results. Building on these working examples, we then approach the actual problem of the emergence of the complete set of TINs in isotropic M-theory limits. Our first example, the quintic threefold, will guide us to a mathematically concrete regularization that involves the wanted TIN and the limit of the known prepotential in the type IIA weak coupling regime towards degeneration limits of the CY. We will then generalize this claim to arbitrary CY threefolds by proposing the regularization

$$Y_{ijk}^{(0)} = - \lim_{t_i \to t_{i,0}} \left[\left(\partial_{t_i} \partial_{t_j} \partial_{t_k} \mathcal{F}_0 \big|_{\text{weak}} - \kappa_{t_it_jt_k} \right) - \text{Div} \right] . \tag{6.2}$$

Here the limit is towards a suitable co-dimension h_{11} degeneration point $t_{i,0}$ that allows us to meaningfully isolate the total divergence Div from (6.1) such that after minimal subtraction no constant other than the explicit $\kappa_{t_it_jt_k}$ remains. (6.2) also shows how the periods at degeneration loci, which naturally carry information about GV invariants, will come into play. How the exact regularization formula comes about will be explained and motivated carefully in the course of this chapter.

Finally, our speculative proposal will then be challenged by looking at two CY threefolds with two Kähler moduli, namely the elliptic fibration $\mathbb{P}^4_{1,1,1,6,9}[18]$ featuring two intersecting conifold loci and the $K3$-fibration $\mathbb{P}^4_{1,1,2,2,6}[12]$, whose discriminant loci we have already encountered in section 5.2.2. The immediate question one might raise is which of the singular loci in a given moduli space can actually do the job. While in all our examples conifold points yield perfectly valid results, other singularities require further refinement. Understanding what exactly distinguishes conifold points in that regard or how the regularization can be potentially modified to universally cover all types of singularities would certainly be an interesting future path of work.

6.1 Preliminaries

To set the stage, we recall some relevant facts and definitions. Once again we consider type IIA string theory on a CY threefold X yielding $N = 2$ supergravity in four dimensions. Our goal is to obtain the classical cubic term in the prepotential from a one-loop Schwinger-like integral in the isotropic M-theory limit, defined via $g_s \to \lambda g_s\,, t_I \to \lambda^{2/3} t_I$ as before. Integrating out the light towers of states with mass scale below or at the species scale we arrive at the prescription of Gopakumar and Vafa. The corresponding Schwinger integral reads

$$\mathcal{F}_0(t) = \sum_{(n,\beta)\neq(0,0)} \alpha_0^\beta \int_0^\infty \frac{ds}{s^3} e^{sZ'_n(\beta)}\,, \tag{6.3}$$

where $Z'_n(\beta) = \frac{2\pi i}{g_s}(\beta \cdot t - n)$ is the central charge of the supersymmetry algebra. The infinite sum over KK momenta n already appeared in chapter 4 and was taken care of by ζ-function regularization. For a general compact CY, the $D2$-$D0$ bound states wrap all 2-cycles in $H_2(X, \mathbb{Z})$ and therefore another infinite sum needs to be performed.

6.1.1 Zero-point Yukawa couplings

Let us for the moment focus on threefolds with $h_{11} = 1$ and a single TIN κ. We can extract information about the cubic term from (6.3) by taking the third derivative with respect to t, giving the Yukawa coupling

$$Y_{ttt} = \frac{g_s^2}{(2\pi i)^3}\partial_t^3 \mathcal{F}_0(t) = \frac{1}{g_s} \sum_{(n,\beta)\neq(0,0)} \alpha_0^\beta \beta^3 \int_0^\infty ds\, e^{\frac{2\pi i}{g_s}s(\beta t+n)}\,. \tag{6.4}$$

After carrying out the integral we can split the sum as $\sum_{(\beta,n)\neq(0,0)} = \sum_{n\neq 0,\beta=0} + \sum_{n\in\mathbb{Z},\beta\neq 0}$. While the first sum doesn't contribute, the second can be simplified with the cotangent identity (A.12). After a few algebraic steps we arrive at

$$Y_{ttt} = \frac{1}{2}\sum_{\beta\neq 0} \alpha_0^\beta \beta^3 \left(1 + \frac{e^{2\pi i t}}{1 - e^{2\pi i t}}\right)\,, \tag{6.5}$$

where the second term in the brackets is related to the non-perturbative terms in the prepotential, i.e. the Li_3-terms. Note that the factor $1/2$ is precisely the formal triple intersection number for the single two-cycle of the resolved conifold. The whole expression, including the factor $1/2$, is reminiscent of the thermal energy-density of a gas of particles in four dimensions with energy levels β, degeneracy of states $\alpha_0^\beta \beta^2$ and temperature $T = i/t$. The first term in the brackets is then the (divergent) zero point energy, which in the following we call the *zero-point Yukawa coupling* denoted as $Y_{ttt}^{(0)}$. This object is the focus of our interest.

Typically the (genus-zero) GV invariants scale exponentially for large β, $\alpha_0^\beta \sim e^{\gamma\beta}$ with $\gamma > 0$. However, for $h_{11} > 1$ there can also be sub-towers of wrapped D2-branes leading to a weaker scaling such as $\alpha_0^\beta \sim e^{\gamma\sqrt{\beta}}$, or even to a polynomial behavior as for KK towers. One can then take a corresponding infinite distance limit so that only these towers contribute. The degeneracy of α_0^β is then closely tied to the type of limit as shown in table 6.1, where we listed the limits relevant for the following discussion. In any case,

limit	degeneracy of α_0^β
M-theory	$\exp(\gamma\beta)$
emergent string	$\exp\big(\gamma\sqrt{\beta}\big)$
6D decompactification	1

Table 6.1: Infinite distance limits and degeneracy of genus-zero GV invariants.

the first sum in (6.5) will be divergent regardless of these limits. Hence, emergence means that upon regularization, the t-independent first term in the bracket in (6.5) should give rise to the self-intersection number

$$Y_{ttt}^{(0)} := \frac{1}{2} \sum_{\beta=1}^{\infty} \alpha_0^\beta\, \beta^3 \Big|_{\mathrm{reg}} \overset{!}{=} \kappa_{ttt}\,. \tag{6.6}$$

Note that here the left-hand side is a quantity to be computed in the strong coupling regime, $g_s \gg 1$, i.e. within M-theory, whereas the right hand side is the usual integer-valued tree-level Yukawa coupling arising at weak string coupling, $g_s \ll 1$.

Therefore, the quest is to *define* a regularization such that the zero-point Yukawa coupling, $Y_{ttt}^{(0)}$, matches with the classical contribution in the perturbative string regime. Recall that $Y_{ttt}^{(0)}$ diverges due to the sum over wrapped $D2$-$D0$ bound states in the M-theory regime where, in contrast to string perturbation theory, we currently have no formalism to perform the integral and get to a finite result right away. Indeed, such a computation shall be part of the microscopic description of M-theory, which is lacking at present. Therefore, here we can only head for a properly working regularization scheme, like that of minimal subtraction and ζ-function regularization proposed for the computation of the R^4 terms in M-theory in [147].

For towers of $D2$-branes with a polynomial scaling of the GV invariants, we can still apply ζ-function regularization, but for the exponentially degenerate towers other methods need to be developed. Our approach is to move from simpler to more complicated cases and in the process refine and adapt the regularization method. Let us start by presenting a regularization method for infinite sums that already demonstrates our main strategy.

6.1.2 Regularization via modular forms

Suppose we are dealing with sums of the form

$$S_n = \sum_{k=1}^{\infty} k^n \,, \tag{6.7}$$

which are usually regularized with the ζ-function $\zeta(-n)$. It is instructive to recall how this works in practice. One introduces a regulator $q = \exp(-2\pi/\Lambda)$, with $\Lambda \gg 1$, such that for finite Λ the sum is finite. Then, one takes the limit $\Lambda \to \infty$ to isolate the divergence. In practice, the conventional way is to introduce q such that derivatives of the geometric series arise, namely

$$S_n = \lim_{\Lambda\to\infty} \sum_{k=1}^{\infty} k^n q^k = \lim_{\Lambda\to\infty} \left[(q\,\partial_q)^n \left(\frac{q}{1-q} \right) \right] , \tag{6.8}$$

for $|q| < 1$. One can then expand the right-hand side for large Λ to obtain

$$S_n = \lim_{\Lambda\to\infty} \left[n! \frac{\Lambda^{n+1}}{(2\pi)^{n+1}} + C_n + \mathcal{O}(\Lambda^{-1}) \right] . \tag{6.9}$$

The divergence appears only at $(n+1)$-th order and can be minimally subtracted, leaving us with the regularized value $C_n \equiv \zeta(-n)$.

Let us now arrive at the same result with an alternative method that exploits properties of modular forms to extract the singular term. We assume n to be odd in what follows. The idea is to express the above sum in terms of an Eisenstein series, which requires us to introduce the regulator q in a different fashion, namely

$$S_{2n-1} = -2 \lim_{\Lambda\to\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k^{2n-1}\, q^{lk} \,. \tag{6.10}$$

The normalization (-2) is chosen in order to compensate with the factor $\zeta(0) = -1/2$ arising from the sum over l in the limit $q \to 1$. We define the summation index $m = lk$ and then replace the second sum with a sum over the divisors of m, giving

$$S_{2n-1} = -2 \lim_{\Lambda\to\infty} \sum_{m=1}^{\infty} \sigma_{2n-1}(m)\, q^m \qquad \text{with} \quad \sigma_n(m) = \sum_{k|m} k^n \,. \tag{6.11}$$

This form of S_{2n-1} can now be directly related to the Eisenstein series

$$E_{2n}(\tau) = 1 + c_{2n} \sum_{m=1}^{\infty} \sigma_{2n-1}(m)\, q^m \,, \tag{6.12}$$

with

$$c_{2n} = \frac{(2\pi i)^{2n}}{(2n-1)!\,\zeta(2n)} = \frac{2}{\zeta(1-2n)}\,. \tag{6.13}$$

In this way, we arrive at

$$S_{2n-1} = \lim_{\Lambda\to\infty}\left[-\frac{2}{c_{2n}}\left(E_{2n}\left(\frac{i}{\Lambda}\right)-1\right)\right]. \tag{6.14}$$

These Eisenstein series are modular forms of degree $2n$, so that under a modular S-transformation they transforms as

$$E_{2n}\left(-\frac{1}{\tau}\right) = \tau^{2n}E_{2n}(\tau)\,. \tag{6.15}$$

By applying this transformation rule to the first term in (6.14), we find the behavior

$$E_{2n}\left(\frac{i}{\Lambda}\right) = (-1)^n\Lambda^{2n} + \mathcal{O}(e^{-2\pi\Lambda})\,. \tag{6.16}$$

A couple of comments are in order here. First, the singularity that has just been isolated has the same degree as the one occurring in the former example with a slightly different regularization scheme. Second, after minimally subtracting this divergence, the leading term is already exponentially suppressed in Λ and from the Eisenstein series one does not get any constant contribution surviving the limit $\Lambda\to\infty$. Hence, the only such contribution comes from the second term in (6.14) and gives the regularized value

$$S_{2n-1} = \frac{2}{c_{2n}} = \zeta(1-2n)\,. \tag{6.17}$$

For our purposes, the advantage of this regularization scheme is that it allows to explicitly isolate the divergence from the constant term and from the contributions vanishing in the limit $\Lambda\to\infty$. We will see that this method, and a generalization thereof, can be successfully applied to regularize the sum over GV invariants as in (6.6).

6.2 Large base limits of fibered CY threefolds

Before we discuss the isotropic M-theory limit, where the full homology lattice contributes to the Schwinger integral, we look at simpler cases where only part of it contributes. This restriction is of course only allowed if it is correlated with taking an appropriate infinite distance limit, in which the left out GV invariants give rise to heavy towers of $D2$-branes with mass scale above the species scale. Even though for such limits one will not have one-loop emergence of the full prepotential, integrating out the light towers of $D2$-branes will provide part of it and one can still approach the problem of what the meaning of the zero point Yukawa coupling is.

6.2.1 Elliptically fibered CY

First, we consider CY threefolds with a sublattice of GV invariants that essentially grow like KK towers (see e.g. the examples in [190]). For such cases, we want to identify an infinite distance limit such that all wrapped $D2$-brane states with exponential degeneracy are among the heavy states that are not integrated out. As we will see, this allows us to carry out the sum over the homology lattice by just using ζ-function regularization.

The CY manifolds that feature the desired property are elliptically fibered. As the maybe simplest example, we consider the CY threefold obtained by resolving the $\mathbb{Z}_3$-singularity of a degree 18 hypersurface in $\mathbb{P}^4_{(1,1,1,6,9)}[18]$ with Hodge numbers $(h_{21}, h_{11}) = (272, 2)$. The Mori cone data specifying the resolved threefold are given in table 6.2, where column p gives the zeroth component of each $l^{(a)}$ vector. The threefold is an

	p	z_1	z_2	z_3	z_4	z_5	z_6
$l^{(1)}$	-6	0	0	0	2	3	1
$l^{(2)}$	0	1	1	1	0	0	-3

Table 6.2: Data specifying elliptically fibered CY.

elliptic fibration with base $\mathbb{P}^2$, i.e. $T^2 \to X \xrightarrow{\pi} \mathbb{P}^2$. Coordinate divisors are generally specified by conditions of the type $z_i = 0$. To be more concrete, $z_6 = 0$ gives us $E = \mathbb{P}^2$, the base of the fibration, and $z_1 = 0$ gives $L = \pi^*(l)$, where l is a curve in the base and $\pi^*(l)$ its respective pullback divisor.

Defining $H = 3L + E$, the intersection numbers of (H, L) were discussed in [191] and read

$$H^3 = 9\,, \quad H^2 \cdot L = 3\,, \quad H \cdot L^2 = 1\,, \quad L^3 = 0\,. \tag{6.18}$$

The Poincaré dual two-forms are denoted by (w_H, w_L) so that we expand the Kähler form as $J = \tau_1 w_H + \tau_2 w_L$. Hence, for the total volume we obtain

$$\mathcal{V}_6 \sim \int J^3 = 9\tau_1^3 + 9\tau_1^2\tau_2 + 3\tau_1\tau_2^2\,. \tag{6.19}$$

Moreover, the fiber class $f \in H_2(X, \mathbb{Z})$ is given by $L \cap L$ and one can easily show that

$$\mathcal{V}_2(f) = \int_{L\cap L} J = \tau_1\,. \tag{6.20}$$

The second Mori-cone generator is given by the curve $l = L \cap E$ of volume

$$\mathcal{V}_2(l) = \int_{l=L\cap(H-3L)} J = \tau_2\,. \tag{6.21}$$

Integrating out lightest towers of states

Our goal is to take an asymptotic limit where only those $D2$-branes wrapping the elliptic fiber f are considered as perturbative states, since only these GV invariants are constant in this particular case, namely

$$\alpha_0^{(n_1,0)} = 540 \qquad \text{for} \quad n_1 \geq 1\,, \tag{6.22}$$

which is equal to minus the Euler characteristic of this CY. This can be achieved by the non-isotropic rescaling $\tau_2 \to \lambda\tau_2$, $g_s \to \lambda g_s$ with $\lambda \to \infty$ and τ_1 fixed, so that $M_{\mathrm{pl},4}$ stays fixed. Hence a $D2$-brane wrapping an effective curve $C = n_1 f + n_2 l$ bound to $D0$-branes is 1/2 BPS and has mass

$$m_{D0-D2} = \frac{M_s}{g_s}\left|n_1 t_1 + n_2 t_2 + m\right| = \frac{M_{\mathrm{pl},4}}{\mathcal{V}_6^{1/2}}\left|n_1 t_1 + n_2 t_2 + m\right| \tag{6.23}$$

with $t_i = b_i + i\tau_i$. Since $\mathcal{V}_6 \sim \lambda^2$, it is obvious that the lightest towers of states are $D0$-branes and $D2$-branes wrapping solely the class f, i.e. having $n_2 = 0$ and mass

$$m \sim \frac{M_{\mathrm{pl},4}}{\lambda}\,. \tag{6.24}$$

Since $\alpha_0^{(n_1,0)} = 540$ is constant, we are dealing with two light multiplicative KK towers [129], which yield the species scale

$$\tilde{\Lambda} = M_{\mathrm{pl},4}^{\frac{1}{2}}\Delta m^{\frac{1}{2}} \sim \frac{M_{\mathrm{pl},4}}{\lambda^{1/2}}\,. \tag{6.25}$$

In the classification of [93] this limit translates into a J-Class A limit in the vector moduli space of M-theory eventually leading to a further decompactification to six dimensions (this F-theory like limit was previously called type III$_c$ in [192] and also studied in [193]). Let us mention that among the light states there is also a light string, given by an $NS5$-brane wrapping the divisor L. One can show that there is no 4-cycle with a volume scaling like $\tau_1^2 \sim \lambda^0$ and the divisor volumes scale like

$$\mathcal{V}_4(H) = \int_H J\wedge J = (3\tau_1+\tau_2)^2 \sim \lambda^2\,, \quad \mathcal{V}_4(L) = \int_L J\wedge J = \tau_1(3\tau_1+\tau_2) \sim \lambda\,. \tag{6.26}$$

For the tension of the $NS5$ wrapped on L one finds $T_{NS5} = \frac{M_s^2}{g_s^2}\mathcal{V}_4(L)$ and hence $M_{\mathrm{str}} = \frac{M_s}{g_s}\sqrt{\mathcal{V}_4(L)} \sim \frac{M_{\mathrm{pl},4}}{\lambda^{1/2}} \sim \tilde{\Lambda}$. As explained in [93], this string is only a weakly coupled critical string, if the threefold admits an additional $K3$ or T^4 fibration. Therefore, we are not in the isotropic M-theory limit but in a limit where the lightest towers are bound states of $D0$- and $D2$-branes (wrapped on f) signaling a decompactification to 6D.

Using formula (6.2), we can now explicitly integrate out the light $D2$-$D0$ bound states which gives

$$\begin{aligned} Y_{t_1t_1t_1} &= \sum_{k=1}^{\infty}\alpha_0^{(k,0)}\,k^3\left(\frac{1}{2}+\frac{q^k}{1-q^k}\right) = \frac{540}{2}\sum_{k=1}^{\infty}k^3 + 540\sum_{k=1}^{\infty}k^3\,\frac{q^k}{1-q^k} \\ &= \frac{9}{4}\left(1+240\sum_{k=1}^{\infty}k^3\,\frac{q^k}{1-q^k}\right) = \frac{9}{4}E_4(t_1)\,, \end{aligned} \tag{6.27}$$

with $q = \exp(2\pi i t_1)$ and where we used $\sum_{k=1}^{\infty} k^3 = \zeta(-3) = 1/120$. From this calculation we do not obtain the triple-intersection number $\kappa_{t_1 t_1 t_1} = 9$, but since we are not taking the M-theory limit with isotropic co-scaling of all compact directions, it was also not expected a priori. However, we see that the divergent zero-point Yukawa coupling, after ζ-function regularization provides precisely the constant term in $E_4(t_1)$ that makes it a modular form of degree 4 (under modular transformation of t_1, the size of the toroidal fiber). While the appearance of a modular form is expected for such an elliptic fibration, the actual point here is to show how this result arises in the non-isotropic large coupling limit from integrating out the light $D2$-$D0$ towers of states. All this we take as a first indication that also for compact CYs the regularized zero-point Yukawa coupling yields physically meaningful results. Note that for getting this result, the generic factor 1/2 in the first line of (6.27) was essential.

This computation is expected to generalize to every torus-fibered CY. One example is the CY $\mathbb{P}^4_{(1,1,1,3,6)}[12]$ with $(h_{21}, h_{11}) = (165, 3)$, which is a fibration over $\mathbb{P}^2$ with two sections. Geometrically, having multiple sections means that there are two distinct copies of the base sitting inside the CY which both intersect every fiber once at different points of the elliptic curve. At the level of the fiber we therefore have additional marked points that vary smoothly across the base. Algebraically, we then no longer identify all tori related by $SL(2,\mathbb{Z})$ transformations but only by those transformations that do not change the position (or the group structure) of said marked points.

Coming back to our example, the GV invariants that count $D2$'s only wrapping the fiber are given by $\alpha_0^{(2k,0,0)} = 324 = -\chi(X)$ and $\alpha_0^{(2k-1,0,0)} = 216$ with $k \geq 1$. For the relevant Yukawa coupling we obtain

$$
\begin{aligned}
Y_{t_1 t_1 t_1} &= \sum_{k\geq 1} k^3 \alpha_0^{(k,0,0)} \left(\frac{1}{2} + \frac{q^k}{1-q^k} \right) \\
&= 216 \sum_{k\geq 1} k^3 \left(\frac{1}{2} + \frac{q^k}{1-q^k} \right) + 4 \cdot 216 \sum_{k\geq 1} k^3 \left(\frac{1}{2} + \frac{q^{2k}}{1-q^{2k}} \right) \\
&= \frac{9}{10} \Big(E_4(t_1) + 4 E_4(2t_1) \Big) ,
\end{aligned}
\tag{6.28}
$$

which is a modular form of weight four for $\Gamma_0^+(2)$. Here $\Gamma_0(2)$ denotes the congruence subgroup[1] and $\Gamma_0^+(2)$ its extension by the Fricke involution ω_2, where ω_N is defined as

$$
\omega_N = \begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix} \in PSL(2,\mathbb{R}) , \qquad N \in \mathbb{N} . \tag{6.30}
$$

[1]Concretely, we consider elliptic fibrations with a cyclic section of order N, which picks a specific cyclic group of N marked points on the fiber. The modular group preserving this cyclic group as a set is

$$
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\} . \tag{6.29}
$$

.

and induces the transformation $\tau \to -1/(N\tau)$ (see e.g. [194]). For torus fibrations with more sections, the cyclic properties of the GV invariants become more involved. For instance, the CY $\mathbb{P}^4_{(1,1,1,3,3)}[9]$ with $(h_{21}, h_{11}) = (112, 4)$ has three sections and the GV invariants with non-trivial fiber class are given by $\alpha_0^{(3k,0,0)} = 216$ and $\alpha_0^{(3k-1,0,0)} = \alpha_0^{(3k-2,0,0)} = 162$ with $k \geq 1$. Nevertheless, the computation works similarly and yields

$$Y_{t_1t_1t_1} = \frac{27}{40}\Big(E_4(t_1) + 9E_4(3t_1)\Big)\,, \tag{6.31}$$

which is now a modular form of weight four for $\Gamma_0^+(3)$.

6.2.2 $K3$-fibered CY

Our next example is a $K3$-fibration over the base $\mathbb{P}^1$. More specifically, we study the three-parameter CY threefold obtained from a hypersurface of degree 24 in $\mathbb{P}^4_{(1,1,2,8,12)}[24]$ with the singularities resolved. The exceptional divisors come from blow-ups of a singular $\mathbb{Z}_2$-curve and an exceptional $\mathbb{Z}_4$ point. The CY has been extensively studied, see e.g. [179, 195–197] for some earlier works. We are particularly interested in the duality to the heterotic string on $K3 \times T^2$ in the limit where the base $\mathbb{P}^1$ grows large [196]. We will use this dictionary to extract explicit expressions for the GV invariants of 2-cycles in the $K3$-fiber. Concerning our conventions, we closely follow [198, 199].

In table (6.3) we provide the Mori cone vectors for the resolution $\mathbb{P}^4_{(1,1,2,8,12)}[24]$ from [179][2]. We denote the type IIA complexified Kähler moduli by t_i $(i = 1, 2, 3)$, where t_2

	p	z_1	z_2	z_3	z_4	z_5	z_6	z_7
$l^{(1)}$	-6	0	0	0	2	3	1	0
$l^{(2)}$	0	1	1	0	0	0	0	-2
$l^{(3)}$	0	0	0	1	0	0	-2	1

Table 6.3: Data specifying $K3$-fibered CY.

measures the size of the base. In the limit of large base $t_2 \to \infty$ the prepotential reads[3]

$$\mathcal{F}_0^{\text{IIA}} = \mathcal{F}_0^{\text{IIA,cubic}} + \frac{1}{(2\pi i)^3} \sum_{n_1,n_3=0}^{\infty} \alpha_0^{(n_1,0,n_3)} \operatorname{Li}_3(q_1^{n_1} q_3^{n_3}) + \mathcal{O}\left(e^{2\pi i t_2}\right)\,, \tag{6.32}$$

where world-sheet instantons wrapping also the base become negligible. The cubic term reads

$$\mathcal{F}_0^{\text{IIA,cubic}} = \frac{4}{3}t_1^3 + t_1^2 t_2 + 2t_1^2 t_3 + t_1 t_2 t_3 + t_1 t_3^2 \tag{6.33}$$

[2]Ordering of elements has been adapted to our chosen conventions for $\mathbb{P}^4_{1,1,2,2,6}[12]$ and $\mathbb{P}^4_{1,1,1,6,9}[18]$. Note that there is also a typing error in $l^{(1)}$ in [179], which was already pointed out in [200].

[3]Here we have set the prefactor $(2\pi i)^3/g_s^2 = 1$.

and provides the classical (tree-level) contribution in the weakly coupled type IIA theory with $g_s^{\mathrm{IIA}} \ll 1$.

This type IIA model is dual to the heterotic STU-model, which has been studied in [168]. Here S denotes the (one-loop corrected) axio-dilaton and (T, U) are the Kähler and complex structure moduli of the T^2. These three heterotic moduli are related to complexified Kähler moduli on the type IIA side as

$$t_1 = iU\,, \quad t_2 = iS\,, \quad t_3 = i(T-U)\,. \tag{6.34}$$

The Kähler cone is $\mathrm{Im}(t_i) > 0$ which means that on the heterotic side we are in the chamber $\mathrm{Re}(T) > \mathrm{Re}(U)$. The prepotential is given by

$$\begin{aligned}\mathcal{F}_0^{\mathrm{het}} = i\Bigg(-\mathcal{F}_0^{\mathrm{het,cubic}} - \frac{1}{(2\pi)^3}\sum_{k,l=0}^{\infty} 2c_1(kl)\mathrm{Li}_3(e^{-2\pi(kT+lU)}) \\ - \frac{2}{(2\pi)^3}\mathrm{Li}_3(e^{-2\pi(T-U)}) + O\left(e^{-2\pi S}\right)\Bigg)\,,\end{aligned} \tag{6.35}$$

where the coefficients $c_1(n)$ are defined via

$$\frac{E_4 E_6}{\eta^{24}}(q) = \sum_{n=-1}^{\infty} c_1(n) = \frac{1}{q} - 240 - 141444q + \dots\,. \tag{6.36}$$

The cubic part is

$$\mathcal{F}_0^{\mathrm{het,cubic}} = STU + \frac{1}{3}U^3 + T^2 U\,. \tag{6.37}$$

Note that only the first term is at tree-level in g_s^{het}, whereas the two S-independent ones and all remaining S-independent terms from (6.35) make the full one-loop correction. The remaining unknown terms come from heterotic $NS5$-brane instantons. One can check that the type IIA and heterotic cubic terms directly match reflecting the duality $\mathcal{F}_0^{\mathrm{IIA}} = \mathcal{F}_0^{\mathrm{het}}$.

Moreover, exploiting this duality the GV invariants in the class

$$C = i(kT + lU) = (k+l)t_1 + kt_3 \tag{6.38}$$

can be directly read-off from (6.35)

$$\alpha_0^{(k+l,0,k)} = -2c_1(kl) \qquad \text{for} \quad k,l \geq 0 \ \text{ and } (k,l) = (1,-1)\,. \tag{6.39}$$

Note that l can also take negative values without violating the Kähler cone condition. Hence, we are in a situation where we happen to know the subset of GV invariants for $D2$-branes not wrapping the large base explicitly in terms of a modular form. Note that this implies that these GV invariants grow like $c_1(n) \sim \exp(4\pi\sqrt{n})$ for large n, which is correlated with the emergent string limit that we are taking.

Yukawa couplings from Schwinger integrals

All these preliminary considerations were in some weakly coupled limit. Now we analyze how the computation could be carried out in the co-scaled infinite distance limit $\tau_2 \to \lambda\tau_2$, $g_s^{\rm IIA} \to \sqrt{\lambda} g_s^{\rm IIA}$, where the 4D Planck scale remains constant. In this limit, the light towers of states with a mass scale below the species scale $\tilde{\Lambda} \sim M_{\rm pl}/\sqrt{\lambda}$ are strings arsing from $NS5$-branes wrapping the $K3$-fiber and $D2$-$D0$ bound states wrapping 2-cycles of the $K3$. The contribution of these $D2$-$D0$ bound states to $\mathcal{F}_0$ is given by (6.3).

To be concrete, let us discuss the zero point Yukawa-coupling $Y^{(0)}_{UUU}$. To evaluate it, one first needs to determine all GV invariants $\alpha_0^{(n_1,0,n_3)}$ for 1/2 BPS $D2$-branes wrapping the 2-cycles in the $K3$-fiber. In general this is a horrendous problem that only a "CY-demon" can solve by pure counting. However, as just explained, we are in the fortunate situation that we know these GV invariants already from computations in the weakly coupled regime. Hence, the "CY-demon" will eventually find the same numbers. Therefore, for the expression for the Yukawa coupling (6.5) the demon will write

$$Y_{UUU} = Y^{(0)}_{UUU} + 2\cdot 240 \sum_{l=1}^{\infty} l^3 \, \frac{q_2^l}{(1-q_2^l)} - \sum_{k,l=1}^{\infty} 2c_1(kl)\, l^3 \frac{q_1^k q_2^l}{(1-q_1^k q_2^l)} - 2\frac{q_1}{q_1-q_2}\,, \tag{6.40}$$

where $q_1 = \exp(-2\pi T)$, $q_2 = \exp(-2\pi U)$ and we have used $c_1(-1) = 1$ and $c_1(0) = -240$. One realizes that the second term on the right hand side can be expressed via the Eisenstein series

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \tag{6.41}$$

as $2(E_4(q_2)-1)$. According to (6.5), the diverging zero point Yukawa coupling is given by

$$\begin{aligned} Y^{(0)}_{UUU} &= 240 \sum_{l=1}^{\infty} l^3 - \sum_{k,l=1}^{\infty} c_1(kl)\, l^3 - c_1(-1)\cdot(-1)^3 \\ &= -2\cdot 240 \sum_{l,d=1}^{\infty} l^3 + 2 \sum_{k,l,d=1}^{\infty} c_1(kl)\, l^3 + 2\sum_{d=1}^{\infty} c_1(-1)\cdot(-1)^3\,, \end{aligned} \tag{6.42}$$

where, as for the example in section 6.1.2, we introduced the sum $\sum_{d=1}^{\infty} 1 = \zeta(0) = -1/2$. Following the same example, we can now regularize this expression by introducing dummy variables $p_1, p_2 < 1$ with $p_1 < p_2$ and eventually take the limit $p_1, p_2 \to 1$. Thus, we get

$$\begin{aligned} Y^{(0)}_{UUU}\Big|_{\rm reg} &:= \lim_{p_1,p_2\to 1} \left[-2\cdot 240 \sum_{l,d=1}^{\infty} l^3\, p_2^{ld} + 2 \sum_{k,l,d=1}^{\infty} c_1(kl)\, l^3\, p_1^{kd}\, p_2^{ld} - 2\sum_{d=1}^{\infty} p_1^d\, p_2^{-d} \right] \\ &= \lim_{p_1,p_2\to 1} \left[2 - 2E_4(p_2) + 2\sum_{k,l=1}^{\infty} c_1(kl) l^3 \frac{p_1^k\, p_2^l}{(1-p_1^k\, p_2^l)} + 2\frac{p_1}{(p_1-p_2)} \right]. \end{aligned} \tag{6.43}$$

Note that the geometric series on the far right-hand side will only be convergent if $p_1 < p_2$. Clearly, after these steps the regularized expression has the same form as the non zero-point Yukawa coupling in (6.40) but it is important to note that we are not taking the $p_1, p_2 \to 0$ limit, where this expression nicely converges but the opposite $p_1, p_2 \to 1$ limit. In order to perform this limit and to isolate the singularity, we employ a useful mathematical relation, the so-called Harvey-Moore identity [201]

$$-E_4(p_2) + \sum_{k,l,d=1}^{\infty} c_1(kl)\, l^3\, p_1^{kd} p_2^{ld} + \frac{p_1}{p_1 - p_2} = \frac{E_4(p_1)E_6(p_1)}{\eta^{24}(p_1)} \frac{E_4(p_2)}{j(p_2) - j(p_1)}\,, \tag{6.44}$$

where the right hand side is a modular form of bi-degree $(-2, 4)$.

As for the pedagogical toy example in section 6.1.2, we now write $p_1 = \exp(-2\pi/\Lambda_1)$, $p_2 = \exp(-2\pi/\Lambda_2)$ with $\Lambda_1 < \Lambda_2$ and apply a modular S-transformation to the right-hand side of (6.44). In this way, we get

$$\begin{aligned} Y^{(0)}_{UUU} &= \lim_{\Lambda_1,\Lambda_2\to\infty} \left(2 - 2\frac{\Lambda_2^4}{\Lambda_1^2} \frac{E_4(i\Lambda_1)E_6(i\Lambda_1)}{\eta^{24}(i\Lambda_1)} \frac{E_4(i\Lambda_2)}{j(i\Lambda_2) - j(i\Lambda_1)} - \mathrm{Div}\right) \\ &= \lim_{\Lambda_1,\Lambda_2\to\infty} \left(2 - 2\frac{\Lambda_2^4}{\Lambda_1^2}\left(1 + O(e^{-2\pi(\Lambda_2-\Lambda_1)})\right) - \mathrm{Div}\right). \end{aligned} \tag{6.45}$$

Here we realize the compelling pattern that in the limit, the second term in (6.45) contains only a diverging term and exponentially suppressed terms. Hence, after minimally subtracting the divergent Λ_2^4/Λ_1^2 term and taking the limit we arrive at the final result

$$Y^{(0)}_{UUU} = 2\,, \tag{6.46}$$

which is indeed the cubic term in the one-loop heterotic Yukawa coupling (6.37). Recall that this corresponds to a tree-level TIN in the type IIA dual model.

Thus, what we have effectively done here is the following: since we do not have a "CY demon", we used the knowledge of the GV invariants derived in the weakly coupled region from the dual heterotic string (or as we will see later from type IIA mirror symmetry) to define the regularized value of the zero point Yukawa coupling, arising from the Schwinger integral over $D0$-$D2$ bound states (in the strongly coupled region), as

$$Y^{(0)}_{UUU} := -\lim_{U,T\to 0}\left[\left(i\,\partial_U^3 \mathcal{F}_0^{\mathrm{het}} - \kappa_{UUU}\right) - \mathrm{Div}\right], \tag{6.47}$$

where Div denotes all terms that diverge in the limit. We note that the cubic term κ_{UUU} can be obtained from the opposite limit

$$\kappa_{UUU} = \lim_{U,T\to\infty} i\,\partial_U^3 \mathcal{F}_0^{\mathrm{het}}\,. \tag{6.48}$$

The definition (6.47) means that one gets precisely $Y^{(0)}_{UUU} = \kappa_{UUU}$, if the $U, T \to 0$ limit of $\partial_U^3 \mathcal{F}_0^{\mathrm{het}}$ itself only gives divergent and vanishing terms, i.e. no further constant terms. Thus, it is satisfactory that the proposed regularization involves the TIN in an explicit way and does not rely on any seemingly miraculous cancellations of constants.

One could now use this definition to evaluate the remaining three zero point Yukawa couplings $Y^{(0)}_{UUT}$, $Y^{(0)}_{UTT}$ and $Y^{(0)}_{TTT}$. Instead of doing this in detail, we just mention that $Y^{(0)}_{TTT} = 0$ can be obtained from an analogous computation in the chamber $\mathrm{Re}(U) > \mathrm{Re}(T)$ and an analytic continuation of the $\mathrm{Li}_3(e^{-2\pi(T-U)})$-term in (6.35). The couplings $Y^{(0)}_{UUT}$ and $Y^{(0)}_{UTT}$ would require a different technique, as the Harvey-Moore formula cannot be applied directly.

To summarize, for an emergent string limit we have successfully invoked a regularization of the zero point Yukawa couplings so that the Schwinger integration over the light towers of $D2$-$D0$ bound states at strong coupling reproduced a subset of the classical (tree-level) TINs. Even though we are not yet taking the isotropic M-theory limit, this computation serves as proof of principle of how this regularization can be defined. Of course we are not yet getting full emergence as the term κ_{STU} is at (heterotic $NS5$-brane) string tree-level and not captured by the one-loop Schwinger integral. The next step is to move forward to the isotropic M-theory limit, where we expect the full weakly coupled type IIA tree-level prepotential to be contained in the Schwinger one-loop integral at strong coupling.

6.3 Isotropic M-theory limit of CYs

Now we want to generalize these computations to the ultimate case, namely to isotropic M-theory limits where the full homology lattice contributes and where the GV invariants generically scale exponentially as $\exp(\gamma\beta)$.

Let us note that the example from the previous section shares one essential feature with the resolved conifold, namely that the 2-cycles wrapped by the light $D2$-branes are allowed to shrink to zero size (in string units). As is obvious from our construction, this feature was crucial in defining the regularization of (6.6) via the limit $t \to 0$ and employing the modular properties of the Yukawa couplings to isolate the divergence and subtract it. However, as already observed in the seminal work [23] and briefly reviewed in 5.3.3, on the Quintic threefold, the exponential degeneracy of the GV invariants is correlated with the appearance of a divergence of the prepotential not at zero but at a finite value $t = t_c$ set by the location of the conifold singularity. Note that this is not the minimal value of $\mathrm{Im}(t)$ which is at the Landau-Ginzburg point. Indeed, the Yukawa-couplings are regular at that point so that it does not seem reasonable to regularize our infinite sums by taking the limit to this point.

As we discussed already in section 5.4, the conifold point is nothing else than the quantum corrected singular point in the middle of the classical phase diagram of a GLSM. Thus, in a certain sense we are indeed taking a limit towards zero size, namely towards vanishing FI parameter $\xi = 0$. Since due to quantum effects this is not really possible, the most natural thing to do is to take the limit towards its quantum corrected value.

6.3.1 Emergence of TINs on CYs with $h_{11} = 1$

For concreteness, let us now consider CYs with one Kähler modulus and therefore no fibration structure. In this case, there is no obvious toroidal structure present and one does not expect the Yukawa coupling to contain any modular forms. As mentioned, the Yukawa couplings in the large radius chart still have a divergence of the form

$$Y_{ttt} \sim \frac{1}{(t-t_c)\log^2(t-t_c)} + \dots\,, \tag{6.49}$$

which is not at zero but at the location of the conifold point $t = t_c$. The same holds for the total list of 14 CYs with $h_{11} = 1$ that we provide here for convenience (see e.g. [202])

$$\begin{aligned}
&\mathbb{P}_{1^4,4,6}[12,2]_{2.302}\,,\ \mathbb{P}_{1^3,2,5}[10]_{2.099}\,,\ \mathbb{P}_{1^2,2^2,3^2}[6,6]_{1.874}\,,\ \mathbb{P}_{1^4,4}[8]_{1.695}\,,\ \mathbb{P}_{1^3,2^2,3}[6,4]_{1.563}\,,\\
&\mathbb{P}_{1^4,2}[6]_{1.421}\,, \mathbb{P}_{1^5,3}[6,2]_{1.334}\,,\ \mathbb{P}_{1^4,2^2}[4,4]_{1.255}\,,\ \mathbb{P}_{1^5}[5]_{1.208}\,,\ \mathbb{P}_{1^5,2}[4,3]_{1.115}\,,\\
&\mathbb{P}_{1^5}[4,2]_{1.029}\,,\ \mathbb{P}_{1^6}[3,3]_{0.975}\,,\ \mathbb{P}_{1^7}[3,2,2]_{0.891}\,,\ \mathbb{P}_{1^8}[2,2,2,2]_{0.807}\,,
\end{aligned} \tag{6.50}$$

where the lower index is the value of $\mathrm{Im}(t_c)$ at the conifold singularity. As we have reviewed in section 5.3.3, in [23] this divergence was utilized to derive the asymptotic behavior of the GV invariants

$$\alpha_0^\beta \sim \frac{e^{2\pi\beta\,\mathrm{Im}(t_c)}}{\beta^3\log(\beta)^2}\,. \tag{6.51}$$

Crucially, for defining a regularization of the zero point Yukawa couplings, the divergence at the conifold singularity encodes information about the exponential degeneracy of the GV invariants. Therefore, it does not make sense to define the regularization of the zero point Yukawa coupling via a $t \to 0$ limit, but rather as

$$Y_{ttt}^{(0)} := -\lim_{t\to t_c}\left[\sum_{n=1}^{\infty}\alpha_0^n\,n^3\frac{q^n}{1-q^n} - \mathrm{Div}\right] = -\lim_{t\to t_c}\left[\left(\partial_t^3\mathcal{F}_0\big|_{\text{weak}} - \kappa_{ttt}\right) - \mathrm{Div}\right]. \tag{6.52}$$

Here $q = \exp(2\pi i t)$ and we have set $(2\pi i)^3/g_s^2 = 1$. As usual, before taking the limit to the conifold point, we minimally subtract the divergence for $(t - t_c) \to 0$. To explain the logic, recall that the computation is now done in the strong coupling M-theory limit, where we integrate out the full tower of 1/2-BPS $D2$-$D0$ bound states. Of course, only a "CY demon" can do this in practice, but we are lucky in that we happen to know the exact result from the weakly coupled regime via non-renormalization theorems and mirror symmetry. Thus, the result that the "CY-demon" obtains after carrying out this infinite sum should agree with the weakly coupled result in the large complex structure regime $\partial_t^3\mathcal{F}_0\big|_{\text{weak}}$ minus the triple intersection number κ_{ttt}, that is not contained in the sum $\sum_{n=1}^{\infty}\alpha_0^n\,n^3q^n/(1-q^n)$.

Now, we have arrived at an expression that is very similar to the other regularized expressions (6.14) and (6.47) encountered so far, where we were dealing with modular forms. Instead of applying a modular S-transformation, next we have to evaluate

$\partial_t^3 \mathcal{F}_0\big|_{\text{weak}}$ around the conifold point and disentangle the divergent, the constant and the suppressed contributions in the $(t - t_c) \to 0$ limit. We have succeeded in defining a working regularization, if the following *regularization condition* holds:

> *The $t \to t_c$ limit of $\partial_t^3 \mathcal{F}_0|_{\text{weak}}$ only gives divergent and vanishing terms, i.e. no further constant terms. Then* (6.52) *yields precisely $Y_{ttt}^{(0)} = \kappa_{ttt}$.*

Example: Quintic

Let us consider the Quintic, i.e. $\mathbb{P}_4[5]$, as a concrete example, where we can rely on well known results. The weakly coupled Yukawa coupling $\partial_t^3 \mathcal{F}_0\big|_{\text{weak}}$ was already computed in the seminal paper [23] using mirror symmetry. Here we are just collecting a couple of useful relations obtained in [23] and also in [203, 204] and refer the reader to the original literature. The mirror of the Quintic has one complex structure modulus, which is given by the deformation of the degree five hypersurface constraint

$$\sum_{i=1}^{5} z_i^5 - (5\psi)\, z_1 z_2 z_3 z_5 z_5 = 0 \tag{6.53}$$

in $\mathbb{P}^4$. The conifold singularity is located at $\psi = 1$. A basis of local periods can be determined as solutions to the Picard-Fuchs equation and in the large complex structure regime can be expressed as

$$\begin{aligned}
\pi_1^{\text{LCS}} &= \omega_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!\, z^n}{(n!)^5} \\
\pi_2^{\text{LCS}} &= \omega_0 \log z + z\, \sigma_1(z) \\
\pi_3^{\text{LCS}} &= \omega_0 \log^2 z + 2z\, \sigma_1(z) \log z + z\, \sigma_2(z) \\
\pi_4^{\text{LCS}} &= \omega_0 \log^3 z + 3z\, \sigma_1(z) \log^2 z + 3z\, \sigma_2(z) \log z + z\, \sigma_3(z)\,,
\end{aligned} \tag{6.54}$$

with $z = 1/(5\psi)^5$ and $|z| < 5^{-5}$. The $\sigma_i(z) = \sum_{n=0}^{\infty} a_{i,n} z^n$ are infinite series in z, starting with a constant. Their precise form will not be important for our purposes. The next step is to map these periods to an integral symplectic basis, which allows to read off the mirror map and the prepotential via[4]

$$\Pi_{\text{LCS}} = \begin{pmatrix} X_0 \\ X_1 \\ F_1 \\ F_0 \end{pmatrix} = \omega_0 \begin{pmatrix} 1 \\ t \\ \partial_t \mathcal{F}_0 \\ 2\mathcal{F}_0 - t\partial_t \mathcal{F}_0 \end{pmatrix}. \tag{6.55}$$

Here $t = iT$ denotes the complexified Kähler modulus of the Quintic that enjoys an expansion

$$t = \frac{X_1}{X_0} = \frac{1}{2\pi i} \log z + \mathcal{O}(z)\,. \tag{6.56}$$

[4]To prevent confusion between the prepotential and period entries, we denote the B-cycle periods by F_I instead of $\mathcal{F}_I$ from now on.

Inverting this mirror map one can express the prepotential around the large complex structure point as

$$\mathcal{F}_0 = \frac{5}{6}t^3 + \frac{25i\zeta(3)}{2\pi^3} + \text{instantons}\,. \tag{6.57}$$

The radius of convergence of the large complex structure patch is precisely $|\psi| = 1$ so that the conifold singularity lies on the boundary. Therefore, for determining the limit of the Yukawa coupling (6.52) this patch is not convenient and one needs to know the periods in the conifold patch and how the two patches are correctly glued together. For the Quintic this problem has been solved e.g. in [204] and here we just recall their results up to the level of detail needed for our purposes.

In the conifold patch, it is convenient to introduce the coordinate $u = 1 - \psi^{-5}$ which vanishes at the conifold locus. Then one can solve the Picard-Fuchs equation in this coordinate, obtain a basis of periods and finally determine a transition matrix to an integral symplectic basis Π_{C}, which on the overlap with the LCS patch matches the former basis Π_{LCS}. This integral symplectic basis takes the form

$$\begin{aligned} X_0 &= -\rho_1(u)\frac{u}{2\pi i}\log u + \rho_2(u) \\ X_1 &= \rho_3(u) \\ F_1 &= \rho_4(u) \\ F_0 &= u\,\rho_1(u)\,, \end{aligned} \tag{6.58}$$

where the $\rho_i(u) = \sum_{n=0}^{\infty} b_{i,n}u^n$ are infinite series in u. Note that the logarithmic term in X_0 leads to the monodromy $X_0 \to X_0 - F_0$ around the conifold.

Now we can read off the Kähler t in terms of the conifold coordinate u, via

$$t = \frac{X_1}{X_0} = \frac{b_{4,0}}{b_{3,0}} \sum_{m=0}^{\infty} (u\log u)^m\, c_m(u)\,, \tag{6.59}$$

where $c_m(u)$ denote infinite series in u with $c_{0,0} = 1$. Using the concrete values $b_{4,0} = i1.29357...$, $b_{3,0} = 1.07073...$ we obtain in the $u \to 0$ limit $t_c = iT_c = i1.20812....$ As mentioned already in 5.3.3, at leading order we get

$$t - t_c = t_c\, c_{1,0}\, u\log u + \ldots\,. \tag{6.60}$$

Next we recall that the Yukawa coupling can be expressed as [23]

$$\partial_t^3 \mathcal{F}_0\big|_{\text{weak}} = \frac{1}{\omega_0^2}\kappa_{\psi\psi\psi}\frac{1}{(dt/d\psi)^3} \tag{6.61}$$

with the B-side Yukawa coupling

$$\kappa_{\psi\psi\psi} = \left(\frac{2\pi i}{5}\right)^3 \frac{5\psi^2}{1-\psi^5}\,. \tag{6.62}$$

Using $u = 1 - \psi^{-5}$ and the general structure of the periods at the conifold, one realizes that the building blocks in (6.61) can be expressed as

$$\begin{aligned} \frac{dt}{du} &= \log u \left(\frac{d_{-1}(u)}{\log u} + \sum_{n=0}^{\infty} (u \log u)^n \, d_n(u) \right) \\ \frac{1}{\omega_0^2} &= \sum_{n=0}^{\infty} (u \log u)^n \, e_n(u) \\ \frac{\kappa_{\psi\psi\psi}}{(\partial_\psi u)^3} &= -\frac{1}{25} \frac{1}{u(1-u)^3} \,, \end{aligned} \tag{6.63}$$

where the $d_n(u)$ and $e_n(u)$ are again infinite series in u. Putting everything together and expanding for small u, we find the general expansion

$$\partial_t^3 \mathcal{F}_0 \big|_{\text{weak}} = \frac{1}{u \log^3 u} \left(\sum_{n,k=0}^{\infty} \frac{(u \log u)^n}{\log^k u} a_{n,k}(u) \right) . \tag{6.64}$$

Consistent with [23], one sees that the leading divergence is

$$\partial_t^3 \mathcal{F}_0 \big|_{\text{weak}} \sim \frac{1}{u \log^3 u} + \ldots \sim \frac{1}{(t - t_c) \log^2 (t - t_c)} + \ldots \,, \tag{6.65}$$

where we used (6.60). However, for $n = 0$ all the constant terms in $a_{0,k}(u) = a_{0,k,0} + O(u)$ lead to singular terms

$$\partial_t^3 \mathcal{F}_0 \big|_{\text{weak}} = \sum_{k=0}^{\infty} \frac{a_{0,k,0}}{u \log^{k+3} u} + \text{Reg} \,, \tag{6.66}$$

which according to our philosophy are minimally subtracted. With all the remaining terms being regular, the final and essential question is whether they all go to zero or whether there remains a constant contribution. That this is not the case can be straightforwardly seen by noting that the cancellation of the $\log u$ term requires $n = k + 3$, for which still a term $u^{k+2} a_{k+3,k}(u)$ remains that goes to zero for $u \to 0$. Hence there is no constant term and our proposed regularization (6.52) indeed yields

$$\mathrm{Y}_{ttt}^{(0)} = - \lim_{t \to t_c} \left[\left(\partial_t^3 \mathcal{F}_0 \big|_{\text{weak}} - \kappa_{ttt} \right) - \text{Div} \right] = \kappa_{ttt} = 5 \,. \tag{6.67}$$

We expect the computation to be completely analogous for the list of 14 CYs with $h_{11} = 1$ presented in (6.50).

Having a closer look at the computation one realizes that the avoidance of a constant contribution in the limit $t \to t_c$ can be traced back to the appearance of the $u \log u$ term in the period X_0, i.e. that $\log u$ is always accompanied by a linear factor u. This is of course a consequence of the monodromy around the conifold point.

It could be that our findings are just an artifact of the too simple choice of CY manifolds with one Kähler modulus. Therefore, next we generalize the computation to CYs with two Kähler moduli. Moreover, we also analyze whether the *regularization* *condition* is only true for approaching conifold singularities or whether it also holds for more general degeneration limits. For $h_{11} = 2$ it also happens that the codimension one degeneration loci intersect in more than one point so that the question arises which point to choose and also along which family of path it should be approached in taking the limit. To approach these questions, in the following sections we consider the two CYs $\mathbb{P}^4_{1,1,1,6,9}[18]$ and $\mathbb{P}^4_{1,1,2,2,6}[12]$.

6.3.2 Emergence of TINs on $\mathbb{P}^4_{1,1,1,6,9}[18]$

We start with the elliptically fibered CY manifold from section 6.2.1, namely $\mathbb{P}^4_{1,1,1,6,9}[18]$ with Hodge numbers $(h_{11}, h_{21}) = (2, 272)$. The two complex structure moduli of the mirror dual CY are parametrized by the deformations of the hypersurface constraint

$$P = z_1^{18} + z_2^{18} + z_3^{18} + z_4^{3} + z_5^{2} - 18\psi\, z_1 z_2 z_3 z_4 z_5 - 3\phi\, z_1^6 z_2^6 z_3^6\,. \tag{6.68}$$

Introducing the combinations

$$\bar{x} = -\frac{1}{2^2 3^8}\frac{\phi}{\psi^6}\,, \qquad \bar{y} = -\frac{1}{\phi^3}\,, \tag{6.69}$$

the two conifold degeneration loci of the manifold are

$$\begin{aligned} \Delta_1 &= (1-\bar{x})^3 - \bar{x}^3\bar{y} = 0\,,\\ \Delta_2 &= 1 + \bar{y} = 0\,. \end{aligned} \tag{6.70}$$

The LCS point is at $\bar{x} = \bar{y} = 0$ and one can solve the corresponding Picard-Fuchs equations in the vicinity of this point, determine an integral symplectic basis of periods $\Pi_I^{\rm LCS}$, $I = 1, \ldots, 6$, such that the mirror map is

$$t_1 = \frac{\Pi_2^{\rm LCS}}{\Pi_1^{\rm LCS}} = \frac{1}{2\pi i}\log(\bar{x}) + \ldots\,, \qquad t_2 = \frac{\Pi_3^{\rm LCS}}{\Pi_1^{\rm LCS}} = \frac{1}{2\pi i}\log(\bar{y}) + \ldots\,. \tag{6.71}$$

Here t_1 measures the size of the toroidal fiber and t_2 the size of the curve $\mathbb{P}^1 \subset \mathbb{P}^2$. Inverting the mirror map, the prepotential comes out as

$$\mathcal{F}_0|_{\rm weak} = \frac{3}{2}t_1^3 + \frac{3}{2}t_1^2 t_2 + \frac{1}{2}t_1 t_2^2 + \frac{135 i\zeta(3)}{4\pi^3} + \text{instantons}\,. \tag{6.72}$$

We are particularly interested in the Yukawa couplings. On the mirror dual side, they are globally defined in terms of the coordinates $(\bar{x}, \bar{y})$ and read [179, 191]

$$\begin{aligned} \kappa_{\bar{x}\bar{x}\bar{x}} &= \frac{9i}{8\pi^3}\frac{1}{\bar{x}^3((1-\bar{x})^3 - \bar{x}^3\bar{y})}\,, & \kappa_{\bar{x}\bar{x}\bar{y}} &= \frac{i}{8\pi^3}\frac{3(1-\bar{x})}{\bar{x}^2\bar{y}((1-\bar{x})^3 - \bar{x}^3\bar{y})}\,,\\ \kappa_{\bar{x}\bar{y}\bar{y}} &= \frac{i}{8\pi^3}\frac{(1-\bar{x})^2}{\bar{x}\bar{y}^2((1-\bar{x})^3 - \bar{x}^3\bar{y})}\,, & \kappa_{\bar{y}\bar{y}\bar{y}} &= \frac{i}{24\pi^3}\frac{1-3\bar{x}+3\bar{x}^2}{\bar{y}^2(1+\bar{y})((1-\bar{x})^3 - \bar{x}^3\bar{y})}\,. \end{aligned} \tag{6.73}$$

For regularizing the zero point Yukawa couplings (6.1) we now want to take the limit to a point in the complex structure moduli space at which the CY develops a singularity, which intuitively encodes the asymptotic growth of the GV invariants in all directions of the homology lattice $H_2(X,\mathbb{Z})$. The natural choice are the intersection points of the two conifold loci (6.70). According to [191], there exist three such points, where the two points $\bar{x}_0 = (3 \pm \sqrt{3}i)/6$ and $\bar{y}_0 = -1$ are obvious and the third one at $\bar{w}_0 = 1/\bar{x}_0 = 0$, $\bar{y}_0 = -1$ is actually a triple intersection with another orbifold singularity D_0 at $\psi = 0$. This is shown on the left in figure 6.1, which is essentially taken from [191].

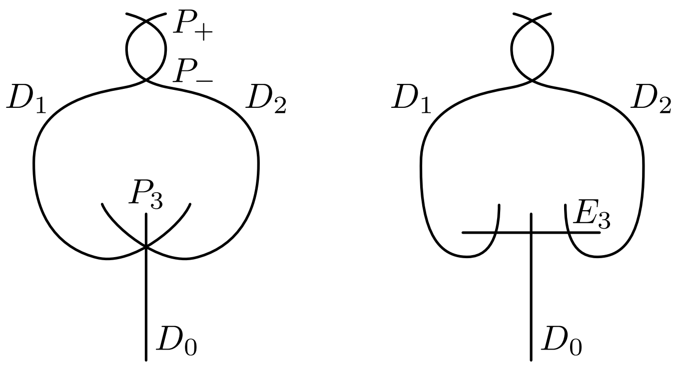


Figure 6.1: Schematic view of the intersections of degeneration loci $D_1 = \{\Delta_1 = 0\}$, $D_2 = \{\Delta_2 = 0\}$, D_0 and their resolution.

To get a smooth moduli space with normal crossings of the degeneration loci, one needs to perform a resolution of the triple intersection. This is done with the standard methods discussed in section 5.2.2, leading to a divisor E_3 which has normal crossings with the three degeneration loci as shown on the right in figure 6.1.

Emergence at the point P_+

First, let us consider the intersection point P_+ with $\bar{x}_0 = (3+\sqrt{3}i)/6$ and $\bar{y}_0 = -1$, where as shown in [183] it is convenient to introduce the local coordinates

$$x_1 = 1 - \frac{\bar{x}}{\bar{x}_0}\,, \qquad x_2 = \left(1 - \frac{\bar{x}}{\bar{x}_0}\right)^{-1}\left(1 - \frac{\bar{y}}{\bar{y}_0}\right) \tag{6.74}$$

for which the two conifold degeneration loci become

$$\Delta_1 = \tfrac{(1-i\sqrt{3})}{2} x_1\big(1 + O(x_i)\big)\,, \qquad \Delta_2 = x_1 x_2\,. \tag{6.75}$$

As was the case for $\mathbb{P}^4_{1,1,2,2,6}[12]$, the intersection of the two conifold loci (6.70) corresponds to the quantum corrected singular point in the middle of the classical phase diagram (for the FI parameters) of the GLSM. Note that (x_1, x_2) should be regarded rather as (complex) polar coordinates around P_+, where x_1 is the radial direction and x_2 related to the angle, which is evident from

$$\frac{\bar{y} - \bar{y}_0}{\bar{x} - \bar{x}_0} = \frac{\bar{y}_0}{\bar{x}_0} x_2 = \tan\phi \tag{6.76}$$

and that P_+ is reached for $x_1 = 0$ independent of x_2. Like for radial coordinates this means that the map is not invertible at P_+.

Then, our proposal is that the TINs $\kappa_{t_it_jt_k}$ are given by the regularization

$$Y^{(0)}_{t_it_jt_k} = -\lim_{\substack{t_1\to t_{1,c}\\ t_2\to t_{2,c}}} \Big[\Big(\partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F}_0\big|_{\text{weak}} - \kappa_{t_it_jt_k}\Big) - \text{Div}\Big] = \kappa_{t_it_jt_k}\,. \tag{6.77}$$

Therefore, to actually evaluate the expression (6.77) we need to know the third derivatives of the prepotential in a vicinity of the intersection point P_+ of the two conifold loci. For that purpose, we first solve the Picard-Fuchs equations in local coordinates around the point $(x_1, x_2) = (0, 0)$ and then determine an integral symplectic basis which is required to be the continuation of the integral symplectic basis around the LCS point $(\bar{x}, \bar{y}) = (0, 0)$. That means that on the overlap of two local charts around these points, they have to coincide. To determine the numerical transition matrices with high enough precision, we have computed all period sets up to total order 40 in the respective local variables (x_1, x_2).

Similar computations have been performed in [93, 200] and for our purposes we have been following this general recipe to arrive at the following results[5], whose detailed derivation will be reported in [205]. Eventually, the periods around the point $(x_1, x_2) = (0, 0)$ take the schematic form (see appendix C.2 for detailed information)

$$\begin{aligned}
X_0 &= x_1 P_0 \log(x_1) + Q_0\\
X_1 &= x_1x_2 P_1 \log(x_1x_2) + Q_1\\
X_2 &= x_1x_2 P_2 \log(x_1x_2) + x_1\tilde{P}_2\log(x_1) + Q_2\\
F_2 &= x_1x_2 \hat{P}_2 \log(x_1x_2) + \hat{Q}_2\\
F_1 &= x_1 \hat{P}_1 \log(x_1) + \hat{Q}_1\\
F_0 &= x_1 \hat{P}_0 \log(x_1) + \hat{Q}_0\,,
\end{aligned} \tag{6.78}$$

where the $P_i, Q_i, \hat{P}_i, \hat{Q}_i, \tilde{P}_i$ are polynomials in x_1, x_2. The periods clearly feature the expected logarithmic terms at the locations of the two conifold singularities (6.70).

Then, the two Kähler moduli enjoy the expansions

$$t_i = \sum_{n=0}^{\infty}\big(x_1\log x_1\big)^n a_n^{(i)} + (x_1x_2)\log(x_1x_2)\sum_{n=0}^{\infty}\big(x_1\log x_1\big)^n\, b_n^{(i)} \qquad i = 1, 2\,, \tag{6.79}$$

where $a_n^{(i)}, b_n^{(i)}$ are polynomials in x_1, x_2, where however an x_2^n term is always accompanied by at least one factor of x_1. In the limit $x_1 = 0$, the values of t_i are determined by the constant terms in $a_0^{(i)}$ which give

$$t_1^{(0)} \approx 0.137 + 0.991i\,, \qquad\qquad t_2^{(0)} \approx 0.545 + 0.427i. \tag{6.80}$$

In figure 6.2 we show a parametric plot of $(\text{Im}(t_1), \text{Im}(t_2))$ around the singularity at $(t_1^{(0)}, t_2^{(0)})$. This shows that also in the Kähler moduli space the singularity can be approached from various directions. Now we are ready to compute the Yukawa couplings

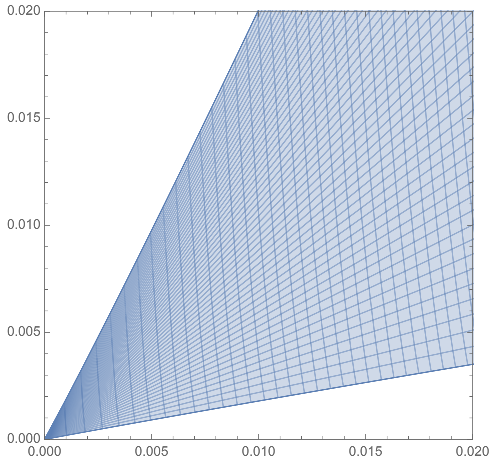


Figure 6.2: Parametric plot of $(\mathrm{Im}(t_1 - t_1^{(0)}), \mathrm{Im}(t_2 - t_2^{(0)}))$ for varying (x_1, x_2) with $\mathrm{Im}(x_i) = 0$ and $0 \leq \mathrm{Re}(x_1) \leq 0.1$, $0.01 \leq \mathrm{Re}(x_2) \leq 2.0$.

(6.73). For that purpose we first transform them to the new coordinates (x_1, x_2) and then to the Kähler moduli (t_1, t_2) via the relation (5.69). Using (6.73), for the $\kappa_{x_\alpha x_\beta x_\gamma}$ we find the behavior

$$\begin{aligned} \kappa_{x_1x_1x_1} &= \frac{1}{x_1} P_1(x_1, x_2)\,, \qquad & \kappa_{x_1x_1x_2} &= P_2(x_1, x_2)\,, \\ \kappa_{x_1x_2x_2} &= x_1^2 P_3(x_1, x_2)\,, \qquad & \kappa_{x_2x_2x_2} &= \frac{x_1^2}{x_2} P_4(x_1, x_2)\,, \end{aligned} \tag{6.81}$$

where the $P_i(x_1, x_2)$ are polynomials in (x_1, x_2) having a non-vanishing constant term.

For computing the partial derivatives $\partial x_\alpha(t)/\partial t_j$ one needs to the invert the mirror map, which for the periods (6.78) does not allow for an iterated procedure. To circumvent this problem, we assume that the x_i are non-vanishing and apply the inverse function theorem for holomorphic functions to determine the derivatives $\partial x_\alpha/\partial t_j$ implicitly. To do so, we need to invert the Jacobian of the mirror map

$$J_{\mathrm{mir}}(x_1, x_2) = \begin{pmatrix} \frac{\partial t_1}{\partial x_1} & \frac{\partial t_1}{\partial x_2} \\ \frac{\partial t_2}{\partial x_1} & \frac{\partial t_2}{\partial x_2} \end{pmatrix}\,, \tag{6.82}$$

allowing us to evaluate (5.69), where the right hand side is an expression depending on the complex structure coordinates (x_1, x_2), for which we eventually take the limit to P_+, i.e. $x_1 \to 0$ for fixed x_2. More generally, we consider the limit $x_1 \to 0$ along a generic path $x_2(x_1) = c + O(x_1)$ with the constant $c \neq 0$.

Similar to the computation for the Quintic, we have to keep track of the leading singularities and the combinations of logarithmic factors with linear terms in the x_1. Doing this, we realize that we can express the derivatives of the Kähler moduli as

$$\frac{\partial t_i}{\partial x_1} = (\log x_1)\, \mathcal{G}^{(i1)}(x)\,, \qquad \frac{\partial t_i}{\partial x_2} = (x_1 \log x_1)\, \mathcal{G}^{(i2)}(x) \tag{6.83}$$

[5] We are grateful to Rafael Álvarez-García who helped us learning how to perform these computations.

with non-singular functions $\mathcal{G}^{(ij)}(x)$ having a general expansion

$$\mathcal{G}^{(i,j)}(x_1,x_2)=\sum_{m,n,p=0}^{\infty}\frac{(x_1\log x_1)^m}{\log^p(x_1)}\left(\frac{\log x_2}{\log x_1}\right)^n a^{(ij)}_{m,n,p}\,, \tag{6.84}$$

where the $a^{(ij)}_{m,n,p}$ are polynomials in (x_1,x_2) having a non-vanishing constant term. In (6.83) most of the $a^{(ij)}_{m,n,p}$ are actually vanishing but in the next step we have to invert such expressions and then, like for the Quintic, we generate the full infinite series. Note that in the limit $x_1\to 0$ and fixed x_2 all terms in (6.84) are controlled, i.e.

$$(x_1\log(x_1))^m\to 0\,,\qquad \log^{-p}(x_1)\to 0\,,\qquad \left(\frac{\log x_2}{\log x_1}\right)^n\to 0\,. \tag{6.85}$$

Now, inverting the Jacobian and using (6.81) we insert everything into (5.69), to realize that we can write the final results as a sum of four terms

$$\begin{aligned}\partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F}_0|_{\mathrm{weak}}=&\frac{1}{x_1\log^3(x_1)}\,\mathcal{G}_1^{(ijk)}(x)+\frac{1}{x_1\log^3(x_1)}\,\mathcal{G}_2^{(ijk)}(x)\\&+\frac{1}{\log^3(x_1)}\,\mathcal{G}_3^{(ijk)}(x)+\frac{1}{x_1\log^3(x_1)}\,\mathcal{G}_4^{(ijk)}(x)\,.\end{aligned} \tag{6.86}$$

Here the $\mathcal{G}_n^{(ijk)}(x)$ are again functions of the type (6.84). The first, the second and the last term show the familiar conifold singularity. The third term directly vanishes in the $x_1\to 0$ limit. As for the Quintic, it is now evident that independent of the "angle" x_2, in the limit $x_1\to 0$ there are no constant terms arising from the expression (6.86). There are only divergent terms, which we subtract and terms that go to zero. Hence, applying the regularization (6.77) one correctly obtains the four TINs

$$Y^{(0)}_{t_1t_1t_1}=9\,,\qquad Y^{(0)}_{t_1t_1t_2}=3\,,\qquad Y^{(0)}_{t_1t_2t_2}=1\,,\qquad Y^{(0)}_{t_2t_2t_2}=0\,. \tag{6.87}$$

One might be worried about taking the limit towards the degeneration point P_+ along a different family of paths and getting a different result. This can indeed be the case and we will come back to this issue in due course. Moreover, we have obtained the desired results by taking the limit towards the intersection point P_+, which after resolving the moduli space is just one of the potential degeneration points (see [191]). For P_- we expect that the computation goes through completely analogous to P_+. However, for handling the triple intersection point $P_3=D_1\cap D_2\cap D_0$ one has to perform the resolution shown already in figure 6.1. How this is done locally is explained in [183]. Since we have already identified a working intersection point, we are not doing these tedious computations for this CY but will present an example for $\mathbb{P}^4_{1,1,2,2,6}[12]$.

6.3.3 Emergence of TINs on $\mathbb{P}^4_{1,1,2,2,6}[12]$

As a second example with two Kähler moduli let us consider the CY manifold $\mathbb{P}^4_{1,1,2,2,6}[12]$ with Hodge numbers $(h_{11},h_{21})=(2,128)$. The GLSM data specifying the resolved threefold are given in table 5.3 (see [179]). The threefold is a $K3$-fibration with base $\mathbb{P}^1$, where

$z_1 = 0$ defines a divisor L of topology $K3$ and $z_3 = 0$ a second divisor H, so that for this basis one gets the TINs [180]

$$H^3 = 4\,, \quad H^2 \cdot L = 2\,, \quad H \cdot L^2 = 0\,, \quad L^3 = 0\,. \tag{6.88}$$

Expanding the Kähler form as $J = t_1\,\omega_H + t_2\,\omega_L$ one finds that t_2 measures the size of the $\mathbb{P}^1$ base and that the Kähler cone is $t_1, t_2 > 0$.

As $\mathbb{P}^4_{1,1,2,2,6}[12]$ is a $K3$-fibration there exists a heterotic dual model so that the corresponding emergent string limit is approached for large radius of the base. Then the two complex structure moduli of the mirror dual CY are parametrized by the deformations of the hypersurface constraint

$$P = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi\, z_1 z_2 z_3 z_4 z_5 - 2\phi\, z_1^6 z_2^6\,, \tag{6.89}$$

where it is customary to introduce the combinations

$$\overline{x} = -\frac{1}{2^5 3^3}\frac{\phi}{\psi^6}\,, \qquad \overline{y} = \frac{1}{\phi^2}\,. \tag{6.90}$$

Recall that in terms of these the two degeneration loci are given by

$$\begin{aligned} \Delta_1 &= (1-\bar{x})^2 - \bar{x}^2\bar{y} = 0\,, \\ \Delta_2 &= 1 - \bar{y} = 0\,, \end{aligned} \tag{6.91}$$

where only the first one is a conifold singularity. The LCS point is at $\overline{x} = \overline{y} = 0$ and one can solve the corresponding Picard-Fuchs equations in the vicinity of this point, determine an integral symplectic basis of periods $\Pi_I^{\rm LCS}$, $I = 1, \ldots, 6$, such that the mirror map is

$$t_1 = \frac{\Pi_2^{\rm LCS}}{\Pi_1^{\rm LCS}} = \frac{1}{2\pi i}\log(\overline{x}) + \ldots\,, \qquad t_2 = \frac{\Pi_3^{\rm LCS}}{\Pi_1^{\rm LCS}} = \frac{1}{2\pi i}\log(\overline{y}) + \ldots\,. \tag{6.92}$$

Inverting the mirror map, the prepotential enjoys the familiar instanton expansion

$$\mathcal{F}_0|_{\rm weak} = +\frac{2}{3}t_1^3 + t_2 t_1^2 + \frac{63 i\zeta(3)}{4\pi^3} + \text{instantons}\,. \tag{6.93}$$

The exact B-side Yukawas were provided in [180, 191], which in terms of $(\bar{x}, \bar{y})$ read [179]

$$\begin{aligned} \kappa_{\bar{x}\bar{x}\bar{x}} &= \frac{2i}{\pi^3}\frac{1}{4\bar{x}^3((1-\bar{x})^2 - \bar{x}^2\bar{y})}\,, & \kappa_{\bar{x}\bar{x}\bar{y}} &= \frac{i}{2\pi^3}\frac{1-\bar{x}}{2\bar{x}^2\bar{y}((1-\bar{x})^2 - \bar{x}^2\bar{y})}\,, \\ \kappa_{\bar{x}\bar{y}\bar{y}} &= \frac{i}{8\pi^3}\frac{2\bar{x}-1}{\bar{x}\bar{y}(1-\bar{y})((1-\bar{x})^2 - \bar{x}^2\bar{y})}\,, & \kappa_{\bar{y}\bar{y}\bar{y}} &= \frac{i}{32\pi^3}\frac{2(1-\bar{x}+\bar{y}-3\bar{x}\bar{y})}{\bar{y}^2(1-\bar{y})^2((1-\bar{x})^2 - \bar{x}^2\bar{y})}\,. \end{aligned} \tag{6.94}$$

Resolution of moduli space

For the following analysis we need to recall information about the resolution of the complex structure moduli space (see section 5.2.2). For the emerging string limit, we are interested in the tangential intersection point between the conifold locus D_1 and the LCS locus $D_{(0,1)}$.

As shown in figure 5.5 on the right-hand side, its resolution involves two exceptional divisors with now pairwise normal crossings. The local coordinates (x_1, x_2) around the three intersections $D_{(0,1)} \cap E_2$, $D_1 \cap E_2$ and $E_1 \cap E_2$ were originally provided [197] and are given in table 5.1

For the M-theory limit, the obvious intersection point P_{12} of two degeneration loci D_1 and D_2 is at $(\overline{x}, \overline{y}) = (1/2, 1)$, but there exists a second point P_3 at $(\overline{w} = \overline{x}^{-1}, \overline{y}) = (0, 1)$, where they intersect also with a third orbifold locus $D_{(-1,0)}$ located at $\psi = 0$. Resolving this triple intersection introduces a divisor E_3 which has normal crossings with the three degeneration loci as shown on in the middle of figure 5.5. To describe the resolved complex structure moduli space around P_3, one needs three charts where each chart provides the local coordinates (x_1, x_2) for the normal intersection of the divisor E_3 with one of the three degeneration loci D_1, D_2 and $D_{(-1,0)}$. Note that in all three charts, the point P_3 with $\bar{w}_0 = 1/\bar{x}_0 = 0$ and $\bar{y}_0 = 1$ is reached for $x_2 = 0$ and x_1 can be regarded as an angular variable. The periods at the relevant points of this CY are listed in appendix C.1.

Emergent heterotic string limit

For large t_2 this model has a weakly coupled heterotic dual on $K3 \times T^2$, where the heterotic 4d complexified dilaton and the Kähler parameter of the T^2 factor are related to the two Kähler moduli of the type IIA CY as

$$t_2 = iS\,, \qquad t_1 = iT\,. \tag{6.95}$$

Hence, $\bar{y} \to 0$ corresponds to the weak coupling limit on the heterotic side. In the heterotic prepotential

$$\mathcal{F}_0^{\text{het}} = i\bigg(- ST^2 - \Big(\frac{2}{3}T^3 + O(e^{-2\pi T})\Big) + O(e^{-2\pi S})\bigg)\,, \tag{6.96}$$

the first term is at tree-level, the second one the one-loop correction and the last one a sum over heterotic $NS5$-brane instantons. Now, let us analyze how (part of) the prepotential is generated in strongly coupled type IIA limits.

First, we take the emergent string limit on the type IIA side, i.e. as in section 6.2.2 we co-scale the size of base like $t_2 \to \lambda t_2$ and the type IIA string coupling as $g_s^{\text{IIA}} \to \lambda^{1/2} g_s^{\text{IIA}}$ so that the 4D Planck-scale remains constant. This is the same limit as for the STU model so that for the same reason not the full set of Yukawa-couplings is emerging from integrating out wrapped $D2$-$D0$ bound states at one loop. In fact, emergence in this case means that the TIN $\kappa_{t_1t_1t_1} = 4$ is given by the regularization of the zero point Yukawa coupling

$$Y^{(0)}_{t_1t_1t_1} = -\lim_{t_1 \to t_{1,c}} \lim_{t_2 \to \infty} \left[\Big(\partial^3_{t_1} \mathcal{F}_0\big|_{\text{weak}} - \kappa_{t_1t_1t_1} \Big) - \text{Div} \right]. \tag{6.97}$$

This is nothing else than the constant piece in the one-loop correction to the Yukawa coupling (6.96) on the dual heterotic side. The first limit $t_2 \to \infty$ guarantees that before regularizing the zero point Yukawa coupling via the limit $\lim_{t_1 \to t_{1,c}}$, all $D2$-branes wrapping the large base $\mathbb{P}^1$ decouple.

This means that on the mirror dual side, we need to take the limit to the point $(\bar{x}, \bar{y}) = (1, 0)$, which is the intersection of the conifold locus and the LCS locus in $\bar{y}$. Therefore, to actually evaluate the expression (6.97) we need to know $\partial^3_{t_1}\mathcal{F}_0\big|_{\text{weak}}$ in a vicinity of this point in complex structure moduli space. After resolving the tangency at this point, we can use the local coordinates in table 5.1.

Let us consider the chart around $P_1 = D_{(0,1)} \cap E_2$. Again, one first needs to solve the Picard-Fuchs equations in local coordinates and then to determine an integral symplectic basis which is required to be the continuation of the integral symplectic basis around the LCS point $(\overline{x}, \overline{y}) = (0, 0)$. The resulting periods take the form

$$\begin{aligned}
X_0/\omega_0 &= 1 \\
X_1/\omega_0 &= P_1(x_1, x_2) \\
X_2/\omega_0 &= \frac{1}{2\pi i}\log\bigl(x_1 x_2^2\bigr) + \frac{F_2/\omega_0}{2\pi i}\log(x_2) + P_2(x_1, x_2) \\
F_2/\omega_0 &= \sqrt{x_2}P_3(x_1, x_2) \\
F_1/\omega_0 &= -\frac{X_1/\omega_0}{\pi i}\log\bigl(x_1 x_2^2\bigr) + P_4(x_1, x_2) \\
F_0/\omega_0 &= -\frac{1}{\pi i}\log\bigl(x_1 x_2^2\bigr) - \frac{F_2/\omega_0}{2\pi i}\log\bigl(x_1 x_2^2\bigr) + P_5(x_1, x_2)\,,
\end{aligned} \tag{6.98}$$

where each $P_i(x_1, x_2)$ is a power series in $(x_1, \sqrt{x_2})$. Indeed the periods X_2, F_2 feature the expected conifold behavior. In the vicinity of $(x_1, x_2) = (0, 0)$, the leading order behavior of the Kähler moduli is

$$\begin{aligned}
t_1 &= \frac{X_1}{\omega_0} = i + c_0\sqrt{x_2} + \dots\,, \\
t_2 &= \frac{X_2}{\omega_0} = \frac{1}{2\pi i}\log\bigl(x_1 x_2^2\bigr) + d_0 + d_1\sqrt{x_2}\log(x_2) + \dots\,,
\end{aligned} \tag{6.99}$$

with c_i and d_i series in x_1 and $\sqrt{x_2}$. Next, one can determine the Jacobian and insert its inverse into the expression for the third derivative of the prepotential. Eventually, in the weak heterotic string coupling limit $x_1 \to 0$ this yields an expansion

$$\partial^3_{t_1}\mathcal{F}_0 = \frac{a_{-1}}{\sqrt{x_2}} + a_0\log\left(\frac{x_2}{\Lambda_0}\right) + \sum_{n=1}^{\infty} a_n\log\left(\frac{x_2}{\Lambda_n}\right)x_2^{n/2}\,, \tag{6.100}$$

where the a_n and Λ_n are real with $a_{-1} \approx 4.826$, $a_0 \approx 1.910$ and $\Lambda_0 \approx 0.320$. The first term diverges for the conifold limit $x_2 \to 0$ and scales as

$$Y^{(0)}_{t_1t_1t_1} \sim \frac{1}{(t_1 - i)} + \dots\,. \tag{6.101}$$

In appendix B, we will relate this to the asymptotic behavior of the GV invariants. All the terms on the right hand side of (6.100) go to zero, while the middle term also diverges though also containing a constant term that is however ambiguous[6]. A very similar

[6]The constant piece in $\log(\epsilon A)$ depends on the scale, i.e. it can be changed by rescaling $\epsilon \to \lambda\epsilon$ and then taking $\epsilon \to 0$ and minimally subtract the divergences.

situation was also encountered in the regularization of the Schwinger integral for the R^4-term in 8D and the one-loop topological free energy in 4D, where such an ambiguous behavior was correlated with the appearance of conformal symmetry [147]. Hence, we consider such ambiguous constants contained in log-factors not as counter examples but as a sign that we are on the right track and might just not yet perform the computation in the best suitable chart. We indicate this by writing $Y^{(0)}_{t_1t_1t_1} = 4^*$.

To substantiate this idea we next repeat the above analysis in the chart around $P_1' = D_1 \cap E_2$. The periods obey very similar relations and are given by

$$\begin{aligned} X_0/\omega_0 &= 1 \\ X_1/\omega_0 &= R_1(x_1, x_2) \\ X_2/\omega_0 &= \frac{1}{2\pi i}\log\left(x_2^2\right) + \frac{F_2/\omega_0}{2\pi i}\log(x_2) + R_2(x_1, x_2) \\ F_2/\omega_0 &= \sqrt{x_2}\, R_3(x_1, x_2) \\ F_1/\omega_0 &= -\frac{X_1/\omega_0}{\pi i}\log\left(x_2^2\right) + R_4(x_1, x_2) \\ F_0/\omega_0 &= -\frac{1}{\pi i}\log\left(x_2^2\right) - \frac{F_2/\omega_0}{\pi i}\log(x_2) + R_5(x_1, x_2)\,. \end{aligned} \tag{6.102}$$

Here each series $R_i(x_1, x_2)$ has the expansion

$$R_i(x_1, x_2) = \sum_{n=0}^{\infty} (x_1\sqrt{x_2}\log(x_1))^n r_n^{(i)}(x_1, x_2)\,. \tag{6.103}$$

The $r_n^{(i)}$ are polynomials in x_1 and $\sqrt{x_2}$ and start with a constant. In the vicinity of $(x_1, x_2) = (0, 0)$, the leading terms in the Kähler moduli are

$$\begin{aligned} t_1 &= \frac{X_1}{\omega_0} = i + c_0\sqrt{x_2} + c_1 x_1\sqrt{x_2}\log(x_1) + \dots\,, \\ t_2 &= \frac{X_2}{\omega_0} = \frac{1}{2\pi i}\log\left(x_2^2\right) + d_0 + d_1\sqrt{x_2}\log(x_2) + \dots\,. \end{aligned} \tag{6.104}$$

Also here c_i and d_i denote series in $x_1, \sqrt{x_2}$. Once again we want to evaluate $\partial^3_{t_1}\mathcal{F}_0$ in the heterotic string weak coupling limit $\bar{y} \to 0$. In the present chart the coordinates satisfy the relation $\bar{x}^2\bar{y} = x_2^2(1 - x_1)$. In the limit $x_1 \to 1$ we reach the boundary of this chart, where we cannot trust the above periods any more. This suggests that we should rather take $x_2 \to 0$ (keeping x_1 small and constant), which according to (6.104) already ensures that $t_2 \to i\infty$ and $t_1 \to t_1^{(0)} = i$. The expansion of $\partial^3_{t_1}\mathcal{F}_0$ for small x_1, x_2 yields

$$\partial^3_{t_1}\mathcal{F}_0 = \frac{f_0(x_1)}{x_1\sqrt{x_2}\log^3(x_1)} + \frac{f_1(x_1) + f_2(x_1)\log(x_2)}{x_1\log^3(x_1)} + \dots\,, \tag{6.105}$$

where $f_i(x_1)$ are functions of x_1 and ellipses denote terms that vanish for $x_2 \to 0$. We find the same leading singularity in the variable x_2 as in (6.100). Moreover, there is a series of $\log(x_2)$-divergent terms that are paired with constants similar to the middle term in (6.100). However, unlike in the previous case these constants all depend on x_1, but treating x_1 as a constant we essentially get the same result as for the chart $D_{(0,1)} \cap E_2$.

However, the absence of a bare, i.e. not x_1 dependent, constant in (6.105) makes it possible to define a certain family of paths towards the singularity where x_1 scales as well. If one for example chooses $x_1 = z\,\sqrt{x_2}$ (with z constant), no constant term remains after minimally subtracting the divergences and taking the limit $x_2 \to 0$. Hence, the absence of a bare constant allowed us to find a family of paths that indeed yields the desired value of the regularized Yukawa coupling $Y^{(0)}_{t_1t_1t_1} = 4$.

M-theory limit I

So far all the limits we were considering for the two CYs with two Kähler moduli were approaching conifold singularities. For these the proposed *regularization condition* was satisfied. In this respect the M-theory limit for $\mathbb{P}^4_{1,1,2,2,6}[12]$ is interesting, as it will involve taking the limit to the intersection $\Delta_1 \cap \Delta_2$, where Δ_2 is not a conifold locus, but a strong coupling locus. The obvious intersection point P_{12} is at $(\overline{x}, \overline{y}) = (1/2, 1)$, but, as we have seen, there exists a second point P_3 at $(\overline{w} = \overline{x}^{-1}, \overline{y}) = (0, 1)$, where they intersect also with a third orbifold locus at $\psi = 0$. The question is whether all four TINs do indeed arise from the regularization of the corresponding zero-point Yukawa couplings $Y^{(0)}_{ijk}$.

To evaluate the limit towards P_{12}, we need to determine the integral symplectic basis of periods around the intersection point $(\overline{x}, \overline{y}) = (1/2, 1)$. For that purpose it is useful to introduce the local coordinates

$$x_1 = 1 - 2\overline{x}\,, \qquad x_2 = \frac{1-\overline{y}}{1-2\overline{x}}\,, \tag{6.106}$$

where like for the coordinates of $\mathbb{P}^4_{1,1,1,6,9}[18]$ around P_+, x_2 is an angular variable and the singularity is reached by taking the limit $x_1 \to 0$ and $x_2 = c \neq 0$ kept constant. Then the two degeneration loci are at $\Delta_1 = x_1\big(1 + O(x_i)\big)$ and $\Delta_2 = x_1x_2$. The periods are

$$\begin{aligned}
X_0/\omega_0 &= 1\\
X_1/\omega_0 &= -\frac{1}{2}X_2 + S_1(x)\\
X_2/\omega_0 &= \sqrt{x_1x_2}\,S_2(x)\\
F_2/\omega_0 &= \frac{1}{2}F_1 - \frac{1}{2\pi i}X_2\log(x_2) + \sqrt{x_1x_2}\,S_3(x)\\
F_1/\omega_0 &= S_4(x)\\
F_0/\omega_0 &= x_1S_5(x)\,,
\end{aligned} \tag{6.107}$$

where (x) denotes dependence of (x_1, x_2) and each series S_i has the general expansion

$$S_i(x) = \sum_{n=0}^{\infty}(x_1\log(x_1))^n s_n^{(i)}(x) \tag{6.108}$$

and the $s_n^{(i)}$ are polynomials in (x_1, x_2) starting with a constant. In the present case the two Kähler moduli share common terms and enjoy the expansion

$$t_1 = S_1(x_1,x_2) - \frac{\sqrt{x_1 x_2}}{2} S_2(x_1,x_2)\,, \qquad t_2 = \sqrt{x_1 x_2}\, S_2(x_1,x_2)\,. \tag{6.109}$$

In particular, for both S_1 and S_2 each x_2^n term is again always accompanied by at least one factor of x_1. In the limit $x_1 \to 0$ one obtains

$$t_1^{(0)} \approx 1.334i\,, \qquad t_2^{(0)} = 0\,. \tag{6.110}$$

Note that in contrast to the $\mathbb{P}^4_{1,1,1,6,9}[18]$ case, here we see only the typical conifold behavior in x_1. The other degeneration at Δ_1 only leads to power-law behavior. Close to $x_1 = 0$ the two degeneration loci are mapped to $t_2 = -2(t_1 - t_1^{(0)})$ and $t_2 = 0$ respectively. This means that close to $x_1 = 0$, the mirror dual of the LCS phase is the cone

$$C_{\text{LCS}} = \Big\{\text{Im}(t_2) > 0, \text{Im}(t_1) > t_1^{(0)} - \frac{1}{2}\text{Im}(t_2)\Big\}. \tag{6.111}$$

Compared to the $\mathbb{P}^4_{1,1,1,6,9}[18]$ there is an essential difference when approaching the singular point in the Kähler moduli space. From (6.109) one can directly infer that for x_2 constant and small x_1 one gets

$$\frac{\Delta t_2}{\Delta t_1} = -2 + O(x_1) \tag{6.112}$$

so that for any fixed value of x_2 one approaches the singularity in a tangential manner. This also evident from the parametric plot in figure 6.3, where we observe that the shaded region is inside the cone C_{LCS}.

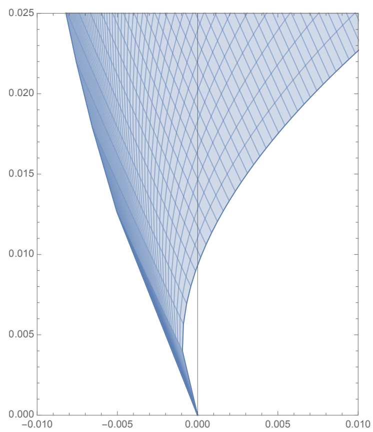


Figure 6.3: Parametric plot of $(\text{Im}(t_1 - t_1^{(0)}), \text{Im}(t_2))$ for varying (x_1, x_2) with $\text{Im}(x_i) = 0$ and $0 \le \text{Re}(x_1) \le 0.1$, $0.1 \le \text{Re}(x_2) \le 1$.

Now we collect the remaining ingredients. Transforming the Yukawa couplings (6.94) to the new coordinates yields

$$\begin{aligned} \kappa_{x_1x_1x_1} &= \frac{x_2}{x_1^2} P_1(x_1,x_2)\,, \qquad & \kappa_{x_1x_1x_2} &= \frac{1}{x_1} P_2(x_1,x_2)\,, \\ \kappa_{x_1x_2x_2} &= \frac{1}{x_2} P_3(x_1,x_2)\,, \qquad & \kappa_{x_2x_2x_2} &= \frac{x_1}{x_2^2} P_4(x_1,x_2)\,. \end{aligned} \tag{6.113}$$

Once again, the P_i are polynomials in (x_1, x_2) with non-vanishing constant term.

Next, we need to determine the leading behavior of the partial derivatives $\partial x_\alpha/\partial t_j$. The derivatives of the mirror map scale as

$$\begin{aligned} \frac{\partial t_1}{\partial x_1} &= \log(x_1)\,\mathcal{G}_1(x) + \sqrt{\frac{x_2}{x_1}}\,\mathcal{H}_1(x)\,, & \frac{\partial t_1}{\partial x_2} &= \sqrt{\frac{x_1}{x_2}}\,\mathcal{G}_2(x)\,, \\ \frac{\partial t_2}{\partial x_1} &= \sqrt{\frac{x_2}{x_1}}\,\mathcal{G}_3(x)\,, & \frac{\partial t_2}{\partial x_2} &= \sqrt{\frac{x_1}{x_2}}\,\mathcal{G}_4(x)\,, \end{aligned} \tag{6.114}$$

with $\mathcal{G}_i$ and $\mathcal{H}$ all regular and finite in the limit $x_1 \to 0$ with x_2 constant. The inverse of the Jacobian matrix again yields the derivatives $\partial x_\alpha/\partial t_j$ as functions of (x_1, x_2). Quite remarkably, the leading-order term of the Jacobian determinant is not a constant, as one would naively expect from (6.114). The relations (6.109) among the Kähler moduli lead to cancellations, so that the determinant scales as

$$\det J_{\text{mir}} = \sqrt{x_1}\log(x_1)\,\mathcal{G}_{\text{det}}(x)\,, \tag{6.115}$$

where $\mathcal{G}_{\text{det}}$ is also regular in the limit we take.

Finally, we combine all the ingredients to obtain the Yukawa couplings on the A-side. Evaluating (6.77) one obtains that for two out of the four regularized zero point Yukawa couplings one indeed finds no constant in the limit of $\partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F}_0|_{\text{weak}}$ so that

$$Y^{(0)}_{t_1t_1t_1} = 4\,, \qquad Y^{(0)}_{t_1t_1t_2} = 2\,. \tag{6.116}$$

However, for the other two, actually vanishing TINs the regularization yields finite constant contributions so that

$$Y^{(0)}_{t_1t_2t_2} \approx 0.24\,, \qquad Y^{(0)}_{t_2t_2t_2} \approx 0.36\,. \tag{6.117}$$

Hence, the proposed regularization method is only successful in half of the cases. For the two correctly regularized zero-point Yukawa couplings (6.116), the generic scaling of their maximal divergence for $x_1 \to 0$ is

$$Y^{(0)} \sim \frac{1}{x_1\log^3(x_1)} + \ldots \sim \frac{1}{\left(t_1 - t_1^{(0)} + \frac{1}{2}t_2\right)\log^2\left(t_1 - t_1^{(0)} + \frac{1}{2}t_2\right)} + \ldots\,. \tag{6.118}$$

In appendix B we investigate whether this behavior can be directly related to the asymptotic growth of the GV invariants.

Looking at figure 6.3, one might suspect that the mismatch of (6.117) is related to the non-generic, tangential limit that we are taking. Then we should be able to improve the results by finding more generic paths towards the degeneration locus. In the concrete case, such paths can be found by balancing the two contributions for $\partial t_1/\partial x_1$ in (6.114). Approaching the singularity along paths with

$$\sqrt{x_1x_2} = -zx_1\log(x_1)\,, \tag{6.119}$$

we find that indeed the family of paths is now more generic but the four resulting regularized zero point Yukawa couplings

$$Y^{(0)}_{t_1t_1t_1} = 4\,, \qquad Y^{(0)}_{t_1t_1t_2} = 2^\star\,, \qquad Y^{(0)}_{t_1t_2t_2} = 0^\star\,, \qquad Y^{(0)}_{t_2t_2t_2} \approx 0.36 + \frac{0.15}{z} \tag{6.120}$$

are still not all correct. Here the star again indicates that this is only up to conformal scaling. As in the previous section about the emergent heterotic string limit, the presence of bare constants in (6.117) only allows us to make some of them ambiguous but not to get rid of them completely.

M-theory limit II

Next we consider taking the limit towards the singular point P_3 located at $(\bar{x} = \infty\,, \bar{y} = 1)$, where the two degeneration loci Δ_1 and Δ_2 intersect with the singular curve $D_{(-1,0)}$ that arises from an orbifold action on the moduli space coordinates. After the required resolution, there are three points of intersection $\mathcal{P} \in \{P_{13}, P_{23}, P_{03}\}$.

We will now analyze the point P_{03}, which is the intersection of $D_{(-1,0)}$ with the exceptional divisor E_3 separating the curves parametrized by Δ_1 and Δ_2. To obtain the integral symplectic basis we solve the Picard-Fuchs equations centered around each of the three above points and match the periods in suitable transition regions step by step. Around $(x_1, x_2) = (0, 0)$, the periods take the form

$$\begin{aligned}
X_0 &= (x_1x_2)^{1/6}A_0 + \sqrt{x_1x_2}\log(x_2)B_0\\
X_1 &= (x_1x_2)^{1/6}A_1 + \sqrt{x_1x_2}\log(x_2)B_1\\
X_2 &= x_1^{1/6}x_2^{1/2}C_2\\
F_0 &= (x_1x_2)^{1/6}\hat{A}_0 + \sqrt{x_1x_2}\log(x_2)\hat{B}_0\\
F_1 &= (x_1x_2)^{1/6}\hat{A}_1 + \sqrt{x_1x_2}\log(x_2)\hat{B}_1\\
F_2 &= (x_1x_2)^{1/6}\hat{A}_2 + \sqrt{x_1x_2}\log(x_2)\hat{B}_2\,,
\end{aligned} \tag{6.121}$$

where $A_i, B_i, \hat{A}_i, \hat{B}_i$ denote polynomials in $(x_1^{1/3}, x_2^{1/3})$ and each $x_1^{n/3}$ is paired at least with a factor of $x_2^{1/3}$. C_2 is a polynomial in $(x_1^{1/3}, x_2)$ and contains bare x_1-terms. The Kähler moduli are given by the expansions

$$\begin{aligned}
t_1 &= \sum_{n=0}^{\infty}\left(x_1^{1/3}x_2^{1/3}\log(x_2)\right)^n a_n = t_1^{(0)} + x_2^{1/3}c_0 + x_1^{1/3}x_2^{1/3}\log(x_2)\,c_1 + \dots\,,\\
t_2 &= x_2^{1/3}\sum_{n=0}^{\infty}\left(x_1^{1/3}x_2^{1/3}\log(x_2)\right)^n b_n = t_2^{(0)} + x_2^{1/3}d_0 + x_1^{1/3}x_2^{2/3}\log(x_2)\,d_1 + \dots\,.
\end{aligned} \tag{6.122}$$

Here a_n, b_n, c_n, d_n are also series in $(x_1^{1/3}, x_2^{1/3})$. In the limit $x_2 \to 0$ we obtain[7]

$$t_1^{(0)} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\,, \qquad t_2^{(0)} = 0\,. \tag{6.123}$$

[7] Of course, for $t_1^{(0)}$ we only obtain numerical values that are suspiciously close (up to order $O(10^{-5})$) to the presented ones.

This limiting value is common for all three points from $\mathcal{P}$ when x_2 is taken to zero, so that indeed we have three patches around a common intersection point. This is also shown in figure 6.4.

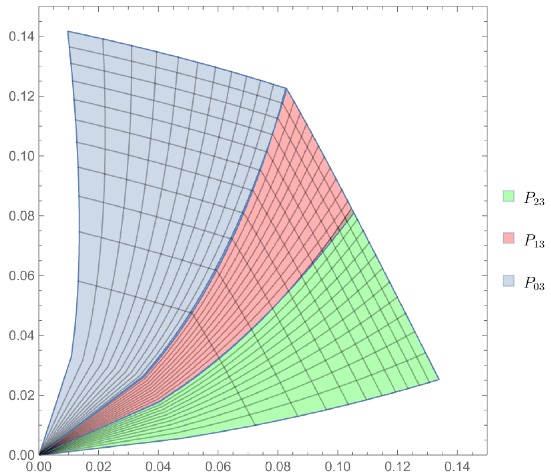


Figure 6.4: Parametric plot of $(\mathrm{Im}(t_1 - t_1^{(0)}), \mathrm{Im}(t_2))$, evaluated in the patches around $\{P_{23}, P_{13}, P_{03}\}$. We vary the respective coordinates (x_1, x_2) with $\mathrm{Im}(x_1) = 0$, $\mathrm{Im}(x_2) = 0$. For P_{23} we have $0.1 < \mathrm{Re}(x_1) < 1$ and $0 < \mathrm{Re}(x_2) < 0.1$, for P_{13} it's $0.01 < \mathrm{Re}(x_1) < 1$ and $0 < \mathrm{Re}(x_2) < 0.05$ and for P_{03} we choose $0.1 < \mathrm{Re}(x_1) < 0.5$ and $0 < \mathrm{Re}(x_2) < 0.1$.

To determine the Yukawa couplings we proceed as for the previous cases. Transforming the B-side Yukawas to the coordinates of the patch around P_{03} yields

$$\begin{aligned} \kappa_{x_1x_1x_1} &= \frac{x_2}{x_1} P_1(x_1, x_2)\,, & \kappa_{x_1x_1x_2} &= \frac{1}{x_1} P_2(x_1, x_2)\,, \\ \kappa_{x_1x_2x_2} &= \frac{1}{x_2} P_3(x_1, x_2)\,, & \kappa_{x_2x_2x_2} &= \frac{x_1}{x_2^2} P_4(x_1, x_2)\,, \end{aligned} \tag{6.124}$$

where each P_i is a polynomial in (x_1, x_2). We compute the partial derivatives $\partial x_\alpha / \partial t_j$ again by using the derivatives of the mirror map, which are given by

$$\begin{aligned} \frac{\partial t_1}{\partial x_1} &= \frac{x_2^{1/3}}{x_1^{2/3}} \log(x_2)\, \mathcal{G}_1(x)\,, \\ \frac{\partial t_1}{\partial x_2} &= \frac{\mathcal{G}_2(x)}{x_2^{2/3}} + \frac{x_1^{1/3}}{x_2^{2/3}} \log(x_2)\, \mathcal{H}_2(x) + \frac{x_1^{2/3}}{x_2^{1/3}} \log^2(x_2)\, \mathcal{I}_2(x) + x_1 \log^3(x_2) \mathcal{J}_2(x)\,, \\ \frac{\partial t_2}{\partial x_1} &= \frac{x_2^{1/3}}{x_1^{2/3}}\, \mathcal{G}_3(x)\,, \\ \frac{\partial t_2}{\partial x_2} &= \frac{\mathcal{G}_4(x)}{x_2^{2/3}} + \frac{x_1^{1/3}}{x_2^{1/3}} \log(x_2)\, \mathcal{H}_4(x) + x_1^{2/3} \log^2(x_2)\, \mathcal{I}_4(x)\,, \end{aligned} \tag{6.125}$$

where each of the $\mathcal{G}_i, \mathcal{H}_i, \mathcal{I}_i, \mathcal{J}_i$ are regular and finite in the limit $x_2 \to 0$. The leading divergence of the Jacobian determinant scales like

$$\det J_{\rm mir} = \frac{\log(x_2)}{x_1^{2/3} x_2^{1/3}}\, \mathcal{G}_{\rm det}\,, \tag{6.126}$$

with $\mathcal{G}_{\rm det}$ also finite and non-divergent as before.

Finally, we combine all these results to evaluate the A-side zero point Yukawa couplings. We observe that their divergence only appears from the scaling $X_0 \sim (x_1 x_2)^{1/6}$ of the fundamental period. Following the natural family of paths with $x_1 = c \neq 0$ and $x_2 \to 0$, we find no constant for two out of four regularized zero point Yukawa couplings, namely

$$Y^{(0)}_{t_1 t_1 t_1} = 4\,, \qquad Y^{(0)}_{t_1 t_1 t_2} = 2\,. \tag{6.127}$$

For the remaining two Yukawa couplings we get no bare constants, but a series of x_1-dependent terms that survive in the specified limit. Hence, we essentially arrive that the same result as for the M-theory limit I.

However, in contrast to the M-theory limit I, the absence of a bare constant and the experience from the emergent heterotic string limit suggest that one can define another family of paths that works. One such family is $x_1 = z\, x_2$ (z constant), which ensures that both Kähler moduli in (6.122) scale as $x_2^{1/3}$ at leading order. In this case all regularized zero point Yukawa couplings carry no constant and really give the TINs

$$Y^{(0)}_{t_1 t_1 t_1} = 4\,, \qquad Y^{(0)}_{t_1 t_1 t_2} = 2\,, \quad Y^{(0)}_{t_1 t_2 t_2} = 0\,, \qquad Y^{(0)}_{t_2 t_2 t_2} = 0\,. \tag{6.128}$$

The generic scaling of their divergence for $x_2 \to 0$ is

$$Y^{(0)} \sim \frac{1}{x_2^{1/3} \log^2(x_2)} \sim \frac{1}{(t_i - t_i^{(0)}) \log^2(t_i - t_i^{(0)})}\,, \tag{6.129}$$

at least for three out of four zero point Yukawa couplings.

The lesson we draw from this example is that the proposed regularization procedure is, maybe not surprisingly, sensitive to the actual co-dimension two degeneration point and the family of paths taken towards it. In contrast to the previous $\mathbb{P}^4_{1,1,1,6,9}[18]$ CY, in this example, we had to approach not the point P_{12} but P_3 to successfully obtain all four triple intersection numbers via our proposed regularization method of the zero point Yukawa couplings. The art is to find a maximal codimension degeneration point so that all regularized zero point Yukawa couplings do not contain any bare constant. Then, it seems to be possible to identify a family of paths so that the proposal is satisfied and the so defined regularization procedure gives the wanted set of TINs. The intuitive picture behind this is that by approaching the complex codimension 2 degeneration locus, the limit has to be sensitive to the asymptotic growth of the GV invariants along all directions of the two-dimensional homology lattice. From the few examples studied so far, we could not yet identify a clear pattern that allows us to decide beforehand what the best suited patch around each degeneration point is.

Chapter 7

Conclusions and Outlook

In this thesis we investigated the Emergence Proposal, which posits that terms in the low-energy effective action of a QG theory arise as a quantum effect after integrating out light towers of states. In its original formulation, emergence was envisioned as a generic property of QG theories and was mostly explored within EFT setups, where one imposes a hard energy cutoff on the states that are being integrated out. The motivation of our work was to leave the EFT perspective behind and reproduce exact couplings from the effective action by explicitly integrating out *full, infinite* towers of states. Our computations led us to a sharpened version of the Emergence Proposal, according to which emergence is naturally realized in the infinite distance M-theory limit $M_* R_{11} \gg 1$, where the eleventh dimension decompactifies while the Planck scale stays fixed. Specifically, one needs to integrate out the full towers of states whose mass scale is parametrically not larger than the 11D Planck scale, i.e. the species scale. These are transverse $M2$- and $M5$-branes with momentum along the eleventh direction ($D0$-branes) and along any potentially present compact direction. In this M-theory limit, the *entire* effective action is expected to emerge via quantum effects. This refinement of the Emergence Proposal is supported by the BFSS matrix model, a conjectured (but so far incomplete) non-perturbative formulation of M-theory. There the effective gravitational potential between a pair of graviton states arises as a pure quantum effect and matches the classical expectation. Nevertheless, it remains unclear how matrix theory explicitly connects to the above ideas, given that the matrix model limit and isotropic M-theory limit are different, at least at first sight.

Even though a microscopic description of M-theory is still out of reach, we were able to gather evidence by focusing on certain BPS-protected amplitudes. These only receive contributions from short BPS multiplets, which can be reliably described in terms of their counterparts in weakly coupled string theory and admit a controlled extrapolation to strong coupling. Concretely, we studied CY compactifications of type IIA string theory, leading to an effective 4D $N = 2$ supergravity description at low energies. Supersymmetry constraints organize the kinetic terms of vector multiplet fields into a single holomorphic function known as the (genus-zero) prepotential. Gopakumar and Vafa showed that this function (as well as all higher-genus prepotentials) can be expressed in terms of a one-loop Schwinger integral, in which bound states of $D0$- and $D2$-branes are integrated out. In the isotropic M-theory limit, these are precisely the states that are expected to reproduce the full amplitude according to the M-theoretic Emergence Proposal. To evaluate the Schwinger integral, it was necessary to find a pragmatic regularization that reproduces the classically expected result. As a first check, we considered the case where the CY is given by the non-compact resolved conifold, featuring only one 2-cycle with the topology of a sphere. Since the $D2$-branes can wrap this sphere only once, the infinite sum over $D2$-brane states reduced to just a single term. The remaining divergence was handled by using the analytic properties of the Riemann ζ-function, enabling us to derive exact expressions for the full genus-zero and genus-one prepotentials. These results were contrasted with an emergent string limit for a $K3$-fibered CY, which can be understood in terms of a weakly coupled dual heterotic model. In that case the type IIA prepotential only yielded the string one-loop correction from the heterotic point of view, showing that in these limits the full prepotential is not emerging just from Schwinger integrals.

To give stronger support for the Emergence Proposal, we then generalized our analysis to compact CY threefolds, with the aim of reproducing the classical Yukawa couplings (the TINs of the CY) from a Schwinger integral. Compared to the previous case, one has to deal with a second infinite sum over the homology lattice $H_2(X, \mathbb{Z})$ encoding multi-wrappings of $D2$-branes on effective curve classes. This typically leads to an exponentially growing number of BPS states, making the regularization problem significantly more challenging. After concretizing the question and taking some lessons from a toy model, we first studied infinite-distance limits other than the isotropic M-theory limit, in which only a subset of the homology lattice contributes. Generalizing our findings to the isotropic M-theory limit, we formulated a mathematically concrete (but admittedly, still speculative) proposal for the regularization of the infinite sum over $D2$-brane states. In a nutshell, we regularize by expressing these sums in terms of the world-sheet instanton contributions to the Yukawa couplings, which are computed at large volume, and then taking a small radius limit towards a degeneration locus in the CY moduli space. For actually evaluating such an expression we needed to determine the CY periods in a chart around the location of the singularity. This information was obtained by solving the Picard-Fuchs equations and then continuously gluing these solutions to the periods in the well known LCS/large volume regime. With only a few concrete examples studied in this paper, the proposal should be regarded as a working hypothesis to be explored in more detail in future work. Nevertheless, our investigation revealed that the M-theoretic Emergence Proposal could be a powerful guide for uncovering so far hidden structures in the relation between the asymptotic behavior of GV invariants and degenerations of CYs. One could entertain the picture that the moduli spaces of CY manifolds encode so much information about QG that their infinite distance degenerations are consistent with the ESC and that their finite distance degenerations are consistent with the M-theoretic Emergence Proposal.

There are several directions in which our research can be extended. Let us outline a few of them in the following:

- our proposed regularization formula for the zero-point Yukawa couplings could be further tested by constructing a larger dataset. One could continue with the remaining CY hypersurfaces with few Kähler moduli and classify the finite distance degenerations according to their output. Based on the examples analyzed so far, CY manifolds that do not feature pure intersections of conifold loci appear to be more intricate. Within the same setup, one could also try to obtain the linear term in the genus-one prepotential from the associated Schwinger integral.

- a puzzling issue appears for the Enriques CY $X = (K3 \times T^2)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts via a free action on the $K3$ and an inversion of the complex coordinate $z \to -z$ on the T^2. The free quotient of $K3$ is the Enriques surface $\mathcal{E}$ with Euler characteristic $\chi(\mathcal{E}) = c_2(\mathcal{E}) = 12$, leading to a CY with Hodge numbers $(h_{11}, h_{21}) = (11, 11)$. This CY is a $K3$-fibration over a base $\mathbb{P}^1$, with the fibration reducing to $\mathcal{E}$ over the four $\mathbb{Z}_2$ fixed points in the base. In [206] we argued that the genus one prepotential [207, 208] emerges from the appropriate Schwinger integral. However, since the genus-zero Gopakumar–Vafa invariants of X vanish [207], our regularization method for obtaining the Yukawa couplings does not extend to this CY.

- it was also advertised [74, 209] that tree-level potentials could be emergent. A convenient setting in which this idea could be tested is provided by type IIA CY compactifications, where one considers the worldvolume theory of a stack of $D4$-branes supported on $\mathbb{R}^2 \subset \mathbb{R}^4$ and wrapping a Lagrangian submanifold $L \subset \text{CY}$. The low-energy effective action has F-terms supported by the 2D worldvolume of the $D4$'s, giving rise to brane-localized superpotentials that describe the open string sector. It was shown [210] that these terms can be similarly encoded by a one-loop Schwinger integral where particle-like BPS-excitations are integrated out[1]. In this case they originate from $D0$-$D2$ bound states where the $D2$-branes wrap curves whose boundaries lie on L (for more details see [154, 212]). In [171] it was proposed that a contour deformation of the original Schwinger integral could reproduce the full non-perturbative amplitudes.

- certainly, a more challenging task would be to demonstrate the emergence of non-BPS-protected couplings. A good example are the kinetic terms of the hypermultiplet fields in type IIA on CY threefolds (recall equation (2.97)). Unlike the F-terms discussed previously, the metric of the hypermultiplet moduli space receives perturbative and non-perturbative corrections in g_s, the latter coming from Euclidean $NS5$- and $D2$-brane instantons in type IIA (for a review see e.g. [213, 214]). And since these terms receive contributions from BPS and non-BPS states alike, complicated bound states can form and it is no longer clear which states should be integrated out in the first place.

Of course, a fully convincing validation of the M-theoretic Emergence Proposal requires the quantization of M-theory. In such a formulation, neither gravity nor spacetime geometry would (presumably) be fundamental. Following the logic of the BFSS matrix model,

[1]Here the BPS states are counted by Donaldson-Thomas-type invariants, see e.g. [211].

spacetime would arise only in a special limit of the theory, while gravity would emerge after integrating out the microscopic degrees of freedom. BPS-protected amplitudes would then be special instances where the bound states of fundamental degrees of freedom can be approximated by BPS-brane configurations. One could try to refine the BFSS model, for example by including the so far missing $M5$-branes at a fundamental level. In [215], it was suggested that non-associative cubic matrices might be used to model the degrees of freedom of $M5$-branes. This idea is closely related to the general expectation [216] that the fundamental objects in M-theory should be described using higher homotopy structures (generalizations of algebraic structures such as Lie algebras). These concepts may eventually lead to a background- and frame-independent formulation of M-theory, allowing us to finally leave the Platonic cave of BPS-protected observables.

Appendix A

Special functions and identities

In this appendix, we collect some useful functions and relations that we employed in the derivation of the prepotential and the genus-one free energy of the resolved conifold in section 4.4 and further considerations in section 6.2.2.

ζ-functions

We recall the definition of the Riemann ζ-function and its first derivative which, for $\mathrm{Re}(s) > 1$, can be written as

$$\zeta(s) = \sum_{n\geq 1} n^{-s}\,, \qquad \zeta'(s) = -\sum_{n\geq 1} \frac{\log(n)}{n^s}\,. \tag{A.1}$$

Outside $\mathrm{Re}(s) > 1$ an analytic continuation is understood. The trivial zeros of the ζ-function are located at the negative integers, while for its derivative evaluated at these points we have

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1)\,, \quad 0 \neq n \in \mathbb{N} \tag{A.2}$$

Some particular values that are of importance in our calculations are

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(-2) = 0, \quad \zeta'(0) = -\frac{1}{2}\log(2\pi)\,, \quad \zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}\,, \tag{A.3}$$

where $\zeta(3) = 1,2020569\ldots$ is Apéry's constant. The Riemann ζ-function is generalized by the Hurwitz ζ-function, $\zeta(s,z)$. For $\mathrm{Re}(s) > 1$ and $z \neq 0, -1, -2, \ldots$, such function and its derivative with respect to s are represented by the series

$$\zeta(s,z) = \sum_{n\geq 0} (n+z)^{-s}, \qquad \zeta'(s,z) = -\sum_{n\geq 0} (n+z)^{-s} \log(n+z)\,, \tag{A.4}$$

otherwise one can perform an analytic continuation for $s \neq 1$.

Polylogarithms

We recall the definition of a polylogarithm

$$\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}\,, \tag{A.5}$$

which is valid for any complex number s and for $|z| < 1$. The natural logarithm is recovered for the specific value $s = 1$,

$$\mathrm{Li}_1(z) = -\log(1-z)\,, \tag{A.6}$$

while for $\mathrm{Re}(s) > 1$ the ζ-function corresponds to $z = 1$

$$\mathrm{Li}_s(1) = \zeta(s)\,. \tag{A.7}$$

Useful identities

Recalling that

$$\log\left(\frac{\sin(\pi z)}{\pi z}\right) = \log\left[\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right]\,, \tag{A.8}$$

one can prove the identity

$$\sum_{n\geq 1}\log\left(1-\frac{z^2}{n^2}\right) = \log\left(\frac{e^{i\pi z}-e^{-i\pi z}}{2\pi i z}\right) = -i\pi z - \log(2\pi i z) + \log\bigl(1-e^{2\pi i z}\bigr)\,, \tag{A.9}$$

which is used to obtain (4.77). Furthermore, integrating (A.9) twice and using

$$\int_0^A dz\,\mathrm{Li}_n(e^{2i\pi z}) = \frac{1}{2\pi i}\left(-\zeta(n+1) + \mathrm{Li}_{n+1}(e^{2\pi i A})\right)\,, \tag{A.10}$$

one can derive the curious identity

$$\begin{aligned}\sum_{n\geq 1}\Big[\left(\frac{z^2+n^2}{2}\right)\log\left(1-\frac{z^2}{n^2}\right) + 2zn\,\mathrm{arctanh}\left(\frac{z}{n}\right) - \frac{3}{2}z^2\Big] = \\ \frac{1}{4\pi^2}\mathrm{Li}_3(e^{2\pi i z}) - \frac{i\pi z}{12} + \frac{3}{4}z^2 - \frac{i\pi z^3}{6} - \frac{1}{2}z^2\log(-2\pi i z) - \frac{\zeta(3)}{4\pi^2}\,,\end{aligned} \tag{A.11}$$

which is used to obtain (4.84). In section 6.1.1 we also need the cotangent identity

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} \tag{A.12}$$

to derive the formula for zero-point Yukawa couplings.

Modular forms

The modular group maps a complex modulus T according to

$$iT \to \frac{iaT + b}{icT + d} \quad \text{with} \quad ad - bc = 1\,, \quad a, b, c, d \in \mathbb{Z}\,. \tag{A.13}$$

The notation with $\tau = iT$ is also very common. A modular form $F_r(T)$ of weight r is a holomorphic function from the upper half plane transforming as

$$F_r(T) \to (icT + d)^r F_r(T) \tag{A.14}$$

under modular transformations. One example is the modular invariant j-function, defined for $\mathrm{Re}(T) > 0$ as

$$j(iT) = \frac{E_4^3(iT)}{\eta^{24}(iT)} = \frac{E_6^2(iT)}{\eta^{24}(iT)} + 1728\,. \tag{A.15}$$

It is a combination of modular forms of weight 4 and 6, called Eisenstein series, which, for $q = e^{-2\pi T}$, are given by

$$E_4(iT) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}\,, \quad E_6(iT) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \tag{A.16}$$

and the Dedekind η-function

$$\eta(iT) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\,, \tag{A.17}$$

which is of weight $1/2$, so that the j-function is of weight 0 and hence called the j-invariant. One of its properties that is particularly useful in our context is given by Borcherds' product formula [217]:

$$j(iT) - j(iU) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}\,, \tag{A.18}$$

where $p = e^{-2\pi T}$ and $q = e^{-2\pi U}$ and the coefficients c are given by the expansion

$$j(iT) - 744 = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 196884 q + 21493760 q^2 \ldots\,. \tag{A.19}$$

The infinite product converges only for $|p|, |q| < e^{-2\pi}$ and $p \neq q$, while the infinite series converges for $|q| < 1$, but the relations may be extended on the entire complex plane via analytic continuation. Taking the logarithm of (A.18) we obtain

$$\log(j(iT) - j(iU)) = 2\pi T + \sum_{m>0, n\in\mathbb{Z}} c(mn) \log\bigl(1 - e^{-2\pi(mT + nU)}\bigr)\,. \tag{A.20}$$

While $T = U$ is a branch singularity of the above expression, one can already recognize the functional behaviors justifying the equivalence of

Appendix B

Singularities and Gopakumar-Vafa invariants

In section 5.3.3 we reviewed the correlation between the asymptotic growth of GV invariants and the conifold singularity of the quintic. Here we try to generalize the argument to CYs with Kähler moduli. For the CY $\mathbb{P}^4_{1,1,1,6,9}[18]$ we have seen that there are two intersecting conifold loci so that we expect that the GV invariants $\alpha_0^{(n_1,n_2)}$ counting BPS 2-cycles with volume $n_1 t_1 + n_2 t_2$ scale exponentially along both directions. Employing the software package `CYTools` [218, 219], we were determining the GV invariants up to total order $n_1 + n_2 \leq 200$ with the lowest ones listed in table B.1.

	0	1	2	3	4	5	6
0	*	3	-6	27	-192	1695	-17 064
1	540	-1080	2700	-17 280	154 440	-1 640 520	19 369 800
2	540	143 370	-574 560	5 051 970	-57 879 900	751 684 050	-10 500 261 120
3	540	204 071 184	74 810 520	-913 383 000	13 593 850 920	-218 032 516 800	3 630 383 423 100
4	540	21 772 947 555	-49 933 059 660	224 108 858 700	-2 953 943 334 360	51 350 781 706 785	-967 920 854 160 960
5	540	1 076 518 252 152	7 772 494 870 800	-42 712 135 606 368	603 778 002 921 828	-11 035 406 089 270 080	224 651 517 028 866 252
6	540	33 381 348 217 290	31 128 163 315 047 072	4 047 949 393 968 960	-90 433 961 251 273 800	2 000 248 139 674 298 880	-45 689 218 327 425 589 920

Table B.1: Gopakumar-Vafa invariants of $\mathbb{P}^4_{1,1,1,6,9}[18]$.

However, these data were not yet sufficient to fix the precise form of the argument of the exponential function. Hence, also the potential appearance of the values (6.80) of the Kähler moduli (t_1, t_2) at the point P_+ were not yet apparent.

For the other CY of interest, namely $\mathbb{P}^4_{1,1,2,2,6}[12]$ there is an intersection of a conifold locus and another degeneration locus at $t_2 = 0$. In table B.2 we list the lowest GV invariants $\alpha_0^{(n_1,n_2)}$, which vanish for $n_2 > n_1$ and satisfy $\alpha_0^{(n_1,n_2)} = \alpha_0^{(n_1,n_1-n_2)}$.

	0	1	2	3	4	5
0	0	2	0	0	0	0
1	2496	2496	0	0	0	0
2	223 752	1 941 264	223 752	0	0	0
3	38 637 504	1 327 392 512	1 327 392 512	38 637 504	0	0
4	9 100 224 984	861 202 986 072	2 859 010 142 112	861 202 986 072	9 100 224 984	0
5	2 557 481 027 520	540 194 037 151 104	4 247 105 405 354 496	4 247 105 405 354 496	540 194 037 151 104	2 557 481 027 520

Table B.2: Gopakumar-Vafa invariants of $\mathbb{P}_{1,1,2,2,6}[12]$.

In fact, we have computed all GV invariants up to total order $n_1 + n_2 \leq 200$. Here the situation turns out to be a bit clearer than for the previous CY. Inspection reveals that for fixed n_1 the GV invariants seem to follow a Gaussian distribution with a maximum at $n_2 = [n_1/2]$ and width $\sigma^2 \sim n_1$. Looking more closer, after making an educated ansatz, we determine the form of the GV invariants as

$$\alpha_0^{(n_1,n_2)} \sim e^{2\pi n_1 \mu} \, \frac{1}{n_1^{\rho+3}} \, \log^\sigma(n_1) \exp\left(-2\pi\lambda \frac{(n_2 - n_1/2)^2}{n_1}\right) \tag{B.1}$$

with $\mu = \lambda = 4/3$ and $\sigma = 0$. We notice that $\mu = 4/3$ precisely matches the value (6.110) of $\mathrm{Im}(t_1)$ at the degeneration point P_1. Moreover, for $n_2 = 0$ one can fit the GV data very well with $\rho = 0$. Just sticking to the first column in B.2, the exponential rate is $\mu - \lambda/4 = 1$, which happens to be the value (6.99) of $\mathrm{Im}(t_1)$ at the emergent string limit $D_\infty \cap D_1$. Performing a similar computation as for the Quintic just for this column we would get the singular behavior

$$Y_{t_1 t_1 t_1} \sim \frac{1}{t_1 - i} + \ldots\,, \tag{B.2}$$

which indeed matches for the emergent string limit.

Taking also the other columns into account, the data reveal that ρ is not constant but follows a plateau like behavior that in the following we approximate via the function

$$\rho \approx \frac{3}{4} \tanh\left(\kappa \frac{n_2(n_1 - n_2)}{n_1^2}\right)\,, \tag{B.3}$$

where κ is a not too small parameter to guarantee a broad plateau. It is clear that the function can only depend on n_2/n_1 and must be invariant under the reflection $n_2 \to n_1 - n_2$. Following the previous computation for the quintic, one could try to estimate the four Yukawa couplings via the double sum

$$Y_{t_1^l t_2^{3-l}} \sim \sum_{n_1,n_2=1}^{\infty} n_1^l \, n_2^{3-l} \, \alpha_0^{(n_1,n_2)} \, e^{-2\pi(t_1 n_1 + t_2 n_2)} \tag{B.4}$$

with $l = 0, \ldots, 3$. Again, we can approximate the infinite sums via integrals over continuous variables $(n_1, n_2) \to (y_1, y_2)$. We did not succeed in analytically solving the appearing double integral. However, after introducing polar coordinates $y_1 = r\sin\phi$, $y_2 = r\cos\phi$ the integral over r can be carried out analytically. The result can then be numerically integrated over the angle $0 \leq \phi \leq \pi/4$ for various choices of t_1 and t_2 approaching zero. First, one realizes that (B.4) seems to diverge for $\Delta t = t_1 - 4/3 + t_2/2$ to zero. The precise functional dependence cannot be uniquely fixed but we observe that the numerics is well consistent with a divergence

$$Y_{t_1^l t_2^{3-l}} \sim \frac{1}{\Delta t \log^2(\Delta t)} + \ldots\,. \tag{B.5}$$

This can be inferred from figure B.1, which should then be a straight line.

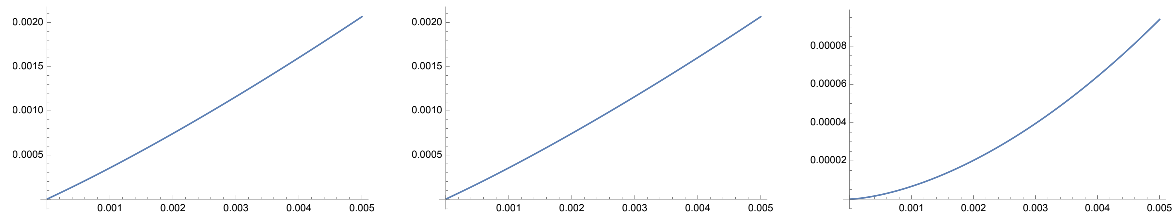


Figure B.1: Plots of $(\log^2(\Delta t)\, Y_{111})^{-M}$ as a function of Δt for $\kappa = 10$. Left $M = 2/3$, middle $M = 1$, right $M = 3/2$.

From (6.109) we find at leading order

$$\Delta t = t_1 - t_1^{(0)} + t_2/2 \sim x_1 \log(x_1) + \dots \tag{B.6}$$

so that we realize that (B.5) is consistent with the leading divergence (6.118) found at the degeneration point P_1. In the same manner it is consistent with (6.129) for the point P_2. Having just this single example we cannot decide whether this is really the correct procedure but it made it evident that the asymptotic behavior of the GV invariants encodes information about the degeneration loci, which is the underlying structure of our proposed regularization procedure for the zero-point Yukawa couplings.

Appendix C

Period data

In this appendix, we gather results on the periods of the two CY threefolds $\mathbb{P}^4_{1,1,2,2,6}[12]$ and $\mathbb{P}^4_{1,1,1,6,9}[18]$. The focus is on obtaining an integral symplectic basis near degeneration loci in the interior of moduli space. In chapter 6 we use the data to obtain expressions for the quantum corrected Yukawa couplings near these degeneration points.

C.1 Periods of $\mathbb{P}^4_{1,1,2,2,6}[12]$

We start with the $K3$-fibered CY $\mathbb{P}^4_{1,1,2,2,6}[12]$ with $(h_{11}, h_{21}) = (2, 128)$ and recall some essential facts. Using the LCS coordinates (x, y), the Picard-Fuchs operators are

$$\begin{aligned} \mathcal{L}_1 &= \theta_x^2(\theta_x - 2\theta_y) - 8x(6\theta_x + 1)(6\theta_x + 3)(6\theta_x + 5)\,, \\ \mathcal{L}_2 &= \theta_y^2 - y(2\theta_y - \theta_x)(2\theta_y - \theta_x + 1) \end{aligned} \tag{C.1}$$

where $\theta_x = x\frac{d}{dx}$ and $\theta_y = y\frac{d}{dy}$. It is convenient to use the rescaled coordinates

$$\bar{x} = 2^6 3^3 x\,, \quad \bar{y} = 4y\,. \tag{C.2}$$

The vanishing loci of the discriminants

$$\begin{aligned} \Delta_1 &= (1 - \bar{x})^2 - \bar{x}^2\bar{y} = 0\,, \\ \Delta_2 &= 1 - \bar{y} = 0 \end{aligned} \tag{C.3}$$

correspond to the loci C_{con} and C_1 in the paper [180]. After resolving and compactifying the moduli space by normal crossing divisors, we can choose appropriate local coordinates around different divisor intersections in the moduli space. For $\mathbb{P}^4_{1,1,2,2,6}[12]$ the procedure was discussed in section 5.2. We will provide the period data for the LCS at the divisor intersection $D_{(1,0)} \cap D_{(0,1)}$ as well as the following loci with local coordinates (x_1, x_2):

$$\begin{aligned} &P_1 = D_{(0,1)} \cap E_2 : \left(\frac{\bar{x}^2\bar{y}}{(1-\bar{x})^2},\, 1 - \bar{x}\right)\,, \quad P_{12} = D_1 \cap D_2 : \left(1 - 2\bar{x},\, \frac{1-\bar{y}}{1-2\bar{x}}\right)\,, \\ &P_1' = D_1 \cap E_2 : \left(1 - \frac{\bar{x}^2\bar{y}}{(1-\bar{x})^2},\, 1 - \bar{x}\right)\,, \quad P_{03} = D_{(-1,0)} \cap E_3 : \left(\frac{1}{\bar{x}(1-\bar{y})},\, 1 - \bar{y}\right)\,. \end{aligned} \tag{C.4}$$

Solving the Picard-Fuchs equations at a locus Δ leads to a local basis of periods, denoted as π^{Δ}. One can rotate to an integral symplectic basis

$$\Pi^{\Delta} = \mathcal{T}_{\Delta}\pi^{\Delta} \quad \text{with} \quad \mathcal{T}_{\Delta} \equiv \mathcal{T}_{\pi^{\Delta}\to\Pi^{\Delta}} \tag{C.5}$$

by means of the transition matrix $\mathcal{T}_{\Delta}$. At the LCS, the matrix can be determined by comparing the local basis to the one derived from the classical prepotential. At other points, a numerical transition matrix is computed by matching two sets of periods that are fixed in one coordinate, such that their regions of convergence agree (for more details see section 5.2.4). Computing the numerical matrices with sufficient precision required evaluating the local periods to total orders between 30 and 50.

Comment on conventions

We would like to emphasize that the results presented in this appendix rely on slightly different conventions for the prepotential compared to those adopted in the main text. These conventions have *no impact* on our final results since the periods just change by a symplectic transformation. Among the results below, only the transition matrices are affected. Concretely, here we work with the prepotential

$$\mathcal{F}_0(t) = -\frac{1}{6}\kappa_{ijk}t^it^jt^k + \frac{1}{2}a_{ij}t^it^j + b_it^i + \frac{c}{2} + \frac{1}{(2\pi i)^3}\sum_{\beta\in H_2(Y,\mathbb{Z})} \mathrm{Li}_3(e^{2\pi i\beta\cdot t})\,. \tag{C.6}$$

Here $i = 1,\ldots,h_{11}$ and the constants b_i and c are those provided in (5.55), i.e. only the sign of the cubic term is different. As in the main text, we order our integral symplectic basis of periods such that their symplectic product is represented by the matrix

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1}_3 \\ -\mathbb{1}_3 & 0 \end{pmatrix}. \tag{C.7}$$

Periods at the LCS

The local basis of periods is given by (note that here $x_1 = x$ and $x_2 = y$)

$$\pi_1^{\mathrm{LCS}} = w_1\,, \tag{C.8}$$
$$(2\pi i)\,\pi_2^{\mathrm{LCS}} = w_2 + w_1\log(x_1)\,, \tag{C.9}$$
$$(2\pi i)\,\pi_3^{\mathrm{LCS}} = w_3 + w_1\log(x_2)\,, \tag{C.10}$$
$$(2\pi i)^2\,\pi_4^{\mathrm{LCS}} = w_4 - 2w_2\log(x_1) + w_1\log^2(x_1)\,, \tag{C.11}$$
$$(2\pi i)^2\,\pi_5^{\mathrm{LCS}} = w_5 - w_3\log(x_1) - w_2\log(x_2) + w_1\log(x_1)\log(x_2)\,, \tag{C.12}$$
$$\begin{aligned}(2\pi i)^3\,\pi_6^{\mathrm{LCS}} &= w_6 - 3w_4\log(x_1) - 3w_5\log(x_1) - 3w_2\log^2(x_1) - \frac{3}{2}w_3\log^2(x_1)\\ &\quad + w_1\log^3(x_1) - \frac{3}{2}w_4\log(x_2) - 3w_2\log(x_1)\log(x_2)\\ &\quad + \frac{3}{2}w_1\log^2(x_1)\log(x_2)\,,\end{aligned} \tag{C.13}$$

where the power series $(w_i)_{1\leq i\leq 6}$ are given by[1]

$$w_1 = 1 + 120\,x_1 + 83160\,x_1^2 + \dots\,, \tag{C.14}$$
$$w_2 = 744\,x_1 + 562932\,x_1^2 - x_2 + 120\,x_1x_2 - \frac{3}{2}\,x_2^2 + \dots\,, \tag{C.15}$$
$$w_3 = 240\,x_1 + 249480\,x_1^2 + 2\,x_2 - 240\,x_1x_2 + 3\,x_2^2 + \dots\,, \tag{C.16}$$
$$w_4 = -553536\,x_1^2 - 2\,x_2 - 1728\,x_1x_2 - \frac{11}{2}x_2^2 + \dots\,, \tag{C.17}$$
$$w_5 = -1248\,x_1 - 1480896\,x_1^2 + 2\,x_2 + 1968\,x_1x_2 + \frac{13}{2}\,x_2^2 + \dots\,, \tag{C.18}$$
$$w_6 = -7488\,x_1 - 4356288\,x_1^2 - 6\,x_2 - 5184\,x_1x_2 - \frac{27}{4}x_2^2 + \dots\,. \tag{C.19}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{\mathrm{LCS}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -\frac{13}{6} & 0 & 0 & 2 & 2 & 0 \\ \frac{63i\zeta(3)}{2\pi^3} & -\frac{13}{6} & -1 & 0 & 0 & \frac{2}{3} \end{pmatrix}. \tag{C.20}$$

Periods at P_1

The local basis of periods is given by

$$\pi_1^{P_1} = w_1\,, \tag{C.21}$$
$$(2\pi i)\,\pi_2^{P_1} = w_2 + w_1\log(x_1) + 2w_1\log(x_2)\,, \tag{C.22}$$
$$\pi_3^{P_1} = w_3\,, \tag{C.23}$$
$$(2\pi i)\,\pi_4^{P_1} = w_4 + w_3\log(x_1)\,, \tag{C.24}$$
$$\pi_5^{P_1} = w_5\,, \tag{C.25}$$
$$(2\pi i)\,\pi_6^{P_1} = w_6 + w_5\log(x_1) + 2w_5\log(x_2)\,, \tag{C.26}$$
$$\tag{C.27}$$

where the power series $(w_i)_{1\leq i\leq 6}$ are given by

$$w_1 = 1 + \frac{354010}{6830747}x_2 + \frac{9795}{709688}x_2^2 + \dots\,, \tag{C.28}$$
$$w_2 = \frac{225835352}{525967519}x_2 + \dots\,, \tag{C.29}$$
$$w_3 = \sqrt{x_2} - \frac{1}{16}x_1\sqrt{x_2} - \frac{15}{1024}x_1^2\sqrt{x_2} + \frac{23}{54}x_2^{3/2} + \frac{23}{288}x_1x_2^{3/2} + \frac{2689}{9720}x_2^{5/2} + \dots\,, \tag{C.30}$$
$$w_4 = \sqrt{x_2} + \frac{1}{16}x_1\sqrt{x_2} - \frac{1}{512}x_1^2\sqrt{x_2} + \frac{161}{162}x_2^{3/2} - \frac{115}{288}x_1x_2^{3/2} + \frac{126383}{145800}x_2^{5/2}\dots\,, \tag{C.31}$$

[1] For each distinct solution to the index equations, we generally present the expansion up to and including total order 2.

$$w_5 = x_2 + \frac{77}{108}x_2^2 + \frac{77}{216}x_1x_2^2 + \frac{4081}{7290}x_2^3 \dots\,, \tag{C.32}$$
$$w_6 = x_2 + \frac{787}{324}x_2^2 + \frac{325}{648}x_1x_2^2 + \frac{285403}{109350}x_2^3 \dots\,. \tag{C.33}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_1} = \begin{pmatrix} 1.21 & 0 & -0.318 & 0 & 0.126 & 0 \\ 1.21i & 0 & 0 & 0 & 0.0841i & 0 \\ 0.105i & 1.21 & -0.0587i & -0.318 & -0.167i & 0.126 \\ 0 & 0 & 0.637 & 0 & 0 & 0 \\ 0.287 & -2.41i & 1.38 & 0 & -0.584 & -0.168i \\ -2.49i & -2.41 & -0.777i & 0 & 0.277i & -0.252 \end{pmatrix}. \tag{C.34}$$

Periods at P_{12}

The local basis of periods is given by

$$\pi_1^{P_{12}} = w_1\,, \tag{C.35}$$
$$\pi_2^{P_{12}} = w_2\,, \tag{C.36}$$
$$\pi_3^{P_{12}} = w_3\,, \tag{C.37}$$
$$(2\pi i)\,\pi_4^{P_{12}} = w_4 + w_1\log(x_1)\,, \tag{C.38}$$
$$\pi_5^{P_{12}} = w_5\,, \tag{C.39}$$
$$(2\pi i)\,\pi_6^{P_{12}} = w_6 + w_5\log(x_1) + w_5\log(x_2)\,, \tag{C.40}$$

where the power series $(w_i)_{1\le i\le 6}$ are given by

$$w_1 = 1 - \frac{5}{144}x_1^2 + \dots\,, \tag{C.41}$$
$$w_2 = x_1 + \frac{41}{72}x_1^2 + \frac{3005}{7776}x_1^3 + \frac{41}{144}x_1^2x_2 + \dots\,, \tag{C.42}$$
$$w_3 = \frac{1}{2}x_1^2 + \frac{133}{216}x_1^3 + \frac{29857}{46656}x_1^4 + x_1x_2 - \frac{1}{4}x_1^2x_2 + \frac{43}{288}x_1^3x_2 + \frac{1433}{2592}x_1^2x_2^2 + \dots\,, \tag{C.43}$$
$$w_4 = \frac{44594}{2835} - \frac{37}{567}x_1 + \dots\,, \tag{C.44}$$
$$w_5 = \sqrt{x_1}\sqrt{x_2} + \frac{149}{432}x_1^{3/2}x_2^{3/2} \dots\,, \tag{C.45}$$
$$w_6 = -x_1^{3/2}\sqrt{x_2} - \frac{1}{2}x_1^{5/2}\sqrt{x_2} + \frac{229}{648}x_1^{3/2}x_2^{3/2} \dots\,. \tag{C.46}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_{12}} = \begin{pmatrix} 0.267 & -0.0195 & 0.0200 & 0.318i & 0 & 0 \\ 1.42i & 0.152i & 0.0818i & 0 & -0.159i & 0 \\ 0 & 0 & 0 & 0 & 0.318i & 0 \\ 2.53 & 0.486 & 0.161 & -0.637i & -0.479 & 0.318i \\ 4.98 & 0.978 & 0.315 & -1.38i & 0 & 0 \\ -5.50i & -1.02i & -0.385i & -0.777 & 0.0531i & 0 \end{pmatrix}. \tag{C.47}$$

Periods at P_1'

The local basis of periods is given by

$$\pi_1^{P_1'} = w_1\,, \tag{C.48}$$
$$(2\pi i)\,\pi_2^{P_1'} = w_2 + w_1\log(x_2)\,, \tag{C.49}$$
$$\pi_3^{P_1'} = w_3\,, \tag{C.50}$$
$$(2\pi i)\,\pi_4^{P_1'} = w_4 + w_3\log(x_1)\,, \tag{C.51}$$
$$\pi_5^{P_1'} = w_5\,, \tag{C.52}$$
$$(2\pi i)\,\pi_6^{P_1'} = w_6 + w_5\log(x_2)\,, \tag{C.53}$$

where the power series $(w_i)_{1\le i\le 6}$ are given by

$$w_1 = 1 - \frac{5}{144}x_2^2 + \ldots\,, \tag{C.54}$$
$$w_2 = -\frac{1}{2}x_1 - \frac{1}{4}x_1^2 + \frac{9227311}{20492241}x_2 + \frac{7543169}{38323152}x_2^2 + \ldots\,, \tag{C.55}$$
$$w_3 = x_1\sqrt{x_2} + \frac{15}{32}x_1^2\sqrt{x_2} + \frac{315}{1024}x_1^3\sqrt{x_2} + \frac{23}{144}x_1^2x_2^{3/2} + \ldots\,, \tag{C.56}$$
$$w_4 = -16x_1\sqrt{x_2} + \frac{19}{64}x_1^3\sqrt{x_2} - \frac{736}{81}x_1x_2^{3/2} + \frac{46}{27}x_1^2x_2^{3/2} - \frac{172096}{18225}x_1x_2^{5/2} + \ldots\,, \tag{C.57}$$
$$w_5 = x_2 + \frac{77}{72}x_2^2 - \frac{77}{216}x_1x_2^2 + \frac{4081}{2916}x_2^3 + \ldots\,, \tag{C.58}$$
$$w_6 = x_2 - \frac{1}{2}x_1x_2 - \frac{1}{4}x_1^2x_2 + 2\,x_2^2 - \frac{1249}{1296}x_1x_2^2 + \frac{6829}{2187}x_2^3 + \ldots\,. \tag{C.59}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_1'} = \begin{pmatrix} 1.21 & 0 & -0.0208 & 0.113i & 0.189 & 0 \\ 1.21i & 0 & 0 & 0 & 0.147i & 0 \\ 0.105i & 2.41 & -0.0563i & 0 & -0.0308i & 0.377 \\ 0 & 0 & 0.0415 & -0.225i & 0 & 0 \\ 0.287 & -4.83i & 0.0899 & -0.488i & -0.322 & -0.586i \\ -2.49i & -4.83 & -0.0507i & -0.275 & -0.113i & -0.754 \end{pmatrix}. \tag{C.60}$$

Periods at P_{03}

The local basis of periods is given by

$$\pi_1^{P_{03}} = w_1\,, \tag{C.61}$$
$$\pi_2^{P_{03}} = w_2\,, \tag{C.62}$$
$$\pi_3^{P_{03}} = w_3\,, \tag{C.63}$$
$$(2\pi i)\,\pi_4^{P_{03}} = w_4 + w_3\log(x_2)\,, \tag{C.64}$$
$$\pi_5^{P_{03}} = w_5\,, \tag{C.65}$$
$$\pi_6^{P_{03}} = w_6\,, \tag{C.66}$$

where the power series $(w_i)_{1\leq i\leq 6}$ are given by

$$w_1 = x_1^{1/6}x_2^{1/6} + \frac{7}{96}x_1^{1/6}x_2^{7/6} + \frac{1}{384}x_1^{7/6}x_2^{7/6} + \frac{1729}{46080}x_1^{1/6}x_2^{13/6} + \dots\,, \tag{C.67}$$

$$\begin{aligned} w_2 &= x_1^{1/6}\sqrt{x_2} - \frac{1}{12}x_1^{7/6}\sqrt{x_2} - \frac{49}{1440}x_1^{13/6}\sqrt{x_2} + \frac{55}{192}x_1^{1/6}x_2^{3/2} \\ &\quad + \frac{5}{576}x_1^{7/6}x_2^{3/2} + \frac{21505}{129024}x_1^{1/6}x_2^{5/2} + \dots\,, \end{aligned} \tag{C.68}$$

$$w_3 = \sqrt{x_1}\sqrt{x_2} + \frac{3}{16}\sqrt{x_1}x_2^{3/2} + \frac{9}{256}x_1^{3/2}x_2^{3/2} + \frac{105}{1024}\sqrt{x_1}x_2^{5/2} + \dots\,, \tag{C.69}$$

$$\begin{aligned} w_4 &= \sqrt{x_1}\sqrt{x_2} - \frac{9}{16}x_1^{3/2}\sqrt{x_2} - \frac{729}{2240}x_1^{5/2}\sqrt{x_2} + \frac{13}{16}\sqrt{x_1}x_2^{3/2} \\ &\quad + \frac{9}{64}x_1^{3/2}x_2^{3/2} + \frac{247}{512}\sqrt{x_1}x_2^{5/2} + \dots\,, \end{aligned} \tag{C.70}$$

$$\begin{aligned} w_5 &= x_1^{5/6}\sqrt{x_2} + \frac{5}{24}x_1^{11/6}x_2^{1/2} + \frac{605}{4032}x_1^{17/6}\sqrt{x_2} + \frac{7}{96}x_1^{5/6}x_2^{3/2} \\ &\quad + \frac{25}{1152}x_1^{11/6}x_2^{3/2} + \frac{1729}{46080}x_1^{5/6}x_2^{5/2} + \dots\,, \end{aligned} \tag{C.71}$$

$$w_6 = x_1^{5/6}x_2^{5/6} + \frac{55}{192}x_1^{5/6}x_2^{11/6} + \frac{125}{1536}x_1^{11/6}x_2^{11/6} + \frac{21505}{129024}x_1^{5/6}x_2^{17/6} + \dots\,. \tag{C.72}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_{03}} = \begin{pmatrix} 1.47-0.85i & -0.192i & -0.390+1.343i & 0.780 & -0.318i & 0.01068+0.00617i \\ 1.70i & -0.254i & -0.431i & -0.390 & 0.0880i & -0.01234i \\ 0 & 0.699i & -0.481i & 0 & 0.142i & 0 \\ 1.70i & -0.611+0.384i & 0.84-2.69i & -1.56 & -0.124+0.636i & -0.01234i \\ -0.49+3.68i & 0.833i & 1.30-5.82i & -3.38 & 1.378i & -0.0035-0.0267i \\ 2.07-2.07i & 0.469+0.085i & -3.28-0.81i & 0.13+1.90i & 0.776-0.029i & -0.0151+0.0672i \end{pmatrix}. \tag{C.73}$$

To obtain this matrix, one must also compute the local periods at the loci P_{23} and P_{13} and glue them to the integral symplectic basis in the order $P_{12} \to P_{23} \to P_{13} \to P_{03}$. The intermediate periods did not lead to better results in the analysis of section 6.3.3 and are therefore omitted here for brevity.

C.2 Periods of $\mathbb{P}^4_{1,1,1,6,9}[18]$

We now consider the elliptically fibered CY $\mathbb{P}^4_{1,1,1,6,9}[18]$ with $(h_{11}, h_{21}) = (2, 272)$. Here, the zero sets of the discriminants give rise to *two* conifold loci intersecting at two distinct points (see figure 6.1). We can use the same techniques from toric geometry to set up the Picard–Fuchs system, find local coordinates near degeneration points, and solve for the periods. Using the LCS coordinates (x, y), the Picard-Fuchs operators for $\mathbb{P}^4_{1,1,1,6,9}[18]$ are

$$\begin{aligned} \mathcal{L}_1 &= \theta_x(\theta_x - 3\theta_y) - 12x(6\theta_x + 5)(6\theta_x + 1)\,, \\ \mathcal{L}_2 &= \theta_y^3 - y(\theta_x - 3\theta_y - 2)(\theta_x - 3\theta_y - 1)(\theta_x - 3\theta_y) \end{aligned} \tag{C.74}$$

with $\theta_x = x\frac{d}{dx}$ and $\theta_y = y\frac{d}{dy}$ as before. It is convenient to use the rescaled coordinates

$$\bar{x} = 2^4 3^3 x\,, \quad \bar{y} = 3^3 y\,. \tag{C.75}$$

The vanishing loci of

$$\begin{aligned} \Delta_1 &= (1-\bar{x})^3 - \bar{x}^3\bar{y} = 0\,, \\ \Delta_2 &= 1+\bar{y} = 0 \end{aligned} \tag{C.76}$$

correspond, in the notation of [191], to the conifold loci C_{con} and B_{con}, respectively. After resolving and compactifying the moduli space by normal crossing divisors, we can choose appropriate local coordinates around different divisor intersections in the moduli space. We will be mostly interested in one of the two intersections of the conifold loci, located at $(\bar{x},\bar{y}) = ((3+\sqrt{3}i)/6, -1) \equiv (\bar{x}_0, \bar{y}_0)$ and denoted as P_+ in chapter 6. As an intermediate step we will also need the periods at $P_2 = D_{(1,0)} \cap D_2$ located at $(\bar{x},\bar{y}) = (0,-1)$, where the LCS divisor intersects $D_2 = \{\Delta_2 = 0\}$. The local coordinates (x_1, x_2) at the these two points are

$$P_2 = D_{(1,0)} \cap D_2 : \Big(\bar{x}, 1+\bar{y}\Big)\,, \quad P_+ = D_1 \cap D_2 : \left(1-\frac{\bar{x}}{\bar{x}_0}, \frac{1-\frac{\bar{y}}{\bar{y}_0}}{1-\frac{\bar{x}}{\bar{x}_0}}\right)\,. \tag{C.77}$$

Once again, the symplectic product is represented by the matrix (C.7) and we use the convention (C.6) for the prepotential.

Periods at the LCS

The local basis of periods is given by (note that here $x_1 = x$ and $x_2 = y$)

$$\pi_1^{\text{LCS}} = w_1\,, \tag{C.78}$$

$$(2\pi i)\,\pi_2^{\text{LCS}} = w_2 + w_1\log(x_1)\,, \tag{C.79}$$

$$(2\pi i)\,\pi_3^{\text{LCS}} = w_3 + w_1\log(x_2)\,, \tag{C.80}$$

$$\begin{aligned}(2\pi i)^2\,\pi_4^{\text{LCS}} &= w_4 - 2w_2\log(x_1) - \frac{2}{3}w_3\log(x_1) + w_1\log^2(x_1) - \\ &\quad - \frac{2}{3}w_2\log(x_2) + \frac{2}{3}w_1\log(x_1)\log(x_2)\,, \end{aligned} \tag{C.81}$$

$$(2\pi i)^2\,\pi_5^{\text{LCS}} = w_5 - 2w_3\log(x_2) + w_1\log^2(x_2)\,, \tag{C.82}$$

$$\begin{aligned}(2\pi i)^3\,\pi_6^{\text{LCS}} &= w_6 - 3w_4\log(x_1) - \frac{1}{3}w_5\log(x_1) - 3w_2\log^2(x_1) - w_3\log^2(x_1) \\ &\quad + w_1\log^3(x_1) - w_4\log(x_2) - 2w_2\log(x_1)\log(x_2) - \frac{2}{3}w_3\log(x_1)\log(x_2) \\ &\quad + w_1\log^2(x_1)\log(x_2) - \frac{1}{3}w_2\log^2(x_2) + \frac{1}{3}w_1\log(x_1)\log^2(x_2)\,, \end{aligned} \tag{C.83}$$

where the power series $(w_i)_{1\le i\le 6}$ are given by

$$w_1 = 1 + 60\,x_1 + 13860\,x_1^2 + \dots\,, \tag{C.84}$$

$$w_2 = 312\,x_1 + 77652\,x_1^2 + 2\,x_2 - 60\,x_1x_2 - 15\,x_2^2 + \dots\,, \tag{C.85}$$

$$w_3 = 180\,x_1 + 62370\,x_1^2 - 6\,x_2 + 180\,x_1x_2 + 45\,x_2^2 + \dots\,, \tag{C.86}$$

$$w_4 = -134784\,x_2^2 - 2\,x_2 + \frac{47}{2}x_2^2 + \dots\,, \tag{C.87}$$

$$w_5 = -1080\,x_1 - 436590\,x_1^2 + 18\,x_2 - \frac{423}{2}x_2^2 + \dots\,, \tag{C.88}$$

$$w_6 = -720\,x_1 - 156330\,x_1^2 - 4\,x_2 + 120\,x_1x_2 + \frac{39}{2}x_2^2 + \dots\,. \tag{C.89}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{\mathrm{LCS}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{17}{4} & 0 & 0 & \frac{9}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & 0 & 0 & \frac{3}{2} & 0 & 0 \\ \frac{135i\zeta(3)}{2\pi^3} & -\frac{17}{4} & -\frac{3}{2} & 0 & 0 & \frac{3}{2} \end{pmatrix}. \tag{C.90}$$

Periods at P_2

The local basis of periods is given by

$$\pi_1^{P_2} = w_1\,, \tag{C.91}$$

$$(2\pi i)\,\pi_2^{P_2} = w_2 + w_1\log(x_1)\,, \tag{C.92}$$

$$(2\pi i)^2\,\pi_3^{P_2} = w_3 + w_2\log(x_1) - \frac{1}{2}w_1\log^2(x_1)\,, \tag{C.93}$$

$$(2\pi i)^3\,\pi_4^{P_2} = w_4 + 6w_3\log(x_1) - 3w_2\log^2(x_1) + w_1\log^3(x_1)\,, \tag{C.94}$$

$$\pi_5^{P_2} = w_5\,, \tag{C.95}$$

$$(2\pi i)\,\pi_6^{P_2} = w_6 + w_5\log(x_2)\,, \tag{C.96}$$

where the power series $(w_i)_{1\le i\le 6}$ are given by

$$w_1 = 1 + \frac{5}{36}x_1 + \frac{385}{5184}x_1^2 + \dots\,, \tag{C.97}$$

$$w_2 = \frac{31}{36}x_1 + \frac{1823}{3456}x_1^2 - \frac{1}{3}x_2 - \frac{5}{108}x_1x_2 - \frac{1}{6}x_2^2 + \dots\,, \tag{C.98}$$

$$w_3 = \frac{5}{36}x_1 + \frac{10183}{20736}x_1^2 - \frac{31}{108}x_1x_2 + \frac{1}{18}x_2^2 + \dots\,, \tag{C.99}$$

$$w_4 = -\frac{5}{3}x_1 - \frac{965}{1152}x_1^2 + \frac{1}{9}x_2^2 + \dots\,, \tag{C.100}$$

$$w_5 = x_2 + \frac{11}{18}x_2^2 + \frac{5}{216}x_1x_2^2 + \frac{109}{243}x_2^3 + \dots\,, \tag{C.101}$$

$$w_6 = \frac{5}{4}x_1x_2 + \frac{2695}{2304}x_1^2x_2 - \frac{5}{12}x_1x_2^2 + \frac{7}{12}x_2^3 + \dots\,. \tag{C.102}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_2} = \begin{pmatrix} 1.00 & 0 & 0 & 0 & 0 & 0 \\ 0.0206i & 1.00 & 0 & 0 & -0.0628i & -0.0919 \\ 0.500+0.463i & 0 & 0 & 0 & 0.188i & 0.276 \\ -4.24-0.26i & -1.50-1.57i & -9.00 & 0 & 0 & 0 \\ -1.50-0.01i & -0.500-0.525i & -3.00 & 0 & -0.0306+0.0314i & 0.0459 \\ -0.75+1.84i & -4.26+0.26i & 1.50+1.57i & 1.50 & 0.0153-0.0172i & -0.0459 \end{pmatrix}. \tag{C.103}$$

Periods at P_+

The local basis of periods is given by[2]

$$\begin{aligned}
\pi_1^{P_+} &= w_1\,, && \text{(C.104)}\\
\pi_2^{P_+} &= w_2\,, && \text{(C.105)}\\
(2\pi i)\,\pi_3^{P_+} &= w_3 + w_2\log(x_1)\,, && \text{(C.106)}\\
\pi_4^{P_+} &= w_4\,, && \text{(C.107)}\\
(2\pi i)\,\pi_5^{P_+} &= w_5 + w_4\log(x_1) + w_4\log(x_2)\,, && \text{(C.108)}\\
\pi_6^{P_+} &= w_6\,, && \text{(C.109)}
\end{aligned}$$

where the power series $(w_i)_{1\le i\le 6}$ are given by

$$w_1 = 1 + \left(\frac{385}{31104} - \frac{265i}{10368\sqrt{3}}\right)x_1^3 - \left(\frac{5}{216} + \frac{5i}{216\sqrt{3}}\right)x_1^2x_2 + \dots\,, \tag{C.110}$$

$$\begin{aligned}
w_2 = x_1 &+ \left(\frac{59}{72} - \frac{i}{4\sqrt{3}}\right)x_1^2 + \left(\frac{9925}{15552} - \frac{149i}{432\sqrt{3}}\right)x_1^3\\
&+ \left(\frac{1}{6} - \frac{i}{6\sqrt{3}}\right)x_1x_2 + \left(\frac{53}{216} - \frac{5i}{216\sqrt{3}}\right)x_1^2x_2 \dots\,,
\end{aligned} \tag{C.111}$$

$$\begin{aligned}
w_3 = &-\left(\frac{14633}{93312} - \frac{601i}{2592\sqrt{3}}\right)x_1^3 + \left(\frac{53}{432} + \frac{67i}{432\sqrt{3}}\right)x_1^2x_2\\
&+ \left(\frac{1}{108} - \frac{i}{36\sqrt{3}}\right)x_1x_2^2 + \dots\,,
\end{aligned} \tag{C.112}$$

$$w_4 = -\frac{48i\sqrt{3}}{5}x_1x_2 + \left(\frac{1}{9} - \frac{269i}{15\sqrt{3}}\right)x_1^2x_2^2 + \dots\,, \tag{C.113}$$

$$w_5 = x_1^3 + \left(\frac{1}{2} - \frac{i}{2\sqrt{3}}\right)x_1^3x_2 - \left(\frac{1}{6} + \frac{163i}{10\sqrt{3}}\right)x_1^2x_2^2 + \dots\,, \tag{C.114}$$

$$\begin{aligned}
w_6 = x_1^2 &+ \left(\frac{311}{216} - \frac{7i}{12\sqrt{3}}\right)x_1^3 + \left(\frac{1}{3} - \frac{i}{3\sqrt{3}}\right)x_1^2x_2\\
&+ \left(\frac{269}{432} - \frac{149i}{432\sqrt{3}}\right)x_1^3x_2 + \dots\,.
\end{aligned} \tag{C.115}$$

The transition matrix to the symplectic basis is

$$\mathcal{T}_{P_+} = \begin{pmatrix}
1.097+0.076i & -0.0490-0.0485i & -0.239+0.413i & -0.00088+0.00230i & 0 & 0.0018+0.0302i\\
0.076+1.097i & 0.004+0.144i & 0 & 0.00198-0.00089i & -0.00553i & -0.0229-0.0416i\\
0.565+0.510i & -0.0296-0.0111i & -0.239+0.413i & -0.01242+0.00122i & 0.0166i & -0.00517+0.00074i\\
1.45-3.46i & 2.17+0.08i & 2.15-3.72i & 0.00157+0.00028i & 0 & -0.407+0.093i\\
0.396-1.080i & 0.727+0.034i & 0.72-1.24i & -0.00140-0.00193i & 0.00276i & -0.1368+0.0262i\\
-3.17-4.58i & -0.18-1.97i & -2.16-1.25i & 0.00879+0.00171i & -0.00276i & 0.201+0.356i
\end{pmatrix}. \tag{C.116}$$

[2]Compared to the periods presented in section 6.3.2, we use a different ordering of the local periods and slightly different coefficient normalizations. This does not affect the final result.

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