On perturbative corrections to the Einstein-Hilbert term in Type-IIB orientifolds

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München 2025

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Dissertation
der Fakultät für Physik
Ludwig–Maximilians–Universität
München

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München, 04/Sep/2025

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Zweitgutachter: Prof. Dr. Dieter Lüst
Datum der mündlichen Prüfung: 06/Nov/2025

Abstract

We study quantum corrections to the Einstein-Hilbert term in Type-IIB orientifolds using string perturbation theory. We adopt two different approaches: one is genus-1 3-point amplitude and the other one is genus-$\frac{3}{2}$ 2-point amplitude.

In first approach, we begin with revisiting the Heterotic genus-1 3-graviton amplitude and derive a kinematic structure before coordinate integration that differs from previous reports. We then extend the calculation to the Type-I string, including the "pinched-off" contributions which are previously neglected. We find that even after including these contributions, the resulting genus- 1 correction breaks the expected gravitational kinematic structure, indicating that the string amplitude calculation remains incomplete and requires further study. This involves assignment of picture number on the surfaces, and a new technique "vertical integration" should be considered to deal with the calculation of amplitudes. This procedure may introduce potential new contributions. Unfortunately, the application of "vertical integration" is still under research. We have to leave this topic to our future study.

In second approach, using the concept of relevant modular transformations, we determine the moduli spaces of all genus-$\frac{3}{2}$ Riemann surfaces, correcting an earlier result about the fundamental domain. With this knowledge, it should in principle be possible to derive genus-$\frac{3}{2}$ amplitude corrections.

Zusammenfassung

Wir untersuchen Quantenkorrekturen zum Einstein-Hilbert-Term in Type-IIB Orientifolds unter Verwendung der String Störungsrechnung. Wir verfolgen dabei zwei verschiedene Ansätze: zum einen die 3-Punkt Amplitude vom Genus-1 und zum anderen die 2-Punkt Amplitude vom Genus-$\frac{3}{2}$.

Im ersten Ansatz beginnen wir mit einer erneuten Betrachtung der Heterotic 3-Graviton Amplitude vom Genus-1 und leiten eine kinematische Struktur vor der Koordinatenintegration ab, die sich von früheren Berichten unterscheidet. Anschließend erweitern wir die Berechnung auf den Type-I String, einschließlich der zuvor vernachlässigten "pinched-off" Beiträge. Wir stellen fest, dass selbst nach Einbeziehung dieser Beiträge die resultierende Korrektur der Genus-1 die erwartete gravitative kinematische Struktur bricht, was darauf hindeutet, dass die Berechnung der String Amplitude unvollständig bleibt und weiterer Untersuchungen bedarf. Dies beinhaltet die Zuweisung von Bildnummern auf den Weltflächen, und es sollte eine neue Technik, die "vertikale Integration", in Betracht gezogen werden, um die Berechnung der Amplituden zu bewältigen. Dieses Verfahren kann potenzielle neue Beiträge einbringen. Leider befindet sich die Anwendung der "vertikale Integration" noch in der Forschung. Wir müssen dieses Thema für unsere zukünftige Studie zurückstellen.

Im zweiten Ansatz bestimmen wir unter Verwendung des Konzepts relevanter modularer Transformationen die Moduli-Räume aller Riemannschen Flächen vom Genus-$\frac{3}{2}$ und korrigieren damit ein früheres Ergebnis über die fundamentale Domäne. Mit diesem Wissen sollte es grundsätzlich möglich sein, Amplitudenkorrekturen der Genus-$\frac{3}{2}$ abzuleiten.

Acknowledgments

I would like to express my deepest gratitude to everyone who supported me during my doctoral studies.

First and foremost, I am profoundly grateful to my supervisor, Michael Haack. Throughout the many difficulties I faced, his unwavering patience, encouragement, and scientific guidance were indispensable. No matter how bad my condition was, he always made time to discuss ideas with me and to help me move forward. His belief in me and his thoughtful mentorship made it possible for me to continue my work when my anxiety made it almost impossible to carry on.

My heartfelt thanks go to my wife, Chuyun Meng. Her steady emotional support, compassion, and care helped me through the darkest periods and played a crucial role in my gradual recovery. I could not have completed this journey without her by my side.

I also wish to thank my parents for being my steadfast foundation: their understanding, patience, and unconditional support gave me the space and strength to persist. Finally, I am grateful to my colleagues, friends, and everyone who offered help, encouragement, or a listening ear along the way.

To all of you who helped me—whether by academic advice, practical assistance, or simply kindness—thank you. This thesis is as much yours as it is mine.

Contents

Chapter 1

Introduction

We can think of String Theory as the extension of Quantum Field Theory, which means that it is the quantum theory of extended dynamical objects (strings/branes). With the help of the tools and techniques from 2 dimensional conformal field theory and only a few fundamental assumptions we can arrive at a beautiful theory of unified interactions. Since the spectrum of string theory contains a massless spin-2 particle which can be naturally interpreted as the graviton, it leads to the possibility that string theory could be a promising candidate of the theory of quantum gravity.

Lorentz invariance and anomaly free request that superstring theory should possess 10 dimensional space-time degrees of freedom. To make the theory phenomenologically interesting one has to reduce the dimension of the target space down to 4 dimension, which brings up the concept of compactification, or in a more general manner, appropriate choice of internal conformal field theory. Additional condition of 4D space-time minimal supersymmetry leads to the discovery of the Calabi-Yau compactification which can preserve 4D space-time $\mathcal{N} = 2$ supersymmetry. Orientifold with D-branes could further reduce half of the supercharges, thus preserves $\mathcal{N} = 1$ supersymmetry. However, compactification creates many moduli, which is undesired by phenomenological and cosmological reasons. Then one further develops the method of moduli stabilization which considers non-trivial background values for various massless bosonic excitations. See e.g.[43][22][57] for reviews.

One most important contribution of string theory boils down to that it provides the new *methodology* of discovering a new theory by studying the mathematical structures and consistency conditions *first*, then refining them to get a more realistic theory consistent with real world data, which reverses the traditional methodology upside down.

String theorists are often confronted with strong criticism about not making reasonable/testable predictions. In order to make the theory consistent with real world data, one would have to understand low energy effective theory in details, especially the quantum corrections. Keeping this in mind, one would require a computable method to deal with this. String perturbation theory plays a central role in the computation of such corrections to the effective action of certain model, which could greatly improve our understanding of the specific model. This is one of the primary motivation of the study of string perturbation theory. In string perturbation theory, one can already make use of all the experiences and developments from quantum field theory of point particles and conformal field theory.

We know that Calabi-Yau compactification and orientifold are necessary for constructing a low energy effective theory with 4 dimensional target space and $\mathcal{N}=1$ supersymmetry, which is meaningful from a phenomenological perspective. $\mathcal{N}=1$ supersymmetry in 4D is phenomenologically important because it can possess chirality which is necessary for Standard Model; it can avoid large numbers of extra light particles in conflict with experiments; it has a reasonable SUSY breaking scale to hide superpartners; it can possibly match gauge coupling unification patterns observed in Standard Model. CY compactification leads to Kähler moduli as well as other moduli in 4D effective theory[1]. The moduli are Kähler moduli T, complex structure moduli U, complexified dilaton D, and D-brane moduli ϕ typically in Type IIB orientifolds. Kähler moduli and other moduli have kinetic terms in the 4D effective action. The metric of the kinetic term of the Kähler moduli is the Kähler metric. None of the moduli was observed yet in experiments. Moduli stabilization, as a technical mechanism, was introduced to make the moduli massive, thus to avoid inconsistency with experimental results. However experience tells that it is important to understand the string effective action at least at the 1-loop level or even higher-loop level before attempt of moduli stabilization[2]. We would only focus on our interest in quantum corrections, thus would not go into details of compactification, moduli stabilization and derivation of effective action.

A simple introduction to the effective action and quantum corrections is given in §4.1 and §4.2. The kinetic terms of the effective action of 4-d supergravity in string frame[3], up to 1-loop order, is[4]

$$S_4 = \frac{1}{\kappa_4^2}\int d^4x\sqrt{-h}\left[(e^{-2\Phi_4}+\delta E)\frac{1}{2}R+\left(\tilde{G}^{(0)}+\tilde{G}^{(1)}\right)\partial_\mu\tau^{(0)}\partial^\mu\tau^{(0)}\right]+\dots . \quad (1.1)$$

[1]This is explained in standard textbooks, e.g.[44][23].

[2]This is discussed in e.g.[27][12][14]

[3]See §4.1 for the definition of string and Einstein frame.

[4]In this and next paragraphs we are following [14] and [47].

$\tau^{(0)}$ is the tree level form of the imaginary part of the Kähler moduli T. δE is the correction to the Einstein-Hilbert term, including tree level α' corrections, 1-loop g_s corrections, and possibly higher loop corrections, R is the Ricci scalar, Φ_4 is 4 dimensional dilaton, $\tilde{G}^{(0)}$ is the tree level moduli space (Kähler) metric including α' corrections and $\tilde{G}^{(1)}$ is the 1-loop contributions to the string frame moduli space (Kähler) metric.

1-loop corrections to the Einstein-Hilbert term contribute to the quantum corrections to the moduli Kähler metric in minimally supersymmetric toroidal Type-IIB orientifolds with D-branes, which can be observed from the 1-loop contributions to the moduli space (Kähler) metric in Einstein frame

$$\begin{aligned} G^{(1)}_{T\bar{T}}(T) =& e^{2\Phi_4}\tilde{G}^{(1)}(\tau) + 12\left(\frac{\partial \Phi_4}{\partial \tau^{(0)}}\right)^2 \delta E e^{2\Phi_4} + 6\frac{\partial \Phi_4}{\partial \tau^{(0)}}\frac{\partial \delta E}{\partial \tau^{(0)}} e^{2\Phi_4} \\ & - \delta E e^{4\Phi_4}\tilde{G}^{(0)}(\tau) + \frac{1}{2\tau^3}\delta\tau - \frac{1}{2\tau^2}\frac{\partial \delta\tau}{\partial \tau} + \dots \end{aligned} \tag{1.2}$$

where $\delta\tau$ is 1-loop correction to $\tau^{(0)}$. This is because when we go from String frame to Einstein frame, Einstein-Hilbert term would be involved in the correction to the moduli Kähler metric. Besides, the Kähler metric and Kähler potential also show up in most terms of the low energy effective action, see from §4.1. Therefore, to study the loop behavior of the theory, understanding of the correction to the Einstein-Hilbert term is inevitable. This motivates the calculation of the correction to the Einstein Hilbert term, in order to better understand the effective action.

It's always difficult to compute multi-point and higher-genus-contributions from string amplitudes. Multi-point contributions involve more operators, and thus they are highly non-trivial from the perspective of operator calculation. Moreover, higher-genus-surfaces also include those surfaces with non-trivial geometrical properties which cause difficulties for analyzing or calculating certain math objects like fundamental region or period matrix etc. Moreover, beyond the understanding of the effective theory, there are more obstacles like moduli stabilization and string model building of Standard Model. In the end we would have to adapt all our understanding with experimental results. These are out of the reach of this thesis, so we would not engage in any detail of them.

Overview

The main concern of this thesis is to extend the existing methods of string perturbation theory from the 3-point torus graviton amplitude (of Heterotic string) to all 4 genus-1 surfaces (of Type-I string): Torus, Annulus, Klein bottle, Möbius strip, and from genus-1 2-point Einstein-Hilbert term correction to the next order (genus-$\frac{3}{2}$) correction. 2-point amplitudes directly contribute to kinetic term of Kähler moduli, but they may have ambiguities. Meanwhile 3-point amplitudes could "strengthen" 2-point amplitudes by providing independent, complementary information that makes the loop corrections more robust and unambiguous. We are interested in confirming old results as well as developing calculating techniques.

Pioneering researches indicate that 1-loop corrections to Einstein-Hilbert term in Heterotic theory are absent[52][10]. Corrections of N = 2 models in Type-I theory were then calculated in [7][9]. Thereafter the calculation were generalized to N = 1 orientifolds in [53][36].

First we studied the 3-point amplitude. One-loop 3-point 4D Heterotic graviton amplitudes have been calculated in e.g.[40]. A special technique called "pinched-off" integration has been applied by Minahan[58] to extract correct kinematic structure from higher order kinematic terms. This technique was also used in e.g.[56][17]. When going from Heterotic to Type-I, one would have to include all 1-loop surfaces other than torus: Cylinder, Klein bottle and Möbius strip. Then it is necessary to represent the integrals on the 1-loop surfaces by the integrals on the torus by lifting technique, which is used in e.g.[7].

We inherited Minahan's approach[58] and tried to generalize the 3-point amplitude calculation in [40] to Type-I theory. The graviton 3-point amplitude in Type-I was already studied in [9] without application of Minahan's approach[58], so called "Pinched-off integration". In our calculation, we include extra contributions from pinched-off integration, and Taylor expansion was also considered during the pinched-off integration. We first reproduce the Heterotic kinematic structure calculation, then lift the result in Heterotic to Type-I by the lifting technique on the covering torus[7]. We have found unusual conclusion in Type-I and expect to study it in future with the help of the idea of factorization and the method called "vertical integration" from string field theory.

Then we turned to the study of 2-point amplitude calculation of correction to the Einstein-Hilbert term. In order to study the phenomenologically interesting model Type-IIB orientifold, people have tried to study the corrections to the moduli in different scenarios, like [15][16][14]. The importance of the correction to the Einstein-Hilbert term was mentioned in [47].

After 1-loop calculation, people's attention turned to next order: Euler characteristic -1 (or so called genus-$\frac{3}{2}$). This is because in some cases 1-loop corrections could be vanishing[28][35] or logarithmically suppressed[1], which makes genus-$\frac{3}{2}$ corrections the leading order. Besides, genus-$\frac{3}{2}$ corrections also show up in a specific configuration of a supersymmetry breaking[10]. There a surface with a hole and a handle was studied. As a follow up research, a surface with 3 holes has been studied in [5].

The concepts of involution and taking square root from the double cover were applied to go from 1-loop calculation to higher-genus-calculation, e.g. in the study of bosonic amplitudes[21] and fermionic amplitudes[10][63]. Consistency of the modular transformations with the involution introduces the concept of Relative modular transformation[18][19]. A further consistency requirement of the modular transformations with diffeomorphisms proposes the idea of Relevant modular transformation[5].

We followed the path of [47] and [54], tried to explore the calculation in genus-$\frac{3}{2}$. We begin with the double cover of the genus-$\frac{3}{2}$ surfaces, which is 2-torus. Involution and taking square root from double cover were introduced[21][10][63]. We discussed and clarified the concept of Relevant modular transformation[5], and pointed out the necessity of it. We extended the result of the moduli space in [5] to all 5 genus-$\frac{3}{2}$ surfaces.

Part I

Preliminaries

Chapter 2

String Theory

As the extension of quantum field theory, string theory shares many fundamental characteristics with quantum field theory. It encompasses a very broad and in-depth range of content. We introduce some basic knowledge of string theory to prepare for advanced topics in the following chapters. Starting with the fundamental dynamical object and thinking of the string theory as a mathematical theory: field X being the map from world-sheet to the target space, we work on 2 dimensional world-sheet. Considering all symmetries and quantizing the theory one concludes that this is a 2D conformal field theory with critical target space dimension $D = 26$. Joining and splitting of the open and closed string represents the interaction of the theory. Therefore the interactions of the theory are encoded in the topology of the world-sheet. By the path integral method we get the scattering amplitudes from summing over all topologies of the world-sheet.

By including supersymmetry, bosonic string extends to superstring which has a 10 dimensional target space. Classifying the left/right-movers as well as orientability, and imposing anomaly free condition and other consistency conditions, one restricts superstring to 5 different theories. Besides, open string can end on dynamical object, which is called D-brane. From unorientable world-sheets one also introduce the concept of cross-cap and O-plane.

Compactification is then applied to reduce the dimension of String Theory to 4 which meets with the dimension of our real universe.

In this chapter we follow closely to [61], [62] and [23].

2.1 Bosonic String Theory

Just like Quantum Mechanics studies 0 dimensional particle's world-line, in String Theory we study how 1 dimensional strings propagate on world-sheets. One describes the motion of string in D dimensional space-time by the function $X^\mu(\tau,\sigma)$, $\mu = 0\dots D-1$, τ and σ are the world-sheet coordinates. Of course the physics should be independent of parameterization. To begin with, no supersymmetry is involved thus we focus on bosonic theory now.

Worldsheet action and symmetries

The start point of a good quantum theory would always be the action and all possible symmetries of the action. We take the area of the world-sheet swept out by the string as the action. Then follow the similar procedure as in Quantum Mechanics to avoid square root difficulty, we deduce that, the simplest Pincaré-invariant, reparameterization invariant action would be the Polyakov action:

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma (-h)^{\frac{1}{2}}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{2.1}$$

where h is the world-sheet metric, $h = \det h_{ab}$ and Σ is the world-sheet.

There would be an extra symmetry of the action which is the Weyl invariance

$$\begin{aligned} X'^\mu(\tau,\sigma) &= X^\mu(\tau,\sigma), \\ h'_{ab}(\tau,\sigma) &= \exp(2\Omega(\tau,\sigma))h_{ab}(\tau,\sigma) \qquad \forall\Omega(\tau,\sigma). \end{aligned} \tag{2.2}$$

One defines the energy momentum tensor as the variation of the action:

$$T^{ab}(\tau,\sigma) = 4\pi i h^{-\frac{1}{2}}\frac{\delta}{\delta h_{ab}}S_P \tag{2.3}$$

$$= -\frac{1}{\alpha'}(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}h^{ab}\partial_c X^\mu \partial^c X_\mu). \tag{2.4}$$

The equations of motion are

$$T_{ab} = 0, \tag{2.5}$$

$$\nabla^2 X^\mu = 0. \tag{2.6}$$

To eliminate possible surface terms in world-sheets with boundaries while keeping Poincaré invariance, we impose Neumann boundary conditions

$$n^a\partial_a X_\mu = 0 \qquad \text{on } \partial\Sigma, \tag{2.7}$$

where n^a is the normal unit vector to $\partial\Sigma$. Or one can impose periodic boundary conditions that the fields X^μ, $\partial^\sigma X^\mu$ and h_{ab} are periodic.

If we relax Poincaré invariance, we can also impose Dirichlet boundary conditions

$$X^\mu(\tau, 0) = 0, \qquad X^\mu(\tau, \ell) = y^\mu, \tag{2.8}$$

where y^μ is a constant and $\ell = 2\pi\sqrt{\alpha'}$ is the string length. $\alpha' = \frac{1}{2\pi T}$ is the Regge slope and T is the string tension.

Light cone gauge In light cone gauge the theory can be expressed by physical degrees of freedom only. One defines space-time coordinates as (X^+, X^-, X^i), $i = 2 \cdots D-1$, with

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1). \tag{2.9}$$

The gauge is fixed by setting

$$X^+ = \frac{2\pi\alpha'}{\ell} p^+ \tau, \tag{2.10}$$

where $p^+ = -p_- = -\partial\mathcal{L}/\partial\dot{X}^-$.

Path Integral Quantization

The concept of path integral is well-known in quantum field theory. The basic idea of path integral is that the amplitude of quantum mechanics is determined by the integral of all possible paths from the initial state to the final state weighted by the exponential of the action

$$\langle\mathcal{O}\rangle = \langle f|\,\mathcal{O}\,|i\rangle = \int_{X_i}^{X_f} dX \exp\{iS\}\mathcal{O} \tag{2.11}$$

where $\mathcal{O}$ is product of local operators.

Euclidean path integrals Euclidean path integrals are often more well-defined because of the damping behavior of the exponential in 2 dimension. In string perturbation theory most calculation will be carried out in the Euclidean formalism. To define the Euclidean amplitudes, we take the time $t = X^0 = -iu$ for real u. The equivalence of Euclidean to Minkowski can be easily derived from analytical continuation on the complex plane.

Conformal Field Theory

Strings on the world-sheet is a 2 dimensional field theory equipped with conformal symmetry. The necessary mathematical tool needed is the conformal field theory.

Conformal field theory is much more well-defined in Euclidean metric. Making a wick rotation $\tau : \tau \to -i\tau$, we can define the complex coordinates

$$w = \tau - i\sigma, \qquad \bar{w} = \tau + i\sigma, \tag{2.12}$$

or

$$z = \exp(-iw) = \exp(-i\tau + \sigma). \tag{2.13}$$

A particular useful gauge in conformal field theory ,which is called conformal gauge, is defined by

$$ds^2 = \Omega^2(-d\tau^2 + d\sigma^2) = -\Omega^2 d\sigma^+ \sigma^-, \tag{2.14}$$

that the two dimensional metric is conformally flat.

Primary fields $\phi(z, \bar{z})$ (conformal fields) are the basic objects in a conformal field theory. It transforms under conformal transformation as tensor:

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}`) = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}). \tag{2.15}$$

h and $\bar{h}$ are conformal weights under analytic and anti-analytic transformations.

Operator Product Expansion In perturbation theory we are mainly interested in the expectation value of operators, especially the behavior of it in the limit that two operators are approaching each other. Operator Product Expansion is the systematic tool to describe the limit. Formally it can be presented as

$$\mathcal{A}_i(z_1)\mathcal{A}_j(z_2) = \sum_k c_{ij}^k \mathcal{A}_k(z_2), \tag{2.16}$$

where $z_1 \to z_2$, and $\mathcal{A}_k(z)$ is a basis for the set of local operators.

Since we are often interested in the limits, the OPE at small separation is dominated by the singular terms.

Normal Ordering We need normal ordering to well define the product of operators at the same point:

$$: X^{\mu_1}(z_1, \bar{z}_1) \dots X^{\mu_n}(z_n, \bar{z}_n) := X^{\mu_1}(z_1, \bar{z}_1) \dots X^{\mu_n}(z_n, \bar{z}_n) + \sum \text{subtractions}, \quad (2.17)$$

where "subtractions" are all ways of choosing from $X^{\mu_1}(z_1, \bar{z}_1) \dots X^{\mu_n}(z_n, \bar{z}_n)$ any number of pairs of operators and replacing each pair with the singular term of the OPE of the pair of operators.

Mode expansion In $z = \exp(-iw) = \exp(-i\sigma^1 + \sigma^2)$ coordinate, one can expand the holomorphic field $\partial X^\mu(z)$ into modes:

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}, \quad (2.18)$$

or equivalently using the residue theorem we have

$$\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dw}{2\pi} z^m \partial X^\mu(z). \quad (2.19)$$

We have similar expressions for anti-holomorphic field $\bar{\partial} X(\bar{z})$ with $\tilde{\alpha}_m^\mu$.

State-Operator correspondence

In w coordinate we must specify the boundary condition, or equivalently speaking the initial state $|\mathscr{A}\rangle$ as $\operatorname{Im} w \to -\infty$.

When transformed to z coordinate, $\operatorname{Im} w = -\infty$ maps to $z = 0$, and equivalently the initial state $|\mathscr{A}\rangle$ in w can be represented by a local operator $\mathscr{A}$ in $z = 0$, or so-called Vertex operator.

One can naturally define the initial state as

$$|\mathscr{A}\rangle = \lim_{z \to 0} \mathscr{A}(z)|0\rangle = \mathscr{A}(0)|0\rangle, \quad (2.20)$$

where by Cauchy's integral formula and C_0 is a contour surrounding the origin $z = 0$

$$\mathscr{A}(0) = \oint_{C_0} \frac{dz}{2\pi i} \frac{1}{z} \mathscr{A}(z). \quad (2.21)$$

This shows 1-1 correspondence between the state $|\mathscr{A}\rangle$ and the operator $\mathscr{A}(z)$. Thus it is called the state-operator correspondence.

As an important example, using (2.19), Cauchy's differentiation formula and holomorphicity of the fields we get the correspondence

$$\alpha^{\mu}_{-m}|0\rangle \cong \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!}\partial^m X^{\mu}(0), \qquad m>0. \tag{2.22}$$

Initial states can be constructed from the ground state $|1\rangle$ by acting with the creation operators $\alpha^{\mu}_{-m}(m>0)$ and $\tilde{\alpha}^{\mu}_{-m}(m>0)$. The corresponding operator is the normal ordered product of the local corresponding operators $\partial^m X^{\mu}(0)$ of α^{μ}_{-m} and the anti-holomorphic analogous part.

Gauge-Fixing and Moduli

Locally one can always fix the gauge of the path integral with (diff×Weyl) transformations, because the dimension of (diff×Weyl) group matches the degree of freedom of the metric. One could use the Faddeev-Popov method to fix the gauge, details could be found in the standard textbooks like [61, §3.3]. Here we would like to mention the expansion of diff×Weyl transformations of the metric

$$\begin{aligned}\delta h_{ab} &= 2\delta\omega h_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a \\ &= (2\delta\omega - \nabla_c\delta\sigma^c)h_{ab} - 2(P_1\delta\sigma)_{ab},\end{aligned} \tag{2.23}$$

where the operator P_1 maps vectors into traceless symmetric 2-tensors,

$$(P_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - h_{ab}\nabla_c\delta\sigma^c). \tag{2.24}$$

But globally there is a small mismatch between the metric space and the gauge group. The remnant is the moduli space. Moduli are the variation of the metrics which are orthogonal to diff×Weyl transformations, while conformal killing vectors (CKVs) are infinitesimal diff×Weyl transformations which do not change the metric. Then we find that moduli correspond to the kernel of $P_1^\dagger$ and CKVs to the kernel of P_1. By Riemann-Roch theorem, we have

$$\mu - \kappa = -3\chi \tag{2.25}$$

with $\mu = \dim\ker P_1^\dagger$ and $\kappa = \dim\ker P_1$. And κ vanishes for $\chi<0$, while μ vanishes for $\chi>0$.

The gauge-fixed n-point amplitude would be

$$A_n(k_1,\dots,k_n) = \sum_{\text{topologies}} \int_F \frac{d^\mu t}{n_R}\int [d\phi db dc]\exp(-S_m - S_g - \lambda\chi)$$

World-sheet	Euler number	dim ker P_1	dim ker $P_1^\dagger$
Sphere	2	6	0
Disk	1	3	0
Projective plane	1	3	0
Torus	0	2	2
Cylinder	0	1	1
Möbius strip	0	1	1
Klein bottle	0	1	1

Table 2.1: Moduli and CKV numbers of $g \leq 1$ surfaces

$$\times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_{t_k}\hat{h}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \hat{h}(\sigma_i)^{\frac{1}{2}} \mathscr{V}_i(k_i, \sigma_i), \tag{2.26}$$

where F stands for Fundamental domain, t are the moduli, b and c are the Faddeev-Popov ghosts, S_m and S_g are the matter and ghost actions respectively, $\hat{h}$ is the fiducial metric, $\mathscr{V}$ are the vertex operators, $(a,i) \in f$ are the set of coordinates of the vertex operators fixed by the conformal killing vectors, n_R is the finite order of a possible residual discrete group of symmetries. This amplitude is valid also for superstring if we include the superconformal field theory and $\beta\gamma$ ghost system.

χ is the Euler characteristic and $g_s = e^\lambda$ works as the string coupling constant. λ is the constant background value of the dilaton D while D is the trace part of the massless tensor spectrum. Therefore we can think of the amplitude A_n as the perturbative expansion in genus-$g = 1 - \chi/2$.

2.2 Superstring Theory

The bosonic string theory has tachyons as well as no place for fermions. To solve this, supersymmetry on world-sheet is imposed to extend the bosonic string theory to superstring theory which includes tachyon-free theories and accommodates fermionic degrees of freedom. Supersymmetry is the maximal extension of the Poincaré Symmetry.

Superaction and Supersymmetry The complete superstring action is

$$S = -\frac{1}{8\pi} \int d\sigma d\tau e \left(\frac{2}{\alpha'} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\sqrt{\frac{2}{\alpha'}} \partial_\beta X_\mu - \frac{i}{4} \bar{\chi}_\beta \psi_\mu \right) \right), \tag{2.27}$$

where $e = |\det e_\alpha^a| = \sqrt{-h}$, e_α^a is the zwei-bein for describing spinors on curved manifolds, ψ is the superpartner of X, χ is the gravitino. This action is invariant under the supersymmetry

$$\begin{aligned}\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu, \\ \delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha\left(\frac{2}{\alpha'}\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha\psi^\mu\right), \\ \delta_\epsilon e_\alpha^a &= \frac{i}{2}\bar{\epsilon}\rho^a\chi_\alpha \\ \delta_\epsilon\chi_\alpha &= 2D_\alpha\epsilon,\end{aligned} \tag{2.28}$$

where $\epsilon(\sigma,\tau)$ is a Majorana spinor parameterizing supersymmetry transformations and D_α a covariant derivative with torsion.

Type-II String We are considering closed strings. The world-sheet free action of the Type-II String in the light cone (l.c.) gauge is given by

$$S_{l.c.} = -\frac{1}{2\pi}\int d\sigma d\tau(\partial_+ X^i\partial_- X^i - i\psi^i\partial_-\psi^i - i\bar{\psi}^i\partial_+\bar{\psi}^i). \tag{2.29}$$

The bosonic field X satisfies periodic boundary condition. Fermionic field ψ can be either periodic (Ramond sector) or antiperiodic (Neveu-Schwarz sector) on the left and on the right. One has to perform the GSO projection in each sector to get the superstring with space-time supersymmetry.

The fermionic oscillators are defined by $\sqrt{2}b_m = \psi^{2m-1} + i\psi^{2m}$, $m = 1,\ldots,4$, which satisfy the usual anticommutation relations

$$\{b_m, b_n^\dagger\} = \delta_{mn}, \qquad \{b_m, b_n\} = 0, \qquad \{b_m^\dagger, b_n^\dagger\} = 0. \tag{2.30}$$

There is a NS and R sector for both of the left and right-movers. The relative choice of the GSO projection for the right-movers and for the left-movers is significant and leads to two different sectors of the closed superstring theory. One can keep either fermions of the same chirality or of opposite chirality in the two sectors. Depending on the choice, we get either Type-IIA theory (non-chiral) or Type-IIB theory (chiral):

$$\begin{aligned}&\text{Type-IIA}: \qquad (\mathbf{8v}\oplus\mathbf{8s})\otimes(\mathbf{8v}\oplus\mathbf{8c}) \\ &\text{Type-IIB}: \qquad (\mathbf{8v}\oplus\mathbf{8c})\otimes(\mathbf{8v}\oplus\mathbf{8c}),\end{aligned} \tag{2.31}$$

which is the massless spectrum of Type-IIA or IIB respectively.

Heterotic String Since the left and right-moving sectors can be treated independently, the Heterotic string is constructed by the left-moving sector of the 26 dimensional bosonic string combined with the right-moving sector of the 10- dimensional superstring. 16 compactified left-moving bosonic fields live in the internal space which is a 16 dimensional torus, and we are left with a 10 dimensional string theory. Modular invariance of the one-loop partition function constrains the internal 16 dimensional momentum lattice to be an even self-dual Euclidean lattice, and further implies the gauge group resulting from torus compactification to be either $E_8 \times E_8$ or $SO(32)$.

Type-I String Type-I theory is an orientifold[1] of Type-IIB theory with orientifold symmetry group

$$\mathbb{Z}_2 = \{1, \Omega\}. \tag{2.32}$$

Closed String Sector: The closed string sector of Type-I theory contains unoriented strings that are invariant under orientation-reversal. The massless states are simply the states of Type-IIB that are invariant under Ω. We know that only g_{ij}, ϕ, B'_{ij} (R-R 2-form), and a symmetric combination of the two gravitini survive the projection.

Open String Sector: Open string sector arises from the addition of D-branes that are required to cancel the charge of the orientifold plane. Orientation reversal is a purely world-sheet symmetry, so it leaves the entire 9 dimensional space invariant. Therefore, the orientifold plane is a O_9-plane. It turns out to have -32 units of charge w.r.t. the 10-form non-propagating field from the R-R sector. This charge can be canceled by adding 32 Dirichlet D_9-branes which each has unit charge. The world-volume theory of the D_9-branes gives rise to gauge group $U(32)$ but only an $SO(32)$ subgroup is invariant under the action of Ω.

Type-I supergravity theory is anomaly free only if the gauge group is $SO(32)$ or $E_8 \times E_8$. It is satisfying that the spectrum determined by requiring world-sheet consistency is automatically anomaly free

Spin Structure Spinor defined on a genus-g Riemann surface could have either periodic or anti-periodic boundary conditions along $2g$ non-contractible homology basis. Then there are 2^{2g} possible spin structures for a genus-g surface. A spin structure is called even (odd) if the number of zero modes of chiral Dirac operator is even (odd), and this number modulo two is a topological invariant and additive when two surfaces are glued together.

[1] cf. §2.3.2

2.2.1 Ghost system

Fadeev-Popov quantization introduces ghost systems which simplifies the calculation of the conformal field theory.

bc **ghosts** *bc* ghosts are obtained from fixing the gauge of the reparameterization and Weyl invariance (world-sheet metric)[2]. In other word, the ghost part is equivalent to the vector laplacian $\det'^{1/2}(P_1^\dagger P_1)$ which is a result of the Jacobian of decomposition of the metric space into diff×Weyl space and moduli space. In conformal gauge, one has

$$\Delta_{\mathrm{FP}} = \int dbdc\, e^{-S_{\mathrm{ghost}}[b,c]} \tag{2.33}$$

$$S_{\mathrm{ghost}}[b,c] = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}) \tag{2.34}$$

with conformal weights $h(b) = 2$, $h(c) = -1$, $c_{b,c} = -26$ and the OPEs

$$b(z)c(w) \sim \frac{1}{z-w} + \cdots \qquad b(z)b(w) = c(z)c(w) = \mathcal{O}(z-w). \tag{2.35}$$

We can trade dim ker $P_1^\dagger$ number (dimension of the conformal Killing group) of the integration of the position of the vertex operators for the same number of $c\bar{c}$ fields. For example, at tree level (sphere), we have

$$|\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle|^2 = |(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2. \tag{2.36}$$

Derivation of this result could be found in [23, §6.2].

$\beta\gamma$ **superghosts** Similarly, $\beta\gamma$ superghosts are obtained from fixing the gauge of the supersymmetry (world-sheet gravitino)[3]. One has

$$S_{\mathrm{superghost}}[\beta,\gamma] = \frac{1}{2\pi}\int d^2z(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}), \tag{2.37}$$

with conformal weights $h(\beta) = \frac{3}{2}$, $h(\gamma) = -\frac{1}{2}$, $c_{\beta,\gamma} = 11$ and the OPEs

$$\gamma(z)\beta(w) \sim \frac{1}{z-w} + \cdots, \qquad \beta(z)\beta(w) = \gamma(z)\gamma(w) = \mathcal{O}(1). \tag{2.38}$$

[2] cf. [23, §6.2]
[3] cf. [23, §8.4]

Bosonization

Bosonization means that a conformal field theory of $2n$ fermions with specific boundary conditions and a conformal field theory of n bosons compactified on a torus share the same correlation functions and thus are equivalent. Bosonization greatly simplifies the treatment of ghost systems and is important for the construction of the covariant vertex operators with ghost systems. We follow [23, §13.1] in this section.

For 2 fields ψ_1 and ψ_2, bosonization can be defined as

$$\Psi^{\pm}(z) = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2)(z) \tag{2.39}$$

$$\Psi^{\pm}(z) =: e^{\pm\phi(z)} : \tag{2.40}$$

$$V_{\Lambda}(z) =: e^{i\Lambda\cdot\phi(z)} : \tag{2.41}$$

where ϕ is the bosonized boson, V is the vertex operator and Λ is the lattice vector of the compactified space.

It contains a $U(1)$ current algebra

$$\begin{aligned} j(z) =: \Psi^+\Psi^- := i\epsilon\partial\phi, \qquad & j(z)j(w) = \frac{1}{(z-w)^2} + \cdots, \\ j(z)\Psi^+(w) = \frac{\Psi^+(w)}{z-w} + \cdots, \qquad & j(z)\Psi^-(w) = -\frac{\Psi^-(w)}{z-w} + \cdots, \end{aligned} \tag{2.42}$$

We parameterize the statistics by $\epsilon = 1$ for Fermi statistics and $\epsilon = -1$ for Bose statistics.

As an example, we take a first order action

$$S = \frac{1}{2\pi}\int d^2z b\bar{\partial}c \tag{2.43}$$

that b has conformal weight λ and c has conformal weight $1-\lambda$. To bosonize the first order system, we identify $b = \Psi^+$ and $c = \Psi^-$. The energy-momentum tensor is

$$\begin{aligned} T =& -\lambda : b\partial c : +(1-\lambda) : (\partial b)c : \\ =& \frac{1}{2}(: (\partial b)c : - : b\partial c :) + \frac{1}{2}\epsilon Q(: bc :) \end{aligned} \tag{2.44}$$

with a background charge $Q = \epsilon(1-2\lambda)$.

We get the $U(1)$ current and bosonize it as

$$j(z) = - : b(z)c(z) := i\epsilon\partial\phi(z) \tag{2.45}$$

with

$$\phi(z)\phi(w) \sim \epsilon \ln(z-w) \tag{2.46}$$

and

$$T(z)j(w) = \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} + \cdots . \tag{2.47}$$

The action now turns into

$$S = -\frac{1}{8\pi}\int d^2z\sqrt{h}(\epsilon h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + QR\phi) \tag{2.48}$$

where R is the scalar curvature. The energy-momentum tensor is now

$$T^{(j)} = \epsilon\bigg(\frac{1}{2} : jj : -\frac{1}{2}Q\partial j\bigg). \tag{2.49}$$

We simply list the algebra:

$$j(z)e^{q\phi(w)} = \frac{q}{z-w}e^{q\phi(w)} + \cdots , \tag{2.50}$$

$$T^{(j)}(z)e^{q\phi(w)} = \bigg[\frac{\frac{1}{2}\epsilon q(q+Q)}{(z-w)^2} + \frac{\partial_w}{z-w}\bigg]e^{q\phi(w)} + \cdots . \tag{2.51}$$

Conformal ghost bosonization In $\epsilon = 1$ case, the bc system can be bosonized as

$$\begin{gathered} b(z) = e^{i\phi(z)}, \qquad c(z) = e^{-i\phi(z)}, \\ \phi(z)\phi(w) \sim \ln(z-w). \end{gathered} \tag{2.52}$$

Superconformal ghost bosonization In $\epsilon = -1$ case, bosonization is more complicated. We have

$$\begin{gathered} \beta(z) = e^{-\phi(z)}\partial\xi(z), \qquad \gamma(z) = \eta(z)e^{\phi(z)}, \\ \phi(z)\phi(w) \sim -\ln(z-w), \end{gathered} \tag{2.53}$$

where ξ and η form a fermionic first order system with central charge $c = -2$ and conformal weight of ξ is $h = 0$. One could further bosonize the $\xi\eta$ system as

$$\begin{gathered} : \xi(z)\eta(z) := \partial\chi(z), \qquad \chi(z)\chi(w) \sim \ln(z-w), \\ \eta(z) = e^{-\chi(z)}, \qquad \xi(z) = e^{\chi(z)}. \end{gathered} \tag{2.54}$$

Finally we get

$$\beta(z) = e^{-\phi(z)}e^{\chi(z)}\partial\chi(z), \qquad \gamma(z) = e^{-\chi(z)}e^{\phi(z)}. \tag{2.55}$$

Picture number After bosonization of the superconformal ghosts, the bosonized system obtained a picture charge as a new quantum number.[4] Using the relation between scalar curvature R and Euler character χ

$$\chi = \frac{1}{4\pi} \int d^2 z \sqrt{h} R, \tag{2.56}$$

we see from the above action (2.48) that it restricts the background picture charge of the states to be $-Q\chi/2$ with $Q = \epsilon(1-2h)$ and $\chi = 2(1-g)$. Thus we have to assign a total picture charge of $-Q\chi/2$ to vertex operators in order to get a non-vanishing correlation function. In the case of sphere and $\beta\gamma$ ghost system as an example, we have conformal weight $h = 3/2$, antisymmetry $\epsilon = -1$ and Euler number $\chi = 2$, thus the total picture charge of vertex operators in a correlation function should be -2. Another example is a torus with $\beta\gamma$ system requiring a background picture charge of $-\epsilon(1-2h) \cdot 2(1-g)/2 = 0$. A state with picture charge $q = -1$ is called canonical, because when one surface was factorized into 2 surfaces by a plumbing connecting them, to satisfy the correct background picture charge of each factor surface, vertex operators on the two ends of the plumbing require the canonical picture charge.

Picture Changing operator The picture changing operator (PCO) P_{+1} was defined through

$$V_{q+1} = P_{+1} V_q = [Q, 2\xi V_q], \tag{2.57}$$

where V_q is vertex operator with picture number q, and ξ is from bosonization of superconformal ghosts. P_{+1} carries 1 unit of picture charge and its action on V_q would raise the picture number of the operator by 1 unit.

2.2.2 D-Branes

From the construction of open string, we require a soliton in space-time to let the open string end on it. We consider a p dimensional hyperplane (Dp-brane) along the directions X^1,...,X^p. Take the longitudinal coordinates X^μ, $\mu = 0, \ldots, p$ to satisfy NN boundary conditions, and the transverse coordinates X^m, $m = p+1, \ldots, 9$ to satisfy DD boundary conditions. Open strings are allowed to end on the p dimensional hyperplane which can be viewed as a p-brane at a location determined by the zero mode of the coordinates X^m. This construction shares all features of a BPS soliton. Parallel branes preserve the space-time supersymmetry, while anti-branes (with opposite charge) and branes at angles will break some supersymmetry.

[4] cf. [51, §4.20]

Chan-Paton factor If there are n identical parallel D-branes, then the open string can begin on a D-brane labeled by i and end on one labeled by j. The label of the D-brane is called the Chan-Paton index at each end. Denote a general state in the open string sector by $|\psi, ij\rangle\lambda_{ij}$ with i, j Chan-Paton indices, λ_{ij} is the Chan-Paton factor, ψ is the state of the world-sheet fields, and $\lambda^\dagger = \lambda$ due to the reality of the string wave function. The massless excitation of the open string give rise to a supersymmetric $U(n)$ gauge theory on the worldvolume.

2.3 Discrete Symmetries

The approach of this part of introduction is to illustrate the main ingredients of the general procedure of orientifolds.

2.3.1 Orbifolds

An orbifold $\mathcal{M}' = \mathcal{M}/G$ is obtained from a manifold $\mathcal{M}$ on which a discrete isometry G acts. $\mathcal{M}'$ is singular near the fixed points. Strings moving on a target space $\mathcal{M}$ led to the concept of orbifolds in conformal field theory. Orbifold can be used to construct a new theory T' from an existing theory T by taking the orbifold action G on T and get $T' = T/G$.

$\mathbb{Z}_N$-orbifold on Torus $\mathcal{T}^D$ (D even) We will need these orbifolds in later sections. We start with complexifing the coordinates:

$$Z^j = \frac{1}{\sqrt{2}}(X^{2j-1} + iX^{2j}), \qquad Z^{*j} = \frac{1}{\sqrt{2}}(X^{2j-1} - iX^{2j}), \tag{2.58}$$

with the orbifold action acting as

$$G : Z^j \mapsto e^{2\pi i v_j} Z^j, \qquad Z^{*j} \mapsto e^{-2\pi i v_j} Z^{*j}, \qquad j = 1, \ldots, D/2 \tag{2.59}$$

and $v_j = k_j/N$ for $k_j \in \mathbb{Z}$ is called twist vector. $e^{\pm 2\pi i v_j}$ is the eigenvalue of the single generator $\theta \in SO(D)$ ($\theta^N = 1$) of $\mathbb{Z}_N$ in the vector representation. Spectrum of T is reduced to states that are invariant under G. Torus can be represented as R^D/Λ, we observe that θ acts crystallographically on the torus lattice Λ. Thus θ must have all integer entries in the lattice basis. Then we know that

$$\mathrm{Tr}\,\theta = \sum_{j=1}^{D/2} 2\cos(2\pi v_j), \tag{2.60}$$

$$\chi(\theta) = \det(1-\theta) = \prod_{j=1}^{D/2} 4\sin^2(\pi v_j) \tag{2.61}$$

must both be integers. $\chi(\theta)$ is the number of fixed points of θ by the Lefschetz fixed point theorem. Ignoring possible factor with $v_j = 0 \mod 1$ in (2.61) gives the number of fixed tori.

Twisted sectors

Due to the orbifold action, different points in the covering manifold $\mathcal{M}$ are equivalent in the quotient manifold $\mathcal{M}/G$ under the orbifold action. Thus strings which are not closed in $\mathcal{M}$ could be closed in $\mathcal{M}/G$. And this induces the concept of twisted sectors, which means strings are closed in these sectors but not closed in $\mathcal{M}$.

We impose the boundary conditions of complexified boson

$$Z^j(\sigma^0, \sigma^1 + 2\pi) = e^{2\pi i k v_j} Z^j(\sigma^0, \sigma^1) \tag{2.62}$$

$$Z^j(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = e^{2\pi i \ell v_j} Z^j(\sigma^0, \sigma^1) \tag{2.63}$$

where $\tau = \tau_1 + i\tau_2$ is the modulus on the torus. $k \in \mathbb{Z}$ ($ell \in \mathbb{Z}$) means that the boson is in the k-th (ℓ-th) twisted sector along one of the 2 periodic directions of the torus.

Complex fermions on a $\mathbb{Z}_N$ toroidal orbifold satisfy the twisted boundary conditions

$$\psi^j(\sigma^0, \sigma^1 + 2\pi) = -e^{+2\pi i\alpha} e^{2\pi i k v_j} \psi(\sigma^0, \sigma^1), \tag{2.64}$$

$$\psi^j(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = -e^{+2\pi i\beta} e^{2\pi i \ell v_j} \psi(\sigma^0, \sigma^1) \tag{2.65}$$

with $\alpha, \beta \in \{0, 1/2\}$ representing the spin structure.

Twisted sectors are essential for modular invariance. The partition functions of twisted sectors and more other details can be found in app.D.

2.3.2 Orientifolds

The orientation-reversal action Ω (also called world-sheet parity) is defined as

$$\Omega : (\tau, \sigma) \to (\tau, \ell - \sigma). \tag{2.66}$$

It reverses the orientation of the strings. Like orbifolds, a new theory T' could also be constructed from an existing theory T by Ω as $T' = T/\Omega$. Orientation-reversal

Ω could break half of the space-time supersymmetry, thus is a practical tool for the construction of minimal space-time SUSY string theory in 4 dimension. More details of orientifold symmetry could be found in app.C.

Typically people are interested in a $\mathbb{Z}_N$ orbifold of toroidally compactified Type-IIB theory and then orientifold it further by a symmetry $\mathbb{Z}_2 = \{1, \Omega\}$. If the orbifold group $\mathbb{Z}_N$ is generated by θ, then the total orientifold symmetry is $G = \{1, \theta, \ldots, \theta^{N-1}, \Omega, \Omega\theta, \ldots, \Omega\theta^{N-1}\}$ or symbolically, $G = \mathbb{Z}_N \cup \Omega\mathbb{Z}_N$, cf. (C.1).

Tadpole Cancellation and Orientifold Planes

There is a consistency requirement for orientifolds that is analogous to the requirement of modular invariance for the torus. This is the requirement of 'tadpole cancellation'.

There exists non-vanishing 1-point functions, or so-called tadpoles, on orientifolds. Cancellation of all tadpoles is necessary for obtaining a stable string vacuum. This requirement is very restrictive and it more or less completely determines when and how the open string should be added.

Physically, nonzero tadpoles imply that the equations of motion of some massless fields are not satisfied. They occur for the following reason. The planes that are left invariant by an orientation-reversal symmetry is called the orientifold plane. Like a D-brane, an orientifold plane is a p dimensional hyperplane which couples to an R-R $(p+1)$-form which we generically refer to as C_{p+1}. The charge of the orientifold plane can be calculated by looking at the R-R tadpole. If the orientifold plane has a nonzero charge then it acts as a source term in the equations of motion for the $(p+1)$-form field C_{p+1}. The field lines must start and end on charge sources in a compact space, and the net charge must vanish on the compact space. The negative charge of a p dimensional orientifold plane in a compact transverse space can only be neutralized by adding the right-number of Dp-branes so that Gauss law is satisfied and all tadpoles cancel.

Chapter 3

String Perturbation Theory

Similar to quantum field theory, we need to study string perturbation theory to gain further insight into string theory. Phenomenologically interesting string amplitudes also require string perturbation theory for calculations.

3.1 Basic concepts of String Perturbation theory

The string amplitude (2.26) could be abbreviated as

$$A_n = \sum_{g=0}^{\infty} A_n^{(g)} \tag{3.1}$$

to emphasize the perturbation form of the theory. $Z^{(g)} = A_0^{(g)}$ is the genus-g partition function.

Two dimensional oriented surfaces without boundary are topologically completely characterized by the genus-g. If we extend the surface to include not only handles but also boundaries and cross-caps, then two dimensional surfaces would be topologically characterized only by $\chi = 2 - 2h - b - c$, with h handles, b holes and c cross-caps, and genus-$g = 1 - \chi/2 = h + \frac{1}{2}b + \frac{1}{2}c$. Then each term in (3.1) will be weighted by

$$g_s^{-\chi+n_c+\frac{1}{2}n_o} = (e^{\lambda})^{-\chi+n_c+\frac{1}{2}n_o}, \tag{3.2}$$

where n_c is the number of closed string vertex operators and n_o is the number of open string vertex operators. λ turns out to be the constant background value of the dilaton field D. Due to the expectation of small λ we can apply perturbation techniques to the theory, but convergence will be ignored in this work.

Riemann Surfaces

A 2-real dimensional (1-complex dimensional) complex manifold is called a Riemann surface. Only in 2-real dimension there is a one-to-one correspondence between Riemann Surfaces and Riemann Manifolds mod Weyl transformation. Also for the same reason of the above correspondence, we know that on Riemann surfaces, a conformal structure is the same as a complex structure.

3.2 Correlation Functions

By Wick's theorem, all correlation functions can be expressed by propagators (also known as two-point functions) $\langle \phi_1 \phi_2 \rangle$ where ϕ_1 and ϕ_2 are the fields in the theory. Other than two-point functions, we would also need to know the exact forms of the vertex operators to calculate the correlation functions.

3.2.1 Vertex Operators

We use the state-operator correspondence to obtain the vertex operators. In the major work of this thesis we need graviton vertex operators, so we give the exact forms of them in the following. We observe that gravitons are only present in excitations of closed strings.

The superstring massless vertex operator[23]

$$V_{(-1,-1)}(k,\epsilon) =: \epsilon_{\mu\nu}(k)\bar{V}^{\mu}_{(-1)}(k,\bar{z})V^{\nu}_{(-1)}(k,z) :, \tag{3.3}$$

with

$$V^{\mu}_{(-1)}(k,z) = e^{-\phi}\psi^{\mu}(z)e^{ik\cdot X(z)}, \tag{3.4}$$

is obtained from the state $\epsilon_{\mu\nu}(k)b^{\mu}_{-1/2}\bar{b}^{\nu}_{-1/2}\left|k\right\rangle$ with normalization condition $\epsilon^{(G)}_{\mu\nu}\epsilon^{(G)\mu\nu} = 1$, where G stands for graviton.

Zero picture vertex operator is

$$V_{(0,0)}(k,\epsilon) =: \epsilon_{\mu\nu}\bar{V}^{\mu}_{(0)}(k,\bar{z})V^{\nu}_{(0)}(k,z) : \tag{3.5}$$

with

$$V^{\mu}_{(0)}(k,z) = \sqrt{\frac{2}{\alpha'}}\left(i\partial X^{\mu} + \frac{\alpha'}{2}(k\cdot\psi)\psi^{\mu}\right)e^{ik\cdot X(z)}. \tag{3.6}$$

Heterotic massless vertex operators are

$$V_{(-1)}(k,\epsilon) =: \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k)i\bar{\partial}X^{\mu}(\bar{z})e^{-\phi}\psi^{\nu}(z)e^{ik\cdot X(z,\bar{z})} :, \tag{3.7}$$

$$V_{(0)}(k,\epsilon) =: \frac{2}{\alpha'}\epsilon_{\mu\nu}(k) i\bar{\partial}X^{\mu}(\bar{z})[i\partial X^{\nu}(z) + \frac{\alpha'}{2}(k\cdot\psi)\psi^{\nu}(z)]e^{ik\cdot X(z,\bar{z})} : . \tag{3.8}$$

The polarization tensor $\epsilon_{\mu\nu}(k)$ represents the wave function of the massless string excitation. BRST invariance imposes the on-shell conditions as $k^{\mu}\epsilon_{\mu\nu} = \epsilon_{\mu\nu}k^{\nu} = 0$ and $k^2 = 0$. We can decompose $\epsilon_{\mu\nu}$ into 3 irreducible parts, which are symmetric and traceless part (graviton $h_{\mu\nu}$, $\epsilon^{G}_{\mu\nu}$), anti-symmetric part (anti-symmetric tensor $B_{\mu\nu}$, $\epsilon^{B}_{\mu\nu}$) and transverse diagonal part (dilaton D, $\epsilon^{D}_{\mu\nu}$). We have the decomposition and the on-shell conditions as

$$\begin{aligned} \epsilon^{G}_{\mu\nu} = \epsilon^{G}_{\nu\mu}, \qquad & \epsilon^{G}_{\mu\nu}\eta^{\mu\nu} = k^{\mu}\epsilon^{G}_{\mu\nu} = 0, \qquad \text{(graviton)} \\ \epsilon^{B}_{\mu\nu} = -\epsilon^{B}_{\nu\mu}, \qquad & k^{\mu}\epsilon^{B}_{\mu\nu} = 0, \qquad \text{(antisymmetric tensor)} \\ \epsilon^{D}_{\mu\nu} = \frac{1}{\sqrt{d-2}}(\eta_{\mu\nu} - k_{\mu}\bar{k}_{\nu} - \bar{k}_{\nu}k_{\mu}), \qquad & k^{\mu}\epsilon^{D}_{\mu\nu} = 0, \qquad \text{(dilaton)}. \end{aligned} \tag{3.9}$$

In this work, we would calculate the graviton amplitudes, thus the graviton vertex operator is needed.

3.2.2 Two-point functions

One would have to use mathematical tricks to derive or guess the Green's function which satisfies the differential equation with the differential operator in the action. Details could be found in [61] and [62].

Propagators on Sphere

The bosonic Green's function $P_{\mathcal{S}}{}'$ satisfies the differential equation

$$-\frac{1}{2\pi\alpha'}\nabla^2 P_{\mathcal{S}}{}'(\sigma_1,\sigma_2) = \sum_{I\neq 0} X_I(\sigma_1)X_I(\sigma_2) = h^{-\frac{1}{2}}\delta^2(\sigma_1-\sigma_2) - X_0^2. \tag{3.10}$$

The solution on sphere to the equation is given by (3.47), but in most cases we would only need the simplified form:

$$\langle X^{\mu}(z_1)X^{\nu}(z_2)\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z_1-z_2). \tag{3.11}$$

Similarly the fermionic Green's function $S_{\mathcal{S}}{}'$ satisfies the differential equation

$$-\frac{1}{2\pi}\mathcal{D}S_{\mathcal{S}}{}'(\sigma_1,\sigma_2) = \delta^2(\sigma_1-\sigma_2) - \text{zero modes} \tag{3.12}$$

where $\mathcal{D}$ is dirac operator. And the solution on sphere is

$$S_{\mathcal{S}}{}'(z_1,z_2)=\langle\psi^\mu(z_1)\psi^\nu(z_2)\rangle=\frac{\eta^{\mu\nu}}{z_1-z_2}. \tag{3.13}$$

Meanwhile, by the same calculation as X field, two-point functions of bosonized field ϕ of superconformal ghosts and bosonized field ϕ^μ of fermion ψ^μ are

$$\begin{aligned}\langle\phi(z_1)\phi(z_2)\rangle&=-\ln(z_1-z_2),\\ \langle\phi^\mu(z_1)\phi^\nu(z_2)\rangle&=-\,\delta^{\mu\nu}\ln(z_1-z_2).\end{aligned} \tag{3.14}$$

One-loop propagators

It is useful to list the propagators in genus-1 case. The propagators on the torus are

$$P_{\mathcal{T}}(z,w)=\langle X(z)X(w)\rangle_{\mathcal{T}}=-\frac{\alpha'}{2}\ln\left|\frac{\vartheta_1(z-w|\tau)^2}{\vartheta_1'(0|\tau)}\right|+\frac{\pi(z_2-w_2)^2}{2\tau_2}, \tag{3.15}$$

$$S_{\mathcal{T}}(s;z,w)=\langle\psi(z)\psi(w)\rangle^s_{\mathcal{T}}=i\alpha'\frac{\vartheta_s(z-w|\tau)}{\vartheta_1(z-w|\tau)}\frac{\vartheta_1'(0|\tau)}{\vartheta_s(0|\tau)}. \tag{3.16}$$

Other 1-loop surfaces can be derived through method of images/involution[25][26][7], we list here:

$$P_\sigma(z,w)=P_{\mathcal{T}}(z,w)+P_{\mathcal{T}}(z,I_\sigma(w)), \tag{3.17}$$

$$\langle\partial_iX_i\partial_jX_j\rangle=\frac{\alpha'\pi}{2\tau_2}+\partial_i\partial_jP_\sigma(z_i,z_j), \tag{3.18}$$

$$\langle\bar\partial_iX_i\partial_jX_j\rangle=-\frac{\alpha'\pi}{2\tau_2}+\bar\partial_i\partial_jP_\sigma(z_i,I_\sigma(z_j)), \tag{3.19}$$

$$\langle\psi(z)\psi(w)\rangle_\sigma=S^{even}_{\mathcal{T}}(s;z,w), \tag{3.20}$$

$$\langle\psi(z)\bar\psi(\bar w)\rangle_\sigma=S^{even}_{\mathcal{T}}(s;z,I_\sigma(w)), \tag{3.21}$$

$$\langle\bar\psi(\bar z)\bar\psi(\bar w)\rangle_\sigma=\bar S^{even}_{\mathcal{T}}(\bar s;\bar z,\bar w), \tag{3.22}$$

where $\sigma=\mathcal{A},\mathcal{K},\mathcal{M}$ represents the one-loop surface and I_σ is the involution (3.57). "even" means even spin structure. ϑ functions are defined in app.A.

Arbitrary Genus

The detailed derivation of the propagators on arbitrary surface can be found in [68], [38] and [21]. We only give the results here.

The fermionic two-point function in even spin structures is

$$\langle\bar{\psi}(z)\psi(w)\rangle_{even} = S_{\Sigma}^{even}(z,w) = \frac{1}{E(z,w)}\frac{\vartheta[s](z-w)}{\vartheta[s](0)} \tag{3.23}$$

where $E(z,w)$ is the prime form[38] with $E(z,w) \sim z-w$ as $z \sim w$. The fermionic two-point function with odd spin structure is more complicated due to the zero modes:

$$\langle\bar{\psi}(z)\psi(w)\rangle_{odd} = S_{\Sigma}^{odd}(z,w) = h_s(z)h_s(w)p_{F,s\,odd}(z,w) \tag{3.24}$$

with $h_s^2(z) = \sum_i \partial_i\vartheta[s](0)\omega_i(z)$ is a holomorphic $\frac{1}{2}$-differential. ω_i is the basis of holomorphic 1-forms (see from [68] for details of the 1-forms), and

$$p_{F,s\,odd}(z,w) = \frac{1}{E(z,w)}\frac{\sum\partial_i\vartheta[s](z-w)\omega_i(y)}{\sum\partial_i\vartheta[s](0)\omega_i(y)}. \tag{3.25}$$

And the bosonic two-point function is

$$\begin{aligned}\langle X(z)X(w)\rangle = P_{\Sigma}(z,w) = -\ln F + \frac{1}{A}\int d^2y h^{\frac{1}{2}}(y)\big(\ln F(z,y) + \ln F(y,w)\big)\\ -\frac{1}{A^2}\int\int d^2x d^2y h^{\frac{1}{2}}(x)h^{\frac{1}{2}}(y)\ln F\end{aligned} \tag{3.26}$$

with $A = \int d^2z h^{1/2}$, and

$$F(z,w) = \exp\left[-2\pi\left(\mathrm{Im}\int_w^z\right)\omega(\mathrm{Im}\,\Omega)^{-1}\left(\mathrm{Im}\int_z^z\omega\right)\right]|E(z-w)|^2 \tag{3.27}$$

where Ω is the period matrix.

We notice the asymptotic behaviors of the propagators while $z \sim w$ are

$$P_{\Sigma}(z,w) \sim -\ln(z-w), \qquad S_{\Sigma}(z,w) \sim \frac{1}{z-w} \qquad (z \sim w). \tag{3.28}$$

This can be easily deduced, because the propagators are localized while $z \sim w$, thus they ignore the global geometry properties.

3.3 Perturbative Amplitudes

String Perturbation Theory studies the correlators of vertex operators in quantum conformal field theory with interactions. From the bosonic result (2.26), we extend it to the superstring as

$$A_n(k_1,\ldots,k_n) = \sum_{\text{topologies}}\int_F \frac{d^\mu t}{n_R}\int[d\phi db dc d\beta d\gamma]\exp(-S_m - S_{ghost} - S_{superghost} - \lambda\chi)$$

$$\times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} B \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \hat{h}(\sigma_i)^{\frac{1}{2}} \mathscr{V}_i(k_i, \sigma_i) \tag{3.29}$$

where B is the superfield of ghost insertion of b ghost and β ghost. The picture charges of $\mathscr{V}_i$ sum up to $-\chi$, which is -2 in sphere, 0 in 1-loop, 1 in 3/2-loop and 2 in 2-loop. Since the B ghost integration would be absorbed into a moduli related term which is independent of the vertex operators, we will focus only on the calculation of the correlation functions of the vertex operators and ignore other integration parts.

3.3.1 Moduli Space

Starting from the metric space M_h, which is the space of all metrics on the surface Σ_g, we need to gauge away the redundancy of the system. First step is gauging the Weyl transformation plus diffeomorphisms connected to the identity Weyl$\times$ Diff$_0$, and the result is the Teichmüller space

$$\mathcal{T}_g = \frac{M_h}{\text{Weyl} \times \text{Diff}_0}. \tag{3.30}$$

There is further redundancy in the disconnected diffeomorphisms. We have to reduce the Teichmüller space $\mathcal{T}_g$ to moduli space $\mathcal{M}_g$

$$\mathcal{M}_g = \frac{M_h}{\text{Weyl} \times \text{Diff}} = \frac{\mathcal{T}_g}{\text{MCG}} \tag{3.31}$$

with the mapping class group MCG, which is

$$\text{MCG} = \frac{\text{Diff}}{\text{Diff}_0}. \tag{3.32}$$

For genus-$g \geq 2$ Riemann surfaces there are no conformal killing vectors but $3g-3$ complex moduli, whose number is identical to the complex dimension of the moduli space.

Choose $2g$ linear independent cycles $a_i, b_i (i = 1, \cdots, g)$ on the surfaces as a canonical homology basis with the property

$$\begin{aligned} (a_i, a_j) &= (b_i, b_j) = 0, \\ (a_i, b_j) &= -(b_i, a_j) = \delta_{ij}. \end{aligned} \tag{3.33}$$

(a, b) means the intersecting pairing of two homology cycles. Then one defines the Abelian differentials $\omega_i, \bar{\omega}_i$ as

$$\int_{a_i} \omega_j = \delta_{ij}, \tag{3.34}$$

and the period matrix Ω_{ij} is determined as

$$\Omega_{ij} = \int_{b_i} \omega_j. \tag{3.35}$$

The dimension of Ω_{ij} coincides with the dimension of the moduli space for $g = 0, 1, 2, 3$, thus it can be used to parameterize conformally inequivalent Riemann surfaces for $g \leq 3$.

Modular transformations are the disconnected diffeomorphisms which act non-trivially on the given homology basis, and when $g > 1$ it is a subgroup of the MCG. Modular transformations can be presented as a $2g \times 2g$ matrix in

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.36}$$

where a, b are the homology basis and A, B, C, D are $g \times g$ matrices. To preserve the property (3.33), the $2g \times 2g$ matrix should be an element of the symplectic group $Sp(2g, \mathbb{Z}) = \{M \in M_{2g\times 2g}(\mathbb{Z}) : M^\top J M = J\}$,

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \tag{3.37}$$

Under the modular transformation, the Abelian differentials and the period matrix transform as

$$\omega'_j = \omega_k (C\Omega + D)^{-1}_{kj} \tag{3.38}$$

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1}. \tag{3.39}$$

The generators of modular transformations are called Dehn twist. They act along the canonical homology basis. We have two generators acting on a_i and b_i for each handle, and one generator for each cycle $a_i^{-1} a_{i+1}$ linking two consecutive handles. All the Dehn twist matrices generate the whole $Sp(2g, \mathbb{Z})$.

It is worth mentioning that there exists non-trivial disconnected diffeomorphisms twist around trivial cycles on the surface so that they do not affect the homology basis. These transformations form the Torelli group which is the quotient of the mapping class group and the modular group $Sp(2g, \mathbb{Z})$. In one-loop case the Torelli group is trivial.

Moduli space is trivial for tree level, and on torus as an example, there is only 1 complex moduli τ, cf. Table 2.1. The modular group $SL(2, \mathbb{Z})$, which is identical to $Sp(2,\mathbb{Z})$ in 1-loop, is generated by two transformations $\tau \to \tau+1$ and $\tau \to \frac{\tau}{\tau+1}$. And they corresponds to two Dehn twists

$$D_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad D_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.40}$$

along two canonical homology basis a and b[1]. Instead, one often uses

$$\begin{aligned} T: &\quad \tau \to \tau + 1, \\ S: &\quad \tau \to -\frac{1}{\tau} \end{aligned} \tag{3.41}$$

as the generators of the modular group.

Another example is $g = 2$ showed in Figure 3.1, where the generators of $Sp(4,\mathbb{Z})$ are given by the Dehn twists:

$$D_{a_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{b_1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.42}$$

$$D_{a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad D_{b_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{a_1{}^{-1}a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}. \tag{3.43}$$

We see that the action of Dehn twist, taking D_{a_1} as example, could be expressed as cutting the genus-2 torus along a_1, twisting along the a_1 cycle, and gluing back two ends together. We would need $g = 2$ moduli transformations when we discuss $g = 3/2$ correction because $g = 3/2$ amplitudes are derived from $g = 2$ amplitudes by involution.

3.3.2 Tree level amplitudes

When we talk about m-loop ($m \geq 1$), we study the m-th order amplitudes $Z^{(g)}$ of (3.1) on genus-$g = m$ surfaces. Tree level is a bit different. It means surfaces with positive Euler number, which are genus-$g = 0$ surfaces with a possible hole or cross-cap, thus means 0 and $\frac{1}{2}$-loops.

[1] cf. [23, Figure 6.8]

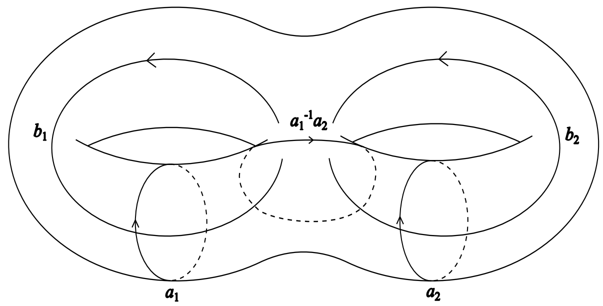


Figure 3.1: Homology basis of genus-2 torus

Since in tree level $\chi > 0$, the moduli number μ vanish. According to Riemann-Roch theorem (2.25), sphere ($h = b = c = 0$) has 6 CKVs, both disc ($h = c = 0, b = 1$) and projective plane ($h = b = 0, c = 1$) have 3 CKVs.

As in the quantum field theory, expectation values of vertex operators are the most basic quantities in string theory. There are several different methods of calculating the expectation values. We would extensively use the path integral method in the main calculation of this work, thus we only give the brief introduction to bosonic path integral calculation.[2]

Path Integral Calculation of Bosonic Expectation values

Begin with the generating functional

$$Z_B[J] = \left\langle \exp\left(i \int d^2\sigma J(\sigma) \cdot X(\sigma)\right)\right\rangle, \tag{3.44}$$

$J_\mu(\sigma)$ arbitrary. If we expand $X^\mu(\sigma)$ in terms of the eigenstates X_I of $\nabla^2 X_I = -\omega_I^2 X_I$, we can make $Z_B[J]$ quadratic and express it as

$$Z_B[J] = i(2\pi)^d \delta^d(J_0)\left(\det{}' \frac{-\nabla^2}{4\pi^2\alpha'}\right)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}\int d^2\sigma d^2\sigma' J(\sigma)\cdot J(\sigma')G'(\sigma,\sigma')\right) \tag{3.45}$$

with $J_0 = \int d^2\sigma J(\sigma) X_0$ and the Green's function $G'(\sigma_1, \sigma_2) = \sum_{I\neq 0} \frac{2\pi\alpha'}{\omega_I^2} X_I(\sigma_1)X_I(\sigma_2)$. Prime of a function means excluding the zero mode contribution ($I \neq 0$).

[2]We mainly follow [61] in this section

Sphere

We use bosonic string on sphere as the simplest example. We take $J(\sigma) = \sum_{i=1}^{n} k_i \delta^2(\sigma - \sigma_i)$ and from the generating functional (3.44) we get the expectation value of n tachyon vertex operators on sphere S_2

$$\begin{aligned} A^n_{S_2}(k,\sigma) &= \left\langle \left[e^{ik_1\cdot X(\sigma_1)}\right]_r \left[e^{ik_2\cdot X(\sigma_2)}\right]_r \cdots \left[e^{ik_n\cdot X(\sigma_n)}\right]_r \right\rangle \\ &= iC^X_{S_2}(2\pi)^d \delta^d(\sum_i k_i) \exp\Bigg(-\sum_{\substack{i,j=1\\ i<j}}^{n} k_i \cdot k_j G(\sigma_i,\sigma_j)\Bigg) \end{aligned} \tag{3.46}$$

where $C^X_{S_2}$ is a topology-related constant and $\delta^d(\sum_i k_i)$ is the momentum conservation condition. Then we have to solve the Green's function G based on the specific topology of the surface and including the renormalized Green's function for self-contraction as well, which is

$$G(\sigma_1,\sigma_2) = -\frac{\alpha'}{2}\ln|z_{12}|^2 + \delta_{12}\Bigg(\frac{\alpha'}{2}\ln|z_{12}|^2 + \alpha'\omega(z,\bar z)\Bigg), \tag{3.47}$$

where ω is the Weyl factor.

With higher vertex operators than tachyons in the expectation value

$$\left\langle \prod_{i=1}^{n} \left[e^{ik_i\cdot X(z_i,\bar z_i)}\right]_r \prod_{j=1}^{p} \partial X^{\mu_j}(z'_j) \prod_{k=1}^{q} \bar\partial X^{\nu_k}(\bar z''_k) \right\rangle_{S_2}, \tag{3.48}$$

we will need to contract all X derivatives either with another derivative or with an exponential, then multiply the contraction result with the tachyon result to get the final result. Be aware that the contractions are also related to the topology of the surface.

Generalizations to other tree level surfaces are straightforward with restricting the coordinates and solving related contractions and Green's functions. A disk involves boundary operators while both a disk and a projective plane involve the method of image.

3.3.3 One-loop amplitudes

From (3.1) we know that 1-loop amplitudes are defined on genus-1 surfaces. There are 4 Riemann surfaces with genus-1. In 1-loop surfaces the quadratic differentials and CKVs are constants. A detailed derivation of both the bosonic and fermionic 1-loop partition functions of Type-IIB orientifold needed in this work could be found in app.D.

Torus The torus $\mathcal{T}$ is the only closed oriented genus-1 surface with 1 complex modulus $\tau = \tau_1 + i\tau_2$ and 2 CKVs. It is described as

$$(\sigma^1, \sigma^2) \cong (\sigma^1 + 2\pi, \sigma^2) \cong (\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2), \tag{3.49}$$

which can be thought of as rotate the end of a cylinder of circumference 2π and length $2\pi\tau_2$ by an angel of $2\pi\tau_1$ and glue the two ends together. We can think of torus as a closed string propagating along a loop, which in operator method gives the partition function

$$\langle 1 \rangle_{\mathcal{T}} \equiv Z_{\mathcal{T}}(\tau) = \mathrm{Tr}_{\text{closed}} \left[\exp(2\pi i\tau_1 P - 2\pi\tau_2 H) \right], \tag{3.50}$$

with P the Momentum operator and H the Hamiltonian.

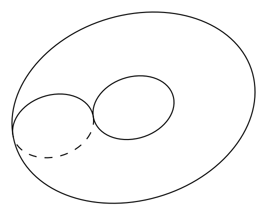


Figure 3.2: Torus

Annulus (Cylinder) The annulus $\mathcal{A}$ has 1 real modulus t and 1 CKV, and it is described as

$$0 \le \sigma_1 \le \pi, \qquad (\sigma^1, \sigma^2) \cong (\sigma^1, \sigma^2 + 2\pi i t). \tag{3.51}$$

We can think of a cylinder[3] as an open string propagating along a loop, with the partition function

$$\langle 1 \rangle_{\mathcal{A}} \equiv Z_{\mathcal{A}}(\tau) = \mathrm{Tr}_{\text{open}} \left[\exp(-2\pi t H)) \right]. \tag{3.52}$$

Klein Bottle The Klein Bottle $\mathcal{K}$ has 1 real modulus t and 1 CKV, and it is described as

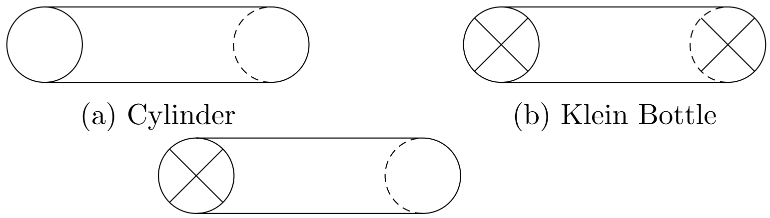
$$(\sigma^1, \sigma^2) \cong (\sigma^1 + 2\pi, \sigma^2) \cong (-\sigma^1, \sigma^2 + 2\pi t), \tag{3.53}$$

and the partition function is

$$Z_{\mathcal{K}} = \mathrm{Tr}_{closed}[\Omega \exp(-2\pi t H)], \tag{3.54}$$

where Ω is the orientation-reversal operator (2.66).

[3] Cylinder and annulus are conformally equivalent, so we often use them interchangeably.



(a) Cylinder (b) Klein Bottle (c) Möbius Strip

Figure 3.3: 1-loop surfaces represented in tree channel diagram

Möbius Strip The Möbius Strip $\mathcal{M}$ has 1 real modulous t and 1 CKV, and it is described as

$$0 \leq \sigma^1 \leq \pi, \qquad (\sigma^1, \sigma^2) \cong (-\sigma^1 + \pi, \sigma^2 + 2\pi t). \tag{3.55}$$

and the partition function is

$$Z_{\mathcal{M}} = \mathrm{Tr}_{open}[\Omega \exp(-2\pi t H)]. \tag{3.56}$$

Tree channel diagram We can always represent the fundamental domain of Klein Bottle (Möbius Strip) in a tree channel diagram between two cross-caps or one cross-cap and one hole, respectively. See Figure 3.3a, 3.3b and 3.3c. Then the string boundary states can be used to compute the tree level diagram.

Involution $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{M}$ surfaces can be constructed from double-covering torus under anti-conformal involutions[4]

$$I_{\mathcal{A}}(z) = I_{\mathcal{M}}(z) = 1 - \bar{z}, \qquad I_{\mathcal{K}}(z) = 1 - \bar{z} + \tau/2, \tag{3.57}$$

with $\tau = \tau_1 + i\tau_2$ be the modular parameter of the torus. And the fundamental regions of the involutions are chosen to be

$$\mathcal{A} : z \in [0, \frac{1}{2}] \times [0, \tau_2] \qquad \mathcal{M} : z \in [\frac{1}{2}, 1] \times [0, \tau_2] \qquad \mathcal{K} : z \in [0, 1] \times [0, \frac{\tau_2}{2}]. \tag{3.58}$$

The relative modular parameters of the tori are:

$$\tau_{\mathcal{A}} = \frac{it}{2} \qquad \tau_{\mathcal{M}} = \frac{1}{2} + i\frac{t}{2} \qquad \tau_{\mathcal{K}} = 2it. \tag{3.59}$$

[4]Here we closely follow the Appendix of [7].

Chapter 4

1-Loop correction to the Einstein-Hilbert term in Type-IIB orientifolds

4.1 Low energy effective action

Low energy approximation of String Theory is always a crucial tool of string phenomenology. We would of course assume that particle physics and general relativity are emergent concepts of the low energy approximation of String Theory. To relate the quantum field theory to low energy approximation of String Theory, we have to find the low energy effective action first. The idea is that the low energy action should reproduce the amplitudes of massless string scattering. This can be done in a perturbative fashion[23]. It's always easy to write down the free action $\mathcal{L}_{2point}$ of massless particles. Then try to add $\mathcal{L}_{3point}$ to reproduce 3-point functions of massless string scattering. We can already relate various string constants to the coupling constants of the effective action. Next level is the 4-point. Massless contribution of 4-point amplitudes are generated by $\mathcal{L}_{3point}$, while massive contributions can be expanded and described by the $\mathcal{L}_{4point}$, see from [23, §16.3] for details. Higher order terms can be carried out order by order with the same fashion.

There is another method to restrict the effective action, which is using the spacetime symmetries such as supersymmetry or coordinate invariance. Then 10 dimensional effective supergravity theories of the string theories with maximal supersymmetry could be fixed in this routine. In most cases both methods would be combined to find the effective action. We would not dive into the derivation of the low energy effective actions here.

We consider only the massless modes, that the effective action is the supergravity action, the derivation of the relevant effective action would not be presented in this work. In the string low energy approximation, the supergravity action depends on three functions: the holomorphic superpotential $W(\Phi)$; an arbitrary holomorphic function $f_{ab}(\Phi)$ replacing the gauge coupling g_a^{-2}; the Kähler potential $K(\Phi, \Phi^*)$ which is a general function of the superfields.[1] To demonstrate, the purely bosonic part of the Lagrangian density is

$$\begin{aligned}\mathcal{L}_{bos} \propto \frac{1}{2\kappa^2} R - K_{,\bar{i}j} D_\mu \phi^{i*} D^\mu \phi^j - \frac{1}{4}\,\mathrm{Re}(f_{ab}(\phi)) F^a_{\mu\nu} F^{b\mu\nu} \\ -\frac{1}{8}\,\mathrm{Im}(f_{ab}(\phi)) \epsilon^{\mu\nu\sigma\rho} F^a_{\mu\nu} F^b_{\sigma\rho} - V(\phi, \phi^*),\end{aligned} \tag{4.1}$$

and R is Ricci scalar. The potential is

$$V(\phi, \phi^*) = \exp\big(\kappa^2 K\big)\big(K^{\bar{i}j} W^*_{;i} W_{;j} - 3\kappa^2 W^* W\big) + \frac{1}{2} f_{ab} D^a D^b. \tag{4.2}$$

Here $K^{\bar{i}j}$ is the inverse matrix to $\partial_{\bar{i}} \partial_j K$ and

$$W_{;i} = \partial_i W + \kappa^2 \partial_i K W \tag{4.3}$$

$$\mathrm{Re}(f_{ab}(\phi)) D^b = -2\xi_a - K_{,i} t^a_{ij} \phi^j. \tag{4.4}$$

where ξ_a is a Fayet–Iliopoulos parameter for a $U(1)$ symmetry. The negative term proportional to κ^2 in $V(\phi, \phi^*)$ is a supergravity effect.

The kinetic term for the scalars is field-dependent. The second derivative

$$K_{,\bar{i}j} = \frac{\partial^2 K(\phi, \phi^*)}{\partial \phi^{i*} \partial \phi^j}, \tag{4.5}$$

in the form of Kähler metric, plays the role of a metric for the space of scalar fields.

Einstein Frame We know that it is always possible to transform the effective action by field redefinition without changing the physics. So different effective actions can reproduce the same string amplitudes if they only differ by field redefinition. By convention we call an effective action in "String Frame" when the action has an overall factor of $e^{-2\Phi 2}$; we call it in "Einstein Frame" when the dilaton and the graviton decouple by field redefinition. In Einstein frame we have a purely gravitational term.

[1] We follow [62, §B] closely here.

[2] Actually frame is defined by the metric. String frame metric is exactly the same as the metric in Polyakov action.

4.2 Corrections to the Type-IIB Einstein-Hilbert term

Our basic set-up is the 4D effective supergravity theory constructed from 10D Type-IIB orientifold compactified on an internal $T^6/\mathbb{Z}_N$ space. $T^6/\mathbb{Z}_N$ has singularities as a Calabi-Yau space, thus it is the limit of a real Calabi-Yau space. This leads to a 4D $\mathcal{N}=1$ supergravity which is simple enough to be tactable and phenomenologically interesting. Calculations of n-point string amplitudes contribute to the effective supergravity action. We are not interested in the details of compactification, so we skip the introduction to compactification and just mention that the corrections in the 4D theories came from compactification from 10D to 4D.

Terms of order $(k^2)^n (n > 2)$ in the kinematic tensor structure of the pure graviton string amplitudes correspond to the R^n term (Riemann tensor) in the effective action in low energy limit. For example, an R^2 term in the effective action would give an amplitude of order k^4. No k^4 or k^6 term exists in 3 or 4-point tree level amplitudes of type II theories, thus there is no R^2 or R^3 term in the type II effective actions at tree level.[3]

However, in the tree level 4-point type II amplitudes with 4 massless NS-NS bosons

$$-\frac{i\kappa^2\alpha'^3}{4}\frac{\Gamma(-\frac{1}{4}\alpha's)\Gamma(-\frac{1}{4}\alpha't)\Gamma(-\frac{1}{4}\alpha'u)}{\Gamma(1+\frac{1}{4}\alpha's)\Gamma(1+\frac{1}{4}\alpha't)\Gamma(1+\frac{1}{4}\alpha'u)}K_{\mathrm{c}}(e_1,e_2,e_3,e_4) \tag{4.6}$$

with kinematic structure

$$K_{\mathrm{c}}(e_1,e_2,e_3,e_4) = t^{\mu_1\nu_1\ldots\mu_4\nu_4}t^{\rho_1\sigma_1\ldots\rho_4\sigma_4}\prod_{j=1}^{4} e_{j\mu_j\rho_j}k_{j\nu_j}k_{j\sigma_j}, \tag{4.7}$$

tensor t representing the kinematic structure[4] and s, t, u being Mandelstam variables, one observes that there exists k^8 terms, which correspond to the R^4 term in the effective action if one contracts $e_{\mu\rho}k_\nu k_\sigma$ with t to get $R_{\mu\nu\sigma\rho}/4\kappa$[62]. The ratio of gamma functions can be expanded as

$$-\frac{64}{\alpha'^3 stu} - 2\zeta(3) + \mathcal{O}(\alpha') \tag{4.8}$$

with

$$\zeta(k) = \sum_{m=1}^{\infty}\frac{1}{m^k}. \tag{4.9}$$

[3]Details of tree level R^n terms could be found in [62, §12.4]

[4]Tensor t is often called t^8 and the exact form could be found in [62, (12.4.25)].

The first term in the amplitude is proportional to κ^2 without α' dependence and it arises from the Einstein-Hilbert term. The second term of zeta function is an R^4 term.

In [44, §9.2.3], the authors claimed that "The open-string amplitude with four external massless states has the same overall kinematic factor K for both tree and one-loop amplitudes (and probably multiloop amplitudes as well)". Thus we also expect that the kinematic factors stay the same for tree and one-loop amplitudes in closed string case, and this is indeed shown in 10D in [45].

There exists $R\wedge R\wedge R$ term inside the R^4 term that turns into the Euler number χ of the internal space after compactification to 4 dimensions[8], by using the relation

$$\int_{X_6} R\wedge R\wedge R = \frac{1}{3!(2\pi)^3}\chi. \tag{4.10}$$

This makes the R^4 term in 10 dimensions into an R correction term to the Einstein-Hilbert term in 4 dimensions as

$$\int_{M_4}\left[e^{-2\phi_4}+\frac{\chi}{(2\pi)^3}\left(2\zeta(3)\frac{e^{-2\phi_4}}{\mathcal{V}}+\frac{2\pi^2}{3}+\dots\right)\right]\sqrt{-h}R, \tag{4.11}$$

where $\mathcal{V}$ is the volume of the internal Calabi-Yau space and ϕ_4 is the 4 dimensional dilaton, be aware that α' is set to 2.

From (4.11) we see that the tree level plus torus correction to the Einstein-Hilbert term is

$$(\delta E)_{S_2+\mathcal{T}} = \frac{\chi}{(2\pi)^3}\left(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}}+\frac{2\pi^2}{3}\right). \tag{4.12}$$

Be aware that since we would like to consider the orientifold, we should add a factor of 1/2 to the torus contribution of the above term because of the orientifold projection[47], which gives

$$(\delta E)_{S_2+\mathcal{T}} = \frac{\chi}{(2\pi)^3}\left(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}}+\frac{\pi^2}{3}\right) \qquad \text{(Orientifold)} \tag{4.13}$$

in orientifolds. 1-loop corrections of Annulus, Klein Bottle and Möbius strips in Type-IIB orientifold have been calculated already in [47].

Our focus in this thesis is to follow the work in [47] and try to extend it to higher genus and more-point amplitudes. Following the setup in [47], we would concentrate on the Kähler moduli metric of the T^2 torus in $K3\times T^2$. The Kähler moduli in 4D effective theory arise from CY compactification[5], and the moduli metric is the metric

[5]See from [22] for review.

in the kinetic term of the moduli in the 4D effective action. Upon compactification to 4 dimensions, the quantum corrected kinetic term of tree level modulus $\tau^{(0)}$ coupled to gravity in string frame and up to 1-loop order is given by[6]

$$S_4 = \frac{1}{\kappa_4^2}\int d^4x\sqrt{-h}\left[(e^{-2\Phi_4}+\delta E)\frac{1}{2}R+\left(\tilde{G}^{(0)}+\tilde{G}^{(1)}\right)\partial_\mu\tau^{(0)}\partial^\mu\tau^{(0)}\right]+\ldots, \quad (4.14)$$

where δE is the correction to the Einstein-Hilbert term, including tree level α' corrections, 1-loop g_s corrections. $\tilde{G}^{(0)}$ is the tree level moduli space metric including α' corrections and $\tilde{G}^{(1)}$ is the 1-loop contributions to the string frame moduli space metric. The next order to genus-1 correction is genus-$\frac{3}{2}$. Genus-$\frac{3}{2}$ correction $\delta E^{(\frac{3}{2})}$ is in higher genus-terms (dots in (4.14)). Furthermore,

$$\kappa_4^{-2} = (2\pi\sqrt{\alpha'})^6\kappa_{10}^{-2} = (\pi\alpha')^{-1} \quad (4.15)$$

and

$$e^{-2\Phi_4} \equiv e^{-2\Phi_{10}}t_1t_2t_3 = \sqrt{\sigma^{(0)}\tau_1^{(0)}\tau_2^{(0)}\tau_3^{(0)}}, \quad (4.16)$$

where $e^{-2\Phi_{10}}$ is the 10 dimensional dilaton and

$$\sigma^{(0)} = e^{-\Phi_{10}}t_1t_2t_3, \qquad \tau_i^{(0)} = e^{-\Phi_{10}}t_i. \quad (4.17)$$

t_i are the dimensionless torus volumes measured with the string frame metric. The definition of the Kähler variables in general gets quantum corrected

$$\tau = \tau^{(0)} + \delta\tau, \quad (4.18)$$

where $\delta\tau$ is a moduli dependent function.

Starting from (4.14) and performing a Weyl transformation to go to Einstein frame, one observes that the quantum correction to the metric of quantum corrected Kähler modulus T (with imaginary part τ), is given, up to 1-loop order, by

$$\begin{aligned} G^{(1)}_{T\bar{T}}(T) =& e^{2\Phi_4}\tilde{G}^{(1)}(\tau) + 12\left(\frac{\partial\Phi_4}{\partial\tau^{(0)}}\right)^2\delta E e^{2\Phi_4} + 6\frac{\partial\Phi_4}{\partial\tau^{(0)}}\frac{\partial\delta E}{\partial\tau^{(0)}}e^{2\Phi_4} \\ & - \delta E e^{4\Phi_4}\tilde{G}^{(0)}(\tau) + \frac{1}{2\tau^3}\delta\tau - \frac{1}{2\tau^2}\frac{\partial\delta\tau}{\partial\tau} + \ldots. \end{aligned} \quad (4.19)$$

We can see that δE showed up in different terms. Therefore we can conclude that δE does play an important role in the quantum correction to Kähler metric. Technical details of calculation of partition functions and analysis of surfaces are given in app.D.1 and app.D.2 respectively.

[6]This section and the following three sections are cited from [54].

4.3 Graviton 1-loop 2-point function

In this section we derive some general formulas needed for computing 1-loop correction to the Planck mass in $\mathcal{N}=1$ Type-IIB toroidal orientifolds. Tadpole-free condition was discussed in [2]). We are going to discuss general features in this section, and apply them to $K3\times T^2$ space in §5. Here we follow closely to [47, §3].

Begin with an amplitude of two gravitons (with momenta p_i and polarization tensors ε_i)

$$\langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle = \sum_{\sigma\in\{\mathcal{T},\mathcal{K},\mathcal{A},\mathcal{M}\}} \langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle_\sigma, \tag{4.20}$$

where the vertex operators are given by

$$V_g(p,\varepsilon) = -\frac{2g_c}{\alpha'}\varepsilon_{\mu\nu}\Big(i\partial X^\mu + \frac{\alpha'}{2}p\cdot\psi\psi^\mu\Big)\Big(i\bar\partial X^\nu + \frac{\alpha'}{2}p\cdot\bar\psi\bar\psi^\nu\Big)e^{ip\cdot X} \tag{4.21}$$

with $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu}=1$. Using on-shell, transversality and tracelessness conditions

$$p_1^2 = p_2^2 = p_1\cdot p_2 = p_{1\mu}\varepsilon_1^{\mu\nu} = p_{2\mu}\varepsilon_2^{\mu\nu} = \eta_{\mu\nu}\varepsilon_1^{\mu\nu} = \eta_{\mu\nu}\varepsilon_2^{\mu\nu} = 0, \tag{4.22}$$

the amplitude (4.20) has to be proportional to the only remaining contraction, i.e.

$$\langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle = AiV_4 g_c^2 p_2^\mu \varepsilon_{1\mu\nu}\eta^{\nu\lambda}\varepsilon_{2\lambda\rho}p_1^\rho + \mathcal{O}(p^4). \tag{4.23}$$

We have to compare this to the relevant term in the action which leads to linearized Einstein equations, because both of them represent the 2-point amplitude of gravitons. We read off

$$S = \frac{M_P^2}{2}\int d^4x\Big(-\frac{1}{2}h_{\mu\nu,\rho}h^{\nu\rho,\mu}\Big), \tag{4.24}$$

where

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.25}$$

for a symmetric fluctuation $h_{\mu\nu}$. $h_{\mu\nu}$ and $\varepsilon_{\mu\nu}$ have the relation in momentum space showed by the vertex operator (4.21)

$$h_{\mu\nu} = -4\pi g_c \varepsilon_{\mu\nu}e^{ip\cdot X}. \tag{4.26}$$

Using (4.14), we have

$$M_P^2 = \frac{1}{\kappa_4^2}(e^{-2\Phi_4} + \delta E). \tag{4.27}$$

Thus we compare (4.23) with

$$-\frac{1}{4}\kappa_4^2 \int d^4x \delta E h_{\mu\nu,\rho} h^{\nu\rho,\mu}. \tag{4.28}$$

And we get

$$\delta E = \frac{\kappa_4^2}{8\pi^2} A = \frac{\alpha'}{8\pi} A. \tag{4.29}$$

The amplitude A gets contributions from all 1-loop surfaces, i.e. $\mathcal{T}$, $\mathcal{K}$, $\mathcal{A}$, $\mathcal{M}$.

4.4 One-loop surfaces

A detailed analysis of one-loop surfaces can be found in app.D.2. Torus and Sphere contributions are already given in (4.13).

4.4.1 Contributions from $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$

Here we closely follow the calculation in [36]. Neglecting the momentum conservation δ function arising from the bosonic zero mode integration we have

$$\begin{aligned} A_\sigma = -\frac{1}{8N(4\pi^2\alpha')^2} \sum_{s=\text{even}} \int_0^\infty \frac{dt}{t^3} \sum_{k=0}^{N-1} Z_\sigma^{(\ell)}(\tau_\sigma, s) \int_\sigma d^2\nu_1 \int_\sigma d^2\nu_2 \\ \Big(\langle \bar\partial X_1 \bar\partial X_2 \rangle_\sigma (\langle \psi_2 \psi_1 \rangle_\sigma^s)^2 + \langle \partial X_1 \bar\partial X_2 \rangle_\sigma (\langle \psi_2 \bar\psi_1 \rangle_\sigma^s)^2 \\ + \langle \bar\partial X_1 \partial X_2 \rangle_\sigma (\langle \bar\psi_2 \psi_1 \rangle_\sigma^s)^2 + \langle \partial X_1 \partial X_2 \rangle_\sigma (\langle \bar\psi_2 \bar\psi_1 \rangle_\sigma^s)^2 \Big) \end{aligned} \tag{4.30}$$

where σ stands for the different world-sheet topologies $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$, with world-sheet parameters $\tau_\mathcal{K} = 2it$, $\tau_\mathcal{A} = \frac{it}{2}$, $\tau_\mathcal{M} = \frac{1}{2} + \frac{it}{2}$. $Z_\sigma^{(\ell)}(\tau_\sigma, s)$ is the contribution (D.48) to the partition function from the θ^ℓ element inserted sector. The spin structure sum only runs over the even spin structures s. Note that there is no contribution to A_σ from eight fermion terms, cf. [14, §3.4].

From [36], we use

$$(\langle \psi_2(\nu)\psi_1(0) \rangle_\sigma^s)^2 = -\partial_\nu^2 \ln \vartheta_1(\nu,\tau) + \partial_v^2 \frac{\vartheta_s(v,\tau)}{\vartheta_s(0,\tau)}\bigg|_{v=0}. \tag{4.31}$$

It is the sum of a spin structure independent term with a spin structure dependent term. The contribution to A_σ involving the first term in (4.31) (the spin structure independent term) does not survive the sum over spin structures in the supersymmetric case. On the other hand, the spin structure dependent term does not depend on the vertex operator position and, thus can be taken out of the ν integrals. Besides, provided that it does depend on the vertex operator position, this is the same for $(\langle\psi_2\psi_1\rangle_\sigma^s)^2$, $(\langle\psi_2\bar{\psi}_1\rangle_\sigma^s)^2$, $(\langle\bar{\psi}_2\psi_1\rangle_\sigma^s)^2$ and $(\langle\bar{\psi}_2\bar{\psi}_1\rangle_\sigma^s)^2$. Take care of the relative minus signs arising from conventions, the resulting ν integral can be solved using [7]

$$\int_\sigma d^2\nu_1 \int_\sigma d^2\nu_2 \Big(\langle\bar{\partial}X_1\bar{\partial}X_2\rangle_\sigma - \langle\partial X_1\bar{\partial}X_2\rangle_\sigma - \langle\bar{\partial}X_1\partial X_2\rangle_\sigma + \langle\partial X_1\partial X_2\rangle_\sigma\Big) = \frac{\alpha'\pi\,\mathrm{Im}(\tau_\sigma)}{2}. \tag{4.32}$$

Taking into account (4.29), we finally achieve

$$\begin{aligned}
(\delta E)_\sigma &= -\frac{\alpha'}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2}\partial_v^2 \sum_{s=\text{even}} \int_0^\infty \frac{dt}{t^3}\sum_{\ell=0}^{N-1} Z_\sigma^{(\ell)}(\tau_\sigma,s)\frac{\vartheta_s(v,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)}\frac{\alpha'\pi\,\mathrm{Im}(\tau_\sigma)}{2}\Bigg|_{v=0} \\
&= -\frac{(\alpha')^2}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^3}\frac{\pi\,\mathrm{Im}(\tau_\sigma)}{2}\sum_{\ell=0}^{N-1}\partial_v^2\sum_{s=\text{even}} Z_\sigma^{(\ell)}(\tau_\sigma,s)\frac{\vartheta_s(v,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)}\Bigg|_{v=0} \\
&= -\frac{(\alpha')^2}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^3}\frac{\pi\,\mathrm{Im}(\tau_\sigma)}{2}\sum_{\ell=0}^{N-1}\sum_{s=\text{even}} Z_\sigma^{(\ell)}(\tau_\sigma,s)\frac{\vartheta_s''(0,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)}.
\end{aligned} \tag{4.33}$$

The lattice sums can be done after performing the spin-structure summation. Thus the sum over spin-structure in (4.33) can be performed using (D.48) and (D.49) for the partition function. Then we need the formula (cf. [15, (130)])

$$\sum_{s=\text{even}} Z_s^{(\ell)}\frac{\vartheta_s''(0)}{\vartheta_s(0)} = \sum_{i=1}^{3}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}. \tag{4.34}$$

With this, (4.33) reads

$$\begin{aligned}
(\delta E)_\sigma = &-\frac{\pi(\alpha')^2}{32N(4\pi^2\alpha')^2}\int_0^\infty\frac{dt}{t^2}\frac{\mathrm{Im}(\tau_\sigma)}{t}\sum_{\ell=0}^{N-1} CP_\sigma\tilde{\chi}_\sigma\sin(\pi\gamma_3) \\
&\cdot\left(\prod_{j=1}^{2} f(\gamma_j)\right)\sum_{i=1}^{3}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}.
\end{aligned} \tag{4.35}$$

4.5 Supersymmetric sectors

Here we discuss the contributions from different supersymmetric sectors.

4.5.1 $\mathcal{N}=1$ sectors

Following [14, §3.8-3.11], $\mathcal{N}=1$ sectors contribution to the Planck mass is

$$(\delta E)^{(\mathcal{N}=1)} = \sum_\sigma (\delta E)_\sigma^{(\mathcal{N}=1)} = -\frac{\pi(\alpha')^2}{64N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\sum_\sigma \sum_{\ell\in\{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)}. \tag{4.36}$$

Here

$$\sigma^{(\ell)} = \tilde{e}_\sigma \tilde{\chi}_\sigma \sin(\pi\gamma_3)\left(\prod_{j=1}^{2} f(\gamma_j)\right)\hat{\sigma}^{(\ell)} \qquad \text{for } \ell \in \{\mathcal{N}=1\} \tag{4.37}$$

with

$$\tilde{e}_\sigma = \begin{cases} 1 & \text{for } \mathcal{A},\mathcal{M} \\ 4 & \text{for } \mathcal{K} \end{cases} \tag{4.38}$$

and

$$\hat{\sigma}^{(\ell)} = \sum_{i=1}^{3} \frac{\vartheta'\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}{\vartheta\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}. \tag{4.39}$$

For later use, we also introduce

$$e_\sigma = \begin{cases} 1 & \text{for } \mathcal{A} \\ 4 & \text{for } \mathcal{M},\mathcal{K} \end{cases} \quad , \tag{4.40}$$

From (4.38)-(4.40) and Table D.2, we have

$$\begin{aligned}
\mathcal{K}_u^{(\ell)} &= 16\sin(2\pi\ell v_3)\sin(2\pi\ell v_1)\sin(2\pi\ell v_2)\hat{\mathcal{K}}_u^{(\ell)}, \\
\mathcal{K}_t^{(\ell)} &= 4\tilde{\chi}(\theta^{N/2},\theta^\ell)\sin(2\pi\ell v_3)\hat{\mathcal{K}}_t^{(\ell)}, \\
\mathcal{A}_{99}^{(\ell)} &= 4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{A}}_{99}^{(\ell)}, \\
\mathcal{A}_{55}^{(\ell)} &= 4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{A}}_{55}^{(\ell)}, \\
\mathcal{A}_{95}^{(\ell)} &= 2\sin(\pi\ell v_3)\hat{\mathcal{A}}_{95}^{(\ell)}, \\
\mathcal{M}_9^{(\ell)} &= -4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{M}}_9^{(\ell)}, \\
\mathcal{M}_5^{(\ell)} &= -4\sin(\pi\ell v_3)\cos(\pi\ell v_1)\cos(\pi\ell v_2)\hat{\mathcal{M}}_5^{(\ell)}.
\end{aligned} \tag{4.41}$$

Note that for odd N there is no contribution from $\mathcal{K}_t$, $\mathcal{A}_{55}$, $\mathcal{A}_{95}$ and $\mathcal{M}_5$.

Making use of (A.10) and the fact that the even/odd spin structure ϑ functions are even/odd functions of their argument, together with the super-symmetry condition $\sum_i v_i = 0$, we can get

$$\begin{aligned} \hat{\sigma}^{(qN\pm\ell)} &= \pm\hat{\sigma}^{(\ell)} \qquad \text{for all } \sigma, \qquad \hat{\sigma}^{(\frac{qN}{2}\pm\ell)} = \pm\hat{\sigma}^{(\ell)} \qquad \text{for } \mathcal{K}, \\ \sigma^{(qN\pm\ell)} &= \sigma^{(\ell)} \qquad \text{for all } \sigma, \qquad \sigma^{(\frac{qN}{2}\pm\ell)} = \sigma^{(\ell)} \qquad \text{for } \mathcal{K}. \end{aligned} \tag{4.42}$$

q is an arbitrary integer. These identities allow the individual sectors to be related to each other.

For $\mathcal{N}=1$ sectors with $h_i = 0$, the t-integral in (4.36) can be performed using [14, (115)- (117)], i.e. (assuming $0<\gamma<1$ for $\mathcal{A}$ and $\mathcal{K}$, and $0<\gamma<1/2$ for $\mathcal{M}$)

$$\begin{aligned} I_{\mathcal{A}/\mathcal{K}}(\gamma) &= \int_{\frac{1}{e_\sigma\Lambda}}^{\infty} \frac{dt}{t^2}\frac{\vartheta_1'(\gamma,\tau_\sigma)}{\vartheta_1(\gamma,\tau_\sigma)} \\ &= e_\sigma\pi(1-2\gamma)\Lambda^2 + e_\sigma\frac{\pi}{24}[\psi'(\gamma)-\psi'(1-\gamma)], \end{aligned} \tag{4.43}$$

$$\begin{aligned} I_{\mathcal{M}}(\gamma) &= \int_{\frac{1}{4\Lambda}}^{\infty} \frac{dt}{t^2}\frac{\vartheta_1'(\gamma,\frac{1}{2}+\frac{it}{2})}{\vartheta_1(\gamma,\frac{1}{2}+\frac{it}{2})} \\ &= 8\pi(1-4\gamma)\Lambda^2 + \frac{\pi}{12}[\psi'(\gamma)-\psi'(1-\gamma)-\frac{1}{2}\psi'(\frac{1}{2}+\gamma)+\frac{1}{2}\psi'(\frac{1}{2}-\gamma)]. \end{aligned} \tag{4.44}$$

Here $\psi'(x)$ denotes the trigamma function, i.e. the derivative of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$.

The t-integral of terms with $h_i = \pm 1/2$, appearing in $\mathcal{K}_t$ and $\mathcal{A}_{95}$, is given in app.E where we find (for $0<\gamma<1$)

$$\tilde{I}_{\mathcal{A}/\mathcal{K}}(\gamma) = \int_{\frac{1}{e_\sigma\Lambda}}^{\infty} \frac{dt}{t^2}\frac{\vartheta_4'(\gamma,\tau_\sigma)}{\vartheta_4(\gamma,\tau_\sigma)} = e_\sigma\pi(1-2\gamma)\Lambda^2 - e_\sigma\frac{\pi}{48}[\psi'(\gamma)-\psi'(1-\gamma)]. \tag{4.45}$$

Furthermore, the t-integral for $\mathcal{M}$ when $\gamma > \frac{1}{2}$ is computed in app.E.1 where we find (for $\frac{1}{2}<\gamma<1$)

$$\begin{aligned} \tilde{I}_{\mathcal{M}}(\gamma) &= \int_{\frac{1}{4\Lambda}}^{\infty} \frac{dt}{t^2}\frac{\vartheta_1'(\gamma,\frac{1}{2}+\frac{it}{2})}{\vartheta_1(\gamma,\frac{1}{2}+\frac{it}{2})} \\ &= 8\pi(3-4\gamma)\Lambda^2 - \frac{\pi}{24}\Big[\psi'(\gamma-\frac{1}{2})-\psi'(\frac{3}{2}-\gamma)+2\psi'(1-\gamma)-2\psi'(\gamma)\Big]. \end{aligned} \tag{4.46}$$

4.5.2 $\mathcal{N} \geq 2$ sectors

$\mathcal{N} = 2$ sectors are characterized by the fact that along exactly one torus (say the n-th torus) h_n vanishes and $\gamma_n + g_n$ is integer. Thus one needs to take the limit of (4.35)

$$(-2\sin\pi(\gamma_n+g_n))\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)} \to \frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)}{\eta^3}\mathcal{L}^{[n,M/W]}$$
$$= (-2\pi)(-1)^{\gamma_n+g_n}\mathcal{L}^{[n,M/W]}. \qquad (4.47)$$

To summarize, the $\mathcal{N} = 2$ sector contribution is given by

$$(\delta E)^{(\mathcal{N}=2)} = \sum_\sigma (\delta E)_\sigma^{(\mathcal{N}=2)} = -\frac{\pi(\alpha')^2}{64N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\sum_\sigma \sum_{\ell\in\{\mathcal{N}=2\}} CP_\sigma \sigma^{(\ell)}. \qquad (4.48)$$

Here

$$\sigma^{(\ell)} = \pi\tilde{e}_\sigma\tilde{\chi}_\sigma D_\sigma^{(\ell)}\mathcal{L}^{[n,M/W]} \quad \text{for } k \in \{\mathcal{N}=2\}, \qquad (4.49)$$

and the constant factor $D_\sigma^{(\ell)}$ is given by

$$D_\sigma^{(\ell)} = (-1)^{\gamma_n+g_n}\prod_{i\neq n}^{3} f(\gamma_i) \qquad (4.50)$$

with $f(\gamma_3) = -2\sin\pi\gamma_3$. n depends on ℓ and σ.

Let us express (C.22) and (C.23) collectively as

$$\mathcal{L}^{[n,M/W]} = \frac{C^{[n,M/W]}}{t}\sum_{m^1,m^2} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[n,M/W]}}, \qquad (4.51)$$

where

$$C^{[n,M/W]} = \begin{cases}\frac{V_n}{4\pi^2\alpha'} & \text{for M (momentum sum)}\\ \frac{4\pi^2\alpha'}{V_n} & \text{for W (winding sum)}\end{cases} \qquad (4.52)$$

and

$$g_{ab}^{[n,M/W]} = \begin{cases}g_{ab}^{[n]} & \text{for M (momentum sum)}\\ g^{[n]ab} & \text{for W (winding sum)}\end{cases}, \qquad (4.53)$$

i.e. $g_{ab}^{[n,W]}$ is the inverse matrix of $g_{ab}^{[n,M]}$.

Now we split $\mathcal{L}^{[n,M/W]}$ as

$$\begin{aligned}\mathcal{L}^{[n,M/W]} =& \frac{C^{[n,M/W]}}{t}\left(1+\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[n,M/W]}}\right)\\ =& \frac{C^{[n,M/W]}}{t} + \mathcal{L}'^{[n,M/W]}\end{aligned} \tag{4.54}$$

with

$$\mathcal{L}'^{[n,M/W]} = \frac{C^{[n,M/W]}}{t}\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[n,M/W]}}. \tag{4.55}$$

Then we have

$$\int_{\frac{1}{e_\sigma\Lambda}}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[n,M/W]} = \frac{C^{[n,M/W]}e_\sigma^2\Lambda^2}{2} + \int_0^\infty\frac{dt}{t^2}\mathcal{L}'^{[n,M/W]}. \tag{4.56}$$

Here we set $\Lambda = \infty$ in the second term on the right hand side since it is finite in the limit $\Lambda = \infty$. It can be evaluated using (app.§E)

$$\begin{aligned}\Gamma^{[n,M/W]} \equiv& \int_0^\infty \frac{dt}{t^3}\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[n,M/W]}}\\ =& \begin{cases}\frac{(4\pi^2\alpha')^2}{\pi^2 V_n^2}E_2(U^{[n]}) & \text{for M (momentum sum)}\\ \frac{V_n^2}{\pi^2(4\pi^2\alpha')^2}E_2\Big(-\frac{1}{U^{[n]}}\Big) & \text{for W (winding sum)}\end{cases},\end{aligned} \tag{4.57}$$

where $U^{[n]}$ is the complex structure of the n-th torus and E_2 is a non-holomorphic Eisenstein series, cf. (E.7).

For $\mathcal{N}=4$ sectors h_i vanish and $\gamma_i + g_i$ are integer along all three tori. Thus, the numerator of (4.35) has a triple zero which can not be balanced by the simple zero in the denominator. Consequently the $\mathcal{N}=4$ sectors do not contribute.

Part II

Loop corrections to the Einstein-Hilbert term

Chapter 5

$\mathcal{O}(k^2)$ 1-loop 3-point graviton amplitude

We are interested in confirming the calculation of corrections through graviton 3-point amplitudes, because it could help to provide complementary information as well as to clarify ambiguities in 2-point calculations. Graviton 3-point amplitude was studied in [9] without application of Minahan's approach (see §5.1.2), apparently their result was incomplete. We would like to reproduce $\mathcal{O}(k^2)$ graviton 3-point amplitudes of Heterotic theory[40] where pinched-off integration was considered, and extend it to Type-I theory.

As we have mentioned in §4.1, looking for the effective action requires to compare the amplitudes calculated from string theories with the amplitudes calculated from the effective action. Expanding the Einstein-Hilbert term in the effective action around the flat metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we get the first non-vanishing contribution of gravity at 3-point level[40]:

$$\begin{aligned} \sqrt{g}R|_{h^3} &= h_{\mu\nu}\,(h,^{\mu\nu} h) + 2h_{\mu\nu},^{\sigma} h^{\nu\rho},^{\mu} h_{\rho\sigma} && (5.1) \\ &\to (k_2\epsilon_1 k_2)\,(\epsilon_2\epsilon_3) + 2\,(k_3\epsilon_2\epsilon_3\epsilon_1 k_2) + \text{cyclic perm.} && (5.2) \end{aligned}$$

$h_{\mu\nu}$ is required to satisfy the harmonic gauge condition $\Box h_{\mu\nu} = 0, \partial^\mu h_{\mu\nu} = 0$ and the tracelessness condition $h = 0$, to be able to correspond to string amplitudes. We expect that at tree level and 1-loop level the calculation of string amplitudes gives exactly the above kinematic structure $(k_2\epsilon^1 k_2)\,(\epsilon^2\epsilon^3) + 2\,(k_3\epsilon^2\epsilon^3\epsilon^1 k_2) +$ cyclic perm. at $\mathcal{O}(k^2)$ order.

We apply the pinched-off integration and Taylor expansion in order to extract extra $\mathcal{O}(k^2)$ contributions from higher k order terms. We first try to reproduce the Heterotic kinematic structure calculation in [40]. This was done in 2 different routine: one by direct calculation of vertex operator contractions, and the other one by operator product expansion calculation. The exact calculations were completed in collaboration with Dr. Harold Erbin and Dr. Jin U Kang. Then we lift the result in Heterotic to Type-I by the lifting technique (see from e.g.[7][9]). For simplicity, we rename the two-point functions $P(z_i, z_j) \equiv P_{ij}$ and $S(z_i, z_j) \equiv S_{ij}$ in this chapter.

5.1 Preliminary

We follow the notation in [23] that terms involving $\bar{z}$ are left-moving (anti-holomorphic) and terms involving z are right-moving (holomorphic).

5.1.1 Modular invariance and Transversality

We would require modular invariance condition[1]

$$k_1 \cdot k_2 + k_2 \cdot k_3 + k_3 \cdot k_1 = 0. \tag{5.3}$$

Polarization transversality condition[2].

$$k^\mu \epsilon_{\mu\nu} = 0 \tag{5.4}$$

and momentum conservation

$$k_1 + k_2 + k_3 = 0 \tag{5.5}$$

would be imposed after pinched-off integration as we will mention in the following §5.1.2.

5.1.2 Pinched-off Integration

The kinematic structure of $\mathcal{O}(k^2)$ contribution in 3-point 1-loop calculation does not match (5.2). However we do not expect such a mismatch between string calculation and effective action calculation. To get the exact gravity kinematic structure (5.2) at 1-loop level in string amplitude, one has to consider contributions from $\mathcal{O}(k^4)$ terms in 1-loop calculation. A special technique called "pinched-off" integration (used in e.g.[58][56][17]) would be used to extract extra $\mathcal{O}(k^2)$ contributions from $\mathcal{O}(k^4)$ terms.

[1]The Koba–Nielsen factor in the amplitude picks up an extra phase under modular transformation on torus, and modular invariance requires the extra phase to vanish. Therefore it imposes the modular invariance condition.[58, p.56]

[2]Transversality condition is imposed by BRST invariance as an on-shell conditions $k^\mu \epsilon_{\mu\nu} = \epsilon_{\mu\nu} k^\nu = 0$ and $k^2 = 0$[23, p.591]

Pinched off integration arises in the limit $z_{ij} \to 0$ of 3-point amplitude, which means that two points of 3 are pinched towards each other. Making use of (3.28) and taking the limit $z_{ij} \to 0$, one obtains

$$\text{When } z_{ij} \to 0: \qquad P_{ij} = -\frac{\alpha'}{2} \ln |\chi_{ij}|^2 \to -\frac{\alpha'}{2} \ln |z_{ij}|^2, \qquad |\chi_{ij}|^2 \to |z_{ij}|^2, \tag{5.6a}$$

$$\langle : \prod_j e^{ik_j \cdot X_j} : \rangle = \prod_{i<j} |\chi_{ij}|^{\alpha' k_i \cdot k_j} \to \prod_{i<j} |z_{ij}|^{\alpha' k_i \cdot k_j} \qquad \text{(Koba–Nielsen factor)}, \tag{5.6b}$$

$$\bar{\partial}_i P_{ij} \to \frac{\alpha'}{2\bar{z}_{ij}}, \qquad \partial_i P_{ij} \to \frac{\alpha'}{2 z_{ij}}. \tag{5.6c}$$

In this chapter we use χ_{ij} as a convenient abbreviation of the variable of the logarithmic function in the 1-loop bosonic propagator(3.15).

Due to momentum conservation, we have the condition $k_i \cdot k_j = 0$ such that certain $\mathcal{O}(k^4)$ terms containing $k_i \cdot k_j$ vanish. Referring to the technique introduced by Minahan [58, p. 50], we relax the condition $k_i \cdot k_j = 0$ of momentum conservation, and impose the conditions only after we have done the integration. Then the integral over the region $|z_{ij}| < \epsilon$ can yield a pole in $k_i \cdot k_j$, if we assume that $\alpha' |k_i \cdot k_j| \ll \left|\frac{1}{\ln \epsilon}\right|$:

$$\int_{|z_{ij}|<\epsilon} d^2 z_{ij} \frac{|z_{ij}|^{\alpha' k_i \cdot k_j}}{|z_{ij}|^2} \simeq \frac{2\pi}{\alpha' k_i \cdot k_j}, \tag{5.7}$$

and if $k_i \cdot k_j$ is analytically continued to a region where the integral is convergent. One immediately finds that the integration cancels possible $k_i \cdot k_j$ (order k^2) in $\mathcal{O}(k^4)$ terms, and thus the cancellation results in finite value $\mathcal{O}(k^2)$ terms.

Double pinched-off integration There cannot be a double pinched-off integration contribution to k^2 order, because of the factorization of Riemann surface[65, §8], see Figure 5.1. Due to the tiny integration region $|z_{ij}| < \epsilon$, the surface with double pinched-off integration would be factorized[69][65] into a torus and a sphere with 3 punctures, and they are connected by a string propagator (plumbing fixture) with degeneration parameter $q \to 0$. Since tadpole is vanishing in Type-I on torus, analysis in [69] and [65, §8] shows that double pinched-off integration could be absorbed into a field redefinition and its contribution to k^2 order is unphysical.

5.1.3 Taylor expansion trick

It is possible to find more singular terms in the pinching limit on which one could apply pinched-off integration, with the help of Taylor expansion. One could use these

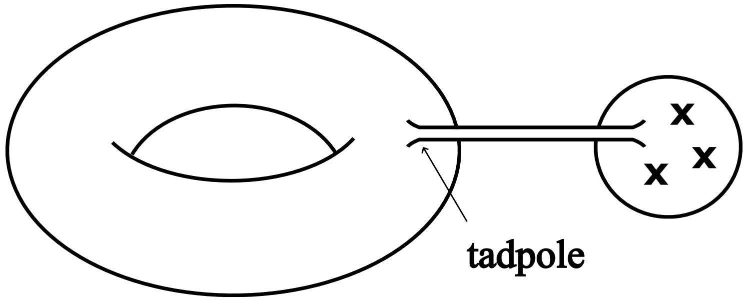


Figure 5.1: Double pinching factorization
x represents the insertion of a vertex operator

momentum terms to cancel corresponding momentum terms in the denominators in the amplitude.

More concretely, one can Taylor-expand

$$\begin{aligned}\lim_{z_l\to z_j} P(\bar{z}_i-\bar{z}_j) &= P((\bar{z}_i-\bar{z}_l)+(\bar{z}_l-\bar{z}_j))\\ &= \lim_{z_l\to z_j}(P_{il}+(\bar{z}_l-\bar{z}_j)\bar{\partial}_{\bar{z}_i-\bar{z}_l}P_{il}+\cdots)\\ &\sim \lim_{\bar{z}_l\to \bar{z}_j}(P_{il}+\bar{z}_{lj}\bar{\partial}_i P_{il}),\end{aligned}\tag{5.8}$$

and we use the propagator with $\bar{z}$ because this would be used mostly in the later calculation.

It can be much more tedious if we list all the middle steps of the calculation in the following sections. So we omit the tedious middle steps and instead give an example of how (5.8) works, to provide readers an intuitive understanding:

$$\begin{aligned}\lim_{z_{ij}\to 0}\partial_i P_{ij}\bar{\partial}^2_{ij}P_{ij}(\bar{\partial}_l P_{li}-\bar{\partial}_l P_{lj}) &= \lim_{z_{ij}\to 0}\frac{1}{z_{ij}}\frac{1}{\bar{z}^2_{ij}}(\bar{\partial}_l P_{li}-\bar{\partial}_l P_{lj})\\ &= \lim_{z_{ij}\to 0}\frac{1}{|z_{ij}|^2}\frac{1}{\bar{z}_{ij}}(\bar{\partial}_l P_{li}-\bar{\partial}_l P_{lj})\\ &= -\lim_{z_{ij}\to 0}\frac{\alpha'}{2}\frac{1}{|z_{ij}|^2}\frac{1}{\bar{z}_{ij}}(\bar{z}_{ij}\bar{\partial}_l\bar{\partial}_j P_{lj}) \qquad \text{using (5.8)}\\ &= -\lim_{z_{ij}\to 0}\frac{\alpha'}{2}\frac{1}{|z_{ij}|^2}\bar{\partial}_{lj}P_{lj}.\end{aligned}\tag{5.9}$$

We see that the term $\lim_{z_{ij}\to 0}\frac{1}{|z_{ij}|^2}$ required for pinched-off integration(5.7) shows up in the above expansion. Be aware that, besides Taylor expansion trick, we will also use (5.32) throughout the calculations in this chapter, as well as (5.33) and (5.34) after pinched-off integration, in order to transform the momentum terms into the form of (5.2). The terms relevant to the kinematic structure (5.2) should have the kinematic structures in the form of $(k_j\epsilon_i k_j)\,(\epsilon_j\epsilon_l)$ or $(k_k\epsilon_i\epsilon_j\epsilon_l k_i)$.

5.2 Heterotic String

We consider here the Heterotic string that the world-sheet is oriented without boundaries, and the corresponding tree level and one-loop level surfaces are sphere and torus.

Vertex Operator The zero ghost picture massless vertex operator of Heterotic String is

$$V^g_{(0)}(z,\bar{z},k,\epsilon) = \quad : \frac{2}{\alpha'}\epsilon_{\mu\nu}(k) i\bar{\partial}X^\mu(\bar{z})\left(i\partial X^\nu(z) + \frac{\alpha'}{2}(k\cdot\psi)\psi^\nu(z)\right)e^{ik\cdot X(z,\bar{z})} :, \tag{5.10}$$

and the -1 ghost picture massless vertex operator of Heterotic String is

$$V^g_{(-1)}(z,\bar{z},k,\epsilon) = \quad : \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k) i\bar{\partial}X^\mu(\bar{z})e^{-\phi}\psi^\nu(z)e^{ik\cdot X(z,\bar{z})} : . \tag{5.11}$$

5.2.1 Tree level

As discussed in §3.3, the ghost picture charge of tree level surfaces would be -2, thus we would need vertex operators in zero and -1 ghost picture. The tree level 3-point amplitude is

$$A_3^{(0)} = \langle c\bar{c}V_{-1}(z_1)\, c\bar{c}V_{-1}(z_2)\, c\bar{c}V_0(z_3)\rangle_{\Sigma_0} \tag{5.12}$$

with z_1, z_2 and z_3 fixed. Inserting the vertex operators (5.10) and (5.11) gives (normal ordering omitted)

$$\begin{aligned} A_3^{(0)} = &\left(\frac{2}{\alpha'}\right)^2 \epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}\langle e^{-\phi_1}e^{-\phi_2}\rangle\langle c_1\bar{c}_1c_2\bar{c}_2c_3\bar{c}_3\rangle \\ &\times \left\langle i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle \\ &\times \Bigg(\left\langle i\partial X_3^{\nu_3}e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}e^{ik_3\cdot X_3}\right\rangle \langle\psi_1^{\nu_1}\psi_2^{\nu_2}\rangle \\ &\quad + \frac{\alpha'}{2}\left\langle e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}e^{ik_3\cdot X_3}\right\rangle \langle\psi_1^{\nu_1}\psi_2^{\nu_2}\, k_3\cdot\psi_3\,\psi_3^{\nu_3}\rangle\Bigg). \end{aligned} \tag{5.13}$$

The ϕ correlator is

$$\langle e^{-\phi_1} e^{-\phi_2} \rangle = \frac{1}{z_{12}}, \tag{5.14}$$

and the ghost correlator yields

$$\langle c_1 \bar{c}_1 c_2 \bar{c}_2 c_3 \bar{c}_3 \rangle = |z_{12}|^2 |z_{23}|^2 |z_{13}|^2. \tag{5.15}$$

Using (B.2e) with an auxiliary variable ρ, we have (right-moving)

$$\begin{aligned}\left\langle \prod_{i=1}^{3} i\partial X^{\mu_i}(z_i) e^{ik_i\cdot X(z_i)} \right\rangle = \prod_{i=1}^{3} \frac{\partial}{\partial \rho_{i,\mu_i}} &\left[\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 \left(\sum_{i<j} \frac{\rho_i \cdot \rho_j}{z_{ij}^2} + \sum_{i\neq j} \frac{k_i \cdot \rho_j}{z_{ij}} \right)^2 \right. \\ &\left. + \frac{1}{3!}\left(\frac{\alpha'}{2}\right)^3 \left(\sum_{i\neq j} \frac{k_i\cdot \rho_j}{z_{ij}} \right)^3 \right] \prod_{i<j} (z_{ij})^{\frac{\alpha'}{2} k_i \cdot k_j} \Bigg|_{\rho_i = 0}\end{aligned} \tag{5.16}$$

by expanding the exponential in series and keeping only terms which will contribute after taking derivatives and setting the ρ_i to 0. Bar is omitted for simplicity, and we will take it back later. Then the expression simplifies to

$$\begin{aligned}\left\langle \prod_{i=1}^{3} i\partial X^{\mu_i}(z_i) e^{ik_i\cdot X(z_i)} \right\rangle = \frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 &\left[\frac{\eta^{\mu_1\mu_2}}{z_{12}^2} \left(\frac{k_1^{\mu_3}}{z_{13}} + \frac{k_2^{\mu_3}}{z_{23}} \right) + \frac{\eta^{\mu_1\mu_3}}{z_{13}^2} \left(\frac{k_1^{\mu_2}}{z_{12}} - \frac{k_3^{\mu_2}}{z_{23}} \right) \right. \\ &- \frac{\eta^{\mu_2\mu_3}}{z_{23}^2} \left(\frac{k_2^{\mu_1}}{z_{12}} + \frac{k_3^{\mu_1}}{z_{13}} \right) \\ &\left. - \frac{\alpha'}{2} \left(\frac{k_1^{\mu_3}}{z_{13}} + \frac{k_2^{\mu_3}}{z_{23}} \right) \left(\frac{k_1^{\mu_2}}{z_{12}} - \frac{k_3^{\mu_2}}{z_{23}} \right) \left(\frac{k_2^{\mu_1}}{z_{12}} + \frac{k_3^{\mu_1}}{z_{13}} \right) \right] \prod_{i<j} z_{ij}^{\frac{\alpha'}{2} k_i\cdot k_j}.\end{aligned} \tag{5.17}$$

Using momentum conservation and transversality $k_i^\mu \epsilon_{i,\mu\nu} = 0$, we can further simplify the expression. For $\mathcal{O}(\alpha'^0)$ part, one has as an example

$$\frac{k_1^{\mu_3}}{z_{13}} + \frac{k_2^{\mu_3}}{z_{23}} = \frac{-(k_2^{\mu_3} + \cancel{k_3^{\mu_3}}) z_{23} + k_2^{\mu_3} z_{13}}{z_{13} z_{23}} = k_2^{\mu_3} \frac{z_{13} - z_{23}}{z_{13} z_{23}} = k_2^{\mu_3} \frac{z_{12}}{z_{13} z_{23}} \tag{5.18}$$

such that

$$\frac{\eta^{\mu_1\mu_2}}{z_{12}^2} \left(\frac{k_1^{\mu_3}}{z_{13}} + \frac{k_2^{\mu_3}}{z_{23}} \right) = \frac{\eta^{\mu_1\mu_2} k_2^{\mu_3}}{z_{12} z_{13} z_{23}}. \tag{5.19}$$

Similarly one has

$$\frac{k_1^{\mu_2}}{z_{12}} - \frac{k_3^{\mu_2}}{z_{23}} = k_1^{\mu_2} \frac{z_{13}}{z_{12} z_{23}}, \qquad \frac{k_2^{\mu_1}}{z_{12}} + \frac{k_3^{\mu_1}}{z_{13}} = k_3^{\mu_1} \frac{z_{23}}{z_{12} z_{13}} \tag{5.20}$$

and

$$\frac{\eta^{\mu_1\mu_3}}{z_{13}^2}\left(\frac{k_1^{\mu_2}}{z_{12}}-\frac{k_3^{\mu_2}}{z_{23}}\right)=\frac{\eta^{\mu_1\mu_3}k_1^{\mu_2}}{z_{12}z_{13}z_{23}},\qquad -\frac{\eta^{\mu_2\mu_3}}{z_{23}^2}\left(\frac{k_2^{\mu_1}}{z_{12}}+\frac{k_3^{\mu_1}}{z_{13}}\right)=\frac{\eta^{\mu_2\mu_3}k_3^{\mu_1}}{z_{12}z_{13}z_{23}}. \tag{5.21}$$

Sum up these 3 terms we get

$$t^{\mu_1\mu_2\mu_3}=\eta^{\mu_1\mu_2}k_3^{\mu_2}+\eta^{\mu_2\mu_3}k_3^{\mu_1}+\eta^{\mu_3\mu_1}k_1^{\mu_2} \tag{5.22}$$

in (A.12).

$\mathcal{O}(\alpha')$ term could also be simplified as

$$\begin{aligned}\left(\frac{k_1^{\mu_3}}{z_{13}}+\frac{k_2^{\mu_3}}{z_{23}}\right)\left(\frac{k_1^{\mu_2}}{z_{12}}-\frac{k_3^{\mu_2}}{z_{23}}\right)\left(\frac{k_2^{\mu_1}}{z_{12}}+\frac{k_3^{\mu_1}}{z_{13}}\right)&=-k_2^{\mu_3}\frac{z_{12}}{z_{13}z_{23}}k_1^{\mu_2}\frac{z_{13}}{z_{12}z_{23}}k_3^{\mu_1}\frac{z_{23}}{z_{12}z_{13}}\\&=-\frac{k_1^{\mu_2}k_2^{\mu_3}k_3^{\mu_1}}{z_{12}z_{13}z_{23}}.\end{aligned} \tag{5.23}$$

Gathering the above results we obtain (left-moving)

$$\left\langle\prod_{i=1}^{3}i\bar{\partial}X^{\mu_i}(\bar{z}_i)e^{ik_i\cdot X(\bar{z}_i)}\right\rangle\propto\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2\frac{T^{\mu_1\mu_2\mu_3}}{\bar{z}_{12}\bar{z}_{13}\bar{z}_{23}} \tag{5.24}$$

where

$$T^{\mu_1\mu_2\mu_3}=\eta^{\mu_1\mu_2}k_2^{\mu_3}+\eta^{\mu_2\mu_3}k_3^{\mu_1}+\eta^{\mu_3\mu_1}k_1^{\mu_2}+\frac{\alpha'}{2}k_3^{\mu_1}k_1^{\mu_2}k_2^{\mu_3} \tag{5.25}$$

is defined in (A.12). Similarly one derives (right-moving)

$$\left\langle i\partial X^{\mu_3}(z_3)\prod_{i=1}^{3}e^{ik_i\cdot X(z_i)}\right\rangle\propto-\frac{\alpha'}{2}\frac{k_1^{\mu_3}z_{12}}{z_{13}z_{23}}\prod_{i<j}(z_{ij})^{\frac{\alpha'}{2}k_i\cdot k_j}. \tag{5.26}$$

Using (3.13), fermion part is

$$\begin{aligned}\langle\psi^{\nu_1}(z_1)\psi^{\nu_2}(z_2)k_3\cdot\psi\psi^{\nu_3}(z_3)\rangle&=k_{3\rho}\langle\psi^{\nu_1}(z_1)\psi^{\nu_2}(z_2)\psi^{\rho}(z_3)\psi^{\nu_3}(z_3)\rangle\\&\propto\frac{-\eta^{\nu_2\nu_3}k_3^{\nu_1}+\eta^{\nu_1\nu_3}k_3^{\nu_2}}{z_{13}z_{23}}.\end{aligned} \tag{5.27}$$

Summing up all the correlation functions and using momentum conservation and transversality of ϵ, finally we obtain

$$A_3^{(0)}=\frac{\alpha'}{4}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}T^{\mu_1\mu_2\mu_3}t^{\nu_1\nu_2\nu_3}\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j}. \tag{5.28}$$

Imposing on-shell condition $k_i \cdot k_j = 0$ we have $\prod_{i<j} |z_{ij}|^{\alpha' k_i \cdot k_j} = 1$. Then at order k^2 we get

$$A_3^{(0)}\Big|_{k^2} = \frac{\alpha'}{4}\left[(\epsilon_1^\top \epsilon_2)(k_2\epsilon_3 k_2) + (k_2\epsilon_3\epsilon_2^\top \epsilon_1 k_3) + (k_2\epsilon_3\epsilon_1^\top \epsilon_2 k_1) + \text{cyclic perms}\right]. \quad (5.29)$$

One immediately finds that this amplitude does share the same kinematic structure as the expanded Einstein-Hilbert term (5.2) in the effective action[3].

5.2.2 Direct Calculation of the 1-loop kinematic structure

As discussed in §2.2.1, the ghost picture charge of 1-loop level surfaces would be 0, thus we would need vertex operator only in zero ghost picture. The 1-loop level 3-point amplitude is

$$A_3^{(1)} = \left\langle \prod_{i=1}^{3} \int d^2 z_i \, V_0(z_i, \bar{z}_i) \right\rangle_{\Sigma_1} . \quad (5.30)$$

Inserting the vertex operators (5.10) gives the correlation function (normal ordering omitted)

$$\begin{aligned}
G_3^{(1)} &= \left(\frac{2}{\alpha'}\right)^3 \epsilon_1^{\mu_1\nu_1}\epsilon_2^{\mu_2\nu_2}\epsilon_3^{\mu_3\nu_3} \left\langle \prod_{i=1}^{3} \left[i\bar\partial X_i^{\mu_i} \left(i\partial X_i^{\nu_i} + \frac{\alpha'}{2} k_i \cdot \psi_i \psi_i^{\nu_i} \right) e^{ik_i\cdot X_i} \right] \right\rangle && (5.31\text{a})\\
&= \left(\frac{2}{\alpha'}\right)^3 \epsilon_1^{\mu_1\nu_1}\epsilon_2^{\mu_2\nu_2}\epsilon_3^{\mu_3\nu_3} \left[\left\langle \prod_i i\bar\partial X_i^{\mu_i} i\partial X_i^{\nu_i} e^{ik_i\cdot X_i} \right\rangle \right.\\
&\quad + \frac{\alpha'}{2} \sum_j \left\langle \prod_{\ell\neq j} i\partial X_\ell^{\nu_\ell} \prod_i i\bar\partial X_i^{\mu_i} e^{ik_i\cdot X_i} \right\rangle \langle k_j \cdot \psi_j \psi_j^{\nu_j} \rangle\\
&\quad + \left(\frac{\alpha'}{2}\right)^2 \sum_\ell \left\langle i\partial X_\ell^{\nu_\ell} \prod_i i\bar\partial X_i^{\mu_i} e^{ik_i\cdot X_i} \right\rangle \left\langle \prod_{j\neq\ell} k_j \cdot \psi_j \psi_j^{\nu_j} \right\rangle\\
&\quad \left. + \left(\frac{\alpha'}{2}\right)^3 \left\langle \prod_i i\bar\partial X_i^{\mu_i} e^{ik_i\cdot X_i} \right\rangle \left\langle \prod_i k_i \cdot \psi_i \psi_i^{\nu_i} \right\rangle \right] && (5.31\text{b})\\
&\equiv G_3^{0f} + G_3^{2f} + G_3^{4f} + G_3^{6f}. && (5.31\text{c})
\end{aligned}$$

The superscript nf stands for the number of fermions. We observe that there is the cyclic permutation symmetry of the correlation function, thus it is only necessary to compute 1 specific order and derive other orders by cyclic permutation.

[3] We notice that ϵ_i are symmetric in gravitons, thus the transposition could be ignored when it is compared to (5.2).

Spin structure independent term is vanishing due to the sum over the spin structures[40], therefore $G_3^{(0f)} = 0$. And $G_3^{(2f)} = 0$ because the 2 fermions are normal ordered and cannot be contracted with anything. We are left with $G_3^{(1)} = G_3^{(4f)} + G_3^{(6f)}$.

In the following computation, the contractions between the exponentials $e^{ik_i \cdot X_i}$ would not be indicated, because they always contribute the same factor $\prod_{i<j} |\chi_{ij}|^{\alpha' k_i \cdot k_j}$.

As a reminder, we have used modular invariance (5.3) to transform the momentum terms throughout all calculations in this chapter, in order to get the desired forms of the momentum terms. For example,

$$\frac{k_i \cdot k_j}{k_l \cdot k_i} + \frac{k_j \cdot k_l}{k_l \cdot k_i} = -\frac{k_l \cdot k_i}{k_l \cdot k_i} = -1. \tag{5.32}$$

We have also used momentum conservation and transversality (5.4) after pinched-off integration to get the desired forms of the kinematic structures. For example,

$$k_l \epsilon_i \epsilon_j \epsilon_l k_i = -k_j \epsilon_i \epsilon_j \epsilon_l k_i, \tag{5.33}$$

and

$$(k_j \epsilon_l k_i)(\epsilon_i \epsilon_j) = -(k_i \epsilon_l k_i)(\epsilon_i \epsilon_j) \tag{5.34}$$

Pinched-off integration contracting rules We observe that to get extra $\mathcal{O}(k^2)$ terms from pinched-off integration, the following rules would be helpful when contracting operators:[4]

1. Pinched-off integration (5.7) would require $\frac{1}{z_{ij}}$ and $\frac{1}{\bar{z}_{ij}}$ in the denominator of one term in the amplitude. It is only possible to derive $\frac{1}{z_{ij}}$ or $\frac{1}{\bar{z}_{ij}}$ from either $\langle \psi_i \psi_j \rangle$ or $\langle \partial X_i e^{ik_j \cdot X_j} \rangle$ in the pinching limit $|z_{ij}| < \epsilon \ll 1$.

2. We require at least 1 pair of fermions to be contracted in the form $\langle k_i \cdot \psi_i k_j \cdot \psi_j \rangle = k_i \cdot k_j \langle \psi_i \psi_j \rangle$. Since in the $6f$ terms there are only 3 of $k_i \cdot \psi_i$, then one can have at most 1 pair of $k_i \cdot k_j$. In the $4f$ terms we have no choice but take the only possible contraction $k_i \cdot k_j \langle \psi_i \psi_j \rangle$. We have this rule because we want to cancel the pole from the pinched-off integration.

3. In the $6f$ terms, we require one boson to be contracted with one exponential to contribute $1/\bar{z}_{ij}$, which has 2 ways: $\langle \bar{\partial} X_i e^{ik_j \cdot X_j} \rangle$ or interchanging z_i and z_j. And we also require one pair of fermions contraction $\langle \psi_i \psi_j \rangle$ to contribute $1/z_{ij}$.

[4] Beaware that the rules work for both Heterotic and Type-I strings.

4. To get the tensor structure $(k_j\epsilon_i k_j)(\epsilon_j\epsilon_k)$, one should notice that $(\epsilon_j\epsilon_k)$ already restricts part of the contractions to be $\langle\bar\partial X_j^{\mu_j}\bar\partial X_k^{\mu_k}\rangle\langle\psi_j^{\nu_j}\psi_k^{\nu_k}\rangle$. In other words, one should look for terms with $\eta^{\mu_j\mu_k}\eta^{\nu_j\nu_k}$.

5. The number of momenta k equals the number of imaginary signs i, which will be needed when considering the overall sign.

We present two examples of the tensor structure $(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)$:

$$\epsilon_{1,\mu_1\nu_1}\bar\partial X_1^{\mu_1} ik_1\cdot\psi_1\psi_1^{\nu_1}e^{ik_1\cdot X_1}\epsilon_{2,\mu_2\nu_2}\bar\partial X_2^{\mu_2} ik_2\cdot\psi_2\psi_2^{\nu_2}e^{ik_2\cdot X_2}\epsilon_{3,\mu_3\nu_3}\bar\partial X_3^{\mu_3}\partial X_3^{\nu_3}e^{ik_3\cdot X_3} \tag{5.35}$$

With the z_{13} pinched-off integration, the correlation function of the above contractions of a $4f$ term to the k^2 order turns into

$$\begin{aligned}\int d^2z_1\int d^2z_2\int d^2z_3 G^1_{4f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)} &\sim -\frac{\alpha'}{2}\int d^2z_1\int d^2z_2\int d^2z_3(k_1\cdot k_2)(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)\\ &\qquad S_{12}^2\bar\partial^2P_{12}\frac{|\chi_{13}|^{\frac{1}{2}k_1\cdot k_3}}{|z_{13}|^2}|\chi_{23}|^{\frac{1}{2}k_2\cdot k_3}|\chi_{12}|^{\frac{1}{2}k_1\cdot k_2}\\ &\overset{\text{POI}}{\sim} -\pi\int d^2z_1\int d^2z_2\frac{k_1\cdot k_2}{k_1\cdot k_3}(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)\\ &\qquad S_{12}^2\bar\partial^2P_{12}|\chi_{23}|^{\frac{1}{2}k_2\cdot k_3}|\chi_{12}|^{\frac{1}{2}k_1\cdot k_2},\end{aligned} \tag{5.36}$$

and "POI" stands for "Pinched-Off Integration". We recall that P_{ij} is the 1-loop bosonic propagator here and χ_{ij} is the variable of the logarithm function in P_{ij}.

$$\epsilon_{1,\mu_1\nu_1}\bar\partial X_1^{\mu_1} ik_1\cdot\psi_1\psi_1^{\nu_1}e^{ik_1\cdot X_1}\epsilon_{2,\mu_2\nu_2}\bar\partial X_2^{\mu_2} ik_2\cdot\psi_2\psi_2^{\nu_2}e^{ik_2\cdot X_2}\epsilon_{3,\mu_3\nu_3}\bar\partial X_3^{\mu_3} ik_3\cdot\psi_3\psi_3^{\nu_3}e^{ik_3\cdot X_3} \tag{5.37}$$

Using $S_{13}\simeq z_{13}^{-1}$ and $S_{12}\approx -S_{23}$ when $|z_{13}|<\epsilon$, and applying the z_{13} pinched-off integration, the correlation function of the above contractions of a $6f$ term to the k^2 order turns into

$$\begin{aligned}\int d^2z_1\int d^2z_2\int d^2z_3 G^1_{6f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)} &\sim \frac{\alpha'}{2}\int d^2z_1\int d^2z_2\int d^2z_3(k_2\cdot k_3)(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)\\ &\qquad S_{12}S_{13}S_{23}\bar\partial^2P_{12}\frac{|\chi_{13}|^{\frac{1}{2}k_1\cdot k_3}}{\bar z_{31}}|\chi_{23}|^{\frac{1}{2}k_2\cdot k_3}|\chi_{12}|^{\frac{1}{2}k_1\cdot k_2}\\ &\overset{\text{POI}}{\sim} \pi\int d^2z_1\int d^2z_2\frac{k_2\cdot k_3}{k_1\cdot k_3}(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)\end{aligned}$$

$$S_{12}^2\bar{\partial}^2 P_{12}|\chi_{23}|^{\frac{1}{2}k_2\cdot k_3}|\chi_{12}|^{\frac{1}{2}k_1\cdot k_2}. \tag{5.38}$$

In the following we start the calculation, we do it in a compact way, try to deal with the whole correlation function together, and omit some tedious middle steps.

Computation of the 4-fermion term

Using correlation functions in app.B, consider G_3^{4f} term with $\ell = 1$ in (5.31b):

$$G_3^{4f}\Big|_{\ell=1} = \frac{2}{\alpha'}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\left\langle k_2\cdot\psi_2\psi_2^{\nu_2}k_3\cdot\psi_3\psi_3^{\nu_3}\right\rangle \tag{5.39a}$$

$$\begin{aligned}
=&\frac{2}{\alpha'}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}k_{2\rho}k_{3\sigma}\\
&\times\Big[\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + \left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\\
&+\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + \left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\\
&+\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + \left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\\
&+(2\leftrightarrow 3)\\
&+\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + \left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\\
&+\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + \left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\\
&+\left\langle i\partial X_1^{\nu_1} i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1} i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2} i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle + (2\leftrightarrow 3)\Big]\\
&\times\left[\langle\psi_2^\rho\psi_2^{\nu_2}\psi_3^\sigma\psi_3^{\nu_3}\rangle + \langle\psi_2^\rho\psi_2^{\nu_2}\psi_3^\sigma\psi_3^{\nu_3}\rangle\right]
\end{aligned} \tag{5.39b}$$

$$\begin{aligned}
=&\frac{2}{\alpha'}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}k_{2\rho}k_{3\sigma}\left(-\eta^{\rho\sigma}\eta^{\nu_2\nu_3}+\eta^{\rho\nu_3}\eta^{\sigma\nu_2}\right)S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}.\\
&\times\Bigg\{\left(\frac{\alpha'}{2}\right)^2\left(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}}\right)\Bigg[-\left(-\frac{k_1^{\mu_3}}{\bar{z}_{13}}-\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\eta^{\mu_1\mu_2}\bar{\partial}_{12}^2P_{12}-\left(-\frac{k_1^{\mu_2}}{\bar{z}_{12}}+\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\eta^{\mu_1\mu_3}\bar{\partial}_{13}^2P_{13}\\
&-\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\eta^{\mu_2\mu_3}\bar{\partial}_{23}^2P_{23}\Bigg]\\
&-\left(\frac{\alpha'}{2}\right)^2\eta^{\nu_1\mu_2}\partial_1\bar{\partial}_2P_{12}\left[-\frac{k_2^{\mu_1}}{\bar{z}_{12}}\frac{k_1^{\mu_3}}{\bar{z}_{13}}-\frac{k_2^{\mu_1}}{\bar{z}_{12}}\frac{k_2^{\mu_3}}{\bar{z}_{23}}-\frac{k_3^{\mu_1}}{\bar{z}_{13}}\frac{k_1^{\mu_3}}{\bar{z}_{13}}-\frac{k_3^{\mu_1}}{\bar{z}_{13}}\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right]
\end{aligned}$$

$$
-\left(\frac{\alpha'}{2}\right)^2 \eta^{\nu_1\mu_3}\partial_1\bar{\partial}_3 P_{13}\left[-\frac{k_3^{\mu_1}}{\bar{z}_{13}}\frac{k_1^{\mu_2}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\frac{k_3^{\mu_2}}{\bar{z}_{23}}-\frac{k_2^{\mu_1}}{\bar{z}_{12}}\frac{k_1^{\mu_2}}{\bar{z}_{12}}+\frac{k_2^{\mu_1}}{\bar{z}_{12}}\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right]
$$

$$
+\eta^{\nu_1\mu_2}\eta^{\mu_1\mu_3}\partial_1\bar{\partial}_2 P_{12}\bar{\partial}_{13}^2 P_{13}+\eta^{\nu_1\mu_3}\eta^{\mu_1\mu_2}\partial_1\bar{\partial}_3 P_{13}\bar{\partial}_{12}^2 P_{12}\Big\}\,. \tag{5.39c}
$$

After some simplifications one finds

$$\boxed{\begin{aligned} G_3^{4f}\Big|_{\ell=1} =& -\frac{\alpha'}{2}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}\left(-k_2\cdot k_3\eta^{\nu_2\nu_3}+k_2^{\nu_3}k_3^{\nu_2}\right)S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j} \\ &\times\left\{\left(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}}\right)\left[-\left(\frac{k_1^{\mu_3}}{\bar z_{13}}+\frac{k_2^{\mu_3}}{\bar z_{23}}\right)\eta^{\mu_1\mu_2}\bar\partial_{12}^2P_{12}\right.\right. \\ &+\left.\left(-\frac{k_1^{\mu_2}}{\bar z_{12}}+\frac{k_3^{\mu_2}}{\bar z_{23}}\right)\eta^{\mu_1\mu_3}\bar\partial_{13}^2P_{13}+\left(\frac{k_2^{\mu_1}}{\bar z_{12}}+\frac{k_3^{\mu_1}}{\bar z_{13}}\right)\eta^{\mu_2\mu_3}\bar\partial_{23}^2P_{23}\right] \\ &+\frac{\alpha'\pi}{2\tau_2}\left(\frac{k_2^{\mu_1}}{\bar z_{12}}+\frac{k_3^{\mu_1}}{\bar z_{13}}\right)\left[\eta^{\nu_1\mu_2}\left(\frac{k_1^{\mu_3}}{\bar z_{13}}+\frac{k_2^{\mu_3}}{\bar z_{23}}\right)+\eta^{\nu_1\mu_3}\left(\frac{k_1^{\mu_2}}{\bar z_{12}}-\frac{k_3^{\mu_2}}{\bar z_{23}}\right)\right] \\ &\left.+\frac{2\pi}{\alpha'\tau_2}\left[\eta^{\nu_1\mu_2}\eta^{\mu_1\mu_3}\bar\partial_{13}^2P_{13}+\eta^{\nu_1\mu_3}\eta^{\mu_1\mu_2}\bar\partial_{12}^2P_{12}\right]\right\} \end{aligned}} \tag{5.40}$$

Note that the result is invariant under $2\leftrightarrow 3$ as expected. The contributions $\ell=2$ and $\ell=3$ are obtained by permuting cyclically $(1,2,3)$ (one should note that the factor in the first square bracket is invariant, as expected since the anti-holomorphic sector is identical in all three cases).

To be even more explicit, we can perform the contractions between the Lorentz indices to identify the different tensor structures:

$$\begin{aligned} G_3^{4f}\Big|_{\ell=1} =& -\frac{\alpha'}{2}\epsilon_{1,\mu_1\nu_1}\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)_{\mu_2\mu_3}+\left(\epsilon_3k_2\right)_{\mu_3}\left(\epsilon_2k_3\right)_{\mu_2}\right)S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j} \\ &\times\left\{\left(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}}\right)\left[-\left(\frac{k_1^{\mu_3}}{\bar z_{13}}+\frac{k_2^{\mu_3}}{\bar z_{23}}\right)\eta^{\mu_1\mu_2}\bar\partial_{12}^2P_{12}+\left(-\frac{k_1^{\mu_2}}{\bar z_{12}}+\frac{k_3^{\mu_2}}{\bar z_{23}}\right)\eta^{\mu_1\mu_3}\bar\partial_{13}^2P_{13}\right.\right. \\ &+\left.\left(\frac{k_2^{\mu_1}}{\bar z_{12}}+\frac{k_3^{\mu_1}}{\bar z_{13}}\right)\eta^{\mu_2\mu_3}\bar\partial_{23}^2P_{23}\right] \\ &+\frac{\alpha'\pi}{2\tau_2}\left(\frac{k_2^{\mu_1}}{\bar z_{12}}+\frac{k_3^{\mu_1}}{\bar z_{13}}\right)\left[\eta^{\nu_1\mu_2}\left(\frac{k_1^{\mu_3}}{\bar z_{13}}+\frac{k_2^{\mu_3}}{\bar z_{23}}\right)+\eta^{\nu_1\mu_3}\left(\frac{k_1^{\mu_2}}{\bar z_{12}}-\frac{k_3^{\mu_2}}{\bar z_{23}}\right)\right] \end{aligned}$$

$$+\frac{2\pi}{\alpha'\tau_2}\left[\eta^{\nu_1\mu_2}\eta^{\mu_1\mu_3}\bar{\partial}_{13}^2 P_{13}+\eta^{\nu_1\mu_3}\eta^{\mu_1\mu_2}\bar{\partial}_{12}^2 P_{12}\right]\Bigg\} \tag{5.41a}$$

$$\begin{aligned}
=&-\frac{\alpha'}{2}S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg\{\left(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}}\right)\Bigg[-\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\left(-k_2\cdot k_3\left(\epsilon_1^\top\epsilon_2\epsilon_3^\top\right)_{\nu_1\mu_3}+\left(\epsilon_3 k_2\right)_{\mu_3}\left(\epsilon_1^\top\epsilon_2 k_3\right)_{\nu_1}\right)\bar{\partial}_{12}^2 P_{12}\\
&+\left(-\frac{k_1^{\mu_2}}{\bar{z}_{12}}+\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\epsilon_1\right)_{\mu_2\nu_1}+\left(\epsilon_1^\top\epsilon_3 k_2\right)_{\nu_1}\left(\epsilon_2 k_3\right)_{\mu_2}\right)\bar{\partial}_{13}^2 P_{13}\\
&+\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\epsilon_{1,\mu_1\nu_1}\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)+\left(k_3\epsilon_2^\top\epsilon_3 k_2\right)\right)\bar{\partial}_{23}^2 P_{23}\Bigg]\\
&+\frac{\alpha'\pi}{2\tau_2}\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\Bigg[\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\left(-k_2\cdot k_3\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)_{\mu_1\mu_3}+\left(\epsilon_3 k_2\right)_{\mu_3}\left(\epsilon_1\epsilon_2 k_3\right)_{\mu_1}\right)\\
&+\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)_{\mu_2\mu_1}+\left(\epsilon_1\epsilon_3 k_2\right)_{\mu_1}\left(\epsilon_2 k_3\right)_{\mu_2}\right)\Bigg]\\
&+\frac{2\pi}{\alpha'\tau_2}\left[\left(-k_2\cdot k_3\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)+\left(k_2\epsilon_3^\top\epsilon_1\epsilon_2 k_3\right)\right)\bar{\partial}_{13}^2 P_{13}\right.\\
&\left.+\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)+\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3 k_2\right)\right)\bar{\partial}_{12}^2 P_{12}\right]\Bigg\}
\end{aligned} \tag{5.41b}$$

$$\begin{aligned}
=&-\frac{\alpha'}{2}S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg\{\left(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}}\right)\Bigg[-\bar{z}_{13}^{-1}\left(-k_2\cdot k_3\left(\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1\right)_{\nu_1}+\left(k_1\epsilon_3 k_2\right)\left(\epsilon_1^\top\epsilon_2 k_3\right)_{\nu_1}\right)\bar{\partial}_{12}^2 P_{12}\\
&-\bar{z}_{23}^{-1}\left(-k_2\cdot k_3\left(\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2\right)_{\nu_1}+\left(k_2\epsilon_3 k_2\right)\left(\epsilon_1^\top\epsilon_2 k_3\right)_{\nu_1}\right)\bar{\partial}_{12}^2 P_{12}\\
&-\bar{z}_{12}^{-1}\left(-k_2\cdot k_3\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1\right)_{\nu_1}+\left(\epsilon_1^\top\epsilon_3 k_2\right)_{\nu_1}\left(k_1\epsilon_2 k_3\right)\right)\bar{\partial}_{13}^2 P_{13}\\
&+\bar{z}_{23}^{-1}\left(-k_2\cdot k_3\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1\right)_{\nu_1}+\left(\epsilon_1^\top\epsilon_3 k_2\right)_{\nu_1}\left(k_3\epsilon_2 k_3\right)\right)\bar{\partial}_{13}^2 P_{13}\\
&+\bar{z}_{12}^{-1}\left(k_2\epsilon_1\right)_{\nu_1}\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)+\left(k_3\epsilon_2^\top\epsilon_3 k_2\right)\right)\bar{\partial}_{23}^2 P_{23}\\
&+\bar{z}_{13}^{-1}\left(k_3\epsilon_1\right)_{\nu_1}\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)+\left(k_3\epsilon_2^\top\epsilon_3 k_2\right)\right)\bar{\partial}_{23}^2 P_{23}\Bigg]\\
&+\frac{\alpha'\pi}{2\tau_2}\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\Bigg[\bar{z}_{13}^{-1}\left(-k_2\cdot k_3\left(\epsilon_1\epsilon_2\epsilon_3^\top k_1\right)_{\mu_1}+\left(k_1\epsilon_3 k_2\right)\left(\epsilon_1\epsilon_2 k_3\right)_{\mu_1}\right)\\
&+\bar{z}_{23}^{-1}\left(-k_2\cdot k_3\left(\epsilon_1\epsilon_2\epsilon_3^\top k_2\right)_{\mu_1}+\left(k_2\epsilon_3 k_2\right)\left(\epsilon_1\epsilon_2 k_3\right)_{\mu_1}\right)\\
&+\bar{z}_{12}^{-1}\left(-k_2\cdot k_3\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)_{\mu_1}+\left(\epsilon_1\epsilon_3 k_2\right)_{\mu_1}\left(k_1\epsilon_2 k_3\right)\right)\\
&+\bar{z}_{23}^{-1}\left(k_2\cdot k_3\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)_{\mu_1}-\left(\epsilon_1\epsilon_3 k_2\right)_{\mu_1}\left(k_3\epsilon_2 k_3\right)\right)\Bigg]\\
&+\frac{2\pi}{\alpha'\tau_2}\left[-k_2\cdot k_3\left(\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)\bar{\partial}_{13}^2 P_{13}+\left(\epsilon_1\epsilon_3\epsilon_2^\top\right)\bar{\partial}_{12}^2 P_{12}\right)\right.\\
&\left.+\left(k_3\epsilon_2^\top\epsilon_1^\top\epsilon_3 k_2\right)\bar{\partial}_{13}^2 P_{13}+\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3 k_2\right)\bar{\partial}_{12}^2 P_{12}\right]\Bigg\}
\end{aligned} \tag{5.41c}$$

$$
\begin{aligned}
=&-\frac{\alpha'}{2}S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg\{(\frac{k_2^{\nu_1}}{z_{12}}+\frac{k_3^{\nu_1}}{z_{13}})\Bigg[\left(\frac{\left(\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1\right)_{\nu_1}}{\bar z_{13}}+\frac{\left(\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2\right)_{\nu_1}}{\bar z_{23}}\right)k_2\cdot k_3\bar\partial_{12}^2P_{12}\\
&-\left(\frac{(k_1\epsilon_3k_2)}{\bar z_{13}}+\frac{(k_2\epsilon_3k_2)}{\bar z_{23}}\right)\left(\epsilon_1^\top\epsilon_2k_3\right)_{\nu_1}\bar\partial_{12}^2P_{12}\\
&+\left(\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1\right)_{\nu_1}}{\bar z_{12}}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1\right)_{\nu_1}}{\bar z_{23}}\right)k_2\cdot k_3\bar\partial_{13}^2P_{13}\\
&+\left(-\frac{(k_1\epsilon_2k_3)}{\bar z_{12}}+\frac{(k_3\epsilon_2k_3)}{\bar z_{23}}\right)\left(\epsilon_1^\top\epsilon_3k_2\right)_{\nu_1}\bar\partial_{13}^2P_{13}\\
&+\left(\frac{(k_2\epsilon_1)_{\nu_1}}{\bar z_{12}}+\frac{(k_3\epsilon_1)_{\nu_1}}{\bar z_{13}}\right)\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)+\left(k_3\epsilon_2^\top\epsilon_3k_2\right)\right)\bar\partial_{23}^2P_{23}\Bigg]\\
&+\frac{\alpha'\pi}{2\tau_2}\left(\frac{k_2^{\mu_1}}{\bar z_{12}}+\frac{k_3^{\mu_1}}{\bar z_{13}}\right)\Bigg[-\left(\frac{\left(\epsilon_1\epsilon_2\epsilon_3^\top k_1\right)_{\mu_1}}{\bar z_{13}}+\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)_{\mu_1}}{\bar z_{12}}+\frac{\left(\epsilon_1\epsilon_2\epsilon_3^\top k_2\right)_{\mu_1}}{\bar z_{23}}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top\right)_{\mu_1}}{\bar z_{23}}\right)k_2\cdot k_3\\
&+\left(\frac{(k_1\epsilon_3k_2)}{\bar z_{13}}+\frac{(k_2\epsilon_3k_2)}{\bar z_{23}}\right)(\epsilon_1\epsilon_2k_3)_{\mu_1}+\left(\frac{(k_1\epsilon_2k_3)}{\bar z_{12}}-\frac{(k_3\epsilon_2k_3)}{\bar z_{23}}\right)(\epsilon_1\epsilon_3k_2)_{\mu_1}\Bigg]\\
&+\frac{2\pi}{\alpha'\tau_2}\left[-k_2\cdot k_3\left(\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)\bar\partial_{13}^2P_{13}+\left(\epsilon_1\epsilon_3\epsilon_2^\top\right)\bar\partial_{12}^2P_{12}\right)\right.\\
&\left.+\left(k_3\epsilon_2^\top\epsilon_1^\top\epsilon_3k_2\right)\bar\partial_{13}^2P_{13}+\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3k_2\right)\bar\partial_{12}^2P_{12}\right]\Bigg\}
\end{aligned}
\tag{5.41d}
$$

$$
\begin{aligned}
=&-\frac{\alpha'}{2}S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg\{\left(\frac{\left(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1\right)}{z_{12}\bar z_{13}}+\frac{\left(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2\right)}{z_{12}\bar z_{23}}+\frac{\left(k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1\right)}{|z_{13}|^2}+\frac{\left(k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2\right)}{z_{13}\bar z_{23}}\right)k_2\cdot k_3\bar\partial_{12}^2P_{12}\\
&-\left(\frac{(k_1\epsilon_3k_2)}{\bar z_{13}}+\frac{(k_2\epsilon_3k_2)}{\bar z_{23}}\right)\left(\frac{\left(k_2\epsilon_1^\top\epsilon_2k_3\right)}{z_{12}}+\frac{\left(k_3\epsilon_1^\top\epsilon_2k_3\right)}{z_{13}}\right)\bar\partial_{12}^2P_{12}\\
&+\left(\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1k_2\right)}{|z_{12}|^2}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1k_2\right)}{z_{12}\bar z_{23}}+\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1k_3\right)}{z_{13}\bar z_{12}}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1k_3\right)}{z_{13}\bar z_{23}}\right)k_2\cdot k_3\bar\partial_{13}^2P_{13}\\
&+\left(-\frac{(k_1\epsilon_2k_3)}{\bar z_{12}}+\frac{(k_3\epsilon_2k_3)}{\bar z_{23}}\right)\left(\frac{\left(k_2\epsilon_1^\top\epsilon_3k_2\right)}{z_{12}}+\frac{\left(k_3\epsilon_1^\top\epsilon_3k_2\right)}{z_{13}}\right)\bar\partial_{13}^2P_{13}\\
&+\left(\frac{(k_2\epsilon_1k_2)}{|z_{12}|^2}+\frac{(k_3\epsilon_1k_2)}{z_{12}\bar z_{13}}+\frac{(k_2\epsilon_1k_3)}{z_{13}\bar z_{12}}+\frac{(k_3\epsilon_1k_3)}{|z_{13}|^2}\right)\left(-k_2\cdot k_3\left(\epsilon_2\epsilon_3^\top\right)+\left(k_3\epsilon_2^\top\epsilon_3k_2\right)\right)\bar\partial_{23}^2P_{23}\\
&+\frac{\alpha'\pi}{2\tau_2}\Bigg[-\left(\frac{\left(k_2\epsilon_1\epsilon_2\epsilon_3^\top k_1\right)}{\bar z_{12}\bar z_{13}}+\frac{\left(k_2\epsilon_1\epsilon_2\epsilon_3^\top k_2\right)}{\bar z_{12}\bar z_{23}}+\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top k_2\right)}{\bar z_{12}^2}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top k_2\right)}{\bar z_{12}}\right)k_{23}\cdot k_3\\
&-\left(\frac{\left(k_3\epsilon_1\epsilon_2\epsilon_3^\top k_1\right)}{\bar z_{13}^2}+\frac{\left(k_3\epsilon_1\epsilon_2\epsilon_3^\top k_2\right)}{\bar z_{13}\bar z_{23}}+\frac{\left(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top k_3\right)}{\bar z_{12}\bar z_{13}}-\frac{\left(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top k_3\right)}{\bar z_{13}\bar z_{23}}\right)k_2\cdot k_3
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{(k_1\epsilon_3k_2)}{\bar z_{13}}+\frac{(k_2\epsilon_3k_2)}{\bar z_{23}}\right)\left(\frac{(k_2\epsilon_1\epsilon_2k_3)}{\bar z_{12}}+\frac{(k_3\epsilon_1\epsilon_2k_3)}{\bar z_{13}}\right)\\
&+\left(\frac{(k_1\epsilon_2k_3)}{\bar z_{12}}-\frac{(k_3\epsilon_2k_3)}{\bar z_{23}}\right)\left(\frac{(k_2\epsilon_1\epsilon_3k_2)}{\bar z_{12}}+\frac{(k_3\epsilon_1\epsilon_3k_2)}{\bar z_{13}}\right)\Bigg]\\
&+\frac{2\pi}{\alpha'\tau_2}\left[-k_2\cdot k_3\left(\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)\bar\partial_{13}^2P_{13}+\left(\epsilon_1\epsilon_3\epsilon_2^\top\right)\bar\partial_{12}^2P_{12}\right)\right.\\
&\left.+\left(k_3\epsilon_2^\top\epsilon_1^\top\epsilon_3k_2\right)\bar\partial_{13}^2P_{13}+\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3k_2\right)\bar\partial_{12}^2P_{12}\right]\Bigg\}
\end{aligned}
\tag{5.41e}
$$

Finally, we obtain the expanded result

$$
\boxed{
\begin{aligned}
G_3^{4f}\Big|_{\ell=1}=&-\frac{\alpha'}{2}S_{23}^2\times\prod_{i<j}|\chi_{ij}|^{\alpha'k_i\cdot k_j}\\
&\times\Bigg\{k_2\cdot k_3\Bigg[\left(\frac{(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1)}{z_{12}\bar z_{13}}+\frac{(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2)}{z_{12}\bar z_{23}}+\frac{(k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top k_1)}{|z_{13}|^2}+\frac{(k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top k_2)}{z_{13}\bar z_{23}}\right)\bar\partial_{12}^2P_{12}\\
&+\left(\frac{(k_1\epsilon_2\epsilon_3^\top\epsilon_1k_2)}{|z_{12}|^2}-\frac{(k_3\epsilon_2\epsilon_3^\top\epsilon_1k_2)}{z_{12}\bar z_{23}}+\frac{(k_1\epsilon_2\epsilon_3^\top\epsilon_1k_3)}{z_{13}\bar z_{12}}-\frac{(k_3\epsilon_2\epsilon_3^\top\epsilon_1k_3)}{z_{13}\bar z_{23}}\right)\bar\partial_{13}^2P_{13}\\
&-\left(\epsilon_2\epsilon_3^\top\right)\left(\frac{(k_2\epsilon_1k_2)}{|z_{12}|^2}+\frac{(k_3\epsilon_1k_2)}{z_{12}\bar z_{13}}+\frac{(k_2\epsilon_1k_3)}{z_{13}\bar z_{12}}+\frac{(k_3\epsilon_1k_3)}{|z_{13}|^2}\right)\bar\partial_{23}^2P_{23}\\
&-\frac{2\pi}{\alpha'\tau_2}\left(\frac{(k_2\epsilon_1\epsilon_2\epsilon_3^\top k_1)}{\bar z_{12}\bar z_{13}}+\frac{(k_2\epsilon_1\epsilon_2\epsilon_3^\top k_2)}{\bar z_{12}\bar z_{23}}+\frac{(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top k_2)}{\bar z_{12}^2}-\frac{(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top k_2)}{\bar z_{12}\bar z_{23}}\right)\\
&-\frac{2\pi}{\alpha'\tau_2}\left(\frac{(k_3\epsilon_1\epsilon_2\epsilon_3^\top k_1)}{\bar z_{13}^2}+\frac{(k_3\epsilon_1\epsilon_2\epsilon_3^\top k_2)}{\bar z_{13}\bar z_{23}}+\frac{(k_1\epsilon_2\epsilon_3^\top\epsilon_1^\top k_3)}{\bar z_{12}\bar z_{13}}-\frac{(k_3\epsilon_2\epsilon_3^\top\epsilon_1^\top k_3)}{\bar z_{13}\bar z_{23}}\right)\Bigg]\\
&-\left(\frac{(k_1\epsilon_3k_2)}{\bar z_{13}}+\frac{(k_2\epsilon_3k_2)}{\bar z_{23}}\right)\left[\left(\frac{(k_2\epsilon_1^\top\epsilon_2k_3)}{z_{12}}+\frac{(k_3\epsilon_1^\top\epsilon_2k_3)}{z_{13}}\right)\bar\partial_{12}^2P_{12}+\frac{2\pi}{\alpha'\tau_2}\left(\frac{(k_2\epsilon_1\epsilon_2k_3)}{\bar z_{12}}+\frac{(k_3\epsilon_1\epsilon_2k_3)}{\bar z_{13}}\right)\right]\\
&-\left(\frac{(k_1\epsilon_2k_3)}{\bar z_{12}}+\frac{(k_3\epsilon_2k_3)}{\bar z_{23}}\right)\left[\left(\frac{(k_2\epsilon_1^\top\epsilon_3k_2)}{z_{12}}+\frac{(k_3\epsilon_1^\top\epsilon_3k_2)}{z_{13}}\right)\bar\partial_{13}^2P_{13}+\frac{2\pi}{\alpha'\tau_2}\left(\frac{(k_2\epsilon_1\epsilon_3k_2)}{\bar z_{12}}+\frac{(k_3\epsilon_1\epsilon_3k_2)}{\bar z_{13}}\right)\right]\\
&+\left(\frac{(k_2\epsilon_1k_2)}{|z_{12}|^2}+\frac{(k_3\epsilon_1k_2)}{z_{12}\bar z_{13}}+\frac{(k_2\epsilon_1k_3)}{z_{13}\bar z_{12}}+\frac{(k_3\epsilon_1k_3)}{|z_{13}|^2}\right)\left(k_3\epsilon_2^\top\epsilon_3k_2\right)\bar\partial_{23}^2P_{23}\\
&+\frac{2\pi}{\alpha'\tau_2}\left[-k_2\cdot k_3\left(\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)\bar\partial_{13}^2P_{13}+\left(\epsilon_1\epsilon_3\epsilon_2^\top\right)\bar\partial_{12}^2P_{12}\right)\right.\\
&\left.+\left(k_3\epsilon_2^\top\epsilon_1^\top\epsilon_3k_2\right)\bar\partial_{13}^2P_{13}+\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3k_2\right)\bar\partial_{12}^2P_{12}\right]\Bigg\}.
\end{aligned}
}
\tag{5.42}
$$

Note that all factors in the bracket are of order $\mathcal{O}\left(k^4\right)$ except for the last two lines which are of order $\mathcal{O}\left(k^2\right)$.

Extra tensor structure Beyond the two tensor structures related to the expanded Einstein-Hilbert term at $\mathcal{O}(k^2)$, two more tensor structures $\left(k_2\epsilon_1\epsilon_2k_1\right)\left(k_1\epsilon_3k_1\right)$ and $\left(\epsilon_1\epsilon_2\epsilon_3\right)$ still appear in the correlation functions. The former contributes to $\mathcal{O}(k^4)$

order and the latter vanishes after imposing momentum conservation, thus they would not be considered in the following calculation.

Computation of the 6-fermion term

In this section, we want to compute

$$\begin{aligned} G_3^{6f} =&\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}\left\langle i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1}i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2}i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle \\ &\times\left\langle k_1\cdot\psi_1\psi_1^{\nu_1}k_2\cdot\psi_2\psi_2^{\nu_2}k_3\cdot\psi_3\psi_3^{\nu_3}\right\rangle . \end{aligned} \tag{5.43}$$

We have

$$\begin{aligned} G_3^{6f} =&\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}k_{1,\rho_1}k_{2,\rho_2}k_{3,\rho_3} \\ &\times\left[\left\langle i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1}i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2}i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle\right. \\ &+\left\langle i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1}i\bar{\partial}X_2^{\mu_2}e^{\dagger k_2\cdot X_2}i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle+\text{ cyclic perms} \\ &\left.+\left\langle i\bar{\partial}X_1^{\mu_1}e^{ik_1\cdot X_1}i\bar{\partial}X_2^{\mu_2}e^{ik_2\cdot X_2}i\bar{\partial}X_3^{\mu_3}e^{ik_3\cdot X_3}\right\rangle+\text{ perms }\right] \\ &\times\left[\left\langle\psi_1^{\rho_1}\psi_1^{\nu_1}\psi_2^{\rho_2}\psi_2^{\nu_2}\psi_3^{\rho_3}\psi_3^{\nu_3}\right\rangle+\left\langle\psi_1^{\rho_1}\psi_1^{\nu_1}\psi_2^{\rho_2}\psi_2^{\nu_2}\psi_3^{\rho_3}\psi_3^{\nu_3}\right\rangle-(\rho_2\leftrightarrow\nu_2)\right. \\ &\left.-(\rho_1\leftrightarrow\nu_1)\right] \end{aligned} \tag{5.44a}$$

$$\begin{aligned} =&\frac{\alpha'}{2}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}k_{1,\rho_1}k_{2,\rho_2}k_{3,\rho_3}S_{12}S_{13}S_{23}\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j} \\ &\times\left[-\eta^{\mu_1\mu_2}\bar{\partial}_{12}^2P_{12}\left(-\frac{k_1^{\mu_3}}{\bar{z}_{13}}-\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)-\eta^{\mu_1\mu_3}\bar{\partial}_{13}^2P_{13}\left(\frac{k_3^{\mu_2}}{\bar{z}_{23}}-\frac{k_1^{\mu_2}}{\bar{z}_{12}}\right)\right. \\ &-\eta^{\mu_2\mu_3}\bar{\partial}_{23}^2P_{23}\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right) \\ &\left.-\left(\frac{\alpha'}{2}\right)^2\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\right] \\ &\times\left[\eta^{\rho_1\rho_2}\left(\eta^{\nu_1\rho_3}\eta^{\nu_2\nu_3}-\eta^{\nu_1\nu_3}\eta^{\nu_2\rho_3}\right)-\eta^{\rho_1\nu_2}\left(\eta^{\nu_1\rho_3}\eta^{\rho_2\nu_3}-\eta^{\nu_1\nu_3}\eta^{\rho_2\rho_3}\right)\right. \\ &\left.-\eta^{\nu_1\rho_2}\left(\eta^{\rho_1\rho_3}\eta^{\nu_2\nu_3}-\eta^{\rho_1\nu_3}\eta^{\nu_2\rho_3}\right)+\eta^{\nu_1\nu_2}\left(\eta^{\rho_1\rho_3}\eta^{\rho_2\nu_3}-\eta^{\rho_1\nu_3}\eta^{\rho_2\rho_3}\right)\right] \end{aligned} \tag{5.44b}$$

And we find

$$\boxed{\begin{aligned}G_3^{6f} =&\frac{\alpha'}{2}\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}S_{12}S_{13}S_{23}\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg[\eta^{\mu_1\mu_2}\bar{\partial}_{12}^2P_{12}\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)+\eta^{\mu_1\mu_3}\bar{\partial}_{13}^2P_{13}\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\\
&\quad-\eta^{\mu_2\mu_3}\bar{\partial}_{23}^2P_{23}\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\\
&\quad-\left(\frac{\alpha'}{2}\right)^2\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\Bigg]\\
&\times[k_1\cdot k_2\,(\eta^{\nu_2\nu_3}k_3^{\nu_1}-\eta^{\nu_1\nu_3}k_3^{\nu_2})+k_1\cdot k_3\,(\eta^{\nu_1\nu_2}k_2^{\nu_3}-\eta^{\nu_2\nu_3}k_2^{\nu_1})\\
&\quad+k_2\cdot k_3\,(\eta^{\nu_1\nu_3}k_1^{\nu_2}-\eta^{\nu_1\nu_2}k_1^{\nu_3})+k_1^{\nu_3}k_2^{\nu_1}k_3^{\nu_2}-k_1^{\nu_2}k_2^{\nu_3}k_3^{\nu_1}]\end{aligned}} \tag{5.45}$$

We can further simplify the tensor structure:

$$\begin{aligned}G_3^{6f} =&\frac{\alpha'}{2}S_{12}S_{13}S_{23}\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg[\eta^{\mu_1\mu_2}\bar{\partial}_{12}^2P_{12}\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)+\eta^{\mu_1\mu_3}\bar{\partial}_{13}^2P_{13}\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\\
&\quad-\eta^{\mu_2\mu_3}\bar{\partial}_{23}^2P_{23}\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\\
&\quad-\left(\frac{\alpha'}{2}\right)^2\left(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}}\right)\left(\frac{k_1^{\mu_2}}{\bar{z}_{12}}-\frac{k_3^{\mu_2}}{\bar{z}_{23}}\right)\left(\frac{k_2^{\mu_1}}{\bar{z}_{12}}+\frac{k_3^{\mu_1}}{\bar{z}_{13}}\right)\Bigg]\\
&\times\Big[k_1\cdot k_2\left(\left(\epsilon_1k_3\right)_{\mu_1}\left(\epsilon_2\epsilon_3^\top\right)_{\mu_2\mu_3}-\left(\epsilon_2k_3\right)_{\mu_2}\left(\epsilon_1\epsilon_3^\top\right)_{\mu_1\mu_3}\right)\\
&\quad+k_1\cdot k_3\left(\left(\epsilon_3k_2\right)_{\mu_3}\left(\epsilon_1\epsilon_2^\top\right)_{\mu_1\mu_2}-\left(\epsilon_1k_2\right)_{\mu_1}\left(\epsilon_2\epsilon_3^\top\right)_{\mu_2\mu_3}\right.\\
&\quad+k_2\cdot k_3\left(\left(\epsilon_2k_1\right)_{\mu_2}\left(\epsilon_1\epsilon_3^\top\right)_{\mu_1\mu_3}-\left(\epsilon_3k_1\right)_{\mu_3}\left(\epsilon_1\epsilon_2^\top\right)_{\mu_1\mu_2}\right.\\
&\quad+\left(\epsilon_3k_1\right)_{\mu_3}\left(\epsilon_1k_2\right)_{\mu_1}\left(\epsilon_2k_3\right)_{\mu_2}-\left(\epsilon_2k_1\right)_{\mu_2}\left(\epsilon_3k_2\right)_{\mu_3}\left(\epsilon_1k_3\right)_{\mu_1}\Big]\end{aligned} \tag{5.46}$$

$$\begin{aligned}=&\frac{\alpha'}{2}S_{12}S_{13}S_{23}\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
&\times\Bigg\{\bar{\partial}_{12}^2P_{12}(\frac{k_1^{\mu_3}}{\bar{z}_{13}}+\frac{k_2^{\mu_3}}{\bar{z}_{23}})\Big[k_1\cdot k_2\left(\left(k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top\right)_{\mu_3}-\left(k_3\epsilon_2^\top\epsilon_1\epsilon_3^\top\right)_{\mu_3}\right)\\
&\quad+k_1\cdot k_3\left(\left(\epsilon_1\epsilon_2^\top\right)\left(\epsilon_3k_2\right)_{\mu_3}-\left(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top\right)_{\mu_3}\right)\end{aligned}$$

$$
\begin{aligned}
&+ k_2\cdot k_3\left(\left(k_1\epsilon_2^\top\epsilon_1\epsilon_3^\top\right)_{\mu_3} - \left(\epsilon_1\epsilon_2^\top\right)\left(\epsilon_3 k_1\right)_{\mu_3}\right)\\
&+ \left(k_2\epsilon_1^\top\epsilon_2 k_3\right)\left(\epsilon_3 k_1\right)_{\mu_3} - \left(k_3\epsilon_1^\top\epsilon_2 k_1\right)\left(\epsilon_3 k_2\right)_{\mu_3}\Big]\\
&+ \bar\partial_{13}^2 P_{13}\left(\frac{k_1^{\mu_2}}{\bar z_{12}} - \frac{k_3^{\mu_2}}{\bar z_{23}}\right)\Big[k_1\cdot k_2\left(\left(\epsilon_2\epsilon_3^\top\epsilon_1 k_3\right)_{\mu_2} - \left(\epsilon_1\epsilon_3^\top\right)\left(\epsilon_2 k_3\right)_{\mu_2}\right)\\
&+ k_1\cdot k_3\left(\left(k_2\epsilon_3^\top\epsilon_1\epsilon_2^\top\right)_{\mu_2} - \left(\epsilon_2\epsilon_3^\top\epsilon_1 k_2\right)_{\mu_2}\right)\\
&+ k_2\cdot k_3\left(\left(\epsilon_1\epsilon_3^\top\right)\left(\epsilon_2 k_1\right)_{\mu_2} - \left(k_1\epsilon_3^\top\epsilon_1\epsilon_2^\top\right)_{\mu_2}\right)\\
&+ \left(k_2\epsilon_1^\top\epsilon_3 k_1\right)\left(\epsilon_2 k_3\right)_{\mu_2} - \left(k_3\epsilon_1^\top\epsilon_3 k_2\right)\left(\epsilon_2 k_1\right)_{\mu_2}\Big]\\
&- \bar\partial_{23}^2 P_{23}\left(\frac{k_2^{\mu_1}}{\bar z_{12}} + \frac{k_3^{\mu_1}}{\bar z_{13}}\right)\Big[k_1\cdot k_2\left(\left(\epsilon_2\epsilon_3^\top\right)\left(\epsilon_1 k_3\right)_{\mu_1} - \left(\epsilon_1\epsilon_3^\top\epsilon_2 k_3\right)_{\mu_1}\right)\\
&+ k_1\cdot k_3\left(\left(\epsilon_1\epsilon_2^\top\epsilon_3 k_2\right)_{\mu_1} - \left(\epsilon_2\epsilon_3^\top\right)\left(\epsilon_1 k_2\right)_{\mu_1}\right)\\
&+ k_2\cdot k_3\left(\left(\epsilon_1\epsilon_3^\top\epsilon_2 k_1\right)_{\mu_1} - \left(\epsilon_1\epsilon_2^\top\epsilon_3 k_1\right)_{\mu_1}\right)\\
&+ \left(k_3\epsilon_2^\top\epsilon_3 k_1\right)\left(\epsilon_1 k_2\right)_{\mu_1} - \left(k_1\epsilon_2^\top\epsilon_3 k_2\right)\left(\epsilon_1 k_3\right)_{\mu_1}\Big]\\
&- \left(\frac{\alpha'}{2}\right)^2\left(\frac{k_1^{\mu_3}}{\bar z_{13}} + \frac{k_2^{\mu_3}}{\bar z_{23}}\right)\left(\frac{k_1^{\mu_2}}{\bar z_{12}} - \frac{k_3^{\mu_2}}{\bar z_{23}}\right)\left(\frac{k_2^{\mu_1}}{\bar z_{12}} + \frac{k_3^{\mu_1}}{\bar z_{13}}\right)\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}\\
\times&\left[k_1\cdot k_2\left(\eta^{\nu_2\nu_3}k_3^{\nu_1} - \eta^{\nu_1\nu_3}k_3^{\nu_2}\right) + k_1\cdot k_3\left(\eta^{\nu_1\nu_2}k_2^{\nu_3} - \eta^{\nu_2\nu_3}k_2^{\nu_1}\right)\right.\\
&\left.+k_2\cdot k_3\left(\eta^{\nu_1\nu_3}k_1^{\nu_2} - \eta^{\nu_1\nu_2}k_1^{\nu_3}\right) + k_1^{\nu_3}k_2^{\nu_1}k_3^{\nu_2} - k_1^{\nu_2}k_2^{\nu_3}k_3^{\nu_1}\right]\Big\}
\end{aligned}
\tag{5.47}
$$

$$
\begin{aligned}
=&\frac{\alpha'}{2}S_{12}S_{13}S_{23}\times\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}\\
\times&\Big\{\bar\partial_{12}^2 P_{12}\left(\frac{k_1^{\mu_3}}{\bar z_{13}} + \frac{k_2^{\mu_3}}{\bar z_{23}}\right)\Big[\left(\epsilon_1\epsilon_2^\top\right)\left(k_1\cdot k_3\left(\epsilon_3 k_2\right)_{\mu_3} - k_2\cdot k_3\left(\epsilon_3 k_1\right)_{\mu_3}\right)\\
&+ k_1\cdot k_2\left((k_3\epsilon_1^\top\epsilon_2\epsilon_3^\top)_{\mu_3} - (k_3\epsilon_2^\top\epsilon_1\epsilon_3^\top)_{\mu_3}\right)\\
&- k_1\cdot k_3\left(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top\right)_{\mu_3} + k_2\cdot k_3\left(k_1\epsilon_2^\top\epsilon_1\epsilon_3^\top\right)_{\mu_3}\\
&+ \left(k_2\epsilon_1^\top\epsilon_2 k_3\right)\left(\epsilon_3 k_1\right)_{\mu_3} - \left(k_3\epsilon_1^\top\epsilon_2 k_1\right)\left(\epsilon_3 k_2\right)_{\mu_3}\Big]\\
&+ \bar\partial_{13}^2 P_{13}\left(\frac{k_1^{\mu_2}}{\bar z_{12}} - \frac{k_3^{\mu_2}}{\bar z_{23}}\right)\Big[\left(\epsilon_1\epsilon_3^\top\right)\left(-k_1\cdot k_2\left(\epsilon_2 k_3\right)_{\mu_2} + k_2\cdot k_3\left(\epsilon_2 k_1\right)_{\mu_2}\right)\\
&+ k_1\cdot k_3\left(k_2\epsilon_1^\top\epsilon_2\epsilon_3^\top\right)_{\mu_3} + k_2\cdot k_3\left(k_1\epsilon_2^\top\epsilon_1\epsilon_3^\top\right)_{\mu_3}\\
&+ k_1\cdot k_2\left(\epsilon_2\epsilon_3^\top\epsilon_1 k_3\right)_{\mu_2} - k_2\cdot k_3(k_1\epsilon_3^\top\epsilon_1\epsilon_2^\top)_{\mu_2}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \left(k_2\epsilon_1^\top \epsilon_3 k_1\right)\left(\epsilon_2 k_3\right)_{\mu_2} - \left(k_3\epsilon_1^\top \epsilon_3 k_2\right)\left(\epsilon_2 k_1\right)_{\mu_2}\Big] \\
&\quad - \bar\partial_{23}^2 P_{23}\left(\frac{k_2^{\mu_1}}{\bar z_{12}} + \frac{k_3^{\mu_1}}{\bar z_{13}}\right)\Big[k_1\cdot k_2\left((\epsilon_2\epsilon_3^\top)(\epsilon_1 k_3)_{\mu_1} - (\epsilon_1\epsilon_3^\top\epsilon_2 k_3)_{\mu_1}\right) \\
&\qquad + k_1\cdot k_3\left((\epsilon_1\epsilon_2^\top\epsilon_3 k_2)_{\mu_1} - (\epsilon_2\epsilon_3^\top)(\epsilon_1 k_2)_{\mu_1}\right) \\
&\qquad + k_2\cdot k_3\left((\epsilon_1\epsilon_3^\top\epsilon_2 k_1)_{\mu_1} - (\epsilon_1\epsilon_2^\top\epsilon_3 k_1)\right) \\
&\qquad + \left(k_3\epsilon_2^\top\epsilon_3 k_1\right)\left(\epsilon_1 k_2\right)_{\mu_1} - \left(k_1\epsilon_2^\top\epsilon_3 k_2\right)\left(\epsilon_1 k_3\right)_{\mu_1}\Big] \\
& - \left(\frac{\alpha'}{2}\right)^2\left(\frac{k_1^{\mu_3}}{\bar z_{13}} + \frac{k_2^{\mu_3}}{\bar z_{23}}\right)\left(\frac{k_1^{\mu_2}}{\bar z_{12}} - \frac{k_3^{\mu_2}}{\bar z_{23}}\right)\left(\frac{k_2^{\mu_1}}{\bar z_{12}} + \frac{k_3^{\mu_1}}{\bar z_{13}}\right)\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3} \\
&\quad \times\big[k_1\cdot k_2(\eta^{\nu_2\nu_3}k_3^{\nu_1} - \eta^{\nu_1\nu_3}k_3^{\nu_2}) + k_1\cdot k_3(\eta^{\nu_1\nu_2}k_2^{\nu_3} - \eta^{\nu_2\nu_3}k_2^{\nu_1}) \\
&\qquad + k_2\cdot k_3(\eta^{\nu_1\nu_3}k_1^{\nu_2} - \eta^{\nu_1\nu_2}k_1^{\nu_3}) + k_1^{\nu_3}k_2^{\nu_1}k_3^{\nu_2} - k_1^{\nu_2}k_2^{\nu_3}k_3^{\nu_1}\big]\big\}\,.
\end{aligned} \tag{5.48}
$$

The spin sum in maximal supersymmetry (in particular $D = 10$) implies that the amplitude vanishes[13, §3.3]. The simplest would be to perform the spin sum here, before analysing further the different contributions. However, this breaks the uniformity of the formulas so we don't do it now.

3 graviton scattering, $\mathcal{O}\left(k^2\right)$ contribution

Now we focus on 3 gravitons amplitude, which means only symmetric polarization tensors should be considered when necessary.

Since we are trying to check the tensor structure (5.2) of the Einstein-Hilbert term, we would like to study the $\mathcal{O}\left(k^2\right)$ terms in 3 gravitons amplitude, which could be derived from two sources: 1) contractions giving k^2, 2) k^4 with pinched-off integration. We observe that the fermion terms and the products $\partial X e^{ik\cdot X}$ contribute at order k^4 in (5.31b). Note that we set $|\chi_{ij}|^{\alpha' k_i\cdot k_j} = 1$ (except when considering pinching) for simplicity since they would contribute to higher powers of the momenta. The terms relevant to the kinematic structure (5.2) should have the kinematic structures in the form of $(k_j\epsilon_i k_j)(\epsilon_j\epsilon_l)$ or $(k_k\epsilon_i\epsilon_j\epsilon_l k_i)$.

$\mathcal{O}(k^2)$ contributions without pinched-off integration It is easy to derive $\mathcal{O}(k^2)$ contributions without pinched-off integration from the last two lines of (5.42):

$$
\begin{aligned}
G_3^{4f}\Big|_{\text{no pinched-off}} &= -\frac{\alpha'}{2}S_{23}^2 \times \prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j} \\
&\quad \times\left\{\frac{2\pi}{\alpha'\tau_2}\left[-k_2\cdot k_3\left(\left(\epsilon_1\epsilon_2\epsilon_3^\top\right)\bar\partial_{13}^2 P_{13} + \left(\epsilon_1\epsilon_3\epsilon_2^\top\right)\bar\partial_{12}^2 P_{12}\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(k_3\epsilon_2^\top\epsilon_1^\top\epsilon_3 k_2\right)\bar{\partial}_{13}^2 P_{13} + \left(k_3\epsilon_2^\top\epsilon_1\epsilon_3 k_2\right)\bar{\partial}_{12}^2 P_{12}\big]\big\} \\
&+ \text{cyclic permutations.}
\end{aligned} \tag{5.49}
$$

We notice that the second line of the above result vanishes after imposing momentum conservation (5.5).

Koba-Nielsen factor Pinched-off integration would use one $|\chi_{mn}|^{\alpha' k_m\cdot k_n}$. The remaining non-pinching part of the Koba–Nielsen factor could be expanded as

$$
\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j}_{non-pinching} = 1 - 2\alpha'\sum_{i<j} k_i\cdot k_j P_{ij} + \text{terms of higher } k \text{ order}, \tag{5.50}
$$

which is vaild only if $|z_{ij}| > \epsilon$ (non-pinching) because of the singularity of P_{ij}[40]. Only the leading term after the above expansion in the amplitude still keeps in $\mathcal{O}(k^2)$ order. Because of this, we take $\prod_{i<j}|\chi_{ij}|^{\alpha' k_i\cdot k_j} \sim 1$ in our $\mathcal{O}(k^2)$ calculation.

4-Fermion The terms relevant to the kinematic structure (5.2) from 4-fermion contribution (5.42) are

$$
\begin{aligned}
&\frac{\alpha'}{2}k_2\cdot k_3\Bigg[\left(\frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_1}{z_{12}\bar{z}_{13}} + \frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{12}\bar{z}_{23}} + \frac{k_3\epsilon_1\epsilon_2\epsilon_3 k_1}{|z_{13}|^2}\right)\bar{\partial}_{12}^2 P_{12} \\
&\quad + \left(\frac{k_1\epsilon_2\epsilon_3\epsilon_1 k_2}{|z_{12}|^2} + \frac{k_1\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{13}\bar{z}_{12}} - \frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{13}\bar{z}_{23}}\right)\bar{\partial}_{13}^2 P_{13} \\
&\quad - (\epsilon_2\epsilon_3)\left(\frac{k_2\epsilon_1 k_2}{|z_{12}|^2} + \frac{k_3\epsilon_1 k_3}{|z_{13}|^2}\right)\bar{\partial}_{23}^2 P_{23}\Bigg] \\
&+\frac{\alpha'}{2}k_3\cdot k_1\Bigg[\left(\frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_2}{z_{23}\bar{z}_{21}} + \frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{23}\bar{z}_{31}} + \frac{k_1\epsilon_2\epsilon_3\epsilon_1 k_2}{|z_{21}|^2}\right)\bar{\partial}_{23}^2 P_{23} \\
&\quad + \left(\frac{k_2\epsilon_3\epsilon_1\epsilon_2 k_3}{|z_{23}|^2} + \frac{k_2\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{21}\bar{z}_{23}} - \frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{21}\bar{z}_{31}}\right)\bar{\partial}_{21}^2 P_{21} \\
&\quad - (\epsilon_3\epsilon_1)\left(\frac{k_3\epsilon_2 k_3}{|z_{23}|^2} + \frac{k_1\epsilon_2 k_1}{|z_{21}|^2}\right)\bar{\partial}_{31}^2 P_{31}\Bigg] \\
&+\frac{\alpha'}{2}k_1\cdot k_2\Bigg[\left(\frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_3}{z_{31}\bar{z}_{32}} + \frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{31}\bar{z}_{12}} + \frac{k_2\epsilon_3\epsilon_1\epsilon_2 k_3}{|z_{32}|^2}\right)\bar{\partial}_{31}^2 P_{31} \\
&\quad + \left(\frac{k_3\epsilon_1\epsilon_2\epsilon_3 k_1}{|z_{31}|^2} + \frac{k_3\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{32}\bar{z}_{31}} - \frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{32}\bar{z}_{12}}\right)\bar{\partial}_{32}^2 P_{32} \\
&\quad - (\epsilon_1\epsilon_2)\left(\frac{k_1\epsilon_3 k_1}{|z_{31}|^2} + \frac{k_2\epsilon_3 k_2}{|z_{32}|^2}\right)\bar{\partial}_{12}^2 P_{12}\Bigg]
\end{aligned}
$$

$$+\frac{\alpha'}{2}\frac{2\pi}{\alpha'\tau_2}\left[\left(k_3\epsilon_2\epsilon_1\epsilon_3k_2\right)\bar{\partial}_{13}^2P_{13}+\left(k_3\epsilon_2\epsilon_1\epsilon_3k_2\right)\bar{\partial}_{12}^2P_{12}+\left(k_1\epsilon_3\epsilon_2\epsilon_1k_3\right)\bar{\partial}_{21}^2P_{21}\right.$$
$$\left.+\left(k_1\epsilon_3\epsilon_2\epsilon_1k_3\right)\bar{\partial}_{23}^2P_{23}+\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)\bar{\partial}_{32}^2P_{32}+\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)\bar{\partial}_{31}^2P_{31}\right]. \tag{5.51}$$

6-Fermion with Taylor expansion The terms relevant to the kinematic structure (5.2) from 6-fermion contribution (5.46), with application of Taylor expansion trick §5.1.3, are

$$\begin{aligned}
&-\frac{1}{z_{12}}\left(-\bar{\partial}_{12}^2P_{12}\right)\left(-k_1^{\mu_3}\bar{\partial}_3P_{31}-k_2^{\mu_3}\bar{\partial}_3P_{32}\right)\Big\{\left(\epsilon_1\epsilon_2\right)\Big[\left(k_1\cdot k_3\right)\left(\epsilon_3k_2\right)_{\mu_3}-\left(k_2\cdot k_3\right)\left(\epsilon_3k_1\right)_{\mu_3}\Big]\\
&+\left(k_1\cdot k_2\right)\Big[\left(k_3\epsilon_1\epsilon_2\epsilon_3\right)_{\mu_3}-\left(k_3\epsilon_2\epsilon_1\epsilon_3\right)_{\mu_3}\Big]-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3\right)_{\mu_3}+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3\right)_{\mu_3}\Big\}\\
&-\frac{1}{z_{31}}\left(-\bar{\partial}_{13}^2P_{13}\right)\left(-k_3^{\mu_2}\bar{\partial}_2P_{23}-k_1^{\mu_2}\bar{\partial}_2P_{21}\right)\Big\{\left(\epsilon_1\epsilon_3\right)\Big[\left(k_2\cdot k_3\right)\left(\epsilon_2k_1\right)_{\mu_2}-\left(k_1\cdot k_2\right)\left(\epsilon_2k_3\right)_{\mu_2}\Big]\\
&+\left(k_1\cdot k_3\right)\Big[\left(k_2\epsilon_3\epsilon_1\epsilon_2\right)_{\mu_2}-\left(\epsilon_2\epsilon_3\epsilon_1k_2\right)_{\mu_2}\Big]+\left(k_1\cdot k_2\right)\left(\epsilon_2\epsilon_3\epsilon_1k_3\right)_{\mu_2}-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2\right)_{\mu_2}\Big\}\\
&-\frac{1}{z_{23}}\left(-\bar{\partial}_{23}^2P_{23}\right)\left(-k_2^{\mu_1}\bar{\partial}_1P_{12}-k_3^{\mu_1}\bar{\partial}_1P_{13}\right)\Big\{\left(\epsilon_2\epsilon_3\right)\Big[\left(k_1\cdot k_2\right)\left(\epsilon_1k_3\right)_{\mu_1}-\left(k_1\cdot k_3\right)\left(\epsilon_1k_2\right)_{\mu_1}\Big]\\
&+\left(k_2\cdot k_3\right)\Big[\left(\epsilon_1\epsilon_3\epsilon_2k_1\right)_{\mu_1}-\left(\epsilon_1\epsilon_2\epsilon_3k_1\right)_{\mu_1}\Big]-\left(k_1\cdot k_2\right)\left(\epsilon_1\epsilon_3\epsilon_2k_3\right)_{\mu_1}+\left(k_1\cdot k_3\right)\left(\epsilon_1\epsilon_2\epsilon_3k_2\right)_{\mu_1}\Big\}
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
=&-\frac{1}{z_{12}}\bar{\partial}_{12}^2P_{12}\left(\bar{\partial}_3P_{31}-\bar{\partial}_3P_{32}\right)\{\left(\epsilon_1\epsilon_2\right)\left[\left(k_1\cdot k_3\right)\left(k_1\epsilon_3k_2\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3k_1\right)\right]\\
&+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\}\\
&-\frac{1}{z_{31}}\bar{\partial}_{13}^2P_{13}\left(\bar{\partial}_2P_{23}-\bar{\partial}_2P_{21}\right)\{\left(\epsilon_1\epsilon_3\right)\left[\left(k_2\cdot k_3\right)\left(k_3\epsilon_2k_1\right)-\left(k_1\cdot k_2\right)\left(k_3\epsilon_2k_3\right)\right]\\
&+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)\}\\
&-\frac{1}{z_{23}}\bar{\partial}_{23}^2P_{23}\left(\bar{\partial}_1P_{12}-\bar{\partial}_1P_{13}\right)\{\left(\epsilon_2\epsilon_3\right)\left[\left(k_1\cdot k_2\right)\left(k_2\epsilon_1k_3\right)-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1k_2\right)\right]\\
&+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\}
\end{aligned} \tag{5.53}$$

Using Taylor expansion trick(5.8):

$$\begin{aligned}
=&\frac{\alpha'}{2}\frac{1}{\left|z_{12}\right|^2}\frac{1}{\bar{z}_{12}}\left(+\bar{z}_{12}\bar{\partial}_3\bar{\partial}_2P_{32}\right)\{\left(\epsilon_1\epsilon_2\right)\left(k_1\epsilon_3k_1\right)\left[-\left(k_1\cdot k_3\right)-\left(k_2\cdot k_3\right)\right]\\
&+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\}\\
&+\frac{\alpha'}{2}\frac{1}{\left|z_{31}\right|^2}\frac{1}{\bar{z}_{31}}\left(+\bar{z}_{31}\bar{\partial}_2\bar{\partial}_1P_{21}\right)\{\left(\epsilon_1\epsilon_3\right)\left(k_3\epsilon_2k_3\right)\left[-\left(k_2\cdot k_3\right)-\left(k_1\cdot k_2\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ (k_1 \cdot k_3) \left[(k_2 \epsilon_3 \epsilon_1 \epsilon_2 k_3) - (k_3 \epsilon_2 \epsilon_3 \epsilon_1 k_2) \right] + (k_1 \cdot k_2) (k_3 \epsilon_2 \epsilon_3 \epsilon_1 k_3) - (k_2 \cdot k_3) (k_1 \epsilon_3 \epsilon_1 \epsilon_2 k_3) \} \\
&+ \frac{\alpha'}{2} \frac{1}{|z_{23}|^2} \frac{1}{\bar{z}_{23}} \left(+ \bar{z}_{23} \bar{\partial}_1 \bar{\partial}_3 P_{13} \right) \{ (\epsilon_2 \epsilon_3) (k_2 \epsilon_1 k_2) \left[- (k_1 \cdot k_2) - (k_1 \cdot k_3) \right] \\
&+ (k_2 \cdot k_3) \left[(k_2 \epsilon_1 \epsilon_3 \epsilon_2 k_1) - (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_1) \right] - (k_1 \cdot k_2) (k_2 \epsilon_1 \epsilon_3 \epsilon_2 k_3) + (k_1 \cdot k_3) (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_2) \}
\end{aligned} \tag{5.54}
$$

$$
\begin{aligned}
= &+ \frac{\alpha'}{2} \frac{1}{|z_{12}|^2} \bar{\partial}_{23}^2 P_{23} \{ (k_1 \cdot k_2) (\epsilon_1 \epsilon_2) (k_1 \epsilon_3 k_1) + (k_1 \cdot k_2) \left[(k_3 \epsilon_1 \epsilon_2 \epsilon_3 k_1) - (k_3 \epsilon_2 \epsilon_1 \epsilon_3 k_1) \right] \\
&- (k_1 \cdot k_3) (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_1) + (k_2 \cdot k_3) (k_1 \epsilon_2 \epsilon_1 \epsilon_3 k_1) \} \\
&+ \frac{\alpha'}{2} \frac{1}{|z_{13}|^2} \bar{\partial}_{12}^2 P_{12} \{ (k_1 \cdot k_3) (\epsilon_1 \epsilon_3) (k_3 \epsilon_2 k_3) + (k_1 \cdot k_3) \left[(k_2 \epsilon_3 \epsilon_1 \epsilon_2 k_3) - (k_3 \epsilon_2 \epsilon_3 \epsilon_1 k_2) \right] \\
&+ (k_1 \cdot k_2) (k_3 \epsilon_2 \epsilon_3 \epsilon_1 k_3) - (k_2 \cdot k_3) (k_1 \epsilon_3 \epsilon_1 \epsilon_2 k_3) \} \\
&+ \frac{\alpha'}{2} \frac{1}{|z_{23}|^2} \bar{\partial}_{13}^2 P_{13} \{ (k_2 \cdot k_3) (\epsilon_2 \epsilon_3) (k_2 \epsilon_1 k_2) + (k_2 \cdot k_3) \left[(k_2 \epsilon_1 \epsilon_3 \epsilon_2 k_1) - (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_1) \right] \\
&- (k_1 \cdot k_2) (k_2 \epsilon_1 \epsilon_3 \epsilon_2 k_3) + (k_1 \cdot k_3) (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_2) \} .
\end{aligned} \tag{5.55}
$$

6-Fermion without Taylor expansion The terms relevant to the kinematic structure (5.2) from 6-fermion contribution (5.46), without application of Taylor expansion trick §5.1.3, are

$$
\begin{aligned}
&\frac{\alpha'}{2} \bar{\partial}_{12}^2 P_{12} \left(\frac{k_1^{\mu_3}}{|z_{13}|^2} - \frac{k_2^{\mu_3}}{|z_{23}|^2} \right) \Big\{ (\epsilon_1 \epsilon_2) \Big[(k_1 \cdot k_3) (\epsilon_3 k_2)_{\mu_3} - (k_2 \cdot k_3) (\epsilon_3 k_1)_{\mu_3} \Big] \\
&+ (k_1 \cdot k_2) \Big[(k_3 \epsilon_1 \epsilon_2 \epsilon_3)_{\mu_3} - (k_3 \epsilon_2 \epsilon_1 \epsilon_3)_{\mu_3} \Big] - (k_1 \cdot k_3) (k_2 \epsilon_1 \epsilon_2 \epsilon_3)_{\mu_3} + (k_2 \cdot k_3) (k_1 \epsilon_2 \epsilon_1 \epsilon_3)_{\mu_3} \Big\} \\
&+ \frac{\alpha'}{2} \bar{\partial}_{13}^2 P_{13} \left(- \frac{k_1^{\mu_2}}{|z_{12}|^2} + \frac{k_3^{\mu_2}}{|z_{23}|^2} \right) \Big\{ (\epsilon_1 \epsilon_3) \Big[(k_2 \cdot k_3) (\epsilon_2 k_1)_{\mu_2} - (k_1 \cdot k_2) (\epsilon_2 k_3)_{\mu_2} \Big] \\
&+ (k_1 \cdot k_3) \Big[(k_2 \epsilon_3 \epsilon_1 \epsilon_2)_{\mu_2} - (\epsilon_2 \epsilon_3 \epsilon_1 k_2)_{\mu_2} \Big] + (k_1 \cdot k_2) (\epsilon_2 \epsilon_3 \epsilon_1 k_3)_{\mu_2} - (k_2 \cdot k_3) (k_1 \epsilon_3 \epsilon_1 \epsilon_2)_{\mu_2} \Big\} \\
&+ \frac{\alpha'}{2} \bar{\partial}_{23}^2 P_{23} \left(\frac{k_2^{\mu_1}}{|z_{12}|^2} - \frac{k_3^{\mu_1}}{|z_{13}|^2} \right) \Big\{ (\epsilon_2 \epsilon_3) \Big[(k_1 \cdot k_2) (\epsilon_1 k_3)_{\mu_1} - (k_1 \cdot k_3) (\epsilon_1 k_2)_{\mu_1} \Big] \\
&+ (k_2 \cdot k_3) \Big[(\epsilon_1 \epsilon_3 \epsilon_2 k_1)_{\mu_1} - (\epsilon_1 \epsilon_2 \epsilon_3 k_1)_{\mu_1} \Big] - (k_1 \cdot k_2) (\epsilon_1 \epsilon_3 \epsilon_2 k_3)_{\mu_1} + (k_1 \cdot k_3) (\epsilon_1 \epsilon_2 \epsilon_3 k_2)_{\mu_1} \Big\}
\end{aligned} \tag{5.56}
$$

$$
\begin{aligned}
= &\frac{\alpha'}{2} \bar{\partial}_{12}^2 P_{12} \left(\frac{1}{|z_{13}|^2} + \frac{1}{|z_{23}|^2} \right) \{ (\epsilon_1 \epsilon_2) (k_1 \epsilon_3 k_1) \left[- (k_1 \cdot k_3) - (k_2 \cdot k_3) \right] \\
&+ (k_1 \cdot k_2) \left[(k_3 \epsilon_1 \epsilon_2 \epsilon_3 k_1) - (k_3 \epsilon_2 \epsilon_1 \epsilon_3 k_1) \right] - (k_1 \cdot k_3) (k_2 \epsilon_1 \epsilon_2 \epsilon_3 k_1) + (k_2 \cdot k_3) (k_1 \epsilon_2 \epsilon_1 \epsilon_3 k_1) \} \\
&+ \frac{\alpha'}{2} \bar{\partial}_{13}^2 P_{13} \left(\frac{1}{|z_{12}|^2} + \frac{1}{|z_{23}|^2} \right) \{ (\epsilon_1 \epsilon_3) (k_3 \epsilon_2 k_3) \left[- (k_2 \cdot k_3) - (k_1 \cdot k_2) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ (k_1 \cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3) - (k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right] + (k_1 \cdot k_2)(k_3\epsilon_2\epsilon_3\epsilon_1k_3) - (k_2 \cdot k_3)(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\} \\
&+ \frac{\alpha'}{2}\bar{\partial}_{23}^2 P_{23}\left(\frac{1}{|z_{12}|^2} + \frac{1}{|z_{13}|^2}\right)\{(\epsilon_2\epsilon_3)(k_2\epsilon_1k_2)\left[-(k_1 \cdot k_2) - (k_1 \cdot k_3)\right] \\
&+ (k_2 \cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1) - (k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right] - (k_1 \cdot k_2)(k_2\epsilon_1\epsilon_3\epsilon_2k_3) + (k_1 \cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\}
\end{aligned}
\tag{5.57}
$$

$$
\begin{aligned}
=&\frac{\alpha'}{2}\bar{\partial}_{12}^2 P_{12}\left(\frac{1}{|z_{13}|^2} + \frac{1}{|z_{23}|^2}\right)\{(k_1 \cdot k_2)(\epsilon_1\epsilon_2)(k_1\epsilon_3k_1) + (k_1 \cdot k_2)\left[(k_3\epsilon_1\epsilon_2\epsilon_3k_1) - (k_3\epsilon_2\epsilon_1\epsilon_3k_1)\right] \\
&- (k_1 \cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3k_1) + (k_2 \cdot k_3)(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\} \\
&+ \frac{\alpha'}{2}\bar{\partial}_{13}^2 P_{13}\left(\frac{1}{|z_{12}|^2} + \frac{1}{|z_{23}|^2}\right)\{(k_1 \cdot k_3)(\epsilon_1\epsilon_3)(k_3\epsilon_2k_3) + (k_1 \cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3) - (k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right] \\
&+ (k_1 \cdot k_2)(k_3\epsilon_2\epsilon_3\epsilon_1k_3) - (k_2 \cdot k_3)(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\} \\
&+ \frac{\alpha'}{2}\bar{\partial}_{23}^2 P_{23}\left(\frac{1}{|z_{12}|^2} + \frac{1}{|z_{13}|^2}\right)\{(k_2 \cdot k_3)(\epsilon_2\epsilon_3)(k_2\epsilon_1k_2) + (k_2 \cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1) - (k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right] \\
&- (k_1 \cdot k_2)(k_2\epsilon_1\epsilon_3\epsilon_2k_3) + (k_1 \cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\}\,.
\end{aligned}
\tag{5.58}
$$

Gathering all relevant terms, we get

$$
\begin{aligned}
\frac{\alpha'}{2}k_2 \cdot k_3 \Bigg[&\left(\frac{k_2\epsilon_1\epsilon_2\epsilon_3k_1}{z_{12}\bar{z}_{13}} + \frac{k_2\epsilon_1\epsilon_2\epsilon_3k_2}{z_{12}\bar{z}_{23}} + \frac{k_3\epsilon_1\epsilon_2\epsilon_3k_1}{|z_{13}|^2}\right)\bar{\partial}_{12}^2 P_{12} \\
&+ \left(\frac{k_1\epsilon_2\epsilon_3\epsilon_1k_2}{|z_{12}|^2} + \frac{k_1\epsilon_2\epsilon_3\epsilon_1k_3}{z_{13}\bar{z}_{12}} - \frac{k_3\epsilon_2\epsilon_3\epsilon_1k_3}{z_{13}\bar{z}_{23}}\right)\bar{\partial}_{13}^2 P_{13} \\
&- (\epsilon_2\epsilon_3)\left(\frac{k_2\epsilon_1k_2}{|z_{12}|^2} + \frac{k_3\epsilon_1k_3}{|z_{13}|^2}\right)\bar{\partial}_{23}^2 P_{23}\Bigg] \\
&+ \frac{\alpha'}{2}k_3 \cdot k_1 \Bigg[\left(\frac{k_3\epsilon_2\epsilon_3\epsilon_1k_2}{z_{23}\bar{z}_{21}} + \frac{k_3\epsilon_2\epsilon_3\epsilon_1k_3}{z_{23}\bar{z}_{31}} + \frac{k_1\epsilon_2\epsilon_3\epsilon_1k_2}{|z_{21}|^2}\right)\bar{\partial}_{23}^2 P_{23} \\
&+ \left(\frac{k_2\epsilon_3\epsilon_1\epsilon_2k_3}{|z_{23}|^2} + \frac{k_2\epsilon_3\epsilon_1\epsilon_2k_1}{z_{21}\bar{z}_{23}} - \frac{k_1\epsilon_3\epsilon_1\epsilon_2k_1}{z_{21}\bar{z}_{31}}\right)\bar{\partial}_{21}^2 P_{21} \\
&- (\epsilon_3\epsilon_1)\left(\frac{k_3\epsilon_2k_3}{|z_{23}|^2} + \frac{k_1\epsilon_2k_1}{|z_{21}|^2}\right)\bar{\partial}_{31}^2 P_{31}\Bigg] \\
&+ \frac{\alpha'}{2}k_1 \cdot k_2 \Bigg[\left(\frac{k_1\epsilon_3\epsilon_1\epsilon_2k_3}{z_{31}\bar{z}_{32}} + \frac{k_1\epsilon_3\epsilon_1\epsilon_2k_1}{z_{31}\bar{z}_{12}} + \frac{k_2\epsilon_3\epsilon_1\epsilon_2k_3}{|z_{32}|^2}\right)\bar{\partial}_{31}^2 P_{31} \\
&+ \left(\frac{k_3\epsilon_1\epsilon_2\epsilon_3k_1}{|z_{31}|^2} + \frac{k_3\epsilon_1\epsilon_2\epsilon_3k_2}{z_{32}\bar{z}_{31}} - \frac{k_2\epsilon_1\epsilon_2\epsilon_3k_2}{z_{32}\bar{z}_{12}}\right)\bar{\partial}_{32}^2 P_{32} \\
&- (\epsilon_1\epsilon_2)\left(\frac{k_1\epsilon_3k_1}{|z_{31}|^2} + \frac{k_2\epsilon_3k_2}{|z_{32}|^2}\right)\bar{\partial}_{12}^2 P_{12}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\pi}{\tau_2}\left[(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar\partial_{13}^2P_{13}+(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar\partial_{12}^2P_{12}+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar\partial_{21}^2P_{21}\right.\\
&\left.+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar\partial_{23}^2P_{23}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar\partial_{32}^2P_{32}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar\partial_{31}^2P_{31}\right]\\
&+\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial_{23}^2P_{23}\left\{(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3k_1)+(k_1\cdot k_2)\left[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)\right]\right.\\
&\left.-(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\right\}\\
&+\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar\partial_{12}^2P_{12}\left\{(k_1\cdot k_3)\,(\epsilon_1\epsilon_3)\,(k_3\epsilon_2k_3)+(k_1\cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right]\right.\\
&\left.+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)-(k_2\cdot k_3)\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\right\}\\
&+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar\partial_{13}^2P_{13}\left\{(k_2\cdot k_3)\,(\epsilon_2\epsilon_3)\,(k_2\epsilon_1k_2)+(k_2\cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right]\right.\\
&\left.-(k_1\cdot k_2)\,(k_2\epsilon_1\epsilon_3\epsilon_2k_3)+(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}\\
&+\frac{\alpha'}{2}\bar\partial_{12}^2P_{12}\left(\frac{1}{|z_{13}|^2}+\frac{1}{|z_{23}|^2}\right)\left\{(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3k_1)+(k_1\cdot k_2)\left[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)\right]\right.\\
&\left.-(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\right\}\\
&+\frac{\alpha'}{2}\bar\partial_{13}^2P_{13}\left(\frac{1}{|z_{12}|^2}+\frac{1}{|z_{23}|^2}\right)\left\{(k_1\cdot k_3)\,(\epsilon_1\epsilon_3)\,(k_3\epsilon_2k_3)+(k_1\cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right]\right.\\
&\left.+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)-(k_2\cdot k_3)\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\right\}\\
&+\frac{\alpha'}{2}\bar\partial_{23}^2P_{23}\left(\frac{1}{|z_{12}|^2}+\frac{1}{|z_{13}|^2}\right)\left\{(k_2\cdot k_3)\,(\epsilon_2\epsilon_3)\,(k_2\epsilon_1k_2)+(k_2\cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right]\right.\\
&\left.-(k_1\cdot k_2)\,(k_2\epsilon_1\epsilon_3\epsilon_2k_3)+(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}. \qquad (5.59)
\end{aligned}
$$

Obviously, the $\epsilon_i\epsilon_j$ terms of the 4-fermion and the 6-fermion terms without Taylor expansion trick cancel each other leading to

$$
\begin{aligned}
&\frac{\alpha'}{2}k_2\cdot k_3\left[\left(\frac{k_2\epsilon_1\epsilon_2\epsilon_3k_1}{z_{12}\bar z_{13}}+\frac{k_2\epsilon_1\epsilon_2\epsilon_3k_2}{z_{12}\bar z_{23}}+\frac{k_3\epsilon_1\epsilon_2\epsilon_3k_1}{|z_{13}|^2}\right)\bar\partial_{12}^2P_{12}\right.\\
&\qquad+\left(\frac{k_1\epsilon_2\epsilon_3\epsilon_1k_2}{|z_{12}|^2}+\frac{k_1\epsilon_2\epsilon_3\epsilon_1k_3}{z_{13}\bar z_{12}}-\frac{k_3\epsilon_2\epsilon_3\epsilon_1k_3}{z_{13}\bar z_{23}}\right)\bar\partial_{13}^2P_{13}\\
&\qquad+\frac{\alpha'}{2}k_3\cdot k_1\left[\left(\frac{k_3\epsilon_2\epsilon_3\epsilon_1k_2}{z_{23}\bar z_{21}}+\frac{k_3\epsilon_2\epsilon_3\epsilon_1k_3}{z_{23}\bar z_{31}}+\frac{k_1\epsilon_2\epsilon_3\epsilon_1k_2}{|z_{21}|^2}\right)\bar\partial_{23}^2P_{23}\right.\\
&\qquad+\left(\frac{k_2\epsilon_3\epsilon_1\epsilon_2k_3}{|z_{23}|^2}+\frac{k_2\epsilon_3\epsilon_1\epsilon_2k_1}{z_{21}\bar z_{23}}-\frac{k_1\epsilon_3\epsilon_1\epsilon_2k_1}{z_{21}\bar z_{31}}\right)\bar\partial_{21}^2P_{21}\\
&\qquad+\frac{\alpha'}{2}k_1\cdot k_2\left[\left(\frac{k_1\epsilon_3\epsilon_1\epsilon_2k_3}{z_{31}\bar z_{32}}+\frac{k_1\epsilon_3\epsilon_1\epsilon_2k_1}{z_{31}\bar z_{12}}+\frac{k_2\epsilon_3\epsilon_1\epsilon_2k_3}{|z_{32}|^2}\right)\bar\partial_{31}^2P_{31}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{k_3\epsilon_1\epsilon_2\epsilon_3k_1}{|z_{31}|^2}+\frac{k_3\epsilon_1\epsilon_2\epsilon_3k_2}{z_{32}\bar{z}_{31}}-\frac{k_2\epsilon_1\epsilon_2\epsilon_3k_2}{z_{32}\bar{z}_{12}}\right)\bar{\partial}_{32}^2P_{32}\\
&+\frac{\pi}{\tau_2}\left[(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}_{13}^2P_{13}+(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}_{12}^2P_{12}+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar{\partial}_{21}^2P_{21}\right.\\
&\left.+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar{\partial}_{23}^2P_{23}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar{\partial}_{32}^2P_{32}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar{\partial}_{31}^2P_{31}\right]\\
&+\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar{\partial}_{23}^2P_{23}\left\{(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3k_1)+(k_1\cdot k_2)\,[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)]\right.\\
&\left.-(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\right\}\\
&+\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar{\partial}_{12}^2P_{12}\left\{(k_1\cdot k_3)\,(\epsilon_1\epsilon_3)\,(k_3\epsilon_2k_3)+(k_1\cdot k_3)\,[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)]\right.\\
&\left.+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)-(k_2\cdot k_3)\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\right\}\\
&+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}_{13}^2P_{13}\left\{(k_2\cdot k_3)\,(\epsilon_2\epsilon_3)\,(k_2\epsilon_1k_2)+(k_2\cdot k_3)\,[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)]\right.\\
&\left.-(k_1\cdot k_2)\,(k_2\epsilon_1\epsilon_3\epsilon_2k_3)+(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}\\
&+\frac{\alpha'}{2}\bar{\partial}_{12}^2P_{12}\left(\frac{1}{|z_{13}|^2}+\frac{1}{|z_{23}|^2}\right)\left\{(k_1\cdot k_2)\,[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)]\right.\\
&\left.-(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\right\}\\
&+\frac{\alpha'}{2}\bar{\partial}_{13}^2P_{13}\left(\frac{1}{|z_{12}|^2}+\frac{1}{|z_{23}|^2}\right)\left\{(k_1\cdot k_3)\,[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)]\right.\\
&\left.+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)-(k_2\cdot k_3)\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\right\}\\
&+\frac{\alpha'}{2}\bar{\partial}_{23}^2P_{23}\left(\frac{1}{|z_{12}|^2}+\frac{1}{|z_{13}|^2}\right)\left\{(k_2\cdot k_3)\,[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)]\right.\\
&\left.-(k_1\cdot k_2)\,(k_2\epsilon_1\epsilon_3\epsilon_2k_3)+(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}.
\end{aligned}
\tag{5.60}
$$

Excluding terms in (5.60) that are not relevant to pinched-off integration, we obtain the following contributions proportional to $1/|z_{ij}|^2$:

$$
\begin{aligned}
\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar{\partial}_{23}^2P_{23}\,\{&(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_3\epsilon_1k_2)+(k_3\cdot k_1)\,(k_1\epsilon_2\epsilon_3\epsilon_1k_2)\\
&+(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3k_1)+(k_1\cdot k_2)\,[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)]\\
&-(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\\
&+(k_1\cdot k_3)\,[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)]\\
&+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)-(k_2\cdot k_3)\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\\
&+(k_2\cdot k_3)\,[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)]
\end{aligned}
$$

$$
\begin{aligned}
&-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\} && (5.61)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial_{23}^2P_{23}\left\{-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)\right.\\
&+\left(k_1\cdot k_2\right)\left(\epsilon_1\epsilon_2\right)\left(k_1\epsilon_3k_1\right)+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]\\
&-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)+\left(k_1\cdot k_3\right)\left(k_1\epsilon_2\epsilon_3\epsilon_1k_2\right)\\
&+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)\\
&\left.+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_3\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\right\} && (5.62)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial_{23}^2P_{23}\left\{+\left(k_1\cdot k_2\right)\left(\epsilon_1\epsilon_2\right)\left(k_1\epsilon_3k_1\right)\right.\\
&+2\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]\\
&+2\left(k_1\cdot k_3\right)\left(k_1\epsilon_2\epsilon_3\epsilon_1k_2\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)+2\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\\
&\left.+2\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_3\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+2\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\right\} && (5.63)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial_{23}^2P_{23}\left\{+\left(k_1\cdot k_2\right)\left(\epsilon_1\epsilon_2\right)\left(k_1\epsilon_3k_1\right)+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]\right.\\
&+\left(k_1\cdot k_3\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)+\left(k_2\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)\\
&\left.+2\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)+2\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\right\}, && (5.64)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar\partial_{12}^2P_{12}\left\{\left(k_2\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_2\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)\right.\\
&\quad+\left(k_1\cdot k_3\right)\left(\epsilon_1\epsilon_3\right)\left(k_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&\quad+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)\\
&\quad+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]\\
&\quad-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\\
&\quad+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\\
&\quad\left.-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\right\} && (5.65)\\
&=\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar\partial_{12}^2P_{12}\left\{\left(k_1\cdot k_2\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)\right.\\
&\quad+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)\\
&\quad+\left(k_1\cdot k_3\right)\left(\epsilon_1\epsilon_3\right)\left(k_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&\quad-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\\
&\quad+\left(k_2\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\\
&\quad\left.-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\right\} && (5.66)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar{\partial}_{12}^2 P_{12}\left\{2\left(k_1\cdot k_2\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)+2\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)\right.\\
&+\left(k_1\cdot k_3\right)\left(\epsilon_1\epsilon_3\right)\left(k_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&+2\left(k_1\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)\\
&+2\left(k_2\cdot k_3\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)+2\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right) \qquad (5.67)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar{\partial}_{12}^2 P_{12}\left\{+\left(k_1\cdot k_3\right)\left(\epsilon_1\epsilon_3\right)\left(k_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\right.\\
&+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_1\epsilon_3k_2\right)+\left(k_2\cdot k_3\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)\\
&\left.+2\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)+2\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\right\}, \qquad (5.68)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}_{13}^2 P_{13}\left\{\left(k_3\cdot k_1\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)+\left(k_1\cdot k_2\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)\right.\\
&+\left(k_2\cdot k_3\right)\left(\epsilon_2\epsilon_3\right)\left(k_2\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\\
&-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\\
&+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]\\
&-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)\\
&+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&\left.+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)\right\} \qquad (5.69)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}_{13}^2 P_{13}\left\{-\left(k_1\cdot k_2\right)\left(k_2\epsilon_1\epsilon_3\epsilon_2k_3\right)+\left(k_1\cdot k_2\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)\right.\\
&+\left(k_1\cdot k_2\right)\left[\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)-\left(k_3\epsilon_2\epsilon_1\epsilon_3k_1\right)\right]+\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)\\
&+\left(k_1\cdot k_3\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\\
&+\left(k_1\cdot k_3\right)\left[\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)\right]\\
&+\left(k_2\cdot k_3\right)\left(\epsilon_2\epsilon_3\right)\left(k_2\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\\
&\left.+\left(k_2\cdot k_3\right)\left(k_1\epsilon_2\epsilon_1\epsilon_3k_1\right)-\left(k_2\cdot k_3\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2k_3\right)\right\} \qquad (5.70)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}_{13}^2 P_{13}\left\{2\left(k_1\cdot k_2\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)+\left(k_1\cdot k_2\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+2\left(k_1\cdot k_2\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_3\right)\right.\\
&+2\left(k_1\cdot k_3\right)\left(k_2\epsilon_3\epsilon_1\epsilon_2k_3\right)-\left(k_1\cdot k_3\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1k_2\right)+2\left(k_1\cdot k_3\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3k_2\right)\\
&+\left(k_2\cdot k_3\right)\left(\epsilon_2\epsilon_3\right)\left(k_2\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\\
&\left.+2\left(k_2\cdot k_3\right)\left(k_3\epsilon_2\epsilon_1\epsilon_3k_2\right)\right\} \qquad (5.71)\\
=&\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}_{13}^2 P_{13}\left\{+\left(k_2\cdot k_3\right)\left(\epsilon_2\epsilon_3\right)\left(k_2\epsilon_1k_2\right)+\left(k_2\cdot k_3\right)\left[\left(k_2\epsilon_1\epsilon_3\epsilon_2k_1\right)-\left(k_2\epsilon_1\epsilon_2\epsilon_3k_1\right)\right]\right.\\
&+\left(k_1\cdot k_2\right)\left(k_3\epsilon_1\epsilon_2\epsilon_3k_1\right)+\left(k_1\cdot k_3\right)\left(k_1\epsilon_2\epsilon_3\epsilon_1k_2\right)
\end{aligned}
$$

$$+2\,(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1 k_3)+2\,(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3 k_2)\}\,. \tag{5.72}$$

Summing up all contributions (including non-pinching contributions) gives

$$\begin{aligned}
&\frac{\alpha'}{2}k_2\cdot k_3\left(\frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_1}{z_{12}\bar z_{13}}+\frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{12}\bar z_{23}}\right)\bar\partial^2_{12}P_{12}+\frac{\alpha'}{2}k_3\cdot k_1\left(\frac{k_2\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{21}\bar z_{23}}-\frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{21}\bar z_{31}}\right)\bar\partial^2_{21}P_{21}\\
&\quad+\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial^2_{23}P_{23}\,\{+(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3 k_1)+(k_1\cdot k_2)\,[(k_3\epsilon_1\epsilon_2\epsilon_3 k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3 k_1)]\\
&\quad+(k_1\cdot k_3)\,(k_2\epsilon_3\epsilon_1\epsilon_2 k_3)+(k_2\cdot k_3)\,(k_3\epsilon_1\epsilon_2\epsilon_3 k_1)\\
&\quad+2\,(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3 k_2)+2\,(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3 k_1)\}\\
&\quad+\frac{\pi}{\tau_2}\left((k_1\epsilon_3\epsilon_2\epsilon_1 k_3)\,\bar\partial^2_{23}P_{23}+(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)\,\bar\partial^2_{32}P_{32}\right)\\
&\quad+\frac{\alpha'}{2}k_2\cdot k_3\left(\frac{k_1\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{13}\bar z_{12}}-\frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{13}\bar z_{23}}\right)\bar\partial^2_{13}P_{13}+\frac{\alpha'}{2}k_1\cdot k_2\left(\frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_3}{z_{31}\bar z_{32}}+\frac{k_1\epsilon_3\epsilon_1\epsilon_2 k_1}{z_{31}\bar z_{12}}\right)\partial^2_{31}P_{31}\\
&\quad+\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar\partial^2_{12}P_{12}\,\{+(k_1\cdot k_3)\,(\epsilon_1\epsilon_3)\,(k_3\epsilon_2 k_3)+(k_1\cdot k_3)\,[(k_2\epsilon_3\epsilon_1\epsilon_2 k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1 k_2)]\\
&\quad+(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_1\epsilon_3 k_2)+(k_2\cdot k_3)\,(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)\\
&\quad+2\,(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1 k_3)+2\,(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3 k_1)\}\\
&\quad+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3 k_2)\,\bar\partial^2_{12}P_{12}+(k_1\epsilon_3\epsilon_2\epsilon_1 k_3)\,\bar\partial^2_{21}P_{21}\right)\\
&\quad+\frac{\alpha'}{2}k_3\cdot k_1\left(\frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_2}{z_{23}\bar z_{21}}+\frac{k_3\epsilon_2\epsilon_3\epsilon_1 k_3}{z_{23}\bar z_{31}}\right)\bar\partial^2_{23}P_{23}+\frac{\alpha'}{2}k_1\cdot k_2\left(\frac{k_3\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{32}\bar z_{31}}-\frac{k_2\epsilon_1\epsilon_2\epsilon_3 k_2}{z_{32}\bar z_{12}}\right)\bar\partial^2_{32}P_{32}\\
&\quad+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar\partial^2_{13}P_{13}\,\{+(k_2\cdot k_3)\,(\epsilon_2\epsilon_3)\,(k_2\epsilon_1 k_2)+(k_2\cdot k_3)\,[(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3 k_1)]\\
&\quad+(k_1\cdot k_2)\,(k_3\epsilon_1\epsilon_2\epsilon_3 k_1)+(k_1\cdot k_3)\,(k_1\epsilon_2\epsilon_3\epsilon_1 k_2)\\
&\quad+2\,(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1 k_3)+2\,(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3 k_2)\}\\
&\quad+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3 k_2)\,\bar\partial^2_{13}P_{13}+(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)\,\bar\partial^2_{31}P_{31}\right).
\end{aligned}\tag{5.73}$$

Note that the terms in the first and third row add up by using (5.32), (5.33), (5.34), and Taylor expansion trick (5.8) on the first row, and similarly for the 6th and 8th row, and the 11th and 13th. Using modular invariance (5.3) and transversality (5.4) these observations result in

$$\begin{aligned}
&\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar\partial^2_{23}P_{23}\,\{(k_1\cdot k_2)\,(\epsilon_1\epsilon_2)\,(k_1\epsilon_3 k_1)+(k_1\cdot k_2)\,[(k_3\epsilon_1\epsilon_2\epsilon_3 k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3 k_1)]\\
&\quad+2\,(k_1\cdot k_3)\,(k_2\epsilon_3\epsilon_1\epsilon_2 k_3)+2\,(k_2\cdot k_3)\,(k_3\epsilon_1\epsilon_2\epsilon_3 k_1)\\
&\quad+2\,(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3 k_2)+2\,(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3 k_1)\}\\
&\quad+\frac{\pi}{\tau_2}\left((k_1\epsilon_3\epsilon_2\epsilon_1 k_3)\,\bar\partial^2_{23}P_{23}+(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)\,\bar\partial^2_{32}P_{32}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar{\partial}^2_{12}P_{12}\left\{(k_1\cdot k_3)(\epsilon_1\epsilon_3)(k_3\epsilon_2k_3)+(k_1\cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right]\right.\\
&+2(k_1\cdot k_2)(k_3\epsilon_2\epsilon_1\epsilon_3k_2)+2(k_2\cdot k_3)(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\\
&\left.+2(k_1\cdot k_2)(k_3\epsilon_2\epsilon_3\epsilon_1k_3)+2(k_2\cdot k_3)(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\right\}\\
&+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}^2_{12}P_{12}+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar{\partial}^2_{21}P_{21}\right)\\
&+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}^2_{13}P_{13}\left\{(k_2\cdot k_3)(\epsilon_2\epsilon_3)(k_2\epsilon_1k_2)+(k_2\cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right]\right.\\
&+2(k_1\cdot k_2)(k_3\epsilon_1\epsilon_2\epsilon_3k_1)+2(k_1\cdot k_3)(k_1\epsilon_2\epsilon_3\epsilon_1k_2)\\
&\left.+2(k_1\cdot k_2)(k_3\epsilon_2\epsilon_3\epsilon_1k_3)+2(k_1\cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}\\
&+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}^2_{13}P_{13}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar{\partial}^2_{31}P_{31}\right)
\end{aligned}
\tag{5.74}
$$

$$
\begin{aligned}
=&\frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\bar{\partial}^2_{23}P_{23}\left\{(k_1\cdot k_2)(\epsilon_1\epsilon_2)(k_1\epsilon_3k_1)+(k_1\cdot k_2)\left[(k_3\epsilon_1\epsilon_2\epsilon_3k_1)-(k_3\epsilon_2\epsilon_1\epsilon_3k_1)\right]\right.\\
&\left.-2(k_1\cdot k_2)(k_1\epsilon_3\epsilon_1\epsilon_2k_1)-2(k_1\cdot k_2)(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\right\}\\
&+\frac{\pi}{\tau_2}\left((k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar{\partial}^2_{23}P_{23}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar{\partial}^2_{32}P_{32}\right)\\
&+\frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\bar{\partial}^2_{12}P_{12}\left\{(k_1\cdot k_3)(\epsilon_1\epsilon_3)(k_3\epsilon_2k_3)+(k_1\cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2k_3)-(k_3\epsilon_2\epsilon_3\epsilon_1k_2)\right]\right.\\
&\left.-2(k_1\cdot k_3)(k_1\epsilon_2\epsilon_1\epsilon_3k_1)-2(k_1\cdot k_3)(k_3\epsilon_1\epsilon_3\epsilon_2k_3)\right\}\\
&+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}^2_{12}P_{12}+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\,\bar{\partial}^2_{21}P_{21}\right)\\
&+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\bar{\partial}^2_{13}P_{13}\left\{(k_2\cdot k_3)(\epsilon_2\epsilon_3)(k_2\epsilon_1k_2)+(k_2\cdot k_3)\left[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)\right]\right.\\
&\left.-2(k_2\cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3k_2)-2(k_2\cdot k_3)(k_3\epsilon_2\epsilon_3\epsilon_1k_3)\right\}\\
&+\frac{\pi}{\tau_2}\left((k_3\epsilon_2\epsilon_1\epsilon_3k_2)\,\bar{\partial}^2_{13}P_{13}+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\,\bar{\partial}^2_{31}P_{31}\right).
\end{aligned}
\tag{5.75}
$$

Fermionic propagators would be absorbed into spin summation $Z^{\text{int}}_{s=1}$ (see from e.g.[16][40]) and would not affect the coordinate integration. Performing pinched-off integration and using (5.32), (5.33) and (5.34) again gives the kinematic structure (using $\int_{\mathcal{T}} d^2z = 2\tau_2$)

$$
\begin{aligned}
&\pi\bar{\partial}^2_{23}P_{23}\left\{(\epsilon_1\epsilon_2)(k_1\epsilon_3k_1)+(k_3\epsilon_2\epsilon_1\epsilon_3k_1)+(k_3\epsilon_1\epsilon_2\epsilon_3k_1)+2(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\right\}\\
+&\pi\bar{\partial}^2_{12}P_{12}\left\{(\epsilon_1\epsilon_3)(k_3\epsilon_2k_3)+(k_3\epsilon_2\epsilon_3\epsilon_1k_2)+(k_2\epsilon_3\epsilon_1\epsilon_2k_3)+2(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\right\}\\
+&\pi\bar{\partial}^2_{13}P_{13}\left\{(\epsilon_2\epsilon_3)(k_2\epsilon_1k_2)+(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)+2(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\right\}.
\end{aligned}
\tag{5.76}
$$

Using (5.33), as well as the fact that ϵ are symmetric for gravitons, to transform and simplify the above result (5.76), we arrive at

$$\begin{aligned} &\pi\bar{\partial}_{23}^2 P_{23}\left\{\left(k_1\epsilon_3 k_1\right)\left(\epsilon_1\epsilon_2\right)-2\left(k_3\epsilon_2\epsilon_3\epsilon_1 k_2\right)\right\} \\ &+\pi\bar{\partial}_{12}^2 P_{12}\left\{\left(k_3\epsilon_2 k_3\right)\left(\epsilon_3\epsilon_1\right)-2\left(k_2\epsilon_1\epsilon_2\epsilon_3 k_1\right)\right\} \\ &+\pi\bar{\partial}_{13}^2 P_{13}\left\{\left(k_2\epsilon_1 k_2\right)\left(\epsilon_2\epsilon_3\right)-2\left(k_1\epsilon_3\epsilon_1\epsilon_2 k_3\right)\right\} \\ &+\left(\pi\bar{\partial}_{23}^2 P_{23}-\pi\bar{\partial}_{12}^2 P_{12}\right)\left(k_1\epsilon_3\epsilon_1\epsilon_2 k_3\right) \\ &+\left(\pi\bar{\partial}_{13}^2 P_{13}-\pi\bar{\partial}_{23}^2 P_{23}\right)\left(k_2\epsilon_1\epsilon_2\epsilon_3 k_1\right) \\ &+\left(\pi\bar{\partial}_{12}^2 P_{12}-\pi\bar{\partial}_{23}^2 P_{23}\right)\left(k_3\epsilon_2\epsilon_3\epsilon_1 k_2\right). \end{aligned} \tag{5.77}$$

$\left(\bar{\partial}_{st}^2 P_{st}-\bar{\partial}_{tu}^2 P_{tu}\right)\times\prod_{i<j}\left|\chi_{ij}\right|^{\alpha' k_i\cdot k_j}$ will be vanishing after coordinate integration because the value of $\int d^2z_1 d^2z_2 d^2z_3 \bar{\partial}_{st}^2 P_{st}\times\prod_{i<j}\left|\chi_{ij}\right|^{\alpha' k_i\cdot k_j}$ does not depend on s and t. So we could get rid of these terms and obtain the final result

$$\boxed{\begin{aligned} &\pi\bar{\partial}_{23}^2 P_{23}\left\{\left(k_1\epsilon_3 k_1\right)\left(\epsilon_1\epsilon_2\right)-2\left(k_3\epsilon_2\epsilon_3\epsilon_1 k_2\right)\right\} \\ &+\pi\bar{\partial}_{12}^2 P_{12}\left\{\left(k_3\epsilon_2 k_3\right)\left(\epsilon_3\epsilon_1\right)-2\left(k_2\epsilon_1\epsilon_2\epsilon_3 k_1\right)\right\} \\ &+\pi\bar{\partial}_{13}^2 P_{13}\left\{\left(k_2\epsilon_1 k_2\right)\left(\epsilon_2\epsilon_3\right)-2\left(k_1\epsilon_3\epsilon_1\epsilon_2 k_3\right)\right\}. \end{aligned}} \tag{5.78}$$

We see that this result differs from the gravity kinematic structure (5.2) by a relative sign.

5.2.3 OPE Calculation of the 1-loop kinematic structure

Setup

We keep α' explicit and use notations $\partial X_i^{\nu_i}=\partial X^{\nu_i}\left(z_i,\bar{z}_i\right)\equiv J_i^{\nu_i}, \bar{\partial} X_i^{\mu_i}=\bar{\partial} X^{\mu_i}\left(z_i,\bar{z}_i\right)\equiv \bar{J}_i^{\mu_i}, \psi_i^{\nu_i}=\psi^{\nu_i}\left(z_i\right)$ and $z_{ij}\equiv z_i-z_j$. For the graviton vertex operators for Heterotic string theory we introduce (with $k_i^2=0$)

$$V_i^{\mu_i\nu_i}\equiv\frac{2}{\alpha'}: i\bar{J}_i^{\mu_i}\left(iJ_i^{\nu_i}+\frac{\alpha'}{2}k_i\cdot\psi_i\psi_i^{\nu_i}\right)e^{ik_iX_i}:, \tag{5.79}$$

so that the full (graviton) vertex operator at zero picture reads

$$V_i=\epsilon_{i,\mu_i\nu_i}V_i^{\mu_i\nu_i}. \tag{5.80}$$

Our aim is to evaluate the following 3-point function with pinching-off integration at order of k^2:

$$\left\langle V_l\int_{\left|z_i-z_j\right|<\varepsilon}d^2z_i V_i V_j\right\rangle=\left\langle V_l^{\mu_l\nu_l}\int_{\left|z_i-z_j\right|<\varepsilon}d^2z_i V_i^{\mu_i\nu_i}V_j^{\mu_j\nu_j}\right\rangle\epsilon_{l,\mu_l\nu_l}\epsilon_{i,\mu_i\nu_i}\epsilon_{j,\mu_j\nu_j}. \tag{5.81}$$

Here ε is infinitesimal so that $V_i^{\mu_i\nu_i}V_j^{\mu_j\nu_j}$ can be replaced by the OPE, and because of

$$\int_{|z_i-z_j|<\varepsilon} d^2z_i \frac{|z_{ij}|^{\alpha' k_i\cdot k_j}}{z_{ij}^n \bar{z}_{ij}^m} \simeq \frac{2\pi}{\alpha' k_i\cdot k_j}\delta_{n1}\delta_{m1} \tag{5.82}$$

only following terms in the OPE can give non-vanishing contribution to the 3-point function:

$$\mathrm{OPE}\left(V_i^{\mu_i\nu_i}V_j^{\mu_j\nu_j}\right) \quad \ni \quad \frac{|z_{ij}|^{\alpha' k_i\cdot k_j}}{|z_{ij}|^2} O_j^{\mu_i\nu_i\mu_j\nu_j}, \tag{5.83}$$

where operator $O_j^{\mu_i\nu_i\mu_j\nu_j}$ is further constrained to be of the form

$$O_j^{\mu_i\nu_i\mu_j\nu_j} = t_{\alpha\beta\gamma}^{\mu_i\nu_i\mu_j\nu_j}\left(k_i,k_j\right) : \bar{J}_j^\alpha \psi_j^\beta \psi_j^\gamma e^{i(k_i+k_j)\cdot X_j} : . \tag{5.84}$$

Here $t_{\alpha\beta\gamma}^{\mu_i\nu_i\mu_j\nu_j}\left(k_i,k_j\right)$ are k-dependent OPE (tensor) coefficients. The above restriction to the form of the operators follows from the following requirements:

- Counting the weight on the both sides of the OPE and the weight of the $\frac{|z_{ij}|^{\alpha' k_i\cdot k_j}}{|z_{ij}|^2}$ tells us that operator $O_j^{\mu_i\nu_i\mu_j\nu_j}$ must have weight $\left(1+\frac{\alpha' k_i\cdot k_j}{2}, 1+\frac{\alpha' k_i\cdot k_j}{2}\right)$.
- Operator $O_j^{\mu_i\nu_i\mu_j\nu_j}$ must contain at least 2 fermions, since otherwise the spin-summation with the partition function makes the 3-point function vanish.

In the Heterotic case, the operator satisfying the above two requirements is only $\bar{J}_j\psi_j\psi_j e^{i(k_i+k_j)\cdot X_j}$ (for $k_i^2 = 0 = k_j^2$). In type I case, the allowed operators include $J_j\bar{\psi}_j\bar{\psi}_j e^{i(k_i+k_j)\cdot X_j}$ and $\psi_j\psi_j\bar{\psi}_j\bar{\psi}_j e^{i(k_i+k_j)\cdot X_j}$.

Results of the OPE

Here we determine $O_j^{\mu_i\nu_i\mu_j\nu_j}$ in (5.83). The result (up to order $\mathcal{O}\left(k^3\right)$) reads

$$\begin{aligned} O_j^{\mu_i\nu_i\mu_j\nu_j} = -\frac{i\alpha'}{2} : &\left[\eta^{\mu_i\mu_j} k_i\cdot\bar{J}_j + k_j^{\mu_i}\bar{J}_j^{\mu_j} - k_i^{\mu_j}\bar{J}_j^{\mu_i}\right] \\ &\left[\eta^{\nu_i\nu_j}\left(k_i\cdot\psi_j\right)\left(k_j\cdot\psi_j\right) + \left(k_i\cdot k_j\right)\psi_j^{\nu_i}\psi_j^{\nu_j}\right. \\ &\left.-\left(k_i^{\nu_j}\psi_j^{\nu_i} - k_j^{\nu_i}\psi_j^{\nu_j}\right)\left(\left(k_i+k_j\right)\cdot\psi_j\right)\right] e^{i(k_i+k_j)\cdot X_j} :, \end{aligned} \tag{5.85}$$

so the OPE coefficients $t_{\alpha\beta\gamma}^{\mu_i\nu_i\mu_j\nu_j}\left(k_i,k_j\right)$ in (5.84)) are

$$\begin{aligned} t_{\alpha\beta\gamma}^{\mu_i\nu_i\mu_j\nu_j}\left(k_i,k_j\right) = -\frac{i\alpha'}{2}&\left[\eta^{\mu_i\mu_j}k_{i\alpha} + k_j^{\mu_i}\eta_\alpha^{\mu_j} - k_i^{\mu_j}\eta_\alpha^{\mu_i}\right] \\ \left[\eta^{\nu_i\nu_j}k_{i\beta}k_{j\gamma} + \left(k_i\cdot k_j\right)\eta_\beta^{\nu_i}\eta_\gamma^{\nu_j}\right. &\left. - \left(k_i^{\nu_j}\eta_\beta^{\nu_i} - k_j^{\nu_i}\eta_\beta^{\nu_j}\right)\left(k_i+k_j\right)_\gamma\right] \end{aligned} \tag{5.86}$$

3-point function from pinched-off singularity

Using the above OPE results, we can compute following 3-point function at the k^2 order originating from pinched-off singularity (partition function and spin-sums are suppressed but implied here):

$$\begin{aligned}
&\int d^2 z_l \int d^2 z_j \left\langle V_l \int_{|z_i - z_j|<\varepsilon} d^2 z_i V_i V_j \right\rangle = \int d^2 z_l \int d^2 z_j \left\langle V_l \int_{|z_i - z_j|<\varepsilon} d^2 z_i \mathrm{OPE}\left(V_i V_j\right) \right\rangle \\
=&\frac{2i\pi}{\alpha' k_i \cdot k_j} \epsilon_{l,\mu_l\nu_l} \epsilon_{i,\mu_i\nu_i} \epsilon_{j,\mu_j\nu_j} t^{\mu_i\nu_i\mu_j\nu_j}_{\alpha\beta\gamma}\left(k_i, k_j\right) \\
&\int d^2 z_l \int d^2 z_j \left\langle : \bar{J}^{\mu_l}_l k_l \cdot \psi_l \psi^{\nu_l}_l e^{ik_l\cdot X_l} :: \bar{J}^{\alpha}_j \psi^{\beta}_j \psi^{\gamma}_j e^{i(k_i+k_j)\cdot X_j} : \right\rangle \\
=&\frac{2i\pi}{\alpha' k_i \cdot k_j} \epsilon_{l,\mu_l\nu_l} \epsilon_{i,\mu_i\nu_i} \epsilon_{j,\mu_j\nu_j} t^{\mu_i\nu_i\mu_j\nu_j}_{\alpha\beta\gamma}\left(k_i, k_j\right) \\
&\int d^2 z_l \int d^2 z_j \left\langle : k_l \cdot \psi_l \psi^{\nu_l}_l :: \psi^{\beta}_j \psi^{\gamma}_j : \right\rangle \left\langle : \bar{J}^{\mu_l}_l e^{ik_l\cdot X_l} :: \bar{J}^{\alpha}_j e^{i(k_i+k_j)\cdot X_j} : \right\rangle \\
=&K_{ijl} \int d^2 z_l \int d^2 z_j \left(S_{lj}\right)^2 \bar{\partial}^2_l P_{lj} e^{-k_l\cdot(k_i+k_j)P_{lj}}
\end{aligned} \tag{5.87}$$

Here K_{ijl} is the kinematic factor ($\sim k^2$) given by

$$\begin{aligned}
K_{ijl} &\equiv -\frac{2i\pi}{\alpha' k_i \cdot k_j} \epsilon_{l,\mu_l\nu_l} \epsilon_{i,\mu_i\nu_i} \epsilon_{j,\mu_j\nu_j} t^{\mu_i\nu_i\mu_j\nu_j}_{\alpha\beta\gamma} \eta^{\mu_l\alpha} \left(k^{\gamma}_l \eta^{\nu_l\beta} - k^{\beta}_l \eta^{\nu_l\gamma}\right) \\
&= -\frac{2i\pi}{\alpha' k_i \cdot k_j} \epsilon_{i,\mu_i\nu_i} \epsilon_{j,\mu_j\nu_j} \left(k^{\gamma}_l \epsilon^{\alpha\beta}_l - k^{\beta}_l \epsilon^{\alpha\gamma}_l\right) t^{\mu_i\nu_i\mu_j\nu_j}_{\alpha\beta\gamma}
\end{aligned} \tag{5.88}$$

The kinematic factor K_{ijl} given above can be explicitly computed by substituting (5.86) for $t^{\mu_i\nu_i\mu_j\nu_j}_{\alpha\beta\gamma}$ and we obtain

$$K_{ijl} = -\frac{\pi}{\left(k_i \cdot k_j\right)} \left(T^{(ijl)}_1 + T^{(ijl)}_2 + T^{(ijl)}_3\right) \tag{5.89}$$

where the 3 different tensor structures are given by (with notational abbreviation $\left(\epsilon^{\top}_i \epsilon_j\right) \equiv \mathrm{Tr}\left(\epsilon^{\top}_i \epsilon_j\right)$)

$$\begin{aligned}
T^{(ijl)}_1 =& \left(\epsilon^{\top}_i \epsilon_j\right)\left[\left(k_l \cdot k_j\right)\left(k_i \epsilon_l k_i\right) - \left(k_l \cdot k_i\right)\left(k_i \epsilon_l k_j\right)\right] - \left(k_i \cdot k_j\right)\left[\left(\epsilon^{\top}_l \epsilon_i\right)\left(k_i \epsilon_j k_l\right) + \left(\epsilon^{\top}_l \epsilon_j\right)\left(k_j \epsilon_i k_l\right)\right] \\
&+ \left(k_l \cdot \left(k_i + k_j\right)\right)\left[\left(\epsilon^{\top}_l \epsilon_j\right)\left(k_j \epsilon_i k_j\right) + \left(\epsilon^{\top}_l \epsilon_i\right)\left(k_i \epsilon_j k_i\right)\right] \\
=& \left(\epsilon^{\top}_i \epsilon_j\right)\left[\left(k_l \cdot k_j\right)\left(k_i \epsilon_l k_i\right) - \left(k_l \cdot k_i\right)\left(k_i \epsilon_l k_j\right)\right] \\
&- \left(k_i \cdot k_j\right)\left[\left(\epsilon^{\top}_l \epsilon_i\right)\left(k_i \epsilon_j \left(k_l + k_i\right)\right) + \left(\epsilon^{\top}_l \epsilon_j\right)\left(k_j \epsilon_i \left(k_l + k_j\right)\right)\right]
\end{aligned} \tag{5.90a}$$

$$
\begin{aligned}
T_2^{(ijl)} =& (k_l \cdot k_j)\left[\left(k_j\epsilon_i\epsilon_j^\top\epsilon_l k_i\right) - \left(k_i\epsilon_j\epsilon_i^\top\epsilon_l k_i\right)\right] + (k_l \cdot k_i)\left[\left(k_i\epsilon_j\epsilon_i^\top\epsilon_l k_j\right) - \left(k_j\epsilon_i\epsilon_j^\top\epsilon_l k_j\right)\right] \\
&+ (k_i \cdot k_j)\left[\left(k_i\epsilon_l\epsilon_i^\top\epsilon_j k_l\right) + \left(k_j\epsilon_i\epsilon_l^\top\epsilon_j k_l\right) - \left(k_i\epsilon_l\epsilon_j^\top\epsilon_i k_l\right) + \left(k_i\epsilon_j\epsilon_l^\top\epsilon_i k_l\right)\right] \\
&- (k_l \cdot (k_i + k_j))\left[\left(k_i\epsilon_l\epsilon_i^\top\epsilon_j k_i\right) - \left(k_i\epsilon_l\epsilon_j^\top\epsilon_i k_j\right) + \left(k_j\epsilon_i\epsilon_l^\top\epsilon_j k_i\right) + \left(k_i\epsilon_j\epsilon_l^\top\epsilon_i k_j\right)\right] \\
=& (k_l \cdot k_j)\left[\left(k_j\epsilon_i\epsilon_j^\top\epsilon_l k_i\right) - \left(k_i\epsilon_j\epsilon_i^\top\epsilon_l k_i\right)\right] + (k_l \cdot k_i)\left[\left(k_i\epsilon_j\epsilon_i^\top\epsilon_l k_j\right) - \left(k_j\epsilon_i\epsilon_j^\top\epsilon_l k_j\right)\right] \\
&+ (k_i \cdot k_j)\left[\left(k_i\epsilon_l\epsilon_i^\top\epsilon_j\left(k_l + k_i\right)\right) + \left(k_j\epsilon_i\epsilon_l^\top\epsilon_j\left(k_l + k_i\right)\right)\right. \\
&\left.- \left(k_i\epsilon_l\epsilon_j^\top\epsilon_i\left(k_l + k_j\right)\right) + \left(k_i\epsilon_j\epsilon_l^\top\epsilon_i\left(k_l + k_j\right)\right)\right]
\end{aligned}
\tag{5.90b}
$$

$$
\begin{aligned}
T_3^{(ijl)} =& \left(k_i\epsilon_l\left(k_i + k_j\right)\right)\left[\left(k_i\epsilon_j^\top\epsilon_i k_l\right) - \left(k_l\epsilon_j^\top\epsilon_i k_j\right)\right] \\
&- \left(k_j\epsilon_i k_j\right)\left(k_l\epsilon_j^\top\epsilon_l\left(k_i + k_j\right)\right) - \left(k_i\epsilon_j k_i\right)\left(k_l\epsilon_i^\top\epsilon_l\left(k_i + k_j\right)\right) \\
&+ \left(k_i\epsilon_j k_l\right)\left(k_j\epsilon_i^\top\epsilon_l\left(k_i + k_j\right)\right) + \left(k_j\epsilon_i k_l\right)\left(k_i\epsilon_j^\top\epsilon_l\left(k_i + k_j\right)\right)
\end{aligned}
\tag{5.90c}
$$

For the second equalities of (5.90a) and (5.90b) , we have used the modular invariance condition $k_i \cdot k_j + k_j \cdot k_l + k_i \cdot k_l = 0$, i.e. $(k_l \cdot (k_i + k_j)) = -k_i \cdot k_j$. The permutation symmetry is still to be taken into account, which will be done later. T_3 is a problematic tensor structure as this should not appear in the final result. It is actually vanishing, which is shown in the following.

Simplifications of tensor structures (5.90a)-(5.90c)

At this stage, let us note following points:

a. So far we have relaxed the momentum conservation, i.e. $p = k_i + k_j + k_l \neq 0$ with $p^2 = 0$, and p is treated as an infinitesimal parameter. Then it follows that $k_i \cdot k_j = -p \cdot k_l$ and $\epsilon_l\left(k_i + k_j\right) = \epsilon_l p$, so these quantities are of linear order in p.

b. From the point a above, (5.89) can be written as $K \sim \frac{\left(\sum_i T_i\right)}{(p \cdot k_l)}$. Since we will take $p \to 0$ at the end and since p appears in the denominator of (5.89), we need to extract only the linear order in p for each T_i as the higher orders would not contribute to $\lim_{p\to 0} K$. Note that the last line of (5.90a) and the last two lines of (5.90b) are of order p^2 due to point a, so they do not contribute.

c. Combining both points above, we will set $p = 0$ (i.e. $k_i + k_j + k_l = 0$) for the coefficients of $k_i \cdot k_j$ and $\epsilon_j\left(k_i + k_j\right)$, in order to extract the contribution at the linear order in p.

Using the prescription given at point c above, (5.90a)-(5.90c) simplify as

$$
T_1^{(ijl)}(\mathcal{O}(p)) = \left(k_i \cdot k_j\right)\left(\epsilon_i^\top\epsilon_j\right)\left(k_i\epsilon_l k_j\right), \tag{5.91a}
$$

$$T_2^{(ijl)}(\mathcal{O}(p)) = -\left(k_i \cdot k_j\right)\left[\left(k_i \epsilon_j \epsilon_i^\top \epsilon_l k_j\right) + \left(k_j \epsilon_i \epsilon_j^\top \epsilon_l k_i\right)\right], \tag{5.91b}$$

$$T_3^{(ijl)}(\mathcal{O}(p)) = 0. \tag{5.91c}$$

Thus (5.89) to this order reads

$$K_{ijl} = \pi\left[-\left(\epsilon_i^\top \epsilon_j\right)\left(k_i \epsilon_l k_j\right) + \left(k_i \epsilon_j \epsilon_i^\top \epsilon_l k_j\right) + \left(k_j \epsilon_i \epsilon_j^\top \epsilon_l k_i\right)\right]. \tag{5.92}$$

Note that the above is symmetric w.r.t. $i \leftrightarrow j$, i.e. $K_{ijl} = K_{jil}$, which can be seen from

$$\left(\epsilon_i^\top \epsilon_j\right) = \left(\epsilon_j^\top \epsilon_i\right) \tag{5.93}$$

$$\left(k_i \epsilon_l k_j\right) = \left(k_j \epsilon_l k_i\right). \tag{5.94}$$

Symmetrization

The overall result of 3-point function must be symmetric w.r.t. all permutations as $\langle V_l V_i V_j \rangle$ is. The permutations can be decomposed into permutation w.r.t. $i \leftrightarrow j$ and the cyclic permutations. The symmetry w.r.t. cyclic permutations is broken in (5.87) by the OPE taken in the way of (5.87) (c.f. 1st line). On the other hand, (5.87) is symmetric w.r.t. $i \leftrightarrow j$, which can be seen from the symmetry of (5.92) and the integral in the last line of (5.87). In particular, the integral in the last line of (5.87) is invariant w.r.t. ($i \leftrightarrow j$) since z_j is a dummy variable integrated over and can be renamed as z_i (also the index j of P s). Moreover, in the exponential we can use momentum conservation and set $k_l \cdot (k_i + k_j) = 0$, which eliminates the exponential and hence the index i. Therefore, the integral becomes a common factor invariant w.r.t. any permutations of the indices, so that one needs to symmetrize only factor K in (5.87), which we will do now. The upshot is that we have to symmetrize (5.87) by adding only cyclic permutations of K. This can be summarized as follows:

$$\begin{aligned}
&\iiint d^2 z_l d^2 z_j d^2 z_i \left\langle V_l V_i V_j \right\rangle \\
&\to \int d^2 z_l \iint d^2 z_j d^2 z_i \left\langle V_l \left[\mathrm{OPE}\left(V_i V_j\right)\right]\right\rangle + 2 \text{ cyclic permutations} \\
&\to \left(\int d^2 z_l \int d^2 z_j \left\langle V_l \int_{|z_i - z_j| < \varepsilon} d^2 z_i \, \mathrm{OPE}\left(V_i V_j\right)\right\rangle\right) + 2 \text{ cyclic permutations} \\
&= (5.87) + 2 \text{ cyclic permutations} \\
&= K_{ijl} \int d^2 z_l \int d^2 z_j \left(S_{lj}\right)^2 \bar{\partial}_l^2 P_{lj} + 2 \text{ cyclic permutations}
\end{aligned} \tag{5.95}$$

Here in the 2nd line it is assumed that the integration region over z_i and z_j are restricted to that where the separation between z_i and z_j are smaller than the distance to z_l, i.e. $|z_i - z_j| < \min(|z_i - z_l|, |z_j - z_l|)$, in order to allow us to take the OPE. The 2 cyclic permutations in the 2nd line correspond to different choices of 2 vertex operators for taking OPE.

Using (5.92), the final result after pinched-off integration is given by

$$\boxed{\begin{aligned}&\iiint d^2 z_1 d^2 z_2 d^2 z_3 \left\langle V_1 V_3 V_2 \right\rangle\Big|_{\text{pinched-off}} = \\ &\pi \left[\left[(\epsilon_3\epsilon_2)(k_2\epsilon_1 k_2) + (k_2\epsilon_1\epsilon_2\epsilon_3 k_1) + (k_2\epsilon_1\epsilon_3\epsilon_2 k_3)\right] \int d^2 z_1 \int d^2 z_2 \left(S_{12}\right)^2 \bar{\partial}_1^2 P_{12}\right] \\ &+ 2 \text{ cyclic permutations.}\end{aligned}} \tag{5.96}$$

Transpose $\top$ is omitted because we only consider gravitons. In the last line the exponential has been set to 1 since only quadratic order in momenta is concerned. We substituted lij with 123 to make the result easier comparable with the former result. Again, fermionic propagators S_{12}^2 would be absorbed into spin summation $Z_{s=1}^{\text{int}}$ (see from e.g.[16][40])

We see that the kinematic structure of the above result (5.96) contains only the part with pinched-off integration. It agrees with the pinched-off integration part of the result from previous section §5.2.2. So OPE calculation works as a double check and confirms the correctness of our calculation.

5.2.4 Discussion

We notice that $\bar{\partial}_{ij}^2 P(\bar{z}_i, \bar{z}_j)$ is a total derivative. Due to the reason explained in 5.2.2, the Koba-Nielsen factor was expanded and we only take the leading part 1. The coordinate integration of the amplitude is now an integration over a total derivative on torus, which means that the whole Heterotic 1-loop 3-point correction is vanishing anyway. This guarantees that the 1-loop will not break the gravity kinematic structure (5.2) of the effective theory. This is different for Type I theory to which we turn now in the next section.

5.3 Type I string

We consider the type I string compactified on $T^6/\mathbb{Z}_N$ orientifold with $D = 4$ non-compact dimensions.

Vertex Operator The difference between Type-I and Heterotic 3-point graviton amplitudes is about the vertex operators. In Heterotic case we only have to consider left-moving fermions, see (5.10) and (5.11). But for Type-I theory, the vertex operators in the $(0,0)$ and $(-1,-1)$ pictures are

$$V^g_{(0,0)}(z,\bar{z},k,\epsilon) =: \frac{2}{\alpha'}\epsilon_{\mu\nu}(k)\left(i\bar{\partial}X^\nu(\bar{z}) + \frac{\alpha'}{2}(k\cdot\bar{\psi})\bar{\psi}^\nu(\bar{z})\right)\left(i\partial X^\nu(z) + \frac{\alpha'}{2}(k\cdot\psi)\psi^\nu(z)\right)e^{ik\cdot X(z,\bar{z})}: \tag{5.97}$$

$$V^g_{-1,-1}(z,\bar{z},k,\epsilon) =: \epsilon_{\mu\nu}(k)\mathrm{e}^{-\phi-\bar{\phi}}\psi^\mu\bar{\psi}^\nu e^{\mathrm{i}k\cdot X}:. \tag{5.98}$$

Useful references are [7][6][14].

5.3.1 Tree level

The tree level 3-point amplitude reads

$$A_3^{(0)} = \langle c\bar{c}V_{-1,-1}(z_1)\, c\bar{c}V_{-1,-1}(z_2)\, c\bar{c}V_{0,0}(z_3)\rangle_{\Sigma_0} \tag{5.99}$$

The computation is completely similar to the one of the holomorphic sector of the Heterotic string and yields

$$A_3^{(0)} \sim \epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\epsilon_{3,\mu_3\nu_3}t^{\mu_1\mu_2\mu_3}t^{\nu_1\nu_2\nu_3} \times \prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j} \tag{5.100}$$

where we recall

$$t^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}k_2^{\mu_3} + \eta^{\mu_2\mu_3}k_3^{\mu_1} + \eta^{\mu_3\mu_1}k_1^{\mu_2} \tag{5.101}$$

Note that there are no α' corrections.

The result has the same tensor structure as (5.28) at leading order in α' in the Heterotic tree level amplitude, thus we know already that the above amplitude again shares the same kinematic structure as the expanded Einstein-Hilbert term (5.2).

There is no disk-level $\mathcal{O}(k^2)$ contribution in Type-I. The 2-point function in Type-I was discussed in [11, §2][30][31][67][59][48][49], and the 3-point function in Type-I was discussed in [20, §B].

5.3.2 1-loop kinematic structure in Type-I

From the 3 graviton amplitudes of both Type-I and Heterotic, we know that the calculation of Type-I 3 graviton amplitude would be pretty similar to Heterotic. Thus we do not repeat the tedious calculation in §5.2 but change the strategy. We directly focus on the two specific tensor structures $\left(k_j\epsilon^i k_j\right)\left(\epsilon^j\epsilon^l\right)$ and $\left(k_l\epsilon^j\epsilon^l\epsilon^i k_j\right)$, and present only differences to Heterotic calculation.

Following the same argument in §2.2.1 as Heterotic string, we would again only need vertex operators in zero ghost picture for genus-1 surfaces. The 1-loop level 3-point amplitude in Type-I theory is

$$G_3^{(1)} = \left\langle \prod_{i=1}^{3} \int d^2 z_i \, V_{0,0}(z_i, \bar{z}_i) \right\rangle_{\Sigma_1} . \tag{5.102}$$

We rename $V_{0,0}(z_i, \bar{z}_i)$ as V_i for simplicity in the following.

String theory has four types of surfaces at 1-loop level, or $\chi = 0, g = 1$, i.e. $\mathcal{O}\left(g_s^2\right)$. These are the torus $\mathcal{T}$, the annulus (or cylinder) $\mathcal{A}$, the Möbius strip $\mathcal{M}$ and the Klein bottle $\mathcal{K}$.

8-Fermion contribution in Type-I

We first observe the possible contractions of 8 (and even more) fermions in Type-I which do not exist in Heterotic 3 graviton amplitude. In this case we will get momentum of order k^4 from fermions and we must do the pinched-off integration to cancel k^2. There are two different choices of picking fermions among vertex operators for a certain 8 fermion contraction contributing to $\mathcal{O}(k^2)$ with pinched-off integration:

1. In the first case, the contraction of 8 fermions consists of 4-fermions coming from one vertex operator and 2 fermions coming from each of the remaining 2 vertex operators. All the other operators in the contraction are bosons. W.l.o.g. we assign 4-fermions to Vertex operator V_1 as an example, then the term we are going to contract would be

$$k_1 \cdot \bar{\psi}_1 \bar{\psi}_1^{\mu} k_1 \cdot \psi_1 \psi_1^{\nu} e^{ik_1 \cdot X_1} \bar{\partial} X_2^{\mu} k_2 \cdot \psi_2 \psi_2^{\nu} e^{ik_2 \cdot X_2} \bar{\partial} X_3^{\mu} k_3 \cdot \psi_3 \psi_3^{\nu} e^{ik_3 \cdot X_3} . \tag{5.103}$$

   Since a contraction of a boson with an exponential would bring in an extra k, see (B.1c), we cannot contract any boson with the exponentials here. The only source of the $\frac{1}{|z_{ij}|^2}$ term required by the pinched-off integration must be the fermion contractions. Due to $|z_{ij}|^2 = z_{ij}\bar{z}_{ij}$, one needs 1 pair of $\langle \psi_i \psi_j \rangle$ and 1 pair of $\langle \bar{\psi}_i \bar{\psi}_j \rangle$ to get $1/z_{ij}$ and $1/\bar{z}_{ij}$ when $z_i \to z_j$. However, we have only one pair of fermions each from V_2 and V_3, and in the same vertex operator the two fermions are either both left-moving or right-moving, it's impossible to get $1/z_{ij}$ and $1/\bar{z}_{ij}$ simultaneously from either V_2 or V_3. Therefore, we can exclude this case from $\mathcal{O}(k^2)$ contribution.

2. In the second case, the contraction has left and right-moving bosons within the same vertex operator, and 8 fermions are in the remaining 2 vertex operators. W.l.o.g. we choose this vertex operator to be V_3 as an example, then the contraction term would be

$$k_1 \cdot \bar{\psi}_1 \bar{\psi}_1^\mu k_1 \psi_1 \psi_1^\nu e^{ik_1\cdot X_1} k_2 \cdot \bar{\psi}_2 \bar{\psi}_2^\mu k_2 \psi_2 \psi_2^\nu e^{ik_2\cdot X_2} \bar{\partial} X_3^\mu \partial X_3^\nu e^{ik_3\cdot X_3}. \tag{5.104}$$

Here we observe that $\bar{\partial} X_3$ cannot be contracted with ∂X_3, thus both of them have to be contracted with the exponentials from V_1 and V_2. However this disobeys the statement we have made in the last paragraph that bosons should not be contracted with exponentials, which would give an extra k. Therefore we can also exclude this case from k^2 contribution.

To conclude, from the above argument, 8 fermion contractions should not be included when considering $\mathcal{O}(k^2)$ contribution.

10-and more Fermion contributions in Type-I Since 10 and more fermion contributions in the amplitude of Type-I have 5 or more momenta already, while the pinched-off integration can only cancel two momenta, we can ignore these contractions when only considering $\mathcal{O}(k^2)$ in Type-I theory.

6-Fermion contribution in Type-I

By a simple observation we claim that in 6-fermion contractions each vertex operator must contribute one and only one pair of fermions, otherwise one will have to contract fermions located at the same position, which is forbidden. Furthermore, we should mention that the tensor structure will be invariant when we exchange left-moving and right-moving components of the contractions, since the polarization tensor $\epsilon_{\mu\nu}$ is symmetric. These two observations show that one can already use most of the results from the Heterotic 1-loop 3 graviton calculation. We should follow the rules listed in §5.2.2 as well. These will greatly simplify our work, and we can already get an anti-holomorphic copy from Heterotic calculation by changing all left (right)-moving components in the Heterotic case to right (left)-moving, respectively.

G_3^{4f} and G_3^{6f} correlation functions in Type-I

Here we want to discuss the possible variants of G_3^{6f} correlation functions when we go to Type-I from Heterotic. W.l.o.g we again take $G^1_{6f,(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2)}$ in (5.37) as an example, and rewrite it in Type-I:

$$\epsilon_{1,\mu_1\nu_1}\bar{\partial}X_1^{\mu_1} ik_1\cdot\psi_1\psi_1^{\nu_1}e^{ik_1\cdot X_1}\epsilon_{2\mu_2\nu_2}\bar{\partial}X_2^{\mu_2} ik_2\cdot\psi_2\psi_2^{\nu_2}e^{ik_2\cdot X_2}\epsilon_{3\mu_3\nu_3}\bar{\partial}X_3^{\mu_3} ik_3\cdot\psi_3\psi_3^{\nu_3}e^{ik_3\cdot X_3}. \quad (5.105)$$

$$G^{1,LLL}_{6f,(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2)} \propto -\frac{1}{16}\int_\sigma d^2z_1\int_\sigma d^2z_2\int_\sigma d^2z_3 (k_2\cdot k_3)(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2)$$
$$\langle\psi_1\psi_2\rangle_\sigma\langle\psi_2\psi_3\rangle_\sigma\langle\psi_1\psi_3\rangle_\sigma\langle\bar{\partial}X_1\bar{\partial}X_2\rangle_\sigma\bar{\partial}P^\sigma_{31}|\chi^\sigma_{13}|^{\frac{1}{2}k_1\cdot k_3}|\chi^\sigma_{23}|^{\frac{1}{2}k_2\cdot k_3}|\chi^\sigma_{12}|^{\frac{1}{2}k_1\cdot k_2}. \quad (5.106)$$

Here LLL in the superscript of G_{6f} means that all three pairs of fermions are left-moving, in the sequence of V_1, V_2, V_3. R refers to right-moving. We will use this notation throughout Type-I calculation.

We notice that if all 3 superscripts of L and R of V_1, V_2 and V_3 for G_3^{4f} and G_3^{6f} correlation functions are switched to the other moving (e.g. $G^{LLR}\to G^{RRL}$), the amplitude is invariant: $G^{LLR}=G^{RRL}$.

1-loop Type-I theory has 3 more topologies other than torus. $\partial^2\langle XX\rangle_\sigma \neq \partial^2 P_\sigma$ [7, (A.4)] for $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{M}$. Therefore the coordinate integration is not performed on a total derivative any more. This will break the vanishing result of $\mathcal{O}(k^2)$ with pinching singularity in Heterotic case.

Lifting to covering Torus

To get a consistent result, we need to lift the integrals from 1-loop surfaces $\sigma=\mathcal{A},\mathcal{K},\mathcal{M}$ to the covering torus using the following equation backwards[7]

$$\int_{\mathcal{T}} d^2z f(z) = \left[\int_\sigma d^2z + \int_{I_\sigma(z)} d^2z\right] f(z) = \int_\sigma d^2z\left[f(z)+f(I_\sigma(z))\right], \quad (5.107)$$

where $I(\sigma)$ is the image of involution on the double cover of σ, see (A.13) or [7, (A.1)].

In order to make use of the full power of the lifting technique, we need to revert all the amplitudes to the original form with all the propagators NOT expressed explicitly by functions. For example, we should rewrite (5.106) as

$$G^{1,LLL}_{6f,(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2)} = -\frac{\alpha'}{2}\int_\sigma d^2z_1\int_\sigma d^2z_2\int_\sigma d^2z_3 (k_2\cdot k_3)(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2)$$

$$
\begin{aligned}
&\langle\psi_1\psi_2\rangle_\sigma\langle\psi_2\psi_3\rangle_\sigma\langle\psi_1\psi_3\rangle_\sigma\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\langle\bar\partial X_3X_1\rangle_\sigma\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j}\\
=&-\frac{\alpha'}{2}\int_\sigma d^2z_1\int_\sigma d^2z_2\int_\sigma d^2z_3(k_2\cdot k_3)(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)f_1^{LLL}(z_1,z_2,z_3)\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j}.
\end{aligned}
\tag{5.108}
$$

We know that we have 7 additional variants of this amplitude in Type-I, which are $G^{1,LLR}$, $G^{1,LRL}$, $G^{1,LRR}$, $G^{1,RLL}$, $G^{1,RLR}$, $G^{1,RRL}$, $G^{1,RRR}$. Using the fact that [7, (A.3)]

$$
\langle X(z_i)X(z_j)\rangle_\sigma=\langle X(z_i)X(I_\sigma(z_j))\rangle_\sigma=\langle X(I_\sigma(z_i))X(z_j)\rangle_\sigma\qquad(\sigma=\mathcal{A},\mathcal{K},\mathcal{M}),
\tag{5.109}
$$

and [14, (185), (186)] (noticing the i)

$$
\langle\psi(z_i)\bar\psi(z_j)\rangle_\sigma=iS_\sigma(z_i,I_\sigma(z_j)),
\tag{5.110}
$$

$$
\langle\bar\psi(z_i)\psi(z_j)\rangle_\sigma=iS_\sigma(I_\sigma(z_i),z_j),
\tag{5.111}
$$

we find for example that

$$
\begin{aligned}
&f_1^{LLR}(z_1,z_2,z_3)\\
&=\langle\psi_1\psi_2\rangle_\sigma\langle\psi_2\bar\psi_3\rangle_\sigma\langle\psi_1\bar\psi_3\rangle_\sigma\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\langle\partial X_3X_1\rangle_\sigma\\
&=\langle\psi_1\psi_2\rangle_\sigma iS_\sigma(z_2,I_\sigma(z_3))iS_\sigma(z_1,I_\sigma(z_3))\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\frac{\partial}{\partial z_3}\langle X_3X_1\rangle_\sigma\\
&=-\langle\psi_1\psi_2\rangle_\sigma iS_\sigma(z_2,I_\sigma(z_3))iS_\sigma(z_1,I_\sigma(z_3))\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\frac{\partial}{\partial(1-z_3)}\langle X(z_3)X(z_1)\rangle_\sigma\\
&=\langle\psi_1\psi_2\rangle_\sigma S_\sigma(z_2,I_\sigma(z_3))S_\sigma(z_1,I_\sigma(z_3))\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\frac{\partial}{\partial\overline{I_\sigma(z_3)}}\langle X(z_3)X(z_1)\rangle_\sigma\\
&=\langle\psi_1\psi_2\rangle_\sigma S_\sigma(z_2,I_\sigma(z_3))S_\sigma(z_1,I_\sigma(z_3))\langle\bar\partial X_1\bar\partial X_2\rangle_\sigma\frac{\partial}{\partial\overline{I_\sigma(z_3)}}\langle X(I_\sigma(z_3))X(z_1)\rangle_\sigma\\
&=f_1^{LLL}(z_1,z_2,I_\sigma(z_3)),
\end{aligned}
\tag{5.112}
$$

where in the 4th line the partial derivative is $\partial/\partial(1-z_3+\frac{\tau}{2})=\partial/\partial\overline{I_{\mathcal{K}}(z_3)}$ for $\mathcal{K}$. This means that we can pair all these variants to lift to the covering torus by using (5.107)

$$
\begin{aligned}
G^{1,LLL}_{6f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)}+G^{1,LLR}_{6f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)}=&-\frac{1}{4}\int_\sigma d^2z_1\int_\sigma d^2z_2\int_\sigma d^2z_3(k_2\cdot k_3)(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)\\
&\Big(f_1^{LLL}(z_1,z_2,z_3)+f_1^{LLR}(z_1,z_2,z_3)\Big)\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4}\int_\sigma d^2z_1 \int_\sigma d^2z_2 \int_\sigma d^2z_3 (k_2\cdot k_3)(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2) \\
&\quad \Big(f_1^{LLL}(z_1,z_2,z_3) + f_1^{LLL}(z_1,z_2,I_\sigma(z_3))\Big)\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j} \\
&= -\frac{1}{4}\int_\sigma d^2z_1 \int_\sigma d^2z_2 \int_\mathcal{T} d^2z_3 (k_2\cdot k_3)(k_1\epsilon_3 k_1)(\epsilon_1\epsilon_2) \\
&\quad f_1^{LLL}(z_1,z_2,z_3)\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j}. \qquad (5.113)
\end{aligned}
$$

We observe that the integrand of this sum is exactly the same as $G^{1,LLL}_{6f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)}$ except for the domain. Thus we can repeat the above lifting procedure and finally obtain the result that

$$
\begin{aligned}
&G_{6f}^{1,LLL}+G_{6f}^{1,LLR}+G_{6f}^{1,LRL}+G_{6f}^{1,LRR}+G_{6f}^{1,RLL}+G_{6f}^{1,RLR}+G_{6f}^{1,RRL}+G_{6f}^{1,RRR}|_{(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2),(\mathcal{A},\mathcal{K},\mathcal{M})} \\
&\quad = -\frac{1}{4}\int_\mathcal{T} d^2z_1 \int_\mathcal{T} d^2z_2 \int_\mathcal{T} d^2z_3 (k_2\cdot k_3)(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2) f_1^{LLL}(z_1,z_2,z_3)\prod_{i<j}\left|\chi_{ij}^\sigma\right|^{\frac{1}{2}k_i\cdot k_j} \\
&\quad = G^{1,LLL}_{6f,(k_1\epsilon_3k_1)(\epsilon_1\epsilon_2)}|_\mathcal{T}. \qquad (5.114)
\end{aligned}
$$

Moreover, we can apply this procedure to all G_{4f} and G_{6f} correlation functions, and reduce/lift them all to the torus correlation functions with all left-moving fermions.

But we would like to point out that, G_{4f} correlation functions has only 2 pairs of fermions, thus actually they have only two superscript of L or R in Type-I. We notice that

$$
\begin{aligned}
&\int_\sigma d^2z_3 \left\langle \bar\partial X_1 \bar\partial X_3\right\rangle_\sigma \left\langle \bar\partial X_2 \partial X_3\right\rangle_\sigma \\
&\quad = \int_\sigma d^2z_3 \bar\partial_1 \frac{\partial}{\partial \bar z_3}\left\langle X_1 X(z_3)\right\rangle_\sigma \bar\partial_2 \frac{\partial}{\partial z_3}\left\langle X_2 X(z_3)\right\rangle_\sigma \\
&\quad = \int_\sigma d^2z_3 \bar\partial_1 \frac{\partial}{\partial(-I_\sigma(z_3))}\left\langle X_1 X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma \bar\partial_2 \frac{\partial}{\overline{\overline{\partial(-I_\sigma(z_3))}}}\left\langle X_2 X(z_3)\right\rangle_\sigma \\
&\quad = \int_\sigma d^2z_3 \bar\partial_1 \frac{\partial}{\partial(-I_\sigma(z_3))}\left\langle X_1 X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma \bar\partial_2 \frac{\partial}{\overline{\overline{\partial(-I_\sigma(z_3))}}}\left\langle X_2 X\left(I_\sigma(z_3)\right)\right\rangle_\sigma \\
&\quad = \int_\sigma d^2z_3 \bar\partial_1 \frac{\partial}{\partial(I_\sigma(z_3))}\left\langle X_1 X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma \bar\partial_2 \frac{\partial}{\partial\overline{\overline{I_\sigma(z_3)}}}\left\langle X_2 X\left(I_\sigma(z_3)\right)\right\rangle_\sigma \\
&\quad = \int_\sigma d^2z_3 \left\langle \bar\partial X_1 \partial X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma \left\langle \bar\partial X_2 \bar\partial X\left(I_\sigma(z_3)\right)\right\rangle_\sigma \qquad \text{and vice versa} \quad (5.115)
\end{aligned}
$$

by using the invariance under $I_\sigma(z_j)$ of the bosonic correlator $\left\langle X_i X(z_j)\right\rangle_\sigma|_{i\neq j} = \left\langle X_i X\Big(I_\sigma(z_j)\Big)\right\rangle_\sigma|_{i\neq j}$. Therefore we can apply (5.107) to lift the G_{4f} correlation functions on $\mathcal{A},\mathcal{K},\mathcal{M}$ to $\mathcal{T}$, for example:

$$\begin{aligned}
&\int_\sigma d^2z_3 \left\langle \bar\partial X_1 \bar\partial X_3\right\rangle_\sigma \left\langle \bar\partial X_2 \partial X_3\right\rangle_\sigma + \int_\sigma d^2z_3 \left\langle \bar\partial X_1 \partial X_3\right\rangle_\sigma \left\langle \bar\partial X_2 \bar\partial X_3\right\rangle_\sigma \\
&\qquad = \int_\sigma d^2z_3 \left\langle \bar\partial X_1 \partial X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma \left\langle \bar\partial X_2 \bar\partial X\Big(I_\sigma(z_3)\Big)\right\rangle_\sigma + \int_\sigma d^2z_3 \left\langle \bar\partial X_1 \partial X_3\right\rangle_\sigma \left\langle \bar\partial X_2 \bar\partial X_3\right\rangle_\sigma \\
&\qquad = \int_{\mathcal{T}} d^2z_3 \left\langle \bar\partial X_1 \partial X_3\right\rangle_\sigma \left\langle \bar\partial X_2 \bar\partial X_3\right\rangle_\sigma . \qquad (5.116)
\end{aligned}$$

We conclude that the lifting procedure would also lift 4-fermion correlation functions on 1-loop surfaces to the corresponding correlation function on the covering torus.

Summing up 1-loop Type-I 3-point amplitudes

From the conclusion of "Lifting to covering Torus" §5.3.2, we know that lifting procedure imposes no extra factor, thus the result of Type-I will be exactly the same as the result of Heterotic, except that we should use the correlators (3.18) and (3.19) in Type-I for 1-loop surfaces $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{M}$. Be aware that as in Heterotic, fermion propagators would again be absorbed into spin summation $Z^{\text{int}}_{s=1}$. 1-loop 3 graviton Type-I amplitude receives contributions only from $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{M}$[5]. We summarize the kinematic structure result up to the overall factor:

$$\begin{aligned}
G_\sigma \propto & \frac{\alpha'}{2}\frac{1}{|z_{12}|^2}\left(\frac{\alpha'\pi}{2\tau_2} + \bar\partial^2_{23}P_\sigma(\bar z_2,\bar z_3)\right)\{(k_1\cdot k_2)(\epsilon_1\epsilon_2)(k_1\epsilon_3 k_1) + (k_1\cdot k_2)\left[(k_3\epsilon_1\epsilon_2\epsilon_3 k_1) - (k_3\epsilon_2\epsilon_1\epsilon_3 k_1)\right] \\
& + 2(k_1\cdot k_3)(k_2\epsilon_3\epsilon_1\epsilon_2 k_3) + 2(k_2\cdot k_3)(k_3\epsilon_1\epsilon_2\epsilon_3 k_1) \\
& +2(k_1\cdot k_3)(k_2\epsilon_1\epsilon_2\epsilon_3 k_2) + 2(k_2\cdot k_3)(k_1\epsilon_2\epsilon_1\epsilon_3 k_1)\} \\
& - \frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2} + \partial_2\bar\partial_1 P_\sigma(z_2, I_\sigma(z_1))\right)(k_1\epsilon_3\epsilon_2\epsilon_1 k_3)\left(\frac{\alpha'\pi}{2\tau_2} + \bar\partial^2_{23}P_\sigma(\bar z_2,\bar z_3)\right) \\
& - \frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2} + \partial_3\bar\partial_1 P_\sigma(z_3, I_\sigma(z_1))\right)(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)\left(\frac{\alpha'\pi}{2\tau_2} + \bar\partial^2_{32}P_\sigma(\bar z_3,\bar z_2)\right) \\
& + \frac{\alpha'}{2}\frac{1}{|z_{13}|^2}\left(\frac{\alpha'\pi}{2\tau_2} + \bar\partial^2_{12}P_\sigma(\bar z_1,\bar z_2)\right)\{(k_1\cdot k_3)(\epsilon_1\epsilon_3)(k_3\epsilon_2 k_3) + (k_1\cdot k_3)\left[(k_2\epsilon_3\epsilon_1\epsilon_2 k_3) - (k_3\epsilon_2\epsilon_3\epsilon_1 k_2)\right] \\
& + 2(k_1\cdot k_2)(k_3\epsilon_2\epsilon_1\epsilon_3 k_2) + 2(k_2\cdot k_3)(k_2\epsilon_1\epsilon_3\epsilon_2 k_1)
\end{aligned}$$

[5]Type-I theory also contains torus amplitude. However torus amplitude would exactly the same as in Heterotic, thus it is vanishing.

$$
\begin{aligned}
&+2\,(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)+2\,(k_2\cdot k_3)\,(k_1\epsilon_2\epsilon_1\epsilon_3k_1)\}\\
&-\frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2}+\partial_1\bar\partial_3P_\sigma(z_1,I_\sigma(z_3))\right)(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\left(\frac{\alpha'\pi}{2\tau_2}+\bar\partial^2_{12}P_\sigma(\bar z_1,\bar z_2)\right)\\
&-\frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2}+\partial_2\bar\partial_3P_\sigma(z_2,I_\sigma(z_3))\right)(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\left(\frac{\alpha'\pi}{2\tau_2}+\bar\partial^2_{21}P_\sigma(\bar z_2,\bar z_1)\right)\\
&+\frac{\alpha'}{2}\frac{1}{|z_{23}|^2}\left(\frac{\alpha'\pi}{2\tau_2}+\bar\partial^2_{13}P_\sigma(\bar z_1,\bar z_3)\right)\{(k_2\cdot k_3)\,(\epsilon_2\epsilon_3)\,(k_2\epsilon_1k_2)+(k_2\cdot k_3)\,[(k_2\epsilon_1\epsilon_3\epsilon_2k_1)-(k_2\epsilon_1\epsilon_2\epsilon_3k_1)]\\
&+2\,(k_1\cdot k_2)\,(k_3\epsilon_1\epsilon_2\epsilon_3k_1)+2\,(k_1\cdot k_3)\,(k_1\epsilon_2\epsilon_3\epsilon_1k_2)\\
&+2\,(k_1\cdot k_2)\,(k_3\epsilon_2\epsilon_3\epsilon_1k_3)+2\,(k_1\cdot k_3)\,(k_2\epsilon_1\epsilon_2\epsilon_3k_2)\}\\
&-\frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2}+\partial_1\bar\partial_2P_\sigma(z_1,I_\sigma(z_2))\right)(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\left(\frac{\alpha'\pi}{2\tau_2}+\bar\partial^2_{13}P_\sigma(\bar z_1,\bar z_3)\right)\\
&-\frac{2}{\alpha'}\left(-\frac{\alpha'\pi}{2\tau_2}+\partial_3\bar\partial_2P_\sigma(z_3,I_\sigma(z_2))\right)(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\left(\frac{\alpha'\pi}{2\tau_2}+\bar\partial^2_{31}P_\sigma(\bar z_3,\bar z_1)\right).
\end{aligned}
\tag{5.117}
$$

As in Heterotic theory, we notice that both $\partial_i\bar\partial_jP_\sigma(z_i,I_\sigma(z_j))$ and $\bar\partial^2_{ij}P_\sigma(\bar z_i,\bar z_j)$ are total derivatives, thus they vanish upon coordinate integration on the covering torus. Thus we could simplify our life and get rid of all terms including $\partial_i\bar\partial_jP_\sigma(z_i,I_\sigma(z_j))$ or $\bar\partial^2_{ij}P_\sigma(\bar z_i,\bar z_j)$. Then we perform the pinched-off integration as well as an extra coordinator integration to obtain the non-vanishing contribution (using again $\int_{\mathcal{T}}d^2z=2\tau_2$)

$$
\begin{aligned}
\alpha'\pi^2\,\{\,&(\epsilon_1\epsilon_2)\,(k_1\epsilon_3k_1)+(k_3\epsilon_2\epsilon_1\epsilon_3k_1)+(k_3\epsilon_1\epsilon_2\epsilon_3k_1)+2\,(k_2\epsilon_1\epsilon_3\epsilon_2k_1)\\
&+(\epsilon_1\epsilon_3)\,(k_3\epsilon_2k_3)+(k_3\epsilon_2\epsilon_3\epsilon_1k_2)+(k_2\epsilon_3\epsilon_1\epsilon_2k_3)+2\,(k_1\epsilon_3\epsilon_2\epsilon_1k_3)\\
&+(\epsilon_2\epsilon_3)\,(k_2\epsilon_1k_2)+(k_2\epsilon_1\epsilon_2\epsilon_3k_1)+(k_2\epsilon_1\epsilon_3\epsilon_2k_1)+2\,(k_3\epsilon_2\epsilon_1\epsilon_3k_2)\}\,.
\end{aligned}
\tag{5.118}
$$

Using (5.33) to simplify the above contribution, we arrive at the final result

$$
\boxed{
\begin{aligned}
\alpha'\pi^2\,\{\,&(k_1\epsilon_3k_1)\,(\epsilon_1\epsilon_2)-2\,(k_3\epsilon_2\epsilon_3\epsilon_1k_2)\\
&+(k_3\epsilon_2k_3)\,(\epsilon_3\epsilon_1)-2\,(k_2\epsilon_1\epsilon_2\epsilon_3k_1)\\
&+(k_2\epsilon_1k_2)\,(\epsilon_2\epsilon_3)-2\,(k_1\epsilon_3\epsilon_1\epsilon_2k_3)\,\}\,.
\end{aligned}}
\tag{5.119}
$$

5.3.3 Discussion

We double-checked the gravity kinematic structure in the Heterotic 1-loop 3-point graviton amplitude from [40]. We derived the kinematic structure (5.78) different from theirs. We extended the calculation to Type-I 1-loop 3-point graviton amplitude. We already knew that the calculation to Type-I 1-loop 3-point graviton amplitude in [9] was incomplete because they did not consider pinched-off integration. However, even including pinched-off integration, non-vanishing 1-loop correction (5.119) from our Type-I calculation still breaks the gravity kinematic structure

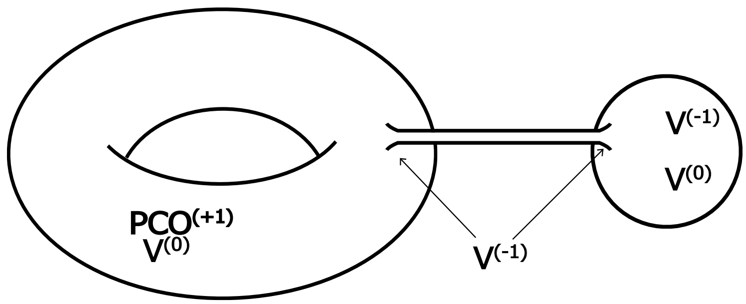


Figure 5.2: Factorization of pinched-off integration contribution

(5.2), which shows that the string amplitude calculation is incomplete. Further studies are necessary to solve this problem.

In the pinching limit, according to [69, §2.5.1, §4.4, §4.5], the 1-loop surface that we are working on should be factorized into a sphere with two vertex operators on it connected to a 1-loop surface with 1 vertex operator on it by a propagator. We illustrate the image in Figure 5.2. Actually in the calculation we never take into account the effect of PCOs (Picture Changing Operators). The so called "vertical integration" technique[64, §3.6] provides a potential method toward the solution. In order to make the picture numbers on both sphere and 1-loop surface consistent in Figure 5.2, we would have to move 1 PCO from the vertex operator on the sphere factor to the vertex operator on the torus factor in the pinching limit [69, §6.4.6][65, §8] [6]. We see that in the Figure 5.2, the 2 vertex operators at the 2 punctures connected by the plumbing are forced to have canonical picture number[7], thus the torus and the sphere could have the correct total picture numbers of 0 and -2 respectively, as discussed in §2.2.1. This modification of distribution of picture numbers involves "vertical integration" and introduces potential new contributions. Unfortunately, the application of "vertical integration" is still under research. We have to leave this topic to our future study.

To complete the 3-point 1-loop correction calculation, 1-loop partition functions of Type-IIB $T^6/\mathbb{Z}_N$ orientifolds would be necessary. These are introduced in app.C, app.D.1 and app.D.2. They can also be found in e.g.[7][47].

[6] The superscripts in Figure 5.2 stand for the picture number of the corresponding vertex operator or PCO.

[7] We are dealing with gravitons here, so we are in the NS sector and the picture numbers of the vertex operators on the two punctures at the 2 ends of the plumbing should be canonical. RR sector leads to a different total picture number, therefore requires the insertion of an extra PCO on the plumbing.[37, §17.2.2]

Chapter 6

Preliminary: Basics about Genus-$\frac{3}{2}$ correction

The natural idea of the next step of the loop correction would be higher order of perturbation series. Genus-$\frac{3}{2}$ is the next higher order to the 1-loop. Genus-$\frac{3}{2}$ surfaces are surfaces with boundaries (open surfaces) plus unoriented surfaces. Generally, one associates a closed oriented surface $\bar{\Sigma}$ which is a double cover of Σ to obtain an open or unoriented surface Σ. Involution reduces $\bar{\Sigma}$ to Σ. We want to show that one can express the determinants, differentials and classical action on Σ in terms of the corresponding quantities on $\bar{\Sigma}$. And we get genus-$\frac{3}{2}$ surfaces by the involution on 2-torus. This allows us to utilize the well-studied properties of genus-2 surfaces. [1]

6.1 Involution on surfaces

A conformal structure $[g]$ determines an almost complex structure J ($J^2 = -1$) by $J^a_b = \sqrt{g} g^{ac} \epsilon_{cb}$, ϵ_{cb} is the Levi-Civita symbol. And on 2 dimensional Riemann surfaces Σ, J^a_b determines a complex structure $J = idz \otimes \frac{\partial}{\partial z} - id\bar{z} \otimes \frac{\partial}{\partial \bar{z}}$. One can classify Riemann surfaces Σ into three cases:

Oriented surface with boundary $\partial\Sigma$ [38] Take Σ^* to be the copy of Σ with opposite orientation. The double cover of oriented surface Σ with boundary is a closed oriented surface $\bar{\Sigma}$ without boundary, which is obtained by attaching Σ^* to Σ along their corresponding boundaries. The involution $I : \bar{\Sigma} \to \bar{\Sigma}$ is orientation reversing and maps 1 to 1 from Σ to Σ^*. We can extend the almost complex structure

[1] In this chapter we closely follow [21].

J on Σ to an almost complex structure $\bar{J}$ on $\bar{\Sigma}$ by letting $\bar{J}_p = J_p$ for $p \in \Sigma$ and $\bar{J}_p = -(I \circ J \circ I^{-1})_p$ for $p \in \Sigma^*$. By construction that I is orientation reversing, I is anti-conformal and its fixed point set is $\partial\Sigma$, and $\Sigma = \bar{\Sigma}/I$.

Unoriented surface without boundary [3] In this case, an unoriented surface Σ without boundary has a compact oriented double cover $\bar{\Sigma}$. The corresponding anti-conformal involution I interchanges two points of $\bar{\Sigma}$ which corresponds to the same point on Σ, but the image $I(p) \in \Sigma^*$ lies above $p \in \Sigma$. In other words, this could be considered effectively as gluing an orientation-reversed copy Σ^* of Σ along the cross-caps of Σ. This explains the construction of the double cover, and I has no fixed point. We can lift the conformal structure on Σ to $\bar{\Sigma}$. And since $\bar{\Sigma}$ is oriented, the conformal structure naturally determines a complex structure on $\bar{\Sigma}$.

Unoriented surface with boundary [3] For Σ unoriented with boundary, we take the complex double cover, which is obtained by doubling the Σ (effectively along the cross-cap) to get an oriented double cover O with twice as many boundaries as Σ as the anti-conformal involution in the unoriented surface without boundary, then identifying each boundary with its image under I. There is another possible double cover B across the boundaries, which is unoriented and has no boundaries. We define the quadruple of Σ to be Q, which is the oriented double cover of B and it is boundaryless. Q will be needed in §6.3.

The anti-conformal involution I transforms canonical homology basis of the Riemann surfaces a-cycles to a, and b-cycles to b, and it is an orientation-reversing diffeomorphism due to anti-conformal property. We express the involution I as

$$Ia_i = \Gamma_{ij} a_j, \tag{6.1}$$

where elements of Γ_{ij} are integers such that $\Gamma^2 = 1$. Because of the orientation-reversing property, the involution I preserves the intersection pairing but changes its sign, which can be expressed as

$$Ib_i = -\Gamma_{ji} b_j. \tag{6.2}$$

And then we take a normalized basis of holomorphic differentials on $\bar{\Sigma}$

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad \int_{b_i} \omega_j = \Omega_{ij}, \tag{6.3}$$

where Ω is the period matrix. It gives

$$I\omega_i = \Gamma_{ji} \bar{\omega}_j, \qquad \Omega = -\Gamma^\top \bar{\Omega} \Gamma, \tag{6.4}$$

where bar means complex conjugate.

6.2 Moduli space

Period matrix Ω as defined in (3.35) could be used to identify the moduli space of genus-$\frac{3}{2}$ and 2 surfaces. We give the partition function of one boson on Σ and the moduli space measure σ_Σ as an example: [4]

$$A_\Sigma(X_i) = \int_{\mathcal{M}(\Sigma)} \sigma_\Sigma \int \Big(\prod dX_i\Big) e^{-S_{\mathrm{cl}}[X_i]},$$
$$\sigma_\Sigma = \left(\frac{\det' \Delta_{\Sigma,g}}{\int_\Sigma \sqrt{g}}\right)^{-\frac{1}{4}} (\det\Big(P_1^\dagger P_1\Big)_{\Sigma,g})^{\frac{1}{2}} d\mu_{\mathrm{WP}}(\Sigma), \tag{6.5}$$

where $\mathcal{M}(\Sigma)$ is the moduli space. The measure of the moduli space is separated into 3 parts: $d\mu_{\mathrm{WP}}(\Sigma)$ is the Weil-Petersson measure of the moduli space defined in (6.24); $\Delta_{\Sigma,g}$ is the scalar Laplacian $\partial^\dagger\partial$ acting on X; $(P_1^\dagger P_1)_{\Sigma,g}$ is the vector Laplacian as P_1 acts on vector fields, see (2.24). $S_{\mathrm{cl}}[X_i]$ is the classical action.

In the path integral, we integrate over the moduli space of conformal structures, which indicates that Σ and $\bar{\Sigma}$ no longer have fixed conformal structures. Define the action of I on the space $C(\bar{\Sigma})$ of almost complex structures J by

$$J \mapsto -I_* \circ J \circ I_*, \tag{6.6}$$

which is anti-conformal. Then $\bar{\Sigma}$ could be a double cover of Σ iff J of $\bar{\Sigma}$ is in the fixed point set $C^I(\bar{\Sigma})$ under I.

The Teichmüller and moduli spaces of $\bar{\Sigma}$ are[29]

$$T(\bar{\Sigma}) = C(\bar{\Sigma})/\mathrm{Diff}_0(\bar{\Sigma}), \qquad \mathcal{M}(\bar{\Sigma}) = C(\bar{\Sigma})/\mathrm{Diff}(\bar{\Sigma}), \tag{6.7}$$

where $\mathrm{Diff}(\bar{\Sigma})$ is the group of orientation-preserving diffeomorphisms of $\bar{\Sigma}$ and $\mathrm{Diff}_0(\bar{\Sigma})$ is the subgroup of $\mathrm{Diff}(\bar{\Sigma})$ connected to the identity. And

$$G(\bar{\Sigma}) = \mathrm{Diff}(\bar{\Sigma})/\mathrm{Diff}_0(\bar{\Sigma}) \tag{6.8}$$

is the mapping class group.

We identify the real tangent space to $T(\bar{\Sigma})$ at $[J]$ with the space of real Beltrami differentials $\mu_{[J]} \equiv \mu_a^b dx^a \otimes \partial/\partial x^b$, and the real cotangent space $T^*(\bar{\Sigma})$ at $[J]$ with the space of real quadratic differentials $\phi_{[J]} \equiv \phi_{ab} dx^a \otimes dx^b$. $T(\bar{\Sigma})$ and $T^*(\bar{\Sigma})$ are complex manifolds due to the almost complex structure[60]

$$\mu_{[J]} \mapsto \big(J\mu_{[J]}\big) \equiv J_a^b \mu_b^c dx^a \otimes \frac{\partial}{\partial x^c},$$

$$\phi_{[J]} \mapsto \left(J\phi_{[J]}\right) \equiv J^b_a \phi_{cb} dx^a \otimes dx^c. \tag{6.9}$$

One could do the same construction on Σ, but if Σ is unoriented, then one takes $C(\Sigma)$ to be the space of conformal structures. If Σ has boundaries, $\mathrm{Diff}_0(\Sigma)$ takes each boundary component into itself[66].

$T(\Sigma)$ should be identified with a slice of $T(\bar{\Sigma})$ which is the fixed point set $T^I(\bar{\Sigma})$ under I. The mapping class group $G(\Sigma)$ is naturally identified with the relative modular group[2] $G(\bar{\Sigma}, I) \equiv \mathrm{Diff}^I(\bar{\Sigma})/\mathrm{Diff}^I_0(\bar{\Sigma})$ where Diff^I is the group of diffeomorphisms which commute with I. Therefore

$$\mathcal{M}(\Sigma) \cong T^I(\bar{\Sigma})/G(\bar{\Sigma}, I). \tag{6.10}$$

From the above constructions, we are able to do the following steps: First, express the string integrand for Σ as a form on $T(\Sigma)$; Second, rewrite the determinants, differentials, and classical action in σ_Σ in terms of quantities defined on $\bar{\Sigma}$; Then, identify the resulting expression as a form on $T^I(\bar{\Sigma}) \in T(\bar{\Sigma})$; Finally, check that this form is invariant under the relative modular group $G(\bar{\Sigma}, I)$.

After showing that the string integrand descends to a form on $T^I(\bar{\Sigma})/G(\bar{\Sigma}, I) \cong \mathcal{M}(\Sigma)$, we are able to write the amplitude A_Σ for Σ in terms of a real slice of $T(\bar{\Sigma})$.

6.3 Determinants

Now we proceed to the treatment of determinants of the string integrand. First we deal with $\det'\Delta_{\Sigma,g}$. $\det'$ always stands for the determinant without zero modes while zero modes are taken care of in the ghost system. We want to express $\det'\Delta_{\Sigma,g}$ in terms of the scalar determinant on $\bar{\Sigma}$. In order to do so, we try to express $\det'\Delta^{\pm}_{\bar{\Sigma},g}$ in terms of $\det'\Delta_{\bar{\Sigma},g}$ first, where $\det'\Delta^{+}_{\bar{\Sigma},g}$ and $\det\Delta^{-}_{\bar{\Sigma},g}$ denote the laplacian $\Delta_{\bar{\Sigma},g}$ restricted to functions even $(+)$ and odd $(-)$ under I. One notices that functions odd under I do not include zero modes.

We introduce a quantity[32]:

$$R_{\bar{\Sigma},I}(J) \equiv \det{}' \operatorname{Im}\Omega^+/\det{}' \operatorname{Im}\Omega^- = \det\left[(1+\Gamma)\operatorname{Im}\Omega + (1-\Gamma)(\operatorname{Im}\Omega)^{-1}\right], \tag{6.11}$$

where $\operatorname{Im}\Omega^{\pm} = (1\pm\Gamma)\operatorname{Im}\Omega(1\pm\Gamma)$ and Γ is defining the involution $Ia_i = \Gamma_{ij}a_j$.

We claim that

$$\left(\frac{\det'\Delta^{+}_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}}\sqrt{g}}\right)/\det\Delta^{-}_{\bar{\Sigma},g} = R_{\bar{\Sigma},I}(J)^{-1} \tag{6.12}$$

[2]Details of relative modular group would be discussed in §6.5

up to a multiplicative constant independent of g. The proof is given in [21][3], and the central idea is to calculate the variation of the equation. Using this equation together with the relation

$$\det{}'\Delta_{\bar{\Sigma},g} = \det{}'\Delta^{+}_{\bar{\Sigma},g} \det \Delta^{-}_{\bar{\Sigma},g}, \tag{6.13}$$

one derives

$$\begin{aligned} \frac{\det{}'\Delta^{+}_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} &= \left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{-\frac{1}{2}}, \\ \det \Delta^{-}_{\bar{\Sigma},g} &= \left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{+\frac{1}{2}}. \end{aligned} \tag{6.14}$$

Next we need to relate $\det{}'\Delta_{\Sigma,g}$ to $\det{}'\Delta^{\pm}_{\bar{\Sigma},g}$[21]. If Σ is oriented with boundary,

$$\det \Delta^{D}_{\Sigma,g} = \det \Delta^{-}_{\bar{\Sigma},g}, \qquad \det{}'\Delta^{N}_{\Sigma,g} = \det{}'\Delta^{+}_{\bar{\Sigma},g}, \tag{6.15}$$

where D and N denote Dirichlet and Neumann boundary conditions respectively. If Σ is unoriented with boundary,

$$\det \Delta^{D}_{\Sigma,g} \det{}'\Delta^{N}_{\Sigma,g} = \det{}'\Delta^{+}_{Q,g}, \qquad \det{}'\Delta^{N}_{\Sigma,g} = \det{}'\Delta^{+}_{\bar{\Sigma},g}, \tag{6.16}$$

where Q is the quadruple of Σ. Combining equations (6.12), (6.15) and (6.16) one gets

	$\det \Delta^{D}_{\Sigma,g}$	$\det{}'\Delta^{N}_{\Sigma,g} / \int_{\Sigma} \sqrt{g}$
oriented	$\left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{\frac{1}{2}}$	$\left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{-\frac{1}{2}}$
unoriented	$\frac{(\det{}'\Delta_{Q,g})^{\frac{1}{2}} (R_{Q,I}(J))^{\frac{1}{2}}}{(\det{}'\Delta_{\bar{\Sigma},g})^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{\frac{1}{2}}}$	$\left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{-\frac{1}{2}}$

If Σ is unoriented without boundaries, one can similarly derive

$$\frac{\det{}'\Delta_{\Sigma,g}}{\int_{\bar{\Sigma}} \sqrt{g}} = \left(\frac{\det{}'\Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g}} \right)^{\frac{1}{2}} (R_{\bar{\Sigma},I}(J))^{-\frac{1}{2}}. \tag{6.17}$$

[3] be aware of the typo in [21], actually all Σ in [21, (4.1)] should have a bar

Vector laplacian One also has to consider the vector laplacian in the amplitude (6.5)

$$(P_1^\dagger P_1)^a_b = (\Delta^c\Delta_c + \frac{1}{2}R)\delta^a_b. \tag{6.18}$$

With a symmetric tensor metric g one has

$$\det\Big(P_1^\dagger P_1\Big)_{\bar\Sigma,g} = \det\Big(P_1^\dagger P_1\Big)^+_{\bar\Sigma,g}\det\Big(P_1^\dagger P_1\Big)^-_{\bar\Sigma,g}, \tag{6.19}$$

where the + and − sign denote the vector laplacians restricted to even and odd vector fields under I. Using the fact that J anticommutes with the involution and commutes with the vector laplacian, one derives that

$$\det\Big(P_1^\dagger P_1\Big)^+_{\bar\Sigma,g} = \det\Big(P_1^\dagger P_1\Big)^-_{\bar\Sigma,g}, \tag{6.20}$$

so

$$\det\Big(P_1^\dagger P_1\Big)^+_{\bar\Sigma,g} = \left(\det\Big(P_1^\dagger P_1\Big)_{\bar\Sigma,g}\right)^{\frac{1}{2}} \qquad \text{and} \qquad \det\Big(P_1^\dagger P_1\Big)_{\Sigma,g} = \det\Big(P_1^\dagger P_1\Big)^+_{\bar\Sigma,g}, \tag{6.21}$$

thus

$$\det\Big(P_1^\dagger P_1\Big)_{\Sigma,g} = \left(\det\Big(P_1^\dagger P_1\Big)_{\bar\Sigma,g}\right)^{\frac{1}{2}}. \tag{6.22}$$

As an example, the string partition function of one boson on unoriented Σ with boundary is

$$A_\Sigma(X_i) = \int_{T^I/G} M_{\bar\Sigma} {R^{\pm}_{\bar\Sigma,I}}^{\frac{1}{2}} \int (\prod dX_i) e^{-S_{\mathrm{cl}}[X_i]}. \tag{6.23}$$

Remarks:

1. The sign in exponent of $R_{\bar\Sigma,I}$ is chosen w.r.t. the following: positive sign is for open string (Neumann) boundary conditions or cross caps; negative sign is for closed string (Dirichlet) boundary conditions.
2. $T^I(\bar\Sigma)/G(\bar\Sigma, I) \cong \mathcal{M}(\Sigma)$ is the moduli space of Σ.
3. The classical action $S_{\mathrm{cl}}[X_i]$, which is only present if there are external string states, can be explicitly given in terms of abelian differentials on $\bar\Sigma$ and parameterized boundary curves X_i.

The Weil-Petersson measure with a symmetric metric g can be factorized[33]:

$$d\mu_{\mathrm{WP}}(\bar{\Sigma}) \equiv \frac{\det\langle\phi_i, \mu_j\rangle}{(\det\langle\phi_i, \phi_j\rangle)^{\frac{1}{2}}} \prod_{i=1}^{6g-6} dm_i = d\mu_{\mathrm{WP}}(\Sigma) \wedge J d\mu_{\mathrm{WP}}(\Sigma), \tag{6.24}$$

where $\{\phi_i\}$ is any basis for the space of real quadratic differentials on $\bar{\Sigma}$, $\{\mu_i\}$ is any basis of Beltrami differentials corresponding to tangent vectors d/dm_i to Teichmüller space, and $\langle\cdot,\cdot\rangle$ is the Petersson pairing. Since g is symmetric, the space 0f quadratic differentials $\{\phi_i\}$ splits into subspaces $\{\phi_i\} = \{S_i, JS_i\}$ where S_i (JS_i) is even (odd) under I respectively.

From the factorization of $d\mu_{\mathrm{WP}}(\bar{\Sigma})$ we can derive the measure $M_{\bar{\Sigma}}$ with help of $\sigma_{\bar{\Sigma}} = M_{\bar{\Sigma}} \wedge JM_{\bar{\Sigma}}$:

$$M_{\bar{\Sigma}} = \left\{ \frac{\det\left(P_1^\dagger P_1\right)^{\frac{1}{2}}_{\bar{\Sigma},g}}{\det\langle\phi_i, \phi_j\rangle^{\frac{1}{2}}} \left(\frac{\det \Delta_{\bar{\Sigma},g}}{\int_{\bar{\Sigma}} \sqrt{g} \det \operatorname{Im}\Omega}\right)^{-\frac{1}{2}} \right\}^{\frac{1}{2}} (\det \operatorname{Im}\Omega)^{-\frac{1}{4}} \det\langle S_i, \mu_j\rangle \prod_{i=1}^{3g-3} dm_i. \tag{6.25}$$

Then the determinant of the partition function of one boson on Σ is determined.

6.4 Fermions under Involution

Above method of taking square root from the double cover also applies to fermionic determinants giving rise to the fermionic partition functions[10]. In the following, we consider only the right moving part of fermions as a simple illustration since left and right movers can have different spin structures.

The determinant of Dirac operator $\det'_\Sigma D$ on the certain surfaces can be expressed by the determinant of Dirac operator $\det'_{\bar{\Sigma}} D$ on the double cover of that surface[63], which is similar to the bosonic determinant. Here we give a detailed explanation of this argument.

The key should be the point "There is therefore a one-to-one correspondence between even and odd eigenfunctions with the same eigenvalue." in [63]. We know that

$$\det{}'_{\bar{\Sigma}} D = \det{}'_{\bar{\Sigma}} D^+ \det{}'_{\bar{\Sigma}} D^- \tag{6.26}$$

where $+$ and $-$ represent that D is restricted to the even and odd states under involution I respectively. For every eigenstate T of $\det'_{\bar{\Sigma}} D$ which is even under Involution, it is always possible to construct an unique $(*T)$ that is an eigenstate with the same eigenvalue and is odd under Involution, and vice versa[63]. This indicates

$$\det{}'_{\bar{\Sigma}} D^+ = \det{}'_{\bar{\Sigma}} D^-. \tag{6.27}$$

By definition, one easily finds that

$$\det{}'_{\Sigma} D = \det{}'_{\bar{\Sigma}} D^+ \qquad \text{or} \qquad \det{}'_{\Sigma} D = \det{}'_{\bar{\Sigma}} D^- \tag{6.28}$$

w.r.t. certain spin structure and involution. Then the claim

$$\det{}'_{\bar{\Sigma}} D = (\det{}'_{\Sigma} D)^{\frac{1}{2}} \tag{6.29}$$

is straightforward.

Refering to [63], we can explicitly write down the determinant up to a numerical constant:

$$\det{}'_{\Sigma} D = \left(\frac{\det{}'_{\bar{\Sigma}} \Delta}{\int_{\bar{\Sigma}} \sqrt{g} \det \operatorname{Im} \Omega} \right)^{-\frac{1}{4}} \left| \vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (t) \right|, \tag{6.30}$$

where $\vec{a}$ and $\vec{b}$ are the twists which represent the spin structures, and $\vartheta[\substack{\vec{a} \\ \vec{b}}]$ is the standard theta function with characteristics.

Fermionic partition function includes a sum of theta functions over spin structures with coefficient c:

$$\begin{aligned} Z_F \propto \sum_{spin} c \left[\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right] \det{}'_{\Sigma} D &= \sum_{\vec{a},\vec{b}} c \left[\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right] \left(\frac{\det{}'_{\bar{\Sigma}} \Delta}{\int_{\bar{\Sigma}} \sqrt{g} \det \operatorname{Im} \Omega} \right)^{-\frac{1}{4}} \left| \vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (t) \right| \\ &= \sum_{\vec{a},\vec{b}} c \left[\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right] \left(\frac{\det{}'_{\bar{\Sigma}} \Delta}{\int_{\bar{\Sigma}} \sqrt{g} \det \operatorname{Im} \Omega} \right)^{-\frac{1}{4}} \sum_{\vec{n} \in \mathbb{Z}^g} e^{-\pi(\vec{n}+\vec{a})t(\vec{n}+\vec{a})^\top + 2i\pi(\vec{n}+\vec{a})\vec{b}^\top}. \end{aligned} \tag{6.31}$$

The coefficients c of higher loops could be determined by factorizing the double covering surface into 1-loop surfaces. For example, a genus-2 torus $\mathcal{T}$ could be factorized into two separate tori $\mathcal{T}_1$ and $\mathcal{T}_2$. c of $\mathcal{T}$ as well as the relative signs between c could be determined up to a normalizing constant by expressing $c = c_{\mathcal{T}_1} c_{\mathcal{T}_2}$ and imposing invariance under modular transformation, in the standard way. Due to the involution, the coefficients c in genus-$\frac{3}{2}$ surface are inherited from the coefficients in its genus-2 torus double cover.

6.5 Relative Modular Group

All modular transformations M that preserve the involution I in the sense that $I' = MIM^{-1} \equiv I$ form the so called "relative modular group", which is a subgroup of the modular group $Sp(2g, \mathbb{Z})$ and is dependent on the individual surface. Only relative modular transformations survive the involution on the double covering surface, and thus they are relevant for the space after involution. Here we consider the general case that genus-g is arbitrary.[4]

We know that the spin-structures can be described by the periodicity/anti-periodicity when a fermions goes around the homology basis/cycles. We define the canonical homology basis as a_i winding around handles and b_i winding around holes (see Figure 3.1) with symplectic intersection form

$$\begin{aligned} J(a_i, b_j) &= -J(b_j, a_i) = \delta_{ij}, \\ J(a_i, a_j) &= J(b_i, b_j) = 0. \end{aligned} \tag{6.32}$$

The symplectic form could be represented by the $2g$ dimensional homology basis vector

$$\mathbf{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix}, \qquad \mathbf{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} (n_{th} \text{ component})$$

as

$$\mathbf{e}_m^\top J \mathbf{e}_n = J_{mn} \tag{6.33}$$

where J_{mn} is the matrix element of J. We can give the explicit form of J as

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}_{2g\times 2g} \tag{6.34}$$

According to §6.1, the involution I is anti-conformal, thus I reverses the relative angle between intersecting homology basis on the double cover $\bar{\Sigma}$. Then the involution matrix I acting on arbitrary $\mathbf{e}$ satisfies

$$-J_{mn} = (I\mathbf{e}_m)^\top J(I\mathbf{e}_n) = \mathbf{e}_m^\top I^\top J I \mathbf{e}_n \quad \text{and} \quad -J_{mn} = \mathbf{e}_m^\top(-J)\mathbf{e}_n \qquad \Rightarrow I^\top J I = -J. \tag{6.35}$$

[4] This section closely follows [18] and [19]

The action on the a-cycles determines the behavior of the abelian differentials by preserving $\int_{a_i} w_j = \delta_{ij}$. We take the general form of involution matrix as

$$I = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}_{(2g\times 2g)}, \tag{6.36}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $g \times g$ matrices of integers and satisfy

$$\mathcal{C}^\top \mathcal{A} = \mathcal{A}^\top \mathcal{C}, \qquad \mathcal{D}^\top \mathcal{B} = \mathcal{B}^\top \mathcal{D}, \qquad \mathcal{C}^\top \mathcal{B} - \mathcal{A}^\top \mathcal{D} = \mathbf{1} \tag{6.37}$$

on account of (6.34) and (6.35). Period matrix Ω compatible with I should satisfy

$$\bar{\Omega} = I(\Omega) = (\mathcal{C} + \mathcal{D}\Omega)(\mathcal{A} + \mathcal{B}\Omega)^{-1} \tag{6.38}$$

while $\int_{a_i} w_j = \delta_{ij}$ should be preserved under I, because of (6.4).

One can choose a proper homology basis to simplify both the involution matrix and the corresponding period matrix Ω. And it's always possible to reduce the involution matrix to the triangular form

$$I = \begin{pmatrix} \mathbf{1} & 0 \\ \Delta & -\mathbf{1} \end{pmatrix}, \tag{6.39}$$

where the matrix Δ is symmetric on account of (6.37). Δ determines the real part of the period matrix Ω as in (7.3)[55]. Such a basis which has this triangular form of I is named "identity basis".

As a basic rule to obtain the identity basis, the homology a-cycles should be invariant under involution, and the homology b-cycles should intersect a-cycles, thus their orientation would be reversed under involution and therefore the sign of the symplectic intersection pairing $J(a, b)$ would be flipped.

By the classification theorem of closed surfaces[50], any connected closed surface should be homeomorphic to one of the 3 families of surfaces:

- an oriented surface
- an oriented surface glued with 1 cross-cap: factorized into an oriented surface with 1 boundary and a Möbius strip, glued along the boundaries
- an oriented surface glued with 2 cross-caps: factorized into an oriented surface with 2 boundaries and 2 Möbius strips, glued along the boundaries

Therefore we do not have to consider surfaces with 3 or more cross-caps. We factorize the surfaces in above way because we want to associate the cross-caps of a surface with homology cycles on the double cover of the surface. The construction of the double covering Cylinder of a Möbius strip is illustrated in Figure 6.1. We see in Figure 6.1 how a cross-cap of a Möbius strip is associated with a cycle A on the double covering Cylinder. Equivalently we can say that the cross-cap on a surface Σ is associated with the homology cycle A on the double cover $\bar{\Sigma}$, which is shown in Figure 6.2.

If a surface Σ has a total number n of boundaries plus cross-caps, then we take $n-1$ of them to be associated with $n-1$ a-cycles of identity basis on the double cover $\bar{\Sigma}$, and this fixes the positions of these $n-1$ a-cycles of identity basis on $\bar{\Sigma}$. For remaining a-cycles on $\bar{\Sigma}$ which are not associated with boundaries or cross-caps on Σ, we take them to be a subset of the canonical homology basis on $\bar{\Sigma}$ that is not associated with any boundaries/cross-caps on Σ (for example a_1, a_2 in Figure 6.3b). So if $n=1$, e.g. Σ is a torus with a hole or a cross-cap, no a-cycle on the double covering Torus $\bar{\Sigma}$ will be associated with a boundary or cross-cap on Σ, see Figure 6.3b. A detailed illustration of the identity basis of genus-$\frac{3}{2}$ surfaces could be found in the following "Genus-$\frac{3}{2}$" section, paragraph "Identity basis" (§6.5).

All the transformations M in the modular group $Sp(2g,\mathbb{Z})$ that preserve I with

$$I' = MIM^{-1} \equiv I \tag{6.40}$$

are of the form[5]

$$M_R = \begin{pmatrix} A & 0 \\ C & (A^{-1})^\top \end{pmatrix}_{(2g\times 2g)} \tag{6.41}$$

with $A \in \mathrm{GL}(g,\mathbb{Z})$ and

$$2C = \Delta A - (A^{-1})^\top \Delta, \qquad C \in \mathbb{Z}^{g\times g}. \tag{6.42}$$

To avoid ambiguity, from now on we always call relative modular transformations M_R, and general modular transformations M. One should notice that on genus-$g>1$ surfaces, the relative modular transformation M_R may mix neighboring tori.

Genus-1

Acting involution on a torus one could get 3 topologically different surfaces: Annulus, Möbius strip and Klein bottle. The involution matrix under identity basis are:

$$I_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_M = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad I_K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.43}$$

[5]cf. app.F for more details.

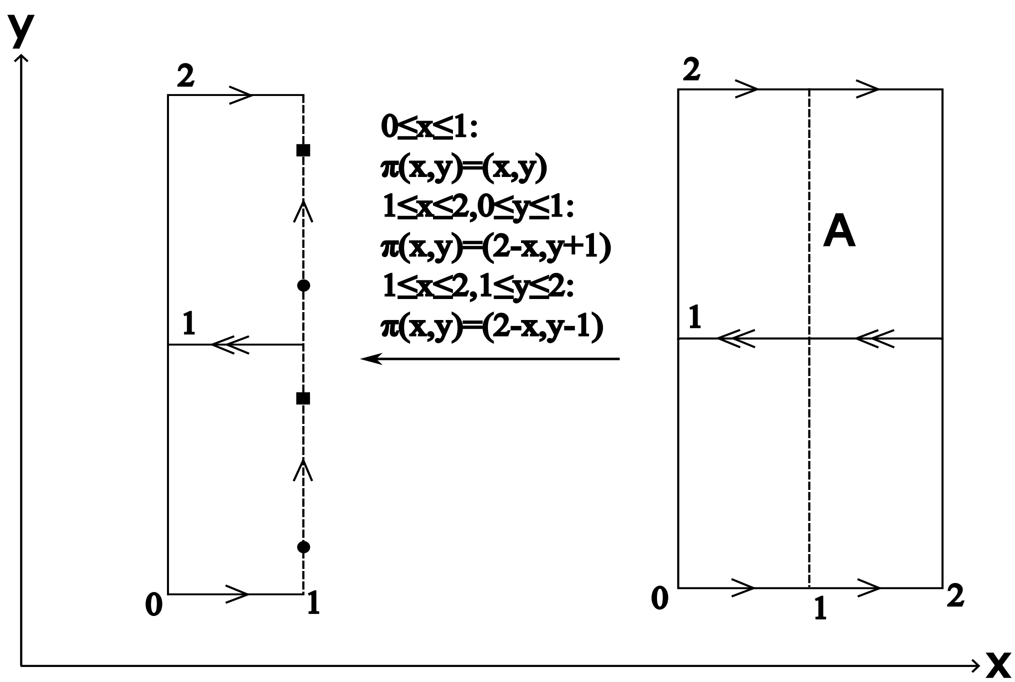


Figure 6.1: Double cover of Möbius strip
Left: Möbius strip, Right: double covering Cylinder
the dotted line A is a cycle on double covering Cylinder
π is the projecting operator from double covering Cylinder to Möbius strip

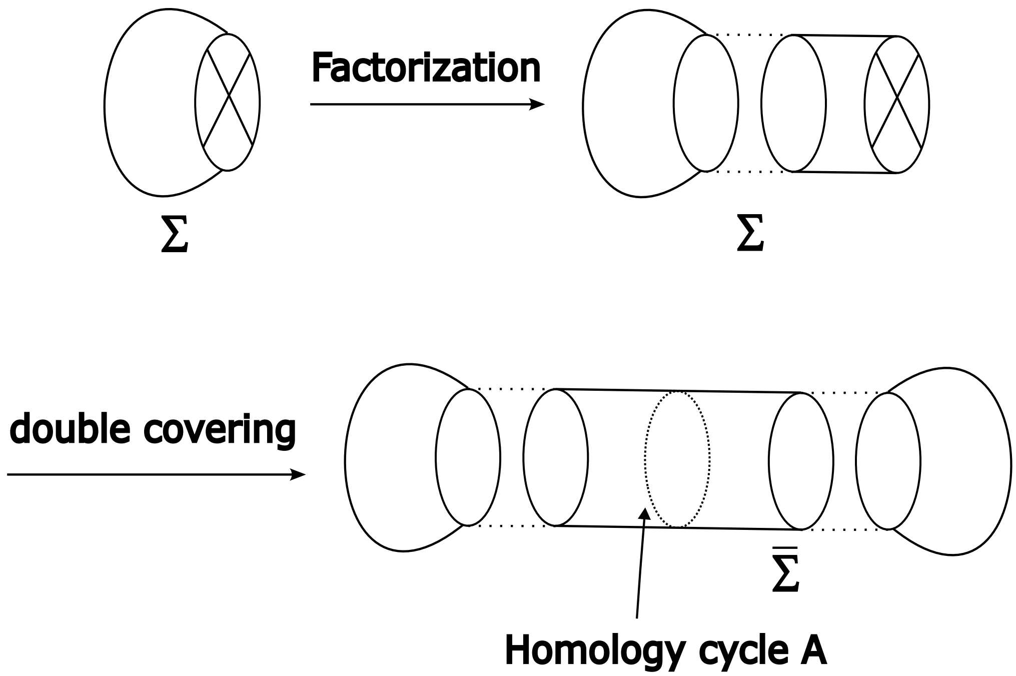


Figure 6.2: Correspondence between cross-cap and homology cycle

It is worth to mention that here the identity basis and involution matrix are treated under tree-channel, or in other word, we are representing the involution matrix by its action on identity basis. This is different from what has been done in [7] and [18]. There the involution is acting on the complex z-plane of a torus. Identity basis always means working in tree-channel.

Genus-$\frac{3}{2}$

In the case of genus-$\frac{3}{2}$ in which we are interested, there are 5 topologically different surfaces. Choosing the identity basis we get the Δ matrices for (030) the "pair of pants", (110) the "torus with a hole" and (101) the "torus with a cross-cap":[18]

$$\Delta_{(030)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Delta_{(110)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Delta_{(101)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.44}$$

where the three integers (hbc) represent the number of handles (h), boundaries (b) and cross-caps (c). The dimension of the Δ being 2 is because we always consider the homology basis to be on the double cover $\bar{\Sigma}$ (in this case, genus-2 torus) as we have discussed previously.

Identity basis We illustrate the identity basis a_1, a_2, b_1, b_2 of genus-$\frac{3}{2}$ surfaces on the double covering genus-2 torus in Figure 6.3. We split all 5 surfaces into 2 different cases.

First case is the surfaces without handle as in Figure 6.3a. Here the homology basis is identity basis and a-cycle or combination of a-cycles is always associated with a boundary or a cross-cap, and b-cycle always intersects the corresponding a-cycle.

Second case is the surfaces with handle as in Figure 6.3b. Here we take the canonical homology basis a_1, a_2, b_1, b_2 in (b). In this basis we can treat the modular transformations almost the same as the modular transformations on a single torus, thus could simplify the calculation. Identity basis of the surfaces with handle is shown in Figure 6.3c. One can see the relation between the identity basis and the canonical basis.

For (012) "Klein bottle with a hole", we have to distinguish among three positions of the boundary relative to the cross-caps, which leads to three Δ matrices:

$$\Delta^{(1)}_{(012)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Delta^{(2)}_{(012)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta^{(3)}_{(012)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{6.45}$$

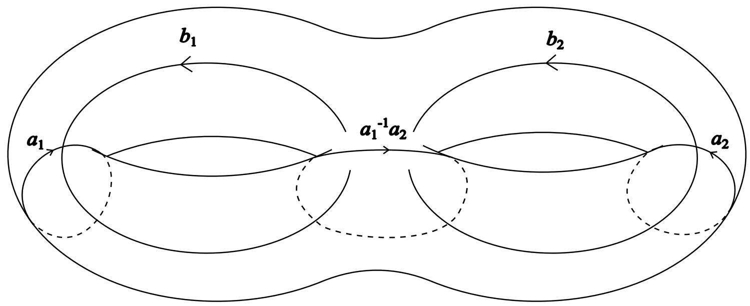
$\Delta^{(1)}_{(012)}$ could be derived from $\Delta^{(2)}_{(012)}$ by acting modular transformation M on the Involution I with the components of M being $A^{-1} = 1 + \sigma^{+}$ and $C = 0$, while $\Delta^{(3)}_{(012)}$ could be derived from $\Delta^{(1)}_{(012)}$ by acting M on I with the components of M being $A = \sigma^{1}$ and $C = 0$, where σ are pauli matrices. Be aware that these two M are not relative modular transformations since they do not preserve Δ hence do not preserve I. We have to distinguish them because Δ is chosen based on a specific choice of homology basis which is "identity basis". The two involutions $I^{(1)}$ and $I^{(3)}$ correspond to two different "identity bases" w.r.t. the different positions of the homology basis relative to the original positions of the homology basis in $\Delta^{(2)}_{(012)}$ case.

Similarly, for (021) "Möbius strip with a hole", one has to distinguish among three positions of the cross-cap relative to the holes, which leads to:

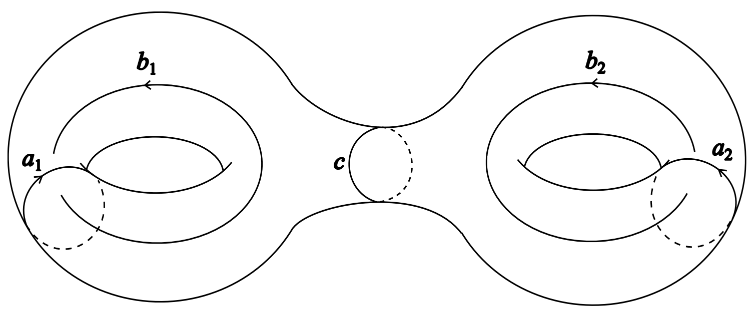
$$\Delta^{(1)}_{(021)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Delta^{(2)}_{(021)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Delta^{(3)}_{(021)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.46}$$

The rank of the relative modular group of a genus-$\frac{3}{2}$ surface is 2.[18] This can be easily seen from the form of the relative modular transformation matrix, that the relative modular transformation is only determined by upper left 2 block. The relative modular group of (012) and (021) surfaces in $\Delta^{(2)}_{(012)}$ and $\Delta^{(2)}_{(021)}$ bases are generated by the same set of two generators:

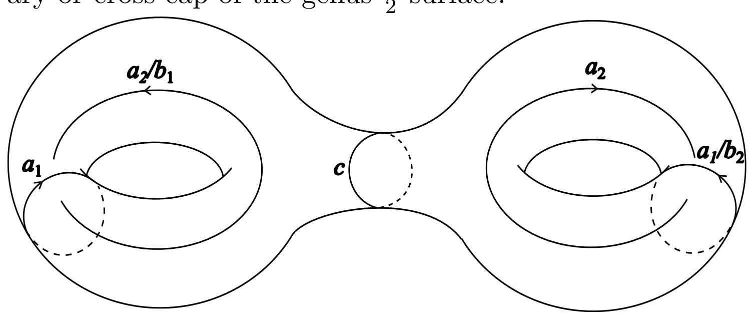
$$G^{(2)} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \qquad \text{and} \qquad G'^{(2)} = \begin{pmatrix} T^2 & 0 \\ C & (T^{-2})^\top \end{pmatrix} \tag{6.47}$$



(a) Identity Basis of genus-$\frac{3}{2}$ surfaces without handle. a_1, a_2 and $a_1^{-1}a_2$ are associated with the boundaries or cross-caps of the genus-$\frac{3}{2}$ surface.



(b) Canonical Basis of genus-$\frac{3}{2}$ surfaces with handle. c cycle is associated with the boundary or cross-cap of the genus-$\frac{3}{2}$ surface.



(c) Identity Basis of genus-$\frac{3}{2}$ surfaces with handle.

Figure 6.3: Identity Basis of genus-$\frac{3}{2}$ surfaces on the double cover

where

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{6.48}$$

is the generator of $SL(2,\mathbb{Z})$. The off-diagonal block C should be fixed by (6.42) with $\Delta^{(2)}_{(012)}$ and $\Delta^{(2)}_{(021)}$ respectively[46] as

$$C_{(012)} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_{(021)} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.49}$$

Chapter 7

Moduli Space of Genus-$\frac{3}{2}$ surfaces

Our ultimate goal regarding genus-$\frac{3}{2}$ surface would be extending the genus-1 results in app.D to genus-$\frac{3}{2}$ surfaces. However we have to understand the moduli space and fundamental domain of genus-$\frac{3}{2}$ surfaces first, and this is the major target of this chapter. Both bosonic and fermionic partition functions of genus-$\frac{3}{2}$ were studied in the former Chapter §6.

7.1 Fundamental Domain of Genus-$\frac{3}{2}$ surfaces

On the genus-2 Riemann surface, one uses period matrix of the surface to identify conformally inequivalent surfaces, or in other words, to represent moduli space. Since one can relate all the quantities (moduli spaces, determinants of scalar, vector and Dirac operators etc.) of genus-$\frac{3}{2}$ to those quantities of the double covering genus-2 surfaces, we could represent all the quantities of genus-$\frac{3}{2}$ surfaces by the quantities of the double covering genus-2 surfaces. For example, the homology basis of genus-$\frac{3}{2}$ surfaces are represented by the homology basis of the double covering genus-2 surface, and the modular transformations and involutions are 4 matrices acting on 4 homology basis (a_1, a_2, b_1, b_2) as in genus-2[1].

7.1.1 General procedure

To begin with, we need to understand how to obtain the period matrix and how to derive the moduli space from the period matrix. We are working with genus-$g = \frac{3}{2}$ surfaces.

[1]cf. Figure 3.1 for illustration of the genus-2 canonical homology basis

First Step: The period matrix $\Omega = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ compatible with the involution $I = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ satisfies

$$\bar{\Omega} = I(\Omega) = (\mathcal{D}\Omega + \mathcal{C})(\mathcal{B}\Omega + \mathcal{A})^{-1}. \tag{7.1}$$

So the involution I would restrict the form of the period matrix Ω.

Second Step: We apply relative modular transformation7.1.2 $M_R = \begin{pmatrix} A & 0 \\ C & (A^{-1})^\top \end{pmatrix}$ to the period matrix Ω and get

$$\begin{aligned} \Omega' &= M_R(\Omega) \\ &= \Big[(A^{-1})^\top \Omega + C\Big] \cdot A^{-1} \\ &= (A^{-1})^\top \Omega A^{-1} + CA^{-1}. \end{aligned} \tag{7.2}$$

From this, one can see how the relative modular transformations act on the period matrix, which encodes the moduli space on genus-$g \leq 2$ surfaces .

Third Step: After getting the 3 moduli from first step, we can then impose the action of relative modular transformations as well as the positivity of the imaginary part of the period matrix on the moduli space from second step to get the fundamental domain.

Last Step: One would have to check further possible restrictions (relevant modular transformation etc.) on the modular group, as well as to study the degeneration limits of the surfaces. These would be discussed in later section §7.1.2.

Identity Basis

In the identity basis $I = \begin{pmatrix} \mathbf{1} & 0 \\ \Delta & -\mathbf{1} \end{pmatrix}$, then

$$\bar{\Omega} = I(\Omega) = (-\Omega + \Delta) \cdot \mathbf{1} = -\Omega + \Delta \quad \Rightarrow \quad \mathrm{Re}\{\Omega\} = \frac{1}{2}\Delta. \tag{7.3}$$

One gets the constant real part of Ω, and the imaginary part of Ω is the matrix of Siegel half space moduli:

$$\mathrm{Im}\{\Omega\} = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}. \tag{7.4}$$

The positivity of the imaginary part of the period matrix requires

$$\mathbf{v}^\top \,\mathrm{Im}\{\Omega\}\mathbf{v} > 0, \qquad \mathbf{v} := (a, b)^\top \in \mathbb{R}^2$$

$$\Rightarrow t_{11}|a|^2 + t_{22}|b|^2 + t_{12}(ab + ba) \geq 0$$
$$\Rightarrow \begin{cases} t_{11} \geq 0 \\ t_{22} \geq 0 \\ t_{11}t_{22} \geq t_{12}^2 \end{cases}. \tag{7.5}$$

We will discuss (012) and (021) (handles, boundaries, cross-caps) surfaces first. Other 3 surfaces (030), (110) and (101) will be discussed later. And we know that the generators for these two cases are in the same form (cf. generators (6.47) and [19, p.391]):

$$G_{(021)} = G_{(012)} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad G'_{(021)} = G'_{(012)} = \begin{pmatrix} T^2 & 0 \\ C & (T^{-2})^\top \end{pmatrix}. \tag{7.6}$$

So it's easy to calculate Ω' based on Ω.

Genus-1 Genus-1 surfaces are simple examples. Applying the above procedure and using (6.43)

$$\Delta_A = \Delta_K = 0, \Delta_M = 1, \tag{7.7}$$

defining $\tau = r + it$ ($r, t \in \mathbb{R}$) as the period, we derive

$$\bar{\tau} = -\tau + \Delta \to \tau = \frac{\Delta}{2} + it \tag{7.8}$$

and the moduli of Annulus, Möbius strip and Klein bottle are

$$\tau_A = \tau_K = it, \qquad \tau_M = \frac{1}{2} + it. \tag{7.9}$$

There is no modular transformation on these genus-1 surfaces up to the sign, thus the moduli space of these surfaces is just $t \in \mathbb{R}^+$.

7.1.2 Relevant Modular Transformation

In [5], the authors mentioned that one should only consider the "Relevant Modular Transformations" (cf. [5, (5.3)]) which are the relative modular transformations that act at most as permutations of the boundaries and cross-caps.

We would like to emphasize that this is a crucial point if one wants to analyze the moduli space of string theory. This is because, in String Theory, the moduli space is defined by modding out the diffeomorphisms which are not connected to identity. Thus one knows that the modular transformations as diffeomorphisms should not change the topology of the manifold (in our case, Riemann Surfaces), which implies that a single connected component is always mapped by diffeomorphisms to a single connected component, and boundaries/cross-caps should always be mapped to boundaries/cross-caps respectively.[2] This leads to the "relevant modular transformation" condition that the relevant modular transformations should only at most permute the boundaries and cross-caps, and a single boundary/cross-cap can never be mapped to more than 1 boundary/cross-cap, and the number of boundaries/cross-caps should not be changed respectively. Since in identity basis, homology a-cycles on the double cover $\bar{\Sigma}$ are associated with boundaries/cross-caps on the surface Σ, homology a-cycles on $\bar{\Sigma}$ should also comply with the relevant modular transformation condition as well as relative modular transformations, in the way that homology a-cycles should at most be permuted and will never be mixed under relevant modular transformations.

This will imply a strong restriction to the modular transformations we are looking for on genus-$\frac{3}{2}$ surfaces. (030) surface is already discussed in [5]. A simple observation directly shows that $G'_{(012)}$, which is one of the two generators of (012), will be ruled out by the relevant modular transformation condition. This is due to the fact that the element T^2 in $G'_{(012)}$ will act on the a-cycles of the identity basis as

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1^2 a_2 \end{pmatrix}. \tag{7.10}$$

We see that $a_1^2 a_2$ is no longer a homology cycle around a boundary or cross-cap, neither a connected component of the surface. Besides, we would like to point out that the generator $G_{(012)}$ plus the two modular transformations $M_{(2)\to(1)}$ and $M_{(1)\to(3)}$[3] would already be enough to generate the permutation group of the boundaries and cross-caps. Thus we don't have to worry about any other generator. One has the same argument for (021).

[2]When we consider the cross-caps on an unoriented surface, its modular transformations are always described by the relative modular transformations on the oriented double cover of the unoriented surface. The unoriented surface could effectively be obtained by cutting the double cover along homology cycles, and glue cross-caps onto it. Thus those relative modular transformations are acting on the homology cycles on the double cover, which should also be mapped into a connected component rather than a mixture of homology cycles. A cross-cap can only be glued to a single connected component.

[3]cf. (7.26) and (7.31)

7.1.3 Moduli Space of (012) and (021) surfaces

We pay attention to the $\Delta^{(2)}$ matrices first, and prove the results of the period matrices of $\Delta^{(1)}$ and $\Delta^{(3)}$ matrices later in this section.

Delta Matrix $\Delta^{(2)}_{(012)}$

We have

$$\Delta^{(2)}_{(012)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.11}$$

And we know from [19, (9)] that C in (7.6) is the Pauli matrix σ_1. Following the first step in §7.1.1 we get

$$\bar{\Omega} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{d} \end{pmatrix} = -\Omega + \Delta^{(2)}_{(012)} = -\begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-a & -b \\ -b & 1-d \end{pmatrix}. \tag{7.12}$$

Thus we find $\Omega = \begin{pmatrix} \frac{1}{2} + it_{11} & it_{12} \\ it_{12} & \frac{1}{2} + it_{22} \end{pmatrix}$. Then following the second step in §7.1.1 we get the action of the relative modular transformations

$$\begin{aligned}
G_{(012)} : \quad & \begin{pmatrix} \frac{1}{2} + it'_{11} & it'_{12} \\ it'_{12} & \frac{1}{2} + it'_{22} \end{pmatrix} \\
&= G_{(012)}(\Omega) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + it_{11} & it_{12} \\ it_{12} & \frac{1}{2} + it_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} + it_{22} & it_{12} \\ it_{12} & \frac{1}{2} + it_{11} \end{pmatrix} \qquad (7.13) \\
&\Rightarrow G_{(012)} : \quad \begin{cases} t_{11} \to t'_{11} = t_{22} \\ t_{22} \to t'_{22} = t_{11}, \\ t_{12} \to t'_{12} = t_{12} \end{cases} \qquad (7.14)
\end{aligned}$$

and

$$\begin{aligned}
G'_{(012)} : \quad & \begin{pmatrix} \frac{1}{2} + it'_{11} & it'_{12} \\ it'_{12} & \frac{1}{2} + it'_{22} \end{pmatrix} \\
&= G'_{(012)}(\Omega) \\
&= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + it_{11} & it_{12} \\ it_{12} & \frac{1}{2} + it_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \frac{1}{2} + it_{11} - 4it_{12} + 4it_{22} & it_{12} - 2it_{22} \\ it_{12} - 2it_{22} & \frac{1}{2} + it_{22} \end{pmatrix} \tag{7.15}$$

$$\Rightarrow G'_{(012)} : \quad \begin{cases} t_{11} \to t'_{11} = t_{11} - 4t_{12} + 4t_{22} \\ t_{22} \to t'_{22} = t_{22} \\ t_{12} \to t'_{12} = t_{12} - 2t_{22} \end{cases}, \tag{7.16}$$

which presents how the two generators $G_{(012)}$ and $G'_{(012)}$ act on the moduli.

$G'_{(012)}$ is not a relevant modular transformation7.1.2, so we just get rid of it. Applying $G_{(012)} : t_{11} \leftrightarrow t_{22}$ as well as the positivity of the imaginary part of the period matrix (7.5), one could choose the fundamental domain: (cf. [5])

$$0 \le t_{11} \le t_{22} \le \infty \qquad \text{and} \qquad t_{12}^2 \le t_{11}t_{22}. \tag{7.17}$$

Delta Matrix $\Delta^{(2)}_{(021)}$

We have

$$\Delta^{(2)}_{(021)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{7.18}$$

But now we need to fix the off diagonal block C of the generators by using

$$2C = \Delta A - (A^{-1})^\top \Delta, \tag{7.19}$$

and then we can get

$$G_{(021)} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \tag{7.20}$$

$$G'_{(021)} = \begin{pmatrix} T^2 & 0 \\ C & (T^{-2})^\top \end{pmatrix} \qquad \text{where} \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.21}$$

Following the first step in §7.1.1 we get

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{d} \end{pmatrix} = -\begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-a & 1-b \\ 1-b & 1-d \end{pmatrix}. \tag{7.22}$$

Thus we find $\Omega = \begin{pmatrix} \frac{1}{2} + it_{11} & \frac{1}{2} + it_{12} \\ \frac{1}{2} + it_{12} & \frac{1}{2} + it_{22} \end{pmatrix}$. Then following the second step in §7.1.1 we get the action of the relevant modular transformation (Here we have already applied the relevant modular transformation condition and have got rid of $G'_{(021)}$, $G'_{(021)}$ is again not relevant modular transformation.)

$$G_{(021)} : \quad \begin{pmatrix} \frac{1}{2} + it'_{11} & \frac{1}{2} + it'_{12} \\ \frac{1}{2} + it'_{12} & \frac{1}{2} + it'_{22} \end{pmatrix} \quad = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + it_{11} & \frac{1}{2} + it_{12} \\ \frac{1}{2} + it_{12} & \frac{1}{2} + it_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+it_{22} & \frac{1}{2}+it_{12} \\ \frac{1}{2}+it_{12} & \frac{1}{2}+it_{11} \end{pmatrix} \tag{7.23}$$

$$\Rightarrow G_{(021)}: \quad t_{11} \leftrightarrow t_{22}. \tag{7.24}$$

Exactly as for (012), using the positivity of the imaginary part of the period matrix and the relevant modular transformation $G_{(021)}$, one can choose the fundamental domain: (cf. [5])

$$0 \le t_{11} \le t_{22} \le \infty \qquad \text{and} \qquad t_{12}^2 \le t_{11}t_{22}. \tag{7.25}$$

Other Δ Matrices

We know that there are four more Δ matrices for (021) and (012) cases, each has two: $\Delta^{(1)}$ and $\Delta^{(3)}$. Here we are going to prove that the extra Δ matrices will have the same moduli and modular transformations as the $\Delta^{(2)}$ case.

We begin with $\Delta^{(1)}$. The transformation of Involution matrix I in which we take $A^{-1} = 1+\sigma^+$ of a $Sp(4,\mathbb{Z})$ transformation $M = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^\top \end{pmatrix}$ (cf. [18])

$$I^{(1)} = M_{(2)\to(1)} I^{(2)} M^{-1}_{(2)\to(1)}, \qquad M_{(2)\to(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{7.26}$$

brings $\Delta^{(2)}$ to $\Delta^{(1)}$. We will use M instead of $M_{(2)\to(1)}$ or $M_{(1)\to(3)}$ in the following when there is no ambiguity. However, one can actually consider here the M transformation as a change of coordinate (homology basis). From our notation, we have the homology basis in the vector form:

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = v^{(2)}. \tag{7.27}$$

But one should notice that after the M transformation, the basis vector $v^{(2)}$, the period matrix Ω and the relative modular group $G_{(012)}$ and $G'_{(012)}$ are also transformed like $v^{(2)} \to v^{(1)} = Mv^{(2)}$, $\Omega^{(2)} \to \Omega^{(1)} = M(\Omega^{(2)})$ and $G^{(2)} \to G^{(1)} = MG^{(2)}M^{-1}$. Thus following the standard procedure in §7.1.1, we find that two key equations are:[4]

$$\overline{\Omega}^{(1)} = I^{(1)}(\Omega^{(1)}) = MI^{(2)}M^{-1}(M(\Omega^{(2)})) = MI^{(2)}(\Omega^{(2)}) = M(\overline{\Omega}^{(2)})$$

[4]We need to mention that these equations are only valid for M in the block diagonal form as is the case in (7.26). Thus this proof does not hold for general surfaces.

$$\Rightarrow \overline{\Omega}^{(1)} = M(\overline{\Omega}^{(2)}), \tag{7.28}$$

$$\Omega'^{(1)} = G^{(1)}(\Omega^{(1)}) = MG^{(2)}M^{-1}(M(\Omega^{(2)})) = MG^{(2)}(\Omega^{(2)}) \tag{7.29}$$

$$\Rightarrow \Omega'^{(1)} = M(\Omega'^{(2)}). \tag{7.30}$$

Therefore, we can easily see that for $\Delta^{(1)}$, the above two equations show that the action of the relative modular transformations $G^{(1)}$ on the moduli of $\Omega^{(1)}$ is exactly the same as that of $\Delta^{(2)}$. Then we can safely say that $\Delta^{(1)}$ has the same fundamental domain as $\Delta^{(2)}$ up to the modular transformation $M_{(2)\to(1)}$.

Following the same argument for $\Delta^{(3)}$ with

$$M_{(1)\to(3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{7.31}$$

we claim that $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ have the same fundamental domain up to the modular transformations $M_{(2)\to(1)}$ and $M_{(1)\to(3)}$.

7.1.4 Moduli Space of (030) surface

Restricting the (030) fundamental domain by fixed points

Due to the property of the modular transformations of (030) surface as shown in [5][5]:

$$(G)^2 = \mathbf{1}, \quad (G')^2 = \mathbf{1} \quad \text{and} \quad (GG')^3 = \mathbf{1}, \tag{7.32}$$

where

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.33}$$

and

$$G' = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \tag{7.34}$$

are the two generators of the relevant modular group, and the relevant modular transformations have fixed point sets. The boundary of fundamental domain should be the union set of the fixed points of all relevant modular transformations plus other restrictions. But thanks to the nilpotent property (7.32), the number of different

[5] We have used a different notation of generators than [5], in order to be compatible with other generators in this thesis.

relevant modular transformations is limited. We have to find all the fixed points of all relevant modular transformations G, G', GG', $GG'GG'$, $G'G$, $G'GG'G$, $G'GG'$, $GG'G$ and so on. Here one observes that:

(1) G only exchanges t_{11} and t_{22}, and the fixed point set of GM, where M is any modular transformation except G, will be the intersection of the fixed point sets of G and M;

(2) Also because G exchanges t_{11} and t_{22}, the fixed points of MG will be the intersection of the fixed point sets of M and G with t_{11} and t_{22} exchanged;

(3) Since $GG'GG' = (GG')^{-1}$, $GG'GG'$ will have the same fixed point set as GG', this argument also holds for $G'GG'G$.

Using these properties, we find that the fixed point set of all these modular transformations should be the intersection of the fixed point sets of G, G' and $GG'G$. The fixed point sets of all other modular transformations will be the subset of the fixed point set of these three modular transformations.

The fixed points of G satisfy the condition

$$t'_{11} = t_{22} \stackrel{!}{=} t_{11} \quad \Rightarrow \quad t_{11} = t_{22}. \tag{7.35}$$

The fixed points of G' satisfy the condition

$$t'_{12} = -t_{12} - t_{22} \stackrel{!}{=} t_{12} \quad \Rightarrow \quad t_{12} = -\frac{1}{2}t_{22}. \tag{7.36}$$

The fixed points of $GG'G$ satisfy the condition

$$t'_{12} = -t_{12} - t_{11} \stackrel{!}{=} t_{12} \quad \Rightarrow \quad t_{12} = -\frac{1}{2}t_{11}. \tag{7.37}$$

Combining all these conditions, choosing $t_{11} \leq t_{22}$ according to (7.35), and choosing $-\frac{1}{2}t_{11} \leq t_{12}$, also including the positive definiteness of imaginary part of period matrix, we obtain the fundamental domain:

$$-\frac{1}{2}t_{11} \leq t_{12} \leq \sqrt{t_{11}t_{22}}, \quad 0 \leq t_{11} \leq t_{22} \leq \infty. \tag{7.38}$$

The above fundamental domain is different from [5, (5.5)]

$$-\sqrt{t_{11}t_{22}} \leq t_{12} \leq 0 \leq t_{11} \leq t_{22} < \infty. \tag{7.39}$$

But we observe that apply G' and $G \circ G'$ on any point $t = (t_{11}, t_{12}, t_{22})^\top$ in (7.39), one of the two transformed points

$$t'_1 = G'(t) = \begin{pmatrix} t_{11} + t_{22} + 2t_{12} \\ -t_{12} - t_{22} \\ t_{22}, \end{pmatrix} \quad \text{or} \quad t'_2 = G \circ G'(t) = \begin{pmatrix} t_{22} \\ -t_{12} - t_{22} \\ t_{11} + t_{22} + 2t_{12} \end{pmatrix} \tag{7.40}$$

will still be in (7.39). This means that (7.39) was incorrect.

7.1.5 Moduli Spaces of (110) and (101) surfaces

According to §6.5 and Figure 6.3, we know that (110) and (101) surfaces have the same homology basis as well as the same involution, such that they have the same moduli space. It is easy to see that (a_1, b_1) and (a_2, b_2) in Figure 6.3b each form a canonical homology basis of a torus. One could see that the relative modular transformations on the double cover of (110) or (101) consist of the double copy of the modular transformations on the torus of (110) or (101), which acts simultaneously on (a_1, b_1) and (a_2, b_2^{-1}) and are image of each other under involution. Also we see that the canonical homology basis does not involve the boundary/cross-cap of the surface, thus would not impose the relevant modular transformation condition.

One could relate identity basis to canonical basis. However, on genus-$\frac{3}{2}$ surfaces with a handle, identity basis introduces extra difficulties on deriving the fundamental domain and stops one from making use of the knowledge of the modular group of torus. From the above observation, one realizes that the canonical basis could be more useful since the modular transformations are the same as those on torus. This is already studied in [10]. So in this section we consider the canonical basis as in Figure 6.3b. In the canonical basis, the involution represented in matrix is

$$I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{7.41}$$

and the period matrix satisfies

$$\bar{\Omega} = I(\Omega) = -\sigma^1 \Omega \sigma^1. \tag{7.42}$$

One derives

$$\Omega = \begin{pmatrix} t_1 + it_2 & -it_{12} \\ -it_{12} & -t_1 + it_2 \end{pmatrix}. \tag{7.43}$$

We rename the moduli as

$$\Omega = \begin{pmatrix} \tau & -il \\ -il & -\bar{\tau} \end{pmatrix}. \tag{7.44}$$

to make it consistent with [10].

We already know that the two generators of the modular group of a torus are S and T transformations, and S interchanges a with b while T shifts b by a[10]. Extending the two generators to double torus and representing them in matrix notation and preserving symplectic form and involution, one obtains[6]

$$S\vec{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1' = b_1 \\ a_2' = b_2^{-1} \\ b_1' = a_1^{-1} \\ b_2' = a_2 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{7.45}$$

and

$$T\vec{V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1' = a_1 \\ a_2' = a_2 \\ b_1' = a_1 + b_1 \\ b_2' = b_2 + a_2^{-1} \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{7.46}$$

The actions of the two generators on the moduli are

$$\begin{aligned} S: \quad & \begin{pmatrix} \tau' & -il' \\ -il' & -\bar{\tau}' \end{pmatrix} \\ &= S(\Omega) \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & -il \\ -il & -\bar{\tau} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\bar{\tau}}{l^2-|\tau|^2} & \frac{il}{l^2-|\tau|^2} \\ \frac{il}{l^2-|\tau|^2} & \frac{-\tau}{l^2-|\tau|^2} \end{pmatrix} \end{aligned} \tag{7.47}$$
$$\Rightarrow S: \quad \begin{cases} \tau \to \frac{\bar{\tau}}{l^2-|\tau|^2} \\ l \to -\frac{l}{l^2-|\tau|^2}, \end{cases} \tag{7.48}$$

and

$$T: \quad \begin{pmatrix} \tau' & -il' \\ -il & -\bar{\tau}' \end{pmatrix}$$

[6] We used a different S transformation than [10] to preserve the symplectic form of the basis.

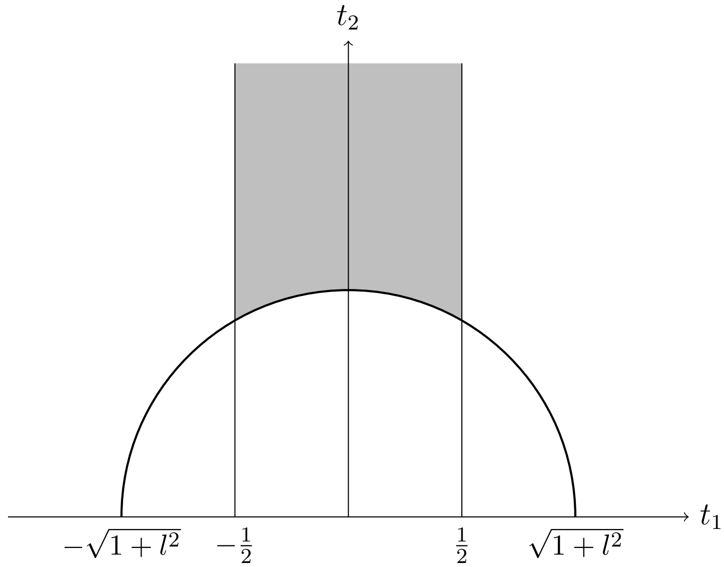


Figure 7.1: Fundamental Domain of (012) and (021)

$$
\begin{aligned}
&= T(\Omega) \\
&= \mathbb{1}\begin{pmatrix} \tau & -il \\ -il & -\bar{\tau} \end{pmatrix}\mathbb{1} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \tau+1 & -il \\ -il & -\tau \bar{+} 1 \end{pmatrix}
\end{aligned} \tag{7.49}
$$

$$
\Rightarrow T: \quad \begin{cases} \tau \to \tau + 1 \\ l \to l \end{cases}. \tag{7.50}
$$

Positive definiteness of the period matrix implies that $t_2 > 0$ and $-\det\Omega = |\tau|^2 - l^2 > 0$. Using (7.48) and (7.50), one can derive the fundamental domain as shown in Figure 7.1. The fundamental domain consists of $-\frac{1}{2} < t_1 < \frac{1}{2}$ and $|t_1 + it_2|^2 > 1 + l^2$[10]. One observes that t_{12} represents the length of the dividing geodesics c in Figure 6.3b[10].

Involution implies a further restriction that $t_{12} > 0$ by using Riemann bilinear relation to relate t_{12} with $\mathrm{Im}\{\Omega\}$ under involution and the positive definiteness of the period matrix, details could be found in [10].

7.2 Discussion

We discussed the concept of "Relevant Modular Transformation"[5] in detail. We applied this concept to extend former research on the moduli spaces of genus-$\frac{3}{2}$ surfaces, and we found the moduli spaces of all genus-$\frac{3}{2}$ surfaces. We pointed out that the fundamental domain found in [5, (5.5)] was incorrect. With the knowledge of determinants from Chapter §6 and moduli spaces from this Chapter, we should be able to derive genus-$\frac{3}{2}$ amplitudes from genus-2 amplitudes. It is then possible to further perform calculations of genus-$\frac{3}{2}$ corrections. However the integration over the moduli spaces could be highly non-trivial. A numerical result as in [54] may involve extra effort.

Chapter 8

Conclusion

This thesis extends perturbative corrections to Einstein-Hilbert term in Type-IIB orientifolds in new directions and highlights unresolved challenges.

We revisited the Heterotic genus-1 3-graviton amplitude. We generalized the calculation to include all four genus-1 surfaces in Type-I theory, incorporating pinched-off contributions. These new 1-loop corrections break the previously expected gravitational kinematic structure, indicating missing contributions in the calculation. A reassessment of picture number involves "vertical integration" technique, and it may introduce potential new contributions to amplitudes.

We also studied genus-$\frac{3}{2}$ surfaces. We determine the moduli spaces of genus-$\frac{3}{2}$ surface via relevant modular transformations, correcting earlier results and setting the stage for computing genus-$\frac{3}{2}$ amplitudes.

Overall, this work deepens our understanding of quantum corrections to Einstein-Hilbert term in Type-IIB orientifolds at 1 and higher genus. This could further improve understanding of the low energy effective action in string theory. The findings emphasize the importance of vertical integration and modular analysis in string perturbation theory. Future work will focus on: 1. implementing the vertical-integration technique in genus-1 3-point amplitudes calculation; 2. deriving the explicit form of the involution acting on the coordinates, getting genus-$\frac{3}{2}$ Green's function, performing the moduli and coordinates integrals for genus-$\frac{3}{2}$ amplitudes, and completing the genus-$\frac{3}{2}$ 2-point calculation.

Part III

Appendices

Appendix A

Useful formula[1]

Abbreviations

$$\epsilon_{i,\mu\nu} \equiv \epsilon_{\mu\nu}(k_i), \quad X_i \equiv X^\mu(z_i, \bar{z}_i), \quad z_{ij} = z_i - z_j, \quad q = e^{2\pi i\tau}. \tag{A.1}$$

ϑ functions

$$\vartheta[^{\vec{\alpha}}_{\vec{\beta}}](\vec{\nu}, G) = \sum_{\vec{n}\in\mathbb{Z}^N} e^{i\pi(\vec{n}+\vec{\alpha})^\top G(\vec{n}+\vec{\alpha})} e^{2\pi i(\vec{\nu}+\vec{\beta})^\top(\vec{n}+\vec{\alpha})}, \tag{A.2}$$

$$\vartheta_1 = -\vartheta[^{\frac{1}{2}}_{\frac{1}{2}}](\nu,\tau) = 2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n), \tag{A.3a}$$

$$\vartheta_2 = \vartheta[^{\frac{1}{2}}_{0}](\nu,\tau) = 2e^{\pi i\tau/4}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^n), \tag{A.3b}$$

$$\vartheta_3 = \vartheta[^{0}_{0}](\nu,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}}), \tag{A.3c}$$

$$\vartheta_4 = \vartheta[^{0}_{\frac{1}{2}}](\nu,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-zq^{n-\frac{1}{2}})(1-z^{-1}q^{n-\frac{1}{2}}), \tag{A.3d}$$

where $z = e^{2\pi i\nu}$.

η function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \left[\frac{\partial_\nu\vartheta_1(0,\tau)}{-2\pi}\right]^{\frac{1}{3}}, \tag{A.4}$$

[1] Partly follow [23] and [47].

and

$$\frac{\vartheta[^{\alpha}_{\beta}](0,\tau)}{\eta(\tau)} = e^{2\pi i\alpha\beta} q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta})(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta}). \tag{A.5}$$

Poisson re-summation

$$\vartheta[^{\vec{0}}_{\vec{0}}](0, itG^{-1}) = \sqrt{G}t^{-N/2}\vartheta[^{\vec{0}}_{\vec{0}}](0, it^{-1}G) \tag{A.6}$$

Modular transformation S for annulus and Klein bottle

$$\vartheta[^{\alpha}_{\beta}](\nu,\tau) = (-i\tau)^{-1/2} e^{2\pi i\alpha\beta - \pi i\nu^2/\tau}\vartheta[^{-\beta}_{\ \alpha}](\nu/\tau, -1/\tau). \tag{A.7}$$

Modular transformation ST^2S for Möbius

$$\vartheta[^{\alpha}_{\beta}](\nu,\tau) = (1-2\tau)^{-1/2} e^{2\pi i\beta^2} e^{-\pi i\nu^2/(\tau-1/2)}\vartheta[^{\alpha+2\beta}_{\ \ \beta}](\frac{\nu}{1-2\tau}, \frac{\tau}{1-2\tau}). \tag{A.8}$$

General Modular transformation S and T for ϑ-functions and η-function

$$\vartheta[^{\alpha}_{\beta}](\tau+1) = e^{-\pi i(\alpha^2-\alpha)}\vartheta[_{\alpha+\beta-\frac{1}{2}}^{\ \ \alpha}](\tau), \tag{A.9a}$$

$$\vartheta[^{\alpha}_{\beta}](-\frac{1}{\tau}) = \sqrt{-i\tau}e^{2\pi i\alpha\beta}\vartheta[^{-\beta}_{\ \alpha}](\tau) \qquad |\arg\sqrt{-i\tau}| < \frac{\pi}{2}, \tag{A.9b}$$

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \tag{A.9c}$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau). \tag{A.9d}$$

Shifts in characteristics

$$\begin{aligned} \vartheta[^{\alpha+1}_{\ \ \beta}](\nu,\tau) &= \vartheta[^{\alpha}_{\beta}](\nu,\tau), \\ \vartheta[_{\beta+1}^{\ \ \alpha}](\nu,\tau) &= e^{2\pi i\alpha}\vartheta[^{\alpha}_{\beta}](\nu,\tau). \end{aligned} \tag{A.10}$$

ν-periodicity formula

$$\vartheta[^{\alpha}_{\beta}](\nu + a\tau + b, \tau) = e^{-2\pi iab}e^{-\pi ia^2\tau}e^{-2\pi ia(\nu+b)}\vartheta[^{\alpha+a}_{\beta+b}](\nu,\tau). \tag{A.11}$$

Gravity

$$\begin{aligned}
\sqrt{g}R\big|_{h^3} &= h_{\mu\nu}h_{\rho\sigma}\partial^{\mu\nu}h^{\rho\sigma} + 2h^{\rho\sigma}\partial_\sigma h^{\mu\nu}\partial_\mu h_{\nu\rho} \\
&\to (k_2\cdot\epsilon_1\cdot k_2)(\epsilon_2\cdot\epsilon_3) + 2(k_3\cdot\epsilon_2\cdot\epsilon_3\cdot\epsilon_1\cdot k_2) + \text{cyclic perms}, && \text{(A.12a)} \\
t^{\mu_1\mu_2\mu_3} &= \eta^{\mu_1\mu_2}k_3^{\mu_2} + \eta^{\mu_2\mu_3}k_3^{\mu_1} + \eta^{\mu_3\mu_1}k_1^{\mu_2}, && \text{(A.12b)} \\
T^{\mu_1\mu_2\mu_3} &= t^{\mu_1\mu_2\mu_3} + \frac{\alpha'}{2}k_3^{\nu_1}k_1^{\nu_2}k_2^{\nu_3}. && \text{(A.12c)}
\end{aligned}$$

Involution One-loop surfaces $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{K}$ can be defined as quotients of tori under different involutions[7, (A.1)]

$$I_{\mathcal{A}}(z) = I_{\mathcal{M}}(z) = 1 - \bar{z}, \qquad I_{\mathcal{K}}(z) = 1 - \bar{z} + \frac{\tau}{2}, \tag{A.13}$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter of the defining torus.

$\mathbb{Z}_N$ **actions in** $D = 4$ In the table are the twist vectors for different $\mathbb{Z}_N$ orbifold Type-IIB string models on T^6 ($D = 4$ space-time dimensions with 6 compact dimensions).

$\mathbb{Z}_3$	$\frac{1}{3}(1,1,2)$	$\mathbb{Z}_6'$	$\frac{1}{6}(1,-3,2)$	$\mathbb{Z}_8'$	$\frac{1}{8}(1,-3,2)$
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$	$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$	$\mathbb{Z}_{12}$	$\frac{1}{12}(1,-5,4)$
$\mathbb{Z}_6$	$\frac{1}{6}(1,1,-2)$	$\mathbb{Z}_8$	$\frac{1}{8}(1,3,-4)$	$\mathbb{Z}_{12}'$	$\frac{1}{12}(1,5,-6)$

Cited from [2, Table 2]. Only $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_6'$, $\mathbb{Z}_7$, $\mathbb{Z}_{12}$ models are tadpole-free, which is discussed in [2].

Appendix B

OPE and CFT correlation functions

It would be helpful to show the relation between the OPE and correlation functions.

correlation functions on genus-g surfaces

$$\langle X(z)X(w)\rangle = P_g(z,w), \tag{B.1a}$$

$$\langle \psi(z)\psi(w)\rangle = S_g(z,w), \tag{B.1b}$$

$$\left\langle \partial X(z) e^{ikX(w)}\right\rangle = ik\partial_z P_g(z,w) e^{ikX(w)}. \tag{B.1c}$$

The correlation functions refer to §3.2.2.

OPE and correlation functions on flat space

$$X(z)X(w) \sim -\frac{\alpha'}{2}\ln(z-w), \tag{B.2a}$$

$$\partial X(z)\partial X(w) \sim -\frac{\alpha'}{2}\cdot\frac{1}{(z-w)^2}, \tag{B.2b}$$

$$\partial X(z) e^{ik\cdot X(w)} \sim -ik\cdot\frac{\alpha'}{2}\cdot\frac{e^{ik\cdot X(w)}}{z-w}, \tag{B.2c}$$

$$e^{ik\cdot X(z)} e^{ik'\cdot X(w)} \sim \frac{e^{i(k+k')\cdot X(w)}}{(z-w)^{\frac{\alpha'}{2}k\cdot k'}}, \tag{B.2d}$$

$$\left\langle \prod_i e^{i(k_i\cdot X(z_i)+\rho_i\cdot\partial X(z_i))}\right\rangle = \exp\left(\frac{\alpha'}{2}\sum_{i<j}\frac{\rho_i\cdot\rho_j}{(z_i-z_j)^2} + \frac{\alpha'}{2}\sum_{i\neq j}\frac{k_i\cdot\rho_j}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^{\frac{\alpha'}{2}k_i\cdot k_j}, \tag{B.2e}$$

$$e^{q\phi(z)}e^{q'\phi(w)} \sim \frac{e^{(q+q')\phi(w)}}{(z-w)^{qq'}}, \tag{B.2f}$$

$$\psi(z)\psi(w) \sim \frac{1}{z-w}. \tag{B.2g}$$

Torus correlation functions On torus we have:

$$\langle \partial X(z)\partial X(w)\rangle = \partial_z\partial_w P_{\mathcal{T}}(z,w), \tag{B.3a}$$

$$\langle \partial X(z)\bar{\partial} X(w)\rangle = \partial_z\partial_{\bar{w}} P_{\mathcal{T}}(z,w) = -\frac{\alpha'}{2}\cdot\frac{\pi}{\tau_2}. \tag{B.3b}$$

Appendix C

Orientifold Ω symmetry

There are two distinct orientifold groups possible:

$$Y_N = \{1, \Omega, \theta^k, \Omega_k\}, \quad k = 1, 2, \ldots, N, \quad \theta^k \equiv e^{2\pi ik/N}, \quad \Omega_k \equiv e^{2\pi ik/N}\Omega \tag{C.1}$$

and

$$W_N = \{1, \theta^{2k-2}, \Omega_{2k-1}\}, \quad k = 1, 2, \ldots, \frac{N}{2}, \quad N\,\text{even}. \tag{C.2}$$

Ω action and CP factors All conventions follow [2, §2]. We now elaborate the action of the orientifold groups on the states in the open string sector, on D-branes. A generic state can be written as $\lambda_{ij}|X, ij\rangle$ where i, j label the end points of the open strings, λ is a CP matrix, and X collectively labels the world-sheet oscillators that are involved in that state.

The orientifold elements have two possible actions on a generic D-brane state. In addition to the obvious action on the oscillator states, they also act on the CP indices with a matrix representation of the orientifold group. It is generated via matrices γ_θ

$$\theta^k : |X, ij\rangle \to \epsilon_k(\gamma_k)_{ii'}|\theta^k \cdot X, i'j'\rangle(\gamma_k^{-1})_{j'j}, \tag{C.3}$$

$$\Omega_k : |X, ij\rangle \to \epsilon_{\Omega_k}(\gamma_{\Omega_k})_{ii'}|\theta^k \cdot X, j'i'\rangle(\gamma_{\Omega_k}^{-1})_{j'j}, \tag{C.4}$$

where $\epsilon_k, \epsilon_{\Omega_k}$ are signs. Note that the Ω_k elements interchange also the string end points. The group property $\theta^k = (\theta_1)^k$ and $\theta_N = 1$ implies

$$\gamma_k = \pm(\gamma_1)^k, \quad (\gamma_k)^N = \pm 1. \tag{C.5}$$

Furthermore, the condition that Ω^2

$$\Omega^2 : |X, ij\rangle \to \epsilon_\Omega^2(\gamma_\Omega(\gamma_\Omega^\top)^{-1})_{ii'}|X, i'j'\rangle(\gamma_\Omega^\top\gamma_\Omega^{-1})_{j'j}, \tag{C.6}$$

is equal to the identity requires that

$$\gamma_\Omega = \zeta \gamma_\Omega^\top, \quad \zeta^2 = 1. \tag{C.7}$$

Note that the adjoint action on the CP indices implies that the representation of the orientifold group on the CP sector is defined up to a sign.

To evaluate the trace of partition functions under Ω, we require the action of the orientation reversal on the bosonic oscillators

$$\Omega\alpha_k^\mu\Omega^{-1} = \bar{\alpha}_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = \alpha_k^\mu, \qquad \text{(Closed String)} \tag{C.8}$$

$$\Omega\alpha_k^\mu\Omega^{-1} = (-1)^k\alpha_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = (-1)^k\bar{\alpha}_k^\mu, \qquad \text{(NN boundary condition)} \tag{C.9}$$

$$\Omega\alpha_k^\mu\Omega^{-1} = (-1)^{k+1}\alpha_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = (-1)^{k+1}\bar{\alpha}_k^\mu, \qquad \text{(DD boundary condition)}, \tag{C.10}$$

and Ω also transforms ND boundary conditions to DN ones.

For the fermionic ones, we have

$$\Omega\psi_r\Omega^{-1} = \bar{\psi}_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = -\psi_r, \qquad \text{(Closed String)} \tag{C.11}$$

$$\Omega\psi_r\Omega^{-1} = (-1)^r\psi_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = (-1)^r\bar{\psi}_r, \qquad \text{(NN boundary condition)} \tag{C.12}$$

$$\Omega\psi_r\Omega^{-1} = (-1)^{r+1}\psi_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = (-1)^{r+1}\bar{\psi}_r, \qquad \text{(DD boundary condition)}. \tag{C.13}$$

The extra minus sign in (C.11) is inserted in order for the product $\psi_r\bar{\psi}_r$ to be orientation invariant. This choice does not affect the GSO-invariant states.

Moreover, we should notice that only the left-right symmetric sectors (NS-NS and R-R) survive the Ω projection.

Lattice Sum on T^D under Ω We only have the lattice sum in the case of that there is fixed tori, i.e. $\chi(\theta^m) = 0$, or equivalently, ℓv_j is integer or half-integer. Otherwise there is no windings nor momenta in the compactified dimensions. And we need to compute the traces of the lattice states, which is what we are going to do to here: Lattice Sum. We use complex torus coordinates to represent the coordinates of the compact dimensions, thus we complexify the momenta and windings

$$M_j = m_{2j-1} + im_{2j} \qquad j = 1 \ldots \frac{D}{2}, \tag{C.14}$$

$$N_j = n_{2j-1} + in_{2j} \qquad j = 1 \ldots \frac{D}{2}. \tag{C.15}$$

This is allowed because if we observe the mode expansion of X^i

$$X^i(\sigma,\tau) = x^i + \alpha' p^i \tau + LR\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\alpha_n^i e^{-in(\tau-\sigma)} + \bar{\alpha}_n^i e^{-in(\tau+\sigma)}), \qquad \text{(C.16)}$$

we see that the momenta $m_i = p_i \cdot R$ and windings $n_i = L$ follow the same θ^ℓ transformation as X^i. Therefore there will be no problem to complexify those parameters.

The orientation reversal acts on momenta and windings as

$$\Omega|M_j, N_j\rangle = |M_j, -N_j\rangle, \qquad \text{(C.17)}$$

then only momenta survive the trace when no $\mathbb{Z}_N$ element θ^ℓ is inserted

$$\langle M_j, N_j|\Omega|M_j, N_j\rangle = \prod_{i=1}^{D} \delta_{N_j,0}. \qquad \text{(C.18)}$$

On the other hand, due to (D.10) and (D.11), we can get

$$\theta^\ell|M_j, N_j\rangle = |e^{2\pi i\ell v_j} M_j, e^{2\pi i\ell v_j} N_j\rangle, \qquad \text{(C.19)}$$

we observe that the state survives the θ^ℓ action after trace only when ℓv_j is integer, because m_i and n_i have to be integers.

Furthermore,

$$\Omega\theta^\ell|M_j, N_j\rangle = |e^{2\pi i\ell v_j} M_i, e^{2\pi i(\ell v_j - \frac{1}{2})} N_i\rangle. \qquad \text{(C.20)}$$

We can easily see that the state survives the $\Omega\theta^\ell$ action after trace only when ℓv_j is integer or half-integer. However, momenta and windings will not simultaneously survive the $\Omega\theta^\ell$ action after trace. If ℓv_j is integer, then momentum survives. If ℓv_j is half-integer, then winding number survives.

[51, §4.18.5] gives the details of the calculation. The j_0 current of L_0 is changed due to the toroidal compactification, which results in a lattice sum over the internal momenta and windings, cf. [24, §4.2.2]. The general result is

$$Z_{\text{lattice}}^{\text{torus}} = \frac{\sqrt{g}}{\ell_s^2(\sqrt{\tau_2}\eta)^2} \sum_{\vec{m},\vec{n}} e^{[\pi(g_{ij}+B_{ij})/\tau_2\ell_s^2](m^i+n^i\tau)(m^j+n^j\bar{\tau})}, \qquad \text{(C.21)}$$

g_{ij} is the metric of the 2-torus in the target space, B_{ij} is antisymmetric constant background value of the two-index antisymmetric tensor over the 2-torus. We won't consider B in our calculation, thus set $B_{ij} = 0$. We define $V_j = \sqrt{g}$ to be the regularized volume of the torus. j stands for the j-th coordinate of the torus. G is the determinant of the metric g_{ij}.

Since in the following sections, momentum and winding won't simultaneously appear in the partition function. After performing a Poisson re-summation, we summarize and rewrite the momentum/winding sum along the j-th torus with volume V_j and metric $g^{[j]}_{ab}$ from (C.21) as

$$\mathcal{L}^{[j,M]} = \frac{V_j}{4\pi^2\alpha' t} \sum_{m^1,m^2} e^{-\frac{\pi}{t} m^a m^b g^{[j]}_{ab}}, \tag{C.22}$$

$$\mathcal{L}^{[j,W]} = \frac{4\pi^2\alpha'}{V_j t} \sum_{n^1,n^2} e^{-\frac{\pi}{t} n_a n_b g^{[j]ab}}. \tag{C.23}$$

These sums are expressed in the closed string channel. Details could be found in [61, (8.2.9)].

Twisted Sectors Here we need to consider the insertion of the Orientifold element Ω. We know that only left-right symmetric states will survive the Ω insertion after trace. Using the results from app.D.1.1, we can easily see that only when $s = n + kv_j$ and $t = n - kv_j$ (k is the k-th twisted sector) are the same index set, the state is left-right symmetric. This is equivalent to requiring kv_j is integer or half-integer for all j. However, this could only be possible for $k = 0$ or $\frac{N}{2}$. Then we know that for twisted sectors of Klein bottles, only the $\frac{N}{2}$-th twisted sector survives.

Appendix D

Calculation of the 1-loop partition function of Type-IIB $T^6/\mathbb{Z}_N$ orientifolds

In this appendix we mainly study the calculation of the partition function of Type-IIB $T^6/\mathbb{Z}_N$ orientifolds.[1] We decompose the partition function into bosonic and fermionic part, then further consider untwisted and twisted sectors. All the notations and main calculations follow [23]. For orbifold Γ, we have $SO(D)$ generators θ and twist vector $\vec{v}$. We always use the light-cone gauge.

D.1 1-loop partition function of Type-IIB $T^6/\mathbb{Z}_N$ orientifolds

First we directly give the general partition function for the surface σ:

$$\begin{aligned}\langle 1\text{-loop}\rangle_\sigma = Z_\sigma &= \mathrm{Tr}_{NS,R}^{U+T\,or\,D-\mathrm{branes}}[\frac{1+\Omega}{2}\cdot P\cdot\frac{1+(-1)^F}{2}e^{-2\pi i\tau H}] \\ &= \frac{V_{10-D}}{2\cdot(10-D)N(4\pi^2\alpha')^{(10-D)/2}}\int_0^\infty\frac{dt}{t^{D/2}}\sum_{k,\ell}\sum_{s=\mathrm{even}}Z_\sigma[\theta^k,\theta^\ell](\tau,s)\end{aligned} \tag{D.1}$$

[1]This section is cited from [54]

with

$$P = \frac{1}{N}\sum_{\ell=0}^{N-1}\theta^\ell, \tag{D.2}$$

$$Z_{\mathcal{A}}[\theta^\ell](\tau_{\mathcal{A}}, s) = Z_{99}[\theta^\ell](\tau_{\mathcal{A}}, s) + Z_{55}[\theta^\ell](\tau_{\mathcal{A}}, s) + Z_{95}[\theta^\ell](\tau_{\mathcal{A}}, s), \tag{D.3}$$

$$Z_{\mathcal{M}}[\theta^\ell](\tau_{\mathcal{M}}, s) = Z_9[\theta^\ell](\tau_{\mathcal{M}}, s) + Z_5[\theta^\ell](\tau_{\mathcal{M}}, s), \tag{D.4}$$

$$Z_{\mathcal{K}}[\theta^\ell](\tau_{\mathcal{A}}, s) = Z_{\text{untwisted}}[\mathbb{1}, \theta^\ell](\tau_{\mathcal{K}}, s) + \sum_{k=1}^{N-1} Z_{\text{twisted}}[\theta^k, \theta^\ell](\tau_{\mathcal{K}}, s), \tag{D.5}$$

$$Z_{\mathcal{T}}[\theta^\ell](\tau_{\mathcal{A}}, s) = Z_{\text{untwisted}}[\mathbb{1}, \theta^\ell](\tau_{\mathcal{T}}, s) + \sum_{k=1}^{N-1} Z_{\text{twisted}}[\theta^k, \theta^\ell](\tau_{\mathcal{T}}, s). \tag{D.6}$$

Here τ_σ is defined as

$$\tau_{\mathcal{T}} = it, \qquad \tau_{\mathcal{K}} = 2it, \qquad \tau_{\mathcal{A}} = \frac{it}{2}, \qquad \tau_{\mathcal{M}} = \frac{1}{2} + \frac{it}{2}. \tag{D.7}$$

$\frac{1+\Omega}{2}$ is the orientifold projection and P is the $\mathbb{Z}_N$ symmetry projection. Spin structures can be expressed in (α, β) or s. And we should also notice that there is no twisted sectors for $\mathcal{A}$ and $\mathcal{M}$, because both of the two surfaces can be considered as open string in loop-channel, thus have no twisted sector.

We are considering here 1-loop amplitudes, i.e. Euler Number $\chi = 0$ surfaces. Therefore σ should be taken to be Torus, Annulus, Klein bottle or Möbius strip. For the Torus and Klein bottle, we use U/T to label the untwisted/twisted sectors, respectively. As we know, while Annulus and Möbius strip are the propagators of the closed strings propagating between two D-branes, they are also equivalent to closed 1-loop amplitudes of open strings with end-points on the two D-branes, by closed-open duality. In this sense, we can calculate the amplitudes using open string theory. We use D-branes to label where the open strings are attached.

D.1.1 Bosonic partition function

We will compute the bosonic partition function of type II string compactified on a toroidal $\mathbb{Z}_N$ orbifold first.

Non-compact dimension

For non-compact dimension, the computation is standard. We have the partition function

$$Z = \frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}$$

for each non-compact dimension.

Compact dimension

Mode expansion We use the complexified coordinates

$$\begin{aligned} Z^j &= \frac{1}{\sqrt{2}}(X^{2j-1} + iX^{2j}) \\ Z^{*j} &= \frac{1}{\sqrt{2}}(X^{2j-1} - iX^{2j}) \end{aligned}$$

and we have:

$$\begin{aligned} \theta^\ell Z^j \theta^{-\ell} &= e^{2\pi i \ell v_j} Z^j \\ \theta^\ell Z^{*j} \theta^{-\ell} &= e^{-2\pi i \ell v_j} Z^j \end{aligned} \tag{D.8}$$

v_j is the twist vector which is determined by the crystallographical structure.
The mode expansions are

$$Z^j(\sigma^0, \sigma^1) = z_0^j + \alpha' \frac{M^j}{R}\sigma^0 + N^j R \sigma^1 + i\sqrt{\frac{\alpha'}{2}} \sum_s \frac{\alpha_s^j}{s} e^{-is(\sigma^0 - \sigma^1)} + i\sqrt{\frac{\alpha'}{2}} \sum_t \frac{\bar{\alpha}_t^j}{t} e^{-it(\sigma^0 + \sigma^1)}. \tag{D.9}$$

M^j and N^j are complexified internal momenta and winding numbers respectively. W.l.o.g. we consider the right-mover. We can find that

$$\begin{aligned} \theta^\ell \alpha_n^j \theta^{-\ell} &= e^{2\pi i \ell v_j} \alpha_n^j \\ \theta^\ell \bar{\alpha}_n^j \theta^{-\ell} &= e^{2\pi i \ell v_j} \bar{\alpha}_n^j \end{aligned} \tag{D.10}$$

for Z^j and

$$\begin{aligned} \theta^\ell \alpha_n^{*j} \theta^{-\ell} &= e^{-2\pi i \ell v_j} \alpha_n^{*j}, \\ \theta^\ell \bar{\alpha}_n^{*j} \theta^{-\ell} &= e^{-2\pi i \ell v_j} \bar{\alpha}_n^{*j} \end{aligned} \tag{D.11}$$

for Z^{*j}.

Imposing

$$Z^j(\sigma^0, \sigma^1 + 2\pi) = e^{2\pi i k v_j} Z^j(\sigma^0, \sigma^1), \tag{D.12}$$

which is valid for a complex boson in the k-th twisted sector, fixes the frequencies of the mode expansion to $s = n + kv_j$ and $t = n - kv_j$ with n integer. Furthermore, z_0^j must satisfy $(1 - e^{2\pi i k v_j}) z_0^j = 0 \quad mod \quad 2\pi\Lambda$ (Λ is the torus coordinates lattice), i.e. it must be a fixed point of the orbifold action and, therefore, states in the twisted sectors are localized at the fixed points.
For the complex conjugate Z^{*j} there is an analogous expansion with coefficients $\alpha^{*j}_{n-kv_j} = (\alpha^j_{-n+kv_j})^\dagger$ for the right-movers, $\bar{\alpha}^{*j}_{n+kv_j} = (\bar{\alpha}^j_{-n-kv_j})^\dagger$ for the left-movers and $z_0^{*j} = (z_0^j)^\dagger$ for the center-of-mass position. Canonical quantization results in the following commutator relations for the oscillators

$$\begin{aligned} [\alpha^i_{m+kv_i}, \alpha^{*j}_{n-kv_j}] &= (m + kv_i)\delta^{ij}\delta_{m+n,0}, \\ [\bar{\alpha}^i_{m-kv_i}, \bar{\alpha}^{*j}_{n+kv_j}] &= (m - kv_i)\delta^{ij}\delta_{m+n,0}. \end{aligned} \tag{D.13}$$

The creation operators are $\alpha^j_{-n+kv_j}$, $n > 0$ and $\alpha^{*j}_{-n-kv_j}$, $n \geq 0$ for the right-movers and $\bar{\alpha}^j_{-n-kv_j}$, $n > 0$ and $\bar{\alpha}^{*j}_{-n+kv_j}$, $n \geq 0$ for the left-movers. Here we consider the case where $0 < kv_j < 1$. The occupation number operators are

$$\begin{aligned} N^j_R &= \sum_{n=-\infty}^{\infty} : \alpha^j_{n+kv_j} \alpha^{*j}_{-n-kv_j} :, \\ N^j_L &= \sum_{n=-\infty}^{\infty} : \bar{\alpha}^j_{n+kv_j} \bar{\alpha}^{*j}_{-n-kv_j} :, \end{aligned}$$

with normal ordering $: \quad :$. Note that the eigenvalues of N_L and N_R in the twisted sectors are multiples of $1/N$.

$Z_B[\mathbb{1}, \mathbb{1}]$ **untwisted sector** For untwisted sector ($k = 0$)

$$\begin{aligned} L^j_0(\mathbb{1}) &= \frac{1}{2}(p^j_R)^2 + N^j_R(k = 0) \\ \bar{L}^j_0(\mathbb{1}) &= \frac{1}{2}(p^j_L)^2 + N^j_L(k = 0), \end{aligned}$$

p^j_L and p^j_R are the Kaluza-Klein momenta for the left and right movers on the (compact) j-th dimensions. L_0 without j is just the sum of L^j_0 over j.

The bosonic partition function is

$$\begin{aligned} Z^{\text{untwisted}}_{\text{bosonic}} &= Z_B[\mathbb{1},\mathbb{1}] = \operatorname{Tr}\Big(q^{L_0-\frac{1}{12}}\bar{q}^{\bar{L}_0-\frac{1}{12}}\Big) \\ &= \frac{1}{|\eta(\tau)|^{2D}} \sum_{\mathbf{m}_R,\mathbf{m}_L\in\Lambda^*} \sum_{\mathbf{n}_R,\mathbf{n}_L\in\Lambda} q^{\frac{1}{2}(\mathbf{m}+\frac{1}{2}\mathbf{n})^2}\bar{q}^{\frac{1}{2}(\mathbf{m}-\frac{1}{2}\mathbf{n})^2}, \end{aligned}$$

$\mathbf{m}$ is quantized momentum and $\mathbf{n}$ is winding number. Λ^* is the dual lattice of the torus coordinates lattice.

$\mathbb{Z}_N$ projection For $\ell \neq 0$ twisted sectors, i.e. for complex bosons which satisfy the boundary conditions

$$Z^j(\sigma^0+2\pi\tau_2,\sigma^1+2\pi\tau_1) = e^{2\pi i l v_j} Z^j(\sigma^0,\sigma^1), \tag{D.14}$$

we need to evaluate the trace with an θ^ℓ insertion. Since we assume that θ^ℓ leaves no directions unrotated, thus neither quantized momenta nor windings survive the trace. θ^ℓ is $\mathbb{Z}_N$ group element insertion. We only need to consider states obtained from the Fock vacuum by acting with creation operators for which the complex coordinates are eigenvectors of θ^l. The Fock vacuum is defined to be invariant under θ

$$|n_1^j,n_2^j,\dots,n_1^{*j},n_2^{*j},\dots\rangle := (\alpha^j_{-1})^{n_1}(\alpha^j_{-2})^{n_2}\dots(\alpha^{*j}_{-1})^{n_1^*}(\alpha^{*j}_{-2})^{n_2^*}\dots|0\rangle.$$

$Z[\mathbb{1},\theta^\ell]$ sector Then, for instance, for the right movers in Z^j, using (D.10) and (D.11), we find the contribution

$$\begin{aligned} \operatorname{Tr}\Big(\theta^\ell q^{L_0^j(\mathbb{1})-\frac{1}{12})}\Big) &= q^{-\frac{1}{12}} \sum_{n_m^j,n_m^{*j}} \langle n_1^j,n_2^j,\dots,n_1^{*j},n_2^{*j},\dots|\theta^\ell q^{L_0^j(\mathbb{1})}|n_1^j,n_2^j,\dots,n_1^{*j},n_2^{*j},\dots\rangle \\ &= q^{-\frac{1}{12}}(1+qe^{2\pi i\ell v_j}+qe^{-2\pi i\ell v_j}+\dots) \end{aligned} \tag{D.15}$$

where the first term is the contribution from the vacuum, the second and third terms from states obtained by acting with α^j_{-1} and α^{*j}_{-1} on the vacuum, and so on. It is not hard to see that the whole expansion can be cast into the form

$$\begin{aligned} \operatorname{Tr}\Big(\theta^\ell q^{L_0^j(\mathbb{1})-\frac{1}{12})}\Big) &= q^{-\frac{1}{12}} \sum_{all\, n_m,n_m^*} \Big(\prod_m (q^m e^{2\pi i\ell v_j})^{n_m}(q^m e^{-2\pi i\ell v_j})^{n_m^*}\Big) \\ &= q^{-\frac{1}{12}} \prod_m \Big(\sum_a (q^m e^{2\pi i\ell v_j})^a \sum_b (q^m e^{-2\pi i\ell v_j})^b\Big) \end{aligned}$$

$$= q^{-\frac{1}{12}} \prod_{m=1}^{\infty} (1 - q^m e^{2\pi i \ell v_j})^{-1} (1 - q^m e^{-2\pi i \ell v_j})^{-1}$$

$$= -2\sin(\ell\pi v_j) \frac{\eta(\tau)}{\vartheta \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} - \ell v_j \end{bmatrix} (\tau)}. \tag{D.16}$$

The last step is derived by using the definitions of ϑ and η functions, cf. (A.3a) and(A.4).

Taking into account left and right-movers for all compact coordinates we obtain

$$Z[\mathbb{1}, \theta^\ell] = \mathrm{Tr}^U (\theta^\ell q^{L_0 - \frac{1}{12}} \bar{q}^{\bar{L}_0 - \frac{1}{12}}) = \chi(\theta^\ell) \left| \prod_{j=1}^{D/2} \frac{\eta(\tau)}{\vartheta \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} - \ell v_j \end{bmatrix} (\tau)} \right|^2$$

$\mathbb{1}$ means untwisted and θ^l means $\mathbb{Z}_N$ element inserted.

Since P defined in (D.2) must act crystallographically on the torus lattice and since $\mathbf{L} = n_i \mathbf{e}_i$ with integer coefficients n_i, in the lattice basis θ must be a matrix of integers. Hence the quantities

$$\mathrm{Tr}\,\theta^\ell = \sum_{j=1}^{D/2} 2\cos(2\pi\ell v_j) \quad \text{and} \quad \chi(\theta^l) = \prod_{j=1}^{D/2} 4\sin^2(\pi\ell v_j) \tag{D.17}$$

must be integers. In fact, by the Lefschetz fixed point theorem, $\chi(\theta^\ell)$ is the number of fixed points of θ^ℓ, and this can be explained as the result of the crystallographical structure.

General Bosonic Partition Function Using modular transformations of ϑ and η functions, we can get the partition functions of twisted sectors

$$S : \tau \to -\frac{1}{\tau}, \tag{D.18}$$

$$S\Big(Z[\mathbb{1}, \theta^k]\Big) = \chi(\theta^k) \left| \prod_{j=1}^{D/2} \frac{\eta(-\frac{1}{\tau})}{\vartheta \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} - k v_j \end{bmatrix} (-\frac{1}{\tau})} \right|^2$$

$$= \chi(\theta^k) \left| \prod_{j=1}^{D/2} \frac{\eta(\tau)}{\vartheta \begin{bmatrix} \frac{1}{2} + k v_j \\ \frac{1}{2} \end{bmatrix} (\tau)} \right|^2$$

$$= \chi(\theta^k)(q\bar{q})^{-\frac{D}{24}+E_k}\left|\prod_{j=1}^{D/2}\prod_{n=1}^{\infty}(1-q^{n-1+\{kv_j\}})^{-1}(1-q^{n-\{kv_j\}})^{-1}\right|^2$$
$$= Z[\theta^k, \mathbb{1}],$$

where $Z[\theta^k, \mathbb{1}]$ means θ^k twisted sector and no $\mathbb{Z}_N$ element inserted, and (cf. [23, (10.166)])

$$E_k^j = \frac{1}{2}\{kv_j\}(1-\{kv_j\}), \tag{D.19}$$

$$E_k = \sum_{j=1}^{D/2}\frac{1}{2}\{kv_j\}(1-\{kv_j\}) \tag{D.20}$$

is the vacuum expectation value of L_0 in the twisted Fock vacuum which is annihilated by all positive oscillator modes. We define $0 \le \{x\} < 1$ as the fractional value of $x : \{x\} = x - \lfloor x \rfloor$. (cf. [23, p.304-305])

We can continue generating pieces of the partition function by employing modular transformations (A.9a)-(A.9d). The general result can be easily found to be

$$Z[\theta^k, \theta^\ell] = \chi(\theta^k, \theta^\ell)\left|\prod_{j=1}^{D/2}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}+kv_j\\ \frac{1}{2}+\ell v_j\end{bmatrix}(\tau)}\right|^2, \qquad (k\ell v_j \notin \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2}) \tag{D.21}$$

$\chi(\theta^k, \theta^\ell)$ is the number of simultaneous fixed points of θ^k and θ^ℓ. This formula is valid when θ^k leaves no fixed directions, otherwise a sum over momenta and windings could appear. In addition, $\chi(\theta^k, \theta^l)$ should be replaced by $\tilde{\chi}(\theta^k, \theta^l)$, the number of fixed points in the sub-lattice effectively rotated by θ^k. χ and $\tilde{\chi}$ differ because when kv_j=integer, the expansion of $\vartheta[{}^{\frac{1}{2}+kv_j}_{\frac{1}{2}-lv_j}]/\eta$ has a prefactor $(2\sin\pi\ell v_j)$, as follows from the product representation of the ϑ-function. Therefore the actual coefficient in the expansion of (D.21) is $\tilde{\chi}(\theta^k, \theta^l) = \chi(\theta^k, \theta^\ell)/\prod_{j,kv_j\in\mathbb{Z}} 4\sin^2\pi\ell v_j$.

Summary The bosonic piece of the partition function of the type II string compactified on a symmetric $\mathbb{Z}_N$ orbifold is:

$$Z_B[\theta^k, \theta^\ell] = \left(\frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}\right)^{8-D}\tilde{\chi}(\theta^k, \theta^\ell)\left|\prod_{j=1}^{D/2}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}+kv_j\\ \frac{1}{2}+\ell v_j\end{bmatrix}(\tau)}\right|^2, \qquad (k\ell v_j \notin \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2}) \tag{D.22}$$

D is the number of compact dimension.

Number of Fixed points χ and $\tilde{\chi}$

From [39, (A.4)] we know

$$\tilde{\chi}(1,\theta^n) = 1, \qquad \tilde{\chi}(\theta^m,\theta^n) = \chi(\theta^m,\theta^n) \quad \text{if } \chi(\theta^m) \neq 0,$$
$$\tilde{\chi}(\theta^m,\theta^n) = \hat{\chi}(\theta^m,\theta^n) = \chi(\theta^m,\theta^n) / \prod_{j,mv_j\in\mathbb{Z}} 4\sin^2 \pi n v_j \quad \text{if } \chi(\theta^m) = 0, \tag{D.23}$$

where $\chi(\theta^m,\theta^n)$ is the number of simultaneous fixed points of θ^m and θ^n. If θ^m leaves fixed tori, i.e. $\chi(\theta^m) = 0$, we must use $\hat{\chi}(\theta^m,\theta^n)$ which is the number of simultaneous fixed points in the subspace actually rotated by θ^m. This is the same as we discussed above.

As we see in app.C, only $\theta^{N/2}$-twisted sector will survive, thus we are only interested in $\chi(\theta^{N/2},\theta^n)$ cases.

From [42, p.4], we see that the $\mathbb{Z}_N$ orbifold group action is generated by

$$\theta : z^j \to e^{2\pi i v_j} z^j, \tag{D.24}$$

with twist vector $\vec{v}$.

From [34, p.301], we can conclude that (using $\chi_{g,h}$ to represent arbitrary $\chi(\theta^m,\theta^n)$) if e is the identity element of $\mathbb{Z}_N$,

$$\chi_{e,g} = \chi(F_g) = \det(1-g) = \chi(\theta^\ell) = \prod_{j=1}^{D/2} 4\sin^2(\pi \ell v_j). \tag{D.25}$$

Since x is a fixed point of gh, if it is a fixed point of g and a fixed point of h, one sees that

$$\chi_{g,h} = \chi_{g,gh}. \tag{D.26}$$

Similarly,

$$\chi_{g,h} = \chi_{g^{-1},h} \tag{D.27}$$

since the fixed point sets of g and g^{-1} are identical. This is also true for h and h^{-1}, thus we have

$$\chi_{g,h} = \chi_{g,h^{-1}}. \tag{D.28}$$

Moreover, the number is symmetric under exchanging g and h, so we have

$$\chi_{g,h} = \chi_{h,g}. \tag{D.29}$$

Using all these facts we can evaluate all terms of the form χ_{θ^m,θ^n}.

D.1.2 Fermionic partition function

Now we come to the fermionic part. Since the Torus compactification has no action on fermionic degrees of freedom, we don't have to distinguish compact and non-compact dimensions. Also be aware that the twist vectors of fermion v_i is different from the twist vectors of compactified bosons v_j, because fermions are not compactified thus they are in different dimension than the bosonic case. However, the twist vectors of fermions won't change the uncompactified dimensions of fermions, therefore we take those non-compact dimensional components of the twist vectors to be 0.

Fermion

We now compute the one-loop partition function of a complex fermion with twisted boundary conditions.

We define $\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)$ and $\bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2)$. W.l.o.g, we observe the action of the right-mover

$$S = \frac{i}{\pi}\int d^2\sigma\bar{\psi}\partial_+\psi \tag{D.30}$$

with energy-momentum tensor

$$T = \frac{i}{2}(\bar{\psi}\partial_-\psi + \psi\partial_-\bar{\psi}). \tag{D.31}$$

Again, using mode expansion and canonical quantization, we can get the Hamiltonian $H = L_0 - \frac{c}{24}$ with

$$L_0 = \sum_{m=1}^{\infty}\left\{\left(m+\alpha-\frac{1}{2}\right)\bar{b}_{-m-\alpha+\frac{1}{2}}b_{m+\alpha-\frac{1}{2}} + \left(m-\alpha-\frac{1}{2}\right)b_{-m+\alpha+\frac{1}{2}}\bar{b}_{m-\alpha-\frac{1}{2}}\right\} + \frac{\alpha^2}{2} \tag{D.32}$$

and $c = 1$ for one complex fermion. α is the parameter of the twisted boundary condition defined in below.

Then we impose the twisted boundary conditions. For toroidal spatial direction

$$\begin{aligned}\psi(\sigma^0, \sigma^1 + 2\pi) &= -e^{+2\pi i\alpha}\psi(\sigma^0, \sigma^1),\\ \bar{\psi}(\sigma^0, \sigma^1 + 2\pi) &= -e^{-2\pi i\alpha}\psi(\sigma^0, \sigma^1).\end{aligned}$$

For toroidal time direction

$$\begin{aligned}\psi(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) &= -e^{+2\pi i\beta}\psi(\sigma^0, \sigma^1),\\ \bar{\psi}(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) &= -e^{-2\pi i\beta}\psi(\sigma^0, \sigma^1).\end{aligned}$$

The minus signs correspond to path-integral with anti-periodic boundary conditions. Periodic boundary conditions request to insert $(-1)^F$. $\alpha, \beta \in \{0, \frac{1}{2}\}$ are spin structures, namely α stands for NS or R sectors, and β stands for $(-1)^F$ inserted or not. But we still need to implement the β-twist (i.e. GSO projection) on operators, i.e. we look for an operator P_{GSO} which satisfies

$$\begin{aligned} P_{\text{GSO}} b_{n+\alpha+\frac{1}{2}} P_{\text{GSO}}^{-1} &= e^{2\pi i\beta} b_{n+\alpha+\frac{1}{2}}, \\ P_{\text{GSO}} \bar{b}_{n+\alpha+\frac{1}{2}} P_{\text{GSO}}^{-1} &= e^{-2\pi i\beta} \bar{b}_{n+\alpha+\frac{1}{2}}, \end{aligned}$$

and thus the GSO projection is implemented by P_{GSO}.
This operator is easily found to be

$$P_{\text{GSO}} = e^{2\pi i\beta(N-\bar{N})},$$

where $N, \bar{N}$ are the number operators

$$N = \sum_{\eta>0} b_{-\eta}\bar{b}_{\eta}, \qquad \bar{N} = \sum_{\eta>0} \bar{b}_{-\eta} b_{\eta}. \tag{D.33}$$

The partition function in the α, β sector is

$$\begin{aligned} Z[{}^{\alpha}_{\beta}](\tau) &= \text{Tr}\Big(P_{\text{GSO}} q^{L_0 - \frac{1}{24}}\Big) \\ &= \text{Tr}\Big(e^{2\pi i\beta(N-\bar{N})} q^{L_0 - \frac{1}{24}}\Big) \\ &= q^{\frac{\alpha^2}{2} - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\alpha-\frac{1}{2}} e^{-2\pi i\beta})(1 + q^{n-\alpha-\frac{1}{2}} e^{+2\pi i\beta}) \\ &= e^{2\pi i\alpha\beta} \frac{\vartheta \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} (\tau)}{\eta(\tau)}, \end{aligned} \tag{D.34}$$

cf. (A.1) for the definition of q. We can get the full result by adding the left and right-mover part.

Partition function

Now let us consider the orbifold symmetry, which imposes additional boundary conditions

$$\psi^j(\sigma^0, \sigma^1 + 2\pi) = -e^{+2\pi i\alpha} e^{2\pi i k v_i} \psi^j(\sigma^0, \sigma^1),$$

$$\psi^j(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = -e^{+2\pi i\beta} e^{2\pi i \ell v_i} \psi^j(\sigma^0, \sigma^1). \tag{D.35}$$

And the partition function on the j-th complex compact dimension is

$$Z_F^j[\theta^k, \theta^\ell] = \mathrm{Tr}_{(NS\oplus R)\otimes(NS\oplus R)} \left(P_{\mathrm{GSO}} \theta^\ell q^{L_0^j(\theta^k) - \frac{1}{24}} \bar{q}^{\bar{L}_0^j(\theta^k) - \frac{1}{24}} \right). \tag{D.36}$$

The trace is over the left and right NS and R sectors for the fermions. This is equivalent to summing over $\alpha \in \{0, \frac{1}{2}\}$. Similarly, the GSO projection amounts to summing over $\beta \in \{0, \frac{1}{2}\}$.

Using the result from app.D.1.2, we get the partition function of fermion

$$Z_F[\theta^k, \theta^\ell] = \frac{1}{4} \left| \sum_{\alpha,\beta} s_{\alpha\beta}(k,\ell) \prod_{j=1}^{4} \frac{\vartheta \begin{bmatrix} \alpha + k v_j \\ -\beta - \ell v_j \end{bmatrix}}{\eta} \right|^2, \tag{D.37}$$

$s_{\alpha\beta}(k,\ell)$ is the spin structure coefficients. By convention we take $s_{00}(k,\ell) = 1$. Imposing modular invariance, notice that $\sum v_i = 0$, we check

$$s_{00}(k,\ell) = -s_{\frac{1}{2}0}(k,\ell) = 1, s_{0\frac{1}{2}}(k,\ell) = -e^{i\pi k \sum v_i} = -1 = \mp s_{\frac{1}{2}\frac{1}{2}}(k,\ell) \tag{D.38}$$

leads to a modular invariant partition function.

Note that $k = N$ should give the same solution as $k = 0$. This gives, once more, the condition $\sum v_i = 0$. Note further that the sign of $s_{\frac{1}{2}\frac{1}{2}}(k,\ell)$ is not fixed by modular invariance. Choosing opposite (equal) signs in the left and right-movers corresponds to orbifold compactifications of Type-IIA (B) strings (as one can see by looking at the $k = \ell = 0$ sector).

D.1.3 D-branes on $T^D/\mathbb{Z}_N$

This section refers to [51, §9.14.3] and [2, §2.2]

The tadpole of Klein bottle amplitudes will be canceled by the insertion of D_9-branes filling all ten dimensions. Through T-duality, we can further see the existence of D_5-branes because T-duality transforms D_9-branes to D_5-branes. And the tadpole must be canceled by the addition of D_5-branes as well. After that, we need O-planes to cancel D-brane charges over compact space.

D_5-branes will be stretching in the six non-compact dimensions. The orbifold now acts on the transverse positions of the branes. Thus, there are two options to consider.

We may consider a group of branes sitting at a fixed point of the orbifold action. In such a case there is no further restriction on the transverse position. We may also consider a group of branes at a generic position x^i on T^D. Orbifold invariance imposes that we also include a mirror brane group at the position $-x^i$.

In the orientifold we are considering, the D_5-branes will have vanishing twisted tadpoles and therefore will not be fractional. Fractional means branes which are fixed to the orbifold fixed points. This means we won't have to worry about those fixed branes.

In order to accommodate the orbifold action on the CP factors of D_9-and D_5-branes we must introduce matrices $\gamma_{\theta/\Omega,9}$ and $\gamma_{\theta/\Omega,5}$. They satisfy the constraints (C.5)-(C.7) coming from the orbifold group property.

For the trace of the CP factors, using (C.4) we may evaluate the trace as in [51, (5.3.24)]

$$\sum_{ij}\langle i,j|\Omega|i,j\rangle = \sum_{iji'j'}\langle i,j|j',i'\rangle(\gamma_\Omega)_{ii'}(\gamma_\Omega^{-1})_{j'j} = \mathrm{Tr}\big[\gamma_\Omega^\top\gamma_\Omega^{-1}\big]. \tag{D.39}$$

And we have similar results for θ

$$\sum_{ij}\langle i,j|\theta^k|i,j\rangle) = \sum_{iji'j'}\langle i,j|j',i'\rangle(\gamma_{\theta^k})_{ii'}(\gamma_{\theta^k}^{-1})_{j'j} = \mathrm{Tr}\big[\gamma_{\theta^k}^\top\gamma_{\theta^k}^{-1}\big]. \tag{D.40}$$

Fixing signs According to the detailed discussion in [51, §7.3], in the NS sector there is an ϵ phase for each of the 9-9 and 5-5 strings as follows

$$\Omega|9-9,p;ij\rangle_{NS} = \epsilon_{99}(\gamma_{\Omega,9})_{ii'}|9-9,p;j'i'\rangle_{NS}(\gamma_{\Omega,9})^{-1}_{j'j}, \tag{D.41}$$

$$\Omega|5-5,p;ij\rangle_{NS} = \epsilon_{55}(\gamma_{\Omega,5})_{ii'}|5-5,p;j'i'\rangle_{NS}(\gamma_{\Omega,5})^{-1}_{j'j}. \tag{D.42}$$

Similar arguments as in [51, §7.3] fix

$$\epsilon_{99}^2 = \epsilon_{55}^2 = -1, \qquad \gamma_{\Omega,5/9} = \zeta_{5/9}\gamma_{\Omega,5/9}^\top, \qquad \zeta_5^2 = \zeta_9^2 = 1. \tag{D.43}$$

In the 5-9, 9-5 sectors, however, we may write

$$\Omega|5-9,p;ij\rangle_{NS} = \epsilon_{59}(\gamma_{\Omega,5})_{ii'}|9-5,p;j'i'\rangle_{NS}(\gamma_{\Omega,9})^{-1}_{j'j}, \tag{D.44}$$

$$\Omega|9-5,p;ij\rangle_{NS} = \epsilon_{59}(\gamma_{\Omega,9})_{ii'}|5-9,p;j'i'\rangle_{NS}(\gamma_{\Omega,5})^{-1}_{j'j}, \tag{D.45}$$

Imposing $\Omega^2 = 1$ we obtain

$$\epsilon_{59}^2\zeta_5\zeta_9 = 1. \tag{D.46}$$

The phase ϵ_{59} captures the transformation properties under Ω of the $SO(D)$ twisted spinor as well of the NS open string vacuum. If two 9-5 states interact, they may produce a 5-5 or a 9-9 state. Thus, a nontrivial coupling of two 9-5 states to the massless 9-9 or 5-5 states should be allowed. This implies that $\epsilon_{59}^2 = -1$. Therefore from (D.46), the CP projection is opposite for D_5-branes compared to that of D_9-branes,

$$\zeta_5\zeta_9 = -1. \tag{D.47}$$

Boundary conditions We have to notice that in the case of effective open string surfaces of Annulus and Möbius strips, due to the boundary conditions, we have the general properties: NN directions have only momenta, DD directions have only windings, and DN have none of both.

D_5**-branes** Due to tadpole cancellation, only $\mathbb{Z}_{even}$ Type-IIB orbifold has D_9-branes filling the space and D_5-branes transversal to 1-st and 2-nd tori and parallel to (wrapped around) 3-rd torus. So it means D_5-branes only exist for $\mathbb{Z}_{even}$ models.

D.2 Analysis of the 4 Euler Number $\chi = 0$ surfaces in 1-loop correction of Type-IIB $T^6/\mathbb{Z}_N$ orientifolds

We continue our discussion of the 1-loop partition function in app.D and present the details of the partition functions of 4 one-loop surfaces in this section. Since the calculation of the partition function of the surfaces is related to the twist vector v_j of a certain $\mathbb{Z}_N$ group, we'll give the general idea first, then give the examples in detailed orientifolds in the following sections.[2]

From now on we'll concentrate on phenomenally interesting $D = 4$ case and we would like to study the properties of 1-loop surfaces needed in 2-point calculation of 1-loop corrections.

D.2.1 Partition Function

This section follows [47, §3] closely.

Using the results of partition functions we derived in app.D.1, we get the general partition functions of the 3 different $\chi = 0$ surfaces except torus

$$Z_\sigma^{(\ell)}(\tau_\sigma, s) = (-2\pi) C P_\sigma \tilde{\chi}_\sigma (-2\sin(\pi\gamma_3)) \left(\prod_{j=1}^{2} f(\gamma_j) \right) Z_s^\vartheta(\gamma_i, h_i, g_i) \tag{D.48}$$

[2] This section is cited from [54]

with $Z_s^\vartheta(\gamma_i, h_i, g_i)$ being the ϑ-dependent part of the partition function given by

$$Z_s^\vartheta(\gamma_i, h_i, g_i) = \eta_{\alpha\beta} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \vartheta\begin{bmatrix}\alpha + h_1\\ \beta+\gamma_1+g_1\end{bmatrix} \vartheta\begin{bmatrix}\alpha + h_2\\ \beta+\gamma_2+g_2\end{bmatrix} \vartheta\begin{bmatrix}\alpha\\ \beta+\gamma_3\end{bmatrix}}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} \vartheta\begin{bmatrix}\frac{1}{2} + h_1\\ \frac{1}{2}+\gamma_1+g_1\end{bmatrix} \vartheta\begin{bmatrix}\frac{1}{2} + h_2\\ \frac{1}{2}+\gamma_2+g_2\end{bmatrix} \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}}, \tag{D.49}$$

where the spin structure relation between s and (α, β) can be found in Table D.1. And $\vartheta'[\frac{1}{2}, \frac{1}{2}] \equiv -2\pi\eta^3$, cf. (A.4). σ stands for the surfaces of Klein bottle $\mathcal{K}$, Annulus $\mathcal{A}$ and Möbius strip $\mathcal{M}$, with world-sheet parameters $\tau_{\mathcal{K}} = 2it$, $\tau_{\mathcal{A}} = \frac{it}{2}$, $\tau_{\mathcal{M}} = \frac{1}{2} + \frac{it}{2}$. More details can be found in [7]. CP_σ stands for the corresponding Chan-Paton factor of the open string world-sheets and $CP = 1$ for the Klein bottle, cf. app.C. Values for CP_σ, $\tilde{\chi}_\sigma$, γ_i, $f(\gamma_j)$, h_i and g_i can be found in Table D.2. Formula (D.48) holds for all tadpole-free $\mathbb{Z}_N$ Type-IIB orientifolds. Orientifolds with even N have D_5-branes wrapped around the third torus leading to the distinction of γ_3 in (D.48). And therefore the 3-rd torus always has NN boundary condition no matter whether it is attached to D_9 or D_5-branes.

We choose

$$\operatorname{tr}\left(\gamma_{\Omega_\ell,5}^{-1}\gamma_{\Omega_\ell,5}^{\top}\right) = -\operatorname{tr}\gamma_{2\ell,5} \tag{D.50}$$

and

$$\operatorname{tr}\left(\gamma_{\Omega_\ell,9}^{-1}\gamma_{\Omega_\ell,9}^{\top}\right) = \operatorname{tr}\gamma_{2\ell,9}. \tag{D.51}$$

The minus sign is due to the Gimon and Polchinski action of Ω, cf. [2, §2.3 and (2.41)].

s	1	2	3	4
$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}$
η_s	-1	-1	$+1$	-1

Table D.1: Spin structures

D.2.2 $\mathcal{N} \geq 2$ sectors

In these cases $(-2\sin(\pi\gamma_3))\prod_{j=1}^{2} f(\gamma_j)$ vanishes. $\mathcal{N} = 2$ sectors are characterized by that along exactly one i_{th}-torus, h_i vanishes and $\gamma_i + g_i$ is integer. $\mathcal{N} = 4$ sectors are characterized by that along all three torus, all three h_i vanish and all three $\gamma_i + g_i$ are

σ	CP	$\tilde{\chi}$	γ_i	$f(\gamma_i)$ (i=1 or 2)	h_1	h_2	g_1	g_2
$\mathcal{K}_u$	1	1	$2\ell v_i$	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{K}_t$	1	$\tilde{\chi}(\theta^{N/2}, \theta^\ell)$	$2\ell v_i$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathcal{A}_{99}$	$(\mathrm{tr}\,\gamma_{\ell,9})^2$	1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{A}_{55}$	$(\mathrm{tr}\,\gamma_{\ell,5})^2$	1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{A}_{95}$	$(\mathrm{tr}\,\gamma_{\ell,9})(\mathrm{tr}\,\gamma_{\ell,5})$	2	ℓv_i	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathcal{M}_9$	$\mathrm{tr}\,\gamma_{2\ell,9}$	-1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{M}_5$	$\mathrm{tr}\,\gamma_{2\ell,5}$	-1	ℓv_i	$2\cos(\pi\gamma_i)$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$

Table D.2: Refer to [47]. $\mathcal{K}_u$ and $\mathcal{K}_t$ denote the Klein bottle contributions with untwisted and $\theta^{N/2}$-twisted closed strings running in the loop. $\tilde{\chi}(\theta^{N/2}, \theta^\ell)$ denotes the number of simultaneous fixed points of $\theta^{N/2}$ and θ^ℓ. The CP factors corresponding to the D_5-branes assume that all D_5-branes are sitting at the fixed point at the origin of the compact transverse space, details cf. [2, §2.3]. Derivation of these constants in the table will be explained in the following subsections.

integer. In these cases, (D.48) has a well defined limit $\frac{1}{\eta^2}$ of singular part, but one has to include internal momenta or windings, therefore we should substitute these singular part with momentum/winding lattice sum (C.22) and (C.23).

For $\mathcal{A}$ and $\mathcal{M}$ the momentum sum $\mathcal{L}^{[j,M]}$ appears if the j-th torus is parallel to the branes whereas the winding sum $\mathcal{L}^{[j,W]}$ appears if the j-th torus is transversal to the branes, and this actually is related to the boundary conditions of the open strings attached to the D-branes. For $\mathcal{K}$ the situation is as follows: If γ_j is even, the corresponding torus is not reflected. The orientation reversal Ω, however, reverses the winding modes. Thus only the momentum modes survive. On the other hand, if γ_j is odd, the corresponding torus is reflected (i.e. kv_j is half-integer). Combined with Ω, this leaves the winding modes along this torus invariant. The terms "momentum" and "winding" are used here referring to the open string channel.

D.2.3 Torus

Topologically Torus is the 1-loop closed string amplitude, without Orientifold symmetry Ω action.

This part is just the Type-IIB orbifold thus is trivial as (3.50) and has no tadpole.

D.2.4 Klein bottle

Topologically Klein bottle is the 1-loop closed string amplitude, with Orientifold symmetry Ω action.

In the operator form, the partition function of Klein bottle is

$$\Lambda_{\mathcal{K}} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}^{U+T} \left[\frac{\Omega}{2} \cdot \frac{1}{N} \sum_{\ell=0}^{N-1} \theta^\ell \cdot \frac{1+(-1)^F}{2} e^{-2\pi(2it)(L_0 - c/24)} \right] \tag{D.52}$$

Be aware that Ω can act on bosonic and fermionic oscillators as described in (C.8)-(C.13). Ω projects out NS-R and R-NS sectors. The action of Ω on the bosonic and fermionic oscillators results in a nonzero contribution in the trace only if the state has the same left and right oscillators. This effectively sets $L_0 + \bar{L}_0 \to 2L_0$ for such symmetric states and causes the final amplitude to have a modular parameter 2τ instead of τ.

Also, since Ω exchanges θ^k with θ^{N-k}, we only have twisted strings with $k = 0$ and $k = \frac{N}{2}$, N even.

CP factors Since Klein bottle is not attached to D-branes, thus the CP factor is 1.

γ_i Due to the Ω action, $L_0 + \bar{L}_0 \to 2L_0$ will also double the γ_i. This can be easily seen from the calculation of (D.16).

Untwisted sector

$\tilde{\chi}$ **and** $f(\gamma_i)$ Since Ω action leaves only left-right symmetric states, from (D.16) we can see that we no longer have $4\sin^2(\pi\ell v_j)$ for $f(\gamma_i)$, but only have $-2\sin(2\pi\ell v_j)$.

Lattice sum cf. app.D.2.2

$$\gamma_i = \text{even-integer}, \quad i = 1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[j,M]} \tag{D.53}$$

$$\gamma_i = \text{odd-integer}, \quad i = 1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[j,W]} \tag{D.54}$$

Twisted sector

From the paragraph "Twisted Sectors" in app.C we know that only $\frac{N}{2}$-twisted sector is allowed.

h_i $\mathcal{K}_t$ is $\theta^{N/2}$-twisted, thus kv_j = half integer. And this is equivalent to shifting the α of ϑ functions in the T^4 direction (1-st and 2-nd tori) by h_i, cf. (D.21).

$\tilde{\chi}$ **and** $f(\gamma_i)$ As we discussed after (D.21), here $\frac{N}{2} \cdot v_j$ is integer, thus we have $\tilde{\chi}(\theta^{N/2}, \theta^{\ell})$ for $\tilde{\chi}$.

Lattice sum cf. app.D.2.2

$$\gamma_3 = \text{even-integer} \quad : \quad \frac{-2\sin\pi\gamma_3}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[i,M]} \tag{D.55}$$

$$\gamma_3 = \text{odd-integer} \quad : \quad \frac{-2\sin\pi\gamma_3}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[i,W]} \tag{D.56}$$

D.2.5 Annulus

Annulus surface represents closed string propagates between two D-branes, without Orientifold symmetry Ω action. Topologically and effectively we can consider it as the 1-loop open string amplitude, without Orientifold symmetry Ω action.

In the operator form, the amplitude is

$$\Lambda_{\mathcal{A}} = \int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}_{NS,R}^{99+55+95+59}\left[\frac{1}{2}\cdot\frac{1}{N}\sum_{\ell=0}^{N-1}\cdot\frac{1+(-1)^F}{2}e^{-2\pi(\frac{it}{2})(L_0-c/24)}\right] \tag{D.57}$$

Now we need to consider D-branes. According to earlier discussion about tadpole cancellation in app.D.1.3, we know that we would only consider D_9 and D_5-branes. Follow the discussion in app.D.1.3 and [51, §9.14.3], we have non-trivial CP factors in the partition function for Annulus.

Recall that open string boundary conditions on compactified dimensions have the results: NN directions have only momenta. DD only windings, and DN none of the above.

$\mathcal{A}_{99}$

CP factors $\mathcal{A}_{99}$ is attached to two D_9-branes. Therefore we have the CP factor as square of $tr\gamma_{9,k}$.

Lattice sum Here we have NN boundary conditions in the T^4 directions of $\mathcal{A}_{99}$, and also NN boundary conditions in the 3-rd torus. Then the compact directions have only momenta. And we need to substitute

$$\gamma_i = \text{integer},\ i = 1,2,3: \quad \frac{-2\sin(\pi\gamma_i)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[i,M]} \tag{D.58}$$

$\mathcal{A}_{55}$

CP factors $\mathcal{A}_{55}$ is attached to two D_5-branes. Therefore we have the CP factor as square of $tr\gamma_{5,k}$.

Lattice sum Here we have DD boundary conditions in the T^4 directions of $\mathcal{A}_{55}$, and NN boundary conditions in the 3-rd torus. Then the T^4 compact directions have only windings. And we need to substitute

$$\gamma_i = \text{integer},\ i = 1,2: \quad \frac{-2\sin(\pi\gamma_i)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[i,W]} \tag{D.59}$$

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin(\pi\gamma_3)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[3,M]} \tag{D.60}$$

$\mathcal{A}_{95}$

CP factors $\mathcal{A}_{95}$ is attached to one D_5-brane and one D_9-brane. Therefore we have the CP factor as the product of $tr\gamma_{5,k}$ and $tr\gamma_{9,k}$.

h_i **and** $f(\gamma_i)$ $\mathcal{A}_{95}$ has Dirichlet-Neumann boundary conditions along 1-st and 2-nd torus. And the presence of 4 DN directions effectively $\mathbb{Z}_2$-twist the T^4 space (1-st and 2-nd torus), cf. [62, §13.4]. This is equivalent to the $\theta^{N/2}$-twisted sector in app.D.2.4. Therefore we have the same h_i and $f(\gamma_i)$ as in app.D.2.4.

$\tilde{\chi}$ $\mathcal{A}_{95}$ actually has two orientation, which are $\mathcal{A}_{95}$ and $\mathcal{A}_{59}$. Thus this contribute a factor of 2 to the partition function.

Lattice sum Here we have ND boundary conditions in the T^4 directions of $\mathcal{A}_{55}$, and NN boundary conditions in the 3-rd torus. Then the T^4 compact directions have no momentum or windings. And we need to substitute

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin(\pi\gamma_3)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[3,M]} \tag{D.61}$$

D.2.6 Möbius strip

Möbius strip surface represents closed string propagates between D-brane and orientifold plane, with Orientifold symmetry Ω action. Topologically and effectively we can consider it as the 1-loop open string amplitude, with Orientifold symmetry Ω action.

In the operator form, the amplitude is

$$\Lambda_{\mathcal{M}} = \int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}^{9+5}_{NS,R}\left[\frac{\Omega}{2}\cdot\frac{1}{N}\sum_{\ell=0}^{N-1}\cdot\frac{1+(-1)^F}{2}e^{-2\pi(\frac{1}{2}+\frac{it}{2})(L_0-c/24)}\right]. \tag{D.62}$$

Be aware that Ω in the $\mathrm{Tr}\left[\Omega q^{L_0-c/24}\right]$ is equivalent to adding a minus sign to q because of the action of Ω on L_0, cf. (C.8)-(C.13). This is equivalent to substitute the torus parameter τ in the partition functions with the half-shifted torus parameter

$$\tau_M = \frac{1}{2}+\frac{it}{2}, \tag{D.63}$$

as we have mentioned before about world-sheet parameters, cf. (D.7).

Since Ω changes the orientation of the string, 9-5 strings do not contribute to the trace. For the same reason, only strings starting and ending on the same D_5-brane contribute after $\mathbb{Z}_2$ projection.

$\mathcal{M}_9$

Lattice sum Open strings on $\mathcal{M}_9$ has NN boundary condition, thus only K-K momentum states survive.

$$\gamma_i = \text{integer}, \quad i=1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[i,M]} \tag{D.64}$$

CP factors $\mathcal{M}_9$ is attached to D_9-branes, and it has Ω action, thus we have $CP_{\mathcal{M}_9} = \mathrm{tr}\big(\gamma^{-1}_{\Omega_\ell,9}\gamma^{\top}_{\Omega_\ell,9}\big) = tr\gamma_{2\ell,9}$, cf. [2, (2.36)].

$\tilde{\chi}$ Due to the Ω action on the fermionic state for NN boundary condition (cf. (C.12)) and the Ω action on the vacuum states (cf. [51, (7.3.10) and (7.3.16)]), we have $\Omega(\psi^{\mu}_{\frac{1}{2}}|0\rangle) \propto -\psi^{\mu}_{\frac{1}{2}}|0\rangle$, i.e. we have an extra minus sign in $\tilde{\chi}$, also cf. [41, (3.11),(3.12)].

$\mathcal{M}_5$

Lattice sum Open strings on $\mathcal{M}_5$ has DD boundary condition, thus only winding states survive.

$$\gamma_i = \text{half-integer}, \quad i = 1,2: \quad \frac{2\cos\pi\gamma_i}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}} \to \frac{(-1)^i}{\eta^3}\mathcal{L}^{[i,W]} \tag{D.65}$$

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin\pi\gamma_3}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[3,M]} \tag{D.66}$$

g_i Because now we have DD boundary conditions for T^4 directions, according to (C.13), the T^4 directions have an extra minus sign. This is equivalent to an insertion of $\theta^{N/2}$ element in the trace, and thus equivalent to shifting the β in ϑ functions in the T^4 direction (1-st and 2-nd tori) by g_i.

$f(\gamma_i)$ Due to the insertion of $\theta^{N/2}$, this will shift the $\sin(\pi\gamma_j)$ in $f(\gamma_j)$ for $\pi/2$, or equivalently shift γ_j to $\gamma_j + g_j$, and thus turns $-\sin$ into cos function for each of the 1-st and 2-nd tori.

CP factors and $\tilde{\chi}$ $\mathcal{M}_5$ is attached to D_5-branes, and it has Ω action, thus we have $CP_{\mathcal{M}_5} = \mathrm{tr}\big(\gamma^{-1}_{\Omega_\ell,5}\gamma^{\top}_{\Omega_\ell,5}\big) = -tr\gamma_{2\ell,5}$, cf. [2, (2.41)]. But here we take $CP_{\mathcal{M}_5} = tr\gamma_{2\ell,5}$, thus we move the minus sign to $\tilde{\chi}$, which means we get $\tilde{\chi} = -1$.

Appendix E

t-integrals

We need to evaluate t-integrals in §4.5.1 and §4.5.2. A more detailed derivation of t-integrals could be found in [47, §C], we only list the results here. app.E.1 follows [54].

$\mathcal{N}=1$ sector t-integral Here we deal with the t-integral of $\mathcal{N}=1$ sectors (see §4.5.1). The integral to be evaluated is (assuming $0<\gamma<1$)

$$I=\int_{\frac{1}{e_\sigma\Lambda}}^{\infty}\frac{dt}{t^2}\frac{\vartheta_4'(\gamma,\tau_\sigma)}{\vartheta_4(\gamma,\tau_\sigma)} \tag{E.1}$$

with $\sigma=\mathcal{K},\mathcal{A}$ and $\tau_\sigma=\frac{ie_\sigma t}{2}$ (e_σ was defined in (4.40)). And the result is

$$\int_{\frac{1}{e_\sigma\Lambda}}^{\infty}\frac{dt}{t^2}\frac{\vartheta_4'(\gamma,ie_\sigma t/2)}{\vartheta_4(\gamma,ie_\sigma t/2)}=e_\sigma\pi(1-2\gamma)\Lambda^2-e_\sigma\frac{\pi}{48}[\psi'(\gamma)-\psi'(1-\gamma)]. \tag{E.2}$$

$\mathcal{N}=2$ sector t-integral Here we deal with the t-integrals of $\mathcal{N}=2$ sector (see §4.5.2). The integrals to be evaluated are

$$\begin{aligned}\Gamma[n,M/W]&=\int_0^\infty\frac{dt}{t^3}\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}}e^{-\frac{\pi}{t}m^am^bg_{ab}^{[n,M/W]}}\\&=\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}}\int_0^\infty\frac{dt}{t^3}e^{-\frac{\pi}{t}m^am^bg_{ab}^{[n,M/W]}}\\&=\frac{1}{\pi^2}\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}}\frac{1}{\left(m^am^bg_{ab}^{[n,M/W]}\right)^2}.\end{aligned} \tag{E.3}$$

The metric $g_{ab}^{[n,M/W]}$ is given by (4.53). Using (4.53) and the expression for $g_{ab}^{[n]}$ in terms of the complex structure $U^{[n]} = U_1^{[n]} + iU_2^{[n]}$ of n-th torus, i.e.

$$g_{ab}^{[n]} = \frac{\sqrt{\det g^{[n]}}}{U_2^{[n]}} \begin{pmatrix} 1 & U_1^{[n]} \\ U_1^{[n]} & |U^{[n]}|^2 \end{pmatrix}, \tag{E.4}$$

one can write

$$g_{ab}^{[n,M/W]} = \begin{cases} \frac{\sqrt{\det g^{[n]}}}{U_2^{[n]}} \begin{pmatrix} 1 & U_1^{[n]} \\ U_1^{[n]} & |U^{[n]}|^2 \end{pmatrix} & \text{for M (momentum sum)} \\ \frac{1}{U_2^{[n]}\sqrt{\det g^{[n]}}} \begin{pmatrix} 1 & \tilde{U}_1^{[n]} \\ \tilde{U}_1^{[n]} & |\tilde{U}^{[n]}|^2 \end{pmatrix} & \text{for W (winding sum)} \end{cases} \tag{E.5}$$

with $\tilde{U}^{[n]} = \tilde{U}_1^{[n]} + i\tilde{U}_2^{[n]} = -(U^{[n]})^{-1}$ (i.e. $\tilde{U}_1^{[n]} = -U_1^{[n]}/|U^{[n]}|^2$ and $\tilde{U}_2^{[n]} = U_2^{[n]}/|U^{[n]}|^2$). The result is

$$\Gamma^{[n,M/W]} = \begin{cases} \frac{(4\pi^2\alpha')^2}{\pi^2 V_n^2} E_2(U^{[n]}) & \text{for M (momentum sum)} \\ \frac{V_n^2}{\pi^2(4\pi^2\alpha')^2} E_2(-(U^{[n]})^{-1}) & \text{for W (winding sum)} \end{cases}, \tag{E.6}$$

where $E_s(U)$ is the non-holomorphic Eisenstein series

$$E_s(U) = \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} \frac{U_2^s}{|m^1 + m^2 U|^{2s}}. \tag{E.7}$$

E.1 t-integral for $\mathcal{M}$ with $\gamma > \frac{1}{2}$

When $\frac{1}{2} < \gamma < 1$ for $\mathcal{M}$, we need to do the integral

$$\tilde{I}_{\mathcal{M}} = \int_0^\infty \frac{dt}{t^2} \frac{\vartheta_1'(\gamma, \tau_{\mathcal{M}})}{\vartheta_1(\gamma, \tau_{\mathcal{M}})}, \tag{E.8}$$

here $\tau_{\mathcal{M}} = \frac{it}{2} + \frac{1}{2}$. We substitute $\gamma' = \gamma - \frac{1}{2}$ for γ, and this transforms the original integral to

$$\tilde{I}_{\mathcal{M}} = \int_0^\infty \frac{dt}{t^2} \frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})}. \tag{E.9}$$

By following the similar calculation in [14, §M.2], we perform ST^2S modular transformations:

$$\tau_{\mathcal{M}} = \frac{it}{2} + \frac{1}{2} \to -\frac{1}{\tau_{\mathcal{M}}} \to -\frac{1}{\tau_{\mathcal{M}}} + 2 \to \left(\frac{1}{\tau_{\mathcal{M}}} - 2\right)^{-1} = 2il - \frac{1}{2} =: l_{\mathcal{M}}. \tag{E.10}$$

Here $l = \frac{1}{4t}$. The result of ST^2S modular transformation (A.8) is

$$\frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})} \overset{l=\frac{1}{4t}}{=} -16\pi\gamma' l + 4il \frac{\vartheta_2'(4i\gamma' l, 2il - \frac{1}{2})}{\vartheta_2(4i\gamma' l, 2il - \frac{1}{2})}. \tag{E.11}$$

Using the representation of ϑ_2'/ϑ_2 for $|\operatorname{Im}(z)| < \operatorname{Im}(\tau_\sigma)$

$$\begin{aligned}\frac{\vartheta_2'(z)}{\vartheta_2(z)} &= -\pi \tan \pi z + 4\pi \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 - q^n} \sin 2\pi n z \\ &= -\pi \tan \pi z + 4\pi \sum_{n,m=1}^{\infty} (-1)^n q^{nm} \sin 2\pi n z\end{aligned} \tag{E.12}$$

we get

$$\begin{aligned}\tilde{I}_{\mathcal{M}} &= \int_{\frac{1}{4\Lambda}}^{\infty} \frac{dt}{t^2} \frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})} \\ &= 4 \int_0^{\Lambda} dl \left(-16\pi\gamma' l + 4il \frac{\vartheta_2'(4i\gamma' l, 2il - \frac{1}{2})}{\vartheta_2(4i\gamma' l, 2il - \frac{1}{2})} \right) \\ &= -16\pi \int_0^{\Lambda} dl\; l \Big(4\gamma' - \tanh(4\pi\gamma' l) + 4 \sum_{n,m=1}^{\infty} (-1)^{n(m+1)} e^{-4\pi l n m} \sinh(8\pi n \gamma' l) \Big).\end{aligned} \tag{E.13}$$

Following the similar calculation as [14, (397),(398)]:

$$\begin{aligned}I_1 &= \sum_{m,n=1}^{\infty} \int_0^{\infty} dl\; l (-1)^{n(m+1)} e^{-4\pi l n m} \sinh(8\pi n \gamma' l) \\ &= \sum_{m,n=1}^{\infty} (-1)^{n(m+1)} \frac{m\gamma'}{4n^2\pi^2(4\gamma'^2 - m^2)^2} \\ &= \sum_{m=1}^{\infty} \frac{m\gamma' \operatorname{Li}_2((-1)^{m+1})}{4\pi^2(4\gamma'^2 - m^2)^2}.\end{aligned} \tag{E.14}$$

Note that the integral converges provided that $2|\gamma'| \le m$ (which is true now because $\gamma' = \gamma - \frac{1}{2}$). Now we split the sum into sums over even and odd m:

$$I_1 = \sum_{k=1}^{\infty} \left[\frac{(2k)\gamma' \operatorname{Li}_2(-1)}{4\pi^2(4\gamma'^2 - (2k)^2)^2} \right] + \sum_{k=0}^{\infty} \left[\frac{(2k+1)\gamma' \operatorname{Li}_2(1)}{4\pi^2(4\gamma'^2 - (2k+1)^2)^2} \right]$$

$$= \frac{1}{1536}\Big[\psi'(1+\gamma') - \psi'(1-\gamma')\Big] + \frac{1}{768}\Big[\psi'(\frac{1}{2}-\gamma') - \psi'(\frac{1}{2}+\gamma')\Big]. \qquad \text{(E.15)}$$

All together we arrive at

$$= 8\pi(1-4\gamma')\Lambda^2 - \frac{\pi}{24\gamma'^2} - \frac{\pi}{24}\Big[\psi'(1+\gamma') - \psi'(1-\gamma') + 2\psi'(\frac{1}{2}-\gamma') - 2\psi'(\frac{1}{2}+\gamma')\Big]$$
$$= 8\pi(3-4\gamma)\Lambda^2 - \frac{\pi}{24}\Big[\psi'(\gamma-\frac{1}{2}) - \psi'(\frac{3}{2}-\gamma) + 2\psi'(1-\gamma) - 2\psi'(\gamma)\Big]. \qquad \text{(E.16)}$$

Appendix F

Derivation of relative modular transformation

Here we present a simple derivation of the lower triangular form of relative modular transformation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}) \tag{F.1}$$

which preserves the involution

$$I = \begin{pmatrix} \mathbb{1} & 0 \\ \Delta & -\mathbb{1} \end{pmatrix}. \tag{F.2}$$

Taking $\vec{V} := (a_1, \cdots, a_g, b_1, \cdots, b_g)^\top$ to be the vector of the homology basis. The symplectic preserving condition (F.1) is

$$M\vec{V}(M\vec{V})^\top = MJM^\top = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} A^\top & C^\top \\ B^\top & D^\top \end{pmatrix} \equiv \vec{V}\vec{V}^\top = J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} AB^\top - BA^\top = 0 \\ AD^\top - BC^\top = \mathbb{1} \\ CB^\top - DA^\top = -\mathbb{1} \\ CD^\top - DC^\top = 0 \end{cases}. \tag{F.3}$$

From the involution preservation condition

$$MI\vec{V} = MIM^{-1}M\vec{V} = I'M\vec{V} \equiv IM\vec{V}$$

$$\Rightarrow I' = MIM^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \Delta & -\mathbb{1} \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \equiv I = \begin{pmatrix} \mathbb{1} & 0 \\ \Delta & -\mathbb{1} \end{pmatrix} \tag{F.4}$$

where $M^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ and

$$MM^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \mathbb{1}, \tag{F.5}$$

as well as the symplectic preserving condition(F.3), one finds that

$$B = B' = 0, A' = A^{-1}, D = (A^{-1})^\top = D'^{-1}, C' = -A^\top C A^{-1}. \tag{F.6}$$

The general form of M is

$$M = \begin{pmatrix} A & 0 \\ C & (A^{-1})^\top \end{pmatrix} \tag{F.7}$$

with

$$2C = \Delta A - (A^{-1})^\top \Delta. \tag{F.8}$$

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