Symmetries and extended operators in affine Rozansky-Witten models

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Abstract

Topological quantum field theories (TQFTs) play a prominent role in modern mathematical physics, providing connections between geometry, topology and quantum field theory. Many important examples of TQFTs originate from supersymmetric field theories, in which the observables invariant under a fermionic symmetry generator define a topological subsector, known as the topologically twisted theory. These twisted theories often allow exact computations of manifold invariants and admit a categorical formulation as functors between suitable categories. However, many physically motivated TQFTs still lack a consistent functorial description, particularly in the presence of non-local operators, such as line or surface defects. This thesis addresses this gap for Rozansky-Witten models with flat targets, a prototypical family of three-dimensional twisted theories with extended supersymmetry $\mathcal{N} = 4$.

The first part of this work provides the necessary background. We review topological twists in two and three dimensions and present the Landau-Ginzburg and Rozansky-Witten models, including boundary conditions compatible with these twists. We then introduce the mathematical background on higher categories, outline the categories that describe the Landau-Ginzburg and Rozansky-Witten models and give a concise overview of the cobordism hypothesis.

The second part presents an algorithmic construction of the TQFT functor for Rozansky-Witten models with flat targets, starting from the category that encodes these models. Since such non-compact targets lead to infinite-dimensional state spaces, we truncate the three-dimensional theory to two dimensions. This effectively ignores the correlation functions of local operators. After giving a complete categorical description of the category of truncated affine Rozansky-Witten models, we equip this category with a class of defects that encode symmetry actions. Using the cobordism hypothesis with defects, we compute the state spaces of genus-g Riemann surfaces decorated with symmetry defect networks, recovering results known from physics literature. Finally, we discuss the gauging of the relevant global symmetries via the orbifold construction.

Zusammenfassung

Topologische Quantenfeldtheorien (TQFTs) spielen eine prominente Rolle in der modernen mathematischen Physik und stellen Verbindungen zwischen Geometrie, Topologie und Quantenfeldtheorie her. Viele wichtige Beispiele von TQFTs stammen aus supersymmetrischen Feldtheorien, in denen die Observablen, die unter einem fermionischen Symmetriegenerator invariant sind, einen topologischen Subsektor bilden, bekannt als die topologisch getwistete Theorie. Diese getwisteten Theorien erlauben häufig exakte Berechnungen von Mannigfaltigkeitsinvarianten und besitzen eine kategoriale Formulierung als Funktoren zwischen geeigneten Kategorien. Dennoch fehlt für viele physikalisch motivierte TQFTs eine konsistente funktorielle Beschreibung, insbesondere im Zusammenhang mit nicht-lokalen Operatoren wie Linien- oder Flächendefekten. Diese Arbeit schließt diese Lücke für Rozansky-Witten-Modelle mit flachen Zielräumen, einer prototypischen Familie dreidimensionaler getwisteter Theorien mit erweiterter Supersymmetrie $\mathcal{N} = 4$.

Der erste Teil dieser Arbeit liefert den notwendigen Hintergrund. Wir behandeln topologische Twists in zwei und drei Dimensionen und präsentieren die Landau-Ginzburg- und Rozansky-Witten-Modelle, einschließlich der mit diesen Twists kompatiblen Randbedingungen. Anschließend führen wir die mathematischen Grundlagen zu höheren Kategorien ein, skizzieren die Kategorien, die die Landau-Ginzburg- und Rozansky-Witten-Modelle beschreiben, und geben einen knappen Überblick über die Cobordismus-Hypothese.

Der zweite Teil stellt eine algorithmische Konstruktion des TQFT-Funktors für Rozansky-Witten-Modelle mit flachen Zielräumen vor, ausgehend von der Kategorie, die diese Modelle kodiert. Da solche nicht-kompakten Zielräume zu unendlichdimensionalen Zustandsräumen führen, trunkieren wir die dreidimensionale Theorie auf zwei Dimensionen. Dies ignoriert effektiv die Korrelationsfunktionen lokaler Operatoren. Nach einer vollständigen kategorialen Beschreibung der Kategorie der trunkierten affinen Rozansky-Witten-Modelle statten wir diese Kategorie mit einer Klasse von Defekten aus, die Symmetriewirkungen kodieren. Mithilfe der Cobordismus-Hypothese mit Defekten berechnen wir die Zustandsräume von genus-g Riemannschen Flächen, die mit Symmetriedefektnetzwerken dekoriert sind, und reproduzieren damit bekannte Ergebnisse aus der physikalischen Literatur. Schließlich diskutieren wir die Eichung der relevanten globalen Symmetrien mittels der Orbifold-Konstruktion.

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Chapter 1

Introduction and Summary

Symmetry plays a fundamental role in physics, providing a framework that helps to uncover the underlying laws of nature. A symmetry in a physical system refers to a transformation that leaves the system invariant. In quantum physics, symmetry is formalised using the mathematical language of groups, algebras, and their representations. Quantum systems are described by state vectors in Hilbert spaces, and symmetries correspond to group actions on these state vectors.

A particular kind of symmetry is supersymmetry, which is a theoretical framework in particle physics that postulates a symmetry between fermions and bosons. Supersymmetry is formalised by the *supersymmetry algebra*, an extension of the Poincaré algebra that includes additional fermionic generators Q. These generators anti-commute to momenta and the algebra is schematically represented as:

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu. \tag{1.0.1}$$

Supersymmetric field theories are then quantum field theories whose action S is invariant under the action of a supersymmetry algebra.

Topological quantum field theories (TQFTs) are quantum field theories invariant under a particularly large symmetry, namely invariance under smooth deformations of the spacetime metric. A large class of TQFTs arises from supersymmetric field theories. These TQFTs were introduced by Witten in [53] and they typically satisfy the following properties:

- Have fermionic supersymmetry generators, namely fermionic supercharges $\{Q_i\}$ that annihilate the action: $Q_i \cdot S = 0$.
- A distinguished linear combination Q of these generators squares to 0: $Q^2 = 0$.
- The stress-energy tensor of the theory is Q-exact: $T = Q \cdot T'$.

Then there is a subset[1] of the observables of the theory given by Q-invariant observables $Q \cdot \mathcal{O}$ modulo Q-exact ones $\mathcal{O} = Q \cdot \mathcal{O}'$ called *topologically twisted* sector of the theory

[1] In fact, it is not just a set, but is endowed with an algebraic structure, typically an associative unital algebra.

or simply topologically twist. The topologically twisted sector has the feature that the corresponding correlation functions depend only on the global (topological) properties of the space on which they are defined. In particular, these correlation functions do not depend on the metric of the space and are thus independent of local smooth deformations.

In certain dimensions, a necessary condition for the existence of topological sectors in supersymmetric theories is *extended supersymmetry*, namely $\mathcal{N}$ copies of the supersymmetry algebra 1.0.1. In 3 dimensions, the required amount of supersymmetry is $\mathcal{N} = 4$, the case on which this thesis is focused. The corresponding algebra has an explicit expression:

$$\{Q^I_\alpha, Q^J_\beta\} = \delta^{IJ}\gamma^\mu_{\alpha\beta}P_\mu, \quad I, J = 1, \ldots, 4. \tag{1.0.2}$$

Picking a 3-dimensional sigma model with $\mathcal{N} = 4$ supersymmetry and holomorphic symplectic target (a $4n$-dimensional manifold that locally looks like $T^*\mathbb{C}^n$) and twisting with a specific linear combination of the supercharges $Q = \lambda^\alpha_I Q^I_\alpha$, gives rise to the Rozansky-Witten models [47]. These are TQFTs whose observables capture 3-manifold invariants that depend on the geometry of the holomorphic symplectic target. Boundary conditions of the Rozansky-Witten models compatible with Q are also known to support a class of TQFTs in 2 dimensions, the Landau-Ginzburg models [36].

Topologically twisted theories are often solvable, namely correlation functions can be exactly computed without perturbative approximations. This is an extremely rare feature of quantum field theories, since we can usually only approximate correlation functions using perturbation theory. Furthermore, they have the remarkable property that correlation functions compute directly certain topological invariants of smooth manifolds using path integral methods. Conversely, developments of category theory in mathematics offer a framework for rigorously defining and studying TQFTs and possibly novel computational methods to extract physical quantities of interest. The current thesis is working towards this precise goal.

Moreover, TQFTs offer one of the few examples of quantum field theories that admit a rigorous mathematical definition that avoids the difficulties of defining an infinite-dimensional path integral. The axiomatic definition of TQFTs was given by Atiyah, expressed in the language of *categories* and *functors*. A category consists of

- a collection of *objects*,
- a set of arrows called *morphisms* for every pair of objects.

Morphisms can be composed, composition is associative and each object has an identity morphism, which acts as neutral element of the composition. In broad terms, a category is a formal framework that encodes a family of mathematical objects and the structure preserving mappings (morphisms) between them. For instance, the category of vector spaces has vector spaces as objects and linear maps as morphisms. A functor, on the other hand, is a structure preserving map between categories: it maps objects to objects and morphisms to morphisms, preserving compositions and identity morphisms. In other words, it encodes a relation between possibly different mathematical structures.

Motivated by the path integral formulation, one can express TQFT as a functor from a geometric category to an algebraic category (see [15] for a review on the path integral motivation). More precisely, Atiyah defined a TQFT in n dimensions as the following assignment:

- To every closed oriented $(n-1)$-dimensional manifold (spatial slice) M_{n-1}, we assign a vector space $\mathcal{Z}(M_{n-1})$, the Hilbert space of quantum states.
- To disjoint unions of manifolds $M_{n-1} \sqcup N_{n-1}$ we assign the tensor product of state spaces $\mathcal{Z}(M_{n-1} \sqcup N_{n-1}) = \mathcal{Z}(M_{n-1}) \otimes \mathcal{Z}(N_{n-1})$ and to the empty set we assign the complex numbers $\mathcal{Z}(\varnothing) = \mathbb{C}$.
- Assume a pair of $(n-1)$-manifolds, M_{n-1} and N_{n-1}, an n-dimensional "spacetime" manifold X_n with boundaries with "in" boundary $\partial_{\text{in}} X = M_{n-1}$ and "out" boundary $\partial_{\text{out}} X = N_{n-1}$. Then the TQFT assigns to X_n a linear map $\mathcal{Z}(X_n) : \mathcal{Z}(M_{n-1}) \to \mathcal{Z}(N_{n-1})$, interpreted as a propagation between Hilbert spaces $\mathcal{Z}(M_{n-1})$ and $\mathcal{Z}(N_{n-1})$ along X_n.
- Gluing spacetimes is assigned to composition of linear maps. Similarly, cutting along a spatial slice corresponds to propagating through the intermediate states.

These data in the language of category theory can be summarised as follows: a TQFT is a *symmetric monoidal functor* from the category of bordisms to the category of vector spaces.

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{or}}_{n,n-1} \longrightarrow \mathrm{Vect} \ . \tag{1.0.3}$$

This can be thought as an abstraction describing assignment of a Hilbert space of quantum states to a certain smooth manifold on which the theory lives. The "topological" property is encoded in the data of the category of bordisms, whose underlying geometric data are invariant under smooth deformations. We can also equip bordisms with additional topological structure, giving rise to a corresponding "type" of TQFT. In the present thesis we are interested in oriented bordisms and TQFT will usually imply oriented TQFT, hence the superscript in the domain of $\mathcal{Z}$ in (1.0.3).

Under these definitions, one can view a closed n-manifold X_n as a bordism $X_n : \varnothing \longrightarrow \varnothing$, which, under the action of $\mathcal{Z}$, is assigned to a linear map $\mathbb{C} \longrightarrow \mathbb{C}$, namely a complex number $\mathcal{Z}(X_n)$. By the axioms above, this number is invariant under diffeomorphisms of X_n and is therefore topological invariant. Thus $\mathcal{Z}(X_n)$ corresponds to the *partition function* of the physical theory on X_n and should match the one computed by standard path integral techniques.

One might try to exploit the TQFT cutting and gluing axioms, in order to define TQFTs using a set of geometric building blocks subject to a certain relations. In low dimensions this indeed gives some classifying results. For $n = 1$, the assignment $\mathcal{Z} : \mathrm{Bord}^{\mathrm{or}}_{1,0} \longrightarrow \mathrm{Vect}$ is equivalent to specifying a finite-dimensional vector space, while for $n = 2$ the TQFT $\mathcal{Z} : \mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathrm{Vect}$ contains data equivalent to that of a commutative Frobenius algebra, namely a vector space equipped with a multiplication and a non-degenerate inner product.

However, in higher dimensions the number of these building blocks is infinite and thus this classification is not possible.

Higher categories

Atiyah's definition of TQFTs only incorporated "spatial" spacetime slices, or more precisely only codimension 1 submanifolds. However, this picture does not capture fundamental physical structures, for example local operators and their operator product, non-local operators, such as Wilson lines and defects, boundary conditions and interfaces. In order to bring the definition closer to our physical intuition, we would like to extend it in two ways:

1. enable cutting and glueing the spacetime along submanifolds of any dimension and
2. allow the computation of expectation values of lower dimensional observables, such as local fields, line and surface operators and related objects.

Both of these directions can be (independently) formalised by passing from categories to *higher categories*. In particular, an n-category contains

- a collection of objects,
- arrows between objects, the *1-morphisms*
- arrows between 1-morphisms, the *2-morphisms*
- and in general arrows between $k-1$-morphisms, the *k-morphisms*, for $k \leq n$.

These morphisms are equipped with various "directions" of composition. However, in most cases of interest, the appropriate n-categories are *weak*, meaning that compositions are associative and unital only up to isomorphism and these isomorphisms are required to satisfy certain coherence relations. As one could imagine, the precise description of such weak higher categories gets very complicated very quickly as n increases. In fact, for $n > 3$ explicit descriptions become impractically complex.[2]

Extended TQFTs

We can now use the language of higher categories to slice manifolds along submanifolds of any dimension. For non-extended TQFTs, we considered only n-manifolds and closed $(n-1)$-manifolds and assigned to them morphisms and objects of an algebraic category respectively. We can *extend down* by one dimension, by considering also $(n-1)$-manifolds with boundary, closed $(n-2)$-manifolds and assigning objects of a 2-category to $(n-2)$-manifolds, 1-morphisms to $(n-1)$-manifolds and 2-morphisms to n-manifolds. In a similar fashion, one can *extend down to points*, by considering submanifolds (with corners) of any dimension and assigning

[2] An alternative, more abstract approach is that of (∞, n)-categories where morphisms are associative and unital up to homotopy.

- objects of a symmetric monoidal n-category to points,
- 1-morphisms to 1-dimensional manifolds,
- and more generally k-morphisms to k-manifolds, for $k \leq n$.

These assignments still need to be endowed with a notion of a monoidal (tensor) product. In the language of higher categories, we have defined a bordism n-category as our source category and considered a symmetric monoidal n-functor from the bordism n-category to some symmetric monoidal n-category $\mathcal{B}$. This n-functor is a *fully extended TQFT with values in* $\mathcal{B}$.

At first it seems that this definition is too complicated, as it includes substantial amount of algebraic data and assigns very abstract invariants to lower dimensional manifolds. However, it turns out that the situation can be greatly simplified thanks to the *cobordism hypothesis* of Baez and Dolan ([1]): a fully extended TQFT is completely determined by the object of the n-category $\mathcal{B}$ it assigns to a point. Conversely, any object in $\mathcal{B}$ that satisfies certain dualisability assumptions defines a fully extended TQFT. The criterion of dualisability of objects can be thought as a higher categorical analogue of dualisability of vector spaces, which is true if and only if the vector spaces are finite-dimensional.

The cobordism hypothesis was proven by Lurie using the technology of (∞, n)-categories. In 2 dimensions it is possible to express these results using weak 2-categories (*bicategories*). This was done by Schommer-Pries and Pstragowski: Schommer-Pries gave a generators-and-relations presentation of the bordism category in 2 dimensions [50] and Pstragowski used this presentation, in order to prove the cobordism hypothesis in the language of symmetric monoidal bicategories [44].

Defect TQFTs

There is another generalisation of closed TQFTs, which allow for the presence of lower dimensional *defects* within the manifolds on which the theory is defined. The geometric properties of the defects are encoded in a set of *defect data* $\mathbb{D}$ and the corresponding TQFT, as formally defined in [17], can be expressed as

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \longrightarrow \mathrm{Vect}\,. \tag{1.0.4}$$

It is expected that n-dimensional defect TQFTs give rise to a certain (weak) n-category [33]: objects should correspond to (n-dimensional) theories and k-morphisms to codimension k defects. These statements have been proven for dimensions up to 3 [13, 19]. For these dimensions and for "strict enough" categories, it is possible to define a graphical calculus [2, 39], where computations can be pictorially represented using *string diagrams*. The graphical calculus is a rigorously defined visual and intuitive method to represent objects, morphisms and various directions of composition, greatly simplifying possibly complicated formulas. Furthermore, evaluation of string diagrams in category theory in these cases corresponds precisely to correlators in the TQFT, namely to the values assigned to decorated bordisms by the TQFT.

Specifically for $n = 3$, a defect TQFT is equivalent to an appropriate 3-category, which has 3-dimensional theories as objects, surface defects as 1-morphisms between 3-dimensional theories, line defects as 2-morphisms between surface defects and point defects as 3-morphisms between line defects. Such 3-categories encoding Rozansky-Witten models were described in [35]. Considering only identity defects and employing the cobordism hypothesis, [7] recovered the extended Rozansky-Witten TQFT with target manifold $T^*\mathbb{C}^n$. However there is a subtlety: the theories with non-compact target have infinite-dimensional state spaces, which implies that objects are not fully dualisable. A solution to this problem is to disregard the local fields of the theory: we can define the truncated 2-category which has

- the objects of the 3-category as objects,
- the 1-morphisms of the 3-category as 1-morphisms,
- isomorphism classes of 2-morphisms of the 3-category as 2-morphisms.

In this category objects are indeed fully dualisable and we can define an extended TQFT with that category as target. This procedure amounts physically to neglecting the correlation functions containing local fields.

A particularly interesting class of defects are symmetry defects [17, 23]. Using the language of category theory, symmetry defects can be employed to encode group actions, allowing to generalise and abstract the notion of symmetry: we view symmetries as morphisms in a category with certain composition properties, rather than merely group elements. This categorical perspective opens new avenues for exploring symmetries in a highly structured and generalized manner.

The main result of this thesis is the construction of extended defect TQFTs valued in the Kapustin-Rozansky category $\mathcal{RW}^{\text{aff}}$ of [35] using the cobordism hypothesis with defects. Chapter 2 provides a review of the necessary background, organised into a physics part and a mathematics part. In the physics section, we outline topological twists and the TQFTs relevant to this work, namely the Landau-Ginzburg and Rozansky-Witten models. In the mathematics section, we introduce the basics of higher category theory and graphical calculus, describe matrix factorisations and the category $\mathcal{LG}_k$ of Landau-Ginzburg models and review the cobordism hypothesis without defects and the cobordism hypothesis with defects in 2 dimensions, expressed in the language of bicategories.

In section 3, we present the category of truncated affine Rozansky-Witten models and equip it with a class of defects encoding their symmetries. We then employ the cobordism hypothesis with defects, together with 3-dimensional graphical calculus to construct state spaces in the presence of symmetry defect networks. Physically, this procedure corresponds to turning on non-trivial flat background gauge connections. Finally, we explore gauging these symmetries using the orbifold construction of [17].

Chapter 2

Background

2.1 Physics background - Topological twists

In this section we study topological twists. We begin by reviewing supersymmetry algebras and the procedure of supersymmetric twists, before focusing on the $2d$ $\mathcal{N}=(2,2)$ and $3d$ $\mathcal{N}=4$ algebras, leading to the B-twist in 2 dimensions and the Rozansky-Witten twist in 3 dimensions. Our treatment will primarily follow known physics arguments, often deliberately avoiding mathematical formalism. We then briefly review Landau-Ginzburg and Rozansky-Witten models in the presence of boundaries and defects, providing physical motivation for their categorical description.

2.1.1 Topological twists

Cohomological TQFTs

In physics literature, the term topological field theory usually refers to a quantum field theory defined on a manifold $\mathcal{M}$, possibly equipped with a metric h, whose correlation functions of physical observables are independent of h or the operator insertion points. TQFTs fall in two basic categories. On one hand, in *Schwarz type* TQFTs, the theory has a Lagrangian description, with an action functional (and thus also correlation functions) that is manifestly independent of h. The prototypical example of such theories is Chern-Simons theory. On the other hand, *cohomological* or *Witten type* TQFTs are defined using metric-dependent action, but we consider a subsector of the physical observables of the theory whose correlation functions are indeed metric-independent. In this chapter, we will study this kind of TQFT's.

Cohomological TQFTs are characterized by the presence of a nilpotent operator Q that plays the role of a differential. Physical observables are identified with the Q-cohomology. The fundamental ingredients and properties of a cohomological TQFT follow from standard path integral arguments, cf. [53, Sect. 3]:

- The theory has a fermionic symmetry with a nilpotent generator Q: $Q^2 \coloneqq \frac{1}{2}\{Q,Q\} = 0$.

- Correlation functions of Q-closed operators are independent of insertion points.
- Correlation functions of Q-exact operators vanish.
- The energy-momentum tensor is Q-exact.

We then define the TQFT as the subset of the observables of the original theory in the cohomology of Q. The multiplication of observables of the original theory (an operator product expansion for instance) descends to Q-cohomology, inducing a well-defined algebra structure on the cohomology classes.

Supersymmetry algebras

Many cases of cohomological TQFTs originate from supersymmetric field theories which are indeed equipped with fermionic generators. Supersymmetric field theories are quantum field theories whose symmetry algebra extends the Poincaré Lie algebra to a superalgebra with fermionic generators. Let us first recall the basic constructions of supersymmetry algebras, following the treatment of [22] that is as general as possible in terms of spacetime dimension.

Let $V_{\mathbb{R}} = \mathbb{R}^n$ be a real vector space of dimension n and $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. The Poincaré algebra is the semidirect product $\mathfrak{so}(V) \ltimes V$. In other words, it is the direct sum $\mathfrak{so}(V) \oplus V$ with Lie bracket

$$[A, B] = AB - BA,\ [A, v] = A \cdot v,\ [v, u] = 0, \quad A, B \in \mathfrak{so}(V),\ v, u \in V. \tag{2.1.1}$$

Let us fix a spinorial representation (ρ, Σ) of $\mathfrak{so}(V)$. Recall that in odd dimensions there is a distinguished fundamental representation S, while in even dimensions there is a pair of distinguished fundamental representations S_+ and S_-. This means that spinorial representations can be written as direct sums of the fundamental representations:

$$\begin{aligned} \Sigma &= S^{\oplus \mathcal{N}} \cong S \otimes W, \quad n \text{ odd}, \\ \Sigma &= S_+^{\oplus \mathcal{N}_+} \oplus S_-^{\oplus \mathcal{N}_-} \cong (S_+ \otimes W_+) \oplus (S_- \otimes W_-), \quad n \text{ even}, \end{aligned} \tag{2.1.2}$$

where W, W_+, W_- are "multiplicity spaces" with dimensions $\mathcal{N}, \mathcal{N}_+, \mathcal{N}_-$ respectively. Thus $\mathcal{N}$ in even dimensions and $\mathcal{N}_+, \mathcal{N}_-$ in odd dimensions completely fixes Σ. Let us also equip Σ with a non-degenerate symmetric $\mathfrak{so}(V)$-equivariant pairing $\Gamma : \mathrm{Sym}^2(\Sigma) \to V$. Then we can define the *supertranslation Lie algebra* as the direct sum $\mathfrak{A} = V \oplus \Sigma$, with the only non-trivial bracket given by Γ. We can then form the *super-Poincaré algebra* as the semidirect product $\mathfrak{so}(V) \ltimes \mathfrak{A}$, with non-trivial brackets

$$[A, \psi] = \rho(A) \cdot \psi,\ \{\psi, \chi\} = \Gamma(\psi, \chi), \quad A \in \mathfrak{so}(V),\ \psi, \chi \in \Sigma. \tag{2.1.3}$$

Given a spinorial representation, the pairing Γ is unique up to scale. Thus the choice of Σ (and thus the numbers $\mathcal{N}$ or $\mathcal{N}_+, \mathcal{N}_-$) defines the super-Poincaré algebra and we talk about the $\mathcal{N}$- or $(\mathcal{N}_+, \mathcal{N}_-)$-extended super-Poincaré algebra[1].

[1] The presence of the pairing Γ further restricts the relation between $\mathcal{N}_+$ and $\mathcal{N}_-$ in certain even dimensions, see [22, Prop. 3.2]

There is another way to act on the supertranslation algebra, the *R-symmetry* group G_R. This is the group of ($\mathfrak{so}(V)$-equivariant) outer automorphisms of the algebra of supertranslations that leaves the even part invariant. We can extend the super-Poincaré algebra by the Lie algebra of the R-symmetry group $\mathfrak{g}_R$, which defines the *supersymmetry algebra* $(\mathfrak{so}(V) \oplus \mathfrak{g}_R) \ltimes \mathfrak{A}$.

Topological twists

Twisting means to consider observables that are invariant under the action of a generator Q that lives in the odd part of the supersymmetry algebra. Since we want these invariants to be independent of the reference frame, we want to construct a nilpotent generator Q that transforms as a scalar in an "improved" spin group. For this, we define a *twisting homomorphism* $\phi : \mathrm{Spin}(V) \to G_R$, embedding the spin group into the improved spin group $(\mathrm{id}, \phi) : \mathrm{Spin}(V) \to \mathrm{Spin}(V) \times G_R =: \mathrm{Spin}(V)'$. In this group, some of the fermionic generators are rearranged to scalars. Then we pick a generator that transforms trivially under $\mathrm{Spin}(V)'$ and satisfies $\{Q, Q\} = 0$. If we can find all translation generators in the image of Q, in other words if the map

$$\{Q, \cdot\} : \Sigma \to V \tag{2.1.4}$$

is surjective, the resulting theory is topological.

The $2d$ $\mathcal{N} = (2, 2)$ A- and B-models

The Lorentz group in 2 dimensions is $\mathrm{SO}(2) \cong \mathrm{U}(1)$ and we denote it by $\mathrm{U}(1)_E$. The supertranslation algebra reads

$$\{Q_\pm, \overline{Q}_\pm\} = 2P_\pm, \tag{2.1.5}$$

with $P_\pm = -i\partial_\pm$. The supersymmetry algebra has an R-symmetry group $G_R = \mathrm{U}(1)_A \times \mathrm{U}(1)_V$ whose factors are respectively called *axial* and *vector* R-symmetry. The vector R-symmetry acts diagonally on Q_+, Q_-, while the axial R-symmetry acts antidiagonally. In detail, the action of spin and R-symmetries on generators reads (see also [29, Sect. 16.2], reviewed at [3, Sect. 4.1])

	Q_+	Q_-	$\overline{Q}_+$	$\overline{Q}_-$
$\mathrm{U}(1)_E$	-1	1	-1	1
$\mathrm{U}(1)_A$	-1	1	1	-1
$\mathrm{U}(1)_V$	-1	-1	1	1
$\mathrm{U}(1)_E \times \mathrm{U}(1)_A$	-2	2	0	0
$\mathrm{U}(1)_E \times \mathrm{U}(1)_V$	-2	0	0	2

(2.1.6)

In $\mathcal{N} = (2, 2)$ models we can select to twist with either the axial or vector R-symmetry, meaning that the twisting homomorphism $\mathrm{U}(1)_E \to \mathrm{U}(1)_E \times G_R$ is defined by the diagonal embedding into either U(1) factor $\mathrm{U}(1)_E \to \mathrm{U}(1)_E \times \mathrm{U}(1)_R$, with $\mathrm{U}(1)_R = \mathrm{U}(1)_V$ or $\mathrm{U}(1)_A$. The nilpotent supercharges compatible with these twisting homomorphisms are

$$Q_A = \overline{Q}_+ + Q_-, \quad Q_B = \overline{Q}_+ + \overline{Q}_-\,, \tag{2.1.7}$$

which define the A- and B-twists respectively. As we can see from 2.1.6, Q_A (resp. Q_B) is a scalar under the improved $\mathrm{U}(1)_E \times \mathrm{U}(1)_V$ (resp. $\mathrm{U}(1)_E \times \mathrm{U}(1)_A$) spin group. Furthermore, it has a well-defined remaining R-charge factor 1 defined by $\mathrm{U}(1)_A$ (resp. $\mathrm{U}(1)_V$), which defines a $\mathbb{Z}$ grading on the algebra of observables. The other supercharges $Q_+, \overline{Q}_-$ (resp. Q_+, Q_-) are then reorganised into a vector representation under the improved spin group and make the action 2.1.4 indeed surjective:

$$\begin{aligned} \{Q_A, Q_+\} &= 2P_+, \quad \{Q_A, \overline{Q}_-\} = 2P_-, \\ \{Q_B, Q_+\} &= 2P_+, \quad \{Q_B, Q_-\} = 2P_-. \end{aligned} \tag{2.1.8}$$

The $3d$ $\mathcal{N}=4$ supersymmetry algebra and topological twists

In 3 dimensions the spin group is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, which we denote again by $\mathrm{SU}(2)_E$ and the fundamental spinor representation is $S = \mathbb{C}^2$. The $\mathcal{N}=4$ supertranslation algebra has the spinor representation $\Sigma = S \otimes W = \mathbb{C}^2 \otimes \mathbb{C}^4$ as odd part. The algebra has an R-symmetry group $\mathrm{SO}(4) \cong \mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$, under which the supercharges transform in addition to the spatial rotation group $\mathrm{SU}(2)_E$. Thus $W = \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ has the structure of a bifundamental representation, which we denote as $H \otimes C$ for clarity. As a consequence, the supercharges $Q \in S \otimes H \otimes C$ carry three spinor indices: $\alpha = 1,2$ the Lorentz spinor index, $A = 1,2$ the $\mathrm{SU}(2)_H$ index and $\dot{A} = 1,2$ the $\mathrm{SU}(2)_C$ index. The invariant pairing $\mathrm{Sym}^2(S) \to V$ of $\mathfrak{so}(3)$ in components reads $\gamma^\mu_{\alpha\beta}$, obtained by the Pauli matrices contracted with the antisymmetric symbol $\epsilon^{\alpha\beta}$. Thus the supertranslation algebra in component form reads (see [24, Sect. A.3] for precise component expressions)

$$[Q^{A\dot{A}}_\alpha, Q^{B\dot{B}}_\beta] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\gamma^\mu_{\alpha\beta}P_\mu. \tag{2.1.9}$$

Similar to $2d$ $\mathcal{N}=(2,2)$ case, there are two possible topological twists, one for each R-symmetry $\mathrm{SU}(2)$ factor: the Rozansky-Witten twist and its mirror. The twisting homomorphisms are also obtained in a similar way, namely the diagonal embedding $\mathrm{SU}(2)_E \to \mathrm{SU}(2)_E \times \mathrm{SU}(2)_R$ with $\mathrm{SU}(2)_R = \mathrm{SU}(2)_C$ or $\mathrm{SU}(2)_H$.

The Rozansky-Witten twist corresponds to the diagonal embedding $\mathrm{SU}(2)_E \to \mathrm{SU}(2)_E \times \mathrm{SU}(2)_C$. The compatible topological supercharge transforms as a scalar in the tensor product $S \otimes C$. One possible choice is to pick a definite H spin, for example

$$Q_{RW} = \delta^\alpha_{\dot{A}} Q^{1\dot{A}}_\alpha = Q^{1\dot{1}}_1 + Q^{1\dot{2}}_2 \tag{2.1.10}$$

or, more generally, rotate in the H-space giving a $\mathbb{CP}^1$ family of twists:

$$Q^{(\zeta)}_{RW} = \frac{1}{\sqrt{1+|\zeta|^2}} \delta^\alpha_{\dot{A}} (Q^{1\dot{A}}_\alpha + \zeta Q^{2\dot{A}}_\alpha), \quad \zeta \in \mathbb{CP}^1. \tag{2.1.11}$$

The mirror to Rozansky-Witten twist is defined with H exchanged with C. These twists make the action of nilpotent supercharges surjective. For explicit expressions, see for example [3, Sect. 5.1].

Note that both twists can be viewed as a deformation of a holomorphic twist. The image of the corresponding nilpotent supercharge is 2-dimensional, as explained in detail in [9, Sect. 2.1.2].

2.1.2 Landau-Ginzburg models

The bulk model

Landau-Ginzburg models ([52], reviewed at [29, Chap. 13]) are $\mathcal{N} = (2,2)$ supersymmetric sigma models $\phi : \Sigma \to X$, where Σ is a 2-dimensional worldsheet and X a Riemannian manifold. The defining feature of Landau-Ginzburg action is that it contains a superpotential term. The action functional is schematically

$$S = \int_\Sigma d^2\sigma \left(|\partial\phi|^2 + \left|\frac{\partial W}{\partial \phi}\right|^2 + \text{fermionic terms} \right). \tag{2.1.12}$$

Landau-Ginzburg models are B-twistable if the axial R-symmetry is unbroken at the quantum level. This is the case whenever the target space X has vanishing first Chern class $c_1(X) = 0$, as for Calabi-Yau manifolds. In this work, we restrict to flat targets $X = \mathbb{C}^n$, so this requirement is trivially satisfied. The B-twist is performed using the supercharge $Q_B = \overline{Q}_+ + \overline{Q}_-$, which remains scalar after the twist and defines the BRST operator of the topological theory. Furthermore, to preserve vector R-symmetry, the superpotential W must carry vector R-charge 2. This holds if W is a quasi-homogeneous polynomial of degree 2 with respect to the vector R-charges assigned to the scalar fields $\phi \in X$:

$$W(\lambda^{q_i}\phi^i) = \lambda^2 W(\phi^i). \tag{2.1.13}$$

As mentioned after (2.1.7), the vector R-symmetry acts on the space of fields and induces a $\mathbb{Q}$ grading. The physical observables of the model, namely those that lie in Q_B-cohomology, are holomorphic functions of the fields ϕ^i. The space of observables, called the *chiral ring* has the structure of a commutative algebra and is given by the Jacobi ring

$$\mathcal{R} = \frac{\mathbb{C}[\phi^i]}{(\partial_i W)}. \tag{2.1.14}$$

Boundaries and defects

Introducing boundaries in B-twisted Landau-Ginzburg models [8, 26, 34] breaks translation symmetry perpendicular to the boundary and consequently half of supersymmetry. In order for the boundary theory to retain the topological character of the bulk, we require that the preserved supersymmetry includes Q_B. This imposes two basic conditions: first, the introduction of fermionic boundary degrees of freedom and second, the ability to factorise the superpotential in the form

$$W = \sum_{i=1}^{k} p_i q_i, \tag{2.1.15}$$

where p_i, q_i are polynomials in the fields ϕ and k some positive integer. These data define the structure of *matrix factorisations* of W, which we will present in more detail in section 2.2.2.

One can generalise the notion of boundaries to that of *defects*, namely interfaces separating world-sheet regions with possibly different superpotentials W and V. Defects that preserve B-type supersymmetry, and thus topological, are described analogously to boundaries by matrix factorisations of the difference of superpotentials $W - V$ [10]. Topological defects admit well-defined fusion products among themselves and with boundaries, giving rise to rich mathematical structures in the language of bicategories. For Landau-Ginzburg models, these structures were formalised in [14] and we will briefly review them in section 2.2.3.

2.1.3 Rozansky-Witten models

In this section we review Rozansky-Witten models and their boundaries, following primarily the original constructions of [36, 47] and the complementary treatments of [11, 18, 25].

Rozansky-Witten models are topological twists of $\mathcal{N} = 4$ supersymmetric sigma models $\phi : \Sigma \to X$, where Σ is a smooth 3-manifold and X is a hyperkähler target. In this work, we focus on sigma models with flat targets $X = T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$, in which case the model simply reduces to a theory of free hypermultiplets.

Before twisting, the scalar components $\phi, \overline{\phi}$ of the hypermultiplets transform as a doublet of the $\mathrm{SU}(2)_H$ factor of the R-symmetry, a property that is preserved after twisting. The fermionic components, on the other hand, transform in the untwisted theory as doublets of the other R-symmetry factor $\mathrm{SU}(2)_C$, which becomes part of the modified Lorentz group $\mathrm{SU}(2)_{E'} = \mathrm{SU}(2)_E \times \mathrm{SU}(2)_C$ under the twisting homomorphism. Rearranging the fermions into representations of $\mathrm{SU}(2)_{E'}$, we obtain a scalar η and a one-form field χ. Moreover, both bosonic and fermionic fields take values in the target $X = T^*\mathbb{C}^n$ and transform under the flavour symmetry group $\mathrm{Sp}(n)$ which is the structure group of X.

From the (classical) BRST transformation of the hypermultiplet components [47, Eq. 2.22], one finds that the Q-closed local operators are generated by the fermionic fields η with coefficients in the scalar fields ϕ. Upon choosing a complex structure on X, the algebra of local operators can be identified with the Dolbeault cohomology of X. In the flat space, this reduces to the space of holomorphic functions on $\mathbb{C}^{2n}$.

To compute the state spaces $\mathcal{H}_{\Sigma_g}$ assigned to a genus-g Riemann surface Σ_g, we quantise the theory on $M = \Sigma_g \times \mathbb{R}$. The path integral localises to constant bosonic maps $\phi : M \to X$, around which we expand. There are $2n$ fermionic zero modes arising from the scalar fermions η and in addition $2n \cdot b_1(\Sigma_g) = 4ng$ fermionic zero modes of the one-form fermions χ. Using their commutation relations [47, Eq. 5.5-5.6], these can be split into $2ng$ creation and $2ng$ annihilation operators, spanning a fermionic Fock space.

Quantising on a sphere $\Sigma_g = S^2$, for a compact target X, there are no fermionic zero modes of χ. The Hilbert space reduces to the Fock space generated by $2n$ fermions η, which corresponds to the Dolbeault cohomology

$$\mathcal{H}_{S^2} = H^{\bullet}_{\overline{\partial}}(X, \mathcal{O}_X) = H^{0,\bullet}(X), \tag{2.1.16}$$

which matches the algebra of local operators obtained from classical analysis. For higher genus surfaces, the result takes the form

$$\mathcal{H}_{\Sigma_g} = H^{\bullet}_{\bar{\partial}}(X, (\wedge^{\bullet} TX)^{\otimes g}). \tag{2.1.17}$$

However, if X is non-compact, the resulting state spaces are infinite dimensional. These have been studied in [11, 25]. In the case of interest $X = T^*\mathbb{C}^n$, quantisation on the sphere S^2 yields the space of constant holomorphic functions on X, recovering again the space $\mathbb{C}[\phi]$ of local operators obtained from classical analysis. For higher genus surfaces, the state spaces are generated by the $2ng$ one-form fermions that act as creation operators. The Hilbert spaces are then given by

$$\mathcal{H}_{\Sigma_g} = \mathbb{C}[\phi] \otimes \wedge^{\bullet}(\mathbb{C}^{2ng}). \tag{2.1.18}$$

Stressing the fact that we are working in the context of *graded* vector spaces, where the product is graded-commutative, this state space can also be written as

$$\mathcal{H}_{\Sigma_g} = \mathbb{C}[\phi] \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}[1])^{\otimes 2ng}. \tag{2.1.19}$$

Boundary conditions

Introducing a boundary breaks part of the original 3d $\mathcal{N} = 4$ supersymmetry algebra. We seek boundary conditions that preserve the largest possible subalgebra while retaining the topological features of the theory. The first goal can be achieved by preserving either a 2d $\mathcal{N} = (2, 2)$ or $\mathcal{N} = (0, 4)$ subalgebra. To satisfy the second condition, we must ensure that the Rozansky-Witten BRST charge (2.1.10) is contained in the preserved subalgebra. This requirement selects the $\mathcal{N} = (2, 2)$ subalgebra, as it contains supercharges of both chiralities. The same conclusion also applies to the mirror Rozansky-Witten twist, since its twisting supercharge necessarily has mixed chirality. Therefore, based purely on supersymmetry considerations, we expect the boundary to support a 2d $\mathcal{N} = (2, 2)$ topological field theory.

As shown in [36], such boundary conditions are characterised by complex Lagrangian submanifolds $Y \subset X$. In particular one can choose a superpotential W that generates the Lagrangian Y, in the sense that Y is the graph of dW:

$$Y = \{(q, p) \in T^*\mathbb{C}^n \mid p = \partial W/\partial q\}. \tag{2.1.20}$$

The effective theory on the boundary is precisely the one of a B-twisted Landau-Ginzburg model.

One can generalise boundaries to surface defects, which separate theories with possibly different target spaces X_1 and X_2. Surface defects can be understood via the folding trick: a neighbourhood of the defect is reinterpreted as a boundary condition in a theory with target $X_1^* \times X_2$, where star denotes parity reversal. A boundary is then a special case of a surface defect, separating a theory with target X from the trivial theory, whose target is empty.

Due to the topological nature of the theory, surface defects can be brought arbitrarily close together without affecting the physics. This allows one to define a fusion product of surface defects, which endows them with a monoidal structure. Furthermore, surfaces can support line operators, which themselves have well-defined monoidal structure. Together, these structures are naturally encoded in the structure of a higher category [35, 36], a more detailed description of which will be presented in the following chapter.

2.2 Math background - Categories, TQFTs and cobordism hypothesis

This chapter introduces basic notions of higher category theory, together with some examples that will play an important role in our constructions presented in later sections. We assume that the reader is familiar with the fundamental definitions of categories and functors. These can be found in standard textbooks, for instance [41].

2.2.1 Higher categories and functors

Basic definitions

In this section, we are following the definitions from [4, 14, 30]. A bicategory can be thought of as a category with an additional layer of structure. In addition to objects and morphisms, it contains "morphisms between morphisms", the *2-morphisms*. Furthermore, a bicategory relaxes the strict associativity and unity axioms of an ordinary category, introducing the *associator* and *unitor* isomorphisms. In other words, composition is no longer strictly associative and unital, but these properties are mediated by appropriate invertible 2-morphisms, satisfying coherence conditions. In particular:

Definition 2.1: A *bicategory* $\mathcal{B}$ consists of the following data:

- *Objects:* $\mathcal{B}$ is equipped with a class $|\mathcal{B}|$ of objects.
- *Morphism categories:* For every pair X, Y of objects in $\mathcal{B}$, there is a (small) category $\mathcal{B}(X, Y)$ of morphisms. Its objects are called 1-morphisms and its morphisms are called 2-morphisms. Composition of 2-morphisms ϕ, ψ in $\mathcal{B}(X, Y)$ is called *vertical composition* and denoted by $\psi \circ \phi$. Identities of 1-morphisms f are denoted by 1_f.
- *Horizontal composition:* For every triple X, Y, Z of objects in $\mathcal{B}$, there is a functor

  $$c_{XYZ} : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \tag{2.2.1}$$

  called horizontal composition. For 1-morphisms $f \in \mathcal{B}(X, Y)$, $g \in \mathcal{B}(Y, Z)$, we denote

  $$c_{XYZ}(f, g) =: g \otimes f \tag{2.2.2}$$

for 2-morphisms $\phi : f \to f'$, $\psi : g \to g'$ we denote the action of the composition functor by

$$c_{XYZ}(\phi, \psi) =: \psi \otimes \phi. \tag{2.2.3}$$

- *Identity 1-morphism:* For every object X in $\mathcal{B}$, there is an identity 1-morphism $1_X \in \mathcal{B}(X, X)$.

- *Associator:* For every triple of composable 1-morphisms f, g, h, there is a natural isomorphism

$$\alpha_{f,g,h} : (h \otimes g) \otimes f \to h \otimes (g \otimes f). \tag{2.2.4}$$

- *Unitors:* For every 1-morphism $f \in \mathcal{B}(X, Y)$, there is a pair of natural isomorphisms

$$\lambda_f : 1_Y \otimes f \to f, \quad \rho_f : f \otimes 1_X \to f, \tag{2.2.5}$$

called the *left unitor* and *right unitor* respectively.

The above data satisfy the following axioms:

(i) *Unity axiom:* For 1-morphisms $f \in \mathcal{B}(X, Y)$, $g \in \mathcal{B}(Y, Z)$, the following diagram commutes:

$$\begin{array}{ccc} (g \otimes 1_Y) \otimes f & \xrightarrow{\alpha_{g,1_Y,f}} & g \otimes (1_Y \otimes f) \\ & {\scriptstyle \rho_g \otimes 1_f} \searrow \qquad \swarrow {\scriptstyle 1_g \otimes \lambda_f} & \\ & g \otimes f & \end{array} . \tag{2.2.6}$$

(ii) *Associativity axiom:* For composable 1-morphisms f, g, h, k, the following diagram commutes:

$$\begin{array}{ccc} ((k \otimes h) \otimes g) \otimes f & \xrightarrow{\alpha_{k\otimes h,g,f}} & (k \otimes h) \otimes (g \otimes f) \\ \downarrow {\scriptstyle \alpha_{k,h,g} \otimes 1_f} & & \downarrow {\scriptstyle \alpha_{k,h,g\otimes f}} \\ (k \otimes (h \otimes g)) \otimes f & & k \otimes (h \otimes (g \otimes f)) \\ & {\scriptstyle \alpha_{k,h\otimes g,f}} \searrow \qquad \nearrow {\scriptstyle 1_k \otimes \alpha_{h,g,f}} & \\ & k \otimes ((h \otimes g) \otimes f) & \end{array} . \tag{2.2.7}$$

Remark 2.2: Some remarks on the definition of a bicategory:

1. Functoriality of horizontal composition implies that it preserves identities and (vertical) compositions, namely:

$$1_g \otimes 1_f = 1_{g\otimes f}, \tag{2.2.8}$$

for $f \in \mathcal{B}(X,Y)$, $g \in \mathcal{B}(Y,Z)$ and

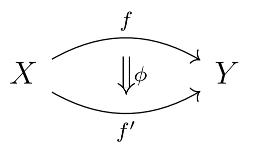
$$(\psi' \circ \psi) \otimes (\phi' \circ \phi) = (\phi' \otimes \psi') \circ (\phi \otimes \psi), \tag{2.2.9}$$

for appropriately composable 2-morphisms ϕ, ϕ', ψ, ψ'.

2. We display 1-morphisms $f \in \mathcal{B}(X,Y)$ as

$$f : X \longrightarrow Y$$

and 2-morphisms $\phi : f \to f'$ as



3. The associator is a natural transformation between the functors

$$\alpha_{XYZW} : c_{XYW} \circ (c_{YZW} \times 1_{\mathcal{B}(X,Y)}) \to c_{XZW} \circ (1_{\mathcal{B}(Z,W)} \times c_{XYZ}) \tag{2.2.10}$$

for objects X, Y, Z, W and for 1-morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, namely there is an isomorphism with components

$$(\alpha_{XYZW})_{f,g,h} : (h \otimes g) \otimes f \to h \otimes (g \otimes f).$$

For ease of notation, we suppress the object indices and write $\alpha_{f,g,h}$, as in (2.2.4). Naturality implies that for 2-morphisms $\phi : f \to f'$, $\psi : g \to g'$, $\zeta : h \to h'$, the following diagram commutes:

$$\begin{array}{ccc} (h \otimes g) \otimes f & \xrightarrow{\alpha_{f,g,h}} & h \otimes (g \otimes f) \\ \Big\downarrow{\scriptstyle (\zeta\otimes\psi)\otimes\phi} & & \Big\downarrow{\scriptstyle \zeta\otimes(\psi\otimes\phi)} \\ (h' \otimes g') \otimes f' & \xrightarrow{\alpha_{f',g',h'}} & h' \otimes (g' \otimes f') \end{array} \; . \tag{2.2.11}$$

The associativity axiom equates the two possible ways to map between $((k\otimes h)\otimes g)\otimes f$ and $k \otimes (h \otimes (g \otimes f))$.

4. Similarly, naturality of unitors implies commutativity of the following diagram for all 2-morphisms $\alpha : f \to f'$:

$$\begin{array}{ccccc} 1_Y \otimes f & \xrightarrow{\lambda_f} & f & \xleftarrow{\rho_f} & f \otimes 1_X \\ \downarrow{\scriptstyle (1_{1_Y})\otimes\phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi\otimes(1_{1_X})} \\ 1_Y \otimes f' & \xrightarrow{\lambda_{f'}} & f' & \xleftarrow{\rho_{f'}} & f' \otimes 1_X \end{array} . \tag{2.2.12}$$

The unity axiom equates the two possible ways to "horizontally squeeze" a unitor 1-morphism between two 1-morphisms.

2-categories

Definition 2.3: A *2-category* is a bicategory whose associator α and unitors λ, ρ are identities.

Remark 2.4: In 2-categories, horizontal composition of 1- and 2-morphisms is *strictly associative*, namely:

$$(h \otimes g) \otimes f = h \otimes (g \otimes f), \tag{2.2.13}$$

on the level of 1-morphisms, as follows from (2.2.4) and

$$(\zeta \otimes \psi) \otimes \phi = \zeta \otimes (\psi \otimes \phi), \tag{2.2.14}$$

on the level of 2-morphisms, as follows from (2.2.11). Furthermore, horizontal composition is *strictly unital*, namely:

$$1_Y \otimes f = f = f \otimes 1_X, \tag{2.2.15}$$

on the level of 1-morphisms, as follows from (2.2.5) and

$$1_{1_Y} \otimes \phi = \phi = \phi \otimes 1_{1_X}, \tag{2.2.16}$$

as follows from (2.2.12).

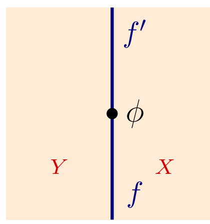
String diagrams for 2-categories

String diagrams provide a pictorial and intuitive way to represent equations between 2-morphisms in a bicategory. The graphical calculus was established in [32] for the 2-dimensional case and in [2] for 3 dimensions. Our presentation follows [12, 39, 49].

For the graphical calculus to be well-defined, the underlying categories must be sufficiently strict. In 2 dimensions, the requirement is satisfied for 2-categories. However, since every bicategory is equivalent to a strict 2-category [31, 43] we may use the string diagrams in the bicategorical setting as well, keeping track of unitors (but omitting associators). We thus interpret 2-morphism equations "modulo coherence". In 3 dimensions, the situation is analogous, but more technically more involved. The relevant coherence result states that the "most strict" 3-categories are the so-called *Gray categories*, for which the graphical calculus is well-defined [2]. We will not delve further into these coherence results and refer the reader to the cited references for details.

In the string diagram formalism, objects are represented by 2-dimensional regions in the plane, while 1-morphisms are drawn as boundary lines between these regions. These lines are required to be progressive in the sense of [2, Def. 2.8], meaning that they are smooth and the tangent vectors at all points have a non-trivial vertical component. By convention, we read the lines representing 1-morphisms as mapping from right to left. Finally, 2-morphisms are represented by vertices where possibly several lines meet. Our convention is to read the vertices labelling 2-morphisms as mapping from bottom to top.

To illustrate our conventions, consider objects X, Y in a bicategory $\mathcal{B}$ and 1-morphisms $f, f' : X \to Y$. Then we represent a 2-morphism $\phi : f \to f'$ by



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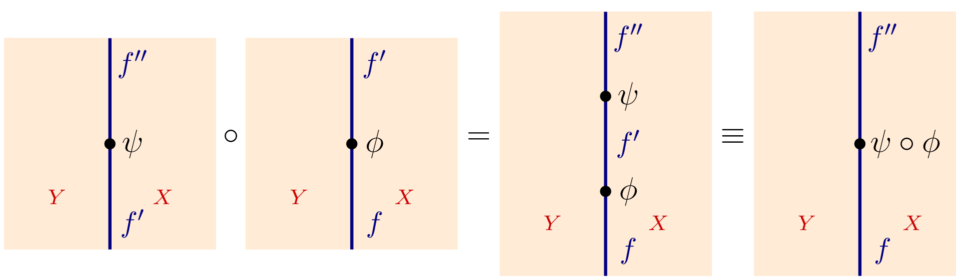
Vertical composition of 2-morphisms is denoted by vertically stacking the diagrams, glueing them along the intermediate line:



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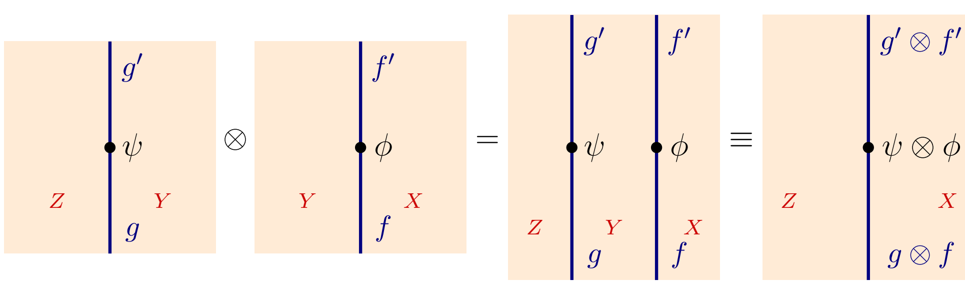
while horizontal composition of 2-morphisms, as in (2.2.3), is denoted by putting the diagrams side by side:



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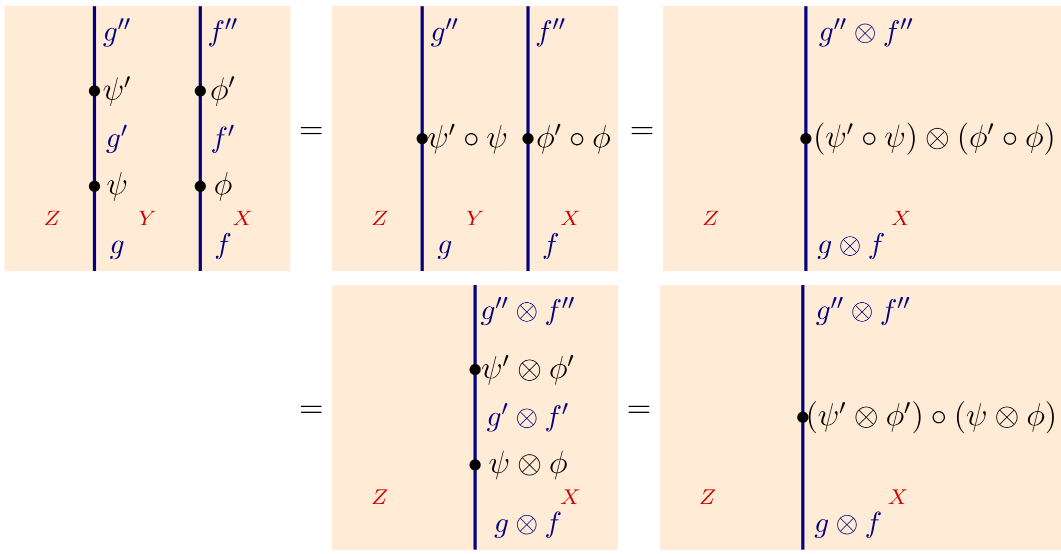
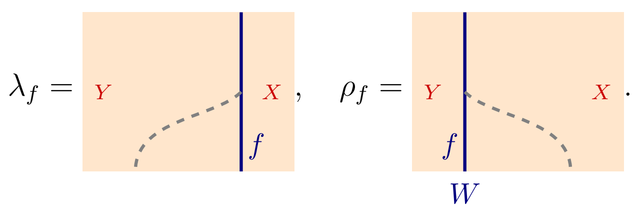
In the language of graphical calculus, certain bicategorical axioms can simply be interpreted as the expressing that different ways of composing 2-morphisms yield the same result. For instance, functoriality of horizontal composition (2.2.9) states that the two possible ways of composing the following diagram result in an equal 2-morphism:



(2.2.17)

Using coherence, all possible ways (bracketings) to read diagrams in a bicategory are equal. Therefore, it is enough to read them as diagrams in the corresponding 2-category. We will however sometimes denote the unitor lines explicitly, especially when they are important for clarity of computations. For instance, we represent the unitor 2-morphisms (2.2.5) by



(2.2.18)

Adjunctions and pivotality

Definition 2.5: Let $\mathcal{B}$ be a bicategory and let $f : X \to Y$, $g : Y \to X$ be 1-morphisms. An *adjunction* between f and g is a pair of 1-morphisms f, g together with a pair of 2-morphisms

$$\mathrm{ev} : f \otimes g \to 1_Y, \quad \mathrm{coev} : 1_X \to g \otimes f, \tag{2.2.19}$$

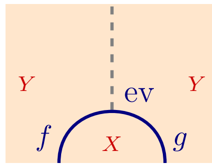
called evaluation and coevaulation 2-moprhisms repsectively, such that the compositions

$$f \xrightarrow{\rho_f^{-1}} f \otimes 1_X \xrightarrow{1_f \otimes \mathrm{coev}} f \otimes (g \otimes f) \xrightarrow{\alpha^{-1}_{f,g,f}} (f \otimes g) \otimes f \xrightarrow{\mathrm{ev} \otimes 1_f} 1_Y \otimes f \xrightarrow{\lambda_f} f$$

$$g \xrightarrow{\lambda_g^{-1}} 1_X \otimes g \xrightarrow{\mathrm{coev} \otimes 1_g} (g \otimes f) \otimes g \xrightarrow{\alpha_{g,f,g}} g \otimes (f \otimes g) \xrightarrow{1_g \otimes \mathrm{ev}} 1_Y \otimes f \xrightarrow{\lambda_f} f \tag{2.2.20}$$

evaluate to identities. We then say that f is *left adjoint* to g and g is *right adjoint* to f.

Using string diagrams, we display the evaluation and coevaluation 2-morphisms as



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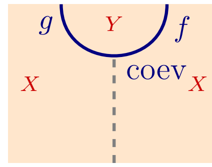

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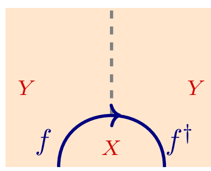
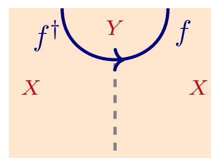
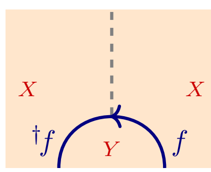
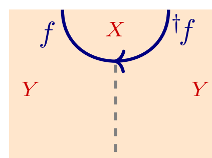
Proposition 2.6: Adjoints are unique up to unique isomorphism compatible with adjunction maps.

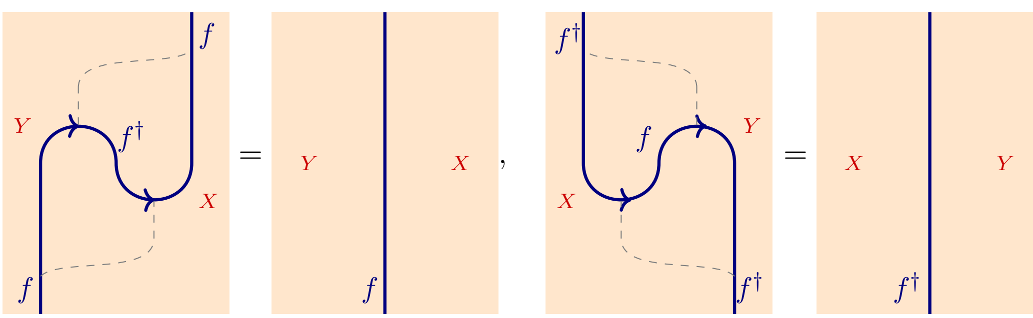
Definition 2.7: A bicategory $\mathcal{B}$ has *left adjoints* (resp. *right adjoints*), if every 1-morphism f in $\mathcal{B}$ admits a left adjoint (resp. a right adjoint).

By proposition 2.6, left and right adjoints of a 1-morphism f are unique up to unique isomorphism. We denote them by ${}^\dagger f$ and $f^\dagger$, respectively. The adjunction 2-morphisms are then expressed in the string diagram language:

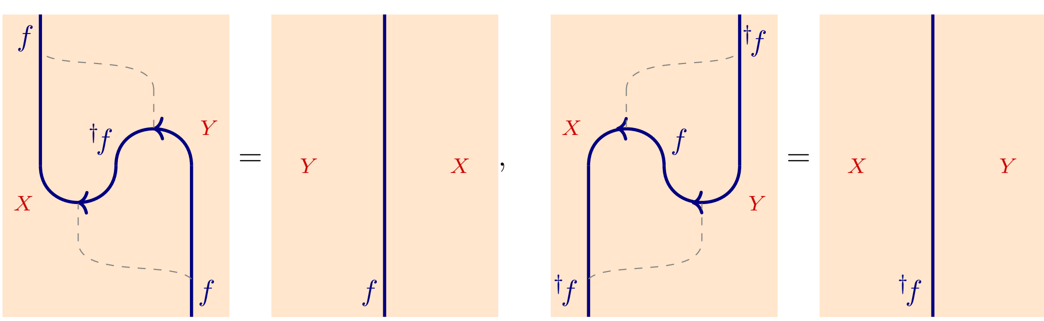
$\widetilde{\mathrm{ev}}_f =$  , $\widetilde{\mathrm{coev}}_f =$  , (2.2.21)

$\mathrm{ev}_f =$  , $\mathrm{coev}_f =$  (2.2.22)

which satisfy the so-called *Zorro moves*:



(2.2.23)

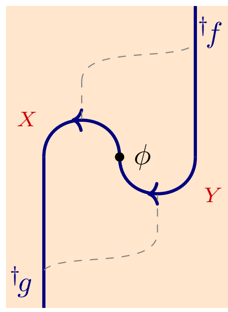


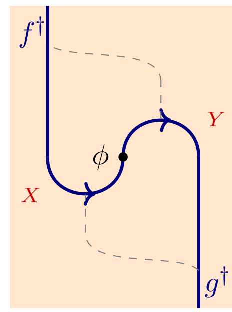
(2.2.24)

For 1-morphisms $f, g : X \longrightarrow Y$ and a 2-morphism $\phi : f \longrightarrow g$, the left and right duals of ϕ

$$^{\dagger}\phi : {}^{\dagger}g \longrightarrow {}^{\dagger}f, \quad \phi^{\dagger} : g^{\dagger} \longrightarrow f^{\dagger}$$

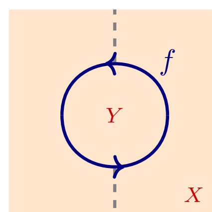
are respectively defined as

$^{\dagger}\phi =$ 

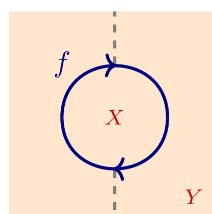
, $\phi^{\dagger} =$ 

. (2.2.25)

In case the left and right adjoints of f coincide, we can define the *left* and *right quantum dimensions*

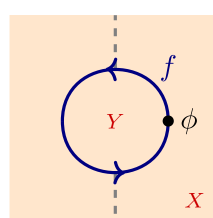
$\dim_{\mathrm{l}}(f) = \mathrm{ev}_f \circ \widetilde{\mathrm{coev}}_f =$ 

, (2.2.26)

$\dim_{\mathrm{r}}(f) = \widetilde{\mathrm{ev}}_f \circ \mathrm{coev}_f =$ 

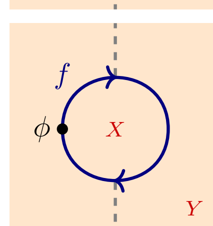
(2.2.27)

and for a 2-endomorphism of f $\phi : f \longrightarrow f$, the *left* and *right traces* of ϕ

$\mathrm{tr}_{\mathrm{l}}(\phi) = \mathrm{ev}_f \circ (1_f \otimes \phi) \circ \widetilde{\mathrm{coev}}_f =$ 

,

(2.2.28)

$\mathrm{tr}_{\mathrm{r}}(\phi) = \widetilde{\mathrm{ev}}_f \circ (\phi \otimes 1_f) \circ \mathrm{coev}_f =$ 

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2.2.2 Matrix factorisations

Let $\mathcal{R}$ be a commutative ring.

Definition 2.8: Given a polynomial $f \in \mathcal{R}$, a *matrix factorization* of f is given by a free $\mathbb{Z}_2$-graded $\mathcal{R}$-module $X = X_0 \oplus X_1$ together with an odd $\mathcal{R}$-module map $d_X : X \to X$, such that

$$(d_X)^2 \equiv d_X \circ d_X = f \cdot \mathrm{id}_X. \tag{2.2.29}$$

We denote this matrix factorisation as (X, d_X) and we call d_X the *differential* corresponding to this matrix factorisation.

Picking a basis for X, we often write the differential in its matrix form:

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \tag{2.2.30}$$

Definition 2.9: The *shift* of a matrix factorisation (X, d_X) is defined by

$$(X[1])_i = X_{i+1}, \quad d^i_{X[1]} = -d_X^{i+1}, \tag{2.2.31}$$

with $i = 0, 1 \bmod 2$. In matrix form:

$$d_{X[1]} = \begin{pmatrix} 0 & -d_X^0 \\ -d_X^1 & 0 \end{pmatrix}. \tag{2.2.32}$$

Let (X, d_X), (Y, d_Y) be matrix factorisations of $f \in \mathcal{R}$.

Definition 2.10: The *direct sum* or *coproduct* of $(X, d_X), (Y, d_Y)$ is a matrix factorisation of $f \in \mathcal{R}$ with underlying module $X \oplus Y$ and differential $d_{X\oplus Y}(a + b) = d_X(a) + d_Y(b)$.

The module maps $X \to Y$ form a free $\mathbb{Z}_2$-graded $\mathcal{R}$-module $\mathrm{Hom}_{\mathcal{R}}(X, Y)$ which is equipped with a differential:

$$\begin{aligned} \delta_{X,Y} : \mathrm{Hom}_{\mathcal{R}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathcal{R}}(X, Y) \\ \phi &\longmapsto d_Y \circ \phi - (-1)^{|\phi|} \phi \circ d_X. \end{aligned} \tag{2.2.33}$$

Definition 2.11: The *homotopy category of matrix factorisations* $\mathrm{hmf}(\mathcal{R}, f)$:

- Objects are finite-rank matrix factorisations of f.
- For matrix factorisations (X, d_X), (Y, d_Y), morphisms are elements in even cohomology of $\delta_{X,Y}$.

In detail, morphisms in this category are even module maps $\phi = \phi_0 + \phi_1$, with $\phi_i : X_i \to Y_i$ which commute with the differentials:

$$d_Y \circ \phi = \phi \circ d_X. \tag{2.2.34}$$

Furthermore, $\phi_1 \sim \phi_2$ if $\phi_1 - \phi_2 = d_Y \circ \psi + \psi \circ d_X$ for some odd module map ψ.

Tensor products of matrix factorisations

Lemma 2.12: Let $(X, d_X) \in \text{hmf}(\mathcal{R}, f)$, $(Y, d_Y) \in \text{hmf}(\mathcal{R}, g)$. There is a matrix factorisation in $\text{hmf}(\mathcal{R}, f+g)$, called the *tensor product* matrix factorization, whose underlying module is $X \otimes_{\mathcal{R}} Y$ and its differential is

$$d_{X\otimes Y} = d_X \otimes_{\mathcal{R}} 1_Y + 1_X \otimes_{\mathcal{R}} d_Y. \tag{2.2.35}$$

Proof. The proof is, for example, a combination of lemmata 2.4.2, 2.6.8 of [42]. □

Let k be a commutative ring and $\underline{x} \equiv \{x_1, \dots, x_n\}$ be an ordered set of variables. From now on, we restrict our discussion to matrix factorisations of polynomials $f \in \mathcal{R} = k[\underline{x}]$.

In the categorical description of Landau-Ginzburg models and affine Rozansky-Witten models, of particular importance is the following form of tensor product.

Definition 2.13: Let $\underline{x}, \underline{y}, \underline{z}$ be ordered sets of variables, $f \in k[\underline{x}], g \in k[\underline{y}], h \in k[\underline{z}]$ and matrix factorisations $(X, d_X) \in \text{hmf}(k[\underline{x}, \underline{y}], f-g)$, $(Y, d_Y) \in \text{hmf}(k[\underline{y}, \underline{z}], g-h)$. The *tensor product over* $k[\underline{y}]$ is a matrix factorisation of $f-h$ with underlying module $X \otimes_{k[\underline{y}]} Y$ and differential $d_X \otimes_{k[\underline{y}]} 1_Y + 1_X \otimes_{k[\underline{y}]} d_Y$.

Remark 2.14: We note the following about this tensor product construction:

- The tensor product of modules and module maps over $k[\underline{y}]$ is in fact given by 2.12 (for $\mathcal{R} = k[\underline{x}, \underline{y}, \underline{z}]$), precomposed with appropriate extensions of scalars and postcomposed with restriction of scalars. The detailed definition can be found in [21, Sect. 12] (see also definitions 2.4.6, 2.6.10 in [42]).

- This construction does not a priori yield an element of $\text{hmf}(k[\underline{x}, \underline{z}], f-h)$, since the resulting modules are generically $k[\underline{x}, \underline{z}]$ modules of infinite rank. However, it is homotopy equivalent to a finite rank factorisation as shown in [37, Sect. 4] or in [21, Sect. 12]. Explicit examples can also be found in [10]. In order to consider the tensor product as a functor with target $\text{hmf}(k[\underline{x}, \underline{z}], f-h)$, we need to complete this category to contain direct summands of finite rank.

Definition 2.15: $\text{hmf}(k[\underline{x}], f)^\omega$ is the idempotent closure of $\text{hmf}(k[\underline{x}], f)$ and consists of direct summands of finite rank matrix factorisations.

Koszul matrix factorisation

Assume a polynomial $f \in k[\underline{x}]$ can be written as a sum

$$f = \sum_{i=1}^{r} p_i q_i =: \underline{p} \cdot \underline{q} \tag{2.2.36}$$

and let $\underline{p} = (p_1, \dots, p_r)$, $\underline{q} = (q_1, \dots, q_r)$ denote the lists of polynomials corresponding to this presentation.

Definition 2.16: The *Koszul matrix factorisation* corresponding to the lists $(\underline{p}, \underline{q})$ is defined as the tensor product factorisation

$$[\underline{p}, \underline{q}] = \bigotimes_{i=1}^{r} [p_i, q_i] \tag{2.2.37}$$

defined as $[p_i, q_i] = (K_i, d_{K_i})$ with

$$K_i = k[\underline{x}] \oplus k[\underline{x}], \quad d_{K_i} = \begin{pmatrix} 0 & p_i \\ q_i & 0 \end{pmatrix} \tag{2.2.38}$$

and the tensor product defined over $k[\underline{x}]$ in the sense of definition 2.12.

It is particularly convenient to express the Koszul factorisation in terms of the exterior algebra of $k[\underline{x}]^{\oplus r}$. In particular, if $\{\theta_i\}$ is a basis of $k[\underline{x}]^{\oplus r}$, we define $K(\underline{p}, \underline{q})$ to be the module

$$K(\underline{p}, \underline{q}) = \bigwedge \Big(\bigoplus_{i=1}^{r} k[\underline{x}] \cdot \theta_i \Big). \tag{2.2.39}$$

The space of $k[\underline{x}]$ module maps $K(\underline{p}, \underline{q}) \to K(\underline{p}, \underline{q})$ is a $k[\underline{x}]$ algebra generated by $\{\theta_i, \theta_i^*\}$ with the relations $\theta_i^2 = (\theta_i^*)^2 = 0, \ \theta_i \theta_j^* + \theta_i^* \theta_j = \delta_{ij}$. Then the module map

$$d_{K(\underline{p}, \underline{q})} = \sum_{i=1}^{r} (p_i \cdot \theta_i + q_i \cdot \theta_i^*) \tag{2.2.40}$$

clearly satisfies to $(d_{K(\underline{p}, \underline{q})})^2 = f \cdot 1_{K(\underline{p}, \underline{q})}$ and the Koszul factorisation can be written as

$$[\underline{p}, \underline{q}] = \big(K(\underline{p}, \underline{q}), d_{K(\underline{p}, \underline{q})} \big). \tag{2.2.41}$$

This form has the advantage that taking tensor products over $k[\underline{x}]$ amounts to simply summing the differentials of the individual factorisations.

Graded factorisations

For the constructions of chapter 3, we need to generalise to the case of graded matrix factorisations. We first quickly review some basic facts about graded rings and modules.

We consider graded polynomial rings $k[\underline{x}]$ with degrees in some Abelian group A ($A = \mathbb{Z}$ or $\mathbb{Q}$ or Cartesian products thereof in our cases of interest) assigned to all x_i:

$$\deg(x_i) \in A. \tag{2.2.42}$$

The A-degrees of ring elements are compatible with ring multiplication:

$$\deg(r_1 r_2) = \deg(r_1) + \deg(r_2). \tag{2.2.43}$$

Furthermore, we consider polynomials $f \in k[\underline{x}]$ that are homogeneous in x_i and thus have fixed degree

$$\deg(f) = d. \tag{2.2.44}$$

We use graded free $k[\underline{x}]$ modules X to construct graded matrix factorisations. A-gradings of elements $a \in X$ are compatible with $k[\underline{x}]$ multiplication:

$$\deg(r \cdot a) = \deg(r) + \deg(a). \tag{2.2.45}$$

We denote the degree r part of an A-graded module as X_r, so that $X = \bigoplus_r X_r$. Recall that we consider free $k[\underline{x}]$ modules X, so that X is a direct sum of copies isomorphic to $k[\underline{x}]$. We can then define the A-degree of every rank-1 module as the A-degree of $1 \in k[\underline{x}]$ viewed as an element of X.

If a module map $\phi : X \to Y,\ a \mapsto \phi(a)$ satisfies $\deg(\phi(a)) - \deg(a) = d_\phi$ for all $a \in X$, we say that ϕ has degree $\deg(\phi) = d_\phi$. Note that this is *not* a morphism in the category of graded modules, which is required to preserve the gradings.

Finally, for an A-graded module $X = \bigoplus_r X_r$, we define the A-*shifted module* $X\{m\}$ as

$$\big(X\{m\}\big)_r = X_{m+r}. \tag{2.2.46}$$

The tensor product of graded modules $X = \bigoplus_r X_r,\ Y = \bigoplus_q Y_q$ has the decomposition $X \otimes Y = \bigoplus (X \otimes Y)_r$ with

$$(X \otimes Y)_r = \bigoplus_q X_{r-q} \otimes Y_q. \tag{2.2.47}$$

It then follows that the shift of tensor product of graded modules

$$\begin{aligned}\big((X \otimes Y)\{m\}\big)_r &= \big(\bigoplus_{r'} (\bigoplus_q X_{r'-q} \otimes Y_q)\{m\}\big)_r \\ &= \bigoplus_q X_{r+m-q} \otimes Y_q \\ &= \bigoplus_q X_q \otimes Y_{r+m-q}\end{aligned} \tag{2.2.48}$$

has the distributive property

$$(X \otimes Y)\{m\} = (X\{m\}) \otimes Y = X \otimes (Y\{m\}). \tag{2.2.49}$$

Definition 2.17: Let $k[\underline{x}]$ be an A-graded polynomial ring and $f \in k[\underline{x}]$ with A-grading $2d_f$. An *A-graded matrix factorisation* of f is a matrix factorisation of f whose underlying module is an A-graded $k[\underline{x}]$ module and whose differential has degree d_f.

Definition 2.18: The A-*shift* $\{m\}$ of a matrix factorisation (X, d_X) is

$$(X, d_X)\{m\} = (X\{m\}, d_X)\,. \tag{2.2.50}$$

Applying this definition to the tensor product factorisation, together with (2.2.49)

$$\big((X,d_X)\otimes(Y,d_Y)\big)\{m\} = \big((X,d_X)\{m\}\big)\otimes(Y,d_Y) = (X,d_X)\otimes\big((Y,d_Y)\{m\}\big) \quad (2.2.51)$$

The A-shift $\{m\}$ of a matrix factorisation commutes with the ($\mathbb{Z}_2$-) shift [1] defined in 2.2.31. We can thus combine the shifts in a $\mathbb{Z}_2\times A$ grading as

$$(X,d_X)\{(m,a)\} = (X[m]\{a\},(-1)^m d_X), \quad m\in\mathbb{Z}_2, a\in A. \quad (2.2.52)$$

Example 2.19: A Koszul module as defined in 2.2.39 for $r=1$ is

$$K = k[\underline{x}]\cdot 1\oplus k[\underline{x}]\cdot\theta = K_0\oplus K_1. \quad (2.2.53)$$

Then we have inclusions $k[\underline{x}]\hookrightarrow K_0, 1\mapsto 1\cdot 1$ and $k[\underline{x}]\hookrightarrow K_1, 1\mapsto 1\cdot\theta$ and we define the A-degree of K_0 as the A-degree of $1\cdot 1\in K_0$ and the A-degree of K_1 as the A-degree of $1\cdot\theta\in K_1$. Note that the module subscripts here denote only the initial $\mathbb{Z}_2$ grading of the factorisation. Let us fix the degree of K_0 to 0 and define the degree of K_1 as $\deg(\theta)$. The polynomial $f=pq$ has degree

$$\deg(f) = \deg(p)+\deg(q) =: 2d_f \quad (2.2.54)$$

The differential of this matrix factorisation can be written as

$$d = p\cdot\theta + q\cdot\theta^*, \quad (2.2.55)$$

and is homogeneous with respect to its A-degree with

$$\deg(d) = \frac{\deg(f)}{2} \quad (2.2.56)$$

which implies

$$\deg(\theta) = d_f - \deg(p). \quad (2.2.57)$$

The graded Koszul factorisation $K(p,q)$ thus has the form

$$K(p,q) = k[\underline{x}]\oplus k[\underline{x}]\{d_f-\deg(p)\}, \quad (2.2.58)$$

which means that the gradings of the summands are determined by the degree of f and the polynomial p.

Let us now relate the Koszul factorisation $[q,p]$ to the factorisation $[p,q]$. The differential of $[q,p]$ is $d = q\cdot\theta + p\cdot\theta^*$ and the underlying module decomposes as

$$\begin{aligned} K(q,p) &= k[\underline{x}]\oplus k[\underline{x}]\{d_f-\deg(q)\} \\ &= (k[\underline{x}]\{\deg(p)-d_f\}\oplus k[\underline{x}])[1] \\ &= K(p,q)[1]\{d_f-\deg(p)\} \end{aligned} \quad (2.2.59)$$

where we used (2.2.54). Therefore, using the bigrading (2.2.52), the relation between matrix factorisations reads

$$[q,p] \cong [p,q]\{(1, d_f - \deg(p))\} \tag{2.2.60}$$

which directly generalises to higher rank Koszul factorisations by the property (2.2.51):

$$[\underline{q},\underline{p}] \cong [\underline{p},\underline{q}]\{\sum_{i=1}^{r}(1, d_f - \deg(p_i))\} =: [\underline{p},\underline{q}]\{-\Phi_{\underline{p}}\} \tag{2.2.61}$$

where we defined the bidegree $\Phi_{\underline{p}} \in \mathbb{Z}_2 \times A$

$$\Phi_{\underline{p}} := \sum_i \left(1, \deg(p_i) - \frac{\deg(f)}{2}\right) \tag{2.2.62}$$

Properties of Koszul factorisations

By the definitions of Koszul factorisations and their shifts, one can derive a set of particularly useful properties. Here we simply list them and their proofs can be found in [7, App. A]:

$$[\underline{q},\underline{p}] \cong [\underline{p},\underline{q}]\{-\Phi_{\underline{p}}\}, \tag{2.2.63}$$
$$[z\underline{p}, z^{-1}\underline{q}] \cong [\underline{p},\underline{q}],\ z \in \mathbb{C}^\times, \tag{2.2.64}$$
$$[\underline{p},\underline{q}] \otimes [\underline{p}',\underline{q}'] \cong [\underline{p}+\underline{p}',\underline{q}] \otimes [\underline{p}',\underline{q}'-\underline{q}], \tag{2.2.65}$$
$$[\underline{b}-\underline{a},\underline{p}] \otimes P(\underline{a},\underline{b},\underline{x}) \cong P(\underline{a},\underline{a},\underline{x}), \tag{2.2.66}$$

where, for the last property, $\underline{p}$ is a list of polynomials in $\mathbb{C}[\underline{a},\underline{b},\underline{x}]$ and $P(\underline{a},\underline{b},\underline{x})$ is a matrix factorisation of $W(\underline{a},\underline{b},\underline{x})$ such that the sum $W(\underline{a},\underline{b},\underline{x}) + (\underline{b}-\underline{a})\cdot\underline{p}$ does not depend on $\underline{b}$. We call then the $\underline{b}$ list *internal* and property (2.2.66) can be thought as eliminating the internal list $\underline{b}$.

2.2.3 Bicategory of Landau-Ginzburg models

We conclude this section with a review of the bicategory $\mathcal{LG}_k$ of Landau-Ginzburg models, a fundamental component of the category of affine Rozansky-Witten models developed in Section 2.1.3. Most of the statements in this paragraph were developed in [14].

Before we proceed with the description of $\mathcal{LG}_k$, we need to define a specific type of Koszul matrix factorisations which will define unit 1-morphisms. Let $\underline{x},\underline{y}$ be identical copies of a list of variables of length n, a polynomial $W \in k[\underline{x}]$ and let us explicitly denote its dependence on variables $\underline{x}$ by $W(\underline{x}) \equiv W(x_1,\dots,x_n)$. First, we define the *divided difference operator* by

$$\Delta_i W(\tbinom{x}{y}) := \frac{W(y_1,\dots,y_{i-1},x_i,x_{i+1},\dots,x_n) - W(y_1,\dots,y_{i-1},y_i,x_{i+1},\dots,x_n)}{x_i - y_i} \tag{2.2.67}$$

(observe that $x_i = y_i$ is a root of the numerator, so the expression is well-defined as a polynomial) and we denote the list of these polynomials as $\underline{\Delta}W(\binom{x}{y}) \coloneqq \{\Delta_i W(\binom{x}{y})\}_{i=1,\dots,n}$. Secondly, we observe that

$$(\underline{x} - \underline{y}) \cdot \underline{\Delta}W(\binom{x}{y}) \equiv \sum_{i=1}^{n} (x_i - y_i)\Delta_i W(\binom{x}{y}) = W(\underline{x}) - W(\underline{y}). \tag{2.2.68}$$

Thus, for any $W \in k[\underline{x}]$, we can define a Koszul factorisation 2.16 $1_W \in \mathrm{hmf}(k[\underline{x}, \underline{y}], W(\underline{x}) - W(\underline{y}))^\omega$:

$$1_W = [\underline{x} - \underline{y}, \underline{\Delta}W(\binom{x}{y})], \tag{2.2.69}$$

which effectively "renames" the variable list $\underline{y}$ to its copy $\underline{x}$. This factorisation is precisely the unit 1-morphism we will use in $\mathcal{LG}_k$.

Since we will usually restrict our attention to 1-morphisms in this category, disregarding 2-morphisms, we provide only a brief overview of the definitions of 2-morphisms, which is Prop. 2.7 of [14]:

Proposition 2.20: The following data constitutes a bicategory, called the *bicategory of Landau-Ginzburg models* $\mathcal{LG}_k$ over the ring k

- *Objects*: Pairs $(\underline{x}, W)$ with $\underline{x} \equiv \{x_1, \dots, x_n\}$ an ordered set of variables and $W \in k[\underline{x}]$ a potential[2].
- *Category of 1-morphisms* $(\underline{x}, W) \longrightarrow (\underline{y}, V)$: The category $\mathrm{hmf}(k[\underline{x}, \underline{y}], V - W)^\omega$.
- *Horizontal composition*: The tensor product over intermediate polynomial ring $k[\underline{y}]$ defined in 2.13.
- *Identity 1-morphism and unitors*: Given by the Koszul factorisation (2.2.69). The unitors are defined in equation 2.17 of [14] and their inverses in 4.14, 4.16 of the same paper.
- *Associator*: Given by the natural isomorphism $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$, with tensor products over the appropriate intermediate variables.

Remark 2.21: The physical intuition behind this category is that we treat bulk Landau-Ginzburg models, described by superpotentials, as objects, interfaces preserving B-type supersymmetry as 1-morphisms, described by matrix factorisations of differences of superpotentials and local operators or point defects as "defect changing operators", described by morphisms of matrix factorisations. This is a rigorous definition which agrees with the generic picture of a defect TQFT described informally at [33].

[2]This is a technical condition ensuring that the critical locus of W defines a variety with well-behaved algebraic and geometric properties, including the "physical" assumption that the critical points are isolated, cf. [14, Def. 2.4] for a precise definition.

2.2.4 Symmetric monoidal bicategories

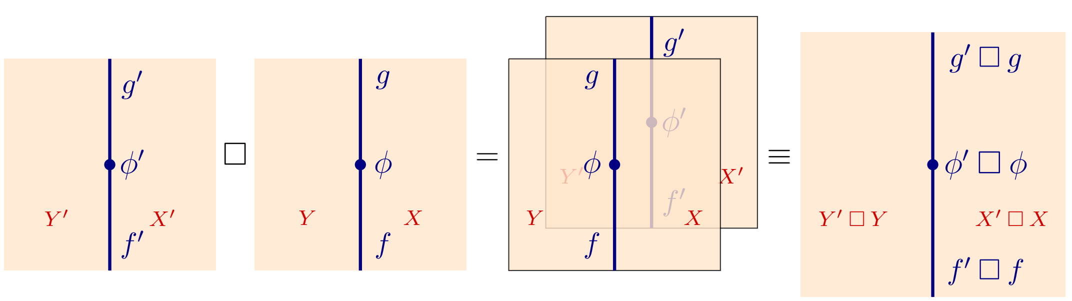
Definition

Another layer of structure that is important for the statements of cobordism hypothesis is the symmetric monoidal structure. We go quickly through an (incomplete) list of data that constitute the definition of a symmetric monoidal bicategory, whose rigorous statement can be found in [50, Sect. 2.3]. Let $\mathcal{B}$ be a bicategory. The following data equip $\mathcal{B}$ with a symmetric monoidal structure:

- A monoidal product $\square : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$
- A monoidal unit $1 \in \mathcal{B}$ together with left and right 1-unitors $1 \square X \longrightarrow X$, $X \square 1 \longrightarrow X$ respectively. Unity properties like (2.2.6) (with respect to the monoidal product $\square$) are relaxed from strict equalities (commutative diagrams) to appropriate invertible modifications.
- An associator $a_{X,Y,Z} : (X \square Y) \square Z \to X \square (Y \square Z)$, whose pentagon axiom (2.2.7) is also upgraded to an invertible modification, called *pentagonator*.
- A brading $b_{X,Y} : X \square Y \longrightarrow Y \square X$. The *symmetry* of the braiding in the usual setup is upgraded to an invertible modification called *syllepsis* σ, while the hexagon axioms (compatibility of braiding with associator) is witnessed by invertible modifications called R and S.
- Compatibility axioms among all pieces of data above observed by appropriate relations between these modifications.

String diagrams

Since symmetric monoidal bicategories are equivalent to some Gray category with duals, we can use the 3-dimensional diagrammatic calculus of [2] to represent relations in symmetric monoidal bicategories. The monoidal product is then represented as "stacking surfaces" in the third dimension:



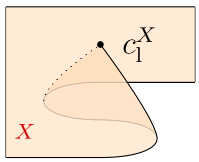
where we read the direction of $\square$ composition from front to back. We will also use black frames for our three-dimensional string diagrams from now on, in order to improve the diagram's legibility.

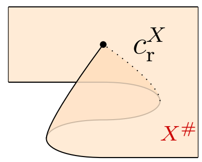
Duals in symmetric monoidal bicategories and full dualisability

Definition 2.22: An object $X \in \mathcal{B}$ is called *right dualisable* if there exists an object $X^\# \in \mathcal{B}$, together with adjunction 1-morphisms

$$\widetilde{\mathrm{ev}}_X : X \,\Box\, X^\# \longrightarrow 1, \quad \widetilde{\mathrm{coev}}_X : 1 \longrightarrow X^\# \,\Box\, X \tag{2.2.70}$$

and *cusp isomorphisms* $c_\mathrm{l}^X, c_\mathrm{r}^X$ which categorify the Zorro moves (2.2.24):



,


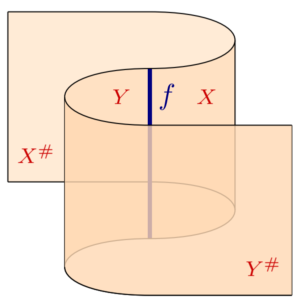
. (2.2.71)

The definition of a left dualisable object is defined in a similar fashion. However, in the context of symmetric monoidal bicategories, a right dual object is also left dual with adjunction 1-morphisms defined using the braiding:

$$\mathrm{ev}_X = \widetilde{\mathrm{ev}}_X \circ b_{X^\#,X}, \quad \mathrm{coev}_X = b_{X,X^\#} \circ \widetilde{\mathrm{coev}}_X \tag{2.2.72}$$

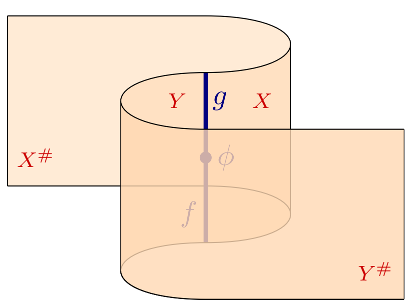
We thus do not distinguish between left and right dual objects and we call $X^\#$ the dual object of X.

The right dual of a 1-morphism $f : X \longrightarrow Y$ is defined as (recall (2.2.25))

$f^\# =$


(2.2.73)

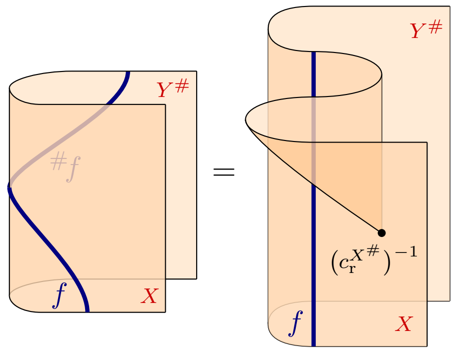
and the right dual $\phi^\#$ of a 2-morphism $\phi : f \longrightarrow g$ with $f, g : X \longrightarrow Y$

$\phi^\# =$


. (2.2.74)

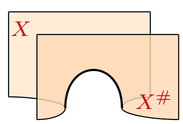
One can similarly define the left dual 1-morphism $f^\#$ and 2-morphism $\phi^\#$. Furthermore, there are canonical isomorphisms between $f^\#$ and ${}^\#f$ induced by the braiding and relation (2.2.72).

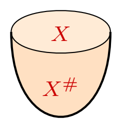
Using the cusp isomorphisms, we can define 2-isomorphisms that map between f and ${}^{\#}f$, which in the string diagram representation "move the 1-morphism between the front and back page". These read:



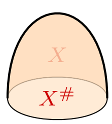
(2.2.75)

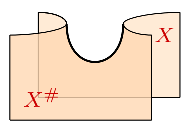
Finally, we call an object X *fully dualisable*, if it is dualisable and its adjunction morphisms $\widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X$ have both left and right adjoints. For a fully dualisable object $X \in \mathcal{B}$, we can then pick the adjunction 2-morphisms of adjunction 1-morphisms and denote them as as cups, caps, saddles and inverted saddles:

$\mathrm{ev}_{\widetilde{\mathrm{ev}}_u} =$ 

, $\mathrm{coev}_{\widetilde{\mathrm{ev}}_u} =$ 

, (2.2.76)

$\widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_u} =$ 

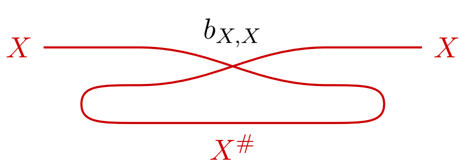
, $\widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_u} =$ 

. (2.2.77)

Some technical details

- According to the original definition of full dualisability, *all* 1-morphisms admit both adjoints. But this implies that we have to verify an infinite tower of adjunction data. However, [44, Thm. 3.9] determines a minimal set of conditions for full dualisability. In particular, it is enough to consider the *Serre automorphism*

$$S_X = (1_X \,\square\, \widetilde{\mathrm{ev}}_X) \otimes (b_{X,X} \,\square\, 1_{X^{\#}}) \otimes (1_X \,\square\, \widetilde{\mathrm{ev}}_X^{\dagger})$$

$=$ 

. (2.2.78)

Then all adjoints of $\widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X$ can be computed from $\widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X$ themselves, the braiding and the Serre automorphism.

- If there is an isomorphism $\lambda_X : S_X \xrightarrow{\cong} 1_X$, the Serre isomorphism is called *trivialisable*. In this case, we can compute an isomorphism between left/right adjoints of $\widetilde{\mathrm{ev}}_X$ and coev_X and left/right adjoints of $\widetilde{\mathrm{coev}}_X$ and ev_X. This effectively implies that we can "glue" cups with caps and saddles with inverted saddles using these isomorphisms.

- Let X be a dualisable object. Then the object together with its dual, the adjunction maps and the cusps form the *duality data*

$$(X, X^{\#}, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X, c_{\mathrm{l}}^X, c_{\mathrm{r}}^X). \tag{2.2.79}$$

If X is fully dualisable, then the duality data, together with the adjunction 2-morphisms of its adjunction 1-morphisms, the Serre automorphism and the 2-isomorphisms

$$\phi : S_X^{-1} \otimes S_X \longrightarrow 1_X, \quad \psi : S_X \otimes S_X^{-1} \longrightarrow 1_X \tag{2.2.80}$$

form the *full duality data*

$$(X, X^{\#}, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X, c_{\mathrm{l}}^X, c_{\mathrm{r}}^X, S_X, S_X^{-1}, \mathrm{ev}_{\widetilde{\mathrm{ev}}_X}, \mathrm{ev}_{\widetilde{\mathrm{coev}}_X}, \mathrm{coev}_{\widetilde{\mathrm{ev}}_X}, \mathrm{coev}_{\widetilde{\mathrm{coev}}_X}, \phi, \psi). \tag{2.2.81}$$

2.2.5 Extended TQFTs and cobordism hypothesis

Cobordism hypothesis recap

The cobordism hypothesis of Baez and Dolan [1] classifies extended n-dimensional TQFTs (see review in A.1) with target a symmetric monoidal n-category $\mathcal{B}$. The statement of the cobordism hypothesis was formalised in the language of (∞, n)-categories by Lurie [40]. Since in this text we focus on two dimensions and work within the framework of bicategories, we give a sketch of Lurie's statement. Specifically, we intentionally use vague terminology, such as "categories", "functors" and so on, without specifying whether they are strict, higher or (∞, n)-categorical.

Theorem 2.23 (Cobordism Hypothesis, framed version)**:** The category of extended n-dimensional TQFTs is equivalent to the category of fully dualisable objects.

In particular, one direction of the equivalence is given by the evaluation of the TQFT on a point. This statement essentially means that *an extended TQFT is fully determined by its value on points.*

Let us clarify some of the ingredients:

- The category of TQFTs is formally the category of symmetric monoidal functors from the appropriate extended bordism category (with framing) to $\mathcal{B}$.

- The category of fully dualisable objects $\mathcal{B}^{\mathrm{fd}}$ is the subcategory of $\mathcal{B}$ consisting of only the fully dualisable objects. Furthermore, we consider the maximal sub-groupoid $(\mathcal{B}^{\mathrm{fd}})^{\times}$, meaning the subcategory whose k-morphisms are invertible, for all $1 \leq k \leq n$.

In this version of the statement we consider *framed* TQFTs, namely is equipped with a framing structure. If we wish to equip the bordism category with other tangential structures (such as orientation or spin), then the appropriate version of the cobordism hypothesis involves passing from $\left(\mathcal{B}^{\mathrm{fd}}\right)^{\times}$ to homotopy fixed points of some group action on $\left(\mathcal{B}^{\mathrm{fd}}\right)^{\times}$. This is explained in detail in [40, Sect. 2.4] and reviewed in [49, Sect. 8]. In particular, in the oriented case, which is the case of interest for us, we consider homotopy fixed points of $\mathrm{SO}(n)$ action.

In two dimensions the statement obtains a particularly convenient form in the language of symmetric monoidal bicategories due to [27, 28] and [44]. In particular, considering $\left(\mathcal{B}^{\mathrm{fd}}\right)^{\times}$ is equivalent to the category of coherent full dualisability data. The coherence conditions (swallowtail and cusp-counit) are spelled out in [44] and reviewed also in the string diagram formalism in [5, Sect 2.3]. On the other hand, passing to $\mathrm{SO}(2)$ fixed point amounts to picking a trivialisation of the Serre automorphism (2.2.78) [27, 28]. The 2-dimensional oriented cobordism hypothesis can then be stated as follows:
An extended TQFT $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{B}$ is equivalently described by a pair (X, λ_X) of a fully dualisable object $X \in \mathcal{B}$ together with chosen coherent full duality data (2.2.81) and a 2-isomorphism $\lambda_X\colon S_X \longrightarrow 1_X$. Evaluating $\mathcal{Z}$ on any 2-morphisms $[\Sigma]$ in $\mathrm{Bord}^{\mathrm{or}}_{2,1,0}$ then amounts to

- choosing a representative bordism Σ,
- choosing a generic embedding of Σ into the cube $[0,1]^{\times 3}$ compatible with the graphical calculus of $\mathrm{Bord}^{\mathrm{or}}_{2,1,0}$, and
- interpreting the resulting diagram in the graphical calculus for $\mathcal{B}$.

Cobordism hypothesis with defects

We fix defect data $\mathbb{D}$ and a 2-dimensional extended defect TQFT $\mathcal{Z}$ (see review in A.2) with values in a symmetric monoidal bicategory $\mathcal{B}$. The cobordism hypothesis with defects aims to describe the action of $\mathcal{Z}$ with data internal to $\mathcal{B}$. Since the bordism category with defects with only one object X and trivial stratification is equivalent to the oriented bordism category (A.2.2), the ordinary cobordism hypothesis implies that it is described by a fully dualisable object $\mathcal{Z}(X)$. It remains then to describe the action of $\mathcal{Z}$ on nontrivially stratified bordisms. An answer was put forward in the context of (∞, n)-categories in [40, Sect. 4.3] as the "cobordism hypothesis with singularities", which restricts to the standard cobordism hypothesis in the case of trivial stratifications. Lurie also provides strong evidence for the validity of the cobordism hypothesis with singularities/defects. To our knowledge a detailed, complete proof has however not been published – neither for arbitrary (∞, n)-categories nor in a setting of (weak) 2-categories relevant for our purposes here.

Absent an established theorem, we base our applications in subsequent sections on the assumption that the following version of the cobordism hypothesis with singularities holds:
An extended defect TQFT $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D}) \longrightarrow \mathcal{B}$ is equivalently described by

(0) pairs $(\mathcal{Z}(X), \lambda_X)$ of fully dualisable objects $\mathcal{Z}(X) \in \mathcal{B}$ together with chosen coherent full duality data as in the previous paragraph and 2-isomorphisms $\lambda_X \colon S_{\mathcal{Z}(X)} \longrightarrow 1_{\mathcal{Z}(X)}$ for all $X \in D_2$,

(1) 1-morphisms $\mathcal{Z}(f) \colon \mathcal{Z}(s(f)) \longrightarrow \mathcal{Z}(t(f))$ for all $f \in D_1$ that have coherently isomorphic left and right adjoints ${}^\dagger\mathcal{Z}(f) \cong \mathcal{Z}(f)^\dagger$ with chosen adjunction data,

(2) 2-morphisms $\mathcal{Z}(\phi) \colon 1_{\mathcal{Z}(s(f_1))} \longrightarrow \bigotimes_{i=1}^m \mathcal{Z}(f_i)^{\varepsilon_i}$ (where $Z^+ := Z$ and $Z^- := Z^\dagger$) if $j(\phi) = [(f_1, \varepsilon_1), \dots, (f_m, \varepsilon_m)]$, and $\mathcal{Z}(\phi) \in \mathrm{End}_{\mathcal{B}}(1_{\mathcal{Z}(j(\phi))})$ if $j(\phi) \in D_2$, for all $\phi \in D_0$.

Evaluating $\mathcal{Z}$ on any 2-morphisms $[\Sigma]$ in $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$ then amounts to

- choosing a representative defect bordism Σ,
- choosing a generic embedding of Σ into the cube $[0,1]^{\times 3}$ compatible with the graphical calculus of $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$,
- replacing every label $\xi \in D_j$ of j-strata by $\mathcal{Z}(\xi)$ for all $j \in \{0,1,2\}$, and
- interpreting the resulting diagram in the graphical calculus for $\mathcal{B}$.

By restricting this procedure to boundaries one also obtains the action of $\mathcal{Z}$ on objects and 1-morphisms.

In short, the 2-dimensional cobordism hypothesis with defects precisely instructs us to interpret defect bordisms as 2-morphisms in the target $\mathcal{B}$, and analogously in higher dimensions. That their diagrammatic evaluation is independent of the choices made is a non-trivial statement.

Chapter 3

Symmetries and defects

Symmetry defects encode symmetries of the bulk theories. We introduce natural trivalent junctions for these symmetry defects which can be combined into symmetry defect networks. Inserting such networks corresponds to turning on non-trivial flat background gauge fields. We then compute state spaces associated with surfaces decorated by such networks. In particular, for closed genus-g surfaces we obtain (3.2.20). We also show that the spate spaces are invariant under local changes of the network corresponding to Pachner moves between triangulations. Finally, we compute the category of line operators in the twisted sectors of the theory, namely line operators exhibiting non-trivial holonomy on cycles around them.

3.1 Symmetry defects in affine Rozansky-Witten models

3.1.1 Category of affine Rozansky-Witten models

In this section, we study the 3-category $\mathcal{RW}^{\text{aff}}$ describing affine Rozansky-Witten models, that is, we restrict to models with target spaces $T^*\mathbb{C}^n$. This 3-category was first described in [35, 36] and later reviewed and supplemented with a symmetric monoidal structure and gradings in [5, 7].

Here we provide a schematic description of $\mathcal{RW}^{\text{aff}}$, without giving a fully rigorous definition of all data of a 3-category. The axioms of a weak 3-category are too elaborate present in detail within the scope of this thesis and much of the 3-categorical structure will in any case be disregarded when we truncate to two dimensions later on.

- *Objects* in this category are ordered sets of variables $(x_1, \ldots, x_n)$ of length $n \in \mathbb{Z}_{\geq 0}$. We will often omit the length of the list n and write

$$\underline{x} \coloneqq (x_1, \ldots, x_n). \tag{3.1.1}$$

  We will also call them *bulk variables*.

- *1-morphisms*: For objects $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_m)$, a 1-morphism $\underline{x} \to \underline{y}$ is given by another list of variables $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, together with a polynomial $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$. We will write these 1-morphisms as $(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})) \equiv (\underline{\alpha}; W)$.

  Given a triple of objects $\underline{x}, \underline{y}, \underline{z}$ and 1-morphisms $\big(\underline{\alpha}; W\big) : \underline{x} \to \underline{y}$, $\big(\underline{\beta}; V\big) : \underline{y} \to \underline{z}$, their horizontal composition is given by

$$\big(\underline{\beta}; V\big) \circ \big(\underline{\alpha}; W\big) = (\underline{\alpha}, \underline{y}, \underline{\beta}; W(\underline{a}, \underline{x}, \underline{y}) + V(\underline{b}, \underline{y}, \underline{z})). \tag{3.1.2}$$

  We will call the $\underline{\alpha}$ variables *surface variables.*

- For $\underline{x}, \underline{y}$ lists of length n the *unit 1-morphism* is (recall notation from (2.2.36))

$$1_{\underline{x}} = \big(\underline{\alpha}; \underline{\alpha} \cdot (\underline{y} - \underline{x})\big) : \underline{x} \longrightarrow \underline{y}, \tag{3.1.3}$$

  where $\underline{\alpha}$ is a list of surface variables of length n and treating $\underline{x}, \underline{y}$ as copies of the same object in $\mathcal{RW}^{\text{aff}}$.

- *2-morphisms*: Given 1-morphisms $(\underline{\alpha}_1; W_1), (\underline{\alpha}_2; W_2) : \underline{x} \to \underline{y}$, a 2-morphism

$$(\underline{\alpha}_1; W_1) \longrightarrow (\underline{\alpha}_2; W_2) \tag{3.1.4}$$

  is given by a matrix factorisation (see section 2.2.2) $(X, d_X) \in \text{hmf}(\mathbb{C}[\underline{\alpha}_1, \underline{\alpha}_2, \underline{x}, \underline{y}], W_2 - W_1)^{\omega}$.

  Vertical composition of composable 2-morphisms is given by the tensor product matrix factorization over the intermediate *surface* variable, as defined in 2.13. Horizontal composition of is given by tensor product factorisation over the intermediate *bulk* variable.

- *3-morphisms*: Given by morphisms of matrix factorizations.

Similar to the discussion 2.21, the definition of this 3-category follows the physical intuition of [33]. Objects in $\mathcal{RW}^{\text{aff}}$ are lists of variables (3.1.1). These variables correspond physically to the free bulk hypermultiplets in affine Rozansky-Witten models.

1-morphisms in $\mathcal{C}$ are physically interpreted as interfaces (surfaces) between Rozansky-Witten models. These surfaces support a supersymmetric Landau-Ginzburg theory, described by additional variables $\underline{\alpha}$, together with the restrictions of the bulk variables $\underline{x}, \underline{y}$ and a superpotential W. Note in formula (3.1.2) that the bulk variables $\underline{y}$ become surface variables in the composition 1-morphism. This agrees with the physical picture of composition of surfaces as 1-morphisms between bulk 3-dimensional theories: the bulk variables are "squeezed" between the fused surfaces and they become surface variables as well.

1-morphisms in $\mathcal{RW}^{\text{aff}}$ have duals. We postpone the definition of duality data, as well as unitors, to present together with the string diagrams of the truncated 2-category.

In order to keep track of the R- and flavour U(1) symmetries of the model, we equip the polynomial rings and matrix factorisations with gradings valued in the Abelian group $\mathbb{Q} \times \mathbb{Q}$, where the first factor corresponds to the R-symmetry and the second factor to the flavour symmetry. Preserving the model's symmetries requires assigning bidegrees $(1,-1)$ to the bulk variables x_i, which correspond to free hypermultiplets or, equivalently, the base coordinates of $T^*\mathbb{C}^n$ (cf. [7, App. B3]). Furthermore, the standard Landau-Ginzburg R-symmetry preservation condition (see for example [29, Sect. 13.2]) demands that the superpotential carries bidegree $(2,0)$.

We combine $\mathbb{Z}_2$ with $\mathbb{Q}\times\mathbb{Q}$ matrix factorisation grading in a $\mathbb{Z}_2\times\mathbb{Q}\times\mathbb{Q}$ grading, defining a total shift functor as in (2.2.52):

$$(X,d_X)\{(m,r,q)\} = (X[m]\{(r,q)\},(-1)^m d_X), \quad m\in\mathbb{Z}_2, r,q\in\mathbb{Q}. \tag{3.1.5}$$

For manipulation of Koszul factorisations, it is particularly convenient to define a shift (recall (2.2.62)) related to a list of variables $\underline{u}=(u_1,\dots,u_n)$ of length n with bidegree (r_{u_i},q_{u_i}):

$$\Phi_{\underline{u}} = \sum_{i=1}^{n}(1,r_{u_i}-1,q_{u_i}). \tag{3.1.6}$$

In particular, for lists of bulk variables of length n, this evaluates to

$$\Phi_{\underline{x}} = (n,0,-n). \tag{3.1.7}$$

Remark 3.1: The 1-morphism categories $\underline{x}\longrightarrow\underline{y}$ are essentially the bicategory $\mathcal{LG}_k$ of [14] with a few subtle differences.

- The role of the base ring k here is played by $\mathbb{C}[\underline{x},\underline{y}]$, that is, the bulk variables are "observers". More precisely, the 2-morphisms (3.1.4) are elements of $\operatorname{hmf}(k'[\underline{\alpha}_1,\underline{\alpha}_2],W_2-W_1)^\omega$ with $k'=\mathbb{C}[\underline{x},\underline{y}]$, $W_1\in k'[\underline{\alpha}_1], W_2\in k'[\underline{\alpha}_2]$.

- Divided differences of potentials (2.2.67) can now refer to specific list of variables, generalising to
$$\underline{\Delta}f(\cdots,\tbinom{\underline{x}}{\underline{y}},\cdots). \tag{3.1.8}$$
Its use will be evident when we define unitor and adjunction 2-morphisms in the truncated category.

- Tensor products can now be defined over intermediate surface *or* bulk variables. Recalling the discussion 2.14, the technical difference between the two is that the resulting polynomial does not lie in the image of any ring inclusion (otherwise stated, there is no polynomial cancellation and thus all variables are present in the sum of potentials) and there is no implicit postcomposition with restriction of scalars.

- We relax the condition that polynomials W in $\operatorname{hmf}(k[\underline{x}],W)^\omega$ are potentials. In particular, identity 1-morphisms and the 1-morphisms considered in 3.1.2 are generically not potentials. This results in certain technical problems addressed in [51] related to adjunction 3-morphisms. However, since we truncate to 2 dimensions, they are not relevant in the scope of this thesis.

Truncated 2-category

Having described the 3-category $\mathcal{RW}^{\text{aff}}$, one might attempt to apply the cobordism hypothesis to construct a fully extended 3-dimensional TQFT valued in $\mathcal{RW}^{\text{aff}}$. According to the discussion in section 2.2.5, this would require fully dualisable objects in $\mathcal{RW}^{\text{aff}}$. However, computing the duals of adjunction morphisms only works up to dimension 2. In particular, adjunction 2-morphisms of adjunction 1-morphisms of objects fail to admit duals. This is a consequence of the non-compactness of the target $T^*\mathbb{C}^n$, which leads to infinite dimensional state spaces, as discussed in 2.1.3, and hence prevents a definition of a closed 3-dimensional TQFT valued in Vect.

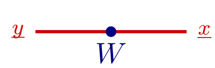
Nevertheless, we can still exploit extendability down to one dimension by truncating the theory and disregarding the correlation functions of non-trivial local operators. This yields a well-defined 2-category with fully dualisable objects, which can be used to define a fully extended 2-dimensional TQFT and allows us to evaluate the state spaces of Rozansky-Witten models.

Definition 3.2: Let $\mathcal{B}$ be a 3-category.The *truncated* bicategory $\mathrm{T}(\mathcal{B})$ has

- the objects in $\mathcal{B}$ as objects,
- the 1-morphisms in $\mathcal{B}$ as 1-morphisms
- and isomorphism classes of 2-morphisms in $\mathcal{B}$ as 2-morphisms.

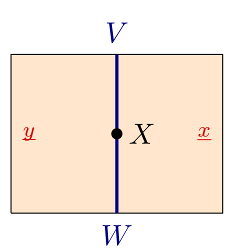
This defines indeed a bicategory according to [48, Prop. 5.2.1]. Thus the truncated category $\mathrm{T}(\mathcal{RW}^{\text{aff}})$ has the same objects and 1-morphisms as $\mathcal{RW}^{\text{aff}}$ and the same 2-morphisms modulo 3-isomorphisms. In practice we "ignore a layer of structure", as we identify isomorphic line defects (in the 3-dimensional picture) and we do not have to keep track of complicated isomorphisms in matrix factorisation categories. For brevity, we denote the truncated category as $\mathcal{C} := \mathrm{T}(\mathcal{RW}^{\text{aff}})$.

Since we are now in the realm of symmetric monoidal bicategories, we can use the graphical calculus of [2]. Our conventions for reading string diagrams are right to left, bottom to top and front to back. A 1-morphism $W := \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big) : \underline{x} \longrightarrow \underline{y}$ is represented by

$W =$ 

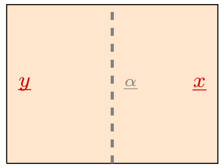
. (3.1.9)

For 1-morphisms $W := \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big), V := \big(\underline{\beta}; V(\underline{\beta}, \underline{x}, \underline{y})\big) : \underline{x} \longrightarrow \underline{y}$, a 2-morphism $(X, d_X) : W \longrightarrow V$ is represented as

$(X, d_X) =$ 

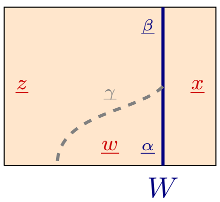
. (3.1.10)

Whenever necessary, we will also label the surface variables $\underline{\alpha}, \underline{\beta}, \ldots$ used in the definition of 1-morphisms. For instance, the unit 1-morphism of (3.1.3) is denoted by[1]

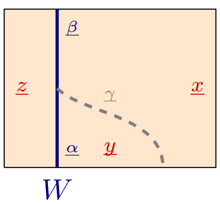


. (3.1.11)

Let $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$ be lists of pairwise same length, thus considered as copies of the same object in $\mathcal{C}$. For any 1-morphism $W = \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{z})\big) : \underline{x} \longrightarrow \underline{z}$ we define the unitor 2-morphisms λ_W, ρ_W:

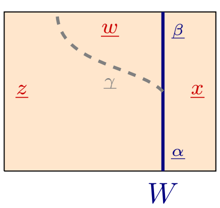
$\lambda_W =$


$$= \left[\underline{z} - \underline{w}, -\underline{\gamma} + \underline{\Delta} W\left(\underline{\alpha}, \underline{x}, \left(\begin{smallmatrix}\underline{z}\\ \underline{w}\end{smallmatrix}\right)\right)\right] \otimes \left[\underline{\beta} - \underline{\alpha}, \underline{\Delta} W\left(\left(\begin{smallmatrix}\underline{\beta}\\ \underline{\alpha}\end{smallmatrix}\right), \underline{x}, \underline{z}\right)\right]\{-\Phi_{\underline{w}}\}, \tag{3.1.12}$$

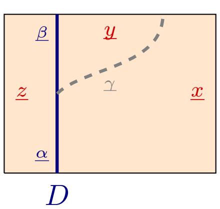
$\rho_W =$


$$= \left[\underline{x} - \underline{y}, \underline{\gamma} + \underline{\Delta} W\left(\underline{\alpha}, \left(\begin{smallmatrix}\underline{x}\\ \underline{y}\end{smallmatrix}\right), \underline{z}\right)\right] \otimes \left[\underline{\beta} - \underline{\alpha}, \underline{\Delta} W\left(\left(\begin{smallmatrix}\underline{\beta}\\ \underline{\alpha}\end{smallmatrix}\right), \underline{x}, \underline{z}\right)\right]\{-\Phi_{\underline{y}}\}. \tag{3.1.13}$$

Their inverses are then

$\lambda_W^{-1} =$


$$= \left[\underline{w} - \underline{z}, -\underline{\gamma} + \underline{\Delta} W\left(\underline{\alpha}, \underline{x}, \left(\begin{smallmatrix}\underline{z}\\ \underline{w}\end{smallmatrix}\right)\right)\right] \otimes \left[\underline{\beta} - \underline{\alpha}, \underline{\Delta} W\left(\left(\begin{smallmatrix}\underline{\beta}\\ \underline{\alpha}\end{smallmatrix}\right), \underline{x}, \underline{w}\right)\right], \tag{3.1.14}$$

$\rho_W^{-1} =$


$$= \left[\underline{y} - \underline{x}, \underline{\gamma} + \underline{\Delta} W\left(\underline{\alpha}, \left(\begin{smallmatrix}\underline{y}\\ \underline{x}\end{smallmatrix}\right), \underline{z}\right)\right] \otimes \left[\underline{\beta} - \underline{\alpha}, \underline{\Delta} W\left(\left(\begin{smallmatrix}\underline{\beta}\\ \underline{\alpha}\end{smallmatrix}\right), \underline{y}, \underline{z}\right)\right]. \tag{3.1.15}$$

Collecting from [7, Sect. 2], [5, Sect. 4], we give a quick overview of the most important statements about $\mathcal{C}$.

[1] Or, more precisely, the unit 2-morphism of the unit 1-morphism, which we identify with the unit 1-morphism.

- The truncated category $\mathcal{C}$ can be equipped with a monoidal structure $\square$ ([7, Prop. 2.1, 2.2]). The action of $\square$ on objects is concatenation of lists: for $\underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_m)$, we have

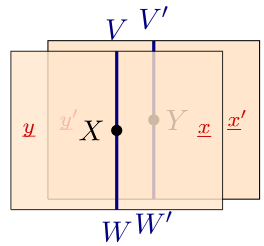
$$\underline{x} \,\square\, \underline{y} = (x_1, \dots, x_n, y_1, \dots, y_m). \tag{3.1.16}$$

It also acts on 1-morphisms as addition of polynomials and on 2-morphisms as tensor product over $\mathbb{C}$.

$$\begin{gathered}\big(\underline{\alpha}'; W'(\underline{\alpha}', \underline{x}', \underline{y}')\big) \,\square\, \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big) = \big(\underline{\alpha}' \,\square\, \underline{\alpha}; W'(\underline{\alpha}', \underline{x}', \underline{y}') + W(\underline{\alpha}, \underline{x}, \underline{y})\big) \\ (Y, d_Y) \,\square\, (X, d_X) = (Y, d_Y) \otimes_{\mathbb{C}} (X, d_X)\end{gathered} \tag{3.1.17}$$

The monoidal product of 2-morphisms is represented in graphical calculus as

$$(Y, d_Y) \,\square\, (X, d_X) = \tag{3.1.18}$$



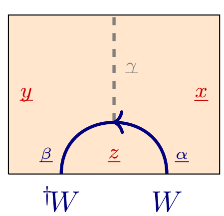
The monoidal product, together with appropriate structure morphisms, endows $\mathcal{C}$ with a symmetric monoidal structure.

- 1-morphisms in $\mathcal{C}$ have adjoints. Let $W = \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big) : \underline{x} \longrightarrow \underline{y}$. We define the left and right adjoints:

$$W^\dagger = {}^\dagger W := \big(\underline{\alpha}; -W(\underline{\alpha}, \underline{x}, \underline{y})\big) : \underline{y} \longrightarrow \underline{x} \tag{3.1.19}$$

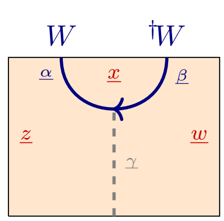
The adjunction 2-morphisms corresponding to the left adjoint are

$\mathrm{ev}_W =$



$$= \left[\underline{y} - \underline{x}, \underline{\gamma} + \underline{\Delta} W(\underline{\beta}, \left(\begin{smallmatrix}\underline{y}\\ \underline{x}\end{smallmatrix}\right), \underline{z})\right] \otimes \left[\underline{\beta} - \underline{\alpha}, \underline{\Delta} W(\left(\begin{smallmatrix}\underline{\beta}\\ \underline{\alpha}\end{smallmatrix}\right), \underline{x}, \underline{z})\right]\{-\Phi_{\underline{z}}\}\,, \tag{3.1.20}$$

$\mathrm{coev}_W =$



$$= \left[\underline{z} - \underline{w}, -\underline{\gamma} + \underline{\Delta} W(\underline{\alpha}, \underline{x}, \left(\begin{smallmatrix}\underline{z}\\ \underline{w}\end{smallmatrix}\right))\right] \otimes \left[\underline{\alpha} - \underline{\beta}, \underline{\Delta} W(\left(\begin{smallmatrix}\underline{\alpha}\\ \underline{\beta}\end{smallmatrix}\right), \underline{x}, \underline{w})\right]. \tag{3.1.21}$$

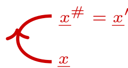
The adjunction 2-morphisms corresponding to the right adjoints are defined as

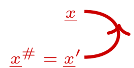
$$\widetilde{\mathrm{ev}}_W := \mathrm{ev}_{W^\dagger}, \quad \widetilde{\mathrm{coev}}_W := \mathrm{coev}_{W^\dagger}, \tag{3.1.22}$$

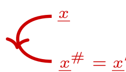
using $W^{\dagger\dagger} = W$. In particular, $\mathcal{C}$ has a pivotal structure.

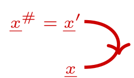
- Objects in $\mathcal{C}$ are self-dual:
$$\underline{x}^{\#} = {}^{\#}\underline{x} = \underline{x} \tag{3.1.23}$$
with adjunction 1-morphisms defined by

$$\begin{aligned}
\mathrm{ev}_{\underline{x}} &= \text{[diagram 1]} := \big(\underline{\alpha};\, \underline{\alpha}(\underline{x} - \underline{x}')\big) : \underline{x}' \,\square\, \underline{x} \longrightarrow \varnothing\,, \\
\mathrm{coev}_{\underline{x}} &= \text{[diagram 2]} := \big(\underline{\alpha};\, \underline{\alpha}(\underline{x} - \underline{x}')\big) : \varnothing \longrightarrow \underline{x} \,\square\, \underline{x}'\,, \\
\widetilde{\mathrm{ev}}_{\underline{x}} &= \text{[diagram 3]} := \big(\underline{\alpha};\, \underline{\alpha}(\underline{x}' - \underline{x})\big) : \underline{x} \,\square\, \underline{x}' \longrightarrow \varnothing\,, \\
\widetilde{\mathrm{coev}}_{\underline{x}} &= \text{[diagram 4]} := \big(\underline{\alpha};\, \underline{\alpha}(\underline{x}' - \underline{x})\big) : \varnothing \longrightarrow \underline{x}' \,\square\, \underline{x}\,.
\end{aligned} \tag{3.1.24}$$



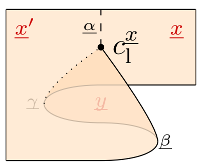




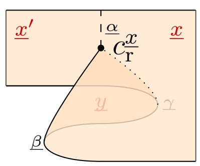


and the cusp 2-isomorphisms

$$c^{\underline{x}}_{\mathrm{l}} = \text{[diagram 5]} = \big[\underline{\alpha} + \underline{\beta}, \underline{x}' - \underline{y}\big] \otimes \big[\underline{y} - \underline{x}, \underline{\alpha} - \underline{\gamma}\big]\{-\Phi_{\underline{x}}\}\,, \tag{3.1.25}$$

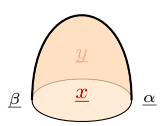


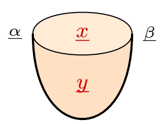
$$c^{\underline{x}}_{\mathrm{r}} = \text{[diagram 6]} = \big[\underline{\alpha} + \underline{\beta}, \underline{y} - \underline{x}\big] \otimes \big[\underline{x}' - \underline{y}, \underline{\alpha} - \underline{\gamma}\big]\{-\Phi_{\underline{x}}\}\,. \tag{3.1.26}$$

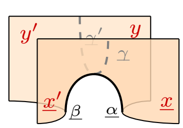


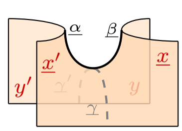
Furthermore, as explained above, all 1-morphisms have adjoints, thus objects in $\mathcal{C}$ are fully dualisable. The adjunction 2-morphisms of adjunction 1-morphisms follow according to (3.1.20), (3.1.21)

$$\begin{aligned}
\mathrm{ev}_{\mathrm{coev}_{\underline{x}}} &= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \text{[diagram 7]} = \big[\underline{\beta} - \underline{\alpha}, \underline{y} - \underline{x}\big]\{-2\Phi_{\underline{x}}\}\,, \\
\mathrm{coev}_{\widetilde{\mathrm{ev}}_{\underline{x}}} &= \widetilde{\mathrm{coev}}_{\mathrm{coev}_{\underline{x}}} = \text{[diagram 8]} = \big[\underline{\alpha} - \underline{\beta}, \underline{x} - \underline{y}\big]\,, \\
\mathrm{ev}_{\mathrm{ev}_{\underline{x}}} &= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{coev}}_{\underline{x}}} = \text{[diagram 9]} = \begin{matrix} & \big[\underline{x}' - \underline{x}, \underline{\gamma} + \underline{\beta}\big] \\ \otimes & \big[\underline{y}' - \underline{y}, \underline{\gamma}' - \underline{\beta}\big] \\ \otimes & \big[\underline{\beta} - \underline{\alpha}, \underline{x} - \underline{y}\big] \end{matrix}\,, \\
\mathrm{coev}_{\widetilde{\mathrm{coev}}_{\underline{x}}} &= \widetilde{\mathrm{coev}}_{\mathrm{ev}_{\underline{x}}} = \text{[diagram 10]} = \begin{matrix} & \big[\underline{x} - \underline{x}', \underline{\gamma} + \underline{\alpha}\big] \\ \otimes & \big[\underline{y} - \underline{y}', \underline{\gamma}' - \underline{\alpha}\big] \\ \otimes & \big[\underline{\alpha} - \underline{\beta}, \underline{y} - \underline{x}\big] \end{matrix}\,.
\end{aligned} \tag{3.1.27}$$









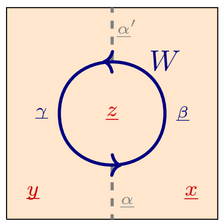
Therefore, as proven in [7, Thm. 3.4], we can construct an extended oriented TQFT whose target is $\mathcal{C}$. According to [7, Prop. 3.5], this TQFT assigns to genus g surfaces Σ_g the state spaces

$$\mathcal{Z}(\Sigma_g) = \big((\mathbb{C} \oplus \mathbb{C}[1]) \otimes (\mathbb{C} \oplus \mathbb{C}[1])\big)^{\otimes ng} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}, \underline{\alpha}] \,. \tag{3.1.28}$$

Useful formulas

For later use, we mention a few useful formulas proven in [5, Sect. 4]. For a 1-morphism $W = \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{z})\big) : \underline{x} \longrightarrow \underline{z}$ and a 2-morphism $\phi : W \longrightarrow W$, the *left quantum dimension* of W is

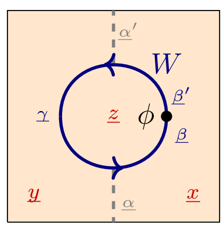
$$\dim_{\mathrm{l}}(W) = \mathrm{ev}_W \cdot \mathrm{coev}_{W^\dagger} =$$



$$= \big[\underline{\alpha} - \underline{\alpha}', \underline{x} - \underline{y}\big] \otimes \big[0, \partial_{\underline{\gamma}} W(\underline{\gamma}, \underline{x}, \underline{z})\big] \otimes \big[0, \underline{\alpha}' + \underline{\Delta} W(\underline{\gamma}, \big(\begin{smallmatrix}\underline{y} \\ \underline{x}\end{smallmatrix}\big), \underline{z})\big] \{\Phi_{\underline{x}} - \Phi_{\underline{z}}\} \tag{3.1.29}$$

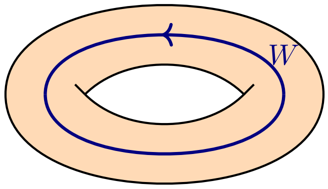
and the *left trace* of ϕ is

$$\mathrm{tr}_{\mathrm{l}}(\phi) = \mathrm{ev}_W \cdot (1_W \otimes \phi) \cdot \mathrm{coev}_{W^\dagger} = \tag{3.1.30}$$



$$= \phi(\beta, \beta, \underline{x}, \underline{z}) \otimes \big[\underline{\alpha}' - \underline{\alpha}, \underline{y} - \underline{x}\big] \otimes \big[0, \underline{\alpha}' + \underline{\Delta} W(\underline{\beta}, \big(\begin{smallmatrix}\underline{y} \\ \underline{x}\end{smallmatrix}\big), \underline{z})\big] \{\Phi_{\underline{x}} - \Phi_{\underline{z}}\} \,.$$

The decorated torus evaluates to



$$= \big[\underline{0}, \partial_{\underline{\delta}} W(\underline{\delta}, \underline{x}, \underline{x})\big] \otimes \big[\underline{0}, \partial_{\underline{x}} W(\underline{\delta}, \underline{x}, \underline{x})\big] \,, \tag{3.1.31}$$

which for the case of the identity defect (3.1.3) computes

$$\big((\mathbb{C} \oplus \mathbb{C}\{1, 0, -1\}) \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}\{1, 0, 1\})\big)^{\otimes n} \mathbb{C}[\underline{x}, \underline{\delta}] \,, \tag{3.1.32}$$

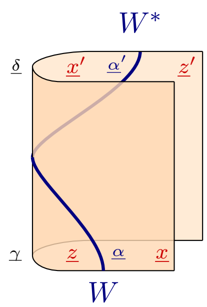
reproducing the partition function of the torus without defects (3.1.28).

For a 1-morphism $W = \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{z})\big)$, we will use a dual that is isomorphic, but not strictly equal to the #-dual (2.2.73):

$$W^* = {}^*W = \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{z})\big), \tag{3.1.33}$$

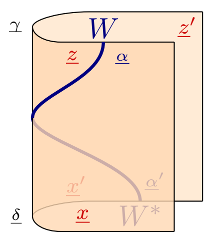
that is, we use the same form of W, but with "reversed orientation". This form of dual greatly simplifies computations, together with other nice properties explained in detail in [5, Sect. 4.2.3]. With this dual, the bending isomorphisms can be computed

$$\widetilde{\Omega}_W =$$



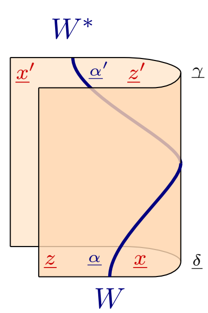
$$= \begin{array}{l} \big[\underline{\alpha}' - \underline{\alpha}, \underline{\Delta} W\big(\big(\begin{smallmatrix}\underline{\alpha}'\\ \underline{\alpha}\end{smallmatrix}\big), \underline{x}, \underline{z}\big)\big]\{-\Phi_{\underline{z}}\} \\ \otimes\ \big[\underline{x}' - \underline{x}, -\underline{\delta} + \underline{\Delta} W\big(\underline{\alpha}', \big(\begin{smallmatrix}\underline{x}'\\ \underline{x}\end{smallmatrix}\big), \underline{z}'\big)\big] \\ \otimes\ \big[\underline{z}' - \underline{z}, \underline{\gamma} + \underline{\Delta} W\big(\underline{\alpha}', \underline{x}, \big(\begin{smallmatrix}\underline{z}'\\ \underline{z}\end{smallmatrix}\big)\big)\big], \end{array} \tag{3.1.34}$$

$$\widetilde{\Omega}_D^{-1} =$$



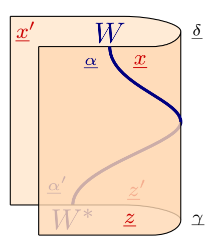
$$= \begin{array}{l} \big[\underline{\alpha} - \underline{\alpha}', \underline{\Delta} W\big(\big(\begin{smallmatrix}\underline{\alpha}'\\ \underline{\alpha}\end{smallmatrix}\big), \underline{x}, \underline{z}\big)\big]\{-\Phi_{\underline{x}}\} \\ \otimes\ \big[\underline{x} - \underline{x}', -\underline{\delta} + \underline{\Delta} W\big(\underline{\alpha}', \big(\begin{smallmatrix}\underline{x}'\\ \underline{x}\end{smallmatrix}\big), \underline{z}'\big)\big] \\ \otimes\ \big[\underline{z} - \underline{z}', \underline{\gamma} + \underline{\Delta} W\big(\underline{\alpha}', \underline{x}, \big(\begin{smallmatrix}\underline{z}'\\ \underline{z}\end{smallmatrix}\big)\big)\big], \end{array} \tag{3.1.35}$$

$$\widetilde{\Omega}'_W =$$



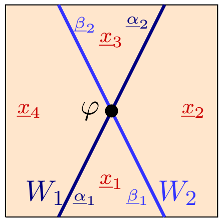
$$= \begin{array}{l} \big[\underline{\alpha}' - \underline{\alpha}, \underline{\Delta} W\big(\big(\begin{smallmatrix}\underline{\alpha}'\\ \underline{\alpha}\end{smallmatrix}\big), \underline{x}, \underline{z}\big)\big]\{-\Phi_{\underline{x}}\} \\ \otimes\ \big[\underline{x}' - \underline{x}, -\underline{\delta} + \underline{\Delta} W\big(\underline{\alpha}', \big(\begin{smallmatrix}\underline{x}'\\ \underline{x}\end{smallmatrix}\big), \underline{z}\big)\big] \\ \otimes\ \big[\underline{z}' - \underline{z}, \underline{\gamma} + \underline{\Delta} W\big(\underline{\alpha}', \underline{x}', \big(\begin{smallmatrix}\underline{z}'\\ \underline{z}\end{smallmatrix}\big)\big)\big], \end{array} \tag{3.1.36}$$

$$\big(\widetilde{\Omega}'_W\big)^{-1} =$$



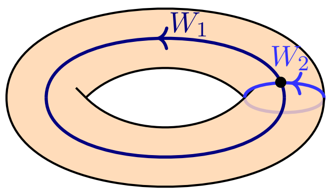
$$= \begin{array}{l} \big[\underline{\alpha} - \underline{\alpha}', \underline{\Delta} W\big(\big(\begin{smallmatrix}\underline{\alpha}'\\ \underline{\alpha}\end{smallmatrix}\big), \underline{x}, \underline{z}\big)\big]\{-\Phi_{\underline{z}}\} \\ \otimes\ \big[\underline{x} - \underline{x}', -\underline{\delta} + \underline{\Delta} W\big(\underline{\alpha}', \big(\begin{smallmatrix}\underline{x}'\\ \underline{x}\end{smallmatrix}\big), \underline{z}\big)\big] \\ \otimes\ \big[\underline{z} - \underline{z}', \underline{\gamma} + \underline{\Delta} W\big(\underline{\alpha}', \underline{x}', \big(\begin{smallmatrix}\underline{z}'\\ \underline{z}\end{smallmatrix}\big)\big)\big]. \end{array} \tag{3.1.37}$$

Finally, assuming W_1, W_2 1-endomorphisms of $\underline{x}$, let $\varphi : W_1 \otimes W_2 \longrightarrow W_2 \otimes W_1$ be described by the matrix factorisation P:



$$= P(\underline{\alpha}_1, \underline{\alpha}_2, \underline{\beta}_1, \underline{\beta}_2, \underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4)\,. \tag{3.1.38}$$

Using this patch, the bending isomorphisms from above and the adjunction 2-morphisms (3.1.20), (3.1.21), we can compute the partition function of a torus with W_1, W_2 cycles wrapped around it to be



$$\cong P(\underline{\alpha}, \underline{\alpha}, \underline{\beta}, \underline{\beta}, \underline{x}, \underline{x}, \underline{x}, \underline{x})\,. \tag{3.1.39}$$

3.1.2 Symmetries of affine Rozansky-Witten models

Rozansky–Witten models with target spaces $T^*\mathbb{C}^n$ exhibit an $\mathrm{Sp}_{2n}(\mathbb{C})$-symmetry, acting linearly on the target manifold. Recall that $\mathrm{Sp}_{2n}(\mathbb{C})$ is generated by the three subgroups

$$D := \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_n(\mathbb{C}) \right\}, \tag{3.1.40}$$

$$N := \left\{ \begin{pmatrix} 1\!\!1_n & B \\ 0 & 1\!\!1_n \end{pmatrix} \,\middle|\, B^\dagger = B \right\}, \tag{3.1.41}$$

$$L := \left\{ 1\!\!1_{2n},\, \Omega := \begin{pmatrix} 0 & 1\!\!1_n \\ -1\!\!1_n & 0 \end{pmatrix} \right\}. \tag{3.1.42}$$

Given objects $\underline{x}, \underline{y} \in \mathcal{C}$ of length n,

$$I_A := \Big(\underline{\alpha}; \underline{\alpha} \cdot (\underline{x} - A\underline{y})\Big) : \underline{y} \longrightarrow \underline{x} \text{ for } A \in \mathrm{GL}_n(\mathbb{C}) \tag{3.1.43}$$

are the 1-isomorphisms implementing the symmetry generators in the subgroup $D \subset \mathrm{Sp}_{2n}(\mathbb{C})$. Here we use $(A\underline{x})_i = \sum_j A_{ij} x_j$ to denote the left action of the matrix A on a list of variables $\underline{x}$. Similarly, $(\underline{\alpha}A)_i = \sum_j \alpha_j A_{ji}$ denotes the right action on $\underline{\alpha}$.

On the other hand, elements in $N \subset \mathrm{Sp}_{2n}(\mathbb{C})$ give rise to 1-isomorphisms

$$N_B := \big(\underline{\alpha};\, \underline{\alpha} \cdot (\underline{y} - \underline{x}) + \underline{\alpha} \cdot B\underline{\alpha}\big) : \underline{x} \longrightarrow \underline{y} \tag{3.1.44}$$

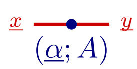
for Hermitian $(n \times n)$-matrices B, while the 1-isomorphism corresponding to Ω is the "Legendre transformation" 1-morphism, defined in [35, Sect. 2.3]:

$$J := (\varnothing; \underline{x} \cdot \underline{y}) : \underline{x} \longrightarrow \underline{y}\,. \tag{3.1.45}$$

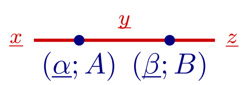
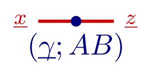
It is straightforward to verify that the composition of the respective 1-isomorphisms indeed satisfies the required relations: $I_{A_1} \circ I_{A_2} \cong I_{A_1 A_2}$, $N_{B_1} \circ N_{B_2} = N_{B_1+B_2}$ and $J \circ J \cong D_{-\mathbb{1}_n}$. By composing 1-isomorphisms I_A, N_B and J, we therefore obtain 1-isomorphisms $D(g)$ for any $g \in \mathrm{Sp}_{2n}(\mathbb{C})$, which in particular satisfy $D(g_1) \circ D(g_2) \cong D(g_1 g_2)$.

We note however that N_B (for $B \neq 0$) and J are not compatible with the U(1)-flavour symmetry. As discussed in Section 3.1.1, compatibility with this symmetry requires the polynomials W defining 1-morphisms in $\mathcal{C}$ to be homogeneous of flavour charge 0. Since we assign U(1)-flavour charge $q_{x_i} = -1$ to all the bulk variables x_i, the potentials appearing in the definition of N_B for $B \neq 0$ and J cannot have flavour charge 0. Thus, only the subgroup $D \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is realized in $\mathcal{C}$. The full symmetry group $\mathrm{Sp}_{2n}(\mathbb{C})$ can only be realised if we give up on the U(1)-flavour symmetry.[2]

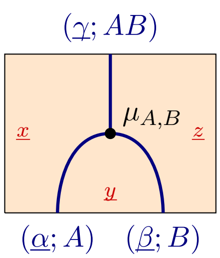
In the following, we shall preserve the flavour symmetry and only consider symmetry defects I_A, $A \in \mathrm{GL}_n(\mathbb{C})$. We represent them graphically as

$I_A :=$  . (3.1.46)

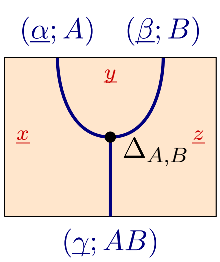
They indeed satisfy $I_A \circ I_B \cong I_{AB}$ for all $A, B \in \mathrm{GL}_n(\mathbb{C})$,

 $\cong$  , (3.1.47)

as witnessed by the 2-isomorphisms



$$= \mu_{A,B} := \left[B\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A\right] \otimes \left[\underline{\gamma} - \underline{\alpha}, \underline{x} - AB\underline{z}\right]\{-\Phi_{\underline{y}}\}, \tag{3.1.48}$$



$$= \Delta_{A,B} := \left[\underline{y} - B\underline{z}, \underline{\beta} - \underline{\gamma}A\right] \otimes \left[\underline{\alpha} - \underline{\gamma}, \underline{x} - A\underline{y}\right]. \tag{3.1.49}$$

These isomorphisms are related to structure morphisms of $\mathcal{C}$. To explain this we first introduce some notation. Consider any 1-morphism $D\colon \underline{x} \longrightarrow \underline{y}$ given by $D = (\alpha; W(\underline{\alpha}, \underline{x}, \underline{y}))$,

[2] This is of course not surprising. After all the flavour symmetry acts with opposite signs on base respectively fibre variables of $T^*\mathbb{C}^n$. Hence only those $\mathrm{Sp}_{2n}(\mathbb{C})$-transformations which do not mix base and fibre variables can be compatible with the flavour symmetry. And these are exactly the ones in the subgroup $D \subset \mathrm{Sp}_{2n}(\mathbb{C})$.

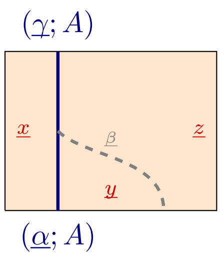
where the length of $\underline{x}$ is n. Then for any $B \in \mathrm{GL}_n(\mathbb{C})$ we define the *right twist* of D by B as

$$(D)_B := \big(\alpha; W(\alpha, B\underline{x}, \underline{y})\big)\,. \tag{3.1.50}$$

This twist also acts on 2-morphisms. Namely, let $E = (\alpha; V(\underline{\beta}, \underline{x}, \underline{y}))\colon \underline{x} \longrightarrow \underline{y}$ be another 1-morphism and $\phi\colon D \longrightarrow E$ a 2-morphism represented by a matrix factorisation $P(\underline{\alpha}, \underline{\beta}, \underline{x}, \underline{y})$ of $V(\underline{\beta}, \underline{x}, \underline{y}) - W(\alpha, \underline{x}, \underline{y})$. Then the right twist $(\phi)_B\colon (D)_B \longrightarrow (E)_B$ is represented by $P(\underline{\alpha}, \underline{\beta}, B\underline{x}, \underline{y})$.

We can check that $(I_A)_B = I_{AB}$. Indeed, applying $(-)_B$ to the right unitor (cf. (3.1.13))

$$\rho_{I_A} =$$



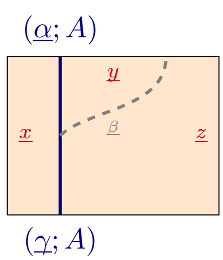
$$= \big[\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A\big] \otimes \big[\underline{\gamma} - \underline{\alpha}, \underline{x} - A\underline{z}\big]\{-\Phi_{\underline{y}}\} \tag{3.1.51}$$

yields a 2-morphism $(\rho_{I_A})_B\colon I_A \circ I_B \longrightarrow I_{AB}$, which coincides with the 2-isomorphism $\mu_{A,B}$:

$$(\rho_{I_A})_B = \big[B\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A\big] \otimes \big[\underline{\gamma} - \underline{\alpha}, \underline{x} - AB\underline{z}\big]\{-\Phi_{\underline{y}}\} = \mu_{A,B}. \tag{3.1.52}$$

Similarly, twisting the inverse $(\rho_{I_A})^{-1}$ of the right unitor (cf. (3.1.15))

$$(\rho_{I_A})^{-1} =$$



$$= \big[\underline{y} - \underline{z}, \underline{\beta} - \underline{\gamma}A\big] \otimes \big[\underline{\alpha} - \underline{\gamma}, \underline{x} - A\underline{y}\big]\,, \tag{3.1.53}$$

one obtains the 2-morphism $(\rho_{I_A}^{-1})_B\colon I_{AB} \longrightarrow I_A \circ I_B$

$$(\rho_{I_A}^{-1})_B = \big[\underline{y} - B\underline{z}, \underline{\beta} - \underline{\gamma}A\big] \otimes \big[\underline{\alpha} - \underline{\gamma}, \underline{x} - A\underline{y}\big] = \Delta_{A,B} \tag{3.1.54}$$

which coincides with $\Delta_{A,B}$.[3]

The isomorphisms $\mu_{A,B}$ and $\Delta_{A,B}$ are also related to the adjunction 2-morphisms of I_A. For this we need the fact that $I_{A^{-1}} \cong (I_A)^\dagger$. Namely, for $I_A = (\underline{\alpha}; \underline{\alpha} \cdot (\underline{x} - A\underline{y}))\colon \underline{y} \longrightarrow \underline{x}$, we get from (3.1.19) that

$$(I_A)^\dagger = \big(\underline{\alpha};\, -\underline{\alpha} \cdot (\underline{x} - A\underline{y})\big) = \big(\underline{\alpha};\, \underline{\alpha} \cdot A(\underline{y} - A^{-1}\underline{x})\big)\colon \underline{x} \longrightarrow \underline{y}\,. \tag{3.1.55}$$

[3] The maps $\mu_{A,B}$ and $\Delta_{A,B}$ can equally be obtained by left-twisting the left unitors λ_{I_B} and its inverse $(\lambda_{I_B})^{-1}$ by A, respectively.

This is isomorphic to $I_{A^{-1}} = (\underline{\alpha}'; \underline{\alpha}' \cdot (\underline{y} - A^{-1}\underline{x})) \colon \underline{x} \longrightarrow \underline{y}$ by means of the isomorphism $\chi_A \colon (I_A)^\dagger \longrightarrow I_{A^{-1}}$ given by the Koszul factorisation

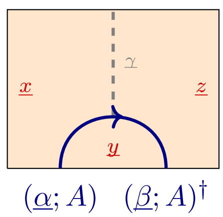
$$\chi_A = \left[\underline{\alpha}' - \underline{\alpha}A,\, \underline{y} - A^{-1}\underline{x}\right] . \tag{3.1.56}$$

Using this isomorphism one finds

$$\mu_{A,A^{-1}} = \widetilde{\mathrm{ev}}_{I_A} \cdot (1_{I_A} \circ \chi_A^{-1}) \,, \quad \Delta_{A,A^{-1}} = (1_{I_A} \circ \chi_A) \cdot \mathrm{coev}_{I_A} \ . \tag{3.1.57}$$

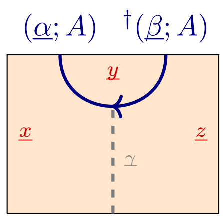
Indeed, using the adjunction 2-morphisms

$$\widetilde{\mathrm{ev}}_{I_A} =$$



$$= \left[\underline{x} - \underline{z}, \underline{\gamma} - \underline{\alpha}\right] \otimes \left[\underline{\alpha} - \underline{\beta}, -(\underline{z} - A\underline{y})\right]\{-\Phi_{\underline{y}}\} \,, \tag{3.1.58}$$

$$\mathrm{coev}_{I_A} =$$



$$= \left[\underline{x} - \underline{z}, -\underline{\gamma} + \underline{\alpha}\right] \otimes \left[\underline{\alpha} - \underline{\beta}, \underline{z} - A\underline{y}\right] \tag{3.1.59}$$

from Section 3.1.1, as well as the properties of matrix factorisations (2.2.63)–(2.2.66) combined with the degree shifts (3.1.6), we compute

$$\begin{aligned} \widetilde{\mathrm{ev}}_{I_A} \cdot (1_{I_A} \circ \chi_A^{-1}) =& \left[\underline{x} - \underline{z}, \underline{\gamma} - \underline{\alpha}\right] \otimes \left[\underline{\alpha} - \underline{\beta}, -(\underline{z} - A\underline{y})\right] \\ & \otimes \left[\underline{\beta}A - \underline{\beta}', \underline{y} - A^{-1}\underline{z}\right]\{-\Phi_{\underline{y}}\} \\ \cong& \left[\underline{x} - \underline{z}, \underline{\gamma} - \underline{\alpha}\right] \otimes \left[\underline{\alpha} - \underline{\beta}, -(\underline{z} - A\underline{y})\right] \\ & \otimes \left[\underline{\beta} - \underline{\beta}'A^{-1}, A\underline{y} - \underline{z}\right]\{-\Phi_{\underline{y}}\} \\ \cong& \left[\underline{x} - \underline{z}, \underline{\gamma} - \underline{\alpha}\right] \otimes \left[\underline{\alpha} - \underline{\beta}'A^{-1}, -(\underline{z} - A\underline{y})\right]\{-\Phi_{\underline{y}}\} \\ \cong& \left[\underline{\gamma} - \underline{\alpha}, \underline{x} - \underline{z}\right] \otimes \left[A^{-1}\underline{z} - \underline{y}, \underline{\beta}' - \underline{\alpha}A\right]\{-\Phi_{\underline{y}}\} \\ =& \mu_{A,A^{-1}} \ . \end{aligned} \tag{3.1.60}$$

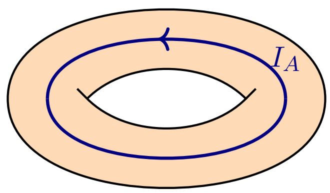
This yields the first relation in (3.1.57). The second one can be checked in a similar fashion.

3.2 State spaces

3.2.1 Defect networks

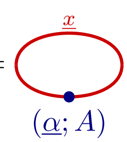
In this section we calculate the state spaces associated to genus-g closed surfaces Σ_g with networks of symmetry defects I_A. The vertices of the networks are given by (or built out of) the trivalent junctions μ and Δ, cf. (3.1.48) and (3.1.49), respectively. Inserting such a fixed choice of network can be interpreted as turning on a flat background gauge field, where a defect I_A on the surface introduces a gluing of the respective flat gauge connections on either side of the defect via A. Note that there are many different defect networks which in this way describe the same gauge connection. We will argue in Section 3.2.2 below that the state spaces of two different defect networks associated to the same background gauge field are indeed isomorphic.

We first compute state spaces on surfaces with symmetry defects wrapped along non-contractible cycles. Let us start with a torus with defect I_A wrapped around the a-cycle (corresponding to a background gauge field with holonomy A around the b-cycle):



(3.2.1)

As noted in (3.1.30), the state space associated to this surface corresponds to the quantum dimension $\dim_l(X_A)$ of the 1-morphism[4]

$$X_A =$$


. (3.2.2)

Applying formula (3.1.31) for $W(\underline{\alpha}, \underline{x}, \underline{y}) = \underline{\alpha}\cdot(\underline{x} - A\underline{y})$ and using the fact that the shifts $\Phi_{\underline{x}}$, $\Phi_{\underline{\alpha}}$ defined in (3.1.6) satisfy $\Phi_{\underline{x}} + \Phi_{\underline{\alpha}} = 0$, together with the property of matrix factorisations (2.2.63), we arrive at

$$\dim_l(X_A) = \big[(\mathbb{1} - A)\underline{x},\, \underline{0}\big] \otimes \big[\underline{\alpha}(\mathbb{1} - A),\, \underline{0}\big]\,. \tag{3.2.3}$$

This matrix factorisation can be simplified. The matrix $(\mathbb{1} - A)$ acts linearly on $\underline{x}$ and $\underline{\alpha}$. Hence via (2.2.65) all the linear combinations of variables x_i and α_i in the image of $(\mathbb{1} - A)$ and $(\mathbb{1} - A)^{\mathrm{T}}$, respectively, can be eliminated by setting them to zero. This uses up all the matrix factorisations with non-zero differentials. Only $\dim(\ker(\mathbb{1} - A))$-many pairs of rank-1 matrix factorisations with zero differential remain. This leaves $\dim(\ker(\mathbb{1} - A))$-many pairs of x- and α-variables not in the image of $(\mathbb{1} - A)$, respectively $(\mathbb{1} - A)^{\mathrm{T}}$, which we identify with the variables invariant under the A-action. We denote these invariant variables by $\underline{x}^{\mathrm{inv}}$ and $\underline{\alpha}^{\mathrm{inv}}$, respectively. Since $\dim_l(X_A)$ is a matrix factorisation of 0, it is equal in $\mathcal{C}$ to its cohomology, which can be written as

$$\dim_l(X_A) = \big((\mathbb{C} \oplus \mathbb{C}\{1, 0, -1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1, 0, 1\})\big)^{\dim\ker(\mathbb{1} - A)} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}^{\mathrm{inv}}, \underline{\alpha}^{\mathrm{inv}}]\,. \tag{3.2.4}$$

[4] In this diagram we only label the variables remaining in (3.1.31) after removing intermediate variables. Likewise, in the diagrams in the rest of the section, we only show the names of the internal variables when they are necessary for some computation.

Note that for a generic $A \in \mathrm{GL}_n(\mathbb{C})$ for which $\ker(\mathbb{1} - A) = 0$, one obtains the trivial state space

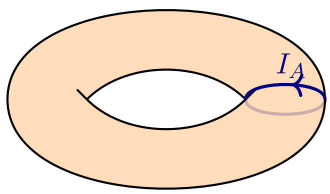
$$\dim_{\mathrm{l}}(X_A) = \mathbb{C}, \tag{3.2.5}$$

while for $A = \mathbb{1}$ one recovers the state space

$$\dim_{\mathrm{l}}(X_A) = \big((\mathbb{C} \oplus \mathbb{C}\{1,0,-1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1,0,1\})\big)^n \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}, \underline{\alpha}] \tag{3.2.6}$$

of the torus without defect insertion, cf. (3.1.32).

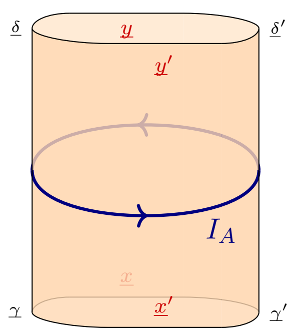
We can do a similar computation to evaluate the state space of a torus with defect wrapping the other cycle:



(3.2.7)

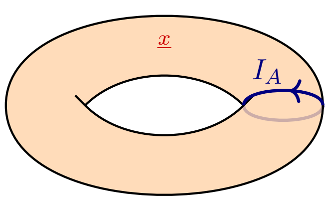
Since the tori are related by a modular transformation, they belong to the same bordism class. Hence this computation should give the same result as (3.2.3). However, in the graphical calculus, this is a completely different computation and it will provide us with a consistency check of our construction. We compute this state space as a left trace $\mathrm{tr}_{\mathrm{l}}(\phi)$ (cf. (2.2.28)) of the 2-morphism

$\phi :=$



$$= \begin{array}{ll} & \big[\underline{0}, \underline{x} - A\underline{y}\big] \otimes \big[\underline{0}, \underline{\delta}' - \underline{\gamma}' A\big] \\ \otimes & \big[\underline{y}' - \underline{y}, \underline{\delta} - \underline{\delta}'\big] \\ \otimes & \big[\underline{x} - \underline{x}', \underline{\gamma} - \underline{\gamma}'\big]\{-\Phi_{\underline{x}}\}\,. \end{array} \tag{3.2.8}$$

To arrive at the expression on the right-hand side, the bending isomorphisms (3.1.34) and (3.1.36) were used. Inserting ϕ into the general formula (3.1.30) for the left trace, we obtain

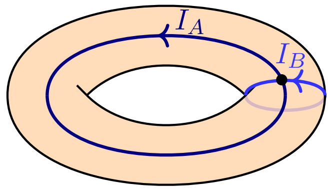


(3.2.9)

$$\begin{aligned} &\cong \big[\underline{0}, \underline{x} - A\underline{x}\big] \otimes \big[\underline{0}, \underline{\gamma}' - \underline{\gamma}' A\big] \otimes \big[\underline{x}' - \underline{x}, \underline{\gamma} - \underline{\gamma}'\big] \otimes \big[\underline{x} - \underline{x}', \underline{\gamma} - \underline{\gamma}'\big]\{-\Phi_{\underline{x}}\} \\ &\cong \big[(\mathbb{1} - A)\underline{x}, \underline{0}\big] \otimes \big[\underline{\gamma}(\mathbb{1} - A), \underline{0}\big]\,. \end{aligned}$$

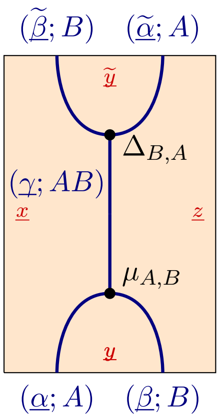
In the last step we used the two rightmost matrix factorisations to eliminate the variables $\underline{x}'$ and $\underline{\gamma}'$. Hence, we indeed arrive at the same result as (3.2.3).

Next, let us wrap symmetry defects on both cycles of the torus:



(3.2.10)

We choose $A, B \in \mathrm{GL}_n(\mathbb{C})$ such that $AB = BA$. This allows us in particular to define the junction by resolving it into trivalent junctions of the type (3.1.48) and (3.1.49), i. e. by defining the 4-valent junction to be



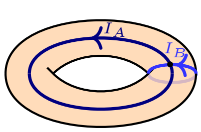
$$\cong \big[\widetilde{\underline{y}} - A\underline{z}, \widetilde{\underline{\alpha}} - \widetilde{\underline{\beta}} B\big] \otimes \big[B\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha} A\big] \otimes \big[\widetilde{\underline{\beta}} - \underline{\alpha}, \underline{x} - AB\underline{z}\big]\{-\Phi_{\underline{y}}\}\,. \quad (3.2.11)$$

Inserting this junction 2-morphism into the general formula (3.1.39) for the intersection of defects on a torus, we obtain

$$\big[\underline{x} - A\underline{x}, \underline{\alpha} - \underline{\beta} B\big] \otimes \big[B\underline{x} - \underline{x}, \underline{\beta} - \underline{\alpha} A\big] \otimes \big[\underline{\beta} - \underline{\alpha}, \underline{x} - AB\underline{x}\big]\{-\Phi_{\underline{x}}\} \quad (3.2.12)$$

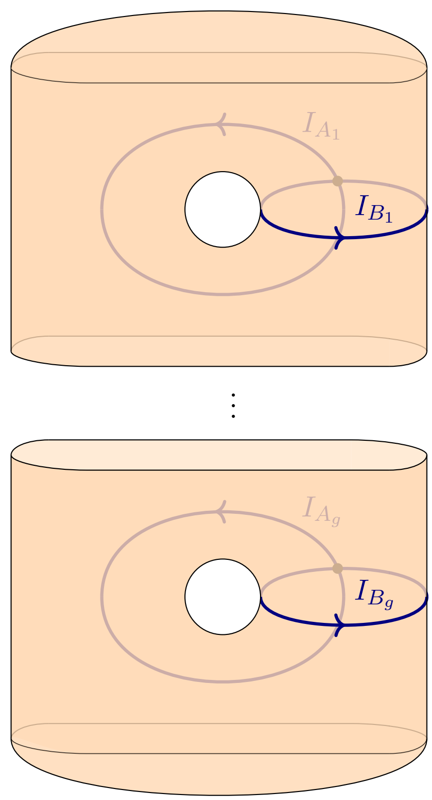
$$\cong \big[(\mathbb{1} - A)\underline{x}, \underline{\alpha}(\mathbb{1} - B)\big] \otimes \big[(B - \mathbb{1})\underline{x}, \underline{\alpha}(\mathbb{1} - A)\big]\{-\Phi_{\underline{x}}\}\,. \quad (3.2.13)$$

As before, we can eleminate all but the variables $\underline{x}^{\mathrm{inv}}$ and $\underline{\alpha}^{\mathrm{inv}}$ which are invariant under both the A- and B-action, i. e. those which lie in the common kernel $V_{A,B} := \ker(\mathbb{1} - A) \cap \ker(\mathbb{1} - B)$, and end up with



$$= \big((\mathbb{C} \oplus \mathbb{C}\{1, 0, -1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1, 0, 1\})\big)^{\dim V_{A,B}} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}^{\mathrm{inv}}, \underline{\alpha}^{\mathrm{inv}}]\,. \quad (3.2.14)$$

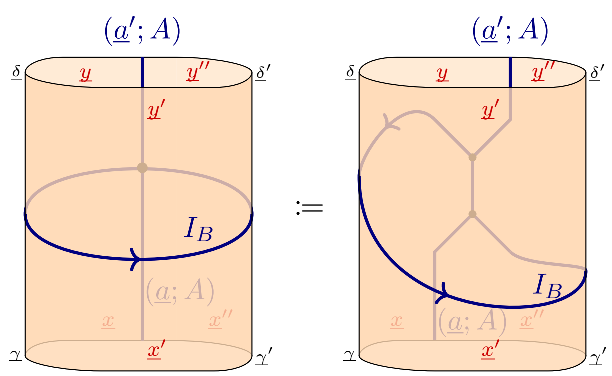
Having calculated the state space of a torus with symmetry defects wrapped around the two homology cycles, we now turn our attention to general closed surfaces Σ_g of genus g, with symmetry defects wrapping all the homology cycles:



(3.2.15)

As before we assume that $A_I B_I = B_I A_I$ for all $I \in \{1, \dots, g\}$, and we use (3.2.11) as the respective junction fields. This amounts to turning on a non-trivial flat gauge background with holonomies A_I, B_I around the respective cycles.

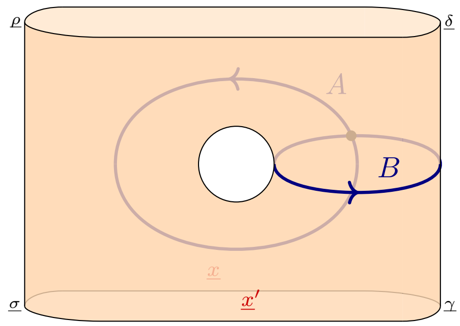
In order to calculate the state space, we first evaluate the decorated cylinder



(3.2.16)

$$= \left[\underline{y}' - \underline{y}, \underline{\delta} - \underline{\gamma} B\right] \otimes \left[\underline{y}'' - \underline{y}', \underline{\delta}' - \underline{\gamma} B\right] \otimes \left[\underline{x}' - \underline{x}'', \underline{\gamma}' - \underline{\gamma}\right]$$
$$\otimes \left[\underline{y} - A\underline{y}'', \underline{\alpha}' - \underline{\gamma} B\right] \otimes \left[B\underline{y}'' - \underline{x}'', \underline{\gamma} - \underline{\alpha} A\right] \otimes \left[\underline{\gamma} - \underline{\alpha}, \underline{x} - AB\underline{y}''\right]\{-2\Phi_{\underline{x}}\}\,.$$

This can be used to evaluate the decorated handle element:



$$= \left[\underline{\rho} - \underline{\sigma}, \underline{x}' - \underline{x}\right] \otimes \left[\underline{\delta} - \underline{\gamma}, \underline{x} - \underline{x}'\right] \\ \otimes \left[(1\!\!1 - A)\underline{x}, \underline{\gamma}(1\!\!1 - B)\right] \otimes \left[(B - 1\!\!1)\underline{x}, \underline{\gamma}(1\!\!1 - A)\right]\{2\Phi_{\underline{x}}\}\,. \tag{3.2.17}$$

Composing g copies of the handle element and eliminating the internal variables yields

$$\left[\underline{\rho} - \underline{\sigma}, \underline{x}' - \underline{x}\right] \otimes \left[\underline{\delta} - \underline{\gamma}, \underline{x} - \underline{x}'\right] \\ \otimes \bigotimes_{I=1}^{g} \left[(1\!\!1 - A_I)\underline{x}, \underline{\gamma}(1\!\!1 - B_I)\right] \otimes \left[(B_I - 1\!\!1)\underline{x}, \underline{\gamma}(1\!\!1 - A_I)\right]\{2g\Phi_{\underline{x}}\}\,. \tag{3.2.18}$$

Pre- and post-composing with the cup and cap 2-morphisms from (3.1.27), respectively, we arrive at

$$\bigotimes_{I=1}^{g} \left[(1\!\!1 - A_I)\underline{x}, \underline{\gamma}(1\!\!1 - B_I)\right] \otimes \left[\underline{\gamma}(1\!\!1 - A_I), (B_I - 1\!\!1)\underline{x}\right]\{3(g-1)\Phi_{\underline{x}}\}\,. \tag{3.2.19}$$

As before, we can eliminate all but the variables $\underline{x}^{\text{inv}}$ and $\underline{\gamma}^{\text{inv}}$ which are invariant under all A_I and B_I, which then automatically sets the differentials in the remaining matrix factorisations to zero. One obtains the cohomology

$$\big((\mathbb{C} \oplus \mathbb{C}\{1,0,-1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1,0,1\})\big)^{n(g-1)+\dim V_{A_\bullet,B_\bullet}} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}^{\text{inv}}, \underline{\alpha}^{\text{inv}}]\,, \tag{3.2.20}$$

where

$$V_{A_\bullet,B_\bullet} = \bigcap_{I=1}^{g} \big(\ker(1\!\!1 - A_I) \cap \ker(1\!\!1 - B_I)\big) \tag{3.2.21}$$

is the subspace of $\mathbb{C}^n$ which is invariant under all the A_I and B_I, $I \in \{1, \dots, g\}$.

Note that if A_I and B_I are all trivial, no variables can be eliminated, and one recovers the state space associated to a genus-g surface without defect network, as in [7, Prop. 3.5], which we recall for future reference:

$$\big((\mathbb{C} \oplus \mathbb{C}\{1,0,-1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1,0,1\})\big)^{ng} \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}, \underline{\alpha}]\,. \tag{3.2.22}$$

The space is isomorphic as a vector space to the exterior algebra on $2ng$ fermions with coefficients in $\mathbb{C}[\underline{x}, \underline{\gamma}]$, as discussed in [7, App. B.2]. This in particular agrees with the original result for state spaces for free Rozansky–Witten models in [47] (see also [11, 18] for a recent treatment).

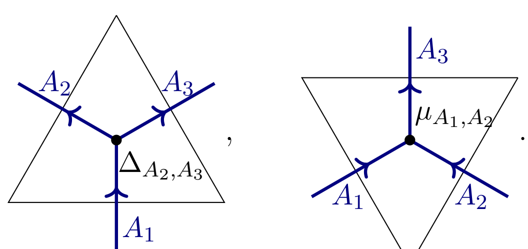
If at least one of A_I, B_I is non-trivial, variables can be eliminated and the matrix factorisation reduced. If for instance the A_I, B_I are chosen such that $V_{A_\bullet,B_\bullet} = 0$ (which for instance is the case if one of the A_I, B_I has no eigenvalue 1), the state space (3.2.20) becomes

$$\big((\mathbb{C} \oplus \mathbb{C}\{1,0,-1\}) \otimes (\mathbb{C} \oplus \mathbb{C}\{1,0,1\})\big)^{n(g-1)} . \tag{3.2.23}$$

It is the space generated by $2n(g-1)$ fermions but without bosonic degrees of freedom. This agrees with the state space of a theory of n free hypermultiplets with corresponding non-trivial gauge background derived by other methods in [18]: turning on a generic non-trivial gauge connection modifies the BV differential on the state space associated to Σ_g by canonical quantisation in such a way that the $2n$ chiral fields corresponding to the $\underline{x}$ and $\underline{\alpha}$ become exact, and $2n$ of the fermions disappear from the kernel of the BV differential.

3.2.2 Triangulation independence

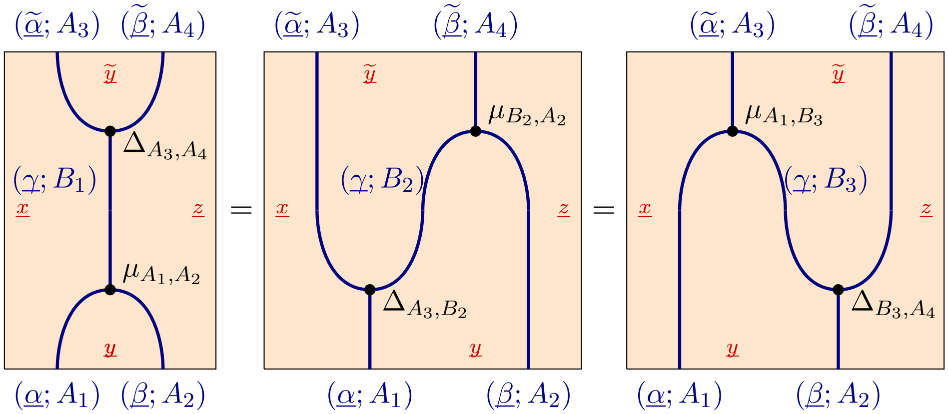
As alluded to above, insertion of a network of symmetry defects with 3-junctions defined in (3.1.48) and (3.1.49) amounts to introducing a non-trivial flat background gauge field. In fact, any such background can be modelled by inserting such a network on the Poincaré dual of a triangulation of the surface. For this, one assigns the two 2-morphisms (3.1.48), (3.1.49) to the oppositely oriented triangles



(3.2.24)

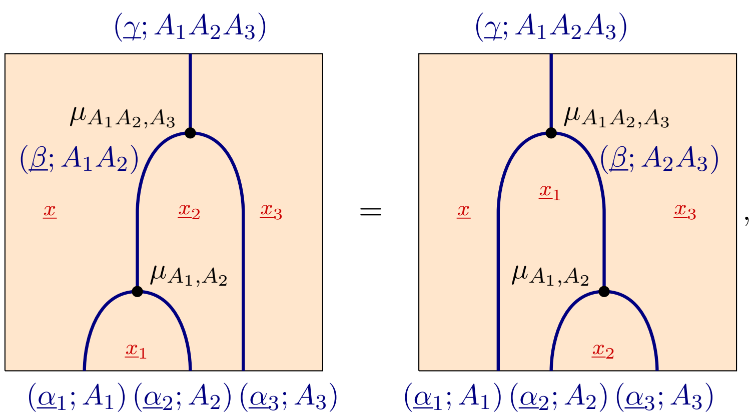
Parallel transport along an edge of a triangle is given by the respective group element (or its inverse, depending on the relative orientation) corresponding to the symmetry defect it intersects. Note that the conditions on the vertices ($A_1 = A_2A_3$ for Δ on the left, and $A_1A_2 = A_3$ for μ on the right) ensure that holonomies around contractible cycles vanish.

Of course the state space associated to a surface with gauge background should only depend on the isomorphism class of the respective bundle with connection, and not on a choice of triangulation. That this is in fact true in our construction follows from the following relations satisfied by the 3-junctions (3.1.48) and (3.1.49):



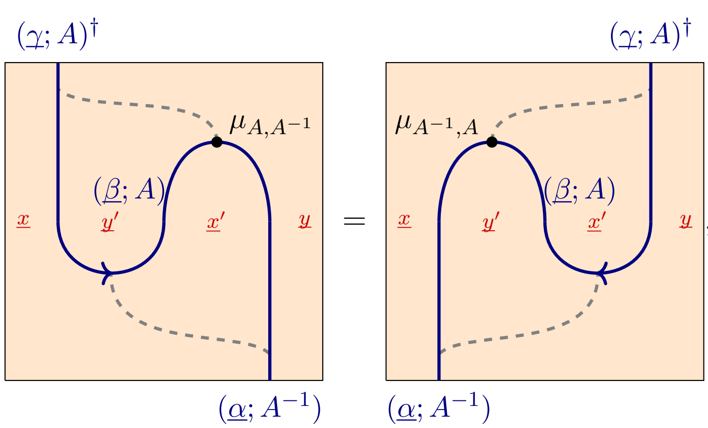
(3.2.25)

in case $A_1A_2 = A_3A_4$, as well as

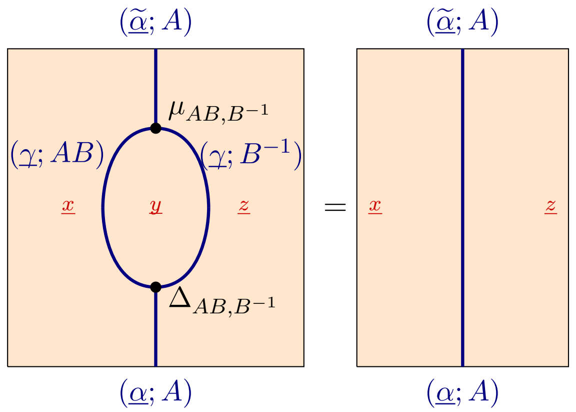


, (3.2.26)

its vertically reflected version involving Δ, and



, (3.2.27)



(3.2.28)

Thus, state spaces are invariant under the local changes (3.2.25)–(3.2.28) of the symmetry defect network. These local changes exactly correspond to the Pachner moves of triangulations, and hence, the state spaces do not depend on the triangulations.

As an aside we remark that relations (3.2.25)–(3.2.28) are "component versions" of Frobenius, (co)associativity, symmetry and separability relations, respectively, which appear in the generalised orbifold construction (see [16, 23] for the 2-dimensional case of relevance here, and [17] for the basic theory in arbitrary dimension): given a Δ-separable symmetric Frobenius algebra $\mathcal{A}$ in some pivotal 2-category, one can insert $\mathcal{A}$-defect networks on any bordism and show that the corresponding correlators are independent of the choice of network (in the interior). Indeed, in case we restrict to a finite (sub)group, we can gauge the symmetry by summing over all group elements. A direct sum completion of the relations (3.2.25)–(3.2.28) can then be used to construct a Δ-separable symmetric Frobenius algebra and thus apply the orbifold construction. See for instance [16, Sect. 7] for the example of a Landau–Ginzburg orbifold. We do not perform the gauging in this paper, see however [11] for results on state spaces for gauged theories obtained by other means.

In the following we prove the first identity of (3.2.25). The other relations follow in a similar way. The leftmost diagram of (3.2.25) evaluates to

$$\begin{aligned}&\big[\widetilde{\underline{y}} - A_4\underline{z}, \widetilde{\underline{\beta}} - \underline{\gamma}A_3\big] \otimes \big[\widetilde{\underline{\alpha}} - \underline{\gamma}, \underline{x} - A_3\widetilde{\underline{y}}\big] \\ &\qquad \otimes \big[A_2\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A_1\big] \otimes \big[\underline{\gamma} - \underline{\alpha}, \underline{x} - B_1\underline{z}\big]\{-\Phi_{\underline{y}}\} \\ \cong &\big[\widetilde{\underline{y}} - A_4\underline{z}, \widetilde{\underline{\beta}} - \underline{\alpha}A_3\big] \otimes \big[\widetilde{\underline{\alpha}} - \underline{\alpha}, \underline{x} - A_3\widetilde{\underline{y}}\big] \otimes \big[A_2\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A_1\big]\{-\Phi_{\underline{y}}\}\,, \end{aligned} \tag{3.2.29}$$

while the one in the middle yields

$$\begin{aligned}&\big[A_2\underline{z} - \underline{y}, \underline{\beta} - \underline{\gamma}B_2\big] \otimes \big[\widetilde{\underline{\beta}} - \underline{\gamma}, \widetilde{\underline{y}} - B_2\underline{y}\big] \\ &\qquad \otimes \big[\widetilde{\underline{y}} - B_2\underline{y}, \underline{\gamma} - \underline{\alpha}A_3\big] \otimes \big[\widetilde{\underline{\alpha}} - \underline{\alpha}, \underline{x} - A_3\widetilde{\underline{y}}\big]\{-\Phi_{\underline{y}}\} \\ \cong &\big[A_2\underline{z} - \underline{y}, \underline{\beta} - \widetilde{\underline{\beta}}B_2\big] \otimes \big[\widetilde{\underline{y}} - B_2\underline{y}, \widetilde{\underline{\beta}} - \underline{\alpha}A_3\big] \otimes \big[\widetilde{\underline{\alpha}} - \underline{\alpha}, \underline{x} - A_3\widetilde{\underline{y}}\big]\{-\Phi_{\underline{y}}\}\,. \end{aligned} \tag{3.2.30}$$

These two expressions are isomorphic if

$$\big[\widetilde{\underline{y}} - A_4\underline{z}, \widetilde{\underline{\beta}} - \underline{\alpha}A_3\big] \otimes \big[A_2\underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha}A_1\big] \cong \big[A_2\underline{z} - \underline{y}, \underline{\beta} - \widetilde{\underline{\beta}}B_2\big] \otimes \big[\widetilde{\underline{y}} - B_2\underline{y}, \widetilde{\underline{\beta}} - \underline{\alpha}A_3\big] \tag{3.2.31}$$

which in turn is a consequence of the general relation of matrix factorisations

$$\bigotimes_i \left[p_1^{(i)}, p_0^{(i)}\right] \otimes \left[q_1^{(i)}, q_0^{(i)}\right] \cong \bigotimes_i \left[p_1^{(i)} + \sum_j T_{ij} q_1^{(i)}, p_0^{(i)}\right] \otimes \left[q_1^{(i)}, q_0^{(i)} - \sum_j q_1^{(i)} T_{ji}\right] \quad (3.2.32)$$

where T_{ij} are arbitrary linear transformations. This is a direct generalisation of property (2.2.65). Applying this formula to the left-hand side of (3.2.31) for $T = B_2$, and using the conditions $A_1 = A_3 B_2$, $A_4 = B_2 A_2$, we obtain

$$\begin{aligned} &\left[\widetilde{\underline{y}} - A_4 \underline{z}, \widetilde{\underline{\beta}} - \underline{\alpha} A_3\right] \otimes \left[A_2 \underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha} A_1\right] \\ \cong &\left[\widetilde{\underline{y}} - B_2 A_2 \underline{z} + B_2 (A_2 \underline{z} - \underline{y}), \widetilde{\underline{\beta}} - \underline{\alpha} A_3\right] \otimes \left[A_2 \underline{z} - \underline{y}, \underline{\beta} - \underline{\alpha} A_3 B_2 - (\widetilde{\underline{\beta}} - \underline{\alpha} A_3) B_2\right] \\ \cong &\left[\widetilde{\underline{y}} - B_2 \underline{y}, \widetilde{\underline{\beta}} - \underline{\alpha} A_3\right] \otimes \left[A_2 \underline{z} - \underline{y}, \underline{\beta} - \widetilde{\underline{\beta}} B_2\right], \end{aligned} \quad (3.2.33)$$

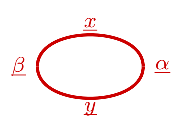
which implies (3.2.31).

3.2.3 Twisted sector line operarators

Given any extended d-dimensional TQFT $\mathcal{Z}$ with values in some d-category $\mathcal{D}$, one can extract information about its k-dimensional defects from what $\mathcal{Z}$ assigns to the sphere S^{d-k-1}. More precisely, the (higher) category of k-dimensional defects is the (higher) Hom category of $(d-k-1)$-morphisms $\mathcal{D}(\mathcal{Z}(\varnothing_{d-k-1}), \mathcal{Z}(S^{d-k-1}))$ between what $\mathcal{Z}$ associates to the $(d-k-1)$-dimensional empty set and what it assigns to S^{d-k-1}, see e. g. [33].

In our truncation of a 3-dimensional theory, taking values in the 2-category $\mathcal{C}$, we do not see the full category of line operators, but only the vector space $\mathcal{C}(\mathcal{Z}(\varnothing_1), \mathcal{Z}(S^1))$ of its isomorphism classes. With this caveat we however continue to use the phrase "category of line operators", especially since $\mathcal{C}$ is the homotopy 2-category of the 3-category $\mathcal{RW}^{\text{aff}}$ of [35], and our extended TQFTs $\mathcal{Z}_n$ are expected to lift to 3-dimensional ones valued in $\mathcal{RW}^{\text{aff}}$ (over $\mathbb{C}[\underline{x}, \underline{\alpha}]$, not over $\mathbb{C}$).

As computed in [7], the image of the circle under $\mathcal{Z}$ is



$$= \left(\underline{\alpha}, \underline{\beta}, \underline{x}, \underline{y};\ \underline{\beta} \cdot (\underline{y} - \underline{x}) + \underline{\alpha} \cdot (\underline{x} - \underline{y})\right), \quad (3.2.34)$$

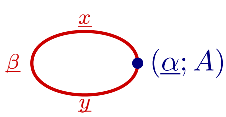
and the image of $\varnothing_1$ is the zero potential. Hence the category of line defects is given by the homotopy category of matrix factorisations of the potential $W = (\underline{\beta} - \underline{\alpha}) \cdot (\underline{y} - \underline{x})$. Indeed, in this way we recover the description of bulk line operators in the initial 3-dimensional target category of [35], where they arise as line operators on an invisible surface defect. As the identity surface defect corresponds to the 1-morphism $(\underline{\beta}; \underline{\alpha} \cdot (\underline{x} - \underline{y}))$, the latter are indeed given by matrix factorisations of $(\underline{\beta} - \underline{\alpha}) \cdot (\underline{y} - \underline{x})$. By Knörrer periodicity, (3.2.34) is equivalent to $(\underline{\alpha}, \underline{x}; 0)$, or the homotopy category of matrix factorisations of 0, which in turn is equivalent to $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}, \underline{\alpha}]$-modules. This agrees with [45, 46].[5]

[5] Following [33, 36], we stress that a priori and for general target there is only a $\mathbb{Z}_2$-grading on this category, which in our special case (due to the preserved symmetries) can be upgraded to a $\mathbb{Z}$-grading which is however not expected for general Rozansky–Witten models.

In Sections 3.2.1–3.2.2, we considered closed 2-dimensional bordisms with arbitrary networks of symmetry defects, corresponding to fixed background gauge fields. Cutting discs out of such bordisms gives rise to circles dressed by group elements, marking the points where the symmetry defects end. The image of such circles under the extended TQFT encodes the information about line operators in a fixed gauge background with prescribed monodromy. This provides a description of "twisted sector line operators", in close analogy to the well-known twisted sector point operators, often also refered to as disorder (point or line) operators, see [20] for a discussion in the context of gauge symmetries.

Note that circles X_A marked by group elements A (recall (3.2.2)) can be connected by bordisms dressed with symmetry defects, where a defect I_A has to end on the marked point. For example, there is a pair-of-pants connecting two circles marked by A and B, respectively, to one marked by the composition of the group elements AB, $X_A \circ X_B \longrightarrow X_{AB}$. Such a structure is expected physically, since different twisted line operators can be merged whenever this is compatible with the twist.

Concretely, our extended TQFT associates the 1-morphism

$X_A =$ 

$= \big(\underline{\alpha}, \underline{\beta}, \underline{x}, \underline{y}; \underline{\beta} \cdot (\underline{y} - \underline{x}) + \underline{\alpha} \cdot (\underline{x} - A\underline{y})\big)$ (3.2.35)

to the circle with an insertion of the symmetry defect I_A. The category whose Grothendieck group is $\mathcal{C}(\mathcal{Z}(\varnothing_1), \mathcal{Z}(X_A))$ is then given by the homotopy category of (equivalence classes of) matrix factorisations

$$\mathcal{L}_A := \operatorname{hmf}\left(\mathbb{C}[\underline{\alpha}, \underline{\beta}, \underline{x}, \underline{y}], W_A\right)^{\omega} \tag{3.2.36}$$

of the potential

$$W_A := \underline{\beta} \cdot (\underline{y} - \underline{x}) + \underline{\alpha} \cdot (\underline{x} - A\underline{y}) = (\underline{\alpha} - \underline{\beta}) \cdot (\underline{x} - \underline{y}) + \underline{\alpha}(\mathbb{1} - A)\underline{y} \,. \tag{3.2.37}$$

It corresponds to the category of line defects in the sector twisted by the defect I_A of the underlying theory.

Indeed, as in the untwisted case discussed above, this agrees with the perspective of the full 3-category $\mathcal{RW}^{\mathrm{aff}}$ of affine Rozansky–Witten models of [35]. Here the line operators of the twisted sector correspond to line defects between the identity surface defect and the respective symmetry defect. The category of these line operators is given by $\mathcal{RW}^{\mathrm{aff}}(I_{\mathbb{1}}, I_A)$ which is nothing but $\mathcal{L}_A$, where now the potential W_A is obtained as the difference of the superpotential associated to I_A and the one associated to the identity defect.

The category $\mathcal{L}_A$ can be further simplified: substituting $\underline{\gamma} = \underline{\alpha} - \underline{\beta}$ and $\underline{z} = \underline{x} - \underline{y}$, one obtains $\operatorname{hmf}(\mathbb{C}[\underline{\alpha}, \underline{\gamma}, \underline{y}, \underline{z}],\ \underline{\gamma} \cdot \underline{z} + \underline{\alpha} \cdot (\mathbb{1} - A)\underline{y})^{\omega}$ which in turn, by virtue of Knörrer periodicity, is equivalent to the category $\operatorname{hmf}(\mathbb{C}[\underline{\alpha}, \underline{y}], \underline{\alpha} \cdot (\mathbb{1} - A)\underline{y})^{\omega}$. In fact, Knörrer periodicity can be further used to eliminate variables α and y in the image of $\mathbb{1} - A$. This uses up the entire potential and only $\dim(\ker(\mathbb{1} - A))$-many variables $\underline{y}^{\mathrm{inv}}$ and $\underline{\alpha}^{\mathrm{inv}}$ remain. Thus $\mathcal{L}_A$ is equivalent to

$$\operatorname{hmf}\left(\mathbb{C}[\underline{\alpha}^{\mathrm{inv}}, \underline{y}^{\mathrm{inv}}], 0\right) \cong \operatorname{mod}^{\mathbb{Z}_2}\left(\mathbb{C}[\underline{\alpha}^{\mathrm{inv}}, \underline{y}^{\mathrm{inv}}]\right) . \tag{3.2.38}$$

If A is generic in the sense that $\ker(\mathbb{1}-A)=0$, then all variables can be eliminated and the category is equivalent to $\mathrm{sVect}_{\mathbb{C}}$. This agrees with the discussion of holonomy line defects in the theory of free hypermultiplets in [18].

3.3 Orbifold construction

3.3.1 Review of orbifold construction

Let us recall some generalities about orbifolds. In the physics and string theory context, orbifolding proceeds in the following steps:

- Start with a TQFT with a symmetry described a finite group G, defined on a manifold M. The fields of the theory may transform under some G representation.
- Gauge the G action, i.e. quotient by G and retain only the observables that are invariant under the group action.
- Include the twisted sectors, consisting of field configurations with non-trivial G monodromies around non-contractible loops in M. In the case of closed string theory, this corresponds to allowing strings to "*close up to a G action*".

In [23] the authors formulated the orbifold construction in the language of defects, initially in the context of two-dimensional conformal field theory and later generalised to TQFTs. While this construction is inspired by orbifolds associated with group actions,it is not restricted to them. This leads to what is commonly referred to as the *generalised orbifold* construction, whose outline we provide now.

Consider a defect TQFT (A.2.4)

$$\mathcal{Z}_{\mathrm{def}} : \mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}, \tag{3.3.1}$$

with $\mathbb{D}=(D_0,D_1,D_2,s,t,j)$. We define the *orbifold data* as a tuple $\mathcal{A}=(a,A,\mu,\Delta)$, with

- $a \in \mathcal{D}_{\mathcal{Z}_{\mathrm{def}}}$ an object in the bicategory assigned to $\mathcal{Z}_{\mathrm{def}}$,
- $A \in \mathcal{D}_{\mathcal{Z}_{\mathrm{def}}}(a,a)$, a 1-endomorphism of a,
- 2-morphisms $\mu : A\otimes A \longrightarrow A$, $\Delta : A \longrightarrow A\otimes A$, playing the role of *multiplication* and *comultiplication* respectively.

The 2-morphisms μ and Δ are required to satisfy the "*bubble omission*" and "*crossing*" conditions [23, Eq. (5)]. These conditions can be resolved using the duality data of A in a series of equations [16, Eq.'s (3.10-15)], which are equivalent to stating that A together with μ,Δ has the structure of a separable symmetric Frobenius algebra [16, Prop. (3.4)].

Then we can construct a closed two-dimensional TQFT (without defects), the *orbifold TQFT*

$$\mathcal{Z}_{\mathcal{A}}^{\mathrm{orb}} : \mathrm{Bord}_{2,1}^{\mathrm{or}} \longrightarrow \mathrm{Vect}, \tag{3.3.2}$$

which is labelled by the defect data $\mathcal{A}$ and defined as follows.

- Action of objects: A disjoint union of circles gets mapped to the assignment of A-decorated circles under $\mathcal{Z}_{\text{def}}$.

- Action on morphisms: Consider morphism $X : M \longrightarrow N$ in $\text{Bord}^{\text{or}}_{2,1}$. We first decorate M with a "fine enough" A-defect network t, where only the trivalent junctions μ and Δ are allowed. We call this decorated bordism $M^{A,t}$. Then we define $\mathcal{Z}^{\text{orb}}_{\mathcal{A}}(M) = \mathcal{Z}_{\text{def}}(M^{A,t})$, namely the bordism in the orbifolded theory is assigned to the image of decorated bordism $M^{A,t}$ under the unorbifolded theory.

Using that A carries the structure of a separable symmetric Frobenius algebra and that the defect network is fine enough, one finds that correlation functions become independent of the chosen network. Hence, the construction is well-defined as an oriented closed TQFT.

It turns out that the insertion of a defect network can be equivalently expressed using a triangulation of the surface the theory is defined on. Then the defect network corresponds to the Poincaré dual of the triangulation. This observation is key to generalising the orbifold construction to higher dimensions, as explained in [17].

3.3.2 Orbifolds with arbitrary target category

For group-type defects, the most natural choice for orbifold datum $\mathcal{A}$ is a certain type of direct sum over the group defects, whose definition we will make explicit shortly. This construction is known for Landau-Ginzburg models (see [16, Sect. 7] or [6]). Here we attempt to follow a similar approach for an orbifold construction in our TQFT valued in the truncated Rozansky-Witten category $\mathcal{C} = \mathrm{T}(\mathcal{RW}^{\text{aff}})$. In particular, having established the "component" Frobenius relations in section 3.2.2, we aim to construct a candidate orbifold datum, by first completing our category to include direct sums and then summing over group defects. However, this construction encounters an obstruction due to the absence of a $\mathbb{C}$ action on the direct sum completed category. Understanding whether and how this obstruction can be overcome remains an interesting question for future work.

Direct sum completion

Let $\underline{x}, \underline{y} \in \mathcal{C}$ be objects of the same length n. Assume also a finite number of 1-morphisms (suppressing surface variables to lighten notation) $(\underline{\alpha}_i;\ W_i) \equiv W_i : \underline{x} \longrightarrow \underline{y}$ for $i = 1, \ldots, p$. We complete $\mathcal{C}$ to include direct sums of 1-morphisms which we define to be *finite ordered lists of potentials*

$$\{W_1, \ldots, W_p\},$$

with $W_i \in \mathcal{C}(\underline{x}, \underline{y})$. We denote this direct sum completed category as $\overline{\mathcal{C}}^{\oplus}$. We will also suppress the index set labels, denoting $\{W_i\}_{i=1,\ldots,p} \equiv \{W_i\}$.

Let $\{W_i\}, \{V_j\} : \underline{x} \longrightarrow \underline{y}$ be 1-morphisms in $\overline{\mathcal{C}}^{\oplus}$. Then, for every component 1-morphism W_i, V_i in the original truncated category $\mathcal{C}$, we have 2-morphisms described by matrix factorisations of $W_i \longrightarrow V_j$ which are elements of $\text{hmf}\left(\mathbb{C}[\underline{\alpha}_i, \underline{\beta}_j, \underline{x}, \underline{y}], V_j - W_i\right)^{\omega}$. We define the 2-morphisms in $\overline{\mathcal{C}}^{\oplus}$ to be elements of the product category

$$\times_{i,j}\,\mathrm{hmf}\left(\mathbb{C}[\underline{\alpha}_i, \underline{\beta}_j, \underline{x}, \underline{y}], V_j - W_i\right)^{\omega}.$$

In other words, 2-morphisms in the direct sum completed category can be viewed as "matrices of matrix factorizations":

$$\{f_i^j\} \in \overline{\mathcal{C}}^{\oplus}(\{W_i\}, \{V_j\}), \quad \text{with} \quad f_i^j \in \mathrm{hmf}\left(\mathbb{C}[\underline{\alpha}_i, \underline{\beta}_j, \underline{x}, \underline{y}], V_j - W_i\right)^{\omega}.$$

Compositions of 2-morphisms

Let $\{W_i\}$, $\{V_j\}$, $\{Z_k\} : \underline{x} \longrightarrow \underline{y}$ be 1-morphisms in $\overline{\mathcal{C}}^{\oplus}$, 2-morphisms $\{f_i^j\} \in \overline{\mathcal{C}}^{\oplus}(\{W_i\}, \{V_j\})$, $\{g_j^k\} \in \overline{\mathcal{C}}^{\oplus}(\{V_j\}, \{Z_k\})$. Then we have compositions $g_j^k \otimes f_i^j \in \mathrm{hmf}\left(\mathbb{C}[\underline{\alpha}_i, \underline{\gamma}_k, \underline{x}, \underline{y}], Z_k - W_i\right)^{\omega}$, where the tensor product is taken over the intermediate variables $\underline{\beta}_j$, which are eliminated. We define the horizontal composition in $\overline{\mathcal{C}}^{\oplus}$ as the direct sum of these factorizations over all intermediate steps, namely the assignment:

$$(\{g_j^k\}, \{f_i^j\}) \mapsto \{h_i^k\} \in \overline{\mathcal{C}}^{\oplus}(\{W_i\}, \{Z_k\}).$$

Here, $h_i^k \in \mathrm{hmf}\left(\mathbb{C}[\underline{\alpha}_i, \underline{\gamma}_k, \underline{x}, \underline{y}], Z_k - W_i\right)^{\omega}$ is given by

$$h_i^k = \bigoplus_j g_j^k \otimes f_i^j,$$

with $\oplus$ the usual direct sum of matrix factorizations. Therefore composition is simply matrix multiplication of matrices of matrix factorizations. We can write them as matrices where the lower (source) index is the column index and the upper (target) index is the row index.

We define the identity 2-morphism $1_{\{W_i\}} \in \overline{\mathcal{C}}^{\oplus}(W_i, W_j)$ as $\{1_{W_i}\delta_{ij}\}$. In other words, it is a matrix of matrix factorisations, with identity factorizations on the diagonal and trivial factorizations everywhere else.

Direct sum and monoidal product

Let $\boxtimes$ be the composition of 1-morphisms in $\mathcal{C}$. Then we define a monoidal product $\boxtimes_{\oplus}$ in $\overline{\mathcal{C}}^{\oplus}$ as

$$(\bigoplus_i W_i) \boxtimes_{\oplus} (\bigoplus_j V_j) := \bigoplus_{i,j} W_i \boxtimes V_j. \tag{3.3.3}$$

The monoidal product $\boxtimes_{\oplus}$ acts on morphisms as follows: let $f = \{f_{i_1}^{i_2}\} \in \overline{\mathcal{C}}^{\oplus}(\{W_{1,i_1}\}, \{W_{2,i_2}\})$, $g = \{g_{j_1}^{j_2}\} \in \overline{\mathcal{C}}^{\oplus}(\{V_{1,j_1}\}, \{V_{2,j_2}\})$. Then we define

$$f \boxtimes_{\oplus} g : \bigoplus_{i_1,j_1} W_{1,i_1} \boxtimes V_{1,j_1} \to \bigoplus_{i_2,j_2} W_{2,i_2} \boxtimes V_{2,j_2}$$

as

$$(f \boxtimes_{\oplus} g)_{i_1 j_1}^{i_2 j_2} := f_{i_1}^{i_2} \boxtimes g_{j_1}^{j_2}$$

It is straightforward to verify that this definition of monoidal product is functorial and we provide a detailed proof in proposition B.2 of the appendix.

Associativity

Let $\{W_{1,i_1}\},\ \{W_{2,i_2}\},\ \{W_{3,i_3}\}$ be 1-morphisms in $\overline{\mathcal{C}}^{\oplus}$. Then we have

$$(\{W_{1,i_1}\} \boxtimes_{\oplus} \{W_{2,i_2}\}) \boxtimes_{\oplus} \{W_{3,i_3}\} = \{W_{1,i_1} \boxtimes W_{2,i_2}\} \boxtimes_{\oplus} \{W_{3,i_3}\} = \{(W_{1,i_1} \boxtimes W_{2,i_2}) \boxtimes W_{3,i_3}\}$$

and

$$\{W_{1,i_1}\} \boxtimes_{\oplus} (\{W_{2,i_2}\} \boxtimes_{\oplus} \{W_{3,i_3}\}) = \{W_{1,i_1}\} \boxtimes_{\oplus} \{W_{2,i_2} \boxtimes W_{3,i_3}\} = \{W_{1,i_1} \boxtimes (W_{2,i_2} \boxtimes W_{3,i_3})\}$$

The associator is thus induced by the associator of $\mathcal{C}$: let

$$\alpha_{W_{1,i_1},W_{2,i_2},W_{3,i_3}} : (W_{1,i_1} \boxtimes W_{2,i_2}) \boxtimes W_{3,i_3} \xrightarrow{\cong} W_{1,i_1} \boxtimes (W_{2,i_2} \boxtimes W_{3,i_3})$$

be the associator in $\mathcal{C}$. Then the associator in $\overline{\mathcal{C}}^{\oplus}$ has components $\{\alpha_{W_{1,i_1},W_{2,i_2},W_{3,i_3}}\}$ acting "diagonally" (ie supressing deltas). The pentagon follows componentwise from the pentagon in $\mathcal{C}$.

Orbifold data and Frobenius algebra

With the newly defined direct sum on 1-morphisms in $\mathcal{C}$, we define our candidate for orbifold datum. Let G be a finite group and pick

- an object $\underline{x} \in \mathcal{C}$,
- a 1-morphism $A_G = \{I_A\}_{A \in G}$, with I_A defined as in (3.1.46),
- $\mu = \{\mu^C_{A,B}\}_{A,B,C \in G}$, $\Delta = \{\Delta^{A,B}_C\}_{A,B,C \in G}$ defined as

  $$\mu^C_{A,B} = \mu_{A,B}\ \delta_{C,AB},\ \ \Delta^{A,B}_C = \Delta_{A,B}\ \delta_{AB,C} \tag{3.3.4}$$

  with $\mu_{A,B}$, $\Delta_{A,B}$ from definitions (3.1.48) and (3.1.49) respectively.

Then, using equations (3.2.25) - (3.2.28), it is straightforward to show that Frobenius and symmetry relations hold for A_G. The detailed computations can be found in propositions B.5, B.6.

However separability does not hold with the above definitions of μ and Δ. Instead, we obtain

$$\mu \otimes \Delta = \{\bigoplus_{A' \in G} 1_{I_A} \delta_{A,B}\}_{A,B \in G}, \tag{3.3.5}$$

as shown in proposition B.7, rather than the desired $1_{A_G} = \{1_{I_A}\ \delta_{A,B}\}_{A,B \in G}$. Interpreting the direct sum as copies of matrix factorisations, we see that the right-hand side of separability equation contains $|G|$ copies of each 1_{I_A} component. One can try to modify either μ or Δ by inserting a 2-morphism ψ on one of the legs of, say, Δ. Unfortunately, since there is no $\mathbb{C}$ action in this category, it is not possible to invert such 2-morphisms componentwise, thus the orbifold construction fails under these definitions.

Chapter 4

Conclusions and outlook

In this work, we employed the cobordism hypothesis with defects to construct the extended defect TQFT

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D}) \longrightarrow \mathcal{C}$$

in terms of data internal to $\mathcal{C}$, where $\mathcal{C}$ is the homotopy 2-category of Rozansky-Witten models with with flat targets $T^*\mathbb{C}^n$. For the case of defects that realise group actions corresponding to target space symmetries, we explicitly computed the invariants assigned by $\mathcal{Z}$ to closed surfaces Σ_g of genus g. The results recover those obtained with path integral or canonical quantisation techniques in physics literature. Furthermore, given a symmetry defect decoration of Σ_g that corresponds to a triangulation of Σ_g, we showed that the corresponding state spaces are invariant under Pachner moves. This is equivalent to the local conditions of (co)associativity, separability and Frobenius.

We also used the image of the category of 1-morphisms between the empty set and a circle decorated by a symmetry defect to recover the category of twisted line operators. In the full 3-dimensional theory this lifts to the category of line defects between the identity surface and a symmetry surface defect and is described by a homotopy category of matrix factorisations, which reduces to $\mathrm{sVect}_{\mathbb{C}}$ for generic symmetry generator.

Since we have triangulation independence in terms of line defects corresponding to symmetry generators, it is natural to attempt to gauge the symmetry with the generalised orbifold construction. This requires restricting the symmetry to a finite group, completing the category by allowing direct sums and summing over all group elements. The main obstruction in this construction is separability or "bubble removal", which produces $|G|$ copies of the orbifolding defect. In contrast to standard orbifold constructions with target Vect, this procedure cannot be "regularized", for example, by rescaling μ or Δ by a factor $|G|$. If such a redefinition were possible, it would provide the necessary ingredients to construct an orbifold TQFT for the truncated Rozansky-Witten models.

It would be interesting to further investigate ways to circumvent this obstruction to orbifolding. One option would be to pass to a completed version of the truncated target category in which rescalings by complex numbers become meaningful and separability holds. Such an enlargement could allow the generalised orbifold construction to be consistently defined while remaining within the truncated setting.

Appendix A

Bordism categories and TQFTs

In this section, we present the bordism categories considered in this text, restricting attention to two dimensions, which constitutes our main focus. We complete the discussion by stating the functorial definitions of the corresponding TQFTs.

A.1 Oriented TQFTs

Closed oriented bordism category

As a warmup, we begin with a brief review of the classic definition of 2d closed oriented TQFTs, which can also be found in standard references, like [15, 38]. The closed bordism category is defined as follows:

- *Objects*: finite disjoint unions Σ of oriented circles S^1.
- *Morphisms* $\Sigma_0 \longrightarrow \Sigma_1$: oriented surfaces with in boundary Σ_0 and out boundary Σ_1.

The most convenient presentation of the closed bordism category is in terms of generators and relations. The morphisms can then be decomposed into the basic building blocks: cups, caps, pairs of pants (both straight and inverse). The relations between these generators are given by (3.26)-(3.29) of [15].

Let $\mathcal{C}$ be a symmetric monoidal category.

Definition A.1: A *2-dimensional oriented closed TQFT* with values in $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathcal{C}. \tag{A.1.1}$$

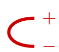
The most usual case is $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$. In this case, the images of the bordism category generators under the functor endow the vector space with the structure of an algebra and a coalgebra, while the relations translate into the algebra being (co)associative, (co)unital, Frobenius and commutative. Thus, 2-dimensional closed oriented TQFTs are in one-to-one correspondence with commutative Frobenius algebras. This serves as the primary example of a *classification result* for TQFTs.

Extended bordism category

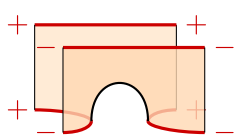
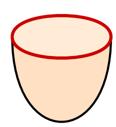
The extended bordism category is defined as follows:

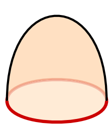
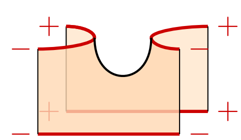
- *Objects*: finite disjoint unions of points with orientation (positive or negative).
- *1-morphisms*: oriented 1-dimensional bordisms, whose orientation is compatible with source and target points
- *2-morphisms*: diffeomorphism classes of 2-dimensional compact oriented bordisms with corners.

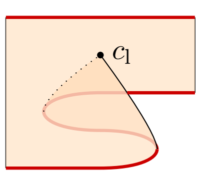
Again, it is convenient to describe this category via a generators-and-relations presentation. In the extended case, obtaining such a presentation is a highly non-trivial result. In particular, it was shown in [50] that the category is generated by $+$ and $-$ as objects, the elbows

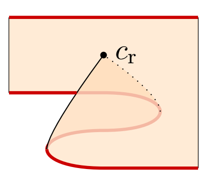
$\widetilde{\mathrm{ev}}_+ :=$  $: + \sqcup - \longrightarrow \varnothing$, $\widetilde{\mathrm{coev}}_+ :=$  $: \varnothing \longrightarrow - \sqcup +$ (A.1.2)

as 1-morphisms and

 $=: \mathrm{ev}_{\widetilde{\mathrm{ev}}_+}$,  $=: \mathrm{coev}_{\widetilde{\mathrm{ev}}_+}$, (A.1.3)

 $=: \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+}$,  $=: \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_+}$, (A.1.4)



,


. (A.1.5)

as 2-morphisms. The relations can be found in figure 3.8 of [50] and express that

- $+$ and $-$ are mutually dual,
- $\mathrm{coev}_+ :=$  $\cong$  (where the left-hand side involves the braiding $b_{-,+}$ which by definition is the mapping cylinder over the swap diffeomorphism $- \sqcup + \longrightarrow + \sqcup -$) is left and right adjoint to $\widetilde{\mathrm{ev}}_+$,
- $\mathrm{ev}_+ :=$  $\cong$  is left and right adjoint to $\widetilde{\mathrm{coev}}_+$,

- the cusps (A.1.5) are invertible, and compatible with the adjunction 1-morphisms (A.1.3)–(A.1.4).

Let $\mathcal{B}$ be a symmetric monoidal bicategory

Definition A.2: A *2-dimensional extended oriented TQFT* with values in $\mathcal{B}$ is a symmetric monoidal 2-functor

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{B}. \tag{A.1.6}$$

Relation to closed bordism category

We can recover the ordinary closed bordism category by "forgetting" the points $+, -$ and considering only endomorphisms of the unit object $\varnothing$. This is given by a functor Ω:

$$\mathrm{Bord}^{\mathrm{or}}_{2,1} \cong \Omega(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}) \coloneqq \mathrm{End}_{\mathrm{Bord}^{\mathrm{or}}_{2,1,0}}(\varnothing). \tag{A.1.7}$$

A.2 Extended defect TQFTs

Bordism category

We give a quick review of the defect bordism category. For full definition in n dimensions see [17] and reviews for 2 dimensions in [12] and [5, Sect. 3.1].

A defect bordism is a stratified bordism whose strata take values in finite label sets in a compatible way. This is encoded in the *defect data* $\mathbb{D} = (D_0, D_1, D_2, s, t, j)$, where

- D_j are sets which define the labels of j-dimensional strata,
- s, t are maps $D_1 \longrightarrow D_2$, the *source* and *target* maps, which define the source and target of every D_1 element (or *line defect*),
- j are junction maps, which define the neighbourhood of every D_0 element (or *point defect*). In other words, the junction maps define how line defects are allowed to meet locally. Formally, they map

$$j \colon D_0 \longrightarrow D_2 \sqcup \bigsqcup_{m \geqslant 1} \big((D_1 \times \{\pm\}) \times_{D_2} \cdots \times_{D_2} (D_1 \times \{\pm\})\big)/\mathbb{Z}_m. \tag{A.2.1}$$

  Here the product encodes which lines end in the point and with which orientation and the quotient accounts for symmetry under cyclic permutations. D_2 is target in case no line ends to the point defect (neighbourhood is a 2-dimensional stratum).

In fact, D_2 can be thought as a set 2-dimensional *theories* or *phases*, D_1 as a set of *line defects* between theories and D_0 as a set of *point defects* on which line defects are allowed to end.

The defect bordism category is defined as follows:

- Objects are objects of $\mathrm{Bord}^{\mathrm{or}}_{2,1,0}$ (finite disjoint union of points with orientation), together with labels from D_2 for each point.
- 1-morphisms are 1-morphisms of $\mathrm{Bord}^{\mathrm{or}}_{2,1,0}$, together with stratifications labelled by D_2 for 1-strata and D_1 for 0-strata, compatible with source and target maps s, t.
- 2-morphisms are diffeomorphism classes of stratified surfaces with corners, where the 1-strata are allowed to end transversally to the boundary, j-strata are labelled by D_j and labels. All labels have to be compatible with s, t and j maps.

Intuitively one can think of $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$ (see also [12, Eq. 2.10]) as the (pivotal) 2-category of closed TQFTs, line defects and local operators. Here local operators are understood in the topological sense as contractible neighbourhoods.

Relation to other bordism categories

Consider a subcategory of $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})$ where 1- and 2-morphisms are trivially stratified and an element $X \in D_2$ which labels all objects, 1- and 2-morphisms. We denote this subcategory by $\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})\Big|_X$. Then, by ignoring the labels X, the defect bordism category reduces to the oriented bordism category

$$\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})\Big|_X \cong \mathrm{Bord}^{\mathrm{or}}_{2,1,0}. \tag{A.2.2}$$

As in the case of extended bordisms, one can recover the non-extended bordism category by considering the endomorphisms of the unit object $\varnothing$:

$$\mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \cong \Omega(\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})) := \mathrm{End}_{\mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D})}(\varnothing). \tag{A.2.3}$$

Let $\mathcal{C}$ be a symmetric monoidal category and $\mathcal{B}$ a symmetric monoidal 2-category.

Definition A.3: A *2-dimensional defect TQFT* with values in $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}. \tag{A.2.4}$$

Definition A.4: A *2-dimensional extended defect TQFT* with values in $\mathcal{B}$ is a symmetric monoidal 2-functor

$$\mathcal{Z} : \mathrm{Bord}^{\mathrm{def}}_{2,1,0}(\mathbb{D}) \longrightarrow \mathcal{B}. \tag{A.2.5}$$

Appendix B

Orbifolds

B.1 Collection of proofs

Here we present in detail proofs for the statements of section 3.3.2

Proposition B.1: The direct sum of 1-morphisms in $\overline{\mathcal{C}}^{\oplus}$ as defined in 3.3.2 is a coproduct.

Proof. The direct sum of $\big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big)$ and $\big(\underline{\beta}; V(\underline{\beta}, \underline{x}, \underline{y})\big)$ is a list

$$\Big(\big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big),\ \big(\underline{\beta}; V(\underline{\beta}, \underline{x}, \underline{y})\big)\Big) \tag{B.1.1}$$

with canonical injections

$$\begin{aligned} i_W &= \begin{pmatrix} I_W \\ 0_{W(\underline{\alpha},\underline{x},\underline{y})\to V(\underline{\beta}',\underline{x},\underline{y})} \end{pmatrix}, \\ i_V &= \begin{pmatrix} 0_{V(\underline{\beta},\underline{x},\underline{y})\to W(\underline{\alpha}',\underline{x},\underline{y})} \\ I_V \end{pmatrix}, \end{aligned} \tag{B.1.2}$$

where we denote as usual the identity matrix factorisations $I_W = \big[\underline{\alpha}' - \underline{\alpha}, \underline{\Delta} W(\binom{\underline{\alpha}'}{\underline{\alpha}}, \underline{x}, \underline{y})\big]$ and $I_V = \big[\underline{\beta}' - \underline{\beta}, \underline{\Delta} V(\binom{\underline{\beta}'}{\underline{\beta}}, \underline{x}, \underline{y})\big]$ and the trivial matrix factorisation

$$0_{W(\underline{\alpha},\underline{x},\underline{y})\to V(\underline{\beta}',\underline{x},\underline{y})} = \big[\underline{1}, V(\underline{\beta}', \underline{x}, \underline{y}) - W(\underline{\alpha}, \underline{x}, \underline{y})\big]. \tag{B.1.3}$$

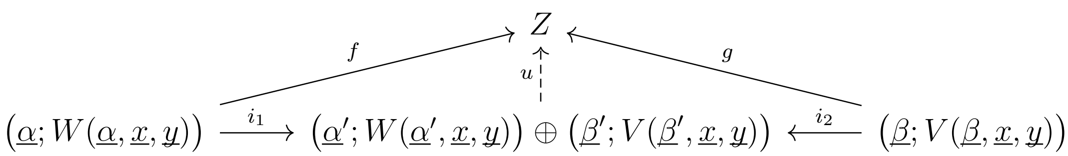
Consider then an arbitrary 1-morphism $Z : \underline{x} \longrightarrow \underline{y}$ in $\mathcal{C}$ and any 2-morphisms $f : \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big) \to Z,\ g : \big(\underline{\beta}; V(\underline{\beta}, \underline{x}, \underline{y})\big) \to Z$. These can be factored through the direct sum using the morphism $u : \big(\underline{\alpha}; W(\underline{\alpha}, \underline{x}, \underline{y})\big) \oplus \big(\underline{\beta}; V(\underline{\beta}, \underline{x}, \underline{y})\big) \to Z$ defined as

$$u = \{f, g\}$$

or we can write in matrix form

$$u = \begin{pmatrix} f & g \end{pmatrix}$$

This map makes the following diagram commute.



Indeed, we have for the left triangle (and similarly for the right triangle):

$$u \otimes i_1 = \begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} I_W \\ 0_{W(\underline{\alpha},\underline{x},\underline{y}) \to V(\underline{\beta},\underline{x},\underline{y})} \end{pmatrix} = (f \otimes I_W) \oplus \big(g \otimes 0_{W(\underline{\alpha},\underline{x},\underline{y}) \to V(\underline{\beta},\underline{x},\underline{y})}\big) \cong f \otimes I_W \cong f, \tag{B.1.4}$$

where we used that the fusion product of a trivial factorization with another factorization is again trivial. □

Proposition B.2: The monoidal product $\boxtimes_\oplus$ (3.3.3) is functorial:

$$(f' \otimes f) \boxtimes_\oplus (g' \otimes g) = (f' \boxtimes_\oplus g') \otimes (f \boxtimes_\oplus g),$$

for any 2-morphisms $f : \{W_{1,i_1}\} \longrightarrow \{W_{2,i_2}\}$, $f' : \{W_{2,i_2}\} \longrightarrow \{W_{3,i_3}\}$, $g : \{V_{1,j_1}\} \longrightarrow \{V_{2,j_2}\}$, $g' : \{V_{2,j_2}\} \longrightarrow \{V_{3,j_3}\}$ in $\overline{\mathcal{C}}^{\oplus}$.

Proof.

$$\begin{aligned}
\big((f' \otimes f) \boxtimes_\oplus (g' \otimes g)\big)^{i_3 j_3}_{i_1 j_1} &= (f' \otimes f)^{i_3}_{i_1} \boxtimes (g' \otimes g)^{j_3}_{j_1} \\
&= \Big(\bigoplus_{i_2} ((f')^{i_3}_{i_2} \otimes f^{i_2}_{i_1})\Big) \boxtimes \Big(\bigoplus_{j_2} ((g')^{j_3}_{j_2} \otimes g^{j_2}_{j_1})\Big) \\
&= \bigoplus_{i_2,j_2} ((f')^{i_3}_{i_2} \otimes f^{i_2}_{i_1}) \boxtimes ((g')^{j_3}_{j_2} \otimes g^{j_2}_{j_1}) \\
&= \bigoplus_{i_2,j_2} ((f')^{i_3}_{i_2} \boxtimes (g')^{j_3}_{j_2}) \otimes (f^{i_2}_{i_1} \boxtimes g^{j_2}_{j_1}) \\
&= \bigoplus_{i_2,j_2} (f' \boxtimes_\oplus g')^{i_3 j_3}_{i_2 j_2} \otimes (f \boxtimes_\oplus g)^{i_2 j_2}_{i_1 j_1} \\
&= \Big((f' \boxtimes_\oplus g') \otimes (f \boxtimes_\oplus g)\Big)^{i_3 j_3}_{i_1 j_1}
\end{aligned}$$

The first line is the definition of product, the second line is given by definition of composition, the third line is distributivity of horizontal composition in $\mathcal{C}$, the fourth line is functoriality of horizontal composition in the homotopy category of matrix factorisations, the fifth line is definition of product (backwards) and the sixth line is again definition of composition.

□

Proposition B.3: A_G is a unital algebra

Proof.

$$
\begin{aligned}
& & \mu \otimes (\eta \boxtimes_{\oplus} 1_{A_G}) \cong \lambda_{A_G} \\
\iff & & \{\mu^C_{A,B}\} \otimes (\{\eta^{A'}\} \boxtimes \{1_{I_{B'}}\})\delta_{A,A'}\delta_{B,B'} \cong \{\lambda_{I_A}\} \\
\iff & & \sum_{A,B} \mu_{A,B} \otimes (\eta^A \boxtimes 1_{I_B}) \cong \sum_A \lambda_{I_A} \\
\iff & & \sum_{A,B} \mu_{A,B} \otimes (\delta_{A,e} 1_{I_e} \boxtimes 1_{I_h}) \cong \sum_A \lambda_{I_A} \\
\iff & & \sum_{h} \mu_{e,h} \otimes (1_{I_e} \boxtimes 1_{I_h}) \cong \sum_A \lambda_{I_A}
\end{aligned}
$$

It remains to show that

$$\mu_{e,B} \otimes (1_{I_e} \boxtimes 1_{I_B}) \cong \lambda_{I_B}$$

By functoriality of the horizontal composition, we have

$$\mu_{e,B} \otimes (1_{I_e} \boxtimes 1_{I_B}) \cong \mu_{e,B} \otimes (1_{I_e \boxtimes I_B}) \cong \mu_{e,B}$$

and the rest follows by (3.1.52). For the right unit the proof is similar and it boils down to showing

$$\mu_{A,e} \cong \rho_{I_A},$$

which is direct application of (3.1.52). □

Proposition B.4: Associativity holds for orbifold datum candidate:

$$\mu \otimes (\mu \boxtimes 1_A) = \mu \otimes (1_A \boxtimes \mu) \tag{B.1.5}$$

Proof. The multiplication

$$\mu = \{\mu^C_{A,B}\}$$

is given by

$$\mu^C_{A,B} = \mu_{A,B}\delta_{C,AB}$$

and the unit

$$1_{A_G} = \{(1_{A_G})^B_A\} = \{1_{I_A}\delta_{A,B}\}$$

Then the proof follows using component associativity relations (3.2.26)

$$\begin{aligned}
\mu \otimes (\mu \boxtimes_{\oplus} 1_{A_G}) &= \{\mu^C_{A,B}\} \otimes (\{\mu^{C'}_{A',B'}\} \boxtimes_{\oplus} \{(1_{A_G})^{C''}_{A''}\}) \\
&= \{\bigoplus_{A,B} \mu^C_{A,B} \otimes (\mu^A_{A',B'} \boxtimes (1_{A_G})^B_{A''})\} \\
&= \{\delta_{A'B'A'',C}\ \mu_{A'B',A''} \otimes (\mu_{A',B'} \boxtimes 1_{I_{A''}})\} \\
&= \{\delta_{A'B'A'',C}\ \mu_{A',B'A''} \otimes (1_{I_{A'}} \boxtimes \mu_{B',A''})\} \\
&= \{\bigoplus_{A,B} \mu^C_{A,B} \otimes ((1_{A_G})^A_{A'} \boxtimes \mu^B_{B',A''})\} \\
&= \{\mu^C_{A,B}\} \otimes (\{(1_{A_G})^{C'}_{A'}\} \boxtimes_{\oplus} \{\mu^{C''}_{B',A''}\}) \\
&= \mu \otimes (1_{A_G} \boxtimes_{\oplus} \mu)
\end{aligned}$$

□

Proposition B.5: Frobenius relations hold for orbifold data $(\underline{x}, A_G, \mu, \Delta)$.

Proof. We want to show the relation

$$\Delta \otimes \mu = (1_A \boxtimes \mu) \otimes (\Delta \boxtimes 1_A). \tag{B.1.6}$$

We have shown in (3.2.25)

$$\Delta_{A,BC} \otimes \mu_{AB,C} = (1_A \boxtimes \mu_{B,C}) \otimes (\Delta_{A,B} \boxtimes 1_C)$$

The left hand side evaluates to

$$\begin{aligned}
\Delta \otimes \mu &= \Big(\frac{1}{|G|} \sum_{A,B} \Delta_{A,B}\Big) \otimes \Big(\sum_{A',B'} \mu_{A',B'}\Big) \delta_{AB,A'B'} \\
&= \frac{1}{|G|} \sum_{A,B,B'} \Delta_{A,B} \otimes \mu_{ABB'^{-1},B'} \\
&= \frac{1}{|G|} \sum_{A,B,C} \Delta_{A,BC} \otimes \mu_{AB,C}
\end{aligned}$$

and the RHS

$$\begin{aligned}
(1_A \boxtimes \mu) \otimes (\Delta \boxtimes 1_A) &= (\sum_A 1_A \boxtimes \sum_{B,C} \mu_{B,C}) \otimes (\sum_{A',B'} \Delta_{A',B'} \boxtimes \sum_{C'} 1_C) \delta_{A,A'} \delta_{B,B'} \delta_{C,C'} \\
&= \sum_{A,B,C} (1_A \boxtimes \mu_{B,C}) \otimes (\Delta_{A,B} \boxtimes 1_C) \\
&= \Delta \otimes \mu
\end{aligned}$$

□

Proposition B.6: Symmetry holds for orbifold data $(\underline{x}, A_G, \mu, \Delta)$

Proof. Follows from (3.2.27) □

Proposition B.7: For orbifold data $(\underline{x}, A_G, \mu, \Delta)$, a modified separability property holds:

$$\mu \otimes \Delta = \{\bigoplus_{A'} 1_{I_A}\ \delta_{A,B}\}_{A,B\in G} \tag{B.1.7}$$

Proof.

$$\begin{aligned}
\mu \otimes \Delta &= \{(\mu \otimes \Delta)^A_B\}_{A,B\in G} \\
&= \{\bigoplus_{A',B'} \mu^A_{A',B'} \otimes \Delta^{A',B'}_B\}_{A,B\in G} \\
&= \{\bigoplus_{A',B'} \mu_{A',B'} \otimes \Delta_{A',B'}\ \delta_{A,A'B'}\delta_{B,A'B'}\}_{A,B\in G} \\
&= \{\bigoplus_{A',B'} 1_{I_{A'B'}}\ \delta_{A,A'B'}\delta_{B,A'B'}\}_{A,B\in G} \\
&= \{\bigoplus_{A'} 1_{I_A}\ \delta_{A,B}\}_{A,B\in G}
\end{aligned}$$

□

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