Potentials on scalar field space and the Swampland

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Abstract

The Distance Conjecture stands as one of the foundational pillars of the Swampland programme. As such, it serves as a pivotal element within a broader set of principles that aim to identify the effective field theories coupled to gravity which can be completed to a theory of quantum gravity in the UV. By now, there is significant evidence of its validity. Furthermore, the insights and implications of the Distance Conjecture, along with closely related variants, have recently been used to put forward phenomenological predictions that can be tested in the near future. However, the Distance Conjecture in its original formulation is a statement on moduli space, i.e. on the field space spanned by massless moduli fields. In realistic scenarios, these fields are subject to a scalar potential, lifting the formerly flat moduli space to a more generic scalar field space with potential. Aiming to use the Distance Conjecture in these settings, one therefore has to carefully extend its applicability, taking care of novel features that may not arise within the paradigm of a flat moduli space.

The goal of the present thesis is to investigate some of these novel aspects arising in the presence of a scalar potential on field space, as well as implications from (generalised) dualities and the role of non-geometric backgrounds. We give a concise introduction to the Swampland program, focussing on the Distance Conjecture and related variants. In particular, we provide a brief summary of the current state of the art with respect to the Distance Conjecture in the presence of a potential as well as the connection with dualities. We then investigate the implications of (generalised) T-duality for the Distance Conjecture encountering aspects like topology change and missing towers at infinite distance. This will lead to a proposed amendment to the Distance Conjecture in the presence of diverging scalar potentials. We furthermore discuss non-geometric backgrounds as well as generalised T-duals and the way they fit within the Swampland program. As an alternative approach to the problem of the potential we utilise geometric flow equations, in particular (generalised) Ricci flow. We explain how the fixed points and singularities of such flows can offer a different angle on the problem of diverging potentials and missing towers encountered before, leading to an extension of the Ricci Flow Conjecture to also include fluxes. We define some novel flows and, after establishing their well-posedness, outline their potential applications to study related issues like scale-separation in 10D string vacua. Lastly, we thoroughly discuss the closely related problem of defining a suitable (formal) notion of distance in the presence of a potential. We close with a brief outlook on work in progress that aims to address this problem within the framework of optimal transport.

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Zusammenfassung

Die Distanzvermutung (engl. Distance Conjecture) ist eine der tragenden Säulen des Swampland-Programms. Als solche, ist sie von zentraler Bedeutung innerhalb einer Sammlung von Konzepten, mit dem Ziel gravitative effektive Feldtheorien zu identifizieren, die zu einer konsistenten Theorie der Quantengravitation vervollständigt werden können. Zahlreiche Hinweise stützen mittlerweile ihre Gültigkeit, und Erkenntnisse aus der Distanzvermutung sowie abgeleitete Varianten wurden verwendet, um phänomenologische Vorhersagen zu entwickeln, die zeitnah experimentell überprüfbar sind. Ursprünglich bezieht sich die Distanzvermutung jedoch auf den Modulraum, also den Feldraum, der von masselosen Feldern aufgespannt wird. In realistischen Szenarien unterliegen diese Felder einem Skalarpotential, wodurch der zuvor flache Modulraum zu einem allgemeineren Skalarfeldraum mit Potential erweitert wird. Die Verwendung der Distanzvermutung in diesem Kontext erfordert daher eine sorgfältige Erweiterung ihrer Anwendbarkeit, wobei neue Aspekte berücksichtigt werden müssen, die im Zusammenhang mit flachen Modulräumen nicht auftreten.

Ziel dieser Arbeit ist es, solche neuartigen Aspekte zu untersuchen, die durch ein Skalarpotential im Feldraum entstehen, sowie Implikationen aus (verallgemeinerten) Dualitäten und die Rolle nicht-geometrischer Räume zu analysieren. Wir geben eine kompakte Einführung in das Swampland-Programm, mit besonderem Fokus auf die Distanzvermutung und deren Varianten, ergänzt durch eine kurze Zusammenfassung des aktuellen Forschungsstands in Gegenwart eines Potentials und den Zusammenhang mit Dualitäten. Darauf aufbauend, werden die Konsequenzen von (verallgemeinerter) T-Dualität für die Distanzvermutung untersucht, einschließlich topologieverändernden Prozesse und fehlender Zustände in unendlicher Distanz. Dies führt zu einem Vorschlag für eine Modifikation der Distanzvermutung bei divergierenden Skalarpotentialen. Ebenso werden nicht-geometrische Räume sowie verallgemeinert T-duale Konfigurationen und deren Einbindung in das Swampland-Programm behandelt. Als alternative Herangehensweise werden geometrische Flussgleichungen, insbesondere der (verallgemeinerte) Ricci-Fluss, analysiert. Fixpunkte und Singularitäten solcher Flüsse eröffnen eine neue Perspektive auf divergierende Potentiale und fehlende Zustände und resultieren in einer verallgemeinerten Ricci-Fluss-Vermutung. Wir definieren neue Flüsse, überprüfen ihre Wohldefiniertheit und diskutieren mögliche Anwendungen, etwa zur Untersuchung der Skalen-Trennung von zehndimensionalen stringtheoretischen Lösungen. Abschließend behandeln wir das eng verwandte Problem der Definition einer geeigneten Abstandsfunktion in Gegenwart eines Potentials. Die Arbeit schließt mit einem Ausblick auf laufende Untersuchungen, der diese Problematik anhand von Optimalem Transport adressiert.

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Contents

Introduction

"Trying to quantise the gravitational field"
– B. DeWitt to W. Pauli, as a response to being asked what he is working on.

"That is a very important problem – but it will take someone really smart!"
– W. Pauli.

Institute of Advanced Study, November 1949 [1]

I.1. Einstein's 8th symphony: Quantum Gravity

When Franz Schubert died in 1828, at the age of only 31, he had completed an overwhelming amount of well over 1000 pieces, including over 600 songs, countless piano pieces, and seven symphonies. Among them his intonation of *Erlkönig*, *Ave Maria*, the *Trout Quintet* and several impromptus. However, there is one piece in Schubert's magnum opus that plays a distinguished role: his unfinished 8th symphony, *Die Unvollendete* (ger. The Unfinished). Only two of the four conjectured movements are completed, the third is fragmented, and the fourth absent. In particular, besides all his masterpieces, precisely this unfinished piece is still performed on a regular basis. One could argue that this is due to the fact that together with Beethoven's *Eroica*, the symphony in the unusual "black key" B minor marks the beginning of the first romantic symphonies, moving away from the clear structures of the classical period. While this is certainly partially true, there is an arguably even greater factor: the intriguing mystery of the unfinished and humanity's urge for completion. Still today historians and musicians are wondering what would have been Schubert's ideas and visions and trying to unravel the full picture behind what Schubert left unfinished.

The urge towards a full and complete understanding is surely not unique to music. It can be described as a fundamental drive in our very human nature, and in particular, it is this desire to obtain a complete understanding of our world that has driven modern physics for centuries. We have come a long way in this quest, with the development of *Quantum Mechanics* [2–4] and *Special Relativity* [5,6] paving the road for *Quantum Field Theory* (QFT) [7–9], peaking in the formulation of the *Standard Model* of particle physics. Unifying the three fundamental forces of electromagnetism and the weak and strong force, the Standard Model is able to describe an overwhelming amount of processes to an unprecedented level of accuracy. However, several open questions remain, like the smallness of the cosmological constant, i.e. the cosmological constant problem [10,11], dark matter [12,13] or neutrino oscillation [14]. Most notably, there is Einstein's theory of *General Relativity* (GR) [15,16], at the heart of which lies the Einstein Field Equation [15]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \,,$$

together with its action principle, given in terms of the Einstein-Hilbert action [15,17]

$$\mathcal{S}_{\mathrm{EH}} = \frac{1}{2\kappa} \int \mathrm{d}^4 x \sqrt{-g}\, (R - 2\Lambda) + \int \mathrm{d}^4 x \sqrt{-g}\, \mathcal{L}_M \,.$$

Despite General Relativity's great success in accurately describing (classical) gravitational phenomena, it turns out to be impossible to consistently combine GR with the Standard Model into a unified and complete (quantum) theory of all the known interactions. Before going into any more details as to why this incompatibility arises, we may ask ourselves why this is even desirable.

Curiously, the necessity of a quantised theory of gravity was first raised by Einstein himself in the year 1916, many years before Schrödinger postulated his famous wave equation and quantum mechanics was far from being fully developed: [18]

696 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Gleichwohl müßten die Atome zufolge der inneratomischen Elektronenbewegung nicht nur elektromagnetische, sondern auch Gravitationsenergie ausstrahlen, wenn auch in winzigem Betrage. Da dies in Wahrheit in der Natur nicht zutreffen dürfte, so scheint es, daß die Quantentheorie nicht nur die Maxwellsche Elektrodynamik, sondern auch die neue Gravitationstheorie wird modifizieren müssen.

Einstein concludes his short argument, a gravitational analogue of Bohr's quantisation condition for the electron, with the words *"...therefore, it seems, that the quantum theory will have to modify, not only Maxwell's electrodynamics, but also the new theory of gravitation."* There are many similar heuristic arguments suggesting that indeed there is a need for a quantum theory of gravity; see for example [19] for a scientific philosopher's perspective on some of these heuristic arguments. However, the strongest and most robust evidence comes from the theory of black holes and their experimental confirmation through the observation of gravitational waves [20].

In a semi-classical approach, problems arising from non-renormalizability [21, 22] of GR when treated as a QFT can be somewhat evaded by considering the theory as a ultraviolet (UV)-incomplete effective theory of quantum gravity. The role of quantum effects for gravitational setups has been demonstrated in this framework. In fact, this has led to many significant results in cosmology [23, 24] and the theory of black holes, most notably the celebrated area law of the Bekenstein-Hawking entropy [25–27]

$$S_{BH} = \frac{A}{4G_N} ,$$

at the heart of black hole thermodynamics. At the same time, the semi-classical description cannot provide the microscopic picture behind this formula. While at the horizon of the black hole curvature is comparably small, the black hole curvature as described within GR diverges at the singularity in the centre of the black hole, and a semi-classical approach cannot be used to understand this apparent singularity. Hence, any answer to these issues seems to require a UV-complete description of gravity that can describe phenomena at the smallest scales, taking into account the laws of quantum mechanics. This sought after theory is *Quantum Gravity* (QG).

In a broader sense, we see that like Schubert's 8$^{\text{th}}$, General Relativity can be seen as Einstein's *Unvollendete*. GR is a beautiful and powerful framework that accurately describes the nature of gravity over a vast range of scales, with semi-classical gravity having fragments of its third movement written. While the fourth and last movement, quantum gravity, is still not fully worked out, the search for a theory of quantum gravity has already led to many important insights, including the above-mentioned black hole thermodynamics à la Bekenstein-Hawking [25–27], the Hamiltonian formulation of GR of Arnowitt, Deser, and Misner [28] or the holographic principle of 't Hooft [29] and Susskind [30] and its realisation via Maldacena's AdS/CFT correspondence [31]. Nevertheless, a full complete picture is still missing, and this is one of the biggest tasks, but also mysteries, left for current and future physicists.

There is a distinguished candidate for this last step towards a full theory of quantum gravity describing our universe, and this is *String Theory*.

I.2. String theory as a theory of everything?

Trying to perform a perturbative quantisation of classical GR, one soon encounters ultraviolet (UV) divergences in the resulting loop diagrams that render the theory *non-renormalizable* [21, 22]. This is due to the fact that when going to arbitrarily high energies, the associated interaction vertices of *gravitons*, which are the resulting gauge bosons of the gravitational interaction, become coincident and the Feynman diagrams diverge. However, it was realised that there is a remarkably simple, and yet somewhat revolutionary way to circumvent this problem. The interaction can be smeared out and hence the divergence softened by introducing one-dimensional extended objects as the fundamental degrees of freedom of the quantum theory. These objects are in turn called *strings* and their resulting theory leads to the notion of *String Theory* [32–37]. These strings come in two variants, *open* and *closed* strings, and have an associated characteristic length scale l_s. While the former can be thought of as one-dimensional objects with two endpoints, closed strings can be thought of as loops propagating in spacetime. Similarly to how interactions in quantum field theory are perturbatively described by Feynman diagrams that depict the possible worldline configurations of a point particle, interactions of strings are described by diagrams in terms of worldsheets and hence the perturbative expansion is given in terms of worldsheet topologies. In particular, this means that divergences coming from local singularities at the interaction vertices are smoothened out by the extended nature of the string, and this is why there are no pathological divergences in the UV. Classical point-like particles can then be thought of as vibrational modes of the string. In particular, the theory contains the aforementioned graviton, a massless spin-2 particle, which is the gauge boson of gravitational interaction at the quantum level. String theory then offers a microscopic picture for black hole thermodynamics and correctly reproduces the aforementioned Bekenstein-Hawking entropy [38]. At low energies, the theory reduces to GR and the open strings give rise to Yang-Mills theories. Hence, string theory unifies gravitational and gauge interactions, providing a *unified theory of all the (known) forces*.

Whether this makes string theory THE theory of quantum gravity or *Theory of Everything* is not clear yet. However, it is clear that string theory is a consistent theory of quantum gravity. It is the so far only fully consistently developed QG we have, and there are strong hints pointing towards the fact that it is the unique theory of gravity compatible with our (current) expectations of a theory of quantum gravity. This is known as *String Universality* [39–44]. In any case – at the very least – it is a valuable testing ground for collecting intuition of properties a generic QG theory might, or might not, posses. These ideas and concepts can then be contrasted with other (more general) physical arguments, and this line of thought leads to the *Swampland Program* [45].

I.3. The Swampland program

A naive estimate for the scale at which quantum gravitational effects become important can be made on dimensional grounds. In particular, such a scale should involve the relevant fundamental constants leading to the Planck length

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \simeq 1.6 \times 10^{-33}\,\text{cm}\,,$$

serving as a first estimate for the string length l_s. This is an extremely tiny scale, and hence, if this was the end of the story, the detection of imprints of quantum gravitational effects would seem hopeless. However, as we will see, the string scale l_s can be parametrically smaller than the Planck length $l_s \ll l_P$, such that, for example, Kaluza-Klein (KK) modes originating from extra dimensions can, in principle, be detected.

Even more importantly, and somewhat surprisingly, even if quantum gravitational effects are highly suppressed, high-energy (UV) physics can have crucial impacts on low-energy (IR) physics, once quantum gravity is properly included in the discussion. This goes by the name of UV/IR mixing [46–52] and is in conflict with a traditional Wilsonian approach to Effective Field Theories (EFT), where the UV physics is assumed to have no, or only highly suppressed, impacts on the IR physics. This effect is inherently due to the presence of (quantum) gravity, and hence gravity is in some sense special. Once taken into account, great care is needed to determine the consistency and validity of EFTs that seem perfectly fine in the absence of gravity. In particular, the actual cut-off of the EFT might be lower than naively expected, as captured by the so-called species scale [53–57], which quantifies the scale at which quantum gravitational effects become important, hence serving as the maximum cut-off. Subsequently, it has become clear in more recent years that not every possible effective field theory can be consistently coupled to gravity in the UV. This idea was first formulated in [45] by Vafa and goes by the name of *Swampland program* [58–62]. The Swampland program then aims to identify the low-energy EFTs that can be consistently coupled to gravity and hence lead to a UV-complete theory of quantum gravity. This is done by the formulation of a growing web of so-called *Swampland Conjectures*. If a theory passes (all) these conjectures, it is in the *Landscape*, while if it fails to satisfy them, it belongs to the *Swampland*.

The conjectures are motivated by general principles we believe a theory of quantum gravity should possess, and further substantiated by black hole arguments and experience or testing in string theory. As such, some of them are powerful enough to derive phenomenologically interesting predictions that can be tested experimentally in the near future [63–66]. Hence, they may offer a glimpse into the possible imprints of quantum gravitational effects on our physical world and, most importantly, help to deepen our understanding of quantum gravity.

I.4. Outline

The main goal of this thesis is to highlight some issues and novel features that arise when studying aspects of the Swampland program in the presence of a non-vanishing scalar potential on field space, along with implications of generalised dualities and non-geometry. Hereby, the thesis is split into two main parts.

Part I serves as an introduction to some of the main concepts heavily used in the main Part II. In Chapter 1 we collect some selected basic principles of bosonic string theory. Chapter 2, in turn, offers a detailed discussion of the role of dualities, in particular T-duality, and generalisations thereof. In Chapter 3 we introduce the concept of Double Field Theory as well as non-geometric fluxes and spaces, respectively. Finally, Chapter 4 contains a brief introduction to the theory of geometric flows, in particular, Ricci flow, and its generalisation and applications in string theory and mathematics. The reader familiar with these topics may skip these chapters and directly move to Part II and only return to them when referred to from the main chapters for further details.

In Part II, starting with Chapter 5, we offer a brief introduction to the Swampland program. We focus on aspects of the Swampland Distance Conjecture (SDC) and related concepts. In particular, in view of later chapters, we discuss the issues arising in the presence of a non-trivial scalar potential and the use of generalised dualities and non-geometric backgrounds. We provide a short overview of the current state-of-the-art in the field and some recent developments. The remaining chapter then aims to offer some possible approaches developed by the author and collaborators to tackle the aforementioned questions.

Chapter 6 starts with a discussion of the implications of (topological) T-duality and the associated mapping of states for the SDC. Working with the example of the three-sphere with H-flux, we discover a lack of tower of states for the SDC and relate it to a divergence in the potential on scalar field space. This ultimately leads us to propose an amendment to the SDC. We derive a general reduction formula applicable to generic 10 vacua and discuss some further examples.

In Chapter 7 we explain how to include certain non-geometric spaces and their formulation in terms of β-supergravity into the discussion of the SDC and subsequently move to the example of the T-duality chain introduced in Chapter 3.

Chapter 8 is again dedicated to a discussion of dualities, now focussing on the realisation of non-Abelian T-duality via Poisson-Lie T-duality. The dual space resulting from the S^3 example under consideration turns out to be non-geometric, leading to a combined treatment of generalised dualities and non-geometry. We close the chapter with an outlook on genuine Poisson-Lie T-duality and the Distance Conjecture.

In Chapter 9 we shift the focus to the use of geometric flows within the Swampland, discussing a generalisation of the Ricci Flow Conjecture to include (NSNS) fluxes and applications to field spaces with potentials. The emerging picture offers an alternative perspective on the divergent potential discussed in Chapter 6. This further motivates the problem of distances in the presence of a potential alluded to in Chapter 5.

Starting from the results obtained in Chapter 9, Chapter 10 sets out to generalise Ricci flow to also include RR-fluxes, thus leading to flow for Type II supergravity. After proofing the well-posedness of the flow, we discuss the extension to 11D, the inclusion of source contributions, as well as several examples.

Finally, Chapter 11 is devoted to an in-depth discussion of the issues of a proper notion of distance or length in the presence of a potential. Reflecting the results obtained in the previous chapters as well as recent literature, we are led to propose an alternative more general approach via the theory of Optimal Transport.

Publications

Chapters 6 to 11 in Part II of the present thesis closely follow the work published by the author and collaborators during his doctoral studies:

[67]: S. Demulder, D. Lüst and T. Raml, *Topology change and non-geometry at infinite distance*, *JHEP* 06 (2024) 079, [2312.07674].

[68]: S. Demulder, D. Lüst and T. Raml, *Navigating string theory field space with geometric flows*, *JHEP* 05 (2025) 030, [2412.10364].

The following papers were also published during the authors doctoral studies, but are only very briefly touched upon in Chapter 2 of this thesis:

[69]: S. Demulder and T. Raml, *Poisson-Lie T-duality defects and target space fusion*, *JHEP* 11 (2022) 165, [2208.04662].

[70]: R. Blumenhagen, A. Paraskevopoulou and T. Raml, *Non-associative Algebras of Cubic Matrices and their Gauge Theories*, *JHEP* 09 (2025) 003, [2504.02942].

Parts of the discussion in Chapters 5 and 11 and are based on work in progress:

[71]: S. Demulder, D. Lüst, C. Montella and T. Raml, *Cost functions and optimal transportation on scalar field space*, (work in progress).

Furthermore, Chapters 6-8 and 10 contain additional details and novel results by the author that have not been published so far.

Notation & Conventions

We work in the "mostly plus" metric signature such that $\eta_{\mu\nu} = \text{diag}(-,+,\dots,+)$. For G a Riemannian metric, we denote by $\langle\cdot,\cdot\rangle_G$ the inner product induced by G (with M, N p-tensors)

$$\langle M, N\rangle_G = M_{I_1\dots I_p} N_{J_1\dots J_p} G^{I_1 J_1} \dots G^{I_p J_p} \,, \tag{0.1}$$

and its induced norm by $\|M\|^2 = \langle M, M\rangle$. If G is Lorentzian, the above is no longer positive-definite in general, and we denote the "square" of a rank-p tensor F_p with

$$|F_p|^2 = \frac{1}{p!} F_{I_1\dots I_n} F_{J_1\dots J_n} G^{I_1 J_1} \dots G^{I_p J_p} \,, \tag{0.2}$$

such that $\|M\|^2 = p!\,|M|^2$. In contrast, $(F_p)^2_{IJ}$ we define without normalisation, i.e.

$$(F_n)^2_{IJ} = F_{IL_1\dots I_{p-1}} F_{JM_1\dots M_{p-1}} G^{L_1 M_1} \dots G^{L_{p-1} M_{p-1}} \,. \tag{0.3}$$

Furthermore, we adopt natural units in which

$$c = \hbar = k_B = \epsilon_0 = 1 \,, \tag{0.4}$$

but mostly keep explicit Newton's gravitational constant

$$G_N = 6.7087(10) \times 10^{-39} (\text{GeV})^{-2} \,, \tag{0.5}$$

such that the (4-dimensional) Planck mass is given as

$$M_P = \frac{1}{\sqrt{G_N}} \simeq 10^{19}\text{GeV} \,. \tag{0.6}$$

The standard Einstein-Hilbert action is of the form

$$\mathcal{S}^{(4)}_{EH} = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-g} R(g) = \frac{\bar{M}_P^2}{2} \int \mathrm{d}^4x \sqrt{-g} R(g) \,, \tag{0.7}$$

with the *reduced Planck mass* $\bar{M}_P = \sqrt{1/(8\pi G_N)}$. Similarly in D dimensions, we define the (reduced) Planck mass $\bar{M}_{P,D}$ as the coefficient in front of the Ricci scalar of the action

$$\mathcal{S}^{(D)}_{EH} = \frac{\bar{M}_{P,D}^{(D-2)}}{2} \int \mathrm{d}^Dx \sqrt{-g} R(g) \,. \tag{0.8}$$

which is in the *Einstein frame*, in contrast to the *string frame* defined later in Eq. (1.11).

For differential geometry conventions and identities, see Appendix E.

Part I.

String Theory & Co.

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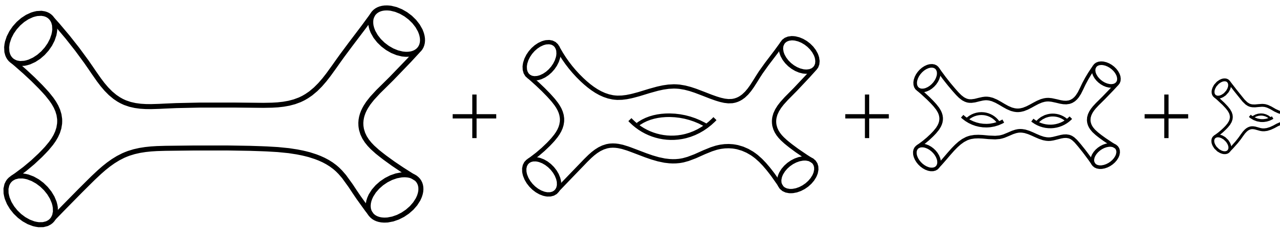


CHAPTER 1

Selected elements of string theory

As of today, string theory is our most promising candidate for obtaining a complete theory describing all the known interactions of our universe. In fact, it is the so far only fully developed consistent theory of quantum gravity which furthermore elegantly reproduces the know gravitational interaction governed by Einstein's General Relativity in its low-energy limit. The theory is built on an elegant and extremely rich underlying mathematical structure. Being defined as a theory in 10 spacetime dimensions, the way in which lower dimensional (effective) descriptions arise, that at least resemble the Standard Model of Particle Physics, is rather involved. As such, it is clear that any self-contained introduction to string theory and its phenomenological implications goes well beyond the scope of the introductory chapter of a doctoral thesis. Therefore, in the present work, we will assume some familiarity of the reader with the basic concepts of string theory. The following chapter should rather be viewed as refresher on some selected aspects that will be particularly important in the course of this thesis. For complete introductions to the (basics) of string theory, we refer to the excellent standard references [32–37].

1.1. The string worldsheet non-linear σ-model

The bosonic string in Minkowski spacetime is described most conveniently in terms of the so-called Polyakov action [72]

$$\mathcal{S}_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}h^{\alpha\beta}\eta_{IJ}\partial_\alpha X^I\partial_\beta X^J\,, \tag{1.1}$$

with $h_{\alpha\beta}$ the (auxiliary) metric on the worldsheet Σ, parametrised by τ and σ. The metric h is not be confused with a non-trivial spacetime metric G_{IJ} which we introduce below. Besides being *Poincaré invariant*, *reparametrisation invariant* and (classically) *Weyl invariant*, the action is crucially *renormalizable*. Due to the symmetries mentioned above, there is enough freedom to choose the so-called conformal gauge in which $h_{\alpha\beta} = \eta_{\alpha\beta}$. In particular, the Polyakov action with flat target space classically is a 2d *Conformal Field Theory* (CFT) of D massless free scalar fields $X^\mu(\tau,\sigma)$ (see [73] for an introduction).

The Polyakov action (1.1) is merely a special case of the more general notion of a *non-linear σ-model* (NLSM), which can be very naturally[1] generalised to a generic manifold M with metric tensor G_{IJ}, such that the action reads

$$\mathcal{S}_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}h^{\alpha\beta}G_{IJ}\partial_\alpha X^I\partial_\beta X^J\,. \tag{1.2}$$

Here, X^I are target space embedding fields, i.e. maps $X^I : \Sigma \to M$ from the worldsheet into the target space M. Although at first glance this might seem like a mild generalisation, since the theory is no longer quadratic in X, the action now describes an *interacting* field theory. We can expand around a classical solution at $X(\sigma) = X_0$, by writing $X^I(\sigma) = X_0^I + Y^I(\sigma)$ such that we obtain[2]

$$G_{IJ}\partial_\alpha X^I\partial_\beta X^J = \left(G_{IJ}(X_0) + G_{IJ,K}(X_0)Y^K + \frac{1}{2}G_{IJ,LK}(X_0)Y^LY^K + \dots\right)\partial_\alpha Y^I\partial_\beta Y^J\,. \tag{1.3}$$

In the above expression we can view the $G_{IJ,K\dots}(X_0)$ as coupling constants, determining the kinetic term and an infinite tower of higher-order interactions for the fields Y^I. Since the fields Y^I have (worldsheet) dimension zero, all the interaction terms have an associated coupling of dimension zero, and hence the NLSM (1.2) is *renormalizable* by power counting. We will return to this fact in more detail in Chapter 4. We can also determine the theory's range of validity as a weakly coupled perturbative theory. In particular, there is a characteristic (target space) length scale associated with the expansion, the radius of curvature r_c, which is roughly set by $\partial_X G \sim r_c^{-1}$. Rescaling

[1] Indeed, writing $G_{IJ} = \eta_{IJ} + f_{IJ}(X)$ with f a perturbation around flat Minkowski, we can think of this as a string moving in a background of a coherent state of gravitons, corresponding to a (superposition of) wave functions $f_{IJ} \sim \zeta_{IJ}e^{ik\cdot X}$; cf. [33,35].

[2] Indices after a comma denote derivatives with respect to this index.

$Y^I \to \sqrt{\alpha'} Y^I$ such that the new Y^I has target space (mass) dimension zero, we see that the effective (dimensionless) coupling becomes $\sqrt{\alpha'}/r_c$. Hence, whenever the characteristic scale of the target space geometry, given by r_c is much larger than the string scale, we have a well-defined perturbation theory[3], measured by powers of α'. Furthermore, since this also implies that the associated energy scales are very low, we can safely ignore the extended nature of the string in the limit $\alpha' \to 0$ and move to a description in terms of a *low-energy effective action* of point-like particles. Hence, the expansion in α' measures the "stringiness" with which we describe the geometry.

We stressed before that the NLSM exhibits *classical Weyl invariance*. However, generically this symmetry is broken by quantum effects. Consider first the case of a flat target space with metric $\eta_{\mu\nu}$. Without going into any details, note that the breakdown of Weyl invariance is captured by a non-vanishing trace of the energy momentum tensor $T^\alpha{}_\alpha$, which at the end of the day reads

$$T^\alpha{}_\alpha = -\frac{D-26}{12} R(h) \,, \tag{1.4}$$

where $R(h)$ is the Ricci scalar of the worldsheet metric h, D the dimension of the target space M and the factor of 26 comes from the bc ghost system that has to be included, cf. [32–37]. Hence, we see that the *Weyl anomaly* vanishes exactly when $D = 26$,which is the critical dimension for the bosonic string.

One can repeat the above calculation for a non-flat target space metric G_{IJ}, for example by *covariant background field expansion*. In particular, computing the relevant diagrams that contribute to the trace of T at leading order, one obtains an additional anomaly contribution

$$T^\alpha{}_\alpha = -\frac{1}{2} R_{IJ}(G) \partial_\alpha X^I \partial^\alpha X^J \equiv -\frac{1}{2\alpha'} \hat{\beta}^G_{IJ} \partial_\alpha X^I \partial^\alpha X^J \,. \tag{1.5}$$

The coefficient $\hat{\beta}^G_{IJ}$ is called *Weyl anomaly coefficient* and measures the theory's failure of being Weyl invariant. In particular, we see that the theory is Weyl invariant if and only if $R_{IJ}(G) = 0$, hence the target space is Ricci flat. That is, a consistent background needs to fulfil the vacuum *Einstein equation*.

We use the notation $\hat{\beta}$, in order to distinguish it from the β-function of the associated RG flow of the NLSM, which is closely related but, in general, does not coincide with the anomaly coefficient. This will be discussed in more detail in Section 4.2.2.

[3]String theory is defined as a double expansion in both α' and the *string coupling* $g_s = e^\Phi$, with the latter an expansion in string loops, corresponding to the genus of the associated string worldsheet. Hence g_s sets the strength of string interactions and a well-defined perturbative expansion also requires small string coupling, which in contrast to α' a free parameter, is dynamically set by (the vacuum expectation value of) the dilaton Φ.

1.1.1. Including the Kalb-Ramond field and dilaton

The NLSM action (1.2) is in fact not the only possible (bosonic) contribution, that is compatible with reparametrisation invariance and (power counting) renormalisability[4]. There are two more contributions that one can add, such that the full NSNS non-linear σ-model action reads

$$\mathcal{S}_\sigma = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\Big(\sqrt{-h}h^{\alpha\beta}G_{IJ}\partial_\alpha X^I\partial_\beta X^J + \epsilon^{\alpha\beta}B_{IJ}\partial_\alpha X^I\partial_\beta X^J + \alpha'\sqrt{-h}R(h)\Phi\Big)\,. \tag{1.6}$$

Here we introduced the Kalb-Ramond two-form field B as well as the dilaton scalar field Φ. The latter actually explicitly breaks Weyl invariance. However, on dimensional grounds, it enters at first order in α' and hence can be considered a loop effect. It will turn out that its tree-level contributions exactly cancel the loop-level contributions of G and B. Repeating the Weyl anomaly calculation, i.e. calculating also the loop contributions coming from B and Φ, one finds

$$T^\alpha{}_\alpha = -\frac{1}{2\alpha'}\hat{\beta}^G_{IJ}\partial_\alpha X^I\partial^\alpha X^J - \frac{1}{2\alpha'}\hat{\beta}^B_{IJ}\epsilon^{\alpha\beta}\partial_\alpha X^I\partial_\beta X^J - \frac{1}{2}\hat{\beta}^\Phi R(h)\,, \tag{1.7}$$

with anomaly coefficients

$$\begin{aligned}
\hat{\beta}^G_{IJ} &= \alpha'\left(R_{IJ}(G) - \frac{1}{4}H_{IKL}H_J{}^{KL} + 2\nabla_I\nabla_J\Phi\right) + \mathcal{O}(\alpha'^2)\,,\\
\hat{\beta}^B_{IJ} &= \alpha'\left(\frac{1}{2}\nabla^L H_{LIJ} - \nabla^L\Phi H_{LIJ}\right) + \mathcal{O}(\alpha'^2)\,,\\
\hat{\beta}^\Phi &= \frac{D-26}{6} - \alpha'\left(\frac{1}{2}R(G) + \frac{1}{24}H_{IJL}H^{IJL} + 2\nabla_L\Phi\nabla^L\Phi - 2\nabla^2\Phi\right) + \mathcal{O}(\alpha'^2)\,.
\end{aligned} \tag{1.8}$$

Note that the anomaly coefficient for the dilaton (working with $D = 26$) is first-order in α' and hence can indeed be considered a higher-order loop effect. The theory is Weyl invariant up to leading order in α', if the above coefficients vanish. We already mentioned that in the absence of H and Φ, the vanishing of the anomaly coefficient for G precisely gives the Einstein equation in the vacuum and therefore it is suggestive to view the associated Einstein-Hilbert action as the low-energy effective action descending from the stringy NLSM. Amazingly, even in the presence of B and Φ, we can write down such an effective action, and it takes the relatively simple form

$$\mathcal{S}^{(D)} = \frac{1}{2\tilde{\kappa}_D}\int \mathrm{d}^D x\sqrt{-G}e^{-2\Phi}\Big(-\frac{2(D-26)}{3\alpha'} + R(G) - \frac{1}{12}H_{IJK}H^{IJK} + 4\partial_I\Phi\partial^I\Phi + \mathcal{O}(\alpha')\Big). \tag{1.9}$$

[4] These are all operators of dimension two, hence resulting in a dimensionless coupling constant. There is in fact a dimension zero operator that can be added to the action, which corresponds to a coupling to a background tachyon. It breaks Weyl invariance, but actually plays no significant role and is therefore omitted here; cf. [74].

Indeed, varying the action with respect to G, B and Φ, the resulting equations of motion are precisely given by (1.8). The action (1.9) and its interpretation as a low-energy effective action will play a central role in this thesis and will be discussed in more detail in the following.

Remark

- For the superstring (in the RNS (Ramond–Neveu–Schwarz) formulation [75,76]), the Polyakov action is supplemented by an additional Dirac term that describes D free Mayorana fermions on the world sheet in the vector representation of the Lorentz group $SO(D-1,1)$. The total action reads

$$\mathcal{S} = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\sigma \left(\partial_\alpha X_I \partial^\alpha X^I + \bar{\psi}_I \rho^\alpha \partial_\alpha \psi^I\right) , \tag{1.10}$$

  with ρ^α two-dimensional Dirac matrices obeying the usual Dirac algebra. This leads to a $\mathcal{N} = (1,1)$ supersymmetric worldsheet NLSM. The resulting theory goes by the name of *Superstring Theory*, and turns out to be well defined in the critical dimension $D = 10$. We will not discuss the superstring here further, but refer to the standard literature [32–37]

1.2. RR sector and low-energy effective theories

So far we have only covered the NSNS sector of the bosonic string, which gives rise to space-time bosons. However, the superstring bosonic spectrum on target space also includes states from the RR sector. These cannot be directly coupled to the worldsheet non-linear σ-model, at least in the RNS-formulation[5]. We will not discuss this issue here but directly move to their *low-energy effective theories*. Whenever we work at energies far below the string scale, we can safely omit[6] the massive string states and work with an effective description in terms of 10D supergravity.

1.2.1. Type II supergravity

We briefly summarise the bosonic sector of Type II *Supergravity* (SUGRA) theories, which will be used in the later Part II of this thesis. For the remaining supergravity theories, as well as the fermionic sector and supersymmetry, we again refer to the standard literature [32–36, 82].

[5]On has to use either the Pure Spinor formalism [77,78] or the Green-Schwarz formalism [79–81].
[6]This also requires weak curvature, so we can omit higher α' corrections; cf. discussion below (1.3).

NSNS sector

Working in the critical $D = 26$ or $D = 10$ dimensions for the bosonic or superstring respectively, the NSNS sector, common to all[7] supergravity theories reads

$$\mathcal{S}^{(D)} = \frac{1}{2\tilde{\kappa}_D^2} \int \mathrm{d}^D x \sqrt{-G} e^{-2\Phi} \left(R(G) - \frac{1}{12} H_{IJK} H^{IJK} + 4\partial_I \Phi \partial^I \Phi \right) , \tag{1.11}$$

where we omit higher-order contributions in α'. The constant $\tilde{\kappa}_{10}$ is not determined a priori and can be changed by rescaling of Φ. After pairing up with the vacuum expectation value of Φ (and going to the Einstein frame), $\tilde{\kappa}_D$ sets the gravitational coupling constant $\kappa_D = \tilde{\kappa}_D e^{\Phi_0}$. The present form of the action, with the factor $e^{-2\Phi}$ in front of $R(G)$, is referred to as the *string frame* action.

RR sector

Type II string theory comes in two different variants; Type IIA and Type IIB theories. Accordingly, there are two supergravity actions which are also denoted by IIA and IIB. In particular, the massless sector of Type IIA string theory or supergravity includes bosonic states from the tensor product of two Ramond ground states, which give rise to RR form fields of odd degree, C_1, C_3. Hence, Type IIA supergravity includes these fields and their associated field strengths F_2 and F_4, which are defined as

$$F_2 = \mathrm{d}C_1 \, , \qquad F_4 = \mathrm{d}C_3 - H \wedge C_1 \, , \tag{1.12}$$

and similar for Type IIB with fields C_0, C_2, C_4 and associated field strengths F_1, F_3, F_5.

Adding the contributions to the NSNS sector, the Type II supergravity action reads

$$\mathcal{S}_{\mathrm{II}} = \frac{1}{2\tilde{\kappa}_{10}^2} \int \mathrm{d}^{10} x \sqrt{-G} \left[e^{-2\Phi} \left(R(G) - \frac{1}{2}|H|^2 + 4|\partial\Phi|^2 \right) - \frac{1}{2} \sum_p \left(1 - \frac{\delta_{p5}}{2} \right) |F_p|^2 \right] , \tag{1.13}$$

where $p \in \{2, 4\}$ for IIA and $p \in \{1, 3, 5\}$ for IIB. The five-form flux F_5 has to be supplemented with a self-duality constraint, rendering the Type IIB action strictly speaking a pseudo-action and leading to the additional factor of 1/2 in (1.13). This is only the dynamical part of the action. There are also topological Chern-Simons (CS) terms $\mathcal{S}_{\mathrm{CS}}$ and localised source contributions $\mathcal{S}_{\mathrm{source}}$, for example from D$p$-branes or orientifold O$p$-planes, that have to be added. The former read

$$\mathcal{S}_{\mathrm{CS}}^{\mathrm{IIA}} = -\frac{1}{4\tilde{\kappa}_{10}^2} \int B_2 \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3 \, , \qquad \mathcal{S}_{\mathrm{CS}}^{\mathrm{IIB}} = -\frac{1}{4\tilde{\kappa}_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \, . \tag{1.14}$$

[7] In the Type I theory, the Kalb-Ramond field B is projected out.

The source contributions in turn, modify the resulting *Bianchi identities* (see below) and are crucial for *tadpole cancellation* [83] (see for example [32]). Furthermore, it is clear that they modify the equations of motions, and we make use of these contributions when discussing geometric flows in Chapter 10. Hence, we list some relevant terms, as well as the deformation of Type IIA – *massive* Type IIA – and corresponding equations of motions in Appendix A.

Equations of motion

The equations of motion can be obtained by variation of the action (1.13). For the Type IIA theory, neglecting possible source contributions, those coming from the variation of G and Φ read

$$0 = R_{IJ} - \frac{1}{4}H^2_{IJ} + 2\nabla_I\nabla_J\Phi - e^{2\Phi}\left(\frac{1}{2}(F_2^2)_{IJ} + \frac{1}{12}(F_4^2)_{IJ} - \frac{1}{4}g_{IJ}\left(|F_2|^2 + |F_4|^2\right)\right), \tag{1.15}$$

$$0 = 2\nabla^2\Phi - 2|\nabla\Phi|^2 + \frac{1}{2}R - \frac{1}{4}|H|^2\,, \tag{1.16}$$

while the equation of motion for B is

$$0 = \mathrm{d}(e^{-2\Phi} \star H) - F_2 \wedge \star F_4 - \frac{1}{2}F_4 \wedge F_4\,. \tag{1.17}$$

The variations with respect to the fluxes give

$$0 = \mathrm{d}\star F_2 + H \wedge \star F_4 = 0\,, \tag{1.18}$$

$$0 = \mathrm{d}\star F_4 + H \wedge F_4 = 0\,. \tag{1.19}$$

These equitations of motion are supplemented with the *Bianchi identities*

$$\mathrm{d}H = 0\,, \qquad \mathrm{d}F_2 = 0\,, \qquad \mathrm{d}F_4 = H \wedge F_2\,. \tag{1.20}$$

The Type IIB equations are analogous, and we refer to the literature.

Remarks

- Strictly speaking, the equation for G in (1.15) is obtained only after variation and subsequent substitution of the dilaton equation of motion (1.16) in order to eliminate contributions $G_{IJ}R$ and $G_{IJ}|H|^2$. On-shell this fact is not important, but it will matter in Chapter 10 when discussing flow equations.
- It is not hard to see that upon setting the RR fluxes to zero, the equations of motion for G, B, Φ indeed match the condition of vanishing Weyl anomaly coefficients

$\hat{\beta}_{IJ} = 0$. This is immediate for G and Φ. For later use in Chapter 9 and 10 we note[8]

$$0 = \star(-2\mathrm{d}\Phi \wedge \star H + \mathrm{d} \star H) = -2\iota_{(d\Phi)^\sharp} H - \mathrm{d}^* H = -2\hat{\beta}^B \,. \tag{1.21}$$

- Type IIB supergravity action in the Einstein frame admits the particularly nice, $SL(2,\mathbb{R})$-invariant rewriting (with $\tau = C_0 + ie^{-\Phi}$ the *axio-dilaton*)

$$\mathcal{S}_{\mathrm{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left(R(G) - \frac{1}{2} \frac{\partial_M \tau \partial^M \tau}{\mathrm{Im}(\tau)^2} - \frac{1}{2} \frac{|G_3|^2}{\mathrm{Im}\{\tau\}} - \frac{1}{4} |F_5|^2 \right) - \mathcal{S}_{\mathrm{CS}} \,. \tag{1.22}$$

We will return to it in Section 2.1, when discussing *S-duality*.

1.2.2. 11D supergravity

Supersymmetry places very tight restrictions on the form an action can have in D dimensions, while satisfying certain requirements like causality or not having particles with spin greater than two. In fact, it turns out that in $D = 11$ there is a unique supergravity action which reads [84]

$$\mathcal{S}_{\mathrm{11D}} = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x \sqrt{-G} \left(R(G) - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int C_3 \wedge F_4 \wedge F_4 \,, \tag{1.23}$$

with $F_4 = \mathrm{d}C_3$. Compactifying on a circle, one obtains Type IIA supergravity [85]. Furthermore, the action can be viewed as the low-energy limit of another theory of quantum gravity that has not been discussed so far, namely *M-theory*. This will be briefly discussed in Chapter 2. The equations of motion[9] read

$$0 = R_{MN} - \frac{1}{12} (F_4)^2_{ij} + \frac{1}{6} |F_4|^2 \,, \qquad 0 = d \star F_4 + \frac{1}{2} F_4 \wedge F_4 \,, \tag{1.24}$$

together with the trivial Bianchi identity $\mathrm{d}F_4 = 0$.

1.3. Lower-dimensional effective theories

So far, all our considerations have been performed in the critical dimension of $D = 26$ or $D = 10$, for the bosonic and superstring, respectively. However, in order to connect string theory as a theory of quantum gravity and their low-energy effective theories discussed in the previous Section to conventional physics in four dimensions, one needs to explain the seemingly absent additional 6 dimensions. One way to achieve this is

[8] Here we used the identities $\iota_X \star \omega = (-1)^k \star (X^\beta \wedge \omega)$ for $\omega \in \Omega^k(M)$. This gives $\star(\mathrm{d}\Phi \wedge \star H) = (-1)^{(n-3)} \iota_{(\mathrm{d}\Phi)^\sharp} \star \star H = (-1)^{4(n-3)} \iota_{(\mathrm{d}\Phi)^\sharp} H = \iota_{(\mathrm{d}\Phi)^\sharp} H$; see also Appendix E for conventions.

[9] In fact, this is the trace reversed form of the "Einstein equation" for g.

through so-called *compactification* and successive *dimensional reduction* in order to obtain a lower-dimensional (low-energy) *effective theory*[10]. This is an extremely important but also vastly broad subject, so we will only briefly sketch some important aspects. More details can be found in the standard textbook references [32–37] and [82,86]. A textbook more tailored to string phenomenology and model-building is [87] as well as [88–90]

1.3.1. Compactifications

The basic idea of compactification is rather simple. Starting form a D-dimensional (total) space, we assume that (at least locally) we can write the geometry as

$$\mathcal{M}_D = \mathcal{M}_d \times K_n \,, \qquad d = (D-n)\,, \tag{1.25}$$

with the n-dimensional compact manifold K_n and the (generically non-compact maximally symmetric) $d = (D-n)$ dimensional manifold $\mathcal{M}^d$. We call the former the *compactification* or *internal* manifold, and the latter the *external* manifold. The full configuration, of course, still needs to satisfy the equations of motion, and this puts strong restrictions on the possible geometries. In the absence of fluxes, and assuming that at the product is global, such that the total metric takes the form

$$\mathrm{d}s^2 = G_{IJ}\mathrm{d}X^I\mathrm{d}X^J = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + h(y)_{mn}\mathrm{d}y^m\mathrm{d}y^n \,, \tag{1.26}$$

this implies that both spaces, in particular K_n need to be Ricci flat $R_{mn} = 0$. Here external directions and the associated metric are denoted with x and $g(x)$ while the internal ones are given by y and $h(y)$,which is not to be confused with the worldsheet metric of the NLSM (1.6) introduced before. The simplest examples of such internal manifolds, which will be discussed in Chapter 2, are given by the circle S^1 or more generally n-dimensional tori T^n. These are special examples of *Calabi-Yau* manifolds, which play a paramount role in string theory compactifications. We will not discuss Calabi-Yau manifolds here but refer to [32] for an introduction. Instead we will sketch the idea on the example of the circle, which will reveal a crucial property of string theory compactifications, the notion of *T-duality*.

The closed string compactified on a circle

We split the $D = 26$ dimensional space into a 25-dimensional external part and a circle S^1_r of radius r along the X^{25} direction. The closed string can then wrap the compact direction, resulting in a nontrivial winding number $w \in \mathbb{Z}$, relaxing the periodicity

[10] Not every effective theory, is a low-energy effective theory, see Remark in Section 1.3.2.

relation to

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi r w\,, \qquad \omega \in \mathbb{Z}\,. \tag{1.27}$$

A standard mode expansion leads to

$$X^{25}(\sigma, \tau) = x^{25} + \frac{\alpha' n}{r}\tau + w r \sigma + \text{osc.}\,, \tag{1.28}$$

where we have taken into account the winding mode contribution as well as the fact that momentum on a compact space is quantised according to $p^{25} = n/r$ with $n \in \mathbb{Z}$ the momentum modes. The higher Fourier or oscillator modes are not important in the following and are denoted by *osc.* Comparing with the mode expansion for the left- and right-moving components, we see that for the compact direction X^{25}, we now have

$$p_L = \frac{n}{r} + \frac{wr}{\alpha'}\,, \qquad p_R = \frac{n}{r} - \frac{wr}{\alpha'}\,. \tag{1.29}$$

From the viewpoint of an observer in the remaining 25 non-compact directions the mass evaluates to

$$m^2 = -\sum_{\mu=0}^{24} \frac{n^2}{r^2} + \frac{w^2 r^2}{\alpha'^2} + \text{osc.} \tag{1.30}$$

The mass formula is invariant under the transformation[11]

$$r \longrightarrow \tilde{r} = \frac{\alpha'}{r}\,, \qquad m \longleftrightarrow w\,, \tag{1.31}$$

i.e., invariant under inversion of the radius and simultaneous exchange of winding and momentum modes. For the example at hand, it is then rather easy to see that T-duality extends to a duality of the full spectrum. Applying the transformation rules (1.31) to the centre-of-mass momenta p_L, p_R we see that T-duality acts as $(p_L^{25}, p_R^{25}) \longrightarrow (p_R^{25}, -p_L^{25})$, which can be extended to the full mode expansion and hence also the oscillator modes by

$$X_L^{25} \longrightarrow X_L^{25}\,, \qquad X_R^{25} \longrightarrow -X_R^{25}\,. \tag{1.32}$$

This is a $\mathbb{Z}_2$ symmetry that only acts on the right-moving sector of the string in a non-trivial way. However, the NLSM[12] corresponding to theories with $\widetilde{X}^{25} = X_L^{25} - X_R^{25}$ can be shown to be fully equivalent to the one of $X^{25} = X_L^{25} + X_R^{25}$ and hence T-duality really is a proper symmetry of the full bosonic string theory.

[11] The oscillator states, although not given here, are trivially invariant under this transformation.
[12] In the case of a toroidal target space this defines a CFT and hence the results is exact.

Remark

- The action of T-duality as a $\mathbb{Z}_2$ symmetry of the form (1.32) also carries over to the open string. From this one can in turn deduce that starting from an open string with *Neumann boundary conditions*[13], the dual open string $\tilde{X}^{25} = X_L^{25} - X_R^{25}$ has *Dirichlet boundary conditions* and vice versa. This shows that although breaking Poincaré invariance, Dirichlet boundary conditions cannot be neglected, and therefore T-duality necessarily leads to the notion of *D-branes* [91] (see [92,93] for a comprehensive introduction). In fact, since T-duality along a compact direction exchanges the two boundary conditions, the Dp-branes are mapped to $D(p+1)$ or $D(p-1)$-branes [91], depending on whether the compact dualised direction corresponds to a transverse or longitudinal direction of the brane, respectively.

1.3.2. Field theory limit and dimensional reduction

In case we are only interested in the low-energy effective theory of the above example, we can decouple the stringy contributions. This *field theory limit* amounts to sending $\alpha' \to 0$, such that we can indeed omit all oscillator contributions, as well as the winding states which have no analogue in a point-particle-like description and become infinitely heavy and hence decouple. If we are furthermore interested in length scales much larger than r, effectively taking the limit $r \to 0$, the remaining infinite tower of momentum modes is truncated to the massless zero mode $n = 0$. The resulting procedure is known as *Kaluza-Klein* (KK)reduction [94–96]. It is an example of a *dimensional reduction*. More generally for some given compactified manifolds K_n, dimensional reduction amounts to keeping only the zero modes of the (Fourier) expansion along the compact directions, whereas in the compactified theory a priori all, in particular also the higher, massive modes are kept in the lower-dimensional (effective) description.

Note that there are different notions of effective actions and reductions or truncations, and one should be careful in differentiating between them. Besides the already mentioned notion of Kaluza-Klein reductions there are *Scherk-Schwarz* reductions [97]. Furthermore, there is the concept of *(consistent) truncations*[14] and *effective actions*. Neither have to give rise to a valid *low-energy effective action*, correctly describing the (4d) physics below a given cutoff Λ, nor is a generic low-energy effective action, for example, a consistent truncation. See [98–100] and [101–103].

[13] Neumann boundary conditions require $\partial_\sigma X^I = 0$ at $0, \pi$, i.e. at the ends of the (open) string. In contrast, Dirichlet boundary conditions read δX^I at $0, \pi$, such that the ends of the string are fixed in some hypersurface $X^I = c^I$ in target space.

[14] A truncation of some D-dimensional theory, to a (lower) d-dimensional theory in terms of a reduced (truncated) set of degrees of freedoms or fields is called *consistent* if any solution in d-dimensions, can be lifted to full D-dimensional solutions of the full theory [98–100].

1.3.3. Moduli fields and moduli stabilisation

The equations of motion, or anomaly coefficients, put tight restrictions on the possible internal manifolds. However, there is still a rather extensive freedom on the precise shape and size of the internal compact manifold, even for a fixed topology. This freedom is parametrised by some (yet) unfixed free parameters that control exactly the size and shape of the geometry. These parameters manifest themselves after compactification as massless scalar fields, which are called *moduli fields* or simply *moduli* and each of them controls some certain aspect of the geometry[15].

This is highly problematic for several reasons. First, at the level of the dimensionally reduced effective action in $d = 4$, this would result in several massless scalar bosons, schematically

$$\mathcal{S}^{(4)} = \frac{\mathcal{V}_{\text{int}}}{2\kappa_{10}^2} \int \mathrm{d}^4x\sqrt{-g}\left(R(g) - \gamma_{ab}\partial_\mu\varphi^a\partial^\mu\varphi^b\right), \tag{1.33}$$

with $\mathcal{V}_{\text{int}}$ the volume of the internal space, $\mathcal{V}_{\text{int}} = \int_K \mathrm{d}^n y\sqrt{h}$. This would lead to deviations from Newton's law that are excluded [65] by experiment. Second, since these parameters are free, there is no singled-out vacuum expectation value. However, some of these scalars control the overall volume of the compact space, which hence is not fixed by the theory. This is problematic since too small or large internal spaces would run into conceptual or phenomenological problems, respectively. In particular, we argued in Section 1.1, that the effective dimensionless coupling is set by $\sqrt{\alpha'}/r_c$, where r_c is the characteristic length scale ("radius") of the geometry. Hence, to have a well-defined perturbative expansion and be able to neglect higher curvature corrections, we should have $r_c^2 \gg \alpha'$. On the other hand, if the size of the compact space is too large, this is incompatible with observations and is therefore ruled out by experiments [107].

It is therefore clear that in order to obtain phenomenologically meaningful models, which at the same time are under computational control, it is important to stabilise the moduli φ^a at some finite vacuum expectation value $\langle\phi^a\rangle$, the *string vacuum*, such that they are massive enough not to be in contrast with current observations. This is done by the introduction of a *scalar potential* $V(\varphi)$

$$\mathcal{S}^{(4)} = \frac{\mathcal{V}_{\text{int}}}{2\kappa_{10}^2} \int \mathrm{d}^4x\sqrt{-g}\left(R(g) - \gamma_{ab}(\varphi)\partial_\mu\varphi^a\partial^\mu\varphi^b - V(\varphi)\right), \tag{1.34}$$

and the resulting procedure is known as *moduli stabilisation*. These potentials arise

[15] The dilaton Φ, common to all String Theories is an exception. It is the distinguished (modulus) field, the vacuum expectation value of which sets the string coupling g_s. There are also moduli associated with RR-fields or open string moduli which we will not discuss here. See, for example, [104] for a compact introduction of moduli fields. Further discussions can be found in [87, 89, 105, 106] and the standard textbooks.

due to additional objects that have to be introduced to the 10D theory, which upon dimensional reduction source the potential,

$$V(\varphi) \supset \text{curvature, fluxes, (localised) sources...} \,. \tag{1.35}$$

The precise details and ingredients of the compactification model crucially influence the physics of the effective theory, for example, through the value and nature of the 4d cosmological constant Λ [108–110] or the way in which supersymmetry is broken [110–113]. Generating such potentials in a well-defined and well-behaved manner represents an active field of research of highest interest.

A particularly nice way to introduce such a potential is by using background fluxes, and the resulting procedure is known as *flux compactification*. This is again an extremely deep and rich subject, so we merely summarise the main idea and refer to the literature for a self-contained introduction; see [62, 88, 89, 106]. While for a pure gravitational background determined solely by some metric tensor G_{MN}, the possible solutions are restricted to Ricci flat backgrounds, once we introduce background NSNS or RR-flux, the space of solutions is greatly enlarged. In particular, we are no longer bound to Calabi-Yau (Ricci flat) manifolds, but to more general structures, in particular *generalised Calabi-Yau* spaces, which are tightly connected to the notion of *generalised geometry* to be briefly discussed in Chapter 3. We will see in detail how these terms enter the potential in Chapter 6, so we refer to this Section and the extensive literature on flux compactification listed earlier. Furthermore, we note that there are further exotic fluxes that have not been discussed so far, that have received quite some attention within the framework of moduli stabilisation. These are the so-called *non-geometric* fluxes [114–116] which we will be introduced in Chapter 3 and further discussed in Chapter 7. The easiest string-theoretic example of a true moduli space, i.e. without a potential, is the example of the KK reduction of the closed string on a circle.

Moduli spaces and their associated metric

For a theory with n moduli fields $\{\varphi^a\}$, the space spanned by these fields is called the *moduli space* $\mathcal{M}$ of the theory. This space is generically non-compact [117] and comes with a natural notion of associated metric $G_{ab}(\varphi)$, which is given in terms of the coefficient of the kinetic term γ_{ab} in (1.34), $G_{ab}(\varphi) = \gamma_{ab}(\varphi)$. Consequently, there exists a notion of distance on moduli space that can be viewed as a distance between physically different configurations of the underlying theory. This notion of distance will be paramount for the discussion of the Swampland program, in particular the Swampland Distance Conjecture and will be discussed in Chapter 5.

We already mentioned that there is a particularly intuitive modulus that corresponds to the overall volume of the compact space. The simplest example of such a (true) modulus is the radius of the circle compactification discussed before. Start from the (NSNS sector) low-energy effective action in terms of a ($D = d + 1$)-dimensional supergravity and compactify the theory on a circle, assuming the metric is of the form

$$\mathrm{d}s^2 = r^{\frac{-2}{d-2}} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + r^2 \mathrm{d}y^2 \,, \tag{1.36}$$

with r the radius of S^1. Allowing for r to be a function of the external coordinates x^μ, $r = r(x)$ and going to the Einstein frame (see the next Section) the resulting action reads

$$\mathcal{S}^{(d)} = \frac{\bar{M}_{P,d}^{(d-2)}}{2} \int \mathrm{d}^d x \sqrt{-g} \left(R(g) - \frac{d-1}{d-2} \frac{1}{r^2} \partial_\mu r \partial^\mu r \right) . \tag{1.37}$$

Hence we can identify $\gamma = \frac{d-1}{d-2}\frac{1}{r^2}$ as the metric on moduli space, which one might naively identify with the half-line $r \in [0, \infty)$. However, at the level of string theory, we have T-duality, which relates circles of radius r with $\tilde{r} = \alpha'/r$. These configurations are physically equivalent, and the true moduli space of inequivalent theories is given by

$$r \in (0, r_*] \;, \quad \left(\text{or equivalently } r \in [r_*, \infty)\right) \;, \tag{1.38}$$

where $r_* = \sqrt{\alpha'}$ is the self-dual radius.

Generically, the naive moduli space $\mathcal{M}$ as inferred from the massless scalar fields of the (effective) action gets reduced by nontrivial symmetries or dualities that relate points in moduli space. These correspond to physically equivalent theories when lifted to the level of string theory. In particular, in case there is a certain redundancy, for example due to gauge or duality symmetry parametrised by some group K, the true moduli space of inequivalent configurations $\widehat{\mathcal{M}}$ is

$$\widehat{\mathcal{M}} = \mathcal{M}/K \,. \tag{1.39}$$

In particular, in the present example we have

$$\widehat{\mathcal{M}} = (0, r_*] \cong \mathbb{R}_{>0}/\mathbb{Z}_2 \,, \tag{1.40}$$

where the $\mathbb{Z}_2$ corresponds to the T-duality inversion $r \to \alpha'/r$. We will encounter the natural extension to the case of a compactification on n-tori in Chapter 2.

When there is a non-zero scalar potential $V(\varphi)$, some of the directions in moduli space get lifted or stabilised and (at least some of) the fields become massive. In this case, the resulting field space is no longer called moduli space but simply *scalar field space* and the massive fields scalar fields. In this case the existence and physical meaning of a proper distance in the presence of a potential, induced by the metric on field space, is much more involved and the problem of defining and understanding such a distance in the presence of a potential is one of the main goals of this thesis. We will further review the resulting challenges in Chapter 5 and propose some possible implications and resolutions in the remaining chapters of the main Part II of this thesis.

We close this section on moduli spaces with a word of caution. There are several different notions of moduli spaces in the context of theoretical physics and mathematics. In the context of string compactifications, as explained above, we denote as moduli the massless scalar fields parametrising the internal geometry or coupling of our theory. Up to possible symmetries, these span the moduli space, or, in the presence of a scalar potential, the scalar field space of the theory. Relatedly, the word moduli space can also more generally denote the space of (inequivalent) physical configurations or solutions of a system. The most closely related and also most relevant notion of moduli space in mathematics is the *moduli space of Riemannian metrics* $\widehat{\mathcal{M}}(M)$ on some given manifold M, which we will discuss in detail in Chapter 4. As a special case in 2D, this contains the *moduli space of Riemannian surfaces* of genus g, denoted $\widehat{\mathcal{M}}_g$ which is defined as [33](or [118] for a mathematically rigorous introduction)

$$\widehat{\mathcal{M}}_g = \frac{\mathcal{M}(M_g)}{\mathrm{Diff}(M_g) \ltimes \mathrm{Weyl}(M_g)} , \tag{1.41}$$

where $\mathcal{M}(M_g)$ is the space of all Riemannian metrics for the manifold M_g of fixed genus g and $\ltimes$ denotes the *semidirect product*, cf. [119]. Since string theory is an expansion in worldsheet topologies[16] (genus), this space is of paramount importance as a measure for the Polyakov path integral, cf. [32,33].

Remarks

- In (1.34) there is a factor $\mathcal{V}_{int} = \int_{K_n} \mathrm{d}^n y \sqrt{h}$ in front of the action. Comparing with the definition of the reduced (4D) Planck mass given in (0.7) we obtain

$$\bar{M}_P^2 = \frac{\mathcal{V}_{\mathrm{int}}}{\kappa_{10}^2} = \frac{M_s^2}{\pi g_s^2} \frac{\mathcal{V}_{\mathrm{int}}}{l_s^6} . \tag{1.42}$$

  where we used that $\kappa_{10}^2 = \frac{(4\pi^2\alpha')^4}{4\pi} g_s^2$, $l_s = 2\pi\sqrt{\alpha'}$ and $M_s = \frac{1}{\alpha'}$. For large (in units of l_s) internal spaces, the string scale (and also 10D Planck scale) can be much lower than M_P. In the formal limit of taking $\mathcal{V}_{\mathrm{int}} \to \infty$, the (4D) Planck mass goes to infinity, and hence the gravitational coupling goes to zero, and gravity becomes

[16] We ignore worldsheets with boundaries which are important for the open string, as well as marked surfaces relevant for discussion of string scattering amplitudes.

negligible. The limit $M_P \to \infty$ is the so-called gravity *decoupling limit*[17]. On the other hand, for stringy sized extra dimensions, we roughly have $M_P \sim M_s$.

- (Perturbative) string theory is defined as a double expansion in the string coupling g_s as well as the σ-model coupling $\sqrt{\alpha}/r$. Hence in general, the action (1.34) and, in particular, the scalar potential receive perturbative quantum corrections induced by higher-loop effects and higher-curvature corrections to the effective action; see [87–89,106,110,120].

- Besides the perturbative corrections mentioned above, there are further non-perturbative corrections that arise, for example, from (worldsheet) instantons, D-branes wrapping certain internal cycles or gaugino condensation. We refer to [32,87] and the reviews [62,88,106,110,120].

- The corrections listed above play a crucial role for moduli stabilisation in certain models like the *KKLT* scenario [121] or the *Large Volume Scenario* (LVS) [122], where stabilisation is achieved not at leading but higher order. In contrast, in *DGKT*-like [123] models[18], all moduli may be stabilised without the need for non-perturbative or higher order contributions. Nevertheless, depending on the parametric regime (in particular small volume) correction can become important and can introduce for example stability issues; see [125,126] and references therein for a recent discussion in view of the Swampland program (cf. Chapter 5).

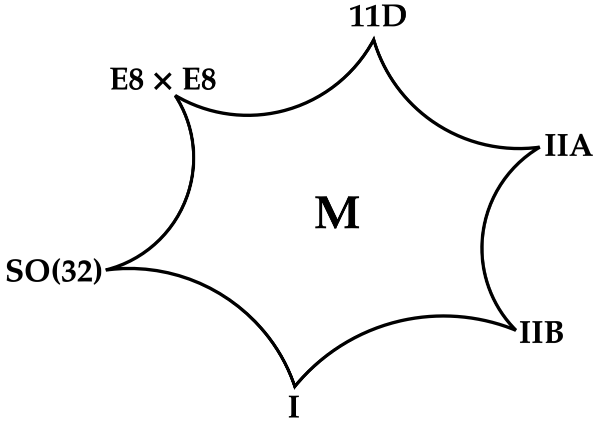
- For the case of $\mathcal{N} = 1$ 4d supergravity, there is a very convenient description of the scalar potential and metric on field space. In particular the non-topological (F-term) part of V can be derived from the so-called holomorphic superpotential W and from the Kähler potential K such that we have

$$V \supset V_F = e^K \left(G^{i\bar{j}} D_i W D_{\bar{j}} \overline{W} - 3|W|^2\right), \tag{1.43}$$

with $G_{i\bar{j}}$ the Kähler metric and $D_i W$ the Kähler covariant derivative and we have set $\kappa_4 = 1$. For details and further conventions, we refer to [32].

[17] Generically $\mathcal{V}_{\text{int}} \to \infty$ also generates an infinite tower of light KK modes and so the resulting theory can not be viewed as decoupled from gravity but as an additional dimensions opening up; cf. Chapter 5.

[18] These vacua were almost simultaneously found in [124].



CHAPTER 2

Dualities

In the introduction, we advertised string theory as the – possibly unique – candidate for a theory of quantum gravity. However, it is well-known that at the level of the full superstring theory, there are five distinct formulations that differ in particular by their particle content and gauge groups. The resolution of this apparent contradiction leads to the notion of *dualities*. These can be seen as a slightly generalised notion of symmetry, relating possibly distinct theories that are, however, *physically equivalent*. Dualities played a fundamental role in the development of string theory and they marked the beginning of the so-called *second superstring revolution*, which lead to the unification of these five different superstring theories. They are an essential part of the theories structure and provide invaluable tools in perforating explicit calculations in otherwise hopeless situations. Apart from the already mentioned T-duality, we will introduce the notion of S- and U-duality as well as an outlook on the conjectured 11D non-perturbative limit of Type IIA string theory, known as M-theory. Thereafter we will give a detailed discussion of T-duality on topological non-trivial spaces and in the presence of fluxes, as well as several generalised notions of T-duality adapted to non-Abelian isometries. Lastly we comment on the relation to topological defects as well as the role of generalised dualities for string theory and possible implications for the Swampland program to be discussed later in Part II.

2.1. Dualities as a unifying tool: From 5 to 1

We saw that T-duality on a circle inverts the radius of the circle while exchanging winding and momentum modes of the string. These are both states of the perturbative spectrum of the closed string, and T-duality maps a weakly coupled perturbative theory with light degrees of freedom given by the momentum modes to an equivalent weakly coupled theory with light degrees of freedom given by the winding modes. In particular, T-duality is a map at fixed order in the string loop expansion. We will see below that this behaviour carries over more generally and hence T-duality should be considered a *perturbative* duality.

However, it is already known from standard QFT that there are dualities that relate weakly coupled theories with strongly coupled descriptions of the same or even a different theory, the prime example being the *Montonen–Olive duality* [127]. The resulting dualities are called *strong-weak (coupling)* dualities and such a duality also exists in string theory where it goes by the name of *S-duality*. Similarly to T-duality it represents an invaluable tool in understanding the different corners, or realisations of string theory. S-duality can be very compactly motivated on the example of the Type IIB theory: at the quantum level, the global $SL(2,\mathbb{R})$ symmetry of the Type IIB supergravity action (1.22) is broken to $SL(2,\mathbb{Z})$. The axio-dilaton $\tau = C_0 + ie^{-\phi}$ then transforms under $SL(2,\mathbb{Z})$ through fractional linear transformations

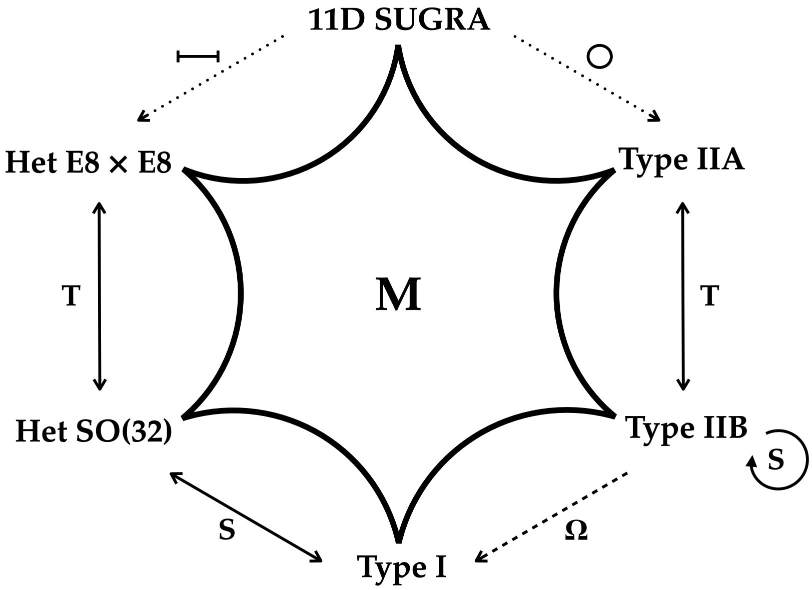
$$\tau \longrightarrow \tilde{\tau} = \frac{a\tau + b}{c\tau + d}\,, \qquad a,b,c,d \in \mathbb{Z}\,, \quad ad - bc = 1\,. \tag{2.1}$$

Now taking $C_0 = 0$ and considering the special element S with $a = d = 0$ and $c = -b = 1$ we obtain $\tau \to -1/\tau$, which in terms of the string coupling constant $g_s = e^{\Phi}$ translates to

$$g_s \longrightarrow \frac{1}{g_s}\,. \tag{2.2}$$

Therefore, the transformation is mapping a theory at weak string coupling to a theory at strong coupling and vice versa. Together with the element T defined by $a = b = d = 1, c = 0$, the elements S and T generate the full $SL(2,\mathbb{Z})$, which is the *S-duality group* of Type IIB string theory. Hence, Type IIB is in fact *self-dual* under S-duality.

In general S-duality will, however, not relate the strong and weakly coupled sectors of the *same* theory, but of different theories. For example, it turns out that the weakly coupled Type I theory is S-dual to the strongly coupled heterotic $SO(32)$ and vice versa. Furthermore, since T-duality exchanges the Neumann and Dirichlet boundary conditions, it maps Dp-branes to D$(p+1)$ or D$(p-1)$ branes, depending on whether the duality is applied along a transverse or longitudinal direction, respectively. This extends to the RR charges such that odd-degree forms are exchanged with even-degree form fields, and hence T-duality along a single compact direction maps Type IIA string theory to Type IIB and the other way around. Similarly, T-duality along a compact circle relates the two heterotic theories. In total, we obtain a web of dualities that can be summarised by the following diagram [85, 128] below.



Looking at the above diagram, several comments are in order. First note that the arrow from Type IIB to the Type I theory, with label Ω, denotes an orientifold projection resulting in the unoriented Type I theory. Furthermore, we added one more spike, i.e. one more theory to the star. This is 11-dimensional supergravity, which compactified on a circle, at the level of the low-energy effective actions, turns out to be equivalent to Type IIA (supergravity) theory. Similarly, when compactified on an interval (equivalently $S^1/\mathbb{Z}_2$) the theory reduces to the effective theory of the heterotic string with gauge group $E_8 \times E_8$; see [32, 37] for details.

Indeed, we see that all the different string theories, or supergravities, are related by an intricate web of dualities which unify [85, 128] the seemingly distinct theories. Even more, the diagram seems to suggest that all the different realisations are in some sense merely different (asymptotic) corners [85] of some Mother theory in the middle of the star, which we denoted by the symbol M. There are in fact strong hints that such a Mysterious theory exists, and it goes by the name of M-theory. We will briefly motivate this (conjectured) 11-dimensional theory in the following section.

Note that the word *duality*, although wildly used, has no clear definition and is frequently used together with the word symmetry, often even synonymously. Furthermore, it might or might not coincide with formal mathematical notions of duality operations. In the present thesis, we adopt the following notion of duality:

> *A duality, is a transformation between possibly different theories that are, however, physically equivalent. In particular, the physical spectrum, the equations of motion, the correlation functions, etc. are left invariant.*

In particular, we do not require a duality to leave the action invariant or to map a theory to the *same* theory. If a duality is a map between the same theories, we call it a *self-duality*, which is often used synonymously[1] with *symmetry*. Lastly, note that the combination of T- and S-dualities leads to the notion of U-duality [128]; see also [129].

For general overviews of dualities in supergravity and string theory as well as further references, see [32, 37, 130–133]. The canonical references tailored to T-duality and its generalised notions are [134–136].

2.1.1. M(atrix)-Theory

A possible way to think about the aforementioned 11-dimensional theory is to view its compactification on a circle to 10D as the (T-dual of the) strong coupling limit of the Type IIA string theory. In the low-energy limit the 11D theory is described by 11-dimensional supergravity; cf. Section 1.2.2. Since the theory includes a three-form field C_3, there are electrically and magnetically charged branes under this field, which are the $M2$ and $M5$ branes, respectively. Upon compactifying on a circle, these objects, together with the metric tensor, result in the bosonic field content of the Type IIA theory. Furthermore, the radius of the compact circle is identified with the coupling constant of the Type IIA theory [85]

$$r_{11} = l_s g_s \,, \qquad \left(equivalently : g_s = (r_{11} M_{P,11})^{\frac{3}{2}} \right) \tag{2.3}$$

such that the strong coupling limit of the latter corresponds to a decompatification limit[2] of the 11D theory. Note that M-theory does not include (fundamental) strings and since by definition the theory sits at strong coupling, there is no well-defined description in terms of a perturbative expansion. One can, however, learn more about the non-perturbative sector using the duality relations outlined above, for example, by

[1] Note that depending on the definition of symmetry used, a self-duality can be a slightly more general concept. For example, $SL(2,\mathbb{R})$ is a proper symmetry of the Type II supergravity action. At the quantum level it is broken to $SL(2,\mathbb{Z})$ and there is no full description in terms of a local action functional. Hence, requiring a symmetry to leave invariant the action, Type IIB string theory is merely self-dual under the S-duality group, which in that sense does not correspond to a proper symmetry.

[2] We keep this fact in mind, as (infinite) limits like this are of mayor interest in the Swampland program, in particular the Swampland Distance Conjecture.

making use of BPS protected non-perturbative objects like D0-branes. This allows us to extract information that goes well beyond 11-dimensional supergravity; see [37] for some easy examples.

An exact non-perturbative formulation of the microscopic degrees of freedom of M-theory is still lacking. One proposal for such a non-perturbative theory is *Matrix theory*, and in particular the *BFSS Matrix model* [137], see [138–140] for a review. Despite being the most promising formulation we have so far, the formulation comes with its own shortcomings, most notably the obscured role of the $M5$-branes in comparison to the $M2$-branes. In particular, the absence of (longitudinal) $M5$-brane currents can be traced back to the fact that the fundamental degrees of freedom in terms of matrices are associative and the current, which evaluates to [141]

$$\mathcal{M}^{+-ijkl} \sim \mathrm{STr}\left([X^i,[X^j,X^k]] + \text{cyclic}\right) , \tag{2.4}$$

hence vanishes by means of the Jacobi identity.

From the point of view of the Swampland program (cf. Chapter 5), in particular the M-theoretic Emergence Proposal (see [142] for a review) this, however, hints towards the fact, that the BFFS model is not yet a complete realisation of M-theory at the quantum level. Motivated by this fact, the author and collaborators in [70] explored the possibility of evading the problem related to the Jacobi identity, by introducing so-called *cubic matrices*. Defining appropriate products, these objects, together with usual (bi)-matrices form a non-associative algebra that can be regarded as a two-term truncated L_∞ algebra giving rise to a fundamental identity between the two and the three-bracket. Besides providing a simple class of concrete examples for such algebras a generalised construction of Yang-Mills theories, topological BF theory and generalised IKKT models was presented. We will not cover these topics here but refer the reader to the publication [70] and the references therein.

Lastly, we note that by construction the resulting theory features non-associativity. From the point of view of a particle theory, (spacetime) non-commutativity and non-associativity seem very exotic. However, we will see that non-commutativity and non-associativity appear almost naturally once string theory or more generally quantum gravity comes into play and this will be discussed in more detail in Chapter 3 when introducing *non-geometric* backgrounds and their associated fluxes.

For a general introduction to M-theory (and U-duality) see [129,143,144]; for a focus on the matrix model formulation [140,145,146].

2.2. A closer look on T-duality

We briefly summarised the fundamental role that dualities, in particular T-duality, played in the development of string theory. It can be shown that T-duality is a proper duality of the full (perturbative) quantum theory and hence provides a pivotal tool in many areas of the field. It will also play a major role in Part II of this thesis, and hence we devote the remainder of this chapter to a more in-depth treatment.

After a short review of T-duality on the n-torus we discuss abelian T-duality on curved backgrounds, which are by far the most rigorous and best understood notions of T-duality. We then move to generalised notions of T-duality. The role of the latter at the quantum level is not yet fully clear, and there are several subtleties that need to be investigated [147]. It is beyond the scope of this short introduction to the subject to give an in-depth discussion of these issues here, but we refer to the discussion and literature provided at the end of this Chapter. For us, generalised T-dualities will be mostly a solution generating technique[3] in order to generate new, generically less symmetric backgrounds and the implications for the Swampland program, in particular the Swampland Distance Conjecture, cf. Chapter 5.

2.2.1. Toroidal compactifications

The circle example of Chapter 1.6 can be easily generalised to the n-torus. Imposing the periodicities $X^I = X^I + 2\pi w^I$, with w^I the winding numbers, one can show that Hamiltonian of the zero modes in the compact directions can be written as

$$H = \frac{1}{2}(p_L^2 + p_R^2) = \frac{1}{2} Z^T \mathcal{H} Z \,, \tag{2.5}$$

with the $2d \times 2d$ matrix $\mathcal{H} \in O(d,d,\mathbb{R})$ called the generalised metric and Z a vector of momentum n_i and winding numbers w^i, which read

$$\mathcal{H} = \mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} , \qquad Z = \begin{pmatrix} m \\ n \end{pmatrix} . \tag{2.6}$$

The momenta $p_{L/R}$ form an even self-dual Lorentzian lattice $\Gamma^{(d,d)}$ [148] such that

$$p_L^2 - p_R^2 = 2 w^I n_I \in 2\mathbb{Z} \,, \tag{2.7}$$

and all such lattices are related by $O(d,d,\mathbb{R})$ rotations. Every point in the lattice corresponds to a different target space geometry with generically different spectra. However, since the momenta $p_{L/R}$ transform in the vector representation of $O(d,d,\mathbb{R})$,

[3] See also the remark at the end of the Chapter.

the terms $p_{L/R}^2$ are invariant under $O(d,\mathbb{R})$ such that the Hamiltonian is invariant under the maximal compact subgroup $O(d,\mathbb{R}) \times O(d,\mathbb{R})$. Extending the argument to the oscillator modes shows that the moduli space of toroidal compactifications is locally isomorphic to [134, 148, 149]

$$\frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \times O(d,\mathbb{R})} \,. \tag{2.8}$$

The remaining question is whether there exists a subgroup H of transformations of the above coset that leaves the physical system, in particular the spectrum, invariant and therefore relates two physically equivalent[4] theories. Since $\mathcal{H}$ transforms under $O \in O(d,d,\mathbb{R})$ according to $\mathcal{H}' = O^T \mathcal{H} O$, it can worked out explicitly by acting on $\mathcal{H}$, that such a subgroup is given by $H = O(d,d,\mathbb{Z})$. Therefore, the true moduli space of inequivalent theories reads [134]

$$\frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \times O(d,\mathbb{R})} \Big/ O(d,d,\mathbb{Z}) \,. \tag{2.9}$$

The subgroup $O(d,d,\mathbb{Z})$ precisely generalises the simple $r \to 1/r$ duality of the circle compactification and describes T-duality for toroidal backgrounds. For a discussion of the generating elements of $O(d,d,\mathbb{Z})$ and their physical interpretation in terms of diffeomorphisms, gauge as well as duality transformations, see, for example, [134].

2.2.2. Curved backgrounds and the Buscher rules

The closed string on a (curved) background is described by the worldsheet non-linear σ-model (1.6). Now assume that the action (1.6) is invariant under a diffeomorphism generated by the (nowhere vanishing, globally defined) Killing vector k^μ, i.e. under the global transformation

$$\delta_\epsilon X^I = \epsilon k^I(X) \,. \tag{2.10}$$

In particular, this means that the background has to satisfy (ω a one-form)

$$\mathcal{L}_k G = 0 \,, \qquad \mathcal{L}_k \Phi = 0 \,, \qquad \mathcal{L}_k B = \mathrm{d}\omega \,. \tag{2.11}$$

Working in adapted coordinates such that $k = (1, 0, \ldots, 0)$ we can use the above invariance to choose all background fields independent of the isometry direction X^0. We can then gauge the global symmetry (2.10), i.e. consider the local transformation

$$\delta_\epsilon X^I = \epsilon(X) k^I(X) \,, \tag{2.12}$$

[4] Full equivalence of the physical theories requires not only an automorphism on the space of states but also a perfect mapping of correlation functions under the transformation. We refer to [134] for a thorough discussion of the associated gauge symmetry that establishes the equivalence.

and introduce a gauge field A and covariant derivative $D_\alpha X^I$ such that

$$\delta A_\alpha(X) = \partial_\alpha \epsilon(X) + A_\alpha \epsilon\,, \quad \partial_\alpha X^I \to \mathrm{D}_\alpha X^I = \partial_\alpha X^I - k^I A_\alpha\,. \tag{2.13}$$

Then we arrive at[5] the action [150, 151]

$$\begin{aligned} \mathcal{S}_\sigma = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \Big(& \sqrt{-h} h^{\alpha\beta} G_{IJ} \mathrm{D}_\alpha X^I \mathrm{D}_\beta X^J + \epsilon^{\alpha\beta} B_{IJ} \partial_\alpha X^I \partial_\beta X^J \\ & + \epsilon^{\alpha\beta}\left(A_\alpha k^I B_{IJ} \partial_\beta X^J - 2 A_\alpha \omega_I \partial_\beta X^I \right) + \epsilon^{\alpha\beta} \chi \partial_\alpha A_\beta + \alpha' \sqrt{-h} R(h) \Phi \Big)\,, \end{aligned} \tag{2.14}$$

where we have introduced the Lagrange multiplier field χ. Integrating out the Lagrange multiplier leads to A being pure gauge, $A = \mathrm{d}\xi$. Using the gauge invariance (2.13), A can then be set to zero, leading back to the original action. On the other hand, integrating out the gauge field A leads to an action that is equivalent to (1.6), now with the background quantities $\{G, B, \Phi\}$ replaced by $\{\tilde{G}, \tilde{B}, \tilde{\Phi}\}$, where the latter are given by [152, 153]

$$\begin{aligned} \tilde{G}_{00} &= \frac{1}{G_{00}}\,, & & \\ \tilde{G}_{0I} &= \frac{B_{0I}}{G_{00}}\,, & \tilde{B}_{0I} &= \frac{G_{0I}}{G_{00}}\,, \\ \tilde{G}_{IJ} &= G_{IJ} - \frac{G_{0I}G_{0J} - B_{0I}B_{0J}}{G_{00}}\,, & \tilde{B}_{IJ} &= B_{IJ} - \frac{G_{0I}B_{0J} - G_{0I}B_{0J}}{G_{00}}\,, \end{aligned} \tag{2.15}$$

together with the shift of the dilaton according to

$$\tilde{\Phi} = \Phi - \frac{1}{2}\log G_{00}\,. \tag{2.16}$$

The coordinate X^0 is now replaced by the Lagrange multiplier χ, which is then usually denoted $\tilde{X}^0$, such that the metric is given in the basis $\{\tilde{X}^0, X^1, \ldots, X^D\}$ and it can be shown [147] that $\tilde{X}^0$ (or χ) again corresponds to a compact direction. These relations are known as the famous *Buscher rules*. The transformation rule for the dilaton, being a higher-order α' contribution, is calculated separately, for example, using a path integral argument [152, 153]. The procedure can be generalised to backgrounds with n abelian isometries, which is equivalent to an iterative application of the above procedure.

Remarks

- The treatment outlined above was purely local. Whenever the worldsheet Σ is allowed to have nontrivial topology, the procedure has to be performed with some care. This includes the role of the Lagrange multiplier [147, 154–156], topological

[5] Only the metric part can be gauged by minimal coupling. The Kalb-Ramond field is more involved to take care of and requires the introduction of additional terms, cf. [150, 151].

terms and large gauge transformations [147, 154], and global properties of B and ω [157, 158]. Further references can be found in [134, 154] or in [159].

- While in the toroidal case, T-duality can be shown to be a symmetry of the partition function (cf. [134] and references therein) and hence a (perturbative) duality of the full quantum theory, curved backgrounds are more involved. The anomaly coefficients $\hat{\beta}$ of the NLSM, are invariant under the Buscher rules at linear order in α' and this invariance can be extended to higher orders with the transformation rules generically receiving [160] corrections at higher loops. In case the theory admits a CFT description, it was shown in [147, 161] that conformal invariance is preserved and therefore the duality is exact and extends to all orders in α'[6]. This extends to *Wess-Zumino-Witten* (WZW) models (cf. [73] or [162]), the duals of which are orbifolds of the initial theory [163] with an equivalent spectrum.

Example: T-duality of S^3

A particularly nice example to illustrate the Buscher rules is the three-sphere S^3, which can be viewed as the Lie group manifold $SU(2)$. A convenient parametrisation reads

$$\mathrm{d}s^2 = \frac{r^2}{4}(\mathrm{d}\eta^2 + \sin(\theta)^2 \mathrm{d}\xi_1^2) + r^2\big(\frac{1}{2}\cos(\eta)\mathrm{d}\xi_1 - \mathrm{d}\xi_2\big)^2\,, \tag{2.17}$$

with $\eta \in [0, \pi]\,, \xi_1, \xi_2 \in [0, 2\pi)$ and a trivial Kalb-Ramond field and dilaton. This parametrisation highlights the structure of S^3 as a non-trivial $U(1)$-fibration (along ξ_2) over the two-sphere S^2, known as the *Hopf fibration* – also known as *Hopf bundle*, cf. [164]. The associated first Chern class is $c_1(S^3) = 1$, while the fundamental group is trivial $\pi_1(S^3) = 0$. Clearly, (2.11) is satisfied.

T-dualising along the circle fibre using the Buscher rules (2.15) and (2.16) we obtain

$$\widetilde{\mathrm{d}s^2} = \frac{r^2}{4}\left(\mathrm{d}\eta^2 + \sin(\eta)^2 \mathrm{d}\xi_1^2\right) + \frac{1}{r^2}\mathrm{d}\xi_2^2\,, \quad \widetilde{B} = -\frac{1}{2}\cos(\eta)\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2\,, \quad \widetilde{\Phi} = -\log(r)\,. \tag{2.18}$$

Hence we obtain a dual space that is the product of a two-sphere and a circle of inverse radius together with a nontrivial two-form field $\tilde{B}$ and dilaton $\tilde{\Phi}$.

We can see from this example that T-duality on curved backgrounds can introduce non-trivial fluxes and also change the topology[7] [166, 170] of the underlying geometry. In particular $\pi_1(S^3) = 0$, while $\pi_1(S^2 \times S^1) = \mathbb{Z}$. We will return to this important point in the next section and later in Chapter 6 in the context of the Swampland program.

[6] By a careful treatment of global issues of the worldsheet this extends to arbitrary genus and hence all orders in string coupling [147].

[7] Topology change is widely believed to be a universal property of quantum gravity. In particular it is featured by string theory [165–170]; see also [171] and [172] for a more recent account and further references. Note also the close connection with the "Cobordism Conjecture" in Chapter 5.

2.2.3. RR sector, Fourier-Mukai transform and topological T-duality

T-duality can be extended to the full (quantum) theory hence in particular to the remaining bosonic fields in the Ramond-Ramond sector. However, in the RNS formulation, these cannot be coupled to the worldsheet σ-model[8] and hence a derivation à la Buscher is not possible. However, since RR fields couple to D-branes, it is possible to infer their transformations, at least in the case of a trivially fibred circle and $B = 0$, from the transformation rules of the D-branes. T-duality for open strings interchanges Neumann with Dirichlet boundary conditions. Hence, depending on whether we dualise along a direction transverse or perpendicular to the D-brane, we either change a Neumann into Dirichlet direction or vice versa. D-branes however are charged under the RR-fields, in particular a p-form F_p gauge field couples electrically to a $(p-1)$-brane. Hence, in order to obtain a consistent picture under T-duality we can deduce that the potentials $F = \mathrm{d}C$ have to transform as

$$\widetilde{C}_0 = C\,, \qquad \widetilde{C}_I = C_{0I}\,, \qquad \widetilde{C}_{0IJ} = C_{IJ}\,, \qquad \widetilde{C}_{IJK} = C_{0IJK}\,. \tag{2.19}$$

This is the analogue of the simple $r \leftrightarrow 1/r$ rule of T-duality on a circle.

Buscher rules for RR sector

The analogue of the Buscher rules for RR potentials under (abelian) T-duality on a curved background with isometry was derived in several different approaches [173–179]. For Type II theories, they relate (massless) Type IIA theories with Type IIB and, again adopting the isometry direction along X^0, they read [179]

$$\begin{aligned} \widetilde{C}^{(p)}_{0I_2\ldots I_p} &= C^{(p)-1}_{I_2\ldots I_n} - (p-1)\frac{G_{0[I_2}C^{(p-1)}_{0I_3\ldots I_p]}}{G_{00}}\,, \\ \widetilde{C}^{(p)}_{I_1\ldots I_p} &= C^{(p+1)}_{0I_1\ldots I_p} - pB_{0[I_1}C^{(p)}_{0I_2\ldots I_p]}\,. \end{aligned} \tag{2.20}$$

In particular, odd-form potentials are exchanged with even ones, and hence T-duality is indeed a map between Type IIA and Type IIB theories, as mentioned earlier.

The T-duality rules (2.20) for the RR potentials hold for an arbitrary isometric direction. In particular, this holds not only for the case of a trivial product manifold $M \times S^1$ but also for a more general S^1-fibration over some base manifold M, $S^1 \hookrightarrow X$. Let us for the moment restrict ourselves to trivially fibred spaces, potentially with n abelian isometries, i.e. $X = M \times T^n$ and a NSNS three-form H that is trivial in $H^3(X, \mathbb{Z})$ such that it can be globally written as $H = \mathrm{d}B$. In this case, there is a very compact formula that encodes the T-duality for the RR field strengths. In terms of the polyform $\mathfrak{F} = \sum_p F_p$

[8] A possible approach is to work in the Green-Schwarz formulation, cf. [79–81].

and its dual $\tilde{\mathfrak{F}} = \sum_p \tilde{F}_p$ on $\tilde{X} = M \times \tilde{T}^n$ it reads [180]

$$\widetilde{\mathfrak{F}} = \int_{T^n} \mathrm{ch}(\mathcal{P}) e^{\tilde{B}-B} \mathfrak{F} = \int_{T^n} e^{\tilde{B}-B+\sum_i \mathrm{d}\tilde{\theta}_i \wedge \mathrm{d}\theta_i} \mathfrak{F} \equiv \int_{T^n} e^F \mathfrak{F} \,, \tag{2.21}$$

with $\mathrm{ch}(\mathcal{P})$ the Chern character of the complex Poincaré line bundle[9] $\mathcal{P}$ over $\tilde{T}^n \times T^n$ with $U(1)$ connection and curvature $\mathcal{F} = -2\pi i \sum_i \mathrm{d}\tilde{\theta}_i \wedge \mathrm{d}\theta_i$. Here $\theta_i, \tilde{\theta}_i$ are the coordinates on T^n and $\tilde{T}^n$ respectively, and we omitted some pullbacks on B; compare (2.24). The above transformation is known as *Fourier-Mukai transform* and can be interpreted as an isomorphism between H-twisted cohomologies $H^\bullet(X, H)$

$$T_* : H^\bullet(M \times T^n, H) \longrightarrow H^{\bullet+1}(M \times \widetilde{T}^n, \tilde{H}) \,, \tag{2.22}$$

which in turn can be lifted to an isomorphism of (twisted) K-theories classifying the D-brane charges and their transformation under T-duality [180, 182]. In particular, the Fourier-Mukai transform can be thought of an integral transform

$$\Lambda^\bullet T^* X \to \Lambda^\bullet T^* \tilde{X} : \alpha \mapsto (p_2)_*(K \cdot p_1^* \alpha) \,, \tag{2.23}$$

with $p : X \times \tilde{X} \to X$ the projection map onto X and similar for $\tilde{p}$ and K is called the kernel of the integral transform. The Fourier-Mukai hence acts by first pulling back to the correspondence space, convoluting with the kernel K and then pushing forward to $\tilde{X}$. Comparing with (2.21) we see that the kernel is essentially given by the exponential

$$K = e^F \,, \qquad F = \tilde{p}^* \tilde{B} - p^* B + \sum_i \mathrm{d}\tilde{\theta}_i \wedge \mathrm{d}\theta_i \,, \tag{2.24}$$

upon identifying the pushforward map as fibrewise integration and reinstating the pullbacks omitted before. It is then exactly this kernel that gives rise to the isomorphism (2.22) via

$$T_* := \tilde{p}_* \circ e^F \circ p^* : H^\bullet(M \times T^n, H) \longrightarrow H^{\bullet+1}(M \times \tilde{T}^n, \tilde{H}) \,, \tag{2.25}$$

and subsequently the K-theory equivalent. The construction has been generalised to non-trivial circle bundles E and non-trivial H in [182, 183].

Topological T-duality

We know from the previous section that T-duality can also be applied to nontrivial circle fibration over M. In this case, the formula (2.21) is no longer well defined globally but gives the correct Buscher rules at least locally [182]. It has been argued in [182, 183], that indeed the isomorphism (2.25) can be extended to the fibred case; see also [184].

[9] See for example [181] for a definition and discussion focused towards T-duality.

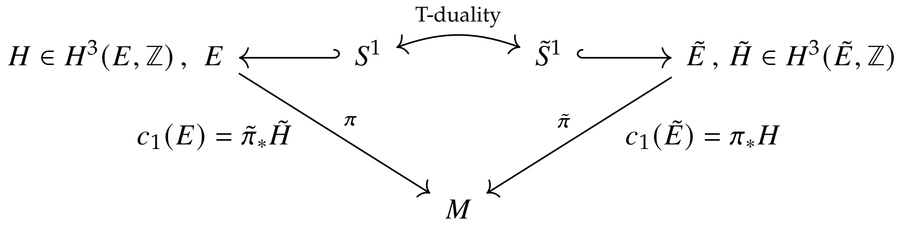


Figure 2.1.: Visualisation of topological T-duality as a isomorphism between H-twisted cohomologies of two dual (generically topologically inequivalent) circle-fibrations E and $\tilde{E}$ with H-flux H and $\tilde{H}$ over a common base M.

Let E be an (oriented) S^1 bundle over M with Chern class $c_1(E) \in H^2(M, \mathbb{Z})$ together with a nontrivial H-flux $H \in H^3(E, \mathbb{Z})$. Then the T-dual space $\tilde{E}$ is again a circle fibration over M with nontrivial H-flux $\tilde{H}$ such that [182]

$$c_1(\tilde{E}) = \pi_* H\,, \qquad c_1(E) = \pi_* \tilde{H}\,, \tag{2.26}$$

where $\pi, \tilde{\pi}$ are the associated projections to the base M. Therefore, loosely speaking, T-duality on topological nontrivial spaces with H flux exchanges flux with topology. Note that in the physics literature it is common to denote the unit of flux k associated with a certain representative in $H^3(E, \mathbb{Z})$ with $[H]$ such that schematically we can write

$$\text{topology: } c_1(E) \quad \longleftrightarrow \quad \text{flux: } [\tilde{H}]\,. \tag{2.27}$$

Furthermore, we see that indeed $c_1(E) \in H^2(M, \mathbb{Z})$ is mapped to $H \in H^3(\tilde{E}, \mathbb{Z})$ and, as mentioned before, the isomorphism (2.22) extends to an isomorphism [182]

$$T_* : H^{\bullet}(E, H) \longrightarrow H^{\bullet+1}(\tilde{E}, \tilde{H})\,. \tag{2.28}$$

The situation is summarised in Figure 2.1. For further details, see [182, 183].

Example: S^3 with H-flux

We have already encountered the example of the three-sphere fitting into these rules. Here S^3 was mapped to the topologically distinct space $S^2 \times S^1$ together with a nontrivial B-field and dilaton. In particular, the T-duality exchanged $c_1(S^3) = 1, [H] = 0$ with $c_1(S^2 \times S^1) = 0, [H] = 1$. We can easily generate another example by supporting $[H] = k$ units of flux on S^3. The resulting theory can be viewed as a $SU(2)$ WZW-model at level k, denoted $SU(2)_k$. We define the B-field as

$$B = -\frac{k}{2} \cos(\eta) \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2\,, \tag{2.29}$$

which can be easily shown to give $[H] = k$. The dilaton is chosen to be trivial.

T-dualising using the Buscher rules gives

$$\widetilde{\mathrm{d}s^2} = \frac{r^2}{4}\left(\mathrm{d}\eta^2 + \sin(\eta)^2\mathrm{d}\xi_1^2\right) + \frac{1}{r^2}\left(\frac{1}{2}k\cos(\eta)\mathrm{d}\xi_1 - 2\mathrm{d}\xi_2\right)^2 , \tag{2.30}$$

together with

$$\widetilde{B} = -\frac{1}{2}\cos(\eta)\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 , \qquad \widetilde{\Phi} = -\log(r) . \tag{2.31}$$

From the dual line element, one can read off[10] that $c_1(\tilde{E}) = k$ and $[\tilde{H}] = 1$, hence realising (2.26). We will see later that his example fits into the more general family of Lens spaces that we discuss in Chapter 6, Section 6.3.

2.2.4. Generalised T-dualities

We provide a minimalistic guide through the main aspects of generalised dualities, restricting to the minimum needed for later considerations in Part II. We point the reader to the relevant literature for details and focus on contrasting the standard approach to non-Abelian T-duality with the more general notion of Poisson-Lie T-duality.

Standard approach to non-Abelian T-duality (NATD)

The gauging procedure outline in the previous section can be generalised to multiple non-Abelian isometries, introducing multiple non-commuting gauge fields A^a as initially worked out in [150,151]. The dual theory can then be again obtained by introducing a set of Lagrange multipliers λ^a, leading to the standard formulation of non-Abelian T-duality (NATD) [154,186–188]. However, the procedure is much less straightforward and suffers from many drawbacks compared to abelian T-duality. In the case where the isometries act without isotropies, hence without any fixed points, one can write down transformations rules for the dual background similar to the Buscher rules (2.15) and (2.16). However, these will in particular depend on the non-Abelian structure constants ${f_{ab}}^c$ of the initial symmetry group. In the case where the isometry group is the whole group manifold under consideration, the transformation rule reads [135,155,189,190]

$$\widetilde{E}_{ab} = \left(E_{ab} + \lambda^c f_{cab}\right)^{-1} , \tag{2.32}$$

given in the Lie algebra basis of (left-invariant) Maurer-Cartan forms L^a defined through $g^{-1}\mathrm{d}g = L^a T_a$ with $g \in G$ an element of the isometry group G. The Killing form has been used to lower the index on ${f_{ab}}^c$. From this, $\tilde{G}$ and $\tilde{B}$ can be obtained as symmetric and antisymmetric parts, respectively.

[10] This can be seen by calculating the associated connection and charge, as nicely explained in [185].

T-duality	Vector fields	Symmetry cond.	Conservation eq.
abelian	$[v_a, v_b] = 0$	$\mathcal{L}_{v_a} E_{ij} = 0$	$\mathrm{d} \star J_a = 0$
non-Abelian	$[v_a, v_b] = f_{ab}{}^c v_c$	$\mathcal{L}_{v_a} E_{ij} = 0$	$\mathrm{d} \star J_a = 0$
Poisson-Lie	$[v_a, v_b] = f_{ab}{}^c v_c$	(2.35)	$\mathrm{d} \star J_a = \tilde{f}^{bc}{}_a J_b \wedge J_c$

Table 2.1.: Summary of properties of the different notions of (generalised) T-duality. The current J_a is defined in (B.1) in Appendix B.

For discussions of some of the issues arising within this approach, like non-compactness or lack of isometries, see [135,154,188] and references therein.

Example: NATD of S^3

Viewed as a group manifold, we have $S^3 \cong SU(2)$ such that $f_{abc} = \epsilon_{abc}$. In the basis of left-invariant forms L^a the metric on S^3 is simply given by $E_{ab} = G_{ab} = r^2\delta_{ab}$, such that we obtain

$$\widetilde{E}_{ab} = (r^2 + \lambda^c \epsilon_{cab})^{-1} , \tag{2.33}$$

which, upon moving to spherical coordinates with $l^2 = \sum_a (\lambda^a)^2$, leads to [187,191]

$$\widetilde{\mathrm{d}s^2} = \frac{\mathrm{d}l^2}{r^2} + \frac{rl^2}{r^2 + l^2}\mathrm{d}s^2(S^2) , \quad \widetilde{B} = \frac{r^3}{r^2 + l^2}\mathrm{vol}(S^2) , \quad \widetilde{\Phi} = \Phi - \frac{1}{2}\log\Big(r^3 + rl^2\Big) , \tag{2.34}$$

where the transformation rule of Φ can be found in [186]. In particular, the dual is non-compact and lacks some of the symmetries of the original model, cf. [187].

Poisson-Lie T-duality

Non-Abelian T-duality is one way to generalise the notion of conventional T-duality for isometry groups that are non-Abelian. We now turn to Poisson-Lie (PL) T-duality, which weakens the requirement of having an isometry even further. In particular, we let go of the isometry requirement $\mathcal{L}_{v_a} E_{ij} = 0$ but instead impose the weaker condition [192,193]

$$\mathcal{L}_{v_a} E_{ij} = \tilde{f}^{bc}{}_a v_b{}^l E_{il} E_{kj} v_c{}^j , \tag{2.35}$$

with $a, b, c, \ldots$ denoting algebra indices while $i, j, k, \ldots$ are curved indices. For the moment, $\tilde{f}^{ab}{}_c$ are simply constants, the role of which will become clear in a second. Let $\{v_a\}$ be a set of vector fields generating some Lie algebra $\mathfrak{g}$ with an associated Lie group G such that the condition (2.35) is satisfied for a given background E. Then the NLSM with background E is called *Poisson-Lie T-dualisabale*, and the consistency relation $[\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] = \mathcal{L}_{[v_a, v_b]}$ implies that $\tilde{f}^{ab}{}_c$ are, in fact, structure constants of a dual algebra $\mathfrak{g}$. This turns $\mathfrak{g}$ (and by duality also $\tilde{\mathfrak{g}}$) into a so called *bialgebra*, hence an algebra admitting a dual algebra such that the mixed Jacobi identity[11] is satisfied. Now, forming the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$, one obtains the co-called Drinfeld double $\mathfrak{d}$. The dual group $\tilde{G}$ naturally encodes the Poisson-Lie T-dual background [192, 193]. The associated Lie group $\mathbb{D} = \exp(\mathfrak{d})$ is also referred to as a Drinfel'd double, which (at least for $\mathbb{D}$ simply connected) splits as $\mathbb{D} = G \bowtie \tilde{G}$; here $\bowtie$ denotes the *Zappa–Szép product*, cf. [194]. For further details, we refer to the reviews [136, 195–197]. The symmetries and conserved currents for the different incarnations of T-duality are summarised in Table 2.1.

The Poisson-Lie symmetric σ-models for a Lie-group G the reads

$$\mathcal{S}_{\text{PL}} = \int_\Sigma L^a (E_0^{-1} + \Pi(g))^{-1}_{ab} L^b \,, \tag{2.36}$$

with $L = g^{-1}\mathrm{d}g$ again the left-invariant Maurer-Cartan forms written in a basis of generators $\{T^a\}$ of the Lie-algebra $\mathfrak{g}$ and group $G \ni g$. Here E_0 is a constant background matrix and $\Pi = \Pi(g)$ is the Poisson-Lie structure bivector of G which can be obtained from the r-matrix solving the classical Yang-Baxter equation[12]. The dual theory is then given by

$$\tilde{\mathcal{S}}_{\text{PL}} = \int_\Sigma \tilde{L}_a [(E_0^{-1} + \tilde{\Pi}(\tilde{g}))^{-1}]^{ab} \tilde{L}_b \,, \tag{2.37}$$

from which one can readily obtain the dual background data $\tilde{G}, \tilde{B}$. Similarly to the abelian case, the dilaton requires a separate treatment. Its transformation rules were derived in [198] (see also [199]) and it turns out that in order to have a PL-symmetric configuration, the dilaton and its dual need to take the form

$$\Phi = \phi_0 + \frac{1}{4} \log\left(\det(G_{ab})\right) \,, \qquad \tilde{\Phi} = \Phi + \frac{1}{4} \log\left(\frac{\det(\tilde{G}_{ab})}{\det(G_{ab})}\right) \,, \tag{2.38}$$

with G_{ab} and $\tilde{G}_{ab}$ again evaluated in the algebra frame.

Abelian and standard non-Abelian T-duality can be obtained as special cases:

- *abelian T-duality:* We have $\mathfrak{d} = \mathfrak{u}(1)^d \oplus \mathfrak{u}(1)^d$ such that $\mathbb{D} = U(1)^d \times U(1)^d$ and since $\Pi = \tilde{\Pi} = 0$ we get the familiar toroidal T-duality rule $E \to 1/E$, cf. Section 2.2.1.
- *NATD:* The Drinfeld double reads $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{u}(1)^d$ so that (at least locally) we have $\mathbb{D} = G \ltimes U(1)^{\dim G}$. In this case $\Pi = 0$ while $\tilde{\Pi} = f_{ab}{}^c \lambda_c$ and we obtain the background $\tilde{E}_{ab} = (E_0^{-1})_{ab} + f_{ab}{}^c \lambda_c$ which is nothing else than (2.32), however crucially with angular coordinates leading to a compact geometry.

[11] The mixed Jacobi identity for structure constants $f_{ab}{}^c$ and their duals $\tilde{f}^{jk}{}_l$ reads $f_{ab}{}^c \tilde{f}^{de}{}_c = f_{ac}{}^d \tilde{f}^{ce}{}_b + f_{ac}{}^e \tilde{f}^{dc}{}_b + f_{cb}{}^e \tilde{f}^{dc}{}_a + f_{cb}{}^d \tilde{f}^{ce}{}_a$.

[12] Alternatively one can explicitly compute Π (and $\tilde{\Pi}$) from the adjoint action for the group elements on the generators $T, \tilde{T}$, see for example [69] Appendix A.

We denote the NATD obtained via PL T-duality of the double $\mathbb{D} = G \ltimes U(1)^{\dim G}$ as NATD^{PL} while NATD without superscript denotes the standard approach.

Remark

- Choosing the Drinfeld double for NATD to be the cotangent bundle $\mathbb{D} = T^*G \cong G \ltimes \mathbb{R}^{\dim G}$ one can recover the non-compact case of standard NATD. For example $T^*SU(2) \cong SU(2) \ltimes \mathbb{R}^3$, with $\tilde{E}$ as in (2.32).

2.2.5. Dualities from defects

There is an interesting relation between (generalised) dualities and so-called *topological defects* [200–206], which is in particular connected to the notion of T-duality via the Fourier-Mukai transform sketched in the previous section.

At the level of the σ-model action (1.6), we can introduce a (topological) defect, separating two a priori unrelated theories and gluing them along a common defect line. In particular, consider a pair of NLSM actions with target space maps $X : \Sigma \to M$ and $\tilde{X} : \tilde{\Sigma} \to \tilde{M}$ and associated background data G, B and $\tilde{G}, \tilde{B}$, which we keep arbitrary for the moment. We then glue the theories by combining the worldsheets $\Sigma \cup \tilde{\Sigma}$ along a *defect line* D. The latter we locate at $\sigma = 0$, so that the map $X(\tau, \sigma)$ is now defined for $\sigma \geq 0$ and $\tilde{X}(\tau, \sigma)$ for $\sigma \leq 0$. It is convenient to define the following product target space map

$$\mathbb{X} = X \times \tilde{X} \, : \, \Sigma \cup \tilde{\Sigma} \to M \times \tilde{M} : (\tau, \sigma) \mapsto \mathbb{X}^I = (X^i, \tilde{X}^i) \, . \tag{2.39}$$

Omitting any details, we define a connection $\mathbb{A}$ on a line bundle taking values in the pushforward of the product map $\mathbb{X}$ restricted to the defect line D. The connection has an associated curvature $\mathcal{F} = \mathrm{d}\mathbb{A}$. We thus have a total worldsheet action [204, 207]

$$\mathcal{S} = \int_\Sigma \mathrm{d}^2\sigma \, L(G, B, X) + \int_{\tilde{\Sigma}} \mathrm{d}^2\sigma \, L(\tilde{G}, \tilde{B}, \tilde{X}) - \frac{1}{2\pi\alpha'} \int_D \mathrm{d}\tau \, \mathbb{A}_I(\mathbb{X}) \partial_\tau \mathbb{X}^I \, , \tag{2.40}$$

with $L(G, B, X)$ the Lagrangian of the bosonic sigma model (1.6), see Figure 2.2.

Varying the complete action and evaluating the location of the defect $\sigma = 0$ imposes equations of motion that generically relate the background data G, B to the quantitates $\tilde{G}, \tilde{B}$ in a way depending on the explicit choice of the defect or connection (curvature). It was realised in [207] for the case of abelian T-duality and later for NATD [191], that starting from a dualisable model G, B and imposing a specific choice of defect associated with curvature $\mathcal{F}$, the equations of motion fix the background $\tilde{G}, \tilde{B}$ to be the (non-)Abelian T-dual of the former. Abelian T-duality corresponds to the choice

$$\mathcal{F} = -\mathrm{d}\tilde{\theta}_i \wedge \theta_i \, . \tag{2.41}$$

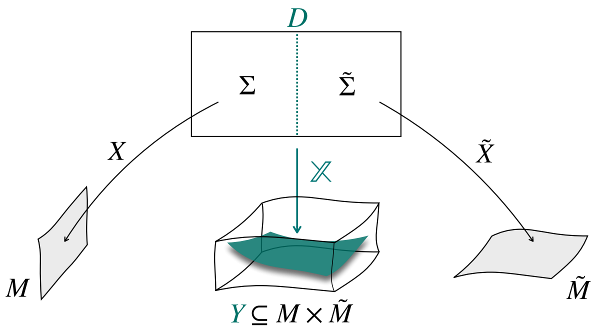


Figure 2.2.: Two worldsheet theories glued along a defect line D. The defect contributions is mapped to a submanifold Y of the target space product manifold $Y \subseteq M \times \tilde{M}$.

In the beginning of this paragraph we hinted towards the connection of defects and the Fourier-Mukai transformation as a transformation of the RR charges. In order to understand this, note that T-duality of a D-brane can be understood as a fusion operation of a defect with a boundary condition[13]. Since the associated D-branes are charged under the RR fields, the duality defect encodes some information about the duality transformation of these charges. It was shown in [191,207] that the two form $\mathcal{F}$ living on the worldvolume of the topological defect encoding (non-) abelian T-duality determines the kernel K of the Fourier-Mukai-like transformation via

$$K = e^F \,, \qquad F = \tilde{p}^*\tilde{B} - p^*B - \mathcal{F} \,. \tag{2.42}$$

Recalling the choice of curvature that leads to the abelian T-duality defect given in (2.41), we indeed obtain the correct kernel for the Fourier-Mukai transformation given in (2.24). It was shown in [191] that this carries over to NATD, where the curvature of the defect connection again gives rise to the correct transformation rules for the RR sector under non-Abelian T-duality.

Finally, in [69] the author and a collaborator derived the defect contribution implementing PL T-duality. This confirmed from a different perspective the proposed Fourier-Mukai kernel for PL T-duality derived in [208] from a completely different approach. We refer to the original publication for details and further references [69].

2.2.6. The role of generalised dualities in string theory

We close this Chapter by addressing some questions and issues concerning generalised T-dualities. A discussion of the potential role of generalised dualities for the Swampland program is delayed until Chapter 5.

[13]See [69] and references therein for details on the fusion process.

Solution generating technique vs quantum symmetry

It is not clear if and which incarnation of generalised T-duality can be lifted to the quantum level and hence serve as a true duality of string theory. It was recently shown [209] that for certain WZW models PL T-duality is indeed valid up to all orders in α' and string coupling and hence indeed extends beyond the classical level. Here, it is worth mentioning that while drastically relaxing the underlying assumptions of abelian T-duality, generalised T-dualities can be shown to remain canonical equivalences [170, 192, 210] and hence transformations between theories with a classically common phase space. The question of whether non-Abelian T-duality can be a (perturbative) symmetry of string theory is a long-standing one [155] and it is believed that generalised T-dualities generically fail to hold at all orders in genus expansion. However, starting from a CFT, the generalised dualities map CFTs to (generically inequivalent) CFTs at least up to second order in α' [211–213], which, at the very least, suggests that they are seen as powerful solution-generating techniques.

Map between scalar field spaces

In view of the Swampland Distance Conjecture, to be introduced in Chapter 5, we are interested in the scalar field space descending from the internal manifold, in particular the associated field space metric and the scalar potential. In contrast to abelian T-duality, it is not clear for generalised T-duality that upon dualising, the metric and scalar potential stay invariant. We will investigate this in Chapter 8.

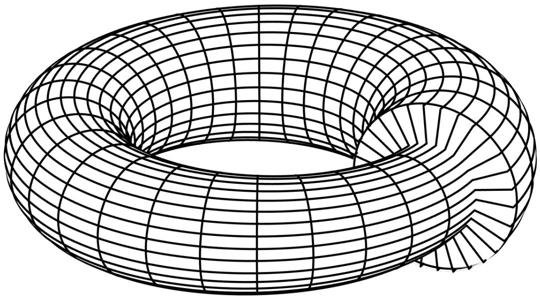
Global structure

The global structure of the generalised T-duals is much less trivial than in the abelian case. In view of our later use as compactification manifolds, we would like the dual space to be compact to perform the dimensional reduction in the standard way. NATD maps a compact space to a non-compact one, since the Lagrange multiplier procedure does not capture the global information about the dual manifold [170]. PL T-duality circumvents this problem, and hence we will follow this procedure[14].

In conclusion, we see that the power of generalised T-duality clearly lies in the ability to generate a plethora of background with properties that challenge and expand the space of known solutions. While the latter are often highly symmetric, their generalised duals are in general much less symmetric while still computationally under control and supported by intuition gained from the initial space[15].

[14] Understanding the global structure is anyway highly nontrivial also in the PL case; see e.g. [208].

[15] Prime examples for this, although not discussed here, are generalised duals of TsT-deformations [214, 215] and the closely related (integrable) $J\bar{J}$-deformations [166, 216, 217] of CFTs. These models feature exotic global properties such as changes in topology along certain directions in moduli space [216].



CHAPTER 3

Generalised geometry, DFT and non-geometric backgrounds

Strings are extended objects that can perceive spacetime geometry rather differently than point particles. Consider, for example, closed string winding modes wrapping nontrivial cycles of the geometry. These do not have a point-particle analogue. Even more striking is the ability to define the theory on configurations that do not fit into the standard paradigm of Riemannian geometry. While in the latter, local patches are glued by diffeomorphisms – and in the presence of fluxes – gauge transformations, in string theory there exist further transformations that leave the theory invariant. These include in particular the dualities discussed in the previous section, which from a string perspective can be put on equal footing with diffeomorphism and gauge transformations. In particular, strings can propagate on spaces with transition functions corresponding to T-dualities or even S-dualities, and such spaces are often called *T-folds* (or *S-folds* respectively) and more generally *non-geometric spaces*. There are even more exotic configurations that lack a local Riemannian description due to the dependence on canonical conjugate winding coordinates or where the left and right moving sectors of the closed string perceive spacetime differently. These spaces are known as *locally non-geometric*. It turns out that non-geometric spaces in general can be described very naturally within the framework of *Double Field Theory*, which we review at the beginning of this chapter before moving to non-geometric spaces.

3.1. Double field theory and generalised geometry

There are several physical motivations for the introduction of the notion of *Generalised Geometry* and *Double Field Theory* (DFT). From a string theory point of view, it is particularly natural to start from the following observation[1]. The closed strings on a circle (parametrised by X) not only gives rise to momentum modes n^I but also winding modes w_I. Upon T-dualising, these winding modes can be viewed as momentum modes along a dual circle with coordinate $\tilde{X}$. Woking towards a full field theoretic description of the closed string on a D-torus it seems natural to write down a theory that – along the compact direction – depends both on the physical coordinates X^I as well as the dual winding coordinates $\tilde{X}_I$, such that the fields are now functions of both these coordinates[2]. Schematically, we can write such an action as

$$\mathcal{S} = \int_\Sigma \mathrm{d}\sigma^2 L(X^i, \tilde{X}_i, Y^\mu)\,, \tag{3.1}$$

where Y^μ is a target space coordinate along the non-compact direction. While in principle, one could now push this idea to the level of the worldsheet action and write down, for example, a doubled σ-model action [219] or even a formulation of String Field Theory (cf. [220] for a review), we are more interested in the low-energy effective description of such an action. In particular we are interested in the action of the NSNS sector, with field content G, B, Φ and its generalisation to the doubled formulaation.

Finding such a description is of course a non-trivial task. Without going into any details on the derivation and focussing on the compact directions which we denote by X^M, the action reads [221]

$$\begin{aligned}\mathcal{S}_{\mathrm{DFT}} = \int \mathrm{d}^{2D}X e^{-2d}\Big(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK}\\ - 2\partial_M d\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\partial_N d\Big)\,.\end{aligned} \tag{3.2}$$

with

$$\mathcal{H} \equiv \mathcal{H}^{MN} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1}\end{pmatrix}\,, \qquad e^{-2d} = \sqrt{G}e^{-2\Phi}\,, \tag{3.3}$$

where $\mathcal{H}$ is the generalised metric that we already encountered in Section 2.2.1. Indices are raised and lowered by use of the $O(D,D)$ "metric"

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0\end{pmatrix}\,, \tag{3.4}$$

[1] See for example [218] for a nice and compact introduction, which we partially follow below.

[2] We have already implicitly hinted at such a doubling when writing the spectrum of the string as $Z^T\mathcal{H}Z$ with Z a $2D$-dimensional zero mode vector in Section 2.2.1.

such that the inverse $\mathcal{H}^{-1}$ is given by

$$\mathcal{H}_{MN} = \eta_{MO}\eta_{PN}\mathcal{H}^{OP} . \tag{3.5}$$

Here, $I = 1, \ldots, 2D$, and the first D indices corresponding to the coordinates X^i and the latter D to their tilde pendants. Likewise, the derivatives ∂_M split as $\partial_M = \partial_i = \partial_{X^i}$ for $i = 1, \ldots, D$ and $\partial_M = \tilde{\partial}^i = \partial_{\tilde{X}_i}$ for $i = D+1, \ldots, 2D$. The action is manifestly $O(D,D)$ invariant[3] and the resulting theory is known as *Double Field Theory* (DFT) [221–224].

In particular, $O(D,D)$-transformations O act on $\mathcal{H}$ via $\widetilde{\mathcal{H}} = O\mathcal{H}O^T$ and on X via $\tilde{X} = OX$, or explicitly in indices

$$\widetilde{\mathcal{H}}^{MN} = O^M{}_I \mathcal{H}^{IJ} O^N{}_J , \qquad \tilde{X}^M = O^M{}_N X^N . \tag{3.6}$$

Since in the action all $O(D,D)$ indices are contracted, the invariance is apparent.

Trying to motivate the action from symmetry principles almost automatically leads to the notion of *Generalised Geometry* [225, 226] which we briefly summarise now. The low-energy effective action (1.9) has gauge symmetries consisting of diffeomorphisms and gauge transformations of B. While the former are generated by vector fields $V \in \Gamma(TM)$, the latter are generated by one-forms $\xi \in \Gamma(T^*M)$. The core essence of generalised geometry then boils down to combining both of them into a single object, the generalised vector X

$$X = V + \xi \in \Gamma(TM \oplus T^*M) , \tag{3.7}$$

where $TM \oplus T^*M$ is called the generalised tangent bundle of M. It is convenient to combine the elements into a $2D$ vector X^M

$$X^M \equiv \begin{pmatrix} \tilde{X}_i \\ X^i \end{pmatrix} = \begin{pmatrix} \xi_i \\ V^i \end{pmatrix} , \tag{3.8}$$

with the index i running from 1 to D. The generalised tangent bundle E can be more generally written as an element of the short exact sequence

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0 , \tag{3.9}$$

such that $E \cong T^*M \oplus TM$ and the sections of E are the generalised vectors X. This simple yet elegant packaging is at the heart of generalised geometry. For the many details omitted here, we refer to the original work [225, 226], see also the excellent review [227] aimed towards applications in string theory.

Note that so far the introduction of the generalised tangent bundle does not amount to the doubling of coordinates of the above action. However, as we will see in a second,

[3] And hence in particular invariant under (abelian) T-duality.

the generalised tangent bundle comes with a natural bilinear form $\langle\cdot,\cdot\rangle$ of signature (D,D) as well as a positive definite metric $\mathcal{H}$, that is, in fact, the generalised metric from above. The former is given for $X = V+\xi, Y = W+\chi \in \Gamma(TM\oplus T^*M)$ by

$$\langle X,Y\rangle = \frac{1}{2}X^T\eta Y = \frac{1}{2}\left(\iota_X\chi + \iota_W\xi\right)\,, \tag{3.10}$$

where η is again given by (3.4). In addition to this bilinear, we would also like to define a *positive-definite* inner product on E. In fact, we already encountered a very convenient choice, namely the generalised metric $\mathcal{H}$ of (3.3). It is convenient to write $\mathcal{H}$ in terms of so-called generalised vielbeins $\mathcal{E}$, that are defined via

$$\mathcal{H} = \mathcal{E}^T\mathbb{1}_{2d}\mathcal{E}\,. \tag{3.11}$$

This object is not unique but only fixed up to $O(2D)$ transformations. A convenient and common choice, is

$$\mathcal{E}_{(e,B)} = \begin{pmatrix} e & 0\\ -e^Tb & e^{-T}\end{pmatrix}\,, \tag{3.12}$$

where e is the standard D-dimensional vielbein defined via $G = e^T\mathbb{1}_d e = e_i{}^a\delta_{ab}e^b{}_j$ and b is given in the vielbein basis $B = e_i{}^a b_{ab}e^b{}_j$. We will shortly see another frame, which is more convenient in certain situations, the so-called *non-geometric frame*.

Lastly, we briefly comment on the symmetries of the doubled action (3.2). The gauge algebra of the action (3.2) and hence its symmetries are encoded into the so-called *C-bracket* $[\cdot,\cdot]_C$ [224], which is the Double Field Theory generalisation of the *Courant bracket* [228] on E, playing a role similar to the Lie bracket for vector fields. In particular, there is a *generalised Lie derivative* $\hat{\mathcal{L}}_X$ such that the gauge transformation of the generalised metric can be written as [221]

$$\delta\mathcal{H}^{MN} = \hat{\mathcal{L}}_X\mathcal{H}^{MN} \quad\Longrightarrow\quad [\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y] = -\hat{\mathcal{L}}_{[X,Y]_C}\,. \tag{3.13}$$

Therefore, the gauge algebra is determined by the C-bracket.

Remarks

- Double field theory, as originally derived [223] from (closed) String Field Theory was first formulated as an expansion up to cubic order in terms of the field $\mathcal{E}_{IJ} = E_{IJ} + e_{IJ}(X,\tilde{X}) + O(e^2)$ with $E = G+B$ the constant background field. Furthermore, there is a formulation [229–231] in terms of generalised fluxes $\mathcal{F}_{ABC}$ that is particularly convenient to connect to gauged supergravity but also non-geometric spaces. We refer to the review [232] and the references therein.

- Later we will be mostly interested in curved backgrounds, in particular group manifolds. Starting from Wee-Zumino-Witten models, the framework of Double Field Theory was generalised to these situations in [233–235] and is known as DFT_{WZW}. The action can be obtained by replacing the $O(D,D)$ derivative ∂_I in $\mathcal{S}_{\text{DFT}}$ by a covariant derivative ∇_I, keeping manifest $O(D,D)$-invariance.

- DFT makes manifest the T-duality invariance of string theory. Combining T-dualities with S-dualities leads to U-duality and the corresponding theory invariant under the U-duality is *Exceptional Field Theory*, see [236–238].

3.1.1. Section condition and the choice of frame

So far, we did not provide a vital piece of information, namely how to obtain the standard NSNS effective action from $\mathcal{S}_{\text{DFT}}$. In particular, we need to select the D physical coordinates among the $2D$ total coordinates, and hence a D-dimensional subspace in the full $2D$-dimensional target space of the action. This is known as choosing a *polarisation* or *section* and the associated constraint as *section condition*. A particular way to choose such a polarisation is by means of the *strong constraint*, which for $\Phi_a(X,\tilde{X})$ generic fields reads

$$\partial_M \Phi_a \partial^M \Phi_b = \partial_i \Phi_a \tilde{\partial}^i \Phi_b + \tilde{\partial}^i \Phi_a \partial_i \Phi_b = 0\,, \qquad \forall a,b\,. \tag{3.14}$$

If this constraint is satisfied, it can be shown that the theory depends at most on D of the coordinates, and hence one can always find a frame in which the fields only depend on X and not on $\tilde{X}$. It is this frame then, in which we can make contact with the standard formulation of NSNS effective action.

The strong constraint, in fact, is not just an arbitrary way to impose the condition. It is, in fact, the natural generalisation of the level matching condition in string theory and is crucial for the consistency of the doubled theory. A particular simple choice is to impose $\tilde{\partial}(\cdot) = 0$, i.e. all fields depend only on the physical coordinates X. This trivially satisfies the strong constraint and one finds that upon this choice the doubled action reduces to the standard NSNS supergravity action

$$\begin{aligned} \mathcal{S}_{\text{DFT}} &= \int \mathrm{d}^{2D} X e^{-2d} \left(\tfrac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \dots\right) \\ & \Big\Downarrow{\scriptstyle \tilde{\partial}=0} \\ \mathcal{S} &= \int \mathrm{d}^D x \sqrt{G} e^{-2\Phi}\left(R(G) - \tfrac{1}{12} H_{ijk}H^{ijk} + 4\partial_i \Phi \partial^i \Phi\right)\,. \end{aligned} \tag{3.15}$$

We saw that the explicit form of $\mathcal{H}$ in terms of the fields G, B is determined by a choice of generalised frame field $\mathcal{E}$. The canonical choice (3.12) leads to the familiar parametrisation of $\mathcal{H}$ in the form of (3.3) in terms of G and B. Although this choice recovers upon the appropriate section condition the familiar geometric NSNS action, from a doubled perspective there is a priori no preferred choice for the decomposition of $\mathcal{H}$ in terms of a specific $\mathcal{E}$. There are backgrounds that, from a string perspective, are perfectly fine but may look ill-defined from a standard Riemannian point of view in terms of G and B. These backgrounds may be plagued by monodromies that are neither diffeomorphism nor gauge transformations, but actually duality transformations. Their target space geometry cannot be described in terms of standard Riemannian geometry, as for example, transition functions also need to include duality transformations; see also Figure 3.1. This leads to the notion of *non-geometric spaces* [114–116, 159, 239]. In these cases, it turns out to be favourable to switch to an alternative description in terms of new fields $\hat{G}$ and β defining a different frame $\mathcal{E}_{\hat{e},\beta}$. Taking the generalised vielbein to be given as [240]

$$\mathcal{E}_{(\hat{e},\beta)} = \begin{pmatrix} \hat{e} & \hat{e}\beta \\ 0 & \hat{e}^{-T} \end{pmatrix}, \tag{3.16}$$

leads to a specific frame, known as *β–frame* that is particularly suited to treat these non-geometric spaces, which will be the topic of the next section.

Comprehensive reviews on DFT can be found in [218, 232, 241].

3.2. Non-geometric spaces and β-gravity

We will see shortly that (at least a special class of) non-geometric spaces are closely related to T-duality. Furthermore, they can feature exotic properties like explicit dependences on winding coordinates. It should therefore not come as a surprise that DFT is a very convenient tool for working with these spaces. In the following section, we will outline how starting from the DFT action one can define a new frame - the *non-geometric frame* - and an associated effective theory [242, 243] with Lagrangian L_β that is particularly suited to investigate these spaces. The latter defines an alternative to the standard NSNS effective action and is often called *β-(super)gravity*, see also [243–245].

3.2.1. Why non-geometry?

From a phenomenological point of view the idea of non-geometry can seem very exotic or even undesirable. However, quite the opposite is true. Moduli stabilisation (cf. Section 1.3) can be conveniently achieved by turning on constant background fluxes in the (internal) geometry of the higher-dimensional theory, for example through the NSNS three-form flux or Ramond-Ramond fluxes. Upon compactification these generate a scalar potential for the effective theory, stabilising (at least some of) the moduli at the minimum of the potential. These fluxes are part of the geometric field content of supergravity or string theory and are therefore *geometric fluxes*. However it turned out that in certain instances these fluxes were not enough to stabilise *all* moduli. Subsequently, it was realised [246–250] that stabilisation could be potentially realised by introducing exotic fluxes that had no apparent geometric origin [247] in the standard formulation of 10D SUGRA or string theory respectively, see also [250, 251]. Whilst achieving the desired moduli stabilisation the origin of these fluxes from a higher-dimensional realisation remained unclear until the authors of [240, 242, 252] managed to resolve the puzzle by approaching the problem via DFT. By starting from the full doubled action, choosing a particular frame and imposing the section condition it was possible to write down a ten-dimensional effective action involving new exotic fluxes, that upon compactification could be matched with the one that were observed before in the context of moduli stabilisation. It is exactly the appearance of these new fluxes that characterise the features of globally and locally non-geometric spaces.

3.2.2. β-gravity

We start by noting that due to the defining relation (3.11), the vielbein given in (3.16) is merely a reparametrisation of the same $\mathcal{H}$, i.e.

$$\mathcal{E}^T_{(e,B)} \mathbb{1}_{2D} \mathcal{E}_{(e,B)} = \mathcal{H} = \mathcal{E}^T_{(\hat{e},\beta)} \mathbb{1}_{2D} \mathcal{E}_{(\hat{e},\beta)} \,. \tag{3.17}$$

The two frames are related through a $O(D-1,1)\times O(1,D-1)$-transformation O, i.e. $\tilde{\mathcal{E}} = O\mathcal{E}$, cf. Appendix C in [243]. The generalised metric in terms of these fields reads

$$\mathcal{H} = \begin{pmatrix} \hat{G} & \hat{G}\beta \\ -\beta\hat{G} & \hat{G}^{-1} - \beta\hat{G}\beta \end{pmatrix} , \tag{3.18}$$

from which by comparison with (3.3) one can deduce the relation[4]

$$(G+B)^{-1} = (\hat{G}^{-1} + \beta) \,. \tag{3.19}$$

The dilaton in the new frame can be defined by requiring the generalised volume element to remain invariant

$$e^{-2\Phi}\sqrt{G} = e^{-2\hat{\Phi}}\sqrt{\hat{G}} \,. \tag{3.20}$$

This field redefinitions were employed in a series of works [240, 242, 245, 252] in order to rewrite the standard NSNS sector in terms of theses new fields. The resulting theory is known as *β-(super)gravity* and is particularly suited to describe non-geometric spaces

[4] Note the index structure for β resulting from this equation, $\beta \equiv \beta^{ij}$. In particular β is not a two-form, but a bivector $\beta = \frac{1}{2}\beta^{ij}\partial_i \wedge \partial_j$, cf. [240]. For explicit transformation rules, see, for example, [240].

and their exotic fluxes. Crucially, it is equivalent to the NSNS action up to a total derivative and it is this new frame, in which non-geometric spaces are well-defined.

Without going into details, the strategy to find this new description or new action can be summarised in the following diagram (adapted from [242])

$$\begin{array}{ccc} L_{DFT}(\mathcal{H}(G,B),d) & \xLeftrightarrow{O\in O(d-1,1)\times O(1,d-1)} & L_{DFT}(\mathcal{H}(\hat{G},\beta),\hat{d}) + \partial_M(\dots) \\ \Big\Downarrow{\scriptstyle \tilde{\partial}=0} & & \Big\Downarrow{\scriptstyle \tilde{\partial}=0} \\ L_{NSNS}(G,B,\Phi) + \partial(\dots) & \xLeftrightarrow{\quad(3.19)\quad} & L_{\beta}(\hat{G},\beta,\hat{\Phi}) + \partial(\dots)\,. \end{array}$$

Hence, the new action can be obtained by first rotating to the new frame in the DFT action and then applying the strong constraint[5], reducing to a D-dimensional theory depending only on the physical coordinates X. The resulting action reads [242, 243]

$$\begin{aligned} \mathcal{S}_\beta = \int \mathrm{d}^D x e^{-2\hat{\Phi}}\sqrt{\hat{G}}\Big(R(\hat{G}) + \check{R}(\hat{G}) - \frac{1}{12}R^{ijk}R_{ijk} + 4(\partial\hat{\Phi})^2 \\ + 4(\beta^{ij}\partial_j\hat{\Phi} - \frac{1}{2}\hat{G}_{pq}\beta^{ij}\partial_j\hat{G}^{pq} + Q_k{}^{ki})^2\Big)\,, \end{aligned} \tag{3.21}$$

where $\check{R}$ is a rather complicated expression, involving the field β and the Q-flux which we define below. It is a scalar under diffeomorphism and can be seen as a kind of Ricci scalar-like tensor associated with a connection $\check{\Gamma}$, cf. remark below and [245]. It reads

$$\check{R} = -\frac{1}{4}\hat{G}_{ij}\hat{G}_{mn}\hat{G}^{kl}Q_k{}^{\ell j}Q_\ell{}^{\ell j} - \frac{1}{2}\hat{G}_{ij}Q_k{}^{\ell j}Q_j{}^{ki} - \hat{G}_{ij}Q_k{}^{ki}Q_\ell{}^{\ell j} + \dots\,, \tag{3.22}$$

where we omitted terms that are further contractions of Q, G and derivatives thereof, as they will vanish and hence not play a role in our discussion in Chapter 7. The full expression can be found, for example, in Eq. (A.8) of [242]. We will see in Chapter 7, that this form of the action is crucial in establishing the correct matching of the potentials of non-geometric spaces and their duals for the Swampland Distance Conjecture. In particular the action explicitly contains the non-geometric fluxes

$$Q_k{}^{ij} = \partial_k\beta^{ij}\,, \qquad R^{ijk} = 3\beta^{\ell[i}\partial_\ell\beta^{jk]}\,. \tag{3.23}$$

Upon compactification, these fluxes will give contributions to the scalar potential that are similar to their NSNS cousins. As indicated in the diagram above, the Lagrangian of the β-gravity theory is equivalent to the NSNS Lagrangian up to a total derivative

$$L_{\mathrm{NSNS}} = L_\beta + \partial\,(\dots)\,. \tag{3.24}$$

This means that the two Lagrangians share the same symmetries and – up to field

[5]This need not be possible always, cf. remarks below.

redefinitions – also the same set of equations of motion, implying that a vacuum of one theory will automatically also be a (local) vacuum of the other one. The explicit form of the total derivative can be found in the appendix of [67], where the matching of the two frames is demonstrated explicitly on the example of S^3 and its $\mathrm{NATD}^{\mathrm{PL}}$.

Remarks

- Finding a suitable connection $\tilde{\Gamma}$ and covariant derivative ∇ turned out to be challenging. It was however defined in [245] together with an associated tensor

$$\check{R} = \check{R}(\beta, \tilde{\Gamma}) , \tag{3.25}$$

  similar to the standard Ricci tensor such that diffeomorphism invariance of the theory is manifest[6]. The formulation in terms of $\check{R}$ will be crucial for later chapters.

- At the level of the full DFT action, the R-flux has another contribution from dual coordinates

$$R^{ijk} = 3\left(\tilde{\partial}^{[i}\beta^{jk]} + \beta^{\ell[i}\partial_\ell\beta^{jk]}\right) . \tag{3.26}$$

  Generically the section condition is then not satisfied and one can not perform the truncation to the physical coordinates.

- Non-geometric properties can also be understood from the failure of the frame fields to be globally defined. See [253] for a detailed discussed of the role of the frame in non-geometric settings and its global properties. One can also translate this to the section condition, which ceases to be defined globally [254]. Since patches are glued with dualities, the section condition has to be adjusted.

- We reserve the word *non-geometry* or *non-geometric background* for the effective theories that fall within the framework described above. There are more general notions of the word non-geometric background or string vacuum. We can think of a (perturbative) *geometric string background* as arising from a worldsheet NLSM admitting a (pseudo-)Riemannian target space manifold with associated differential structure and field content. Then all theories not fitting into these class can be thought of as *non-geometric* in one sense or another. In particular these are generic $2d$ worldsheet SCFTS like rational CFTs [162], in particular Gepner models [255] or also asymmetric orbifolds[7] [258].

[6]While $R^{ijk} = 3\beta^{l[i]}\partial_l\beta^{jk]} = 3\beta^{l[i]}\nabla_l\beta^{jk]}$ is a tensor such that the R^2 term is diffeomorphism invariant, $Q_k^{\ ij}$ is not. Repackaging all the Q terms into $\check{R}$ leads to a covariant object, cf. the discussion of [243].

[7]Asymmetric obribfolds are a somewhat special case as their effective theories in terms of gauged supergravities display exactly the non-geometric fluxes introduced above, cf. [256, 257]

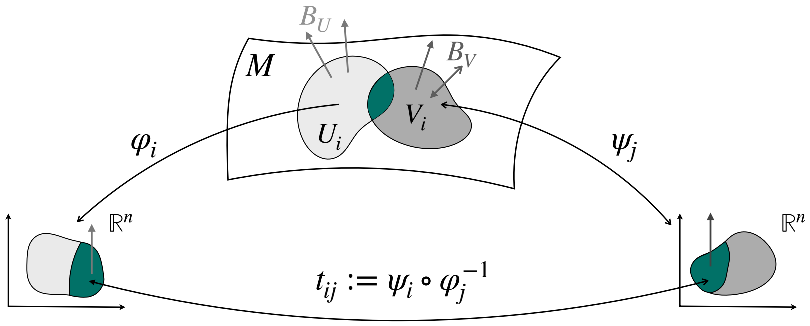


Figure 3.1.: In contrast to standard Riemannian geometry, transition functions t_{ij} are not only allowed to include diffeomorphisms (and gauge transformations of the NSNS form field B) but also dualities. In case of T-duality this leads to the notion of a T-fold.

3.2.3. Monodromies and non-geometry

Non-geometry manifests itself - at least within the form discussed here - as either a T-duality monodromy around some cycle or through the dependence on winding coordinates. At the level of the underlying geometric structure, this translates to transition functions between patches that include T-duality transformations, leading to the notion of a *T-fold*[8]; see also Figure 3.1. The global non-geometric structure obviously can not be detected locally; instead, we have to study non-trivial monodromies around certain non-contractible cycles of the geometry. Monodromies in general are not particular features of non-geometric spaces but generically arise for nontrivial fibrations[9]. Going around the loop in the fibre, $\mathcal{H}$ might not respect a periodicity like $z \sim z + 2\pi$. In contrast, it will transform as

$$\mathcal{H}^{IJ}(z+2\pi) = M^I{}_L \mathcal{H}^{LK}(z) M_K{}^J\,, \tag{3.27}$$

where M is the monodromy matrix associated to the specific cycle. Demanding now that the background gives rise to a valid string background, the matrix M is required to correspond to a symmetry of string theory, i.e. it should lie in the T-duality group $O(D, D, \mathbb{Z})$. Although geometric spaces can have monodromies that lie in the geometric subgroup of $O(D, D, \mathbb{Z})$ corresponding to diffeomorphism or gauge transformations, for non-geometric spaces it is a nontrivial element that is neither of the two and corresponds to a T-duality; see also Appendix C in [261].

For a nice review on aspects of non-geometry, we refer to [159]; see also [239].

[8] See in particular, also [157, 219, 259, 260] for geometric aspects of T-folds, including DFT.

[9] Non-geometric spaces, in particular for example T-folds, viewed as fibrations over some base manifold $\mathcal{B}$, can be characterised in terms of their monodromy properties when going around a cycle in the base. See [259] and references therein.

3.2.4. The T-duality chain

The prototypical example of a non-geometric space can be found by successive T-duality transformations starting from the three-dimensional torus with H-flux. This leads to the so-called *T-duality chain* [262, 263]. As we will use this example in Part II, we introduce some of its basic features here. The detailed examination in view of the Swampland Distance Conjecture is delayed to Chapter 7. The setup can be summarised as

$$\begin{matrix} H\text{-flux} & \xleftrightarrow{x_1} & \text{geometric flux} & \xleftrightarrow{x_2} & Q\text{-flux} & \xleftrightarrow{x_3} & \text{R-flux} \\ H_{ijk} & & f^i{}_{jk} & & Q^{ij}{}_k & & R^{ijk} \end{matrix}\,, \tag{3.28}$$

with arrows denoting T-duality along the corresponding directions. The last one has to be read with care, as it does not correspond to a proper T-duality; cf. discussion below.

T^3 with $[H] = k$

Our starting point is the torus T^3, equipped with k units of H-flux. Denoting the angular coordinates with x^i for $i \in \{1,2,3\}$ the background has the simple form

$$\mathrm{d}s^2 = G_{ij}\mathrm{d}x^i x^j = \sum_i r_i^2(\mathrm{d}x^i)^2\,, \qquad B = kx_3\mathrm{d}x^1 \wedge \mathrm{d}x^2\,. \tag{3.29}$$

In order to have at least two proper isometric directions, we restrict the dilaton to be at most a function of x^3, i.e. $\Phi \equiv \Phi_3 = \Phi_3(x^3)$. The flux associated to B is simply

$$[H] = H_{123} = k\,. \tag{3.30}$$

This background - although not solving the equations of motion and hence not a proper string vacuum - gives rise to a well-defined effective theory when inserted in the effective action $\mathcal{S}$. Since we have two isometries, along x_1 and x_2, we can perform T-duality. Dualising along a single direction, say x_1, leads to the so-called *twisted torus*.

Twisted torus

After T-dualising along the x^1 isometry, the resulting background reads

$$\mathrm{d}s^2 = \sum_{i=1}^{3}(\eta^i)^2\frac{1}{r_1^2}(\mathrm{d}\tilde{x}_1)^2 + r_2^2\Big(1+\frac{k^2(x^3)^2}{r_1^2r_2^2}\Big)(\mathrm{d}x^2)^2 - 2\frac{kx_3}{r_1^2}\mathrm{d}\tilde{x}_1\mathrm{d}x^2 + r_3^2(\mathrm{d}x^3)^2\,, \quad B=0\,, \tag{3.31}$$

with η_i one-forms that read $\eta^1 = 1/r_1(\mathrm{d}\tilde{x}_1 - kx_3\mathrm{d}x^2)$ and $\eta^2 = r_2\mathrm{d}x^2, \eta^3 = r_3\mathrm{d}x^3$ and dual coordinates denoted with $\tilde{x}^i$. The background is subject to the identifications

$$\tilde{x}_1 \sim \tilde{x}_1 + n^1 - n^3kx^2\,, \quad x^2 \sim x^2 + n^2\,, \quad x^3 \sim x^3 + n^3\,, \qquad n^i \in \{0,1\}\,, \tag{3.32}$$

and is in particular compact. Hence, we obtain a purely geometric background with no flux. Sometimes this background is said to have f-flux, which is a slightly misleading way to express that the background has a non-trivial topology giving rise to a flux-like contribution, satisfying the Maurer-Cartan equation

$$\mathrm{d}\eta^1 = f^1_{\ 23}\eta^2 \wedge \eta^3 \,, \tag{3.33}$$

with $f^a_{\ bc}$ the flux number of the geometric flux in the basis of one-forms. In the case at hand (and written in a coordinate basis), it reads $[f] = f^1_{\ 23} = k$. Note that the natural "radius" of the twisted cycle of x^1 is given by $\tilde{r}_1 = 1/r_1$ while $\tilde{r}_{2/3} = r_{2/3}$. Furthermore, the dilaton gets shifted by the usual transformation rule such that it reads

$$\Phi = \Phi_3 - \log(r_1)\,. \tag{3.34}$$

This geometry is an example of the more general notion of a *nilmanifold*.

Q-flux background

Dualising a second time, now along the x^2-direction, we obtain

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 + \frac{k^2(x^3)^2}{r_1^2 r_2^2}\right)^{-1}\left(r_1^{-2}(\mathrm{d}\tilde{x}_1)^2 + r_2^{-2}(\mathrm{d}\tilde{x}_2)^2\right) + r_3^2(\mathrm{d}x^3)^2\,, \\ B &= \frac{k x_3}{r_1^2 r_2^2 + k^2 x_3^2}\mathrm{d}\tilde{x}_1 \wedge \mathrm{d}\tilde{x}_2\,, \quad \Phi = \Phi_3 - \frac{1}{2}\log\left(r_2^2 + \frac{k^2(x^3)^2}{r_1^2}\right). \end{aligned} \tag{3.35}$$

It is easy to see that this configuration is problematic. Going around the x^3-fibre, it is clear[10] that due to the nontrivial denominator, the metric (and similarly the B-field) does not respect the periodicity of the coordinate. Instead, the metric can be viewed as periodic up to T-duality transformations, i.e. transformations in $O(2,2,\mathbb{Z})$. On can check that going around the cycle $x^3 \to x^3 + 2\pi$ it picks up a monodromy

$$M^I{}_J = \begin{pmatrix} \mathbb{1}_3 & 0 \\ P & \mathbb{1}_3 \end{pmatrix} \in O(2,2;\mathbb{Z})\,, \qquad P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\pi \\ 0 & 2\pi & 0 \end{pmatrix}. \tag{3.36}$$

Since M is of lower triangular form, it is neither a diffeomorphism nor a gauge transformation and hence a non-trivial duality element[11] of $O(2,2,\mathbb{Z})$. Therefore going around the base, we have to glue the patches with T-duality, inverting the radius, and hence realising a (non-geometric) T-fold.

[10] Alternatively one can explicitly substitute the background into the action, resulting in a total derivative which renders the theory non single valued, see Appendix B in [240].

[11] For decompositions of $O(D,D)$ in matrix generators and their physical interpretation see [134].

Since we are dealing with a T-fold, the theory is better described in the β-frame. Using the relation (3.19), we readily obtain a new frame in which the geometry is well-defined. It reads

$$\mathrm{d}\hat{s}^2 = r_1^{-2}(\mathrm{d}\tilde{x}_1)^2 + r_2^{-2}(\mathrm{d}\tilde{x}_2)^2 + r_3^2(\mathrm{d}x^3)^2\,, \quad \beta = \begin{pmatrix} 0 & kx^3 & 0 \\ -kx^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \hat{\Phi} = \Phi_3 - \log(r_1 r_2)\,, \tag{3.37}$$

such that the metric has the form of a standard torus with radii $\tilde{r}_{1/2} = 1/r_{1/2}$ and $\tilde{r}_3 = r_3$. Finally, the nontrivial β gives rise to the new Q-flux. The latter is given by $Q_k{}^{ij} = \partial_k \beta^{ij}$ such that we obtain

$$[Q] = Q_3{}^{12} = k\,. \tag{3.38}$$

R-flux background

First, note that a naive application of the Buscher rules would lead to the background

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 + \frac{k^2 (x^3)^2}{r_1^2 r_2^2}\right)^{-1} \left(r_1^{-2}(\mathrm{d}\tilde{x}_1)^2 + r_2^{-2}(\mathrm{d}\tilde{x}_2)^2\right) + r_3^{-2}\,(\mathrm{d}\tilde{x}_3)^2\,, \\ B &= \frac{kx^3}{r_1^2 r_2^2 + k^2 (x^3)^2} \mathrm{d}\tilde{x}_1 \wedge \mathrm{d}\tilde{x}_2\,, \quad \Phi = \Phi_3 - \frac{1}{2}\log\left(r_2^2 + \frac{k^2 (x^3)^2}{r_1^2}\right) - \log(r_3)\,. \end{aligned} \tag{3.39}$$

which besides the obvious illegality of the transformation itself and the problematic monodromy now depends on the coordinate x^3. From the point of view of the new rotated coordinate basis $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ the coordinate x^3 should be viewed as the dual winding coordinates and hence we encounter a background that no longer fulfils the strong constraint. One can argue that we were never allowed to perform such a transformation due to the lack of isometry. However, the same background can be obtained from a DFT perspective by a $O(d, d, \mathbb{Z})$ rotation without any reference to abelian T-duality. This has also been argued in [242] and references therein. Strictly speaking, this background hence should be treated in the doubled formalism. However, since the R-flux itself does not depend on the coordinate x^3, for our purposes it is enough to continue with the above expression. Comparing with the Q-flux background, it is trivial to see that in the β-frame the new space reads

$$\mathrm{d}\hat{s}^2 = r_1^{-2}(\mathrm{d}\tilde{x}_1)^2 + r_2^{-2}(\mathrm{d}\tilde{x}_2)^2 + r_3^2(\mathrm{d}\tilde{x}_3)^2\,, \quad \beta = \begin{pmatrix} 0 & kx^3 & 0 \\ -kx^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \hat{\Phi} = \Phi_3 - \log(r_1 r_2 r_3)\,. \tag{3.40}$$

Recall that before implementing the section condition $\tilde{\partial} = 0$, the R-flux is given by (3.26). Now taking into account that the new physical coordinates read $X^i = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ and the dual $\tilde{X}_i = \{x^1, x^2, x^3\}$, we see that indeed we have a non-vanishing contribution

$$R^{312} = \tilde{\partial}^3 \beta^{12} = k \, . \tag{3.41}$$

Hence, the flux quantum is conserved when mapped along the chain of equation (3.28). The first two background are perfectly well described within Riemannian geometry, while the Q-flux background is only *locally geometric*, i.e. can be described by Riemannian geometry patch-wise; globally T-duality is needed in order to glue the patches. Therefore the space is also called *globally non-geometric*. In contrast, the R-flux background depends on the dual coordinates and is not even locally described within Riemannian geometry and hence an example of a *locally non-geometric* space.

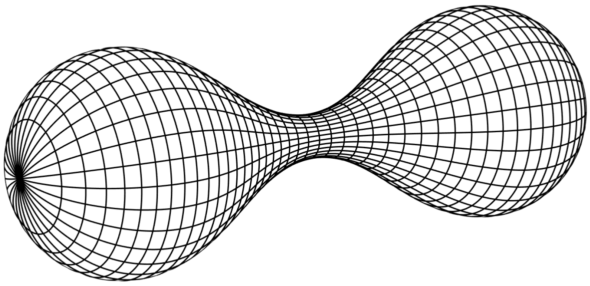
Remarks

- Locally non-geometric space seem to be in tension with a description in terms of point-like particles. To see this start from the torus with H-flux and observe that due to the *Freed-Witten anomaly* [264] there can not be any wrapped $D3$-branes. Translating the situation to the R-flux background via T-duality one arrives at the conclusion that there can not be any $D0$-branes and hence no point particle like states. For a more detailed discussion we refer to [265] or also [266]. Furthermore the background features, besides *non-commutativity* [256, 261, 267–269], even *non-associativity* of the target space coordinates [269, 270].

- Since the spaces in the above chain are all dual[12] to each other, one might argue that the non-geometry of the latter two is merely an artefact. There is however a large family of backgrounds that is not dual to to any geometric space and these can be realised by *asymmetric orbifold constructions* [258]. For the relation of these spaces with non-geometric Q- and R-fluxes see [256, 257]. Since these spaces treat differently left and right movers of the string, they completely lack a target space description and can not be brought to such by any T-duality. They are examples of proper stringy realisation of *truly non-geometric spaces*.

- In the above example we only have either geometric or non-geometric fluxes at each step but not simultaneously. However it was shown, that there are backgrounds in which all of the above fluxes are turned on at the same time, see for example [257] for a construction in terms of asymmetric orbifold. These are again truly non-geometric, i.e. not dual to any purely geometric realisation.

[12]Of course the last step is not a true duality as explained above.



CHAPTER 4

Geometric flows

Partial differential equations (PDE) are the main tools in describing dynamical processes in physical systems. They often come in the form of so-called *flow* or *transport* equations that describe the evolution of some given physical quantity in time. This evolution can be an intuitive evolution of physical quantities in spacetime, like the Heat equation or the Euler equations of fluid dynamics. The can also describe processes in more abstract spaces, like the continuity equation in quantum mechanics, describing the evolution or conservation of the probability current which can be seen as an evolution in the space of probabilities. Geometric flow equations, in addition, describe the evolution of some underlying geometry itself with respect to some abstract notion of "time", which does not necessarily have an interpretation in terms of a physical time. A similar and closely related example is the *Renormalisation Group* (RG) flow, describing the running of some physical coupling with a scale Λ. In the following chapter, we will focus on a particular geometric flow, known as *Ricci flow* and its generalisations. We will give a detailed introduction to its mathematical properties that will be the foundation of the discussion of geometric flows in the context of the Swampland program in Chapter 9 and 10. After a review of standard Ricci flow, we introduce its generalisation including form fields and the relation to the string effective action. After discussing some examples, we close with a short discussion of applications within string theory.

4.1. Basics of Ricci flow and its generalisation

We provide a brief introduction to Ricci flow and its generalisation. We focus on the gradient flow interpretation pioneered by Perelman in [271–273], which led to his celebrated proof of the *Poincaré Conjecture* [274], and the relation to string theory. For a more complete and detailed mathematical treatment, including a treatment of singularities and surgery, we refer to the references listed at the end of the chapter.

4.1.1. Ricci flow

Ricci flow, as introduced by Richard Hamilton, is determined by the partial differential equation (PDE) [275]

$$\frac{\partial}{\partial s} g_{\mu\nu}(s) = -2R_{\mu\nu}(g(s)) \,, \tag{4.1}$$

where $g_{\mu\nu} \in \Gamma(S^2T^*M)$ is a metric tensor on the manifold M and $R_{\mu\nu}(g)$ the Ricci tensor associated to g, which in the following will be simply denoted by $R_{\mu\nu}$ and similar for $R(g) \equiv R$. We explicitly highlighted the dependence on the auxiliary flow parameter[1] s. The equation (4.1) defines a PDE on $g(x,s)$, which has to be supplemented with suitable boundary or initial conditions such that $g(s)$ gives a one-parameter family of metric tensors on M. Ricci flow can be seen[2] as a (non-linear) generalisation of the heat equation, or more general a convection-diffusion-reaction system, cf. [276]. As such, it tends to smoothen out the geometry along the flow by contracting regions with positive sectional curvature $K > 0$ and expanding regions where $K < 0$, cf. Figure 4.1.

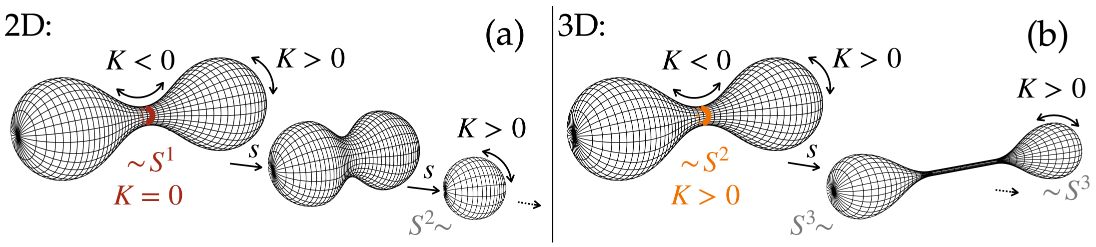


Figure 4.1.: Visualisation of the flow behaviour of curved spaces under Ricci flow for a dumbbell-like geometry in $D = 2$ and $D = 3$. Crucially, in 2D the neck of the geometry is locally a circle S^1 with zero curvature, neither shrinking nor expanding and hence the full geometry evolves towards a sphere. In contrast, in 3D the neck is given by S^2 with $K > 0$ and therefore shrinks, leading to the so-called *neck-pinch*. The precise behaviour, i.e. an emerging singularity or infinitely long neck depends on the initial conditions.

[1] In the mathematical literature the s is often denoted t instead, and viewed as an abstract "time" along the flow. We occasionally also refer to s as (flow) time, but it should not be confused with physical time.

[2] Using *harmonic coordinates* [18] $\{x^\mu\}$ one can locally write (4.1) as $\partial_s g_{\mu\nu} = \Delta g_{\mu\nu} + Q(g^{\mu\nu}, \partial_s g_{\mu\nu})$ with Q quadratic in the arguments.

It was realised in the seminal paper [271] of Perelman, that Ricci flow can be realised as a gradient flow[3] with respect to an associated *entropy functional*

$$\mathcal{F}(g,f) \equiv \int_M \mathrm{d}^n x \sqrt{g}\, e^{-f} \left(R + |\nabla f|^2 \right) , \tag{4.3}$$

with f a smooth real-valued function on M. The scalar f can, up to a factor of two, be identified with the dilaton $f = 2\phi$. In addition, one needs to impose the *generalised unit volume constraint*

$$\int_M \mathrm{d}^n x \sqrt{g}\, e^{-f} = 1 , \tag{4.4}$$

hence keeping the weighted, integrated volume element constant along the flow. Under these conditions, the Ricci flow (4.1) is gradient with respect to the action (4.3) with dilaton f and (4.4) result in an additional equation for f. The combined system is known as *Perelman's combined flow* and reads [271]

$$\frac{\partial g_{\mu\nu}}{\partial s} = -2R_{\mu\nu} , \qquad \frac{\partial f}{\partial s} = -\Delta f + |\nabla f|^2 - R . \tag{4.5}$$

This is actually not the form of the flow directly emerging from the variational procedure but has been gauge fixed by a specific choice of diffeomorphism that decouples the two PDEs. Before fixing the gauge, the system reads

$$\frac{\partial g_{\mu\nu}}{\partial s} = -2R_{\mu\nu} - 2\nabla_\mu \nabla_\nu f + \mathcal{L}_X g_{\mu\nu} , \qquad \frac{\partial f}{\partial s} = -\Delta f - R + \mathcal{L}_X f , \tag{4.6}$$

which clearly represents a coupled system of PDE that is in general much harder to solve[4]. The additional Lie derivative terms are added since the physical metrics are determined only up to diffeomorphisms that enter the flow equations via $\mathcal{L}_X \bullet$. Upon setting $X = \nabla f$, the flow reduces to (4.5). When referring to the Ricci flow or the combined flow, we mostly have in mind the gauge-fixed version (4.5). However, it is important to keep the diffeomorphism ambiguity in mind, as it will appear many times in the discussions to follow and plays a crucial role for establishing the (short-time) existence of Ricci flow.

For later purposes, we note that in the case of an Einstein space, i.e. $R_{\mu\nu} \sim g_{\mu\nu}$, the flow equation can be shown to reduce to a flow equation for the Ricci scalar

$$\partial_s R = \nabla^2 R + 2R_{\mu\nu}R^{\mu\nu} . \tag{4.7}$$

[3] For $v : [0,S] \to \mathbb{R}^n$ and $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ a gradient flow is defined as

$$\partial_t v = -\nabla \mathcal{E}(v) . \tag{4.2}$$

For more abstract settings, e.g. for the case of $g_{\mu\nu}$ relevant here, see Appendix E and Eq. (4.24) and (4.27).

[4] In system (4.1) on the other hand, f is decoupled from the evolution equation of g. Hence, one can first solve for g and then subsequently for f.

4.1.2. Existence of solutions and the space of Riemannian metrics

So far we have introduced a PDE for a given metric tensor g on a manifold M, which we did not specify further. The mathematically inclined reader will therefore immediately ask if such a PDE is well-posed. For an initial condition g_0, do (at least short-lived) solutions always exist and are they unique? Under certain assumptions, these questions can be answered positively, while for others more care is required.

In case the manifold M is equipped with a positive-definite Riemannian metric g, solutions to a given initial condition g_0 always exist, at least for $s \in [0, \epsilon]$, $\epsilon > 0$, and are unique. For closed manifolds M, the existence and uniqueness was first established by Hamilton [275]. Alternatively one can make use of the aforementioned DeTurck-trick [277], using the diffeomorphism freedom to bring the flow into a form in which standard existence and uniqueness theorems from the theory of PDEs, can be applied; cf. [278, 279]. The results were then later extended to complete non-compact manifolds in [280, 281]. For long term existence, i.e. $\forall s > 0$ there have to be imposed further conditions, as for example curvature bounds on g_0.

On the other hand, for Lorentzian signature the situation is in general very different. The indefinite signature of Lorentzian spaces, corresponding to negative diffusion constants and, in particular, negative principal symbols of the associated differential operator, can lead to severe pathologies of the flow. The severity of these pathologies depends both on the underlying manifold as well as the chosen initial conditions. Since in the present thesis we are mostly interested in applying the flow (and its generalisation) to compact Riemannian manifolds, we will not discuss these issues further here but refer to the discussion in [282–284] and references therein.

In the Riemannian case, there is some further mathematical structure that we can introduce that formalises the evolution of g in a more abstract space, namely the so-called *(moduli) space of Riemannian metrics*. In a nutshell, this is the set $\mathcal{M}(M)$ of all Riemannian metrics on the manifold M. It turns out that $\mathcal{M}(M)$ is not only a set, but can be itself viewed as a (infinite-dimensional) manifold. We will not enter the details of dealing with such a highly nontrivial structure but restrict ourselves to some core elements that will be important for later discussions, especially in view of geometric flows. For this first note that a metric tensor g on a manifold M is a section of the second symmetric power of the cotangent bundle of M, i.e. $\Gamma(S^2T^*M)$ such that the induced bilinear form on T_pM is positive definite. One can define a, formally infinite dimensional, vector space $C^\infty(M, S^2T^*M)$ of real symmetric $(0, 2)$ tensor fields on M and equip it with a certain topology, turning it into a *Fréchet manifold*; see, for example, [285, 286]. The set $\mathcal{M}(M)$ is then a subspace of this space,

$$\mathcal{M}(M) < C^\infty(M, S^2T^*M) . \tag{4.8}$$

We already mentioned above that, especially from a physical point of view, we are only interested in metrics up to diffeomorphisms, since it are only these that define physically inequivalent geometries. In fact also form a mathematical point of view this is true and hence the *moduli space of Riemannian metrics* $\widehat{\mathcal{M}}(M)$ on a given manifold M is defined by quotienting out the action of the diffeomorphism group $\mathrm{Diff}(M)$, such that we finally obtain[5]

$$\widehat{\mathcal{M}}(M) = \mathcal{M}(M)/\mathrm{Diff}(M)\,. \tag{4.9}$$

This redundancy of diffeomorphism is reflected in the flow equation by the arbitrary diffeomorphism term in equation (4.6). However, this distinction will not play an important role for us and we will focus on $\mathcal{M}(M)$ rather than $\widehat{\mathcal{M}}(M)$.

Since Ricci flow determines a one-parameter family of metrics $g(s)$, we can view Ricci flow as a path or curve $s \mapsto g(s)$ on $\mathcal{M}(M)$[6]. Accordingly, the natural question concerning the length of such a curve arises. The definition of such a length requires the notion of a metric on the space of metrics itself. In fact, there is a natural Riemannian metric $\mathcal{G}_g^{\mu\nu\alpha\beta}$ on the space of Riemannian metrics $\mathcal{M}(M)$, defining at every point $g \in \mathcal{M}(M)$ a bilinear form denoted $\mathcal{G}_g(v, w)$ via

$$\mathcal{G}_g(v, w) = \int_M \mathrm{d}^D x \sqrt{g}\, \mathcal{G}_g^{\mu\nu\alpha\beta} v_{\mu\alpha} w_{\nu\beta}\,, \tag{4.10}$$

with $v, w \in T_g(\mathcal{M}(M))$. The metric $\mathcal{G}_g$ is given by

$$\mathcal{G}_g^{\mu\nu\alpha\beta} = g^{\mu\alpha} g^{\nu\beta}\,, \tag{4.11}$$

as it was introduced by Ebin [287] as a natural metric on $\mathcal{M}(M)$ and is also known as the *canonical metric*. In fact, this is a special case of the family of metrics[7]

$$\mathcal{G}_{g;\tau}^{\mu\nu\alpha\beta} = \frac{1}{2}\left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - 2\tau g^{\mu\nu} g^{\alpha\beta}\right)\,, \tag{4.12}$$

introduced by DeWitt [288], who was the first to compute geodesics and distances in these settings. The family of metrics is hence known as *DeWitt metric* and the canonical metric corresponds to $\tau = 0$. We will be exclusively interested in the case $\tau = 0$, which we also loosely refer to as DeWitt metric in the following and denote by $\mathcal{G}_g^{\mu\nu\alpha\beta} \equiv \mathcal{G}_{g;0}^{\mu\nu\alpha\beta}$. Having defined a notion of metric we can calculate the induced distance between to points via

$$d(g_0, g_1) = \inf\nolimits_\gamma \left\{ L(\gamma) \,\middle|\, \gamma(0) = g_0\,, \gamma(1) = g_1 \right\}. \tag{4.13}$$

[5]In the mathematical literature usually $\mathcal{M}(M)$ denotes the moduli space itself, i.e. the quotient.

[6]When working with the gauge-fixed flow, i.e. having picked a specific vector field X in (4.6) specifying the diffeomorphism, the flow can be viewed as a path on $\widehat{\mathcal{M}}(M)$ with the "gauge slice" selected by X.

[7]Strictly speaking, $\mathcal{G}_{g;\tau}$ is only positive definite – and hence a proper metric – for $\tau = 0$.

The notion of length $L(\gamma)$ along a path $\gamma(s) : [0,1] \to \mathcal{M}(M)$ then reads

$$L(\gamma) = \int_\gamma \mathrm{d}s \sqrt{\int_M \mathrm{d}^D x \sqrt{g} \mathcal{G}_g^{\mu\nu\alpha\beta} \frac{\mathrm{d}g_{\mu\alpha}}{\mathrm{d}s} \frac{\mathrm{d}g_{\nu\beta}}{\mathrm{d}s}}\,. \tag{4.14}$$

The minimising curves of (4.13), along which the distances are measured, are nothing more than the geodesics of the DeWitt metric. These geodesics are in general highly complicated curves and will not discuss the notion of geodesics further here, but refer to [285, 286, 289] and references therein; we also recommend [284] for a discussion of the above concepts and, in particular, for a in-depth treatment of the geodesics. We will be mostly interested in the distance along the flow trajectories. This distance can be calculated using (4.14) and is generically much easier to obtain than geodesics. However, they will in general not minimise (4.13), i.e. Ricci flow trajectories generically do not correspond to geodesics in $\mathcal{M}(M)$. Considering also an evolution of f, the configuration space is enlarged and the functional is modified. We will delay the discussion of these modifications until later when introducing generalised flows.

Remark

- The aforementioned freedom of adding a diffeomorphism term gives rise to special solutions, called *Ricci solitons* [290]. These are solutions that, broadly speaking, evolve in a self-similar way. In particular, these are geometries that satisfy

$$R_{\mu\nu} + \frac{1}{2}\mathcal{L}_X g_{\mu\nu} = \lambda g_{\mu\nu}\,, \tag{4.15}$$

  which for $\lambda > 0$ are called *shrinking*, for $\lambda < 0$ *expanding* and for $\lambda = 0$ *steady* Ricci solitons. Einstein spaces are special cases of such Ricci solitons; cf. [291].

- There are various modifications of the Ricci flow that, in addition to the Ricci tensor $R_{\mu\nu}$, are sourced by additional terms. These include the *Ricci-Bourguignon flow*, *Yamabe flow* or the *normalised Ricci-flow*; see [292] and references therein.

- Strictly speaking, the length formula (4.14) only makes sense for compact (Riemannian) manifolds. For M non-compact (and possibly Lorentzian), we should renormalise the distance, for example, by dividing by the overall volume $\mathcal{V}(M) = \int_M \mathrm{d}^D x\sqrt{|g|}$ and performing exhaustions by compact subspaces Ω_r so that schematically (note the absolute value)

$$L(g_0, g_1) \sim \lim_{r\to\infty} \int_\gamma \frac{1}{\mathcal{V}(\Omega_r)} \sqrt{\left| \int_{\Omega_r} \mathrm{d}^D x \sqrt{|g|} \mathcal{G}_g^{\mu\nu\alpha\beta} \frac{\mathrm{d}g_{\mu\alpha}}{\mathrm{d}\tau} \frac{\mathrm{d}g_{\nu\beta}}{\mathrm{d}\tau} \right|}\,. \tag{4.16}$$

  The simple volume normalisation is often also used in the compact case.

4.1.3. Simple examples of the Ricci flow

The simplest solution to the Ricci flow equation is found when the geometry is given by an Einstein manifold. For these, the Ricci curvature takes the form

$$R_{\mu\nu}(g) = k g_{\mu\nu} \,, \tag{4.17}$$

for k a real constant. One can now directly compute the resulting flow equation, which reduces to an equation on $k(s)$. However, it is more instructive to look at explicit examples, where k can be related to a geometric quantity that visualises the flow behaviour and allows for a physical interpretation.

Compact & Euclidean: S^n

Particularly useful examples are the n-spheres of radius r for which k in Eq. (4.17) is positive with $k = (n-1)/r^2$. For later purposes, it is illustrative to write the flow equation in terms of the radius r, which will become time-dependent under the flow. Indeed, taking the ansatz $g(s) = r^2(s) g_0$ with g_0 the metric of the round unit sphere (cf. (2.17)), the conventional Ricci flow equation reads

$$\frac{\partial}{\partial s} r(s) = -\frac{(n-1)}{r(s)} \,, \tag{4.18}$$

such that the radius of the sphere shrinks to zero as

$$r(s) = \sqrt{r_0 - 2(n-1)s} \,. \tag{4.19}$$

Non-compact & Lorentzian: AdS_d

The two above examples were compact and, in particular, with Euclidean signature. While existence results for Ricci flow can be generalised to (complete) non-compact Riemannian spaces, we mentioned above, that the Lorentzian case is much more problematic. Working on a case by case basis, in particular with suitable initial conditions, Ricci flow can nevertheless studied also for Lorentzian signature. A particular relevant and rather simple example that will become important later is d-dimensional *Anit-de Sitter space*, denoted AdS_d. In global coordinates, the metric can be written as

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{L^2}\right) \mathrm{d}t^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{d-2} \,, \tag{4.20}$$

with $\mathrm{d}\Omega^2_{d-2}$ the line element of the (unit) $(d-2)$-sphere. AdS is again an example of an Einstein space, with $R_{\mu\nu} = -(d-1)/L^2 g_{\mu\nu}$ and as such is a vacuum solution of the

Einstein equation with the cosmological constant Λ given by

$$\Lambda = -\frac{(d-1)(d-2)}{2L^2} \,. \tag{4.21}$$

Similarly to the compact hyperbolic case, we could calculate the flow behaviour in terms of the AdS radius L. However, in view of later use, it is instructive to describe it in terms of Λ. The associated flow equation and solution then reads

$$\frac{\partial}{\partial s}\Lambda(s) = \frac{2\Lambda^2}{d-2} \qquad \Longrightarrow \qquad \Lambda(s) = \frac{\Lambda_0}{1-\frac{2}{d-2}\Lambda_0 s} \,. \tag{4.22}$$

4.1.4. Generalised Ricci flow

There is a very natural generalisation of the (coupled) Ricci flow equations that emerges from the gradient flow interpretation of Ricci flow and the associated connection to the σ-model β-functions or Weyl anomaly coefficients. This generalised flow, which naturally included the Kalb-Ramond field, was first derived in [293] and is simply referred to as *generalised Ricci flow*; see also [294].

In particular, it was shown in [293] that one can define such a generalised flow, by starting from a *generalised Perelman functional* given by

$$\mathcal{F}(g,H,f) = \int_K \mathrm{d}^n x \sqrt{g} e^{-f} \left(R - \frac{1}{2}|H|^2 + |\nabla f|^2 \right) , \tag{4.23}$$

together with the (unaltered) constant volume constraint (4.4). Of course we readily recognise this as the low-energy effective action (1.11), upon identifying $f = 2\phi$. Promoting the fields to functions of some auxiliary parameter s, we can calculate the "variation" of the action via[8]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F} = \int_K \mathrm{d}^n x \sqrt{g}\, e^{-f} \Bigg[&\left(-R^{\mu\nu} - \nabla^\mu \nabla^\nu f + \frac{1}{4} H^\mu{}_{\kappa\lambda} H^{\nu\kappa\lambda} \right) \frac{\partial g_{\mu\nu}}{\partial s} \\ &+ \left(R - \frac{1}{2}|H|^2 + 2\Delta f - |\nabla f|^2 \right) \left(\frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial s} - \frac{\partial f}{\partial s} \right) + \frac{1}{2} \left(\nabla_k H^{\kappa\mu\nu} - H^{\kappa\mu\nu} \nabla_\kappa f \right) \frac{\partial B_{\mu\nu}}{\partial s} \Bigg] \\ = \int_K \mathrm{d}^n x \sqrt{g}\, e^{-f} \Bigg[&- \hat{\beta}^{\mu\nu}_G \frac{\partial g_{\mu\nu}}{\partial s} - \hat{\beta}^{\mu\nu}_B \frac{\partial B_{\mu\nu}}{\partial s} - S \left(\frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial s} - \frac{\partial f}{\partial s} \right) \Bigg] , \end{aligned} \tag{4.24}$$

where $\hat{\beta}$ are the Weyl anomaly coefficients (1.8) and we introduced the notation

$$S \equiv S(g,h,f) = \left(R - \frac{1}{2}|H|^2 + 2\Delta f - |\nabla f|^2 \right) . \tag{4.25}$$

[8] See [293] for a more detailed calculation.

S can be seen as a *generalised scalar curvature*; see [68, 295]. Now the functional $\mathcal{F}$ is furthermore subject to a constraint on the field f given by (4.4). It is not hard to see that this implies

$$\left(\frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial s} - \frac{\partial f}{\partial s}\right) = 0\,, \tag{4.26}$$

hence, the last contribution in (4.24) vanishes. The remaining two contributions define a gradient flow with flow equations $\hat{\beta}^{g/B}_{\mu\nu}$. It is customary to introduce a factor of 2 in order to match the standard Ricci flow conventions, so that we arrive at [295]

$$\begin{aligned} \frac{\partial}{\partial s}g_{\mu\nu} &= -2\hat{\beta}^{g}_{\mu\nu} = -2R_{\mu\nu} + \frac{1}{2}H^2_{\mu\nu} - 2\nabla_\mu\nabla_\nu f\,, \\ \frac{\partial}{\partial s}B_{\mu\nu} &= -2\hat{\beta}^{B}_{\mu\nu} = \nabla^\lambda H_{\lambda\mu\nu} - H_{\lambda\mu\nu}\nabla^\lambda f\,, \end{aligned} \tag{4.27}$$

and

$$\frac{\partial}{\partial s}f = -R + \frac{1}{4}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \Delta f\,. \tag{4.28}$$

This flow is known as *generalised Perlman* or *combined flow* and from now on, whenever we write $\mathcal{F}$, we refer to the functional (4.23), *together* with the constant volume constraint (4.4). Whenever we talk about the functional only, we will refer to it as $\mathcal{S}$. The last flow equation for f can be computed from (4.26) after plugging in the flow equation for g. Note that the flow equation is *not* given by the $\hat{\beta}$-function for f (respectively $\phi = \frac{1}{2}f$ of (1.8) and therefore an identification of f with ϕ is less direct. We will comment on this in detail in Chapter 9. However, let us note already here that at the end of the flow, i.e. when we are at a fixed point of (g, B), the flow equation of f indeed reduces to $\hat{\beta}^\phi = 0$. Therefore at the very least at the fixed point the identification $f = 2\phi$ is justified. This is nothing else than the statement that whenever $\hat{\beta}^g = \hat{\beta}^B = 0$, this implies also $\hat{\beta}^\phi = 0$, a point that will be important in Section 4.2.2 below.

Similarly to the conventional Ricci flow, we are free to add diffeomorphism terms for the flow of g and f, and an additional term for the flow of B, corresponding to the gauge invariance of B. Using this freedom, the flow can be brought to the form

$$\begin{aligned} \frac{\partial}{\partial s}g_{\mu\nu} &= -2R_{\mu\nu} + \frac{1}{2}H^2_{\mu\nu}\,, \\ \frac{\partial}{\partial s}B_{\mu\nu} &= \nabla^\kappa H_{\kappa\mu\nu}\,, \end{aligned} \tag{4.29}$$

in which f decouples from the other flow equations. The flow equation for f becomes

$$\frac{\partial}{\partial s}f = -R + \frac{3}{2}|H|^2 - \Delta f + |\nabla f|^2\,. \tag{4.30}$$

This system of flow equitations is usually referred to as the *gauge-fixed* generalised Ricci or Perelman flow, or in the case f is omitted, simply *generalised Ricci flow*. We will mostly refer to all the different incarnations loosely as *generalised Ricci flow*, where it should be clear from the context which form of the flow equations is used.

Before giving some instructive examples, we close this short introduction to generalised Ricci flow by pointing out the monotonicity properties of the functional $\mathcal{F}$ which will be crucial in Chapter 9. In particular substituting the definition of the flow equations (4.27) into (4.24) we see that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F} = \int_K \mathrm{d}^n\sqrt{g}\, e^{-f} \left[\left\| -R^{\mu\nu} - \nabla^\mu\nabla^\nu f + \frac{1}{4}H^\mu_{\ \kappa\lambda}H^{\nu\kappa\lambda} \right\|^2 + \left\| \frac{1}{2}\left(\nabla_\kappa H^{\kappa\mu\nu} - H^{\kappa\mu\nu}\nabla_\kappa f\right) \right\|^2 \right] \geq 0\,, \quad (4.31)$$

establishing strict monotonicity along the flow, with zero only at the fixed point.

Remark

- While in the absence of H, the fixed points of the Ricci flow are critical points of $\mathcal{S}$, this is no longer true in general in the presence of H; see the discussion in [68].
- Generalised Ricci flow can be elegantly written using generalised geometry. Defining $\mathcal{G} = \eta\mathcal{H}$ (η and $\mathcal{H}$ defined in Chapter 3) the flow (4.29) can be equivalently stated as

  $$\mathcal{G}^{-1}\frac{\partial}{\partial s}\mathcal{G} = -2\mathcal{R}c(\mathcal{G})\,, \quad (4.32)$$

  with $\mathcal{R}c(\mathcal{G})$ a generalised Ricci tensor, cf. [295]. The (moduli) space of generalised metrics $\mathcal{GM}(M)$, which also includes B will be introduced in Chapter 9.

4.1.5. Simple examples of the generalised Ricci flow

We will apply generalised Ricci flow almost exclusively to internal compact manifolds, with Euclidean signature. Hence we do not list non-compact or Lorentzian examples below. However, note that the behaviour of the compact hyperbolic space generalises completely analogously to, for example, the case of AdS_3 with (RR) flux.

Three-sphere with H-flux.

It is a natural step to support S^3 with k units of H-flux. In three dimensions there is only one possible choice of H-flux, proportional to the volume form

$$H = 2k\,\mathrm{dvol}\,, \quad (4.33)$$

where dvol is the volume form of the unit three-sphere and k is a real number.

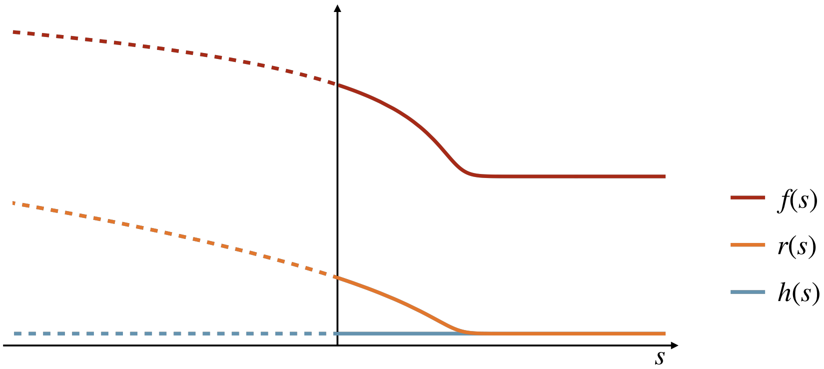


Figure 4.2.: Plot of the flow behaviour of the background S^3 with H-flux under generalised (combined) Ricci flow. The radius (orange) stabilises at the corresponding value of H-flux $[H] = k$ (blue) which stays constant along the flow. The scalar field f (red) converges to a constant such that the unit volume constraint is fulfilled along the flow. Dashed lines correspond to an extension "backwards" in time.

Explicitly, one can easily show[9] that $\partial_s k = 0$. The flow equation for g then reduces to a differential equation for $r(s)$

$$\frac{\partial}{\partial s} r(s) = -\frac{2}{r(s)} + \frac{2k^2}{r(s)^5} \,. \tag{4.34}$$

Taking $k = 0$ gives back the standard Ricci flow of the three-sphere (4.18). The generalised combined flow , including f is then given (in the gauge-fixed frame) as

$$\frac{\partial}{\partial s} r(s) = -\frac{2}{r(s)} + \frac{2k^2}{r(s)^5} \,, \qquad \frac{\partial}{\partial s} k(s) = 0 \,, \qquad \frac{\partial}{\partial s} f(s) = -\frac{6}{r(s)^2} + \frac{6k^2}{r(s)^6} \,. \tag{4.35}$$

The flow behaviour is plotted in Figure 4.2 and will be discussed extensively in Chapter 6 in the context of the (generalised) Swampland Ricci Flow Conjecture.

Three-dimensional hyperbolic space with flux.

Choose an Einstein manifold with k negative, i.e. a compact hyperbolic 3-manifold, $R_{ij}(g) = -2g_{ij}$ with a flux given by the volume form, the flow behaviour is again entirely through the radius $L(t)$ which now is subject to [295]

$$\frac{\partial}{\partial s} L(s) = \frac{2}{L(s)} + \frac{2k^2}{L(s)^5} \,. \tag{4.36}$$

Since both terms now enter with a positive sign, it is clear that there is no non-trivial fixed point and, in contrast to the sphere, even after including the flux the manifolds keeps expanding homothetically forever.

[9] Compute R^+ (cf. appendix of [68]) to see it only has a symmetric part and apply lemma 4.6 of [295].

4.2. Ricci flow in string theory & mathematics

We close this introductory chapter on (generalised) Ricci flow by pointing out some applications & relations of the theory of Ricci flow to physical and mathematical problems. More examples can be found in [283] and [296].

4.2.1. Poincaré Conjecture & Thurston's Geometrisation Conjecture

The by far most famous and celebrated application of Ricci flow is Perelman's proof of the Poincaré Conjecture [274], published in three relatively short articles [271–273] between 2002 and 2003, therefore solving one of the seven Millennium Prize Problems. In fact his work even proofed a stronger conjecture, put forward by Thurston, known as *Geometrisation Conjecture* [297] . Loosely speaking the latter states that every compact orientable three-manifold M admits a canonical decomposition into pieces, each of which belongs to one of eight different canonical geometric structures. In particular, Thurston's conjecture implies the Poincaré Conjecture, namely that any closed three-manifold M with trivial fundamental group $\pi_1(M)$ is homeomorphic to S^3.

The proof of Perelman is highly non-trivial and requires techniques not discussed in this thesis, in particular *surgery*[10]. When written in full detail, the proof covers several hundred pages and hence we refer to the original manuscripts [271–273] as well as [298, 299] that explain the proof in full detail.

4.2.2. Conformal vs. scale invariance

The procedure for obtaining the Weyl anomaly coefficients $\hat{\beta}$, sketched in Chapter 1, is very cumbersome, especially when moving to higher order in α'. In practice it turns out to be much more effective, investigating the UV finiteness of the 2d QFT defined by the NLSM, i.e. computing its *Renormalisation Group* (RG) *β-functions*. These are closely related to the Weyl anomaly coefficients $\hat{\beta}$. However, there is a crucial subtlety arising, which we will briefly discuss in the following. Its resolution will illustrate a very elegant use [300] of (generalised) Ricci flow in the context of string theory.

First note that by classical Weyl invariance we can bring the worldsheet metric to a flat from. In particular, the theory is classically a CFT. However, for a generic background, at the quantum level it suffers from divergent loop diagrams in the UV and the theory needs to be *regularised*. Regularisation procedures, like *dimensional regularisation*, generically break conformal invariance. Moving to $2+\epsilon$ dimensions, the theory

[10] Surgery can be viewed as a kind of regularisation procedure introduced by Perelman in order to deal with the singularities appearing under a generic Ricci flow. It is absolutely curial in order to succeed in proving the Poincaré & Geometrization Conjecture and we refer to [278, 279, 296] for an introduction.

is no longer scale invariant and hence, in particular, conformal invariance is broken. The resulting divergence can be cured by the subtraction of appropriate *counterterms*. These in turn can be absorbed into the fields X^μ together with a renormalisation of the (infinite number of) coupling constants, which for the NLSM are given in terms of $g_{\mu\nu}(X)$. The associated renormalised metric (omitting contributions from B) at first order in α' reads

$$g_{\mu\nu} \to g_{\mu\nu} - \alpha' R_{\mu\nu} \, . \tag{4.37}$$

From this we can identify the RG β-function $\beta^g_{\mu\nu}$ that determines the running of the coupling g with the scale Λ. In particular, one can identify

$$\beta^g_{\mu\nu} = \alpha' R_{\mu\nu} \, , \tag{4.38}$$

such that the running of g with the scale Λ is described by

$$\mu \frac{\partial}{\partial \mu} g_{\mu\nu} \equiv \beta^g_{\mu\nu} = \alpha' R_{\mu\nu} \, . \tag{4.39}$$

If $\beta = 0$, the right-hand side of the equation vanishes and the theory becomes scale-invariant. However, it was realised already in [301] and subsequently further discussed in [302–304], that in fact this requirement is too strong. Physically inequivalent metrics are only defined up to diffeomorphism and, hence, in order to obtain a fixed point of the RG flow it is sufficient to eliminate the counterterm by a diffeomorphism. This translates to

$$R_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \, , \tag{4.40}$$

with the vector field V_μ parametrising the coordinate transformations. Absorbing this factor into the β-function, measuring the theories failure of being scale invariant, can be written

$$\beta^g_{\mu\nu} \equiv R_{\mu\nu} - (\nabla_\mu V_\nu + \nabla_\nu V_\mu) \, . \tag{4.41}$$

The theory is then scale-invariant if $\beta^g_{\mu\nu} = 0$. Comparing with (1.8) we see that this is not the condition for Weyl invariance but *weaker*. In particular, in order to have $\beta^g = 0 \Rightarrow \hat{\beta}^g = 0$ we need that [300, 302–304]

$$V_\mu = -\partial_\mu \Phi \, , \tag{4.42}$$

hence, the arbitrary scalar field has to be given by the gradient of the dilaton Φ.

In fact, while $\beta_{\mu\nu} = 0$ implies [305, 306] *global scalar invariance*, $\hat{\beta}_{\mu\nu} = 0$ implies *local conformal/ Weyl invariance*[11] and hence this raises the question under which conditions

[11] Weyl and conformal invariance can in fact be identified in this case; see [74].

scale invariance implies conformal invariance. Polchinski was the first who, using a rather abstract argument, showed [306] that for 2d NLSM with compact target spaces[12] this is *always* is the case. However, it is very hard to pinpoint the exact necessary and sufficient conditions under which the implication holds[13].

The authors of [300] started from the following observation: taking the RHS of (4.40) to be the operator defining a geometric flow, the resulting (Ricci flow) describes a (steady) *Ricci soliton*, defined in (4.15) with $X = V$. However, it follows from Perelman's seminal work [271, 291], that in fact any steady Ricci soliton is gradient. The gradient function can be identified with the dilaton and hence this gives an alternative proof that scale invariance implies conformal invariance. Building on this insight and using the generalised Ricci flow of [293], the authors of [300] generalised this argument to NLSM including a B-field that applies to all order in perturbation theory. In contrast to the abstract argument of [306], the treatment using Ricci flow is rather direct and concrete, enabling them to define an explicit criteria in terms of a so-called *Lee form* under which (at least in special settings) the implication does or does not hold, confirming the so far known examples of non-compact or non-complete manifolds; see [300] for details.

Remark

- We exclusively discussed the case of 2d QFTs. For $d > 2$, the situation is even less clear. See, for example, [308] for a treatment in $d = 4$. Furthermore, the argument of [300] above was generalised to heterotic σ-models in [309].

- In principle, one can translate between RG flow and Ricci flow by a redefinition of the flow parameter, effectively reversing the flow direction. In particular, introducing the RG scale Λ via $\Lambda = \Lambda_0 e^{-2s}$ the Ricci flow equation

$$\frac{\partial}{\partial s} g_{\mu\nu} = -2R_{\mu\nu}(g) = -2\beta^g_{\mu\nu}\,, \tag{4.43}$$

  takes the familiar form

$$\frac{\partial}{\partial \log(\Lambda)} g_{\mu\nu} = \beta^g_{\mu\nu}\,. \tag{4.44}$$

  Therefore we see that Λ and s are inversely related: while $\Lambda \to +\infty$ corresponds to going towards the UV, $s \to \infty$ describes a flow towards the IR. See [68] for a discussion of the example of the $SU(2)$ principal chiral model.

- For an application of flow equations as a solution generating technique, see [310].

[12] More abstractly, a *unitary* 2d QFT with a discrete spectrum of operator dimensions [306].
[13] Counterexamples lacking some of the above assumptions can be found in [34] and [307].

Part II.

Scalar potentials
&
Swampland program

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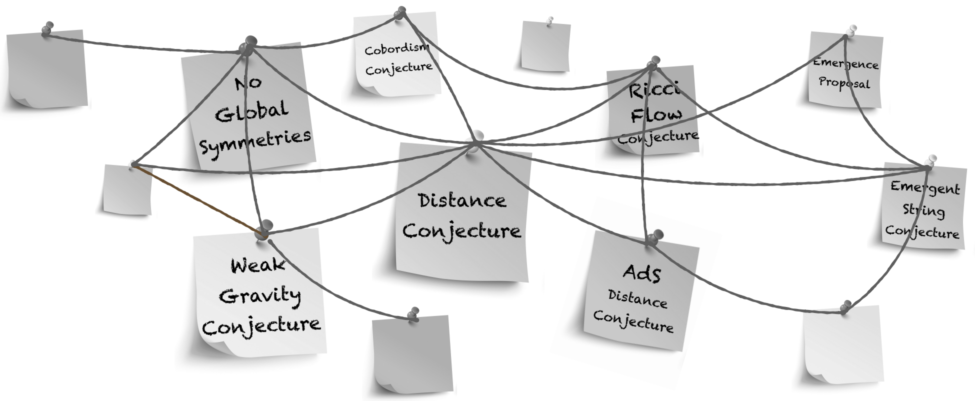


CHAPTER 5

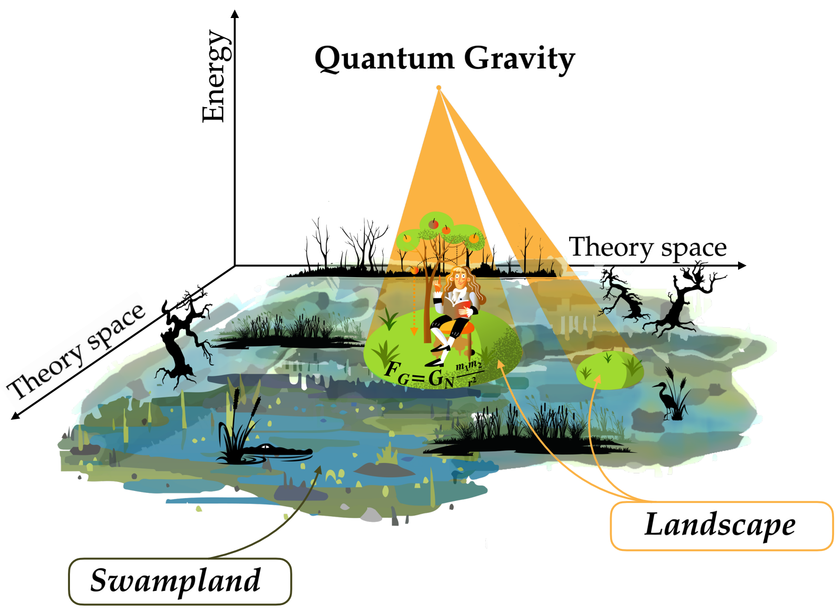
The Swampland program

We have outlined in Chapter 1 how string theory beautifully evades immediate challenges that arise when trying to formulate a quantum theory of gravity from a standard quantum field theory approach. On the other hand, in the lower-energy limit the theory gives rise to a vast amount of possible effective theories descending from the Landscape of string vacua. Although this may seem like quantum gravity looses much – or even all – of its restrictive power in this regime, it has become clear in the last years, that QG anyway puts very thigh constraints on the possible low-energy EFTs that can arise in the IR while being compatible with a theory of quantum gravity in the UV. The set of ideas and principles stemming from this observation is known as the *Swampland program* [45] and it will be the purpose of the following chapter to give a brief introduction to some of its score concepts. We discuss some selected *Swampland Conjectures*, that will be used in the remainder of this thesis. In particular we will focus on the *Distance Conjecture* and related conjectures, as well as questions and issues arising from the presence of non-trivial scalar potentials on field space.

5.1. A new road to quantum gravity

Among the many beautiful features of string theory is its highly restricted nature, descending from the demand of mathematical and physical self-consistency. The vanishing of the Weyl anomaly briefly reviewed at the beginning of Chapter 1, fixing, for instance, the spacetime dimension and leading to the vacuum Einstein equation is only one example. There are many more principles, like tadpole cancellation or unitarity that almost uniquely fix the form of the theory, leading to the famous fact that, unlike a generic EFT, string theory has no free parameters. All parameters, like Yukawa couplings, are fixed by the internal geometry and are by no means arbitrary. Low-energy effective theories are in general much less restrictive and there is an overwhelming amount of possible theories we can write down that seem consistent at the low-energy effective level. However, it has become clear more recently [45,58–62], that once gravity is taken into account the set of *seemingly consistent* EFT gets drastically reduced when requiring them to be completable to a *full consistent theory of quantum gravity* in the UV.

The *Swampland program* [45] can then be defined as the effort of finding suitable criteria – the so-called *Swampland Conjectures* – in order to identify the classically consistent effective field theories that can be completed to a consistent theory of quantum gravity in the UV. Theories that can be coupled to QG in a consistent way are said to belong to the *Landscape*. Conversely, the ones that fail to satisfy at least one of these conditions, and hence *cannot* be completed to a fully consistent theory of quantum gravity belong to the – vastly bigger – *Swampland*.



5.1.1. A web of conjectures

In recent years, the Swampland program has led to an ever-growing net of closely related conjectures. In particular, not only has there been incredible progress in identifying new patterns and principles that led to the proposal of new conjectures, but substantial efforts have been made in order to test and better understand existing Swampland conjectures. This is done either from a top-down perspective, for example working within string-theoretic examples, or from a bottom-up approach, employing basic principles we believe should be realised in a theory of quantum gravity. As such, the relevant literature has become vast and several pedagogical reviews exist by now, covering at least some of the different (aspects of) Swampland conjectures and their refinements and generalisations [58–62]. Hence, below we merely list the **three core conjectures** that can be seen as the foundational pillars upon which many of the more refined and specialised conjectures build. In the rest of the chapter we focus on one of them – the Swampland Distance Conjecture – as well as some of its refinements and generalisations, as these will be our main interest in this thesis:

No Global Symmetries Conjecture [311]. *There are no (exact) global symmetries in a consistent theory of quantum gravity.*

The conjecture has been generalised to broader notions of symmetry, including generalised and non-invertible symmetries [312–314]. The absence of higher-form global symmetries motivates the closely related **Cobordism Conjecture** [315].

(Electric) Weak Gravity Conjecture (WGC) [316]. *In a gravitational theory with $U(1)$ gauge symmetry and gauge coupling e, there is at least one particle of mass m and charge q such that $m^2 \leq 2e^2q^2M_P^2$.*

See [317, 318] for an introduction, further developments and references.

Swampland Distance Conjecture (SDC) [117]. See Section 5.2.1 below.

There exists a several generalisation and refinements, some of which are also discussed in Section 5.2.1 below.

Further conjectures & concepts metioned

- **Emergence Proposal** [319–322]; see also [142] for a recent pedagogical review.
- **Species scale** [53–57]; see [323] and references therein for a recent account in the context of the Swampland program.
- **Dynamical Cobordism** [324–329] & **Cobordism Distance Conjecture** [324].

5.2. The Distance Conjecture and its relatives

Before we dive into a discussion of the Swampland Distance Conjecture, we recall the notion of geometry associated to the moduli space of an effective d-dimensional theory with action

$$\mathcal{S}^{(d)} = \frac{\bar{M}_{P,d}^{d-2}}{2} \int \mathrm{d}^d x \sqrt{-g} \left(R(g) - \gamma_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b \right), \tag{5.1}$$

discussed in Chapter 1. In particular, recall the metric on moduli space $\mathcal{M}$ is defined as the factor γ_{ab} in front of the kinetic term of the scalar field (moduli) in the action (5.1). The geodesic distance between points P, Q on $\mathcal{M}$ descending from γ_{ab} is defined as

$$d(P, Q) \equiv \inf_{\gamma} \int_0^1 \sqrt{\gamma_{ab}(\varphi(s)) \frac{\mathrm{d}\varphi^a}{\mathrm{d}s} \frac{\mathrm{d}\varphi^b}{\mathrm{d}s}} \mathrm{d}s \,, \tag{5.2}$$

with $\gamma(s) : [0, 1] \to \mathcal{M}$ a smooth path on moduli space with $\gamma(0) = P$ and $\gamma(1) = Q$.

5.2.1. The Swampland Distance Conjecture

The Distance Conjecture is a statement about the geometry of the moduli space $\mathcal{M}$ associated with an effective action (5.1) and the consequences of large field displacements in the moduli. In particular, it postulates the appearance of an infinite tower of exponentially light states when approaching points at infinite distance:

Swampland Distance Conjecture (SDC) [117].

1. The moduli space $\mathcal{M}$ of a consistent theory of quantum gravity is non-compact and parametrised by the expectation values of scalar fields ϕ^a, the moduli, which are not subjected to a potential. There are always two points P, Q such that the (geodesic) distance $d(P, Q)$ between them is infinite.

2. When approaching an infinite-distance point Q, there is an infinite tower of states that becomes exponentially light with the geodesic distance

$$m(Q) \sim m(P) e^{-\alpha\, d(P,Q)} \quad \text{when} \quad d(P, Q) \to \infty \,, \tag{5.3}$$

where P is an arbitrary reference point, α is a (positive) constant and m denotes the mass scale of the corresponding states as a function of $P, Q \in \mathcal{M}$.

In particular, in the limit $d(P, Q) \to \infty$ the infinite tower becomes massless, sending the cut-off Λ of the EFT to zero. The behaviour is illustrated in Figure 5.1.

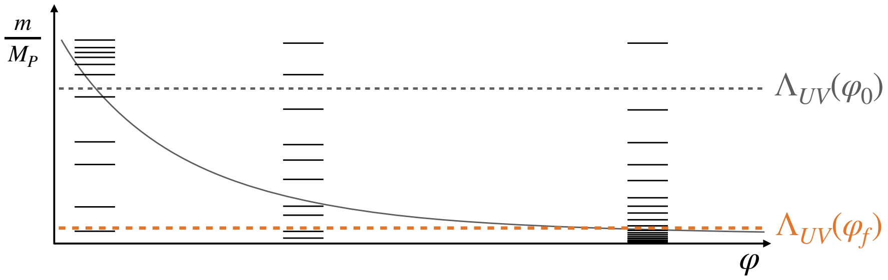


Figure 5.1.: The SDC predicts the breakdown of an EFTs when moving to large (infinite) distances on moduli space as more and more states fall below the theory's UV cut-off Λ. For $\Delta\rho \to \infty$ the infinite tower of massless states invalidates the EFT description.

Note that the metric $\gamma_{ab} = \gamma_{ab}(\varphi)$ is in general a function of the moduli fields. In field-theoretic terms this means that the kinetic term is not *canonically normalised*[1]. However, it is convenient to work with canonically normalised fields ρ^a, in terms of which geodesics are simply straight lines and the distance is given by the proper field displacement $d(P,Q) = \Delta\rho = |\rho^a(1) - \rho^a(0)|$. This implies that the mass formula for large field displacements is exponential in the proper field distance and takes the simple form

$$m(Q) \sim m(P)e^{-\alpha\,\Delta\rho} \quad \text{when} \quad \Delta\rho \to \infty\,. \tag{5.4}$$

At first one might think that there can be many different infinite-distance limits that are different in their physical realisation and meaning. However, quite the contrary seems to be the case. In fact, so far every possible infinite-distance limit of a consistent theory of QG has been shown to belong to one of only two possible scenarios:

- **Decompactification limits:** In a certain duality frame, one or multiple dimensions decompactify with the (leading) tower given by KK modes.
- **Emergent String Limit:** A (unique) critical string becomes asymptotically tensionless and weakly coupled with the leading tower given by string excitation modes. If $d < 10$ there are further KK modes at the same scale, signalling a compactification to $4 \leq d < 10$ spacetime dimensions.

In fact, it was argued in [330] that these are the *only* possible scenarios, and this proposal goes by the name of **Emergent String Conjecture** [330].

There are many more conjectures that are closely related or refinements of the SDC. We will discuss several of them in more detail in the following. A more extensive list can be found at the end of this chapter, as well as in [58–61]. Before moving on, we briefly revisit the example of the circle compactification discussed in Chapter 1.

[1] A scalar field φ is canonically normalised if its kinetic term is of the simple form $\mathcal{L} \supset -\delta_{ab}\partial_\mu\varphi^a\partial^\mu\varphi^b$.

A simple example: circle compactification

In Chapter 1 we discussed the moduli space of the S^1 compactification with action (5.1) and metric on moduli space $\gamma = \frac{d-1}{d-2}\frac{1}{r^2}$ (cf. Eq. (1.37)) such that the canonically normalised field reads

$$\rho = \sqrt{\frac{d-1}{d-2}}\log(r)\,. \tag{5.5}$$

Since the moduli space is one-dimensional (we ignore possible dilaton contributions), the distance between two points on moduli space, or radii, is simply given by

$$d(r_i, r_f) = \sqrt{\frac{d-1}{d-2}}\log\left(\frac{r_f}{r_i}\right), \tag{5.6}$$

as can be inferred from the canonically normalised field or, alternatively, from (5.2). Therefore, we see that there are two infinite-distance limits: $r \to \infty$ and $r \to 0$. Both of them correspond to a decompactification limit or the (T-)dual thereof, respectively. The associated towers predicted by the SDC are given in terms of the Kaluza-Klein momentum modes and the string winding modes. The mass formula for these states was given – in the string frame – in equation (1.30). In the Einstein frame, it reads[2]

$$M^2_{n,w} = r^{\frac{-2}{d-2}}\left(\frac{n}{r}\right)^2 + r^{\frac{2}{d-2}}\left(\frac{wr}{\alpha'}\right)^2, \tag{5.7}$$

such that we find

$$M_{KK}(n) = nr^{-\frac{d-1}{d-2}} = ne^{-\sqrt{\frac{d-1}{d-2}}\rho}\,, \qquad M_{\text{winding}}(w) = wr^{\frac{d-1}{d-2}} = we^{\sqrt{\frac{d-1}{d-2}}\rho}\,. \tag{5.8}$$

We can summarise the behaviour as

$$M(\rho_0 + \Delta\rho) = M(\rho_0)e^{-\alpha|\Delta\rho|}\,, \tag{5.9}$$

with the constant $\alpha = \sqrt{\frac{d-1}{d-2}}$, hence exactly matching the behaviour of (5.4).

Remarks

- The SDC has been examined in a wide range of different settings, and a large body of evidence has been collected in its support [117,320,322,330–361]. It is therefore one of the most well-established conjectures within the Swampland program. For connections to other conjectures see [333,342,362–367] and for connections with (pure) mathematics [320,330,334,343,349,361,366,368–372].

[2] An overall factor $r^{\frac{-2}{d-2}}$ comes from $\tilde{g}^{-1}$ in $m^2 = -p_\mu p^{\tilde{\mu}}$. The additional factor of $(\alpha')^{-2} = M_s^4 = r^{\frac{4}{d-2}}$ for the winding modes stems from the fact that we keep the d-dimensional Planck mass and effective dilaton $\Phi_d = \Phi - \log\left(\sqrt{h}\right)$ fixed, together with $M_{P,d}^{(d-2)} \sim g^{-2}M_s^{(D-2)}$; see also [58].

- The constant α was not specified in the original proposal but is expected to be of order one in Planck units [373, 374] (cf. **rSDC** and **sSDC** in Section 5.3 below).
- The light tower of states is a realisation of light species. The associated species scale Λ_{sp} reads $\Lambda_{\text{sp}} = M_{P,d} N^{-\frac{1}{d-2}} \simeq M_{P,d+1}$ and hence the Planck mass of the higher $(d+1)$-dimensional theory (which for large r is much lower than $M_{P,d}$).

5.2.2. The SDC and dualities

The closed string on a circle is invariant under T-duality, i.e. under the inversion of the radius $r \to \frac{\alpha'}{r}$ and the simultaneous exchange of winding and momentum modes $n \leftrightarrow w$; see Chapter 2. These are precisely the two infinite-distance points of the moduli space and their associated infinite towers of light states. Hence, even if we were unaware of one of the two infinite-distance points, by T-duality we could have immediately predicted the existence of the second point and its associated tower of winding or momentum modes from the first one. Hence, these concepts are closely connected and therefore the SDC is sometimes called a *Duality Conjecture*. In particular, the discussion accompanying the original formulation of the SDC also includes a statement about the finiteness of the volume of moduli space[3]. This notion of finiteness is tightly connected to the existence of dualities in string theory [45]. We refer to [45, 117], but also to the more recent [379] and [380] for a refined treatment. Finiteness, or more concretely "compactifiability" of [379] was then used in [361] to provide even stronger evidence for the SDC. Dualities also play an important role in the recent [359].

Extrapolating from the present example one can hope to use T-duality in order to explore different corners of moduli space and the corresponding towers. In particular, the exchange of string zero modes under T-duality gives a powerful way in understanding the – possibly very complicated – infinite-distance points.

Going beyond the circle

While the exchange of string zero modes is very straightforward for the trivially fibred circle, the exchange is in general much more involved when T-duality acts on nontrivial fibration and in the presence of fluxes. When applied to generic curved spaces (with fluxes), the known simple exchange patterns are greatly modified, leading to a novel narrative in relation to the SDC. We will see in great detail in Chapter 6, that the topology changing nature of T-duality will relate distinct infinite-distance corners of the moduli space, leading, for example, to backgrounds not supporting any winding modes. We will use generalised T-dualities and non-geometry to access an even wider range of dual spaces and hence infinite-distance points in Chapter 7 and 8.

[3] A related statement, put forward in [117], about the geometry of moduli space, concerns the negative scalar curvature at infinite-distance points. See [375–378] for a recent discussion.

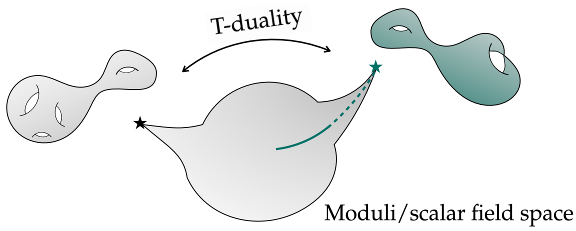


Figure 5.2.: T-duality as a tool for generating related infinite-distance points with very different (geometric) properties and towers of states: change of topology, trivialisation and generation of homology cycles and non-geometry.

5.2.3. Generalised Distance Conjecture & AdS Distance Conjecture

By now, several generalisations of the SDC have been proposed, extending the conjecture to a broader range of applicability. The first such extension that we discuss was put forward in [363] and postulates that the SDC also applies to generic variations of the space-time metric itself, rather than only moduli descending from the internal space:

Generalised Swampland Distance Conjecture (gSDC) [363]. An infinite tower of states with mass scaling

$$m(s_f) \sim m(s_i) e^{-\alpha \Delta_{\mathcal{G}}} , \tag{5.10}$$

arises when considering distances Δ between metrics $G_0 = G(s_i)$ and $G_1 = G(s_f)$ in the space of metrics $\mathcal{M}(M)$.

The gSDC is therefore promoted to a statement on the more abstract space of metrics $\mathcal{M}(M)$, which was already discussed in Section 4.1.2. The natural metric is then the DeWitt metric [288] $\mathcal{G}_G^{MNOP}$ associated with the family of metrics $G(\tau)$ spanning $\mathcal{M}(M)$, introduced in (4.12). The associated distance is

$$\Delta_{\mathcal{G}} \equiv \Delta_{\mathcal{G}}(G_0, G_1) = c \int_{s_i}^{s_f} \mathrm{d}s \left(\frac{1}{\mathcal{V}(M)} \int_M \sqrt{G} G^{MN} G^{OP} \frac{\mathrm{d}G_{MO}}{\mathrm{d}s} \frac{\mathrm{d}G_{NP}}{\mathrm{d}s} \right)^{1/2} , \tag{5.11}$$

which – apart from the volume normalisation – is just the distance L of (4.14).

There is a particularly easy example on which to apply the gSDC: Weyl rescaling $\tilde{g}_{\mu\nu} = e^{2\tau} g_{\mu\nu}$ of the (external) metric $g_{\mu\nu}$. In particular, applying such a rescaling to AdS spaces (see (4.20)), rescales the cosmological constant Λ (cf. (4.21)) via [363]

$$\Lambda = -\frac{1}{2}(d-1)(d-2)e^{-2\tau} . \tag{5.12}$$

Plugging this into the distance formula (5.11), one readily obtains [363, 381]

$$\Delta_g \simeq \tau_f - \tau_i \simeq \log\left(\frac{\Lambda_f}{\Lambda_i}\right) . \tag{5.13}$$

Therefore, $\Lambda_f \to 0$ is at infinite distance and the gSDC postulates an infinite tower of light states, which due to (5.10) scales as [363]

$$m(\Lambda_f) \sim m(\Lambda_i)e^{-\alpha\Delta_g} = m(\Lambda_i)\left(\frac{\Lambda_f}{\Lambda_i}\right)^{\alpha} . \tag{5.14}$$

Hence, this motivates the following conjecture:

AdS Distance Conjecture (ADC) [363]. Any quantum gravity theory formulated on an AdS space with cosmological constant Λ exhibits an infinite tower of massless states in the limit of vanishing $\Lambda \to 0$. In Planck units, the masses scale as

$$m \sim |\Lambda|^{\alpha} , \tag{5.15}$$

with positive constant $\alpha \sim O(1)$.

Remarks

- Requiring $\alpha = \frac{1}{2}$ for supersymmetric AdS vacua leads to the **strong AdS Distance Conjecture** [363], and – as a consequence – places scale separated theories in the Swampland. For arguments in favour, but also challenging this *strong* version and hence also *scale separation*, we refer to [62,126] as well as recent [125,357,382–390] and references therein.

- The ADC can be easily motivated from simple 10D string vacua, like $AdS_5 \times S^5$, in which, due to the lack of scale separation, taking $\Lambda \to 0$ corresponds to taking the radius of S^5 to infinity. This gives rise to a light tower of KK modes on S^5.

5.2.4. Ricci Flow Conjecture

We have just encountered the Generalised Distance Conjecture, extending the SDC to generic variations of the metric of spacetime M itself, hence to a statement on distances on the space of metrics $\mathcal{M}(M)$. On the other hand, we reviewed in Chapter 4 that Ricci flow defines trajectories or paths in exactly this space of metric $\mathcal{M}(M)$. Take, for example, the Ricci flow of AdS spaces as discussed in Section 4.1.3, which reduces to a flow behaviour of the cosmological constant according to (cf. (4.22))

$$\Lambda(s) = \frac{\Lambda_0}{1 - \frac{2}{n-2}\Lambda_0 s} . \tag{5.16}$$

In particular, for $s \to \infty$ we see that ($\Lambda_0 < 0$ for AdS) the cosmological constant vanishes and we approach the Ricci flat Minkowski spacetime, which is the fixed point of the flow. Together with black hole entropy arguments, this led the authors of [381] to propose the following conjecture:

Ricci Flow Conjecture (RFC) [381]. Take a family of metrics $G_{\mu\nu}(s)$ on a manifold M, defining a theory of quantum gravity and satisfying the Ricci flow equation (4.1). Then there exists an infinite tower of states that becomes massless when following the flow to a fixed point $\partial_s g_{\mu\nu}(s)|_{s=s_f} = 0$ at infinite distance.

There are several possibilities proposed in [381] as to how to measure the distance for the conjecture. First, we can again simply use the DeWitt metric $\mathcal{G}_G^{MNOP}$ and the associated induced length (4.14) or (5.11) as the natural distance on $\mathcal{M}(M)$. It was shown in [381], that in the case of AdS the resulting distance is nothing else than the one resulting from the Weyl rescaling[4] (5.12) and hence

$$\Delta_{\mathcal{G}} \simeq \left(1 - \frac{2}{d-2}\Lambda_0 s\right)\Bigg|_{s_i}^{s_f} \simeq \log\left(\frac{\Lambda_f}{\Lambda_i}\right). \tag{5.17}$$

Hence, the RFC immediately and elegantly recovers to the ADC. A more detailed discussion of the RFC and, in particular, the generalisation proposed by the author and collaborators in [68], will be provided in Chapters 9, 10 and 11.

Remarks

- Note that the RFC circumvents certain issues that can arise when applying the gSDC directly to (uncompactified) 10D SUGRA or string vacua, in particular, with Ricci-flat directions; see the discussion in Section 2.1 of [381].

- Partially motivated by the above point, the authors of [381] also proposed other measures of distance along the flow. In particular, one such alternative is to directly define
$$\Delta_R \simeq \ln\left(R_f/R_i\right), \tag{5.18}$$
with R_i the Ricci scalar of the family of metrics $G_{\mu\nu}(s)$ evolving under Ricci flow.

- In the context of black holes, the RFC was extended in [391] to the Einstein-Maxwell action in order to analyse the stability of Reissner-Nordströem black holes. See also [284, 392–396] for other applications of the RFC.

5.3. Potentials on scalar field space

The SDC, in its original formulation, is a statement on moduli space. However, phenomenologically more interesting and viable backgrounds generically depart from this setting as they usually require moduli stabilisation, and hence non-zero scalar potentials. In particular, ingredients like background fluxes and nontrivial topologies

[4]The parameters τ and s are related by the following reparametrisation [381] $\tau = \frac{1}{2}\ln\left(1 - \frac{2}{d-2}\Lambda_0 s\right)$.

generate such potentials and therefore the field spaces of the associated EFTs depart from the paradigm of being spanned by strictly massless moduli fields. Nevertheless, at the very least, one would expect some (generalised) version of the SDC to also apply along (almost) flat directions of the potential, i.e. along some valleys or in asymptotic flat corners. Exploring the conditions under which such extensions of the SDC are applicable is crucial, in particular, in view of possible implications for cosmological models of inflation [374,397–401] and other recent exciting phenomenologically testable predictions that foot on an application of the SDC[5]. It was first argued in [373] and [374] that – at least under some suitable assumptions – the SDC should remain valid:

Refined Swampland Distance Conjecture (rSDC) [373,374]. The SDC, as defined in Section 5.2.1, holds even in the presence of a potential, i.e. for generic scalar fields, with the moduli space replaced by scalar field space.

Furthermore, for $d(P,Q) \gtrsim M_P$, Eq. (5.3) is replaced by the stronger

$$M(Q) < M(P) e^{-\alpha d(P,Q)} \,. \tag{5.19}$$

There are several core questions that arise from the statement of the rSDC:

- What are the paths in field space along which the distances are measured? In the presence of a potential, physical admissible paths and geodesics no longer agree, and we need to define new principles for selecting the path along which to measure distances. Does the SDC also hold for paths that deviate from geodesics?
- Closely connected to the first point, we need to reevaluate the definition of distance or length along the path in field space. Usual geodesics do not encode the "cost" of having a non-zero potential. However, departing from the mathematical well-defined notion of geodesics as length-minimising curves, one might run into troubles with the defining axioms of a distance. Hence, one either has to make sure these issues do not arise or potentially move to a generalised notion of distance that can serve as a sensible measure of length or cost for the SDC.
- Thirdly, infinite-distance points often arise from taking some underlying geometric picture to "extreme" configurations, for example, the limit of vanishing cycles or decompactification. These extrema often translate to distinguished values of the scalar potential, like vanishing or diverging V. While in the former, asymptotically we can neglect the potential, the latter is much less clear.

[5] For example, the *Dark Dimension scenario* [63] predicts the existence of one mesoscopic extra dimension of micrometer size. See [64, 402–410] for further developments and extensions to two mesoscopic dimensions. For explicit string-theoretic realisations, see [411–415] and [416].

Theses three points – together with the role of (generalised) dualities on curved backgrounds and non-geometry – can be viewed as the main motivations and questions discussed in this thesis. Hence, the remainder of this work will be devoted to offering a detailed explanation of several possible approaches to resolving these issues and related problems.

These problems have received considerable attention in the last years and there are several proposals to approach these points in the literate. We close this chapter with a brief section, reviewing some of the existing work and proposals on the SDC in the presence of a potential, before moving to the main chapters of this thesis, discussing the contributions of the author and collaborators.

Remarks

- The SDC in its original formulation does not specify the constant α. It was realised in [373, 374] that in many examples the proper field distance grows at most logarithmically once the field displacement reaches the Planck scale and therefore, in order for the masses to drop faster than any power law, the exponential behaviour has to kick in, once the displacement reaches the Planck length. This leads to the strict inequality in (5.19) and effectively restricts α to be of order one, or more precisely $\frac{1}{\alpha} \lesssim \mathcal{O}(1)$. This restricts the onset of the exponential drop in mass[6], in particular $d(P,Q) \lesssim \mathcal{O}(1)$. See also [337, 371, 418–420].
- It was argued in [348], that there is a sharp lower bound for α, associated with the lightest tower, given by $\alpha \geq 1/\sqrt{d-2}$. This is known as **Sharpened Distance Conjecture** [348]. See also [417] and [421] for a recent account and references.
- Note that also the previously discussed extensions of the SDC, in particular the gSDC, ADC and RFC effectively cover situations in which there is a scalar potential. In particular, changing the cosmological constant by (Weyl) rescaling or Ricci flow, we effectively perform field displacements in the presence of a potential.

5.3.1. Summary of (recent) developments on the SDC and potentials

The study of the SDC in the presence of a potential, and hence the rSDC, goes back to [373] in which so-called axion monodromy [422, 423] and axion alignment [424, 425] scenarios were investigated. In particular, in these models the axion can traverse large – a priori trans-Planckian – field excursions that are in conflict with the behaviour

[6] It was pointed out in [417], that when trying to extract stringent constraints on the EFT under consideration, the rSDC suffers from the fact that although the exponential drop in mass appears at $d(P,Q)$ of order one, for finite $d(P,Q)$ the tower can in principle be still parametrically heavier than M_P.

postulated in (5.19). Hence, invalidating any EFT description beyond some critical field displacement of the order $d(P,Q) \gtrsim M_P$. However, it was argued in [373], that after properly taking into account *backreaction* effects, the proper field distance is bounded in a way, that extending the theory beyond distances of order Planck scale is impossible while retaining a valid effective field theory description. This was later observed in a broader setting in a series of papers [331,369,426,427]. Although not going into details, it is important to understand the way paths and associated distances are defined in these settings. There were to different strategies pursued in [373]. The first, and arguably more natural way, is to restrict the possible trajectories to flat directions, i.e. along the valleys of the potential. We discuss this in more detail in a moment, but first, we briefly outline the second option. The latter amounts to displacing some field $\varphi^1 = \lambda$ from the vacuum, while appropriately adjusting the remaining φ^a , $a \neq 1$ in order to obtain their vacuum expectation values as functions of the displaced field,

$$\frac{\partial}{\partial \varphi^a} V(\varphi) = 0 \Longrightarrow \varphi^a = \varphi^a(\lambda)\,. \tag{5.20}$$

This implies that the naive field space metric γ_{ab} which is itself a function of the fields $\gamma_{ab} = \gamma_{ab}(\varphi^a)$, and which continues to measuring the proper field distance, gets corrected. This is exactly the aforementioned backreaction effect. It turns out that it precisely seems to forbid trans-Planckian excursions, as after some critical value, the proper field distance grows at best logarithmically, forcing the exponential drop in the mass of the tower. For details, we refer to [331,369,426,427]. This behaviour was shown to hold within a relatively broad setup in [369] by studying the asymptotic form of V.

Restricting ourselves to the valley of the potential, we can think of the first scenario as [373] integrating out the massive fields above some cut-off scale Λ, effectively reducing the field space $\mathcal{M}$ at some initial scale Λ_0 to a lower-dimensional field space $\mathcal{M}_\Lambda$ with $\Lambda < \Lambda_0$. Within this field space there might still be non-flat direction, however with an associated energy scale below the cut-off such that it seems reasonable to allow for trajectories also along these directions, viewing them effectively flat. However, there is an apparent problem arising [428]. In the lower-dimensional field or moduli space there are geodesics that, when lifted back to the full field space $\mathcal{M}$ do not correspond to naive geodesics of the associated field space metric. From a point of view of the space $\mathcal{M}$ together with the non-trivial potential, this simply translates to the fact that the valleys of the potential do not correspond to geodesic trajectories of $\mathcal{M}$. Since these geodesics of $\mathcal{M}_\Lambda$ are necessarily of greater length then the ones in $\mathcal{M}$ (see also [366])

$$d_{\mathcal{M}}(P,Q) \leq d_{\mathcal{M}_\Lambda}(P,Q)\,, \tag{5.21}$$

such that these trajectories, corresponding to geodesics in $\mathcal{M}_\Lambda$, might violate the SDC.

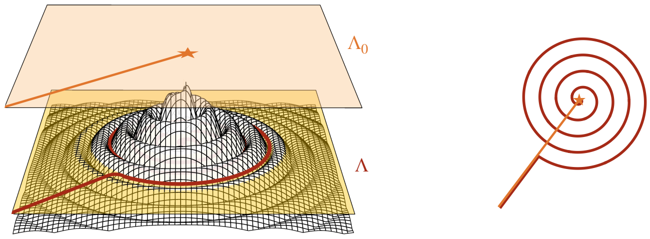


Figure 5.3.: Visualisation of the problematic (possibly infinitely) spiralling potential. Lowering the theories cutoff from Λ_0 to Λ, the trajectories are forced to follow the valley of the potential, which does not agree with the geodesic at Λ.

This issue was investigated[7] in [428] where it was argued that the amount of non-geodesic of such valley trajectories is actually constraint by the SDC and this leads to the **Convex Hull Distance Conjecture** [428], effectively constraining the possible scalar potentials[8] consistent with the expectations of the SDC, or more generally QG.

This fact was later investigated [366] from the perspective of *tameness* [430, 431] and the **Tameness Conjecture** [432]. Consider, for example, the problematic case of a field space with (infinitely) spiralling potential, illustrated in Figure 5.3. If the cutoff Λ is sufficiently large, compared to the scale of the potential, we can consider the straight line to the origin, which a priori can be of finite[9] distance. Then lowering the cutoff Λ far enough, we obtain a reduced moduli space, in which the valley of the spiral is the geodesic, hence giving rise to a "new" infinite-distance point and leading to a potential inconsistency for the SDC. The Tameness Conjecture [432] then, broadly speaking, amounts to postulating that potentials like this cannot arise in physical situations. A related powerful tool for restricting the possible structure of moduli or scalar field spaces is (asymptotic) Hodge theory; see for example, [433].

The scalar potential V also plays a crucial role for cosmological implications, see [52, 400, 403, 404, 434–439] for a (incomplete) recent account. In particular, the **de Sitter Conjecture (dSC)** [362, 440] excluding the existence of (meta-)stable de Sitter vacua as well as the **Trans-Planckian Censorship Conjecture (TCC)** [400]. For the connection of the potential with the **Species Scale** see [399, 441–443]. For other variants of the SDC not mentioned here, like the **Black Hole Entropy DC** [444], the **Gravitino DC** [445], the **Species Scale DC** [446] and other related aspects, we refer to the literature.

The issue concerning the (mathematical) definition of a distance in the presence of a potential, as well as the relevant literature, will be discussed in Chapter 11.

[7] A nice explanation of this fact can also be found in [366].

[8] See also [429] (and the discussion below) for proposals on constraining the scalar potential.

[9] The argument does not necessarily depend on the straight line being of finite distance. Even if the point is at infinite distance before, one can imagine a spiralling potential generating a distance that delays the mass drop to a sub-exponential behaviour in field distance and hence violating the SDC.

CHAPTER 6

Topology change, missing towers and scalar potentials under T-duality

The Swampland Distance Conjecture is closely connected with dualities, in particular T-duality. This is nicely illustrated at the example of the trivially fibred circle, where the infinite-distance points and the associated towers are mapped to each other via T-duality. In this chapter, we investigate how this naive matching carries over to more complicated setups. In particular, we discuss non-trivial topologies and flux contributions, and their role for the infinite-distance points and towers for the SDC as well as their mapping under T-duality. Furthermore, we examine the role of emerging scalar potentials in situation where certain towers are obstructed, leading to and amendment of the SDC. Subsequently, we derive a generic reduction formula for variations that involve the metric and (background) flux contributions. We discuss the role of these flux variations and their relevance for obtaining a consistent T-duality invariant metric on field space. Inspired by the discussions in [382, 447, 448], we outline how T-duality may help in understanding the subtle interplay between off-shell and on-shell considerations when studying the SDC. We close with some further examples as well as a discussion on T-duality for isometries with fixed points and the invariance of the resulting potential on field space.

6.1. The canonical example of S^3 with H-flux

In Chapter 5, we discussed the close connection between the SDC and dualities, in particular, T-duality. This was illustrated on the well-known example of the compactification of the bosonic string on a trivially fibred circle. In this section, we will consider the next simplest example of a background that deviates from this example in several significant ways, namely the three-sphere S^3 with k units of H-flux. Due to the non-trivial topology of curved spaces and the presence of flux, we need to reconsider the existence of the infinite towers of massless states that are dictated by the SDC. In particular, we will encounter missing towers of winding modes as well as non-conserved KK modes. Furthermore, given the fact that we apply T-duality to a circle bundle that is nontrivially fibred over the base manifold and/or with background flux, the resulting T-dual space will display a different topology than the original space. This necessitates a careful analysis of the would-be tower of exponentially light states to establish a consistent picture under T-duality. Another key difference is that the scalar field space is now subject to a potential. Generally, this potential must be taken into account when trying to deduce implications for the SDC. In particular, we will argue that the scalar potential can solve the problem of missing modes, leading us to propose an amendment to the SDC in the presence of (diverging) potentials.

6.1.1. Dimensional reduction

We start illustrating the reduction procedure outlined in section 5.2 on the example of S^3. Hence, we start from the D-dimensional action

$$\mathcal{S} = \frac{1}{2\tilde{\kappa}_D^2} \int \mathrm{d}^D X \sqrt{-G} e^{-2\Phi} \left(R(G) - \frac{1}{12} H_{IJK} H^{IJK} + 4\partial_I \Phi \partial^I \Phi \right) , \tag{6.1}$$

where we assume that the total metric is of the form $G = g(x) \oplus h(y)$ with g a (arbitrary) external metric and h the three-dimensional metric of the three-sphere with radius r as given in (2.17). We denote external coordinates with x^μ, while internal coordinates with y^i. For simplicity, we assume that the only non-vanishing H-flux is supported on S^3 and again given by (2.29) with k units of flux. We formally split the dilaton according to $\Phi = \Phi_0 + \phi(x) + \phi_y(y)$. Now, promoting the radius r to a dynamical field depending on the external coordinates, $r = r(x)$, also h and ϕ_y inherit an implicit dependence on x, i.e. $h(y, r(x)), \phi_y(y, r(x))$. Performing the dimensional reduction and going to the $d = (D-3)$-dimensional Einstein frame we obtain

$$\mathcal{S} = \frac{\mathcal{V}}{2\kappa^2} \int \mathrm{d}^d x \sqrt{-g} \left(R(g) - \frac{4}{d-2} (\partial \phi_d)^2 - \frac{3}{r^2} (\partial r)^2 - e^{\frac{4\phi_d}{d-2}} \left(\frac{k}{2r^6} - \frac{3}{2r^2} \right) \right) . \tag{6.2}$$

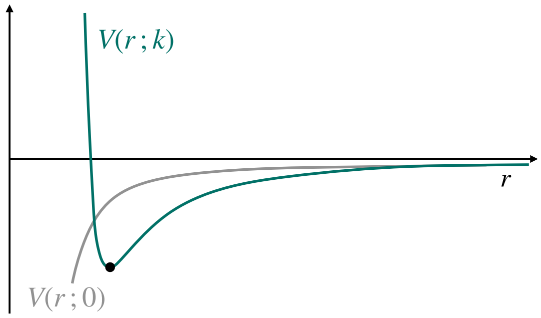


Figure 6.1.: Plot of the effective scalar potential V resulting from dimensional reduction. In green we plot the potential for the sphere with k units of H-flux, sourced by contributions from both the curvature of the sphere as well as a flux contribution. The grey curve depicts the potential with $k = 0$, i.e. just the three-sphere without any flux.

Here we defined the d-dimensional dilaton as $\phi_d = \phi - \frac{1}{2}\log(r^3)$ and $\mathcal{V} \equiv \mathcal{V}_{\text{int}} = 2\pi^2$. In particular, this means that the metric of field space and the potential read

$$\begin{aligned} \gamma_{rr} &= 3r^{-2}\,, \\ V(\phi_d, r) &= e^{\frac{4\phi_d}{d-2}}\left(\frac{k}{2r^6} - \frac{3}{2r^2}\right)\,. \end{aligned} \tag{6.3}$$

It is now an easy exercise to do the same procedure for the T-dual space. Indeed, one recovers precisely the same expression for the potential and metric on field space[1]. However, let us stress that of course now the terms combine in a different non-trivial way since T-duality mixes metric and flux contributions. However, at the end, the T-duality is a proper duality of string theory, and so the final effective actions should be equivalent. Indeed, in the next section we will argue in full generality that the metric on field space as well as the potential are invariant under (abelian) T-duality.

6.1.2. Topology change, zero-modes and mass spectrum

In order to discuss the implications for the SDC we first collect the necessary information on the zero modes on S^3 and its T-dual. This includes in particular the existence and conservation of momentum and winding states, their associated masses, their exchange under topological T-duality, and in particular the behaviour under variations of r.

Although so far we treated the flux number k as a fixed constant, in light of later considerations, we will also examine variations of k. The relevance of this for the SDC will become clear in Section 6.1.5.

[1]There is also the related question of whether to allow also x-dependent $k = k(x)$. This will be discussed in detail in Section 6.1.5.

Topological T-duality

In Chapter 2 the background of S^3 with k units of H-flux was discussed as a particular example for topology change under T-duality. In particular, we observed that the behaviour fits into the general framework of topological T-duality and therefore displays an exchange of flux and topological data according to

$$c_1(S^3) = 1 \leftrightarrow [\tilde{H}] = k\,, \qquad c_1(\tilde{E}) = k \leftrightarrow [H] = 1\,. \tag{6.4}$$

For further details, we refer to Chapter 2.

Existence and conservation of winding and momentum zero modes

The fundamental group of the three-sphere is trivial, $\pi_1(S^3) = 0$, implying that all loops are contractible and therefore no nontrivial winding states can exist on S^3. On the other hand, due to the nontrivial H-flux, the dual space can be shown (cf. Section 6.3) to have a torsion-full fundamental group $\pi_1(\tilde{S^3}) = \mathbb{Z}_k$. Therefore, upon winding k times around the circle fibre, the string can actually unwind and the loop becomes contractible. There are only $k-1$ non-trivial winding states, w_n. Hence (at least for finite k and r) this geometry cannot support an infinite tower of winding states.

Turning to the momentum modes on S^3, there is a priori no obstruction to their existence. However, there is a subtlety concerning their conservation and, therefore, interpretation as states giving rise to an infinite tower of states for the SDC. In particular, we need to take into account the presence of the H-flux, which might lead to non-conservation of momentum modes due to the coupling of winding and momentum zero modes through the Kalb-Ramond field B, analogously to the toroidal case. We will explain this in detail in Section 6.3, as well as in Appendix C. Anticipating this discussion, it is natural to assume that also on the three-sphere with H-flux there might be a problem of non-conserved modes. Although on the S^3 there are no winding modes and this argument does not apply, there is anyway a lack of (infinite) towers for finite r and k. This is hard to see from a perturbative perspective and we will first argue via T-duality and then make use of the exact description in terms of the associated $SU(2)$ WZW model at level k. In particular, on the dual space $\tilde{S}^3$, there are only $\mathbb{Z}_k$ winding states. Since T-duality is expected to exchange winding and momentum states, this hints at the fact that there is indeed a lack of an infinite tower of (conserved) momentum states on S^3, due to the presence of the H-flux. One could try to work out an explicit perturbative argument similar to the one discussed in Chapter 5, however, since in this case one has to deal with spherical harmonics, the calculation is much more involved. Hence, we will take a different path in the following.

For the example at hand, we have the great advantage that the theory can be described as a WZW model[2] at level k, denoted $SU(2)_k$. In particular, the isometry group of the theory is $SO(4) \simeq SU(2)_L \times SU(2)_R$, and hence states are labelled by quantum numbers $j, \bar{j}, m, \bar{m}$. Their associated ranges can be shown to be $j = 0, \frac{1}{2}, 1, \ldots \frac{k}{2}$ and $-j \leq m \leq j$ and similar for the barred quantities, see, for example, [450]. On simply connected group manifolds it is however known that the modular invariant is symmetric [34, 452] and states appearing in the spectrum are the left-right symmetric ones with[3] $j = \bar{j}$ and define states

$$|j, m, \bar{m}\rangle \ . \tag{6.5}$$

In particular, due to the simply-connectedness, there are no twisted sectors of winding states. We can view j as a total angular momentum and $m, \bar{m}$ as the quantum numbers of two (commuting) momentum components J_0^3 and $\bar{J}_0^3$ that generate translation in the directions ξ_1 and ξ_2 respectively; see [451] for more details. Crucially we see that while we have a tower of momentum-like states, similar to the KK modes on a torus, or the states corresponding to the spherical harmonics on the free S^3, the towers are *finite* since $j \leq k/2$, resulting in $k - 1$ different states. From a perturbative perspective this can be viewed as having only $\mathbb{Z}_k$ worth of conserved momentum states. Under T-duality along ξ_2 it can be shown [163] that the WZW model $SU(2)_k$ is mapped to the orbifold theory $SU(2)/\mathbb{Z}_k$ where the $\mathbb{Z}_k$ action is embedded into the right action $U(1)_R$ associated to shifts along ξ_2. States in the latter are labelled

$$|j, m, w\rangle \ , \tag{6.6}$$

with ω the winding number, which is defined only modulo k and hence we see that there is only a finite number of winding states. Furthermore, momentum states are only defined modulo $c_1 = 1$ so that there is only the trivial momentum state. This pattern of exchange of states can be formulated even more generally and extends to T-duality between more general orbifolds of $SU(2)_k$ [453]. This will be explained in Section 6.3. Furthermore, note that also on the dual space, there is only a finite number of KK-like modes, bounded by the level k. This behaviour is summarised in Table 6.2 and is, in particular, consistent with the expected exchange of winding and momentum under (abelian) T-duality, which we discuss in more detail in a moment.

We close this paragraph by discussing the interesting limiting cases of $k, r \to 0$ and $k, r \to \infty$. Here we need to distinguish the case where we keep the theory on-shell, i.e. imposing $k = r^2$ from the one where we vary k and r independently. The latter case

[2] See for example the lecture notes [449], or [34, 450]. Some nice references more tailored to the discussion below are [451] and [452].

[3] In particular particle-like states have $j = \bar{j}$.

S^3_r with $[H] = k$	$r = \text{const.}$	$r \to 0$	$r \to \infty$
$k = \text{const.}$	w : none $n : \mathbb{Z}_k$	w : none; light $n : \mathbb{Z}_{\mathbb{k}}$; heavy	w : none; heavy $n : \mathbb{Z}$; light
$k \to 0$	w : none; const. n : heavy	w : none; light n : heavy	$*$
$k \to \infty$	w : none; const. $n : \mathbb{Z}$; const.	$*$	w : none; heavy $n : \mathbb{Z}$; light

Table 6.1.: Summary of the various limits discussed in section 6.1.2 and the behaviour of the associated towers of states. Simultaneous limits in r and k are taken on-shell, i.e identifying $k = r^2$ along the limit. The limits denoted with $*$ are not covered in the present discussion. In the case where there are no states, "light/heavy" refers to the would-be mass dependence for such states in this limit.

is rather easy to understand from a perturbative point of view. Take, for example, the level k fixed and send $r \to \infty$. In this case the effect of the H-flux or WZ-term becomes less and less important[4]. In the limit $r \to \infty$ we hence obtain the free Principal Chiral Model (PCM) on $SU(2)$ or the string on S^3 while in the limit $r \to 0$ there is only a finite number of states. Keeping on the other hand r fixed and sending $k \to \infty$ corresponds to an infinite flux limit, in which there is no apparent tower of states. The limit $k \to 0$ again corresponds to the limit of a free theory on S^3. However, it will turn out to be of a different nature than the previous ones. Even though there is an infinite tower of states in this theory, given by the string momentum zero modes, we will see that its mass does not depend on k and hence no light tower of states emerges. This is consistent with the fact that this limit does *not* correspond to an infinite-distance limit.

Lastly, we have the on-shell limit keeping k/r^2 constant. Recall that the number of states j grows with k and hence more and more conserved states appear in this limit. Furthermore, since the flux contribution comes with additional inverse powers of g, such that $H^2 \sim k^2/r^{-12}$, this limit also effectively corresponds to a dilute flux limit and hence we are asymptoting the free PCM. The limit $k \to 0$, on the other hand, does not have an obvious tower of states.

Mass spectrum of light states

Understanding the mass dependence of the tower of states when varying the moduli of the internal (curved) geometry is essential to discuss the SDC. Although this is not known for most backgrounds, the KK spectrum of the three-sphere is well understood.

[4]In particular $H/\text{Vol} \ll 1$ and hence one can see the effect of H as a perturbation of the PCM on $SU(2)$. See also the related Appendix C.

	S^3 with $[H] = k$	$\widetilde{S}^3$ with $[H] = 1$
	$c_1 = 1\,, \pi_1 = 0\,, [H] = k$	$c_1 = k\,, \pi_1 = \mathbb{Z}_k\,, [H] = 1$
S^1-fibre	winding : $\mathbb{Z}_1 = \{0\}$ momentum : $\mathbb{Z}_k$	momentum : $\mathbb{Z}_1 = \{0\}$ winding : $\mathbb{Z}_k$
S^2-base	no winding but momentum $\in \mathbb{Z}_k$	

Table 6.2.: Topological and flux data for the three-sphere with Kalb-Ramond flux and its dual as well as the resulting obstructed modes. The mapping of the states is indeed consistent under T-duality and follows the naive exchange of momentum and winding modes familiar from the trivially fibred case. The splitting into fibre and base and the associated assignment of states is inspired by the case of the trivial bundle and should be viewed heuristically. Strictly speaking the towers can not be uniquely associated to a single direction, due to the non-trivial structure of the fibration.

It is given by[5]

$$m^2 = \frac{l(l+2)}{r^2}\,, l = 0, 1, 2, \ldots \tag{6.7}$$

While this is valid for the three-sphere without any flux contribution, in principal we should also take into account the non-vanishing H-flux[6]. However, we will be merely interested in the behaviour near infinite-distance points. In particular for $r \to \infty$ we can use the fact that this contribution will be suppressed in the associated Laplacian problem by inverse powers of $g \sim r^2$ and therefore we can neglect the flux contribution for $r \to \infty$ and use the above scaling behaviour of the KK-modes. Furthermore, once again, one can move to the CFT picture and obtain the mass dependence from the primary fields. Taking into account that the scalar KK modes are the left-right symmetric states $l = (j, j)$ one indeed obtains (6.7), at least for large $k = r^2$.

For the winding modes, obtaining precise relations for the mass dependence is even more challenging. However, here we can build on the familiar - but heuristic - intuition that the tensions or mass of a closed string winding an internal cycle scales with the volume of this cycle. Therefore, winding states are expected to become infinitesimally light when the volume of the cycle goes to zero.

6.1.3. Consistent mapping of states under T-duality

Our previous findings for S^3 with flux and its dual are summarised in Table 6.2. Essentially, T-duality on the Hopf fibre exchanges momentum and winding modes along

[5]While in flat space one can perform a mode expansion, curved spaces require a Hodge decomposition and analysis of the spectrum of the Laplacian problem.

[6]Generically one has to switch to H-twisted cohomology [454] with twisted differential $\mathrm{d}_H = \mathrm{d} - H\wedge$.

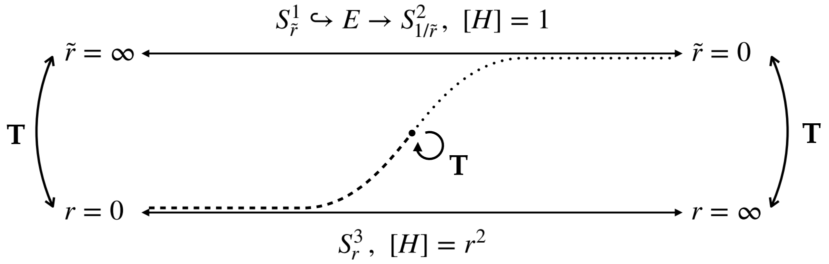


Figure 6.2.: The three-sphere with Kalb-Ramond flux, similarly to the circle, is a non-trivial example of a background with a self duality point, which is located at $r = 1$. Therefore we can chart the full scalar field space, either staying on one duality frame, i.e. moving along the upper or lower line, or change frame "dynamically" as we move across the self duality point.

this direction; however, the precise mapping of states is much more intricate. In particular, since the dual space $\widetilde{S^3}$ has a non-trivial fundamental group $\pi_1 = \mathbb{Z}_k$, it can support non-trivial winding states in the presence of a non-vanishing flux; one expects the momenta on the fibre to be non-conserved due to their coupling through H, see C. However, after winding $k-1$ times, the string can unwind and therefore there can only be a finite amount of winding. This is confirmed from the CFT picture in terms of the orbifold construction. There is a precise formula relating the states and their duals for so-called *Lens spaces*, which we discuss later in this chapter. The above example is merely a special case and hence we refer to section 6.3 for the precise formula.

Not only does the metric and scalar potential on field space for the dual geometry agree with that of the initial S^3 with flux, but the geometry even shares a self-duality point, allowing for a neat interpretation in terms of the SDC; see Figure 6.2.

6.1.4. The role of the scalar potential

Having collected all the necessary information, we can finally turn to a discussion of the Distance Conjecture. Like in the simple circle compactification, there are two infinite distance limits for the radius: $r \to 0$ and $r \to \infty$. In the previous paragraph, we also discussed the dependence of the states on the level or flux number k. We will keep this quantity constant for now and come back to variations associated with k at a later stage in Section 6.1.5. Note that the following reasoning is independent of the duality frame chosen, and therefore we will work with the three-sphere in the following.

Decompactification limit

Taking the limit of large radius $r \to \infty$, i.e. the decompactification limit, we can readily identify the infinite tower of exponentially light states. It is given by the momentum zero modes of the string. The canonically normalised scalar field is

$$\rho = \sqrt{3}\log(r)\,, \tag{6.8}$$

such that the mass of the KK modes scales exponentially with ρ, just like for the circle.

Limit of small radius

Taking the radius to zero $r \to 0$, we immediately encounter an apparent contradiction. There are no winding states which would usually become light in the $r \to 0$ limit and therefore no obvious asymptotically massless tower of states exists[7]. However, upon setting $k = r^2$ and adding a linear dilaton[8], the three sphere with H-flux is a valid string background. By definition, it should therefore be in the Landscape and in particular meet the predictions of the SDC. However, there is one crucial ingredient that we have not taken into account so far, namely the non-trivial scalar potential $V(\phi_d, r)$ sourced by the curvature and non-trivial fluxes on the internal geometry. In particular, we see that while the potential vanishes in the limit of large radii, there is a divergence in V for small $r \to 0$. For large radii we can neglect the potential and treat r as a true modulus and apply the "standard" SDC. For small r on the contrary the potential cannot be neglected and we face the question if and how the SDC can be applied in such a case. In fact, the potential diverges when approaching the problematic point $r = 0$, where we lack an infinite tower of exponentially light states, as predicted by the standard SDC. The resolution of this paradox was originally proposed in [67] by the authors and collaborators: Whenever the internal geometry of an EFT in the Landscape lacks the existence of an infinite tower of states with respect to a "would be infinite-distance point" of the scalar field space, the SDC predicts a diverging scalar potential when approaching this point. This can be seen as an amendment to the SDC for situations with nonzero scalar potentials leading to a modified version of the SDC:

SDC and divergent potentials [67]. *In effective field theories that can be consistently lifted to a theory of quantum gravity in the UV, a divergence in the scalar potential emerges when approaching an infinite locus point for which the target space geometry cannot give rise to a light tower of states. That is, the potential signals pathological infinite-distance loci in the scalar field space.*

Note that the idea of situations with certain inaccessible infinite distance points in moduli space has already been discussed in the literature. See [289] for a similar, however, not directly related obstruction described by the *Negative Distance Conjecture*.

[7] Wrapped branes on S^3, for example, are forbidden by the Freed-Witten anomaly [264].
[8] See also the remark on the next page.

Remarks

- Although the three-sphere with H-flux is a solution to the SUGRA equations of motion for G and B, the dilaton equation has a no-zero contribution that has to be compensated for. At the level of the WZW model, this corresponds to a non-zero vacuum energy. However, the theory can be easily completed to a proper string background by adding, for example, a linear dilaton background. The resulting worldsheet CFT is factorised, and the linear dilaton background merely compensates the vacuum energy by contributing to the central charge, without adding further contributions to the other equations of motion or anomaly coefficients $\hat{\beta}$. Such geometries arise as the near-horizon geometry of a $NS5$-brane, see [451,455].

- The behaviour of the diverging potential at infinite distance seems very similar[9] to the dynamical cobordism scenarios studied in [324–329] as well as the associated Dynamical Cobordism Conjecture [324]. It would be interesting to study this possible relation in more depth.

6.1.5. On-shell vs. off-shell variations & flux contributions

In the last section, we argued that in order to obtain a valid string background, i.e. conformal invariance at least in leading order in α', we have to impose the relation $k = r^2$. This is of course nothing else than the requirement of minimizing the potential (6.3) and hence sitting at a vacuum of the theory, in other words to be on-shell.[10]. Recently, a closely related problem was investigated in [447] in the context of the AdS Distance Conjecture [363]. Then the obvious question of the role of the correlation of k and r for the SDC arises. There are several instances in which insisting on this identification can have a significant impact on the calculation and interpretation. First of all, it is clear that the potential behaves qualitatively differently for $k = r$. Furthermore, if we were to impose this relation already before the compactification procedure, there can be additional contributions to the metric on moduli space since k inherits a x-dependence through r. This will lead to the issue of on- vs. off-shell variations of [447] for the metric on scalar field space. In fact, the authors of [447] introduced what they call an *action metric* that takes into account contributions from the variation of the RR-fluxes. In addition to some conceptual challenges, it was then shown that the resulting contributions are crucial to obtain a positive metric on the space of fields of these families of string vacua; see also [382,448]. We will see that T-duality can help in understanding

[9]We thank Andriana Makridou for pointing this out to us and for several insightful discussions.

[10]This relation does not receive quantum corrections, but is valid to all order in α', as is clear from the WZW description. For a proper string vacuum, one only has to take care of the deficit central charge.

these subtleties also in the case of NSNS flux. Furthermore, at the quantum level the flux number $[H]$ is quantised, and therefore any identification a priori turns the scalar field space into a discrete space. The implications of such a discrete field space and its interpretation were studied in [456]. An alternative approach, utilising for example open string moduli or scalar fields in order to enlarge the field space and hence connect the vacua was studied in [457,458]. We will not discuss these approaches here further but focus on the former aspects and their role for the SDC.

Flux variation and their metric contributions

Following the above considerations, we revisit the compactification procedure that leads to the metric and potential in the field space. Imposing $k = r^2$ while also making the radius dynamical $r = r(x)$ we effectively have to take into account also variations of the flux around its constant vacuum expectation value, i.e. $k = k(x) = r^2(x)$. Hence, this means that we actually compute the metric on field space for an on-shell family of string backgrounds as a function of $r(x)$ along the lines of [447]. Upon substituting $k \to r(x)$, there is one additional problematic term originating from the flux term

$$\frac{1}{4}H_{\mu jk}H^{\mu jk} = \frac{2\cot(\eta)^2}{r^2}(\partial r)^2\,. \tag{6.9}$$

This term is problematic in its own right, due to the missing integration over the internal space. Adding also the contribution coming from the metric, we obtain

$$\begin{aligned}\gamma_{rr} &= \frac{1}{\mathcal{V}}\int_{S^3}\mathrm{d}\eta\mathrm{d}\xi_1\mathrm{d}\xi_2\frac{\sin(\eta)}{4}\frac{1+2\csc(\eta)^2}{r^2}\\ &= \frac{1}{2r^2}\int_0^\pi \mathrm{d}\eta\sin(\eta)(1+2\csc(\eta)^2)\,.\end{aligned} \tag{6.10}$$

However, the left-over integral on η is divergent. We will return to this issue in a second and show that we can eliminate the divergence by using the gauge invariance of H. Before doing so, it turns out to be insightful to examine the situation also from the T-dual perspective. Substituting k with $r(x)$ in (2.30) the dual background takes the simple form

$$\begin{aligned}\widetilde{\mathrm{d}s^2} &= \frac{r^2}{4}(\mathrm{d}\eta^2+\mathrm{d}\xi_1^2+\frac{4}{r^4}\mathrm{d}\xi_2^2-\frac{4}{r^2}\cos(\eta)\mathrm{d}\xi_1\mathrm{d}\xi_2)\,,\\ \widetilde{B} &= -\frac{1}{2}\cos(\eta)\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2\,.\end{aligned} \tag{6.11}$$

The resulting space is a non-trivial circle bundle supported by one unit of H-flux.

In particular, there is no field dependence of the Kalb-Ramond field, and the metric on scalar field space can be calculated in the standard way from the variation of the metric; it coincides with (6.10).

This simple example shows that it is unavoidable to take into account also the variations of the flux contributions to preserve invariance under T-duality. The contributions from the flux are crucial in order to match the expressions of the metric on fields space for a pair of T-dual backgrounds. We will show this in full generality in Section 6.2.

Divergent contribution and gauge invariance

We just argued that T-duality necessitates to also take into account the contribution of the Kalb-Ramond flux for the metric on field space. However, the integral (6.9) we encountered turns out to be divergent. The crucial observation is that, unlike the flux contribution $H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ that enters the potential V, the term $H_{\mu jk}H^{\mu jk}$ is not gauge invariant. In particular, it is sensitive to the explicit choice of Kalb-Ramond field B. Taking inspiration from [447] we can try to use the gauge freedom in B to cure this divergence. Defining B patchwise in a consistent way, we will be able to render the integral finite, and hence obtain a well-defined metric on field space.

First note that due to the chosen structure of B we have $H_{\mu jk} = \partial_\mu B_{ij}$. Now we add a closed term to B such that

$$\hat{B} = B + \zeta \mathrm{d}\alpha(y)\,, \tag{6.12}$$

where ζ is constant and $\alpha(y)$ a one-form on the internal space. Clearly, since we are adding an exact term, this results in the same flux H. Promoting ζ to a function of the external space, i.e. viewing $\zeta = \zeta(x)$ as a scalar field, the resulting flux reads

$$\hat{H} = \mathrm{d}\hat{B} = \mathrm{d}B + \mathrm{d}\zeta(x) \wedge \mathrm{d}\alpha(y)\,, \tag{6.13}$$

or more explicitly

$$\hat{H}_{\mu\nu\lambda} = H_{\mu\nu\lambda}\,, \qquad \hat{H}_{\mu ij} = H_{\mu ij} + \partial_\mu \zeta(x)(\mathrm{d}\alpha)_{ij}\,. \tag{6.14}$$

In particular, whenever we are on-shell, i.e. ζ the modulus is constant, any such choice of Kalb-Ramond field $\hat{B}$ describes the same supergravity solution. However, following the procedure in [447] and allowing for off-shell variations of the action in order to compute the metric on field space, this choice will have a non-trivial contribution to the metric. Appropriately choosing the additional term, it is possible to render the integral finite. We can cover the sphere by two patches corresponding to the northern ($\eta \in [0, 3\pi/4)$) and southern ($\eta \in (\pi/4, \pi]$) hemispheres and define $\hat{B}$ on these as

$$\hat{B}_N = B + r^2 \mathrm{d}(\xi_1 \mathrm{d}\xi_2)\,, \qquad \hat{B}_S = B - r^2 \mathrm{d}(\xi_1 \mathrm{d}\xi_2)\,, \tag{6.15}$$

On-shell, these two choices only differ by an exact term and, therefore, are gauge equivalent. Off-shell however, the relevant flux term is

$$\frac{1}{4}H^N_{\mu jk}H_N^{\mu jk} = \frac{2\tan(\eta/2)^2}{r^2}(\partial r)^2\,, \tag{6.16}$$

and similar for the contribution from B_S. The full flux contribution then reads

$$\gamma^H_{rr} = \frac{1}{8}\left(\int_0^{\pi/2} \mathrm{d}\eta \sin(\eta)\, H^N_{\mu jk}H_N^{\mu jk} + \int_{\pi/2}^{\pi} \mathrm{d}\eta \sin(\eta)\, H^S_{\mu jk}H_S^{\mu jk}\right) = \frac{-1+4\log(2)}{r^2}\,, \tag{6.17}$$

such that the total metric of field space is given by

$$\gamma_{rr} = \gamma^R_{rr} + \gamma^H_{rr} = \frac{1+4\log(2)}{r^2}\,. \tag{6.18}$$

Hence, indeed, by calculating the metric γ from both (2.17) and (2.30) we obtain the same expression (6.10). This carries over to the treatment in terms of the patches. Applying the Buscher rules patch-wise, the transformed patches glue again together smoothly to form a non-trivial circle bundle of S^2. The gauge transformation relating the choices of B gets translated into a diffeomorphism between the new geometric patches. This can be checked explicitly, and one obtains again (6.18) for γ_{rr}; see also [67].

We could not only have identified k with r^2 but also allowed for generic variations of all possible free parameters of the theory, in the example above the flux number k. Our argument using T-duality suggests that this is indeed reasonable if one is to have an invariant metric on field space under T-duality. A similar setup has been investigated in [382, 447, 448] in the context of the AdS Distance Conjecture with non-vanishing RR-fluxes. As was pointed out in these references, in general the fluxes are constrained by the tadpole conditions and hence generically cannot be varied freely. Only in special cases do there exist (infinite) families of vacua such that variations along the flux number can lead to infinite distances. We will return to this issue later.

For completeness and also for later purposes, we also state the metric on field space for generic $k = k(x)$. In this case, we obtain the following expression for the kinetic terms in the action

$$\gamma_{ab}\partial\varphi_a\partial\varphi_b = \frac{3}{r^2}(\partial r)^2 + \frac{2\log(2)-1}{2r^4}(\partial k)^2\,. \tag{6.19}$$

Since k is already canonically normalised, there is only *one* infinite distance, corresponding to $k \to \infty$, while $k = 0$ is at finite distance.

On-shell potentials?

Not only does the on-shell constraint $k = r^2$ introduce non-trivial contributions to the metric on moduli space, but it also modifies the form and interpretation of the scalar potential as a function of r as well as the mass dependence of the tower of states. First, upon identifying $k = r^2$, the scalar potential becomes

$$V(\phi_d, r)|_{k=r^2} = e^{\frac{4\phi_d}{d-2}} \left(\frac{k^2}{2r^6} - \frac{3}{2r^2} \right)\bigg|_{k=r^2} = -\frac{1}{r^2} e^{\frac{4\phi_d}{d-2}} \,. \tag{6.20}$$

This potential is the analogue of the on-shell metric discussed in the previous section. It should be viewed as a potential for the infinite family of vacua given by S^3 with flux number k correlated to the radius via $r = \sqrt{k}$.

We see that there are basically three possible ways to view the scalar field space and its associated potential. First, one can either keep k as a given background constant or promote it to a scalar field. In the second case, there is still the question of whether to impose the on-shell condition $k = r^2$ or keep k as an independent variable:

i) **Keeping k fixed:** In this case, the situation is straightforward. The moduli space metric obtained by varying r as in previous sections gives $\gamma = 3/(2r^2)$. For given but fixed k, the potential with minimum at the vacuum $k = r^2$ reads (omitting ϕ_d)

$$V(r;k) = -\frac{3}{2r^2} + \frac{k^2}{2r^6} \,. \tag{6.21}$$

Asymptotically we have $\lim_{r\to 0} V = +\infty$ and $\lim_{r\to\infty} V = 0$. See Figure 6.3.

ii) **Off-shell scalar field k:** We obtain the two-dimensional[11] moduli space with metric for r and k and potential as above. There are now 4 asymptotic regions: $\lim_{r\to 0} V = +\infty$ and $\lim_{r\to\infty} V = 0$, as well as $\lim_{k\to 0} V = -3/2r^{-2}$ and $\lim_{k\to\infty} V = \infty$. However, according to (6.19) only 3 of them correspond to infinite distances, while $k \to 0$ is at finite distance; cf. Figure 6.3.

iii) **On-shell scalar field k:** The situation is similar to above, only that in the end we have to identify $k = r^2$, effectively reducing to a one-dimensional field space with a potential that reads (again omitting ϕ_d)

$$V(r;k) = -\frac{1}{2r^2} \,. \tag{6.22}$$

This is now a field space for a family of vacua. For every r, we have a valid string background. Hence we look at a certain direction in the previously 2-dimensional field space. This situation corresponds to the red line in Figure 6.3.

[11] We omit the dilaton for the discussion here.

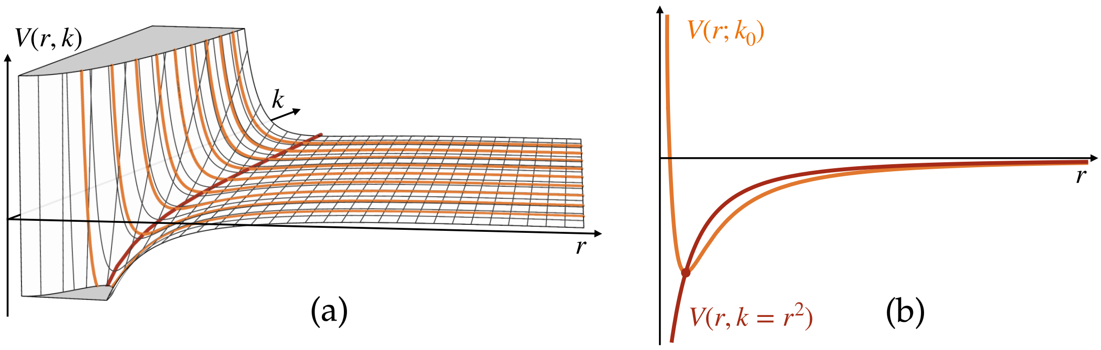


Figure 6.3.: Visualisation of the different scenarios described before. Plot (a) shows the potential as a function of both the radius r as well as the flux number k (which is allowed to vary continuously). The red line represents the valley of the potential, i.e. the line of on-shell vacua. Plot (b) on the right shows the potential as a function of the radius for constant k_0 (orange) as well as the on-shell potential with $k = r^2$ (red).

On vs. off-shell towers and the SDC

Variations of r with fixed k have been extensively discussed above. We also analysed the towers of states and the scalar potential for variable k, as well as the on-shell variations with $k = r^2$. In the former case, there are three infinite distance limits, corresponding to $r \to \{0, \infty\}$ as well as $k \to \infty$. Although we argued that for $r \to \infty$ the theory asymptotes the free PCM with an infinite tower of momentum states becoming light, in the limit $k \to \infty$ there are no states with masses decreasing in k. This is consistent with the fact that exactly in this limit the scalar potential diverges, hence excluding this point from the reasoning of the SDC. The same happens for the limit $r \to 0$, as was already argued before. The limit of $k \to 0$ is at finite distance and corresponds to the free PCM. Since the masses do not scale with k, we do not expect a tower of massless states. This is consistent with the potential remaining finite in this limit.

For the on-shell case, the field space and scalar potential effectively become one-dimensional, and we have two infinite distance points, $r \to \{0, \infty\}$. Again in the small radius limit there is no tower and the potential diverges[12]. For large radii, the situation is somewhat peculiar. Since $k = r^2$ and the number of momentum states is bounded by k, we see that we have more and more conserved states[13] that become light. However, the situation is slightly different from the original picture or formulation of the SDC [117]. In the latter there is an infinite tower of states, the masses of which decrease exponentially along the flow, while here the tower is finite but grows without bound along the limit $k \to \infty$. The cutoff of the effective theory is anyway brought to zero in this limit but in a slightly different manner. We leave the investigation of

[12]Note that in this case the potential diverges to $-\infty$.
[13]In the dual picture this corresponds to more and more winding states becoming light as $\tilde{r}$ shrinks.

this subtle point for future research[14]. Anticipating a forthcoming discussion on non-geometric spaces, we note that for the above arguments to hold, it was essential that we had only one overall volume modulus in terms of the radius r. In particular, moving to the three torus with H-flux, we will see that although the overall situation seems very similar, there are subtleties arising due to the possibility of taking infinite distance limits along distance cycles in specific ways; see Chapter 7.

In closing, we want to stress once more that this was a purely classical consideration. At the quantum level the flux is quantised, and it is not clear that a continuous limit like the above even makes sense. This especially applies to small values of k or r and the limit of $r^2 = k \to 0$. We refer to [456] for a possible discrete approach.

6.2. A general formula for the reduction

We derive a general formula for the metric and potential on field space for generic variations of the metric, fluxes, and dilaton in the bosonic sector of Type II SUGRA that can be used to study, for example, flux variations. We do not assume any additional ingredients such as supersymmetry.

6.2.1. NSNS sector

We start again from the generic D-dimensional effective action

$$\mathcal{S}^{(D)} = \frac{1}{2\tilde{\kappa}_D^2}\int_{M_D} \mathrm{d}^D X\sqrt{-G}e^{-2\Phi}\left(R(G) - \frac{1}{12}H_{IJK}H^{IJK} + 4\partial_I\Phi\partial^I\Phi\right), \tag{6.23}$$

on a D-dimensional manifold M_D which we assume to be of the product form $M_D = M_d \times K_n$ with external space M_d and compact internal manifold K_n. We do not assume supersymmetry nor do we add fermionic or higher-form RR fields for now. The generalisation including the RR sector and possible boundary terms is rather straightforward and we discuss it in a separate step in Section 6.2.3. We take the total metric to be of the form[15]

$$G_{IJ}\mathrm{d}X^I\mathrm{d}X^J = g_{\mu\nu}(x,\varphi_a(x))\mathrm{d}x^\mu\mathrm{d}x^\nu + h_{ij}(y,x,\varphi_a(x))\mathrm{d}y^i\mathrm{d}y^j\,, \tag{6.24}$$

with $\mu = 0,\ldots,d-1$, $i = 1,\ldots,n$ and $D = d+n$. Hence, we allow the internal metric h to depend explicitly on the external coordinates as well as implicitly through some

[14] There is an additional curiosity for such backgrounds when completed to proper string backgrounds by addition of a linear dilaton (and external AdS). In particular, there is a special point $k = 1$ (or $k = 3$ for the bosonic string) where there emerges an infinite tower of massless higher spin states [459–461]. How and if this can be seen as an infinite distances associated to a metric for k coming from a variational procedure like above is not obvious. However, it is potentially related – via the proposed phase transition between black holes and long strings in these examples – to infinite distances as described within [462]. I would like to thank Joaquin Masias for a discussion on this point.

[15] The conventions are slightly different to [67] but more suitable for later generalisation to the RR sector.

scalar fields $\varphi^a(x)$. Decomposing the Ricci scalar $R(G)$ and resumming properly one can show that under the above assumptions it splits as

$$\begin{aligned} R(G) &= R(g) + R(h) + \frac{3}{4}\mathrm{tr}(k_\mu k_\nu)g^{\mu\nu} - \frac{1}{4}\mathrm{tr}(k_\mu)\mathrm{tr}(k_\nu)g^{\mu\nu} - \mathrm{tr}(h^{-1}\Box_g h) \\ &\equiv R(g) + R(h) + \mathcal{J}(g,h)\,, \end{aligned} \tag{6.25}$$

where $k_\mu = h^{-1}\partial_\mu h$ and $\Box_g$ is the d'Alembertian with respect to the metric g. As usual $H_{IJK} = 3\partial_{[I}B_{JK]}$ and we assume that the Kalb-Ramond field and the dilaton respect the block-diagonal structure of the geometry, i.e.

$$\begin{gathered} B_{\mu\nu} = B_{\mu\nu}(x,\varphi^a(x))\,, \quad B_{ij} = B_{ij}(y,x,\varphi^a(x))\,, \quad B_{\mu j} = 0\,, \\ \Phi(X) = \Phi_0 + \phi(x,\varphi^a(x)) + \phi_y(y)\,, \end{gathered} \tag{6.26}$$

with Φ_0 a constant. We assume h can be decomposed into a "internal" y-dependent part and a factor involving the x-dependent scalar fields $\varphi^a(x)$ according to

$$\sqrt{h} \equiv \sqrt{h_0}e^{2\sigma}\,, \tag{6.27}$$

with h_0 at most a function[16] of y^i and $\sigma = \sigma(x,\varphi^a)$. Furthermore, we define the d-dimensional dilaton

$$\phi_d = \phi - \sigma\,. \tag{6.28}$$

After somewhat lengthy computation, splitting the fields into external and internal, as well as going to the Einstein frame via the Weyl rescaling

$$g \to e^{-2\omega}\tilde{g}\,, \qquad \text{with } \omega = \frac{-2\phi_d}{d-2}\,, \tag{6.29}$$

we obtain

$$\begin{aligned} \mathcal{S} = \frac{\widehat{\mathcal{V}}}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-\tilde{g}}\,\Big\{R(\tilde{g}) - \frac{1}{12}H_{\mu\nu\lambda}H^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}}e^{\frac{-8\phi_d}{d-2}} - \frac{4}{d-2}\partial_\mu\phi_d\partial^{\tilde{\mu}}\phi_d \\ - \gamma_{ab}\partial_\mu\varphi_a\partial^{\tilde{\mu}}\varphi_b - \gamma_{a\mu}\partial^{\tilde{\mu}}\varphi_a - V(\varphi_a,\phi_d)\Big\}\,. \end{aligned} \tag{6.30}$$

The metric on field space γ_{ab} is then given by

$$\begin{aligned} \gamma_{ab} &= \frac{1}{4}\widehat{\mathcal{V}}^{-1}\int_K \mathrm{d}^n y\,\sqrt{h_0}\,e^{-2\phi_y}\Big\{\mathrm{tr}(h^{-1}\partial_a h h^{-1}\partial_b h) + \mathrm{tr}(h^{-1}\partial_a B h^{-1}\partial_b B^T)\Big\}\,, \\ \gamma_{a\mu} &= \frac{1}{2}\widehat{\mathcal{V}}^{-1}\int_K \mathrm{d}^n y\,\sqrt{h_0}\,e^{-2\phi_y}\Big\{\mathrm{tr}(h^{-1}\partial_a h h^{-1}\partial_\mu h) + \mathrm{tr}(h^{-1}\partial_a B h^{-1}\partial_\mu B^T)\Big\}\,. \end{aligned} \tag{6.31}$$

[16] In principal one could drop this assumption, however than ϕ_d below is y-dependent and we end up with a warped geometry in the effective action.

The potential $V(\varphi^a, \phi_d)$ is given by

$$\begin{aligned} V(\varphi^a, \phi_d) = &-\widehat{\mathcal{V}}^{-1}\int_K \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y} \left\{ \left(R(h) - \frac{1}{12} H_{ijk}H^{ijk} + 4\partial_i\phi\partial^i\phi \right) e^{\frac{4\phi_d}{d-2}} \right\} \\ &- \frac{1}{2}\widehat{\mathcal{V}}^{-1}\int_K \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y} \left\{ \mathrm{tr}(h^{-1}\partial_\mu h^{-1}\partial^{\tilde{\mu}} h) + \mathrm{tr}(h^{-1}\partial_\mu B h^{-1}\partial^{\tilde{\mu}} B^T) \right\}, \end{aligned} \tag{6.32}$$

with $\widehat{\mathcal{V}} = \int_K \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y}$. The detailed calculation can be found in Appendix D.

The expression γ_{ab} is nothing else than the well-known DeWitt or Candelas-de la Ossa metric for scalar field variations [288, 463, 464] which is in fact the natural metric [464] on the space of parameter of Kähler manifolds in the presence of a non-trivial B-field. Since in general there is a non-zero potential sourced by curvature and flux, (some) scalar fields become massive. Note that we did not take into account any scalar field dependence of the external quantities $R(\tilde{g})$, $H_{\mu\nu\lambda}$, ϕ_d. The procedure has some ambiguity involved at this stage. We defined $g = e^{-2\omega}\tilde{g}$ as well as $\phi_d = \phi - \sigma$, so in general there will be some field dependence also in these external quantities that we should take into account. We will come back to this point later and in particular comment on the strategy explored in [382, 447, 448].

Finally, consider the linear term in $\partial\varphi$ in the field space metric, $\gamma_{a\mu}$, which does not have a clear interpretation and hence should be absent or cancelled by other contributions. Indeed, it was argued in [447] for the case of external AdS, that linear terms like this can be cancelled from variations in the external metric. However, by restricting the scalar field to depend only on a certain subset of the external coordinates, the terms can often be set to zero directly. As an example, consider the linear dilaton background that arises as the near-horizon limit of a NS5-brane. We mentioned before that this is the easiest example for promoting the three-sphere with flux to a full string background without altering the already examined behaviour. It amounts to completing S^3_r with k units of flux to a 10D solution by adding a 7d external Minkowski space $\mathrm{d}s_7^2 = \eta^{(6)}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \mathrm{d}z^2$, together with a linear dilaton[17], given by [455]

$$\Phi = -\frac{z}{\sqrt{q}}\,, \tag{6.33}$$

and on-shell condition $k = q$. Promoting q again to a scalar field, we obtain additional terms (recall that $\Phi = \phi - \sigma$ with $\sigma = \frac{1}{2}\log(r^3)$ for S^3; $\mu = \{\alpha, z\}$)

$$\partial_\mu\Phi\partial^{\tilde{\mu}}\Phi \supset -\frac{z}{q^2}\tilde{g}^{zz}\partial_z q(x) + \frac{3}{\sqrt{q}r}\tilde{g}^{zz}\partial_z r(x) + \ldots\,. \tag{6.34}$$

[17] For $k = q = r^2$ the deficit charge of β^Φ indeed cancels with this contribution from Φ; see also [451].

Therefore, this provides an example of the problematic linear terms mentioned before. However, imposing that q is not a function of z, i.e. $q(x^\mu) \equiv q(x^\alpha)$, the terms in (6.34) can be easily set to zero. In fact, it seems that in the present example, this is the only possibility to make this contribution disappear. Since the external space is Minkowski, there is no apparent way to cancel (6.34) by variations from the external space, nor a natural way to generate such a term from the internal geometry and flux.

6.2.2. T-duality invariance & positivity of the metric and scalar potential

Dropping the integration over the internal manifold as well as constant prefactors, the expression for the metric on field space in the NSNS sector reads

$$\gamma_{ab} = \mathrm{tr}(h^{-1}\partial_a h h^{-1}\partial_b h) + \mathrm{tr}(h^{-1}\partial_a B h^{-1}\partial_b B^T) \,. \tag{6.35}$$

Using the DFT generalised metric $\mathcal{H}$ (cf. Chapter 3)one can express this quantity as

$$\gamma_{ab} = \frac{1}{2}\mathrm{tr}(\mathcal{H}^{-1}\partial_a \mathcal{H}\mathcal{H}^{-1}\partial_b \mathcal{H}) \,. \tag{6.36}$$

Since (abelian) T-duality acts on $\mathcal{H}$ by $O(d,d)$ transformations and using the invariance of the trace, we see that indeed γ_{ab} is T-duality invariant. Hence, this confirms on more general ground what we already argued for in the example of S^3 with H-flux and its dual: the metric is only invariant once we include the variation of the flux or Kalb-Ramond field B. The fully invariant expression is given by (6.35), which we can write compactly as

$$\begin{aligned} \gamma_{ab} &= \langle \partial_a h | \partial_b h \rangle_h + \langle \partial_a B | \partial_b B \rangle_h \\ &= \langle \partial_a (h+B)) | \partial_b (h-B) \rangle_h \\ &= \langle \partial_a E | \partial_b E^T \rangle_h \,, \end{aligned} \tag{6.37}$$

where the inner product is defined in (0.1). It is then immediately clear that in the case of a one-dimensional field space for a scalar field φ we have

$$\gamma_{\varphi\varphi} = \left\|\partial_\varphi h\right\|_h^2 + \left\|\partial_\varphi B\right\|_h^2 \,, \tag{6.38}$$

and therefore the metric is manifest positive definite as is required from a proper metric. In fact, positive definiteness also holds for the multi-dimensional case. To see this note that from (6.37) γ_{ab} is a gram matrix with respect to the norm $\|\cdot\|$. Therefore, it follows that γ_{ab} is positive semidefinite. It is positive definite if and only if all $\partial_a E$ are linearly independent, which in turn is true if the φ^a are linearly independent.

Having established the T-duality invariance of the NSNS sector moduli space metric, it is natural to also investigate the invariance of the potential on field space. Just like the metric, this quantity should be invariant, as T-dual backgrounds are equivalent and should give rise to the same low-energy effective action and moduli or scalar field space, respectively. Again, this follows almost immediately by referring to DFT. Indeed, the effective action (6.23) can be rewritten in terms of the $2n$-dimensional action $\mathcal{S}_{\mathrm{DFT}}$, which upon imposing the section condition reduces to the string effective action; cf. Chapter 3. Since $\mathcal{S}_{\mathrm{DFT}}$ is manifestly $O(d,d)$ invariant, it follows that the string effective action is invariant under abelian T-duality. It is somehow remarkable that the invariance of T-duality, which is defined as a duality at the level of the worldsheet NLSM relating physically equivalent theories, keeps the low-energy effective action invariant. Writing out the individual terms and performing the Buscher rules, it is apparent that the terms combine in a highly nontrivial way into the dual terms in the action.

6.2.3. RR sector

After having dealt with the internal variations of the NSNS sector, let us turn to the RR sector. We take the metric and dilaton as above and focus on the RR fields. The action including the RR-sector reads

$$\mathcal{S} = \frac{1}{2\tilde{\kappa}_D^2} \int \mathrm{d}^D X \sqrt{-G} e^{-2\Phi} \left(R(G) - \frac{1}{2} |H|^2 + 4 |\partial \Phi|^2 - \frac{1}{2} e^{2\Phi} \sum_p \left(1 - \frac{\delta_{p5}}{2}\right) |F_p|^2 \right) . \quad (6.39)$$

Up to this point, the procedure is quite straightforward and analogous to that before, since there are no derivatives of the metric involved. For now, we assume that for all q the potentials C_q are of the form

$$C_{I_1 \dots I_q}(X^I) = C_{\mu_1 \dots \mu_q}(\varphi^a(x), x^\mu) + C_{i_1 \dots i_q}(\varphi^a(x), y^i) . \quad (6.40)$$

This is not the most general setup, but is sufficient for most of the known supergravity solutions. After some manipulations that can be found again in Appendix D we obtain

$$\begin{aligned} \mathcal{S} = \frac{\widehat{\mathcal{V}}}{2\kappa^2} \int \mathrm{d}^d x \sqrt{-\tilde{g}} \Big\{ & R(\tilde{g}) + \dots - \frac{1}{2} e^{2(\Phi_0 + \sigma)} \sum_p \left(1 - \frac{1}{2}\right) e^{2\frac{d-2p}{d-2}\phi_d} |F_p|^2 \\ & - \gamma_{ab}^{RR} e^{2\phi_d} \partial_\mu \varphi^a \partial^{\tilde{\mu}} \varphi^b - \gamma_{a\mu}^{RR} e^{2\phi_d} \partial^{\tilde{\mu}} \varphi^a - V_{RR}(\varphi^a, \phi_d) \Big\} , \end{aligned} \quad (6.41)$$

where we defined

$$\begin{aligned} \gamma_{ab}^{RR} &= \frac{1}{2} \widehat{\mathcal{V}}^{-1} \int_K \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y} \left\{ e^{2(\Phi_0 + \sigma)} \sum_q \frac{1}{q!} \left(1 - \frac{\delta_{p5}}{2}\right) \langle \partial_a C_q | \partial_b C_q \rangle_h \right\} , \\ \gamma_{a\mu}^{RR} &= \widehat{\mathcal{V}}^{-1} \int_K \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y} \left\{ e^{2(\Phi_0 + \sigma)} \sum_q \frac{1}{q!} \left(1 - \frac{\delta_{p5}}{2}\right) \langle \partial_\mu c_q | \partial_a c_q \rangle_h \right\} . \end{aligned} \quad (6.42)$$

The potential contributions reads

$$V_{RR}(\varphi_a,\phi_d) = \frac{1}{2}\widehat{\mathcal{V}}^{-1}\int_K \mathrm{d}^n y\,\sqrt{h_0}\,e^{-2\phi_y}\Big\{e^{2(\Phi_0+\sigma)}e^{2\phi_d}\sum_p\left(1-\frac{\delta_{p5}}{2}\right)\Big(e^{\frac{4\phi_d}{d-2}}\left|f_p\right|^2 + \frac{1}{(p-1)!}\left\langle\partial_\mu c_{p-1}\middle|\partial_\nu c_{p-1}\right\rangle_h \tilde{g}^{\mu\nu}\Big)\Big\}\,. \tag{6.43}$$

Note that while this formula covers the bulk NSNS and RR sector, it does not include contributions from the Chern-Simons term[18], nor source contributions. In general, these terms still have to be added as they will give additional contributions. However, because of their particular (topological) form , they will only contribute to the scalar potential. For us these terms will play only a minor role, and we denote them by $\mathcal{S}_{\text{CS/scource}}$, and their contributions to the scalar potential by $V_{\text{CS/source}}$. For explicit expressions tailored to the examples discussed in Sections 6.4 and 7.3.1 we refer to the Appendices A and D.

Summary of formulas

The full reduced action including both the NSNS and RR sector (and source contributions) to the potential. The effective action reads

$$\mathcal{S}_{\text{eff}} = \frac{\widehat{\mathcal{V}}}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-\tilde{g}}\,\Big\{R(\tilde{g}) - \frac{1}{2}e^{\frac{-8\phi_d}{d-2}}\,|H|^2 - \frac{1}{2}e^{2(\Phi_0+\sigma)}\sum_p\left(1-\frac{\delta_{p5}}{2}\right)e^{2\frac{d-2p}{d-2}\phi_d}\left|F_p\right|^2 - \frac{4}{d-2}\partial_\mu\phi_d\partial^{\tilde{\mu}}\phi_d - \Gamma_{ab}(\phi_d)\partial_\mu\varphi^a\partial^{\tilde{\mu}}\varphi^b - \Gamma_{a\mu}(\phi_d)\partial^{\tilde{\mu}}\varphi^a - V(\varphi^a,\phi_d)\Big\}\,, \tag{6.44}$$

with the total combined expressions

$$\begin{aligned}\Gamma_{ab}(\phi_d) &= \gamma^{NSNS}_{ab} + \gamma^{RR}_{ab}e^{2\phi_d}\\ \Gamma_{a\mu}(\phi_d) &= \gamma^{NSNS}_{a\mu} + \gamma^{RR}_{a\mu}e^{2\phi_d}\\ V(\varphi^a,\phi_d) &= V_{NSNS}(\varphi^a,\phi_d) + V_{RR}(\varphi^a,\phi_d) + V_{\text{CS/source}}(\varphi^a,\phi_d)\,.\end{aligned} \tag{6.45}$$

Separate terms are defined in equation (6.31),(6.32) and (6.42),(6.43) in the previous sections. The metric $\tilde{g}$ is related to the external part g of the D-dimensional product metric G via

$$\tilde{g} = e^{2\omega}g\,, \qquad \omega = \frac{-2\phi_d}{d-2}\,. \tag{6.46}$$

We will see in Chapter 7 that there are further exotic fluxes that can enter the scalar potential, and hence we have to generalise the reduction formula in order to also take these new contributions into account.

[18] Note that working in the democratic formulation, which basically just introduces a factor of 1/2 for the flux terms, there is actually no CS term. See Appendix A and, for example, [227].

Remarks

- As mentioned earlier, there can also be contributions to the metric that arise from varying external quantities, in particular $R(\tilde{g})$. This was extensively studied in a series of articles [382,447,448] in the context of external AdS spaces, and we refer to these references for details. However, in principal, one is not restricted to the form of variations performed in these papers, which is motivated by the specific form of the AdS metric. This leads to the crucial question of which (metric) variations should be allowed for. In particular once we do not restrict ourselves to massless deformations anymore – hence also in the presence of a scalar potential. This point is crucial for fully understanding in particular metrics over full families of vacua and we plan to investigate to this question in the future.

- Generically, the effective action has contributions $\Gamma_{a\mu}$ that are linear in the scalar fields. Such terms have no obvious interpretation in terms of a metric on field space, and it is believed that they should be absent from the start, cancelled among themselves by the separate contributions in $\Gamma_{a\mu}$, or cancelled by the aforementioned external contributions. The former case, which can be often realised by restricting to functions of a certain subset of external coordinates only, was treated in Section 6.2.1. The latter was extensively investigated in [382,447,448].

- In [447,457,458] different procedures from the one used here was used to vary (RR) fluxes in the context of string vacua that have external AdS was explored. In general, it is not expected that the approaches give the same exact metric on field space once generalised to all sectors. We leave a more in depth comparison for future work, but note that both of them come with certain strengths or weaknesses. While the variation presented here depends on the gauge choice for the (B)-filed, it does not necessarily give rise to linear terms in derivatives, cf. Section 6.1.5. For the treatment initiated in [447], there is no such gauge ambiguity due to fixed variations with respect to the singled out AdS radial coordinate z. However, there are generically linear terms that have to be cancelled by other contributions, and it is not clear how to generalise the procedure to more complicated backgrounds, like warped geometries or spaces without a distinguished direction z.

6.3. A family of examples: Lens spaces

The three-sphere turned out to be a highly interesting background for studying the SDC and its interplay with T-duality. There is a generalisation of S^3 that is especially natural in view of topological T-duality. We can think of S^3 as a non-trivial $U(1)$-fibration over the two-sphere S^2 via the Hopf-fibration with first Chern class $c_1(S^3) = 1$ and there is an infinite family of non-trivial circle bundles over S^2 labelled by their Chern class $c_1 = p \in \mathbb{Z}$. These spaces are called Lens spaces [163, 164, 185, 451, 453] and in fact the most natural way to introduce them is by orbifolding S^3 with the discrete group $\mathbb{Z}_n$. For this, note that S^3 can be written in terms of complex coordinates via

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}\,. \tag{6.47}$$

Defining the action of $\mathbb{Z}_n$ on S^3 by [164, 185]

$$(z_1, z_2) \mapsto (\exp(2\pi i l/p) z_1, \exp(2\pi i l/p) z_2)\,, \quad l = 0, 1, \ldots, p-1\,, \tag{6.48}$$

we obtain a background that reads

$$\mathrm{d}s^2 = \frac{r^2}{4}\left(\mathrm{d}\eta^2 + \mathrm{d}\xi_1^2 + \frac{4}{p^2}\mathrm{d}\xi_2^2 - \frac{4}{p}\cos(\eta)\mathrm{d}\xi_1\mathrm{d}\xi_2\right)\,. \tag{6.49}$$

This geometry is known as a lens space[19] $L(p, 1)$ and is often denoted by $S^3/\mathbb{Z}_p$ highlighting the origin as an orbifold of S^3. Lens spaces have first Chern class $c_1(L(p, 1)) = p$ and fundamental group $\pi_1(L(p, 1)) = \mathbb{Z}_p$ for $1 \leq p \in \mathbb{Z}$, hence pure torsion. Here, the torsion-full cycle is given by the non-trivial circle fibre over the base S^2. This fact will be important for the existence of towers of states for the SDC since on these cycles strings can unwind after wrapping the cycle $p - 1$ times. It is clear from the above that initial S^3 is also part of this family, as it can be seen as the lens space $L(1, 1)$ with $\pi_1(S^3) = 0$.

As for the three-sphere the background supports a non-trivial H-flux, given by

$$H = \frac{q}{2}\sin(\eta)\mathrm{d}\eta \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2\,. \tag{6.50}$$

Writing the flux q as $q = \frac{k}{p}$ with $k \in \mathbb{Z}$, it turns out that the background corresponds to an exact CFT in terms of a $SU(2)_k/\mathbb{Z}_p$ WZW model if $r = \sqrt{k} = \sqrt{qp}$.

Topology change and spectrum

Lens spaces $L(m, 1)$ and $L(n, 1)$ for $m \neq n \in \mathbb{Z}$ are not homeomorphic and carry distinct topological invariants. In fact, they are a very elegant family of examples realising the principal of topology change under T-duality via topological T-duality outlined in Section 2.2.3. Combining the metric (6.49) with q units of H-flux and applying T-duality on the circle fibre, one obtains a background with p and q exchanged, i.e. [154, 182]

$$L(p, 1) \quad \text{and} \quad [H] = q \quad \leftrightarrow \quad L(q, 1) \quad \text{and} \quad [H] = p\,. \tag{6.51}$$

[19] There are also Lens spaces $L(p, q)$ with $q \neq 1$, see for example [164, 185].

$S^3/\mathbb{Z}_p$ with $[H]=q$	$S^3/\mathbb{Z}_q$ with $[H]=p$
$c_1 = p\,, \pi_1 = \mathbb{Z}_p$	$c_1 = q\,, \pi_1 = \mathbb{Z}_q$
$w : \mathbb{Z}_p\,, \quad n : \mathbb{Z}_q$	$w : \mathbb{Z}_q\,, \quad n : \mathbb{Z}_p$

Table 6.3.: Momentum and winding modes for a generic Lens space $L(p,1)$ with $[H]=q$ and its dual. The exchange of p and q follows the rules of topological T-duality.

In particular, since we have $c_1(L(p,1)) = p$ and therefore T-duality indeed exchanges the topological invariant with the flux number, as outlined in Section 2.2.3. Performing a T-duality directly at the level of the background along the isometry[20] ξ_2 we obtain

$$\begin{aligned} \widetilde{ds^2} &= \frac{r^2}{4}\Big(d\eta^2 + \sin(\eta)^2\Big) + r^{-2}\Big(\frac{pq}{2}\cos(\eta)d\xi_1 - p\, d\xi_2\Big)\,, \\ \widetilde{B} &= -\frac{p}{2}\cos(\eta)d\xi_1 \wedge d\xi_2\,. \end{aligned} \tag{6.52}$$

This defines a circle fibration with $\tilde{c}_1 = q$ and $[\widetilde{H}] = p$. Upon imposing the previously mentioned relation between r, p and q the background fits again into the form (6.49).

Zero modes and masses

In order to be able to discuss the role of these backgrounds within the Swampland program and the SDC in particular, we need to understand the states and their mass dependence. The situation for a generic lens space $S^2/\mathbb{Z}_p$ is very similar to the one for the T-dual of S^3 with p units of H-flux. Therefore, we keep the discussion minimal and summarise the relevant behaviour in Table 6.3.

Due to the non-trivial fundamental group, there can be winding states. However, since we have torsionful groups of the form $\mathbb{Z}_p$, there can be only $p-1$ distinct or conserved non-trivial winding states. In particular, after winding p times, the loop becomes contractible and the string can unwind. Therefore we are again led to the conclusion that also for Lens spaces the winding modes can not provide an infinite tower of states. Taking into account only the geometry and neglecting the H-flux, there is a priori no reason for an obstruction of a tower of conserved momentum modes. In particular, while the orbifolding projects out states that are not invariant under the $\mathbb{Z}_p$ action, there is still an infinite tower of states that survive the projection. However, when $[H] \neq 0$ the situation is very similar to the one discussed for S^3 with flux and its dual: the H-flux modifies the conservation in a way that there is only a finite conserved tower of KK modes. Under T-duality, these map to the winding states lapelled by $\mathbb{Z}_p$.

[20] Note that in contrast to ξ_2, ξ_1 is not a proper isometric direction along we can dualise. In fact, the corresponding fibre is isomorphic to $U(1)/\mathbb{Z}_p$, as is nicely explained in [185].

This heuristic picture has been worked out and confirmed explicitly at the level of the worldsheet CFT. In particular writing $k = pq$, the WZW orbifold theory $SU(2)_{pq}/\mathbb{Z}_p$ has momentum states n that are defined modulo q and winding states that are defined modulo [453] p, i.e.

$$n \in \mathbb{Z}_q \,, \qquad w \in \mathbb{Z}_p \,. \tag{6.53}$$

Under T-duality along the Hopf fibre, the theory is mapped to $SU(2)_{pq}/\mathbb{Z}_q$ with winding and momentum states

$$\tilde{n} \in \mathbb{Z}_p \,, \qquad \tilde{w} \in \mathbb{Z}_q \,. \tag{6.54}$$

In particular, the theories are fully equivalent at the level of the CFT [163, 453] and explicitly realised the principle of topological T-duality [182]. The mass dependence of the KK spectrum can be obtained by projecting onto the $\mathbb{Z}_p$-invariant states of the three-sphere. In particular, the overall scaling with r remains unchanged, and we still have an infinite number of states, see, e.g. [465]. As argued in Section 6.1 the effect of the flux can again be neglected in the for us relevant regions. The winding modes are expected to scale with the volume of the wrapped cycles and, therefore $m_w^2 \sim r^2$.

Distance Conjecture

Substituting the Lens space background (6.49) into the reducion formula (6.31) and (6.32) we obtain

$$\gamma_{rr} = \frac{3}{r^2} \,, \qquad V(r, \phi_d) = \left(\frac{p^2 q^2}{2r^6} - \frac{3}{2r^2} \right) e^{\frac{4\phi_d}{d-2}} \,. \tag{6.55}$$

Comparing with Section 6.1, we see that the situation for the special case of S^3 with H-flux carries over to the more general family of backgrounds of Lens spaces. While the $r \to \infty$ corner of field space meets the predictions of the SDC, there is a lack of states when approaching $r \to 0$. The situation is again resolved by taking into account the diverging scalar potential and the amendment to the SDC presented in Section 6.1.

Most of the discussion also applies to the special case $\{p = 1, q = 0\}$, which corresponds to the three-sphere without any flux. However, since this space is not a proper string vacuum, the implications for the SDC and the amendment in terms of the diverging potential are less clear or strong than in the case of a proper string background.

6.4. Towards realistic examples: DGKT-like vacua

The (family of) examples discussed so far should be viewed as toy examples in order to gain intuition for the effect of T-duality and the scalar potential for the SDC. The next step is therefore to see how or if this intuition carries over to other more complicated string and M-theory vacua and, in particular, phenomenologically interesting examples. A nice family of vacua to study these effects is the family of 11D vacua given by Freund-Rubin [466] backgrounds. These were also intensively studied in the context of AdS distances and flux variations in [447] and very recently in an algorithm to "backtrack" the underlying brane picture of AdS flux vacua [467]. However, a phenomenologically more interesting setup is that of DGKT-like vacua [123] briefly mentioned in Chapter 1. These theories actually come as a full family of vacua labelled by the F_4 flux quantum $\mathfrak{f}_4$, which is a completely unconstrained parameter, similarly to the RR-flux in Freund-Rubin vacua and provide AdS examples of the four-dimensional $\mathcal{N} = 1$ effective theories mentioned in Chapter 1. These vacua still lack an uplift to full 10D string-theoretic solutions, but have been shown [468] to descend from "approximate" (supersymmetric) massive Type IIA solutions with $SU(3)$ structure and orientifold sources that are not localised but *smeared*, which were found and classified in [469].

We will not discuss whether such smeared solutions are fully consistent beyond the supergravity approximation and hence give rise to genuine string theoretic vacua; see [470] and in particular [125, 126, 471] for a recent account with a special emphasis on scale separation and also Swampland arguments. For us, they merely serve as more realistic examples on which we can test the reduction formula and obtain some intuition on how the potential enters these richer scenarios. However, we briefly comment on some aspects of these vacua and related ones in the context of the SDC in Chapter 7.

We work with the background conventions given in [469] (see also [472]), which at the level of string theory arise in the near-horizon limit of intersecting branes in the presence of orientifold planes. The metric and the dilaton read

$$ds_{10}^2 = L^2 \mathrm{d}s^2_{\mathrm{AdS}_4} + \sum_{i=1}^{6} r_i^2 \mathrm{d}y_i^2\,, \qquad \Phi = \mathrm{const.}\,, \tag{6.56}$$

with L the AdS radius and r_i the radii of the T^6. This is supplemented by constant background fluxes (along internal directions)

$$\begin{aligned} H_{135} = H_{145} = H_{235} = -H_{246} = \mathfrak{h}\,, \\ F_{3456} = F_{1256} = F_{1234} = \mathfrak{f}_4\,, \end{aligned} \tag{6.57}$$

and the non-zero Romans mass $m = \mathfrak{m}$, such that the setup is within *massive* Type IIA SUGRA, see Appendix A. One obtains a string vacuum at $r_i = 1$ if the flux and AdS radius are chosen

$$L = \frac{5}{m} e^{-\Phi}\,, \qquad \mathfrak{h} = -\frac{2}{5} e^{\Phi} \mathfrak{m}\,, \qquad \mathfrak{f}_4 = \frac{3}{5} \mathfrak{m}\,. \tag{6.58}$$

In order to cancel the tadpoles, the theory has to be supplemented with *smeared* orientifold 6-planes ($O6$) along the (internal) directions $(1,3,5)$, $(2,4,5)$, $(2,3,6)$ and $(1,4,6)$

with charge $\mu_6 = \mathfrak{h}\mathfrak{m}$, which will give non-zero contributions to the potential via the source terms; for details, see [469, 471–473] and Appendix A.

Metric and potential on field space

For simplicity, consider the case of a factorised[21] $T^6 = (T^2)^3$.Then, due to the orientifold projection, the only surviving moduli are $r_1, r_2, \tau, \phi_d, \nu, \sigma$, such that(cf. [469])

$$B_{12} = B_{34} = B_{56} = \tau\,, \qquad C_{235} = C_{145} = C_{136} = -\nu\,, \qquad C_{246} = \sigma\,. \tag{6.59}$$

In addition to the standard scalar field, we are interested in variations of the background fluxes like $\mathfrak{h}$, $\mathfrak{f}_4$ and $\mathfrak{m}$. We will discuss these in a second step. Substituting the background data and the allowed scalar fields into our reduction formula ($\mathfrak{h}, \mathfrak{f}_4, \mathfrak{m}$ fixed). For the NSNS sector of the metric γ we obtain

$$\gamma_{ab}^{NSNS}\partial_\mu\varphi^a\partial^{\tilde{\mu}}\varphi^b = 3r_1^{-2}|\partial_\mu r_1|^2 + 3r_2^{-2}|\partial_\mu r_2|^2 + \frac{3}{2}r_1^{-2}r_2^{-2}|\partial_\mu\tau|^2 + 2|\partial_\mu\phi_4|^2\,, \tag{6.60}$$

while for the RR sector we find

$$\gamma_{ab}^{RR}\partial_\mu\varphi^a\partial^{\tilde{\mu}}\varphi^b = \frac{3}{2}r_2 r_1^{-1}e^{2\phi_4}|\partial_\mu\nu|^2 + r_1^3 r_2^{-3}e^{2\phi_4}|\partial_\mu\sigma|^2\,. \tag{6.61}$$

For the potential we get (we only give the potential for the real part of the scalar fields)

$$V(\varphi_a,\phi_4) \sim \mathfrak{h}^2 e^{2\phi_4}\frac{r_1^4+3r_2^4}{r_1^4 r_2^6} - 2\mathfrak{m}\mathfrak{h}e^{3\phi_4}\frac{r_1^2-3r_2^2}{\sqrt{r_1 r_2^3}} + e^{4\phi_4}\frac{3\mathfrak{f}_4^2+\mathfrak{m}^2 r_1^4 r_2^4}{r_1 r_2} + V(\tau,\nu,\sigma,\phi_4)\,. \tag{6.62}$$

It can be checked explicitly that the field space metric and scalar potential above are precisely in agreement with what is obtained by using the $\mathcal{N}=1$ scalar potential formula (1.43) of Section 1.3.3. Hence, this gives a nice consistency check of our general reduction formula (6.44) and (6.45). However, it should be stressed that the formula derived in the previous section holds more general and does not require any supersymmetry.

In view of a later discussion of the SDC, we extract the asymptotic behaviour of the scalar potential. For this first note that the potential is given in terms of the effective $4d$ dilaton ϕ_4, which itself is a function of the radii. We can reinstate the $10d$ dilaton using $\phi_4 = \phi - \sigma$ with $\sigma = \frac{3}{2}\log(r_1 r_2)$. This leads to

$$V(\varphi_a,\phi) \sim \mathfrak{h}^2 e^{2\phi}\frac{r_1^4+3r_2^4}{r_1^7 r_2^9} - 2\mathfrak{m}\mathfrak{h}e^{3\phi}\frac{r_1^2-3r_2^2}{r_1^5 r_2^6} + e^{4\phi}\frac{3\mathfrak{f}_4^2+\mathfrak{m}^2 r_1^4 r_2^4}{r_1^7 r_2^7} + V(\tau,\nu,\sigma,\phi)\,. \tag{6.63}$$

[21] In particular, this can be achieved by considering orbifolds $T^6/\mathbb{Z}_2^2$ like for example in [474]. Considering the even more restricted setup of $T^6/\mathbb{Z}_3^2$ leads to the solutions of [123, 124], where there are no complex structure moduli that survive but a priori three independent volume moduli v_a associated to the T_a^2.

The naive asymptotic limits of this potential, keeping all but one scalar field fixed, then read

$$\begin{aligned} &\lim_{r_1\to 0} V = \infty\,, &&\lim_{r_2\to 0} V = \infty\,, &&\lim_{\phi\to -\infty} V = 0\,, \\ &\lim_{r_1\to \infty} V = 0\,, &&\lim_{r_2\to \infty} V = 0\,, &&\lim_{\phi\to \infty} V = \infty\,. \end{aligned} \tag{6.64}$$

Since the field space is multidimensional, there are, however, also other limits of multiple variables while keeping certain ratios fixed. For example, take the limit of large radii and large string coupling in a way such that $\sigma_4 = const$. This is an infinite distance limit and it is clear then from (6.62) that in this special limit we do not have a vanishing V, but $V \to \infty$ hinting towards the fact that there might be an absence of a tower of states in that specific direction. A detailed discussion of all the different limits and the interpretation in terms of the SDC is beyond the scope of this short section and is left for future work. However, we will comment on this example again in Chapter 7.

Lastly, we discuss the variation of the background fluxes that we have already introduced in Section 6.1.5 for the NSNS flux. We will explain later in Section 7.3 that in fact, due to a non-trivial tadpole, the only parameter that can be freely varied is $\mathfrak{f}_4$, while $\mathfrak{h}$ and $\mathfrak{m}$ are highly constrained by the tadpole condition (7.21). For the former, we define the C_3 potential as

$$\begin{aligned} (C_3)_{235} = (C_3)_{145} = (C_3)_{136} = -\nu(x)\,, \\ (C_3)_{456} = y^3\mathfrak{f}_4(x)\,, \quad (C_3)_{256} = y^1\mathfrak{f}_4(x)\,, \quad (C_3)_{234} = y^1\mathfrak{f}_4(x)\,, \end{aligned} \tag{6.65}$$

where we made a (arbitrary) choice for the components involving $\mathfrak{f}_4$ in order to reproduce the flux (6.57) and promoted $\mathfrak{f}_4$ to a function of the external (AdS) coordinates x^μ. Using the reduction formula (6.42) results in the additional terms

$$\gamma^{RR}_{ab}\partial_\mu\varphi^a\partial^{\tilde{\mu}}\varphi^b \supset 4\pi^3 r_1 r_2^{-1} e^{2\phi_4(x)}|\partial_\mu\mathfrak{f}_4|^2\,, \tag{6.66}$$

where a factor $8\pi^3$ comes from the internal integration that now involves factors of the form $\int_0^{2\pi}\mathrm{d}x^i(x^i)^2$.

Remark

- The DGKT-like background discussed above admits a whole family of solutions, indexed by $\mathfrak{f}_4$ such that all vacuum values can be written in terms of a single parameter. See [382] for a closely related procedure methionine earlier, which also takes into account external variations such that in the end one obtains a (on-shell) metric and potential for the full family of vacua, parametrised in terms of a single scalar field (the AdS radius in that case).

6.5. Local isometries and invariant potentials?

We close this chapter with a short discussion of T-duality for isometries with fixed points and the possible implications for the SDC; see also [475] for a related discussion. In particular, this corresponds to spaces where a local circle fibration degenerates, and hence the space is not globally a principal $U(1)$-bundle. T-dualities on such spaces are arguably the simplest examples that feature topology change under T-duality. However, in contrast to topological T-duality discussed in Chapter 2, this is not due to an exchange of topological invariants and flux quanta, but more due to a certain ill-definedness of the procedure, due to global obstructions in finding a globally well-defined Killing vector field. Take, for instance, the two-dimensional Euclidean plane $\mathbb{R}^2$. Moving to polar coordinates, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}l^2 + l^2 \mathrm{d}\theta^2 \,. \tag{6.67}$$

The metric has an isometry along the θ direction, which, however, degenerates at the origin $r = 0$. Since the associated Killing vector ∂_θ generates rotations around the origin, the latter is a fixed point of the isometry. The dual geometry, naively using the Buscher rules, gives

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}l^2 + \frac{1}{l^2} \mathrm{d}\tilde{\theta}^2 \,, \tag{6.68}$$

which is clearly singular, exactly at the fixed point of the isometry. However, we already saw that strings perceive spacetime very differently and in particular also for the SDC and the associated scalar potential, what matters is the full expression for the potential and not only the part corresponding to the metric. In particular, the Buscher rules, in addition to the change in metric, dictate also a non-trivial dilaton $\tilde{\Phi} = -\log(l)$. Substituting the background into the action, we obtain the Lagrangian[22]

$$V = R(g) = 0 \,, \quad \widetilde{V} = R(\tilde{g}) + 4(\partial\tilde{\Phi})^2 = \frac{-4}{l^2} + \frac{4}{l^2} = 0 \,. \tag{6.69}$$

However, despite this almost trivial example, where the action functionals indeed match, the T-duality invariance for isometries with fixed points is not guaranteed. Take for example S^2_r; the two-sphere can not be written as a globally defined circle fibration, and hence there is no global $U(1)$ isometry. Locally, one can, however, apply the Buscher rules as above and obtain the metric and dilaton. While the sphere gives rise to a non-zero Lagrangian through the Ricci scalar $R(g) = \frac{2}{r^2}$, leading to a non-zero potential $V = 2/r^2$, when regarded as an internal compactification space, the dual potential is easily computed as $\tilde{V} = -\frac{2}{r^2}$ such that there is a sign flip in the potential. Using the

[22] We denote by V the integrand of the potential and omit the volume factor and integration.

general Buscher rules, one can actually show that in $2D$

$$V(G,B,\Phi)=\frac{G'_{00}(G_{11}G'_{00}-2G_{01}G'_{01}+G_{00}G'_{11})+2\det(G)G''_{00}}{2\det(G)^2}=-\widetilde{V}(\tilde{G},\tilde{B},\tilde{\phi})\,, \quad (6.70)$$

using the fact that there is an isometric direction X^0 and hence the initial background can be expressed[23] independently of X^0. Primes denote derivative with respect to X^1. The potentials are zero, and hence agree if the associated metric component along the duality direction is constant

$$G_{00}=const. \quad (6.71)$$

It is not hard, but somewhat tedious to check that this generalised to arbitrary dimensions[24] in the following way. One can show that for arbitrary D,

$$V(G,B,\Phi)\neq\widetilde{V}(\tilde{G},\tilde{B},\tilde{\phi})\,. \quad (6.72)$$

However, if there exists a coordinate system in which G_{00} = const. then the potentials agree

$$G_{00}=const.\Longrightarrow V(G,B,\Phi)=\widetilde{V}(\tilde{G},\tilde{B},\tilde{\phi})\,. \quad (6.73)$$

Note that we can translate the condition of constant G_{00} into a statement of the norm of the Killing vector k. Since we work in adapted coordinates, the latter is

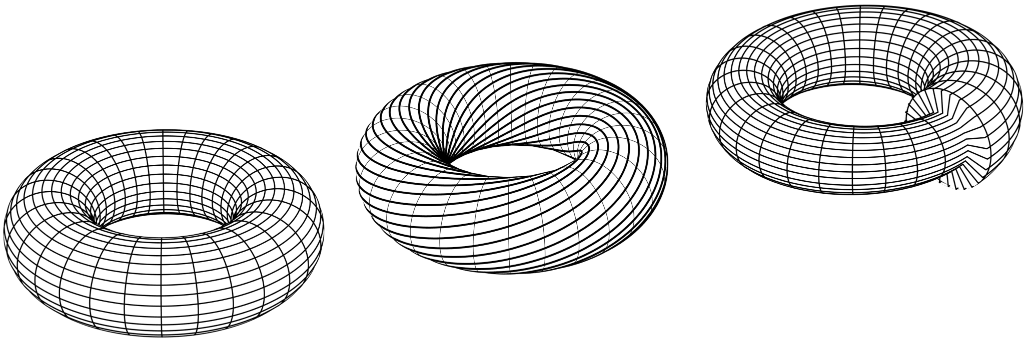
$$|k|^2=G_{ij}k^ik^j=G_{00}\,, \quad (6.74)$$

and hence the statement translate into a condition on the norm of the Killing vector. In particular, if the isometry is not globally defined, the Killing vector is generically not constant and, for example, in the case of a fixed point, degenerates to $|k|=0$, so that the norm cannot be constant globally.

Hence we see that while the Buscher rules give a locally equivalent background also for globally ill-defined S^1 bundles, the effective action captures the global issues and mirrors them through non-invariant scalar potentials under the duality transformation. Note that this reasoning of the failure of the invariance can also be seen from a doubled perspective. In order to perform T-duality, we need a frame in which we can act with the $O(D,D)$ element O_i implementing the factorised T-duality along that direction. If the isometry has a fixed point, this frame degenerates at some point, and hence the argument presented earlier in Section 6.2.2 breaks down.

[23]In adapted coordinates and neglecting subtleties with respect to B, which in 2D is anyway pure gauge.

[24]One can either employ a computer algebra program and check for the relevant dimensions up to $D=11$ or explicitly perform the Buscher rules on all terms in V, checking that the full expressions match. The latter indeed confirms the behaviour for arbitrary D but is very lengthy and is not presented here.



CHAPTER 7

Non-geometric backgrounds and the SDC

At a first glance one might expect that a generic string vacuum has an underlying geometric realisation that can be described by ordinary Riemannian geometry, and hence that non-geometric realisations – introduced in Chapter 3 – are merely exotic corners of the string Landscape. However, quite the contrary seems to be the case. String theory allows for a much bigger configuration space than can be described by standard geometric notions of point particles, and hence we expect [239] that the vast majority of vacua arise from some notion of non-geometry. Geometric configurations are then merely special points within this huge Landscape. A good understanding of these theories and their properties with respect to Swampland conjectures is therefore paramount. In the present chapter, we will investigate a specific notion of non-geometric spaces and their associated fluxes, realised in terms of β-supergravity, and the implications for the SDC. We will start by discussing the appropriately adjusted reduction procedure of the previous chapter and, as a first example, discuss the three-dimensional T-duality chain. After a careful analysis of these backgrounds, unravelling certain missing towers of states, we briefly highlight how some of the issues are circumvented once moving to more realistic examples with additional ingredients leading back to the DGKT-like flux vacuum and its T-dual cousins discussed earlier. We finally comment on possible approaches to truly non-geometric spaces like asymmetric orbifolds, and some recent developments in the literature.

7.1. Reduction of non-geometric backgrounds

The reduction formula derived earlier needs to be modified in case the internal geometry is not a Riemannian manifold but given by some (globally) non-geometric space. Based on the discussion in Part I, we can expect the non-geometric fluxes to give contributions similar to their NSNS pendants. In particular, we have two additional sources for the potential

$$V \supset Q^2, R^2 \,. \tag{7.1}$$

The initial step of the reduction procedure is analogous to the purely geometric setup described in earlier sections. Assuming some suitable factorisation of the total space like in Section 6.2.1 we can split the terms in the action into purely external and internal parts respectively. Additionally there will be mixed terms, coming from the non-trivial dependence of the internal geometry on the external coordinates. At this point in the geometric setting, we would reorganise the terms properly, go to the Einstein frame and finally integrate over the internal geometry. However in the case of a non-geometric background the NSNS background fields are plagued by non-trivial monodromies that preclude us from performing the actual integration and arrive at the reduced $d = (D - n)$-dimensional theory. We will circumvent this problem by going to the non-geometric β-frame introduce in Chapter 3. There is a certain ambiguity in when to switch to the β-frame. Either one starts from the non-geometric frame in D dimensions and then goes back to the standard NSNS frame for the external geometry after the splitting or the other way around. We will choose the second option, i.e. rewriting the internal geometry in the β-frame after the split. This means in particular that the mixed terms which enter the metric on field space are treated in the standard NSNS frame. However, it is easy to see that both procedure are equivalent. In particular, the metric on moduli space is given by $\gamma_{ab} = \frac{1}{2}\mathrm{tr}(\mathcal{H}^{-1}\partial_a\mathcal{H}\mathcal{H}^{-1}\partial_b\mathcal{H})$, with $\mathcal{H}$ invariant under the change of frame as explained in Chapter 3, hence giving equivalent results.

Explicitly this means that after having split the geometry into an internal and external part, we trade the geometric data $\{h, B, \phi_y\}$ for their non-geometric cousins $\{\hat{h}, \beta, \hat{\phi}_y\}$. The resulting geometry has trivial monodromy and can be integrated without any obstacles. Instead of the Lagrangian of the internal manifold in terms of g, B and ϕ_y, we now have the β-gravity Lagrangian given in (3.21). Specialising on globally non-geometric backgrounds, i.e. discarding R-flux contributions, it reads [243]

$$L_\beta = e^{-2\tilde{\phi}_y}\sqrt{\hat{h}}\left(R(\hat{h}) + \check{R}(\hat{h}) + 4(\partial\hat{\phi}_y)^2 + 4(\beta^{ij}\partial_j\hat{\phi}_y + \hat{h}_{pq}\beta^{ij}\partial_j\hat{h}^{pq} + Q_k{}^{ki})^2\right) . \tag{7.2}$$

Now, performing the analogue calculation to Section 6.2.1 one arrives at an expression for the potential on field space that reads ($Q^2 = \hat{h}_{ij}\hat{h}_{mn}\hat{h}^{kl}Q_k^{\ mi}Q_l^{\ nj}$)

$$\widehat{V}(\varphi^a,\phi_d) = -\widehat{\mathcal{V}}^{-1}e^{\frac{4\phi_d}{d-2}}\int \mathrm{d}^n y\,\sqrt{\hat{h}_0}\,e^{-2\hat{\phi}_y}\left\{R(\hat{h})-\frac{1}{4}Q^2-\frac{1}{2}\hat{h}_{ij}Q_k^{\ lj}Q_l^{ki}+4|\partial\hat{\phi}_y|^2+\ldots\right\}, \quad (7.3)$$

where ... denote terms that will not contribute in our concrete examples and can be found in [243]; $\widehat{\mathcal{V}}$ is defined as below (6.32), now in terms of the hatted equivalents.

Note that naively ϕ_d depends on the β-frame through $\hat{\sigma}$. However, under the assumption that also in the non-geometric frame one can split ϕ_y into a sum of a purely field-dependent term and a function of only the coordinates, it follows by using the invariance of $\sqrt{G}e^{2\Phi}$ that, in fact, $\phi_d = \hat{\phi}_d = \phi - \hat{\sigma}$. Furthermore, under the simplifying assumption[1] that $\beta^{km}\partial_m(\cdot) = 0$ the potential reduces to

$$\widehat{V}(\varphi^a,\phi_d) = -\widehat{\mathcal{V}}^{-1}e^{\frac{4\phi_4}{d-2}}\int \mathrm{d}^n y\,\sqrt{\hat{h}_0}\,e^{-2\hat{\phi}_y}\left\{R(\hat{h})-\frac{1}{4}Q^2+4|\partial\hat{\phi}_y|^2\right\}, \quad (7.4)$$

which is particularly similar to the standard geometric expression (6.32). In total we see that also for the non-geometric setting, the relevant formulas are given by (6.44) and (6.45) with $V_0(\varphi^a,\phi_d)$ replaced by $\widehat{V}_0(\varphi^a,\phi_d)$ given by (7.4) or more generally (7.3).

Remark

- Note that the transformation to β-gravity maps to the same vacuum solution and hence we remain at the same point in field space. However, now it is described by a redefined set of fields that, in contrast to the NSNS frame fields, give rise to a well-defined background configuration.

7.1.1. Volume scaling behaviour of potential

Before moving on to discuss some examples, we close the general discussion by referring to a generic formula for the scaling behaviour of the potential in the presence of geometric and non-geometric fluxes. Focussing on the overall volume modulus and (the logarithm of) the 4d dilaton σ, the authors of [240] argued that the potential after compactification from 10D to 4d reads

$$V(\rho,\sigma) = \sigma_4^{-2}\left(-\rho^{-1}V_f^0 + \rho^{-3}V_H^0 + \rho V_Q^0 + \rho^3 V_R^0\right), \quad (7.5)$$

where the fields are defined through

$$h_{ij} = \rho\, h_{ij}^{(0)}, \quad e^{-\Phi} = e^{-\Phi^{(0)}}e^{-\varphi}, \quad \sigma = \rho^{3/2}e^{-\varphi}, \quad (7.6)$$

[1] This holds in particular for simple backgrounds like the T-duality chain. In the example of the NATD of S^3 in Chapter 8 this does *not* hold and we are required to use the more general formula above.

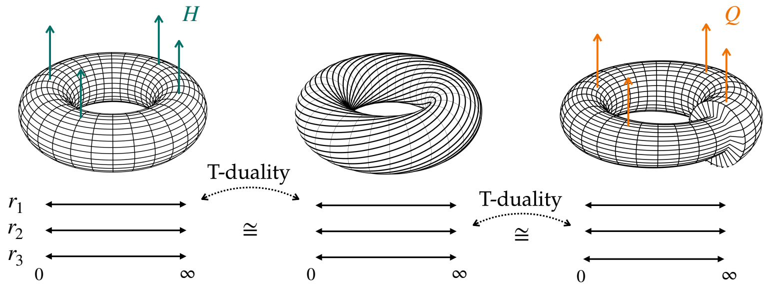


Figure 7.1.: Change in global structure for the backgrounds of the 3D T-duality chain. Schematically illustrated are the T^2 fibres, while the S^1 base is omitted. While the scalar field space and potential are left invariant under the T-duality transformations along the chain, the corresponding target space geometries differ drastically.

and we refer to [240,252] for further details and conventions. With some mild modifications, this formula can also be applied to compactifications to d dimensions. In the case of $\mathcal{N} = 1$ supersymmetric compactifications, the non-geometric fluxes can furthermore be included into the general formula (1.43) discussed in Section 1, see [265].

7.2. 3D T-duality chain in the Swamp?

We investigate the behaviour of the T-duality chain with respect to the SDC. For details, we refer to Chapter 3. Recall that the chain can be written schematically as

$$\begin{array}{ccccccc} H\text{-flux} & \overset{y_1}{\longleftrightarrow} & \text{geometric flux} & \overset{y_2}{\longleftrightarrow} & Q\text{-flux} & \overset{y_3}{\longleftrightarrow} & \text{R-flux} \\ H_{ijk} & & f^i{}_{jk} & & Q^{ij}{}_k & & R^{ijk} \end{array}, \tag{7.7}$$

where for now we omit the last step leading to the exotic R-flux background.

In the following we argue for a tension between non-geometric T-duality chain and the predictions of the SDC: even after consideration of the non-trivial potential the backgrounds seem to lack certain zero modes and associated exponentially light towers required by the SDC. Without further modification, we therefore have to conclude that these backgrounds lie in the Swampland. However, this is not too surprising, since the torus with H-flux is not a proper string background on its own. It has to be embedded into a proper solution that necessarily involves other ingredients such as fluxes or sources. These may come with additional sources for towers of states that we do not account for in this simplest incarnation of the T-duality chain. In Section 7.3.1 we will consider the natural completion of the T-duality chain to full string backgrounds, which is closely related to the DGKT-like example discussed at the end of Chapter 6.

	T^3 with H-flux	Twisted three-torus	$\hat{T}^3$ with Q-flux
Ricci scalar R	–	✓	–
H-flux	✓	–	–
Q-flux	–	–	✓
potential $e^{\frac{-4\phi_d}{d-2}}V(r_i)$	$\frac{k^2}{2}(r_1^2r_2^2r_3^2)^{-1}$	$\frac{k^2}{2}\tilde{r}_1^2(r_2^2r_3^2)^{-1}$	$\frac{k^2}{2}\tilde{r}_1^2\tilde{r}_2^2(r_3^2)^{-1}$

Table 7.1.: Summary of scalar potentials, highlighting the different contributions in each duality frame. Tilde radii $\tilde{r}_i$ denote the natural notion of radius in the corresponding geometry and are inversely related to the ones of the initial torus with radii r_i.

Backgrounds and scalar potentials

Before going into details, we summarise the potentials in Table 7.1. As expected, the potentials agree even though they are sourced by different flux contributions.

Torus with *H*-flux

Take the background as in Section 3 with coordinates y^i for $i \in \{1,2,3\}$ and dilaton at most a function of y^3, i.e. $\phi_y = \phi_3 = \phi_3(y_3)$. Substituting this into our general reduction formula, we obtain the reduced effective action

$$\mathcal{S} = \frac{\widehat{\mathcal{V}}_{\text{int}}}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-g}\Big(R(g) - \frac{4}{d-2}\,|\partial\phi_d|^2 - \sum_i r_i^{-2}\,|\partial r_i|^2 - V(r_i,\phi_d)\Big)\,. \tag{7.8}$$

The scalar potential is given by

$$\begin{aligned} V(r_i,\phi_d) &= -\widehat{\mathcal{V}}_{\text{int}}^{-1} e^{\frac{4\phi_d}{d-2}}\int \mathrm{d}^3y\sqrt{h}e^{-2\phi_y}\left(-\frac{1}{12}\mathcal{H}_{ijk}\mathcal{H}^{ijk} + 4\partial_i\phi_y\partial^i\phi_y\right) \\ &= -V(\phi_3) + \frac{k^2}{2}\frac{1}{r_1^2r_2^2r_3^2}e^{\frac{4\phi_d}{d-2}}\,. \end{aligned} \tag{7.9}$$

In case the internal dilaton reads $\phi_3 = 0$, which we will assume in the following, we have $\widehat{\mathcal{V}}_h = (2\pi)^3$ and $V(\phi_y) = 0$. Clearly, the potential diverges for $r_i \to 0$.

Twisted torus

T-dualising along the y^1 isometry of the T^3 the resulting background is the twisted torus or nilmanifold with metric given in (3.31), which is a purely geometric background without any NSNS or RR-flux. The natural radius of the twisted cycle of $\tilde{y}_1$ is given by $\tilde{r}_1 = 1/r_1$ while $\tilde{r}_{2/3} = r_{2/3}$. The potential is now sourced by different contributions

$$V(r_i,\phi_d) = -\widehat{\mathcal{V}}_{\text{int}}^{-1} e^{\frac{4\phi_d}{d-2}}\int \mathrm{d}\tilde{y}_1\mathrm{d}y^2\mathrm{d}y^3\sqrt{h}e^{-2\tilde{\phi}_y}\left(R(h) + 4\partial_i\phi_y\partial^i\phi_y\right) = \frac{k^2}{2}\frac{1}{r_1^2r_2^2r_3^2}e^{\frac{4\phi_d}{d-2}}\,. \tag{7.10}$$

Q-flux background

Dualising a second time, now along the y^2-direction, the resulting configuration is given by (3.35). In Chapter 3 we already discussed that this background is not well-defined as it features a non-trivial monodromy along the fibre y_3 and is actually a non-geometric T-fold. In order to anyway being able to perform the necessary integration and obtain the reduced effective action, we are therefore forced to go to the non-geometric β-frame. Using the relation (3.19) we showed in Chapter 3 that the resulting background reads

$$\begin{gathered} \mathrm{d}\hat{s}^2 = r_1^{-2}\mathrm{d}\tilde{y}_1^2 + r_2^{-2}\mathrm{d}\tilde{y}_2^2 + r_3^2(\mathrm{d}y^3)^2 \,, \qquad \beta = -k y_3 \partial_1 \wedge \partial_2 \,, \\ \hat{\phi}_y = \phi_3 - \log(r_1 r_2) \,. \end{gathered} \tag{7.11}$$

This new frame is indeed well-defined and free of pathological monodromies with a metric in the form of a standard three-torus with radii $\tilde{r}_{1/2} = 1/r_{1/2}$ and $\tilde{r}_3 = r_3$. The potential is now calculated as

$$V(r_i, \phi_d) = -\widehat{\mathcal{V}}_{\text{int}}^{-1} e^{\frac{4\phi_d}{d-2}} \int \mathrm{d}\tilde{y}_1 \mathrm{d}\tilde{y}_2 \mathrm{d}y^3 \sqrt{h} e^{-2\hat{\phi}_y} \left(4\partial_i \hat{\phi}_4 \partial^i \hat{\phi}_4 - \frac{1}{4} Q_i{}^{jk} Q^i{}_{jk}\right) = \frac{k^2}{2} \frac{1}{r_1^2 r_2^2 r_3^2} e^{\frac{4\phi_d}{d-2}} \,, \tag{7.12}$$

such that the non-trivial part of the potential stems from the Q-flux contribution, which no reads

$$Q_3{}^{12} = -Q_3{}^{21} = -k \,. \tag{7.13}$$

The potential again agrees, but *only* after moving to the β-frame.

Overall we observe that (as expected from the discussion in Section 6.2.2) the potentials, even though sourced by very different fluxes, agree for all the backgrounds in the duality chain. The same holds for the field space and the associated metric. However, although this results in equivalent effective actions, the corresponding internal target spaces, their geometric properties, and the nature of light states differ drastically. This is summarised in Figure 7.1.

Zero modes and mass spectra

We list the presence or absence of the relevant zero modes that would give rise to the towers of states for the SDC for each of the three backgrounds separately. Furthermore, we provide a brief discussion of the individual mass spectra. These turn out to be rather difficult to access, and therefore we mainly restrict ourselves to a rather heuristic discussion, pointing out the challenges encountered. This does not affect our analysis in a negative way since already the very existence or rather absence of modes leads to strong implications for the Distance Conjecture.

Zero modes

i) For the torus with H-flux the situation of momentum modes is similar to the case of S^3 with flux. Also here we encounter non-conservation of momentum due to the H-flux. In fact, momentum is conserved only modulo k. Due to the simplicity of the torus geometry one can even pinpoint this non-conservation quite precisely from the perturbative point of view. Employing the dilute flux approximation one can show [269] (cf. Appendix C for more details) that in the presence of non-trivial winding the momentum modes are no longer conserved. In particular there is an obstruction to conserved momenta of the form

$$\vec{p} \times \vec{w} \,, \tag{7.14}$$

with $\vec{p}, \vec{w}$ the winding and momentum mode vectors. In the torus we can have non-trivial winding $w \in \mathbb{Z}$ along each cycle and thus the momentum zero modes are no longer conserved and can not provide the necessary towers for the SDC. On the other hand, there is no obstruction for winding modes and we have $\pi_1(T^3) = \mathbb{Z}^{\oplus 3}$-worth of winding modes [262]. We will argue below that there are infinite-distance limits with no tower of states but also no diverging potential.

ii) Moving to the dual twisted torus we see that the geometry has a non-trivial homology structure [262] (cf. [263] for a in depth discussion)

$$H_1(T^3_{\text{tw}}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k \,. \tag{7.15}$$

In particular the latter factor corresponds to a torsionful cycle and therefore again to a cycle on which a string can unwind after winding $k-1$ times. This provides an alternative argument for the non-conservation of the momentum zero modes in the torus with H-flux without having to employ any perturbative calculation. Note that this is similar to the non-conservation of $D3$ charges in twisted T^6 backgrounds [476][2]. Naively in the absence of any additional flux one would expect the momentum modes on the twisted torus to form a $\mathbb{Z}^{\oplus 3}$ lattice due to the compact nature of the geometry similar to the standard torus. However the geometry is much more involved due to the structure of the non-trivial fibration.

iii) We finally turn to the Q-flux background. The discussion is conceptually more involved and also less understood due to the unclear role of differential calculus on this background. There are two obvious approaches. One can work in the β-frame where the background is well defined and applying the analogue of

[2]Even though the charges are non-conserved they are anyway BPS [476].

standard Riemannian geometry in terms of the metric $\hat{h}$[3]. Alternatively, one can try to infer the relevant behaviour by use of T-duality. We will choose the second option here and basically reverse our previous argumentation. We infer the properties of this background by consistency under T-duality.

Mass spectrum

i) For large values of r we can neglect the flux contribution and therefore the KK-spectrum has the familiar $1/r$ dependence in this corner of field space. Similarly, the winding modes scale with the volume or radius of the wrapped cycle.

ii) Although the twisted torus is only a T-duality away from the previous background, it already features many significantly different aspects. In particular, it was shown in [101] that the Laplacian spectrum displays an unusual gap between some light states and the rest of the spectrum. The spectrum of the low-energy effective theory can then in principle be computed from the eigenvectors of the Laplacian. However, this is an analysis that goes beyond the scope of this section[4] and we will once again infer the spectrum by use of T-duality, cf. Table 7.2.

iii) As already anticipated in the discussion before, the analysis of the non-geometric frame is not only computationally but also conceptually challenging. Although it is possible in principle to treat the well-defined geometry in the β-frame analogously to the standard torus, it is not clear whether one can interpret the resulting mass dependence as the true physical mass. Defining a Q-twisted differential [245, 263, 477]similar to the H-twisted one for the torus, one can perform a Laplacian analysis. At large values one can again neglect the effect of the flux; however, the physical interpretation remains somewhat unclear.

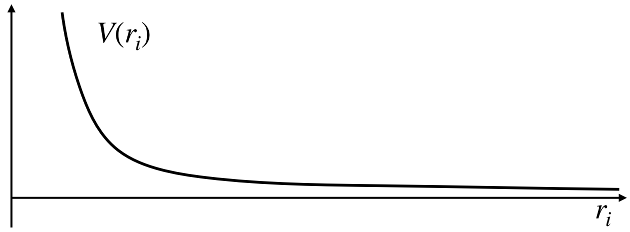
Distance Conjecture

Having collected all the necessary information, we are now in a position to discuss the T-duality chain in light of the SDC. The situation is summarised in Table 7.2. In contrast to the sphere, there are now three independent moduli for every geometry, which correspond to the three independent radii of the three torus r_i for $i \in \{1, 2, 3\}$ as well as the H-flux number k. The infinite distance points again correspond to sending one (or more) of these quantities to either zero or infinity.

The behaviour of the potential for $r_i \to 0$ is fairly similar to the examples encountered here so far. We have a divergence in V for small radii r_i in all the different T-duality

[3] And therefore taking into account the Q-flux, some notion of Q-twisted differential [245, 247, 477].

[4] There are several issues one has to take care of, like for example the appearance of space invaders [98].



		Modes	$r_i \to 0$	$r_i \to \infty$
$T^3_{\mathcal{H}}$, $[\mathcal{H}] = k$	$w:$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	(light, light, light)	(heavy, heavy, heavy)
$\{r_1, r_2, r_3\}$	$p:$	$nc \oplus nc \oplus nc$	(~~heavy~~, ~~heavy~~, ~~heavy~~)	(~~light~~, ~~light~~, ~~light~~)
T^3_{tw}, $[f] = k$	$w:$	$\mathbb{Z}_k \oplus \mathbb{Z} \oplus \mathbb{Z}$	(~~heavy~~,light, light)	(~~light~~,heavy, heavy)
$\{r_1^{-1}, r_2, r_3\}$	$p:$	$nc \oplus nc \oplus \mathbb{Z}$	(light,~~heavy~~, ~~heavy~~)	(heavy,~~light~~, ~~light~~)
T^3_{Q}, $[Q] = k$	$w:$	$\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}$	(~~heavy~~, ~~heavy~~,light)	(~~light~~, ~~light~~,heavy)
$\{r_1^{-1}, r_2^{-1}, r_3\}$	$p:$	$nc \oplus \mathbb{Z} \oplus \mathbb{Z}$	(light, light,~~heavy~~)	(heavy, heavy,~~light~~)

Table 7.2.: The plot illustrates the diverging scalar potential as a function of of the radii r_i, with the others kept fixed. The radii are with respect to the torus with H-flux. The table lists the zero modes and their asymptotic behaviour with respect to the r_i. Crossed ~~expressions~~ denote non-conserved (nc) or only a finite amount of conserved states. For the Q-flux background the expressions are inferred from T-duality.

frames[5]. However, there seems to be no apparent reason for this divergence in terms of missing modes. On the torus, there is no immediate obstruction for the existence of winding modes. This hints at the fact that while the absence of modes for a consistent background seems to imply the emergence of a divergent scalar potential, the converse is not necessarily the case. This aspect requires further investigation elsewhere.

Turning to the limit of large radii r_i, we quickly run into problems. Indeed, looking at Table 7.2 it is apparent that there is no infinite tower of light, *conserved* states and therefore no tower for the SDC. In contrast to the three-sphere with H-flux, where we had only a finite number of states for fixed k and r, now the centre of mass momenta of the string are *all* non-conserved and cannot be interpreted as KK-particles. Only in the strict limit of $r = \infty$ momentum conservation is restored and the tower is massless. However, along the limit of large r we never have proper (particle-like) conserved modes[6]. At the same time, also the potential is regular and even vanishing in this limit. Furthermore, consider the limit of sending both k and one of the radii, say r_1 to infinity, keeping their ratio constant, i.e. k/r_1^2 = const. along the limit. This is clearly

[5]We stress again that that is with respect to the radius r_i associated to the torus with H-flux.

[6]Although we assume in the following that situations like this are not sufficient for the SDC, this aspect deserves a deeper investigation. Even if such a situation turns out to be sufficient, we argue below that there are further infinite-distance limits in the (3D) chain that violate the SDC.

an infinite-distance limit as well; however, since k/r_1^2 are fixed the flux contribution is not even vanishing asymptotically and we never recover conserved momentum states, cf. expansion in Appendix C. Furthermore, the potential stays constant in this limit.

We see that in both cases there is no divergence that would protect us from the absence of states via our proposed amendment to the SDC. We are therefore led to the conclusion [67] that without embedding the solution into a proper string background, i.e. adding fluxes or other ingredients, these backgrounds are in conflict with the SDC and can not arise as the low-energy effective limit of a theory of quantum gravity.

7.3. 6D T-duality chain and locally non-geometric backgrounds

We discuss a natural generalisation of the 3D T-duality chain in 6 dimensions, which builds on the DGKT-like background discussed in Section 6.4. Although not giving a full analysis of all the relevant aspects, we highlight some differences to the 3D example. Furthermore, we give a brief outlook on the last step in the 3D T-duality chain, leading to the *locally* non-geometric R-flux space.

7.3.1. T-duality chain in 6D

The 3D T-duality chain discussed in the last section served as an easy and clear example for discussing the relevant aspects of non-geometric backgrounds for the SDC. However, as mentioned several times, these backgrounds are only toy examples and do not give rise to full string vacua or at least supergravity backgrounds. There is a very natural 6D analogue that can be completed to a supergravity solution and gives rise to a proper 10D T-duality chain, analogous to (7.7). In fact, we already introduced the first background of the chain at the end of Chapter 6, the DGKT-like Type IIA toroidal orientifold with fluxes. We give a brief outlook on how, upon dualising this will lead to a 6D nilmanifold and finally to the higher-dimensional analogue of the Q-flux background and the potential implications for the SDC and Swampland program.

Since these backgrounds involve many more ingredients, the analysis for the SDC becomes significantly more involved. In particular, several things must be taken into account, such as towers of wrapped branes [336,344,478], the role of the smeared orientifold sources [125,389,471,479] or also the issue of scale separation [126] between the internal compactification scale and the external AdS space and the (strong) ADC [363]; see also the list of references in Section 5.2.3. An in-depth analysis of all these points goes way beyond the purpose of this section. The following discussion should be more viewed as an outlook, highlighting some novel features and serving as a guide for further investigation of non-geometric backgrounds within the Swampland program as well as highlighting some novel features.

T^6 with fluxes

This background was introduced in Section 6.4. In contrast to the 3D torus with H-flux we discussed before, this background features not only NSNS-flux but also RR-form flux as well as contributions from (smeared) orientifold sources. In particular, the metric and potential of the reduced action was given by

$$\begin{aligned}\gamma_{ab}\partial_\mu\varphi^a\tilde{\partial}^\mu\varphi^b &= 3r_1^{-2}|\partial r_1|^2 + 3r_2^{-2}|\partial r_2|^2 + 3|\partial\phi_4|^2 + 4\pi^3 r_1 r_2^{-1} e^{2\phi_4(x)}|\partial_\mu \mathfrak{f}_4|^2 + \dots\,, \\ V(\varphi^a,\phi_d) &\sim \mathfrak{h}^2 e^{2\phi_4}\frac{r_1^4+3r_2^4}{r_1^4 r_2^6} - 2\mathfrak{f}_0\mathfrak{h}e^{3\phi_4}\frac{r_1^2-3r_2^2}{\sqrt{r_1 r_2^3}} + e^{4\phi_4}\frac{3\mathfrak{f}_4^2+\mathfrak{f}_0^2 r_1^4 r_2^4}{r_1 r_2} + \dots\,,\end{aligned} \tag{7.16}$$

where we restricted to the real part of the (NSNS sector) scalars and flux variations.

Nilmanifold 4.7

Performing two T-dualities along the directions y^1, y^2 the resulting background is again a nilmanifold (4.7 in the classification of [480])

$$ds_{10}^2 = L^2 \mathrm{d}s^2_{\mathrm{AdS}_4} + \sum_{i=1}^{6}\tilde{r}_i^2 e_i^2\,, \qquad \Phi = \phi - \log(r_1 r_2)\,, \tag{7.17}$$

with e_i a basis of one-forms such that $e_1 = (\mathrm{d}\tilde{y}_1 - (y^6\mathrm{d}y^3 + y^5\mathrm{d}y^4))$, $e_2 = (\mathrm{d}\tilde{y}_2 - (-y^6\mathrm{d}y^4 + y^5\mathrm{d}y^3))$ and $e_i = \mathrm{d}y_i$ for $i = 3,\dots,6$. The radii in this basis are given by $\tilde{r}_1 = \frac{1}{r_1}, \tilde{r}_2 = \frac{1}{r_2}$ and $\tilde{r}_3 = \tilde{r}_5 = r_1$ and $\tilde{r}_4 = \tilde{r}_6 = r_2$. The fluxes are given by

$$H = 0\,, \quad m = 0\,, \qquad F_{12} = \mathfrak{m}\,, \qquad F_{34} = F_{56} = \mathfrak{f}_4\,, \qquad F_{123456} = \mathfrak{f}_4\,, \tag{7.18}$$

which can be easily calculated using the Buscher-like rules for the RR fields given in (2.20), adapted to the massive Type IIA case; see the remark in Appendix A. There is geometric flux but no H-flux. The metric and scalar potential are identical to the ones for T^6 above. In fact, this can be seen very elegantly in the $\mathcal{N} = 1$ formalism, but can be calculated also simply with the reduction formula (6.44) and (6.45). The solution is again completed by $O6$ planes in the directions $(1,3,6)$, $(1,4,5)$, $(2,3,5)$ and $(2,4,6)$.

Q-flux background

After two additional T-dualities along the directions y^3, y^4, the resulting background is the $6D$ analogue of the Q-flux background. After moving to the non-geometric β-frame, in which the resulting background simplifies tremendously, it reads

$$\mathrm{d}s^2_{10} = L^2\mathrm{d}s^2_{\mathrm{AdS}_4}+\sum_{i=1}^{4}\hat{r}_i^2(\mathrm{d}\tilde{y}_i)^2+r_1^2(\mathrm{d}y^5)^2+r_2^2(\mathrm{d}y^6)^2\,, \qquad \hat{r}_1=\hat{r}_3=\frac{1}{r_1}\,,\hat{r}_2=\hat{r}_4=\frac{1}{r_2}\,. \quad (7.19)$$

The NSNS three-form flux is replaced by a non-zero β with components $\beta^{24}=-\beta^{13}=\mathfrak{h}y^6$ and $\beta^{23}=-\beta^{14}=\mathfrak{h}y^5$ resulting in a non-vanishing Q-flux. The RR sector is given by

$$F_{x^3x^4x^5x^6}=F_{x^1x^2x^5x^6}=\mathfrak{f}_4\,, \qquad F_{x^1x^2x^3x^4}=\mathfrak{m}\,, \quad (7.20)$$

together with a non-zero Romans mass $m=\mathfrak{f}_4$. There are $O6$ planes in the directions $(1,3,5),(1,4,6),(2,4,5)$ and $(2,3,6)$. Importantly, similar to the 3D case, the resulting scalar potential is only well defined after moving to the non-geometric β-frame.

Novel features and interesting open questions

Let us highlight some important differences from the 3D toy example and the new aspects to be taken into account for the SDC.

- In the presence of fluxes and sources we do not only have to check the supergravity equations of motion but also the Bianchi identities leading to non-trivial tadpole cancellation conditions. These are highly restrictive and it has been also pointed out in [382] that therefore (on-shell) flux variations are generically possible only in special circumstances, when certain flux quanta are not entering the tadpole conditions. For the T^6 vacuum above, the tadpole conditions reads

$$0=\mathrm{d}F_2=H-j^6\,, \quad (7.21)$$

  where j^6 is the source contribution of the $O6$-planes, cf. [469]. This does not involve the F_4-flux. Furthermore, the theory allows for a full family of vacua labelled by the flux quantum $\mathfrak{f}_4$, which is not constraint by the tadpole condition. On the other hand, H explicitly enters (7.21) and can not be varied arbitrarily on-shell. Hence, the problematic limit of the three torus, sending $k,r_1\to\infty$ while keeping k/r_1^2 fixed, is not allowed, at least not in an on-shell manner.

- There remains the question of the compatibility of the $r_i\to\infty$ limits, for which we encountered a lack of conserved momentum modes. In the presence of RR-fluxes and sources, we cannot perform a simple mode expansion like before, so it is not clear if and in which sense the argument applies in this case. Furthermore, there might be different towers.

- Despite giving rise to (supersymmetric) SUGRA solutions, the validity of the above backgrounds is all but established. First, there is the issue of *smeared* sources and the associated backreaction and control issues when trying to localise them [471,479]. Furthermore, these are prime examples of backgrounds with scale separation, which is in conflict with the *strong ADC*, cf. Section 5.2.3 and [126].

7.3.2. Towards locally non-geometric backgrounds and the SDC?

In the (3D) T-duality chain we have ignored the last step in (7.7), leading to the so called R-flux background. In contrast to the Q-flux background, this is a space that can not even locally be described by Riemannian geometry, therefore called *locally* non-geometric. As mentioned in Chapter 3, these spaces can be better understood from a doubled perspective as such spaces also depend on the dual winding coordinates.

Without going into any details, starting from the 3D Q-flux background and T-dualising along the y^3 direction amounts to viewing y^3 the dual coordinate and $\tilde{y}_3$ the geometric while also inverting the radius, such that the backgrounds reads

$$\begin{gathered} \mathrm{d}\hat{s}^2 = r_1^{-2}(\mathrm{d}\tilde{y}_1)^2 + r_2^{-2}(\mathrm{d}\tilde{y}_2^2 + r_3^{-2}(\mathrm{d}\tilde{y}_3)^2 \,, \qquad \beta = -k y^3 \partial_1 \wedge \partial_2 \,, \\ \hat{\phi}_y = \phi_3 - \log(r_1 r_2 r_3) \,. \end{gathered} \tag{7.22}$$

The dependence on the dual coordinate $\tilde{X}_3 \equiv y^3$ in β in turn leads to a non-vanishing R-flux[7]

$$R^{123} = 3\tilde{\partial}^{[1}\beta^{23]} = -k \,. \tag{7.23}$$

This dependence on the dual winding coordinate renders a standard physical interoperation of the background quite obscure. From a DFT perspective, there is no preferred direction and one could argue that from a stringy point of view one can just as well work in terms of both standard – as well as winding – coordinates (or a mixture of both). However then one should ask about the effective theory of such a configuration after compactification. A naive approach is to indeed treat the coordinate $\tilde{X}_3 = y^3$ just like the other physical coordinates and use our reduction formula for non-geometric backgrounds, now suitably adjusted for the inclusion of the R-flux. In the present case this only modifies the expression for the scalar potential, which now reads

$$\widehat{V}(\varphi^a) = -\widehat{\mathcal{V}}_{\text{int}}^{-1} e^{\frac{4\phi_d}{d-2}} \int \mathrm{d}^3\tilde{y} e^{-2\tilde{\phi}_y} \left\{ R(\hat{h}) - \frac{1}{12} R_{ijk} R^{ijk} + 4(\partial\hat{\phi}_y)^2 + \dots \right\} . \tag{7.24}$$

[7] More precisely we have $R^{ijk} = 3\tilde{D}^{[i}\beta^{jk]}$, cf. [242,252] for details and notation.

Here, dots again denote contractions of $\beta, Q, R, \ldots$ that are zero in the present case. Plugging in the above expressions, we obtain

$$\gamma_{ab}\partial\varphi^a\partial\varphi^b = \sum_i r_i^{-2}\,|\partial r_i| + \frac{4}{d-2}\,|\partial\phi_d|\ , \tag{7.25}$$

and

$$V(r_i,\phi_d) = -\widehat{\mathcal{V}}^{-1} e^{\frac{4\phi_d}{d-2}} \int \mathrm{d}^3\tilde{y}\, e^{-2\tilde{\phi}_y}\left(-\frac{1}{12}R_{ijk}R^{ijk} + 4\partial_i\hat{\phi}_4\partial^i\hat{\phi}_4\right) = \frac{k^2}{2}\frac{1}{r_1^2 r_2^2 r_3^2} e^{\frac{4\phi_d}{d-2}}\ . \tag{7.26}$$

Hence, in treating the winding coordinates as physical for the reduction, we obtain an effective action that is equivalent to the other backgrounds in the duality chain.

Remarks

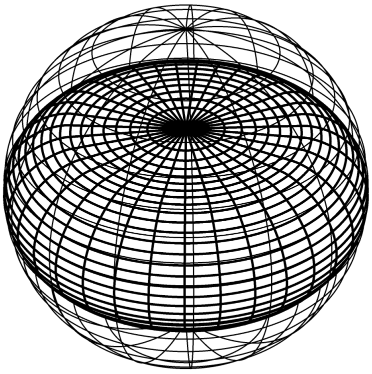
- Flux backgrounds like the ones above, at the level of string theory, can be obtained from intersecting branes, cf. [473]. Non-geometric flux configurations from near-horizon limits of Q- and R-branes and their exotic features were discussed in [481].

- We mentioned before that R-flux spaces are closely related to asymmetric orbifold, which can provide a stringy realisation of these effective theories. Due to their lack of (geometric) target-space description, the metric on field space can in general not be obtained in the standard way by reading off from the kinetic term but from a more formal analysis of the underlying (complex) geometry. See [482] for a first investigation of this kind on the special example of a asymmetric freely acting orbifold.

- Asymmetric orbifolds, at the level of the effective supergravity, are known to give rise to generalised Scherk-Schwarz reductions. Swampland-like criteria on these Scherk-Schwarz reductions and their potential uplift to stringy orbifold constructions were discussed in [483]

- An alternative way to tackle non-geometric spaces and their associated fluxes is the Swampland Tadpole Conjecture [484], which also allows for a discussion of non-geometric backgrounds [485](see also [486]), although most of the existing work has been done for on different notions of non-geometry than the ones discussed here, like Gepner models.

- A further interesting testing ground are Type IIB $T^6/\mathbb{Z}_2$ orientifolds and their T-duals [262]. These backgrounds have been already partially investigated in the context of the SDC in [487] (see also [488]).



CHAPTER 8

Generalised dualities, deformations and the Swampland

It has become clear from the previous chapters that sting dualities are powerful tools for exploring the different infinite-distance points on field space and are in fact tied to the very heart of the Distance Conjecture. When working with curved spaces or fluxes, there is, however, an even bigger family of transformations that can be used to relate points in field space. In the present section, we will consider some of these more general relations that all fall under the term of *generalised T-dualities* and were briefly introduced in Section 2.2.4. We will illustrate the novel aspects with the example of the non-Abelian T-dual of S^3. This example will be particularly interesting as it also connects to the discussion of non-geometric backgrounds of the previous chapter. Furthermore, we will provide an outlook on how to include the currently most general formulation of T-duality, Poisson-Lie T-duality, in the Swampland program, and particularly the SDC. The discussion will be somewhat heuristic and should be taken as a pointer towards possible interesting directions for future investigations. Lastly, we will discuss a particular deformation of the round S^3, which will introduce an interesting playground to check the behaviour of the potential and towers for a direction in field space that does not correspond to a naive volume rescaling but deformation of the geometry.

8.1. Non-abelian T-dual of S^3

As a first step in investigating the role of generalised T-dualities within the Swampland program, we come back to our favourite example of the three-spheres. However, in contrast to before, we will not apply T-duality along some (abelian) isometry but employ non-Abelian T-duality; see Chapter 2. In order to do so, we consider the three-sphere as the non-Abelian Lie group $SU(2)$ and T-dualise with respect to the full $SU(2)$ rather than the Hopf-fibre. As explained in Section 2.2.4, there are different ways to construct non-Abelian T-duals for some given target space geometry. We will focus on the construction through Poisson-Lie T-duality, since this procedure will allow us to circumvent the problem of non-compact Lagrange multipliers and related issues that would arise in the standard approach to NATD, cf. Section 2.2.4. However, in the given example this comes at a price. The resulting space will turn out to be non-geometric, and it will again be crucial to work in the β-frame (cf. Chapter 3) in order to get a consistent picture under (generalised) T-duality for the SDC.

8.1.1. Construction of the non-Abelian dual

In contrast to earlier chapters, we start from the geometry of S^3 without any flux. For reasons that will become clear soon, we denote the non-Abelian T-dual of S^3 by $\widetilde{T}^3$. The construction of the NATD$^{\text{PL}}$ via Poisson-Lie T-duality is via the associated Drinfel'd double, cf. Section 2.2.4. In particular for $SU(2)$ the Drinfel'd double of interest is given by

$$\mathbb{D} = SU(2) \ltimes U(1)^3 \,, \tag{8.1}$$

and the associated Poisson bivectors read

$$\Pi = 0 \,, \qquad \widetilde{\Pi} = \begin{pmatrix} 0 & \theta & -\psi \\ -\theta & 0 & \phi \\ \psi & -\phi & 0 \end{pmatrix} . \tag{8.2}$$

In order to realise the standard geometry of S^3_r we choose $E_0 = r^2 \mathbb{1}_3$ such that the backgrounds read

$$E = \left(E_0^{-1} + \Pi\right)^{-1} = r^2 \mathbb{1}_3 \,, \tag{8.3}$$

$$\widetilde{E} = \left(E_0 + \widetilde{\Pi}\right)^{-1} = \frac{1}{r^2 \mathcal{X}} \begin{pmatrix} r^4 + \phi^2 & \phi\psi - r^2\theta & \theta\phi + r^2\psi \\ \phi\psi + r^2\theta & r^4 + \psi^2 & \theta\psi - r^2\psi \\ \theta\phi - r^2\psi & \theta\psi + r^2\phi & r^4 + \theta^2 \end{pmatrix} , \tag{8.4}$$

with $\mathcal{X} = r^4 + \phi^2 + \psi^2 + \theta^2$ and $\{\phi, \psi, \theta\} \in [0, 2\pi]$ the angular variables parametrising the three independent one-cycles of a topological three torus $\widetilde{T}^3$. The matrices are given in the basis of Maurer-Cartan forms ${L_a}$[1]. Splitting $E, \widetilde{E}$ into symmetric and antisymmetric parts in order to obtain the metric G and B-field we obtain (now in coordinate basis)

$$G = r^2 \begin{pmatrix} 1 & 0 & -\cos(\eta) \\ 0 & 1 & 0 \\ -\cos(\eta) & 0 & 1 \end{pmatrix}, \qquad B = 0\,, \tag{8.5}$$

$$\widetilde{G} = \frac{1}{r^2 \mathcal{X}} \begin{pmatrix} r^4 + \phi^2 & \phi\psi & \theta\phi \\ \phi\psi & r^4 + \psi^2 & \theta\psi \\ \theta\phi & \theta\psi & r^4 + \theta^2 \end{pmatrix}, \qquad \widetilde{B} = \frac{1}{\mathcal{X}} \begin{pmatrix} 0 & -\theta & \psi \\ \theta & 0 & -\phi \\ -\psi & \phi & 0 \end{pmatrix}. \tag{8.6}$$

The metric for G is given in the basis $\{\eta, \xi_1, \xi_2\}$ with ranges as in (2.17) and, therefore, is simply the standard round metric on S^3. In order to establish the full duality, we need to add a dilaton of the form [198]

$$\Phi = \Phi_0 + \frac{1}{4}\log(G) = \frac{3}{2}\log(r)\,, \tag{8.7}$$

where G is to be understood in the algebra basis. The dual dilaton then reads

$$\widetilde{\Phi} = \frac{1}{4}\log\left(\widetilde{G}\right). \tag{8.8}$$

8.1.2. Non-geometry of NATD$^{\mathrm{PL}}$ S^3

The dual background (8.6) is parametrised by three angular coordinates and topologically describes a three-torus. However, it is clear that the metric and B-field do not respect the periodicity of the angular coordinates which are clearly spoilt by the factor $1/\mathcal{X}$ and are therefore ill-defined. As was already explained, this is a strong hint of non-geometry. Indeed, the geometry can be thought of as a T-fold [260]. This is clear when looking at the monodromy of $\mathcal{H}$, revealed under the shift $\varphi^i \to \varphi^i + 2\pi$

$$\mathcal{H}_{IJ}(\phi + 2\pi, \psi + 2\pi, \theta + 2\pi) = M_I{}^L \mathcal{H}_{LK}(\phi, \psi, \theta) M^K{}_J\,, \tag{8.9}$$

with monodromy matrix

$$M^K{}_L = \begin{pmatrix} I_3 & P \\ 0 & I_3 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 2\pi & -2\pi \\ -2\pi & 0 & 2\pi \\ 2\pi & -2\pi & 0 \end{pmatrix}. \tag{8.10}$$

Comparing with the form of a general $O(d,d)$ transformation, see for example [134], we see that this can not be interpreted as a coordinate transformation or gauge transformation of the Kalb-Ramond two-form B. The geometry, when viewed globally, is glued using T-duality transformations rather than standard gauge or diffeomorphism

[1] Since the dual space is just $U(1)^3$ this basis and the standard coordinates basis coincide.

transformations. The background should hence be viewed as a non-geometric T-fold and so the appropriate framework to further analyse the geometry in view of the SDC is the β-frame. It will again be crucial for the consistency under generalised dualities to work within the β-frame and not with the standard NSNS quantities G, B.

8.1.3. NATD$^{\mathrm{PL}}$ in β-frame

We adopt the notation of the previous sections and denote the internal manifold metric by h and the Kalb-Ramond field B. The dual geometry is denoted by $\widetilde{g}, \widetilde{B}$. Switching to the non-geometric frame, we denote the dual simply by $\hat{h}$ (and not $\hat{\widetilde{h}}$) and the bivector by β. Then the background in the β-frame has the particularly simple form

$$\hat{h} = R^{-2}\mathbb{1}_3\,, \quad \beta = \begin{pmatrix} 0 & \theta & -\psi \\ -\theta & 0 & \phi \\ \psi & -\phi & 0 \end{pmatrix}. \tag{8.11}$$

The resulting background in this frame drastically simplified and more importantly is perfectly well-defined and respects the periodicity of the coordinates. Furthermore in this frame it seems reasonable to view the geometry as a three-torus with radius $\hat{r} = 1/r$ and, therefore, denote it by $\hat{T}^3$. Plugging β into the definition of the Q-flux gives

$$Q_1^{\ 23} = 1\,, \tag{8.12}$$

such that the space is supported by one unit of Q-flux. Lastly, the dilaton in the β-frame is (cf. Chapter 3)

$$\hat{\Phi} = \frac{1}{4}\ln(\hat{h}) = -\frac{3}{2}\ln(r)\,. \tag{8.13}$$

Remark

- Working in the framework of Poisson-Lie T-duality has the advantage of obtaining a non-Abelian T-dual of S^3 that is again compact. In contrast, the standard NATD geometry obtained by a Lagrange multiplier procedure gives a non-compact geometry, which in turn is hard to make sense of in view of the SDC but also for holography. However, in [489] a holographic completion of the NATD of S^3 was constructed, starting from a $AdS_3{\times}S_3{\times}T^4$ background. Longing for a well-defined dual CFT, the problematic non-compact direction is segmented into an infinite series of intervals, which then in turn are periodically identified. Although the resulting geometry is effectively compact, it is not obvious how to compare or even match these two forms of the NATD. A first step to connect the two approaches was taken by the author and collaborators in [67], Appendix D, and we refer to the original publication for details.

8.1.4. Scalar potential and metric on field space for NATD$^{\text{PL}}$

After having obtained the relevant internal geometries, we can now move on to the discussion in view of the SDC. For this we first calculate the metric and potential on field space and confirm that indeed the expressions stay invariant. For the NATD$^{\text{PL}}$ it will be crucial to work with the non-geometric formulation, as otherwise we do not obtain perfect agreement of the scalar potential.

Three-sphere S^3

Since our internal manifold before dualisation is simply the standard S^3 without any H-flux, the metric and potential read

$$\gamma_{rr} = \frac{3}{r^2}\,, \qquad V(r,\phi_d) = -\frac{3}{2r^2} e^{\frac{4\phi_d}{d-2}}\,. \tag{8.14}$$

NATD$^{\text{PL}}$ of S^3

Having derived the general reduction formula also for the non-geometric setting in the previous chapter it is now easy to obtain the metric and potential on field space. Plugging the calculated functions into (7.3) we obtain the following result

$$\begin{aligned} \gamma_{rr} &= \frac{3}{r^2}\,, \\ V(r,\phi_d) &= -\widehat{\mathcal{V}}^{-1} e^{\frac{4\phi_d}{d-2}} \int \mathrm{d}\hat{y}\sqrt{\hat{h}} e^{-2\hat{\phi}_y} \left\{ -\frac{1}{4} Q^2 - \frac{1}{2} \hat{h}_{ij} Q_k{}^{lj} Q_l{}^{ki} \right\} = -\frac{3}{2r^2} e^{\frac{4\phi_d}{d-2}}\,. \end{aligned} \tag{8.15}$$

Note that in this example there is an additional term as opposed to the standard Q^2 term. In particular, it can be checked that this background does not meet the simplifying assumption $\beta^{km}\partial_m(\cdot) = 0$, cf. Chapter 3.

More importantly, we see that the scalar potential agrees with that of the NATD$^{\text{PL}}$ S^3. This would have failed if we had not moved to the β-supergravity formulation. Indeed, there would have been a total derivative term that spoils the equivalence. A detailed discussion of this boundary term can be found in the original publication [67], Appendix E. The scalar potential also follows the general scaling behaviour of [240] mentioned in Section 7.1.1 both for S^3 where the contribution comes from geometric flux, as well as for the NATD$^{\text{PL}}$ where the non-geometric Q-flux sources the potential.

8.1.5. Zero modes, mass spectrum and the SDC

We have already examined the spectrum of the three-sphere in great detail in three previous chapters. Therefore, we mainly focus on an analysis of the NATD geometry and the mapping under duality.

Here, the framework of Poisson-Lie T-duality is not only useful for obtaining a compact non-Abelian T-dual that fits within the expectations of a duality within the SDC but also provides a generalised rule [490] to track momentum and winding exchange within these backgrounds, cf. Appendix B and the remark in the next section.

Zero modes

Using the generalised momentum and winding exchange rule reviewed in Appendix B the dualisation with respect to $SU(2)$ gives

$$S^3 \cong SU(2) : w = 0 \text{ and } n \in \mathbb{Z}^{\oplus 3} \longleftrightarrow SU(2)_{\mathrm{NATD}^{\mathrm{PL}}} : w \in \mathbb{Z}^{\oplus 3} \text{ and } n = 0\,. \tag{8.16}$$

While there cannot be nontrivial winding on the three-sphere, there are three copies of infinite towers of quantised momentum modes. Applying non-Abelian T-duality with respect to the full $SU(2)$, the dual geometry is topologically a three torus and therefore has three non-contractible cycles which can give rise to three infinite towers of winding states. More surprisingly, the geometry does not feature any quantised momentum modes at all. This is not only in perfect agreement with the expectations from T-duality of exchanging winding and momentum modes, but can also be understood directly from the geometry. While S^3 has non-Abelian isometries, the dual geometry has no isometries left and, therefore, no apparent conserved momentum states.

Mass spectrum

Having discussed the S^3 in Chapter 6 we directly turn to the NATD$^{\mathrm{PL}}$ geometry. Since the space should be viewed as a *non-Abelian* T-fold we face the same question as in the previous chapter. The background in the standard frame is quite involved and even ill-defined. At the same time the physical meaning of computing the mass spectrum in the non-geometric frame is not completely clear. The equations of motion in the β frame are very similar to the standard torus with H-flux and therefore we expect the momentum states to be non-conserved and scaling with $\hat{r}^{-1}$ and the winding states with $\hat{r}$. With this we indeed conclude that now for $\hat{r} \to 0$ ($r \to \infty$) the winding states become light and provide the infinite tower of states, while the non-conserved momentum modes correspond to the missing tower of winding modes on S^3. The former problem of finding the (Laplacian) spectrum of the NATD$^{\mathrm{PL}}$, i.e. the masses of the KK modes, has been explored in [491]. This was done in the context of the holographic background $AdS_3 \times S^3 \times T^4$, however within the standard approach to non-Abelian T-duality. It was found that at least for large radius $r \to \infty$ the spectrum can be matched to the one of the S^3. Hence at least in this regime the spectrum is consistent under NATD$^{\mathrm{PL}}$.

Background	Potential V	Obstructed modes	Frame
S^3 , $[H] = 0$	$-\frac{3}{2r^2}$	absence of winding	NSNS
NATD$^{\text{PL}}$ of S^3 , $[Q = 1]$	$-\frac{3\hat{r}^2}{2} = -\frac{3}{2r^2}$	non-cons. momentum	β

Table 8.1.: Relevant data of the S^3 and its non-Abelian T-dual. The last column highlights the frame in which the geometry is well-defined. The natural radius $\hat{r}$ of the dual geometry is related to the radius r of the sphere via $\hat{r} = 1/r$.

The relevant properties of S^3 and NATD$^{\text{PL}}$ for the SDC are summarised in Table 8.1. The three-sphere and its NATD fit nicely into the expectations set by the SDC. However, only once the scalar potential and the amendment to the SDC presented in Section 6.1 are taken into account. Indeed, also for the NATD$^{\text{PL}}$ there is a lack of tower of states[2].

8.2. Towards genuine Poisson-Lie T-duality

Motivated by the previous section, it is natural to ask how generalised dualities, in particular proper Poisson-Lie T-duality, fit into this context of the SDC. However, in contrast to the special case of taking the Drinfel'd double to be $\mathfrak{d} = G \times U(1)^{\dim G}$ and hence NATD via PL T-duality, a generic Drinfel'd double associated to G, is not compact and, apart from special cases, the duality maps solutions of the standard supergravity equations of motion to solutions of generalised SUGRA [492, 493], which even lack a description in terms of a d-dimensional action functional [198].

8.2.1. Invariance of metric and potential under PL T-duality

For the reasons mentioned above, in the following we restrict the discussion to the case of NATD$^{\text{PL}}$ with G compact and $\widetilde{G} = U(1)^{\dim(G)}$.

Metric on field space

Specialising to NATD$^{\text{PL}}$, we can work with the standard supergravity formulation and use the formula we obtained in Section 6.2.1. Recall that it was given in terms of the generalised metric as

$$\gamma_{ab} = \frac{1}{2}\text{tr}(\mathcal{H}^{-1}\partial_a\mathcal{H}\mathcal{H}^{-1}\partial_b\mathcal{H}) \,. \tag{8.17}$$

Similarly to abelian T-duality, Poisson-Lie T-duality can be understood [494] as a $O(d,d)$ transformation at the DFT level. However, there is the crucial difference that the

[2]However, note that we are again considering a toy example, which without the addition of further fluxes, etc. does not define a proper supergravity or string vacuum

generalised vielbein and hence, also the $O(d,d)$ elements are not constant anymore but depend on the left-invariant forms or coordinates of the space. This is not a problem since the derivative in (8.17) is with respect to the scalar fields or the external coordinates via the chain rule respectively. Therefore the only problematic contributions which would spoil an invariance can arise if the transformation matrix depends on the scalar fields. However, the way the duality is defined we argue that this does not happen. For this first note that the duality is defined at the level of the underlying group manifold. Moduli or scalar fields arise however at the level of the geometry. They will therefore enter through the constant element E_0. For example, consider the sphere S^3. The canonical bi-invariant metric is inherited from the Killing form of the underlying algebra together with the left invariant forms. In particular, up to normalisation one can write the metric in a local coordinate basis as $h_{ij} = h_{ab}e^a{}_i e^b{}_j = \langle T_a, T_b\rangle e^a{}_i e^b{}_j$ with $\langle\cdot,\cdot\rangle$ the ad-invariant inner product on the algebra. The $e^a{}_i$ are the components of the left-invariant vector fields obtained from the left invariant forms via $\sigma = e^a{}_i T_a \mathrm{d}x^i$. For $SU(2)$ we have $\langle T_a, T_b\rangle = 4\,\mathrm{Tr}\{T_a T_b\}$ and it is clear that the notion of radius r of the geometric sphere simply corresponds to some choice of general normalisation κ of the inner product. Now consider some deformation of the sphere, for example the so-called squashed sphere with metric $g_{ij} = \lambda^1\sigma_1 + \lambda^2\sigma_2 + \lambda^3\sigma_3$. This corresponds to an anisotropic inner product $\langle T_a, T_b\rangle = \kappa_{ab}$. From the definition of the PL symmetric background it is however clear that the radius r or the anisotropic deformation κ_{ab} enter into the definition of E_0. A similar argument holds for B, the parameters of which also enter into E_0. Therefore E_0 is the only quantity carrying a dependence on possible moduli φ and we can again use the $O(d,d)$ invariance to show that the metric on field space is invariant under $\mathrm{NATD}^{\mathrm{PL}}$ and hence also standard non-Abelian T-duality. This explains why the metric on field space stayed invariant in the previous example.

Potential on field space

In the abelian case the invariance of the potential under T-duality is immediate when moving to DFT. For the more general case at hand, this does not carry over directly. As mentioned previously, the $O(d,d)$ transformation that implements the PL T-duality at the level of the DFT action is field dependent. Therefore, it is all but obvious that the action is invariant under this transformation. However, the potential, after going to the appropriate frame, remained invariant under non-Abelian T-duality. Together with the example below, this hints towards such an invariance, but a formal proof is missing[3]. We leave this question as well as the more general setting for future research.

[3] In principal one can simply write out the model directly at the level of the EFT and compare it with the dual model. However, this is very tedious and an alternative approach seems much more inviting.

Remark

- The exchange of winding and momentum states in PL-symmetric models can be elegantly described [490] in terms of the theory on the double $\mathbb{D}$, see also Appendix B. In particular, if the double is given as $\mathbb{D} = G \bowtie \widetilde{G}$ the exchange simplifies to

$$\overbrace{\pi_1(\mathbb{D} = G \bowtie \widetilde{G})}^{\text{winding in } \mathbb{D}}$$
$$\swarrow \qquad\qquad \searrow$$
$$\underbrace{\pi_1(G)}_{\text{winding in } G} \oplus \underbrace{\Big(\pi_1(\widetilde{G})/\pi_2(G)\Big)}_{\text{gen. momentum in } G} \qquad\qquad \underbrace{\pi_1(\widetilde{G})}_{\text{winding in } \widetilde{G}} \oplus \underbrace{\Big(\pi_1(G)/\pi_2(\widetilde{G})\Big)}_{\text{gen. momentum in } \widetilde{G}}$$

For the example of the NATD of S^3 treated in the previous paragraphs, the Drinfel'd double is simply $\mathbb{D} = SU(2) \ltimes U(1)^3$ such that we obtain the winding and momentum exchange summarised in equation (8.16).

8.3. Dualities and deformations

T-dualisable theories need to satisfy very specific conditions and are therefore often highly symmetric. While for generalised T-duality on can greatly loosen the necessary requirements, the underlying (algebraic) structure is nevertheless very rich. Hence, it is not surprising that many of these theories also allow for well-behaved deformations. These generically break even more of the remaining symmetries while keeping the theory under computational control. For us, these deformations are particularly interesting, since they offer a convenient way to introduce new scalar fields corresponding to these geometric deformations so that we can study their implications for the SDC. In particular, we can further examine our proposed amendment to the SDC in the presence of a potential, which will be the main purpose of this section.

8.3.1. The η-deformation of S^3 and its NATD$^{\mathrm{PL}}$

In the previous section, we have looked at the NATD$^{\mathrm{PL}}$ of S^3, which, on top, turned out to be a non-geometric space with Q-flux. The example was therefore not only interesting for studying the role of generalised T-duality within the Swampland program but also the interplay with non-geometry. In the following we will see that the pair of backgrounds even allows for certain deformation that preserved the duality and at the same time provides us with an additional interesting deformation that we can in turn view as an additional scalar field of our effective theory after reduction.

Background expressions

Without going into too many details here, the η-deformation of the $SU(2)$ background corresponds to choosing the constant background matrix E_0 as

$$E_0 = \left(r^{-2}\mathbb{1}_3 - \eta\mathcal{R}\right)^{-1} , \tag{8.18}$$

with $\mathcal{R}$ the associated r-matrix of the model (cf. [136]) and η the deformation parameter. Plugging this into the formulas for E and $\widetilde{E}$ with $\Pi, \widetilde{\Pi}$ as before and separating the metric and Kalb-Ramond parts, we obtain (in the coordinate frame)

$$G = \frac{r^2}{4}\begin{pmatrix} \frac{1}{1+r^4\eta^2} & 0 & 0 \\ 0 & \cos(\zeta)^2 + \frac{\sin(\zeta)^2}{1+r^4\eta^2} & \cos(\zeta) \\ 0 & \cos(\zeta) & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{r^4\eta\sin(\zeta)}{8+8r^4\eta^2} & 0 \\ -\frac{r^4\eta\sin(\zeta)}{8+8r^4\eta^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8.19}$$

while for $\widetilde{G}$ and $\widetilde{B}$ we again find ill-defined expressions that suffer from non-trivial monodromy. However, in the β-frame, we obtain the very simple expressions

$$\widetilde{G} = \begin{pmatrix} \frac{1}{r^2} + r^2\eta^2 & 0 & 0 \\ 0 & \frac{1}{r^2} + r^2\eta^2 & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 & \frac{r^4\eta}{1+r^4\eta^2} + \theta & -\psi \\ -\frac{r^4\eta}{1+r^4\eta^2} - \theta & 0 & \phi \\ \psi & -\phi & 0 \end{pmatrix}. \tag{8.20}$$

Furthermore, note that $\mathrm{d}B = 0$ and therefore B is pure gauge[4].

Metric and scalar potential

We have already argued on general grounds that the metric and potential on field space are indeed invariant under NATD$^{\mathrm{PL}}$. One can easily verify this for the present example and obtain in either case

$$\begin{aligned} \gamma_{ab}\partial\varphi^a\partial\varphi^b &= 3r^{-2}\,|\partial r|^2 + \frac{1}{2}r^4\,|\partial\eta|^2 + \frac{4}{d-2}\,|\partial\phi_d|^2 \,, \\ V(r,\eta,\phi_d) &= -\frac{6+4r^4\eta^2-2r^8\eta^4}{r^2}e^{\frac{4\phi_d}{d-2}} \,. \end{aligned} \tag{8.21}$$

After canonical normalisation the metric reads

$$\gamma_{ab}\partial\varphi^a\partial\varphi^b = |\partial\rho|^2 + \frac{1}{2}e^{4/\sqrt{3}\rho}\,|\partial\eta|^2 + \frac{4}{d-2}\,|\partial\phi_d|^2 \,, \tag{8.22}$$

with $\rho = \sqrt{3}\log(r)$. From this we see that, as usual, there are two infinite-distance points for the radius, associated with $r \to \infty$ and $r \to 0$. However, for the deformation

[4]Since $H^2(S^3, \mathbb{R}) = 0$ any closed form on S^3 is exact and therefore pure gauge.

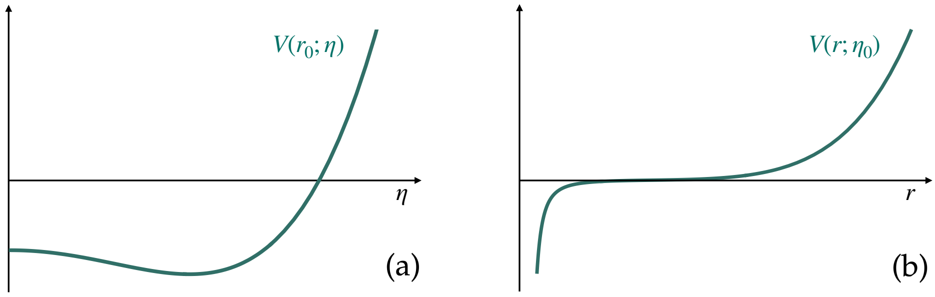


Figure 8.1.: Plotted is the potential of the deformed three-sphere with radius r and deformation parameter η. The left figure (a) shows the potential as a function of η with $r = r_0$ fixed while the right figure (b) depicts the potential as a function of the radius r with fixed $\eta = \eta_0$.

parameter η, we see that only $\eta \to \infty$ leads to an infinite distance, while $\eta \to 0$ is at finite distance. This was actually to be expected, considering the physical meaning of the deformation. The limit $\eta \to 0$ corresponds to a vanishing deformation, and hence in this limit we go back to the unperformed sphere. In contrast, $\eta \to \infty$ corresponds to a "infinitely" deformed sphere, in which the sphere gets squashed more and more until it formally becomes a 2 dimensional disk. While in the former limit we do not expect any breakdown of the EFT and therefore no towers, the latter clearly corresponds to a drastic limit in which, at least naively, we expect a tower of states.

Comparing (8.21) with the expression for the undeformed sphere $V = -6r^{-2}e^{\frac{4\phi_d}{d-2}}$, we see that the behaviour of the potential is drastically modified. Although for $\eta = 0$, the potential vanished for a large radius, we now encounter a divergence for any $\eta \neq 0$. Furthermore, since we are dealing with a higher-dimensional field space, we can also take the limits of small and large η. While for $\eta \to 0$ we simply go back to the potential of the standard undeformed sphere, for large η the potential again diverges. This behaviour is illustrated in Figure 8.1.

Distance Conjecture

The picture we obtain in view of the SDC fits very nicely with our proposed amendment to the SDC. Introducing the deformation parameter η leads to a new direction in scalar field space. The limit $\eta \to 0$, which corresponds to the limit of the round S^3, lies – for constant and finite r – at finite distance and comes with a finite scalar potential. Indeed, we do not expect any breakdown of the associated EFT in this limit and therefore also no tower of light states. In contrast, $\eta \to \infty$ corresponds to an infinite-distance point. Since η measures the deformation of the sphere, this corresponds to a infinitely squashed sphere that effectively becomes a disk. We do not have any tower to our disposal that could arise in this limit[5]. This problematic behaviour is resolved by the divergence of the associated scalar potential, signalling the absence of the tower of states.

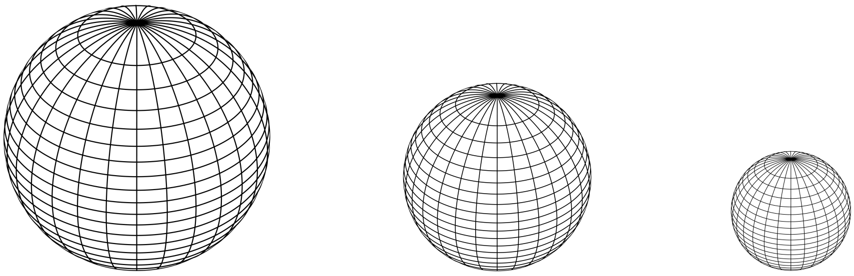
The interpretation for the radius is somewhat more involved. For any $\eta \neq 0$, the deformation breaks at least some of the isometries of the sphere and hence naively due to the breakdown of momentum conservation one might expect the absence of a tower of states. Note that the deformation of the geometry is purely along the S^2 base. Indeed, we can write

$$\mathrm{d}s^2 = \frac{1}{1+r^4\eta^2}\frac{r^2}{4}\left(\mathrm{d}\zeta^2 + \sin(\zeta)^2\mathrm{d}\xi_1^2\right) + \frac{r^2}{4}\left(\cos(\zeta)\mathrm{d}\xi_1 + \mathrm{d}\xi_2\right)^2 , \tag{8.23}$$

from which it is apparent that the Hopf fibre does not get deformed. Hence in fact this suggests that we still have a tower of KK modes associated with the circle fibre. In fact, this highlights an aspect that we have already stressed before. While for consistent backgrounds with missing towers, we argued that there emerges a scalar potential signalling the absence of the tower, the converse seems not necessarily true. In particular, in general it is not clear if a diverging scalar potential in some direction necessarily implies the absence of a tower of states in that very direction or some other kind of inconsistency[6]. In any case the behaviour is consistent with the SDC.

[5]Even if winding modes could potentially serve as towers in such a limit, which is not immediately clear, they are anyway absent for S^3.

[6]Note however the possible connection to Dynamical Cobordism and the Cobordism Distance Conjecture mentioned in the remark in Section 6.1.4.



CHAPTER 9

String theory field space and the generalised Ricci Flow Conjecture

The Swampland Distance Conjecture was originally formulated as a statement on moduli space. Starting from the lift of the conjecture to the space of metrics in terms of the Generalised Distance Conjecture [363] the authors of [381] were led to the Ricci Flow Conjecture. Inspired by the fact that Ricci flow determines an evolution of the spacetime metric and therefore effectively varies the underlying geometry we propose an alternative way to explore the scalar field space of string compactifications by applying this logic to the internal geometry. We study how by evolving the internal geometry under the flow, infinite-distance points in field space can be reached. Aiming to take into account also fluxes, we are naturally lead to consider generalised Ricci flow. This allows us to incorporate fluxes in a mathematically well-defined and easy manner. Hence by lifting the discussion from scalar field space back to the underlying geometry we do not have to face the potential directly. Instead, we obtain a very intuitive geometric evolution dictated by the flow and can measure the length along the paths using the associated entropy functional. This geometric path is then translated into a path on scalar field space. After defining an appropriate measure of length for generalised Ricci flow, we revisit the example of the three-sphere with H-flux and obtain a different viewpoint of the problem of the potential. The resulting picture is in perfect agreement with the conclusion drawn before, provided that we slightly extend the original Ricci Flow Conjecture in order to also take into account the flux contributions.

9.1. Geometric flows on string theory field space

Consider the low-energy effective theory of strings moving on a curved D-dimensional background. As explained in Chapter 1, restricting to the bosonic NSNS sector and allowing for non-trivial three-form flux H and dilaton ϕ, the theory is described by the string effective action

$$\mathcal{S}^{(D)} = \frac{1}{2\tilde{\kappa}_D}\int_{M_D} \mathrm{d}^D x\, \sqrt{-G}\, e^{-2\Phi}\left(R(G) - \frac{1}{2}|H|^2 + 4|\nabla\Phi|^2\right) . \tag{9.1}$$

Similarly to previous chapters, we assume that the total space M_D is the direct product $M_D = M_d \times K_n$ of some external d-dimensional space M_d and a compact internal manifold K_n, such that the metric can be split[1] as $G = \hat{g} \oplus g$. Assuming that also H respects this product structure, the action is factorised as

$$\begin{aligned}\mathcal{S}^{(D)} = \frac{1}{2\tilde{\kappa}_D}\int_{M_D} \mathrm{d}^D x\, \sqrt{-\hat{g}g}\, e^{-2\hat{\phi}-2\phi}\Bigg[&\left(R(\hat{g}) + 4|\nabla\hat{\phi}|^2\right) + \\ &+ \left(R(g) - \frac{1}{2}|H|^2 + 4|\nabla\phi|^2\right)\Bigg] ,\end{aligned} \tag{9.2}$$

where henceforth H is assumed to have legs only in the internal space and we have split the dilaton as $\Phi = \hat{\phi} + \phi$ with $\hat{\phi}$ depending at most on the external coordinates.

Proceeding with the usual compactification procedure discussed in the previous chapters, the second (internal) part would give rise to the metric and potential on the scalar field space for the resulting d-dimensional effective theory. This was discussed in detail in Sections 1.2 and 6. Here we want to take a different route and work directly with the internal part of the action. In particular, we isolate the action functional that captures the internal part of the geometry

$$\mathcal{S} = \int_K \mathrm{d}^n x\, \sqrt{g} e^{-2\phi}\left(R(g) - \frac{1}{2}|H|^2 + 4|\nabla\phi|^2\right) . \tag{9.3}$$

Upon supplementing $\mathcal{S}$ with the generalised unit volume constraint (4.4) this is nothing else than the generating functional of generalised Ricci flow we encountered in Section 4.1.4, and this will be the basis for the considerations in the rest of this chapter.

[1] Note, that in contrast to pervious chapters we denote the internal metric with g and not h in order to match the standard notation in the geometric flow literature; coordinates are denoted by x. There should be no confusion, since most of our discussion will focus solely on the internal geometry.

9.1.1. Generalised Ricci flow and the space of generalised metrics

Recall that upon identifying $f = 2\phi$ and supplementing the action (9.3) with the generalised unit volume constraint

$$\int_K e^{-f} \mathrm{d}V_g = 1\,, \tag{9.4}$$

defines a gradient flow with flow equations [293, 295]

$$\begin{aligned} \frac{\partial}{\partial s} g_{ij} &= -2R_{ij} + \frac{1}{2}H^2_{ij} - 2\nabla_i \nabla_j f\,, \\ \frac{\partial}{\partial s} B_{ij} &= \nabla^k H_{kij} - H_{kij}\nabla^k f\,, \\ \frac{\partial}{\partial s} f &= -R + \frac{1}{4}|H|^2 - \Delta f\,. \end{aligned} \tag{9.5}$$

Here and in the following, we use the notation $\mathrm{d}V_g = \mathrm{d}^n x\sqrt{g}$. The right-hand side of the flow equations for g and B is then nothing more than the Weyl anomaly coefficients $\hat{\beta}^{g/B}$ of the associated NLSM, i.e. $\partial_s g = 2\hat{\beta}^g, \partial_s B = -2\hat{\beta}^B$. The interpretation of f, as explained in Section 4.1.4, is less straightforward and we will return to this point at several instances later. Choosing a suitable diffeomorphism, f can be decoupled from the other flow equations; see Section 4.1.4, in particular Eq. (4.29).

The evolution of a given background $\{g, B, f\}$ under generalised Ricci flow defines a one-parameter family of metrics, Kalb-Ramond fields, and dilaton fields. In fact, in Section 4.1.4 we explained how in the case of the metric this can be viewed as defining a path in the (abstract) space $\mathcal{M}(M)$ of Riemannian metrics on the manifold M. This directly carries over to the combined system $\{g, B, f\}$ evolving under generalised Ricci flow, where the path is now in the space of generalised Riemannian metrics on M, denoted $\mathcal{GM}(M)$ and the configuration space of the dilaton $\mathcal{M}_f(M)$

$$\mathbb{R} \to \mathcal{GM}(M) \times \mathcal{M}_f(M) \cong \mathcal{M}(M) \times \Omega^2(M) \times \mathcal{M}_f(M): \quad s \mapsto (\mathcal{G}(s), f(s))\,. \tag{9.6}$$

Every point along this trajectory corresponds to a generalised metric $\mathcal{G}(g, B)$ (cf. (4.32)) or a point in $\mathcal{GM}$ together with a point in $\mathcal{M}_f$ or dilaton configuration, respectively. Hence, this is the natural generalisation of the geometric scenario described in Section 4.1.2. As already mentioned several times, there is a subtle difference between the string-theoretic dilaton and the scalar field f. We ignore this for the moment and return to an in-depth discussion in Sections 9.2.1 and 9.4. It turns out [495] that the appropriate metric on $\mathcal{GM}(M)$ is still given by the DeWitt metric $\mathcal{G}^{\mu\nu\alpha\beta}$, which was introduced in (4.12). Therefore, distances in $\mathcal{GM}(M)$ are measured according to the

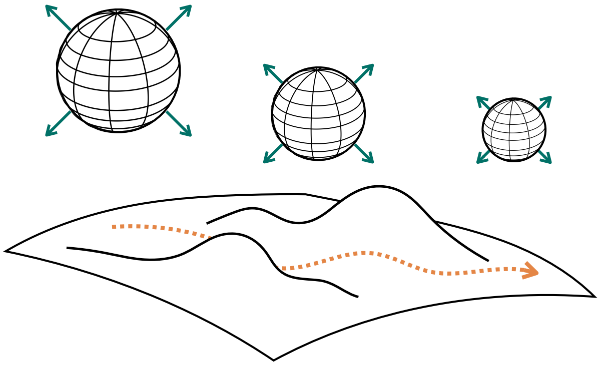


Figure 9.1.: Ricci flow evolves and modifies the (internal) geometry and hence also the associated parameters, which after compactification appear as moduli or, more general, scalar fields. Instead of directly defining a path on field space in the presence of a potential – sourced by the internal geometry and fluxes – we follow the path that is descending from the geometric evolution dictated by (generalised) Ricci flow.

length formula (c a constant)

$$\begin{aligned} L(\gamma) &= c \int_\gamma \mathrm{d}s \sqrt{\int_M \mathrm{d}^D x \sqrt{g}\, \mathcal{G}_g^{\mu\nu\alpha\beta} \left(\frac{\mathrm{d}g_{\mu\alpha}}{\mathrm{d}s} \frac{\mathrm{d}g_{\nu\beta}}{\mathrm{d}s} + \frac{\mathrm{d}B_{\mu\alpha}}{\mathrm{d}s} \frac{\mathrm{d}B^T_{\nu\beta}}{\mathrm{d}s} \right)} \\ &= c \int_\gamma \mathrm{d}s \sqrt{\int_M \mathrm{d}^D x \sqrt{g} \left(\|\partial_s g\|_g^2 + \|\partial_s B\|_g^2 \right)} \,. \end{aligned} \tag{9.7}$$

Geometric flows select paths in the space of (generalised) metrics according to an underlying geometric principle that is formulated in terms of a differential equation. In the case of generalised Ricci flow, formulated as a gradient flow of the Perelman functional, this underlying principle descends directly from the functional $\mathcal{S}$ and the additional constraint. Therefore, a generalised Ricci flow enables us to probe the space of generalised metrics. For true moduli spaces, i.e. in the absence of a potential, there is a distinguished path along which the distance for the SDC should be measured, namely the geodesics. In the presence of a potential, it is no longer obvious that the paths should be measured along geodesics; see discussion in Section (5.3). In the following, we will argue that in these situations there are alternative guiding principles that can select a meaningful path taking into account the effect of the non-trivial potential. In particular, we will work with Ricci flow in order to make progress in the exploration of scalar field spaces with potentials arising from compactifications of higher-dimensional theories. Hereby, the main observation is that, utilising geometric flows, there is no need to work directly on scalar field space. Instead, we can lift the discussion to a purely geometric picture. Working directly at the level of the geometry and fluxes, the (generalised) Ricci flow evolves these quantities and, for some given initial data, defines a unique path in the space of generalised metrics. Instead of directly taking into account the potential, the flow behaviour is therefore dictated by exactly the curvature and flux of the internal manifold which upon the dimensional reduction gives rise to the scalar potential. While the geometry is gradually modified by the flow, the associated scalar field also changes accordingly and sweeps out a path in field space. Therefore, the flow defines a trajectory in the space of generalised metric that is mirrored by a path in scalar field space; this is visualised in Figure 9.1. While the flow does not describe proper string backgrounds along the flow, once hitting a fixed point, the background has vanishing $\hat{\beta}$-functions, and therefore we reach a valid string vacuum. In particular, this offers a precise way in which we can describe off-shell trajectories leading to on-shell configurations at the end of the flow. The off-shell paths follow a very natural functional in this setting, namely the string effective action $\mathcal{S}$, subject to a constraint.

9.2. Measures of distance along the flow

The procedure described in the previous paragraph can only work if we can construct a distance measure along the flow that mirrors the distance on scalar field space in a meaningful way. Hence, it is crucial to have a well-defined notion of distance[2], and the purpose of the following section is to construct such a measure. In particular, we will generalise the existing proposals for a distance along Ricci flow to take into account the presence of the flux. Furthermore, we argue that there is a fundamental definition of distance whenever we have a generating functional at hand which we denote Δ_L. This new definition reduces to the known simple form discussed in Section 5.2.4, i.e. $\Delta_L \simeq \log(\mathcal{F})$ in the fluxless case, but also nicely connects to the generalised DeWitt or Candelas-de la Ossa metric [67, 464] in the presence of H-flux, which was discussed in Chapter 6. This will establish the fact that, at least for gradient flows, we are measuring the distance according to the natural metric on the space of generalised metrics. The role of the flow therefore is mainly in singling out the natural trajectories along which to measure the distance for the SDC or RFC, respectively.

A natural (but not fully satisfactory) guess

Recall from Section 4.1.1 that Ricci flow can be seen as the gradient flow of Perelman's functional $\mathcal{F}(g, f)$, and hence, due to its monotonicity, the functional offers a very natural measure of length along the flow. In particular, guided by the analogy between Perelman's functional and the Gibbons-Hawking entropy, a distance was defined

[2]Whether this is really a genuine distance in the mathematical sense is discussed in Chapter 11.

in [381] through

$$\Delta_{\mathcal{F}} = \log\left(\frac{\mathcal{F}(g,f)|_{s_f}}{\mathcal{F}(g,f)|_{s_i}}\right) . \tag{9.8}$$

The natural choice for generalised Ricci flow is therefore to simply replace $\mathcal{F}(g,f)$ by $\mathcal{F}(g,H,f)$. However, this approach has some drawbacks. Apart from some special cases, it is not clear how to connect this notion of distance to the standard one in terms of the DeWitt metric. The purpose of the following section is to overcome this issue by taking inspiration from the well-known notion of a line integral along a vector field.

9.2.1. A universal definition of length for geometric flows

We start by recalling the basic notion of a line integral. Take a vector field $\mathbf{F}(x)$ and γ a curve in K a (compact) n-dimensional manifold. The line integral along γ is then defined as

$$I_\gamma = \int_\gamma \boldsymbol{F}(\boldsymbol{r}(s)) \cdot \dot{\boldsymbol{r}}(s)\, \mathrm{d}s\,, \tag{9.9}$$

where $r(s)$ is a parametrisation of the curve γ. For geometric flows, in particular generalised Ricci flow, there is a natural candidate for the vector field. Starting from $\mathcal{F}$ the generating functional or Perelman functional we define the vector field $\mathbf{F}$ through the gradient[3] of $\mathcal{F}$, i.e.

$$\boldsymbol{F} = \boldsymbol{\nabla}\mathcal{F} \equiv \int_K \mathrm{d}V_g\, e^{-f}\left((\hat{\beta}_g)^{ij}\boldsymbol{\partial}_{g_{ij}} + (\hat{\beta}_B)^{ij}\boldsymbol{\partial}_{B_{ij}}\right). \tag{9.10}$$

The coefficients $\hat{\beta}^{g/B}$ are then nothing else than the flow equations or Weyl anomaly coefficients of generalised Ricci flow given in Eq. (9.5); see also (4.24). Taking inspiration from the line integral (9.9), we define the length along the flow as[4]

$$\Delta_L = \int_s \frac{1}{\sqrt{2\hat{\mathcal{V}}_K}}\sqrt{|\boldsymbol{F}(\boldsymbol{r}(s))\cdot\dot{\boldsymbol{r}}(s)|}\,\mathrm{d}s = \frac{1}{\sqrt{2\hat{\mathcal{V}}_K}}\int_s |\partial_s\mathcal{F}|^{1/2}\,\mathrm{d}s\,, \tag{9.11}$$

where $\hat{\mathcal{V}}_K = \int_K \mathrm{d}V_g\, e^{-f}$ and $\mathbf{r}(s) = (g_{ij}(s), B_{ij}(s))^T$. The square root was introduced in order to connect to the generalised DeWitt metric discussed in Section 6.2 as will

[3]The gradient is taken with respect to the basis $\left(\frac{\partial}{\partial g_{ij}}, \frac{\partial}{\partial B_{ij}}\right)^T$.

[4]We can pull the volume factor through the integral since it is constant in time by means of the constraint of the generalised Perelman functional. When applying this formula to more general functionals without constraint, it is therefore important to distinguish between the two expressions. Furthermore, note that we introduced an additional factor of $1/\sqrt{2}$ compared to [68] that is purely conventional but will result in an exact match between the distance measured with Δ_L and (5.2) with (6.31).

become clear in a moment. In particula, we can write Δ_L as

$$\begin{aligned}\Delta_L &= \frac{1}{\sqrt{2\hat{\mathcal{V}}_K}} \int_s \mathrm{d}s \left| \int_K \mathrm{d}V_g e^{-f} \left((\hat{\beta}_g)^{ij} \partial_s g_{ij} + (\hat{\beta}_B)^{ij} \partial_s B_{ij} \right) \right|^{1/2} \\ &= \frac{1}{\sqrt{2\hat{\mathcal{V}}_K}} \int_s \mathrm{d}s \left| \int_K \mathrm{d}V_g e^{-f} \left(\tfrac{1}{2} g^{il} g^{jm} \partial_s g_{lm} \partial_s g_{ij} + \tfrac{1}{2} g^{il} g^{jm} \partial_s B_{lm} \partial_s B_{ij} \right) \right|^{1/2} \\ &= \frac{1}{\sqrt{4\hat{\mathcal{V}}_K}} \int_s \mathrm{d}s \left| \int_K \mathrm{d}V_g e^{-f} \left(\|\partial_s g\|_g^2 + \|\partial_s B\|_g^2 \right) \right|^{1/2}, \end{aligned} \tag{9.12}$$

where in the second line we used the definition of the generalised Ricci flow (9.5). We clearly recognise the last line as the generalised DeWitt or Candelas-de la Ossa metric (6.35) of Section 6, evaluated along a path γ selected by the flow. We again observe that, generically, flux contributions enter the notion of distance in a nontrivial way and must not be ignored in general [67,382,447,458]. Furthermore, since we already know that the metric is positive definite, the above expression is strictly positive along the flow and zero only at the fixed points. We will come back to this in Chapter 11.

Relation to $\Delta_{\mathcal{F}}$

The new notion of distance Δ_L connects nicely to the existing distances for Ricci flow,established in [381]. In order to see this, we restrict ourselves to the conventional Perelman flow without H-flux, such that we can write (9.11) as

$$\Delta_L = \frac{1}{\sqrt{2\hat{\mathcal{V}}_K}} \int_s \sqrt{|\partial_s \mathcal{F}|}\, \mathrm{d}t = \frac{1}{2\sqrt{\hat{\mathcal{V}}_K}} \int_s \mathrm{d}s \left| \int_K \mathrm{d}V_g e^{-f}\, \partial_s R(s) \right|^{1/2} . \tag{9.13}$$

Here we stress again that the time-independence of the $\mathrm{d}V_g$ was crucial for pulling the time derivative onto R. Restricting to spaces with spatially constant Ricci curvature with $R_{ij}R^{ij} = \frac{k}{2}R^2$ and k a constant, by equation (4.7) we find that the scalar curvature evolves according to $\partial_s R = kR^2$, such that

$$\Delta_L = \frac{1}{2\sqrt{k}} \int_s \left| \frac{\partial_s R(s)}{R(s)} \right| \mathrm{d}s \simeq \log\left(\frac{R(s_f)}{R(s_i)} \right) \simeq \log\left(\frac{\mathcal{F}(g,f)|_{s_f}}{\mathcal{F}(g,f)|_{s_i}} \right), \tag{9.14}$$

and we recover the proposed distances of [381].

The same carries over to the case with non-trivial H-flux, where under additional conditions on H the distance reduces to

$$\Delta_L \simeq \log\left(\frac{\mathcal{F}(g,H,f)|_{s_f}}{\mathcal{F}(g,H,f)|_{s_i}} \right) . \tag{9.15}$$

We close this first discussion of the distance by stressing once more that in the above we worked with the generalised Perelman flow and the associated functional. This guaranteed that the normalising factor of the dilaton-weighted volume is constant in time and can be pulled out of the time integral. Furthermore, this guaranteed that the generalised Ricci flow is gradient with respect to the functional $\mathcal{F}$, which allowed the elegant connection to the generalised DeWitt metric. The purpose of the next section will be to investigate whether one can relax this constraint. This boils down to studying the flow equation of f and its role as a dilaton. However, these modifications will necessarily come at a price. In general, we will lose the monotonicity of the functional along the flow. However, we investigate how one can still obtain a meaningful notion of distance using a modified version of the above Δ_L. At the same time, it will become clear that the naive choice of $\log(\mathcal{F})$ can become problematic in these settings. The interpretation of the measure Δ_L as a proper distance in the mathematical sense is deferred to Chapter 11.

Role of the scalar field *f*

The scalar field f played a distinguished role in the previous discussion, since it is through f that the unit volume constraint is imposed. This in turn is crucial in order to obtain a well-defined gradient flow, as discussed in Section 4.1.4. The physical interpretation of the constraint is not completely clear, and it might be argued to be somehow artificial. Therefore, also its role in the RFC is unclear. This is even more evident, when noting that the flow behaviour of the volume of the manifold and the dilaton are coupled to each other. In particular, they are inversely correlated and whenever the volume reaches extremal values, f behaves in a way such that the generalised volume element remains constant. It is therefore not clear, seeing f as a genuine dilaton, how to access infinite-distance points while staying at finite string coupling. These limits could be investigated in [381] by considering purely dilatonic flow. We will discuss two different ways of decoupling f from the constraint, bringing it closer to an actual interpretation of a stringy dilaton. Starting from the variation of the action functional given in (9.3), we will see that there are alternative ways to obtain flow equations that preserve at least some of the desirable properties of the flow and the associated distance. We will postpone this detailed discussion to Section 9.4 and focus first on the (modified) Ricci Flow Conjecture in the presence of fluxes and the associated potential on scalar field space.

Remark

- In [496] the authors proposed an interpretation for the constraint. In particular, their analysis suggests a deeper physical principle of the constraint than pure mathematical curiosity. Using the known procedure to obtain the equations of motion of the NSNS sector from the functional with constraint, they extended this procedure to include Ramon-Ramond fields as well as moduli and higher-order string-loop corrections.

9.3. Revisiting S^3 with flux

This chapter would not be complete if we did not come back to our favourite example, S^3 with flux. As was explained earlier, one of the main motivations for studying generalised Ricci flow within the Swampland program is to obtain a different viewpoint on the role of potentials on field space for the SDC. These generically arise when considering fluxes or internal manifolds that have a nontrivial topology and will, in general, also feature divergent directions. For consistent string backgrounds, we argued in Section 6.1.4 that these divergences arise because of a lack of light towers of states required by the SDC and hence effectively screening this would-be infinite-distance point. In the following, we see that there is a nice and complementary interpretation in terms of generalised Ricci flow which will lead to a *generalised Ricci Flow Conjecture*.

We start by quickly recalling the situation we encountered in Chapter 6. The three-sphere cannot support any non-trivial winding and therefore is missing the required infinite tower of light states for $r \to 0$. However, there is a scalar potential, sourced by the curvature and the H-flux supported on the sphere, such that this scalar potential diverges for small r, signalling the lack of a tower and preventing us from visiting this naive infinite-distance point.

9.3.1. Flow behaviour and distance

Trying to understand this example through the lens of generalised Ricci flow, we first need to understand the flow behaviour of the background. The flow was shown in Section 4.1.4 to reduce to an evolution equation for the radius r, which reads

$$\frac{\partial}{\partial t} r(s) = -\frac{2}{r(s)} + \frac{2k^2}{r(s)^5} \,. \tag{9.16}$$

Here, the flux number k, which turns out to be constant in $3D$, is such that the H-flux itself remains fixed for all s. The differential equation (9.16) admits a unique fixed point at $r = \sqrt{k}$ of attractive nature. In particular, for $r < \sqrt{k}$ we have $\partial_s r(s) > 0$ while for $r > \sqrt{k}$, $\partial_s r(s) < 0$ and so in any case the flow is towards the fixed point. Crucially, the inclusion of H has lifted the singularity at the end of the conventional Ricci flow and created a finite stable fixed point $r = \sqrt{k}$. This should be viewed as a first hint that the RFC in the presence of fluxes is richer than the standard one. In particular, we can have fixed points at finite values that do not correspond to singularities or any other "extreme limits" of the underlying geometry. Indeed, we do not expect any tower of massless states to appear when approaching the fixed point $r = \sqrt{k}$, which is exactly the conformal fixed point of the underlying $SU(2)_k$ WZW model.

We are particularly interested in the asymptotics of r. Asymptotically we can neglect the second or first term on the RHS of (9.16) so that the PDE simplifies to

$$\begin{cases} r \gg 1 : & \frac{\partial}{\partial s} r(s) \sim -\frac{2}{r(s)} \quad \Rightarrow \quad r(s) \sim \sqrt{r_0^2 - s}, \\ r \ll 1 : & \frac{\partial}{\partial s} r(s) \sim +\frac{2k^2}{r(s)^5} \quad \Rightarrow \quad r(s) \sim \sqrt[6]{r_0^6 + 6k^2 s} \end{cases}, \quad r(0) = r_0 \,. \tag{9.17}$$

Comparing with (4.34) we observe that for large r the flow asymptotes the one of S^3 without any flux. For small r we have a novel flow behaviour that is modified by the presence of the H-flux. We see that asymptotically we can extend the flow backwards in time, starting at some initial $r_0 = r(s)|_{s=0}$. Starting at some $r_0 \gg 1$ this is possible for all $s < 0$ and therefore the flow can be defined for all $s \in (-\infty, \infty)$. In particular, the radius grows without bound for negative s and therefore the potential approaches zero. On the other hand, starting at some $r_0 \ll 1$ the radius becomes zero after a finite time $s^* = r_0^6/(6k^2)$ where, in particular, the flow equation encounters a singularity. In this limit, the corresponding potential diverges; see Figure 9.2. This means that the solution for large r is *eternal*, while the solution for small radius is merely *immortal* with a maximal interval of existence $s \in (s^*, \infty)$.

Calculating the distance

Before discussing this example in light of the (generalised) Ricci Flow Conjecture, we need to determine the distance Δ_L defined in the previous section. In particular, there are three distinguished points. These are the two limiting cases of zero and infinite radius r, as well as the fixed point $r = \sqrt{k}$.

Recall that the new notion of distance Δ_L can be written in the form

$$\Delta_L = \frac{1}{\sqrt{4\hat{\mathcal{V}}_K}} \int_t \mathrm{d}s \left| \int_K \mathrm{d}V_g e^{-f} \left(\left| g^{-1}\partial_s g \right|^2 + \left| g^{-1}\partial_s B \right|^2 \right) \right|^{1/2} . \tag{9.18}$$

The only time dependence of the metric is through $r(s)$ such that we can easily obtain $\left|g^{-1}\partial_s g\right|^2 = 12(\partial_s r/r)^2$ and therefore

$$\Delta_L = \sqrt{\frac{12}{4}} \int_{s_i}^{s_f} \left| \left(\frac{\partial_s r}{r} \right)^2 \right|^{1/2} \mathrm{d}s = \sqrt{3} \int_{s_i}^{s_f} \frac{\partial_s r}{r} \mathrm{d}s = \sqrt{3} \log \left(\frac{r(s_f)}{r(s_i)} \right) , \tag{9.19}$$

which exactly agrees with the distance obtained directly from the effective action. From this we can easily determine the three distances of interest[5], which evaluate to

$$\begin{cases} \Delta_L(r_0, \sqrt{k}) = \text{const.}\,, & \text{for } r_0 > 0. \\ \Delta_L(r_0, \infty) = \infty\,, & \text{for } r_0 \geq \sqrt{k}. \\ \Delta_L(r_0, 0) = \infty\,, & \text{for } r_0 \leq \sqrt{k}. \end{cases} \tag{9.20}$$

Therefore we have two infinite distance points, both of them reached for $s \to \infty$ and $s \to -\infty$ respectively.

Remarks

i) The flow in (9.16) was calculated from the gauge-fixed flow, in which the scalar f is decoupled from the flow of r. In this gauge the volume is not constant in time and therefore one needs to be especially careful in these cases when evaluating the distance[6] Δ_L. However, for the given example, all the time dependence of g_{ij} is in r and therefore, also f is only a function of s, and not the coordinates. In these cases the two gauges agree and indeed the unit volume constraint is fulfilled in both, such that we did not bother pulling out the volume factor out of the integral.

ii) In Chapter 4, we discussed the nice existence and uniqueness properties of (generalised) Ricci flow. In contrast, backward flow is generally poorly posed[7]. Even though in this case solutions are not guaranteed to exist and may even be unstable, under certain conditions the backward heat flow can anyway be well-posed. These situations are much less studied, but some results are available; see, for example, [497] for uniqueness results under the assumption of curvature bounds. Given the fact that for generalised Ricci flow, even less is known, we will take a pragmatic approach and assume the existence of some backward solution when reasonable and confirm this by numerical computations whenever possible.

9.3.2. The generalised Ricci Flow Conjecture

After having analysed the flow behaviour and the associated distances, we are finally in a position to examine the three-sphere with flux from the perspective of generalised Ricci flow and the RFC or SDC respectively. The flow behaviour is summarised in Figure 9.2. Although being relatively easy and hence tractable, the example features all the interesting possible behaviours we can encounter under a generic flow. In the

[5]Note that in contrast to the fixed point, the other two can only be reached "backwards" in time.

[6]Since $\mathcal{F}$ is gauge invariant the end result does of course not depend on the gauge. Quite simply, one needs to be more careful when performing the integration

[7]This is a familiar problem that for example also appears for the the backwards heat equation

following, we carefully analyse them one by one, while relating to the conclusions of Chapter 6 concerning the role of the scalar potential and the implications for the SDC and the proposed amendment.

(1) $\underline{\boldsymbol{r \to \infty}}$ **:** We pick an initial radius $r_0 > \sqrt{k}$. Taking $s \to -\infty$, the radius grows unbounded and the sphere decompactifies. At the level of the flow equation we have $\lim_{s\to-\infty} \partial_s r(s) = 0$, which means that we approach a fixed point of the flow. Furthermore, according to our analysis in (9.20), this corresponds to an infinite-distance point when following the generalised Ricci flow as measured by Δ_L. The associated infinite light tower of states for the SDC is given by the KK-modes.

(2) $\underline{\boldsymbol{r \to \sqrt{k}}}$ **:** Both for starting at $r_0 > 0$ or $r_0 < 0$ this is a fixed point of the flow where the radius stabilises according to the value of the H-flux. This corresponds to the minimum of the associated potential and is the conformal string vacuum of the associated WZW model at level k. Although this is a fixed point of the flow, i.e. $\lim_{s\to s_*} \partial_s r(s) = 0$, the associated distance in (9.20) is finite. This is perfectly consistent with the fact that we are at a string vacuum and therefore do not expect an infinite tower of light states.

(3) $\underline{\boldsymbol{r \to 0}}$ **:** We start from an initial radius $r_0 < 0$ and extend the flow backwards in time. In contrast to the first situation, we now encounter a singularity, since we have $\lim_{s\to-\infty} = \infty$. The resulting picture fits very nicely with the discussion [67] in Section 6.1. The scalar field space corresponding to the compactification on this background has a nontrivial scalar potential sourced by curvature and flux contributions. While for $r \to \infty$ the potential vanishes and asymptotically we can apply the standard SDC, it diverges for $r \to 0$. In terms of Ricci flow, this limit does not correspond to a fixed point and the Ricci Flow Conjecture does not predict an infinite tower of light states. It was discussed in great length in Section 6.1, that indeed there is no tower for $r \to 0$ due to the missing winding modes on S^3. The potential signalled this absence and in some sense "screened" us from visiting this point. The situation we encounter in this paragraph is the manifestation of this very principle, now seen through the lens of generalised Ricci flow [67].

The above discussion nicely illustrates the impact of the potential for the SDC from another perspective and shows how the addition of H-flux[8] can modify the possible scenarios for the RFC. This motivates the following reformulation of the conjecture.

[8] The extension to RR fluxes is discussed in Chapter 10.

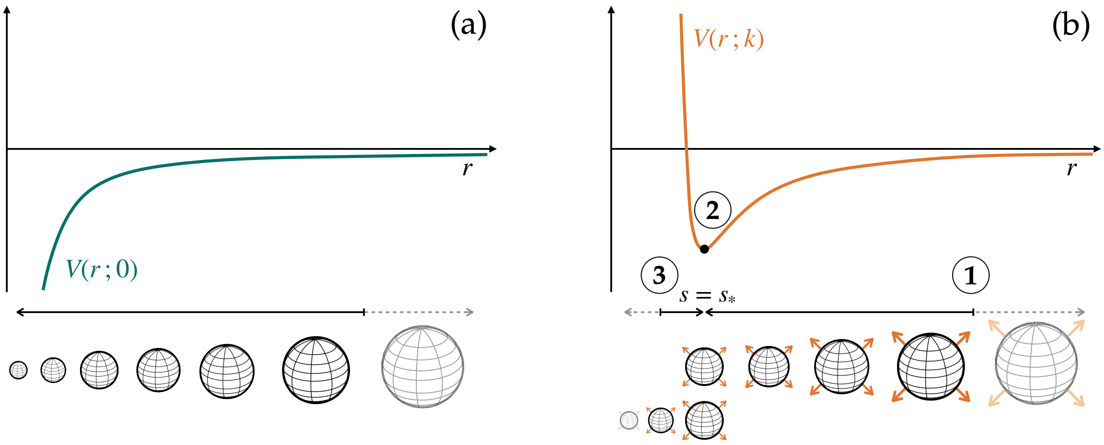


Figure 9.2.: Visualisation of generalised Ricci flow and the associated scalar potential for the three sphere. Panel (a) on the left depicts – for comparison – the situation without flux. The right panel (b), relevant for the discussion here, shows generalised Ricci flow of the sphere with $[H] = k$ units of H-flux and the resulting (modified) scalar potential.

Generalised Ricci Flow Conjecture (gRFC) [67]. *Consider QG on a family of backgrounds determined by the triple $(g_{ij}(s), B_{ij}(s), f(s))$ satisfying the generalised Ricci flow. There exists an infinite tower of states becoming massless along the trajectory of the flow towards a fixed point which in addition lies at infinite distance. The distance between two backgrounds at time s_i and s_f is measured in terms of the generalised Perelman entropy functional $\mathcal{F}(g, B, f)$ by Δ_L.*

We highlight the fact that according to the above conjecture, massless towers of states occur only when approaching a fixed point, which additionally lies at infinite distance. It is apparent from the above discussion that these two things do not necessarily go hand in hand. While case (3) lies at infinite distance, it does not correspond to a fixed point but a singularity and therefore, correctly, no tower of states is predicted.

Finally, note that the gRFC can also be formulated for generalised Ricci flow without f and hence without any reference to a generating functional. The distance can then be measured by the generalised DeWitt metric (6.35) along the trajectory in $\mathcal{GM}(M)$ or by defining an analogue of Δ_R defined in (5.18), i.e, $\Delta_S = \ln\big(R - \frac{1}{2}|H|^2 - 4|\partial\Phi|^2\big)\big|_{s_i}^{s_f}$. However, such a notion of distance does not have the nice monotonicity properties of Δ_L inherited from the functional $\mathcal{F}$.

9.3.3. The gRFC in the T-dual

Especially within the context of string theory, one of the drawbacks of standard Ricci flow is the fact that it is not invariant under T-duality. Starting from a topologically nontrivial purely geometric background with only a metric (and dilaton), the geometry is mapped under (topological) T-duality to a solution with flux and hence cannot be described by standard Ricci flow. It was shown in [498] that the generalised Ricci flow is invariant under this topology-changing process. In order to obtain a consistent picture within string theory, we are therefore forced to consider this more general flow and its associated conjecture. In this paragraph, we illustrate how the generalised Ricci flow and its implications for the SDC and gRFC translate to the T-dual.

Recall that the T-dual of the sphere with k units of H-flux along the Hopf fibre is again given by a circle fibration over $S^1_{1/r} \hookrightarrow E \to S^2_r$, where now the fibre has an associated radius that is proportional to r^{-1}, cf. equation (2.30). Plugging this into the flow equations of generalised Ricci flow, we arrive again at

$$\frac{\partial}{\partial s} r(s) = -\frac{2}{r(s)} + \frac{2k^2}{r(s)^4}\,, \qquad \frac{\partial}{\partial s} h(s) = 0\,, \tag{9.21}$$

hence, as promised, the flow equation stays invariant. However, the geometric realisation and the physical interpretation with respect to the SDC change. We already know from Chapter 6 that the associated scalar potential and metric on field space also remain invariant. What is left to do is therefore to interpret the three distinguished limits and the flow behaviour in the new frame. Again, referring to Figure 9.2, we have

1. **$r \to \infty$:** While on S^3 this corresponds to a decompactification, here only the two-sphere decompactifies while the fibre shrinks because it scales with r^{-1}. Since the flow equations remain invariant, this is again a fixed point of the flow. The infinite tower of light states is provided by the KK modes on S^2. For a discussion of the additional modes and the mapping under T-duality, we refer to Section 6.1.

2. **$r \to \sqrt{k}$:** At the fixed point the geometry is locally a product of a two-sphere with radius $r_{S^2} = \sqrt{k}$ and a circle with associated radius $r_{S^1} = 1/\sqrt{k}$. Since we are at a finite-distance fixed point, there are no towers becoming light.

3. **$r \to 0$:** The sphere shrinks to a point while the circle fibre decompactifies. According to our distance Δ_L, this is an infinite-distance point, however, not a fixed point of the flow. Therefore, we do not expect a proper infinite tower of light states in this limit according to the gRFC. Indeed, we know from Section 6.1 that the conservation of KK modes is spoilt due to the presence of H, cf. Section 6.1.

Indeed, we see that we obtain a fully consistent picture under T-duality.

Lastly, the flow for the T-dual f can be easily extracted from the dual geometric background. Alternatively, recall that in the Buscher rules, the T-dual dilaton is defined in such a way that the weighted volume element $\sqrt{g}e^{-2\phi}$ is invariant under T-duality, such that

$$\tilde{f} = f - \log(\det(g)) \, . \tag{9.22}$$

This $\tilde{f}$ will then automatically fulfil the constant volume constraint.

9.4. Modified flow equations for *f*

We return to the question on the role of the scalar field f, its interpretation as a dilaton, and the related generalised unit volume constraint. In particular, we explore the possibility of discarding the unit-volume constraint, examine the resulting flows, and present an alternative approach to include an explicit dilaton ϕ.

9.4.1. A family of flow equations for *f*

Recall from Section 4.1.4 that the variation of $\mathcal{S}$ gives raise to a term

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S} = \ldots + \left(R - \frac{1}{2}|H|^2 + 2\Delta f - |\nabla f|^2\right)\left(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial f}{\partial s}\right) \, . \tag{9.23}$$

In the absence of the constraint, this contribution does not vanish, but has to be taken care of by some other mechanism. One might contemplate reabsorbing the part proportional to the trace of g in the flow equation of g_{ij}, which would lead to a flow according to the Einstein field equation. However, it is known that this "Einstein flow" is ill-posed and, in general, does not admit even short-time solutions, which also can be seen as a manifestation of the conformal mode problem of gravity. We refer the interested reader to [300] for an insightful discussion and further relevant references.

The formulation of generalised Ricci flow in terms of a gradient of a generating functional naturally provides us with a monotonous functional along the flow. However, as also noted in [295], there is the "curiosity" that the action functional $\mathcal{S}$ remains monotonous along the flow even without the constraint, provided the flow equation for f is modified in a certain way. However, the resulting flow is not gradient with respect to $\mathcal{S}$. Looking at equation (9.23) it is clear that the full variation is manifestly positive if the remaining contribution can be written as a perfect square. Indeed, we would like to write

$$\left(R - \frac{1}{2}|H|^2 + 2\Delta f - |\nabla f|^2\right)\left(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial f}{\partial s}\right) = S(g, B, f) \cdot \frac{\partial_s \mu}{\mu} \, , \tag{9.24}$$

with $S(g, B, f)$ the generalised scalar curvature, Eq. (4.25).

Then imposing the flow equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\mu = \alpha\, S(g,B,f)\,, \tag{9.25}$$

with α a yet free integer we can write the variation of $\mathcal{S}$ as

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S} \sim \int_K \left(\frac{1}{2}|\partial_s g|^2 + \frac{1}{2}|\partial_s B|^2 + \frac{1}{\alpha}\left|\frac{\partial_s \mu}{\mu}\right|^2\right) e^{-f}\mathrm{d}V_g \geq 0\,. \tag{9.26}$$

Therefore, $\mathcal{S}$ increases strictly monotonically if we choose $\alpha \geq 0$. Furthermore, we see that for $\alpha = 0$ we retain the standard constraint $\int_K \mathrm{d}V_g e^{-f} = 1$. The flow equation for f can now be extracted from (9.25). In particular, using the traced flow of g_{ij}

$$\frac{1}{2}g^{ij}\partial_s g_{ij} = -(R + \Delta f - \frac{1}{4}|H|^2)\,, \tag{9.27}$$

we obtain

$$\begin{aligned}
0 &= \alpha\, S - \partial_s \log(\mu) \\
&= \alpha\left(R - \frac{1}{2}|H|^2 + 2\Delta f - |\nabla f|^2\right) + (R + \Delta f - \frac{1}{4}|H|^2) + \partial_s f \\
&= (\alpha+1)R - \frac{\alpha+3}{2}|H|^2 + (2\alpha+1)\Delta f - \alpha|\nabla f|^2 + \partial_s f\,.
\end{aligned} \tag{9.28}$$

Therefore,we obtain a family of flow equations for f, labelled by α, that reads

$$\frac{\partial}{\partial s} f = T_f^{(\alpha)} \equiv -(\alpha+1)R + \frac{\alpha+3}{2}|H|^2 - (2\alpha+1)\Delta f + \alpha|\nabla f|^2\,. \tag{9.29}$$

Choosing $\alpha = 0$ as a consistency check, we indeed get the correct formula for f in the standard case with unit volume constraint. Taking $\alpha = 2$ in analogy to the other flow equations, we obtain

$$-\frac{\partial}{\partial s} f = 3R - \frac{5}{2}|H|^2 + 5\Delta f - 2|\nabla f|^2\,, \tag{9.30}$$

which does not coincide with the dilaton β-function. However, at fixed points

$$\begin{aligned}
0 &= \frac{\partial}{\partial s} g_{ij} = -R_{ij} - \nabla_i\nabla_j f + \frac{1}{4}H_{ikl}H_j^{\ kl}\,, \\
0 &= -\frac{\partial}{\partial s} f = 3R - \frac{5}{2}|H|^2 + 3\Delta f - 2|\nabla f|^2\,.
\end{aligned} \tag{9.31}$$

Upon substituting the traced equation for g into the second, we get

$$0 = 2|H|^2 + 2\Delta f - 2|\nabla f|^2 \equiv \beta^f\,, \tag{9.32}$$

which is precisely the standard dilaton equation of motion $\hat{\beta}^\phi$ of (1.8) (after substituting

$\mathrm{tr}(\hat{\beta}^{g}_{\mu\nu})$ and using $f = 2\phi$). Hence, at the fixed point, f can be identified with an on-shell string-theoretic dilaton. Lastly, there is another curiosity about the above family of flows, since for $\alpha = -1$ the resulting flow is

$$\frac{\partial}{\partial s} f = \frac{1}{6}|H|^2 + \Delta f - |\nabla f|^2 \sim \beta^f \,, \tag{9.33}$$

which directly gives the $\hat{\beta}$-function of the dilaton and therefore it is suggestive to identify $f = 2\phi$. However, this comes at the price of losing monotonicity along the flow, since now the last term in the sum (9.26) appears with a minus sign.

These modified flows will not play a major role in the remainder of this thesis, so we move a discussion of some examples and the SDC to the Appendix F.

9.4.2. Including the dilaton explicitly

In the previous section, by modifying the flow equation we were led to a family of flows that as a special case contained the evolution according to the $\hat{\beta}$-function of the string dilaton. While this is certainly an interesting – maybe even desirable – feature, there was no monotonous functional along the flow available. Indeed, given the fact that the metric and dilaton are already evolving according to the equations of motion of the string effective action, it is natural to accompany this by a flow of the string-theoretic dilaton, completing the picture to the full NSNS sector. Concretely we would therefore like to have a system of flow equations for g, B and ϕ that reads[9]

$$\begin{aligned}
\frac{\partial}{\partial s} g_{ij} &= -2\hat{\beta}^{g}_{ij} = -2R_{ij} + \frac{1}{2}H^2_{ij} - 4\nabla_i\nabla_j\phi \,, \\
\frac{\partial}{\partial s} B_{ij} &= -2\hat{\beta}^{B}_{ij} = \nabla^k H_{kij} - 2H_{kij}\nabla^k\phi \,, \\
\frac{\partial}{\partial s} \phi &= -2\hat{\beta}^{\phi}_{*} = \Delta\phi - 2(\nabla\phi)^2 + \frac{1}{2}|H|^2 \,.
\end{aligned} \tag{9.34}$$

Trying to identify directly $f = 2\phi$ and trying to have $\mathcal{S}$ as a monotonous (or even gradient) functional along the flow leads back to the problems and discussion of the precious section and is therefore not desirable. However, it was realised in [499] that

[9]This system of equations is well-defined in the sense that ϕ can be decoupled from the other flow equations and obeys a Heat-like equation that can be solved a posteriori.

the above system can be realised as the (gauge-fixed) gradient flow of the functional

$$\begin{aligned}\mathcal{F}(g,H,f=2\phi+\tilde{f}) &= \int_K \left(R-\frac{1}{2}|H|^2+|\nabla(2\phi+\tilde{f})|^2\right)e^{-2\phi}e^{-\tilde{f}}\mathrm{d}V_g \\ &= \int_K \left(\left(R-\frac{1}{2}|H|^2+4|\nabla\phi|^2\right)+4\nabla\phi\nabla\tilde{f}+(\nabla\tilde{f})^2\right)e^{-2\phi}e^{-\tilde{f}}\mathrm{d}V_g\,,\end{aligned} \tag{9.35}$$

supplemented by the constraint

$$\int_K e^{-2\phi}e^{-\tilde{f}}\mathrm{d}V_g = 1\,. \tag{9.36}$$

The flow equation for $\tilde{f}$ is obtained from the constraint and reads[10]

$$\frac{\partial}{\partial s}\tilde{f} = \tilde{T}_{\tilde{f}} \equiv -\Delta\tilde{f} - 2\nabla\phi\nabla\tilde{f} - R + \frac{1}{2}|H|^2 - 2\Delta\phi + (\nabla\tilde{f})^2. \tag{9.37}$$

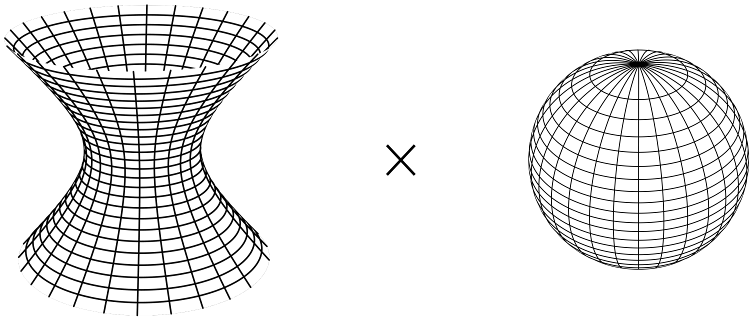
Although this looks very promising, it should be stressed that strictly speaking, only the flow equations for g and B are gradient with respect to $\mathcal{F}$. The dilaton equation $\partial_s\phi$ arises in a somewhat artificial way due to the splitting of f into ϕ and $\tilde{f}$ and will enter the distance along the flow only indirectly through the other flow equations. Therefore, the gradient property must be understood in a weaker sense.

A distance for modified flows?

Although considerations in the preceding section show that there are reasonable modifications to the flow equations that can decouple the dilaton or f from the behaviour of g, the implications for the gRFC and the Distance Conjecture are far from obvious. Indeed, all the modifications discussed come with their own shortcomings as compared to standard generalised Ricci flow.

In particular, loosing the gradient property with respect to $\mathcal{F}$, also the appropriate choice of measure of length becomes less clear. For example, restricting ourselves to the family of flows with $\alpha \geq 0$, we see that the functional is strictly monotonous along the flow. In fact, the summands are positive and only zero at the fixed point of the flow, and the functional $\mathcal{S}$ can be used to define a distance along the flow. On the other hand, allowing for $\alpha = -1$ in principle, the summands in $\partial_s\mathcal{S}$ can cancel each other such that $\partial_s\mathcal{S} = 0$ even at points that do not correspond to fixed points. In view of a proper definition of distance, these points can be problematic, since they lead to a violation of the axiom of strict positivity, i.e. $\Delta(x,y) = 0$ for $x \neq y$. We elaborate further on this in Chapter 11 while a discussion of some examples can be found in the Appendix F.

[10] After acting with a diffeomorphism that decouples f from the other flow equations via $\mathcal{L}_{\nabla\tilde{f}}$ similar to Section 4.1.4.



CHAPTER 10

Geometric flows with RR-fluxes, supergravity flows and product spaces

There are only very few full 10D solutions of Type II supergravity that only involve NSNS flux. Most of them come with additional Ramond-Ramond p-form fluxes and, in more realistic settings descending from string, with (localised) source contributions from D-branes or orientifold planes. Hence, if we want to apply the formalism developed in the previous chapter to internal spaces with more fluxes and source contributions, or even full 10D vacua, we need to include also this contributions into the flow as well as the issues connected to Lorentzian signature. Defining such a flow is highly non-trivial and there are several obstacles to be overcome that were not present in the (compact) NSNS case. We will anyway propose a possible definition for such a flow in 10 as well as 11 dimensions and furthermore establish its well-posedness, at least for the Euclidean case. We give an explicit example for a flow of a full 10D orientifold background towards a Minkowski vacuum. We conclude by highlighting some of the challenges that arise when moving to Lorentzian (product) spaces, in particular, the case of external AdS and propose some possible further approaches in this direction. We also discuss possible applications within the Swampland program, in particular applications for the SDC and RFC as well as related questions, like scale separation or smeared sources, that can potentially be addressed.

10.1. A flow for Type II supergravity

In this section, motivated by the form of the flow equations in the NSNS sector, we introduce a generalised flow that includes also RR p-form fluxes[1]. We are agnostic about the role of the flow as being applied to internal or external spaces (or full product spaces) but first specialise to the Euclidean case. This will enable us – closely following the known proof for the NSNS sector – to establish that the flow is indeed well-defined. We also discuss how the flow in principle can be used to study the implications for the SDC and Swampland program.

10.1.1. Flow equations with RR-fields

We know from the previous chapters that the flow equations for g and B under generalised Ricci flow are nothing else than the (NSNS sector) equations of motion of supergravity. When trying to include p-form RR-fluxes, it therefore seems natural to impose flow equations that mirror this behaviour. In particular in the setting of Type II SUGRA, we choose the flow equations to be precisely the associated equations of motion (1.15), (1.16) and (1.17), such that for (massive) Type IIA theories we obtain the following flow equations[2]

$$\begin{aligned}
\frac{\partial}{\partial s}g_{ij} &= -2\bar{\beta}^{g}_{ij} \equiv -2R_{ij} - 4\nabla_i\nabla_j\phi + \frac{1}{2}H^2_{ij} \\
&\qquad\qquad + 2e^{2\phi}\left[\frac{1}{2}(F_2^2)_{ij} + \frac{1}{12}(F_4^2)_{ij} - \frac{1}{4}g_{ij}\left(m^2 + |F_2|^2 + |F_4|^2\right)\right], \\
\frac{\partial}{\partial s}B_{ij} &= -2\bar{\beta}^{B}_{ij} \equiv \nabla^k H_{kij} - 2H_{kij}\nabla^k\phi + \left(\star(F_2\wedge\star F_4 - \frac{1}{2}F_4\wedge F_4 - m\star F_2)\right)_{ij}, \\
\frac{\partial}{\partial s}\phi &= \begin{cases} -2\bar{\beta}^{\phi}_{0} \equiv 4\Delta\phi - 4|\nabla\phi|^2 + R - \frac{1}{2}|H|^2, \\ -2\bar{\beta}^{\phi}_{*} \equiv 2\Delta\phi - 4|\nabla\phi|^2 + |H|^2 + \frac{1}{2}e^{2\phi}\left((1-\frac{n}{4})|F_2|^2 + (4-n)|F_4|^2 - \frac{n}{2}m^2\right). \end{cases}
\end{aligned} \tag{10.1}$$

Here, $n = g^{ij}g_{ij}$ denotes the dimension of the (Euclidean) space under consideration. These are supplemented by the flow equations for the Ramond-Ramond flux potentials, which read

$$\begin{aligned}
\frac{\partial}{\partial s}C_i &= -2\bar{\beta}^{C_2}_{i} \equiv \nabla^k F_{ki} + \star(H\wedge\star F_4)_i, \\
\frac{\partial}{\partial s}C_{ijm} &= -2\bar{\beta}^{C_3}_{ijm} \equiv \nabla^k F_{kijm} + \star(H\wedge F_4)_{ijm}.
\end{aligned} \tag{10.2}$$

[1] We refer to [500] for a closely related flow that includes higher-form fluxes but can not be applied directly to Type II solutions due to the lack of a dilaton that has to be incorporated into the flow and enters the NSNS and RR sector differently.

[2] Note that there are slightly different normalisation factors compared to [68], due to some normalisation conventions, as well as the sign of the m^2 term corrected.

Analogously, one can also define a flow for Type IIB theories. In principle, the choice of prefactors for the flow equations is arbitrary, but it will be convenient to use the above conventions for later purposes. We stress the slight abuse of notation in denoting the flow equations with $\bar{\beta}^{\bullet}$, which should merely highlight the fact that we use the equations of motion, like in the purely NSNS sector. However, these functions cannot be seen as Weyl anomaly coefficients or RG flow β-functions of a NLSM description.

The reader will wonder why there are two flow equations for the dilaton. This is due to some ambiguity, similar to the one discussed in the remark at the end of Section 1.2. The first equation corresponds to the equation of motion obtained directly from the variation of the action. The latter $\hat{\beta}^{\phi}_{*}$ is obtained by substituting the traced equation of the metric $\bar{\beta}^{g}$. As mentioned before, on-shell – and therefore at the fixed points of the flow – these choices are equivalent while, in general, they lead to different expressions along the flow. It is often customary to work with the latter one due to the decoupling of the Ricci scalar. For us, the following proof does not depend on the choice of $\bar{\beta}^{g}$, so we remain agnostic about the precise choice and denote both by $\bar{\beta}^{\phi}$.

10.1.2. Short time existence and uniqueness

In Section 4.1.1 we discussed the fact that Ricci flow is not strictly parabolic[3], which is connected to the diffeomorphism invariance of the flow equations. Therefore, the flow is a priori not well defined and one has to employ the "DeTurck trick" in order to establish the existence of at least short-lived solutions. This story carries over to the problem at hand. The flow equations (10.1), (10.2) do not define a strictly parabolic system but – following the strategy of [295] – we define the vector field

$$X^k = g^{ij}\left(\Gamma^k_{ij} - (\Gamma_0)^k_{ij}\right) , \tag{10.3}$$

in order to gauge fix the flow equations and remove the ambiguity from the diffeomorphism invariance. No define differential operators denoted $O(g, B, \phi, F_p)$ by taking the right hand side of the flow equations $-2\bar{\beta}^{\bullet}$ corrected by the addition of the Lie derivative of X. Furthermore we add a second Lie derivative with respect to $-\nabla\phi$ in order to bring the flow equations into a from reminiscent of the gauge fixed generalised Ricci flow equations, such that in total the operators read[4]

$$O_{\bullet}(g, H, F_p, \phi) = -2\bar{\beta}^{\bullet} + \mathcal{L}_X(\bullet) - 2\mathcal{L}_{\nabla\phi}(\bullet) \equiv -2\tilde{\beta}^{\bullet} + \mathcal{L}_X(\bullet) . \tag{10.4}$$

[3]A system of PDEs $\partial_t u = Lu$, defined by the differential operator L, is strictly parabolic if the principal symbol of L is positive definite. Loosely speaking, the principal symbol $\sigma(L)$ is captured by the highest-order derivatives in L. For a non-linear L (like the Ricci tensor R_{ij}, viewed as an operator acting on g_{ij}) parabolicity can be established using the linearisation of L; see Appendix E for more details.

[4]The flow equitations for the fluxes F_p denoted by $\hat{\beta}^{F_p}$ are obtained from $\hat{\beta}^{C_{p-1}}$ by derivation $d(\cdot)$.

Here, $\bullet$ denotes all the quantities g, B, F_p, ϕ. For later purposes we also define $\tilde{\beta}^\bullet = \bar{\beta}^\bullet + \mathcal{L}_{\nabla\phi}(\bullet)$. It is important to note at this point that for standard (generalised) Ricci flow the Lie derivative with respect to $\nabla\phi$ would decouple the dilaton from all the other flow equations. Due to the presence of the RR-fluxes that enter the action with a different weighting of the dilaton this is, however, no longer true and this will have a significant impact on our analysis as we will explain shortly. The next step in the procedure is to compute the linearisation of these operators O_i. In order to do so, we introduce the following notation. First, we expand all the fields up to first-order terms, i.e. $g_{ij} = (g_0)_{ij} + h_{ij}(t)$ and similarly for the other fields. We suppress indices and write g_t, H_t, F^p_t, ϕ_t in order to explicitly stress the dependence on the linearisation parameter, and where here we denote $F^p \equiv F_p$. Lastly, we denote

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_t = \hat{g}\,, \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H_t = \hat{H}\,, \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F^p_t = \hat{F}^p\,, \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t = \hat{\phi}\,. \tag{10.5}$$

The resulting PDE are then of second order due to the form on the flow equations. It is immediate that, using the results from the NSNS sector, we obtain [295]

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_g(g_t, \dots) = \Delta\hat{g} + \text{l.o.t.}\,, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_g(H_t, \dots) = \text{l.o.t.}\,, \tag{10.6}$$

where l.o.t denotes terms of lower than leading order in derivatives. Here Δ denotes the "rough" or connection Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$ while we write $\Delta_{dR} = -(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})$ for the Laplace-de Rham Laplacian; see Appendix E. Furthermore, in the above, we only keep quantities explicitly dependent on the linearisation parameter t, while the rest is simply denoted by $\dots$, i.e. $O_g(g_t, \dots) \equiv O_g(g_t, H, F^p, \phi)$. Then since F^p and also ϕ only appear at zeroth order in derivative in O_g, we clearly get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_g(F^p_t, \dots) = \text{l.o.t.}\,, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_g(\phi_t, \dots) = \text{l.o.t.} \tag{10.7}$$

Similarly, one can proceed for all the other O_i and the results read

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_H(g_t, \dots) = \Xi\,, && \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_\phi(g_t, \dots) = \Omega\,, \\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_H(H_t, \dots) = \Delta_{dR}\hat{H} + \text{l.o.t.}\,, && \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_\phi(\phi_t, \dots) = \Delta\hat{\phi}\,, \\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_{F_p}(g_t, \dots) = \Lambda\,, && \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_{F_p}(\phi_t, \dots) = \Sigma\,, \\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_{F_p}(F^p_t, \dots) = \Delta_{dR}\hat{F}^p + \text{l.o.t.}\,,
\end{aligned} \tag{10.8}$$

where $\Xi, \Lambda, \Sigma, \Omega$ are operators, the precise form of which will not be relevant for us as they drop out of the argument. Now by use of the Weizenböck identity (cf. Appendix E, Eq. E.10)

$$\Delta_{dR}\omega = \Delta\omega + \mathrm{Ric}(\omega)\,, \qquad \omega \text{ a } n\text{-form}\,, \tag{10.9}$$

we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_H(H_t,\dots) = \Delta\hat{H} + \text{l.o.t.}\,, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} O_{F_p}(F_t^p,\dots) = \Delta\hat{F}^p + \text{l.o.t.} \tag{10.10}$$

Having collected all the necessary information, we can now obtain the principal symbol σ (in the basis defined by (10.5))

$$\sigma\left(D_{(g,H,\phi,F^p)}(O_g,O_B,O_\phi,O_{F^p})\right)\begin{pmatrix}\hat{g}\\ \hat{H}\\ \hat{\phi}\\ \hat{F}^p\end{pmatrix} = \begin{pmatrix}\Delta & 0 & 0 & 0\\ \Xi & \Delta & 0 & 0\\ \Omega & 0 & \Delta & 0\\ \Lambda & 0 & \Sigma & \Delta\end{pmatrix}\begin{pmatrix}\hat{g}\\ \hat{H}\\ \hat{\phi}\\ \hat{F}^p\end{pmatrix}, \tag{10.11}$$

with the last entry again viewed as a block matrix and similar for the $(4,i)$ and $(i,4)$ entries as column or row vectors, respectively. Then using Silvester's criterion we see that $[\sigma_{ij}]$ is positive-definite, and therefore the system is strictly parabolic. This establishes the existence of short-time solutions for the gauge fixed flow defined in terms of the RHS of the operators in (10.4).

Ultimately, we are interested in the flow defined by $\tilde{\beta}^i$ (or $\bar{\beta}^i$). Therefore in a second step we need to undo the gauge transformation applied in the first step such that we end up with the flow in a form that upon eliminating the RR-fields, reduces to the usual generalised Ricci flow in the NSNS sector. Again following the procedure in [295], define a one-parameter family of diffeomorphisms ψ_s by

$$\frac{\partial}{\partial s}\psi_s = -X(g_s,g_0)\circ\psi_s\,, \quad \text{with } \psi_0 = \mathrm{id}\,. \tag{10.12}$$

Using ψ, we define the pulled-back fields

$$\tilde{g}_s = \psi_s^* g_s\,, \qquad \tilde{F}_s^p = \psi_s^* F_s^p\,, \qquad \tilde{H}_s = \psi_s^* H_s\,, \qquad \tilde{\phi}_s = \psi_s^*\phi_s\,. \tag{10.13}$$

This inverts the DeTurck trick and a short computation following [295] shows[5]

$$\begin{aligned}\partial_s\tilde{g}_s = \partial_s\left(\psi^* g_s\right) &= \psi^*\left(\partial_s g_s\right) + \psi^*\mathcal{L}_{-X(g_s,t_0)}g_s\\ &= \psi^*\left(-2\tilde{\beta}_g(g,H,F^p,\phi) + \mathcal{L}_{X(g_s,g_0)}g)\right) + \psi^*\mathcal{L}_{-X(g_s,g_0)}g_t\\ &= -2\tilde{\beta}_g(\tilde{g},\tilde{H},\tilde{F}^p,\tilde{\phi})\,,\end{aligned} \tag{10.14}$$

[5] Using the identity $\frac{\mathrm{d}}{\mathrm{d}s}(\Psi_s^* T_s) = \Psi^*\left(\mathcal{L}_{X_s}T_s + \frac{\mathrm{d}}{\mathrm{d}t}T_s\right)$ for time-dependent vector fields and tensors (cf. [501], Prop. 22.15).

which trivially then also carries over to $\tilde{H}$, $\tilde{F}_n$ and $\tilde{\phi}$

$$\partial_s \tilde{H}_s = -2\tilde{\beta}_H(\tilde{g}, \tilde{H}, \tilde{F}^p, \tilde{\phi}) , \quad \partial_s \tilde{F}^p = -2\tilde{\beta}_{F^p}(\tilde{g}, \tilde{H}, \tilde{F}^p, \tilde{\phi}) , \quad \partial_s \tilde{\phi}_s = -2\tilde{\beta}_\phi(\tilde{g}, \tilde{H}, \tilde{F}^p, \tilde{\phi}) . \quad (10.15)$$

This shows that the data $\{\tilde{g}, \tilde{H}, \tilde{F}^p, \tilde{\phi}\}$ satisfy a system of geometric flow equations $\tilde{\beta}^i$ that extends the conventional generalised Ricci flow to also include RR-fluxes. Furthermore, by reversing the effect of the diffeomorphism induces by $\nabla\phi$, this established the existence of a family of backgrounds $\{\bar{g}, \bar{H}, \bar{F}^p, \bar{\phi}\}$ that are solutions to the system defined by $\bar{\beta}^i$, hence following the equations of motion of Type IIA supergravity.

The flow equations for the form fields can be obtained by integration, yielding

$$2\partial_s \mathrm{d}\tilde{B}_s = 2\partial_s \tilde{H}_s = \Delta_{\mathrm{dR}}\tilde{H} + \mathrm{d} * (\tilde{F}_2 \wedge \star\tilde{F}_4 + \frac{1}{2}\tilde{F}_4 \wedge \tilde{F}_4 - m \star \tilde{F}_2) , \quad (10.16)$$

such that

$$2\partial_s \tilde{B}_s = -\mathrm{d}^*\tilde{H} + *(\tilde{F}_2 \wedge \star\tilde{F}_4 + \frac{1}{2}\tilde{F}_4 \wedge \tilde{F}_4 - m \star \tilde{F}_2) . \quad (10.17)$$

Similarly, the equations for the RR-fluxes can be integrated. This concludes the discussion and establishes short-time existence of the flow.

Remarks

- We directly imposed the equations of motion as the flow equations, loosing the notion of a gradient flow. One could try to adapt the variation of the action functional performed for the NSNS sector (cf. Section 4.1.4) to the present situation. Indeed, it turns out that all problematic contributions from the RR sector can be repackaged into the flow equation of g, such that

$$S(g, B, f)\left(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial f}{\partial s}\right) , \quad (10.18)$$

  with S the generalised scalar curvature defined in (4.25). This suggests that by again imposing the weighted unit volume constraint, one can generate a gradient flow that results in the flow equations imposed by hand before, except for the dilaton or f, which would be fixed by requiring the vanishing of (10.18). However, this flow equation is problematic because it comes with a negative sign in leading order. This is also the case for the conventional Ricci flow; however, in the standard case, the flow equation for f can be completely decoupled from the rest and then solved a posteriori. In the presence of RR-fluxes, this decoupling is no longer possible. This possibility will therefore not be pursued here further[6].

[6] However, note that the flow might exist under special conditions and hence may also lead to interesting flow behaviours for the SDC.This has to be checked on explicit examples and is left for future work.

10.2. A flow for 11D supergravity

There is an even simpler setting in which we can attempt to include RR-fluxes, namely at the level of 11D supergravity. In particular, in 11D there is no dilaton such that we do not face the problem of the different weighting of the RR fields and the metric that appears in the flow equations (10.1). However, on the other hand, this also means that there is no obvious way to impose the generalised unit volume constraint. So generically the flow will again not be gradient or strictly monotonous. This will have implications for the definition of a distance along the flow, discussed in Section 10.3. For now, recall that the action in 11D reads

$$\mathcal{S} = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x\sqrt{|g|}\left(R - \frac{1}{2}|F_4|^2\right) , \tag{10.19}$$

with κ_{11} the 11-dimensional gravitational coupling. Then analogously to the reasoning in the 10D case, by imposing the equations of motion as flow equations, we obtain[7]

$$\begin{aligned} \frac{\partial}{\partial s} g_{ij} &= -2\bar{\beta}^g_{ij} = -2\left(R_{ij} - \frac{1}{12}(F_4)^2_{ij} + \frac{1}{6}|F_4|^2 g_{ij}\right) , \\ \frac{\partial}{\partial s} C_{ijm} &= -2\bar{\beta}^{C_3}_{ijm} = \left(\nabla^k F_{kijm} + \frac{1}{2} \star (F_4 \wedge F_4)_{ijm}\right) . \end{aligned} \tag{10.20}$$

The short-time existence follows directly from the discussion in the previous section. Note that the symbol $\bar{\beta}^\bullet$ is gain of purely notational nature.

10.2.1. Example: S^4 with RR-flux

There is an obvious example to look at in this setting, which is the analogue of S^3 with H-flux: S^4 with $\mathfrak{f}_4$ units of F_4-flux. Indeed, the flow equations are readily obtained

$$\frac{\partial}{\partial t} r(s) = -\frac{3}{r(s)^2} + \frac{\mathfrak{f}_4^2}{3r(s)^7} , \qquad \frac{\partial}{\partial s}\mathfrak{f}_4 = 0 . \tag{10.21}$$

Note that similarly to the generalised Ricci flow of S^3, the flux does not flow since it is given by the volume form. However, it stabilises the flow of the geometry. The flow is plotted in Figure 10.1 and very closely resembles the one of the S^3 example with H-flux. We delay a discussion with respect to the SDC to the next section (and Chapter 11), where we discuss the problem of a distance for the flow.

[7] Note that a flow of the same form has already been proposed in [310], see also [502]. In these settings, however, the full 11D space is flown rather than only the internal space. This necessarily forces to work in the Lorentzian signature, again raising issues with the short time existence in general.

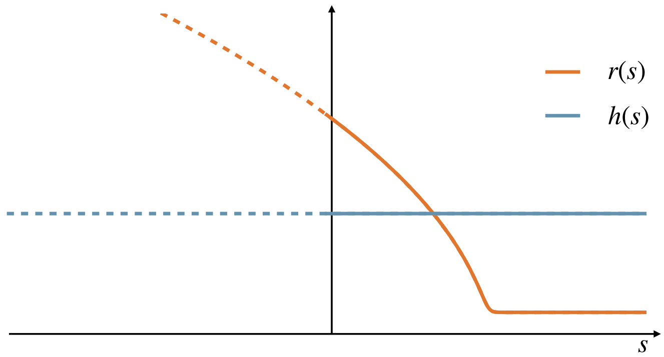


Figure 10.1.: Illustration of the 11D supergravity flow (10.20) on the example of S^4 with $\mathfrak{f}_4 = 3$ units of RR-flux. The flow behaviour is very similar to the one obtained for S^3 with H-flux, with the radius (orange) stabilising at a value set by the flux number (blue). However, the crucial difference is that the given background is not a proper string vacuum by itself, but has to be embedded into a full (11D) solution. Dashed lines correspond to an extension "backwards" in time.

Remark

- In the beginning of this paragraph we commented on the absence of the dilaton and the issue of lacking an obvious way to implement a generalised unit volume constraint. A possible alternative approach[8] could be implemented when considering full product spaces. Introducing a warping factor for the external metric in a way such that the problematic term is eliminated from the variation of the action may circumvent this issue. Upon dimensional reducing this could be seen as a non-trivial dilaton imposing the standard unit volume constraint.

10.3. Notion of distance and the SDC

Having added additional contributions to the flow, we are forced to appropriately adjust our notion of distance in order to correctly take into account these contributions. Once we are equipped with such a distance we can discuss the role or relevance of such flows and some examples for the SDC.

10.3.1. Distances

There are (at least) two obvious choices for distances along the new flow, which are closely related to the distances introduced in Chapter 9.

The first possibility is to again consider the flow solely as a tool for defining distinguished trajectories (off-shell) trajectories in configurations space. The distance along

[8]We would like to thank Chris Blair for a very insightful discussion, pointing out this possibility to us.

the flow can then be calculated using the notion of path length of the associated path γ in this space, which can now be written (schematically) as

$$\mathcal{M}_{\mathrm{II}} \simeq \mathcal{MG}(M) \times \mathcal{M}_{\phi} \times \prod_{p}^{k} \Omega^{p}(M)\,, \tag{10.22}$$

with p even/odd for Type IIA/IIB. A natural guess for a metric on $\mathcal{M}_{\mathrm{II}}$ should then be the analogue of $\Gamma = \gamma^{NSNS} + \gamma^{RR}$ derived Chapter 6, such that a naive guess is

$$\Delta_{\mathrm{II}} \sim \int_{\gamma} \mathrm{d}s \sqrt{\left| \int_{M} \mathrm{d}^{n}x \sqrt{g} e^{-2\phi} \Big(\|\partial_s g\|^2 + \|\partial_s B\|^2 + c_F \sum_{q} \big\|\partial_s C_q\big\|^2 + c_{\phi}\, \|\partial_s \phi\|^2 \Big) \right|}\,, \tag{10.23}$$

where the prefactors $c_{F/\phi}$ have to be worked out explicitly. This is left for future work.

The second possibility is to make use of the fact that there is a canonical choice of a functional along the flow. This is the Type II action $\mathcal{S}_{\mathrm{II}}$ given in (1.13), the equations of motion of which give rise to the flow equations (10.1), (10.2). However, as explained above, the flow is not gradient with respect to $\mathcal{S}_{\mathrm{II}}$ and hence the functional is generally not monotonous along the flow. Therefore, defining a distance according to

$$\Delta_{\mathcal{S}} \simeq \log\left(\frac{\mathcal{S}(g, F_4)|_{s_f}}{\mathcal{S}(g, F_4)|_{s_i}} \right), \tag{10.24}$$

leads to a distance which, in general, violates strict positivity[9]. A similar possibility would be to integrate the (spacetime average) absolute value of $\mathcal{S}$ along the flow. We will not discuss these possibilities here further.

10.3.2. Relevance for SDC

Clearly including the RR-fluxes was an essential and necessary step towards a much broader class of Type II string or SUGRA vacua. After all, there is only a very limited number of purely NSNS vacua. However, we are still not in a position to discuss full 10D vacua since so far we only discussed the Euclidean version of the flow, and the corresponding backgrounds should be viewed only as a (internal) part of a bigger product geometry. Then the only cases we could possibly discuss at this point are vacua that factorise into two separate vacuum solutions, where, for phenomenological reasons, the external part should be maximally symmetric. Focussing in particular on external Minkowski space, this means that we could ignore the external part due to the Ricci-flatness and obtain complete flow behaviour by only looking at the internal part. Unfortunately, it is well known that apart form solutions of the form $M_{d,1} \times X_{10-d-1}$ with X Ricci flat, there cannot be such simple, true 10D Minkowski vacua in the presence

[9] It might even violate positivity, but this can be easily cured by taking the absolute value.

of fluxes without adding negative-tension source contributions, due to the Maldacena-Nuñez no-go theorem [503]. While adding such contributions to the flow equations is in principle possible – as we will briefly discuss in the next section – it often forces us to consider the whole 10D solutions and therefore also the problem of Lorentzian signature and as well as other challenges.

10.3.3. SDC and distance for the S^4 example with RR-flux

We saw in Section 10.2.1 that similar to the NSNS case, the flow behaviour is simply given as a PDE on $r(s)$. Therefore, we can easily calculate the distance[10] using (10.23) which reduces to

$$\Delta_{\mathrm{II}} \simeq \int_{s_i}^{s_f} \mathrm{d}s \frac{\partial_s r(s)}{r(s)} \simeq \ln\left(\frac{r_f}{r_i}\right). \tag{10.25}$$

Hence, there are again two infinite distance points $r \to \{0, \infty\}$.

Only one of these, $r \to \infty$, corresponds to a fixed point, while $r \to 0$ is a singularity. This pattern is very suggestive since indeed on S^4 there are no winding modes and therefore no infinite tower of light states for small radius. However, we are anyway not in a position to make definite statements or interpretations in terms of the SDC or gRFC as explained in the previous section. In fact, the present background can be completed to a solution of 11D SUGRA of Freund-Rubin type [466] by choosing the external space to be AdS_7 with a radius correlated to the one of the internal S_4. Therefore in order to make definite statements for the SDC we would need to take this external contribution into account, possibly by investigating Ricci flow of the full product space. We will not further pursue this topic here in full detail, but we will give a brief outlook on such flows in the next section. A detailed discussion is left for future work.

10.4. Product spaces & Lorentzian signature

We explained before that the Type II flow, although much richer than the NSNS sector generalised Ricci flow, is not sufficient to make a definite statement for the SDC as most internal background configurations with fluxes are not vacua on their own. They have to be supplemented with contributions from the external space. Even more, if we want external Minkwoski or de Sitter space, we need contributions of negative tension like orientifold planes, as required[11] by the Maldacena-Nuñez no-go theorem [503].

In principle, adding such source contributions to the flow is rather straightforward

[10] Alternatively we can also use (10.24) which, at leats asymptotically, results in equivalent distances.

[11] The linear dilaton background of Section 6.2 giving rise to a Minkowski vacuum with internal S^3 and H-flux evades the Maldacena-Nuñez theorem since the dilaton (6.33) is a function of the external space.

when following the recipe for constructing the Type II flow introduced before. We merely add the relevant contributions to the equations of motion. This is certainly true in the absence of non-trivial worldvolume gauge fields and when considering the sources as static objects from the perspective of the flow. In general, however, it can be a more complicated endeavour. Furthermore localised contributions in the form of Dirac-Delta distributions require a particularly careful treatment and for this reason we will restrict the discussion to the so called *smeared approximation*. We already encountered this concept when we discussed the family of DGKT-like examples in Chapter 6 and their T-dual later on. We will again not be concerned with the debated role of their uplifts to string theory vacua but refer to [125,126,470].

For the above reasons, we therefore focus on smeared orientifold contributions in the following outlook. The contributions of O-planes to the action and equations of motion are discussed in Appendix A; see also [470,471,504]. Adding these contributions to the flow equations, the new flow equation for the metric and dilaton read

$$\begin{aligned}\frac{\partial}{\partial s}g_{ij} = -2\bar{\beta}^g_{ij} \equiv& -2R_{ij} - 4\nabla_i\nabla_j\phi + \frac{1}{2}H^2_{ij}\\ &+ 2e^{2\phi}\left[\frac{1}{2}(F_2^2)_{ij} + \frac{1}{12}(F_4^2)_{ij} - \frac{1}{4}g_{ij}\left(m^2 + |F_2|^2 + |F_4|^2\right)\right]\\ &+ \frac{1}{2}e^{\phi}\left(T_{ij} - \frac{1}{2}g_{ij}\sum_{\text{sources}} t_p\right),\end{aligned} \tag{10.26}$$

$$\frac{\partial}{\partial s}\phi = \begin{cases} -2\bar{\beta}^{\phi}_0 \equiv 4\Delta\phi - 4|\nabla\phi|^2 + R - \frac{1}{2}|H|^2 + \frac{1}{2}e^{\phi}\sum_{\text{sources}} t_p\,, \\ -2\bar{\beta}^{\phi}_* \equiv 2\Delta\phi - 4|\nabla\phi|^2 + |H|^2 + \frac{1}{4}e^{\phi}\left(g^{ij}T_{ij} - \frac{n}{2}\sum_{\text{sources}} t_p\right) \\ \qquad +\frac{1}{2}e^{2\phi}\left((1-\frac{n}{4})|F_2|^2 + (4-n)|F_4|^2 - \frac{n}{2}m^2\right)\,. \end{cases} \tag{10.27}$$

where T_{ij} and t_p are defined in the appendix. Similarly, one can define the flow equations for B and the RR-fluxes F_p, which we do not list here. Notice that the source contributions enter in zeroth order in derivatives, so they do not spoil the calculation of the associated principal symbol and the flow remains well-behaved. Note that we have again implicitly assumed a Euclidean signature such that the indices i should be understood as restricted to run only over a subset of some (10D) Lorentzian indices I.

As discussed several times, for Lorentzian signature even conventional Ricci flow turns out to be problematic, and the flow has to be investigated on a case by case basis. This clearly carries over to the flows defined in this chapter and, hence, in case we want to discuss the flow of full product spaces greater care is needed. We will leave a more in-depth investigation of the flow for these cases and in particular its interpretation and implications for the SDC and gRFC for future work. However, there is a specific subset of solutions that mostly evades the problems connected to Lorentzian signature. These are Minkowski vacua of the form $M_{d-1,1} \times K_{D-d}$ for which the geometry is of block diagonal form and *all* non-trivial sources and also dilaton contributions are restricted to the internal geometry, with the exception of O-planes, which are taken to be space filling in the external Minkowski space (and possibly also additionally extended along some internal directions). Then, due to the Ricci flatness of Minkowski space and the specific source setup, no derivative with respect to external coordinates x^μ contributes, in particular not the problematic ∂_t which comes with the negative sign $g_{tt} = -1$. However, note that this does not directly imply that the external flow equations are trivial $\partial_s X_{\mu_1 \dots \mu_d} = 0$ as their equations of motion have terms of the form $g_{IJ}(\dots)$, which are hence generically non-zero. However, we will ignore these contributions, as they will not influence the flow behaviour of the quantities of interest. After finding a solution to the internal flow with non-trivial fixed point, one then has to check that at the fixed point these external contributions also vanish; see also remark below.

10.4.1. Example: $M_{4,1} \times T^5$ with $O4$-planes

A straightforward example illustrating the new flow is the $M_{4,1} \times T^5$ with NSNS H-flux, RR two-form flux F_2 and spacetime filling $O4$-planes belonging to the generalised class of Giddings-Kachru-Polchinsky (GKP)-like solutions [505] found in [470]. In particular, we take

$$\mathrm{d}s = \eta^{(5)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \sum_i^5 (\mathrm{d}y^i)^2 \,, \qquad \Phi = \phi_0 = \text{const.}\,, \tag{10.28}$$

with fluxes given as

$$H = \mathfrak{h}\, \mathrm{d}y^3 \wedge \mathrm{d}y^4 \wedge \mathrm{d}y^5 \,, F_2 = \mathfrak{f}_2\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \,. \tag{10.29}$$

Together with the spacetime filling $O4$-planes this leads to a Minkowski vacuum if

$$\mu_4 = e^{-\phi_0}|H|^2 = e^{\phi_0}|F_2|^2 \quad \overset{\phi_0=0}{\Longrightarrow} \quad \mu_4 = \mathfrak{h}^2 = \mathfrak{f}_2^2 \,. \tag{10.30}$$

In order to discuss the generalised flow (10.26) we introduce an overall volume modulus or radius of a homogenous T^5 which we denote with $r(s)$ and discuss the flow behaviour of $g_{\mu\nu}(s)$ and $\phi(s)$ while keeping $\mathfrak{h}$ and $\mathfrak{f}_2$ fixed. In order to do so consistently, we have to pick our initial conditions compatible with the on-shell relation (10.30) as otherwise we would be forced to take into account also a flow of $\mathfrak{h}(s)$ and $\mathfrak{f}_2(s)$ that is introduced by the RHS of their flow equations being non-zero.

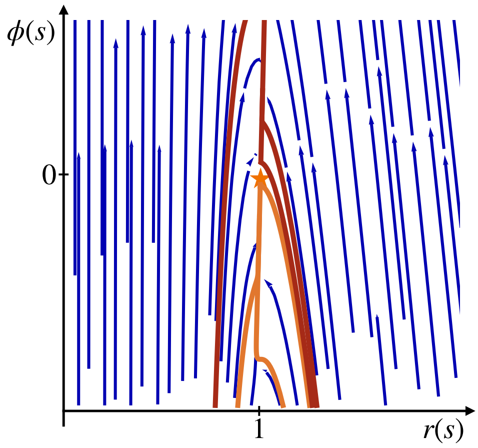


Figure 10.2.: Plot of the flow behaviour of the generalised Type II flow on the Minkowski vacuum with O4-planes discussed in this section. While curves sufficiently close to $r = 1$ and $\phi(s) < 0$ converge to the vacuum, generic initial conditions (at least with the chosen flux numbers and restricting to constant fluxes) are non-convergent (violet, blue).

In doing so, we obtain (the indices refer to the internal directions y^i)

$$\begin{aligned}
\partial_s g_{ii} &= \frac{\sqrt{\mu_4}}{4r(s)^3} e^{2\phi(s)} - \frac{\sqrt{\mu_4}}{4} r(s) e^{\phi(s)}\,, \quad \forall i \in \{1,2,3\}\,,\\
\partial_s g_{jj} &= \frac{\sqrt{\mu_4}}{2r(s)^5} e^{2\phi(s)} - \frac{\sqrt{\mu_4}}{4r(s)^3} e^{2\phi(s)} - \frac{\sqrt{\mu_4}}{4} r(s) e^{\phi(s)}\,, \quad \forall j \in \{4,5\}\,,\\
\frac{\partial}{\partial s}\phi &= \begin{cases} \frac{\sqrt{\mu_4}}{2} + \frac{\sqrt{\mu_4}}{r(s)^6} - \frac{3\sqrt{\mu_4}}{2r(s)^4}\ (= 2\bar{\beta}^{\phi}_1)\,,\\ \frac{6\sqrt{\mu_4}}{r(s)^4} e^{2\phi(2)} - \frac{4\sqrt{\mu_4}}{r(s)^6} - 2\sqrt{\mu_4} e^{\phi(2)}\ (= 2\bar{\beta}^{\phi}_2)\,. \end{cases}
\end{aligned} \tag{10.31}$$

It is clear that indeed $r(s^*) = 1$ is a fixed point of the flow giving rise to the vacuum described above[12]. We can solve the flow equations numerically. It turns out that the flow only converges for special initial values of and, furthermore, the convergence depends on the flow equation chosen for ϕ. The flow is visualised in Figure 10.2, where we plotted the vector field corresponding to the flow of $\{r(s), \phi(s)\}$, together with some selected curves, highlighting the convergence or non-convergence of the flow.

Note that alternatively we could have also taken the dilaton as constant (hence "forgetting" the flow equation for ϕ) but then in order to find proper vacua that fulfil in particular (10.30) we need to allow for a flow of the flux numbers $\mathfrak{f}_2$ and $\mathfrak{h}$. For suitable chosen initial conditions, these should then evolve in a way such that (10.30) is again satisfied for the constant ϕ_0 under consideration. We will not discuss these different aspects further here, but leave a closer investigation for future work.

[12] One can check that then automatically the external equations of motion are satisfied, i.e. $\bar{\beta}^g_{\mu\nu}$.

10.4.2. The SDC and generic vacua

We have only discussed one particular example for the full flow, and in order to gain better intuition concerning, for example, the convergence properties of the flow, a more detailed discussion is needed. This short section should hence be viewed as a proof-of-concept with the main discussion left for future work. In particular, in view of the Swampland program and the SDC, there are many points that need to be investigated: the ability to visit different infinite-distance points, the existence of the relevant towers, and possible implications or modifications for the (generalised) RFC. The ultimate goal or hope is then that an in-depth discussion from this new approach can give new perspectives on debated backgrounds like the DGKT-like AdS vacua discussed in Chapter 6 and 7 and related issues, e.g. the role of smeared sources or scale separation.

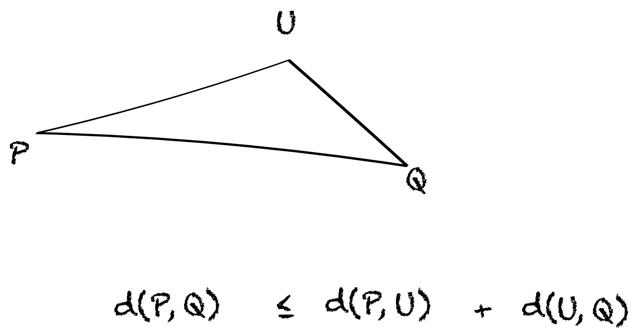
We close this section, highlighting some central questions and further directions relevant for a further investigation of the topic:

- A natural next step would be to consider vacua with external AdS metric, for example the DGKT-like backgrounds discussed earlier. Having a non-flat external space will lead to additional contributions; in particular the "repulsive" nature of AdS in view of Ricci flow, arising from the negative curvature can lead to challenges concerning the convergence properties of the flow to finite fixed points.
- While most details carry over directly to Type IIB, there is a subtlety due to the self-duality constraint of the five-form flux. This is better addressed in the democratic formulation [506].
- A consistent vacuum does not only require the equations of motion to be satisfied but crucially also the *Bianchi identities*, which should also be checked at the fixed points. These issues are also easier to handle within the democratic formulation.
- The flow defined in this chapter motivates a *generalised Ricci-Bourguignon flow*

  $$\partial_s g_{ij}(s) = -2\left(\bar{\beta}^g_{ij} + \rho \mathrm{tr}(\bar{\beta}^g_{ij}) g_{ij} - \frac{\kappa}{4}\bar{\beta}^\phi_0 g_{ij}\right) , \qquad (10.32)$$

  which might have similar applications as the standard Ricci-Bourguignon flow [292, 507] used in [392] for a discussion of product spaces.
- In view of the SDC there remains the definition of a length, the analysis of fixed points and in particular infinite-distance points, as well as the role of the (smeared) sources and related aspects like scale separation. In particular, it would be interesting if a treatment via Ricci flow can shed new light on these issues.

We plan to investigate these examples in the future.



CHAPTER 11

The SDC on field space: distance or cost

The Distance Conjecture in its standard formulation is based on a well-defined notion of distance, induced by the moduli space metric γ_{ab} and its associated geodesics. Measuring distances according to the (shortest) geodesic guarantees that all the mathematical requirements of a proper notion of distance are satisfied. However, from the discussion in the previous chapter it has become clear that our proposed definition of distance in the presence of a potential might violate some of these properties. This is not particular to our approach, but has in fact been observed also in other approaches trying to define a distance in the presence of a potential. In the following chapter, we will provide an in-depth discussion of the challenges that arise for our proposed notions of distance, as well as a comparison with other approaches in the literature. This will lead us to the conclusion that in the presence of a non-zero potential, the requirement of having a proper (mathematical) notion of distance might be too restrictive. Instead, we will give a brief outlook on a possible alternative approach via the theory of Optimal Transport based on work in progress [71].

11.1. What to expect of a distance

Having the underlying higher-dimensional theory at our disposal, in Chapters 9 and 10 we explained how we can evade [68] the problem of the path in the presence of a potential by lifting the problem back to the underlying geometry and using geometric flows. However, in the course of these discussions – and also in Chapter 6 – we have seen that in the presence of fluxes, or more generally, in the presence of a scalar potential, naive measures might not lead to a well-defined (mathematical) notion of a distance any more. For example, by loosing properties like strict positivity or due to the fact that in the presence of a potential, the notion of (naive) geodesics does not necessarily coincide with physical paths anymore. Indeed, it was pointed out, for example in [428], that due to forces introduced by the potential the trajectories are driven to paths that deviate from geodesics; see also the discussion in [373]. Consequently, it is not clear if such a notion of distance, or better length, is on equal footing with the standard geodesic distance on (flat) moduli spaces when it comes to the application of the Distance Conjecture. A deviation from the standard notion of distance was for example also observed in [383], where – inspired by the Maupertuis's principle[1] – a distance in the presence of a generic scalar potential was defined. However, even though passing some nontrivial test and – similar to the RFC – reproducing the ADC, the distance fails to satisfy the *triangle inequality*. A closely related – though not equivalent – proposal was put forward in [384] actually evading some of these problem. However, it comes with other drawbacks as it is, for example – similar to Ricci flow – bound to certain trajectories. Other definitions, like the discrete approach of [456] succeed in satisfying such principles but are difficult to compute in concrete examples. Distances between families of vacua were also discussed in [382, 447, 448] as well as [457, 458, 509]. Furthermore, we saw in Chapter 6 that in the case of a consistent theory with diverging potential, there can be "would-be infinite-distance points" which lack a tower of states [67] and such situations have to be properly accounted for also by a definition of distance in the presence of a potential, or at least by suitable adjustments of the SDC.

Before we go into a more detailed discussion of the distances proposed in this thesis, as well as a short comparison with alternative existing proposals in the literature, we quickly recollect the required axioms of a distance.

11.1.1. Distance axioms

From a mathematical perspective, the requirements for a function $d(\cdot,\cdot)$ to define a distance are strictly set. In particular, for M a set, d is a *distance* (or metric) if it satisfies

[1] See, for example, [508].

the following axioms:

i) *Positivity:* $d(P,Q) > 0$ for $P \neq Q$... *inequivalent points have non-zero distance.*
ii) *Identity of indiscernibles:* $d(P,P) = 0$... *the distance of a point to itself is zero.*
iii) *Symmetry:* $d(P,Q) = d(Q,P)$... *the length form P to Q is the same as from Q to P.*
iv) *Triangle inequality:* $d(P,Q) \leq d(P,U) + d(U,Q)$ ... *generalisation of "a straight line through P and Q is the shortest path".*

Points i) and ii) are sometimes collectively referred to as *strict positivity*.

The SDC in its core formulation is crucially based on the existence of a well-defined notion of (geodesic) distance and associated length-minimising paths. It is therefore worth going through the above axioms and comparing with the situations we countered for the distances or lengths defined in the previous chapters.

Before doing so , there is a comment in order concerning the two major approaches for defining a notion of distance along the flow. The first was through the use of some functional $\mathcal{E}$ (not necessarily the Perelman functional $\mathcal{F}$) that might or might not have nice monotonicity properties along the flow. In the former case, it can be integrated along the flow to give rise to a nice definition of distance, as explained below. In case such a (strong) monotonicity is missing, one can still use the functional to define a distance, for example, through expressions like $\Delta \simeq \log\big(\mathcal{E}(s_f)/\mathcal{E}(s_i)\big)$; cf. Chapters 9 and 10. The second possibility is to consider the flow as a path on a certain configuration space (for example $\mathcal{M}(M)$ or $\mathcal{MG}(M)$) and measure the distance or better length of the associated path γ, using the (natural) metric on the configuration space. In both cases, the Ricci flow serves as a tool to identify the appropriate path. While in the latter case, this is the "only" purpose of the flow, in the first case, the associated functional also enters the distance. This means that in the second case the corresponding potential only influences the selection of the path , while it is not clear in which way the metric on this space captures the potential. In the first case, on the other hand, the geometry and fluxes, and hence – by the correspondence explained in the previous chapters – the distance, receive additional contributions through the generating functional. Hence, in this case the distance itself is "weighted" by contributions that mirror the cost of moving in the presence of a potential. This is a central point, and we will come back to this in Section 11.3. The distance Δ_L is a special case, as we showed in Chapter 9. When using Δ_L, we actually measure the distances in $\mathcal{GM}(M)$ with the natural associated metric given in terms of the generalised DeWitt metric (4.12) and therefore the two approaches are actually equivalent in this case.

Ad i) & ii): For distances defined via geometric flow equations and a (generating) functional, strict positivity of the distance is directly connected to the strict monotonicity properties of the functional along the flow. In particular, for the distance Δ_L defined

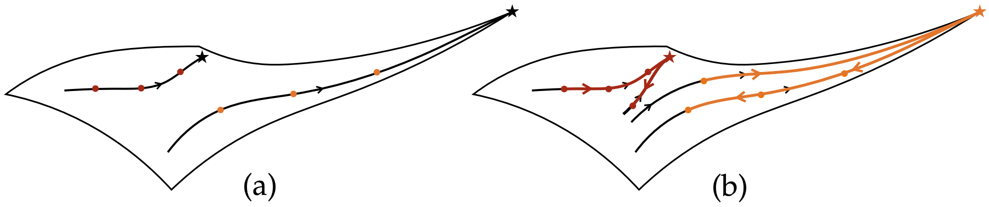


Figure 11.1.: Due to the fact that Ricci flow selects specific trajectories in the configuration space $\mathcal{MG}(M)$, distances are only defined for points on the same orbit. Allowing to connect points on different orbits through fixed points one can assign also distances to points not within the same flow trajectory. In order to obtain sensible results one should only allow such a procedure for fixed points at finite distance.

in (9.11), strict monotonicity is directly inherited from the monotonicity of the generating functional $\mathcal{F}$. As mentioned in Section 9.2.1, the functional is strictly monotonous along the flow and zero only at fixed points. Therefore, the distance Δ_L inherits strict positivity along the flow. However, it is clear that this property is directly correlated with that of $\mathcal{F}$. Once other functionals are used, or the flow modified such that $\mathcal{F}$ is no longer strictly monotonous, also the distance loses the respective property. For example, in Section 9.2.1 – after weakening the unit volume constraint – the resulting flow was not even gradient anymore. The same is true for the supergravity flows including RR-fluxes in the previous chapter. One could weaken the requirement of strict positivity to only require positivity, for which a monotonous function is sufficient while still using the length defined by (9.11). However, in such situations, a nontrivial evolution along the flow might not come hand in hand with an increase of length. Understanding the relevance of such particular cases for the SDC would be an interesting task. Distances measured using the path length in the associated configuration space are strictly positive by definition.

Ad iii): While strict positivity – or its failure – is relatively straightforward and unambiguous to establish, symmetry is subject to interpretation. Geometric flows – in a strict sense – single out a unique direction along the flow, establishing the distance $d(P,Q)$ for two points along the flow trajectory from P to Q. For these points, there is no *different* flow solution running from Q to P as this would conflict with the monotonicity of the functional $\mathcal{F}$. The only way to assign the distance $d(Q,P)$, is via $d(Q,P) = -d(P,Q)$, which we can think of as transverse the flow "backwards[2] in time". In this case, symmetry is satisfied by definition. Similarly, a definition via the path γ on configuration space is symmetric by definition as we can transverse the path in reverse direction.

[2]This backwards evolution should be understood heuristically as running back the flow trajectory define by the "forward" flow and not in the literal sense of backwards Ricci flow. The latter is in general ill-defined as explained also in previous chapters.

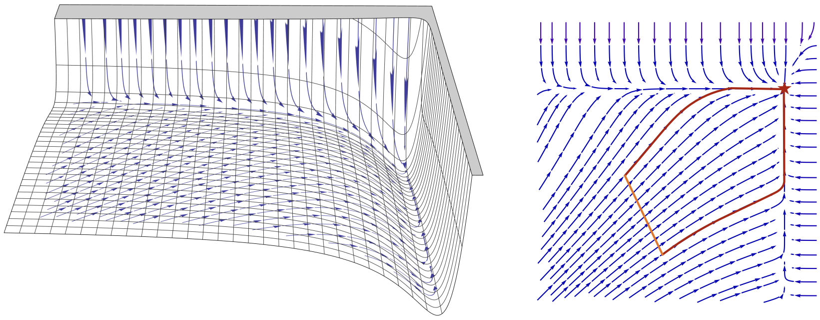


Figure 11.2.: Illustration of the principle explained in point iii) concerning the triangle inequality on the explicit example of $S^3 \times S^3$ with H-flux. By choosing appropriate initial conditions it is possible to generate a flow through every pair of points $\{r_1, r_2\}$, hence covering any point in the configuration space or radii (blue). However in order connect two arbitrary pairs of points, on needs to pass through the fixed point $\{\sqrt{k_1}, \sqrt{k_2}\}$ (at finite distance; red). The orange line corresponds to the shortest geodesic path in the space of radii according to the natural metric on the space of metrics.

Ad iv): Point three highlighted the fact that the definition of paths through (geometric) flow equations raises subtleties concerning the symmetry criteria of the associated distance. This problem is even more present when considering the triangle inequality. For some given initial conditions, the generalised Ricci flow singles out a (unique) path in the space of generalised metric $\mathcal{GM}(M)$. In particular, given some initial point P, in general there is no other flow trajectory that connects P with some other point Q. In general, the only possibility of connecting such points is by finding a common fixed point of the flow (if it exists) and defining the distance piecewise, cf. Figure 11.1 and 11.2. In order to obtain sensible results, such a fixed point has to be of course at finite distance, as otherwise, one would artificially generate infinite-distance points. Furthermore, distances between points P and Q can be measured simply along the geodesics of this generalised DeWitt metric on configuration pace. These will in general not coincide with Ricci flow trajectories, and in particular taking these points into account, one could conclude that the triangle inequality is not satisfied[3]. However, it is not clear how these paths translate to field spaces with non-trivial potentials. In particular they might corresponds to very counter-intuitive paths crossing potentials in a way that follows no first-principle criterion, cf. Figure 5.3. This is in clear contrast with paths defined by the flow that arise as gradient flows of $\mathcal{S}$ or more precisely $\mathcal{F}$. In principle, one could also ask for the triangle inequality to hold for points that can be connected via a flow. Then the triangle inequality is satisfied by construction.

[3] A similar issue arises in [384] where paths are defined via attractor solutions.

There are two other related points or issues that are worth discussing. The first, already alluded to above, is a possible **incompleteness** of reachable points. Usual moduli spaces are given by geodesically complete spaces, and hence measuring distances between points along geodesics is exhaustive. However, while Ricci flow in principle allows for arbitrary points as initial conditions, all other points along the evolution are then fixed by the flow equations. Furthermore, infinite-distance points are often only reachable by extending the flow backward in time, which might not be possible. However, it is worth noting that infinite-distance points usually correspond to certain extrema of the underlying geometry, and these are often covered by Ricci flow.

Lastly, recall that $\mathcal{GM}(M)$ itself comes with a natural metric and hence notion of **geodesics**. In general, these do not coincide[4] and are, generically, incredibly complicated. Furthermore, they lack a clear physical meaning as they do not follow a (action) minimising principle, i.e. they are not on-shell solutions of the equations of motion. It is not clear if and how such distances are covered by the (generalised) SDC.

The above points leave us in a certain dilemma. While for true moduli spaces, there is a well-defined and unambiguous notion of distance, the discussion on Ricci flow as well as arguments already raised in Chapter 5 seem to suggest that once we take into account a non-zero scalar potential, a generalised – somehow weakened – notion of distance might be favoured. We will give a brief perspective in favour [71] of such a more general measure in terms of so-called *cost functions* within the framework of *Optimal transport* in the final Section 11.3. But before that, we briefly contrast our findings with other proposals for distances and the SDC in the presence of a potential.

11.2. Relation to other proposed notions of distance

We briefly comment on some possible connections of our proposed notion of distance to other existing work on the definition of a distance in the presence of a potential or, more generally, on scalar field space.

Before entering any discussion on related approaches, it has become clear from the previous chapter that in the context of geometric flows there are essentially two options. The first is to flow (some part of) the total geometry or string vacuum. Depending on the setup, one can then either flow only the internal geometry and then translate this into a motion on scalar field space. Or flow the total geometry, which translates into a trajectory on the full configuration space, including possible variations of external parameters. This was extensively studied in the present work. The second option, only

[4]Notice that even standard moduli spaces, i.e. in the absence of a potential, allow for several inequivalent measures, leading to possibly different predictions for the SDC; cf. [510]. See also [444] for a discussion on issues arising due to diffeomorphism (non-)invariance of distances on the space of metrics.

briefly mentioned in chapter 5 is to define distances directly with respect to scalar field space of a generic effective action

$$\mathcal{S}_{\text{eff}} = \int \mathrm{d}^d x \sqrt{-g} \left(\frac{\bar{M}_{P,d}^{(d-2)}}{2} R(g) - \frac{1}{2} \gamma_{ab}(\varphi) \partial_\mu \varphi^a \partial^\mu \varphi^b - V(\varphi) \right) . \tag{11.1}$$

Now, for example, if this action descends from compactification on an internal Calabi-Yau, the kinetic term of the scalars is determined by a Kähler metric $\gamma_{ab} \equiv \partial_I \partial_{\bar{J}} \mathcal{K}$ with $\mathcal{K}$ the Kähler potential of the Calabi-Yau. Whenever $V = 0$, one could try to directly flow the moduli space metric γ_{ab}. This would give an alternative way of exploring genuine moduli spaces. Alternatively, one can try to directly define [284] a flow for the system (11.1) that also takes into account the potential V and the dynamical d-dimensional metric g. Such an approach has partially been explored, e.g. in [284], the resulting flow is generally not as nicely behaved as the flows that operate directly at the target space geometry and hence evade explicit potential terms. The Lorentzian signature of g further influences this negatively. However, it is exactly an action of the form (11.1) that is the starting point of the consideration in [383] and [384]. A direct comparison of the definitions of these publications with the ones presented here is therefore nontrivial as it would require a careful analysis of flows on the scalar field space directly.

We nevertheless point out that in particular, in view of AdS spaces and the associated ADC, the proposals of [383] and [384] and a treatment via Ricci flow lead to equivalent results, effectively reproducing the ADC. Indeed, we reviewed in Chapter 5 how by studying Ricci flow of (external) AdS spaces, the Ricci Flow Conjecture directly reproduces the ADC and the associated distance

$$\Delta \simeq \log \left(\frac{\Lambda(t_f)}{\Lambda(t_i)} \right) . \tag{11.2}$$

This distance was also obtained in both [383] and [384].

Similarly to the treatment in [384] which uses attractor solutions to define the trajectories for the SDC in the presence of a potential, in [444, 511, 512] the authors used the attractor mechanism to probe moduli spaces with supersymmetric BPS states. The distance is in turn related to the difference in (squared) ADM mass of the associated black hole. In some sense these approaches are similar to the first scenario described above, flowing the internal space – here a Calabi-Yau – by the means of some evolution equation that gradually changes the internal moduli along the flow. However, crucially, the attractor mechanism as such is not a geometric flow and, in particular, does not have a direct generating functional.

It is therefore clear that so far there is no agreed upon universal[5] proposal for such a distance that is fully satisfactory, or can cover all possible scenarios. Combining all these facts naturally leads to the conclusion that – at least from the point of view of the (generalised) Ricci Flow Conjecture – lengths are rather measured with respect to a minimising cost principle, rather than a genuine distance. This, in turn, motivates a treatment form a (mathematically) more abstract point of view, leading to the framework of *Optimal Transport*, which we turn to in the next and final section of this thesis.

A nice comparison between the proposals in [383] and [384] can be found in the appendix of [384]. For more details on the discrete notion of distance, we refer to [456]. Other proposals of generalised distances that we could not cover in more detail here can be found in [513–516].

11.3. A proper distance on field space via Optimal Transport?

In view of the previous discussion, finding a proper notion of distance in the presence of a potential is very reminiscent of defining a *cost function* for an optimisation problem; see, e.g. [517,518]. In fact, it has been shown recently [500,519] that (generalised) Ricci flow can be realised in such a way, more precisely within the framework of *Optimal Transport* [520] by choosing a suitable generalised cost functional that is closely related to the so-called *Wasserstein distance* on the space of probability measures. The following offers a brief outlook on work in progress of the author and collaborators to appear in the near future [71].

In order to illustrate this some more, take the following toy optimisation problem, illustrated in Figure 11.3. Given two piles of sand of a given volume $\mathcal{V}$ and a specific shape, what is the best or most efficient way to move the (mass distribution of) sand from one pile to the other? Or, formulated differently, what is the *optimal* way of moving a distribution μ supported on (some subset of) $\mathbb{R}$ into a different distribution ν on $\mathbb{R}$ of equal measure. In order to find the optimal way, we need to define a so-called *cost function* $c(x, y)$, quantifying the "cost" of moving the sand from one point x to another point y. In this one-dimensional example we can simply take the Euclidean distance

[5] Note in particular that in general, different notions of distance might lead to different infinite-distance points, and more importantly it might be that finite distance point according to one definition of distance, might be at infinite distance with respect to another definition, causing obvious conceptual problems for the SDC. Such scenarios were examined, for example, in [510] and we refer to it for details. In any case, it raises the question of the existence of a *unique* definition, realising some underlying physical principle.

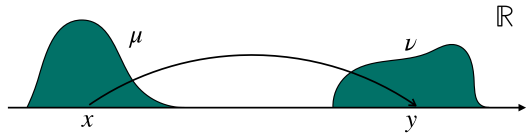


Figure 11.3.: Illustration of mass distributions over the real line. The optimisation problem corresponds to finding a (volume-preserving) map that takes μ into ν in a way that minimises a certain "cost" functional, set by a cost function $c(x, y)$.

$d(x, y)$ as a cost function $c(x, y)$. A map realising the transport of the mass density in a minimal way with respect to the cost function d is then an *optimal transport map*.

Normalising the distributions to unity, we can view them as probability distributions on a metric space M, which in this case is simply $M = \mathbb{R}$. The optimisation problem just sketched can then be rigorously defined within the framework of *Optimal Transport*, for example in terms of the associated *Monge-Kantorovich* problem. We will not enter this here in detail, but focus on a closely related quantity.

In particular, for (M, d) a (separable and complete) metric space denote by $\mathcal{P}(M)$ the space of probability measures on M. On the space $\mathcal{P}(M)$ we can define the *Wasserstein p-distance* W_p between distributions $\rho_0, \rho_1 \in \mathcal{P}(M)$ via

$$W_p(\rho_0, \rho_1) = \inf_{\pi \in \Gamma(\rho_0, \rho_1)} \left(\int_{M \times M} d(x, y)^p \, \mathrm{d}\pi(x, y) \right)^{1/p} , \tag{11.3}$$

where π is a coupling (a joint probability measure) with marginals ρ_0 and ρ_1.

The distance W_p is then exactly the total cost associated with the optimal transport plan minimising the Monge-Kantorovich problem with cost function $c(x, y) = d(x, y)^p$. The Wasserstein distance W_p can therefore be viewed as a measure of how different these distributions are and, crucially, defines a proper distance on the space of probability measures. This raises the following question:

Can we formulate the problem of distances on scalar field space in the presence of a potential within the framework of optimal transport and measure distances using a (generalised) Wasserstein distance?

It is then clear that one of the central aspects of such a proposal is a sensible translation of the field space problem to a problem in terms of probability measures, as well as the identification of a suitable cost function $c(x, y)$ that takes into account the effect of the potential. Indeed, to gain intuition for the latter, we can again look at the toy example introduced at the beginning of this section. While the example was introduced over the one-dimensional space $\mathbb{R}$, let us generalise the setup and consider the distributions on $\mathbb{R}^2$. The mass distribution is then over some area specified by points (x_1, x_2) and a

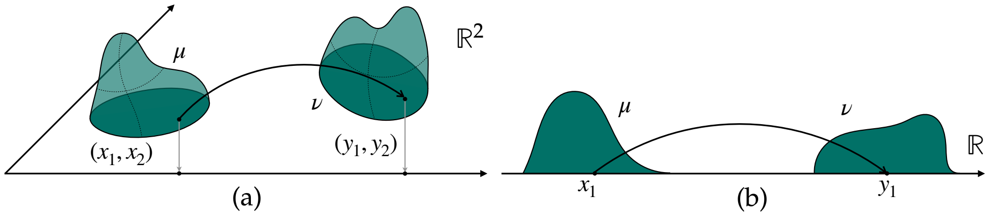


Figure 11.4.: Starting from distributions on a two-dimensional metric space $M = \mathbb{R}^2$ (figure (a)) and projecting onto the one-dimensional subspace $y = 0$, the distance or cost on the resulting space (figure (b)) can be viewed as "ignoring" some contributions from the original example. The inverse problem of finding a cost function that takes into account additional information besides the position x_1 – here the position in x_2 – is the toy analogue of the problem of the scalar potential.

natural cost function is simply the Euclidean distance in two dimensions, which now does not only take into account displacements in $x = x_1$ to y_1, but also displacements from x_2 to y_2. We can project this 2d example onto the real line along x_1 and it is clear that the cost associated to the two scenarios differs in general; see Figure 11.4. Viewing the additional direction as some kind do height, we can formulate this by saying that in the 2*d* example our cost function also takes into account the cost of moving the mass vertically (lifting it at the cost of potential energy), while in the one-dimensional case obtained form projection, we only take into account horizontal displacements. The problem of finding a distance in the presence of a potential can then be seen as the inverse of this toy example. Given a notion of distance or cost on scalar field space, how do we need to modify the cost function in order to take into account the cost of moving in a potential. While for the toy example this amounts to taking the canonical higher-dimensional distance $d(x, y)$, for the scalar field space there seem to be several possible scenarios that also depend on the precise embedding of the field space into a space of probability measures. This will be discussed in the forthcoming publication [71].

The second big question – which in principle has to be addressed first – concerns the embedding of scalar field space into a space of probability measures. We will not enter any details as this is also still subject to current investigations [71]. One possible approach seems to be to work with systems that allow for a formulation in terms of an associated Hamilton-Jacobi problem which can be related in a canonical way to the Wasserstein distance. These arise, for example, in the framework of *(fake) supergravity* [521]. The associated probability distributions in this case can be obtained from the Hamilton-Jacobi system together with a Wentzel–Kramers–Brillouin (WKB) approximation. Additionally we can use the work of [522] and the associated "wave-function of the universe" [523] satisfying the Wheeler-DeWitt equation [524–526] in order to include also a discussion of dynamical spacetimes. We plan to report on these approaches in the forthcoming publication [71].

Part III.

Conclusions

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Summary and future directions

The scalar field space associated to a generic (low-energy) effective theory is not flat, but comes with a nontrivial scalar potential. Therefore, taking into account the implications of such a potential is crucial when formulating Swampland criteria, in particular for the Distance Conjecture. In this work, we investigated this problem from various directions, presenting novel approaches and discussions that allow for an extended application of the SDC.

Focussing on the Distance Conjecture, we performed [67] an in-depth analysis of the implications of topological T-duality for spaces with nonzero curvature and in the presence of fluxes. Carefully tracking the exchange of towers of states under the topology-changing transformation, we explained that due to the topological properties of S^3, there is a lack of towers of states required for the SDC. However, observing a divergence in the scalar potential along this direction, we argued that this problematic direction should be viewed as screened by the potential. We concluded that due to the complicated topological and symmetry properties of curved backgrounds and the presence of fluxes, this should be a more general feature of diverging potentials. This led us to propose an amendment of the SDC presented in Section 6.1.4, extending the SDC to these situations. This was substantiated on further examples, presenting also a first step towards a more realistic examples to check our proposal.

Generic points in the Landscape are expected to correspond to (some form of) non-geometric configuration, rather than a geometric background. As such, they should necessarily be included into the Swampland discussion, and this represents the main motivation of Chapter 7. Starting from a particular realisation of such backgrounds in terms of non-geometric Q- & R-fluxes introduced in Chapter 3, the results of [67] provide a first discussion of such spaces in the context of the Distance Conjecture. We incorporated these spaces into the reduction procedure and showed that a consistent picture for the SDC indeed requires a transformation to the non-geometric β-frame in order to obtain consistent effective actions for the SDC. We explained some issues that arise for the 3D T-duality chain, suggesting it actually lies in the Swampland, and also how some of these issues are resolved by moving to the more complete 6D analogue that can be realised in terms of 10D supergravity solutions.

In Chapter 8 we expanded our discussion to also include a first exploration [67] of the role of generalised T-dualities in the Swampland. It was explained how by using the associated transinformation, one can access an even wider range of (possibly infinite-distance) points on field space through dualities. We employed the framework of Poisson-Lie T-duality to obtain the non-Abelian T-dual of S^3, which turned out to give a highly nontrivial realisation of a non-geometric background, necessitating a careful analysis using β-supergravity. We clarified the (exotic) exchange of zero modes and the consistency under T-duality as well as the duality invariance of the metric and potential on field space, together with some implications for the SDC.

The second major approach toward a treatment of potentials initiated in [68] and introduced here in Chapter 9 was the use of geometric flow equations. In particular, we utilised generalised Ricci flow in order to translate the problem of defining distances on field space to the space of generalised metrics and trajectories defined by generalised Ricci flow. We showed how the Ricci Flow Conjecture of [381] can be extended to flux-supported internal spaces. This suggests a new way to systematically explore scalar field space with potentials or string theory field space, respectively. In particular, if offers an alternative perspective on the divergent potential observed in Chapter 6. Effectively geometrising the exploration on field space, this new approach bypasses the problem of defining paths on field spaces with potentials and the associated challenges by using generalised Ricci flow to single out paths on scalar field space. We refined the existing notion of distance for the RFC and showed that, for gradient flows, it is directly related to the natural distance on the space of generalised metrics, as well as giving a first-principle derivation of the logarithmic dependence proposed in [381]. Furthermore, we discussed the role of the field f or dilaton and proposed some alternative flow equations as well as associated notions of distance.

Building on the intuition gained in previous chapters, in Chapter 10 we further generalised the flows to also include RR-fluxes as well as source contributions, leading to a flow for Type II supergravity. This ultimately also allows us to discuss full 10D (product space) string vacua, at least in the special setting of Minkowski vacua. We explained how these flows can potentially give new perspectives on issues like scale separation or the role of smeared sources.

Finally, Chapter 11 addressed the fundamental question of the definition of a proper notion of distance in the presence of a scalar potential. We contrasted the axioms of a distance with the properties found for the notions of distance defined in earlier chapters, as well as a comparison with other proposals in the recent literature. This led us to the conclusion that a definition in terms of a cost rather than a proper distance might be favoured in these settings. We gave a brief outlook on some work in progress, aiming to realise this idea within the framework of optimal transport.

Altogether, it is fair to say that the role of the scalar potential of the SDC is a deep and intricate problem that requires a careful and thorough treatment of all associated issues. While here we offered some appealing approaches in that direction, there are still many open questions and further aspects that need to be investigated:

Tackling more realistic examples, AdS and scale separation. First of all, the behaviour of the scalar potential should be tested in other – potentially more involved – setups, together with a careful analysis of the involved towers of states in order to understand the precise circumstances under which the proposed divergence in the potential appears. The same holds for the various flow equations introduced here. In particular, it would be very interesting to investigate the behaviour of the generalised flows for nontrivial product spaces, for example also AdS vacua. As explained in Chapter 10, this might lead to new perspectives on hotly debated issues like scale separation [126] or the role of smeared sources [125, 389, 471, 479].

Topology change Topology change is believed to be a universal aspect of quantum gravity, prominently featured in string theory [165–170]; see also [171] and [172]. We explained how this is also highlighted by T-duality extending to a map between manifolds of different topology in the presence of fluxes. These changes can also occur in the framework of deformations and the change of the associated deformation parameters [527]. The moduli space components of spaces with different topology sometimes correspond to moduli space components with different topology [134, 528, 529], and it would be interesting to see how this relates to the SDC but also with[6] dynamical cobordism [324–329].

Flux variations and distances on the space of string theory vacua. Furthermore, starting from configurations that do not satisfy the equations of motion for the fluxes, these new flows can lead to situations in which we can study flux variations from a new perspective. This is particularly interesting also in view of flux variations for families of string solutions that go beyond the rather simple cases of AdS Freund-Rubin or DGKT solutions discussed in Chapter 6. It is not fully clear how to generalise the variational[7] procedure initiated in [447] and further studied in [382, 448] for external AdS spaces – or even more complicated warped spaces – to arbitrary variations (like the ones studied in a slightly different manner in Chapter 6) and (generalised) Ricci flow on product spaces might offer a new perspective on this.

Backreaction and quantum corrections. Having introduced (smeared) sources for our flow equations, it would be interesting to also study their localisation, nontrivial warping, as well as the role of quantum corrections, in particular in view of backgrounds like the DGKT example discussed in Chapter 6; see also [125]. Using the Ricci

[6]We thank Andriana Makridou for a very nice talk at the workshop "Lotus & Swamplandia" in Naxos and a subsequent discussion on this topic in the context of dynamical cobordism.

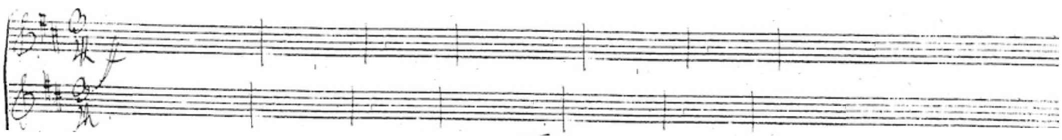
[7]We would like to acknowledge a nice discussion with Nicolò Petri on these, and related, issues.

Flow Conjecture it might be possible to track such corrections by adding appropriate terms to the flow equations. Such corrections have been studied for example using integrability techniques [530]. Flows with higher-loop contributions have also been discussed in [531] starting from the two-loop RG flow of the Heterotic string, which in fact can be seen as a modification of generalised Ricci flow by higher-order corrections or alternatively a flux-modified RG-2 flow [532]. In view of higher-order curvature corrections also the possible connection to [533] seems intriguing.

Generalised Calabi-Yau spaces and geometric flows. Other prominent flows that might have interesting implications for the Swampland program are Kähler-Ricci flow [534] and its generalisation [535], or the Hull-Strominger system [536, 537] and associated flows that have fixed points that solve the anomaly cancellation conditions of the aforementioned system, see, e.g. [538, 539]. The former would open up the possibility of studying $\mathcal{N} = 2$ flux compactifications and generalised Calabi-Yau manifolds from a flow perspective. Similarly, the application of flows for (truly) non-geometric backgrounds, as well as the role of generalised dualities in the context of flows, is still an open question.

A relation to information theory. A last point we would like to mention concerns the issues associated with a proper definition of distance in the presence of a potential and possible connections to an information theoretic distance. In particular, we explained that – especially from a point of view of Ricci flow [500,519] – there is a close connection to optimal transport. While the author is currently investigating this approach in some upcoming work [71], it would be interesting to also connect this with classical information theoretical approaches and possibly connect to the conclusions of [515,516]; see also [540, 541].

In conclusion, we see that much like the unfinished symphony of quantum gravity that motivates this work, several aspects of the Distance Conjecture and the scalar potential remain an unfinished composition, with its final measures still to be written.



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✧ ✧ ✧

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A. Massive Type IIA & sources

Romans mass

Type IIA supergravity allows for a deformation by the introduction of the so-called Romans mass m. The associated theory is called *massive* Type IIA. The action is identical to before, apart from an additional term

$$\mathcal{S}_m = \frac{1}{2\tilde{\kappa}_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\left[-\frac{1}{2}m^2\right], \tag{A.1}$$

and the redefinition of the RR fluxes, which now read

$$F_2 = \mathrm{d}C_1 + mB\,, \qquad F_4 = \mathrm{d}C_3 - H\wedge C_1 + \frac{m}{2}B\wedge B\,. \tag{A.2}$$

Source contributions

In case there are D-brane or O-planes, the action $\mathcal{S}_{\mathrm{II}}$ in (1.13) is supplemented by a source term $\mathcal{S}_{\text{source}}$, which again consists of two parts, a Dirac-Born-Infeld (DBI) [542–544] and a Wess–Zumino (WZ) part [545]. Since the WZ term is topological it will – like the CS term of the bulk action – not contribute to the equations of motion. Specialising to fibred manifolds and setting gauge field contributions on the source worldvolume Σ_{p+1} to zero[8] the DBI term can be written as (we follow the conventions of [480, 504, 506])

$$\mathcal{S}_{\text{DBI}} = -c_p T_p \int e^{-\Phi}\mathrm{d}^{10}x\sqrt{\det(P[g_{IJ}])}\,\delta(\perp)\,, \tag{A.3}$$

where $P[\cdot]$ is a projector onto the directions parallel to the source and $\delta(\perp)$ a delta function that localises the source in the transverse directions. The constant c_p is $c_p = 1$ for a Dp-brane and $c_p = -2^{p-5}$ for an Op-plane, while the tension is $T_p = \frac{\sqrt{\pi}}{\tilde{\kappa}_{10}}(4\pi\alpha')^{\frac{3-p}{2}}$ and $T_p = \mu_P$ the RR charge since they are BPS.

Equations of motion and Bianchi identities for massive Type IIA with sources

The equations of motion for massive Type IIA in the presence of fluxes take the form

$$\begin{aligned} 0 = R_{IJ} + 2\nabla_I\nabla_J\phi - \frac{1}{4}H_{IJ}^2 - e^{2\Phi}\left[\frac{1}{2}(F_2^2)_{IJ} + \frac{1}{12}(F_4^2)_{IJ} - \frac{1}{4}g_{IJ}\left(m^2 + |F_2|^2 + |F_4|^2\right)\right] \\ - \frac{1}{4}e^{\Phi}\left(T_{IJ} - \frac{1}{2}g_{IJ}\sum_{\text{sources}} t_p\right), \end{aligned} \tag{A.4}$$

[8] In particular, we take $\iota^*B - \mathcal{F} = 0$, with ι^* the pullback to the worldvolume Σ_{p+1} and $\mathcal{F}$ the field strength of the worldvolume gauge field.

and for the dilaton

$$0 = 4\Delta\phi - 4|\nabla\phi|^2 + R - \frac{1}{2}|H|^2 + \frac{1}{2}e^{\Phi}\sum_{\text{sources}} t_p\,, \tag{A.5}$$

where T_{IJ} is the energy-momentum tensor associated to $\mathcal{S}_{\text{DBI}}$. Under the above assumptions, it can be written as

$$T_{IJ} = -\frac{2\tilde{\kappa}_{10}^2}{\sqrt{-G}}\sum_{\text{sources}} c_p T_p P[G_{IJ}]\sqrt{\det(P[G_{IJ}])}\,\delta(\perp)\,, \tag{A.6}$$

such that its trace defines t_p via

$$G^{IJ}T_{IJ} \equiv \sum_{\text{sources}} (p+1)t_p\,. \tag{A.7}$$

Similarly, one can obtain the equations of motion for B and the RR-fields, as well as their Bianchi identities. For example, the equation of motion for F_4 now reads

$$\mathrm{d}(\star F_2) + H \wedge \star F_4 = 2\tilde{\kappa}_{10}^2 \sum_{0-\text{sources}} c_0\mu_0\delta_9^{\perp}\,, \tag{A.8}$$

while the Bianchi identity is

$$\mathrm{d}F_2 - H \wedge F_2 = -2\tilde{\kappa}_{10}^2 \sum_{6-\text{sources}} c_6\mu_6\delta_3^{\perp}\,. \tag{A.9}$$

For the remaining fluxes, we refer to the literature; see for example, [504].

Remark

- For massive Type IIA, the RR T-duality rules (2.20) are slightly modified. Define the polyform $C = \sum_n C^{(n)}$ as well as the quantity $\hat{C} = C - mX^0e^B$. Then the T-duality rules are given by the above relations (2.20) but with $C^{(0/2/4)}$ replaced by the hatted equivalent $\hat{C}$ [179].
- In the presence of fluxes and sources it is convenient to work in the so-called *democratic formulation* [506], which explicitly includes also the magnetic dual fluxes, such that, for example, in (massive) Type IIA, we have generically all odd fluxes F_n in the theory, i.e. $n \in \{0, 2, 4, 6, 8, 10\}$, where the Romans mass is identified with F_0; we refer to [506] and the standard textbooks.
- Working with *smeared* sources, at the level of the equations of motion and Bianchi identities, it breaks down to replacing $\delta(\perp)$ with 1 in the equation for G and Φ, while replacing $\delta_p^{\perp}$ with the volume element $\mathrm{d}^{10}x|_{\perp}$.

B. Generalised momentum-winding exchange

T-duality on a circle not only inverts the radius, but also interchanges momentum zero modes with winding states. While in the abelian case this can be shown rigorously, for generalised T-dualities this is more involved. Winding modes might be absent due to lack of non-trivial cycles and the dual background, in general, also lacks some, or all of the isometries of the initial space and therefore also the associated conservation laws. However due to the Poisson-Lie condition (2.35) it turns out that there is a current[9]

$$J = e_a{}^j E_{ij} \partial_+ X^i \mathrm{d}\sigma^+ \tilde{T}^a + e_a{}^j E_{ij} \partial_+ X^i \mathrm{d}\sigma^- \tilde{T}^a \in \tilde{G}\,, \tag{B.1}$$

that, although no longer conserved, turns out to be $\tilde{G}$-flat[10]. The associated charges, in turn, are no longer numbers, but valued in $\tilde{G}$. The flatness of J can be shown to be equivalent to the following unit-monodromy condition

$$\mathcal{P} \exp \int_\gamma J_\sigma(\tilde{g}) = \tilde{e} \in \tilde{G}\,, \tag{B.2}$$

where $\mathcal{P}$ denotes path ordering. It was realised in [193], that the constraint actually provides crucial information about the modes at the quantum level, effectively giving rise to quantisation of momentum and winding modes. In particular, a very similar relation already appears when quantising the free boson NLSM on a circle

$$\mathcal{P} \exp \oint_\gamma J = \tilde{e} \in \widetilde{U(1)}\,, \tag{B.3}$$

which is satisfied due to quantisation of momentum on a compact space. Extrapolating the argument to Poisson-Lie manifolds, the authors in [193] proposed that (B.2) is the appropriate quantisation condition for this more general case, leading to a generalised Narain lattice $\Lambda_{PL} = \pi_1(\mathbb{D})$. Imposing the constraint it is possible to translate the quantisation condition into a homotopy problem of the target space and its dual. In the abelian circle case the current (B.1) is valued in $\widetilde{\mathfrak{u}(1)}$ and the unit monodromy constraint (B.3) projects an infinite number of momentum states onto the unit element. This suggests lifting the current to the universal cover of $U(1)$, the real line $\mathbb{R}$. The quantisation condition or monodromy constraint then selects the allowed quantum states as the one being projected to the unit element. This is schematically depicted in Figure B.1. On the other hand, evaluating the dual current, valued in $\mathfrak{u}(1)$, leads to quantisation of the dual momenta, which can be viewed as the winding modes from the perspective of the initial frame. In total, the winding and momentum states are therefore

[9] The vielbeins $e_a{}^j$ are defined via $g^{-1}\mathrm{d}g = e^a{}_j \mathrm{d}x^j T_a$ for $g \in G$ such that $e_a{}^j e^b{}_j = \delta^b_a$.

[10] See Table 2.1, column "Conservation eq.".

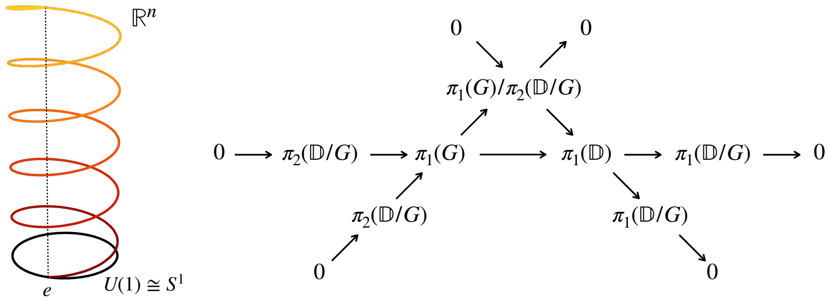

Figure B.1.: An infinite amount of elements in the covering space $\mathbb{R}$, gets projected down to the identity $e \in U(1)$ (left). On the right we visualise how long exact sequences can be decomposed into short exact sequences in a canonical manner. For details see [164].

encoded in the homotopy properties of the double $\mathbb{D} = U(1) \times \widetilde{U(1)}$ via $\pi_1(\mathbb{D}) = \mathbb{Z} \times \mathbb{Z}$, which is indeed the Narain lattice for the boson on the circle. Generalising this to the PL symmetric spaces, the winding and momentum modes are therefore encoded in the Drinfel'd double $\mathbb{D}$ and its homotopy group $\pi_1(\mathbb{D})$. The double $\mathbb{D}$ – viewed as a principal G-bundle – gives rise to a long exact sequence of homotopy groups (see, e.g. [164])

$$0 = \pi_2(\mathbb{D}) \longrightarrow \pi_2(\mathbb{D}/G) \longrightarrow \pi_1(G) \longrightarrow \pi_1(\mathbb{D}) \longrightarrow \pi_1(\mathbb{D}/G) \longrightarrow \pi_0(G) = 0\,. \tag{B.4}$$

Using standard arguments (cf. Figure B.1 or [164]), this can be decomposed into short exact sequences, in particular giving rise to

$$0 \longrightarrow \pi_1(G)/\pi_2(\mathbb{D}/G) \longrightarrow \pi_1(\mathbb{D}) \longrightarrow \pi_1(\mathbb{D}/G) \longrightarrow 0\,, \tag{B.5}$$

which is in fact *split* and hence we obtain an isomorphism

$$\pi_1(\mathbb{D}) = \pi_1(\mathbb{D}/G) \oplus \frac{\pi_1(G)}{\pi_2(\mathbb{D}/G)}\,. \tag{B.6}$$

This can be viewed as the Narain lattice of the PL-symmetric NLSM under the constraint (B.2). Physically, the non-contractible cycles in $\mathbb{D}$ split into quantised momenta and winding modes. The identification depends on which of the groups or spaces is viewed as physical and dual, respectively. The distribution of states is then summarised as

$$\underbrace{\pi_1(\mathbb{D})}_{\text{winding in } \mathbb{D}} \rightsquigarrow \overbrace{\pi_1(\mathbb{D}/G)}^{\text{winding in } \mathbb{D}/G} \oplus \underbrace{(\pi_1(G)/\pi_2(\mathbb{D}/G))}_{\text{non-comm. momentum in } \mathbb{D}/G}\,, \tag{B.7}$$

and similar for $\mathbb{D}/\tilde{G}$. Comparison then yields the generalised momentum-winding exchange. For further details, we refer to [193]. The argument generalises to coset spaces G/H. This will be discussed elsewhere.

C. Dilute fluxes and Laplacian spectrum

We briefly illustrate the idea of the so-called *dilute flux approximation* on the example of the three-torus with H-flux. In particular, assume that the flux is sufficiently weak compared to the overall volume of the geometry, i.e. *"diluted"*

$$\frac{k}{\mathrm{Vol}(\mathrm{T}^3)} \ll 1\,, \tag{C.1}$$

for $[H] = k$ the flux number. As a result, we can view the contributions coming from H as perturbations around the conformal torus fixed point and solve the theory up to leading order in k. The resulting theory is often denoted CFT_H; more details can be found in [267, 269], as well as in [546] and references therein.

Working at the level of the NLSM (1.6), for which the canonical metric and B-field of the flat torus are given by

$$G = \mathrm{diag}(r_1^2, r_2^2, r_3^2)\,, \qquad B = kX^3 \mathrm{d}X_1 \wedge \mathrm{d}X_2\,, \tag{C.2}$$

the equations of motion reads

$$G_{\kappa\nu}\partial^2 X^\nu = k\epsilon_{\kappa\mu\nu}\partial_\sigma X^\mu \partial_\tau X^\nu\,. \tag{C.3}$$

Assuming that we are within the validity range of the approximation, in particular large r_i, this can be solved perturbatively and the solution up to leading order in k reads[11]

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{h}{r_\mu^2}x_H^\mu + p_0^\mu\tau + \frac{h}{r_\mu^2}p_H^\mu\tau + N^\mu\sigma - \sum_{\nu\rho}\frac{h}{2r_\mu^2}\epsilon^\mu_{\nu\rho}p_0^\rho N^\nu\tau^2\,, \tag{C.4}$$

where for once, we are not summing over repeated indices μ. Furthermore, we allowed non-trivial winding N^μ of the string via the usual boundary conditions of the three-torus, $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)+2\pi N^\mu$. Since B is linear in X^μ, the canonical momentum is still given by

$$\Pi_\mu = \frac{1}{2\pi}(G_{\mu\nu}\dot{X}^\nu + B_{\mu\nu}X'^\nu)\,, \tag{C.5}$$

and since B is non-constant, translation invariance generated by Π is generically broken, and there is no conserved momentum. From the expansion (C.4) we can read off

$$\dot{X}^\mu = p_0^\mu + \frac{h}{R_\mu^2}p_H^\mu - \sum_{\nu\rho}\frac{h}{R_\mu^2}\epsilon^\mu_{\nu\rho}p_0^\rho N^\nu\tau\,. \tag{C.6}$$

We see that the centre-of-mass momentum is no longer constant but depends on the

[11] For more details we refer to [261].

worldsheet parameter τ. In vector notation, the last term reads

$$\vec{p}_0 \times \vec{N} = 0 \Longleftrightarrow \vec{p}_0 \parallel \vec{N} \,, \tag{C.7}$$

hence momentum conservation only holds, if momentum and winding vectors are parallel. Since on the torus there is generically non-zero winding and non-zero, arbitrary momentum states, this implies that momentum is non-conserved. This is reminiscent [269] of the *cyclotron effect* of charged point particles moving in a (constant) magnetic field, which in this higher-dimensional situation translates to a constant H.

Attempting to perform a dilute flux approximation like above might give some heuristic ideas of the mass behaviour and conservation properties of states, also for curved spaces. However, several issues arise, rendering such approaches technically much more involved and, in general, not fully trustworthy:

- For curved spaces, a naive flat-space mode expansion is not possible, and the identification of the mass spectrum becomes much harder. In principle, one has to perform a Hodge decomposition and determine the massive spectrum of the KK-tower by solving for the scalar spectrum of the associated Laplacian[12].
- In the presence of fluxes, the above-mentioned Laplacian problem and the Hodge decomposition must be adopted to H-twisted cohomology [454]. In particular, the latter is determined by the nilpotent differential $\mathrm{d}_H = \mathrm{d} - H\wedge$ and the Hodge-decomposition and Laplacian problem should be built form d_H.

D. Derivation of reduction formula

NSNS sector

First, the identity (6.25) is a direct consequence of the assumption (6.24). In particular, there are only four non-zero mixed Christoffel symbols

$$\Gamma^{\alpha}_{\beta a} = \frac{1}{2} g^{\alpha\gamma} \partial_a g_{\alpha\beta} \,,\; \Gamma^{\alpha}_{ab} = -\frac{1}{2} g^{\alpha\gamma} \partial_\gamma h_{ab} \,,\; \Gamma^{a}_{\alpha\beta} = -\frac{1}{2} h^{ab} \partial_b g_{\alpha_\beta} \,,\; \Gamma^{a}_{b\alpha} = \frac{1}{2} h^{ac} \partial_\beta h_{cb} \,. \tag{D.1}$$

such that splitting $R(G)$ into internal and external indices and properly resumming gives

$$R(G) = R(g) + R(h) + \frac{3}{4} \mathrm{tr}(k_\mu k_\nu) g^{\mu\nu} - \frac{1}{4} \mathrm{tr}(k_\mu) \mathrm{tr}(k_\nu) g^{\mu\nu} - \mathrm{tr}(h^{-1} \Box_g h) \,. \tag{D.2}$$

[12] It was pointed out for example in [101], that sometimes the Laplacian spectrum may not coincide with the spectrum of the effective theory. This can be caused by the appearance of so-called *space invaders* [98], arising through the combination of massive modes into massless ones, or by additional fluxes.

Now also using eqs. (6.26) to (6.28) the action splits as (absorbing Φ_0 into κ)

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-g}\,\sqrt{h_0}\, e^{-2\phi_y} e^{-2\phi_d}\Big(R(g) + R(h) + \mathcal{J}(g,h) - \frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda} - \frac{1}{12}\mathcal{H}_{ijk}\mathcal{H}^{ijk} - \frac{1}{4}\mathcal{H}_{\mu jk}\mathcal{H}^{\mu jk} + 4\partial_\mu\phi\partial^\mu\phi + 4\partial_i\phi\partial^i\phi\Big)\,. \tag{D.3}$$

Now using the transformation rule of the Ricci scalar under a Weyl rescaling

$$g \to e^{-2\omega}\tilde{g}\,, \qquad \text{with } \omega = \frac{-2\phi_d}{d-2}\,, \tag{D.4}$$

we can bring the action to the Einstein frame

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}}\,\sqrt{h_0}\, e^{-2\phi_y}\Big\{R(\tilde{g}) - \frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} e^{\frac{-8\phi_d}{d-2}} - \frac{4(d-1)}{d-2}\partial_\mu\phi_d\partial^{\tilde{\mu}}\phi_d + \Big(R(h) - \frac{1}{12}\mathcal{H}_{ijk}\mathcal{H}^{ijk} + 4\partial_i\phi\partial^i\phi\Big)e^{\frac{4\phi_d}{d-2}} - \mathcal{J}(g)e^{-2\omega} - \frac{1}{4}\mathcal{H}_{\mu jk}\mathcal{H}^{\tilde{\mu}jk} + 4\partial_\mu\phi\partial^{\tilde{\mu}}\phi\Big\}\,. \tag{D.5}$$

Note that we partially integrated the contribution of the transformation of the Ricci scalar to obtain the last term. The term $\mathcal{J}(g,h)e^{-2\omega}$ can also be partially integrated[13] to obtain

$$\int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}}\,\sqrt{h_0}\, e^{-2\phi_y}\, \mathcal{J}(g,h)e^{-2\omega} = \int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}}\,\sqrt{h_0}\, e^{-2\phi_y}\Big(-\frac{1}{4}\mathrm{tr}(k_\mu k_\nu)\tilde{g}^{\mu\nu} - \frac{1}{4}\mathrm{tr}(k_\mu)\mathrm{tr}(k_\nu)\tilde{g}^{\mu\nu} - 2\mathrm{tr}(k_\mu)\partial^{\tilde{\mu}}\phi_d\Big)\,. \tag{D.6}$$

Using the identity $\partial(\det(A)) = \det(A)\mathrm{tr}(A^{-1}\partial A)$ we obtain

$$\mathrm{tr}(h^{-1}\partial_\mu h) = 4\partial_\mu\sigma\,, \tag{D.7}$$

such that we end up with

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}}\,\sqrt{h_0}\, e^{-2\phi_y}\Big\{R(\tilde{g}) - \frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} e^{\frac{-8\phi_d}{d-2}} - \frac{4}{d-2}\partial_\mu\phi_d\partial^{\tilde{\mu}}\phi_d + \Big(R(h) - \frac{1}{12}\mathcal{H}_{ijk}\mathcal{H}^{ijk} + 4\partial_i\phi\partial^i\phi\Big)e^{\frac{4\phi_d}{d-2}} - \frac{1}{4}\mathrm{tr}(k_\mu k_\nu)\tilde{g}^{\mu\nu} - \frac{1}{4}\mathcal{H}_{\mu jk}\mathcal{H}^{\tilde{\mu}jk}\Big\}\,. \tag{D.8}$$

Introducing the invariant internal volume

$$\widehat{\mathcal{V}} = \int \mathrm{d}^n y\, \sqrt{h_0}\, e^{-2\phi_y}\,. \tag{D.9}$$

[13] We assume that total derivative terms vanish and hence can be dropped. In principle, this has to be checked for the specific background at hand. In particular, the total derivative might not vanish in the context of non-geometric background, where extra care is needed. See also Appendix E of [67] and Chapter 3.

we can write the action as

$$\mathcal{S} = \frac{\widehat{\mathcal{V}}}{2\kappa^2} \int \mathrm{d}^d x \sqrt{-\tilde{g}} \left\{ R(\tilde{g}) - \frac{1}{12} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} e^{\frac{-8\phi_d}{d-2}} - \frac{4}{d-2} \partial_\mu \phi_d \partial^{\tilde{\mu}} \phi_d + \right.$$
$$\left. - \frac{1}{4} \mathrm{tr}(k_\mu k_\nu) \tilde{g}^{\mu\nu} - \frac{1}{4} \mathcal{H}_{\mu jk} \mathcal{H}^{\tilde{\mu} jk} - V_0(\varphi_a, \phi_d) \right\}, \quad \text{(D.10)}$$

where we defined the scalar potential

$$V_0(\varphi_a, \phi_d) = -\widehat{\mathcal{V}}^{-1} \int \mathrm{d}^n y \sqrt{h_0}\, e^{-2\phi_y} \left\{ \left(R(h) - \frac{1}{12} \mathcal{H}_{ijk} \mathcal{H}^{ijk} + 4 \partial_i \phi \partial^i \phi \right) e^{\frac{4\phi_d}{d-2}} \right\}. \quad \text{(D.11)}$$

Finally, we work out the explicit moduli dependence by splitting the derivative $\partial_\mu(\cdot) = \partial_\mu(\cdot)|_{\varphi=\text{const.}} + (\partial_\mu \varphi^a)\partial^a(\cdot)$ where $\partial_a \equiv \frac{\partial}{\partial \varphi^a}$. This gives, with now all partials ∂_μ understood as $\partial_\mu(\cdot)|_{\varphi=\text{const.}}$, the result

$$\mathcal{S} = \frac{\widehat{\mathcal{V}}}{2\kappa^2} \int \mathrm{d}^d x \sqrt{-\tilde{g}} \left\{ R(\tilde{g}) - \frac{1}{12} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} e^{\frac{-8\phi_d}{d-2}} - \frac{4}{d-2} \partial_\mu \phi_d \partial^{\tilde{\mu}} \phi_d + \right.$$
$$\left. - \gamma_{ab} \partial_\mu \varphi^a \partial^{\tilde{\mu}} \varphi^b - \gamma_{a\mu} \partial^{\tilde{\mu}} \varphi^a - V(\varphi^a, \phi_d) \right\}, \quad \text{(D.12)}$$

with $\gamma_{ab}, \gamma_{a\mu}$ and $V(\varphi^a, \phi_d)$ as in (6.31) and (6.32).

RR sector

We start from the Weyl rescaled action

$$\mathcal{S} = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}} \sqrt{h_0}\, e^{-2\phi_y} \left\{ R(\tilde{g}) + \cdots - \frac{1}{2} e^{2(\Phi-\omega)} \sum_p \left| F_p \right|^2 \right\}. \quad \text{(D.13)}$$

Then using the form (6.40) of the potentials C_p we can write

$$\left| F_p \right|^2 = \left| F_p \right|^2 + \left| f_p \right|^2 + p (f_p)_{\mu a_1 \ldots a_{p-1}} (f_p)_{\nu b_1 \ldots b_{p-1}} \tilde{g}^{\mu\nu} h^{a_1 b_1} \ldots h^{a_{p-1} b_{p-1}}, \quad \text{(D.14)}$$

and therefore we get

$$\mathcal{S} = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}} \sqrt{h_0}\, e^{-2\phi_y} \left\{ R(\tilde{g}) + \cdots - \frac{1}{2} e^{2(\Phi_0+\sigma)} \sum_p \frac{1}{p!} e^{2\frac{d-2p}{d-2}\phi_d} \left| F_p \right|^2 \right. \quad \text{(D.15)}$$
$$\left. - \frac{1}{2} e^{2(\Phi-\omega)} \sum_p \frac{1}{p!} \left| f_p \right| - \frac{1}{2} e^{2\Phi} \sum_p \frac{p}{p!} (f_p)_{\mu a_1 \ldots a_{p-1}} (f_p)_{\nu b_1 \ldots b_{p-1}} \tilde{g}^{\mu\nu} h^{a_1 b_1} \ldots h^{a_{p-1} b_{p-1}} \right\}.$$

Again we work out the explicit moduli dependence by splitting the derivative $\partial_\mu(\cdot) = \partial_\mu(\cdot)|_{\varphi=\text{const.}} + (\partial_\mu \varphi^a)\partial^a(\cdot)$ where $\partial^a \equiv \frac{\partial}{\partial \varphi^a}$. We do not consider variations coming from the purely external fluxes for now. Furthermore the purely internal part can not give contributions as there are no derivatives with respect to x^μ. Therefore we consider only the last term. This gives terms which we recognise as the higher form generalization of the contributions from the Kalb-Ramond 2-form. Plugging these results into the expression for the action we get

$$\begin{aligned}\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x \mathrm{d}^n y \sqrt{-\tilde{g}}\,\sqrt{h_0}\,e^{-2\phi_y}\Big\{R(\tilde{g}) + \cdots - \frac{1}{2}e^{2(\Phi_0+\sigma)}\sum_p \frac{1}{p!}e^{2\frac{d-2p}{d-2}\phi_d}\left|F_p\right|^2 \\ - \frac{1}{2}e^{2(\Phi_0+\sigma)}e^{2\phi_d}\sum_p \frac{2p}{p!}\left\langle \partial_\mu c_{p-1}\middle|\partial_a c_{p-1}\right\rangle_h \left(\partial_\nu \varphi^a \tilde{g}^{\mu\nu}\right) \\ - \frac{1}{2}e^{2(\Phi_0+\sigma)}e^{2\phi_d}\sum_p \frac{p}{p!}\left\langle \partial_a c_{p-1}\middle|\partial_b c_{p-1}\right\rangle_h \left(\partial_\mu \varphi^a \partial_\nu \varphi^b \tilde{g}^{\mu\nu}\right) \\ - \frac{1}{2}e^{2(\Phi_0+\sigma)}e^{2\phi_d}\sum_p \frac{1}{p!}\Big(e^{\frac{4\phi_d}{d-2}}\left|f_p\right|_h^2 + p\left\langle \partial_\mu c_{p-1}\middle|\partial_\nu c_{p-1}\right\rangle_h \tilde{g}^{\mu\nu}\Big)\Big\}\,,\end{aligned} \tag{D.16}$$

leading to the expressions (6.42) and (6.43).

E. Some basics of (parabolic) PDE

We review some basics of (parabolic) PDE, including some differential geometric identities and conventions used in this work. We mostly follow the conventions of [547], except for the relative minus sign of the Laplace-de Rham operator. See also [501].

Differential geometry conventions

For $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative on the n-dimensional manifold M with metric g, we define the co-differential δ as the adjoint $\delta \equiv \mathrm{d}^* : \Omega^{k+1}(M) \to \Omega^k(M)$ satisfying

$$\int_M g(\mathrm{d}^*\alpha, \beta)\mathrm{vol} = \int_M g(\alpha, \mathrm{d}\beta)\mathrm{vol}\,. \tag{E.1}$$

The Hodge dual of a p-form η denoted by $\star\eta$ is given as the unique $(n-p)$-from satisfying

$$\alpha \wedge \star\eta = g(\alpha, \eta)\mathrm{vol}\,, \tag{E.2}$$

for every k-form α, such that we have the following identities (η a p-form)

$$\star\star\,\eta = (-1)^{p(n-p)} sgn(g)\eta\,, \tag{E.3}$$

and the co-differential δ can be written as

$$\delta\eta = (-1)^{n(p+1)+1} sgn(g) \star \mathrm{d} \star \eta\,, \qquad (\Leftrightarrow (\delta\eta)_{i_2\ldots i_p} = -\nabla^k \eta_{k i_2,\ldots,i_p})\,. \tag{E.4}$$

Here ∇ is the Levi-Civita connection on M and we denote by ∇_i also the covariant derivative in the direction $e_i = \frac{\partial}{\partial x^i}$. We can define its formal adjoint ∇^*

$$g(\nabla^*_X \alpha, \beta) = g(\alpha, \nabla_X \beta)\,, \tag{E.5}$$

and with that the *Bochner Laplacian* Δ acting on a p-form η as

$$\Delta\omega = -\nabla^*\nabla\omega = \mathrm{tr}\,\nabla^2\omega\,. \tag{E.6}$$

Acting on a scalar function f, it reduces to the rough or *connection Laplacian* Δ_∇

$$-\nabla^*\nabla f = \mathrm{tr}\nabla^2 f = g^{ij}\nabla_i\nabla_j f = \Delta_\nabla f\,, \tag{E.7}$$

so that we can drop the subscript in Δ. From d and δ we can furthermore define the *Laplace-de Rham operator* or Hodge Laplacian as

$$\Delta_{dR} : \Omega^k(M) \to \Omega^k(m)\,, \quad \Delta_{dR}\omega = -(\mathrm{d}\delta + \delta\mathrm{d})\omega\,. \tag{E.8}$$

The sign choice we adapt here is somewhat unconventional but has the advantage that when acting on scalar functions gives

$$\Delta_{dR} f = -\delta\mathrm{d}f = \nabla^i\nabla_i f = \Delta f\,. \tag{E.9}$$

This can be used to establish the Weizenböck identity (cf. [547])

$$-\Delta_{dR} = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*) = \nabla^*\nabla\omega + \mathrm{Ric}(\omega) \tag{E.10}$$

from which we obtain

$$\Delta_{dR}\omega = \Delta\omega - \mathrm{Ric}(\omega)\,, \tag{E.11}$$

Here $\mathrm{Ric}(\omega)$ is the Weizenböck curvature operator defined through the Ricci tensor. The precise form is not important, only that it is of lower order compared to Δ.

Parabolic PDE and the principal symbol

We recall the definition of parabolic differential operators on curved manifolds, following [548].

Let E, F be smooth vector bundles over a C^∞ manifold M and denote by $C^\infty(E)$ and $C^\infty(F)$ the vector spaces of smooth sections of E and F, respectively. In each local patch, we can write an order k **linear differential operator** $L : C^\infty(E) \to C^\infty(F)$ acting on an element $\psi^\beta \in C^k(M)$ as

$$L(\psi)_\alpha = \sum_{k=0}^{l} (a^l_{\alpha\beta})^{i_1 i_2 \cdots k} \nabla_{i_1 i_2 \cdots k} \psi^\beta \,. \tag{E.12}$$

Here, $\beta = 1, \ldots, p$ enumerates the coordinates $\{\xi^\alpha\}$ on the fibre of E and $\alpha = 1, \ldots, q$ the coordinates $\{\eta^\alpha\}$ on the fibre of F. The coefficients $(a^k_{\alpha\beta})$ are k-tensors.

The **principal symbol** $\sigma(L) : \Pi^*(E) \to \Pi^*(F)$ is a vector bundle homomorphism with $\Pi : T^*M \to M$ the bundle projection[14]. Explicitly, it is defined at $(x, \xi) \in T^*$ as

$$[\sigma(L)(x,\xi)]_{\alpha\beta} = [a^k_{\alpha\beta}(x)]^{i_1 \cdots k} \xi_{i_1} \ldots \xi_{i_k} \,, \tag{E.13}$$

with $\xi_i \in \mathbb{R}^p$. Therefore, since $\nabla_{i_1 \cdots k}\psi \equiv \nabla_{i_1} \ldots \nabla_{i_k}\psi = \partial_{i_1} \ldots \partial_{i_k}\psi$ + loweroderterms, $\sigma(L)$ is effectively obtained by replacing ∂_i with ξ^i in the leading order part of L.

Now L is **elliptic** at a point $x \in M$ if $\sigma(L)(x, \xi)$ is an isomorphism[15] for every $\xi \neq 0$. Furthermore, A is strongly elliptic if there exists a constant $\delta > 0$ such that

$$[\sigma(A)(x,\xi)]_{\alpha\beta} \eta^\alpha \eta^\beta \geq \eta |\xi|^k |\eta|^2 \,, \tag{E.14}$$

where $|\eta|^2 \equiv g^{ij}(x)\xi_i\xi_j$ (for the Riemannian manifold (M, g)) and $\eta_\alpha = h_{\alpha\beta}\eta^\beta$ with $h_{\alpha\beta}$ the Riemannian metric on the fibres. In particular L is **strongly elliptic** if the matrix of the principal symbol is everywhere positive definite.

A **non-linear** differential operator $L : C^\infty(E) \to C^\infty(F)$

$$L\psi = F(x, \psi, \nabla\psi, \ldots, \nabla^k\psi) \,, \tag{E.15}$$

for F differentiable, is said to be **(strongly) elliptic** if the linearised operator DL at x is (strongly) elliptic with respect to ψ.

Finally, a system of PDE is **(strictly) parabolic** if for $\psi_s \in C^1([0, \infty[\, \times C^\infty(E))$ the system

$$\partial_s \psi_s = L_s \psi_s \,, \tag{E.16}$$

is given by a family of smooth (strongly) elliptic operators $L_s : C^\infty(E) \to C^\infty(F)$.

If a system if strongly parabolic, there **exists a unique smooth solution** ψ_s to (E.16) with initial condition $\psi(x, 0) = \Psi(x)$ on $[0, \tau[$ for some $\tau > 0$.

[14] Note $\Pi^*(E)$ is a vector bundle over T^*M with fibre E_x at $(x, \xi) \in T^*$; cf. [278].

[15] This necessitates $p = q$ and one can identify E and F.

Gradient flows

In Chapter 4 we briefly discussed the definition of a **gradient flow**, where we gave a definition for the simple Euclidean case. In a more general setting, following the conventions of [282, 391], we can define a gradient flow in some abstract space X with "coordinates" Ψ^a, metric $G^{AB} \in \Gamma(S^2T^*X)$ and generating functional $\mathcal{E} : X \to \mathbb{R}$ according to [520,549,550]

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi^A = -\mathrm{grad}_G(\mathcal{E}(\Psi)) = -G^{AB}\frac{\delta}{\delta\Psi^B}\mathcal{E}(\Psi)\,. \tag{E.17}$$

In case the coordinates are the metric tensors $\Psi^A \equiv g_{\mu\nu}$, the space X can be identified with the space of Riemannian metric $\mathcal{M}(M)$ and the metric G^{AB} is the DeWitt metric (4.12). Then taking the Perelman functional $\mathcal{F}$ of (4.3) as a generating functional one recovers Ricci flow, or more precisely the combined flow; cf. also [282] and [391]. In this case the gradient is with respect to the basis $\{\partial^A = \partial_{g_{\mu\nu}}\}$. Similar one can then read off the gradient flow of generalised Ricci flow from the variation in Eq. (4.24).

Note that the Wasserstein distance introduced in Chapter 11 can be associated to a certain Riemannian (Wasserstein) metric g^W on the space of probability measures $\mathcal{P}(M)$ such that one can define a gradient according to [520,549,550]

$$\mathrm{grad}_W\mathcal{F}(\rho) \equiv -\mathrm{div}\left(\rho\nabla\frac{\delta}{\delta\rho}\mathcal{E}(\rho)\right)\,, \tag{E.18}$$

with $\rho(x)\mathrm{d}x \in \mathcal{P}(M)$. Using this definition of gradient, together with a suitable generating functional one can hence define gradient flows on the space of probability measures such that the associated distance is the Wasserstein distance (11.3). Appropriately choosing $\mathcal{E}$, one can for example realise the continuity equation as a gradient flow in this way. For details we refer to [520, 549, 550]. The associated differential calculus on $\mathcal{P}(M)$ is known as *Otto calculus* [549].

F. Examples of the modified flows

We illustrate the behaviour of the modified flows on the canonical example of S^3 with H-flux for the special cases of $\alpha = 2$ and $\alpha = -1$, respectively.

For $\alpha = 2$, the flow reads

$$\frac{\partial}{\partial s}f = T_f^{(2)} = -\frac{18}{r(s)^2} + \frac{10k^2}{r(s)^6}\,. \tag{F.1}$$

Since the flow equations for g_{ij} are not influenced by the modification for f, the radius stabilises at $r = \sqrt{k}$ for $s \to \infty$. In this limit $\partial_s f$ is not vanishing but approaching

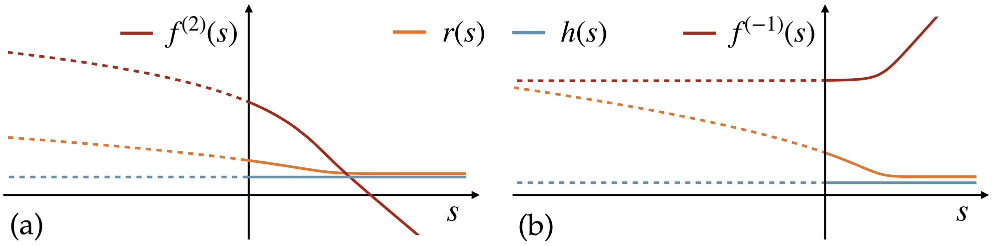

Figure F.1.: The plots show the flow behaviour of the modified flows introduced in this section on the particular example of S^3 with k units of H-flux. As desired, the flow behaviour of r and f is no longer strictly correlated. In particular f diverges both for the flow with $\alpha = 2$ (left) and $\alpha = -1$ (right). Dashed lines correspond to an extension "backwards" in time.

the constant value $-8/k^2$ and therefore this is not a fixed point of the combined flow $\{g_{ij}(s), f(s)\}$. However, extending the flow backward in time for $r_0 > \sqrt{k}$ we reach a fixed point where both $\partial_s g_{ij} = 0$ and $\partial_s f = 0$, cf. Figure F.1, panel a.

On the other hand, for $\alpha = -1$ we obtain

$$\frac{\partial}{\partial s} f = T_f^{(-1)} = \frac{4k^2}{r(s)^6} \,. \tag{F.2}$$

The situation is illustrated in Figure F.1, panel b.

Example including ϕ

Working again with the example of S^3 supported by k units of flux, we obtain

$$\begin{aligned} \frac{\partial}{\partial s}\phi(s) &= \beta^\phi = \frac{2k^2}{r(t)^6} \,, \\ \frac{\partial}{\partial s}\tilde{f}(s) &= \tilde{T}_{\tilde{f}} = -\frac{6}{r(s)^2} + \frac{2k^2}{r(s)^6} \,. \end{aligned} \tag{F.3}$$

We find a situation in which $r \to \sqrt{k}$ but ϕ and $\tilde{f}$ do not approach a constant. However, they diverge such that the generalised volume constraint is satisfied, cf. Figure F.2.

G. A generalised Ricci-Bourguignon flow

There are in principle different choices for the flow equation of ϕ. We explained that this corresponds to taking either the equation of motion descending directly form a variational principle of the action, or – by using the trace of the Einstein equation – replacing the Ricci scalar R. This gives an a priori different flow equitation, that shares however the same fixed points with the initial one. This is not only true of the flow

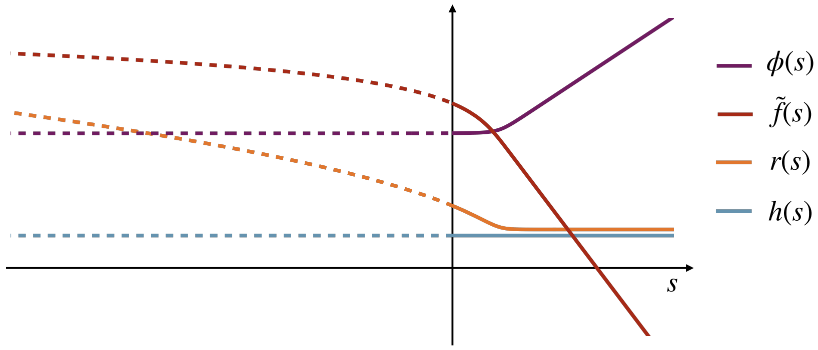

Figure F.2.: Plotted is the combined flow of the system described by (9.35) and (9.37) on the example of S^3 with k units of flux. The two scalars diverge in a way such that the generalised constant volume constraint (9.36) is satisfied.

equation of ϕ, but also g_{ij}; see also the remark at the end of Section 1.2.1. In fact, the form of the equation of motion for g used here is already the one obtained by substituting the dilaton equation of motion, as by this choice we avoid the problems arising from the negative modes of R, rendering the PDE ill-posed. However, these are not the only possibilities. One can create a whole family of flows, all sharing the same fixed point by introducing a flow according to

$$\partial_s g_{ij}(s) = -2\left(\bar{\beta}^g_{ij} + \rho \mathrm{tr}(\bar{\beta}^g_{ij}) g_{ij} - \frac{\kappa}{4}\bar{\beta}^\phi_0 g_{ij}\right) . \tag{G.1}$$

For $\{\rho = 0, \kappa = 0\}$ this is hence the standard flow introduced above, while for $\{\rho = 0, \kappa = 1\}$ it gives the Einstein equation as derived directly from the variational principle; see also [504], Eq. (A.15). Furthermore, in the absence of fluxes and sources, when setting $\kappa = 0$, the flow is reduced to

$$\partial_s g_{ij}(s) = -2\left(R_{ij} + \rho R g_{ij}\right) , \tag{G.2}$$

which is exactly the so-called *Ricci-Bourguignon flow* [292, 507]. The flow (G.1) can therefore be viewed as a generalised version of the Ricci-Bourguignon flow. It would be interesting to see if this flow can have similar relevance for the SDC and RFC as the standard version discussed in [392], in particular when discussing product spaces.

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