Enhanced Area Laws for Entanglement Entropies Corresponding to Half-Filled Lowest Landau Levels

Dissertation

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Otto Leonard Wetzel

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Erstgutachter: Prof. Dr. Peter Müller
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Zusammenfassung

Die Verschränkungsentropie ist ein wichtiges Maß für den Grad an Verschränkung zwischen verschiedenen Teilen eines zusammengesetzten Quantensystems. Für ein System von Teilchen, das in einen beschränkten räumlichen Teilbereich und dessen Komplement aufgeteilt ist, besteht eine gängige Methode zur Untersuchung der entsprechenden bipartiten Verschränkungsentropie darin, ihr Skalierungsverhalten für groß werdende Teilbereiche zu bestimmen. Genauer gesagt ist das asymptotische Verhalten der Verschränkungsentropie für $L \to \infty$, wenn der räumliche Teilbereich mit einem Parameter $L > 0$ skaliert wird, von Interesse. Für d-dimensionale Vielteilchen-Quantensysteme in ihrem Grundzustand wächst die Verschränkungsentropie typischerweise asymptotisch proportional zur (Ober-)Fläche L^{d-1} der Teilregion, was als *Oberflächengesetz* bezeichnet wird. Ist das physikalische System im Grundzustand delokalisiert beziehungsweise elektrisch leitend, können auch sogenannte *verstärkte Oberflächengesetze* beobachtet werden, also Oberflächengesetze mit einem zusätzlichen Verstärkungsterm, welcher ebenfalls mit der Größe des räumlichen Teilbereichs wächst. In den meisten Fällen ist dieser Verstärkungsterm durch einen Logarithmus gegeben, so dass sich die Verschränkungsentropie für große Werte von L wie $L^{d-1} \log L$ verhält. Abgesehen von speziellen eindimensionalen Spinketten sind zum Zeitpunkt dieser Arbeit nach unserem Wissen keine Systeme bekannt, bei denen der Verstärkungsterm schneller als logarithmisch wächst. Das Hauptziel dieser Arbeit liegt darin, eine neue Familie von Beispielen für den Grundzustand eines nichtwechselwirkenden Fermionensystems zu beschreiben, der einem verstärkten Oberflächengesetz der Verschränkungsentropie unterliegt und bei dem der Verstärkungsterm – abhängig von einem Modellparameter – Wachstumsraten stärker als logarithmisch aufweisen kann.

Konkret handelt es sich bei dem von uns betrachteten System um das ideale Fermigas, das aus Teilchen in der zweidimensionalen Ebene $\mathbb{R}^2$ besteht, an die ein senkrechtes konstantes Magnetfeld der Stärke $B > 0$ angelegt ist, wobei ein einzelnes Teilchen durch den Landau-Operator H_B auf $L^2(\mathbb{R}^2)$ beschrieben wird. Als *niedrigstes Landau-Niveau* bezeichnen wir den unendlichdimensionalen Eigenraum zum niedrigsten Eigenwert B von H_B. Die Verschränkungsentropie bezüglich eines voll gefüllten niedrigsten Landau-Niveaus kann über den Spektralprojektor $1_{\{B\}}(H_B)$ ausgedrückt werden und folgt einem strengen Oberflächengesetz ohne Verstärkungsterm. In dieser Arbeit befassen wir uns mit bestimmten *halbgefüllten* Varianten des niedrigsten Landau-Niveaus, womit Unterräume des niedrigsten Landau-Niveaus mit halber Teilchenzahl, d.h. halber Teilchendichte gemeint sind. Solche halbgefüllten niedrigsten Landau-Niveaus sind durch geeignete Teilprojektoren von $1_{\{B\}}(H_B)$ charakterisiert. Wir zeigen, dass anstelle eines strengen Oberflächengesetzes anomale Verstärkungen des Oberflächengesetzes der zugehörigen Verschränkungsentropien auftreten, abhängig von der jeweiligen Art der Halbfüllung. Genauer gesagt konstruieren wir für jedes $\sigma \in [1, 2[$ eine Halbfüllung des niedrigsten Landau-Niveaus, die zu einer Skalierung der Verschränkungsentropie von $L^\sigma \log L$ führt. Für den Fall $\sigma = 1$, der einem gewöhnlichen logarithmisch verstärkten Oberflächengesetz entspricht, sind wir zudem in der Lage, den genauen asymptotischen Koeffizienten zu berechnen, welcher mit dem Koef-

fizienten der Verschränkungsentropie des Grundzustands freier Fermionen in einer Dimension und damit mit der eindimensionalen Widom-Formel zusammenhängt. Schließlich beweisen wir, dass eine Klasse an Halbfüllungen existiert, für die sich die Verschränkungsentropie asymptotisch wie L^2 verhält, was einem für Grundzustände sehr untypischen *Volumengesetz* entspricht.

Summary

Entanglement entropy is an important quantifier for the amount of entanglement between different parts of a composite quantum system. For a system of particles partitioned into a bounded spatial subregion and its complement, one common way of studying the corresponding bipartite entanglement entropy is to determine its scaling behavior as the subregion becomes large. More precisely, if we scale the subregion by a parameter $L > 0$, we are interested in the asymptotic behavior of the entanglement entropy as $L \to \infty$. For d-dimensional many-body quantum systems in their ground state, the entanglement entropy is typically found to grow asymptotically proportional to the (surface) area L^{d-1} of the subregion, which is referred to as an *area law*. When the physical system is delocalized or electrically conductive in its ground state, so-called *enhanced area laws* can also be observed, that is, area laws with an additional enhancement term that also grows with the size of the spatial subregion. In most cases, this enhancement is given by a logarithm, i.e. the entanglement entropy grows like $L^{d-1} \log L$ for large values of L. Apart from specific one-dimensional spin chains, to our knowledge, no systems with enhancements stronger than logarithmic are known at the time of publication of this thesis. The main goal of this thesis is to provide a new family of examples for the ground state of a non-interacting fermion system that is subject to an enhanced area law of the entanglement entropy and where – depending on a model parameter – the enhancement can exhibit growth rates stronger than logarithmic.

Specifically, the system we consider is the ideal Fermi gas consisting of particles in the two-dimensional plane $\mathbb{R}^2$ subject to a perpendicular constant magnetic field of strength $B > 0$, where a single particle is described by the Landau Hamiltonian H_B on $L^2(\mathbb{R}^2)$. We refer to the infinite-dimensional eigenspace of H_B corresponding to its lowest eigenvalue B as the *lowest Landau level*. The entanglement entropy corresponding to the fully-filled lowest Landau level can be written in terms of the spectral projection $1_{\{B\}}(H_B)$ and is known to obey a strict area law without enhancement. In this thesis, we are concerned with certain *half-filled* lowest Landau levels, that is, subspaces of the lowest Landau level with only half the number of particles, in terms of particle density. Such half-filled lowest Landau levels are characterized by suitable sub-projections of $1_{\{B\}}(H_B)$. We show that, instead of a strict area law, anomalous enhancements of the area law of the corresponding entanglement entropies arise, depending on the particular way of half-filling. More precisely, for each $\sigma \in [1, 2[$, we construct a half-filling of the lowest Landau level that gives rise to an entanglement entropy scaling of $L^\sigma \log L$. In the case $\sigma = 1$, corresponding to a regular logarithmically enhanced area law, we are able to calculate the precise asymptotic coefficient, which is related to the coefficient of the ground state entanglement entropy of free fermions in one dimension and therefore to the one-dimensional Widom formula. Finally, we prove that there exists a class of half-fillings for which the entanglement entropy behaves asymptotically as L^2, corresponding to a *volume law* very uncommon for ground states.

Preface

The present thesis is divided into five chapters and two appendices. Chapter 1 provides an introduction to the topics of this thesis. In Chapter 2, we collect some preliminaries for the rest of the thesis. Chapter 3 then contains a detailed exposition of the two main results, the proof of which can be found in the two subsequent Chapters 4 and 5. The first appendix, Appendix A, contains some auxiliary inequalities and general properties. Finally, in Appendix B, we provide some general information on entanglement entropy that might be of independent interest.

The results presented in Chapters 3 to 5 were obtained in scientific collaboration with Peter Müller and will be part of a future publication (see below). This publication will coincide in most part, both in content and writing, with the aforementioned chapters of this thesis. The proofs of the results as well as the first draft of the publication were created by the author of this thesis.

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Contents

1. Introduction

1.1. The notion of entanglement entropy and historical overview of results

Quantum entanglement is one of the most striking features of quantum mechanics and central to the fundamental disparity between classical and quantum physics. When two or more quantum systems are entangled, their joint state is correlated in such a way that the state of each subsystem cannot be fully described independently of the others, even if separated by large distances. This quantum mechanical phenomenon that has no counterpart in classical physics was first discovered by Einstein, Podolsky and Rosen [EPR35] and investigated more profoundly by Schrödinger [Sch35, Sch36] shortly thereafter. It is the object of intensive study in many branches of modern research such as statistical mechanics, quantum information theory, quantum computing and many-body quantum mechanics [HHHH09].

While the notion of entanglement between a small number of well-defined subsystems (such as a pair of qubits in quantum computing) already captures the essential departure from classical correlations, the situation becomes considerably richer and more subtle as the size of the system increases. In extended or many-body systems, one is often less interested in whether entanglement exists at all—since it typically does—than in how it is distributed across different parts of the system. A natural question that arises is how to reasonably quantify the present quantum entanglement. Out of several quantities that try to answer this question [PV07], one of the simplest, yet most important, is the concept of *bipartite entanglement entropy.* It serves as a measure for the degree of entanglement between two parts of a composite quantum system [AFOV08, CCD09].

Given a *bipartition* of a system into two subsystems A and B, consider a pure state ρ describing the full system. The bipartite entanglement entropy is defined as the entropy of the state $\rho_A := \operatorname{tr}_B \rho$ reduced to the subsystem A. Here, $\operatorname{tr}_B(\cdot)$ denotes the partial trace with respect to the subsystem B and can be thought of as discarding any information about the subsystem B encoded in ρ. Moreover, *entropy* refers to the *von Neumann entropy*

$$S(A) = S(A,\rho) := -\operatorname{tr}_A(\rho_A \log \rho_A), \tag{1.1}$$

where $\operatorname{tr}_A(\cdot)$ now is the usual trace on system A, or more generally to the γ*-Rényi entropies*

$$S_\gamma(A) = S_\gamma(A,\rho) := \frac{1}{1-\gamma} \log \operatorname{tr}_A \rho_A^\gamma, \quad \gamma \in]0,\infty[\setminus \{1\}, \tag{1.2}$$

which satisfy $\lim_{\gamma\to 1} S_\gamma(A,\rho) = S(A,\rho)$. We have $S(A) = 0$ if and only if ρ_A is a pure state, in which case no entanglement between A and B is present. In contrast, a large value of $S(A)$ indicates that the two subsystems A and B are significantly entangled. We remark that the role of A and B above is interchangeable: as ρ_A and $\rho_B := \operatorname{tr}_A \rho$ share the same non-zero eigenvalues (counted with multiplicity), their corresponding (von Neumann or γ-Rényi) entropies coincide [BBPS96].

For systems involving many particles, the exact computation of the corresponding entanglement entropies is often not feasible. Even without interactions between the particles, one can in general only hope for upper and lower bounds or asymptotic results. If we partition a quantum system in d-dimensional space $\mathbb{R}^d$ with respect to a bounded spatial subregion $\Lambda \subset \mathbb{R}^d$ and its complement Λ^c, one common way of studying the corresponding bipartite entanglement entropy $S(\Lambda)$ is to determine its asymptotic behavior as Λ becomes large. In other words, if we consider the scaled subregion $\Lambda_L := L \cdot \Lambda$ for some scaling parameter $L > 0$, we are interested in how $S(\Lambda_L)$ behaves asymptotically as $L \to \infty$. It turns out that different types of *scaling laws* can occur [ECP10, Laf16]. Unlike the entropy of a thermal state in statistical mechanics, for ground states of quantum many-body systems, the entanglement entropy is generally not extensive, that is, does not grow proportionally to the volume L^d of Λ_L. Instead, ground state entanglement entropy is typically encountered to exhibit an *area law* $S(\Lambda_L) \sim L^{d-1}$, meaning that it grows proportionally to the (surface) area of the subregion. For some systems, so-called *enhanced area laws* are observed, referring to an area law scaling with an additional enhancement term that also grows with the size of the subregion. In most of the known cases, this enhancement is logarithmic, i.e. we have $S(\Lambda_L) \sim L^{d-1} \log L$.

The rigorous mathematical study of scaling properties of the entanglement entropy has received considerable attention over the last two decades, especially with regard to ground states of non-interacting Fermi gases. An important milestone in these developments was the discovery of a formula that allows to express the entanglement entropy of such ground states Φ purely in terms of the corresponding one-particle Hamiltonian H, see e.g. [Kli06]. Let us consider the special case where the underlying one-particle Hilbert space is given by $L^2(\Gamma)$ for some bounded measurable $\Gamma \subset \mathbb{R}^d$. Due to the Pauli exclusion principle for fermions, the many-body ground state is obtained by adding particles to the system until all stationary states (eigenfunctions) of H with energies up to a *Fermi energy* $E_F \in \mathbb{R}$ are occupied [Sol14]. If $S(\Lambda, \Phi)$ denotes the bipartite entanglement entropy of the (pure) state $\Phi = \Phi(E_F)$ with respect to the spatial subregion $\Lambda \subset \Gamma$ as defined in (1.1), we have the formula

$$S(\Lambda, \Phi) = \operatorname{tr}_{L^2(\Gamma)} h(1_\Lambda 1_{]-\infty, E_F[}(H) 1_\Lambda). \tag{1.3}$$

Here, $1_{]-\infty,E_F[}(H)$ denotes the spectral projection of H called *Fermi projection*, 1_Λ is the operator of multiplication with the corresponding indicator function on $\mathbb{R}^d$ and $h\colon [0,1] \to [0, \log 2]$ is the *von Neumann entropy function* defined by

$$h(\lambda) := -\lambda \log \lambda - (1-\lambda)\log(1-\lambda), \quad \lambda \in [0,1]. \tag{1.4}$$

Throughout this thesis, log denotes the *natural logarithm*. A similar formula as (1.4) holds for the corresponding γ-Rényi entropies (1.2), with the function h replaced by h_γ defined in (2.65) below. A detailed derivation of the one-particle formula (1.3) is provided in Appendix B.

When trying to define entanglement entropy for systems in infinite continuous position space $\mathbb{R}^d$, $d \in \mathbb{N}$, one encounters the following obstacle: For such systems, a many-body ground state Φ in the strict sense as described above may not exist. In particular, this is the case if the spectrum of the corresponding single-particle Hamiltonian is purely essential. One important example is the free Fermi gas in $\mathbb{R}^d$, $d \in \mathbb{N}$ at zero temperature, where the single-particle Hamiltonian is given by the negative Laplacian $-\Delta$ on $L^2(\mathbb{R}^d)$. Another example – which is the system central to this thesis – is the Fermi gas of particles confined to the Euclidean plane $\mathbb{R}^2$ perpendicular to a constant magnetic field, where the single-particle Hamiltonian is the Landau Hamiltonian (2.1). An appropriate way of dealing with this would be restricting the respective Hamiltonian to a finite volume Γ first, in which case one obtains proper eigenfunctions for the resulting restricted Hamiltonian H_Γ to construct a well-defined finite-particle ground state. In the end, one can then let $\Gamma \nearrow \mathbb{R}^d$ and define the entanglement entropy for the full system as the limit

$$\lim_{\Gamma \nearrow \mathbb{R}^d} \operatorname{tr}_{L^2(\Gamma)} h(1_\Lambda 1_{]-\infty,E_F[}(H_\Gamma) 1_\Lambda). \tag{1.5}$$

Mathematically, it is a very difficult task to carry out this limit and to prove that it equals (1.3) with $\Gamma = \mathbb{R}^d$. We circumvent this issue by *defining* entanglement entropy for self-adjoint Hamiltonians on $L^2(\mathbb{R}^d)$ in that way, i.e. we put

$$S(H, E_F, \Lambda) := \operatorname{tr}_{L^2(\mathbb{R}^d)} h(1_\Lambda 1_{]-\infty,E_F[}(H) 1_\Lambda). \tag{1.6}$$

For both of the mentioned examples, that is, free particles and particles in a magnetic field, this quantity is indeed well-defined. The same definition is used for a much larger class of non-interacting Fermi gases in $\mathbb{R}^d$ with more general self-adjoint one-particle Hamiltonians, to some of which we come back below.

The study of asymptotics of traces for $\Lambda \to \mathbb{R}^d$ similar to the right-hand side of (1.6) originated in the early 20th century with the work of Szegő on discrete variants of the involved operators. In [Sze15], he established an asymptotic formula for determinants of truncated Toeplitz matrices as the truncation parameter tends to infinity. Successive publications covered traces of more general test functions of such Toeplitz matrices [Sze20] and provided the next term in the asymptotic expansion [Sze52]. For further reference and discussion of these *classical Szegő asymptotics* in the discrete setting, see the survey article [Kra11].

More relevant to this thesis are the continuous analogues of truncated Toeplitz matrices, namely *truncated Wiener-Hopf operators* of the form

$$T_L(a) = 1_{\Lambda_L} \mathscr{F}^* a \mathscr{F} 1_{\Lambda_L}, \tag{1.7}$$

acting on the space of Schwartz functions $\mathscr{S}(\mathbb{R}^d)$. Here, $\mathscr{F}$ denotes the (unitary) Fourier transform on $L^2(\mathbb{R}^d)$, $a\colon \mathbb{R}^d \to \mathbb{C}$ is a function called *symbol* and $\Lambda_L \subset \mathbb{R}^d$ is some sufficiently regular domain scaled by the parameter $L > 0$. A *Szegő-type asymptotics* for such operators is an asymptotic expansion of the quantity

$$\operatorname{tr}_{L^2(\mathbb{R}^d)} f(T_L(a)) \tag{1.8}$$

as $L \to \infty$ for a preferably large class of test functions f. One can also consider more general *pseudo-differential operators* in place of $T_L(a)$.

Today, there exists a wide range of results regarding such Szegő-type asymptotics. For sufficiently smooth symbols $a\colon \mathbb{R}^d \to \mathbb{R}$ and sufficiently nice Λ, one has a two-term asymptotic expansion

$$\operatorname{tr}_{L^2(\mathbb{R}^d)} f(T_L(a)) = A_0 L^d + A_1 L^{d-1} + o(L^{d-1}) \tag{1.9}$$

as $L \to \infty$ with some explicit coefficients $A_0 = A_0(f, a, \Lambda)$ and $A_1 = A_1(f, a, \partial\Lambda)$, where $\partial\Lambda$ denotes the boundary of Λ. The leading and subleading order terms are called *volume* *term* and *surface term*, respectively. Formula (1.9) was first proved by Kac [Kac54] in dimension $d = 1$ for the case where f is the natural logarithm (yielding an expansion for the determinant of $T_L(a)$) and then extended to higher dimensions in [Wid60, Lin75, Wid74], each time under different assumptions. A very general version of (1.9) allowing for all analytic test functions f was proved in [Wid80], where also matrix-valued symbols were considered. With the increased interest in fermionic entanglement entropy, this result was extended even further to a class of test functions which also includes entropy functions like (1.4), see [Sob17] for the scalar-valued and [FL25] for the matrix-valued case. Depending on the shape of the boundary $\partial\Lambda$ and the smoothness of the test function, in some situations it is possible to determine further lower-order terms in the asymptotic expansion [Roc84, Wid85, Die18, Pfi19].

The situation changes when the smooth symbol is replaced by a discontinuous one, more specifically by one with a jump discontinuity. In this case, instead of (1.7), one considers the truncated Wiener-Hopf operator

$$T_L(a) = 1_{\Lambda_L} \mathscr{F}^* a 1_\Gamma \mathscr{F} 1_{\Lambda_L} \tag{1.10}$$

for some sufficiently regular, bounded domain $\Gamma \subset \mathbb{R}^d$. While it is still assumed that a itself is smooth, a $(d-1)$-dimensional jump discontinuity is introduced by multiplying a with the indicator function 1_Γ. The operator $\mathscr{F}^* a 1_\Gamma \mathscr{F}$ is a generalization of $1_{]-\infty, E_F[}(-\Delta)$, which we will come back to below. In this situation, the corresponding Szegő-type asymptotics has the form

$$\operatorname{tr}_{L^2(\mathbb{R}^d)} f(T_L(a)) = A_0 L^d + W_1 L^{d-1} \log L + o(L^{d-1}) \tag{1.11}$$

as $L \to \infty$, with a different explicit coefficient $W_1 = W_1(f, a, \partial\Lambda, \partial\Gamma)$. We see that the previous area term of (1.9) is now replaced by a logarithmically enhanced area term.

The development of formula (1.11) has a history ranging back to the 1980s and has been substantially slower than in the smooth case (where no indicator 1_Γ is present). In one spatial dimension $d = 1$, a first result was given in [LW80] for the constant symbol $a \equiv 1$. This was followed shortly after by [Wid82], where more general symbols were covered and an improved error term for (1.11) of constant order was provided, possible thanks to a different proof strategy. Moreover, with the latter article, Widom was the first to conjecture both the formula (1.11) for general dimensions $d \in \mathbb{N}$ and the corresponding explicit expressions for the asymptotic coefficients. The initial step towards proving the higher-dimensional case – then referred to as *Widom's conjecture* – was made in [Wid90] by establishing the result for the special case where Γ is a half space. It took more than two additional decades before Widom's conjecture was finally proved for all dimensions $d \in \mathbb{N}$ and all smooth test functions f by Sobolev in his seminal works [Sob13, Sob15]. The formula (1.11) is now known as *Widom's formula* or *Widom-Sobolev formula*. A generalization to the case of matrix-valued symbols was considered in [BM24].

Having discussed the history of Szegő-type asymptotics, we now return to the ground state entanglement entropy of the free Fermi gas at zero temperature as defined by (1.6) with $H := -\Delta$. It was suggested by various authors in the physics literature, notably in [Wol06], that it should be subject to a logarithmically enhanced area law. With the discovery of the single-particle formula (1.3) for fermionic entanglement entropies, the connection to Widom's formula (1.11) became apparent: the operator on the right-hand side of (1.6) is equal to the truncated Wiener-Hopf operator in (1.10) with $\Gamma = \{k \in \mathbb{R}^d : |k|^2 \leqslant E_F\}$ and $a \equiv 1$. The Widom formula in this special case asserts that for every Fermi energy

$E_F > 0$,

$$\operatorname{tr}_{L^2(\mathbb{R}^d)} f(1_{\Lambda_L} 1_{<E_F}(-\Delta) 1_{\Lambda_L}) = N_0(E_F) f(1) |\Lambda| L^d + \Sigma_0(E_F) I(f) |\partial\Lambda| L^{d-1} \log L + o(L^{d-1} \log L) \tag{1.12}$$

as $L \to \infty$, where $1_{<E_F} := 1_{]-\infty, E_F[}$ and the asymptotic coefficients are given by

$$N_0(E_F) := \frac{1}{2\pi} \frac{1}{\Gamma(\frac{d}{2}+1)} (\pi E_F)^{d/2}, \quad \Sigma_0(E_F) := \frac{2}{\Gamma(\frac{d+1}{2})} \left(\frac{E_F}{4\pi}\right)^{(d-1)/2} \tag{1.13}$$

and

$$I(f) := \frac{1}{\pi^2} \int_0^1 \frac{f(\lambda) - \lambda f(1)}{\lambda(1-\lambda)} \, d\lambda. \tag{1.14}$$

At the time when formula (1.3) was discovered, Widom's conjecture had not yet been solved. However, it provided strong evidence [GK06, Gio06, HLS11] that the ground state entanglement entropy of the free Fermi gas should satisfy

$$\begin{aligned} S(-\Delta, E_F, \Lambda_L) &= \operatorname{tr}_{L^2(\mathbb{R}^d)} h(1_{\Lambda_L} 1_{<E_F}(-\Delta) 1_{\Lambda_L}) \\ &= \Sigma_0(E_F) I(h_\gamma) |\partial\Lambda| L^{d-1} \log L + o(L^{d-1} \log L) \end{aligned} \tag{1.15}$$

as $L \to \infty$, where h is the von Neumann entropy function (1.4). A rigorous proof of (1.15) was given by Leschke, Sobolev and Spitzer in [LSS14], who extended the previously mentioned result by Sobolev [Sob13] for smooth test functions to also cover functions merely Hölder continuous at the endpoints $\lambda = 0$ and $\lambda = 1$, covering the entropy functions h_γ. As a crucial ingredient for their proof served Schatten-von Neumann q-quasinorm estimates with $q < 1$ for pseudo-differential operators, which had recently been developed by Sobolev in [Sob14]. In their later publications [LSS17, LSS22], the authors also consider the case of equilibrium states at positive temperature, for which there is no logarithmic enhancement.

The above result for the free Fermi gas led to a wide range of subsequent works covering ground states of more general non-interacting Fermi gases in $\mathbb{R}^d$. A natural generalization lies in moving from free fermions to fermions in an external field, which mathematically amounts to studying the asymptotics of the operator on the right-hand side (1.6) for a more general class of self-adjoint Hamiltonians H.

First, there are various results for fermions in an external *electric field*, which can be described by the single-particle Hamiltonian $H = -\Delta + V$ for some (electrical) potential $V\colon \mathbb{R}^d \to \mathbb{R}$. In [MS20, MS23] a bounded, compactly supported potential V was considered. Based on their intuition that a bounded potential should not yield significantly stronger entanglement-induced correlations, the authors proved that (1.15) stays valid for H in place of $-\Delta$, thereby establishing an enhanced area law with identical asymptotic coefficient. They employed a perturbative approach which is expected to also work for non-compactly supported potentials with decay sufficiently fast at infinity. In [PS18b], a logarithmically enhanced area law was proved in the case of a smooth periodic potential and $d = 1$, provided the Fermi energy E_F falls into the interior of one of the spectral bands of H. The proof strategy for the latter result follows that of the classical one-dimensional Widom formula.

Another series of results of particular importance in the context of this thesis addresses the case of fermions in an external *magnetic field*. In [LSS21], the ideal Fermi gas of non-interacting particles confined to the Euclidean plane $\mathbb{R}^2$ perpendicular to a constant magnetic field was considered. In this situation, a single particle is governed by the Landau Hamiltonian H_B, see (2.1) below, where $B > 0$ is the strength of the magnetic field. The authors showed that the ground state entanglement entropy is subject to a strict area law. In fact, they established the asymptotic formula

$$S_\gamma(H_B, E_F, \Lambda_L) = \operatorname{tr}_{L^2(\mathbb{R}^d)} h_\gamma(1_{\Lambda_L} 1_{<E_F}(H_B) 1_{\Lambda_L}) = |\partial\Lambda| M_\nu(h_\gamma)\sqrt{B}L + o(L) \qquad (1.16)$$

as $L \to \infty$ at any Fermi energy $E_F > B$. The (explicit) coefficient $M_\nu(h_\gamma)$ depends on B and E_F only through the natural number $\nu = \nu(E_F) := \lceil (E_F/B - 1)/2 \rceil - 1 \in \mathbb{N}_0$ capturing the amount of Landau levels occupied in the Fermi gas (here $\lceil \cdot \rceil$ denotes the ceiling function and $\mathbb{N}_0$ the set of natural numbers including 0). This central result was followed by several related works: In [Pfe21] it was shown that (1.16) remains stable under small perturbations of the magnetic field and also under the influence of a small electric field. The article [PS24a] is concerned with a three-dimensional generalization where the particles are able to move freely in the direction of the magnetic field, resulting in a logarithmic enhancement of the area law. The delicate situation of the joint asymptotics with a vanishing magnetic field $B \to 0$ and $L \to \infty$ was considered in [PS24b].

Finally, we also mention the recent publications [FL25, FLS24, BM24, BM25] on the entanglement entropy of free relativistic fermions, where the single-particle Hamiltonian is given by the free Dirac operator.

1.2. Anomalous enhancements of the area law

For all the systems of non-interacting fermions discussed above, we encountered either an area law or a logarithmically enhanced area law of the entanglement entropy. With regards to the physical intuition, one expects an enhancement of the area law typically for states in which there is quantum transport or conductivity at the Fermi surface. Common examples for this are situations where the Fermi energy E_F falls inside the absolutely continuous spectrum of the single-particle Hamiltonian H. The generalized eigenstates in the absolutely continuous spectrum are delocalized and are therefore able to contribute to electrical conductivity. For the opposite situation when E_F falls inside a spectral gap of H or inside a spectral region of complete (Anderson) localization [GK04], no current is able to flow and one expects to encounter a strict area law without any enhancement term. We emphasize that this heuristic for the presence or absence of an enhancement has to be treated with care. For example, the presence of continuous spectrum is not necessary for a logarithmic enhancement to an area law [MPS20].

On the mathematical side, the occurrence of an enhancement to the area law is related to the off-diagonal decay of the integral kernel of the Fermi projection $1_{<E_F}(H)$. If this decay is sufficiently fast, e.g. exponential, we expect no enhancement to be present. In contrast, if the decay is very slow, e.g. like $1/|\cdot|^{(d+1)/2}$, d being the spatial dimension, we expect a logarithmic enhancement to the area law. To give just two examples from the systems discussed so far, it is well-known that the kernel of $1_{<E_F}(-\Delta)$ in (1.15) in one dimension for any $E_F > 0$ satisfies

$$1_{<E_F}\left(-\frac{d^2}{dx^2}\right)(x,y) = \frac{\sin[\sqrt{E_F}(x-y)]}{\pi(x-y)} \sim \frac{1}{|x-y|}, \qquad (1.17)$$

while the kernel of $1_{<E_F}(H_B)$ in (1.16) has Gaussian off-diagonal decay

$$1_{<E_F}(H_B)(x,y) \sim e^{-|x-y|^2} \tag{1.18}$$

for any Fermi energy $E_F > B$, see [LSS21]. With this intuition regarding the occurrence of strict area laws versus enhanced area laws in mind, it is natural to ask the following question:

Are there systems with enhancements to the area law stronger than logarithmic?

To our knowledge, such anomalous enhancements have only been observed for certain one-dimensional spin chains, where any growth rate of the enhancement up to L can be realized [MS16, RRS14, ZAK17]. These systems are of peculiar nature and have been specifically designed to realize the desired anomalous behavior. In this thesis, we will give an affirmative answer to the above question for a quasi-free Fermi gas with a highly degenerate ground state.

As a starting point, we recall the result (1.16) from the end of the previous subsection for the ground state entanglement entropy of the two-dimensional Fermi gas perpendicular to a constant magnetic field of strength $B > 0$. The corresponding single-particle Hamiltonian is the Landau Hamiltonian H_B, whose spectrum consists of evenly spaced, isolated eigenvalues $B(2n+1)$, $n \in \mathbb{N}_0$, each of infinite multiplicity, see Chapter 2.1.3 below. The associated infinitely-degenerate eigenspaces are called *Landau levels*. If P_n denotes the projection onto the n-th Landau level, the Landau Hamiltonian can be written in terms of its spectral decomposition

$$H_B = B \sum_{n\in\mathbb{N}_0} (2n+1) P_n, \tag{1.19}$$

which dates back to Fock [Foc28] and Landau [Lan30]. Consequently, for the Fermi projection $1_{]-\infty,E_F[}(H_B)$ we have

$$1_{]-\infty,E_F[}(H_B) = \sum_{n\in\mathbb{N}_0} 1_{]-\infty,E_F[}(B(2n+1)) P_n = \sum_{n=0}^{\nu} P_n =: P_{\leqslant\nu}, \tag{1.20}$$

where $\nu := \lceil (E_F/B - 1)/2 \rceil - 1$. The amount of occupied Landau levels ν depends on the Fermi energy E_F: The uninteresting case $E_F \leqslant B$, where the Fermi projection is just the zero operator, corresponds to a vanishing number of occupied Landau levels. Likewise, if $(2\nu+1)B < E_F \leqslant (2\nu+3)B$ for some $\nu \in \mathbb{N}_0$, this means that the first $\nu+1$ Landau levels are occupied. A particular Landau level is thus either occupied fully (if E_F is larger than the corresponding eigenvalue) or not at all.

From a physical point of view, in the infinite-particle ground state characterized by the Fermi projection $1_{]-\infty,E_F[}(H_B)$, there is no conductivity or quantum transport present, as it needs additional energy of $2B$ to excite a particle from the highest occupied Landau level to the next. Therefore, the strict area law (1.16) is expected from the heuristics described above. The key idea for the results of this thesis is to consider ground states where a Landau level is only "half-occupied" or "half-filled". If only half of the eigenstates in a particular Landau level are occupied, quantum transport within this Landau level is possible, and we might observe a potential violation of the area law of the entanglement entropy. This idea is in fact not restricted to half-filled lowest Landau levels, but more generally works for different kinds of partial fillings. We here stick to half-fillings for simplicity.

Let us briefly look at the construction of such half-fillings of a Landau level relevant in the context of this thesis. Since we are dealing with an infinite-area ground state and therefore infinitely many particles, we restrict the ground state to some bounded measurable subregion $\Lambda \subset \mathbb{R}^2$ and obtain the corresponding localized Fermi projection $1_\Lambda 1_{]-\infty,E_F[}(H_B)1_\Lambda = 1_\Lambda P_{\leqslant\nu}1_\Lambda$. The (mean) number of particles $N(\Lambda)$ inside Λ is obtained by calculating the trace of this operator, which amounts to integrating the diagonal of its integral kernel:

$$N(\Lambda) = \operatorname{tr}_{L^2(\mathbb{R}^2)}(1_\Lambda P_{\leqslant\nu}1_\Lambda) = \int_\Lambda P_{\leqslant\nu}(x,x)\,dx = (\nu+1)\frac{B}{2\pi}|\Lambda|, \tag{1.21}$$

see [LSS21] for the explicit expression of the integral kernel for general $\nu \in \mathbb{N}_0$ or (2.24) below for the case $\nu = 0$. By raising the Fermi energy E_F and thereby the number ν of occupied Landau levels, the particle number inside the region Λ increases proportionally with ν. For the lowest Landau level, where $\nu = 0$ and $P_{\leqslant 0} = P_0$, the particle number equals $\frac{B}{2\pi}|\Lambda|$. To obtain a half-filled lowest Landau level, we require the particle number instead to be

$$N = \frac{1}{2}\cdot\frac{B}{2\pi}|\Lambda|. \tag{1.22}$$

How can this be achieved? To answer this question, we look for a sub-projection $\tilde{P}_0 \leqslant P_0$ onto a subspace of the lowest Landau level that satisfies

$$\operatorname{tr}_{L^2(\mathbb{R}^2)}(1_\Lambda \tilde{P}_0 1_\Lambda) = \frac{1}{2}\cdot\frac{B}{2\pi}|\Lambda|. \tag{1.23}$$

It turns out that there are many ways of constructing such a sub-projection and not all of them automatically lead to a different scaling behavior of the corresponding entanglement entropies as in the case of a fully occupied Landau level. However, by exploiting a particularly convenient way of parametrizing subspaces of the lowest Landau level as described in Chapter 2.1.4, we are able to construct sub-projections that lead to more interesting asymptotic behavior of the corresponding entanglement entropies: The Landau Hamiltonian in Landau gauge,

$$H_B = -\frac{\partial^2}{\partial x_1^2} + \left(-i\frac{\partial}{\partial x_2} - Bx_1\right)^2, \tag{1.24}$$

has a family of generalized (non-normalizable) eigenfunctions $(\psi_k)_{k\in\mathbb{R}}$ corresponding to the lowest eigenvalue B of the form

$$\psi_k(x_1,x_2) := \frac{e^{ikx_2}}{\sqrt{2\pi}}\left(\frac{B}{\pi}\right)^{1/4} e^{-\frac{B}{2}(x_1-k/B)^2}, \tag{1.25}$$

indexed by the continuous momentum parameter $k \in \mathbb{R}$. Using these eigenfunctions, for each measurable subset $\Gamma \subset \mathbb{R}$, the operator P_Γ with integral kernel

$$p_\Gamma(x,y) = \int_\Gamma \overline{\psi_k(x)}\psi_k(y)\,dk \tag{1.26}$$

is a projection on some subspace V_Γ of the lowest Landau level. We will see in Chapter 3 that for suitable choices of Γ, the projection P_Γ satisfies (1.23) and gives rise to different kinds of enhanced area laws of the corresponding entanglement entropies:

$$\operatorname{tr}_{L^2(\mathbb{R}^2)} h_\gamma(1_{\Lambda_L}\tilde{P}_0 1_{\Lambda_L}) \sim L^\sigma \log L. \tag{1.27}$$

More precisely, for each $\sigma \in [1, 2[$, in the first part of Theorem 3.2.2, we construct a $\Gamma = \Gamma(\sigma)$ leading to an entanglement entropy scaling of $L^\sigma \log L$. In the case $\sigma = 1$, which corresponds to a regular logarithmically enhanced area law, we are able to calculate the precise asymptotic coefficient and thus establish a Szegő-type asymptotics, which is the content of Theorem 3.1.1. The asymptotic coefficient is related to that of the ground state entanglement entropy of free fermions in one dimension, see (1.12) with $d = 1$, and therefore to the one-dimensional Widom formula. Finally, in the second part of Theorem 3.2.2, we consider a certain class of sets Γ for which the entanglement entropy behaves asymptotically as L^2, corresponding to a *volume law* very uncommon for ground states.

Structure of the thesis. The thesis is structured as follows: In Chapter 2, we recall some of the necessary preliminaries and concepts useful to understand the rest of the thesis. The two main results of this thesis are given in Chapter 3, which is followed by Chapters 4 and 5 containing the corresponding proofs in full detail. Appendix A collects some auxiliary estimates and properties of functions that appear in the proofs of the main results, but are sourced out for the sake of structure and readability. Appendix B, which might be of independent interest, provides the proper mathematical definition of entanglement entropy for many-body pure states, and follows up with a rigorous derivation of the previously discussed one-particle formula (B.4.4) for non-interacting Fermi gases.

Some remarks on notation. We follow standard notational conventions and describe the relevant notation upon its first introduction. Large parts of the relevant notation are introduced in Chapter 2. Many times when stating inequalities, we use the letter C for a generic positive constant that may change from line to line. In cases where the dependence of the constant C on the involved variables is not clear from the context, we try to explicitly state the variables the constant depends on after each inequality.

2. Preliminaries

In this chapter, we recall some of the necessary preliminaries and concepts that will be used throughout the rest of the thesis. As most of the content is well-known, we will at many places refrain from providing a proof and instead cite the relevant literature where proofs can be found.

2.1. The Landau Hamiltonian

In this section, we give an overview of the Landau Hamiltonian and its spectral properties, mainly following [FH10, Hel13].

2.1.1. Definition

In physics, the Landau Hamiltonian on $L^2(\mathbb{R}^2)$ is the Schrödinger operator for the energy of a single particle in the plane subject to a perpendicular constant magnetic field of strength $B > 0$. It is given by

$$H_B := (-i\nabla - A)^2 = \left(-i\frac{\partial}{\partial x_1} - A_1\right)^2 + \left(-i\frac{\partial}{\partial x_2} - A_2\right)^2 \tag{2.1}$$

acting on some suitable dense domain $D(H_B)$. Here, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable vector field satisfying

$$\operatorname{curl} A := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B. \tag{2.2}$$

We call A the *vector potential*. We will see below that the spectral properties of H_B depend on A only through B, justifying the notation.

For a given vector potential A, the proper domain of self-adjointness of (2.1) is the *magnetic Sobolev space* $H^2_A(\mathbb{R}^2)$ of second order. It consists of all functions $f \in L^2(\mathbb{R}^2)$ with existing weak derivatives up to the second order such that

$$\left(-i\frac{\partial}{\partial x_1} - A_1\right) f, \quad \left(-i\frac{\partial}{\partial x_2} - A_2\right) f, \quad (-i\nabla - A)^2 f \in L^2(\mathbb{R}^2), \tag{2.3}$$

see [FH10, Chapter 1.2].

2.1.2. Gauge fixing

For the moment, we will write H_A instead of H_B to single out the underlying dependence on the vector potential. There are obviously many different possible choices, called *gauges*, for the vector potential A satisfying (2.2): given a real-valued function $\phi \in C^\infty(\mathbb{R}^2)$, the vector potential $A + \nabla\phi$ also satisfies (2.2), since $\operatorname{curl} \nabla\phi = 0$. Consider the *gauge transformation*

$$U_\phi \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad U_\phi f := e^{i\phi} f. \tag{2.4}$$

A straightforward calculation shows that U is unitary and that

$$U_\phi^* H_A U_\phi = H_{A+\nabla\phi}, \tag{2.5}$$

meaning H_A and $H_{A+\nabla\phi}$ are unitarily equivalent (the latter with the obvious transformed domain). Now assume that $A, \tilde{A} \colon \mathbb{R}^2 \to \mathbb{R}^2$ are two continuously differentiable vector fields both obeying (2.2), and thus

$$\operatorname{curl}(A - \tilde{A}) = 0. \tag{2.6}$$

Since $\mathbb{R}^2$ is simply connected, the Poincaré lemma implies that A and $\tilde{A}$ only differ by a gradient field $\nabla\phi$. By the above, this means that all choices of (continuously differentiable) vector potentials satisfying (2.2) lead to unitarily equivalent Landau Hamiltonians. In particular, all the spectral properties of H_A do not depend on the chosen gauge.

In our spectral analysis we can therefore work with a particular choice of gauge that is suitable for the respective calculations at hand. This process is called *gauge fixing*. Two of the most common gauges are the *symmetric gauge*

$$A_s \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad A_s(x) := \frac{B}{2}(x_2, -x_1), \tag{2.7}$$

and the *Landau gauge*

$$A_L \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad A_L(x) := (0, Bx_1). \tag{2.8}$$

The function

$$\phi \colon \mathbb{R}^2 \to \mathbb{R}, \quad \phi(x_1, x_2) := \frac{B}{2} x_1 x_2 \tag{2.9}$$

satisfies $\nabla\phi(x_1, x_2) = \frac{B}{2}(x_2, x_1)$, so under the corresponding gauge transformation (2.4), we have $U_\phi H_{A_L} U_\phi^* = H_{A_s}$. In this thesis, we will mainly be concerned with the Landau gauge (2.8), due to a particular useful way of describing subspaces of its lowest eigenvalue eigenspace as described in Section 2.1.4.

2.1.3. Spectrum

The following result on the spectrum of H_B is well-known. We give the proof for the convenience of the reader and because we will later make use of the explicit expression of the eigenfunctions in (2.18).

Proposition 2.1.1. *The spectrum of the operator H_B is purely essential and given by evenly spaced eigenvalues of infinite multiplicity, called* Landau levels. *In formulas,*

$$\sigma(H_B) = \sigma_{\text{ess}}(H_B) = \{B(2n + 1) : n \in \mathbb{N}_0\}. \tag{2.10}$$

Consequently, H_B can be written in terms of its spectral decomposition

$$H_B = B \sum_{n \in \mathbb{N}_0} (2n+1) P_n, \tag{2.11}$$

where P_n denotes the projection onto the n-th Landau level.

Proof. We follow [Hel13, Section 10.4.1] and work in the Landau gauge (2.8), i.e.

$$H_{A_L} = -\frac{\partial^2}{\partial x_1^2} + \left(-i\frac{\partial}{\partial x_2} - Bx_1\right)^2. \tag{2.12}$$

We start by performing two unitary transformations, which leave the spectrum of the operator and the respective multiplicities unaltered. For the sake of convenience, we omit the domains of the transformed operators. First, let $\mathscr{F}_2$ denote the (partial) Fourier transform on $L^2(\mathbb{R})$ with respect to the second variable x_2. Upon conjugation, we obtain

$$H_{A_L}^{(2)} := \mathscr{F}_2 H_{A_L} \mathscr{F}_2^* = -\frac{\partial^2}{\partial x_1^2} + (k - Bx_1)^2 \tag{2.13}$$

acting on functions $f = f(x_1, k) \in L^2(\mathbb{R}^2)$. Secondly, let T be the unitary transformation on $L^2(\mathbb{R}^2)$ given by

$$Tf(y,k) := f(y + k/B, k). \tag{2.14}$$

Then

$$H_{A_L}^{(3)} := TH_{A_L}^{(2)} T^* = -\frac{\partial^2}{\partial y^2} + B^2 y^2, \tag{2.15}$$

acting on functions $f = f(y,k) \in L^2(\mathbb{R}^2)$. The operator $H_{A_L}^{(3)}$ does not depend on the variable k, and is a harmonic oscillator in the y-variable, see [FH10, Chapter 3.1]. We are therefore able to write down an explicit orthonormal basis of eigenvectors of this operator: Let ϕ_n be the n-th harmonic oscillator eigenfunction, given by

$$\phi_n(y) := \frac{1}{\sqrt{2^n n!}} \pi^{-\frac{1}{4}} e^{-y^2/2} \mathsf{H}_n(y), \quad n \in \mathbb{N}_0, \tag{2.16}$$

where $\mathsf{H}_n, n \in \mathbb{N}_0$, is the n-th Hermite polynomial defined by

$$\mathsf{H}_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad x \in \mathbb{R}. \tag{2.17}$$

It follows that for each $g \in L^2(\mathbb{R})$ and $n \in \mathbb{N}_0$, the function

$$\varphi_n^{(3)}(y,k) := B^{1/4} g(k) \phi_n(B^{1/2} y) = \frac{1}{\sqrt{2^n n!}} \left(\frac{B}{\pi}\right)^{1/4} g(k) e^{-By^2/2} \mathsf{H}_n(B^{1/2} y) \tag{2.18}$$

is an eigenfunction of the operator (2.15) corresponding to the eigenvalue $B(2n+1)$. In particular, the corresponding eigenspace is infinitely degenerate. Since $(\phi_n)_{n \in \mathbb{N}_0}$ is an orthonormal basis of $L^2(\mathbb{R})$ (cf. [Hel13, Section 1.3]), the eigenvalues of H_B are precisely given by $B(2n+1)$, $n \in \mathbb{N}_0$ and consequently, we have empty discrete spectrum. □

2.1.4. Lowest Landau level

In this subsection, we take a closer look at the lowest Landau level, that is, the eigenspace of the operator H_B corresponding to the eigenvalue B. In Proposition 2.1.2, we construct a class of subspaces suitable to obtain "half-filled" lowest Landau levels as described in the introduction.

We again work with the Landau Hamiltonian in Landau gauge (2.12), in which the specific form of the eigenfunctions will turn out to be very useful. Recall the eigenfunctions (2.18) of the operator $H^{(3)}_{A_L}$ in the proof of Proposition 2.1.1. The eigenfunctions of the lowest Landau level, i.e. where $n = 0$, have the form

$$\varphi^{(3)}(y,k) := \left(\frac{B}{\pi}\right)^{1/4} g(k)e^{-By^2/2} \tag{2.19}$$

with arbitrary $g \in L^2(\mathbb{R})$. To obtain the corresponding eigenfunctions of H_{A_L}, we first change variables back from y to x_1 via the transformation (2.14) and see that the eigenfunctions of $H^{(2)}_{A_L}$ are of the form

$$\varphi^{(2)}(x_1,k) := T^*\varphi(x_1,k) = \varphi(x_1 - k/B, k) = \left(\frac{B}{\pi}\right)^{1/4} g(k)e^{-B(x_1-k/B)^2/2}. \tag{2.20}$$

Finally, we have to take the inverse Fourier transform in the variable k to obtain the eigenfunctions φ of H_{A_L}:

$$\begin{aligned}\varphi(x_1,x_2) := \left(\mathscr{F}_2^*\varphi^{(2)}(x_1,\cdot)\right)(x_2) &= \left(\frac{B}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx_2} g(k) e^{-\frac{B}{2}(x_1-k/B)^2}\,dk \\ &= \int_{\mathbb{R}} \psi_k(x_1,x_2) g(k)\,dk,\end{aligned} \tag{2.21}$$

where $(\psi_k)_{k\in\mathbb{R}}$ is the family of functions $\psi_k \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\psi_k(x_1,x_2) := \frac{e^{ikx_2}}{\sqrt{2\pi}} \left(\frac{B}{\pi}\right)^{1/4} e^{-\frac{B}{2}(x_1-k/B)^2}, \quad k \in \mathbb{R}. \tag{2.22}$$

This leads us to the following proposition.

Proposition 2.1.2 (Lowest Landau level in Landau gauge). *(i) Let $(\psi_k)_{k\in\mathbb{R}}$ be the family of functions defined by* (2.22)*. The lowest Landau level in Landau gauge has the form*

$$V := \ker(H_{A_L} - B) = \left\{ \int_{\mathbb{R}} \psi_k g(k)\,dk : g \in L^2(\mathbb{R}) \right\} \subset L^2(\mathbb{R}^2). \tag{2.23}$$

The orthogonal projection $P \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ onto V has an integral kernel given by

$$p \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}, \quad p(x,y) := \int_{\mathbb{R}} \overline{\psi_k(x)}\psi_k(y)\,dk = \frac{B}{2\pi} e^{-\frac{B}{4}|x-y|^2} e^{-i\frac{B}{2}(y_2-x_2)(x_1+y_1)}. \tag{2.24}$$

(ii) More generally, for each measurable $\Gamma \subset \mathbb{R}$*, the set*

$$V_\Gamma := \left\{ \int_\Gamma \psi_k g(k)\, dk : g \in L^2(\mathbb{R}) \right\} \subset L^2(\mathbb{R}^2) \tag{2.25}$$

is a subspace of the lowest Landau level. The orthogonal projection P_Γ *onto* V_Γ *is given by the integral operator with integral kernel*

$$p_\Gamma \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}, \quad p_\Gamma(x,y) := \int_\Gamma \overline{\psi_k(x)} \psi_k(y)\, dk. \tag{2.26}$$

(iii) For any measurable $\Gamma_1, \Gamma_2 \subset \mathbb{R}$ *with* $\Gamma_1 \cap \Gamma_2 = \emptyset$*, we have*

$$P_{\Gamma_1} P_{\Gamma_2} = 0, \quad P_{\Gamma_1} + P_{\Gamma_2} = P_{\Gamma_1 \cup \Gamma_2}. \tag{2.27}$$

The proof of Proposition 2.1.2 is an easy consequence of the following lemma, which will again be useful later in Chapter 4.

Lemma 2.1.3. *Consider the transformation*

$$U \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}), \quad (U\varphi)(k) := \int_{\mathbb{R}^2} \psi_k(x) \varphi(x)\, dx. \tag{2.28}$$

Then U *is a bounded operator with adjoint* U^* *given by*

$$U^* \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}^2), \quad (U^* g)(x) = \int_{\mathbb{R}} \overline{\psi_k(x)} g(k)\, dk. \tag{2.29}$$

Furthermore, we have the identities

$$P_\Gamma = U^* 1_\Gamma U, \quad UU^* = \mathrm{id}_{L^2(\mathbb{R})}. \tag{2.30}$$

Proof. We first show that U is bounded. Indeed, for fixed $k \in \mathbb{R}$, we get by the Cauchy-Schwarz inequality

$$\begin{aligned}
(U\varphi)(k) &= \int_{\mathbb{R}^2} \psi_k(y) \varphi(y)\, dy \\
&= \left(\frac{B}{\pi}\right)^{1/4} \int_{\mathbb{R}} e^{-\frac{B}{2}(y_1 - k/B)^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iky_2} \varphi(y_1, y_2)\, dy_2\, dy_1 \\
&= \left(\frac{B}{\pi}\right)^{1/4} \int_{\mathbb{R}} e^{-\frac{B}{2}(y_1 - k/B)^2} (\mathscr{F}_2^* \varphi(y_1, \cdot))(k)\, dy_1 \\
&\leqslant C \left(\int_{\mathbb{R}} |(\mathscr{F}_2^* \varphi(y_1, \cdot))(k)|^2\, dy_1 \right)^{1/2},
\end{aligned} \tag{2.31}$$

with a constant $C > 0$ depending on B. By the Tonelli and Plancherel theorems, it follows that

$$\begin{aligned}
\|U\varphi\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \overline{\psi_k(y)} \varphi(y)\, dy \right|^2 dk \\
&\leqslant C \int_{\mathbb{R}} \int_{\mathbb{R}} |(\mathscr{F}_2^* \varphi(y_1, \cdot))(k)|^2\, dy_1\, dk \\
&= C \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(y_1, y_2)|^2\, dy_2\, dy_1 = \|\varphi\|^2_{L^2(\mathbb{R}^2)} < \infty,
\end{aligned} \tag{2.32}$$

since $\varphi \in L^2(\mathbb{R}^2)$ by assumption. U^* is the adjoint of U since

$$\begin{aligned}\langle U\varphi, g\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \overline{(U\varphi)(k)} g(k)\,dk = \int_{\mathbb{R}}\int_{\mathbb{R}^2} \overline{\psi_k(x)\varphi(x)} g(k)\,dx\,dk \\ &= \int_{\mathbb{R}^2} \overline{\varphi(x)} \int_{\mathbb{R}} \overline{\psi_k(x)} g(k)\,dk\,dx = \langle \varphi, U^* g\rangle_{L^2(\mathbb{R}^2)} \end{aligned} \tag{2.33}$$

for all $\varphi \in L^2(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R})$ by Fubini's theorem. The last two identities (2.30) follow from

$$\begin{aligned}(U^* 1_\Gamma U\varphi)(x) &= \int_\Gamma \overline{\psi_k(x)} \int_{\mathbb{R}^2} \psi_k(y)\varphi(y)\,dy\,dk \\ &= \int_{\mathbb{R}^2} \left[\int_\Gamma \overline{\psi_k(x)}\psi_k(y)\,dk\right] \varphi(y)\,dy \\ &= \int_{\mathbb{R}^2} p_\Gamma(x,y)\varphi(y)\,dy = (P_\Gamma\varphi)(x)\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}(UU^* g)(k) &= \int_{\mathbb{R}^2} \psi_k(x) \int_{\mathbb{R}} \overline{\psi_{k'}(x)} g(k')\,dk'\,dx \\ &= \frac{1}{2\pi}\sqrt{\frac{B}{\pi}} \int_{\mathbb{R}} e^{-\frac{B}{2}(x_1+k/B)^2} \int_{\mathbb{R}} e^{ikx_2} \int_{\mathbb{R}} e^{-ik'x_2} e^{-\frac{B}{2}(x_1+k'/B)^2} g(k')\,dk'\,dx_2\,dx_1 \\ &= \sqrt{\frac{B}{\pi}} \int_{\mathbb{R}} e^{-\frac{B}{2}(x_1+k/B)^2} e^{-\frac{B}{2}(x_1+k/B)^2} g(k)\,dx_1 \\ &= g(k)\sqrt{\frac{B}{\pi}} \int_{\mathbb{R}} e^{-Bx_1^2}\,dx_1 = g(k),\end{aligned} \tag{2.35}$$

where we have used the Fourier inversion formula for the third equality. □

Proof of Proposition 2.1.2. Using the previous lemma, we show that for each measurable $\Gamma \subseteq \mathbb{R}$, the projection defined in terms of the kernel (2.26) satisfies $\operatorname{ran} P_\Gamma = V_\Gamma$ and $P_\Gamma^2 = P_\Gamma^* = P_\Gamma$. The second claim is immediate from (2.30). As for the first, note that with the notation from Lemma 2.1.3 we can rewrite

$$V_\Gamma = \left\{U^*(1_\Gamma g) : g \in L^2(\mathbb{R})\right\}. \tag{2.36}$$

Since $P_\Gamma\varphi = U^* 1_\Gamma U\varphi$ and $U\varphi \in L^2(\mathbb{R})$, it follows that $\operatorname{ran} P_\Gamma \subseteq V_\Gamma$. Furthermore, if $\varphi = U^*(1_\Gamma g) \in V_\Gamma$ for some $g \in L^2(\mathbb{R})$, then by (2.30), we have

$$P_\Gamma\varphi = U^* 1_\Gamma U U^*(1_\Gamma g) = U^* 1_\Gamma g = \varphi, \tag{2.37}$$

showing $\operatorname{ran} P_\Gamma = V_\Gamma$. The expression (2.26) for the kernel of the projection $P = P_{\mathbb{R}}$ onto the lowest Landau level is obtained by calculating the Fourier transform of a product of displaced Gaussians:

$$\begin{aligned}p(x,y) &= \int_{\mathbb{R}} \overline{\psi_k(x)}\psi_k(y)\,dk \\ &= \frac{1}{2\pi}\left(\frac{B}{\pi}\right)^{1/2} \int_{\mathbb{R}} e^{ik(y_2-x_2)} e^{-\frac{B}{2}(x_1+k/B)^2} e^{-\frac{B}{2}(y_1+k/B)^2}\,dk \\ &= \frac{B}{2\pi} e^{-\frac{B}{4}|x-y|^2} e^{-i\frac{B}{2}(y_2-x_2)(x_1+y_1)},\end{aligned} \tag{2.38}$$

as desired. □

2.2. Singular values and Schatten-von Neumann classes

In this subsection we recall the definition of the singular values of a compact operator and some of their important properties we will need throughout this thesis. We also introduce the classes of Schatten-von Neumann operators and state some important estimates for such operators. For a more detailed treatment of the material, we refer to [BS87, Chapter 11].

2.2.1. Singular values

Let $\mathscr{H}$ be a (separable) Hilbert space. For a positive compact operator S on $\mathscr{H}$, that is, a compact operator satisfying $\langle \varphi, S\varphi \rangle \geqslant 0$ for all $\varphi \in \mathscr{H}$, we denote by $(\lambda_j(S))_{j=1,\dots,N}$ with $N \in \mathbb{N} \cup \{\infty\}$ the non-increasing sequence of *positive* eigenvalues of S counted with multiplicities.

If T is an arbitrary compact, not necessarily positive, operator on $\mathscr{H}$, $|T| := (T^*T)^{1/2}$ is a positive, compact self-adjoint operator. The positive numbers

$$\mathfrak{s}_j(T) := \lambda_j(|T|) = \sqrt{\lambda_j(T^*T)}, \quad j = 1, \dots, N \tag{2.39}$$

are called *singular values* of T.

An important result about singular values is the *singular value decomposition* of a compact operator T: There exist (finite or infinite) orthonormal families $(\varphi_j)_{j=1,\dots,N}$, $(\psi_j)_{j=1,\dots,N}$ such that

$$T = \sum_{j=1}^{N} \mathfrak{s}_j(T) \langle \varphi_j, \cdot \rangle \psi_j. \tag{2.40}$$

It is immediate from (2.40) that

$$T^* = \sum_{j=1}^{N} \mathfrak{s}_j(T) \langle \psi_j, \cdot \rangle \varphi_j \tag{2.41}$$

and therefore

$$T^*T = \sum_{j=1}^{N} \mathfrak{s}_j^2(T) \langle \varphi_j, \cdot \rangle \varphi_j, \quad TT^* = \sum_{j=1}^{N} \mathfrak{s}_j^2(T) \langle \psi_j, \cdot \rangle \psi_j. \tag{2.42}$$

From (2.42), it follows that the non-zero eigenvalues of the operators TT^* and T^*T coincide: $0 \neq \lambda_j(T^*T) = \lambda_j(TT^*)$.

For the rest of this thesis, in the case $N < \infty$, we put $\mathfrak{s}_j(T) = 0$ for $j > N$, so that we always obtain a sequence $(\mathfrak{s}_j(T))_{j \in \mathbb{N}}$ of singular values.

2.2.2. Schatten-von Neumann classes $\mathfrak{S}_q$

Let T be a compact operator on a Hilbert space $\mathscr{H}$ and $(\mathfrak{s}_j(T))_{j \in \mathbb{N}}$ the sequence of singular values of T. We say that T belongs to the *Schatten–von Neumann class* $\mathfrak{S}_q = \mathfrak{S}_q(\mathscr{H})$ for some $q > 0$, if

$$\|T\|_{\mathfrak{S}_q} := \left(\sum_{j=1}^{\infty} \mathfrak{s}_j(T)^q \right)^{\frac{1}{q}} < \infty. \tag{2.43}$$

For $q \geqslant 1$, (2.43) constitutes a norm on the vector space $\mathfrak{S}_q$, for $q < 1$ only a quasi-norm. We refer to $\|\cdot\|_{\mathfrak{S}_q}$ as the *Schatten–von Neumann q-(quasi-)norm* or only *q-(quasi-)norm*.

In the case $q < 1$, however, one still has the *q-triangle inequality* of the form

$$\|T_1 + T_2\|_{\mathfrak{S}_q}^q \leqslant \|T_1\|_{\mathfrak{S}_q}^q + \|T_2\|_{\mathfrak{S}_q}^q \tag{2.44}$$

for all $T_1, T_2 \in \mathfrak{S}_q$.

Two of the most important examples are the following: The class $\mathfrak{S}_1$ is the standard space of *trace class operators* and $\|\cdot\|_{\mathfrak{S}_1}$ the *trace norm*, while $\mathfrak{S}_2$ is the space of *Hilbert-Schmidt operators* with $\|\cdot\|_{\mathfrak{S}_2}$ being the *Hilbert-Schmidt norm*.

Finally, if we let $\mathfrak{S}_\infty$ denote the space of compact operators on $\mathscr{H}$ and identify the norm $\|\cdot\|_{\mathfrak{S}_\infty}$ with the usual operator norm $\|\cdot\|$, we have the Hölder-type inequality

$$\|T_1 T_2\|_{\mathfrak{S}_p} \leqslant \|T_1\|_{\mathfrak{S}_{p_1}} \|T_2\|_{\mathfrak{S}_{p_2}} \tag{2.45}$$

for any $p_1, p_2 \in \,]0, \infty]$ with $1/p_1 + 1/p_2 = 1/p$ and the usual convention $1/\infty := 0$.

2.2.3. Characterization of trace class operators

The following proposition summarizes some well-known properties of trace-class operators $T \in \mathfrak{S}_1$. Its proof and further properties of trace class operators can be found in [BS87, Chapter 11.2].

Proposition 2.2.1. *(i) A bounded operator T belongs to the trace-class $\mathfrak{S}_1(\mathscr{H})$ if and only if*

$$\sum_{n\in\mathbb{N}} |\langle \varphi_n, T\psi_n \rangle| < \infty \tag{2.46}$$

for all orthonormal families $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{H}$. In this case, the functional

$$\operatorname{tr} T := \sum_{n\in\mathbb{N}} \langle \psi_n, T\psi_n \rangle, \tag{2.47}$$

called trace *of T, converges for any orthonormal basis $(\psi_n)_{n\in\mathbb{N}}$ of $\mathscr{H}$ and does not depend on the choice of the orthonormal basis. Moreover, there exist orthonormal families $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{H}$ such that*

$$\|T\|_{\mathfrak{S}_1} = \sum_{n\in\mathbb{N}} |\langle \varphi_n, T\psi_n \rangle| \,. \tag{2.48}$$

(ii) If T is a bounded positive operator, then $T \in \mathfrak{S}_1(\mathscr{H})$ if the series $\sum_{n\in\mathbb{N}} \langle \varphi_n, T\varphi_n \rangle$ converges for some *orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathscr{H}$ and in this case*

$$\operatorname{tr} T = \operatorname{tr}_{\mathscr{H}} T = \|T\|_{\mathfrak{S}_1} = \sum_{n\in\mathbb{N}} \langle \psi_n, T\psi_n \rangle \tag{2.49}$$

for any *orthonormal basis $(\psi_n)_{n\in\mathbb{N}}$ of $\mathscr{H}$.*

2.2.4. Estimates for singular values and Schatten norms of integral operators

The estimates of this subsection will be used in Chapter 4 in the proof of our first main result, Theorem 3.1.1, where Schatten-von Neumann q-quasi norm bounds with $q < 1$ are needed to lift the asymptotics (3.4) from differentiable to Hölder continuous functions.

In order to state the first estimate, we need the definition of *fractional Sobolev spaces.* Let $I \subset \mathbb{R}$ be an open interval. For $m \in \mathbb{N}_0$, we denote by $H^m(I)$ the usual Sobolev space of all functions $u \in L^2(I)$ with weak derivatives $u^{(j)}$ up to order m such that the norm

$$\|u\|_{H^m(I)} := \left(\sum_{j=0}^{m} \|u^{(j)}\|^2_{L^2(I)} \right)^{1/2} \tag{2.50}$$

is finite.

For a number $\sigma \in]0,1[$ and a function $u \in L^2(I)$, we define

$$[u]_{H^\sigma(I)} := \left(\int_I \int_I \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\sigma}} \, dx \, dy \right)^{1/2}. \tag{2.51}$$

For general $s > 0$ with $s \notin \mathbb{N}$, we define the *fractional Sobolev space* $H^s(I)$ to be the space of all functions $u \in H^m(I)$ for which

$$\|u\|_{H^s(I)} := \left(\|u\|^2_{H^m(I)} + [u^{(m)}]^2_{H^{s-m}(I)} \right)^{1/2} < \infty, \tag{2.52}$$

where $m := \lfloor s \rfloor$ denotes the integer part of s.

Remark 2.2.2. (i) It is straightforward to see that the norm (2.52) is equivalent to the norm

$$\|u\|'_{H^s(I)} := \|u\|_{H^m(I)} + [u^{(m)}]_{H^{s-m}(I)}, \quad u \in H^s(I). \tag{2.53}$$

By generalization of [Leo24, Theorem 5.8] to any $s > 1$ with $s \notin \mathbb{N}$, see [Leo24, Exercise 5.9], it is also equivalent to the norm

$$\|u\|''_{H^s(I)} := \left(\|u\|^2_{L^2(I)} + [u^{(m)}]^2_{H^{s-m}(I)} \right)^{1/2}. \tag{2.54}$$

(ii) For any $0 < s < s'$ and $u \in H^{s'}(I)$ there exists a constant $C = C(I) > 0$ such that

$$\|u\|_{H^s(I)} \leqslant C \|u\|_{H^{s'}(I)}. \tag{2.55}$$

This can be seen in the following way: Write $s = m + \sigma$ and $s' = m' + \sigma'$ with $m, m' \in \mathbb{N}_0$ and $\sigma, \sigma' \in [0,1[$. In the case $m = m'$, by [Leo24, Lemma 2.6] we have $[u^{(m)}]^2_{H^{s-m}(I)} \leqslant C [u^{(m)}]^2_{H^{s'-m}(I)}$, which implies

$$\begin{aligned} \|u\|^2_{H^s(I)} &= \|u\|^2_{H^m(I)} + [u^{(m)}]^2_{H^{s-m}(I)} \\ &\leqslant \|u\|^2_{H^m(I)} + C[u^{(m)}]^2_{H^{s'-m}(I)} = \max\{1, C\} \|u\|^2_{H^{s'}(I)}. \end{aligned} \tag{2.56}$$

In the other case $m' \geqslant m+1$, we in turn have

$$\begin{aligned}\|u\|_{H^s(I)}^2 &= \|u\|_{H^m(I)}^2 + [u^{(m)}]_{H^{s-m}(I)}^2 \leqslant \|u\|_{H^m(I)}^2 + C\|u^{(m)}\|_{H^1(I)}^2 \\ &\leqslant C\|u\|_{H^{m+1}(I)}^2 \leqslant C\|u\|_{H^{m'}(I)}^2 \\ &\leqslant C\left(\|u\|_{H^{m'}(I)}^2 + [u^{(m')}]_{H^{s'-m'}(I)}^2\right) = C\|u\|_{H^{s'}(I)},\end{aligned} \tag{2.57}$$

where for the first inequality we have used the continuous embedding $H^1(I) \subset H^{s-m}$, see [Leo24, Theorem 1.25].

The following is a special case of a much more general estimate from [BS80]. We present it in a form adapted to our purposes.

Lemma 2.2.3. *Let $I \subset \mathbb{R}$ be a bounded interval of length $|I| \geqslant 1$. Let $Z\colon L^2(\mathbb{R}) \to L^2(I)$ be an integral operator whose integral kernel $z\colon I \times \mathbb{R} \to \mathbb{C}$ obeys*

$$N_\gamma(z) := \left[\int_{\mathbb{R}} \|z(\cdot, y)\|_{H^\gamma(I)}^2 \, dy\right]^{\frac{1}{2}} < \infty \tag{2.58}$$

for some $\gamma > 1/2$. Let $(\mathfrak{s}_j(Z))_{j\in\mathbb{N}}$ be the sequence of singular values of Z. Then there exists a constant $C = C(\gamma) > 0$ such that

$$\mathfrak{s}_k(Z) \leqslant C|I|^\gamma k^{-\frac{1}{2}-\gamma} N_\gamma(z) \tag{2.59}$$

for all $k \in \mathbb{N}$. The constant C is independent of I, k and the kernel z.

Proof. Apply [BS80, Corollary 3.5] with I in place of Q, $m = 1$, $p = 2$, $r = 1$, $\alpha = \gamma$. □

The second Schatten-von Neumann class estimate we present in this section requires another definition. For a measurable function $h\colon \mathbb{R} \to \mathbb{C}$ and $\delta > 0$, we put

$$\|h\|_{2,\delta} := \left[\sum_{n\in\mathbb{Z}} \left(\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |h(x)|^2 \, dx\right)^{\frac{\delta}{2}}\right]^{\frac{1}{\delta}} \in [0, \infty] \tag{2.60}$$

and denote by $\ell^\delta(L^2)(\mathbb{R})$ the space of all such functions satisfying $\|h\|_{2,\delta} < \infty$. Then $\|\cdot\|_{2,\delta}$ constitutes a quasi-norm (a norm for $\delta \geqslant 1$) on $\ell^\delta(L^2)(\mathbb{R})$. We refer to $\|\cdot\|_{2,\delta}$ as *lattice (quasi-)norm* or *Birman-Solomyak (quasi-)norm.*

We remark that choosing differently centered unit intervals in (2.60), e.g. $[n, n+1]$ instead of $[n - \frac{1}{2}, n + \frac{1}{2}]$, results in equivalent (quasi-)norms.

Lemma 2.2.4. *Let $\mathscr{F}$ denote the Fourier transform on $L^2(\mathbb{R})$. For every $0 < \delta < 2$ and $f, g \in \ell^\delta(L^2)(\mathbb{R})$, the operator $f\mathscr{F}g\mathscr{F}^*$ is in $\mathfrak{S}_\delta$ and moreover there exists a constant $C = C(\delta) > 0$ such that*

$$\|f\mathscr{F}g\mathscr{F}^*\|_{\mathfrak{S}_\delta} \leqslant C\|f\|_{2,\delta}\|g\|_{2,\delta}. \tag{2.61}$$

Proof. Let $E_{f,g}$ denote the operator on $L^2(\mathbb{R})$ with integral kernel

$$\mathbb{R}^2 \ni (x,y) \mapsto f(x)e^{-ixy}g(y). \tag{2.62}$$

By [BS77, Theorem 11.1], there exists a constant $C = C(\delta) > 0$, independent of f and g, such that

$$\|E_{f,g}\|_{\mathfrak{S}_\delta} \leqslant C\|f\|_{2,\delta}\|g\|_{2,\delta}. \tag{2.63}$$

We note that $f\mathscr{F}g\mathscr{F}^* = E_{f,g}\mathscr{F}^*$. From the unitarity of the Fourier transform and the Hölder-type inequality (2.45) with $p_1 = p = \delta, p_2 = \infty$ we infer

$$\|f\mathscr{F}g\mathscr{F}^*\|_{\mathfrak{S}_\delta} \leqslant \|E_{f,g}\|_{\mathfrak{S}_\delta}, \tag{2.64}$$

so the claim follows from (2.63). □

2.3. Entanglement entropy

This section provides the definition of entanglement entropy that we will work with for the rest of the thesis. It is motivated by the formula (B.50) derived in Appendix B valid for fermionic ground states.

2.3.1. Definition of entanglement entropy

For $\gamma > 0$, we define the *entropy functions*

$$h_\gamma\colon [0,1] \to [0, \log 2], \quad h_\gamma(\lambda) := \begin{cases} -\lambda\log\lambda - (1-\lambda)\log(1-\lambda) & \text{if } \gamma = 1, \\ \dfrac{1}{1-\gamma}\log[\lambda^\gamma + (1-\lambda)^\gamma] & \text{if } \gamma \neq 1 \end{cases} \tag{2.65}$$

with the convention $0\log 0 := 0$. We refer to h_1 as *von Neumann entropy function* and to h_γ with $\gamma \neq 1$ as *γ-Rényi entropy functions*. Note that $\lim_{\gamma\to 1} h_\gamma(\lambda) = h_1(\lambda)$ for all $\lambda \in [0,1]$.

Definition 2.3.1 (Entanglement entropy)**.** For a given Fermi energy $E_F \in \mathbb{R}$ and a parameter $\gamma > 0$, the (bipartite) ground state *entanglement entropy* of the quasi-free Fermi gas governed by the single-particle Hamiltonian H_B with respect to a measurable spatial subregion $\Lambda \subset \mathbb{R}^2$ is defined by

$$S_\gamma = S_\gamma(H_B, E_F, \Lambda) := \operatorname{tr} h_\gamma(1_\Lambda 1_{]-\infty,E_F[}(H_B)1_\Lambda). \tag{2.66}$$

More generally, for any sub-projection $P \leqslant 1_{\{B\}}(H_B)$ of the projection onto the lowest Landau level, we define the entanglement entropy of the ground state corresponding to the partial filling of the lowest Landau level determined by P as

$$S_\gamma = S_\gamma(P, \Lambda) := \operatorname{tr} h_\gamma(1_\Lambda P 1_\Lambda). \tag{2.67}$$

We refer to S_1 as *von Neumann entanglement entropy* and to S_γ with $\gamma \neq 1$ as *γ-Rényi entanglement entropy*.

Remark 2.3.2. (i) Recall the expression $1_{]-\infty,E_F[}(H_B) = P_{\leqslant\nu}$ from (1.20), where $P_{\leqslant\nu}$ is the projection onto the orthogonal sum of the lowest $n+1$ Landau levels. It follows that

$$S_\gamma(H_B, E_F, \Lambda) = \operatorname{tr} h_\gamma(1_\Lambda P_{\leqslant\nu} 1_\Lambda) \tag{2.68}$$

and in particular

$$S_\gamma(H_B, E_F, \Lambda) = \operatorname{tr} h_\gamma(1_\Lambda 1_{\{B\}}(H_B) 1_\Lambda) \tag{2.69}$$

for Fermi energies satisfying $B < E_F \leqslant 3B$, which describe a Fermi gas where every state in the lowest Landau level only is occupied.

(ii) If the sub-projection P satisfies

$$\operatorname{tr}(1_\Lambda P 1_\Lambda) = \frac{1}{2} \cdot \operatorname{tr}(1_\Lambda 1_{\{B\}}(H_B) 1_\Lambda) = \frac{1}{2} \cdot \frac{B}{2\pi} |\Lambda|, \tag{2.70}$$

we refer to $S_\gamma(P, \Lambda)$ as ground state entanglement entropies corresponding to the *half-filling* of the lowest Landau level determined by P. More general partial fillings can be described by projections P satisfying (2.70) with a different factor $\mu \in]0,1[$ than $\frac{1}{2}$.

(iii) Some motivation for the above definitions is provided in the introduction and in Appendix B.

2.3.2. Useful upper and lower bounds

The following estimates are taken from [HLS11, Equation (50)].

Lemma 2.3.3. *Consider the situation of the previous subsection, let P be any sub-projection of $1_{\{B\}}(H_B)$ and abbreviate $Q := 1_\Lambda P 1_\Lambda$. We then have the chain of inequalities*

$$2 \operatorname{tr} Q(1-Q) \leqslant S_2 \leqslant 4\log(2) \operatorname{tr} Q(1-Q) \leqslant S_1 \leqslant 4 \operatorname{tr} \sqrt{Q(1-Q)}. \tag{2.71}$$

Proof. By the properties of the trace, all inequalities follow from the corresponding pointwise bounds

$$2\lambda(1-\lambda) \leqslant h_2(\lambda) \leqslant 4\log(2)\lambda(1-\lambda) \leqslant h_1(\lambda) \leqslant 4\sqrt{\lambda(1-\lambda)}, \quad \lambda \in [0,1]. \tag{2.72}$$

The proof of the third inequality of (2.72) can be found in [Top01, Theorem 1.2]. For the reader's convenience, we provide proofs for the first, second and fourth inequality.

First inequality. For $\lambda \in [0,1]$, we define

$$f(\lambda) := h_2(\lambda) - 2\lambda(1-\lambda) \tag{2.73}$$

and claim that $f(\lambda) \geqslant 0$. A simple calculation shows

$$f'(\lambda) = \frac{4\lambda(1-2\lambda)(1-\lambda)}{\lambda^2 + (1-\lambda)^2} \tag{2.74}$$

for all $\lambda \in [0,1]$, so $f' \geqslant 0$ on $[0,1/2]$ and $f' \leqslant 0$ on $[1/2,1]$. This shows that f is increasing on $[0,1/2]$ and decreasing on $[1/2,1]$. But $f(0) = f(1) = 0$, showing $f(\lambda) \geqslant 0$.

Second inequality. The inequality follows from

$$-\log(1-2\mu) \leqslant 4(\log 2)\mu \tag{2.75}$$

for all $\mu \in [0,1/4]$ by plugging in $\mu = \lambda(1-\lambda)$ with $\lambda \in [0,1]$. To prove (2.75), we note that the function defined by

$$f(\mu) := 4(\log 2)\mu + \log(1-2\mu), \quad \mu \in [0,1/4], \tag{2.76}$$

satisfies

$$f''(\mu) = -\frac{4}{(1-2\mu)^2} < 0, \tag{2.77}$$

and therefore is strictly concave. Since $f(0) = f(1/4) = 0$, we must have $f(\mu) \geqslant 0$ for all $\mu \in [0,1/4]$.

Fourth inequality. Consider two auxiliary functions $f, g\colon [0,1] \to [0,\infty[$ defined by

$$f(\lambda) := -\sqrt{\lambda}\log\lambda, \quad g(\lambda) := -(1-\lambda)\log(1-\lambda), \quad \lambda \in [0,1]. \tag{2.78}$$

Then

$$f'(\lambda) = -\frac{2+\log\lambda}{2\sqrt{\lambda}} \quad \text{and} \quad f''(\lambda) = \frac{\log\lambda}{4\lambda^{3/2}}, \tag{2.79}$$

which implies that f attains a unique maximum at $\lambda_0 = e^{-2}$ with $f(\lambda_0) = 2e^{-1}$. Therefore $f \leqslant 2e^{-1} \leqslant 1$. Secondly, we have

$$g'(\lambda) = \log(1-\lambda) + 1 \leqslant 1 \tag{2.80}$$

for all $\lambda \in [0,1]$, from which we infer

$$g(\lambda) = g(\lambda) - g(0) = \int_0^\lambda g'(t)\,dt \leqslant \lambda \leqslant \sqrt{\lambda}. \tag{2.81}$$

It suffices to show the desired inequality for all $\lambda \in [0,1/2]$ since both sides are symmetric around 1/2. But for $\lambda \in [0,1/2]$ we have $\sqrt{1-\lambda} \geqslant 1/\sqrt{2} \geqslant 1/2$, so that

$$h_1(\lambda) = \sqrt{\lambda}f(\lambda) + g(\lambda) \leqslant \sqrt{\lambda} + \sqrt{\lambda} = 2\sqrt{\lambda} \leqslant 4\sqrt{\lambda(1-\lambda)}, \tag{2.82}$$

as desired. □

3. Main results

We recall the situation of Proposition 2.1.2: Given a measurable subset $\tilde{\Gamma} \subseteq \mathbb{R}$, let $P_{\tilde{\Gamma}}$ be the operator given by the integral kernel (2.26), which is the orthogonal projection of $L^2(\mathbb{R}^2)$ onto the subspace $V_{\tilde{\Gamma}}$ of the lowest Landau level (2.25). Unlike in Chapter 2, we want to single out the dependence of $P_{\tilde{\Gamma}}$ on the magnetic field strength B to keep track of it in the asymptotic coefficients, so we write $P_{B,\tilde{\Gamma}}$ from now on.

The elements of $\tilde{\Gamma}$ have the physical dimension of momentum. In the following it will therefore be convenient to fix some unit $\kappa > 0$ of momentum and consider $\tilde{\Gamma} = \kappa\Gamma$ for a dimensionless measurable subset $\Gamma \subset \mathbb{R}$.

3.1. First result: Szegő-type asymptotics

Our first main result is a Szegő-type asymptotics for the operator $1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2}$ as $L \to \infty$, where $\Gamma \subset \mathbb{R}$ is the union of infinitely many intervals of constant length and $\kappa > 0$ is an arbitrarily chosen momentum parameter. The space of admissible test functions is

$$\begin{aligned}\mathbb{H} := \Big\{ h : [0,1] \to \mathbb{C} \text{ piecewise continuous, } h(0) = 0 \text{ and} \\ \text{Hölder continuous at the endpoints 0 and 1} \Big\}.\end{aligned} \tag{3.1}$$

For $h \in \mathbb{H}$, the functional given by

$$I(h) := \frac{1}{\pi^2} \int_0^1 \frac{h(t) - th(1)}{t(1-t)}\, dt \tag{3.2}$$

is well-defined.

Theorem 3.1.1. *Let*

$$\Gamma := \bigcup_{n \in \mathbb{Z}} [2n, 2n+1] \subset \mathbb{R}. \tag{3.3}$$

For any test function $h \in \mathbb{H}$, we have the two-term asymptotics

$$\begin{aligned}\operatorname{tr} h(1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2}) =& \frac{1}{\pi} h(1) B L^2 + I(h) \frac{B}{\kappa} L \log \sqrt{B} L \\ &+ o_{\frac{\kappa}{\sqrt{B}}}(\sqrt{B} L \log \sqrt{B} L)\end{aligned} \tag{3.4}$$

as $L \to \infty$. The subscript of the error term indicates the dependence on the variables B and κ.

Remark 3.1.2. (i) Our proof will show that all operators $h(1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2})$ with $h \in \mathbb{H}$ are trace class.

(ii) The asymptotic expansion (3.4) depends on L, B and κ only through the dimensionless quantities $\sqrt{B}L$ and $\frac{\kappa}{\sqrt{B}}$. Evidently, it is equivalent to write

$$\operatorname{tr} h(1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2}) = \frac{1}{\pi} h(1) BL^2 + I(h) \frac{B}{\kappa} L \log \kappa L + o_{\frac{\kappa}{\sqrt{B}}}(\kappa L \log \kappa L) \tag{3.5}$$

as $L \to \infty$.

(iii) A simple proof similar to the one of (2.71) shows that the functions h_γ with $\gamma \neq 1$ satisfy

$$h_\gamma(\lambda) \leqslant C \lambda^\gamma (1-\lambda)^\gamma, \quad \lambda \in [0,1], \tag{3.6}$$

and thus $h_\gamma \in \mathbb{H}$ for all $\gamma > 0$. This implies the enhanced area law

$$S_\gamma(P_{B,\kappa\Gamma}, [-L,L]^2) = I(h_\gamma) \frac{B}{\kappa} L \log \sqrt{B} L + o_{\frac{\kappa}{\sqrt{B}}}(\sqrt{B} L \log \sqrt{B} L) \tag{3.7}$$

for all γ-Rényi entanglement entropies (2.67). The asymptotic coefficient can be simplified further using $I(h_\gamma) = (1+\gamma)/(6\gamma)$, see [LSS14].

(iv) By the calculations (4.176) and (4.177) below, the projection $P_{B,\kappa\Gamma}$ with the specific choice (3.3) for Γ satisfies

$$\begin{aligned} \operatorname{tr}(1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2}) &= \frac{B}{\pi} L^2 = \frac{1}{2} \cdot \frac{B}{2\pi} \left| [-L,L]^2 \right| \\ &= \frac{1}{2} \cdot \operatorname{tr}\left(1_{[-L,L]^2} 1_{\{B\}}(H_B) 1_{[-L,L]^2} \right) \end{aligned} \tag{3.8}$$

and therefore corresponds to a half-filled lowest Landau level, cf. Remark 2.3.2. We point out that the Landau level being *half-filled* is not an essential assumption. In fact, using sets Γ similar to the one above, it is not difficult to realize the asymptotics (3.4) with any kind of partial filling of the lowest Landau level with the same proof. The essential property of Γ is rather the sufficient (infinite) amount of gaps.

Theorem 3.1.1 is a consequence of the following version, which reduces the statement to the dimensionless case $B = \kappa = 1$ and to $h(1) = 0$ in (3.4).

Theorem 3.1.3. *Let $\Gamma \subset \mathbb{R}$ be the set defined in* (3.3) *and $P_\Gamma := P_{1,\Gamma}$. Then for any test function $h \in \mathbb{H}$ satisfying $h(1) = 0$, we have*

$$\operatorname{tr} h(1_{[-l,l]^2} P_\Gamma 1_{[-l,l]^2}) = I(h) l \log l + o(l \log l) \tag{3.9}$$

as $l \to \infty$, where the functional I is given by (3.2).

The proof that Theorem 3.1.1 follows from Theorem 3.1.3 is fairly straightforward – most of the work will go into the proof of Theorem 3.1.3. Both proofs are carried out in Chapter 4.

3.2. Second result: Anomalous enhancements of the area law

Our second main result is formulated for the dimensionless case $B = \kappa = 1$ right away and addresses the behavior of $\operatorname{tr} h(1_{[-l,l]^2} P_\Gamma 1_{[-l,l]^2})$ when $\Gamma \subset \mathbb{R}$ consists of intervals of *decreasing* length. We will only focus on one specific test function, namely

$$h\colon [0,1] \to \mathbb{R}, \quad h(\lambda) := \lambda(1-\lambda). \tag{3.10}$$

The quantity $\operatorname{tr} h(1_{[-l,l]^2} P_\Gamma 1_{[-l,l]^2})$ is interpreted as the *particle number fluctuation* in the physics literature. By Lemma 2.3.3, it can be used to bound the 2-Rényi entanglement entropy from above and below. In contrast to the first result, we will only be concerned with asymptotic upper and lower bounds. Note that since $1_{[-l,l]^2}$ and P_Γ are projections,

$$h(1_{[-l,l]^2} P_\Gamma 1_{[-l,l]^2}) = 1_{[-l,l]^2} P_\Gamma 1_{\mathbb{R}^2 \setminus [-l,l]^2} P_\Gamma 1_{[-l,l]^2} = |1_{\mathbb{R}^2 \setminus [-l,l]^2} P_\Gamma 1_{[-l,l]^2}|^2. \tag{3.11}$$

Definition 3.2.1. Let $\alpha \in [0,1]$ be a decay exponent and let $(\xi_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}_0}$ be the real sequences defined by

$$\xi_n = \xi_n(\alpha) := \frac{1}{n^\alpha}, \quad a_n = a_n(\alpha) := \sum_{k=1}^{n} \frac{1}{k^\alpha} \tag{3.12}$$

for each $n \in \mathbb{N}$ and $a_0 := 0$. Finally, we let

$$\Gamma_\alpha := \bigcup_{n\in\mathbb{Z}} I_n(\alpha), \quad I_n(\alpha) := \begin{cases} [2a_n, 2a_n + \xi_{n+1}], & \text{if } n \in \mathbb{N}_0, \\ [-2a_{-n}, -2a_{-n} + \xi_{-n}], & \text{if } n \in -\mathbb{N}. \end{cases} \tag{3.13}$$

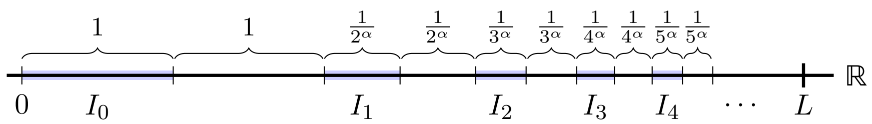

Figure 3.1.: The set Γ_α.

For two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, we write $f(x) = \Theta(g(x))$ as $x \to \infty$ if there exist constants $C_1, C_2, x_0 > 0$ such that

$$C_1|g(x)| \leqslant |f(x)| \leqslant C_2|g(x)| \tag{3.14}$$

for all $x \geqslant x_0$. We are now ready to state our second main result.

Theorem 3.2.2. *Let Γ_α be as defined in* (3.13)*.*

(i) If $\alpha \in [0, \frac{1}{2}[$, then as $l \to \infty$,

$$\operatorname{tr}|1_{\mathbb{R}^2 \setminus [-l,l]^2} P_{\Gamma_\alpha} 1_{[-l,l]^2}|^2 = \Theta(l^{1/(1-\alpha)} \log l). \tag{3.15}$$

(ii) If $\alpha \in [\frac{1}{2}, 1]$, then as $l \to \infty$,

$$\operatorname{tr}|1_{\mathbb{R}^2 \setminus [-l,l]^2} P_{\Gamma_\alpha} 1_{[-l,l]^2}|^2 = \Theta(l^2). \tag{3.16}$$

Here, we use notation (3.14)*.*

Remark 3.2.3. (i) In the case $\alpha = 0$, i.e. where Γ consists of intervals of the same length, (3.15) is a consequence of (3.9) with test function h as in (3.10).

(ii) The factor $l^{1/(1-\alpha)}$ in (3.15) can be interpreted as the number of "band edges" in Γ between $-l$ and l: Indeed, since a_n grows asymptotically like $|n|^{1-\alpha}$ (in the sense of (3.14)), the number of gaps between $-l$ and l is up to a constant given by the number of integers n such that $|n|^{1-\alpha} < l$, or equivalently, $|n| < l^{1/(1-\alpha)}$. As the proof shows, each band edge produces one $\log l$ term, leading to an overall asymptotic behavior of $l^{1/(1-\alpha)} \log l$.

On the other hand, in the case $\alpha \in [\frac{1}{2}, 1]$, the band edges are getting closer to each other more quickly. They do not yield individual contributions anymore but start to interact, resulting in a growth of order l^2, cf. Remark 5.1.2.

(iii) The theorem together with the inequalities (2.71) imply for the corresponding 2-Rényi entanglement entropy (2.67) that

$$S_2(P_{\Gamma_\alpha}, [-l,l]^2) = \begin{cases} \Theta(l^{1/(1-\alpha)} \log l), & \text{if } 0 \leqslant \alpha < \dfrac{1}{2}, \\ \Theta(l^2), & \text{if } \dfrac{1}{2} \leqslant \alpha \leqslant 1. \end{cases} \tag{3.17}$$

For the von Neumann entanglement entropy $S_1(P_{\Gamma_\alpha}, [-l,l]^2)$, one only obtains the corresponding lower bounds using Theorem 3.2.2. Upper bounds require q-Schatten-von Neumann quasinorm bounds with $q < 1$ for the operator $1_{\mathbb{R}^2 \setminus [-l,l]^2} P_{\Gamma_\alpha} 1_{[-l,l]^2}$, which are more difficult to obtain. In this thesis, such a bound is only established in the case where $\alpha = 0$ and Γ is given by (3.3), which is the subject of the first result.

The proofs of Theorems 3.1.3 and 3.2.2 are carried out in Chapters 4 and 5, respectively.

4. Proof of the first main result

4.1. Reduction to a one-dimensional operator

The goal of this chapter is to prove Theorem 3.1.1. We start with the simplified variant Theorem 3.1.3, which requires the most of the work. In the last section we then show how it implies the general statement.

The first step in proving Theorem 3.1.3 consists of reducing the statement (3.9) to the equivalent "one-dimensional" formulation (4.2) below.

Let $U \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ be the transformation (2.28), which satisfies $P_\Omega = U^* 1_\Omega U$ for every measurable $\Omega \subseteq \mathbb{R}$ and $UU^* = \mathrm{id}_{L^2(\mathbb{R})}$ by Lemma 2.1.3. The first equality in particular implies that $U^*U = P_\mathbb{R}$ and therefore that, as an operator on $P_\mathbb{R} L^2(\mathbb{R}^2)$, U is unitary. Thus, for $h \in \mathbb{H}$ and choosing $\Omega = \Gamma$ from (3.3), we obtain

$$\operatorname{tr} h(1_{[-l,l]^2} P_\Gamma 1_{[-l,l]^2}) = \operatorname{tr} h(P_\Gamma 1_{[-l,l]^2} P_\Gamma) = \operatorname{tr} h\big(1_\Gamma U 1_{[-l,l]^2} U^* 1_\Gamma\big). \tag{4.1}$$

Due to $h(0) = 0$, the trace in the middle is effectively only taken over $P_\mathbb{R} L^2(\mathbb{R}^2)$, while the right trace is over $L^2(\mathbb{R})$.

Lemma 4.1.1. *The statement of Theorem 3.1.3 is equivalent to*

$$\operatorname{tr} h(1_\Gamma T_l 1_\Gamma) = I(h) l \log l + o(l \log l) \tag{4.2}$$

as $l \to \infty$ for every $h \in \mathbb{H}$ with $h(1) = 0$, where the functional I is defined by (3.2) *and the operator*

$$T_l := U 1_{[-l,l]^2} U^* \tag{4.3}$$

is an orthogonal projection on $L^2(\mathbb{R})$ for every $l > 0$ with integral kernel given by

$$t_l(k,k') = e^{-(k-k')^2/4} \frac{\sin[l(k-k')]}{\pi(k-k')} f_l\left(\frac{k+k'}{2}\right), \quad k,k' \in \mathbb{R}, \tag{4.4}$$

with $f_l \colon \mathbb{R} \to [0,1]$,

$$f_l(x) := \frac{1}{\sqrt{\pi}} \int_{-l}^{l} e^{-(\xi - x)^2} \, d\xi, \quad x \in \mathbb{R}. \tag{4.5}$$

In (4.4) *and what follows we use the convention that* $\sin(l\,0)/0 := l$.

Proof. In view of (4.1), it remains to verify the expression (4.4) for the kernel of T_l. By definition of U and U^*,

$$\begin{aligned} t_l(k,k') = \int_{[-l,l]^2} \psi_k(x) \overline{\psi_{k'}(x)} \, dx &= \frac{1}{\sqrt{\pi}} \int_{-l}^{l} \frac{e^{ix_2(k-k')}}{2\pi} \, dx_2 \int_{-l}^{l} e^{-[(x_1-k)^2 + (x_1-k')^2]/2} \, dx_1 \\ &= \frac{1}{\sqrt{\pi}} \frac{\sin[l(k-k')]}{\pi(k-k')} \int_{-l}^{l} e^{-[(x_1-k)^2 + (x_1-k')^2]/2} \, dx_1 \end{aligned} \tag{4.6}$$

for any $k, k' \in \mathbb{R}$. Completing the square as

$$\begin{aligned}(x_1-k)^2+(x_1-k')^2 &= 2x_1^2-2x_1(k+k')+k^2+(k')^2\\ &= 2\left(x_1-\frac{k+k'}{2}\right)^2-\frac{(k+k')^2}{2}+k^2+(k')^2 \qquad (4.7)\\ &= 2\left(x_1-\frac{k+k'}{2}\right)^2+\frac{(k-k')^2}{2}, \qquad (4.8)\end{aligned}$$

the claim follows. □

4.2. Asymptotics for polynomial test functions

The first major step in proving the asymptotics (4.2) is establishing it for test functions h being symmetric, resp. anti-symmetric, polynomials defined by

$$\mathscr{s}_J(\lambda) := [\lambda(1-\lambda)]^J \quad \text{and} \quad \mathscr{a}_J(\lambda) := \lambda \mathscr{s}_J(\lambda), \quad J \in \mathbb{N}, \quad \lambda \in [0,1]. \tag{4.9}$$

More precisely, in this section we prove the following

Theorem 4.2.1. *Consider a test function* $h \in \{\mathscr{s}_J, \mathscr{a}_J : J \in \mathbb{N}\}$. *Then*

$$\operatorname{tr} h(1_\Gamma T_l 1_\Gamma) = I(h) l \log l + o(l \log l) \tag{4.10}$$

as $l \to \infty$, *where* T_l *is the orthogonal projection on* $L^2(\mathbb{R})$ *with kernel* (4.4) *and the functional* I *is given by* (3.2).

By linearity, the above theorem establishes (4.2) for all polynomials p with $p(0) = p(1) = 0$. As we will see in Section 4.3, this together with a Schatten quasi-norm estimate for the operator $1_\Gamma T_l 1_{\Gamma^c}$ can be used to prove (4.2) for all test functions $h \in \mathbb{H}_0$ by means of a standard approximation argument. The rest of this section is devoted to the proof of Theorem 4.2.1.

4.2.1. Reduction to contributions from the diagonal

Throughout this section, we will use the abbreviations

$$1_n := 1_{[n,n+1]} \quad \text{and} \quad 1_{\Omega\setminus n} := 1_{\Omega\setminus[n,n+1]} \tag{4.11}$$

for $n \in \mathbb{Z}$ and a measurable set $\Omega \subseteq \mathbb{R}$. Put

$$A_l := 1_\Gamma T_l 1_\Gamma = \sum_{n,m\in 2\mathbb{Z}} 1_n T_l 1_m \quad \text{and} \quad B_l := \sum_{n\in 2\mathbb{Z}} 1_n T_l 1_n, \tag{4.12}$$

so that B_l contains only the diagonal contributions with $n = m$. Note that since T_l is a projection, we have $0 \leqslant A_l \leqslant 1$ and $0 \leqslant B_l \leqslant 1$. This implies that

$$\|\mathscr{s}_1(A_l)\| \leqslant 1 \quad \text{and} \quad \|\mathscr{s}_1(B_l)\| \leqslant 1 \tag{4.13}$$

in operator norm for the first symmetric polynomial, which will be useful later.

The goal of Section 4.2.1 is to show $\operatorname{tr} h(A_l) = \operatorname{tr} h(B_l) + o(l \log l)$ as $l \to \infty$ for $h = \mathscr{s}_J$ and $h = \mathscr{a}_J$ for all $J \in \mathbb{N}$. This will be achieved by the following Lemmas 4.2.2 and 4.2.4. But before, we introduce some notation.

Lemma 4.2.2. *For each $J \in \mathbb{N}$, there exists $C > 0$ such that for all $l \geqslant 1$ we have*

$$\|\mathscr{s}_J(A_l) - \mathscr{s}_J(B_l)\|_{\mathfrak{S}_1} \leqslant Cl. \tag{4.14}$$

A core ingredient for the proof of Lemma 4.2.2 is the following trace-norm estimate from [LW80, Eq. (12)]. We formulate it in the more precise version from [PS18a, Lemma 5.1], where also a proof can be found.

Lemma 4.2.3. *Let $M \subset \mathbb{R}$ be a Borel measurable set. For any $z \in M$, let $p_z, q_z \in L^2(\mathbb{R})$ such that the mappings $z \mapsto p_z$ and $z \mapsto q_z$ are weakly measurable. The operator $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by the form*

$$\langle u, Tv \rangle := \int_M \langle u, p_z \rangle \langle q_z, v \rangle \, dz \tag{4.15}$$

for all $u, v \in L^2(\mathbb{R})$ satisfies the trace-norm estimate

$$\|T\|_{\mathfrak{S}_1} \leqslant \int_M \|p_z\|_2 \|q_z\|_2 \, dz. \tag{4.16}$$

Proof of Lemma 4.2.2. Let $l \geqslant 1$.

Step (i). The case $J \in \mathbb{N} \setminus \{1\}$ is reduced to the case $J = 1$ by a telescoping sum argument. Since

$$X^J - Y^J = \sum_{j=0}^{J-1} X^j (X - Y) Y^{J-j-1}, \tag{4.17}$$

for arbitrary bounded operators X and Y, we infer from (4.13) that

$$\|\mathscr{s}_J(A_l) - \mathscr{s}_J(B_l)\|_{\mathfrak{S}_1} \leqslant J \|\mathscr{s}_1(A_l) - \mathscr{s}_1(B_l)\|_{\mathfrak{S}_1}, \tag{4.18}$$

and it remains to prove the statement for $J = 1$.

Step (ii). In order to deduce the desired estimate in the case $J = 1$ we recall that $T_l^2 = T_l$ so that

$$\mathscr{s}_1(A_l) = A_l(1 - A_l) = 1_\Gamma T_l 1_\Gamma - 1_\Gamma T_l 1_\Gamma T_l 1_\Gamma = 1_\Gamma T_l (1 - 1_\Gamma) T_l 1_\Gamma = 1_\Gamma T_l 1_{\Gamma^c} T_l 1_\Gamma \tag{4.19}$$

and

$$\mathscr{s}_1(B_l) = B_l(1 - B_l) = \sum_{n \in 2\mathbb{Z}} 1_n T_l 1_n - \sum_{n \in 2\mathbb{Z}} 1_n T_l 1_n T_l 1_n = \sum_{n \in 2\mathbb{Z}} 1_n T_l 1_{\mathbb{R} \setminus n} T_l 1_n. \tag{4.20}$$

Combining both calculations yields

$$
\begin{aligned}
\mathcal{J}_1(A_l) - \mathcal{J}_1(B_l) &= \sum_{n,m\in 2\mathbb{Z}} 1_n T_l 1_{\Gamma^c} T_l 1_m - \sum_{n\in 2\mathbb{Z}} 1_n T_l 1_{\mathbb{R}\setminus n} T_l 1_n \\
&= \sum_{n\in 2\mathbb{Z}} \big[1_n T_l (1_{\Gamma^c} - 1_{\mathbb{R}\setminus n}) T_l 1_n\big] + \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} 1_n T_l 1_{\Gamma^c} T_l 1_m \\
&= -\sum_{n\in 2\mathbb{Z}} 1_n T_l 1_{\Gamma\setminus n} T_l 1_n + \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} 1_n T_l 1_{\Gamma^c} T_l 1_m \\
&= -\sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} 1_n T_l 1_m T_l 1_n + \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} 1_n T_l 1_{\Gamma^c} T_l 1_m. \qquad (4.21)
\end{aligned}
$$

Consequently, we obtain the trace-norm estimate

$$
\|\mathcal{J}_1(A_l) - \mathcal{J}_1(B_l)\|_{\mathfrak{S}_1} \leqslant \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} \|1_n T_l 1_m T_l 1_n\|_{\mathfrak{S}_1} + \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} \|1_n T_l 1_{\Gamma^c} T_l 1_m\|_{\mathfrak{S}_1}. \qquad (4.22)
$$

We rewrite the terms in the first sum as a Hilbert–Schmidt norm

$$
\|1_n T_l 1_m T_l 1_n\|_{\mathfrak{S}_1} = \|1_m T_l 1_n\|_{\mathfrak{S}_2}^2 = \int_{[n,n+1]\times[m,m+1]} t_l(x,y)^2 \, dx\, dy, \qquad (4.23)
$$

and since

$$
\bigcup_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} \big([n,n+1]\times[m,m+1]\big) \subset \big\{(x,y)\in\mathbb{R}^2 : |x-y|>1\big\} =: \Omega, \qquad (4.24)
$$

we conclude that

$$
\begin{aligned}
\sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} \|1_n T_l 1_m T_l 1_n\|_{\mathfrak{S}_1} &\leqslant \int_\Omega \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} \Big[f_l\Big(\frac{x+y}{2}\Big)\Big]^2 \, dx\, dy \\
&\leqslant \frac{1}{\pi^2}\int_{\mathbb{R}} e^{-q^2/2}\, dq \int_{\mathbb{R}} f_l^2(Q)\, dQ \leqslant Cl, \qquad (4.25)
\end{aligned}
$$

where the last inequality and the l-independence of the constant C follow from $\|f_l\|_2^2 \leqslant \|f_l\|_1 = 2l$ thanks to $0 \leqslant f_l \leqslant 1$.

Therefore, it remains to show that the second sum in (4.22) is also bounded by a constant multiple of l. To do so, we will apply Lemma 4.2.3: We fix $n, m \in 2\mathbb{Z}$. The integral kernel of the operator $1_n T_l 1_{\Gamma^c} T_l 1_m$ is given by

$$
\mathbb{R}^2 \ni (x,y) \mapsto \int_{\Gamma^c} 1_n(x) t_l(x,z) 1_m(y) t_l(y,z)\, dz. \qquad (4.26)
$$

Therefore, by (4.16),

$$
\begin{aligned}
\|1_n T_l 1_{\Gamma^c} T_l 1_m\|_{\mathfrak{S}_1} &\leqslant \int_{\Gamma^c} \|t_l(\cdot,z)1_n\|_2 \|t_l(\cdot,z)1_m\|_2\, dz \\
&= \int_{\Gamma^c} \|t_l(\cdot,z+n)1_n\|_2 \|t_l(\cdot,z+n)1_m\|_2\, dz, \qquad (4.27)
\end{aligned}
$$

where the equality is due to the translation invariance of Γ^c under even integers. We proceed by first estimating $\|t_l(\cdot, z+n)1_n\|_2$ for fixed $z \in \Gamma^c$. Since $0 \leqslant f_l \leqslant 1$ it follows that

$$\begin{aligned}\|t_l(\cdot, z+n)1_n\|_2^2 &= \int_n^{n+1} \frac{\sin^2[l(x-z-n)]}{\pi^2(x-z-n)^2}\, e^{-(x-z-n)^2/2}\Big[f_l\Big(\frac{x+z+n}{2}\Big)\Big]^2\, dx \\ &\leqslant \int_0^1 \frac{1}{\pi^2(x-z)^2} e^{-(x-z)^2}\, dx. \end{aligned} \tag{4.28}$$

Lemma A.3.1(i) implies the existence of a constant $C > 0$ such that $e^{-(x-y)^2/2} \leqslant Ce^{-y^2/4}$ for every $x \in [0,1]$ and every $y \in \mathbb{R}$. Applying this to the above we get

$$\|t_l(\cdot, z+n)1_n\|_2^2 \leqslant Ce^{-z^2/4} \int_0^1 \frac{1}{(x-z)^2}\, dz = Ce^{-z^2/4}\frac{1}{z(z-1)}. \tag{4.29}$$

Now let us turn to

$$\begin{aligned}&\|t_l(\cdot, z+n)1_m\|_2^2 \\ &= \int_m^{m+1} \frac{\sin^2[l(y-z-n)]}{\pi^2(y-z-n)^2}\, e^{-(y-z-n)^2/2}\Big[f_l\Big(\frac{y+z+n}{2}\Big)\Big]^2\, dy \\ &\leqslant \int_0^1 \frac{1}{\pi^2(y-z-n+m)^2}\, e^{-(y-z-n+m)^2/2}\Big[f_l\Big(\frac{y+z+n+m}{2}\Big)\Big]^2\, dy. \end{aligned} \tag{4.30}$$

For the Gaussian we again use Lemma A.3.1(i) to get

$$e^{-(y-z-n+m)^2/2} \leqslant Ce^{-(z+n-m)^2/4}, \tag{4.31}$$

for a universal constant $C > 0$. The elementary inequality $2ab \leqslant \frac{3}{2}a^2 + \frac{2}{3}b^2$ for $a, b \in \mathbb{R}$ implies

$$\begin{aligned} e^{-(z+n-m)^2/4} &= e^{-z^2/4}e^{-(n-m)^2/4}e^{z(n-m)/2} \\ &\leqslant e^{-z^2/4}e^{-(n-m)^2/4}e^{\frac{3}{8}z^2+\frac{1}{6}(n-m)^2} = e^{z^2/8}e^{-(n-m)^2/12}. \end{aligned} \tag{4.32}$$

Next, we treat the f_l-term in (4.30), this time using Lemma A.3.1(ii) which gives

$$f_l\Big(\frac{y+z+m+n}{2}\Big) \leqslant Cf_l\Big(\frac{z+n+m}{4}\Big), \tag{4.33}$$

where the constant $C > 0$ is independent of $y \in [0,1], z \in \Gamma^c$ and $m, n \in 2\mathbb{Z}$. To simplify the involved expressions, we introduce new summation variables

$$\mu := n-m,\ \nu := n+m, \tag{4.34}$$

so that $\mu \neq 0$ in view of (4.22). Combining (4.32) and (4.31), plugging the result and (4.33) into (4.30), gives

$$\begin{aligned}\|t_l(\cdot, z+n)1_m\|_2^2 &\leqslant Ce^{z^2/8}e^{-\mu^2/12}\Big[f_l\Big(\frac{z+\nu}{4}\Big)\Big]^2 \int_0^1 \frac{1}{(y-z-\mu)^2}\, dy \\ &= Ce^{z^2/8}e^{-\mu^2/12}\Big[f_l\Big(\frac{z+\nu}{4}\Big)\Big]^2 \frac{1}{(z+\mu)(z+\mu-1)}. \end{aligned} \tag{4.35}$$

We recall $z \in \Gamma^c$ and $\mu \in 2\mathbb{Z}$ so that $z + \mu \notin [0,1]$, and the integral in the first line of (4.35) exists.

Now we insert (4.35) and (4.29) into (4.27) and obtain

$$\|1_n T_l 1_{\Gamma^c} T_l 1_m\|_{\mathfrak{S}_1} \leqslant C e^{-\mu^2/24} \int_{\Gamma^c} \frac{e^{-z^2/16}}{\sqrt{z(z-1)(z+\mu)(z+\mu-1)}} f_l\Big(\frac{z+\nu}{4}\Big)\, dz. \tag{4.36}$$

In order to proceed we introduce the unique $\nu_z \in 2\mathbb{Z}$ such that $\hat{z} := z - \nu_z \in]-1, 0[$. Thus, we get

$$\sum_{\nu \in 2\mathbb{Z}} f_l\Big(\frac{z+\nu}{4}\Big) = \sum_{\nu \in \mathbb{Z}} \sum_{\sigma \in \{0,1\}} f_l\Big(\frac{\hat{z}}{4} + \frac{\sigma}{2} + \nu\Big) \leqslant Cl \tag{4.37}$$

with a constant $C > 0$ independent of $l > 0$ and $z \in \Gamma^c$ by Lemma A.1.1(ii). Next we sum (4.36) over $n \neq m \in 2\mathbb{Z}$, use Tonelli's theorem and (4.37). This results in

$$\begin{aligned}\sum_{\substack{n,m \in 2\mathbb{Z} \\ n \neq m}} \|1_n T_l 1_{\Gamma^c} T_l 1_m\|_{\mathfrak{S}_1} &\leqslant Cl \sum_{\substack{\mu \in 2\mathbb{Z} \\ \mu \neq 0}} e^{-\mu^2/24} \int_{\Gamma^c} \frac{e^{-z^2/16}}{\sqrt{z(z-1)(z+\mu)(z+\mu-1)}}\, dz \\ &\leqslant Cl \sum_{\substack{\mu \in 2\mathbb{Z} \\ \mu \neq 0}} e^{-\mu^2/24} \sum_{j \in 2\mathbb{Z}+1} e^{-j^2/32} \mathscr{I}(\mu, j),\end{aligned} \tag{4.38}$$

where we used Lemma A.3.1(i) for the second inequality and introduced the integral

$$\mathscr{I}(\mu, j) := \int_0^1 \frac{dz}{\sqrt{(z-j)(z-j-1)(z-j+\mu)(z-j+\mu-1)}} \leqslant \pi \tag{4.39}$$

for every $0 \neq \mu \in 2\mathbb{Z}$ and every $j \in 2\mathbb{Z}+1$. In order to see the inequality in (4.39), we note that the singularities of the integrand are $\mathbb{Z}$-valued. If a singularity is not located at 0 or 1, it amounts to a factor of the integrand that is bounded from above by 1. Thus, depending on μ and j, the integrand is bounded from above by either 1, $z^{-1/2}$, $|z-1|^{-1/2}$ or $|z(z-1)|^{-1/2}$ because a coincidence of different singularities is not possible at 0 nor at 1 due to the fact that $\mu \neq 0$ is even and j is odd. Taken together, (4.39) and (4.38) imply

$$\sum_{\substack{n,m \in 2\mathbb{Z} \\ n \neq m}} \|1_n T_l 1_{\Gamma^c} T_l 1_m\|_{\mathfrak{S}_1} \leqslant Cl. \tag{4.40}$$

Combining (4.40) and (4.25), we see that both sums in (4.22) are bounded by a constant multiple of l, which proves (4.14) in the case $J = 1$. ☐

Lemma 4.2.4. *For each $J \in \mathbb{N}$,*

$$\|\mathscr{a}_J(A_l) - \mathscr{a}_J(B_l)\|_{\mathfrak{S}_1} = \mathscr{O}(l\sqrt{\log l}) \tag{4.41}$$

as $l \to \infty$.

Proof. We write

$$\begin{aligned}\mathscr{a}_J(A_l) - \mathscr{a}_J(B_l) &= A_l[A_l(1-A_l)]^J - A_l[B_l(1-B_l)]^J + (A_l - B_l)[B_l(1-B_l)]^J \\ &= A_l\big(\mathscr{s}_J(A_l) - \mathscr{s}_J(B_l)\big) + (A_l - B_l)\mathscr{s}_J(B_l)\end{aligned} \tag{4.42}$$

and estimate, using $\|A_l\| \leqslant 1$ and Lemma 4.2.2,

$$\|\mathscr{a}_J(A_l) - \mathscr{a}_J(B_l)\|_{\mathfrak{S}_1} \leqslant Cl + \|A_l - B_l\|_{\mathfrak{S}_2} \|\mathscr{s}_J(B_l)\|_{\mathfrak{S}_2} \tag{4.43}$$

The first factor of the second term is estimated as

$$\|A_l - B_l\|_{\mathfrak{S}_2}^2 = \sum_{\substack{n,m\in 2\mathbb{Z}\\ n\neq m}} \operatorname{tr}(1_n T_l 1_m T_l 1_n) \leqslant Cl, \tag{4.44}$$

where we have used (4.25) for the inequality. For the second factor we combine (4.49) in Lemma 4.2.7 and (4.71) in Lemma 4.2.8 below, which yield

$$\|\jmath_J(B_l)\|_{\mathfrak{S}_2}^2 = \operatorname{tr}\jmath_{2J}(B_l) = I(\jmath_{2J}) l\log l + o(l\log l) = \mathcal{O}(l\log l) \tag{4.45}$$

as $l \to \infty$. ☐

4.2.2. Asymptotics of $\operatorname{tr}\jmath_J(B_l)$ and $\operatorname{tr}\alpha_J(B_l)$

Definition 4.2.5. For $l > 0$ we introduce the bounded operator S_l on $L^2(\mathbb{R})$ by

$$(S_l\varphi)(x) := \int_{\mathbb{R}} s_l(x-y)\varphi(y), \quad \varphi \in L^2(\mathbb{R}),\, x\in\mathbb{R}, \tag{4.46}$$

where $s_l : \mathbb{R}\to\mathbb{R}$ is defined by

$$s_l(x) := \frac{\sin(lx)}{\pi x} e^{-x^2/4} \quad x\in\mathbb{R}. \tag{4.47}$$

We also set

$$X_l := 1_{[0,1]} S_l 1_{\mathbb{R}\setminus[0,1]} S_l 1_{[0,1]}. \tag{4.48}$$

Remark 4.2.6. It is *not* true that $\jmath_1(1_{[0,1]}S_l 1_{[0,1]}) = X_l$ because, unlike T_l, the operator S_l is not a projection.

Lemma 4.2.7. *For every $J\in\mathbb{N}$ we have*

$$\operatorname{tr}\jmath_J(B_l) = l \operatorname{tr} X_l^J + \mathcal{O}((\log l)^J), \tag{4.49}$$

and

$$\operatorname{tr}\alpha_J(B_l) = l \operatorname{tr}\left\{1_{[0,1]}S_l 1_{[0,1]} X_l^J\right\} + \mathcal{O}((\log l)^{J+1}) \tag{4.50}$$

as $l\to\infty$.

Proof. We fix $J\in\mathbb{N}$. To establish (4.49) we recall (4.20) so that

$$\jmath_J(B_l) = [\jmath_1(B_l)]^J = \sum_{n\in 2\mathbb{Z}} \left[1_n T_l 1_{\mathbb{R}\setminus n} T_l 1_n\right]^J. \tag{4.51}$$

Observing $t_l(x,y) = s_l(x-y) f_l(\frac{x+y}{2})$ for the integral kernel (4.4) of T_l, we deduce

$$\begin{aligned}\operatorname{tr}\jmath_J(B_l) &= \sum_{n\in 2\mathbb{Z}} \int_{[n,n+1]^J}\int_{(\mathbb{R}\setminus[n,n+1])^J} \prod_{j=1}^J \Big[s_l(x_j - z_j)s_l(z_j - x_{j+1}) \\ &\qquad\qquad \times f_l\Big(\frac{x_j+z_j}{2}\Big) f_l\Big(\frac{z_j + x_{j+1}}{2}\Big)\Big]\, dz\, dx \\ &= \int_{[0,1]^J}\int_{(\mathbb{R}\setminus[0,1])^J} \prod_{j=1}^J \Big[s_l(x_j - z_j)s_l(z_j - x_{j+1})\Big] \\ &\qquad\qquad \times \sum_{n\in 2\mathbb{Z}} \prod_{j=1}^J \Big[f_l\Big(\frac{x_j+z_j}{2} + n\Big) f_l\Big(\frac{z_j+x_{j+1}}{2} + n\Big)\Big]\, dz\, dx,\end{aligned} \tag{4.52}$$

where we set $x_{J+1} := x_1$ and, for the second equality, performed the change-of-variables $x_j \mapsto x_j + n$, $z_j \mapsto z_j + n$ and used Fubini's theorem. Next, we evaluate the sum in the last line of (4.52) asymptotically for large l. To this end, we introduce $Q_j := (x_{(j+1)/2} + z_{(j+1)/2})/2$ for odd $j \in \{1, \dots, 2J\}$ and $Q_j := (z_{j/2} + x_{j/2+1})/2$ for even $j \in \{1, \dots, 2J\}$. Lemma A.1.1(i) with $\alpha = 2$ then yields

$$\sum_{n\in 2\mathbb{Z}} \prod_{j=1}^{J} f_l\Big(\frac{x_j + z_j}{2} + n\Big) f_l\Big(\frac{z_j + x_{j+1}}{2} + n\Big) = \sum_{n\in 2\mathbb{Z}} \prod_{j=1}^{2J} f_l(Q_j + n) = l + R, \tag{4.53}$$

where the error term $R = R(J, Q_1, \dots, Q_{2J}, l)$ satisfies the bound $|R| \leqslant C(1 + \sum_{j=1}^{2J} |Q_j|)$ with a constant $C > 0$ depending only on J. Inserting (4.53) into (4.52), we obtain two terms. The first term amounts to

$$l \int_{[0,1]^J} \int_{(\mathbb{R}\setminus[0,1])^J} \prod_{j=1}^{J} \Big[s_l(x_j - z_j) s_l(z_j - x_{j+1})\Big]\, dz\, dx = l \ \mathrm{tr}\, X_l^J \tag{4.54}$$

and is the leading term in the claim (4.49). Therefore we have to show that the second term, which—up to a constant—is bounded in absolute value from above by

$$\int_{[0,1]^J} \int_{(\mathbb{R}\setminus[0,1])^J} \Big(1 + \sum_{j=1}^{2J} |Q_j|\Big) \prod_{j=1}^{J} \Big|s_l(x_j - z_j) s_l(z_j - x_{j+1})\Big|\, dz\, dx, \tag{4.55}$$

is of order $\mathscr{O}((\log l)^J)$ as $l \to \infty$. To see this, we argue first that

$$\sum_{j=1}^{2J} |Q_j| = \frac{1}{2}\sum_{j=1}^{J} |x_j + z_j| + \frac{1}{2}\sum_{j=1}^{J} |z_j + x_{j+1}| \leqslant \sum_{j=1}^{J} \big(1 + |z_j|\big) \tag{4.56}$$

and that for every $0 \leqslant x_j \leqslant 1$, $j \in \{1, \dots, J\}$,

$$\begin{aligned}\Big(\sum_{j=1}^{J} \big(1 + |z_j|\big)\Big) \prod_{j=1}^{J} e^{-(x_j - z_j)^2/4 - (z_j - x_{j+1})^2/4} &\leqslant \sum_{j=1}^{J} \big(1 + |z_j|\big) e^{-(x_j - z_j)^2/4 - (z_j - x_{j+1})^2/4} \\ &\leqslant C \sum_{j=1}^{J} \big(1 + |z_j|\big) e^{-z_j^2/4},\end{aligned} \tag{4.57}$$

where we used Lemma A.3.1(i), and the constant is universal. Recalling the definition (4.47) of s_l and taking the sup over all $z_j \in \mathbb{R}$ in (4.57), we deduce that, up to a J-dependent constant, (4.55) is bounded from above by

$$\begin{aligned}&\int_{[0,1]^J} \int_{(\mathbb{R}\setminus[0,1])^J} \prod_{j=1}^{J} \frac{|\sin[l(x_j - z_j)]||\sin[l(z_j - x_{j+1})]|}{|x_j - z_j||z_j - x_{j+1}|}\, dz\, dx \\ &\qquad = \int_{[0,l]^J} \Bigg(\prod_{j=1}^{J} \int_{(\mathbb{R}\setminus[0,l])} \frac{|\sin(x_j - z)||\sin(z - x_{j+1})|}{|x_j - z||z - x_{j+1}|}\, dz\Bigg)\, dx.\end{aligned} \tag{4.58}$$

We claim that for every given $j \in \{1, \dots, J\}$ the z-integral is bounded from above by 9 times

$$\begin{aligned}&\frac{\log(1 + l - x_{j+1}) - \log(1 + l - x_j)}{x_j - x_{j+1}} + \frac{\log(1 + x_j) - \log(1 + x_{j+1})}{x_j - x_{j+1}} \\ &\leqslant \frac{1}{\sqrt{(1 + l - x_{j+1})(1 + l - x_j)}} + \frac{1}{\sqrt{(1 + x_{j+1})(1 + x_j)}} \leqslant 2\sqrt{h_l(x_{j+1}) h_l(x_j)},\end{aligned} \tag{4.59}$$

where

$$h_l(t) := \max\left\{\frac{1}{1+l-t}, \frac{1}{1+t}\right\} = \frac{1}{1+l/2-|l/2-t|}, \quad t \in [0,l]. \tag{4.60}$$

To verify (4.59), we employ the elementary estimate $|\sin t/t| \leqslant 3/(1+|t|$, see Lemma A.2.1 below, split the z-integral into $]-\infty,0]$ and $[l,\infty[$, which resolves the absolute values and evaluate the resulting integrals by a partial fraction decomposition. The first inequality in (4.59) follows from an application of the logarithmic-geometric mean inequality

$$(a-b)/(\log a - \log b) \geqslant \sqrt{ab}, \tag{4.61}$$

which is valid for all $a,b>0$ (see e.g. [Bur87]).

Inserting (4.59) into (4.58) and discarding the factor $(9\cdot 2)^J$, we get as an upper bound

$$\int_{[0,l]^J} \prod_{j=1}^{J} \sqrt{h_l(x_{j+1})}\sqrt{h_l(x_j)}\,dx = \left[\int_0^l h_l(t)\,dt\right]^J. \tag{4.62}$$

By symmetry,

$$\int_0^l h_l(t)\,dt = 2\int_0^{l/2} \frac{1}{1+t}\,dt = 2\log(1+l/2), \tag{4.63}$$

ultimately showing that (4.55) is of order $\mathcal{O}((\log l)^J)$ as $l \to \infty$.

Now we turn to the proof of (4.50). The trace of $\mathcal{a}_J(B_l)$ is given by

$$\sum_{n\in 2\mathbb{Z}} \int_{[n,n+1]^{J+1}} \int_{(\mathbb{R}\setminus[n,n+1])^J} s_l(x_{J+1},x_1) \prod_{j=1}^{J}\Big[s_l(x_j-z_j)s_l(z_j-x_{j+1})\Big] \\ \times f_l\Big(\frac{x_{J+1}+x_1}{2}\Big) \prod_{j=1}^{J}\Big[f_l\Big(\frac{x_j+z_j}{2}\Big)f_l\Big(\frac{z_j+x_{j+1}}{2}\Big)\Big]\,dz\,dx. \tag{4.64}$$

(Note that this time x_{J+1} is not defined as x_1, but rather is an additional integration variable.) By setting $Q_{2J+1} := x_{J+1}+x_1$, we imitate the proof of (4.49), i.e. apply Lemma A.1.1(i) again to obtain two terms, where the first one is

$$l\int_{[0,1]^{J+1}} \int_{(\mathbb{R}\setminus[0,1])^J} s_l(x_{J+1}-x_1) \prod_{j=1}^{J} s_l(x_j-z_j)s_l(z_j-x_{j+1})\,dz\,dx \tag{4.65}$$

which equals $l\ \mathrm{tr}\{1_{[0,1]}S_l 1_{[0,1]}X_l^J\}$, that is, the leading term in (4.50). The second term is bounded in absolute value from above by

$$\int_{[0,1]^{J+1}} \int_{(\mathbb{R}\setminus[0,1])^J} \Big(1+\sum_{j=1}^{2J}|Q_j|\Big)|s_l(x_{J+1}-x_1)| \prod_{j=1}^{J}\Big|s_l(x_j-z_j)s_l(z_j-x_{j+1})\Big|\,dz\,dx. \tag{4.66}$$

We will show that that it is of order $\mathcal{O}((\log l)^{J+1})$ as $l\to\infty$. Using the same reasoning as in (4.56) and (4.57), we find that (4.66) is bounded from above by a J-dependent constant times

$$\int_{[0,l]^{J+1}} \frac{|\sin(x_{J+1}-x_1)|}{|x_{J+1}-x_1|} \left(\prod_{j=1}^{J}\int_{(\mathbb{R}\setminus[0,l])} \frac{|\sin(x_j-z)||\sin(z-x_{j+1})|}{|x_j-z||z-x_{j+1}|}\,dz\right)dx. \tag{4.67}$$

The first factor of the integrand can be estimated with an application of Lemma A.2.1, giving

$$\frac{|\sin(x_{J+1}-x_1)|}{|x_{J+1}-x_1|} \leqslant \frac{3}{1+|x_{J+1}-x_1|}. \tag{4.68}$$

As an upper bound for the the product of z-integrals we again obtain $(9\cdot 2)^J$ times the integrand of the left-hand side of (4.62). Thus, (4.67) is bounded up to a J-dependent constant by

$$\begin{aligned} &\int_{[0,l]^{J+1}} \frac{\sqrt{h_l(x_{J+1})}\sqrt{h_l(x_1)}}{1+|x_{J+1}-x_1|} \prod_{j=2}^{J} \sqrt{h_l(x_{j+1})}\sqrt{h_l(x_j)}\,dx \\ &= (2\log(1+l/2))^{J-1} \int_0^l \int_0^l \frac{\sqrt{h_l(x_{J+1})}\sqrt{h_l(x_1)}}{1+|x_{J+1}-x_1|}\,dx_1\,dx_{J+1}. \end{aligned} \tag{4.69}$$

To estimate the double integral, we apply $ab \leqslant \frac{1}{2}(a^2+b^2)$ to the numerator of the integrand to get

$$\begin{aligned} \int_0^l \int_0^l \frac{\sqrt{h_l(x)}\sqrt{h_l(y)}}{1+|x-y|}\,dx\,dy &\leqslant \int_0^l h_l(y) \int_0^l \frac{1}{|x-y|}\,dx\,dy \\ &= \int_0^l (\log(1+y)+\log(1+l-y))h_l(y)\,dy \\ &\leqslant 2\log(1+l)\int_0^l h_l(y)\,dy = 4\log(1+l)\log(1+l/2). \end{aligned} \tag{4.70}$$

Combining the previous calculations we see that (4.66) is of order $\mathcal{O}((\log l)^{J+1})$ as $l\to\infty$. This finishes the proof. □

4.2.3. Application of the one-dimensional Widom formula

In view of Lemma 4.2.7, we are left with the evaluation of $\operatorname{tr} X_l^J$ and $\operatorname{tr}\left\{1_{[0,1]}S_l 1_{[0,1]}X_l^J\right\}$ asymptotically as $l\to\infty$. This is the content of the next lemma.

Lemma 4.2.8. *For every $J\in\mathbb{N}$, we have*

$$\operatorname{tr} X_l^J = I(\mathscr{b}_J)\log l + o(\log l) \tag{4.71}$$

and

$$\operatorname{tr}\left\{1_{[0,1]}S_l 1_{[0,1]}X_l^J\right\} = I(\mathscr{a}_J)\log l + o(\log l) \tag{4.72}$$

as $l\to\infty$. Here, I is the functional defined in (3.2)*.*

Definition 4.2.9. For $l>0$ we introduce a further bounded operator K_l on $L^2(\mathbb{R})$ by

$$(K_l\varphi)(x) := \int_{\mathbb{R}} k_l(x-y)\varphi(y), \quad \varphi\in L^2(\mathbb{R}),\ x\in\mathbb{R}, \tag{4.73}$$

with integral kernel defined by

$$k_l\colon \mathbb{R}\to\mathbb{R}, \quad k_l(x) := \frac{\sin(lx)}{\pi x}. \tag{4.74}$$

Remark 4.2.10. (i) We have $s_l(x) = k_l(x)e^{-x^2/4}$ for every $x \in \mathbb{R}$.

(ii) It will be convenient to describe the operators S_l and K_l by their action in Fourier space: Let $\mathscr{F}$ denote the (unitary) Fourier transform on $L^2(\mathbb{R})$. Since $\mathscr{F}k_l = \frac{1}{\sqrt{2\pi}}1_{[-l,l]}$ and $\mathscr{F}s_l = \frac{1}{\sqrt{2\pi}}f_l$, we have

$$S_l = \mathscr{F}^* f_l \mathscr{F} \quad \text{and} \quad K_l = \mathscr{F}^* 1_{[-l,l]} \mathscr{F}. \tag{4.75}$$

This means that S_l and K_l act in Fourier space by multiplication with the functions f_l and $1_{[-l,l]}$, respectively. From this, it is also immediate that K_l is an orthogonal projection, while S_l is not.

Proof of Lemma 4.2.8. By the one-dimensional Widom formula [LW80, Proof of Theorem 1], for any polynomial function p with $p(0) = p(1) = 0$, as $l \to \infty$ we have

$$\operatorname{tr} p(1_{[0,1]}K_l 1_{[0,1]}) = I(p)\log l + o(\log l). \tag{4.76}$$

The claim thus follows from (4.76) and Lemma 4.2.11 below. □

Lemma 4.2.11. *Let $J \in \mathbb{N}$. Then for each $l \geqslant 2$,*

$$\|X_l^J - \mathscr{s}_J(1_{[0,1]}K_l 1_{[0,1]})\|_{\mathfrak{S}_1} \leqslant C \tag{4.77}$$

and

$$\|1_{[0,1]}S_l 1_{[0,1]}X_l^J - \mathscr{a}_J(1_{[0,1]}K_l 1_{[0,1]})\|_{\mathfrak{S}_1} \leqslant C\sqrt{\log l} \tag{4.78}$$

with constants independent of l.

Proof. We define operators Y_l and Z_l on $L^2(\mathbb{R})$ by

$$Y_l := 1_{[0,1]}S_l 1_{]-1,0[\cup]1,2[}S_l 1_{[0,1]}, \tag{4.79}$$

$$Z_l := 1_{[0,1]}K_l 1_{]-1,0[\cup]1,2[}K_l 1_{[0,1]}. \tag{4.80}$$

We will successively show

$$\|X_l^J - Y_l^J\|_{\mathfrak{S}_1} \leqslant C, \quad \|Y_l^J - Z_l^J\|_{\mathfrak{S}_1} \leqslant C, \quad \|Z_l^J - \mathscr{s}_J(1_{[0,1]}K_l 1_{[0,1]})\|_{\mathfrak{S}_1} \leqslant C \tag{4.81}$$

with l-independent constants, from which (4.77) follows.

First inequality of (4.81). We start with the case $J = 1$ and observe

$$X_l = Y_l + 1_{[0,1]}S_l 1_{\mathbb{R}\setminus]-1,2[}S_l 1_{[0,1]} = Y_l + |1_{\mathbb{R}\setminus]-1,2[}S_l 1_{[0,1]}|^2, \tag{4.82}$$

so that

$$\operatorname{tr} X_l = \operatorname{tr} Y_l + \|1_{\mathbb{R}\setminus]-1,2[}S_l 1_{[0,1]}\|_{\mathfrak{S}_2}^2. \tag{4.83}$$

But

$$\begin{aligned}\|1_{\mathbb{R}\setminus]-1,2[}S_l 1_{[0,1]}\|_{\mathfrak{S}_2}^2 &= \int_{\mathbb{R}]-1,2[}\int_0^1 \left|\frac{\sin[l(x-y)]}{\pi(x-y)}e^{-(x-y)^2/4}\right|^2 dx\,dy \\ &\leqslant \frac{1}{\pi^2}\int_{\mathbb{R}\setminus]-1,2[}\int_0^1 \frac{1}{(x-y)^2}\,dx\,dy \leqslant C.\end{aligned} \tag{4.84}$$

We point out that the Gaussian factor was estimated from above by 1 and is not needed for the integral to converge. For general $J \in \mathbb{N}$, we use a telescoping sum and the fact that $\|X_l\|, \|Y_l\| \leqslant 1$ in operator norm:

$$\|X_l^J - Y_l^J\|_{\mathfrak{S}_1} \leqslant \sum_{j=0}^{J-1} \|X_l\|^j \|X_l - Y_l\|_{\mathfrak{S}_1} \|Y_l\|^{J-1-j} \leqslant J \|1_{\mathbb{R}\setminus]-1,2[} S_l 1_{[0,1]}\|_{\mathfrak{S}_2}^2 \leqslant C. \tag{4.85}$$

Second inequality of (4.81). Denoting $I :=]-1,0[\cup]1,2[$, we have

$$Y_l - Z_l = 1_{[0,1]} K_l 1_I (S_l - K_l) 1_{[0,1]} + 1_{[0,1]} (S_l - K_l) 1_I S_l 1_{[0,1]}. \tag{4.86}$$

To estimate the trace norm of the above expression, we employ Lemma 4.2.3: We obtain for the first term

$$\begin{aligned}
&\|1_{[0,1]} K_l 1_I (S_l - K_l) 1_{[0,1]}\|_{\mathfrak{S}_1} \\
&\leqslant \int_I \|1_{[0,1]} k_l(\cdot, z)\|_2 (s_l(z,\cdot) - k_l(z,\cdot)) 1_{[0,1]}\|_2 \, dz \\
&= \int_I \left(\int_0^1 \frac{\sin^2[l(x-z)]}{\pi^2 (x-z)^2} \, dx \right)^{1/2} \left(\int_0^1 \frac{\sin^2[l(z-y)]}{\pi^2 (z-y)^2} \left(e^{-(z-y)^2/4} - 1 \right)^2 dy \right)^{1/2} dz \\
&\leqslant \int_I \left(\int_0^1 \frac{1}{\pi^2 (x-z)^2} \, dx \right)^{1/2} \left(\int_0^1 \frac{1}{\pi^2 (z-y)^2} \frac{(z-y)^4}{16} \, dy \right)^{1/2} dz \\
&\leqslant C \int_I \frac{1}{\sqrt{z(z-1)}} \, dz \leqslant C.
\end{aligned} \tag{4.87}$$

For the second inequality we have used $1 - e^{-\xi} \leqslant \xi$ for $\xi > 0$. The second term in (4.86) is treated in the same way, since the additional Gaussian term can just be bounded from above by 1. Together this shows $\|Y_l - Z_l\|_{\mathfrak{S}_1} \leqslant C$. The case of general $J \in \mathbb{N}$ follows with another telescoping argument as in (4.85).

Third inequality of (4.81). As K_l is an orthogonal projection, we have

$$\mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]}) = \left(1_{[0,1]} K_l 1_{\mathbb{R}\setminus[0,1]} K_l 1_{[0,1]}\right)^J \tag{4.88}$$

and the proof follows from that of the first equivalence by replacing S_l with K_l there.

Inequality (4.78). We write

$$\begin{aligned}
1_{[0,1]} S_l 1_{[0,1]} X_l^J = {} & \mathscr{a}_J(1_{[0,1]} K_l 1_{[0,1]}) \\
& + 1_{[0,1]} (S_l - K_l) 1_{[0,1]} \mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]}) \\
& + 1_{[0,1]} S_l 1_{[0,1]} \left(X_l^J - \mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]}) \right),
\end{aligned} \tag{4.89}$$

so there are two error terms to be estimated. In the previous part of the proof we have already shown that $\|X_l^J - \mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]})\|_{\mathfrak{S}_1} \leqslant C$ uniformly in l. Since $\|1_{[0,1]} S_l 1_{[0,1]}\| \leqslant 1$, this implies that the second error term in (4.89) is bounded in trace norm independent of l. For the first error term, we infer from the Cauchy-Schwarz inequality (or Hölder inequality (2.45) with $p = 1, p_1 = p_2 = 2$) that

$$\begin{aligned}
&\left\|1_{[0,1]} (S_l - K_l) 1_{[0,1]} \mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]})\right\|_{\mathfrak{S}_1} \\
&\leqslant \|1_{[0,1]} (S_l - K_l) 1_{[0,1]}\|_{\mathfrak{S}_2} \|\mathscr{s}_J(1_{[0,1]} K_l 1_{[0,1]})\|_{\mathfrak{S}_2} \leqslant C \sqrt{\log l},
\end{aligned} \tag{4.90}$$

where the last inequality is due to

$$\begin{aligned}\|1_{[0,1]}(S_l - K_l)1_{[0,1]}\|_{\mathfrak{S}_2}^2 &= \int_0^1 \int_0^1 \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} \left(e^{-(x-y)^2/4} - 1\right)^2 dx\, dy \\ &\leqslant C \int_0^1 \int_0^1 (x-y)^2\, dx\, dy \leqslant C \end{aligned} \tag{4.91}$$

and the fact that

$$\|\jmath_J(1_{[0,1]}K_l 1_{[0,1]})\|_{\mathfrak{S}_2} = \sqrt{\operatorname{tr} \jmath_{2J}(1_{[0,1]}K_l 1_{[0,1]})} \leqslant C\sqrt{\log l} \tag{4.92}$$

for $l \geqslant 2$ by Widom's formula (4.76). This proves (4.78). □

Proof of Theorem 4.2.1. We combine the results of Lemmas 4.2.2, 4.2.4, 4.2.7 and 4.2.8. □

4.3. A Schatten-von Neumann class estimate

By Lemma 4.1.1 and the previous section, the last step in proving Theorem 3.1.3 amounts to lifting the validity of Theorem 4.2.1 from polynomials p satisfying $p(0) = p(1) = 0$ to all test functions $h \in \mathbb{H}$ with $h(1) = 0$. This can be done with a standard approximation method, which we carry out in Section 4.4. The method as a key ingredient requires a Schatten q-quasi-norm estimate for the operator $1_\Gamma T_l 1_\Gamma$, which we formulate in the next lemma. The rest of the present section is then spent proving this estimate.

Theorem 4.3.1. *Let $q \in \,]0,1[$. There exists a constant $C = C(q) > 0$ such that for all $l \geqslant 2$,*

$$\|1_\Gamma T_l 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant C l \log l. \tag{4.93}$$

Let us briefly describe the strategy to prove Theorem 4.3.1: Firstly, Lemma 4.3.2 allows us to reduce proving (4.93) for the operator T_l with integral kernel t_l (see (4.4)) to proving it for the operator T_l' with integral kernel

$$\mathbb{R}^2 \ni (x,y) \mapsto t_l'(x,y) := e^{-(x-y)^2/4} \frac{\sin[l(x-y)]}{\pi(x-y)} f_l(x). \tag{4.94}$$

This, loosely speaking, amounts to replacing the factor $f_l(\frac{x+y}{2})$ in the kernel t_l by the only x-dependent factor $f_l(x)$. Secondly, we show that we may further drop the Gaussian term in (4.94) at the cost of an error term of order $\mathcal{O}(l)$, leaving us with proving (4.93) for the operator T_l'' with kernel

$$\mathbb{R}^2 \ni (x,y) \mapsto t_l''(x,y) := \frac{\sin[l(x-y)]}{\pi(x-y)} f_l(x). \tag{4.95}$$

Finally, in Lemma 4.3.3 we prove the desired bound for the operator T_l'' by interpreting it as a pseudo-differential operator (4.126) and using a general Schatten-von Neumann class estimate for such operators taken from [Sob14].

Lemma 4.3.2. *Let T'_l be the operator on $L^2(\mathbb{R})$ with integral kernel (4.94). For every $q \in]0,1]$, we have*

$$\|1_\Gamma(T_l - T'_l)1_{\Gamma^c}\|_{\mathfrak{S}_q}^q = o(l) \tag{4.96}$$

as $l \to \infty$.

Proof. Let $l \geqslant 1$. The aim is to utilize Lemma 2.2.3. To that end, we first apply the q-triangle inequality to get

$$\|1_\Gamma(T_l - T'_l)1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant \sum_{n\in 2\mathbb{Z}} \|1_{[n,n+1]}(T_l - T'_l)1_{\Gamma^c}\|_{\mathfrak{S}_q}^q. \tag{4.97}$$

The kernel of the operator $1_{[n,n+1]}(T_l - T'_l)1_{\Gamma^c}$ is given by

$$1_{[n,n+1]}(x)[t_l(x,y) - t'_l(x,y)]1_{\Gamma^c}(y), \quad x,y \in \mathbb{R}. \tag{4.98}$$

Under the scaling transformation V_l given by $V_l f := l^{\frac{1}{2}} f(l\cdot)$ for $f \in L^2(\mathbb{R})$, this operator is unitarily equivalent to the operator $G_{l,n}$ with kernel

$$g_{l,n}(x,y) := 1_{[ln,l(n+1)]}(x) g_l(x,y) 1_{(l\Gamma)^c}(y), \tag{4.99}$$

where

$$g_l(x,y) := \frac{\sin(x-y)}{x-y} e^{-(x-y)^2/(4l^2)} \left[f_l\left(\frac{x+y}{2l}\right) - f_l\left(\frac{x}{l}\right)\right] \quad \text{for } x,y \in \mathbb{R}. \tag{4.100}$$

Considering $G_{l,n}$ as an operator mapping $L^2(\mathbb{R})$ to $L^2([ln, l(n+1)])$, by Lemma 2.2.3, its singular values satisfy

$$\mathfrak{s}_k(G_{l,n}) \leqslant C l^\gamma k^{-\frac{1}{2}-\gamma} N_\gamma(g_{l,n}) \tag{4.101}$$

for all $k \in \mathbb{N}$ and all $\gamma > 1/2$ for which the kernel norm (2.58) is finite. The constant C depends only on γ, which in term is determined solely by q. In particular, C is independent of the kernel $g_{l,n}$. It follows from our estimates below that the kernel norm satisfies $N_\gamma(g_{l,n}) < \infty$ for any $\gamma > 0$. We now choose $\gamma \in]1/q - 1/2, 1/q[$, which implies $\gamma > 1/2$ and that $\mathfrak{s}_k(G_{l,n})^q$ is summable in k by (4.101). Therefore,

$$\|G_{l,n}\|_{\mathfrak{S}_q}^q \leqslant C l^\sigma N_\gamma(g_{l,n})^q \tag{4.102}$$

with $\sigma := \gamma q < 1$, and consequently

$$\|1_\Gamma(T_l - T'_l)1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant C l^\sigma \sum_{n\in 2\mathbb{Z}} N_\gamma(g_{l,n})^q. \tag{4.103}$$

The remaining task is to estimate the sum over n on the right. We fix $n \in 2\mathbb{Z}$. In order to avoid fractional Sobolev norms in the definition (2.58) of the kernel norm, we introduce $J := \lceil \gamma \rceil$, the smallest integer at least as big as γ, and estimate according to Remark 2.2.2(ii) and Lemma A.2.2 as

$$\begin{aligned} N_\gamma(g_{l,n})^q &\leqslant \left[\int_{\mathbb{R}} \|g_{l,n}(\cdot,y)\|_{H^J(I)}^2 \, dy\right]^{\frac{q}{2}} = \left[\int_{(l\Gamma)^c} \int_{ln}^{l(n+1)} \sum_{j=0}^J \left|\frac{\partial^j}{\partial x^j} g_l(x,y)\right|^2 dx\, dy\right]^{\frac{q}{2}} \\ &\leqslant C \sum_{j=0}^J \left[\int_{(l\Gamma)^c} \int_{ln}^{l(n+1)} \frac{e^{-(x-y)^2/(4l^2)}}{(1+|x-y|)^2} \left|\frac{1}{2^j} f_l^{(j)}\left(\frac{x+y}{2l}\right) - f_l^{(j)}\left(\frac{x}{l}\right)\right|^2 dx\, dy\right]^{\frac{q}{2}} \\ &= C \sum_{j=0}^J \left[\int_{\Gamma^c} \int_{n}^{n+1} \frac{l^2 e^{-(x-y)^2/4}}{(1+l|x-y|)^2} \left|\frac{1}{2^j} f_l^{(j)}\left(\frac{x+y}{2}\right) - f_l^{(j)}(x)\right|^2 dx\, dy\right]^{\frac{q}{2}}, \end{aligned} \tag{4.104}$$

where we used the inequality

$$(a+b)^p \leqslant a^p + b^p \tag{4.105}$$

for $p \leqslant 1$ and $a, b \geqslant 0$ several times. The constants depend only on J, which in term depends only on q. Next, we write the set Γ^c as the disjoint union $\Gamma^c = M_n \cup \Gamma^c \setminus M_n$, where

$$M_n :=]n-1, n[\cup]n+1, n+2[, \tag{4.106}$$

that is, we separate Γ^c into points close to the interval $[n, n+1]$ and those with positive distance from the latter. Additionally, by the boundedness of $f_l^{(j)}$ we see that

$$\left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} \right) - f_l^{(j)}(x) \right|^2 \leqslant C \left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} \right) - f_l^{(j)}(x) \right|. \tag{4.107}$$

Thus, in view of (4.105), we can estimate (4.104) by a constant depending only on J times

$$\sum_{j=0}^{J} \left[\int_{M_n} \int_n^{n+1} \frac{l^2 e^{-(x-y)^2/4}}{(1+l|x-y|)^2} \left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} \right) - f_l^{(j)}(x) \right| dx\, dy \right]^{\frac{q}{2}} \tag{4.108}$$

$$+ \sum_{j=0}^{J} \left[\int_{\Gamma^c \setminus M_n} \int_n^{n+1} \frac{l^2 e^{-(x-y)^2/4}}{(1+l|x-y|)^2} \left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} \right) - f_l^{(j)}(x) \right| dx\, dy \right]^{\frac{q}{2}}. \tag{4.109}$$

We will treat the terms (4.108) and (4.109) separately.

Contribution of (4.108) *to* (4.103). We will perform the estimate for the left interval $]n-1, n[$ of M_n only, the right interval is treated identically. Performing the substitutions $x \mapsto x+n$, $y \mapsto y+n$ and estimating the Gaussian $e^{-(x-y)^2/4}$ from above by 1, we arrive at the contribution

$$l^\sigma \sum_{n \in 2\mathbb{Z}} \sum_{j=0}^{J} \left[\int_{-1}^{0} \int_0^1 \frac{l^2}{[1+l(x-y)]^2} \left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} + n \right) - f_l^{(j)}(x+n) \right| dx\, dy \right]^{\frac{q}{2}} \tag{4.110}$$

to an upper bound for (4.103), neglecting a constant depending only on J. If $j = 0$ we estimate the difference in (4.110) by the fundamental theorem of calculus as

$$\begin{aligned} \left| f_l \left(\frac{x+y}{2} + n \right) - f_l(x+n) \right| &\leqslant \int_{(x+y)/2}^{x} |f_l^{(1)}(t+n)|\, dt \\ &\leqslant [x - (x+y)/2] \sup_{z \in]-\frac{1}{2}, 1[} |f_l^{(1)}(z+n)| \\ &\leqslant \sup_{z \in B_1(n)} |f_l^{(1)}(z)| \end{aligned} \tag{4.111}$$

uniformly in $x \in]0,1[$ and $y \in]-1,0[$. For $j \in \{1, \ldots, J\}$ we simply note that for every $x \in]0,1[$ and $y \in]-1,0[$ we have

$$\begin{aligned} \left| \frac{1}{2^j} f_l^{(j)} \left(\frac{x+y}{2} + n \right) - f_l^{(j)}(x+n) \right| &\leqslant \left| f_l^{(j)} \left(\frac{x+y}{2} + n \right) \right| + |f_l^{(j)}(x+n)| \\ &\leqslant 2 \sup_{z \in]-1,1[} \left| f^{(j)}(z+n) \right| \\ &\leqslant 2 \sup_{z \in B_1(n)} \left| f^{(j)}(z) \right|. \end{aligned}$$

Altogether, (4.110) is bounded from above by

$$l^\sigma \left[\int_{-1}^{0}\int_{0}^{1} \frac{l^2}{[1+l(x-y)]^2}\,dx\,dy\right]^{\frac{q}{2}}$$
$$\times \sum_{n\in\mathbb{Z}} \left[\left(\sup_{z\in B_1(n)} |f_l^{(1)}(z)|\right)^{\frac{q}{2}} + \sum_{j=1}^{J}\left(2 \sup_{z\in B_1(n)} |f_l^{(j)}(z)|\right)^{\frac{q}{2}}\right] \leqslant Cl^\sigma(\log(1+l))^{\frac{q}{2}}, \quad (4.112)$$

where we have again used (A.4) for the last inequality, and the constant depends only on q and on J, which in turn also depends only on q.

Contribution of (4.109) *to* (4.103). Since $x \in [n, n+1]$ and $y \in \Gamma^c \setminus M_n$ implies $|x-y| \geqslant 2$, we can estimate

$$\frac{l^2}{(1+l|x-y|)^2} \leqslant \frac{1}{|x-y|^2} \leqslant 1. \quad (4.113)$$

Thus, neglecting a constant depending only on q, (4.109) leads to the contribution

$$l^\sigma \sum_{n\in\mathbb{Z}}\sum_{j=0}^{J}\left[\int_n^{n+1}\int_{\mathbb{R}} e^{-y^2}\left|\frac{1}{2^j}f_l^{(j)}(x+y) - f_l^{(j)}(x)\right| dy\,dx\right]^{\frac{q}{2}} \quad (4.114)$$

to an upper bound for (4.103). Again, we start with term corresponding to $j = 0$. As proper indicator functions are easier to handle than the smoothed out versions f_l, we estimate

$$|f_l(x+y) - f_l(x)|$$
$$\leqslant |f_l(x+y) - 1_{[-l,l]}(x+y)| + |1_{[-l,l]}(x+y) - 1_{[-l,l]}(x)| + |1_{[-l,l]}(x) - f_l(x)|. \quad (4.115)$$

In fact, for the middle term on the right-hand side of (4.115), we note that

$$|1_{[-l,l]}(x+y) - 1_{[-l,l]}(x)| \leqslant \begin{cases} 1_{[-l-y,-l]}(x) + 1_{[l-y,l]}(x) & \text{if } y > 0, \\ 1_{[-l,-l-y]}(x) + 1_{[l,l-y]}(x) & \text{if } y < 0 \end{cases} \quad (4.116)$$

with equality if $|y| < 2l$. Let us focus on the case $y > 0$, the other case is treated analogously. Recalling $(a+b)^p \leqslant a^p + b^p$ for all $p \in [0,1]$ and $a, b \geqslant 0$, it suffices to estimate

$$\left[\int_n^{n+1}\int_0^\infty e^{-y^2}[1_{[-l-y,-l]}(x) + 1_{[l-y,l]}(x)]\,dy\,dx\right]^{\frac{q}{2}}$$
$$\leqslant \left[\int_n^{n+1}\int_0^\infty e^{-y^2}1_{[-l-y,-l]}(x)\,dy\,dx\right]^{\frac{q}{2}} + \left[\int_n^{n+1}\int_0^\infty e^{-y^2}1_{[l-y,l]}(x)\,dy\,dx\right]^{\frac{q}{2}}. \quad (4.117)$$

Notice that the first expression is 0 if $n > -l$ and the second if $n > l$. We start by estimating the n-sum over the second expression. The single term of the n-sum corresponding to $n \in \,]l-1, l]$ is trivially bounded independent of l. If $n \leqslant l-1$ and $x \in [n, n+1]$, then $y \geqslant l-(n+1)$ is necessary for the indicator function to be 1. Hence,

$$\sum_{\substack{n\in\mathbb{Z}:\\ n\leqslant l-1}}\left[\int_n^{n+1}\int_0^\infty e^{-y^2}1_{[l-y,l]}(x)\,dy\,dx\right]^{\frac{q}{2}}$$
$$\leqslant \sum_{n\in\mathbb{Z}} e^{-q(l-n-1)^2/4}\left[\int_n^{n+1}\int_0^\infty e^{-y^2/2}\,dy\,dx\right]^{\frac{q}{2}} \leqslant C, \quad (4.118)$$

where $C > 0$ depends only on q. The n-sum over the first expression on the right-hand side of (4.117) is treated almost identically: The single term corresponding to $n \in \,]-l-1, -l]$ is bounded in l. If $n \leqslant -l-1$ the condition $-l-y \leqslant n+1$, i.e. $y \geqslant -l-n-1$, is necessary for the indicator function to be 1. Therefore,

$$\begin{aligned} &\sum_{\substack{n\in\mathbb{Z}:\\ n\leqslant -l-1}} \left[\int_n^{n+1} \int_0^\infty e^{-y^2} 1_{[-l-y,-l]}(x)\, dy\, dx\right]^{\frac{q}{2}} \\ &\leqslant \sum_{n\in\mathbb{Z}} e^{-q(l+n+1)^2/4} \left[\int_n^{n+1} \int_0^\infty e^{-y^2/2}\, dy\, dx\right]^{\frac{q}{2}} \leqslant C, \end{aligned} \tag{4.119}$$

by the same reasoning as before. This finishes the estimation of the contribution of the middle term on the right-hand side of (4.115) to (4.114). The contributions from the other two terms in (4.115) can be estimated directly using Lemma A.1.1(iv). It remains to treat the terms where $j \in \{1, \dots, J\}$ in (4.114). We again start by estimating

$$\left|\frac{1}{2^j} f_l^{(j)}(x+y) - f_l^{(j)}(x)\right| \leqslant \left|f_l^{(j)}(x+y)\right| + |f_l^{(j)}(x)|. \tag{4.120}$$

The contribution from the second term to (4.114) is bounded with the help of (A.4). For that of the first term, we use inequality (A.22) to deduce the upper bound

$$\begin{aligned} &Cl^\sigma \sum_{n\in\mathbb{Z}} \left[\int_n^{n+1} \int_{\mathbb{R}} e^{-y^2} \left(e^{-(x+y+l)^2/2} + e^{-(x+y-l)^2/2}\right) dy\, dx\right]^{\frac{q}{2}} \\ &= Cl^\sigma \sum_{n\in\mathbb{Z}} \left[\int_n^{n+1} \left[e^{-(x+l)^2/3} + e^{-(x-l)^2/3}\right] dx\right]^{\frac{q}{2}} \\ &\leqslant Cl^\sigma \sum_{n\in\mathbb{Z}} \left[e^{-q(n+l)^2/12} + e^{-q(n-l)^2/12}\right] \leqslant Cl^\sigma, \end{aligned} \tag{4.121}$$

where we applied the convolution identity (A.29) from the first to the second line and Lemma A.3.1(i) from the second to the third. The constant C changes from line to line, but depends only on q. We have thus shown

$$\|1_\Gamma (T_l - T_l') 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant Cl^\sigma \left[\log(1+l)\right]^{\frac{q}{2}} = o(l) \tag{4.122}$$

as $l \to \infty$, thereby completing the proof. □

The following lemma is a consequence of [Sob14, Theorem 4.6].

Lemma 4.3.3. *Let $l \geqslant 2$, consider the symbol* (4.125) *and let $\beta := \lceil q^{-1} \rceil + 1$. Then we have*

$$\|1_{[n,n+1]} T_l'' 1_{[n,n+1]^c}\|_{\mathfrak{S}_q}^q \leqslant C \log l \left(\max_{0\leqslant k\leqslant\beta} \sup_{w\in B_{3/2}(n)} |f_l^{(k)}(w)|\right)^q \tag{4.123}$$

with a constant $C > 0$ depending only on q.

Proof of Lemma 4.3.3. Our aim is to utilize the Schatten-von Neumann estimate for pseudo-differential operators provided by [Sob14, Theorem 4.6]. To that end, for any $l > 0$ and a *symbol* $a \in C(\mathbb{R} \times \mathbb{R})$, we introduce the *pseudo-differential operator* $\operatorname{Op}_l(a)$ by

$$(\operatorname{Op}_l(a)\varphi)(x) := \frac{l}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{il(x-y)\xi} a(x,\xi)\varphi(y)\, dy\, d\xi, \tag{4.124}$$

where φ is a Schwartz function and $x \in \mathbb{R}$. A simple calculation shows that for the specific choice of symbol

$$a_l(x,\xi) := f_l(x) 1_{[-1,1]}(\xi), \quad x, \xi \in \mathbb{R}, \tag{4.125}$$

which itself depends on l, we have

$$\operatorname{Op}_l(a_l) = T_l''. \tag{4.126}$$

By means of the unitary translation operator $U_n f := f(\cdot - n)$, $f \in L^2(\mathbb{R})$, $n \in \mathbb{R}$, the operator

$$1_{[n,n+1]} \operatorname{Op}_l(a_l) 1_{[n,n+1]^c}, \tag{4.127}$$

whose norm we want to estimate, is unitarily equivalent to

$$1_{[0,1]} \operatorname{Op}_l(a_l^{(n)}) 1_{[0,1]^c}, \tag{4.128}$$

where $a_l^{(n)}(x,\xi) := a_l(x+n,\xi)$. Let $\psi = \psi(x), \chi = \chi(\xi) \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $[0,1]$, $\chi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-\frac{1}{2}, \frac{3}{2}] = B_1(\frac{1}{2})$, $\operatorname{supp}\chi \subset [-2,2] = B_2(0)$. Then

$$1_{[0,1]} \operatorname{Op}_l(a_l^{(n)}) 1_{[0,1]^c} = 1_{[0,1]} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{[0,1]^c} \tag{4.129}$$

where $P_{\Omega,l} = \operatorname{Op}_l(1_\Omega)$ and

$$b_l^{(n)}(x,\xi) := \psi(x) f_l(x+n) \chi(\xi). \tag{4.130}$$

The symbol $b_l^{(n)}$ by construction satisfies the conditions of [Sob14, Theorem 4.6] with $\ell = 1$ (different ℓ than our scaling parameter l!) and $\rho = 2$. The theorem provides bounds for operators of the form $1_\Lambda \operatorname{Op}_l(a) P_{\Omega,l} 1_{\Lambda^c}$ with basic domains Λ and Ω, which in one dimension are given by $]0,\infty[$ and $]-\infty,0[$. To obtain such basic domains, we first note

$$\begin{aligned} &\|1_{[0,1]} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{[0,1]^c}\|_{\mathfrak{S}_q}^q \\ &\leqslant \|1_{[0,1]} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{]-\infty,0[}\|_{\mathfrak{S}_q}^q + \|1_{[0,1]} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{]1,\infty[}\|_{\mathfrak{S}_q}^q \\ &\leqslant \|1_{[0,\infty)} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{]-\infty,0[}\|_{\mathfrak{S}_q}^q + \|1_{]-\infty,1]} \operatorname{Op}_l(b_l^{(n)}) P_{]-1,1[,l} 1_{]1,\infty[}\|_{\mathfrak{S}_q}^q. \end{aligned} \tag{4.131}$$

We only treat the first term, the second one is handled identically after applying the unitary translation U_1. To also obtain basic domains in the argument of $P_{]-1,1[,l}$, we start by writing $1_{]-1,1[} = 1_{]-\infty,1[} - 1_{]-\infty,-1]}$ and get

$$\begin{aligned} &\|1_{[0,\infty[} \operatorname{Op}_l(b_l^{(n)}) P_{[-1,1],l} 1_{]-\infty,0[}\|_{\mathfrak{S}_q}^q \\ &\leqslant \|1_{[0,\infty)} \operatorname{Op}_l(b_l^{(n)}) P_{]-\infty,1[,l} 1_{]-\infty,0[}\|_{\mathfrak{S}_q}^q + \|1_{[0,\infty[} \operatorname{Op}_l(b_l^{(n)}) P_{]-\infty,-1],l} 1_{]-\infty,0[}\|_{\mathfrak{S}_q}^q. \end{aligned} \tag{4.132}$$

We again only consider the left term, the other one can be handled the same way. For $\eta \in \mathbb{R}$, we introduce another unitary operator $\tilde{U}_\eta$ by

$$\tilde{U}_\eta \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (\tilde{U}_\eta f)(x) := e^{i\eta x} f(x). \tag{4.133}$$

Conjugating with this operator and $\eta = l$, we get

$$\tilde{U}_l^* \operatorname{Op}_l(b_l^{(n)}) P_{]-\infty,1[,l} \tilde{U}_l = \operatorname{Op}_l(c_l^{(n)}) P_{]-\infty,0[,l} \tag{4.134}$$

with symbol

$$c_l^{(n)}(x,\xi) := b_l^{(n)}(x,\xi+1) = \psi(x) f_l(x+n)\chi(\xi+1). \tag{4.135}$$

Note that $\operatorname{supp}\chi(\cdot+1) \subset [-3,3] = B_3(0)$. Therefore, if we put $\gamma := \lceil 2q^{-1} \rceil + 1$, applying [Sob14, Theorem 4.6] yields

$$\begin{aligned}
&\|1_{[0,\infty)} \operatorname{Op}_l(c_l^{(n)}) P_{]-\infty,0[,l} 1_{]-\infty,0]^c}\|_{\mathfrak{S}_q}^q \\
&\leqslant C \log l \left(\max_{0\leqslant k\leqslant\beta} \sup_{x\in B_1(\frac{1}{2})} \left| \frac{d^k}{dx^k}(\psi(x) f_l(x+n)) \right| \right)^q \left(\max_{0\leqslant r\leqslant\gamma} \sup_{\xi\in B_3(0)} \left| \chi^{(r)}(\xi+1) \right| \right)^q \\
&\leqslant C \log l \left(\max_{0\leqslant k\leqslant\beta} \sup_{x\in B_1(\frac{1}{2})} \left| f_l^{(k)}(x+n) \right| \right)^q \leqslant C \log l \left(\max_{0\leqslant k\leqslant\beta} \sup_{x\in B_{3/2}(n)} \left| f_l^{(k)}(x) \right| \right)^q,
\end{aligned} \tag{4.136}$$

which is the desired inequality. □

Proof of Theorem 4.3.1. We divide our proof into three steps.

Step (i): Replacing T_l by T_l'.

By the q-triangle inequality and Lemma 4.3.2,

$$\|1_\Gamma T_l 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant \|1_\Gamma T_l' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q + \|1_\Gamma (T_l - T_l') 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q = \|1_\Gamma T_l' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q + o(l) \tag{4.137}$$

as $l \to \infty$, so it suffices to prove (4.93) for T_l' instead of T_l.

Step (ii): Replacing T_l' by T_l''.

Recall from Remark 4.2.10(ii) that

$$e^{-x^2/4} \frac{\sin(lx)}{\pi x} = \frac{1}{\sqrt{2\pi}} \mathscr{F} f_l(x), \tag{4.138}$$

so the integral kernel (4.94) of the operator T_l' can be written as

$$f_l(x) e^{-(x-y)^2/4} \frac{\sin[l(x-y)]}{\pi(x-y)} = \frac{1}{\sqrt{2\pi}} f_l(x) \mathscr{F} f_l(x-y). \tag{4.139}$$

while we can write the kernel (4.95) of T_l'' as

$$f_l(x) \frac{\sin[l(x-y)]}{\pi(x-y)} = \frac{1}{\sqrt{2\pi}} f_l(x) \mathscr{F} 1_{[-l,l]}(x-y), \tag{4.140}$$

so that

$$T_l' - T_l'' = f_l \mathscr{F}(f_l - 1_{[-l,l]}) \mathscr{F}^*. \tag{4.141}$$

We can therefore apply Lemma 2.2.4 with $f := f_l$ and $g := f_l - 1_{[-l,l]}$ to obtain

$$\|\operatorname{Op}_{l,0}(p_l) - \operatorname{Op}_l(a_l)\|_{\mathfrak{S}_q}^q \leqslant C \|f_l\|_{2,q}^q \|f_l - 1_{[-l,l]}\|_{2,q}^q, \tag{4.142}$$

where the lattice norms on the right-hand side defined in (2.60). We will estimate them by employing Lemma A.1.1: Firstly, since $|f_l(x)|^2 \leqslant f_l(x)$, inequality (A.3) yields

$$\|f_l\|_{2,q}^q = \sum_{n\in\mathbb{Z}} \left(\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |f_l(x)|^2 \, dx \right)^{\frac{q}{2}} \leqslant \sum_{n\in\mathbb{Z}} \left(\sup_{x\in B_{\frac{1}{2}}(n)} f_l(x) \right)^{\frac{q}{2}} \leqslant Cl. \tag{4.143}$$

Furthermore, since $1_{[-l,l]}$ and $f_l \leqslant 1$, it follows that $|1_{[-l,l]} - f_l|^2 \leqslant 4|1_{[-l,l]} - f_l|$. Consequently, by (A.5),

$$\|f_l - 1_{[-l,l]}\|_{2,q}^q \leqslant C \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f_l(x) - 1_{[-l,l]}(x)| \, dx \right)^{\frac{q}{2}} \leqslant C, \tag{4.144}$$

where we have chosen differently centered intervals for the lattice quasi-norm, cf. the remark after the definition 2.60. We conclude that

$$\|1_\Gamma T_l' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant \|1_\Gamma T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q + \|T_l' - T_l''\|_{\mathfrak{S}_q}^q \leqslant \|1_\Gamma T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q + Cl. \tag{4.145}$$

Step (iii): Estimating $\|1_\Gamma T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q$.

We have the elementary estimates

$$\begin{aligned} \|1_\Gamma T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q &\leqslant \sum_{n\in 2\mathbb{Z}} \|1_{[n,n+1]} T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \\ &\leqslant \sum_{n\in\mathbb{Z}} \|1_{[n,n+1]} T_l'' 1_{[n,n+1]^c}\|_{\mathfrak{S}_q}^q. \end{aligned} \tag{4.146}$$

The claim now follows from Lemma 4.3.3: We combine (4.146) and (4.123) with (A.3) to obtain

$$\|1_\Gamma T_l'' 1_{\Gamma^c}\|_{\mathfrak{S}_q}^q \leqslant C \log l \sum_{k=0}^{\beta} \sum_{n\in\mathbb{Z}} \left(\sup_{w\in B_{3/2}(n)} |f_l^{(k)}(w)| \right)^q \leqslant Cl \log l, \tag{4.147}$$

as desired. □

4.4. Closing the asymptotics: Proof of Theorem 3.1.3

Recall the definitions (3.1) and (3.2) of the test function space $\mathbb{H}$ and the functional $h \mapsto I(h)$, respectively. Let $\mathbb{H}_0$ denote the space of all $h \in \mathbb{H}$ such that $h(1) = 0$.

The goal of this section is to finish the proof of Lemma 4.1.1, that is,

$$\operatorname{tr} h(1_\Gamma T_l 1_\Gamma) = I(h) l \log l + o(l \log l) \tag{4.148}$$

as $l \to \infty$ for every $h \in \mathbb{H}_0$, and thereby ultimately proving Theorem 3.1.3.

We follow the standard approximation method as for example presented in [PS18a, Section 8.2], [LSS14] and [MS23, Proof of Theorem 2.1]. It consists of four steps for test functions of increasing generality.

Step (i). $h \in \mathbb{H}_0$ is a polynomial.
We define *even* and *odd* polynomials $\mathscr{s}_n$ and $\mathscr{a}_n$, $n \in \mathbb{N}$, by

$$\mathscr{s}_n(\lambda) := [\lambda(1-\lambda)]^n, \quad \text{and} \quad \mathscr{a}_n(\lambda) := \lambda \mathscr{s}_n(\lambda), \quad \lambda \in [0,1], \tag{4.149}$$

and note that the set $\{\mathscr{s}_n, \mathscr{a}_n : n \in \mathbb{N}\}$ constitutes a basis of the space of polynomials in $\mathbb{H}_0$ because the linear spans

$$\operatorname{span}\{\mathscr{s}_n, \mathscr{a}_n : n \in \mathbb{N}\} = \operatorname{span}\{\mathscr{s}_1 \operatorname{id}^k : k \in \mathbb{N}_0\} \tag{4.150}$$

coincide. We have shown in Section 4.2 that $\mathscr{a}_n$ and $\mathscr{s}_n$ satisfy (4.148), so Step (i) follows from the linearity of the trace.

Step (ii). $h \in \mathbb{H}_0$ is continuous and differentiable at $t = 0$ and $t = 1$.
Without loss of generality, we may assume that h is real-valued (otherwise we treat real and imaginary part separately). The differentiability conidition at $t = 0$ and $t = 1$ implies that $h(\lambda) = \lambda(1-\lambda)g(\lambda)$ for a continuous real-valued function g. Let $\epsilon > 0$. By the Stone-Weierstraß theorem, we may choose a polynomial $\zeta \colon [0,1] \to \mathbb{R}$ such that

$$\sup_{\lambda \in [0,1]} |g(\lambda) - \zeta(\lambda)| \leqslant \epsilon. \tag{4.151}$$

Denoting $\tilde{\zeta}(t) := \lambda(1-\lambda)\zeta(\lambda)$, we estimate

$$h(t) \leqslant \lambda(1-\lambda)(\zeta(\lambda) + \epsilon) = \tilde{\zeta}(\lambda) + \epsilon\lambda(1-\lambda) \tag{4.152}$$

and

$$h(\lambda) \geqslant \lambda(1-\lambda)(\zeta(\lambda) - \epsilon) = \tilde{\zeta}(\lambda) - \epsilon\lambda(1-\lambda). \tag{4.153}$$

The monotonicity of the trace in combination with (4.152) gives

$$\operatorname{tr} h(1_\Gamma T_l 1_\Gamma) \leqslant \operatorname{tr} \tilde{\zeta}(1_\Gamma T_l 1_\Gamma) + \epsilon \operatorname{tr} \mathscr{s}_1(1_\Gamma T_l 1_\Gamma). \tag{4.154}$$

Since $\tilde{\zeta}$ is a polynomial vanishing at 0 and 1, we get by the previous step

$$\limsup_{l\to\infty} \frac{\operatorname{tr} h(1_\Gamma T_l 1_\Gamma)}{l \log l} \leqslant I(\tilde{\zeta}) + \epsilon I(\mathscr{s}_1) = I(\tilde{\zeta}) + \frac{\epsilon}{\pi^2}. \tag{4.155}$$

Since

$$|I(h) - I(\tilde{\zeta})| = |I(h - \tilde{\zeta})| \leqslant \frac{\epsilon}{\pi^2}, \tag{4.156}$$

we thus obtain

$$\limsup_{l\to\infty} \frac{\operatorname{tr} h(1_\Gamma T_l 1_\Gamma)}{l \log l} \leqslant I(h) + \frac{2\epsilon}{\pi^2}. \tag{4.157}$$

In the same way, (4.153) implies

$$\liminf_{l\to\infty} \frac{\operatorname{tr} h(1_\Gamma T_l 1_\Gamma)}{l \log l} \geqslant I(h) - \frac{2\epsilon}{\pi^2}, \tag{4.158}$$

and since $\epsilon > 0$ was arbitrary, we deduce the validity of (4.148) for our choice of h in this step.

Step (iii). $h \in \mathbb{H}_0$ is continuous and Hölder continuous at $t = 0$ and $t = 1$.
Let $q \in (0,1]$ be the corresponding Hölder-exponent, so that

$$|h(\lambda)| \leqslant C\lambda^q(1-\lambda)^q \tag{4.159}$$

for all $\lambda \in [0,1]$. Let again $\epsilon > 0$ and choose a smooth function ζ_ϵ such that $0 \leqslant \zeta_\epsilon \leqslant 1$ and

$$\zeta_\epsilon(\lambda) := \begin{cases} 1, & \lambda \in [0, \epsilon/2] \cup [1-\epsilon/2, 1], \\ 0, & \lambda \in [\epsilon, 1-\epsilon]. \end{cases} \tag{4.160}$$

Putting $r := q/2$, we observe that

$$|(\zeta_\epsilon h)(\lambda)| \leqslant C[\lambda(1-\lambda)]^q \zeta_\epsilon(\lambda) \leqslant C\epsilon^r [\lambda(1-\lambda)]^r, \tag{4.161}$$

whence

$$\|(\zeta_\epsilon h)(1_\Gamma T_l 1_\Gamma)\|_1 \leqslant C\epsilon^r \|\jmath_1(1_\Gamma T_l 1_\Gamma)\|_r^r = C\epsilon^r \|1_\Gamma T_l 1_{\Gamma^c}\|_q^q. \tag{4.162}$$

Theorem 4.3.1 implies $\|1_\Gamma T_l 1_{\Gamma^c}\|_q^q \leqslant Cl \log l$ with $C = C(q) > 0$ independent of l, so we infer

$$\frac{|\mathrm{tr}[(\zeta_\epsilon h)(1_\Gamma T_l 1_\Gamma)]|}{l \log l} \leqslant C\epsilon^r \tag{4.163}$$

for all $l \geqslant 1$. On the other hand, the function $g_\epsilon := (1-\zeta_\epsilon)h$ vanishes in a neighborhood of 0 and 1 and is consequently differentiable at $t = 0$ and $t = 1$. We may thus apply Step (ii) to g_ϵ and obtain

$$\mathrm{tr}[g_\epsilon(1_\Gamma T_l 1_\Gamma)] = l \log l \, I(g_\epsilon) + o(l \log l), \qquad \text{as } l \to \infty. \tag{4.164}$$

Finally,

$$|I(g_\epsilon) - I(h)| \leqslant C \int_0^\epsilon t^{q-1}(1-t)^{q-1}\, dt + C \int_{1-\epsilon}^1 t^{q-1}(1-t)^{q-1}\, dt \leqslant C\epsilon^q. \tag{4.165}$$

Combining (4.163), (4.164) and (4.165) gives

$$\limsup_{l\to\infty} \left| \frac{\mathrm{tr}[h(1_\Gamma T_l 1_\Gamma)]}{l \log l} - I(h) \right| \leqslant C\epsilon^r + C\epsilon^q \leqslant C\epsilon^r, \tag{4.166}$$

and letting $\epsilon \to 0$ finishes Step (iii).

Step (iv). $h \in \mathbb{H}_0$.
Again assume without loss of generality that h is real-valued. Since h is piecewise continuous and (Hölder) continuous at 0 and 1, we find $\delta \in (0, 1/2)$ such that h is continuous on $[0, 2\delta] \cup [1-2\delta, 1]$. Then for given $\epsilon > 0$, there exist continuous functions $h_1, h_2 \in \mathbb{H}_0$ such that

(1) $h = h_1 = h_2$ on $[0, 2\delta] \cup [1-2\delta, 1]$,

(2) $h_1 \leqslant h \leqslant h_2$,

(3) $\|h_1 - h_2\|_{L^1} < \epsilon$.

These properties imply that

$$|I(h_1) - I(h)|, |I(h_2) - I(h)| \leqslant C\epsilon. \tag{4.167}$$

Furthermore, the monotonicity (2) implies

$$\mathrm{tr}\, h_1(1_\Gamma T_l 1_\Gamma) \leqslant \mathrm{tr}\, h(1_\Gamma T_l 1_\Gamma) \leqslant \mathrm{tr}\, h_2(1_\Gamma T_l 1_\Gamma). \tag{4.168}$$

Finally, by an application of Step (iii) to h_1 and h_2, respectively, we infer

$$\limsup_{l\to\infty} \left| \frac{\mathrm{tr}\, h(1_\Gamma T_l 1_\Gamma)}{l \log l} - 2I(h) \right| \leqslant C\epsilon. \tag{4.169}$$

As $\epsilon > 0$ was arbitrary, this finishes the proof of (3.9) for all $h \in \mathbb{H}_0$ and consequently, as discussed before, for all $h \in \mathbb{H}$.

4.5. Proving the general version

We are left with proving that Theorem 3.1.3 implies Theorem 3.1.1.

Proof of Theorem 3.1.1. For $r > 0$, consider the unitary transformation

$$V_r\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad V_r f(x) := r^{1/2} f(rx). \tag{4.170}$$

It is straightforward to see that

$$V_{\sqrt{B}} 1_{[-L,L]^2} P_{B,\kappa\Gamma} 1_{[-L,L]^2} V^*_{\sqrt{B}} = 1_{[-\sqrt{B}L,\sqrt{B}L]^2} P_{1,\frac{\kappa}{\sqrt{B}}\Gamma} 1_{[-\sqrt{B}L,\sqrt{B}L]^2}. \tag{4.171}$$

Therefore, by considering the dimensionless parameter $l := \sqrt{B}L \to \infty$, (3.4) is equivalent to

$$\operatorname{tr} h(1_{[-l,l]^2} P_{\frac{\kappa}{\sqrt{B}}\Gamma} 1_{[-l,l]^2}) = \frac{1}{\pi} h(1) l^2 + I(h) \frac{\sqrt{B}}{\kappa} l \log l + o_{\frac{\kappa}{\sqrt{B}}}(l \log l) \tag{4.172}$$

For now, we abbreviate $\beta = \beta(\kappa, B) := \frac{\kappa}{\sqrt{B}}$. Define $\tilde{h}\colon [0,1] \to \mathbb{R}$ by

$$\tilde{h}(\lambda) := h(\lambda) - \lambda h(1). \tag{4.173}$$

Then $\tilde{h} \in \mathbb{H}$, $\tilde{h}(1) = 0$ and

$$\operatorname{tr} \tilde{h}(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}) = \operatorname{tr} h(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}) - h(1) \operatorname{tr}(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}). \tag{4.174}$$

To compute the trace of $1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}$, we note that

$$\langle \varphi, 1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2} \varphi \rangle_2 = \langle P_{\beta\Gamma} 1_{[-l,l]^2} \varphi, P_{\beta\Gamma} 1_{[-l,l]^2} \varphi \rangle_2 = \| P_{\beta\Gamma} 1_{[-l,l]^2} \varphi \|^2 \geqslant 0 \tag{4.175}$$

for any $\varphi \in L^2(\mathbb{R}^2)$ and that we may therefore apply the criterion from Proposition 2.2.1. To that end, let $(u_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^2)$. By definition of $P_{\beta\Gamma}$, we calculate

$$\begin{aligned}
\operatorname{tr}(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}) &= \sum_{n\in\mathbb{N}} \langle u_n, 1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2} u_n \rangle \\
&= \sum_{n\in\mathbb{N}} \int_{[-l,l]^2} \overline{u_n(x)} \int_{[-l,l]^2} u_n(y) \int_{\beta\Gamma} \overline{\psi_k(x)} \psi_k(y) \, dk \, dy \, dx \\
&= \int_{\beta\Gamma} \sum_{n\in\mathbb{N}} |\langle 1_{[-l,l]^2} \psi_k, u_n \rangle|^2 \, dk = \int_{\beta\Gamma} \| 1_{[-l,l]^2} \psi_k \|_2^2 \, dk \\
&= \int_{[-l,l]^2} \int_{\beta\Gamma} |\psi_{1,k}(x)|^2 \, dk \, dx = \frac{l}{\pi} \int_{\beta\Gamma} f_l(k) \, dk.
\end{aligned} \tag{4.176}$$

Here, we used Fubini's theorem for the third and the fifth equality, Parseval's identity for the fourth equality and the function f_l defined in (4.5) below. Note that $\int_{\mathbb{R}} f_l(k)\, dk = 2l$ and $f_l(-k) = f_l(k)$ for all $k \in \mathbb{R}$. Furthermore, by definition of Γ, we have the identity $\beta\Gamma \cup (-\beta\Gamma) = \mathbb{R}$. Since $\beta\Gamma \cap (-\beta\Gamma)$ consists only of the endpoints of the corresponding intervals, it has measure zero. Consequently,

$$\frac{l}{\pi} \int_{\beta\Gamma} f_l(k)\, dk = \frac{l}{2\pi} \left[\int_{\beta\Gamma} f_l(k)\, dk + \int_{-\beta\Gamma} f_l(k)\, dk \right] = \frac{l}{2\pi} \int_{\mathbb{R}} f_l(k)\, dk = \frac{1}{\pi} l^2. \tag{4.177}$$

Combining (4.174), (4.176) and (4.177), we get

$$\operatorname{tr} h(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}) = \frac{1}{\pi} h(1) l^2 + \operatorname{tr} \tilde{h}(1_{[-l,l]^2} P_{\beta\Gamma} 1_{[-l,l]^2}). \tag{4.178}$$

It thus suffices to prove

$$\operatorname{tr} h(1_{[-l,l]^2} P_{\frac{\kappa}{\sqrt{B}}\Gamma} 1_{[-l,l]^2}) = I(h) \frac{\sqrt{B}}{\kappa} l \log l + o_{\frac{\kappa}{\sqrt{B}}}(l \log l) \tag{4.179}$$

as $l \to \infty$ for all $h \in \mathbb{H}$ with $h(1) = 0$.

Without loss of generality, we may assume $\beta = 1$, or equivalently $\kappa = \sqrt{B}$. If this is not the case, we replace all occurrences of Γ in the sections below by $\beta\Gamma$. This results in all error terms with a dependence on Γ to also depend on β, which is incorporated in the $o_{\frac{\kappa}{\sqrt{B}}}(l \log l)$ term in (4.179).

The leading order coefficient in (4.179) arises (for the respective polynomials) by putting together (4.49), resp. (4.50) and (4.71), resp. (4.72). Introduce the operators

$$B_{l,\beta} := \sum_{n \in 2\mathbb{Z}} 1_{[\beta n, \beta(n+1)]} T_l 1_{[\beta n, \beta(n+1)]}, \tag{4.180}$$

$$X_{l,\beta} := 1_{[0,\beta]} S_l 1_{\mathbb{R}\setminus[0,\beta]} S_l 1_{[0,\beta]}, \tag{4.181}$$

analogously to the operators B_l and X_l that were defined in (4.12) and (4.48), respectively. The leading order term in (4.49) comes from an application of Lemma A.1.1(i) with $\alpha = 1$ to (4.52), resulting in the term (4.54). Similarly for (4.50), where an application of Lemma A.1.1(i) to (4.64) results in (4.65). To obtain the proper coefficients when Γ is replaced by $\beta\Gamma$, we simply apply Lemma A.1.1(i) with $\alpha = \beta$ instead, from where the almost identical proof leads to

$$\operatorname{tr} \mathscr{s}_J(B_{l,\beta}) = \frac{l}{\beta} \cdot \operatorname{tr} X_{l,\beta}^J + o_\beta(l), \tag{4.182}$$

and

$$\operatorname{tr} \mathscr{a}_J(B_{l,\beta}) = \frac{l}{\beta} \cdot \operatorname{tr} \left\{ 1_{[0,\beta]} S_l 1_{[0,\beta]} X_{l,\beta}^J \right\} + o_\beta(l) \tag{4.183}$$

for any $J \in \mathbb{N}$ as $l \to \infty$. Finally, the leading order terms in (4.71) and (4.72) stem from Lemma 4.2.11 and the one-dimensional Widom formula (4.76). Since

$$\log \beta l = \log l + \log \beta = \log l + \mathcal{O}_\beta(1) \tag{4.184}$$

as $l \to \infty$, the same procedure yields

$$\operatorname{tr} X_{l,\beta}^J = \operatorname{tr} \mathscr{s}_J(C_{l,\beta}) + \mathcal{O}_\beta(1) = I(\mathscr{s}_J) \log l + o_\beta(\log l) \tag{4.185}$$

and

$$\operatorname{tr} \left\{ 1_{[0,\beta]} S_l 1_{[0,\beta]} X_{l,\beta}^J \right\} = \operatorname{tr} \mathscr{a}_J(C_{l,\beta}) + \mathcal{O}_\beta(l) = I(\mathscr{a}_J) \log l + o_\beta(\log l) \tag{4.186}$$

for any $J \in \mathbb{N}$ as $l \to \infty$. Combining (4.182) with (4.185) and (4.183) with (4.186) yields (4.179) for $\mathscr{s}_J$ and $\mathscr{a}_J$ for any $J \in \mathbb{N}$. This generalizes to all admissible test functions just as in the case $\beta = 1$. $\square$

5. Proof of the second main result

The proof of the second main result, Theorem 3.2.2, is divided into two sections, Section 5.1 and 5.2, each dedicated to establishing the appropriate lower and upper bounds for $\operatorname{tr}|1_{\mathbb{R}^2\setminus[-l,l]^2}P_{\Gamma_\alpha}1_{[-l,l]^2}|^2$.

Recall the setting of Definition 3.2.1. We additionally set

$$J_n = J_n(\alpha) := \begin{cases} [2a_n + \xi_{n+1}, 2a_{n+1}], & \text{for } n \in \mathbb{N}_0, \\ [-2a_{-n} + \xi_{-n}, -2a_{-n-1}] & \text{for } n \in -\mathbb{N}. \end{cases} \tag{5.1}$$

Then $(J_n)_{n\in\mathbb{Z}}$ is the collection of intervals "left out" in Γ_α. Note that $I_{-n} = -J_{n-1}$ and $J_{-n} = -I_{n-1}$ for $n \in \mathbb{N}$.

If $\alpha = 0$, we have $\xi_n = 1$ for all $n \in \mathbb{N}$ and consequently $a_n = n$ for all $n \in \mathbb{Z}$. This means we are in the situation of Theorem 3.1.3 which has already been proven, see also Remark 3.2.3(i). For this reason, we assume $\alpha \in]0,1]$ from now on.

In order to prove Theorem 3.2.2, we rewrite the trace as in (4.1), while observing (3.11) for h from (3.10). This gives

$$\operatorname{tr}|1_{\mathbb{R}^2\setminus[-l,l]^2}P_{\Gamma_\alpha}1_{[-l,l]^2}|^2 = \operatorname{tr}|1_{\Gamma_\alpha}T_l 1_{\Gamma_\alpha^c}|^2 = \|1_{\Gamma_\alpha}T_l 1_{\Gamma_\alpha^c}\|_{\mathfrak{S}_2}^2, \tag{5.2}$$

where T_l is the operator (4.3) with integral kernel (4.4).

5.1. Lower bounds

The following lemma proves the lower bound in Theorem 3.2.2 for $\alpha \in]0,1]$.

Lemma 5.1.1. *Let $\Gamma_\alpha \subset \mathbb{R}$ be as in Definition 3.2.1 with $\alpha \in]0,1]$ being the corresponding decay exponent. Then there exist constants $l_0 = l_0(\alpha) \geqslant 2$ and $C = C(\alpha) > 0$ such that*

$$\|1_{\Gamma_\alpha}T_l 1_{\Gamma_\alpha^c}\|_{\mathfrak{S}_2}^2 \geqslant \begin{cases} Cl^{1/(1-\alpha)}\log l, & \text{if } 0 < \alpha < \frac{1}{2}, \\ Cl^2, & \text{if } \frac{1}{2} \leqslant \alpha \leqslant 1 \end{cases} \tag{5.3}$$

for all $l \geqslant l_0$.

Remark 5.1.2. Let us briefly anticipate the core strategy of the proof: Writing

$$\|1_{\Gamma_\alpha}T_l 1_{\Gamma_\alpha^c}\|_2^2 = \sum_{(n,m)\in\mathbb{Z}^2} \mathscr{X}_{n,m}(l) \tag{5.4}$$

with

$$\mathscr{X}_{n,m}(l) := \|1_{I_n} T_l 1_{J_m}\|_2^2 = \int_{I_n} \int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx, \tag{5.5}$$

it will turn out that the leading-order contributions to the asymptotics of the above double sum for large l come from different $(n,m) \in \mathbb{Z}^2$, depending on α:
For $\alpha \in [0, \frac{1}{2}]$, these are the terms where the intervals I_n and J_m touch, that is, where $n = m$ or $n = m-1$. Each of such pairs of intervals yields a $\log l$ contribution, and there are roughly $l^{1/(1-\alpha)}$ many of them inside $[-l, l]^2$.
For $\alpha \in]\frac{1}{2}, 1]$ however, these pairs of intervals will no longer yield individual contributions to the sum. Instead, more terms "near the diagonal" (where m and n are close) will coalesce and give rise to a total contribution of order l^2. The integration rectangles in the first quadrant of the plane are illustrated in Figure 5.1.

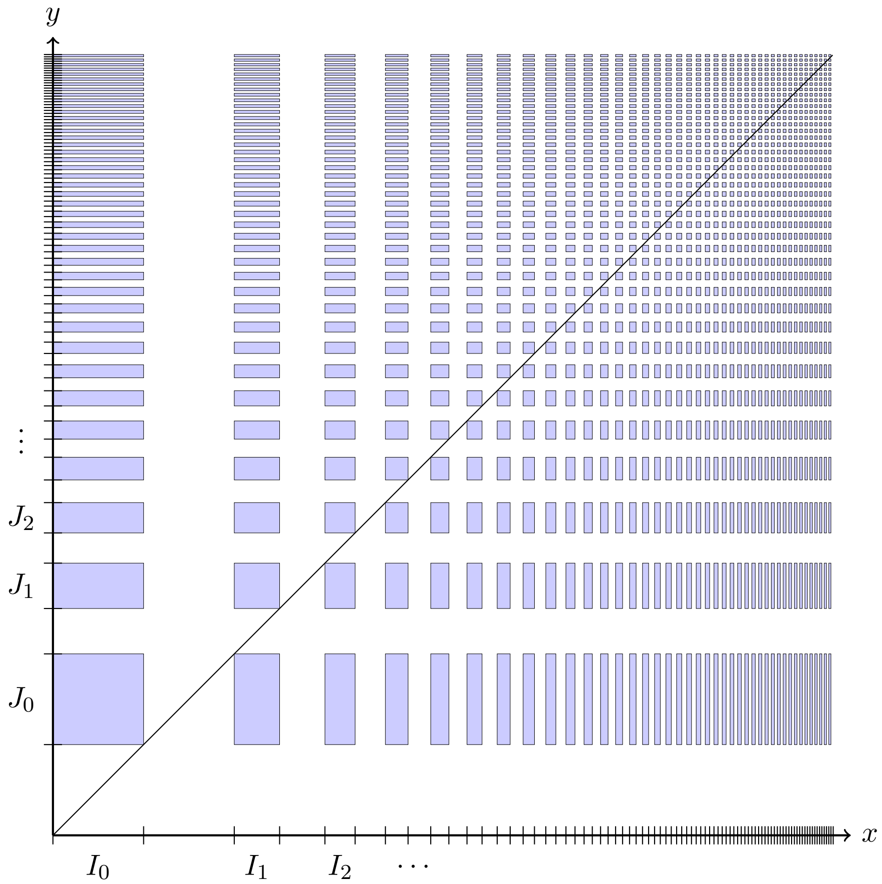

Figure 5.1.: The areas of integration $I_n \times J_m$ in (5.5) where $(n,m) \in \mathbb{N}_0^2$. Here we chose $\alpha = 1$.

Proof of Lemma 5.1.1. Case $\alpha \in]0, \frac{1}{2}]$. According to Remark 5.1.2, we estimate the sum (5.4) from below by keeping just the terms where $m = n \in \mathbb{N}$. We fix $n \in \mathbb{N}$ and perform the substitutions

$$x \mapsto x + 2a_n + \xi_{n+1}, \qquad y \mapsto y + 2a_n + \xi_{n+1} \tag{5.6}$$

to arrive at

$$\mathscr{X}_{n,n}(l) = \int_0^{\xi_{n+1}} \int_{-\xi_{n+1}}^0 \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2} + 2a_n + \xi_{n+1}\right) dy\, dx. \tag{5.7}$$

To simplify the above expression, we start by getting rid of some unnecessary terms. First, note that there exists a constant $C > 0$ such that $f_l \geqslant C1_{[-l,l]}$ for each $l \geqslant 2$. Furthermore, since $(x+y)/2 + \xi_{n+1} \in [-1/2, 3/2]$ independently of $n \in \mathbb{N}$, we have

$$1_{[-l,l]}\left(\frac{x+y}{2} + 2a_n + \xi_{n+1}\right) \geqslant 1_{[-l/2,l/2]}(2a_n) = 1_{[-l,l]}(4a_n) \tag{5.8}$$

for every $l \geqslant 3$. Secondly, the values of $e^{-(x-y)^2/2}$ are bounded away from 0 uniformly in n for $|x|, |y| \in [0, \xi_{n+1}]$. This yields, up to a constant, the expression

$$1_{[-l,l]}(4a_n) \int_0^1 \int_{-1}^0 \frac{\sin^2[(l\xi_{n+1})(x-y)]}{\pi^2(x-y)^2}\, dy\, dx \tag{5.9}$$

as a lower bound for (5.7). For $\lambda > 0$, put

$$g(\lambda) := \int_0^1 \int_{-1}^0 \frac{\sin^2[\lambda(x-y)]}{\pi^2(x-y)^2}\, dy\, dx. \tag{5.10}$$

Our aim is therefore to establish lower bounds for the expression

$$\sum_{n\in\mathbb{N}} 1_{[-l,l]}(4a_n) g(l\xi_{n+1}). \tag{5.11}$$

By comparison to an integral, we have

$$a_{n-1} \leqslant a_n \leqslant \frac{1}{1-\alpha} n^{1-\alpha} \leqslant 2n^{1-\alpha} \tag{5.12}$$

for all $n \geqslant 1$, and thus $1_{[-l,l]}(4a_{n-1}) \geqslant 1_{[-l,l]}(8n^{1-\alpha})$. Since $n \leqslant l^{1/(1-\alpha)}/64$ implies $8n^{1-\alpha} \leqslant l$, we get

$$\sum_{n\in\mathbb{N}} 1_{[-l,l]}(4a_n) g(l/(n+1)^\alpha) = \sum_{\substack{n\in\mathbb{N}:\\ n\geqslant 2}} 1_{[-l,l]}(4a_{n-1}) g(l/n^\alpha) \geqslant \sum_{\substack{n\in\mathbb{N}:\\ 2\leqslant n\leqslant l^{1/(1-\alpha)}/64}} g(l/n^\alpha). \tag{5.13}$$

In order to avoid empty sums, we require l to be so large such that $\lfloor l^{1/(1-\alpha)}/64 \rfloor \geqslant 2$. Note that for $n \leqslant l^{1/(1-\alpha)}/64$ we then have

$$\frac{l}{n^\alpha} \geqslant 64^\alpha l^{1-\frac{\alpha}{1-\alpha}} \geqslant 8. \tag{5.14}$$

For such arguments, the function g can be bounded from below by the natural logarithm, see Lemma A.3.2, which implies that

$$\begin{aligned}
\sum_{\substack{n\in\mathbb{N}:\\ 2\leqslant n\leqslant l^{1/(1-\alpha)}/64}} g(l/n^\alpha) &\geqslant \frac{1}{4} \sum_{n=2}^{\lfloor l^{1/(1-\alpha)}/64 \rfloor} \log(l/n^\alpha) \geqslant \frac{1}{4}(\lfloor l^{1/(1-\alpha)}/64 \rfloor - 1) \log\left(\frac{64^\alpha l}{l^{\alpha/(1-\alpha)}}\right) \\
&\geqslant \frac{1}{2^{10}} l^{1/(1-\alpha)} \begin{cases} \log 8, & \text{if } \alpha = \frac{1}{2}, \\ \frac{1-2\alpha}{1-\alpha} \log l, & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}
\end{aligned} \tag{5.15}$$

Case $\alpha \in]\frac{1}{2}, 1]$. In this case, we estimate the double sum in (5.4) from below by

$$\sum_{n\in\mathbb{N}} \sum_{\substack{m\in\mathbb{N}:\\ n<m<n+n^\alpha}} \mathscr{X}_{n,m}(l) \geqslant \sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} \sum_{\substack{n\in\mathbb{N}:\\ \nu<n^\alpha}} \mathscr{X}_{n,n+\nu}(l) \geqslant \sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} \sum_{\substack{\nu\in\mathbb{N}:\\ \nu<n^\alpha/(4l)}} \mathscr{X}_{n,n+\nu}(l). \tag{5.16}$$

For n, ν in the given ranges and $x \in I_n$, $y \in J_{n+\nu}$, we have

$$y - x \leqslant 2a_{n+\nu+1} - 2a_n = \sum_{k=n+1}^{n+\nu+1} \frac{1}{k^\alpha} \leqslant \frac{\nu+1}{(n+1)^\alpha} < \frac{1}{2l}\frac{n^\alpha}{(n+1)^\alpha} \leqslant \frac{1}{2l}. \tag{5.17}$$

In particular, $e^{-(x-y)^2/2} \geqslant C > 0$ for a constant $C > 0$ independent of n, ν and $l \geqslant 1$. Next, we notice that $n + \nu < n + n^\alpha \leqslant 2n$, which implies

$$x + y \leqslant 2a_{n+1} + 2a_{m+1} \leqslant 4a_{2n+1} \tag{5.18}$$

and thus

$$f_l^2\left(\frac{x+y}{2}\right) \geqslant C 1_{[-l,l]}\left(\frac{x+y}{2}\right) \geqslant 1_{[-l,l]}(2a_{2n+1}). \tag{5.19}$$

Combining the previous inequalities, (5.16) is, up to a constant, bounded from below by

$$\sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} 1_{[-l,l]}(2a_{2n+1}) \sum_{\substack{\nu\in\mathbb{N}:\\ \nu<n^\alpha/(4l)}} \int_{I_n}\int_{J_{n+\nu}} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\, dy\, dx. \tag{5.20}$$

To estimate the remaining double integral, note that (5.17) implies $l(x-y) \leqslant \frac{1}{2} < \frac{\pi}{2}$. Using $\sin(x) \geqslant \frac{2}{\pi}x$ for all $0 \leqslant x \leqslant \frac{\pi}{2}$, this leads to

$$\int_{I_n}\int_{J_{n+\nu}} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\, dy\, dx \geqslant Cl^2 \frac{1}{(n+1)^\alpha}\frac{1}{(n+\nu+1)^\alpha} \geqslant Cl^2\frac{1}{n^{2\alpha}}. \tag{5.21}$$

Therefore, we may bound (5.20) from below further by a constant times

$$\sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} 1_{[-l,l]}(2a_{2n+1}) \sum_{\substack{\nu\in\mathbb{N}:\\ \nu<n^\alpha/(4l)}} l^2\frac{1}{n^{2\alpha}} \geqslant Cl \sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} 1_{[-l,l]}(2a_{2n+1})\frac{1}{n^\alpha}. \tag{5.22}$$

Finally, we distinguish between the cases $\frac{1}{2} < \alpha < 1$ and $\alpha = 1$. In the former case, by (5.12) we have $a_{2n+1} \leqslant rn^{1-\alpha}$ for some $r = r(\alpha) > 0$, implying that $1_{[-l,l]}(2a_{2n+1}) \geqslant 1_{[-l,l]}(2rn^{1-\alpha})$ and hence

$$l \sum_{\substack{n\in\mathbb{N}:\\ n\geqslant(4l)^{1/\alpha}}} 1_{[-l,l]}(2a_{2n+1})\frac{1}{n^\alpha} \geqslant l \sum_{\substack{n\in\mathbb{N}:\\ (4l)^{1/\alpha}\leqslant n\leqslant(l/(2r))^{1/(1-\alpha)}}} \frac{1}{n^\alpha} \geqslant Cl^2, \tag{5.23}$$

provided that l is sufficiently large, since $1/(1-\alpha) > 1/\alpha$ for $\frac{1}{2} < \alpha < 1$. The same follows if $\alpha = 1$ using the bound $a_{2n+1} \leqslant C\log(n)$. □

5.2. Upper bounds

Lemma 5.2.1. *Let $\alpha \in]0,1]$. There exists a constant $C = C(\alpha) > 0$ such that*

$$\|1_\Gamma T_l 1_{\Gamma^c}\|_{\mathfrak{S}_2}^2 \leqslant \begin{cases} Cl^{1/(1-\alpha)}\log l, & \text{if } 0 \leqslant \alpha < \frac{1}{2}, \\ Cl^2, & \text{if } \frac{1}{2} \leqslant \alpha \leqslant 1 \end{cases} \tag{5.24}$$

for all $l \geqslant 2$.

Proof. We recall (5.4) and (5.5), i.e.

$$\begin{aligned}\|1_{\Gamma_\alpha} T_l 1_{\Gamma_\alpha^c}\|_2^2 &= \sum_{(n,m)\in\mathbb{Z}^2} \mathscr{X}_{n,m}(l) \\ &= \sum_{(n,m)\in\mathbb{Z}^2} \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\,dx \end{aligned} \tag{5.25}$$

and split the sum into different sets of indices: $\mathbb{Z}^2 = \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3 \cup \mathscr{E}_4$, where

$$\mathscr{E}_1 := \mathbb{N}_0^2, \quad \mathscr{E}_2 := \{(0,-1)\}, \quad \mathscr{E}_3 := (-\mathbb{N})^2, \tag{5.26}$$

$$\mathscr{E}_4 := ((\mathbb{N}_0 \times -\mathbb{N}) \cup (-\mathbb{N} \times \mathbb{N}_0)) \setminus \mathscr{E}_2. \tag{5.27}$$

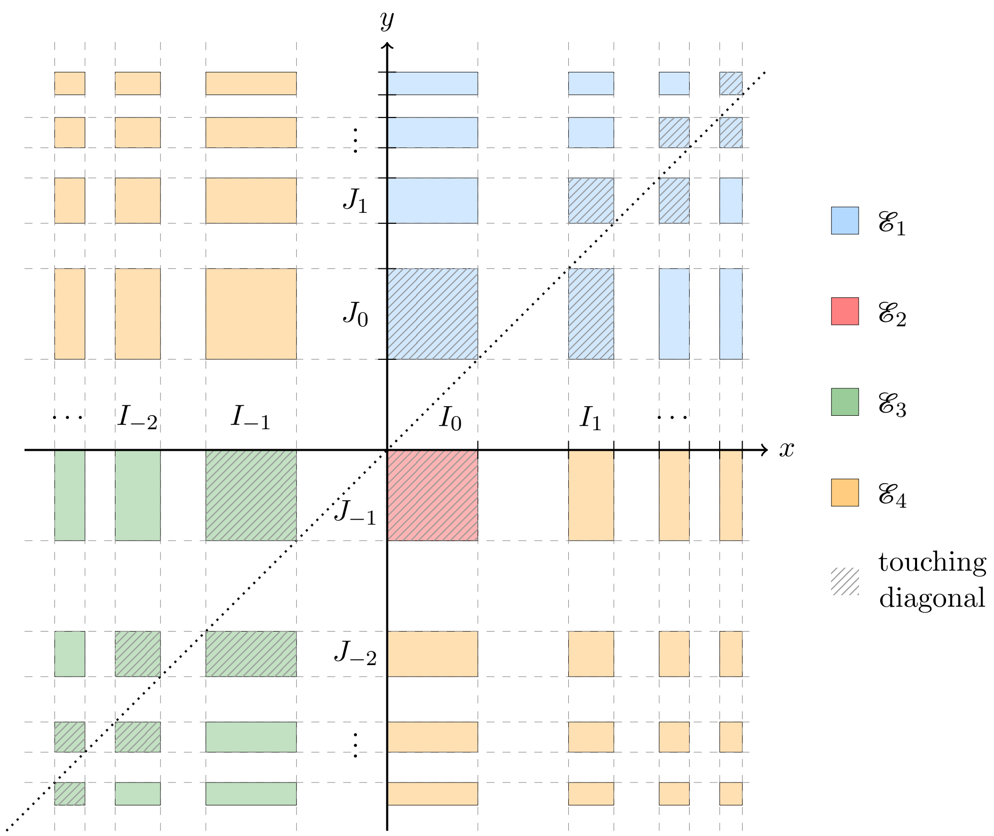

Figure 5.2.: The subsets $\mathscr{E}_1, \ldots, \mathscr{E}_4$.

Let us start by estimating the contributions from $\mathscr{E}_4$. It s straightforward to see that

$$\inf_{(n,m)\in\mathscr{E}_4} \inf_{(x,y)\in I_n\times J_m} |x-y| \geqslant 2. \tag{5.28}$$

It follows that

$$\sum_{(n,m)\in\mathscr{E}_3}\int_{I_n}\int_{J_m}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx$$
$$\leqslant\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx$$
$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}e^{-q^2/2}\,dq\int_{\mathbb{R}}f_l^2(Q)\,dQ\leqslant\frac{\sqrt{2\pi}}{(2\pi)^2}2l. \tag{5.29}$$

As to the contributions from $\mathscr{E}_3$, we argue that

$$\sum_{(n,m)\in\mathscr{E}_3}\int_{I_n}\int_{J_m}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx$$
$$=\sum_{(n,m)\in\mathscr{E}_1}\int_{I_n}\int_{J_m}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx \tag{5.30}$$

because f_l is an even function, $-I_n = J_{-n-1}$, $-J_n = I_{-n-1}$ – see after (5.1) – and the integrand is symmetric under exchanging x and y. Thus it suffices to estimate the terms where $(n,m)\in\mathscr{E}_1\cup\mathscr{E}_2$. These terms include the ones where the corresponding intervals I_n and J_m are close. We further subdivide $\mathscr{E}_1\cup\mathscr{E}_2$ into the "diagonal" part

$$\mathscr{D}:=\{(n,m)\in\mathbb{N}_0^2 : m=n \text{ or } m=n-1\}\cup\{(0,-1)\}, \tag{5.31}$$

the "near the diagonal" part

$$\mathscr{N}:=\{(n,m)\in\mathbb{N}_0^2 : m\leqslant n+n^\alpha \text{ and } n\leqslant m+m^\alpha+1\}\setminus\mathscr{D}, \tag{5.32}$$

and the "far away from the diagonal" part

$$\mathscr{A}:=\mathscr{E}_1\setminus(\mathscr{D}\cup\mathscr{N})=\{(n,m)\in\mathbb{N}_0^2 : m>n+n^\alpha \text{ or } n>m+m^\alpha+1\}. \tag{5.33}$$

(i) We start with estimating the contributions from $\mathscr{A}$. We claim that in this case, the distance $|x-y|$ for $x\in J_m$ and $y\in I_n$ is bounded away from 0 independently of n,m. Assume first that $m>n+n^\alpha$ (which, in particular, implies $m\geqslant n+1$). Notice that

$$|x-y|=x-y\geqslant 2a_m+\xi_{m+1}-2a_n-\xi_{n+1}\geqslant 2a_m-2a_{n+1}+\xi_{n+1}. \tag{5.34}$$

Let $n_0=n_0(\alpha)\in\mathbb{N}$ be such that $\lfloor n_0^\alpha\rfloor\geqslant 2$. It follows for all $n,m\in\mathbb{N}$ satisfying $n\geqslant n_0$ and $m\geqslant n+n^\alpha$ that

$$2a_m-2a_{n+1}+\xi_{n+1}\geqslant\sum_{k=n+2}^{n+\lfloor n^\alpha\rfloor}\frac{1}{k^\alpha}\geqslant(\lfloor n^\alpha\rfloor-1)\frac{1}{(n+\lfloor n^\alpha\rfloor)^\alpha}$$
$$\geqslant\frac{n^\alpha}{4}\frac{1}{(n+n^\alpha)^\alpha}=\frac{1}{4}\left(\frac{1}{1+\frac{1}{n^{1-\alpha}}}\right)^\alpha\geqslant\frac{1}{4}\frac{1}{2^\alpha}>0. \tag{5.35}$$

On the other hand, if $n<n_0$, we simply estimte

$$2a_m-2a_{n+1}+\xi_{n+1}\geqslant\xi_{n+1}>\frac{1}{(n_0+1)^\alpha}. \tag{5.36}$$

In the case $n > m + m^\alpha + 1$ (which, in particular, implies $n \geqslant m+2$), instead of (5.34), we have

$$|x-y| = y - x \geqslant 2a_n - 2a_{m+1} = 2a_n - 2a_{m+2} + 2\xi_{m+2}, \tag{5.37}$$

from where we proceed the same way as before. This proves $|x-y| \geqslant C$ with $C = C(\alpha) > 0$ independent of $(n,m) \in \mathscr{F}_3$. Combining this again with $\sin^2(x) \leqslant 1$, we conclude

$$\begin{aligned}
&\sum_{(n,m)\in\mathscr{F}_3} \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \\
&\leqslant C \sum_{(n,m)\in\mathscr{F}_3} \int_{I_n}\int_{J_m} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \\
&\leqslant C \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \leqslant Cl.
\end{aligned} \tag{5.38}$$

(ii) Let us now turn to the contributions from $\mathscr{D}$, i.e. the contributions close to the diagonal. We first consider the indices $(m,n) \in \mathscr{D}$ where $m = n$, the other case will be treated similarly. As before for the lower bound, we have to estimate

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} \int_{I_n}\int_{J_n} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \\
&= \sum_{n\in\mathbb{N}_0} \int_0^{\xi_{n+1}}\int_{-\xi_{n+1}}^0 \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2} + 2a_n + \xi_{n+1}\right) dy\, dx \\
&\leqslant C \sum_{n\in\mathbb{N}_0} f_l^2(c_{\alpha,n}) \int_0^{\xi_{n+1}}\int_{-\xi_{n+1}}^0 \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} dy\, dx,
\end{aligned} \tag{5.39}$$

where we have estimated the Gaussian via $e^{-(x-y)^2/2} \leqslant 1$ and f_l by writing

$$\frac{x+y}{2} + 2a_n + \xi_{n+1} = 2\log(n+1) + \frac{x+y}{2} + 2(a_n - \log(n+1)) + \xi_{n+1} \tag{5.40}$$

followed by an application of Lemma A.3.1(ii). To estimate the remaining integral in (5.39), we use Lemma 5.2.2(ii) with $n = m$ and get

$$\int_0^{\xi_{n+1}}\int_{-\xi_{n+1}}^0 \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} dy\, dx = g_{l,n,n} \leqslant C\log(1 + l^2\xi_{n+1}^2). \tag{5.41}$$

Combining the above inequalites, we arrive at

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} \int_{I_n}\int_{J_n} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \\
&\leqslant C \sum_{n\in\mathbb{N}_0} f_l^2(c_{\alpha,n}) \log(1 + l^2\xi_{n+1}^2).
\end{aligned} \tag{5.42}$$

Now to the indices $(m,n) \in \mathscr{D}$ where $m = n-1$, i.e. $(n, n-1)$ for $n \in \mathbb{N}_0$. Again using

Lemma 5.2.2(ii), we similarly to before get

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0}\int_{I_n}\int_{J_{n-1}}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx\\
&=\int_{I_0}\int_{J_{-1}}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx\\
&\quad+\sum_{n\in\mathbb{N}}\int_{I_n}\int_{J_{n-1}}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}e^{-(x-y)^2/2}f_l^2\left(\frac{x+y}{2}\right)dy\,dx\\
&\leqslant Cf_l(0)\int_{I_0}\int_{J_{-1}}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx\\
&\quad+C\sum_{n\in\mathbb{N}}f_l(c_{\alpha,n})\int_{I_n}\int_{J_{n-1}}\frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx\\
&=Cf_l(0)g_{l,0,0}+C\sum_{n\in\mathbb{N}}f_l(c_{\alpha,n})g_{l,n,n-1}\\
&\leqslant Cf_l(0)\log(1+l^2)+C\sum_{n\in\mathbb{N}}f_l(c_{\alpha,n})\log(1+l^2\xi_{n+1}\xi_n),
\end{aligned}\tag{5.43}$$

so it is enough to establish an upper bound to (5.42). To that end, we distinguish the cases $\frac{1}{2}<\alpha\leqslant 1$ and $0<\alpha\leqslant\frac{1}{2}$. In the former case, we simply use $f_l\leqslant 1$ and $\log(1+x)\leqslant x$ for $x>0$ to get

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0}f_l^2(c_{\alpha,n})\log\left(1+\frac{l^2}{(n+1)^{2\alpha}}\right)\\
&\leqslant\sum_{n\in\mathbb{N}_0:|n|<l^{1/\alpha}-1}\log\left(1+\frac{l^2}{(n+1)^{2\alpha}}\right)+\sum_{n\in\mathbb{N}_0:|n|\geqslant l^{1/\alpha}-1}\log\left(1+\frac{l^2}{(n+1)^{2\alpha}}\right)\\
&\leqslant C\sum_{n=0}^{\lfloor l^{1/\alpha}\rfloor-1}\log\left(\frac{2l}{(n+1)^{\alpha}}\right)+l^2\sum_{n=\lfloor l^{1/\alpha}\rfloor-1}^{\infty}\frac{1}{(n+1)^{2\alpha}}\\
&\leqslant C\int_0^{\lfloor l^{1/\alpha}\rfloor}\log\left(\frac{2l}{x^{\alpha}}\right)dx+l^2\int_{\lfloor l^{1/\alpha}\rfloor}^{\infty}\frac{1}{x^{2\alpha}}\,dx+\frac{l^2}{\lfloor l^{1/\alpha}\rfloor^{2\alpha}}\\
&\leqslant C\left[x\left(\log\left(\frac{2l}{x^{\alpha}}\right)+\alpha\right)\right]_{x=0}^{\lfloor l^{1/\alpha}\rfloor}+l^2\left[\frac{1}{1-2\alpha}x^{1-2\alpha}\right]_{x=\lfloor l^{1/\alpha}\rfloor}^{\infty}+C\leqslant Cl^{1/\alpha},
\end{aligned}\tag{5.44}$$

which is in accordance with the lower bound for the diagonal.

We remark that in this case, the function f_l did not play a role: the integrability of $x^{-2\alpha}$ for all $1/2<\alpha\leqslant 1$ resulted in a bound for the second sum that grows like $l^{1/\alpha}$ and in particular, not faster than l^2. However, if $0<\alpha\leqslant 1/2$, the aforementioned sum (or integral) would no longer be finite, so in this case we need to utilize the fact that f_l essentially cuts off (up to an error of order e^{-l^2}) the sum where $n^{1-\alpha}>l$, or equivalently, $n>l^{1/(1-\alpha)}$.

Assume now that $0<\alpha\leqslant 1/2$. Recall that if $|x|>2l$ and $|\xi|\leqslant l$,

$$|x-\xi|\geqslant|x|-|\xi|\geqslant\frac{|x|}{2}>l,\tag{5.45}$$

and thus

$$f_l(x) = \int_{-l}^{l} e^{-(x-\xi)^2/4} e^{-(x-\xi)^2/4} e^{-(x-\xi)^2/2} \, \frac{d\xi}{\sqrt{\pi}} \leqslant C e^{-l^2/4} e^{-x^2/8}. \tag{5.46}$$

Note that $n > (2l)^{1/(1-\alpha)}$ guarantees $n^{1-\alpha} > 2l$. Hence

$$\begin{aligned} &\sum_{\substack{n\in\mathbb{N}_0:\\ n>(2l)^{l/(1-\alpha)}}} f_l^2\left(n^{1-\alpha}\right) \log\left(1 + \frac{l^2}{(n+1)^{2\alpha}}\right) \\ &\leqslant C e^{-l^2/2} \log(1+l^2) \sum_{\substack{n\in\mathbb{N}_0:\\ n>(2l)^{l/(1-\alpha)}}} e^{-n^{2-2\alpha}/4} \\ &\leqslant C e^{-l^2/2} \log(1+l^2) \sum_{n=1}^{\infty} e^{-n^{2-2\alpha}/4} \leqslant C e^{-l^2/2} \log(1+l^2). \end{aligned} \tag{5.47}$$

The main contribution thus comes from the sum with indices $n \leqslant (2l)^{1/(1-\alpha)}$, which again is estimated in a straightforward way using $f_l \leqslant 1$:

$$\begin{aligned} &\sum_{\substack{n\in\mathbb{N}_0:\\ n\leqslant(2l)^{1/(1-\alpha)}}} f_l^2\left(n^{1-\alpha}\right) \log\left(1 + \frac{l^2}{(n+1)^{2\alpha}}\right) \\ &\leqslant \sum_{n=1}^{\lfloor (4l)^{1/(1-\alpha)} \rfloor} \log(1+l^2) \leqslant C l^{1/(1-\alpha)} \log l. \end{aligned} \tag{5.48}$$

For $\alpha = 1/2$ we can do better: in that case,

$$\begin{aligned} \sum_{\substack{n\in\mathbb{N}_0:\\ n\leqslant 4l^2}} f_l^2\left(\sqrt{n}\right) \log\left(1 + \frac{l^2}{n+1}\right) &\leqslant C \sum_{n=0}^{4\lfloor l^2 \rfloor} \log\left(\frac{6l^2}{n+1}\right) \\ &\leqslant C \left(\int_1^{4\lfloor l^2\rfloor + 1} \log\left(\frac{6l^2}{x}\right) dx + \log(6l^2) \right) \\ &\leqslant C \left(\left[x\left(\log\left(\frac{6l^2}{x}\right) + 1\right)\right]_{x=1}^{5\lfloor l^2 \rfloor} + \log(6l^2) \right) \\ &= C \left(5\lfloor l^2 \rfloor \log\left(\frac{6l^2}{5\lfloor l^2\rfloor}\right) + 5\lfloor l^2 \rfloor - 1 \right) \leqslant C l^2. \end{aligned} \tag{5.49}$$

This finishes our analysis of the contributions to the sum from $\mathscr{D}$.

Finally we consider the contributions "near the diagonal" from $\mathscr{N}$. By Lemma A.3.3, we have

$$\begin{aligned} &\sum_{(n,m)\in\mathscr{F}_2} \int_{I_n} \int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2 (x-y)^2} e^{-(x-y)^2/2} f_l^2\left(\frac{x+y}{2}\right) dy\, dx \\ &\leqslant \sum_{(n,m)\in\mathscr{F}_2} e^{-(c_{\alpha,n} - c_{\alpha,m})^2} f_l^2\left(\frac{c_{\alpha,n} + c_{\alpha,m}}{2}\right) \int_{I_n} \int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\, dy\, dx. \end{aligned} \tag{5.50}$$

Next, we need some appropriate bounds for the integral term. These are provided by Lemma 5.2.2 below, which we will apply next. We first estimate the terms where either n or m is large, which will turn out to be exponentially small in l because of the effective cutoff coming from the function f_l. Let us consider the case $0 < \alpha < 1$ first. Assume that $n > (4l)^{1/(1-\alpha)}$ or $m > (4l)^{1/(1-\alpha)}$. In both cases,

$$\frac{n^{1-\alpha} + m^{1-\alpha}}{2} > \frac{4l}{2} = 2l. \tag{5.51}$$

Then, by previous calculations and Lemma 5.2.2(i),

$$\begin{aligned}
&\sum_{\substack{(n,m)\in\mathscr{F}_2:\\ n>(4l)^{1/(1-\alpha)}}} f_l^2\left(\frac{n^{1-\alpha}+m^{1-\alpha}}{2}\right) e^{-(n^{1-\alpha}-m^{1-\alpha})^2} \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx \\
&\leqslant \sum_{\substack{n\in\mathbb{N}_0:\\ n>(4l)^{1/(1-\alpha)}}} \sum_{m=0}^{\infty} f_l^2\left(\frac{n^{1-\alpha}+m^{1-\alpha}}{2}\right) e^{-(n^{1-\alpha}-m^{1-\alpha})^2}\,dy\,dx \\
&\leqslant Ce^{-l^2/2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-(n^{1-\alpha}+m^{1-\alpha})^2/32} e^{-(n^{1-\alpha}-m^{1-\alpha})^2} \leqslant Ce^{-l^2/2}.
\end{aligned} \tag{5.52}$$

For symmetry reasons, the same bounds holds for the indices where $m > (4l)^{1/(1-\alpha)}$. If $\alpha = 1$, a similar calculation shows that if $n > e^{4l}$, we have

$$\begin{aligned}
&\sum_{\substack{(n,m)\in\mathscr{F}_2:\\ n>e^{4l}}} f_l^2\left(\frac{\log(n+1)+\log(m+1)}{2}\right) e^{-(\log(n+1)-\log(m+1))^2} \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx \\
&\leqslant Ce^{-l^2/2}.
\end{aligned} \tag{5.53}$$

and accordingly for the case $m > e^{4l}$. It thus remains to treat the indices where $n, m \leqslant (4l)^{1/(1-\alpha)}$ for $0 < \alpha < 1$ and $n, m \leqslant e^{4l}$ for $\alpha = 1$.

This time, by applying Lemma 5.2.2(ii), we get

$$\begin{aligned}
&\sum_{\substack{(n,m)\in\mathscr{F}_2:\\ n,m\leqslant(4l)^{1/(1-\alpha)}}} e^{-(c_{\alpha,n}-c_{\alpha,m})^2} f_l^2\left(\frac{c_{\alpha,n}+c_{\alpha,m}}{2}\right) \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx \\
&\leqslant C \sum_{\substack{(n,m)\in\mathscr{F}_2:\\ n,m\leqslant(4l)^{1/(1-\alpha)}}} e^{-(c_{\alpha,n}-c_{\alpha,m})^2} \log\left(1+\frac{l^2\xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2}\right).
\end{aligned} \tag{5.54}$$

We now further distinguish the cases $n < m \leqslant n + n^\alpha$ and $m + 1 < n \leqslant m + m^\alpha + 1$. In the former case, the task is to estimate

$$\begin{aligned}
&\sum_{\substack{n\in\mathbb{N}_0:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{m\in\mathbb{N}_0:\\ n<m\leqslant n+n^\alpha\\ m\leqslant(4l)^{1/(1-\alpha)}}} e^{-(c_{\alpha,n}-c_{\alpha,m})^2} \log\left(1+\frac{l^2\xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2}\right) \\
&\leqslant \sum_{\substack{n\in\mathbb{N}_0:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{m\in\mathbb{N}_0:\\ n<m\leqslant n+n^\alpha}} \frac{l^2\xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2}.
\end{aligned} \tag{5.55}$$

To get a bound on the double sum in (5.55), we will estimate $d_{n,m}$ in an appropriate way. Note that for $n < m \leqslant n + n^\alpha$, we have

$$\frac{1}{(m+1)^\alpha} < \frac{1}{(n+1)^\alpha} = 2^\alpha \frac{1}{(n+n+2)^\alpha} < 2^\alpha \frac{1}{(n+n^\alpha+1)^\alpha} \leqslant 2^\alpha \frac{1}{(m+1)^\alpha}. \tag{5.56}$$

We also have

$$\begin{aligned} d_{n,m} &= 2a_m - 2a_n + \xi_{m+1} - \xi_{n+1} \\ &= 2a_m - 2a_{n+1} + \xi_{m+1} + \xi_{n+1} \geqslant 2a_m - 2a_{n+1} + \xi_{n+1} \geqslant 0. \end{aligned} \tag{5.57}$$

Hence

$$\begin{aligned} \frac{l^2 \xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2} &\leqslant \frac{l^2}{(n+1)^{2\alpha}\left(1 + l\left(2a_m - 2a_{n+1} + \frac{1}{(n+1)^\alpha}\right)\right)^2} \\ &= \frac{1}{\left(\frac{(n+1)^\alpha}{l} + 1 + (n+1)^\alpha \sum_{k=n+2}^m \frac{1}{k^\alpha}\right)^2}, \end{aligned} \tag{5.58}$$

where we interpret the sum to be 0 if $m = n+1$. We estimate the denominator further by

$$\begin{aligned} \frac{(n+1)^\alpha}{l} + 1 + (n+1)^\alpha \sum_{k=n+2}^{m} \frac{1}{k^\alpha} &\geqslant \frac{n^\alpha}{l} + 1 + \frac{(n+1)^\alpha}{m^\alpha}(m-n-1) \\ &\geqslant \frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}(m-n-1). \end{aligned} \tag{5.59}$$

Thus, for (5.55) we get the bound

$$\begin{aligned} &\sum_{\substack{n\in\mathbb{N}_0: \\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{m\in\mathbb{N}_0: \\ n<m\leqslant n+n^\alpha}} \frac{l^2\xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2} \\ &\leqslant \sum_{\substack{n\in\mathbb{N}_0: \\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{m\in\mathbb{N}_0: \\ n<m\leqslant n+n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}(m-n-1)\right)^2} \\ &\leqslant \sum_{\substack{n\in\mathbb{N}_0: \\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{\nu\in\mathbb{N}_0: \\ 0\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}\nu\right)^2}. \end{aligned} \tag{5.60}$$

If $n = 0$, the inner sum only consists of the $\nu = 0$ term, which is just equal to 1. For $n \geqslant 1$ and $\nu \geqslant 1$, we can compare the inner sum to an integral to obtain an appropriate bound. Indeed,

$$\begin{aligned} \sum_{\substack{\nu\in\mathbb{N}_0: \\ 1\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}\nu\right)^2} &= \sum_{\substack{\nu\in\mathbb{N}_0: \\ 1\leqslant\nu\leqslant n^\alpha}} \int_{\nu-1}^{\nu} \frac{1}{\left(\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}\nu\right)^2}\, dx \\ &\leqslant \int_0^{\lfloor n^\alpha \rfloor} \frac{1}{\left(\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}x\right)^2}\, dx \\ &= \left[-\frac{2^\alpha}{\frac{n^\alpha}{l} + 1 + \frac{1}{2^\alpha}x}\right]_{x=0}^{\lfloor n^\alpha \rfloor} = \frac{\lfloor n^\alpha \rfloor}{\left(\frac{n^\alpha}{l} + 1\right)\left(\frac{n^\alpha}{l} + 1 + \frac{\lfloor n^\alpha \rfloor}{2^\alpha}\right)}. \end{aligned} \tag{5.61}$$

The last term will be estimated depending on α. If $\alpha < \frac{1}{2}$,

$$\frac{\lfloor n^\alpha \rfloor}{\left(\frac{n^\alpha}{l}+1\right)\left(\frac{n^\alpha}{l}+1+\frac{\lfloor n^\alpha \rfloor}{2^\alpha}\right)} \leqslant 2^\alpha, \tag{5.62}$$

and thus

$$\begin{aligned}
&\sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{\nu\in\mathbb{N}_0:\\ 0\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l}+1+\frac{1}{2^\alpha}\nu\right)^2} \\
&= \sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \left(\frac{1}{\frac{n^\alpha}{l}+1} + \sum_{\substack{\nu\in\mathbb{N}_0:\\ 1\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l}+1+\frac{1}{2^\alpha}\nu\right)^2} \right) \\
&\leqslant \sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} (1+2^\alpha) \leqslant Cl^{1/(1-\alpha)}.
\end{aligned} \tag{5.63}$$

If $\alpha \geqslant \frac{1}{2}$, then

$$\frac{\lfloor n^\alpha \rfloor}{\left(\frac{n^\alpha}{l}+1\right)\left(\frac{n^\alpha}{l}+1+\frac{\lfloor n^\alpha \rfloor}{2^\alpha}\right)} \leqslant 2^\alpha \frac{l}{n^\alpha}, \tag{5.64}$$

so that this time

$$\begin{aligned}
&\sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \sum_{\substack{\nu\in\mathbb{N}_0:\\ 0\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l}+1+\frac{1}{2^\alpha}\nu\right)^2} \\
&= \sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \left(\frac{1}{\frac{n^\alpha}{l}+1} + \sum_{\substack{\nu\in\mathbb{N}_0:\\ 1\leqslant\nu\leqslant n^\alpha}} \frac{1}{\left(\frac{n^\alpha}{l}+1+\frac{1}{2^\alpha}\nu\right)^2} \right) \\
&\leqslant (1+2^\alpha)l \sum_{\substack{n\in\mathbb{N}:\\ n\leqslant(4l)^{1/(1-\alpha)}}} \frac{1}{n^\alpha} \leqslant Cl^2.
\end{aligned} \tag{5.65}$$

This finishes the estimation of the contributions from $\mathscr{F}_2$ and thereby the proof. □

Lemma 5.2.2. *Let $l \geqslant 1$ and $n, m \in \mathbb{N}_0$ and put*

$$g_{l,n,m} := \int_{I_n}\int_{J_m} \frac{\sin^2[l(x-y)]}{\pi^2(x-y)^2}\,dy\,dx. \tag{5.66}$$

Then:

(i) If $m \neq n$ and $m \neq n-1$, we have $g_{l,n,m} \leqslant C$ independent of l, n, m.

(ii) Putting

$$d_{n,m} := \begin{cases} 2a_m - 2a_n + \xi_{m+1} - \xi_{n+1} & \textit{if } m \geqslant n, \\ 2a_n - 2a_m - 2\xi_{m+1} & \textit{if } m < n, \end{cases} \tag{5.67}$$

we have

$$g_{l,n,m} \leqslant C \log\left(1 + \frac{l^2\xi_{n+1}\xi_{m+1}}{(1+ld_{n,m})^2}\right). \tag{5.68}$$

Proof. (i) We estimate the enumerator of the integrand using $|\sin(x)| \leqslant 1$. Furthermore, the distance between I_n and J_m must be at least

$$\max\{\xi_{n+1} + \xi_{n+2}, \xi_{m+1} + \xi_{m+2}\} \geqslant 2\max\{\xi_{n+2}, \xi_{m+2}\}, \tag{5.69}$$

so

$$|x - y| \geqslant 2\max\{\xi_{n+2}, \xi_{m+2}\} \tag{5.70}$$

for all $x \in I_n$, $y \in J_m$. If $m > n$, we therefore have

$$g_{l,n,m} \leqslant C \frac{1}{\xi_{n+2}^2} \int_{I_n} \int_{J_m} dx\, dy = C \frac{\xi_{m+1}\xi_{n+1}}{\xi_{n+2}^2} \leqslant C \frac{\xi_{n+1}^2}{\xi_{n+2}^2}, \tag{5.71}$$

while if $n < m - 1$ we get similarly

$$g_{l,n,m} \leqslant C \frac{\xi_{m+1}^2}{\xi_{m+2}^2}. \tag{5.72}$$

Clearly, this expression is bounded by a constant for our respective sequences $\xi_n = n^{-\alpha}$, $0 < \alpha \leqslant 1$.

(ii) We first note that for general real numbers $a < b < c < d$ we have

$$\begin{aligned}
& l^2 \int_a^b \int_c^d \frac{1}{(1 + l(y - x))^2}\, dy\, dx = l \int_a^b \left[-\frac{1}{1 + l(y - x)} \right]_{x=c}^{d} dx \\
& = l \int_a^b \frac{1}{1 + l(c - x)}\, dx - l \int_a^b \frac{1}{1 + l(d - x)}\, dx \\
& = -\left[\log(1 + l(c - y))\right]_a^b + \left[\log(1 + l(d - y))\right]_a^b \\
& = \log\left(\frac{(1 + l(c - a))(1 + l(d - b))}{(1 + l(c - b))(1 + l(d - a))} \right) \\
& = \log\left(1 + \frac{l^2(b - a)(d - c)}{(1 + l(c - b))(1 + l(d - a))} \right) \leqslant \log\left(1 + \frac{l^2(b - a)(d - c)}{(1 + l(c - b))^2} \right).
\end{aligned} \tag{5.73}$$

Now applying inequality (A.36) to $g_{l,n,m}$ and combining with the above, we obtain for $m \geqslant n$

$$\begin{aligned}
g_{l,n,m} & \leqslant C l^2 \int_{2a_n}^{2a_n + \xi_{n+1}} \int_{2a_m + \xi_{m+1}}^{2a_m + 2\xi_{m+1}} \frac{1}{(1 + l|x - y|)^2}\, dy\, dx \\
& \leqslant C \log\left(1 + \frac{l^2 \xi_{n+1}\xi_{m+1}}{(1 + l(2a_m - 2a_n + \xi_{m+1} - \xi_{n+1}))^2} \right),
\end{aligned} \tag{5.74}$$

and similarly for $m < n$. □

A. Auxiliary estimates

A.1. Properties of the function f_l

Lemma A.1.1. *The function f_l given by* (4.5) *has the following properties.*

(i) Let $l, \alpha > 0$, $N \in \mathbb{N}$ and $Q_j \in \mathbb{R}$ for $j = 1, \dots, N$. Then

$$\sum_{n\in\mathbb{Z}} \prod_{j=1}^{N} f_l(Q_j + \alpha n) = \frac{2l}{\alpha} + R, \tag{A.1}$$

where the error term $R = R(N, Q_1, \dots, Q_N, l, \alpha)$ satisfies

$$|R| \leqslant C \left(1 + \sum_{j=1}^{N} |Q_j|\right) \tag{A.2}$$

for some constant $C > 0$ depending only on N and α.

(ii) For every $q \in \,]0, 1]$ and $r > 0$, there exists a constant $C = C(q, r) > 0$ such that for all $l \geqslant 1$,

$$\sum_{n\in\mathbb{Z}} \left(\sup_{w\in B_r(n)} f_l(w) \right)^q \leqslant Cl. \tag{A.3}$$

Furthermore, for every positive integer $k \in \mathbb{N}$ and $q, r > 0$, there exists a constant $C = C(k, q, r) > 0$ such that for all $l \geqslant 1$,

$$\sum_{n\in\mathbb{Z}} \left(\sup_{w\in B_r(n)} |f_l^{(k)}(w)| \right)^q \leqslant C. \tag{A.4}$$

(iii) For every $q \in \,]0, 1]$ there exists a constant $C = C(q) > 0$ such that for all $l \geqslant 1$ we have

$$\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f_l(x) - 1_{[-l,l]}(x)| \, dx \right)^q \leqslant C \tag{A.5}$$

and

$$\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} \int_{\mathbb{R}} e^{-y^2} |f_l(x+y) - 1_{[-l,l]}(x+y)| \, dy \, dx \right)^q \leqslant C. \tag{A.6}$$

Proof. (i) We start by replacing f_l by the indicator function $1_{[-l,l]}$ at the cost of an error term which is uniformly bounded in l. In fact, we infer from a telescoping sum argument, see (4.17), and $0 \leqslant f_l, 1_{[-l,l]} \leqslant 1$ that

$$\left|\prod_{j=1}^{N} f_l(Q_j + x) - \prod_{j=1}^{N} 1_{[-l,l]}(Q_j + x)\right| \leqslant \sum_{j=1}^{N} \left|f_l(Q_j + x) - 1_{[-l,l]}(Q_j + x)\right|. \tag{A.7}$$

for every $x \in \mathbb{R}$. A proof similar to the one of (A.5) shows the existence of a constant $C = C(N, \alpha) > 0$ such that for all $l \geqslant 1$,

$$\sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} \left|f_l(Q_j + \alpha n) - 1_{[-l,l]}(Q_j + \alpha n)\right| \leqslant C. \tag{A.8}$$

In fact, we show that there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |f_l(x + \alpha n) - 1_{[-l,l]}(x + \alpha n)| \leqslant C. \tag{A.9}$$

To see this, fix $x \in \mathbb{R}$ and split the sum according to indices where $x+\alpha n > l$, $x+\alpha n < -l$ and $-l \leqslant x + \alpha n \leqslant l$. In the first case

$$\begin{aligned}\sum_{\substack{n\in\mathbb{Z}:\\ x+\alpha n>l}} |f_l(x + \alpha n) - 1_{[-l,l]}(x + \alpha n)| &= \sum_{\substack{n\in\mathbb{Z}:\\ x+\alpha n>l}} \int_{-l}^{l} e^{-(x+\alpha n-\xi)^2} \frac{d\xi}{\sqrt{\pi}} \\ &\leqslant \sqrt{2} \sum_{\substack{n\in\mathbb{Z}:\\ x+\alpha n>l}} e^{-(x+\alpha n-l)^2/2} \\ &\leqslant \sqrt{2} \sum_{n\in\mathbb{N}_0} e^{-(\alpha n)^2/2} \leqslant C. \end{aligned} \tag{A.10}$$

The other cases work similarly. Thus (A.8) follows and we have

$$\sum_{n\in\mathbb{Z}} \prod_{j=1}^{N} f_l(Q_j + \alpha n) = \sum_{n\in\mathbb{Z}} \prod_{j=1}^{N} 1_{[-l,l]}(Q_j + \alpha n) + R_1 \tag{A.11}$$

with $R_1 = R_1(N, Q_1, \ldots Q_N, l, \alpha)$ satisfying $|R_1| \leqslant C$ with a constant only depending on N and α. For the product of the indicator functions, by Lemma A.1.2 below we have the formula

$$\prod_{j=1}^{N} 1_{[-l,l]}(Q_j + x) = 1_{\bigcap_{j=1}^{N}[-l-Q_j, l-Q_j]}(x) = 1_{[-l-Q_{\min}, l-Q_{\max}]}, \tag{A.12}$$

with $Q_{\min} := \min_{1\leqslant j\leqslant N} Q_j, Q_{\max} := \max_{1\leqslant j\leqslant N} Q_{\max}$. This implies

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \prod_{j=1}^{N} 1_{[-l,l]}(Q_j + \alpha n) &= \# \{n \in \mathbb{Z} : -l - Q_{\min} \leqslant \alpha n \leqslant l - Q_{\max}\} \\ &= \# \left(\mathbb{Z} \cap \left[-\frac{l}{\alpha} - \frac{1}{\alpha} Q_{\min}, \frac{l}{\alpha} - \frac{1}{\alpha} Q_{\max}\right]\right). \end{aligned} \tag{A.13}$$

Clearly, this number is zero, if the right limit of the interval is smaller than the left one, that is, if

$$\hat{Q} := Q_{\max} - Q_{\min} > 2l. \tag{A.14}$$

On the other hand, if the interval in (A.13) is non-empty, i.e. if $\hat{Q} \leqslant 2l$, (A.13) equals the length of the interval plus an error in $[-1,1]$. Taken together, there must exist a number $R_2 = R_2(Q_1, \dots, Q_N, l, \alpha)$ with $|R_2| \leqslant 1$ such that

$$\sum_{n\in\mathbb{Z}} \prod_{j=1}^{N} 1_{[-l,l]}(Q_j + \alpha n) = \left(\frac{2l}{\alpha} - \frac{1}{\alpha}\hat{Q} + R_2\right) 1_{[0,2l]}(\hat{Q}). \tag{A.15}$$

By putting

$$R := R_1 + \frac{2l}{\alpha} 1_{]2l,\infty[}(\hat{Q}) - \frac{\hat{Q}}{2\alpha} 1_{[0,2l]}(\hat{Q}) + R_2 1_{[0,2l]}(\hat{Q}), \tag{A.16}$$

we arrive at the desired identity (A.1) by combining (A.11) with (A.15). The estimate (A.2) follows from

$$\frac{2l}{\alpha} 1_{]2l,\infty[}(\hat{Q}) \leqslant \frac{\hat{Q}}{\alpha} 1_{]2l,\infty[}(\hat{Q}) \leqslant \frac{\hat{Q}}{\alpha} \tag{A.17}$$

and

$$\hat{Q} \leqslant 2 \max_{1\leqslant j\leqslant N} |Q_i| \leqslant 2 \sum_{j=1}^{N} |Q_j|. \tag{A.18}$$

(ii) *Proof of* (A.3). We note that for every $|x| \geqslant 2l$ and $|\xi| \leqslant l$, we have $|x-\xi| \geqslant |x|/2 + |x|/2 - |\xi| \geqslant |x|/2$ and, hence,

$$f_l(x) = \int_{-l}^{l} e^{-(x-\xi)^2/2} e^{-(x-\xi)^2/2} \frac{d\xi}{\sqrt{\pi}} \leqslant e^{-x^2/8} \int_{-l}^{l} e^{-(x-\xi)^2/2} \frac{d\xi}{\sqrt{\pi}} \leqslant \sqrt{2} e^{-x^2/8}. \tag{A.19}$$

Now, we fix $r > 0$ and let $n_0 \in \mathbb{N}$ be the smallest number such that $n_0 > 2l + r$. Then if $w \in B_r(n)$, we have $|w| \geqslant |n| - |w-n| > 2l$ for all $n \in \mathbb{Z}$ with $|n| \geqslant n_0$ and thus the validity of (A.19) for $x = w$. For $|n| < n_0$ and $w \in B_r(n)$ we simply estimate $f_l(w) \leqslant 1$. In summary, we get

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \left(\sup_{w\in B_r(n)} f_l(w)\right)^q &= \sum_{\substack{n\in\mathbb{Z}:\\ |n|<n_0}} \left(\sup_{w\in B_r(n)} f_l(w)\right)^q + \sum_{\substack{n\in\mathbb{Z}:\\ |n|\geqslant n_0}} \left(\sup_{w\in B_r(n)} f_l(w)\right)^q \\ &\leqslant 2n_0 + \sqrt{2} \sum_{\substack{n\in\mathbb{Z}:\\ |n|\geqslant n_0}} e^{-q(|n|-r)^2/8} \leqslant Cl, \end{aligned} \tag{A.20}$$

where we used that $n_0 \leqslant 2l + r + 1$ and obtain a constant $C > 0$ depending on q, r.

Proof of (A.4). Fix $k \in \mathbb{N}$. By differentiating under the integral sign, we observe that for each $x \in \mathbb{R}$

$$\begin{aligned} f_l^{(k)}(x) &= (-1)^k \int_{-l}^{l} \frac{\partial^k}{\partial \xi^k} e^{-(x-\xi)^2} \frac{d\xi}{\sqrt{\pi}} = \frac{(-1)^k}{\sqrt{\pi}} \left[\frac{\partial^{k-1}}{\partial \xi^{k-1}} e^{-(x-\xi)^2}\right]_{\xi=-l}^{l} \\ &= \frac{(-1)^k}{\sqrt{\pi}} \left[\mathsf{H}_{k-1}(x+l) e^{-(x+l)^2} - \mathsf{H}_{k-1}(x-l) e^{-(x-l)^2}\right] \end{aligned} \tag{A.21}$$

with H_k being the k-th Hermite polynomial (2.17), whence

$$\begin{aligned} |f_l^{(k)}(x)| &\leqslant \frac{1}{\sqrt{\pi}} \left(|\mathsf{H}_{k-1}(x+l)| e^{-(x+l)^2} + |\mathsf{H}_{k-1}(x-l)| e^{-(x-l)^2}\right) \\ &\leqslant C \left(e^{-(x+l)^2/2} + e^{-(x-l)^2/2}\right) \end{aligned} \tag{A.22}$$

for a constant $C > 0$ depending only on k. We conclude

$$\sum_{n\in\mathbb{Z}} \left(\sup_{w\in B_r(n)} |f_l^{(k)}(w)| \right)^q \leqslant C \sum_{n\in\mathbb{Z}} \left(\sup_{w\in]-r,r[} \left(e^{-(w+n+l)^2/2} + e^{-(w+n-l)^2/2} \right) \right)^q$$
$$\leqslant C \sum_{n\in\mathbb{Z}} \left(e^{-q(l+n)^2/4} + e^{-q(n-l)^2/4} \right) \leqslant C. \tag{A.23}$$

Here we have used Lemma A.3.1(i) and $(a+b)^q \leqslant a^q + b^q$ for all $a, b \geqslant 0$ and $0 < q \leqslant 1$ in the penultimate step, and the final constant $C > 0$ depends on k, q and r.

(iii) *Proof of* (A.5). We start with those integers $n \in \mathbb{Z}$ of the sum where $n > l$, so that $1_{[-l,l]}(x) = 0$ for $x \in [n, n+1]$. We can then estimate

$$\sum_{n\in\mathbb{Z}:n>l} \left(\int_n^{n+1} \int_{-l}^{l} e^{-(x-\xi)^2} \frac{d\xi}{\sqrt{\pi}} \, dx \right)^q$$
$$\leqslant \sum_{n\in\mathbb{Z}:n>l} e^{-q(n-l)^2/2} \left(\int_n^{n+1} \int_{-l}^{l} e^{-(x-\xi)^2/2} \frac{d\xi}{\sqrt{\pi}} \, dx \right)^q \leqslant 2^{q/2} \sum_{n\in\mathbb{N}_0} e^{-qn^2/2}. \tag{A.24}$$

The case $n < -l-1$ is completely analogous. If $0 \leqslant n \leqslant l-1$, then $x \in [n, n+1]$ ensures $1_{[-l,l]}(x) = 1$. Hence

$$\sum_{0\leqslant n\leqslant l-1} \left(\int_n^{n+1} |f_l(x) - 1| \, dx \right)^q$$
$$= \sum_{0\leqslant n\leqslant l-1} \left(\int_n^{n+1} \int_{\mathbb{R}\setminus[-l,l]} e^{-(x-\xi)^2} \frac{d\xi}{\sqrt{\pi}} \, dx \right)^q$$
$$\leqslant \sum_{0\leqslant n\leqslant l-1} e^{-q(l-n-1)^2/2} \left(\int_n^{n+1} \int_{\mathbb{R}\setminus[-l,l]} e^{-(x-\xi)^2/2} \frac{d\xi}{\sqrt{\pi}} \, dx \right)^q$$
$$\leqslant 2^{q/2} \sum_{n\in\mathbb{N}_0} e^{-qn^2/2}. \tag{A.25}$$

The case $-l \leqslant n < 0$ is again trated analogously. Finally, the two remaining terms of the n-series are uniformly bounded in l because $|f_l(x) - 1_{[-l,l]}(x)| \leqslant 2$.

Proof of (A.6). For a fixed $n \in \mathbb{Z}$, we perform the substitutions $x \mapsto x+n$ and $y \mapsto y-x-n$, so that term to be estimated becomes

$$\left(\int_0^1 \int_{\mathbb{R}} e^{-(y-n-x)^2} |f_l(y) - 1_{[-l,l]}(y)| \, dy \, dx \right)^q. \tag{A.26}$$

Next, we split the y-integration into the regions where $|y| > l$ and $|y| \leqslant l$, respectively. Using the subadditivity $(a+b)^q \leqslant a^q + b^q$ for all $a, b \geqslant 0$ and Lemma A.3.1(i), we obtain

$$\left(\int_0^1 \int_{\mathbb{R}} e^{-(y-n-x)^2} |f_l(y) - 1_{[-l,l]}(y)| \, dy \, dx \right)^q$$
$$\leqslant C \left(\int_{\mathbb{R}\setminus[-l,l]} e^{-(y-n)^2/2} f_l(y) \, dy \right)^q + C \left(\int_{-l}^{l} e^{-(y-n)^2/2} \tilde{f}_l(y) \, dy \, dx \right)^q \tag{A.27}$$

with $\tilde{f}_l(y) := \int_{\mathbb{R}\setminus[-l,l]} e^{-(y-\xi)^2} \frac{d\xi}{\sqrt{\pi}}$. We first estimate the left term (apart from the q-th power) as

$$\begin{aligned}
&\int_{\mathbb{R}\setminus[-l,l]} e^{-(y-n)^2/2} f_l(y)\, dy \\
&\leqslant \int_{\mathbb{R}\setminus[-l,l]} e^{-(y-n)^2/2} e^{-(|y|-l)^2/2} \int_{-l}^{l} e^{-(y-\xi)^2/2}\, \frac{d\xi}{\sqrt{\pi}}\, dy \\
&\leqslant \sqrt{2} \int_0^{\infty} e^{-(y+l-n)^2/2} e^{-y^2/2}\, dy + \sqrt{2} \int_{-\infty}^{0} e^{-(y-l-n)^2/2} e^{-y^2/2}\, dy \\
&\leqslant \sqrt{2\pi} \left(e^{-(n-l)^2/4} + e^{-(n+l)^2/4} \right),
\end{aligned} \tag{A.28}$$

where in the last line we have used the convolution identity

$$(e^{-a(\cdot)^2} * e^{-b(\cdot)^2})(x) = \sqrt{\frac{\pi}{a+b}} e^{-\frac{abx^2}{a+b}}, \quad a, b > 0. \tag{A.29}$$

Taking q-th powers, using again the subadditivity $(a+b)^q \leqslant a^q + b^q$ and summing over $n \in \mathbb{Z}$ yields

$$\sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}\setminus[-l,l]} e^{-(y-n)^2/2} f_l(y)\, dy \right)^q \leqslant (2\pi)^{q/2} \sum_{n\in\mathbb{Z}} \left(e^{-q(n-l)^2/4} + e^{-q(n+l)^2/4} \right) \leqslant C, \tag{A.30}$$

with a constant $C = C(q) > 0$. The second term in (A.27) can be bounded in a similar fashion:

$$\begin{aligned}
&\int_{-l}^{l} e^{-(y-n)^2/2} \tilde{f}_l(y)\, dy \\
&\leqslant \int_{\mathbb{R}\setminus[-l,l]} e^{-(|\xi|-l)^2/2} \int_{-l}^{l} e^{-(y-n)^2/2} e^{-(y-\xi)^2/2}\, dy\, \frac{d\xi}{\sqrt{\pi}} \\
&\leqslant \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}\setminus[-l,l]} e^{-(|\xi|-l)^2/2} \left(e^{-(\cdot)^2/2} * e^{-(\cdot)^2/2} \right) (n-\xi)\, d\xi \\
&= \int_{\mathbb{R}\setminus[-l,l]} e^{-(|\xi|-l)^2/2} e^{-(\xi-n)^2/4}\, d\xi,
\end{aligned} \tag{A.31}$$

from where we are in an almost identical situation as in (A.28). □

Lemma A.1.2. *For $N \in \mathbb{N}$, let $Q_1, \ldots, Q_N \in \mathbb{R}$. Then*

$$\bigcap_{j=1}^{N} [-l - Q_j, l - Q_j] = [-l - Q_{\min}, l - Q_{\max}] \tag{A.32}$$

with $Q_{\min} := \min_{1\leqslant j\leqslant N} Q_j, Q_{\max} := \max_{1\leqslant j\leqslant N} Q_{\max}$, where we put $[a,b] := \emptyset$ for $a > b$.

Proof. It is clear that the intersection is empty if $Q_{\max} - Q_{\min} > 2l$. If not, we get

$$\begin{aligned}
[-l - Q_1, l - Q_1] \cap [-l - Q_2, l - Q_2] &= [\max\{-l - Q_1, -l - Q_2\}, \min\{l - Q_1, l - Q_2\}] \\
&= [-l + \max\{-Q_1, -Q_2\}, l + \min\{-Q_1, -Q_2\}] \\
&= [-l - \min\{Q_1, Q_2\}, l - \max\{Q_1, Q_2\}],
\end{aligned} \tag{A.33}$$

and therefore

$$\begin{aligned}
&[-l - Q_1, l - Q_1] \cap [-l - Q_2, l - Q_2] \cap [-l - Q_3, l - Q_3] \\
&= [-l - \min\{Q_1, Q_2\}, l - \max\{Q_1, Q_2\}] \cap [-l - Q_3, l - Q_3] \\
&= [\max\{-l - \min\{Q_1, Q_2\}, -l - Q_3\}, \min\{l - \max\{Q_1, Q_2\}, l - Q_3\}] \\
&= [-l - \min\{Q_1, Q_2, Q_3\}, l - \max\{Q_1, Q_2, Q_3\}]. \qquad \text{(A.34)}
\end{aligned}$$

The desired formula follows inductively. □

A.2. Useful estimates involving the sinc function

In the following, we will use the usual convention $\sin(0)/0 := 1$.

Lemma A.2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) := \frac{\sin x}{x}. \qquad \text{(A.35)}$$

For each $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we have

$$\left| f^{(k)}(x) \right| \leqslant \frac{3}{1 + |x|}. \qquad \text{(A.36)}$$

Proof. First we observe that we may write $f(x) = \int_0^1 \cos(xt)\, dt$, so that

$$f^{(k)}(x) = \int_0^1 t^k \cos^{(k)}(xt)\, dt \qquad \text{(A.37)}$$

for arbitrary $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Consequently,

$$\left| f^{(k)}(x) \right| \leqslant \int_0^1 t^k\, dt = \frac{1}{k+1} \leqslant 1. \qquad \text{(A.38)}$$

On the other hand, for $x \neq 0$ and $k \geqslant 1$, an integration by parts in (A.37) shows

$$f^{(k)}(x) = \frac{\cos^{(k-1)}(x)}{x} - \int_0^1 k t^{k-1} \frac{\cos^{(k-1)}(x)}{x}\, dt, \qquad \text{(A.39)}$$

from which we deduce that

$$\left| f^{(k)}(x) \right| \leqslant \frac{1}{|x|} + \frac{k}{|x|} \int_0^1 t^{k-1}\, dt = \frac{2}{|x|}. \qquad \text{(A.40)}$$

Clearly, this also holds for $k = 0$. Combining (A.38) and (A.40), we obtain

$$\left| f^{(k)}(x) \right| \leqslant \min\left\{1, \frac{2}{|x|}\right\} \leqslant \frac{3}{1 + |x|}, \qquad \text{(A.41)}$$

as desired. □

Lemma A.2.2. *For $l \geqslant 1$ and $x, y \in \mathbb{R}$, let*

$$g_l(x,y) := \frac{\sin(x-y)}{x-y} e^{-(x-y)^2/(4l^2)} \left[f_l\left(\frac{x+y}{2l}\right) - f_l\left(\frac{x}{l}\right) \right] \tag{A.42}$$

with the usual convention $\sin(0)/0 := 1$*. Then, for all $s \in \mathbb{N}_0$, there is a constant $C = C(s) > 0$ such that*

$$\left| \frac{\partial^s}{\partial x^s} g_l(x,y) \right| \leqslant C \frac{1}{1+|x-y|} e^{-(x-y)^2/(8l^2)} \sum_{\gamma=0}^{s} \left| \frac{1}{2^\gamma} f_l^{(\gamma)}\left(\frac{x+y}{2l}\right) - f_l^{(\gamma)}\left(\frac{x}{l}\right) \right|. \tag{A.43}$$

Proof. By the product rule, the derivative $\frac{\partial^s}{\partial x^s} g_l(x,y)$ equals

$$\sum_{\substack{(\alpha,\beta,\gamma)\in\mathbb{N}_0^3 \\ \alpha+\beta+\gamma=s}} \left[\frac{\partial^\alpha}{\partial x^\alpha} \frac{\sin(x-y)}{x-y} \right] \left[\frac{\partial^\beta}{\partial x^\beta} e^{-(x-y)^2/(4l^2)} \right] \left[\frac{\partial^\gamma}{\partial x^\gamma} \left(f_l\left(\frac{x+y}{2l}\right) - f_l\left(\frac{x}{l}\right) \right) \right]. \tag{A.44}$$

Using the chain rule, we obtain

$$\frac{\partial^\beta}{\partial x^\beta} e^{-(x-y)^2/(4l^2)} = \frac{1}{(2l)^\beta} \mathsf{H}_\beta \left(\frac{x-y}{2l} \right) e^{-(x-y)^2/(4l^2)} \tag{A.45}$$

in terms of the Hermite polynomial H_β of order β (see (2.17)), so that

$$\left| \frac{\partial^\beta}{\partial x^\beta} e^{-(x-y)^2/(4l^2)} \right| \leqslant \frac{1}{(2l)^\beta} e^{-(x-y)^2/(8l^2)} \sup_{z\in\mathbb{R}} |\mathsf{H}_\beta(z) e^{-z^2/2}| \leqslant C e^{-(x-y)^2/(8l^2)}, \tag{A.46}$$

where C only depends on β. Combining (A.44) with (A.46) and inequality (A.36) for the derivatives of the sine kernel yields

$$\begin{aligned} \left| \frac{\partial^s}{\partial x^s} g_l(x,y) \right| &\leqslant C \frac{1}{1+|x-y|} e^{-(x-y)^2/(8l)^2} \sum_{\gamma=0}^{s} \left| \frac{1}{(2l)^\gamma} f_l^{(\gamma)}\left(\frac{x+y}{2l}\right) - \frac{1}{l^\gamma} f_l^{(\gamma)}\left(\frac{x}{l}\right) \right| \\ &\leqslant C \frac{1}{1+|x-y|} e^{-(x-y)^2/(8l)^2} \sum_{\gamma=0}^{s} \left| \frac{1}{2^\gamma} f_l^{(\gamma)}\left(\frac{x+y}{2l}\right) - f_l^{(\gamma)}\left(\frac{x}{l}\right) \right|, \end{aligned} \tag{A.47}$$

with $C > 0$ depending only on s. □

A.3. Further auxiliary estimates

Lemma A.3.1. *Let $r > 0$. Then:*

(i) There exists a constant $C = C(r) > 0$ such that for every $x, y \in \mathbb{R}$ with $|x| \leqslant r$ we have

$$e^{-(x-y)^2} \leqslant C e^{-y^2/2}. \tag{A.48}$$

(ii) There exists a constant $C = C(r) > 0$ such that for every $l > 0$ and every $x, y \in \mathbb{R}$ with $|x| \leqslant r$ we have

$$f_l(x+y) \leqslant C f_l(y/2). \tag{A.49}$$

Proof. (i) As $|x| \leqslant r$, we have for every $y \in \mathbb{R}$

$$e^{-(x-y)^2} \leqslant e^{-y^2} e^{2r|y|} \leqslant e^{-y^2} e^{2r|y|} \leqslant C e^{-y^2/2} \tag{A.50}$$

with $C = e^{2r^2}$ by completing the square in the exponential.

(ii) We use part (i) with the constant from (A.50) to estimate

$$f_l(x+y) \leqslant 2e^{2r^2} \int_{-l}^{l} e^{-(\xi-y)^2/4} \frac{d\xi}{\sqrt{\pi}} \leqslant 2e^{2r^2} f_l(y/2), \tag{A.51}$$

which is the desired inequality. □

Lemma A.3.2. *Let g be the function defined in* (5.10)*. Then*

$$g(\lambda) \geqslant \frac{1}{4} \log \lambda \tag{A.52}$$

for all $\lambda \geqslant 8$.

Proof. We start by substituting $x \mapsto x/\lambda$, $y \mapsto y/\lambda$ and obtain

$$g(\lambda) = \int_0^{\lambda} \int_{-\lambda}^{0} \frac{\sin^2(x-y)}{\pi^2 (x-y)^2} \, dx \, dy. \tag{A.53}$$

A straightforward integration by parts shows

$$\begin{aligned} \int_{-\lambda}^{0} \frac{\sin^2(x-y)}{(x-y)^2} \, dx &= \left[-\frac{\sin^2(x-y)}{x-y} \right]_{x=-\lambda}^{0} + \int_{-\lambda}^{0} \frac{2\sin(x-y)\cos(x-y)}{x-y} \, dx \\ &= \frac{\sin^2 y}{y} - \frac{\sin^2(y+\lambda)}{y+\lambda} + \int_{-\lambda}^{0} \frac{\sin[2(x-y)]}{x-y} \, dx. \end{aligned} \tag{A.54}$$

In view of conducting the y-integration, we calculate

$$\begin{aligned} &\int_0^{\lambda} \int_{-\lambda}^{0} \frac{\sin[2(x-y)]}{x-y} \, dx \, dy = \frac{1}{2} \int_0^{2\lambda} \int_{-2\lambda} \frac{\sin(x-y)}{x-y} \, dx \, dy \\ &= \frac{1}{2} \int_0^{2\lambda} \int_0^{2\lambda} \frac{\sin(x+y)}{x+y} \, dx \, dy = \frac{1}{4} \int_0^{2\lambda} \int_0^{2\lambda} \int_{-1}^{1} e^{i(x+y)k} \, dk \, dx \, dy \\ &= \int_{-1}^{1} \frac{(e^{2i\lambda k} - 1)^2}{(2ik)^2} \, dk = \int_{-1}^{1} \frac{\sin^2 \lambda k}{k^2} e^{2i\lambda k} \, dk \\ &= 2 \int_0^1 \frac{\sin^2 \lambda k}{k^2} \cos(2\lambda k) \, dk = \int_0^1 \frac{(1-\cos(2\lambda k))\cos(2\lambda k)}{k^2} \, dk \\ &= \lambda \int_0^{\lambda} \frac{\cos(2k) - \frac{1}{2} - \frac{1}{2}\cos(4k)}{k^2} \, dk \\ &= \lambda \left(\int_0^{\lambda} \frac{\cos(2k) - 1}{k^2} \, dk + \frac{1}{2} \int_0^{\lambda} \frac{1-\cos(4k)}{k^2} \, dk \right) \\ &= \frac{\lambda}{2} \int_{\lambda/2}^{\lambda} \frac{1-\cos(4k)}{k^2} \, dk \geqslant 0. \end{aligned} \tag{A.55}$$

For $y > 0$, let Ci denote the cosine integral, that is,

$$\mathrm{Ci}(x) := \gamma + \log x + \int_0^x \frac{\cos t - 1}{t}\, dt = -\int_x^\infty \frac{\cos t}{t}\, dt. \tag{A.56}$$

Here, $\gamma \geqslant \frac{1}{2}$ denotes the Euler-Mascheroni constant. One easily checks that the function given by

$$F\colon (0,\infty) \to \mathbb{R}, \quad F(y) := \frac{1}{2}\left(\log y - \mathrm{Ci}(2y)\right) \tag{A.57}$$

satisfies $F'(y) = \sin^2(y)/y$. Furthermore, for $y > 0$ we calculate

$$\begin{aligned} F(\epsilon + y) - F(\epsilon) &= \frac{1}{2}\left(\log(\epsilon + y) - \mathrm{Ci}(2(\epsilon + y)) - \log \epsilon + \mathrm{Ci}(2\epsilon)\right) \\ &= \frac{1}{2}\left(\log(\epsilon + y) - \mathrm{Ci}(2(\epsilon + y)) + \gamma + \log 2 + \int_0^{2\epsilon} \frac{\cos t - 1}{t}\, dt\right) \\ &\xrightarrow{\epsilon \to 0} \frac{1}{2}\left(\log(2y) - \mathrm{Ci}(2y) + \gamma\right). \end{aligned} \tag{A.58}$$

Thus, integrating (A.54) with respect to y and using inequality (A.55) yields

$$\begin{aligned} g(\lambda) &\geqslant \int_0^\lambda \frac{\sin^2 y}{y} - \frac{\sin^2(y+\lambda)}{y+\lambda}\, dy \\ &= F(\lambda) - F(2\lambda) + \lim_{\epsilon \to 0}(F(\epsilon + \lambda) - F(\epsilon)) \\ &= \frac{1}{2}\left(\log \lambda - \mathrm{Ci}(2\lambda) - \log(2\lambda) + \mathrm{Ci}(4\lambda)\right) + \frac{1}{2}\left(\log(2\lambda) - \mathrm{Ci}(2\lambda) + \gamma\right) \\ &= -\mathrm{Ci}(2\lambda) + \frac{1}{2}\mathrm{Ci}(4\lambda) + \frac{1}{2}\log \lambda + \frac{\gamma}{2}, \end{aligned} \tag{A.59}$$

Now $|\mathrm{Ci}(x)| \leqslant \frac{1}{2}$ for each $x \geqslant 1$, so that

$$g(\lambda) \geqslant -\frac{1}{2} - \frac{1}{4} + \frac{1}{2}\log \lambda + \frac{1}{4} = \frac{1}{4}\log \lambda + \frac{1}{4}\log \lambda - \frac{1}{2} \geqslant \frac{1}{4}\log \lambda \tag{A.60}$$

for all $\lambda \geqslant e^2$. Since $e^2 \leqslant 8$, this establishes the claimed inequality. □

Lemma A.3.3. *Let I_n, J_m be the intervals defined in* (3.13) *and* (5.1), *respectively. For $n \in \mathbb{N}_0$, put*

$$c_{\alpha,n} := \begin{cases} n^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ \log(n+1) & \text{if } \alpha = 1. \end{cases} \tag{A.61}$$

Then, for all $n, m \in \mathbb{N}_0$ and $x \in I_n, y \in J_m$, we have

$$e^{-(x-y)^2/2} \leqslant C e^{-(c_{\alpha,n} - c_{\alpha,m})^2} \tag{A.62}$$

and

$$f_l^2\left(\frac{x+y}{2}\right) \leqslant C f_l^2\left(\frac{c_{\alpha,n} + c_{\alpha,m}}{2}\right). \tag{A.63}$$

Proof. Straightforward bounds comparing sums to integrals show that, for $\alpha = 1$ and each $m \in \mathbb{N}$ we have

$$\log(m+1) \leqslant a_m \leqslant 1 + \log(m), \tag{A.64}$$

so that for all $m \in \mathbb{N}_0$,

$$0 \leqslant a_m - \log(m+1) \leqslant 1. \tag{A.65}$$

On the other hand, for $0 < \alpha < 1$ we obtain

$$\frac{1}{1-\alpha}((m+1)^{1-\alpha} - 1) \leqslant a_m \leqslant \frac{1}{1-\alpha}(m^{1-\alpha} - 1) + 1, \tag{A.66}$$

so that

$$0 < \frac{1}{1-\alpha} - 1 \leqslant \frac{1}{1-\alpha} m^{1-\alpha} - a_m \leqslant \frac{1}{1-\alpha}(1 + m^{1-\alpha} - (1+m)^{1-\alpha}) \leqslant \frac{1}{1-\alpha}. \tag{A.67}$$

With these bounds we aim to replace a_m and a_n by their respective asymptotic expressions. We start with the case $\alpha = 1$. For $n, m \in \mathbb{N}_0$ and $x \in I_n$, $y \in J_m$ we write

$$x - y = 2\log(n+1) - 2\log(m+1) + C_{n,m}, \tag{A.68}$$

where $C_{n,m} = (x - 2\log(n+1)) + (2\log(m+1) - y)$. We can now estimate

$$\begin{aligned} |C_{n,m}| &\leqslant |x - 2\log(n+1)| + |y - 2\log(m+1)| \\ &\leqslant |x - 2a_n| + |2a_n - 2\log(n+1)| + |y - 2a_m| + |2a_m + 2\log(m+1)| \\ &\leqslant \xi_{n+1} + 2 + 2\xi_{m+1} + 2 \leqslant 7. \end{aligned} \tag{A.69}$$

On the other hand, if $\alpha = 1$,

$$x - y = \frac{2}{1-\alpha}(n^{1-\alpha} - m^{1-\alpha}) + C_{n,m} \tag{A.70}$$

with

$$C_{n,m} = \left(x - \frac{2}{1-\alpha} n^{1-\alpha}\right) + \left(\frac{2}{1-\alpha} m^{1-\alpha} - y\right). \tag{A.71}$$

Similarly to before,

$$\begin{aligned} |C_{n,m}| &\leqslant \left|x - \frac{2}{1-\alpha} n^{1-\alpha}\right| + \left|y - \frac{2}{1-\alpha} m^{1-\alpha}\right| \\ &\leqslant |x - 2a_n| + \left|\frac{2}{1-\alpha} n^{1-\alpha} - 2a_n\right| + |y - 2a_m| + \left|\frac{2}{1-\alpha} m^{1-\alpha} - 2a_m\right| \\ &\leqslant 3 + \frac{4}{1-\alpha}. \end{aligned} \tag{A.72}$$

We use Lemma A.3.1(i) and $1/(1-\alpha) > 1$ for $0 < \alpha < 1$ to deduce that

$$e^{-(x-y)^2/2} \leqslant \begin{cases} Ce^{-(\log(n+1) - \log(m+1))^2} & \text{if } \alpha = 1, \\ Ce^{-(n^{1-\alpha} - m^{1-\alpha})^2} & \text{if } 0 < \alpha < 1, \end{cases} \tag{A.73}$$

where the constants may depend on α. Equivalently,

$$e^{-(x-y)^2/2} \leqslant Ce^{-(c_{\alpha,n} - c_{\alpha,m})^2}. \tag{A.74}$$

By a similar argument employing Lemma A.3.1(ii), we find

$$f_l^2\left(\frac{x+y}{2}\right) \leqslant Cf_l^2\left(\frac{c_{\alpha,n} + c_{\alpha,m}}{2}\right). \tag{A.75}$$

□

B. Entanglement entropy

In this appendix, we give the precise mathematical definition of bipartite spatial entanglement entropy of a many-body quantum state and derive the convenient formula (B.50) for the case of the ground state of a non-interacting Fermi gas, which makes it possible to express the corresponding entanglement entropy in terms of the underlying single-particle Hamiltonian.

We assume all occuring Hilbert spaces to be separable.

B.1. Tensor products

B.1.1. Definition

We recall here the most important facts about the tensor product of Hilbert spaces. For proofs and further properties, see [Ara18, Chapter 2] and [RS80, Chapter II.2].

Given two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$, the *tensor product* $\mathscr{H}_1 \otimes \mathscr{H}_2$ of $\mathscr{H}_1$ and $\mathscr{H}_2$ is a Hilbert space together with a bilinear map

$$\otimes : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_1 \otimes \mathscr{H}_2, \tag{B.1}$$

such that the inner products satisfy

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{\mathscr{H}_1 \otimes \mathscr{H}_2} = \langle u_1, v_1 \rangle_{\mathscr{H}_1} \langle u_2, v_2 \rangle_{\mathscr{H}_2} \tag{B.2}$$

for each $u_1, v_1 \in \mathscr{H}_1$, $u_2, v_2 \in \mathscr{H}_2$ and such that

$$\mathscr{H}_1 \otimes \mathscr{H}_2 = \overline{\operatorname{span}\{u_1 \otimes u_2 : u_1 \in \mathscr{H}_1, u_2 \in \mathscr{H}_2\}}, \tag{B.3}$$

where the closure is taken with respect to the norm induced by the inner product on $\mathscr{H}_1 \otimes \mathscr{H}_2$. The elements $u_1 \otimes u_2 \in \mathscr{H}_1 \otimes \mathscr{H}_2$ are called *pure tensors*.

The tensor product is unique in the following sense: If $\mathscr{H}_1 \hat{\otimes} \mathscr{H}_2$ is another tensor product satisfying the above conditions, then the map $u_1 \hat{\otimes} u_2 \mapsto u_1 \otimes u_2$ extends to an isometric isomorphism.

If A_1 and A_2 are two bounded linear operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, there exists a unique *tensor product operator* $A_1 \otimes A_2$ on $\mathscr{H}_1 \otimes \mathscr{H}_2$ that acts on pure tensors $u_1 \otimes u_2 \in \mathscr{H}_1 \otimes \mathscr{H}_2$ as

$$A_1 \otimes A_2(u_1 \otimes u_2) = (A_1 u_1) \otimes (A_2 u_2). \tag{B.4}$$

The above construction for the tensor product of two Hilbert spaces can be extended naturally to the tensor product $\bigotimes_{j=1}^N \mathscr{H}_j$ of $N \in \mathbb{N}$ Hilbert spaces with $N \geqslant 3$, and similarly for the tensor product of operators.

B.1.2. Anti-symmetric tensor products

Physically, the N-fold tensor product is used to describe a system of N particles. To describe a system of *indistinguishable* particles, one uses either the symmetric (for *bosons*) or the anti-symmetric (for *fermions*) tensor product. This thesis is concerned with fermion systems, so we introduce only the anti-symmetric tensor product.

Let $N \in \mathbb{N}$, $\mathscr{H}$ be a Hilbert space and $\bigotimes_{j=1}^N \mathscr{H}$ the N-fold tensor product of $\mathscr{H}$ with itself. Let S_N denote the symmetric group, i.e. the set of permutations of $\{1,\dots,N\}$. For each $\sigma \in S_N$, there exists a unique unitary map

$$U_\sigma \colon \bigotimes_{j=1}^N \mathscr{H} \to \bigotimes_{j=1}^N \mathscr{H} \tag{B.5}$$

that acts on pure tensors $u_1 \otimes \cdots \otimes u_N \in \bigotimes_{j=1}^N \mathscr{H}$ as

$$U_\sigma(u_1 \otimes \cdots \otimes u_N) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(N)}. \tag{B.6}$$

Clearly, it satisfies $U_\sigma^{-1} = U_\sigma^* = U_{\sigma^{-1}}$ and $U_\sigma U_\tau = U_{\sigma\tau}$ for all $\sigma, \tau \in S_N$. We define the *anti-symmetrization operator* A_N on $\bigotimes_{j=1}^N \mathscr{H}$ by

$$A_N := \frac{1}{N!} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) U_\sigma. \tag{B.7}$$

The *N-fold anti-symmetric tensor product* of $\mathscr{H}$ with itself is defined by

$$\bigwedge_{j=1}^N \mathscr{H} := A_N \left[\bigotimes_{j=1}^N \mathscr{H} \right]. \tag{B.8}$$

We define the *exterior product* or *Slater determinant* of vectors $\psi_1, \dots, \psi_N \in \mathscr{H}$ as

$$\psi_1 \wedge \ldots \wedge \psi_N := \sqrt{N!} A_N(\psi_1 \otimes \ldots \otimes \psi_N) \in \bigwedge_{j=1}^N \mathscr{H}. \tag{B.9}$$

The Slater determinant has the important property that

$$\psi_{\sigma(1)} \wedge \ldots \wedge \psi_{\sigma(N)} = \operatorname{sgn}(\sigma) \psi_1 \wedge \ldots \wedge \psi_N \tag{B.10}$$

for any $\sigma \in S_N$ and that $\psi_1 \wedge \ldots \wedge \psi_N = 0$ whenever $\psi_i = \psi_j$ for some $i \neq j$ (see [Ara18, Theorem 2.9(v) and Proposition 2.10] for proofs). We will later also need the following

Lemma B.1.1. *Assume that $\varphi_1, \dots, \varphi_N$ is an orthonormal family in $\mathscr{H}$ and put $\mathscr{G} := \operatorname{span}\{\varphi_1, \dots, \varphi_N\}$. If $\psi_1, \dots, \psi_N \in \mathscr{G}$ is another orthonormal family spanning $\mathscr{G}$, there exists $\alpha \in \mathbb{R}$ such that*

$$\psi_1 \wedge \ldots \wedge \psi_N = e^{i\alpha} \varphi_1 \wedge \ldots \wedge \varphi_N. \tag{B.11}$$

Proof. Let $U = (U_{jk})_{1 \leqslant j,k \leqslant N} \in \mathbb{C}^{N \times N}$ be the corresponding unitary transition matrix, i.e.

$$\psi_j = \sum_{k=1}^N U_{jk} \varphi_k \tag{B.12}$$

for each $j = 1, \dots, N$. By definition of the Slater determinant,

$$\begin{aligned}
\psi_1 \wedge \ldots \wedge \psi_N &= \left(\sum_{k=1}^{N} U_{1k}\varphi_k\right) \wedge \ldots \wedge \left(\sum_{k=1}^{N} U_{Nk}\varphi_k\right) \\
&= \sum_{k \in \{1,\dots,N\}^N} U_{1k_1} \cdots U_{Nk_N} \varphi_{k_1} \wedge \ldots \wedge \varphi_{k_N} \\
&= \sum_{\sigma \in S_N} U_{1\sigma(1)} \cdots U_{N\sigma(N)} \varphi_{\sigma(1)} \wedge \ldots \wedge \varphi_{\sigma(N)} \\
&= \sum_{\sigma \in S_N} U_{1\sigma(1)} \cdots U_{N\sigma(N)} \operatorname{sgn}(\sigma) \varphi_1 \wedge \ldots \wedge \varphi_N \\
&= \det U \cdot \varphi_1 \wedge \ldots \wedge \varphi_N .
\end{aligned}$$

where we used that $\varphi_{k_1} \wedge \ldots \wedge \varphi_{k_N} = 0$ if $k_i = k_j$ for some i, j for the third equality, (B.10) for the penultimate and the Leibniz formula for the determinant for the last equality. Since $|\det U| = 1$, the claim follows. □

B.2. Partial traces

B.2.1. Definition of the partial trace

For a given unit vector $f \in \mathscr{H}_2$, define an operator $\Phi(f)$ by its action on pure tensors by

$$\Phi(f)(u \otimes v) := \langle f, v \rangle u. \tag{B.13}$$

Since $\|\Phi(f)\| = 1$, $\Phi(f)$ can be extended to a bounded linear map $\Phi(f)\colon \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1$. Its adjoint is given by

$$\Phi^*(f)\colon \mathscr{H}_1 \to \mathscr{H}_1 \otimes \mathscr{H}_2, \quad \Phi^*(f)u := u \otimes f. \tag{B.14}$$

Lemma B.2.1. *For any trace-class operator $T\colon \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1 \otimes \mathscr{H}_2$ and unit vector $f \in \mathscr{H}_2$, the operator $\Phi(f)T\Phi(f)^*\colon \mathscr{H}_1 \to \mathscr{H}_1$ is trace-class.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ be some arbitrary orthonormal families in $\mathscr{H}_1$. By definition,

$$\sum_{n\in\mathbb{N}} |\langle \varphi_n, \Phi(f)T\Phi(f)^*\psi_n \rangle| = \sum_{n\in\mathbb{N}} |\langle \varphi_n \otimes f, T\psi_n \otimes f \rangle| . \tag{B.15}$$

Note that $(\varphi_n \otimes f)_{n\in\mathbb{N}}, (\psi_n \otimes f)_{n\in\mathbb{N}}$ constitute orthonormal families in $\mathscr{H}_1 \otimes \mathscr{H}_2$, so the claim follows from Proposition 2.2.1(i). □

Proposition B.2.2 (Partial trace)**.** *Let $\mathscr{H}_1, \mathscr{H}_2$ be two Hilbert spaces, $\mathscr{H} := \mathscr{H}_1 \otimes \mathscr{H}_2$ and $T\colon \mathscr{H} \to \mathscr{H}$ a trace-class operator. For every orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{H}_2$, the series*

$$\operatorname{tr}_{\mathscr{H}_2}(T) := \sum_{n\in\mathbb{N}} \Phi(f_n)T\Phi(f_n)^* \tag{B.16}$$

converges in trace norm to some trace-class operator on $\mathscr{H}_1$, which is independent of the choice of the orthonormal basis $(f_n)_{n\in\mathbb{N}}$. It is the unique operator on $\mathfrak{S}_1(\mathscr{H}_1)$ satisfying

$$\operatorname{tr}(T(B \otimes \mathrm{id}_{\mathscr{H}_2})) = \operatorname{tr}(\operatorname{tr}_{\mathscr{H}_2}(T)B) \tag{B.17}$$

for all $B \in \mathscr{B}(\mathscr{H}_1)$, where $\mathscr{B}(\mathscr{H}_1)$ denotes the space of bounded operators on $\mathscr{H}_1$. We call $\operatorname{tr}_{\mathscr{H}_2}(T)$ *the* partial trace *of T.*

Proof. We start with proving the uniqueness assertion. Assume $\xi(T), \xi'(T)$ are two operators on $\mathfrak{S}_1(\mathscr{H}_1)$ satisfying (B.17). In particular, by linearity of the usual trace, this implies

$$\operatorname{tr}((\xi(T) - \xi'(T))B) = 0 \tag{B.18}$$

for all $B \in \mathscr{B}(\mathscr{H}_1)$ and in particular for all compact operators B on $\mathscr{H}_1$. By [BS87, Theorem 11.2.11], the map $S \mapsto \operatorname{tr}(S\,\cdot)$ is an isometric isomorphism from $\mathfrak{S}_1$ to $\mathfrak{S}_\infty^*$, where $\mathfrak{S}_\infty^*$ denotes the dual of the space of compact operators. It follows that $\xi(T) - \xi'(T) = 0$.

By the previous lemma, for each $n \in \mathbb{N}$, the operator $\Phi(f_n)T\Phi(f_n)^*$ is trace class and there exist orthonormal families $(\varphi_{n,m})_{m\in\mathbb{N}}, (\psi_{n,m})_{m\in\mathbb{N}}$ in $\mathscr{H}_1$ such that

$$\begin{aligned}\|\Phi(f_n)T\Phi(f_n)^*\|_1 &= \sum_{m\in\mathbb{N}} |\langle \varphi_{n,m}, \Phi(f_n)T\Phi(f_n)^*\psi_{n,m}\rangle| \\ &= \sum_{m\in\mathbb{N}} |\langle \varphi_{n,m} \otimes f_n, T\psi_{n,m} \otimes f_n\rangle| \,.\end{aligned}$$

The families $(\varphi_{n,m} \otimes f_n)_{n,m\in\mathbb{N}}, (\psi_{n,m} \otimes f_n)_{n,m\in\mathbb{N}}$ are orthonormal in $\mathscr{H}_1 \otimes \mathscr{H}_2$ and T is trace-class, so it follows that

$$\sum_{n\in\mathbb{N}} \|\Phi(f_n)T\Phi(f_n)^*\|_1 = \sum_{n,m\in\mathbb{N}} |\langle \varphi_{n,m} \otimes f_n, T\psi_{n,m} \otimes f_n\rangle| \leqslant \|T\|_1 < \infty. \tag{B.19}$$

Since $\mathfrak{S}_1$ is complete, (B.19) implies convergence of (B.16) to some trace-class operator on $\mathscr{H}_1$. Next, we prove that it satisfies (B.17). Let $B \in \mathscr{B}(\mathscr{H}_1)$. To that end, let $(g_n)_{n\in\mathbb{N}}$ be some orthonormal basis of $\mathscr{H}_1$. Then $(g_m \otimes f_n)_{n,m\in\mathbb{N}}$ is an orthonormal basis of $\mathscr{H}_1 \otimes \mathscr{H}_2$ (see [Ara18, Proposition 2.1(iii)]) and thus, by definition of the trace,

$$\begin{aligned}\operatorname{tr}(T(B \otimes \mathrm{id}_{\mathscr{H}_2})) &= \sum_{n,m\in\mathbb{N}} \langle g_m \otimes f_n, T(B \otimes \mathrm{id}_{\mathscr{H}_2})(g_m \otimes f_n)\rangle \\ &= \sum_{n,m\in\mathbb{N}} \langle g_m \otimes f_n, T(Bg_m \otimes f_n)\rangle \\ &= \sum_{n,m\in\mathbb{N}} \langle g_m, \Phi(f_n)T\Phi(f_n)^*Bg_m\rangle \\ &= \sum_{m\in\mathbb{N}} \left\langle g_m, \sum_{n\in\mathbb{N}} \Phi(f_n)T\Phi(f_n)^*Bg_m \right\rangle = \operatorname{tr}\left(\operatorname{tr}_{\mathscr{H}_2}(T)B\right),\end{aligned} \tag{B.20}$$

showing (B.17). As we showed in the beginning, there is at most one map with this property, so in particular (B.16) must be independent of the chosen orthonormal basis. This finishes the proof. □

B.2.2. Some properties of the partial trace

Proposition B.2.3 (Properties of the partial trace)**.** *The partial trace has the following properties.*

(i) The map
$$\operatorname{tr}_{\mathscr{H}_2} : \mathfrak{S}_1(\mathscr{H}_1 \otimes \mathscr{H}_2) \to \mathfrak{S}_1(\mathscr{H}_1) \tag{B.21}$$
is linear and continuous with respect to the trace norm.

(ii) $T \geqslant 0$ *implies* $\operatorname{tr}_{\mathscr{H}_2} T \geqslant 0$.

(iii) If $T = T_1 \otimes T_2$ *for some trace class operators* T_1, T_2*, we have* $\operatorname{tr}_{\mathscr{H}_2} T = T_1 \operatorname{tr} T_2$.

(iv) $\operatorname{tr}(\operatorname{tr}_{\mathscr{H}_2} T) = \operatorname{tr} T$.

(v) If $T = \langle \psi_1 \otimes \psi_2, \cdot \rangle \varphi_1 \otimes \varphi_2$ *for some* $\psi_1 \otimes \psi_2, \varphi_1 \otimes \varphi_2 \in \mathscr{H}_1 \otimes \mathscr{H}_2$*, then*
$$\operatorname{tr}_{\mathscr{H}_2}(T) = \sum_{n\in\mathbb{N}} \overline{\langle f_n, \psi_2 \rangle} \langle f_n, \varphi_2 \rangle \langle \psi_1, \cdot \rangle \varphi_1 \tag{B.22}$$
for any orthonormal basis $(f_n)_{n\in\mathbb{N}}$ *of* $\mathscr{H}_2$.

Proof. (i) Linearity is clear. Since
$$\|\operatorname{tr}_{\mathscr{H}_2} T\|_1 \leqslant \sum_{n\in\mathbb{N}} \|\Phi(f_n) T \Phi(f_n)^*\|_1 \leqslant \|T\|_1, \tag{B.23}$$
continuity follows from (B.19).

(ii) Let $\varphi \in \mathscr{H}_1$. Then
$$\langle \varphi, \operatorname{tr}_{\mathscr{H}_2} T\varphi \rangle = \sum_{n\in\mathbb{N}} \langle \varphi, \Phi(f_n) T \Phi(f_n)^* \varphi \rangle = \sum_{n\in\mathbb{N}} \langle \varphi \otimes f_n, T \varphi \otimes f_n \rangle \geqslant 0 \tag{B.24}$$
by the positivity of T.

(iii) For $\varphi \in \mathscr{H}_1$, we have
$$\begin{aligned} \operatorname{tr}_{\mathscr{H}_2}(T_1 \otimes T_2)\varphi &= \sum_{n\in\mathbb{N}} \Phi(f_n)(T_1 \otimes T_2)(\varphi \otimes f_n) \\ &= \sum_{n\in\mathbb{N}} \Phi(f_n)(T_1\varphi \otimes T_2 f_n) \\ &= \sum_{n\in\mathbb{N}} \langle f_n, T_2 f_n \rangle T_1 \varphi = T_1 \varphi \operatorname{tr} T_2. \end{aligned} \tag{B.25}$$
(iv) Follows from (B.17) by choosing $B = \mathrm{id}_{\mathscr{H}_1}$.

(v) By definition,
$$\begin{aligned} \operatorname{tr}_{\mathscr{H}_2}(T) &= \sum_{n\in\mathbb{N}} \langle \Phi(f_n) \psi_1 \otimes \psi_2, \cdot \rangle \Phi(f_n) \phi_1 \otimes \phi_2 \\ &= \sum_{n\in\mathbb{N}} \langle \langle f_n, \psi_2 \rangle \psi_1, \cdot \rangle \langle f_n, \varphi_2 \rangle \varphi_1 = \sum_{n\in\mathbb{N}} \overline{\langle f_n, \psi_2 \rangle} \langle f_n, \varphi_2 \rangle \langle \psi_1, \cdot \rangle \varphi_1, \end{aligned} \tag{B.26}$$
finishing the proof. □

B.3. Fermionic Fock space

B.3.1. Definition

For a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}_0}$ of Hilbert spaces, the set

$$\bigoplus_{n\in\mathbb{N}_0} \mathscr{H}_n := \left\{ (\Psi_n)_{n\in\mathbb{N}_0} : \Psi_n \in \mathscr{H}_n \text{ for each } n \in \mathbb{N}_0, \sum_{n\in\mathbb{N}_0} \|\Psi_n\|^2 < \infty \right\} \tag{B.27}$$

equipped with the inner product

$$\langle \Psi, \Phi \rangle := \sum_{n=0}^{\infty} \langle \Psi_n, \Phi_n \rangle, \quad \Psi, \Phi \in \bigoplus_{n\in\mathbb{N}_0} \mathscr{H}_n \tag{B.28}$$

is a Hilbert space, called *infinite direct sum Hilbert space.*

Definition B.3.1 (Fermionic Fock space)**.** Let $\mathscr{H}$ be a Hilbert space. The *fermionic* or *anti-symmetric Fock space* associated to $\mathscr{H}$ is defined by

$$\mathscr{F}_a(\mathscr{H}) := \bigoplus_{n\in\mathbb{N}_0} \bigwedge_{j=1}^{n} \mathscr{H}, \tag{B.29}$$

where we set $\bigwedge_{j=1}^{0} \mathscr{H} := \mathbb{C}$.

B.3.2. Vacuum vector, creation operator and further definitions

We define the *vacuum* vector by

$$\Omega_{\mathscr{H}} := (1, 0, 0, \ldots) \in \mathscr{F}_a(\mathscr{H}). \tag{B.30}$$

For a given $f \in \mathscr{H}$, the *fermionic creation operator* $a^*(f)$ on $\mathscr{F}_a(\mathscr{H})$ is defined by

$$(a^*(f)\Psi)_0 := 0, \quad (a^*(f)\Psi)_n := \sqrt{n} A_n(f \otimes \Psi_{n-1}) \tag{B.31}$$

for each Ψ in the domain

$$D(a^*(f)) := \left\{ (\Psi_n)_{n\in\mathbb{N}_0} \in \mathscr{F}_a(\mathscr{H}) : \sum_{n\in\mathbb{N}} n \|A_n(f \otimes \Psi_{n-1})\|^2 < \infty \right\}. \tag{B.32}$$

It is straightforward to see that for $\varphi_1, \ldots, \varphi_N \in \mathscr{H}$ and $p \geqslant 1$, we have

$$(a^*(\varphi_1) \cdots a^*(\varphi_n) \Omega_{\mathscr{H}})_p = \delta_{np} \varphi_1 \wedge \ldots \wedge \varphi_n. \tag{B.33}$$

With a slight abuse of notation, we just write $a^*(\varphi_1) \cdots a^*(\varphi_n) \Omega_{\mathscr{H}} = \delta_{np} \varphi_1 \wedge \ldots \wedge \varphi_n$.

Lemma B.3.2. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathscr{H}$. Then*

$$\{\Omega_{\mathscr{H}}, a^*(\varphi_{j_1}) \cdots a^*(\varphi_{j_p}) \Omega_{\mathscr{H}} : p \in \mathbb{N}, j_1 < \ldots < j_p, j_i \in \mathbb{N}, i = 1, \ldots, p\} \tag{B.34}$$

is an orthonormal basis of $\mathscr{F}_a(\mathscr{H})$.

Proof. See [Ara18, Theorem 6.11]. □

Finally, we introduce the bounded self-adjoint operator $(-1)^{\mathscr{N}}$ on $\mathscr{F}_a(\mathscr{H})$ by

$$((-1)^{\mathscr{N}}\Psi)^{(n)} := (-1)^n\Psi^{(n)} \tag{B.35}$$

for all $\Psi \in \mathscr{F}_a(\mathscr{H})$ and $n \in \mathbb{N}_0$ (the notation $\mathscr{N}$ should be suggestive for the number of particles).

B.3.3. Fermionic Fock space of a bipartition

For a direct sum $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ of Hilbert spaces, there exists a natural identification

$$\mathscr{F}_a(\mathscr{H}) \cong \mathscr{F}_a(\mathscr{H}_1) \otimes \mathscr{F}_a(\mathscr{H}_2). \tag{B.36}$$

We refer to [Ara18, Theorem 6.12] for a proof of this general result. Here, we will formulate the result for the special case of L^2 spaces:

Let $\Lambda \subset \Gamma \subset \mathbb{R}^d$ be two measurable sets and $\Lambda^c := \Gamma \setminus \Lambda$. We then have $L^2(\Gamma) = L^2(\Lambda) \oplus L^2(\Lambda^c)$ and the lemma below provides a natural identification of the corresponding Fock spaces

$$\mathscr{F}_a(L^2(\Gamma)) \simeq \mathscr{F}_a(L^2(\Lambda)) \otimes \mathscr{F}_a(L^2(\Lambda^c)). \tag{B.37}$$

With slight abuse of notation, we will denote by $a^*(f)$ the creation operator (B.31) on either of the spaces $\mathscr{F}_a(L^2(\Gamma)), \mathscr{F}_a(L^2(\Lambda)), \mathscr{F}_a(L^2(\Lambda^c))$ (with a normalized function $f \in L^2(\Gamma), L^2(\Lambda), L^2(\Lambda^c)$, respectively). Furthermore we consider any function $f \in L^2(\Lambda)$ as a function in $L^2(\Gamma)$ by setting $f(x) = 0$ for all $x \in \Lambda^c$ and similarly for $f \in L^2(\Lambda^c)$.

Lemma B.3.3. *There exists a unique unitary operator*

$$U : \mathscr{F}_a(L^2(\Gamma)) \to \mathscr{F}_a(L^2(\Lambda)) \otimes \mathscr{F}_a(L^2(\Lambda^c)) \tag{B.38}$$

such that

$$U\Omega_{L^2(\Gamma)} = \Omega_{L^2(\Lambda)} \otimes \Omega_{L^2(\Lambda^c)}, \tag{B.39}$$

and, for all $n, m \in \mathbb{N}_0$ and $f_1, \dots f_n \in L^2(\Lambda), g_1, \dots, g_m \in L^2(\Lambda^c)$ we have

$$\begin{aligned} &Ua^*(f_1)\cdots a^*(f_n)a^*(g_1)\cdots a^*(g_m)\Omega_{L^2(\Gamma)} \\ &= a^*(f_1)\cdots a^*(f_n)\Omega_{L^2(\Lambda)} \otimes a^*(g_1)\cdots a^*(g_m)\Omega_{L^2(\Lambda^c)}. \end{aligned} \tag{B.40}$$

Moreover, the following holds: For any functions $f \in L^2(\Lambda), g \in L^2(\Lambda^c)$,

$$Ua^*(f+g)U^* = a^*(f) \otimes \mathrm{id}_{\mathscr{F}_a(L^2(\Lambda^c))} + (-1)^{\mathscr{N}} \otimes a^*(g), \tag{B.41}$$

where $(-1)^{\mathscr{N}}$ is the operator defined in (B.35).

Remark B.3.4. We can write (B.40) in the more suggestive way

$$Uf_1 \wedge \cdots \wedge f_n \wedge g_1 \wedge \cdots \wedge g_m = f_1 \wedge \cdots \wedge f_n \otimes g_1 \wedge \cdots \wedge g_m. \tag{B.42}$$

B.4. Entanglement entropy and the single-particle formula

B.4.1. Definition for a general pure state

Given a trace-class operator T on $\mathscr{F}_a(L^2(\Gamma))$, using the unitary map U from Lemma B.3.3, we obtain an operator UTU^* on $\mathscr{F}_a(L^2(\Lambda)) \otimes \mathscr{F}_a(L^2(\Lambda^c))$. We can thus naturally define the partial trace of T as

$$\operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda^c))}(T) := \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda^c))}(UTU^*), \tag{B.43}$$

where the right hand side is the usual partial trace (B.16).

Before introducing entanglement entropy, we give the general definition of a state.

Definition B.4.1 (States)**.** Let $\mathscr{H}$ be a Hilbert space. A *state* ρ is a positive, self-adjoint trace-class operator with $\operatorname{tr}\rho = 1$. By the spectral theorem for compact operators, it can be written as

$$\rho = \sum_{j\in\mathbb{N}} \lambda_j \langle u_j, \cdot\rangle u_j \tag{B.44}$$

where $(u_j)_{j\in\mathbb{N}}$ is an orthonormal basis of eigenfunctions of ρ and $(\lambda_j)_{j\in\mathbb{N}}$ the corresponding sequence of eigenvalues (counted with multiplicity). A state is called *pure* if

$$\rho = \langle u, \cdot\rangle u \tag{B.45}$$

for some normalized $u \in \mathscr{H}$.

Definition B.4.2 (Entanglement entropy)**.** Let $\Phi \in \mathscr{F}_a(L^2(\Gamma))$ be a many-particle wavefunction corresponding to a pure state $\langle\Phi, \cdot\rangle\Phi$ on the fermionic Fock space $\mathscr{F}_a(L^2(\Gamma))$. We define the state reduced to the subsystem $\mathscr{F}_a(L^2(\Lambda))$ by

$$\rho = \rho(\Phi, \Lambda) := \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda^c))}(\langle\Phi, \cdot\rangle\Phi). \tag{B.46}$$

For $\gamma > 0$, define the (spatially bipartite) *entanglement entropies* of the state ρ with respect to the region Λ as

$$S_\gamma = S_\gamma(\Phi, \Lambda) := \begin{cases} \dfrac{1}{1-\gamma} \log \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda)}(\rho^\gamma) & \text{if } \gamma \neq 1, \\ -\operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda))}(\rho \log \rho) & \text{if } \gamma = 1. \end{cases} \tag{B.47}$$

We refer to $S_1(\Phi, \Lambda)$ as *von Neumann entanglement entropy* and to $S_\gamma(\Phi, \Lambda)$ with $\gamma \neq 1$ as *γ-Rényi entanglement entropy.*

Remark B.4.3. The reduced state ρ defined in (B.46) is indeed a state in the sense of Definition B.4.1 by Proposition B.2.3(ii) and (iv) and therefore the entanglement entropy S_γ is a well-defined number in $[0, \infty]$.

B.4.2. Entanglement entropy of a many-particle fermion ground state

We are most interested in the case of $\langle\Phi,\cdot\rangle\Phi$ being the ground state of a system of N non-interacting fermions.

Let $H\colon D(H)\to L^2(\Gamma)$ be a Hamiltonian on $L^2(\Gamma)$ describing a single particle of the system. Assume that H has purely discrete spectrum, that is, $\sigma(H)$ consists only of isolated eigenvalues of finite multiplicity. Let $(\lambda_j)_{j\in\mathbb{N}}$ be the increasing sequence of eigenvalues (counted with multiplicity) and denote by $(\varphi_j)_{j\in\mathbb{N}}$ the orthonormal sequence of corresponding eigenfunctions. Given a *Fermi energy* $E_F\in\mathbb{R}$, let

$$N = N(E_F) := |\{j\in\mathbb{N} : \lambda_j < E_F\}| < \infty \tag{B.48}$$

be the number of eigenvalues of H less than E_F. The ground state of the Fermi gas with Fermi energy E_F is given by $\langle\Phi,\cdot\rangle\Phi$, with the N-particle wave function

$$\Phi = \Phi(E_F,\Gamma,H) := \left[\prod_{j=1}^{N} a^*(\varphi_j)\right]\Omega_{L^2(\Gamma)} = \varphi_1\wedge\ldots\wedge\varphi_N \in \mathscr{F}_a(L^2(\Gamma)). \tag{B.49}$$

The entanglement entropy of $\langle\Phi,\cdot\rangle\Phi$ can be expressed purely in terms of the one-particle Hamiltonian H:

Theorem B.4.4 (Single-particle formula)**.** *Let $\Lambda\subset\Gamma$ measurable. For each $\gamma>0$, we have*

$$S_\gamma(\Phi,\Lambda) = \operatorname{tr}_{L^2(\Gamma)} h_\gamma(1_\Lambda 1_{]-\infty,E_F[}(H)1_\Lambda), \tag{B.50}$$

where h_γ is given by (2.65)*.*

To prove the single-particle formula, we need the following lemma from linear algebra. It allows us to find an orthonormal basis of the space spanned by the eigenfunctions $\varphi_1,\ldots,\varphi_N\in L^2(\Gamma)$ in terms of functions in $L^2(\Lambda)$ and $L^2(\Lambda^c)$, respectively.

Lemma B.4.5. *Let $\varphi_1,\ldots,\varphi_N$ be an orthonormal family in some Hilbert space $\mathscr{H}$ and $\mathscr{G}:=\operatorname{span}\{\varphi_1,\ldots,\varphi_N\}$. Assume $\mathscr{H}=Y\oplus Y^\perp$ for some closed subspace $Y\subset\mathscr{H}$. Then there exist a unitary map $U\colon\mathscr{G}\to\mathscr{G}$, an orthonormal family $f_1,\ldots,f_N\in Y$, an orthonormal family $g_1,\ldots,g_N\in Y^\perp$ and numbers $d_1,\ldots,d_N\in[0,1]$ such that*

$$U\varphi_j = \sqrt{d_j}f_j + \sqrt{1-d_j}g_j \tag{B.51}$$

for each $j=1,\ldots,N$.

Proof. Let P_Y denote the orthogonal projection onto Y. Note that the operator $P_{\mathscr{G}}P_Y:\mathscr{G}\to\mathscr{G}$ is self-adjoint. By the spectral theorem, we find an orthonormal basis $\psi_1,\ldots,\psi_N\in\mathscr{G}$ of eigenvectors of $P_{\mathscr{G}}P_Y$ and $d_1,\ldots,d_N\in\mathbb{R}$ such that

$$P_{\mathscr{G}}P_Y = \sum_{j=1}^{N} d_j\langle\psi_j,\cdot\rangle\psi_j. \tag{B.52}$$

Since $\|P_{\mathscr{G}}P_Y\| \leqslant 1$, we have $d_j \in [0,1]$ for all $j = 1, \ldots, N$. Assume for the moment that $d_j \in]0,1[$ and put

$$f_j := \frac{1}{\sqrt{d_j}} P_Y \psi_j, \quad g_j := \frac{1}{\sqrt{1-d_j}} P_{Y^\perp} \psi_j \tag{B.53}$$

for each $j = 1, \ldots, N$. We claim that $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ form orthonormal systems in Y and $Y^\perp$, respectively. Indeed, since $P_{\mathscr{G}}\psi_k = \psi_k$, we have

$$\begin{aligned}
\langle f_k, f_j \rangle &= \frac{1}{\sqrt{d_k d_j}} \langle P_Y \psi_k, P_Y \psi_j \rangle \\
&= \frac{1}{\sqrt{d_k d_j}} \langle P_{\mathscr{G}} \psi_k, P_Y \psi_j \rangle \\
&= \frac{1}{\sqrt{d_k d_j}} \langle \psi_k, P_{\mathscr{G}} P_Y \psi_j \rangle = \frac{d_j}{\sqrt{d_k d_j}} \langle \psi_k, \psi_j \rangle = \delta_{kj},
\end{aligned} \tag{B.54}$$

and similarly, using $P_{Y^\perp} = 1 - P_Y$,

$$\begin{aligned}
\langle g_k, g_j \rangle &= \frac{1}{\sqrt{(1-d_k)(1-d_j)}} \langle \psi_k, P_{\mathscr{G}}(1-P_Y)\psi_j \rangle \\
&= \frac{1-d_j}{\sqrt{(1-d_k)(1-d_j)}} \langle \psi_k, \psi_j \rangle = \delta_{kj}
\end{aligned} \tag{B.55}$$

for every $k, j = 1, \ldots, N$. Now if some $d_j = 1$ for some j, it is immediate from (B.52) that $P_{\mathscr{G}}P_Y\psi_j = \psi_j \in \mathscr{G}$ and $P_{\mathscr{G}}P_{Y^\perp}\psi_j = 0$. We then put $f_j := P_Y\psi_j$, $g_j := 0$ instead of (B.53) and the rest works exactly as before. Similarly, we proceed in the case $d_j = 0$ for some j. Finally, define $U\varphi_j := \psi_j$ and extend by linearity to a map $U\colon \mathscr{G} \to \mathscr{G}$. By construction, U is unitary and satisfies (B.51). □

Proof of Theorem B.4.4. We apply Lemma B.4.5 with $\mathscr{H} = L^2(\Gamma)$, $Y = L^2(\Lambda)$, and $\mathscr{G}$ the space spanned by the eigenfunctions $\varphi_1, \ldots, \varphi_N$. We obtain orthonormal families $f_1, \ldots, f_N \in L^2(\Lambda)$, $g_1, \ldots, g_N \in L^2(\Lambda^c)$, numbers $d_1, \ldots, d_N \in [0,1]$ and a unitary map $U\colon \mathscr{G} \to \mathscr{G}$ such that

$$U\varphi_j = \sqrt{d_j} f_j + \sqrt{1-d_j} g_j, \quad j = 1, \ldots, N. \tag{B.56}$$

Next, Lemma B.1.1 allows us to write

$$\begin{aligned}
\Phi &= e^{i\alpha}(\sqrt{d_1} f_1 + \sqrt{1-d_1} g_1) \wedge \ldots \wedge (\sqrt{d_N} f_N + \sqrt{1-d_N} g_N) \\
&= e^{i\alpha} \left[\prod_{j=1}^{N} a^*(\sqrt{d_j} f_j + \sqrt{1-d_j} g_j) \right] \Omega_{L^2(\Gamma)}
\end{aligned} \tag{B.57}$$

for some $\alpha \in \mathbb{R}$. By definitions (B.46) and (B.43), we first have to calculate the reduced state

$$\rho = \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda^c))} \langle \Phi, \cdot \rangle \Phi = \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda^c))} \langle U\Phi, \cdot \rangle U\Phi. \tag{B.58}$$

Using properties (B.39) and (B.41) of U, we obtain

$$
\begin{aligned}
e^{-i\alpha}U\Phi &= U\left[\prod_{j=1}^{N} a^*(\sqrt{d_j}f_j + \sqrt{1-d_j}g_j)\right]\Omega_{L^2(\Gamma)} \\
&= \left[\prod_{j=1}^{N} Ua^*(\sqrt{d_j}f_j + \sqrt{1-d_j}g_j)U^*\right]U\Omega_{L^2(\Gamma)} \\
&= \prod_{j=1}^{N}\Big[a^*(\sqrt{d_j}f_j)\otimes \mathrm{id}_{\mathscr{F}_a(L^2(\Lambda^c))} + (-1)^{\mathscr{N}}\otimes a^*(\sqrt{1-d_j}g_j)\Big]\,\Omega_{L^2(\Lambda)}\otimes\Omega_{L^2(\Lambda^c)} \\
&= \sum_{\nu\in\{0,1\}^N}\prod_{j=1}^{N}\Big[\Big(a^*(\sqrt{d_j}f_j)\otimes\mathrm{id}_{\mathscr{F}_a(L^2(\Lambda^c))}\Big)^{\nu_j} \\
&\qquad\qquad \times\Big((-1)^{\mathscr{N}}\otimes a^*(\sqrt{1-d_j}g_j)\Big)^{1-\nu_j}\Big]\,\Omega_{L^2(\Lambda)}\otimes\Omega_{L^2(\Lambda^c)} \\
&= \sum_{\nu\in\{0,1\}^N}\sqrt{\eta_\nu}\left[\prod_{j=1}^{N} a^*(f_j)^{\nu_j}\left((-1)^{\mathscr{N}}\right)^{1-\nu_j}\right]\otimes\left[\prod_{j=1}^{N} a^*(g_j)^{1-\nu_j}\right]\Omega_{L^2(\Lambda)}\otimes\Omega_{L^2(\Lambda^c)} \\
&= \sum_{\nu\in\{0,1\}^N}\sqrt{\eta_\nu}\left[\prod_{j=1}^{N} a^*(f_j)^{\nu_j}\left((-1)^{\mathscr{N}}\right)^{1-\nu_j}\right]\Omega_{L^2(\Lambda)}\otimes\left[\prod_{j=1}^{N} a^*(g_j)^{1-\nu_j}\right]\Omega_{L^2(\Lambda^c)}, \qquad \text{(B.59)}
\end{aligned}
$$

where we put $\eta_\nu := \prod_{j=1}^{N} d_j^{\nu_j}(1-d_j)^{1-\nu_j}$ for $\nu\in\{0,1\}^N$. With the abbreviations

$$
\Psi_\nu^{(1)} := \left[\prod_{j=1}^{N} a^*(f_j)^{\nu_j}\left((-1)^{\mathscr{N}}\right)^{1-\nu_j}\right]\Omega_{L^2(\Lambda)}, \qquad \text{(B.60)}
$$

$$
\Psi_\nu^{(2)} := \left[\prod_{j=1}^{N} a^*(g_j)^{1-\nu_j}\right]\Omega_{L^2(\Lambda^c)}, \qquad \text{(B.61)}
$$

the above calculation yields

$$
\langle U\Phi,\cdot\rangle U\Phi = \sum_{\substack{\nu\in\{0,1\}^N\\ \tilde\nu\in\{0,1\}^N}}\sqrt{\eta_\nu\eta_{\tilde\nu}}\left\langle\Psi_\nu^{(1)}\otimes\Psi_\nu^{(2)},\cdot\right\rangle\Psi_{\tilde\nu}^{(1)}\otimes\Psi_{\tilde\nu}^{(2)}. \qquad \text{(B.62)}
$$

Extend $g_1,\dots,g_N$ to an orthonormal basis $(g_j)_{j\in\mathbb{N}}$ of $L^2(\Lambda^c)$. Recall that the vectors $\Xi_\mu := \left[\prod_{j\in\mathbb{N}} a^*(g_j)^{\mu_j}\right]\Omega_{L^2(\Lambda^c)}$, where $\mu\in\{0,1\}^{\mathbb{N}}$ with only finitely many terms non-zero, constitute an orthonormal basis of $\mathscr{F}_a(L^2(\Lambda^c))$. In particular, $\langle\Xi_\mu,\Psi_\nu^{(2)}\rangle = 1$ if $\mu_j = 1-\nu_j$ for all $j\in\{1,\dots,N\}$ and 0 otherwise. Thus, using formula (B.22) for the partial trace, we get

$$
\begin{aligned}
&\mathrm{tr}_{\mathscr{F}_a(L^2(\Lambda^c))}\langle U\Phi,\cdot\rangle U\Phi \\
&= \sum_{\substack{\nu\in\{0,1\}^N\\ \tilde\nu\in\{0,1\}^N}}\sqrt{\eta_\nu\eta_{\tilde\nu}}\sum_{\mu\in\{0,1\}^{\mathbb{N}}}\overline{\langle\Xi_\mu,\Psi_\nu^{(2)}\rangle}\langle\Xi_\mu,\Psi_{\tilde\nu}^{(2)}\rangle\langle\Psi_\nu^{(1)},\cdot\rangle\Psi_{\tilde\nu}^{(1)} \\
&= \sum_{\nu\in\{0,1\}^N}\eta_\nu\langle\Psi_\nu^{(1)},\cdot\rangle\Psi_\nu^{(1)}. \qquad \text{(B.63)}
\end{aligned}
$$

Since $(-1)^{\mathscr{N}} a^*(f) = -a^*(f)(-1)^{\mathscr{N}}$, we have

$$\Psi_\nu^{(1)} = (-1)^p \left[\prod_{j=1}^N a^*(f_j)^{\nu_j}\right] \Omega_{L^2(\Lambda)} \tag{B.64}$$

for some p depending on ν, which implies

$$\rho = \sum_{\nu \in \{0,1\}^N} \eta_\nu \left\langle \left[\prod_{j=1}^N a^*(f_j)^{\nu_j}\right] \Omega_{L^2(\Lambda)}, \cdot \right\rangle \left[\prod_{j=1}^N a^*(f_j)^{\nu_j}\right] \Omega_{L^2(\Lambda)}. \tag{B.65}$$

Notice that this is just the spectral representation of the reduced state ρ, making it particularly convenient to calculate functions of its trace: Indeed, complete $f_1, \ldots, f_N$ to an orthonormal basis $(f_j)_{j\in\mathbb{N}}$ of $L^2(\Lambda)$. By Lemma B.3.2,

$$\begin{aligned}&\left\{\Omega_{\mathscr{H}}, a^*(f_{j_1})\cdots a^*(f_{j_p})\Omega_{\mathscr{H}} : p \in \mathbb{N}, j_1 < \ldots < j_p, j_i \in \mathbb{N}, i = i, \ldots, p\right\} \\ &= \left\{ \left[\prod_{j\in\mathbb{N}} a^*(f_j)^{\nu_j}\right] \Omega_{L^2(\Lambda)} : \nu \in \{0,1\}^{\mathbb{N}},\ \nu_j = 0 \text{ for almost every } j \in \mathbb{N} \right\}\end{aligned} \tag{B.66}$$

constitutes a basis of $\mathscr{F}_a(L^2(\Lambda))$. Hence, by the continuous functional calculus and the properties of the trace, for all continuous functions $\zeta\colon [0,1] \to \mathbb{C}$ we see that

$$\operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda)} \zeta(\rho) = \sum_{\nu\in\{0,1\}^N} \zeta(\eta_\nu). \tag{B.67}$$

For the calculation of the entropy $S_\gamma(\langle\Phi,\cdot\rangle\Phi, \Lambda)$, we start with the case $\gamma = 1$, where we notice that

$$\log \eta_\nu = \log \left[\prod_{j=1}^N d_j^{\nu_j}(1-d_j)^{1-\nu_j}\right] = \sum_{j=1}^N \nu_j \log d_j + (1-\nu_j)\log(1-d_j), \tag{B.68}$$

and, for fixed $j \in \{1, \ldots, N\}$,

$$1 = \prod_{\substack{k=1\\k\neq j}}^N (d_k + (1-d_k)) = \sum_{\substack{\nu_l\in\{0,1\},\\ l\in\{1,\ldots,N\}\setminus\{j\}}} \prod_{\substack{k=1\\k\neq j}}^N d_k^{\nu_k}(1-d_k)^{1-\nu_k}. \tag{B.69}$$

Using these identities and (B.67) with $\zeta(x) = -x\log x$, we get

$$\begin{aligned} S_1(\Phi,\Lambda) &= -\operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda))} \rho\log\rho = -\sum_{\nu\in\{0,1\}^N} \eta_\nu \log\eta_\nu \\ &= -\sum_{j=1}^N \sum_{\nu\in\{0,1\}^N} \left[\prod_{k=1}^N d_k^{\nu_k}(1-d_k)^{1-\nu_k}\right] (\nu_j \log d_j + (1-\nu_j)\log(1-d_j)) \\ &= -\sum_{j=1}^N \sum_{\nu_j\in\{0,1\}} \left(d_j^{\nu_j}(1-d_j)^{1-\nu_j}\right)(\nu_j\log d_j + (1-\nu_j)\log(1-d_j)) \\ &= -\sum_{j=1}^N (1-d_j)\log(1-d_j) + d_j\log d_j = \sum_{j=1}^N h_1(d_j). \end{aligned} \tag{B.70}$$

Similarly, if $\gamma \neq 1$, we have the identity

$$\prod_{j=1}^{N}(d_j^\gamma + (1-d_j)^\gamma) = \sum_{\nu\in\{0,1\}^N}\prod_{j=1}^{N}(d_j^\gamma)^{\nu_j}((1-d_j)^\gamma)^{1-\nu_j}, \tag{B.71}$$

so using (B.67) with $\zeta(x) = x^\gamma$ yields

$$\begin{aligned}(1-\gamma)S_\gamma(\langle\Phi,\cdot\rangle\Phi,\Lambda) &= \log \operatorname{tr}_{\mathscr{F}_a(L^2(\Lambda))}(\rho^\gamma) = \log \sum_{\nu\in\{0,1\}^N}\eta_\nu^\gamma \\ &= \log \sum_{\nu\in\{0,1\}^N}\prod_{j=1}^{N}(d_j^\gamma)^{\nu_j}((1-d_j)^\gamma)^{1-\nu_j} \\ &= \sum_{j=1}^{N}\log(d_j^\gamma + (1-d_j)^\gamma) = (1-\gamma)\sum_{j=1}^{N}h_\gamma(d_j).\end{aligned} \tag{B.72}$$

Since the d_j (counted with multiplicity) are by construction (B.52) the eigenvalues of the operator $P_{\mathscr{G}}1_\Lambda$ on $\mathscr{G}$, the eigenvalues of the operator $P_{\mathscr{G}}1_\Lambda P_{\mathscr{G}}$ on $L^2(\Gamma)$ are given by the d_j (counted with multiplicity) and 0. Furthermore, the operators $P_{\mathscr{G}}1_\Lambda P_{\mathscr{G}} = (1_\Lambda P_{\mathscr{G}})^*(1_\Lambda P_{\mathscr{G}})$ and $1_\Lambda P_{\mathscr{G}}1_\Lambda = (1_\Lambda P_{\mathscr{G}})(1_\Lambda P_{\mathscr{G}})^*$ on $L^2(\Gamma)$ share the same non-zero eigenvalues (counted with multiplicity), see (2.42). Thus, the eigenvalues of $1_\Lambda P_{\mathscr{G}}1_\Lambda$ besides 0 are given by the $h(d_j)$ (counted with multiplicity) and $h(0) = 0$. Therefore,

$$\operatorname{tr}_{L^2(\Gamma)} h(1_\Lambda P_{\mathscr{G}}1_\Lambda) = \sum_{j=1}^{N}h(d_j), \tag{B.73}$$

which combined with (B.70), (B.72) gives

$$S_\gamma(\Phi,\Lambda) = \operatorname{tr}_{L^2(\Gamma)} h_\gamma(1_\Lambda P_{\mathscr{G}}1_\Lambda). \tag{B.74}$$

Finally, by the assumptions on H we have $P_{\mathscr{G}} = 1_{]-\infty,E_F[}(H)$, leading to the desired formula (B.50). □

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Eidesstattliche Versicherung

(Gemäß § 8 Abs. 2 Nr. 5 der Promotionsordnung vom 12.07.11)

Hiermit versichere ich an Eides statt, dass die vorliegende Dissertation von mir eigenständig und ohne unerlaubte Hilfe angefertigt wurde.

München, den 31. August 2025

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Otto Leonard Wetzel