On Gravitational Effects at the Quantum and Cosmic Scales

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Zusammenfassung

Die Allgemeine Relativitätstheorie, die in der Differentialgeometrie ausgedrückt ist, hat unser Verständnis des Universums revolutioniert. Sie legte den Grundstein für die berühmte Inflationstheorie, die vorhersagt, dass die großräumigen Strukturen im Universum durch Quantenfluktuationen in der Hintergrundmetrik entstehen. Slow-Roll Inflationsmodelle stellen jedoch eine Herausforderung für das Inflationsparadigma dar, das einst auf der Planckskala begann, während man versuchte, Probleme der Feinabstimmung zu vermeiden, was zu einem Multiversum und einem Verlust der Vorhersagbarkeit der Inflationstheorie führte. Tatsächlich stößt die Gravitationstheorie selbst bei diesen hohen Krümmungsregimen an ihre Grenzen. Andererseits werden Quantenfluktuationen, die unterhalb der Planckschen Skalen der Metrik auftreten, größer als eins, wodurch die klassische geometrische Beschreibung der Raumzeit in diesen Skalen zusammenbricht. Diese Probleme machen unser Verständnis der Physik auf den Planckschen Skalen unvollständig und legen nahe, dass eine alternative Beschreibung des Gravitationsrahmens selbst erforderlich ist.

In dieser Arbeit untersuchen wir zwei grundlegende Probleme innerhalb des Gravitationsrahmens, die strukturell und dynamisch nahe der Planckskala auftreten.

Der erste Teil befasst sich mit dem dynamischen Problem der autauchenden Landschaft und dem Multiversum, das in slow-roll Inflationsmodellen auf Planckschen Skalen auftritt. Wir präsentieren ein Inflationsmodell im Rahmen der Mimetischen Gravitation, einer Erweiterung der Allgemeinen Relativitätstheorie, welche die Metrik durch ein eingeschränktes Skalarfeld ϕ re-parametrisiert. Wir modifizieren die dynamischen Gleichungen bei hohen Krümmungen minimal, indem wir das Inflatonfeld über die invariante Größe $\Box\phi$ an das mimetische Feld koppeln. Dadurch kann die mimetische Inflation auf der Planckskala beginnen, wo die Bedingung der Selbstreproduktion ab $H \sim 1$ verletzt wurde, so dass das Auftreten des Multiversums und die damit verbundenen Probleme erfolgreich umgangen werden. Gleichzeitig vermeidet dieses Modell das Problem der Feinabstimmung in diesem sich aufblähenden Bereich und stellt die Vorhersagbarkeit der Inflationstheorie in der ansonsten riesigen Landschaft wieder

her. Das Modell sagt auch den skalaren Spektralindex n_s und das Tensor-zu-Skalar-Verhältnis r voraus, die mit den aktuellen Beobachtungsdaten für $N = 50$ e-Falten übereinstimmen.

Der zweite Teil der Arbeit befasst sich mit dem strukturellen Problem der Raumzeitbeschreibung der Einsteinschen Gravitation bei Längen kleiner als die Planckskala. Um eine Verbindung zwischen einer klassischen und einer Quantenbeschreibung für die Gravitation zu schaffen, betrachten wir eine neue Formulierung der diskreten Gravitation, in der wir die Gravitation als eine Eichtheorie der Poincare-Gruppe darstellen. Diese Formulierung basiert auf der Definition einer diskreten Mannigfaltigkeit, die aus Elementarzellen in Planckscher Größe besteht, die jeweils mit einem Tangentenraum und Verschiebungsoperatoren ausgestattet sind. Wir erweitern den Tangentenraum um Translationen durch die inhomogene Lorentzgruppe $ISO(d)$. Während ein wesentliches Merkmal der Allgemeinen Relativitätstheorie, die Diffeomorphismusinvarianz, auf dem Gitter verloren geht, ersetzen wir sie, indem wir Translationsinvarianz auferlegen und ihre Parameter durch die Null-Torsions-Bedingung in Beziehung setzen. Dann leiten wir Ausdrücke für Krümmung und Torsion auf dem Gitter ab, zusammen mit Transformationen der Eichfelder, und indem wir einen offensichtlichen kontinuierlichen Grenzwert annehmen, wurden die üblichen Ausdrücke der Differentialgeometrie erfolgreich wiederhergestellt.

Abstract

The theory of General Relativity, written in the language of differential geometry, has revolutionized our understanding of the Universe. It laid the ground for the celebrated Inflationary theory, which predicts that the large-scale structures in the universe are seeded from quantum fluctuations on the background metric. Slow-roll inflationary models, however, pose a challenge to the inflationary paradigm, once set to begin at the Planck scale, while trying to avoid problems of fine-tuning, resulting in a multiverse picture and a loss of predictability of the Inflationary theory. In fact, the gravitational theory itself faces limitations at these high curvature regimes. On the other hand, quantum fluctuations occurring below Planckian scales of the metric become larger than unity, which breaks down the classical geometric description of space-time at these scales. These problems render our understanding of physics at the Planckian scales incomplete, suggesting an alternative description of the gravitational framework itself is needed.

In this thesis we study two fundamental problems within gravitational framework presenting structurally and dynamically close to the Planck scale.

The first part addresses the dynamical problem of emergent landscape and the multiverse exhibited in slow-roll models of Inflation beginning at Planckian scales. We present an Inflationary Model within the Mimetic Gravity framework, an extension of General Relativity that reparametrizes the metric through constrained scalar field ϕ. We minimally modify the dynamical equations at high curvatures by coupling the Inflaton field to the Mimetic field through the invariant quantity $\Box\phi$. This allows Mimetic Inflation to start at the Planck scales, where the self-reproduction condition was violated starting at $H \sim 1$, hence the appearance of the multiverse and its adverse problems are successfully evaded. At the same time, this model avoids the issue of fine-tuning in this inflating patch, restoring the predictability of Inflationary theory in the otherwise vast landscape. The model also predicts scalar spectral index n_s and tensor-to-scalar ratio r which is in agreement with current observational data for $N = 50$ e-folds.

The second part of the thesis addresses the structural problem of space-time description of Einstein's gravity at lengths smaller than the Planck scale. In what foresees a connection between a classical and a quantum description for gravity, we consider a new Discrete Gravity formulation, in which we present gravity as a gauge theory of the Poincare group. This formulation is based on defining a discrete manifold, constructed from Planckian-sized elementary cells, each equipped with a tangent space and displacement operators. We expand the tangent space to include translations through the inhomogeneous Lorentz group $ISO(d)$. While an essential feature of General Relativity, diffeomorphism invariance, is lost on the lattice, we replace it by imposing translational invariance, and relating their parameters through the zero-torsion condition. Then we derive expressions for curvature and torsion on the lattice, together with transformations of the gauge fields, and by taking a manifest continuous limit, the usual expressions of differential geometry were successfully recovered.

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Chapter 1

Inflationary Cosmology: Theory and Observations

1.1 Historical Prelude: A Rundown on Ideas in The Cosmic Plane

It was only when Nicolas Copernicus first challenged Ptolemy's geocentric model in 1543, that the first step towards thinking about the universe on a grande scale was ever taken. After that, the planets were set in motion in their orbits by Johannes Kepler in 1609, and Galileo Galilei then pointed his telescope at Jupiter, an unfulfilled star, and observed its moons in 1610. Later, the idea of an infinite, static universe emerged in Newton's work, *Philosophiæ Naturalis Principia Mathematica* in 1687, which grounded planetary motion in a gravitational central field, and extending this idea to the cosmos at large then set the stage for a long and ongoing discourse on the state, history and future of our universe.

Key Astronomical Observations that shaped Modern Cosmology

Aside from the involved debate on the gravitational stability of an infinite static universe, and Einstein's famous contribution to the discussion through the introduction of a *cosmological constant* Λ, a shift in paradigm was set forward by a sequence of observations. First, the possibility of measuring cosmic distances opened up through Henrietta Leavitt's discovery of the Cepheid Variable Period-Luminosity Relation in 1912, which linked a Cepheid's intrinsic brightness (luminosity) to its pulsation period. With this relationship at hand for the Cepheid, astronomers then could determine the absolute brightness of this 'standard candle' by measuring its period, and comparing it to its apparent brightness, and that allowed them for the first time to calculate reliably, more or less, distances to distant galaxies. This method then laid the foundation for measuring the scale of the universe and discovering cosmic expansion. But before that, Vesto Slipher had to discover the redshift in observed galaxies through studying their spectral lines in 1917 [3], which was the earliest evidence that the universe was expanding. Edwin Hubble then took Leavitt's Cepheid variable method to measure distances to galaxies, and cross examined them with the measured redshifts of Slipher, and came forward with the infamous *Hubble's Law* in 1929 that served as direct observational proof that our universe is expanding, through the linear relationship that reads as *the farther a galaxy is, the faster it moves away from us* [4]. The impact of his discovery rippled through many courses simultaneously, confirming that Einstein's general relativity could also be used to describe the cosmos, at the same time overturning his and his predecessor's static universe model. After the discovery of the expansion of the universe through the receding galaxies, it took two more decades until George Gamow, Alpher and Herman concluded that our universe must have originated from a denser and hotter state, if it has expanded in the past, and still is today. With this inference, the Big Bang model was put forward, and through their seminal work in 1984 [5], Gamow et al predicted the Cosmic Microwave Background (CMB) radiation as leftover heat, a radiation relic, from the early universe. Finally, the discovery of the Cosmic Microwave Background in 1965 [6] laid the foundation for the Big Bang model and all of modern cosmology.

1.2 Friedmann-Robertson-Walker Cosmology and Big Bang Problems

After the discovery of the Cosmic Microwave Background, many studies and experiments emerged, and together with large-scale structure surveys and cosmic expansion measurements served as independent observations which advanced our understanding of our observable universe, and strongly supported the theory of Cosmic Inflation. To introduce Inflation as the theory that solved the standing puzzles at the time, and bring forward the problems with the standard Big Bang model, it will be useful to outline the basics and main concepts in cosmology, following detailed discussions found in [94], [101].

Observations suggest that the universe, as seen on scales larger than 100 Mpc, is homogeneous and isotropic. On smaller scales however, inhomogeneous structures start to appear. As mentioned before, the observable universe expands following Hubble's law. Therefore, the observable region must be described by a general metric that preserves its symmetries, and it can be captured with a metric of the following form, namely the

Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right). \tag{1.1}$$

The spatial part of the metric describes certain hypersurfaces, that scale with the scale factor $a(t)$ which is evolving in time. It allows for different curvatures such that

1. for $k = 0$, the spatial slice describes a flat space,
2. for $k = +1$, it describes a sphere of positive constant curvature,
3. for $k = -1$, it describes a hyperbolic space which has a negative constant curvature.

The spatial surface is in fact an embedding of a 3-dimensional sphere, or pseudo sphere depending on the sign of k, in a higher dimensional space with the appropriate signature. The scale factor a^2 is then positive, and describes the radius of curvature for the curved spaces, however it has no physical meaning for the flat space, and by performing a change of coordinates, it can be reabsorbed into the new variables. From the scale factor, one can define the Hubble rate

$$H = \frac{\dot{a}}{a}, \tag{1.2}$$

and it has a dimension of t^{-1}. It will be more useful to redefine the radial coordinate r into a new variable χ, such that

$$r^2 = \Phi(\chi^2) = \begin{cases} sinh^2\chi & \text{for } k = -1, \\ \chi^2 & \text{for } k = 0, \\ sin^2\chi & \text{for } k = +1. \end{cases} \tag{1.3}$$

The new radial variable χ removes the degeneracy which r has in the positively curved space. Introducing conformal time η into the picture, such that it is given by

$$\eta = \int \frac{dt}{a(t)}, \tag{1.4}$$

the metric described above in terms of the new variables becomes

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + \Phi(\chi)(d\theta^2 + sin^2\theta d\phi^2], \tag{1.5}$$

and in terms of the new coordinates τ and χ, the causal structure of space-time becomes more transparent. With the metric being conformally flat, with an isotropic spatial part, in the $\eta - \chi$ coordinates, the propagation of light will look like that in Minkowski space. It is described by the condition $ds^2 = 0$, which are the radial light geodesics with $\theta, \phi =$ const. In the $\eta - \chi$ plane, this corresponds to light-cones at $\pm 45\,\mathrm{deg}$, such that

$$\chi = \pm\eta + \text{const.} \tag{1.6}$$

Horizons — It is now convenient to define the horizons associated with the maximum duration of time light can propagate from a given point to a reference observer, and vice versa. Considering that the universe could have a finite age, then during a time t, the furthest point in space that can receive a signal sent from a reference observer at the origin will be located at χ_p such that

$$\chi_p = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a}, \tag{1.7}$$

which is called a *particle horizon*, and it is the largest co-moving distance which light could have traveled from the moment of time η_i when the universe originated. The reference observer at the origin will not receive any signal from an event happening at $\chi > \chi_p$, therefore in a sense it is really more like a *causal* horizon. The physical co-moving distance can be obtained from

$$d_p(t) = a(\eta)\chi_p(\eta). \tag{1.8}$$

On the other hand, there is also a maximum separation for which signals sent from a point in space can be received by the observer in the future, and this is called the *event horizon*, defined by

$$\chi_e(\eta) = \eta_{max} - \eta = \int_{t}^{t_{max}} \frac{dt}{a}, \tag{1.9}$$

where at a given moment η, a future observer will not be able to perceive events in the regions beyond χ_e.

This description of the horizons will be important for outlining why the Big Bang model falls short at explaining the observed universe and why one needs a period of inflation instead.

Evolution equations —

From the previous discussions, it is clear that the causal relationship between events at a given moment in time will depend on the scale factor. Then, the evolution of the scale factor $a(t)$ will follow from the Einstein Equations, governing the dynamics of the metric components $g_{\mu\nu}$ which fully describe the gravitational field,

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}. \tag{1.10}$$

From the symmetries of the metric $g_{\mu\nu}$, these equations reduce to ten. For the FRW metric, these further reduce to two key equations, where matter is described by the energy-momentum tensor obeying ${T^{\alpha\beta}}_{;\alpha} = 0$. The energy-momentum tensor for matter at cosmological scales can be captured by that of a perfect fluid which is given by

$$T_{\alpha\beta} = (\varepsilon + p) u_\alpha u_\beta - p g_{\alpha\beta}, \tag{1.11}$$

where p is its pressure, ε is the energy density, and u^α is the 4-velocity. With this matter component, the Einstein equations give

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\varepsilon + 3p), \tag{1.12}$$

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\varepsilon, \tag{1.13}$$

which are the First and Second Friedmann equations respectively. In the first equation, $\ddot{a}/a \equiv \dot{H} + H^2$. In the second equation, k appears as an integration constant, but it carries the meaning of the curvature defined earlier for the different space slice. Combining the first and second Friedmann equations, one gets

$$\dot{\varepsilon} = -3H(\varepsilon + p) \tag{1.14}$$

which is simply an expression of energy conservation, and for a homogeneous and isotropic universe it is a manifestation of the conversation law ${T^\alpha_0}_{,\alpha} = 0$. Now, for the matter described in (1.11), its equation of state can be written as

$$p(\varepsilon) = w\varepsilon \tag{1.15}$$

and w will depend on the properties of matter. Substituting this equation into (1.14), the solution obtained reads

$$\varepsilon \propto a^{-3(1+w)} \tag{1.16}$$

where for non-relativistic matter with $w = 0$ one gets $\varepsilon_m \propto a^{-3}$, and for ultra-relativistic gas or radiation with $w = 1/3$, $\varepsilon_{rad} \propto a^{-4}$. In the standard *Big Bang* picture then, one assumes the universe was filled with matter having $\omega > 0$.

Finally, defining the cosmological parameter $\Omega(t) = \sum_i \Omega_i(t)$, such that for a given matter component,

$$\Omega_i(t) \equiv \frac{\varepsilon_i(t)}{\varepsilon_{cr}(t)}, \tag{1.17}$$

with $\varepsilon_{cr} = \frac{3H^2}{8\pi G}$, and for the curvature component k it is defined as

$$\Omega_k(t) = -\frac{k}{a^2 H^2(t)}, \tag{1.18}$$

then using these definitions, one can rewrite the second Friedmann equation as

$$\Omega_k(t) = \Omega(t) - 1. \tag{1.19}$$

The geometrical properties of the universe will be dictated by the value of $\Omega(t)$, i.e.,

$$\begin{cases} \text{open universe },\, k = -1, & \text{for } \Omega > 1 \\ \text{flat universe },\, k = 0, & \text{for } \Omega = 1 \\ \text{closed universe },\, k = +1, & \text{for } \Omega < 1. \end{cases} \tag{1.20}$$

Problems of Big Bang cosmology — Assuming in the Big Bang scenario that the early universe was filled with either relativistic or non-relativistic matter with $\omega = 0, 1/3$ respectively, in order to explain the observed properties today, the initial requirements will have to be oddly fine-tuned. From here, one can directly describe the first problem arising from Big bang cosmology, the *flatness problem*. From observations today, Ω_k is measured to be almost *zero*, with $\Omega_k \ll 10^{-3}$. However, since the contribution of the curvature is through the factor a^{-2}, in comparison to other components as ordinary matter which scales as a^{-3} and radiation scaling as a^{-4}, Ω_k is expected to dominate today. The fact that it does not, means that the universe must have started initially *flat*, with the cosmological parameter $\Omega(t)$ being close to unity. The question becomes, what would drive Ω_k to be almost negligible in that early phase? While it is possible to simply set the value of k to zero initially, having such a pre-set peculiar value does not seem satisfying without a mechanism that explains it, hence the role of inflation in specifying that mechanism.

The second problem, arises from the fact that the CMB radiation which was measured to be incredibly uniform across the sky by experiments like COBE, WMAP, and Planck, with very tiny fluctuations. Now if one considers that a given form of matter with $w > -1/3$ fills the universe, as it expands, the co-moving distance of light will grow, and can be computed from (1.7) as

$$\chi_p(\eta) = \int_{a_i}^{a} \frac{da}{Ha^2} \sim a^{(1+3w)/2} - a_i^{(1+3w)/2}. \tag{1.21}$$

This horizon grows in proportion to contributions from the first term at the latest moments, hence, as time progresses, new portions of the universe that were formerly outside causal range establish causal contact. Now assuming that at the earliest times, at Planckian times t_{Pl}, the universe was dominated by primordial radiation, then from (1.21) the co-moving distance light could travel is $l_p \sim t^{1/2}$. Comparing the ratio of this distance measured today at t_0 to that at the Planck time from an initial surface, one finds $l_0/l_p \sim 10^{30}$, meaning around 10^{90} initially causally disconnected Hubble volumes must have the homogeneous energy density distribution observed today with a small variation of $\delta\varepsilon/\varepsilon$. The Big Bang model, without inflation, would need a high degree of fine tuning in the initial density of the matter field to justify why the scale of homogeneity already greatly exceeded the initial causality scale, and this is called the *homogeneity or horizon problem*. Another problem, to complete the shortcomings of the Big Bang, is explaining the origin of the initial density perturbations that seed the formation of the large-scale cosmic structures. Following the pattern of temperature variations seen in the CMB, the initial amplitude of these primordial inhomogenieties needs to be on the order of $\delta\varepsilon/\varepsilon \sim 10^{-5}$ on scales corresponding to galactic sizes. This is called the *initial perturbations problem*.

1.3 Predictions from Inflationary Theory

Following these shortcomings that the Big Bang model could not answer, Cosmic Inflation therefore emerged as an extension of the Big Bang that proposed that the universe underwent an exponentially large expansion in a minute fraction of a second right after the Big Bang, with a dominant form of matter having an equation of state $p \approx -\varepsilon$. This equation of state can be realized by scalar-field models, or other self-interacting fields or higher-curvature gravity models, which effectively converge on similar dynamical features as constrained by observations. More broadly, inflation set forth a set of explanations of the problems faced by Big Bang model that were discussed before, and also provided a set of predictions that were confirmed by observation. Independently of a particular model, Inflation stands on a set of pillars which are hereby presented following [8].

General features of Inflationary Models — Taking into account different existing classes of inflationary models, that are accompanied with a graceful exit, there are common features that these models yield, and they factor into a set of *general predictions of Inflation*. These set of predictions are:

1. *Geometry of the Universe is Euclidean:* This means that the spatial curvature of the observable universe is zero, $k = 0$, and $\Omega_k = 1$. This is known from precision measurements of the cosmic microwave background (CMB), where a nearly-flat geometry of the universe is deduced from the the temperature anisotropy spectrum, particularly the angular size of the first acoustic peak (and subsequent peaks) which are sensitive to such geometry [7]. In addition, this flat geometry is confirmed from the Baryon Acoustic Oscillations (BAO) that are measured at late times from large-scale structures.

2. *The amplitude of the gravitational potential Φ depends logarithmically on the physical scale λ:* This is an important prediction of inflation, with the amplitude growing roughly as a function of the logarithm of λ, and getting larger at bigger scales. Approximately the relation can be written as

$$\Phi^2 = \langle \Phi_k^2 k^3 \rangle = \lambda^{1-n_s} \tag{1.22}$$

on the observable scales. This relation has an underlying physical reason, and it is due to the fact that the $n_s - 1$ depends on the equation of state, which has small deviations from $p \approx -\varepsilon$ that allow a graceful exit from inflation. For a steady variation in $\frac{(p+\varepsilon)}{\varepsilon}$, the signature of inflation will be a spectrum which is red-tilted.

3. *Primordial perturbations should be Gaussian:* This means they are precisely characterized by two-point correlation function. Of course, the initial inhomogeneities are only nearly Gaussian, and through the interaction with the background gravitational field they got amplified. Deviations from Gaussianity enter at the second order in perturbation theory for the gravitational potential Φ_g, with the leading non linear term $f_{NL}\Phi_g^2$, which is expected to be of order $f_{NL} \sim O(1)$. Recent analysis from Planck [46] find $|f_{NL}| \lesssim 10$ as theoretically predicted.

4. *Primordial perturbations are Adiabatic:* With high-precision measurements of the CMB, data from WMAP and Planck effectively rule out isocurvature scenarios, confirming that initial perturbations should be adiabetic.

5. *Existence of long-wave Gravitational Waves:* While inflation in general does not specify an exact value for the the tensor-to-scalar ratio r, which will depend on the model considered, one can set a lower-limit for the gravitational waves produced after the spectral index n_S has been determined. The limit, however, is insensitive of the details of a specific inflation model.

Predictions and other requirements from an Inflationary Model — Getting slightly ahead in the chronology of the story, it is worthwhile to mention that, starting from these predictions for a general inflationary model, the main interest in finding a new Inflationary model, as Mimetic Inflation presented in this work, is not only to reproduce these general features, which represent the *minimum* necessary conditions to have a viable inflationary scenario, but also to address and resolve some of the broader questions which put the whole *Inflationary Paradigm* in question, as will be discussed in the next section following [49]. That being said, it is important to highlight that addressing the new challenges presented to Inflation is another criterion for a model to be considered viable, though its robustness in facing the very questions that undermine Inflation, not just as a theory that explains the observations, but as a scientific theory with predictive power, a key concern which had driven cosmologists in recent years away from Inflation.

1.4 Inflationary Theory Facing Trouble

After Planck satellite published its data in 2013 [50], providing precise numerical limits on the available cosmological parameters, it confirmed through its observations the predictions that Inflation set forward, and that were described in the previous section, the spatial geometry of the universe being nearly flat, a spectrum of perturbations that is close to scale-invariance yet having a slight red-tilt, and the close-to Gaussianity of the perturbations. And for the first time, the data came in favor of simpler models, and removed from the pool of candidates exotic models with complex dynamics.

Models Favored by Planck data — By confirming that the observed primordial inhomogeneities are indeed nearly Gaussian, which is predicted by the standard single-field inflaton framework is favored over the multi-field inflationary models and scenarios with elaborate dynamics. Among the simplest single-field models, the data excluded the original 'chaotic' inflation, inflation with exponential potentials, and various other power-law driven inflation, etc... (e.g., [38]-[42]). Then, out of the single-field models, observations favored those having potential with a plateau. Under this category falls a multitude of models that can be recast as such by a transformation to the Einstein frame [50], and at instances by a change of variable. Such models include for example the Starobinsky model [105], models with symmetry-breaking (out of which those with non-minimal coupling currently survive)[54]-[56], certain modified hilltop scenarios [57], and of particular interest, Higgs-like inflation.

However, the celebrated results from Planck did not stand for a long time before serious issues were raised by Ijjas, Steinhardt, and Loeb in their work [49] challenging the single scalar field, with a plateau-like potential inflationary models that the Planck collaboration favored through its data. The authors set forth a number of problems facing these models, and concluded that the data in fact tends to disfavor the inflationary scenarios that were most-motivated being 'exponentially unlikely according to the inner logic of the inflationary paradigm itself', quoting the authors in [49], when one begins inflation at the Planck scale even if plausible initial conditions were taken, and took these challenges as far as touching on the core of the Inflationary theory as a self-consistent scientific theory. These issues will be summarized, with arguments presented in favor of the Inflationary theory, highlighting some arguments following [69], and concluding the way forward.

Problems facing inflation

Likeliness for Inflation to begin — This problem stands as the 'unlikeliness problem' of inflation, which the authors of [49] refer to in regards to their claim that for plateau-like potentials, for example a Higgs-like potential of the form

$$V(\varphi) = \lambda(\varphi^2 - \varphi_0^2)^2 \tag{1.23}$$

inflation can occur only in a narrow range of the parameters that such a model carries, and inflation on the plateau part of the potential, where $|\varphi| < \varphi_0$ occurs exponentially *less* than inflation on the power-law part with $|\varphi| > \varphi_0$, the latter being the models that are disfavored by Planck. This argument is based on the fact that in the power-law part of the potential, the range of values $\Delta\varphi$ is significantly larger, and consequently the maximum number of e-folds, N_{max}, is also significantly larger. Counter arguments to this issue, as presented in [69], can be based on the ambiguity of details related to particle physics dynamics and assumptions made at the Planck-scale, where it is difficult to discern the likeliness of having the scalar-field in a given interval of its possible values or in another, which a valid question to ask in single-field models; or if one accepts that for the available plateau-models, eternal inflation with a multiverse is an inevitable outcome of inflation occurring on that plateau (as will be described in the section on Self-Reproduction), then the problem becomes that of finding the appropriate probability measure, on which no consensus has been reached thus far. Another argument that avoids the given formulation of the unlikeliness problem is looking at slightly more complicated dynamics of the scalar-field, like considering a scenario where tunneling occurs from a metastable state which precedes the desired last number of e-folds occurring on the plateau. This would force one to choose the metastable phase as a reference when comparing with the values of $\Delta\varphi$ and N_{max} on the power-law part, rather than the plateau. As such, the general argument of the unlikeliness problem starts to break down when considering various particularities for each model.

Fine tuning of initial conditions— As Planck data favored single-field inflationary models having a potential characterized by a plateau, the latter characterization arises from the requirement on the tensor-to-scalar perturbations ratio r, whereby the value of its upper bound $r < 0.032$ (which is the most recent value obtained by including data from BAO [43], and the CMB lensing data [44]) restricts the slope of the potential to be small at the moment of time t_I, when the segment of inflation pertaining to observations begins. Then, the inflationary scale which is characterized by the plateau with $V(\varphi_I) \equiv V_I$, where $\varphi_I \equiv \varphi(t_I)$, is then estimated from the ratio r to be orders of magnitude much less than the Planckian scale M_{pl}, particularly $V_I \sim 10^{-12} M_{pl}^4$. While one could assume rather natural initial conditions for different forms of energy $\frac{1}{2}\dot{\varphi} \sim \frac{1}{2}|\nabla\varphi|^2 \sim V$ at Planckian scales as described by chaotic inflation, this assumption fails for the required energies for the plateau V_I. Hence, with the small value of the energy density at the plateau, the fine-tuning problem arises from the fact that starting at the Planck scale one would have to control the initial conditions, such that after the energy density falls to values of order V_I, the Hubble-sized patch could remain nearly uniform when observable inflation starts. The reason for this is that the kinetic terms and gradient terms would dominate over the potential which has a low energy density around t_I, and prevent inflation from beginning unless initial smoothness is somewhat maintained. To give an estimate from [49], starting with an initially homogeneous patch at Planckian scales, for it to remain homogeneous until the onset of inflation requires a scale of homogeneity of order 10^3 Hubble lengths to which its radius grows until the inflation scale is reached, by simply comparing the energy scales $M_{Pl}/M_I \sim 10^3$, which is rather huge. However, this problem can be avoided in a scenario where inflation starts at Planckian scales and while the energy scale shifts to lower values at the plateau where the last 50-60 e-folds of inflation take place, then the initial inhomogeneities will be stretched out to unobservable scales, and the minimal initial condition that remains is that the gradient terms do not prevent inflation from starting in the initial patch, rather than requiring a high degree of homogeneity for a large number of Hubble sized volumes. Several scenarios in which inflation kicks off at Planckian density, then ends on the plateau can be found in (e.g. [37] and references therein), however the problem with such models that begin inflation at the Planck scale is that it leads to eternal inflation and the multiverse. To have inflation begin at the Planck scale then requires these two issues to be addressed simultaneously.

The problem of the Multiverse — Following the Planck-favored plateau-like potentials for Inflationary Models, a common phenomenon with this class of models emerges when trying to kick-start expansion at the Planck scale, whereby this patch of space, once it had entered the accelerated expansion phase, it remains in that inflationary stage forever. This problem was named *Eternal Inflation*, and it can happen at different regions in space, which leads to islands of eternally expanding disconnected universes, resulting in a *multiverse* picture, in which "anything that can happen will happen, and it will happen an infinite number of times" [58]. This then undermines the power of predictability of Inflation as a theory. The reason for that is, since different universes with different physical properties could emerge when starting with a single inflationary model with a unique set of parameters, the model fails to predict the outcome of the universe and uniquely explain its properties, with the emergence of these physically distinct pocket universes. This is a main contradiction that a theory with predictive power faces, and hence testing for this regime in a given model is an important self-consistency check that makes it a viable candidate as an inflationary scenario. For those models which exhibit eternal inflation and the multiverse picture, there were various attempts to find a certain 'measure' to define probabilities for the resulting infinite universes, but these didn't yield any success (e.g. [59]-[63]), and on the contrary some are presented with additional problems such as the Youngness Paradox (e.g. [64, 65]). The only way forward to avoid this fractal of issues is then to find a class of models that evades eternal inflation altogether, through avoiding self-reproduction.

1.5 Self-Reproduction and the Problem of the Multiverse

In this section, the focus will be on demonstrating the technical details involving the phenomenon of self-reproduction, and these arguments will be used later in the analysis of the Mimetic Inflation Model to show how it overcomes this regime by starting inflation at the Planck scale. As mentioned earlier, one is interested in having inflation begin at the Planck scale and restore the natural initial conditions described earlier, to avoid the fine-tuning problem of these conditions needed to start inflation. Hence the goal will be to solve both problems simultaneously, producing a viable candidate model.

Background Equations for a Generic Slow-Roll Model — The action for a general inflationary model driven by a scalar field that is homogeneously distributed is given in the Einstein-Hilbert frame by

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right), \tag{1.24}$$

where Planck units with $8\pi G = 1$ are used hereafter. The dynamics of the scalar field are then analyzed in a flat expanding universe which has its metric defined by

$$ds^2 = dt^2 - a^2\left(t\right)\delta_{ik}dx^i dx^k. \tag{1.25}$$

The energy-momentum tensor for φ can be computed as

$$T_{\mu\nu} = -\frac{1}{2}\frac{\delta S_\varphi}{\delta g_{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V(\varphi)\right), \tag{1.26}$$

then comparing with the $T_{\alpha\beta}$ expression for a perfect fluid in (1.11), the energy density and pressure for the scalar field can be deduced

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{1.27}$$

and here the derivative with respect to the physical time t is denoted by *dot*. Then to mimic the equation of state $p \approx -\varepsilon$, from

$$\omega_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}, \tag{1.28}$$

the potential term must dominate over the kinetic energy,

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi). \tag{1.29}$$

The usual equations of motion for the scale factor $a(t)$ driven by the dynamics of a scalar field φ with a potential $V(\varphi)$ in a (non-modified) General Relativity framework are then

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0, \tag{1.30}$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V\right), \tag{1.31}$$

with $H \equiv \dot{a}/a$ as the Hubble constant, and the derivative with respect to the field φ is denoted by *prime*.

Slow-Roll Conditions — The plateau-like potential favored by the data must then satisfy the following slow-roll conditions,

$$\left(\frac{V'}{V}\right)^2 \ll 1, \qquad \frac{V''}{V} \ll 1, \tag{1.32}$$

for which the time-evolutions equations can be further simplified, where the acceleration term for the scalar field $\ddot{\varphi}$ in (1.30) can be neglected, similarly the kinetic energy term in the Friedmann Equation (1.31) will be negligible compared to the potential energy term, so that the expressions become

$$3H\dot{\varphi} + V' \approx 0, \tag{1.33}$$

$$H^2 \approx \frac{1}{3}V, \tag{1.34}$$

and hence as the field rolls down slowly along the platueau, the universe undergoes exponential expansion dominated by the potential.

Condition for Self-Reproduction — To trace back the physical origin for eternal inflation in a given Planckian-sized space patch, one can define a cosmological timescale $t_H \simeq H^{-1}$ for studying the process, and in this case it will be the time needed for the classical scalar field to decreases its value by an amount

$$\delta\varphi_{cl} \simeq \dot{\varphi}t_H \simeq \frac{\dot{\varphi}}{H}. \tag{1.35}$$

and from (1.33), $\dot{\varphi} \simeq -V'/3H$, the classical scalar field hence decreases during t_H by

$$\delta\varphi_{cl} \sim -\frac{V'}{V}. \tag{1.36}$$

During this time t_H, quantum fluctuation of the field $\delta\varphi_q$ are being super imposed over the classical field value, such that the amplitude of these fluctuations at the Hubble scale H^{-1} is

$$\delta\varphi_q = \pm H. \tag{1.37}$$

In total, at the Hubble scale H^{-1}, the total change of the background field $\Delta\varphi$, for those superimposed quantum fluctuations that are *positive* and hence push the field *upward* its potential, is positive

$$\Delta\varphi = \delta\varphi_{cl} + \delta\varphi_q \simeq H - \frac{V'}{V} > 0, \tag{1.38}$$

given that $H \sim 1$ and from slow-roll conditions $\left(\frac{V'}{V}\right)^2 \ll 1$. This means that, when it is the case, the quantum fluctuations with a positive amplitude are continuously pushing the field up its potential, keeping it from exiting the inflationary regime. This could happen at different patches in space, and the resulting picture is eternally inflating universe bubbles, separated by regions where inflation has ended, hence the emergence of *the Multiverse*. The number of these regions where inflationary expansion continues forever is exponentially large, and so their volume outweighs the domains that have exited inflation and produced a standard matter-dominated Friedmann universe [35, 36], even when considering the different definitions of the measure taken for comparing these volumes (e.g., in [48]).
For this work, the important consideration is the *self-reproduction condition* that drives the situation into an eternally expanding state, namely

$$\Delta\varphi \simeq H - \frac{V'}{V} > 0,$$

which takes into account the slow-roll condition for a usual plateau-model in a non-modified Gravity framework. A possible solution to *violate* this condition will be to modify the slow-roll conditions, which in this case result from the regular set of equations for $a(t)$ and $\varphi(t)$ in (1.30) and (1.31). This can be achieved by obtaining a modified set of equations, and deriving new slow-roll conditions for them, then putting to the test the effect of quantum fluctuations on the classically evolving background. This will be shown in the Mimetic Inflation Model presented in this thesis.

Chapter 2

Mimetic Inflation and Self-Reproduction

In this chapter of the thesis work, the Mimetic framework, mainly through its covariant quantity defining extrinsic curvature, will be employed to construct simple inflationary models that avoid eternal inflation and solve the problem of self-reproduction with its many problems. Core results in this chapter are based on the original published work *Chamseddine, Khaldieh, and Mukhanov (2023)* [1].
In the next section, the focus will be on motivating the Mimetic Gravity framework within which the class of models of *Mimetic Inflation* is developed. In the sections that follow, Mimetic Inflation will be presented through a concrete example, studying the action and its background solutions, then showing how self-reproduction is avoided, and developing its cosmological perturbations leading to its observational predictions.

2.1 Mimetic Gravity Framework

Modified Gravity: Theoretical and Observational Motives

The question on why one appeals to modified gravitational theories stems from several theoretical and observational motivations which suggest that General Relativity, as the current most successful gravitational theory, might not be the complete description of gravity, despite being extremely successful in explaining a wide range of gravitational phenomena, from planetary motion to the evolution of the Universe, and from describing black holes to gravitational waves as ripples in space-tine [66]-[68]. Attempts to extend General Relativity and include it in a broader unifying framework have existed early on since the establishing of this theory. On this theoretical end, modified theories included higher dimensional generalizations such as theories of Kaluza-Klein, additional fields with dynamical degrees of freedom such as Brans-Dicke's Scalar-Tensor theory, and the addition of higher order corrections to the General Relativity action as in Sakharov's proposal, to name a few, aside from the large front that is aimed at finding a Quantum Gravity theory. A comprehensive survey of such theories can be found in many reviews (e.g. [74]). On the observational end, many empirical tensions emerged that revealed many limitations of General relativity, and at the same time provided an evaluation ground of the various modified theories. The 'dark universe' picture of the ΛCDM model which Einstein's equations yield with a dominant proportion over all known forms of matter taking an unusual form of 'dark energy', driving the evolution of the Universe, was the first lead hinting that Einstein's theory might be incorrect at such cosmic scales. Building on this picture, the observed small value for the cosmological constant Λ, which falls short on 120 orders-of-magnitude from what is expected from quantum field theory, poses another serious challenge for ΛCDM. Another component of the proposed dark sector, *dark matter*, faces problems when theoretical simulations are faced with observations. These include the theorized cuspy-density profile of dark matter in its halo contrasted with the observed constant density cores in galaxies [75], the discrepancy between the number of satellite galaxies expected in cosmological simulations with dark matter and the number which is observed [76], and other problems on scales of clusters as Bullet-cluster speed [77], emptier cosmic voids [78], and large-angle CMB anomalies [79]. All these challenges suggest diverging routes from ΛCDM, insofar systematic effects can be resolved. And while there is still no observed explanation from particle dark matter ventures, attempts of modification of the gravitational theory are on the front seat.

Why Mimetic Gravity?

Motivation — Now the question turns to why, out of all other modified gravity theories, is Mimetic Gravity a desirable candidate? To begin, Mimetic gravity is not simply a *modification* to GR, rather a small *extension* that isolates the conformal degree of freedom of the metric, and through imposing a constraint on the isolated conformal factor $\Omega^2(x)$ that is expressed through the derivatives of a scalar-field ϕ. It utilizes a non dynamical degree of freedom, namely the longitudinal gravitational mode, that then 'mimics' the Dark Matter component in the universe without adding new dynamical fields. Since it only involves a reparameterization of the metric while introducing new dynamics, Mimetic gravity is one of the simplest geometrical extensions to GR, in contrast to other Scalar-Tensor gravity theories (e.g., [83, 84]), where the added fields there bring in an external degree of freedom that Einstein's Gravity does not already encompass.

The Original work on Mimetic Gravity — It was first introduced by A. H. Chamseddine and V. Mukhanov in 2013 [82], as *Mimetic Dark Matter*, which offered a geometric explanation for the dark matter component, and since then, many extensions to the original formulation appeared by adding terms involving the field ϕ, either through higher-derivative terms or through a potential $V(\phi)$, or incorporate it into broader gravitational frameworks, and was hence renamed more generally as *Mimetic gravity*.

After discussing the general framework following [82], Mimetic Gravity will be utilized to suggest a new solution to the problem of Self-Reproduction and the Multiverse with their undesirable corollaries.

The Mimetic Framework: Its Equations and Interpretations

Metric Conformal Factor — While General Relativity is not invariant under scale transformations of the metric as

$$\tilde{g}^{\mu\nu}(x) = \Omega^2(x) g^{\mu\nu}(x), \tag{2.1}$$

in Mimetic gravity, the Mimetic field ϕ was introduced to rescale the metric in the following way

$$g_{\mu\nu} = \left(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\tilde{g}_{\mu\nu}, \tag{2.2}$$

where in this case $\tilde{g}_{\mu\nu}$ is an auxiliary metric, and under conformal transformations of $\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}$, the physical metric remains invariant $g_{\mu\nu} \to g_{\mu\nu}$.

The Mimetic Constraint — Taking the inverse of the definition (2.2), the mimetic field ϕ then satisfies the constraint equation

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1. \tag{2.3}$$

which is a first order differential equation. To see how the field ϕ acts as the Dark Matter component, when combined with the longitudinal mode of gravity, the Energy-Momentum tensor will be computed.

Extra Degree of Freedom — To find how the additional degree of freedom behaves, a variation of the action for gravity with a general matter Lagrangian $\mathcal{L}_m$ is performed, taking into account that the variation of the physical metric $g_{\mu\nu}$ will include the variation of field ϕ and the auxiliary metric,

$$S = -\frac{1}{2}\int d^4x\sqrt{-g(\tilde{g}_{\mu\nu},\phi)}[R(g(\tilde{g}_{\mu\nu},\phi)) + \mathcal{L}_m], \tag{2.4}$$

such that such variation leads to, after a few additional steps that can be found in [82], the following field equations

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu}, \tag{2.5}$$

with $T^{\mu\nu}$ as the usual energy momentum tensor for matter, and a new contribution from

$$\tilde{T}^{\mu\nu} = (G - T) g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \phi \partial_\beta \phi. \tag{2.6}$$

The last expression could be reformulated in terms of the Energy-Momentum tensor describing a perfect fluid

$$\tilde{T}^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}, \tag{2.7}$$

where in the expression (2.6), the pressure can be identified to be $p = 0$, while the energy density ρ and the four-velocity u^μ become

$$\rho = G - T, \quad u^\mu = g^{\mu\kappa} \partial_\kappa \phi. \tag{2.8}$$

From here, it can be understood how the extra degree of freedom '*mimics*' dark matter, by imitating the equation of state of *dust*, with $p = 0$ and $\rho = G - T \neq 0$ even when matter is not present. Taking $g^{\mu\kappa}\partial_\kappa \phi$ as the velocity for dust, the constraint equation (2.3) then simply reflects the normalization condition $u^\mu u_\mu = 1$.

The Mimetic Field ϕ — To understand in physical terms the *Mimetic Field* ϕ itself, it is useful to choose a gauge, and the simplest way will be to use the synchronous coordinate system,

$$ds^2 = dt^2 - \gamma_{ik} dx^i dx^k, \tag{2.9}$$

in which the the solution to the first order differential equation (2.3) is simply

$$\phi = t, \tag{2.10}$$

hence, ϕ can be interpreted as the coordinate for 'time' which then defines the slicing of the four-dimensional space-time into spatial hypersurfaces.

Mimetic Dark Matter — Building on the last part, and in a way which will be more practical for the purposes of building an action that integrates the Mimetic framework, it is useful to reformulate the same picture of the physical metric with the isolated conformal degree of freedom in the general action below, which was first shown in [10],

$$S = \int d^4x \sqrt{-g} \Big(-\frac{1}{2} R + \lambda \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1 \Big) + \mathcal{L}_m \Big). \tag{2.11}$$

Therefore, varying this action with respect to ϕ, the solution to the scalar field equation will give

$$\lambda = \frac{const}{\sqrt{det\gamma_{ij}}}, \tag{2.12}$$

where γ_{ij} is the spatial part of the metric as in (2.9), and in a flat Friedmann universe, this expression becomes

$$\lambda = \frac{const}{a^3}. \tag{2.13}$$

This expression represents the dark matter component, and completes the picture of how Mimetic Dark Matter is realized.

Extrinsic curvature κ — Many extensions to Mimetic Gravity are realized through mimetic field ϕ by constructing the covariant quantity, namely $\Box\phi$. In the synchronous coordinate system defined earlier, the physical interpretation of the invariant quantity

$$\Box\phi = g^{\mu\nu}\phi_{;\mu\nu} \equiv \kappa \tag{2.14}$$

becomes more transparent, and for the three-dimensional spatial hypersurface having $\phi = \text{const}$, it represents the trace of its extrinsic-curvature, where

$$\kappa = \frac{1}{2}\frac{\partial}{\partial t}\ln\left(\det\gamma_{ik}\right). \tag{2.15}$$

Thus, by incorporating functions $f(\Box\phi) \equiv f(\kappa)$ into the Lagrangian, the equations of Einstein's gravity can be modified, iterating on the fact that it is done without introducing any new scalar degrees of freedom. This invariant has previously been utilized in works that go beyond the minimal extension, for instance to address cosmological singularities in the universe and singularities in black holes as in [85]-[87], and incorporate the idea of limiting curvature to study asymptotically free solutions as in [88, 89]. The field ϕ can also be implemented directly to obtain different cosmological solutions by adding $V(\phi)$ as in [90], or to further generalize the mimetic framework into $F(R)$ Gravities as in [91], and other general Scalar-Tensor theories [92]. Numerous extensions were developed since the first work on Mimetic Gravity appeared and a comprehensive review of these works until 2016 can be found in [93].

2.2 Action For the Theory and Field Equations

The Action — To Begin, the action for Mimetic Inflation is presented as

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - C\left(\kappa\right)V\left(\varphi\right)\right], \tag{2.16}$$

where reduced Planck units are used such that $8\pi G = 1$, and φ is the usual inflaton field driving the exponential expansion through the potential $V(\varphi)$. Here $V(\varphi)$ is coupled to spatial curvature through a function $C(\kappa)$, which is formally expressed through the d'Alembertian of the mimetic field $\kappa = \Box\phi$, and as such enters the action in a covariant way.

Mimetic Constraint — To get the constraint equation for the mimetic field, this action is varied with respect to the Lagrange multiplier λ, and the previous expression for the constraint (2.3) is recovered

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1,$$

Modified Einstein Equaitions — Now onto the modified Einstein equations, whereby varying the action with respect to the metric gives the usual structure for the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \tilde{T}_{\mu\nu}, \tag{2.17}$$

albeit a modified form for the Energy-Momentum tensor, denoted by $\tilde{T}_{\mu\nu}$, and given by

$$\begin{aligned}\tilde{T}_{\mu\nu} &= 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\left(C - \kappa C'\right)V - g^{\rho\sigma}\partial_\rho\left(C'V\right)\partial_\sigma\phi\right)\\ &+ \left(\partial_\mu\left(C'V\right)\partial_\nu\phi + \partial_\nu\left(C'V\right)\partial_\mu\phi\right) + \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi\right),\end{aligned} \tag{2.18}$$

and here the *prime* indicated the derivative with respect to the argument of the particular function, in case it depends on one variable only, for instance, the coupling function C as it depends on κ and as such $C' \equiv dC/d\kappa$, and similarly for the potential function $V(\varphi)$, where $V' \equiv dV/D\varphi$.

Equations of Motion — For the mimetic field ϕ, one gets the following equation of motion

$$\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\left(2\lambda\partial_\nu\phi + \partial_\nu\left(C'V\right)\right)\right] = 0, \tag{2.19}$$

and that for the inflaton field is therefore obtained by varying (2.16) with respect to φ to get

$$\Box\varphi + CV' = 0, \tag{2.20}$$

where $\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right)$.

2.3 Dynamical Evolution and Background Solutions

To solve the equations of motion obtained, for a flat expanding universe, the metric is expressed by

$$ds^2 = dt^2 - a^2(t)\,\delta_{ik}dx^i dx^k. \tag{2.21}$$

Solution for mimetic field — As expressed before, the solution to the mimetic field will be identified through $\phi = t$ with the time foliation of space-time.

Homogeneous Inflaton Field — In the universe considered, the inflaton field will be a homogeneous one, and only time dependence will be expressed in the field configuration as $\varphi = \varphi(t)$. The inflaton field will evolve then according to its equation of motion

$$\ddot{\varphi} + \kappa\dot{\varphi} + CV' = 0. \tag{2.22}$$

which is recovered from (2.20)

Friedmann Equations — Taking the previous results, $\varphi(t)$ and $\phi = t$, and plugging them into the $0-0$ *Einstein Equation*, it becomes

$$\frac{1}{3}\kappa^2 = 2\lambda + (C - \kappa C')\,V + (C'V)^{\cdot} + \frac{1}{2}\dot{\varphi}^2, \tag{2.23}$$

where $\kappa = 3\dot{a}/a$, and the *dot* denotes the derivative w.r.t. time. The spatial curvature κ is hence understood as the *Hubble constant*, $H = \dot{a}/a$ multiplied by a factor of *three*. To find then the expression for λ, the equation for ϕ (2.19) is solved, and the explicit solution is

$$2\lambda = \frac{B}{a^3} - (C'V)^{\cdot}, \tag{2.24}$$

and here B appears as an integration constant which 'quantifies' the amount of 'mimetic dust', adding to the total value of the amount of *cold dark matter* in the universe, and here is set to *zero* since the universe during inflation is not yet in a phase where regular matter should play any role. Hence, the first Friedmann Equation resulting from the Action of the theory (2.16) simplifies to

$$\frac{1}{3}\kappa^2 = (C - \kappa C')\,V + \frac{1}{2}\dot{\varphi}^2. \tag{2.25}$$

Up till this point, the time-evolution behavior of the scale factor $a(t)$ and the inflaton field $\varphi(t)$ can be fully determined by the equation of motion in (2.22) and the first

Friedman Equation in (2.25).
To find the second Friedmann Equation, the time derivative of the first equation (2.25) is taken, and the expression for the equation of motion for the inflaton (2.22) is used to substitute for the $\ddot{\varphi}$ that appears, one finds that the Hubble constant changes at a rate of

$$3\dot{H} \equiv \dot{\kappa} = -\frac{3}{2}\left(\dot{\varphi}^2 + \left(C'V\right)^{\cdot}\right). \tag{2.26}$$

The previous expression can be rewritten its more explicit form,

$$\dot{H} = -\frac{1}{2}\frac{\dot{\varphi}\left(\dot{\varphi} + C'V'\right)}{1 + \frac{3}{2}C''V}. \tag{2.27}$$

in which the particularities of the *Mimetic Inflation Model* presented in this thesis work will be substituted for the arbitrary functions of the potential and the coupling function in the general theory, to find asymptotical solutions of the homogeneous background, and carry on with the rest of the analysis which are needed in presenting the viability of this model.

2.4 Presenting Mimetic Inflation: A Successful Model

The Potential and Coupling Term for Mimetic Inflation — While there is usually a tradeoff between a model that is 'simple' and one that satisfies all necessary observational requirements *and* solves other problems in its corollaries, in the case of the work presented in this thesis, the problems posed earlier in Section (1.4) can be addressed using the simplest possible model, choosing the following functions for its potential and coupling term that enter the action (2.16),

$$V\left(\varphi\right) = \frac{1}{2}\frac{m^2\varphi^2}{\left(1+\varphi^2\right)}\left(1 + m\varphi^4\right), \tag{2.28}$$

and

$$C\left(\kappa\right) = 1 + \frac{\kappa}{\kappa_0}, \tag{2.29}$$

where κ_0 in the coupling term of the mimetic with the inflaton field is taken to be of the order of the inflaton mass, $\kappa_0 = m$.
To understand the limits of the potential and how they enter into the analysis for the problem of self-reproduction it is useful to study the limits of its analytical expression at hand. For that, a rescaled plot is given in Figure (2.1), which will also be helpful to visually guide the different parts and limits.
For small $\varphi < 1$: the potential reduces to the usual one which describes the self

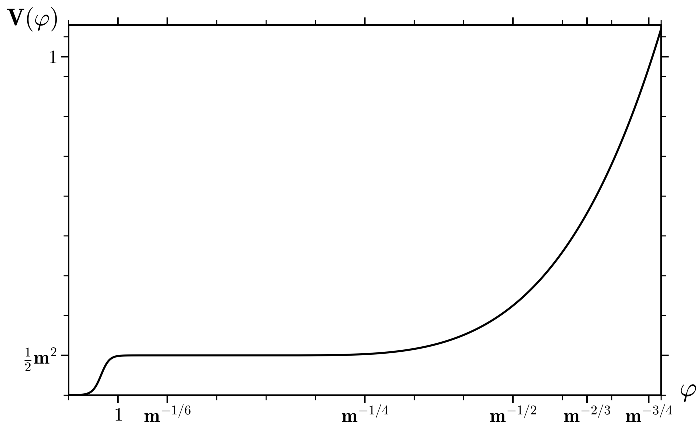

Figure 2.1: A rescaled plot of the potential for the inflaton field $V(\varphi)$

interaction of a massive field,

$$V \simeq \frac{1}{2}m^2\varphi^2. \tag{2.30}$$

For larger $\varphi > 1$ *:* The potential, and for later convenience its derivative, are well approximated as

$$V \simeq \frac{1}{2}m^2\left(1 - \frac{1}{\varphi^2} + m\varphi^4\right), \tag{2.31}$$

$$V' \simeq m^2\left(\frac{1}{\varphi^3} + 2m\varphi^3\right). \tag{2.32}$$

Therefore, for $\varphi > 1$, this approximation will be substituted in the previously obtained expressions for the Friedmann equation (2.25) and the inflaton equation of motion (2.22),

$$3H^2 = V + \frac{1}{2}\dot{\varphi}^2, \tag{2.33}$$

$$\ddot{\varphi} + \kappa\dot{\varphi} + \left(1 + \frac{\kappa}{m}\right)V' = 0, \tag{2.34}$$

to obtain explicitly the expressions,

$$3H^2 = \frac{1}{2}m^2\left(1 - \frac{1}{\varphi^2} + m\varphi^4\right) + \frac{1}{2}\dot{\varphi}^2, \tag{2.35}$$

$$\ddot{\varphi} + \kappa\dot{\varphi} + \left(1 + \frac{\kappa}{m}\right)\left(\frac{m^2}{\varphi^3} + 2m^3\varphi^3\right) = 0, \tag{2.36}$$

keeping in mind that $3H \equiv \kappa$ are used interchangeably.

These expressions will be the main expressions used to demonstrate analytically the resulting inflationary scenario. Since for inflation and its corollaries, including the phenomenon of self-reproduction, the region of interest for the field behavior is $\varphi > 1$, therefore, *from this point onward all the dynamics will be analyzed in the limit* $\varphi > 1$.

2.4.1 Asymptotical Solutions for Mimetic Inflation

Given that the equations of motion for $a(t)$ and $\varphi(t)$ that resulted from the theory action (2.16) are modified, the usual slow-roll parameter analysis in the literature will not be directly applicable to the potential at hand. However, one can re-derive the slow-roll conditions for the current model, starting from the basic arguments for slow-roll, and finding new conditions for which the model at hand reproduces the slow-roll regime of inflation. To begin, the main assumption for slow-roll is $\ddot{\varphi} \ll \kappa\varphi$, and the second condition for slow-roll is usually used in potential-dominated inflation where $H^2 \simeq V$, since the kinetic term $\frac{1}{2}\dot{\varphi}^2$ during slow roll is usually suppressed in comparison to the potential term in the Friedmann equation. In this Mimetic Model, it turns out, that while the field is rolling through different parts of its potential, slow-roll conditions are still met, with a subtle difference, whereby for dynamics involving different parts of the potential, inflation will be dominated by either the potential term (on the plateau) entering in the Friedmann equation, or the kinetic term which will dominate inflation when the inflaton field is rolling along the quartic part of the potential. For the inflationary solution itself, finding an exact analytic solution is not of direct interest for more involved models, rather demonstrating how an inflationary phase is possible and draw from it relevant quantities and asymptotic limit, which will be shown below. The analysis for avoiding self-reproduction will be presented after.

Kinetically Driven Exponential Expansion in Mimetic Inflation

The goal in Mimetic Inflation will to be to begin the inflationary phase in the *kinetically-dominated* region, since as will be shown later, this will be the *lesser degree of fine tuning* required from the model which will simultaneously avoid eternal inflation. For that, an outline for the derivation of the asymptotical solutions for the homogeneous background will be presented, starting from the kinetically-dominant part of the dynamics, which on the Figure (2.1), starts with the scalar field rolling down the quartic part of the potential $\varphi > m^{-1/2}$.

Hence, in the part of the potential where the kinetic term dominates, $\kappa/m > 1$, the effective coupling term reduces to

$$C(\kappa) \simeq \frac{\kappa}{m}, \tag{2.37}$$

and the set of equations of motion for $a(t)$ and φ in (2.33) and (2.34) simplify to

$$\kappa^2 = \frac{3}{2}\dot{\varphi}^2, \quad \ddot{\varphi} + \kappa\left(\dot{\varphi} + V'/m\right) = 0. \tag{2.38}$$

Now, knowing that the scalar field decreases during expansion, hence $\dot{\varphi} < 0$, the Hubble parameter becomes

$$\kappa = -\sqrt{\frac{3}{2}}\dot{\varphi}, \tag{2.39}$$

which is then substituted into the simplified equation of motion for φ in (2.38), which can now be rewritten as

$$\frac{d\dot{\varphi}}{d\varphi} = \sqrt{\frac{3}{2}}\left(\dot{\varphi} + \frac{V'(\varphi)}{m}\right), \tag{2.40}$$

using the definition $\ddot{\varphi} = \dot{\varphi}(d\dot{\varphi}/d\varphi)$.

This first order differential equation of $\dot{\varphi}$ has an explicit solution in terms of φ,

$$\dot{\varphi} = \frac{b}{m}e^{b\varphi}\int^{\varphi} V'(\tilde{\varphi})\, e^{-b\tilde{\varphi}} d\tilde{\varphi}, \tag{2.41}$$

where $b = \sqrt{3/2}$ is defined to simplify the expression. The solution corresponding to the lower limit of the integral is proportional to $Ce^{b\varphi}$, where C is an integration constant, and given that the field φ is decreasing with time, this solution is decaying and can be ignored. Then, integrating the expression (2.41) by parts, one finds the following attractor solution to the equation in (2.40)

$$\dot{\varphi} = -\frac{1}{m}\left(V' + \frac{V''}{b} + \frac{V'''}{b^2} + ...\right). \tag{2.42}$$

Considering the class of power-law potentials, as is the case with the relevant parts of the potential V in Mimetic-Inflation, where $V \propto \varphi^n$, the series in this solution (2.42) will be finite in its terms. The expression above can then be subsequently integrated to find the solution $\varphi(t)$.

Example — To demonstrate an explicit solution for φ, the following simpler potential will be used as an example to give a complete analytical picture,

$$V\left(\varphi\right)=\frac{1}{2}m^{2}\varphi^{2}, \tag{2.43}$$

Equation (2.42) then becomes

$$\dot{\varphi}=-\frac{m}{b}\left(1+b\varphi\right), \tag{2.44}$$

which can be integrated, giving

$$\varphi\left(t\right)=\left(\varphi_{i}+\frac{1}{b}\right)e^{-mt}-\frac{1}{b}, \tag{2.45}$$

with φ_i evaluated at the initial time $\varphi\left(t=0\right)$. At $t=0$, if inflation begins at the Planck scale , then $\dot{\varphi}^2\simeq 1$, and for $m\ll 1$, one gets from (3.75) that $\varphi_i\simeq m^{-1}\gg 1$.

Equations of state — For the above potential, following equations (2.25) and (2.27 one gets

$$\frac{\dot{\kappa}}{\kappa^{2}}=-\frac{1}{1+b\varphi}. \tag{2.46}$$

Recalling earlier $\kappa=3H$, one has

$$H^{2}=\varepsilon/3,\quad \dot{H}=-\frac{1}{2}\left(\varepsilon+p\right), \tag{2.47}$$

with ε as the energy density and p as pressure, one can rewrite the expression in (2.46) as in (e.g., [94])

$$\frac{\varepsilon+p}{\varepsilon}=\frac{2}{1+b\varphi}. \tag{2.48}$$

Then, when $\varphi\gg 1$, the ratio satisfies

$$\left(\varepsilon+p\right)/\varepsilon\ll 1, \tag{2.49}$$

so that the equation of state is

$$p\approx-\varepsilon \tag{2.50}$$

and the universe enters the phase of exponential expansion. Inflation then ends when the field reaches $\varphi<1$, and as it continues to drop to values $|\varphi|\ll 1$, one recovers the ultra-hard equation of state

$$p\approx+\varepsilon, \tag{2.51}$$

where the universe is now matter-dominated.

Then starting with a kinetically-driven Mimetic-Inflation, and integrating (2.39), the solution for the scale factor becomes

$$a = a_i \exp\left(\frac{1}{\sqrt{6}}\left(\varphi_i - \varphi\right)\right), \tag{2.52}$$

and during inflation it grows by a large number of e-folds

$$\frac{a_f}{a_i} \simeq \exp\left(\frac{1}{\sqrt{6}m}\right) \gg 1, \tag{2.53}$$

where $m \ll 1$.

For a Generic Potential in Mimetic-Inflation — The previous analysis can be extended to the whole class of potentials in Mimetic Inflation by finding the requirements on a generic potential $V(\varphi)$ which can drive the inflationary phase outlined above. To rederive the conditions for slow-roll in the case of the Mimetic Inflation, the main assumption that $\ddot{\varphi}$ is neglected in equation (2.34) must hold. Recalling that $\kappa > m$ for $\varphi > 1$, the solution to (2.34) can then be approximated by

$$\dot{\varphi} \approx -\frac{V'}{m}. \tag{2.54}$$

Taking this solution into account, and using (2.39), one finds that the slow-roll condition $|\ddot{\varphi}| \ll |\kappa\dot{\varphi}|$ is valid only when

$$\frac{V''}{V'} \ll 1. \tag{2.55}$$

Then by looking at the how the Hubble scale is changing

$$\frac{|\dot{\kappa}|}{\kappa^2} \simeq \frac{|\ddot{\varphi}|}{|\kappa\dot{\varphi}|} \simeq \frac{V''}{V'}, \tag{2.56}$$

exponential expansion with $|\dot{\kappa}|/\kappa^2 \ll 1$ can occur when the potential satisfies the inequality (2.55).

Clearly, this condition is satisfied *for any power-law potential*, when $\varphi > 1$, and as will be shown in the next part, the self-reproduction condition is met only when the curvature exceeds the Planckian curvature.

2.4.2 Avoiding Self-Reproduction in Mimetic Inflation

Now, the goal is to show how self-reproduction can be avoided as the inflaton field spans all parts of its potential, using conditions that apply to each part separately. Recalling that *the condition for self-reproduction* is for the amplitude of quantum fluctuations to exceed the change in the classical background field,

$$\Delta\varphi_{tot} = \Delta\varphi_{cl} + \Delta\varphi_q \simeq \frac{\dot{\varphi}}{H} + H > 0, \tag{2.57}$$

to evade self-reproduction then, the condition that $\Delta\varphi_{tot} < 0$ must be verified for each part of the potential, considering that a different approximation for the potential is used for $\dot{\varphi}$ in each segment, and similarly H will be evaluated for each case, first when inflation is dominated by the potential, then by the kinetic term. The analysis is shown below, taking the mass

$$m \simeq 10^{-6}, \tag{2.58}$$

which is needed for correct observational estimates.

I. When Mimetic-Inflation is driven by the Potential $V(\varphi)$:

The analysis in this part will be focused on making the case of how the potential-dominated inflation evades self-reproduction.

For the plateau region in the potential, where $1 < \varphi < m^{-1/2}$, $\kappa > m$, recalling that in the slow-roll regime where $\ddot{\varphi}$ is negligible in (2.34), plugging in the potential (2.32) into the simplified expression for $\dot{\varphi}$ already obtained in (2.54), the equation becomes

$$\dot{\varphi} \simeq -\frac{V'}{m} \simeq \left(\frac{m}{\varphi^3} + 2m^2\varphi^3\right). \tag{2.59}$$

In this region, indeed $V > \dot{\varphi}^2$, hence $H^2 \simeq V/3$. Using (2.59), and plugging these back into (2.57) gives

$$\Delta\varphi_{tot} = -\frac{V'}{mH} + H \simeq -\frac{V'}{m\sqrt{V/3}} + \sqrt{\frac{V}{3}} > 0 \tag{2.60}$$

and so, *the condition for self-reproduction for the potential-dominated Mimetic Inflation* can be rewritten as

$$\frac{mV}{3V'} > 1. \tag{2.61}$$

Now, this condition will be tested in each segment of the plateau separately, and show how it is violated.

For region $1 < \varphi < m^{-1/6}$ *:* The potential and its derivative as given in (2.31) and

(2.32) can be further approximated by

$$V \simeq \frac{1}{2}m^2\left(1 - \frac{1}{\varphi^2}\right), \quad V' \simeq m^2\frac{1}{\varphi^3}. \tag{2.62}$$

Testing the self-reproduction condition , $\frac{mV}{V'} \sim \frac{m^3\varphi^3}{m^2} < 1$, so (2.61) is clearly *violated* in the region $1 < \varphi < m^{-1/6}$.

For region $m^{-1/6} < \varphi < m^{-1/4}$: The potential is at the plateau and the second term in its derivative in (2.32) dominates over the first term , giving

$$V \simeq \frac{1}{2}m^2, \quad V' \simeq 2m^3\varphi^3. \tag{2.63}$$

Again, in this region of the potential, $\frac{mV}{V'} \sim \frac{m^3}{m^3\varphi^3} < 1$ *violates* the condition for self-reproduction in (2.61).

For region $m^{-1/4} < \varphi < m^{-1/2}$: The potential and its derivative can be described predominantly by

$$V = \frac{1}{2}m^3\varphi^4, \quad V' = 2m^3\varphi^3, \tag{2.64}$$

and here it can be further deduced that the condition for self-reproduction is *violated* for $\varphi < m^{-1/2}$, keeping in mind that this region is the beginning of the transition into the kinetically-driven inflation region, at $\varphi = m^{-1/2}$, and must be connected with the next region to conclude its behavior at the *limit* of this approximation.

Therefore, for the whole range $1 < \varphi < m^{-1/2}$, and during a typical Hubble time t_H, *the condition for self-reproduction is violated throughout*, and the classical background field $\delta\varphi_{cl} \simeq \dot{\varphi}t_H$ decreases *more* than the quantum fluctuations' amplitude $\delta\varphi_q$ in the Hubble scale during potential-driven inflation, and self-reproduction is successfully avoided. At the limit of this region, the energy density for $\varphi \simeq m^{-1/2}$ has a value of $\varepsilon \simeq m$.

II. When Mimetic-Inflation is Kinetically-driven:

For region $\varphi > m^{-1/2}$: In the quartic part of the potential, the kinetic term dominates over the potential term in (2.33), and drives inflation as was shown earlier. In this region, recalling that effective coupling term is

$$C(\kappa) \simeq \frac{\kappa}{m}, \tag{2.65}$$

and that the set of equations of motion for $a(t)$ and φ in (2.33) and (2.34) simplified to

$$\kappa^2 = 3H^2 = \frac{3}{2}\dot{\varphi}^2, \quad \dot{\varphi} \simeq -V'/m, \tag{2.66}$$

now one can test that self-reproduction is avoided in this kinetically-dominated inflation region. From the equations above, $H \sim -\dot{\varphi}$, and so the condition (2.57) can be rewritten as

$$\Delta\varphi_{tot} = -1 + H > 0. \tag{2.67}$$

It can be clearly seen that *the self-reproduction condition is violated only up to the Planckian scale* $H \sim 1$, which the scalar field reaches at $\varphi \simeq m^{-2/3}$, obtained from the condition

$$H \sim 2m^3\varphi^3 \sim 1. \tag{2.68}$$

Therefore, at the Planck scale, inflation can begin, as was demonstrated, avoiding an eternally self-reproducing universe.

Initial Conditions with a lesser degree of fine-tuning in Mimetic Inflation

Based on the previous result, the limit for which self-reproduction is absent emerges naturally in the model as a desirable cut off. Therefore, *starting inflation at the Planck scale simultaneously solves the problem of fine-tuning of initial conditions and avoids a Universe that is eternally self-reproducing.* It is worth stressing again that these two problems have not been simultaneously solved in any previous work and breaks a core argument against Inflation as a theory with no viable models.

Finally, to conclude this part, it is important to note that all the analysis was performed using the simplest Mimetic-Model of those specified by the action in (2.16), and numerous other models can be developed which are also compatible with current CMB observations. And as will be presented in the following section, the model studied meets every observational constraint, while avoiding the pitfalls of eternally inflating universes.

2.5 Cosmological Perturbations

2.5.1 A General Outline for Perturbation Equations

The goal in this section will be to derive the main equations for the perturbations in the Mimetic Model. For that, a general outline for linearizing the Einstein Equations for small inhomogeneities about the background metric for a flat Friedmann universe will be laid out first, following [94].

Decomposing Perturbations — For perturbations around a background metric, the line element is given by

$$ds^2 = [g_{\mu\nu} + \delta g_{\mu\nu}(x^\kappa)]\, dx^\mu dx^\nu, \tag{2.69}$$

with $|\delta g_{\mu\nu}| \ll |g_{\alpha\beta}|$, and for a flat Friedmann universe, the metric expressed in conformal-time is

$$g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(d\eta^2 - \delta_{ij} dx^i dx^j). \tag{2.70}$$

Now, one can see that the perturbations $\delta g_{\mu\nu}$ can be naturally decomposed into scalar, vector, and tensor modes reflecting how they behave under transformations that leave the metric describing the homogeneous and isotropic universe invariant, i.e., the transformations that belong to the translational and rotational groups. Hence one obtains the components δg_{00}, δg_{0i}, and δg_{ij} which behave respectively as a scalar, vector, and tensor under rotations, and are described by the functions that preserve their property as

$$\delta g_{00} = 2a^2\phi \tag{2.71}$$

$$\delta g_{0i} = a^2(B_{,i} + S_i) \tag{2.72}$$

$$\delta g_{ij} = a^2(2\psi\delta_{ij} + 2E_{ij} + F_{i,j} + F_{j,i} + h_{ij}) \tag{2.73}$$

where ϕ, ψ, E, and B are scalars, S_i and F_i are vectors with zero divergence, and h_{ij} is a traceless transverse tensor. Hence, the independent functions describing the metric perturbations $\delta g_{\mu\nu}$ are in total ten. They have distinct properties based on their type, for instance,

- Scalar modes are generated by the variations in the energy density. In the Mimetic model presented in the thesis, the energy density has new contributions from the Mimetic field and its coupling to the scalar field potential that are distinct from the standard expressions obtained in typical inflationary models, and so its effect will also be reflected in the spectrum of perturbations.

- Vector modes decay rapidly, which makes them largely insignificant in cosmological contexts, and similarly in Mimetic Inflation their behavior will remain unaltered.

- Tensor modes capture the two dynamical degrees of freedom of gravity, and describe propagating gravitational waves.

Gauge-Invariant Variables

Since we are interested in studying the effect of physical or 'real' matter perturbations on the metric, its important to eliminate the fictitious effect of coordinate transformations

$$x^\alpha \to \tilde{x}^\alpha + \xi^\alpha, \tag{2.74}$$

with the infinitesimal quantity ξ^α being a function of space and time, $\xi^\alpha \equiv (\xi^0, \xi^i)$. Skipping over the details of how the previously described decomposition of metric perturbations in (2.71) transforms under these coordinate changes which can be found in references (e.g., [94]), out of the different possibilities of combinations that produce *gauge-invariant* variables, the following are the simplest linear combinations of the perturbation functions

$$\Phi \equiv \frac{1}{a}[a(B - E']', \qquad \Psi \equiv \psi + \frac{a'}{a}(B - E')$$
$$\overline{V}_i = S_i - F_i' \tag{2.75}$$

while the tensor h_{ij} is already gauge invariant, and the *prime* here denotes the derivative with respect to conformal time. After isolating coordinate freedom, the physical degrees of freedom describing the metric perturbations reduce to six, two scalar, two vector, and two in the tensor sector. As will be seen in the equations for the perturbations below, only the tensor degrees of freedom will propagate.

Linearized Einstein Equations — Starting with the field equations, momentarily restoring the factor $8\pi G$,

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}\delta^\mu_\mu R = 8\pi G T^\mu_\nu, \tag{2.76}$$

where the Einstein tensor can computed for the perturbed metric in (2.69),

$$G^\mu_\mu =^{(0)} G^\mu_\nu + \delta G^\mu_\nu + \ \dots\ , \tag{2.77}$$

collecting terms up to linear order in the perturbations into δG^μ_ν, the non-zero components of the unperturbed ${}^{(0)}G^\mu_\nu$ for the conformally flat metric (2.70) can be computed as

$${}^{(0)}G^0_0 = \frac{3\mathcal{H}^2}{a^2}, \qquad {}^{(0)}G^i_j = \frac{1}{a^2}(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij}, \tag{2.78}$$

with $\mathcal{H} \equiv a'/a$. Similarly for the energy-momentum tensor, the background components will be diagonal, such that

$${}^{(0)}T^0_i = 0, \qquad {}^{(0)}T^i_j \propto \delta_{ij}. \tag{2.79}$$

The linearized perturbation equations can be written as

$$\overline{\delta G^{\mu}_{\nu}} = 8\pi G \overline{\delta T^{\mu}_{\nu}}, \tag{2.80}$$

where $\overline{\delta G^{\mu}_{\nu}}$ and $\overline{\delta T^{\mu}_{\nu}}$ are written in gauge-invariant terms, following

$$\begin{aligned}
\overline{\delta T^0_0} &= \delta T^0_0 - ({}^{(0)}T^0_0)'(B - E'), \\
\overline{\delta T^0_i} &= \delta T^0_i - ({}^{(0)}T^0_0 -{}^{(0)} T^k_k/3)(B - E')_{,i} \ , \\
\overline{\delta T^i_j} &= \delta T^i_j - ({}^{(0)}T^i_j)'(B - E'),
\end{aligned} \tag{2.81}$$

with T^k_k denoting the trace. Similar expressions can be obtained for the components of $\overline{\delta G^{\mu}_{\nu}}$.

Following the decomposition of the metric perturbations into scalar, vector and tensor parts, the components of $\overline{\delta T^{\alpha}_{\beta}}$ can be split in a similar way, and since these sectors decouple at linear order, their evolution equations can be studied independently for the corresponding part of $\overline{\delta T^{\alpha}_{\beta}}$, and will be listed below for completeness. Therefore, computing $\overline{\delta G^{\alpha}_{\beta}}$, equations (2.80) are split into

Scalar Perturbations

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G a^2 \overline{\delta T}^0_0 \tag{2.82}$$

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi G a^2 \overline{\delta T}^0_i, \tag{2.83}$$

$$\begin{aligned}
[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{2}\Delta(\Phi - \Psi)]\delta_{ij}& \\
-\frac{1}{2}(\Phi - \Psi)_{,ij} = -4\pi G a^2 \overline{\delta T}^i_j&
\end{aligned} \tag{2.84}$$

Vector Perturbations

$$\Delta \overline{V}_i = 16\pi G a^2 \overline{\delta T}^0_{i(V)} \tag{2.85}$$

$$(\overline{V}_{i,j} + \overline{V}_{j,i})' + 2\mathcal{H}(\overline{V}_{i,j} - \overline{V}_{j,i}) = -16\pi G a^2 \overline{\delta T}^i_{j(V)}. \tag{2.86}$$

Tensor Perturbations

$$(h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij}) = 16\pi G a^2 \overline{\delta T}^i_{j(T)} \tag{2.87}$$

Now, the equations described in this section will be utilized for the Mimetic Inflation Model, and will be analyzed for the modifed Energy-Momentum tensor that was obtained for the model in (2.6).

2.5.2 Cosmological Perturbations in Mimetic Inflation

Newtonian Gauge — The gauge choice for the perturbed metric will be the *conformal Newtonian gauge*, given that for linearized perturbations δT^i_k is diagonal.
Hence,

$$ds^2 = (1+2\Phi)\, dt^2 - a^2(t) \left[(1-2\Phi)\, \delta_{ik} dx^i dx^k - h^{(t)}_{ik} dx^i dx^k \right], \tag{2.88}$$

and here Φ represents the gravitational-potential for the scalar fluctuations, and $h^{(t)}_{ik}$ are the tensor modes described earlier.

Tensor Perturbations in Mimetic-Inflation

In this Mimetic Inflation model, the equation for tensor perturbations is the same as in non-modified general relativity, and so the solution for gravitational waves does not change, and so it is not of interest in this work to reproduce the same result, details of which can be found in [94].

Scalar Perturbations in Mimetic-Inflation

On the other hand, the equations for scalar perturbations in the Mimetic-Inflation Model presented are substantially modified in comparison to the standard picture, since contributions from the perturbation of the Mimetic field $\delta\phi$ will enter the equations, but their analysis will in fact be very similar to the normal case, since all the complicated extra terms in the equations will simplify greatly.
Mimetic field perturbations — To begin, the perturbation of the solution of the Mimetic field in the synchronous coordinate system (2.9) is

$$\phi = t + \delta\phi, \tag{2.89}$$

and plugging this in the constraint equation $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$, one gets for linearized perturbations the following expansion

$$(1-2\Phi)(1+2\dot{\delta\phi} + \mathcal{O}(\dot{\delta\phi}^2)) = 1, \tag{2.90}$$

from which the expression relating the mimetic field perturbations to the gravitational potential is obtained

$$\dot{\delta\phi} = \Phi. \tag{2.91}$$

Similarly, by varying the equation for the d'Alembertian of the Mimetic field with respect to the metric $g^{\mu\nu}$,

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \tag{2.92}$$

one obtains the expression for the variation for the trace of extrinsic curvature κ, given by

$$\delta\kappa \equiv \delta\Box\phi = -3\ddot{\delta\phi} - 3H\dot{\delta\phi} - \frac{\Delta}{a^2}\delta\phi, \tag{2.93}$$

keeping in mind that $H = \kappa/3$ is the Hubble constant.

Energy-Momentum Tensor Perturbations — Considering the modified expression for the Energy-Momentum tensor in (2.18), its gauge-invariant perturbations in the components are given by

$$\overline{\delta T}^0_{\ 0} = 2\delta\lambda + \Phi V'\dot{\varphi} - V'\delta\dot{\varphi} - V''\dot{\varphi}\delta\varphi - \Phi\dot{\varphi}^2 + \dot{\varphi}\delta\dot{\varphi} + V'\delta\varphi, \tag{2.94}$$

$$\overline{\delta T}^0_{\ i} = (2\lambda + (C'V)^{\cdot})\,\delta\phi_{,i} + \delta(C'V)_{,i} + \dot{\varphi}\delta\varphi_{,i}, \tag{2.95}$$

$$\overline{\delta T}^i_{\ k} = \delta^i_k\left(\Phi(-V'\dot{\varphi} + \dot{\varphi}^2) + (V' - \dot{\varphi})\delta\dot{\varphi} + (V' + V''\dot{\varphi})\delta\varphi\right), \tag{2.96}$$

Linearized Perturbation Equations — Using the metric in the Newtonian Gauge defined in (2.88), taking again $8\pi G = 1$, the linearized Einstein Equations become

$$\frac{2}{a^2}\Delta\Phi - 6H(\dot{\Phi} + H\Phi) = \overline{\delta T}^0_{\ 0}, \tag{2.97}$$

$$2(\dot{\Phi} + H\Phi)_{,i} = \overline{\delta T}^0_{\ i}, \tag{2.98}$$

$$2\ddot{\Phi} + 6H\dot{\Phi} + 2(3H^2 + 2\dot{H})\Phi = \overline{\delta T}^i_k. \tag{2.99}$$

0–i Einstein Equation — Ignoring the contribution of mimetic dust by setting $B = 0$ in (2.24), the first term in the $\overline{\delta T}^0_{\ i}$ expression drops out, and the second term can be computed as

$$\begin{aligned}\delta\,(C'V) &= C''V\delta\kappa + C'V'\delta\varphi \\ &= -3C''V\left(\dot{\Phi} + H\Phi + \frac{1}{3a^2}\Delta\delta\phi\right) + C'V'\delta\varphi.\end{aligned} \tag{2.100}$$

One then obtains from the $0-i$ Einstein equation (2.98) the following expression

$$\begin{aligned}\dot{\Phi} + H\Phi &= \frac{1}{2\left(1+\frac{3}{2}C''V\right)}\left[\left(\dot{\varphi}+C'V'\right)\delta\varphi - \frac{C''V}{a^2}\Delta\delta\phi\right] \\ &= -\frac{\dot{H}}{\dot{\varphi}}\delta\varphi - \frac{C''V}{\left(2+3C''V\right)a^2}\Delta\delta\phi, \end{aligned} \tag{2.101}$$

where (2.91) was used to express the derivative of $\delta\phi$, and (2.27) was used to substitute for $\dot{H}$.

Inflaton Perturbations — To linear order in perturbation, the equation for the Inflaton fluctuations is obtained from (2.20) as

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{1}{a^2}\Delta\delta\varphi - 4\dot{\varphi}\dot{\Phi} - 2\left(\ddot{\varphi}+3H\dot{\varphi}\right)\Phi + \delta\left(CV'\right) = 0, \tag{2.102}$$

where $\delta\varphi \equiv \overline{\delta\varphi}$. And similarly as in (2.100), one has

$$\delta\left(CV'\right) = -3C'V'\left(\dot{\Phi} + H\Phi + \frac{1}{3a^2}\Delta\delta\phi\right) + CV''\delta\varphi. \tag{2.103}$$

Now, by deriving the equation of motion for the inflaton (2.20) with respect to time, CV'' can be expressed as

$$CV'' = -\frac{1}{\dot{\varphi}}\left(\dddot{\varphi} + 3H\ddot{\varphi} + 3\dot{H}\dot{\varphi} + 3C'V'\dot{H}\right), \tag{2.104}$$

then plugging this expression into (2.102), together with the expression of $(\dot{\Phi}+H\Phi)$ obtained in (2.101), the equation for the Inflaton perturbations (2.102) becomes

$$\begin{aligned}&\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{1}{a^2}\Delta\left(\delta\varphi + \frac{2C'V' - 4\dot{\varphi}C''V}{2+3C''V}\delta\phi\right) \\ &\qquad - \left(\frac{\dddot{\varphi}}{\dot{\varphi}} + 3H\frac{\ddot{\varphi}}{\dot{\varphi}} - \dot{H}\right)\delta\varphi - 2\left(\ddot{\varphi}+H\dot{\varphi}\right)\Phi = 0. \end{aligned} \tag{2.105}$$

0–0 Einstein Equation — From the $0-0$ Einstein Equation, one can only find the variation in the Lagrange multiplier, $\delta\lambda$, which is not important. And by further taking the time derivative of the $0-0$ equation, and using the linearized Mimetic field equation given in (2.19), one simply recovers again the above equation of motion of the inflaton perturbations. Therefore this equation does not provide any additional insight to the analysis.

i–i Einstein Equation — This equation does not give any useful insight either, since by substituting the expression for $\delta\varphi$ in terms of Φ and $\delta\phi$ from (2.101) into $\overline{\delta T^i_k}$, one finds that the $i-i$ Einstein equation is simply identically satisfied.

From here, the equation for the Mimetic field perturbations in (2.91), the $0-i$ Einstein Equation in (2.101), and the equation for the evolution of the Inflaton perturbations in (2.105) will be the main equations which are enough to find solutions for the three variables $\delta\varphi, \Phi$ and $\delta\phi$.

Plane Wave Solution for Perturbations

Considering plane wave perturbations of the Mimetic field, Inflaton field, and gravitational potential

$$\delta\varphi, \Phi, \delta\phi \propto \exp\left(i\overrightarrow{k}.\overrightarrow{x}\right), \tag{2.106}$$

where $k = |\overrightarrow{k}|$ is the co-moving wave number, and $\overrightarrow{x}$ is the direction of the wave propagation. The relative size of their physical wavelength λ_{ph} with respect to the Hubble curvature scale H^{-1} will affect their evolution, otherwise referred to as the *horizon scale*. While the H^{-1} scale remains somewhat constant during exponential expansion, the physical size of the perturbations $\lambda_{ph} \simeq a/k$ will grow.

The three main equations (2.91), (2.101) and (2.105) will be solved in two cases: in the limit of short-wavelength perturbations which have a physical wavelength $\lambda_{ph} \ll H^{-1}$, and in the limit of $\lambda_{ph} \gg H^{-1}$.

For short-wavelength perturbations — Inside the Hubble scale, the co-moving wave number is $k \gg Ha$. In this case, the equation for the inflaton perturbations (2.105) greatly simplifies to

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + \frac{k^2}{a^2}\delta\varphi_k \simeq 0. \tag{2.107}$$

This simplification follows first from the fact that the gravitational field does not affect the evolution of short-wavelength perturbations, so the term with Φ becomes negligible as the spatial derivative term dominates. Keeping in mind that Φ and $\delta\varphi$ oscillate in this limit, their time-derivatives can be approximated as

$$\dot{\Phi} \sim \frac{k}{a}\Phi, \ \dot{\delta\phi} \sim \frac{k}{a}\delta\phi, \tag{2.108}$$

which are valid in the leading order, and combining them with $\Phi = \dot{\delta\phi}$ from (2.91) into equation (2.101), one finds that

$$\frac{2C'V' - 4\dot{\varphi}C''V}{2+3C''V}\delta\phi \sim \frac{\dot{H}}{(k/a)^2}\delta\varphi, \tag{2.109}$$

and so for the term inside the Laplacian containing perturbations of the mimetic field $\delta\phi$, it can be ignored for $k/a \gg H$ relative to $\delta\varphi$. The last terms in (2.105) that were dropped are smaller by at least a factor of order $Ha/k \ll 1$ relative to the terms that are kept in (2.107).

Hence, the solution to the simplified equation in (2.107) is given by the scalar-field modes

$$\delta\varphi_k \simeq \frac{A_k}{a}\exp\left(\pm ik\int\frac{dt}{a}\right), \tag{2.110}$$

with a constant of integration A_k, which physically start out as *quantum fluctuations* of the vacuum.

For long-wavelength perturbations — For modes that have crossed the Hubble scale, the co-moving wave number is $k \ll Ha$. These inhomogeneities cross the horizon at t_k, where this crossing time is specified from the relation $k \sim H_k a_k$. Then, the spatial derivative terms in the main equations start decaying proportionally to $1/a^2$, and so the Laplacian terms in the equation for the inflation perturbations in (2.101) and (2.105) will be dropped, such that the expressions take the form

$$\begin{aligned}
&\dot{\Phi} + H\Phi \simeq -\frac{\dot{H}}{\dot{\varphi}}\delta\varphi \\
&\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \left(\frac{\dddot{\varphi}}{\dot{\varphi}} + 3H\frac{\ddot{\varphi}}{\dot{\varphi}} - \dot{H}\right)\delta\varphi - 2\left(\ddot{\varphi} + H\dot{\varphi}\right)\Phi \simeq 0.
\end{aligned} \tag{2.111}$$

The exact solutions for the above equations are obtained as

$$\delta\varphi = A\frac{\dot{\varphi}}{a}\int adt, \tag{2.112}$$

$$\Phi = A\frac{d}{dt}\left(\frac{1}{a}\int adt\right), \tag{2.113}$$

and can be simply proven through direct substitution, keeping in mind that A is a constant of integration. The solution for the mimetic field follows from the relation (2.91),

$$\delta\phi = A\frac{1}{a}\int adt. \tag{2.114}$$

These solutions are obtained for arbitrary functions for the coupling term $C(\kappa)$ and the potential $V(\varphi)$ used, and remain valid during and after inflation has ended.

2.5.3 Spectrum of Inhomogenieties

In this section, the spectrum of inhomogeneities that results from the initial quantum fluctuations will be computed for the Mimetic Inflation model presented earlier. Before that, some problems and subtleties regarding initial perturbations will be discussed.

Quantum Fluctuations from Cosmic Inflation: Point of View — The cosmologically relevant perturbations are those that start out below the Hubble scale with $\lambda_{ph} \ll H^{-1}$ and later cross it, then the evolution of these modes can be studied. The amplitude for the quantum minimal fluctuations is fixed on these sub-horizon scales to the smallest value that the uncertainty principle permits, as they can be well defined only on the scales smaller than the Hubble scale, where the space-time manifold can be treated effectively as a flat Minkowski metric. That being said, there are certain technical subtleties that arise from the mathematical description of the initial perturbations and their evolution that gives rise to various conceptual problems that are then debated in the literature, which will be discussed below. From the point of view of cosmic inflation, the most important outcome is that the scalar field will have unavoidable quantum fluctuations that are described at the sub-Hubble scales by the evolution equation given in (2.107), that describe the vacuum fluctuations and preserve the vacuum spectrum.
Classical Inhomogeneities — The first problem is regarding pre-existing inhomogeneities. The initial spectrum of the relevant perturbations on sub-Hubble scales is not - and does not have to be- fine-tuned at the outset, so it can still contain particles ([95]-[97]). But these particles and other present inhomogeneities do not pose a problem since they will be stretched by expansion from the sub-Hubble scales to large unobservable scales and will ultimately become non relevant. The only condition on such inhomogeneities is that they do not prevent the quasi-exponential expansion of inflation from its very beginning. This reinforces the idea that for cosmological perturbations there is *no fine-tuning problem*, even if the region that is undergoing inflation is largely inhomogeneous, granted that the initial inhomogeneities do not prevent this region from entering the inflationary stage or stop it from the very beginning.
The Trans-Planckian Problem and Bunch-Davies Vacuum — These are the two other problems discussed in the literature (e.g. [98]-[100]). Regarding the choice of a Bunch-Davies vacuum, contrary to many arguments, one does not in fact need to assume a Bunch-Davies vacuum, nor a Minkowski vacuum on scales where $\lambda_{ph} < H^{-1}$, where particles are absent, since soon after inflation starts, the inhomogeneities present will be the quantum fluctuations. As highlighted previously, the only condition on the

pre-existing inhomogeneities is that they do not spoil inflation such that the exponential expansion is not terminated from the beginning. When that is the case, the exponential expansion will take care of diluting these inhomogeneities, and so they do not affect the evolution of the universe and become less relevant to it at the observable scales.

Regarding the Trans-planckian problem, it can be formulated for perturbations in inflation analogously to Hawking radiation. This problem appears as an artifact from the method of calculation used to derive these effects. As mentioned, during inflation the inhomogeneities that were initially present on the scales smaller than the Hubble scale will be ‘cleaned up’ as a result of the expansion, and the only remaining fluctuations on scales $\lambda_{ph} < H^{-1}$ will be the quantum fluctuations. These fluctuations can either be described in the static coordinate system which can still be defined within the Hubble scale, and so they resemble the known Minkowski vacuum; alternatively, it is possible to describe the vacuum in a coordinate system which is expanding, such as for the short wavelength perturbations of $\delta\varphi_k$ in (2.110). The reason for choosing expanding coordinates is that one can relate the perturbations on scales that are within the Hubble scale, to those that exceed it, where a static coordinate system can no longer be defined, and this is a purely technical advantage from the point of view of the calculations. For $\lambda_{ph} < H^{-1}$, the spectrum of fluctuations for the scalar field, which describes how amplitude depends on the physical wavelength $\lambda_{ph} \simeq a/k$, stays invariant, and during the evolution that is described for $\delta\varphi_k$ in (2.107), it does not change. So for a massive scalar field, the amplitude $\delta\varphi$ is given by

$$\delta\varphi \equiv \sqrt{\delta\varphi_k^2 k^3} \simeq 1/\lambda_{ph}, \tag{2.115}$$

which becomes $\delta\varphi \simeq H$ on the Hubble scale or at the moment of horizon crossing. Hence, considering perturbation on given physical scale after they have been stretched out by expansion, then following the calculations in expanding coordinates, it may seem as if these perturbations were replaced by those for which the physical wavelength must have started smaller than the Planck scale to so that the invariant vacuum spectrum is preserved. This picture is at the core of the Trans-Planckian problem. But as explained, it is simply a technique used to make the calculations simpler, and can be interpreted rather differently. For the modes that were stretched by expansion from physical scales $\lambda_{ph} < H^{-1}$, new perturbations can be generated naturally by the uncertainty principle with the needed amplitude. In such a way, the vacuum spectrum remains invariant in the static coordinate system at the sub-Hubble scales, and one can overcome the conceptual loophole that perturbations existing on Trans-Planckian scales had to be stretched to some physical scale by expansion. Therefore, the Trans-Planckian problem can be disregarded as an artifact of the calculations, and not associated with any physically real problem, which is also analogous to describing the Minkowski vacuum in Milne coordinates which are expanding.

Perturbations at the Horizon-crossing — *Before Horizon-crossing,* the amplitude of the quantum fluctuations while they were still in the oscillating mode was

$$\delta\varphi \sim |\delta\varphi_k| k^{3/2} \sim \frac{k}{a_k},$$

and during inflation, this amplitude was decaying inversely proportionally to the scale factor. *At the moment of Horizon crossing*, on scales $\lambda_{ph} \simeq H^{-1}$ when $k \sim Ha_k$, it becomes

$$\delta\varphi \simeq H, \tag{2.116}$$

and so the amplitude of the scalar field fluctuations is inversely proportional to the physical scale. These are regarded as the initial conditions for the perturbation with a co-moving wave-number k and at a time t_k which satisfies $k \simeq a(t_k)H(t_k)$, and will be used to fix the initial amplitude of perturbations described in the expanding coordinates in (2.112) on scales that exceed the Hubble scale when curvature becomes relevant.

After Horizon-crossing, for $t > t_k$, with the exponential growth of the scale factor the Hubble parameter remains somewhat constant, and the solution for the perturbation that satisfies the condition $k < Ha$ is given in (2.112). During slow-roll inflation, these solutions are simplified, taking into account $\dot{H} \ll H^2$, and the integral part of the solutions can be expanded as

$$\frac{1}{a}\int a dt = \frac{1}{a}\int \frac{da}{H} = \frac{1}{H}\left(1 + \frac{\dot{H}}{H^2} + ...\right) + \frac{D}{a} \simeq \frac{1}{H}, \tag{2.117}$$

where the decaying mode with the integration constant D is ignored. Now, taking a perturbation with a co-moving wave-number k, and given that the typical amplitude of the quantum fluctuations at time t_k is $\sqrt{\delta\varphi_k^2 k^3} \simeq H_{k=Ha}$, one finds that that the integration constant from the equation of $\delta\varphi$ in (2.112) is given by

$$A \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha}, \tag{2.118}$$

and here the subscript indicates that the given quantity in the parenthesis is to be evaluated at the moment in time when the perturbation gets stretched by inflationary expansion to the physical scale $\lambda_{ph} \simeq H^{-1}$.

For the next part, since the total amplitude of the spectrum is a free parameter of the theory and is to be determined by observations, the coefficients of order $\mathcal{O}(1)$ will be omitted from further estimates.

Spectrum of Gravitational Potential — Substituting the previous result into the solution for the gravitational potential at the scales exceeding the Hubble scale in (2.112), it can be written as

$$\Phi \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha} \frac{d}{dt}\left(\frac{1}{a}\int a dt\right). \tag{2.119}$$

After inflation ends, the time-dependent term on the right becomes simply a constant which is of order one. Therefore, for the perturbations that are of interest, which have left the Hubble scale during inflation, the spectrum for the gravitational potential defined by $\delta_\Phi \equiv \sqrt{\Phi_k^2 k^3}$ is then

$$\delta_\Phi \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha}. \tag{2.120}$$

Spectrum of Gravitational Waves — As have been mentioned earlier, the produced gravitational waves in Mimetic Inflation follow the same consideration as in standard inflation, one can therefore simply use the final result for the gravitational waves spectrum which is given by

$$\delta_h \simeq H_{k=Ha}. \tag{2.121}$$

Now, these results will be used to find the spectrum of perturbations for the Mimetic Model with the potential and coupling term described in (2.28) and (2.29).

Spectrum in Potential-dominated Mimetic Inflation — The observationally relevant phase of inflation occurs in the region where the potential dominates over the kinetic term in equation (2.33), where the scalar field spans

$$1 < \varphi < m^{-1/2}, \tag{2.122}$$

and in this region, apart from the modified expression for $\dot{\varphi}$ for higher curvatures when $k > m$, the perturbations have a similar consideration as in standard potential-driven inflation models. Hence, given that in this region $\dot{\varphi}$ follows equation (2.59), and the potential is approximated by (2.31), the spectrum becomes

$$\delta_\Phi \simeq m\left(\varphi^3 \frac{1+m\varphi^4}{1+2m\varphi^6}\right)_{k=Ha}. \tag{2.123}$$

Rewriting field variables in terms of number of e-folds N — To obtain estimates for predictions of needed observables, it will be useful to express the scalar field $\varphi_{k=Ha}$ in terms of N for a given co-moving wave-number k. For that, the number of e-folds N before the end of inflation is defined as

$$a \simeq a_f e^{-N}, \tag{2.124}$$

and here the scale factor at the end of inflation is defined as a_f. Then, one can define N_k as the number of e-folds at which a perturbation with a given k crosses the Hubble scale before inflation ends, satisfying the condition $k = Ha$, such that it is given by

$$N_k \simeq \ln\left(\frac{Ha_f}{k}\right). \tag{2.125}$$

The span of N which will be considered is $50 < N_k < 60$, and it encompasses the range of scales probed by the CMB observations, while the exact values N will depend on the details of processes like reheating that occur immediately after inflation has ended, and are based on particle physics beyond the Standard Model, which is currently far from being understood.

Now, starting with the

$$H = -\frac{dN}{d\varphi}\dot{\varphi}, \tag{2.126}$$

one arrives at the following relation between N_k and the field values $\varphi_{k=aH}$

$$N = -\int \frac{H}{\dot{\varphi}} d\varphi \simeq \int \frac{\varphi^3 \left(1 + m\varphi^4\right)^{1/2}}{1 + 2m\varphi^6} d\varphi, \tag{2.127}$$

in the potential-dominated region.

2.5.4 Predictions from Mimetic Inflation

To find the predictions for the spectral index and tensor-to-scalar ratio r in Mimetic-Inflation, one can further restrict the observational range to $1 < \varphi < m^{-1/6}$ for the scalar field, taking into account the approximations for the potential and its derivative for this region are given in (2.62).

Spectral Index n_s — Using (2.62) in (2.127), the expression for N can then be integrated to obtain

$$N_k \simeq \varphi^4_{k=Ha}. \tag{2.128}$$

Taking this expression and from (2.123), the spectrum becomes

$$\delta_\Phi \simeq m\varphi^3_{k=Ha} \simeq mN_k^{3/4}, \tag{2.129}$$

and the spectral index can be written as

$$n_s - 1 \equiv \frac{d\ln\delta_\Phi^2}{d\ln k} = -\frac{d\ln\delta_\Phi^2}{dN_k} = -\frac{3}{2N_k}. \tag{2.130}$$

Evaluating the above result for $N_k = 50$ gives

$$n_s = 0.97, \tag{2.131}$$

which is in agreement with the value inferred from observations ($n_s = 0.965 \pm 0.004$ from Planck [45], $n_s = 0.9660 \pm 0.0046$ from Atacama Cosmology Telescope (ACT) data combined with WMAP larger-scale data, $n_s = 0.9709 \pm 0.0038$ from Planck and ACT data combined [47]). To yield the proper amplitude for the gravitational potential in the observable scales, the inflaton mass was set to $m \simeq 10^{-6}$. The expression for the spectral index in (2.130) is in fact valid for the range of scales where $1 < N_k < 10^4$, which is well-beyond the cosmological horizon present today.

Tensor-to-Scalar Ratio r — Using the previous result for δ_Φ, and given that $1 < \varphi < m^{-1/6}$ one gets $\delta_h \simeq m$, the tensor-to-scalar ratio is computed as

$$r \equiv \frac{\delta_h^2}{\delta_\Phi^2} \propto \frac{1}{N^{3/2}}. \tag{2.132}$$

One concludes that this value for $N = 50$ is in agreement with the upper bound on r that is observed today, $r < 0.032$ ([43],[44]). Moreover, the ratio r in Mimetic-Inflation is then more suppressed than in R^2 or Higgs-Inflation scenarios [56], but with a different function for the potential V, one can obtain a Mimetic-Inflation model that agrees with the former at the observed scales.

The Full Spectrum of Perturbations — To complete the picture for the spectrum of perturbations in the whole range of dynamics in Mimetic Inflation, the amplitudes for the spectrum for gravitational potential and gravitational waves is presented in Figure (2.2) which is obtained using numerical computation of the equations for δ_Φ and δ_h as a function of scale corresponding to $\varphi_{k=Ha}$, and the approximate analytic behavior is detailed below.

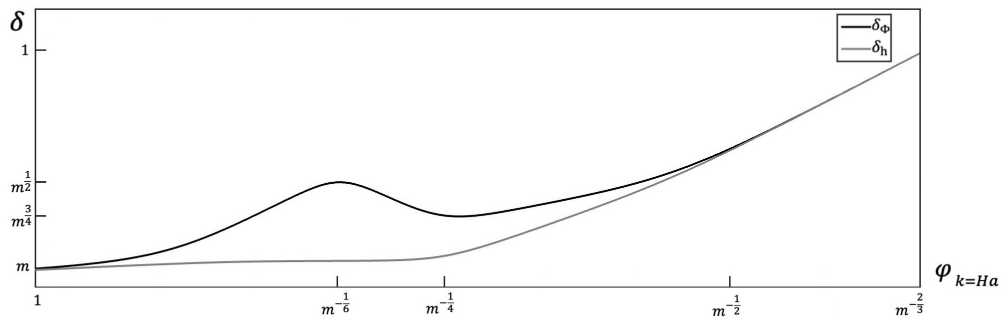

Figure 2.2: Amplitudes of the spectrum for scalar perturbations and gravitational waves

For $m^{-1/6} < \varphi_{k=Ha} < m^{-1/4}$, the perturbation amplitude at these scales decreases as

$$\delta_\Phi \simeq \varphi_{k=Ha}^{-3}, \tag{2.133}$$

and as the scale grows its value goes from $m^{1/2}$ to $m^{3/4}$.
For $m^{-1/4} < \varphi_{k=Ha} < m^{-1/2}$, the amplitude begins to increase again at these larger scales as

$$\delta_\Phi \simeq m\varphi_{k=Ha}. \tag{2.134}$$

At $\varphi \simeq m^{-1/2}$ this amplitude becomes

$$\delta_\Phi \simeq \delta_h \simeq m^{1/2}. \tag{2.135}$$

For $\varphi > m^{-1/2}$, inflation is driven by the kinetic term which dominates over the potential in this region, as discussed in Section (2.4). During this stage, which is the onset of inflation in the Mimetic model considered, the modes that cross the Hubble scale acquire an equal amplitude for the scalar perturbations and gravitational waves, such that

$$\delta_\Phi \simeq \delta_h \simeq m^2\varphi_{k=Ha}^3. \tag{2.136}$$

When the Planck densities are reached for $\varphi \simeq m^{-2/3}$, this amplitude becomes of order one. Otherwise the amplitude is always smaller than unity below the Planck scale, which ensures that self-reproduction does not occur in line with the conditions presented earlier.

Chapter 3

Discrete Gravity

3.1 Introduction

After exploring the evolution of the universe starting with small patches at Planckian scales, a natural question which follows is how does space-time emerge at these scales, and what is the proper framework to describe gravity there, knowing that the Einstein's theory faces serious limitations at Planckian scales. This comes from the observation that at the Planck scale, if one uses normal perturbation theory in Minkowski Space to quantize small perturbations of the gravitational field, these field perturbations become larger than unity, i.e. larger than the Minkowski metric, as soon as one reaches sub-scales of $\ell < \ell_{Pl}$. Therefore, the Minkowskian manifold description inevitably breaks down at these sub-Planckian scales, and it becomes natural to think of an alternative description for the manifold with some form of a minimal structure at this limiting scale. This could be realized by first implementing minimal quanta of geometry which take the form of cell-like structures that discretize the space, and within which points in space are indistinguishable. Then, the next step in recovering gravitational effects would be to associate geometric invariants with these discrete spaces, such as curvature, whereby in the continuous limit where the characteristic length of a discrete cell goes to zero, $\ell_{pl} \to 0$, one obtains the familiar expressions of differential geometry. Hence, in constructing this picture, a *discrete gravity* formalism emerges, where the quanta of geometry give rise to space-time.

From here, this part of the thesis will explore and further develop the new Discrete Gravity framework first introduced in the work of Chamseddine and Mukhanov in 2021 [19]. By studying the fundamental mathematical concepts that the classical and continuous theory of General Relativity bases its description of the space-time manifold on, we will proceed to construct their analogues for the discrete spaces described in this novel framework. In this attempt to write down a discrete theory for gravity, the aim will be to preserve the essential features of the classical theory, which includes symmetries like diffeomorphism invariance. Implementing different symmetries on the discrete manifold will be done through the 'gauging procedure' as inspired by other approaches to discretization as in Lattice Gauge Theory [20]. This will require a gauge treatment of gravity, which would allow one to build curvature and other invariants in the discrete setting, and provides the main advantage for the formalism which is a *manifest* continuous limit to the corresponding notions in General Relativity, in contrast to other discretization methods as in Regge Calculus [31]. In its first formulation [19], Discrete Gravity is initially constructed for a limited symmetry group, the group of rotations in space, since the discretization is considered for *Euclidean* spaces for simplicity. In the work presented in this thesis, the mathematical foundation of this discrete formulation will be expanded as to include the Poincaré group of symmetries, and hence obtain new discrete expressions for the geometric invariants. And even though diffeomorphism invariance stands under question as a requirement to fulfill at the discrete resolution of the theory, while also being absent in the first formulation, this work will use the expanded symmetry group to replace this property with translational invariance as another step in further developing this Discrete Gravity formalism. Core results in this chapter are published in the original work *Chamseddine and Khaldieh (2024)* [2].

In the sections that follow, the mathematical background for a gauge treatment of gravity will be laid out for the Poincaré group considered in this work, followed by a brief outline of ideas from Lattice Gauge theory that inspired the original formulation of Discrete Gravity [19], for which the mathematical structure will be explained and expanded. Then, building up on the main definition of the discrete curvature expression, we find new expressions with the extended symmetry studied in this thesis [2], and later demonstrate the manifest continuous limit for the various expressions obtained.

3.2 Gravity as a Gauge Theory

It is interesting to think that, despite the triumphs of the Gauge theory in particle physics and describing elementary interactions, Gauge theory owes its rise to explorations of Gravity and General relativity. Beginning with Hermann Weyl's invention of the 'Gauge Principle', the conceptual foundation that Weyl laid using the $U(1)$ gauge symmetry, and what became the basis for Quantum Electrodynamics as an Abelian Gauge theory, originally started from his attempt to unify gravitational and electromagnetic interactions [12]. And before Chen-Ning Yang and Robert Mills extended Weyl's idea of gauge symmetry beyond $U(1)$ to groups like $SU(2)$ and $SU(3)$, and set the formal framework for non-Abelian gauge theories, the mathematical elements of non-Abelian Gauge theory were already present the work of Elie Cartan on differential geometry [13], and have foundational ties to the gravitational gauge theories developed.

From here, exploring Gravity as a gauge theory could be attributed to Cartan's geometry, where the gauge theory is not constructed for a Unitary group describing internal symmetries, but rather invokes the Poincaré group $ISO(3,1)$, with Cartan's connections on the base manifold playing the role of the gauge fields, and the torsion as one of the corresponding field-strength components.

As will be also shown, these connections transform according to the same rules as in usual gauge theories, where for a Lie group G, with a Lie algebra $\mathfrak{g}$, a gauge field A_μ will transform under a local gauge transformation $g(x) \in G$ in the following way

$$A_\mu \to \widetilde{A}_\mu = gA_\mu g^{-1} + g\partial_\mu g. \tag{3.1}$$

To elaborate on these notions, the following section will build up the necessary differential geometry elements for gravity in the tetrad formalism, arriving at the transformation rules for gravitational gauge fields as in the relation above, which will be used later in the Discrete Gravity formulation.

3.2.1 The Tetrad Formalism

In order to describe Gravity as a gauge theory of the Poincaré Group, it is necessary to lay down the elements of the tetrad formalism of General Relativity first, and after that rewrite the main equations for curvature in terms of the new variables. The discussion presented below will be based on the methods present in Misner, Thorne and Wheeler [102], and so their notation will be utilized in the equations, in addition to [81] where the treatment was extended to larger tangent spaces.

In General Relativity, gravity is described using the metric tensor $g_{\mu\nu}$, which in the tetrad formalism can be rewritten in terms of the soldering forms e^a_μ which relate the spacetime coordinate basis vectors $\mathbf{e}_\mu$ to local orthonormal tetrad frames $\mathbf{v_a}$.

Starting with a d-dimensional manifold, with coordinate basis $\mathbf{e}_\mu$ that induce the metric through the abstract definition of the metric as a 'bilinear machine',

$$g_{\mu\nu} = \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \mathbf{e}_\mu \bullet \mathbf{e}_\nu, \tag{3.2}$$

A d-dimensional tangent space can then be defined at every point, which is spanned by a set of d vectors $\mathbf{v_a}$, the *vielbeins*, that are linearly independent and orthonormal, such that

$$\eta_{ab} = \mathbf{v_a} \bullet \mathbf{v_b}, \tag{3.3}$$

with η_{ab} being the Minkowski metric. The space-time coordinates are denoted by Greek indices, and tangent space coordinates are denotes by Latin indices. Then, the vielbeins are acted upon by the Lorentz transformations, such that the scalar product is preserved in the following way

$$\tilde{\mathbf{v}}_a = \Lambda_a^b \mathbf{v}_b, \quad \tilde{\mathbf{v}}_a \bullet \tilde{\mathbf{v}}_b = \eta_{ab}, \tag{3.4}$$

where the Lorentz transformations satisfy $\Lambda_a^c \eta_{cd} \Lambda_b^d = \eta_{ab}$. The soldering forms e_μ^a then relate the coordinate basis $\mathbf{e}_\mu$ to the $\mathbf{v_a}$ basis through

$$\mathbf{e}_\mu = e_\mu^a \mathbf{v}_a, \tag{3.5}$$

so that the information about curvature will be stored in the soldering forms,

$$e_{a\mu} = (\mathbf{v}_a \bullet \mathbf{e}_\mu), \tag{3.6}$$

where one can raise and lower the tangent space and manifold indices using the flat metric and the curved manifold metric respectively

$$e_\mu^a = \eta^{ab} e_{b\mu}, \qquad e_a^\mu = g^{\mu\kappa} e_{a\kappa} = g^{\mu\kappa} \eta_{ab} e_\kappa^b. \tag{3.7}$$

An important remark follows form this, where these soldering forms e_μ^b and e_a^μ are regarded as the inverse of each other only in the case when the dimension of manifold is equal to the dimension of the tangent space, and in the case where the dimension of tangent space is expanded, then $e_b^\kappa e_\kappa^a \neq \delta_b^a$. This picture will be useful when working with spinors in curved space, since they 'live' in flat space, and hence can be connected to the curved space through these soldering forms. In the tetrad formalism, the soldering forms will also be important for writing down Lagrangians that are gauge invariant. Now, to obtain the expression of the metric in terms of the soldering forms, one can simply substitute (3.5) in the definition in (3.2) to obtain

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \tag{3.8}$$

Parallel Transport and Torsion — To relate the change in the soldering forms due to the transformation of the vielbeins and basis vectors under the action of parallel transport on the manifold, one starts with the definitions

$$\nabla_\mu \mathbf{e}_\nu = \Gamma^\kappa_{\mu\nu} \mathbf{e}_\kappa, \quad \nabla_\mu \mathbf{v}_a = -\omega_{\mu a}{}^b \mathbf{v}_b, \tag{3.9}$$

where covariant derivative ∇_μ which measures the rate of change of the vectors $\mathbf{e}_\nu$ and $\mathbf{v}_a$ along the basis vector $\mathbf{e}_\mu$ is determined through the affine and spin connections, keeping in mind that the spin connections are antisymmetric in the tangent-space indices due to the metricity condition

$$\nabla\eta_{ab} = \omega_{\mu ab} - \omega_{\mu ba} = 0. \tag{3.10}$$

In the absence of torsion, the affine connections become symmetric in the lower space-time indices

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}. \tag{3.11}$$

Finally, using the above equations for the parallel transported vectors (3.9), the change in the soldering forms can be directly obtained using the definition in (3.6), where ∇_ν acts on $e_{a\mu}$ as a scalar function $\nabla_\nu e_{a\mu} \equiv \partial e_{a\mu}/\partial x^\nu$,

$$\partial_\nu e_{a\mu} = -\omega_{\nu a}{}^b e_{b\mu} + \Gamma^\kappa_{\mu\nu} e_{a\kappa}. \tag{3.12}$$

3.2.2 Invariance under Local Lorentz Group

Considering for now the group of *local* Lorentz transformations $\Lambda_a{}^b \equiv \Lambda_a{}^b(x)$, one can finally arrive at the transformation law for the spin connections, requiring that the theory be invariant under this symmetry group. First, starting with the transformation of the vielbeins under the Lorentz group

$$\widetilde{\mathbf{v}}_a = \Lambda_a{}^b \mathbf{v}_b, \tag{3.13}$$

and then the transformation of the spin connection follows from the definition in (3.9)

$$\widetilde{\omega}_{\mu a}{}^b \widetilde{\mathbf{v}}_b = -\nabla_\mu \widetilde{\mathbf{v}}_a, \tag{3.14}$$

and after substituting $\widetilde{\mathbf{v}}_a$ and $\nabla_\mu \mathbf{v}_a = -\omega_{\mu a}{}^b \mathbf{v}_b$ into the above expression, one obtains

$$\omega_{\mu a}{}^b \;\to\; \widetilde{\omega}_{\mu a}{}^b = \left(\Lambda\omega_\mu\Lambda^{-1}\right)_a^{\;b} + \left(\Lambda\partial_\mu\Lambda^{-1}\right)_a^{\;b}. \tag{3.15}$$

This expression can now be compared with that for a general gauge field transformation given earlier in (3.1), and hence the spin connections transform as *gauge fields* for spacetime rotations under local group action $\Lambda_a{}^b(x)$. Similarly, the transformation law for the soldering forms under local Lorentz transformations is obtained from $\widetilde{\mathbf{v}}_b = \widetilde{e}^\mu_b \mathbf{e}_\mu$, so that

$$e^\mu_b \;\to\; \widetilde{e}^\mu_b = \Lambda_a{}^b e^\mu_a. \tag{3.16}$$

The soldering form e_a^μ is an important quantity in the tertrad or *vielbein* formalism, and under Lorentz Group it simply transforms as a vector. When the group of symmetries is extended from the Lorentz group, or the group of rotations in space-time, to the Poincaré Group which includes translations, the soldering forms will play the role of the *gauge fields* for the translations. Considering the group of local Lorentz transformation is useful for the example that will be shown from the earlier work on Discrete Gravity in [19], whereby only the rotations in space were considered, and which this thesis work aims to expand through including the translations into the group of symmetry and finding extensions and corrections to the obtained equations.

3.2.3 Gauging the Poincaré Group

Euclidean Signature — To put Einstein's theory of gravity on the same footing as a Gauge theory, one needs to consider the full Poincaré group, and even so, it is regarded as the group of global symmetries of spacetime only in the absence of gravity. As the symmetries in the discrete formalism will be considered for *Euclidean* spaces, in this case, the symmetry group considered on the tangent space can be expanded from the group of rotations $SO(d)$, the Euclidean counter part of Lorentz transformations, to the inhomogeneous group $ISO(d)$ in order to include the translational symmetry part of the Poincaré group. The generators of the space-time rotational symmetry and the generators of translations are respectively given by

$$J_{ab} = \zeta^\mu_{[ab]}\partial_\mu = (x_a\delta^\mu_b - x_b\delta^\mu_a)\partial_\mu, \tag{3.17}$$

$$P_a = \zeta^\mu_a\partial_\mu = \delta^\mu_a\partial_\mu. \tag{3.18}$$

Clearly, these symmetries considered for a flat space-time differ from those of General Relativity in the presence of gravity. When factoring in gravity, the curved spacetime will carry the symmetry associated with general coordinate transformations, namely diffeomorphism invariance, in addition to local Lorentz transformations that were previously described for the tangent space. Then, to draw a connection between General Relativity and the gauge theory of the Poincaré group, general coordinate transformations must be replaced by the translational symmetry, which is in principle absent in Einstein's theory, but will be possible through relating the parameters associated with these symmetries. As will be shown later, this will be realized through imposing the zero torsion condition, or in other words taking the curvature component related to translations to be zero.

Gauge Transformations — In Riemannian geometry with Euclidean signature, for $d-$dimensional manifolds, one can consider the transformation of spinors $\psi(x)$ under the Lie group $ISO(d)$. By promoting the global transformations to local ones, with the spin connections $\omega_\mu^{ab}(x)$ acting as the rotational gauge fields, and the soldering forms $e_\mu^a(x)$ as the translational gauge fields, the coupling of the spinors ψ to the gauge fields can be done through defining the connection D_μ for the Dirac action

$$S_D = \int d^4x \sqrt{g}\bar{\psi} i\gamma^a e_a^\mu D_\mu \psi,$$

as

$$D_\mu \psi = (\partial_\mu + \omega_\mu + e_\mu)\psi, \tag{3.19}$$

where

$$\omega_\mu \equiv \omega_\mu^{ab} J_{ab}, \quad e_\mu \equiv e_\mu^a P_a. \tag{3.20}$$

Then, the spinors ψ transform under the action of the gauge group as

$$\psi(x) \to \psi'(x) = e^{\lambda(x)+\zeta(x)}\psi(x), \tag{3.21}$$

where $\lambda(x) = \lambda^{ab}(x)J_{ab}$ and $\zeta(x) = \zeta^a(x)P_a$ are the parameters of the rotations and translations respectively.

Gauge Field Infinitesimal Transformations — The derivative of ψ will transform covariantly as

$$D_\mu \psi \to D'_\mu \psi' = e^{\lambda(x)+\zeta(x)} D_\mu \psi, \tag{3.22}$$

provided that the gauge fields transform infinitesimally through

$$\begin{aligned} \delta {e_\mu}^a &= \partial_\mu \zeta^a + {\omega_\mu}^{ab}\zeta^b - \lambda^{ab} e_\mu^b \,, \\ \delta {\omega_\mu}^{ab} &= \partial_\mu \lambda^{ab} + {\omega_\mu}^{ac}\lambda^{cb} - {\omega_\mu}^{bc}\lambda^{ca}. \end{aligned} \tag{3.23}$$

These gauge fields also transform under a general coordinate transformation $\tilde{x}^\nu = x^\nu + \zeta^\nu$ through the action of the Lie Derivative along ζ^ν, which is the vector field parameter of these transformations, such that

$$\delta' e_\mu^a = \zeta^\nu \partial_\nu e_\mu^a + e_\nu^a \partial_\mu \zeta^\nu, \tag{3.24}$$

$$\delta' \omega_\mu^{ab} = \zeta^\nu \partial_\nu \omega_\mu^{ab} + \omega_\nu^{ab} \partial_\mu \zeta^\nu. \tag{3.25}$$

Curvature as Field Strength — To find the curvature associated with the gauge fields, the commutator of the covariant derivatives is then taken

$$[D_\mu, D_\nu] = R_{\mu\nu}, \tag{3.26}$$

where the curvature components for each gauge field can be deduced from

$$R_{\mu\nu} = \frac{1}{4}R_{\mu\nu}{}^{ab}(\omega)J_{ab} + \frac{1}{2}R^{a}_{\mu\nu}(e)P_{a}, \tag{3.27}$$

with the the generators of the Lie algebra of the rotation and translation groups satisfying the following commutation relations

$$\begin{aligned}
[J_{[ab]}, J_{[cd]}] &= \eta_{bc}J_{[ad]} - \eta_{ac}J_{[bd]} - \eta_{bd}J_{[ac]} - \eta_{ad}J_{[bc]}, \\
[J_{[cd]}, P_{a}] &= P_{c}\eta_{da} - P_{d}\eta_{ca}, \\
[P_{a}, P_{b}] &= 0.
\end{aligned} \tag{3.28}$$

Finally, the curvature associated with the local translations, or Torsion, and that of local rotations are given by

$$\begin{aligned}
R^{a}_{\mu\nu}(e) \equiv T_{\mu\nu}{}^{a} &= \partial_{\mu}e_{\nu}{}^{a} - \partial_{\nu}e_{\mu}{}^{a} + \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b} - \omega_{\nu}{}^{a}{}_{b}e_{\mu}{}^{b}\,, \\
R_{\mu\nu}{}^{ab}(\omega) &= \partial_{\mu}\omega_{\nu}{}^{ab} - \partial_{\nu}\omega_{\mu}{}^{ab} + \omega_{\mu}{}^{ac}\omega_{\nu c}{}^{b} - \omega_{\nu}{}^{ac}\omega_{\mu c}{}^{b}\,.
\end{aligned} \tag{3.29}$$

Relating Diffeomorophisms to Translations — As mentioned earlier, the key to bridging Poincaré gauge theory with Gravity will be through relating the translations to the general coordinate transformations, and this is done by first defining field dependent parameters for translation and rotation as follows

$$\zeta^{a}(x) = \zeta^{\nu}(x)e^{a}_{\nu}(x), \qquad \lambda^{ab}(x) = \zeta^{\nu}(x)\omega^{ab}_{\nu}(x), \tag{3.30}$$

then, by substituting those parameters into the infinitesimal transformation of the soldering form in (3.23), one gets the following relation

$$\delta e^{a}_{\mu} = \partial_{\mu}(\zeta^{\nu}e^{a}_{\nu}) + (\zeta^{\nu}e^{b}_{\nu})\omega^{ab}_{\mu} - (\zeta^{\nu}\omega^{ab}_{\nu})e^{b}_{\mu} \tag{3.31}$$

$$= \delta' e^{a}_{\mu} + \zeta^{\nu}T^{a}_{\mu\nu}. \tag{3.32}$$

where the second line is obtained after adding and subtracting the term $\zeta^{\nu}\partial_{\nu}e^{a}_{\mu}$ after expanding the expression in the first line. With this, the transformation of the soldering form $\delta' e^{a}_{\mu}$ under diffeomorphisms can be directly related to its gauge transformations under translation and rotation, or more generally for all tensors, the diffeomorphism transformations can be generated by translational gauge transformation of the field, with the condition that

$$T_{\mu\nu}{}^{a} = 0.$$

This condition has two important consequences, one is that it allows for the spin connection field ω_μ^{ab} to be fully determined by the soldering forms e_μ^a, and allows to identify the spin connection with the Levi-Civita connection, which is necessarily torsion free in General relativity. Meanwhile, in the presence of torsion the translational and rotational gauge fields can act independently. Hence, in General Relativity extensions, lifting the torsion constraint promotes the spin connection ${\omega_{\mu a}}^b$ to an independent variable, and allows for new dynamics to arise from spin-gravity couplings.

Relation to Einstein-Hilbert Action — In the absence of torsion, the gauge invariant action can be written using the scalar curvature of local rotations as

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{g}R(\omega), \tag{3.33}$$

where the curvature is contracted using the soldering forms

$$R(\omega) = e_a^\mu e_b^\nu {R_{\mu\nu}}^{ab}(\omega). \tag{3.34}$$

It can be shown that this action be equivalent to the Einstein-Hilbert action with

$$R^\rho_{\sigma\mu\nu}(\Gamma) = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\kappa}\Gamma^\kappa\nu\sigma - \Gamma^\rho_{\nu\kappa}\Gamma^\kappa_{\mu\sigma}, \tag{3.35}$$

by relating the curvatures expressions as

$$\begin{aligned} R(\Gamma) &= g^{\rho\mu}R^\nu_{\rho\nu\mu}(\Gamma) \\ &= e_a^\mu e_b^\nu {R_{\mu\nu}}^{ab}(\omega) = R(\omega), \end{aligned}$$

where a detailed proof is found in [81].

3.2.4 Ideas from Lattice Gauge theory

Since the goal in Discrete Gravity will be to implement gravity as a gauge theory on the lattice, which was inspired originally by Lattice Gauge Theory, a brief outline of the latter will be presented following (e.g., [103, 104]), highlighting the elements that will be used in the Discrete Gravity formalism that this part of the thesis work builds on.

First, the setup is given by a cubic Euclidean lattice with a measure a defining the lattice separation, and it represents a natural ultra violet cutoff defined by $a = 1/\Lambda_{UV}$. Points on the lattice will be separated by unit vectors denoted by $\hat{\mu}$, where $\mu = 1, .., d$, which define a $d-$dimensional basis. A 2-dimensional representation of the lattice is given in Figure (3.1).

Considering a general gauge group $SU(N)$, a group element $U(x) \in G$ will be defined

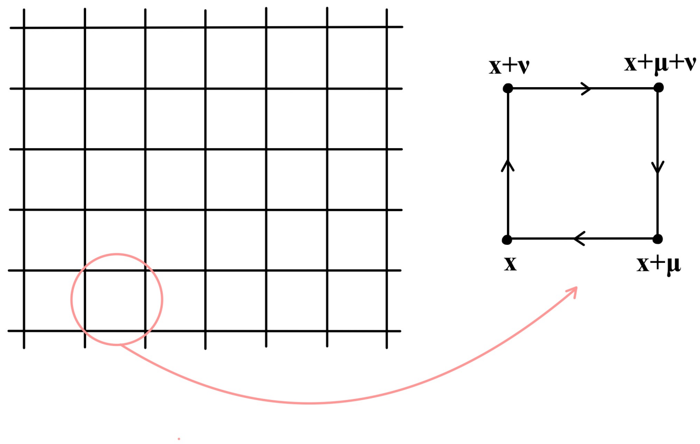

Figure 3.1: A path traversing a plaquette of the lattice with sites connected through links.

on the links between two lattice sites x and $x+\hat{\mu}$, with the spacing a is introduced into the definition as

$$U_\mu(x) = e^{iaA_\mu(x)}, \tag{3.36}$$

and $A_\mu(x)$ simply represent the gauge fields. Then, $U(x)$ transforms under gauge transformations $\Omega(x)$ as

$$U_\mu(x) \;\rightarrow\; \widetilde{U}_\mu(x) = \Omega(x)U_\mu(x)\Omega(x+\hat{\mu}). \tag{3.37}$$

A Wilson loop $U_{\mu\nu}$ can be written for a closed path as shown on the plaquette in Figure(3.1) using the group elements $U(x)$ and their conjugates, which connect two neighboring sites as follows

$$U_\mu(x): \;x \rightarrow x+\hat{\mu}, \quad U^\dagger_\mu(x): \;x \rightarrow x-\hat{\mu}.$$

Then, using the definitions for the link variables given above, one has

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U^\dagger_\mu(x+\hat{\mu})U_\nu(x), \tag{3.38}$$

where path ordering is important for non-commuting elements when considering a non-Abelian group.

To construct a real-valued curvature, one can sum over two loops with opposite orientations, such that the Lattice action can be written as

$$S_{Lattice} = c\mathrm{Tr}\sum_{x} -\frac{1}{2}\left(U_{\mu\nu}(x) + U^{\dagger}_{\mu\nu}(x)\right), \tag{3.39}$$

where c is a constant which will be related to the coupling constant in Yang-Mills, and in the discrete case for a group of dimension N, $c = \frac{\beta}{N}$.

Then by expanding (3.38),

$$U_{\mu\nu}(x) = e^{iaA_{\mu}(x)}e^{iaA_{\nu}(x+\hat{\mu})}e^{-iaA_{\mu}(x+\hat{\nu})}e^{-iaA_{\mu}(x)}, \tag{3.40}$$

one can find its definition in terms of A_{μ} by calculating a few terms in the exponent using the Baker-Campbell-Hausdorff formula

$$e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]+...}, \tag{3.41}$$

where $A_{\mu}(x+\hat{\nu})$ is proportional to lattice spacing as such

$$A_{\mu}(x+\hat{\nu}) \approx A_{\mu}(x) + a\partial_{\nu}A_{\mu}(x) + ... \tag{3.42}$$

Then one obtains for matrix-valued field tensor $\mathcal{F}_{\mu\nu}$

$$U_{\mu\nu}(x) \approx e^{ia^{2}\mathcal{F}_{\mu\nu}(x)}, \tag{3.43}$$

and up to order a^2, one can recover in the continuous limit

$$\mathcal{F}_{\mu\nu} \to F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu}, A_{\nu}]. \tag{3.44}$$

Finally, taking the trace of $U_{\mu\nu}(x)$ gives

$$\mathrm{tr}e^{ia^{2}\mathcal{F}_{\mu\nu}(x)} \approx \mathrm{tr}\frac{a^{4}}{2}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} + ..., \tag{3.45}$$

Hence, the Lattice action can be computed in the continuum limit as

$$S_{Lattice} \to \frac{a^{4}\beta}{4N}\int dx^{4} tr(F_{\mu\nu}F_{\mu\nu}) + ... \tag{3.46}$$

which in the leading order coincides with the Yang-Mills action, with $\frac{\beta}{2N} \equiv \frac{1}{g^2}$.

To this end, it is worth noting that Lattice gauge theory violates rotational invariance, while trying to preserve $U(n)$ gauge invariance, which will be a starting point to address and change in the new Discrete Gravity formalism.

3.3 The Mathematical Basics of Discrete Gravity

We now move to construct the mathematical structures and definitions that will be used in our extension of the Discrete Gravity formulation. These definitions are heavily based on the work by A. Chamseddine and V. Mukhanov in 2021 [19], which laid down the original framework.

3.3.1 Building the Discrete Cells Manifold

Discrete Manifold and Its Dimension — To build a discrete lattice from which the usual space-time manifold will reemerge in the continuous limit, one has to take a step back and walk through how the picture of the continuous manifold which defines the space-time was constructed. In principle, one starts with a set of points that are like sand, and on this set, one starts to define structures on the set, starting with defining the dimensionality $\mathbb{R}^d$ and a (standard) topology $\mathcal{O}_\int$, which together form a topological space $(\mathbb{R}^d, \mathcal{O}_\int)$, and this can be understood as a rubber sheet that can be stretched differently to cover a physical manifold. Only when this topological space becomes equipped with a smooth-Atlas and a connection, that the manifold acquires *shape*, and loses the rubber-sheet description, and then defining the metric gives the manifold *size*. Then, it makes sense to begin with defining the dimensionality of the manifold in the continuous and the discrete cases. Starting with the continuous limit, a d-dimensional manifold is then described as the topological space $(\mathbb{R}^d, \mathcal{O}_\int)$ where the neighborhood of every point in this space can be mapped, via a homeomorphism, to an open subset of the Euclidean space $\mathbb{R}^d$. Now, in the case of the discrete-cells manifold, there is a subtlety around the point-like description, which is in fact replaced by a cell-like structure and now used to define the dimensionality of the discrete manifold. This discrete-cell manifold is built such that it consists of cells of minimal size and elementary volume which can be taken as the Planck volume, and every cell is associated with *one* point only taken to be at the 'center', and within which no smaller resolution is admissible, and hence no points in this cell are physically distinguishable. The shape of the cell is undefined *apriori*, as to avoid the picture of a *fixed* chart or grid, but the important element is that the cells will still have a certain *boundary* with neighboring cells, and it defines the dimensionality of the discrete manifold in the following way:

> A discrete manifold is said to have dimension d such that every cell shares its boundary with $2d$ adjacent cells,

this space is then set to have a *Euclidean signature*.

Cell numbering and functions on the lattice — Naturally, in order to define functions on the discrete cell manifold and operate on them, the cells are numbered with a set of d integers such that

$$n^\mu \equiv \left(n^1, n^2, \cdots, n^d\right) \equiv \mathbf{n}, \tag{3.47}$$

where the numbering of points in two adjacent cells is separated by just *one unit* integer [19]. Looking ahead, this enumeration of the cells will retrieve the usual *coordinate values* in the continuous case, when the volume of the cells will shrink to zero, which will be shown later after introducing a length scale $\ell^\mu \equiv \Delta x^\mu$ into the picture. Now, a set of scalar function $f(n^\mu)$ can be defined on the cell-lattice, so that for every minimal volume these scalar functions assign only a *finite* number of degrees of freedom. Then for a scalar field, for instance, one degree of freedom is assigned in a cell, while two degrees of freedom are assigned for a massless vector-field, etc...
To illustrate this in an example, a two-dimensional lattice is shown in Figure (3.2), and to enumerate this lattice only two integers for n^μ will be needed. Now, taking point M as a reference in the marked cell, it will be enumerated with $n_M = (n^1, n^2)$. The right-neighboring cell with point N will have then the numbering $n_N = (n^1 + 1, n^2)$, if for the right-direction the index is set to $\mu = 1$, and so for the upward-direction $\mu = 2$, and one gets for cell with point P, $n_P = (n^1, n^2 - 1)$. The neighboring cells around M from the opposite directions to N and P will be labeled with $(n^1 - 1, n^2)$ and $(n^1, n^2 + 1)$ respectively. Note that the integers in the enumeration can be positive or negative valued.

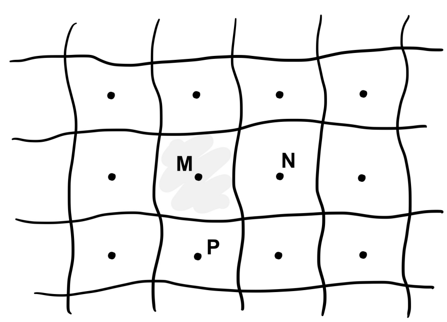

Figure 3.2: Illustration of a discrete lattice in two dimensions.

Lattice structure and Freedom of re-enumeration — A comment is due on the shape of the lattice, which seems from the illustration that is shown that it is somewhat a fixed grid. It should be clear that this is not the case, as the shape of the cells and how they connect through their boundaries in the real sense is not specified, and the goal is to be able to navigate these physical points that are *a priori* scattered like sand as explained earlier, before structure is imposed on this set of points. Here, the structure specified is so far minimal, and there is no avoiding labeling the physical points, which will be navigated through the operators to be defined in the next paragraph. Skipping ahead to when the metric will be defined, which should give meaning to the volume of the manifold, and in this case to the individual cells when it is specified at a point, setting a minimal volume to the cells does not contradict the statement made before on the lattice not having a fixed grid-like structure, as this volume could be occupied by arbitrarily shaped cells, with the only condition being the number of neighbors this cell is supposed to have, as to keep with the consistency of the definition for the dimensionality of the discrete manifold. Figure (3.3) is another illustration accentuating this arbitrariness in the discretization of the 2D Lattice. This picture, together with the freedom of numeration of the cells, is kept as a guide when thinking about the diffeomorphism invariance feature in General Relativity and trying to preserve it, or finding an analogue for it on the lattice.

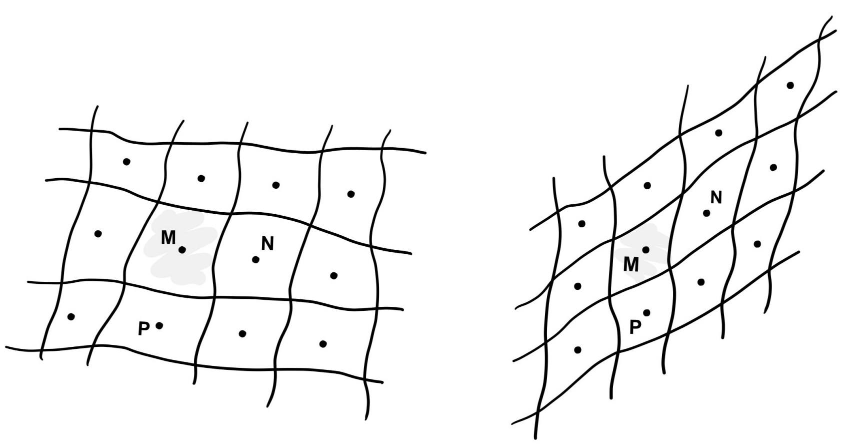

Figure 3.3: Arbitrary choices of discretizing lattices in two dimensions.

Vectors on the Discrete Manifold — In order to construct *tangent vectors* on the cell manifold, one starts by generalizing the notion of a *vector* on the discrete space to an *operator*. Now, one can define in each elementary volume a set of d shift operators denoted by $\mathbf{E}_\beta$, such that they form a basis on a linear space of dimension d, which will allow one to move between the neighboring cells. Hence, these forward-shift or displacement operators will act on the function defined on the cell by shifting its argument, for instance acting on point M in the previous figure (3.2), keeping arbitrary indices as

$$\mathbf{E}_{\nu}(n_M)f(n^{\mu}_M) \equiv f(n^{\mu}_M + \delta^{\mu}_{\nu}), \tag{3.48}$$

where the function has now moved one unit $\delta^{\mu}_{\nu} \equiv 1_{\mu}$ forward in the μ-direction, and if $\mu = 1$, then the shift operator moves the function to cell N so that

$$\mathbf{E_1}(n_M)f(n_M) = f(n_M + 1_1) \equiv f(n_N). \tag{3.49}$$

Similarly, the backward displacement is defined through the *inverse* shift-operators $\mathbf{E}^{-1}_{\nu}$ that act on the function at point M as

$$\mathbf{E}^{-1}_{\nu}\left(n_M\right) f\left(n^{\mu}\right) \equiv f\left(n_M - \delta^{\mu}_{\nu}\right), \tag{3.50}$$

i.e. through $\mathbf{E}^{-1}_{\nu} \equiv \mathbf{E}_{-\nu}$, and if here $\beta = 2$, the function is shifted a unit $\delta^{\mu}_{2} \equiv 1_2$ from point M in the downward direction to point P such that

$$\mathbf{E}^{-1}_{2}\left(n_M\right) f\left(n_M\right) = f\left(n_M - 1_2\right) \equiv f(n_P), \tag{3.51}$$

keeping in mind that shift operator and its inverse must satisfy the following relation at every point (n_*)

$$\mathbf{E}_{\mu}(n_*)\mathbf{E}^{-1}_{\mu}(n_*) = \mathbf{E}^{-1}_{\mu}(n_*)\mathbf{E}_{\mu}(n_*) = 1. \tag{3.52}$$

To complete the formal definition for the linear space of the shift operators, the spaced is equipped with a sum and multiplication, such that for a set of real numbers $(b, c) \in \mathbb{R}$

$$(b\mathbf{E}_{\kappa} + c\mathbf{E}_{\gamma})f\left(n\right) = bf\left(n + 1_{\kappa}\right) + cf\left(n + 1\gamma\right). \tag{3.53}$$

A subtlety arises when applying the shift-operators on a function in sequence, as their action can only be defined when applied to the function and other operators that are in the same cell, and so the order of operation is from *left-to-right* in an expression, as for instance

$$\begin{aligned}
\mathbf{E}_{\kappa}(n)\mathbf{E}^{-1}_{\gamma}(n)f(n) &= \mathbf{E}_{\kappa}(n)\left(\mathbf{E}^{-1}_{\gamma}(n)f(n)\right) \\
&= \mathbf{E}^{-1}_{\gamma}(n + 1_{\kappa})f(n + 1_{\kappa}) \\
&= f(n + 1_{\kappa} - 1_{\gamma}),
\end{aligned} \tag{3.54}$$

and when $\kappa = \gamma$, the identity for the shift-operators in (3.52) is satisfied.

3.3.2 Tangent Operators and the Discrete Metric

Now that every cell is equipped with a d-dimensional linear space of shift operators, d tangent operators can be constructed from those operators defined through central difference as follows [19]

$$\mathbf{e}_\mu(n) \equiv \frac{1}{2}\left(\mathbf{E}_\mu(n) - \mathbf{E}_\mu^{-1}(n)\right). \tag{3.55}$$

This central difference definition will be key for recovering the tangent vector definition on the continuous coordinate lines. In the same linear space, one can define a set of d orthonormal operators, or *vielbeins* $\mathbf{v_a}$, that satisfy the usual orthogonality condition in each cell

$$\mathbf{v}_a \bullet \mathbf{v_b} = \delta_{ab}, \tag{3.56}$$

and these operators carry Latin indices spanning $a, b, .. = 1, 2, ..., d$, with δ^a_b playing the role of the Euclidean η_{ab} metric.
As the theory must preserve local rotational invariance on the discrete manifold, as the space is set to be Euclidean, the vielbeins will be acted upon by the elements of the rotation group, such that

$$\widetilde{\mathbf{v}}_a(n) = \mathcal{R}^b_a(n)\mathbf{v}_b(n), \quad \mathcal{R}^b_a(n) \in SO(n), \tag{3.57}$$

with $\mathcal{R}^b_a(n)$ defined in the vector representation.
Once again, similar to the standard definitions, the tangent operators $\mathbf{e}_\mu$ and the operators $\mathbf{v_a}$ can be connected through the soldering forms $\bar{e}^\mu_a(n)$, which are used to express $\mathbf{v_a}$ as linear combinations of $\mathbf{e}_\mu$ as

$$\mathbf{v}_a(n) = \bar{e}^\mu_a(n)\mathbf{e}_\mu(n), \tag{3.58}$$

$$\mathbf{e}_\mu(n) = e^b_\mu(n)\mathbf{v}_b(n), \tag{3.59}$$

and to write the second expression, the inverse of the soldering form $e^b_\mu(n)$ was used, from which one then gets

$$e_{a\mu}(n) = \mathbf{v}_a(n) \bullet \mathbf{e}_\mu(n), \tag{3.60}$$

keeping in mind that the soldering form and its inverse satisfy

$$e^b_\mu(n)\bar{e}^\mu_a(n) = \delta^b_a. \tag{3.61}$$

And now one arrives at the metric definition, which gives a notion of size to the elementary cell as explained before, and so the metric is defined on the discrete lattice in analogy to how the covariant components in the metric are obtained in (3.2) and so in every cell one has

$$g_{\mu\nu}(n) \equiv \mathbf{e}_\mu(n) \bullet \mathbf{e}_\nu(n) = e^a_\mu(n) e^b_\nu(n) \delta_{ab}. \tag{3.62}$$

In this and the coming parts, Einstein's summation rule was used.

Parallel Transport and Affine Connections — In order to recover the expression for the affine connection, the parallel transport rules will be derived on the discrete manifold. Drawing on parallels with the earlier definition of covariant derivative action on the basis vectors and the vielbeins, the covariant derivative can be deconstructed as follows,

$$\nabla_{\mathbf{e}_\nu} \mathbf{e}_\mu(n) = \lim_{\varepsilon \longrightarrow 0} \frac{\mathbf{e}^{p.t.}_\mu(n + 1_\nu \to n) - \mathbf{e}_\mu(n)}{\varepsilon}, \tag{3.63}$$

where '*p.t.*' is short for 'parallel transported', and by keeping with the minimal length unit set to one, ε can be dropped from the limit, and shall be recovered later when taking the continuous limit, and hence from the earlier definitions of $\nabla_\nu \mathbf{e}_\mu$ and $\nabla_\nu \mathbf{v}_A$, one can write [19]

$$\mathbf{e}^{p.t.}_\mu(n + 1_\nu \to n) = \mathbf{e}_\mu(n) + \Gamma^\gamma_{\mu\nu}(n) \mathbf{e}_\gamma(n), \tag{3.64}$$

$$\mathbf{v}^{p.t.}_a(n + 1_\nu \to n) = (\Omega^{-1}_\nu(n))^b_a \mathbf{v}_b(n), \tag{3.65}$$

with $(\Omega^{-1}_\nu(n))^b_a$ is the inverse of the local spin connection group element

$$\Omega_\nu(n) = \exp\left(\omega^{cd}_\nu(n) J_{cd}\right), \tag{3.66}$$

and ω^{ab}_ν being the spin connection. Keeping in mind the construction of the formalism was based only on the rotational invariance on the cells, the rotation group is thus used in deriving all the preliminary expressions, but later can be generalized to use elements $\Omega_\nu(n)$ from the desired group of symmetries.

Keeping with the group of rotations as the main symmetry of the theory, the generators of the commutations J_{ab} are then appropriately taken in the vector representation

$$(J_{cd})^b{}_a = \frac{1}{2}(\delta^b_c \delta_{da} - \delta_{ac} \delta^b_d), \tag{3.67}$$

satisfying the appropriate commutation relations. Now, given that the scalar product does not change in parallel transport, it follows from (3.60) that

$$e^{p.t.}_{a\mu}(n+1_\nu \to n) = e_{a\mu}(n+1_\nu), \tag{3.68}$$

and from the definitions in (3.64) and (3.65), one can compute

$$\begin{aligned} e_{a\mu}(n+1_\nu) &= \mathbf{v}^{p.t.}_a(n+1_\nu \to n) \bullet \mathbf{e}^{p.t.}_\mu(n+1_\nu \to n) \\ &= (\mathit{\Omega}^{-1}_\nu(n))^b_a \mathbf{v}_a(n) \bullet \Big(\mathbf{e}_\mu(n) + \Gamma^\gamma_{\mu\nu}(n)\mathbf{e}_\gamma(n)\Big) \\ &= (\mathit{\Omega}^{-1}_\nu(n))^b_a e_{b\mu}(n) + (\mathit{\Omega}^{-1}_\nu(n))^b_a \Gamma^\gamma_{\mu\nu}(n) e_{b\gamma}(n). \end{aligned} \tag{3.69}$$

Finally, multiplying the last equality by Ω, the affine connections can be deduced in terms of soldering forms and spin connections as follows [19]

$$\Gamma^\gamma_{\mu\nu}(n)e_{a\gamma}(n) = (\mathit{\Omega}_\nu(n))^b_a e_{b\mu}(n+1_\nu) - e_{a\mu}(n). \tag{3.70}$$

Spinor Representation — For the following, it will be more convenient to use the spinor representation, where the Dirac gamma matrices are defined such that they satisfy the following Clifford algebra

$$\{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}, \tag{3.71}$$

and are assumed to be Hermitian. The generators for the rotation group are then defined through the gamma matrices as

$$J_{ab} = \frac{1}{8}(\gamma_a\gamma_b - \gamma_b\gamma_a), \tag{3.72}$$

with their commutation relations given by

$$[J_{ab}, \gamma_c] = \frac{1}{2}\left(\delta_{bc}\gamma_a - \delta_{ac}\gamma_b\right). \tag{3.73}$$

3.3.3 Curvature and Torsion in $SO(d)$

Curvature — To define the main curvature expression in this lattice formulation, it will be convenient to define the operator [19]

$$\mathit{\Upsilon}_\mu(n) \equiv \mathit{\Omega}_\mu(n)\mathbf{E}_\mu(n), \tag{3.74}$$

which combines the two actions of $\mathbf{E}_\mu(n)$ and $\mathit{\Omega}_\mu(n)$ in a combined *shift and rotate* action, with and $\mathit{\Omega}_\mu(n)$ defined in (3.66), while the generators associated with this group element will depend on the choice of representation. The spinor representation will be used to derive the torsion condition for the associated spinors, and will be easier for the computation for the rotational invariance in the 2-dimensional case, as shown in [19] as an example.

Considering now a path along a plaquette that begins from a given cell n, and traverses a loop around the neighboring cells, just like in the setup pictured earlier for Lattice gauge theory, with the *shift and rotate* operators $\varUpsilon_\mu(n)$ and $\varUpsilon_\nu(n)$ performing the measurement by moving to the neighboring cells in a given direction around the plaquette, so that the curvature will be the difference form the measure over the loop and its opposite orientation. The curvature can hence be defined as [19]

$$R_{\mu\nu}(n) = \frac{1}{2\ell^\mu \ell^\nu}\left(\varUpsilon_\mu(n)\varUpsilon_\nu(n)\varUpsilon_\mu^{-1}(n)\varUpsilon_\nu^{-1}(n) - \varUpsilon_\nu(n)\varUpsilon_\mu(n)\varUpsilon_\nu^{-1}(n)\varUpsilon_\mu^{-1}(n)\right), \quad (3.75)$$

where the length scales introduced ℓ^μ can be different in each direction. This curvature expression agrees with the Yang-Mills definition, with the important note that the curvature measure around the plaquette here is in effect antisymmetric,

$$R_{\mu\nu}\,(n) = -R_{\nu\mu}\,(n)\,. \quad (3.76)$$

The curvature $R_{\mu\nu}\,(n)$ defined above is covariant, and this can be seen through the covariant transformations of the operators $\varUpsilon_\mu(n)$ its encompasses, which will be shown below. Considering that in the discrete space formulation described the invariance is with respect to the local rotations, with the vielbeins $\mathbf{v}_a$ transforming in the vector representation

$$\widetilde{\mathbf{v}}_a(n) = \mathcal{R}_a^b(n)\mathbf{v}_b(n), \quad (3.77)$$

under the action the rotation group $SO(n)$. Then under the action of this group of symmetries, the spin connection group element transforms as

$$\varOmega_\nu(n) \to \widetilde{\varOmega}_\nu(n) = \mathcal{R}(n)\varOmega_\nu(n)\mathcal{R}^{-1}(n+1_\nu), \quad (3.78)$$

where $\mathcal{R}(n)$ in this case must be taken in the spinor representation. From here onwards, in order to simplify the notation, the unit shift in the argument indicating the cell will written as $1_\mu \to \hat{\mu}$, indicating for instance $\mathcal{R}^{-1}(n+1_\nu) \equiv \mathcal{R}^{-1}(n+\hat{\nu})$. Next, to find how the rotation-shift operators $\varUpsilon_\mu(n)$ that were introduced earlier transform under rotations, one can introduce into the expression for $\widetilde{\varOmega}_\nu(n)$ the shift operator by letting it act on a function $f(n+\hat{\nu}) = \mathbf{E}_\nu(n) f(n)$ through

$$\begin{aligned}\widetilde{\Omega}_\nu(n)\Big(f(n+\hat{\nu})\Big) &= \widetilde{\Omega}_\nu(n)\mathbf{E}_\nu(n)f(n)\\ &= \mathcal{R}(n)\Omega_\nu(n)\mathcal{R}^{-1}(n+\hat{\nu})\Big(f(n+\hat{\nu})\Big)\\ &= \mathcal{R}(n)\Omega_\nu(n)\mathbf{E}_\nu(n)\mathcal{R}^{-1}(n)f(n),\end{aligned} \tag{3.79}$$

and from this expression the transformation of the operators $\Upsilon_\mu(n)$ can be found

$$\Upsilon_\mu(n) \to \widetilde{\Upsilon}_\mu(n) = \mathcal{R}(n)\Upsilon_\mu(n)\mathcal{R}^{-1}(n), \tag{3.80}$$

resulting in the covariant transformation of the curvature,

$$\tilde{R}_{\mu\nu} = \mathcal{R}(n)R_{\mu\nu}(n)\mathcal{R}^{-1}(n). \tag{3.81}$$

In an expanded form, the curvature $R_{\mu\nu}(n)$ can be written as

$$R_{\mu\nu}(n) = \frac{1}{2\ell^\mu\ell^\nu}\Big(\Omega_\mu(n)\Omega_\nu(n+\hat{\mu})\Omega_\mu^{-1}(n+\hat{\nu})\Omega_\nu^{-1}(n) - (\mu \leftrightarrow \nu)\Big). \tag{3.82}$$

It is important to note that for the symmetry groups $SO(d)$ considered, for dimensions $d = 2, 3, 4$, the curvature as expressed in (3.82) is in fact an element of the Lie algebra of the respective group which becomes evident when the group elements Ω_μ are expanded,

$$(\exp(\omega_\mu(n))\exp(\omega_\nu(n+\hat{\mu})\exp(-\omega_\mu(n+\hat{\nu}))\exp(-\omega_\nu(n)))\,) - \mu \leftrightarrow \nu \ \in \ \text{Lie Algebra}$$

and so the components for curvature can be obtained from the expression

$$R_{\mu\nu}(n) = R_{\mu\nu}{}^{ab}(n)J_{ab}, \tag{3.83}$$

with the rotation generators J_{ab} taken in the spinor representation presented earlier.

Torsion Condition — In the absence of the translational group in the symmetry considered so far, the zero-torsion condition which allows one to define the spin connections $\omega_\nu^{ab}(n)$ in terms of the $e_{a\nu}$, can be expressed through the affine connection as

$$\Gamma^\gamma_{\mu\nu} - \Gamma^\gamma_{\nu\mu} = 0,$$

which gives the following equation from which one can compute $\omega_\nu^{ab}(n)$, after using the definition obtained in (3.70),

$$\Big((\Omega_\nu(n))_a^b e_{b\mu}(n+\hat{\nu}) - e_{a\mu}(n)\Big) - (\mu \leftrightarrow \nu) = 0. \tag{3.84}$$

In the spinor representation, one can define the curved gamma matrices e_μ through connecting the soldering forms to the Clifford algebra

$$e_\mu(n) \equiv e_\mu^a(n)\,\gamma_a, \tag{3.85}$$

and the condition for zero-torsion can be written for the matrices e_μ in an analogue to the expression in (3.84). Starting from the metricity condition (3.12), it can be written in the following form for e_ν [19],

$$\Upsilon_\mu(n) e_\nu(n) \Upsilon_\mu^{-1}(n) - e_\nu(n) = \Gamma^\rho_{\mu\nu}(n) e_\rho(n), \tag{3.86}$$

with the left hand side obtained from the spin connection group elements $\Omega_\nu(n)$ using the relation

$$e^\omega e_\nu e^{-\omega} = e_\nu + [\omega, e_\nu] + \frac{1}{2!}[\omega,[\omega,e_\nu]] + \cdots \frac{1}{m!}[\omega, \cdots [\omega, e_\nu]] + \cdots . \tag{3.87}$$

keeping in mind that applying $\Upsilon_\mu(n) \equiv \Omega_\mu(n)\mathbf{E}_\mu(n)$ to an intermediate trial function $f(n)$ gives

$$\Upsilon_\mu(n) e_\nu(n) \Upsilon_\mu^{-1}(n) = \Omega_\mu(n) e_\nu(n+\hat{\mu}) \Omega_\mu^{-1}(n). \tag{3.88}$$

Then the expression in (3.86) gives the zero-torsion expression for e_μ as

$$(\Upsilon_\nu(n) e_\mu(n) \Upsilon_\nu^{-1}(n) - e_\mu(n)) - (\mu \leftrightarrow \nu) = 0, \tag{3.89}$$

which can be otherwise be case in the form

$$T^a_{\mu\nu}\gamma_a = 0. \tag{3.90}$$

It is important to note here that the expression for torsion given in (3.89) which was obtained in the original formulation [19] is in fact not complete, as Torsion must be computed when translations enter into the group element $\Omega_\nu(n)$, and will be re-derived fully in the context of diffeomorphism invariance that this part of the thesis studied. But for invariance under rotations, it served as a good approximation to determine the spin connection explicitly from the soldering forms in the given cell and its surrounding when considering the example of SO(2) in [19], from the conditions provided in (3.90).

3.3.4 Actions in Discrete Gravity

Dirac Action — To obtain a discrete expression of the Dirac action, the hermitian Dirac operator must be defined such that the usual equation can be recovered in the continuous manifold limit. Starting with a continuous action that couples gravity to spinors, with the symmetry tangent group $SO(d)$, one has

$$S = \int d^4 x e \psi^\dagger D \psi, \tag{3.91}$$

and the Dirac operator in this continuous case is given by

$$D = \gamma^a e_a^\mu (\partial_\mu + \omega_\mu), \tag{3.92}$$

with $\omega_\mu = \frac{1}{4}\omega_\mu^{ab} J_{ab}$, and J_{ab} was given in (3.72). Then, the first naive attempt at writing the Dirac action is by taking the expression inspired by Lattice gauge theory,

$$\sum_n \sum_\mu \frac{i}{2} \psi^\dagger(n) \gamma^\mu (\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})), \tag{3.93}$$

which can be simply written in the language of the shift operators as

$$\sum_n \sum_\mu \frac{i}{2} \psi^\dagger(n) \gamma^\mu (\mathbf{E}_\mu(n)\psi(n) - \mathbf{E}^{-1}(n)\psi(n)), \tag{3.94}$$

and is not invariant under $SO(d)$. To introduce the spin connections into the picture, the following definition for the inner product was considered,

$$(\psi, \psi) = \sum_n \psi^\dagger(n)\psi(n), \tag{3.95}$$

and taking into account that the spinor $\psi(n)$ must transform under the tangent group as a spinor of $SO(d)$,

$$\psi(n) \to \mathcal{R}(n)\psi(n), \tag{3.96}$$

where

$$\mathcal{R}(n) = \exp\left(\frac{1}{4}\lambda^{ab}(n)\gamma_{ab}\right) \tag{3.97}$$

with the parameters $\lambda^{ab}(n)$ taken to be real. The connection $\Upsilon_\mu(n) = \Omega_\mu(n) E_\mu(n)$ was incorporated into the discrete candidate for the Dirac operator with the ansatz [19]

$$D(n) \equiv i v(n) \bar{e}^\mu(n) (\Upsilon_\mu(n) - \Upsilon_\mu^{-1}(n)), \tag{3.98}$$

with $\bar{e}^\mu(n) \equiv \bar{e}^\mu_b(n)\gamma^b$ connecting the Euclidean γ^b to the discrete manifold through $e^\mu_b(n)$. The density function $v(n)$ will become the determinant of e^b_μ only in the continuous limit, and in the discrete setting it is obtained from the requirement that the Dirac operator D is hermitian with respect to the inner product such that

$$(\psi, D\psi) = (D\psi, \psi), \tag{3.99}$$

one can compute first

$$\begin{aligned}(\psi, D\psi) &= \sum_n \Big(\psi(n), iv(n)\bar{e}^\mu(n)(\Upsilon_\mu(n) - \Upsilon^{-1}_\mu(n))\psi(n)\Big) \\ &= \sum_n iv(n)\psi^\dagger(n)\bar{e}^\mu(n)\left(\Omega_\mu(n)\psi(n+\hat{\mu}) - \Omega^{-1}_\mu(n-\hat{\mu})\psi(n-\hat{\mu})\right),\end{aligned} \tag{3.100}$$

and similarly one obtains for

$$\begin{aligned}(D\psi, \psi) &= \sum_n \Big(iv(n)\bar{e}^\mu(n)(\Upsilon_\mu(n) - \Upsilon^{-1}_\mu(n))\psi(n)\Big)^\dagger \psi(n) \\ &= \sum_n -i\Big(v(n+\hat{\mu})\psi^\dagger(n)\Omega_\mu(n)\bar{e}^\mu(n+\hat{\mu})\psi(n+\hat{\mu}) \\ &\qquad - v(n-\hat{\mu})\psi^\dagger(n)\Omega^{-1}_\mu(n-\hat{\mu})\bar{e}^\mu(n-\hat{\mu})\psi(n-\hat{\mu}),\end{aligned} \tag{3.101}$$

where

$$D^\dagger(n) = i(\Upsilon_\mu(n) - \Upsilon^{-1}_\mu(n))v(n)\bar{e}^\mu(n). \tag{3.102}$$

Therefore, comparing the two expressions (3.100) and (3.101), one finds that $v(n)$ satisfies the equation

$$v(n)\bar{e}^\mu(n)\Omega_\mu(n) = v(n+\hat{\mu})\Omega_\mu(n)\bar{e}^\mu(n+\hat{\mu}), \tag{3.103}$$

From these proposed definitions, the discrete version for the Dirac Action was obtained as [19]

$$S = \sum_n i\psi^\dagger(n)v(n)\bar{e}^\mu(n)(\Upsilon_\mu(n) - \Upsilon^{-1}_\mu(n))\psi(n). \tag{3.104}$$

Discrete Action for Gravity — The discrete action for gravity in Euclidean space is hence written as [19]

$$S = \sum_n v(n)R(n), \tag{3.105}$$

which is gauge invariant, and the scalar curvature can be obtained by contracting with the soldering forms in the usual way

$$R(n) = R_{\mu\nu}{}^{ab}(n) e^{\mu}_{a}(n) e^{\nu}_{b}(n). \tag{3.106}$$

3.4 Poincaré Invariance in Discrete Gravity

In this section, we will develop our new extension [2] of the previously described formulation of Discrete Gravity, where the symmetry group will be expanded to include translations and attempt to replace diffeomorphism invariance with translational invariance on the lattice. New discrete expressions for the curvatures involved will be computed, together with the transformations of the spin connections and soldering forms for the case of a two-dimensional space.

3.4.1 Extension of Tangent Space and Group Contraction

In order to implement Poincaré invariance into the discrete space through the inhomogeneous group $ISO(d)$, one can start with the rotation group in one dimension higher, $SO(d+1)$ instead of $SO(d)$, then perform a contraction. First, recalling the generators of rotation J_{AB} where

$$[J_{AB}, J_{CD}] = \delta_{BC} J_{AD} - \delta_{AC} J_{BD} - \delta_{BD} J_{AC} + \delta_{AD} J_{BC}, \tag{3.107}$$

with indices $A = a, \cdots, d+1$, and $a = 1, \cdots, d$. Isolating the $(d+1)^{th}$ element of J_{AB} in the following way,

$$J_{a(d+1)} = r P_a, \tag{3.108}$$

as to isolate the generator of translation from the higher dimension element of the rotation generators J_{AB}, so the commutation relations for it become

$$[J_{a(d+1)}, J_{b(d+1)}] = -J_{ab}, \tag{3.109}$$

$$[J_{ab}, J_{c(d+1)}] = \delta_{bc} J_{a(d+1)} - \delta_{ac} J_{b(d+1)}, \tag{3.110}$$

and the commutation relations with generators P_a then become

$$[P_a, P_b] = -\frac{1}{r^2} J_{ab}, \tag{3.111}$$

$$[J_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b. \tag{3.112}$$

Effectively, using this contraction method and taking the limit $r \to \infty$, the Poincaré group is thus recovered from the $SO(d+1)$ group. It is worthwhile to mention that this

method was implemented into the geometric construction of *Supergravity* with $N = 1$ as a gauge symmetry of the supersymmetry algebra after it was discovered in 1976, and is considered an elegant way to derive the supergravity action [16].
Now, one can define the begin by defining the spin connection on the extended tangent space with the rotation group $SO(n+1)$, where it can be expanded as

$$\frac{1}{4}\Delta x^{\mu}\omega_{\mu}^{AB}\gamma_{AB} = \frac{1}{4}\Delta x^{\mu}\omega_{\mu}^{ab}\gamma_{ab} + \frac{1}{2r}\Delta x^{\mu}e_{\mu}^{a}(n)\gamma_{a}\gamma, \tag{3.113}$$

where the the sum is over the Latin indices only, and $a, b = 1, ..., d$. For the Clifford algebra basis $\gamma = \gamma_{d+1}$, the following relations are satisfied

$$\gamma^2 = 1, \quad \{\gamma, \gamma^a\} = 0. \tag{3.114}$$

The spin connection group element $\Omega_{\mu}(n)$ in a given cell n in this splitting becomes

$$\Omega_{\mu}(n) = \exp\Big(\omega_{\mu}(n) + \frac{1}{r}e_{\mu}(n)\Big), \tag{3.115}$$

with the following definitions for $\omega_{\mu}(n)$ and $e_{\mu}(n)$

$$\omega_{\mu}(n) = \frac{1}{4}\Delta x^{\mu}\omega_{\mu}^{ab}\gamma_{ab}, \quad e_{\mu}(n) = \frac{1}{2r}\Delta x^{\mu}e_{\mu}^{a}(n)\gamma_{a}\gamma. \tag{3.116}$$

The group element $U(n)$ is defined as

$$U(n) = exp\Big(\lambda(n) + \frac{1}{r}\zeta(n)\Big), \tag{3.117}$$

where

$$\lambda(n) = \frac{1}{4}\lambda^{ab}(n)\gamma_{ab}, \quad \zeta(n) = \frac{1}{2r}\zeta^{a}(n)\gamma_{a}\gamma. \tag{3.118}$$

Recalling that the connection $\Upsilon_{\mu}(n) = \Omega_{\mu}(n)E_{\mu}(n)$ transforms under the action of this group as

$$\Upsilon'_{\mu} = U(n)\Upsilon_{\mu}(n)U^{-1}(n), \tag{3.119}$$

and acting through with the shift operator, one recovers the transformation of $\Omega_{\mu}(n)$ as

$$\Omega'_{\mu}(n) = U(n)\Omega_{\mu}(n)U^{-1}(n+\hat{\mu}), \tag{3.120}$$

and expanding the group elements the expression above becomes

$$e^{\omega'_{\mu}(n)+\frac{1}{r}e'_{\mu}(n)} = e^{\lambda(n)+\frac{1}{r}\zeta(n)}e^{\omega_{\mu}(n)+\frac{1}{r}e_{\mu}(n)}e^{-(\lambda(n+\hat{\mu})+\frac{1}{r}\zeta(n+\hat{\mu}))}. \tag{3.121}$$

Expanding the exponential — Since it is of interest to obtain the transformations of ω_μ and e_μ, one needs to expand fully the expression of the form

$$e^{\omega_\mu(n)+\frac{1}{r}e_\mu(n)} \equiv e^{X+Y}, \tag{3.122}$$

which is in principle complicated as it is given by the Zassenhaus formula

$$e^{X+Y} = e^X e^Y \prod_{n=2}^{\infty} e^{C_n(X,Y)}, \tag{3.123}$$

with $C_n(X,Y)$ being a homogeneous Lie polynomial of degree n in X and Y, and the formula is only known to order 9. But in this case, the result that is of of interest is after the group contraction is applied, which is occurring in the limit $r \to \infty$, and hence one can use the expansion that was obtained by Volkin [22],

$$e^{\omega(n)+\frac{1}{r}e(n)} = e^{\omega(n)}\Big(1+\frac{1}{r}(e-\frac{1}{2!}[\omega,e]+\frac{1}{3!}[\omega,[\omega,e]]-\frac{1}{4!}[\omega,[\omega,[\omega,e]]])+O(\frac{1}{r^2})\Big), \tag{3.124}$$

and in this case it is valid to all orders. Obtaining a closed-form expression for the infinite series given above is in principle possible for $SO(d+1)$ groups, but in the process of derivations it turns out the expressions for $d = 3, 4$ were not transparent. In the following, the procedure will be shown for the space of dimension $d = 2$, as the higher dimensional cases will require further development.

3.4.2 Group Compactification in Two-Dimensional Space

The goal will be now to find discrete expressions for the transformation of the *zweibein* and its Torsion expression in a 2-dimensional space, by contracting the $SO(3)$ group on the tangent space following the previous procedure.

The expansion of a general Lie group element $\Omega_\mu(n)$ in 3−dimensions is well known and is given by the formula

$$e^{\frac{i}{2}\ell^\mu A^i_\mu(n)\sigma_i} = \cos\frac{1}{2}\ell^\mu A_\mu(n) + i\frac{A^i_\mu}{A_\mu}\sin\frac{1}{2}\ell^\mu A_\mu(n)\sigma_i, \tag{3.125}$$

where the field in this case splits as

$$A^i_\mu(n)\sigma_i = \omega_\mu(n)\sigma_3 + \frac{2}{r}e^a_\mu(n)\sigma_a, \tag{3.126}$$

and for $SO(3)$ the group generators are given by

$$\omega_\mu \sigma_3 = \omega_\mu^{ab}\gamma_{ab}, \quad e_\mu^a \sigma_a = \omega_\mu^{a3}\gamma_{a3}. \tag{3.127}$$

The magnitude A_μ is computed as

$$(A_\mu)^2 = (\omega_\mu(n))^2 + \frac{4}{r^2} e_\mu^a e_\mu^a \tag{3.128}$$

with the summation on the Latin indices only, and then by expanding it in a power series in r for the limit $r \to \infty$,

$$\begin{aligned} A_\mu(n) &= \omega_\mu(n)\sqrt{1 + \frac{4}{r^2}\frac{e_\mu^a(n) e_\mu^a(n)}{(A_\mu)^2}} \\ &= \omega_\mu + \mathcal{O}(\frac{1}{r^2}), \end{aligned} \tag{3.129}$$

and so up to order $\mathcal{O}(\frac{1}{r^2})$ the magnitude in (3.125) will be estimated by $A_\mu \simeq \omega_\mu$. The Lie group element (3.115) for the contracted $SO(3)$ group then becomes

$$\Omega_\mu(n) = e^{\frac{i}{2}\ell^\mu \omega_\mu(n)\sigma_3} + \frac{2i}{r}\frac{e_\mu^a(n)}{\omega_\mu(n)} \sin\frac{1}{2}\ell^\mu \omega_\mu(n)\ \sigma_a. \tag{3.130}$$

Transformation of ω_μ and e_μ — Now, the transformations for ω_μ and e_μ^a will be obtained from expanding

$$\Omega'_\mu(n) = e^{\frac{i}{2}\ell^\mu \omega'_\mu(n)\sigma_3} + \frac{2i}{r}\frac{e'^a_\mu(n)}{\omega'_\mu(n)} \sin\frac{1}{2}\ell^\mu \omega'_\mu(n)\ \sigma_a, \tag{3.131}$$

under gauge transformations

$$\Omega'_\mu(n) = U(n)\Omega_\mu(n)U^{-1}(n+\hat{\mu}).$$

A similar expansion as in (3.130) can be written for the group element $U(n)$, where one obtains

$$U(n) = e^{\frac{i}{2}\lambda(n)\sigma_3} + \frac{2i}{r}\frac{\zeta^a(n)}{\lambda(n)} \sin\frac{1}{2}\lambda(n)\ \sigma_a. \tag{3.132}$$

Expanding the right hand side in (3.120) to orders $1/r$, and plugging in (3.132), one gets

$$\Omega'_\mu(n) = \left(e^{\frac{i}{2}\lambda(n)\sigma_3} + \frac{2i}{r}\frac{\zeta^a(n)}{\lambda(n)}\sin\frac{1}{2}\lambda(n)\ \sigma_a\right)\left(e^{\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3} + \frac{2i}{r}\frac{e^a_\mu(n)}{\omega_\mu(n)}\sin\frac{1}{2}\ell^\mu\omega_\mu(n)\ \sigma_a\right)$$
$$\left(e^{-\frac{i}{2}\lambda(n+\hat{\mu})\sigma_3} + \frac{2i}{r}\frac{\zeta^a(n+\hat{\mu})}{\lambda(n+\hat{\mu})}\sin\frac{1}{2}\lambda(n+\hat{\mu})\ \sigma_a\right), \tag{3.133}$$

and separating terms with respect to their r-dependence, one gets

$$\Omega'_\mu(n) = e^{\frac{i}{2}(\lambda(n)-\lambda(n+\hat{\mu})+\ell^\mu\omega_\mu(n))\sigma_3} + \frac{2i}{r}\Big(e^{\frac{i}{2}(\lambda(n)-\lambda(n+\hat{\mu})\sigma_3}\ \bar{e}^a_\mu(n) \tag{3.134}$$
$$+ e^{\frac{i}{2}(\lambda(n+\hat{\mu})-\ell^\mu\omega_\mu(n))\sigma_3}\ \bar{\zeta}^a(n) + e^{-\frac{i}{2}(\lambda(n)+\ell^\mu\omega_\mu(n))\sigma_3}\ \bar{\zeta}^a(n+\hat{\mu})\Big)\sigma_a,$$

and to make the expression more compact, the following definitions were introduced for e^a_μ and $\zeta(n)$

$$\bar{e}^a_\mu(n) \equiv \frac{e^a_\mu(n)}{\omega_\mu(n)}\sin\Big(\frac{1}{2}\ell^\mu\omega_\mu(n)\Big), \qquad \bar{\zeta}(n) \equiv \frac{\zeta^a(n)}{\lambda(n)}\sin\Big(\frac{1}{2}\lambda(n)\Big). \tag{3.135}$$

Now, equating the expansion in (3.134) with the definition in (3.131), for the r-dependent parts one gets

$$\ell^\mu\omega'_\mu(n) = \ell^\mu\omega_\mu(n) + \lambda(n) - \lambda(n+\mu), \tag{3.136}$$

from which an explicit discrete transformation for the spin connection $\omega_\mu(n)$ is obtained with $\ell^\mu \equiv 1$

$$\omega'_\mu(n) = \omega_\mu(n) - \Delta_\mu\lambda. \tag{3.137}$$

To get the transformation for the zweibein e^a_μ, the r-dependent part is now considered

$$\frac{e'^a_\mu}{\omega'_\mu(n)}\sin\frac{1}{2}\ell^\mu\omega'_\mu(n)\ \sigma_a = \Big(e^{-\frac{i}{2}\ell^\mu\Delta_\mu\lambda(n)\sigma_3}\bar{e}^a_\mu(n) + e^{\frac{i}{2}(\lambda(n+\hat{\mu})-\ell^\mu\omega_\mu(n))\sigma_3}\bar{\zeta}^a(n)$$
$$- e^{\frac{i}{2}(\lambda(n)+\ell^\mu\omega_\mu(n))\sigma_3}\bar{\zeta}^a(n+\hat{\mu})\Big)\sigma_a. \tag{3.138}$$

In order to simplify this expression, the rotation parameter $\lambda \to 0$ will be taken as a gauge choice, since for the purpose of this exploration, only the transformation under translation is of interest. Then, the transformation of ω_μ in (3.137) is substituted into (3.138), and an exact solution is then obtained

$$e'^a_\mu(n)\sigma_a = \Big(e^a_\mu(n) + \frac{\omega_\mu(n)}{\sin\frac{1}{2}\ell^\mu\omega_\mu(n)}\Big(e^{-\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3}\zeta_a(n) - e^{\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3}\zeta_a(n+\mu)\Big)\Big)\sigma_a. \tag{3.139}$$

The transformation of the zweibein $e^a_\mu(n)$ can expressed in terms of real fields through implementing the following identity

$$e^{i\alpha\sigma_3}\beta_a\sigma_a = (\cos\alpha\beta^a + \sin\alpha\epsilon^{ab}\beta^b)\sigma_a, \tag{3.140}$$

to finally arrive at the expression for the discrete transformation of $e^a_\mu(n)$

$$e'^a_\mu(n) = e^a_\mu(n) - \ell^\mu\omega_\mu(n)\cot\frac{1}{2}\ell^\mu\omega_\mu(n)\Delta_\mu\zeta(n) - \omega_\mu(n)\epsilon^{ab}\Big(\zeta^b(n) + \zeta^b(n+\mu)\Big). \tag{3.141}$$

In the above expression one has to substitute the solution of $\omega_\mu(n)$ as function of $e^a_\mu(n)$, however, the best one can achieve is to develop a perturbative expansion of the expression as a function of ℓ^μ, as the substitution can only be implemented numerically with the torsion equation being a transcendental equation.

Transformation of the metric $g_{\mu\nu}$ — For completion, the discrete transformation of the metric can be directly computed from (3.141) as

$$\begin{aligned} g'_{\mu\nu}(n) = g_{\mu\nu}(n) &- \ell^\mu\omega_\mu(n)\cot\frac{1}{2}\ell^\mu\omega_\mu(n)\Delta_\mu\zeta^a(n)e^a_\nu(n) \\ &- \ell^\nu\omega_\nu(n)\cot\frac{1}{2}\ell^\nu\omega_\nu(n)\Delta_\nu\zeta^a(n)e^a_\mu(n) \\ &- \omega_\mu(n)\epsilon^{ab}(\zeta^b(n) + \zeta^b(n+\hat{\mu}))e^a_\nu(n) \\ &- \omega_\nu(n)\epsilon^{ab}(\zeta^b(n) + \zeta^b(n+\hat{\nu}))e^a_\mu(n) + O(\zeta^a)^2. \end{aligned} \tag{3.142}$$

3.4.3 Curvature and Torsion in $ISO(d)$

Now the goal will be to compute the curvature expression and torsion for 2-dimensional lattice starting with the tangent group $SO(3)$. Recalling that the curvature expression is

$$\begin{aligned} \Theta_{\mu\nu}(n) &= \frac{1}{2\ell^\mu\ell^\nu}\Big(\varUpsilon_\mu(n)\varUpsilon_\nu(n)\varUpsilon^{-1}_\mu(n)\varUpsilon^{-1}_\nu(n) - \mu\longleftrightarrow\nu\Big) \\ &= \frac{1}{2\ell^\mu\ell^\nu}\Big(\Omega_\mu(n)\Omega_\nu(n+\hat{\mu})\Omega^{-1}_\mu(n+\hat{\nu})\Omega^{-1}_\nu(n) - \mu\longleftrightarrow\nu\Big), \end{aligned}$$

it is convenient to define the element $P_{\mu\nu}$ such that

$$\begin{aligned}
P_{\mu\nu}(n) &\equiv \Omega_\mu(n)\Omega_\nu(n+\hat{\mu})\Omega_\mu^{-1}(n+\hat{\nu})\Omega_\nu^{-1}(n) \qquad (3.143)\\
&= \left(e^{\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3} + \frac{2i}{r}\frac{e_\mu^a(n)}{\omega_\mu(n)}\sin\frac{1}{2}\ell^\mu\omega_\mu(n)\ \sigma_a\right)\\
&\quad \left(e^{\frac{i}{2}\ell^\nu\omega_\nu(n+\hat{\mu})\sigma_3} + \frac{2i}{r}\frac{e_\nu^a(n+\hat{\mu})}{\omega_\nu(n+\hat{\mu})}\sin\frac{1}{2}\ell^\nu\omega_\nu(n+\hat{\mu})\ \sigma_a\right)\\
&\quad \left(e^{-\frac{i}{2}\ell^\mu\omega_\mu(n+\hat{\nu})\sigma_3} - \frac{2i}{r}\frac{e_\nu^a(n+\hat{\nu})}{\omega_\mu(n+\hat{\nu})}\sin\frac{1}{2}\ell^\mu\omega_\mu(n+\hat{\nu})\ \sigma_a\right)\\
&\quad \left(e^{-\frac{i}{2}\ell^\nu\omega_\nu(n)\sigma_3} - \frac{2i}{r}\frac{e_\nu^a(n)}{\omega_\nu(n)}\sin\frac{1}{2}\ell^\nu\omega_\nu(n)\ \sigma_a\right), \qquad (3.144)
\end{aligned}$$

then by expanding this product up to order $\frac{1}{r}$, and separating terms based on r-dependence, one can write $P_{\mu\nu}(n)$ as

$$P_{\mu\nu}(n) = P_{\mu\nu}^{(0)} + \frac{2i}{r}P_{\mu\nu}^{(r)}, \qquad (3.145)$$

with its r-independent term given by

$$P_{\mu\nu}^{(0)}(n) = e^{\frac{i}{2}(\ell^\mu\omega_\mu(n)+\ell^\nu\omega_\nu(n+\hat{\mu})-\ell^\mu\omega_\mu(n+\hat{\nu})-\ell^\nu\omega_\nu(n))\sigma_3}, \qquad (3.146)$$

and similarly the r-related term is

$$\begin{aligned}
P_{\mu\nu}^{(r)} &= \left(e^{\frac{i}{2}(-\ell^\nu\omega_\nu(n+\hat{\mu})+\ell^\mu\omega_\mu(n+\hat{\nu})+\ell^\nu\omega_\nu(n))\sigma_3}\bar{e}_\mu^a(n) - \mu \longleftrightarrow \nu\right)\\
&\quad + \left(e^{\frac{i}{2}(\ell^\mu\omega_\mu(n)+\ell^\mu\omega_\mu(n+\hat{\nu})+\ell^\nu\omega_\nu(n))\sigma_3}\bar{e}_\nu^a(n+\hat{\mu}) - \mu \longleftrightarrow \nu\right)\sigma^a, \qquad (3.147)
\end{aligned}$$

From $\Theta_{\mu\nu}(n)$ one can then deduce the discrete curvature expression, which in the compact notation is written as

$$R_{\mu\nu}(n) = \frac{1}{\ell^\mu\ell^\nu}(P_{\mu\nu}^0 - P_{\nu\mu}^0) = \frac{4i}{\ell^\mu\ell^\nu}\sin\frac{\ell^\mu\ell^\nu}{2}\Big(\Delta_\mu\omega_\nu(n) - \Delta_\nu\omega_\mu(n)\Big)\sigma_3. \qquad (3.148)$$

Finally, the discrete torsion expression can be written compactly using (3.147) as

$$T_{\mu\nu}^a\sigma_a = \frac{1}{\ell^\mu\ell^\nu}(P_{\mu\nu}^{(r)} - P_{\nu\mu}^{(r)}) = \frac{2}{\ell^\mu\ell^\nu}P_{\mu\nu}^{(r)}, \qquad (3.149)$$

which in this case was obtained by including translations into the symmetry group.

3.4.4 Continuous Limits in the Discrete Gravity Formalism

To recollect the picture, the discrete space setup consisted of cells with a minimal volume that was set to unity, with each cell labeled by a set of integers. In order to connect the discrete picture with the continuous limit, the procedure in [19] will be first outlined, then applied to the case with Poincaré invariance explored in this thesis. To get the continuous manifold limit, the volume of the discrete cells must shrink to zero. For that, one can start by transforming the integer labels into coordinate variables

$$x^\mu = \epsilon n^\mu, \tag{3.150}$$

then the continuous limit $\epsilon \to 0$ has to be taken in such a way so that all discrete cells do not shrink into one single point. To fix the obtained points then, a given point x^μ will remain finite in the continuous limit only when $n \to \infty$.

For the Shift Operator — The definition introduced earlier for the shift operator acting on a function then becomes

$$\mathbf{E}_\nu(n)f(n) = f(n + 1_\nu) \;\to\; \mathbf{E}_\nu(x)f(x) = f(x + \epsilon_\nu), \tag{3.151}$$

were now the coordinate $x \equiv (x^1, ..., x^d)$ replaces the cell numbering and $\epsilon_\nu \equiv \epsilon 1_\nu$ indicates the infinitesimal change in a given d coordinate, such that

$$\mathbf{E}_m(x)f(x) = f(x^1, ..., x^m + \epsilon_m, ..., x^d).$$

For the Tangent Operators — Using the above definition in the action of the tangent operator on a function $f(x)$, and restoring ϵ for the finite difference definition, the tangent operator becomes the partial differential operator in the limit $\epsilon \to 0$

$$\mathbf{e}_\nu(x)f(x) = \lim_{\epsilon\to 0} \frac{f(x+\epsilon_\nu) - f(x-\epsilon_\nu)}{2\epsilon_\nu} = \frac{\partial}{\partial x_\nu} f(x), \tag{3.152}$$

i.e., on a given coordinate line of the continuous manifold the operator $\mathbf{e}_\nu(x)$ is simply the tangent vector [19]

$$\mathbf{e}_\nu(x) \equiv \frac{\partial}{\partial x_\nu}. \tag{3.153}$$

For Parallel Transport — Restoring ϵ in the parallel transport expression (3.64),

$$\mathbf{e}_\mu^{p.t.}(x+\epsilon_\nu \to x) = \mathbf{e}_\mu(x) + \Gamma^\gamma_{\mu\nu}(x)\mathbf{e}_\gamma(x)\epsilon_\nu, \tag{3.154}$$

which simply recovers the known expression for the covariant derivative, as inspired earlier in the derivation through taking the limit $\epsilon \to 0$

$$\lim_{\epsilon \to 0} \frac{\mathbf{e}_\mu^{p.t.}(x+\epsilon_\nu \to x) - \mathbf{e}_\mu(x)}{\epsilon_\nu} = \Gamma^\gamma_{\mu\nu}(x)\mathbf{e}_\gamma, \tag{3.155}$$

hence,

$$\nabla_\nu \mathbf{e}_\mu = \Gamma^\gamma_{\mu\nu}\mathbf{e}_\gamma. \tag{3.156}$$

Similarly, by expanding the spin connection group element in ϵ

$$\Omega_\nu(x) = \exp\left(\epsilon_\nu \omega_\nu^{cd}(x) J_{cd}\right),$$

one obtains for $\Omega_\nu(x)$ and its inverse

$$\Omega_\nu^{-1}(x) = 1 + \epsilon_\nu \omega_\nu^{cd} J_{cd}, \quad \Omega_\nu^{-1}(x) = 1 - \epsilon_\nu \omega_\nu^{cd} J_{cd}, \tag{3.157}$$

ignoring higher order terms in ϵ and plugging in the expression

$$\mathbf{v}_a^{p.t.}(x+\epsilon_\nu \to x) = (\Omega_\nu^{-1}(x))_a^b \mathbf{v}_b(x), \tag{3.158}$$

and rearranging then taking the limit in similar steps as for the tangent operator, one obtains the needed covariant derivative expression for the vielbein

$$\nabla_\nu \mathbf{v}_a = \omega_{\nu a}{}^b \mathbf{v}_b. \tag{3.159}$$

For the Curvature — In the limit $\epsilon \to 0$, the curvature expression can be expanded using the group elements obtained in (3.157)

$$\begin{aligned} R_{\mu\nu}(x) &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon_\mu \epsilon_\nu}\left(\Omega_\mu(x)\Omega_\nu(x+\epsilon_\mu)\Omega_\mu^{-1}(x+\epsilon_\nu)\Omega_\nu^{-1}(x) - (\mu \leftrightarrow \nu)\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon_\mu \epsilon_\nu}\Big((1+\epsilon_\mu\omega_\mu(x))(1+\epsilon_\nu\omega_\nu(x+\epsilon_\mu)(1-\epsilon_\mu\omega_\mu(x+\epsilon_\nu)(1-\epsilon_\nu\omega_\nu(x)) \\ &\qquad - (\mu \leftrightarrow \nu)\Big), \end{aligned} \tag{3.160}$$

and in the continuous limit one recovers the curvature in the form of

$$R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu], \tag{3.161}$$

where its components can be deduced from $R_{\mu\nu}(x) \equiv R_{\mu\nu}{}^{ab} J_{ab}$

$$R_{\mu\nu}{}^{cd}(x) = \partial_\mu \omega_\nu^{cd} - \partial_\nu \omega_\mu^{cd} + \omega_\mu^{cl} \omega_{\nu l}{}^{d} - \omega_\nu^{cl} \omega_{\mu l}{}^{d}. \tag{3.162}$$

which is the usual expression for the curvature associated with the spin connections.

For the case with Poincaré invariance

Now the same method is applied on the discrete expressions obtained from gauging the Poincaré group. Starting with the transformation of the soldering forms,

$$e'^a_\mu(x) = e^a_\mu(x) - \epsilon_\mu \omega_\mu(x) \cot \frac{1}{2} \epsilon_\mu \omega_\mu(x) \Delta_\mu \zeta(x) - \omega_\mu(x) \epsilon^{ab} \Big(\zeta^b(x) + \zeta^b(x + \epsilon_\mu) \Big),$$

in the continuous limit $\epsilon_\mu \to 0$, the e^a_μ transformation simply becomes

$$e'^a_\mu \to e^a_\nu - \partial_\mu \zeta^a - \omega_\mu \epsilon^{ab} \zeta^b. \tag{3.163}$$

For the curvature expression (3.148) obtained with $ISO(2)$ as the tangent group,

$$R_{\mu\nu}(x) = \frac{4i}{\epsilon_\mu \epsilon_\nu} \sin \frac{\epsilon_\mu \epsilon_\nu}{2} \Big(\Delta_\mu \omega_\nu(x) - \Delta_\nu \omega_\mu(x) \Big) \sigma_3,$$

the curvature components in the continuous limit are obtained in a straightforward way, however in rather more simplified form for the 2-dimensional space considered

$$R_{\mu\nu} \to \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \tag{3.164}$$

The torsion computed in (3.149) can be written explicitly as

$$\begin{aligned} T^a_{\mu\nu} \sigma_a = \frac{2}{\epsilon_\mu \epsilon_\nu} \Bigg(& e^{\frac{i}{2}(-\epsilon_\nu \omega_\nu(x+\hat{\mu}) + \epsilon_\mu \omega_\mu(x+\hat{\nu}) + \epsilon_\nu \omega_\nu(x))\sigma_3} \frac{e^a_\mu(x)}{\omega_\mu(x)} \sin\Big(\frac{1}{2} \epsilon_\mu \omega_\mu(x)\Big) \\ & + \Big(e^{\frac{i}{2}(\epsilon_\mu \omega_\mu(x) + \epsilon_\mu \omega_\mu(x+\hat{\nu}) + \epsilon_\nu \omega_\nu(x))\sigma_3} \frac{e^a_\nu(x+\epsilon_\mu)}{\omega_\nu(x+\epsilon_\mu)} \sin\Big(\frac{1}{2} \epsilon_\nu \omega_\nu(x+\epsilon_\mu)\Big) - (\mu \longleftrightarrow \nu) \Bigg) \sigma_a, \end{aligned} \tag{3.165}$$

then expanding for small ϵ, the expression becomes

$$\begin{aligned}T^a_{\mu\nu}\sigma_a =&\frac{2}{\epsilon_\mu\epsilon_\nu}\Big(\frac{1}{2}\epsilon_\mu e^b_\mu(x)(\delta^a_b+\epsilon^{ab}(-\epsilon_\nu\omega_\nu(x+\epsilon_\mu)+\epsilon_\mu\omega_\mu(x+\epsilon_\nu)+\epsilon_\nu\omega_\nu(x))\\ &-\frac{1}{2}\epsilon_\mu e^b_\mu(x+\epsilon_\nu)(\delta^a_b+\epsilon^{ab}(\epsilon_\mu\omega_\mu(x)+\epsilon_\nu\omega_\nu(x+\epsilon_\mu)+\epsilon_\nu\omega_\nu(x))-(\mu\longleftrightarrow\nu)\Big)\sigma_a\\ =&\frac{1}{\epsilon_\nu}\Big((e^a_\mu(x)-e^a_\mu(x+\epsilon_n u))\\ &+\frac{1}{2}\epsilon^{ab}(\epsilon_\mu\omega_\mu(x+\epsilon_\nu)-\epsilon_\nu\omega_\nu(x+\epsilon_\mu)-\epsilon_\mu\omega_\mu(x)-\epsilon_\nu\omega_\nu(x+\epsilon_\mu))\Big)-(\mu\longleftrightarrow\nu),\end{aligned}$$

and from here, the components for torsion in the continuous limit $\epsilon\to 0$ are recovered as

$$T^a_{\mu\nu}\to(\partial_\mu e^a_\nu-\partial_\nu e^a_\mu+\epsilon^{ab}\omega_\mu e_{\nu b}-\epsilon^{ab}\omega_\nu e_{\mu b}). \tag{3.166}$$

For the Discrete Curvature Action — Finally, returning to the Discrete Action in [19], replacing the sum with an integral over the line element,

$$\epsilon\sum\longrightarrow\int dx,$$

from the following simple relation

$$\lim_{\epsilon\to 0}\sum_n\epsilon f(n)=\int f(x)dx, \tag{3.167}$$

the discrete action for gravity (3.105) in is therefore recovered below in the continuous limit

$$S=\int\det\Big(e^b_\mu\Big)R(x)dx^1...dx^d. \tag{3.168}$$

Therefore, this completes the proof that in this construction, the Discrete Gravity formalism considered successfully recovers the essential differential geometry elements in the continuous limit and in a manifest way.

3.4.5 Torsion corrections from the extended symmetry

Comparing now the torsion that was obtained in this thesis work by requiring invariance under the Poincaré group $ISO(d)$, with the old one [19] shown in Section (3.3.3) by requiring invariance under group of rotations $SO(d)$, which is quoted below

$$T^{a\,(old)}_{\mu\nu}(n) = \frac{1}{\ell^\mu}(\Upsilon_\mu(n)e_\nu(n)\Upsilon^{-1}_\mu(n) - e_\nu(n)) - \frac{1}{\ell^\nu}(\Upsilon_\nu(n)e_\mu(n)\Upsilon^{-1}_\mu(n) - e_\mu(n)), \quad (3.169)$$

it can be written in an expanded form by using the definition of a Lie group element in (3.125) into

$$\Upsilon_\mu(n)e_\nu(n)\Upsilon^{-1}_\mu(n) = e^{\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3}e^a_\nu(n+\hat{\mu})\sigma_a e^{-\frac{i}{2}\ell^\mu\omega_\mu(n)\sigma_3}, \quad (3.170)$$

hence for the sake of comparison the old torsion (3.169) takes the form

$$T^{a\,(old)}_{\mu\nu}(n) = \frac{1}{\ell^\mu}(\cos\ell^\mu\omega_\mu(n)e^a_\nu(n+\hat{\mu}) + \sin\ell^\mu\omega_\mu(n)\epsilon^{ab}e^b_\nu(n+\hat{\mu}) - e^a_\nu(n)) - (\mu\leftrightarrow\nu). \quad (3.171)$$

Rewriting now the new torsion expression obtained in (3.149) more explicitly as

$$\begin{aligned} T^a_{\mu\nu}\sigma_a =& \frac{2}{\ell^\mu\ell^\nu}\Big[e^{\frac{i}{2}(-\ell^\mu\ell^\nu\Delta_\mu\omega_\nu(n)+\ell^\mu\omega_\mu(n+\hat{\nu}))\sigma_3}\Big(\bar{e}^a_\mu(n) \quad (3.172)\\ & - e^{i(-\frac{1}{2}\ell^\mu\ell^\nu\Delta_\nu\omega_\mu(n)+\ell^\nu\omega_\nu(n+\hat{\mu}))\sigma_3}\bar{e}^a_\mu(n+\hat{\nu})\Big) - (\mu\longleftrightarrow\nu)\Big]\sigma_a, \end{aligned}$$

one finds that the torsion obtained from $ISO(d)$ includes corrections that extend to the zweibein which in this approach becomes rescaled

$$e^a_\mu(n) \;\to \bar{e}^a_\mu(n)\frac{\sin\left(\frac{1}{2}\ell^\mu\omega_\mu(n)\right)}{\omega_\mu(n)}e^a_\mu(n)$$

with the scaling factor being in fact small $\frac{\sin\frac{1}{2}\ell^\mu\omega_\mu(n)}{\frac{1}{2}\ell^\mu\omega_\mu(n)} < 1$. Other corrections that depend on the factor

$$e^{\frac{i}{2}\ell^\mu\ell^\nu\Delta_\mu\omega_\nu(n)\sigma_3}$$

can also be regarded as small for fine lattices. The most important corrections then come from the factor

$$e^{\frac{i}{2}(\ell^\mu\omega_\mu(n+\nu)-\ell^\nu\omega_\nu(n+\mu))\sigma_3}$$

which appears in the zero torsion condition. These corrections are important to include in computations, especially when implementing numerically this condition for solving the spin connections as functions of the soldering forms.

3.4.6 Can the Translational Invariance replace Diffeomorphism Invariance on a Discrete Lattice?

In the process of producing this work, there were many attempts at recreating a discrete version for the general coordinate transformations (GCT) explicitly without trying to appeal to the gauge-like formalism, where as a starting point, the idea was to utilize the shift operators in matrix combinations that would act on the functions which are defined on the lattice to produce in the continuous limit the proper equations of coordinate transformations, but the main problem which was not quite resolved was that the mixing of the action of shift operators would not output the function at the correct cell position.
Alternatively, and appealing to the idea that the the the GCT parameter with spacetime indices ζ^{μ} can be connected to the translation parameter of the group of translations with tangent space indices ζ^{a} through the soldering forms

$$\zeta^{a} = \zeta^{\mu} e^{a}_{\mu},$$

following the previous discussion in the continuous limit, it was possible to relate the transformations of the translational gauge fields δe^{a}_{μ} to those under general coordinate transformations $\delta' e^{a}_{\mu}$ through the relation quoted below

$$\delta e^{a}_{\mu} = \delta' e^{a}_{\mu} + \zeta^{\nu} T^{a}_{\mu\nu},$$

hence, finding the right expression for torsion in the discrete setting and enforcing the condition $T^{a}_{\mu\nu} = 0$ allows for appropriately relating the symmetries of General Relativity with the Poincaré group.
To this end, while in the first work [19] the authors tried to relate the freedom in enumerating the lattice as the remnant of this essential feature of general relativity, the torsion condition was derived without invoking the translation group into the picture. In this thesis work we took another conceptually step in that direction, considering replacing diffeomorphism invariance with the translational invariance, and relating their parameters through defining torsion as the curvature associated with the translational gauge field on the lattice

$$R_{\mu\nu}{}^{a}(e^{a}_{\mu}) = 0,$$

which was absent in the original formulation as the group of symmetries on the lattice was restricted to rotational invariance. In our approach, this procedure was done more explicitly invoking the translation group into the tangent space symmetry defined for the theory, and the torsion expression was derived more systematically, giving the properly corrected expression for torsion.

Alas, it remains an open discussion whether our alternative realization of diffeomorphism invariance on the lattice provides the final answer in discrete spaces, while its remnant, the freedom in choosing a slicing for the discrete lattice, remains inherent in the formalism as long as the connection between the cells keeps with the definition of the dimensionality of the discrete manifold.

Chapter 4

Conclusions and Future Outlook

In this thesis work we studied two fundamental problems that occur near the Planckian scales, starting first from the early Universe dynamics, down to the structural foundations of the gravitational theory. Studying how the universe evolves starting with a small patch at the Planck scale, which is the focus of the first part of the thesis, lends itself to the question of how space-time can emerge at these scales, knowing that the geometric description of the gravitational theory faces problems below Planck lengths. The natural framework to address this emergent problem then presents itself in a Discrete Gravity approach, which the second part of the thesis explored and expanded on both mathematical and conceptual ends.

In the first part of the thesis work, we showed how the long-standing problem of eternal inflation and the self-reproduction of the universe were evaded, and at the same time, the issue of fine-tuning of initial conditions was avoided in a class of models where inflation starts at the Planck curvature. This was achieved within a minimally modified gravity framework, Mimetic Gravity, by coupling the mimetic ϕ field to the Inflaton potential $V(\varphi)$, and obtaining modified dynamical equations that allowed for the violation of self-reproduction conditions. The coupling function $C(\kappa)$ in the Lagrangian of the Mimetic Model was taken to be linear in κ for large curvatures, ensuring that the kinetic energy of the inflaton field has the dominant contribution in the energy density, driving inflation at the onset, and generating scalar perturbations and gravitational waves with a comparable amplitude at these scales. On the observable scales, for the model studied, the tensor-to-scalar ratio of these perturbations agrees with the current bound on $r < 0.032$, and the computed spectral index n_s agrees with current measurements. Interestingly, the Lagrangian that was used modified Einstein's gravity only at high curvatures, where it is already known to face limitations, and the known degrees of freedom for gravity were not altered in this process, as the mimetic framework does not add any new dynamical degrees of freedom through the mimetic field, but rather adds a 'dust' component through its constrained degree of freedom. In fact, quickly after inflation begins, the dust component becomes negligible, but later it can be restored again to describe the dark matter component of the universe. While we have studied in this thesis a particular inflationary scenario for a specified potential, the Lagrangian laid out describes a whole class of inflationary models that can be explored given that the coupling function can be arbitrary at low curvatures. Hence this work restored the ground for predictability of the Inflationary theory which was debated earlier in [49] as discussed in the first chapter. The resolution of the fine-tuning problem and avoiding the Multiverse has not been shown in any other work.

The next question to address on this end would be: *how can coupling to the Mimetic field be further exploited in cosmological settings?* — The Lagrangian used in this work can be generalized by adding another potential $\tilde{V}(\varphi)$ that is different from the coupled one, or by having an extra coupling term to the scalar kinetic term, such as $g^{\mu\nu}D(\kappa)\partial_\mu\phi\partial_\nu\phi$, so that even when the potential of the field vanishes, the new coupling term can mimic different equations of state depending on the power of κ in the coupling. This could potentially address the Dark Energy problem today through an effective equation of state. Indeed, we are currently investigating this model as an extension of the Mimetic inflation model presented here. On the other hand, there are other interesting variants to be explored, such as a direct coupling of the scalar field to the mimetic field ϕ through its derivative of the form $g^{\mu\nu}E(\kappa)\partial_\mu F(\varphi)\partial_\nu\phi$, from which new and simple inflationary scenarios can be constructed. Hence, this new type of coupling through κ in the Lagrangian opens many possibilities for further investigation of the early and late-time evolution of the universe.

Turning to the structural end of issues associated with the Planckian scale, in the second part of the thesis, we explored a novel Discrete Gravity framework. Therein, we took a new step in advancing the conceptual and computational aspects of this formalism, by implementing the idea of having gravity as a gauge theory of the Poincaré group into the new discrete lattice formulation, inspired by Lattice gauge theory. This allowed for obtaining new modified expressions for the curvatures associated with the gauge fields, after imposing translational invariance. We achieved that by expanding the tangent space to include translations together with the rotational symmetry through the inhomogeneous rotational group $ISO(d)$, which was recovered by performing a contraction on the higher-dimension $(d+1)$ of the group $SO(d+1)$, taking its radius to infinity. As a result, we obtained the full discrete torsion expression, which is needed to calculate the spin connection elements, and allows to compute numerically the curvature for defined surfaces, as was done using the old definitions in [23, 24, 25]. This was done by considering two-dimensional spaces. By comparing our new curvature and torsion expressions to the old ones obtained in [19], the results show that the modifications are functions of higher order in lattice size. We also derived the discrete transformations for the spin connections and vielbein in the example for $ISO(d)$ invariance in two-dimensions. Finally, while diffeomorphism invariance is still not made manifest in discrete spaces, and was only reflected in the freedom in renumbering the discrete cells and choosing them, we took a new step in replacing this invariance by translational invariance, which was absent in the old formulation. It is noteworthy that in the used Discrete Gravity formalism we succeeded in recovering manifest continuous limits for the obtained expressions, which is a main advantage over other discrete approaches in literature.

Looking forward, as this Discrete Gravity formalism is being developed and expanded, more work is anticipated for finding the obtained expressions in two dimensions for higher dimensional spaces, and recovering discrete expressions for other mathematical relations found in General Relativity, for instance a form for the Bianchi identities for curvature as a next step. On the other hand, in line with our idea of expanding the tangent space for the discrete manifold, a worthwhile question becomes: *is it possible to achieve unification on the lattice?* — The unification of gravity with other gauge interactions on the lattice comes as a further step to lattice gauge theory that this Discrete Gravity formulation may allow for. By taking the current symmetries on the tangent space, a similar extension on the symmetry groups can be performed to include the other gauge groups, which is worthwhile to explore since this approach was already shown to be fruitful in the continuous case as done in the work in [106]. Then, spinors defined on the lattice would have an additional rotational $SO(d)$ or $ISO(d)$ symmetry, and the main challenge there would be in applying the Baker-Campbell-Hausdorf formula for the Lie groups. Another problem to address on this end would be the doubling of fermions in Euclidean spaces which still remains in this discrete formulation, as it is the case for other approaches too.

Another fundamental question that this discrete formulation opens to is: *could Discrete Gravity be a new competing approach to Quantum Gravity?* — This bigger scheme of reconciling gravity with the quantum theory has followed competing approaches, many centered around discretization, for example, those starting with discretizing the path integral as in Regge Calculus [31] and Dynamical Triangulations [32]. However, these face problems in taking the continuum limit, which is important for recovering the limit to the classical theory with a continuous manifold. In contrast, a manifest continuum limit is a main advantage of the framework adopted in this thesis, which makes it a viable alternative. On the other hand, using this formalism, quantization can be applied to the space where discretization is applied to the space-like hypersurfaces, leaving the time coordinate continuous, which is a natural approach to take as seen in canonical quantum gravity. The goal would be then to replace quantum field theory with one that incorporates gravity and instead has a finite number of degrees of freedom, which appears naturally in the description of the elementary volumes in discrete gravity that carry a finite number of such degrees of freedom, possibly giving an inherent UV cut-off in the theory. Another step in this direction can be taken in promoting the continuous indices carried by field variables to integers on the lattice, and this in a sense would bring quantum field theory closer to quantum mechanics. This direction then requires finding a proper Hamiltonian description of this discrete formalism.

Alas, the dynamical and structural problems addressed in this thesis that start from the Planckian scale have opened the door to several connected and interesting problems interpolating between: *i)* challenges to the gravitational theory of GR at the classical level, which motivated the search for a modified gravity theory at the high curvature limit from observational and theoretical standpoints, and solving long standing problems in its most celebrated theory of Inflation; *ii)* to another set of challenges pertaining to the classical description of space-time, which breaks down at the smallest scales, requiring novel frameworks, such as Discrete Gravity, which builds on a gauge theoretical description of Gravity on the lattice, and invites deeper questions on reconciling gravity with quantum mechanics.

Bibliography

[1] Chamseddine, A. H., Khaldieh, M., Mukhanov, V. . Mimetic inflation and self-reproduction. Journal of Cosmology and Astroparticle Physics, **11** (2023) 022.

[2] Chamseddine, A. H., Khaldieh, M. *Poincare invariance in discrete gravity.* Journal of High Energy Physics, **4** (2025) 1-15.

[3] Slipher, Vesto. *Nebulae*. Proceedings of the American Philosophical Society **56** (1917) 403-409.

[4] Hubble, E. *A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae*. Proceedings of the National Academy of Sciences of the United States of America **15** (1929) 168-73.

[5] Alpher, Ralph A., Hans Bethe, and George Gamow. *The origin of chemical elements.* Physical Review **73** (1948) 803.

[6] A. A. Penzias, R. W. Wilson, *A measurement of excess antenna temperature at 4080 Mc/s.* Astrophys. J. **142** (1965) 419–421.

[7] de Bernardis, P., Ade, P., Bock, J. et al. *A flat Universe from high-resolution maps of the cosmic microwave background radiation.* Nature **404**(2000) 955–959.

[8] V. Mukhanov, *Quantum cosmological perturbations: predictions and observations.* Eur. Phys. J. **73** (2013) 2486.

[9] A. H. Chamseddine and V. Mukhanov, *Mimetic Dark Matter,* JHEP **1311** (2013) 135.

[10] A. Golovnev, *On the recently proposed Mimetic Dark Matter*. Phys. Lett. B **728** (2014) 39-40.

[11] R. Utiyama, *Invariant theoretical interpretation of interaction,* Phys. Rev. **101** (1956) 1597.

[12] H. Weyl, *Gravitation und Elektrizität.* Sitzungsber. Akademie der Wissenschaften Berlin (1918) 465-480.

[13] E. Cartan, *Les récentes généralisations de la notion d'espace.* Bull. Sci. Math., (1924) 825-861. E. Cartan, *Les espaces à connexion conforme.* Ann. Polon. Math (1923) 171-221. E. Cartan *Sur les variétés à connexion projective* Bull. Soc. Math. France (1924) 205-241.

[14] T. W. B. Kibble, *Lorentz invariance and the gravitational field,* J. Math. Phys. **2** (1961) 212.

[15] Y. M. Cho, *Higher - Dimensional Unifications of Gravitation and Gauge Theories,* J. Math. Phys. **16** (1975) 2029.

[16] A. H. Chamseddine and P. C. West, *Supergravity as a Gauge Theory of Supersymmetry,* Nucl. Phys. B **129** (1977) 39.

[17] P. van Nieuwenhuizen, *Supergravity,* Phys. Rept. **68** (1981) 189.

[18] E. Inonu and Eugene Wigner, *On the Contraction of groups and their represenations,* Proc. Nat. Acad. Sci. **39** (1953) 510.

[19] A. H. Chamseddine and V. Mukhanov, *Discrete Gravity,* JHEP **11** (2021) 013.

[20] K. G. Wilson, *Confinement of Quarks,* Phys. Rev. D **10** (1974) 2445.

[21] F. Casas, A. Murua, M. Nadinic, *Efficient computation of the Zassenhaus formula,* Computer Physics Communications **183** (2012) 2386.

[22] H. C. Volkin, *Iterated commutators and functions of operators,* Nasa Technical Note **D-4857** (1968).

[23] A. H. Chamseddine, O. Malaeb and S. Najem, *Scalar Curvature in Discrete Gravity, Eur. Phys. J. C.* **82** (2022) 7, 651.

[24] A. H. Chamseddine, O. Malaeb and S. Najem, *Curvature Tensor in Discrete Gravity, Eur. Phys. J. C.* **83** (2023) 10, 896.

[25] A. H. Chamseddine, O. Malaeb and S. Najem, *Black Hole in Discrete Gravity, Eur. Phys. J. C.* **84** (2024) 3, 284.

[26] A. H. Chamseddine and V. Mukhanov, *Riemann Tensor in Discrete Gravity with Symmetries of the Continuous Tensor,* to appear.

[27] Terence Tao, *Expansion in finite simple groups of Lie type* Graduate Studies in Mathematics, AMS **164** (2015).

[28] K. Miller. *The calculus of finite differences and difference equations.* Dover Publications (1964).

[29] Weinberg, S. . *Ultraviolet divergences in quantum theories of gravitation.* General relativity (1979).

[30] t Hooft, G., Veltman, M. . *One-loop divergencies in the theory of gravitation.* Annales de l'institut Henri Poincaré. Section A, Physique Théorique **20** (1974) 69-94.

[31] T. Regge, *General Relativity without Coordinates,* Nuova Cim. **19** (1961) 558.

[32] Ambjorn, J., Jurkiewicz, J., Loll, R. . Lattice quantum gravity-an update. (2011) arXiv preprint arXiv:1105.5582.

[33] A. H. Chamseddine, G. Felder and J. Fröhlich, *Gravity in Noncommutative Geometry,* Comm. Math. Phys. **155**, (1993) 205.

[34] J. Fröhlich, *Regge calculus and discretized gravitational functional integrals*, published in 'Non-Perturbative Quantum Field Theory: Mathematical Aspects and Applications', Selected Papers of Jurg Fröhlich, World Scientific (1992).

[35] A. Vilenkin, *Birth of inflationary universes* Phys.Rev. **D27** (1983) 2848

[36] A. D. Linde, *Eternally Existing Self-Reproducing Chaotic Inflationary Universe* Phys. Lett. **B175** (1986) 395.

[37] Linde, Andrei. *On the problem of initial conditions for inflation.* Foundations of Physics **48** (2018) 1246-1260.

[38] Lucchin, Francesco, and Sabino Matarrese. *Power-law inflation.* Physical Review D **32.6** (1985) 1316.

[39] A. D. Linde, *Chaotic Inflation*, Phys. Lett. **B129** (1983) 177–181.

[40] A. D. Linde, *Initial conditions for inflation*, Phys. Lett. **B162** (1985) 281–286.

[41] Barrow, John D. *Graduated inflationary universes.* Physics Letters B **235** (1990): 40-43.

[42] Muslimov, A. G. *On the scalar field dynamics in a spatially flat Friedman universe.* Classical and Quantum Gravity **7.2** (1990) 231.

[43] S. Alam *et al. Completed SDSS-IV extended Baryon Oscillation Spectroscopic Survey: Cosmological implications from two decades of spectroscopic surveys at the Apache Point Observatory* Phys. Rev. D **103** (2021) 083533

[44] Planck Collaboration VIII *Planck 2018 results. VIII. Gravitational lensing* Astronomy and Astrophysics (2020) A8, 641

[45] Aghanim, Nabila, et al. "Planck 2018 results-VI. Cosmological parameters." Astronomy and Astrophysics (2020) A6, 641.

[46] Y. Akrami , et al. *Planck 2018 results-IX. Constraints on primordial non-Gaussianity.* Astronomy and Astrophysics (2020) A9, 641.

[47] T. Louis, et al. *The Atacama Cosmology Telescope: DR6 Power Spectra, Likelihoods and* Λ *CDM Parameters.* arXiv preprint arXiv:2503.14452 (2025).

[48] Sergei Winitzki, *Eternal inflation* (2008) World Scientific.

[49] A. Ijjasa, P. J. Steinhardt and A. Loeb, *Inflationary paradigm in trouble after Planck2013* Phys.Lett. **B723** (2013) 261.

[50] Ade, Peter AR, et al. *Planck 2013 results. XXII. Constraints on inflation.* Astronomy and Astrophysics **571** (2014) A22.

[51] Linde, Andrei D. *A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems.* Physics Letters B **108.6** (1982) 389-393.

[52] Albrecht, Andreas, and Paul J. Steinhardt. *Cosmology for grand unified theories with radiatively induced symmetry breaking.* Physical Review Letters **48.17** (1982) 1220.

[53] Olive, K.A *Inflation* Physics Reports **190** (1990) 307.

[54] Salopek, D. S., J. R. Bond, and James M. Bardeen. *Designing density fluctuation spectra in inflation.* Physical Review D **40.6** (1989) 1753.

[55] Fakir, Redouane, and William G. Unruh. *Improvement on cosmological chaotic inflation through nonminimal coupling.* Physical Review D **41.6** (1990) 1783.

[56] F. Bezrukov, M. Shaposhnikov, *The Standard Model Higgs boson as the inflaton.* Physics Letters **B659** (2008) 703.

[57] Steinhardt, Paul J., and Michael S. Turner. *Prescription for successful new inflation.* Physical Review D **29.10** (1984) 2162.

[58] Guth, Alan H. *Eternal inflation and its implications.* Journal of Physics A: Mathematical and Theoretical 40 **25** (2007) 6811.

[59] Garriga, Jaume, and Alexander Vilenkin. *Prescription for probabilities in eternal inflation.* Physical Review D **64.2** (2001) 023507.

[60] Garriga, Jaume, et al. *Probabilities in the inflationary multiverse.* Journal of Cosmology and Astroparticle Physics **01** (2006) 017.

[61] Aguirre, Anthony, Steven Gratton, and Matthew C. Johnson. *Hurdles for recent measures in eternal inflation.* Physical Review D—Particles, Fields, Gravitation, and Cosmology **75.12** (2007) 123501.

[62] Vilenkin, Alexander. *A measure of the multiverse.* Journal of Physics A: Mathematical and Theoretical **40.25** (2007) 6777.

[63] Winitzki, Sergei. *Predictions in eternal inflation.* Inflationary Cosmology. Berlin, Heidelberg: Springer Berlin Heidelberg, (2008) 157-191.

[64] Linde, Andrei, Dmitri Linde, and Arthur Mezhlumian. *Do we live in the center of the world?.* Physics Letters B **345.3** (1995) 203-210.

[65] Guth, Alan H. *Inflation and eternal inflation.* Physics Reports **333** (2000) 555-574.

[66] M. Ishak, *Testing general relativity in cosmology*, Living Rev. Rel. **22** (2019) 1.

[67] Abbott, R., et al. *Tests of general relativity with GWTC-3.* arXiv:2112.06861 (2021).

[68] C.M. Will, *Theory and experiment in gravitational physics, 2nd edition*, Cambridge University Press (2018).

[69] Guth, Alan H., David I. Kaiser, and Yasunori Nomura. *Inflationary paradigm after Planck 2013.* Physics Letters B **733** (2014) 112-119.

[70] A. Guth, V. Vanchurin, *Eternal Inflation, Global Time Cutoff Measures, and a Probability Paradox* (2011) arXiv:1108.0665.

[71] Ken D. Olum, *Is there any coherent measure for eternal inflation?* Phys. Rev. **D86,** (2012) 063509.

[72] Ben Freivogel, *Making predictions in the multiverse*, Class. Quant. Grav. **28**, (2011) 204007.

[73] J. Martin , R. H. Brandenberger. *Trans-Planckian problem of inflationary cosmology.* Physical Review D **12** (2001) 123501.

[74] Clifton, Timothy, et al. *Modified gravity and cosmology.* Physics reports **513** (2012) 1-189.

[75] W. de BLok. *The Cusp-Core Problem.* Advances in Astronomy, **0910** (2010) 789293.

[76] J. S. Bullock. *Notes on the Missing Satellites Problem.* Local Group Cosmology **20** (2013) 95.

[77] J. Lee, E. Komatsu *Bullet Cluster: A Challenge to LCDM Cosmology.* Astrophysics Journal **718** (2010) 60.

[78] Peebles, P. J. E. *The void phenomenon.* The Astrophysical Journal **557** (2001) 495.

[79] Copi, Craig J., et al. *Large-Angle Anomalies in the CMB.* Advances in Astronomy **2010.1** (2010) 847541.

[80] V. Mukhanov, *Inflation without selfreproduction* Fortschritte Der Physik, **63**, (2015) 36.

[81] A. Chamseddine, and V. Mukhanov. *Gravity with de Sitter and unitary tangent groups.* Journal of High Energy Physics **2010.3** (2010) 1-18.

[82] A. Chamseddine, V. Mukhanov, *Mimetic dark matter* JHEP, **1311** (2013) 1.

[83] Brans, *Carl H. The roots of scalar-tensor theory: an approximate history.* arXiv preprint gr-qc/0506063 (2005).

[84] C. Brans, R. H. Dicke. *Mach's principle and a relativistic theory of gravitation.* Physical review 124 **3** (1961) 925.

[85] A. Chamseddine, V. Mukhanov, *Resolving cosmological singularities* JCAP. **77** (2017) 009.

[86] A. Chamseddine, V. Mukhanov, *Nonsingular black hole* The European Physical Journal C. **77** (2017) 1.

[87] Chamseddine, A., Mukhanov, V. & Russ, T. *Black hole remnants* JHEP. **10** (2019) 1.

[88] Chamseddine, A., Mukhanov, V. & Russ, T. *Non-Flat Universes and Black Holes in Asymptotically Free Mimetic Gravity* Fortschritte Der Physik. **68** (2020) 1900103.

[89] A. H. Chamseddine, V. Mukhanov, T. B. Russ. *Asymptotically free mimetic gravity.* The European Physical Journal C **79** (2019) 1-6.

[90] A. H. Chamseddine, V. Mukhanov, A. Vikman . *Cosmology with mimetic matter.* Journal of Cosmology and Astroparticle Physics, **06** (2014) 017.

[91] S. I. Nojiri , S. D. Odintsov. *Mimetic F (R) gravity: inflation, dark energy and bounce.* Modern Physics Letters A 29 **40**, (2014) 1450211.

[92] F. Arroja, N. Bartolo, P. Karmakar, S. Matarrese. T*he two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier.* Journal of Cosmology and Astroparticle Physics, **09** (2015) 051.

[93] L. Sebastiani, S. Vagnozzi, R. Myrzakulov. *Mimetic gravity: a review of recent developments and applications to cosmology and astrophysics.* Advances in High Energy Physics **1** (2017) 3156915.

[94] V. Mukhanov, *Physical foundations of cosmology.* (2005) Cambridge university press.

[95] S. Goldwirth *Inhomogeneous initial conditions for inflation.* Physical Review D 43 **10** (1991) 3204.

[96] W.E. East, M. Kleban, A. Linde and L. Senatore, *Beginning inflation in an inhomogeneous universe* JCAP **09** (2016) 010.

[97] R. Brandenberger, *Initial conditions for inflation — A short review*, Int. J. Mod. Phys. D **26** (2016) 1740002.

[98] T. S. Bunch, P. Davies, *Quantum field theory in de Sitter space: Renormalization by point splitting*, Proc. R. Soc. A 360 (1978) 117.

[99] J. Martin , R. H. Brandenberger. *Trans-Planckian problem of inflationary cosmology.* Physical Review D 63 **12** (2001) 123501.

[100] R. H. Brandenberger, J. Martin. *Trans-Planckian issues for inflationary cosmology.* Classical and Quantum Gravity **11** (2013) 113001.

[101] L. Senatore . *Lectures on inflation.* New Frontiers in Fields and Strings: TASI 2015 Proceedings of the 2015 Theoretical Advanced Study Institute in Elementary Particle Physics. (2017) 447-543.

[102] Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler. *Gravitation.* Macmillan (1973).

[103] Tong, David. *Gauge theory.* Lecture notes, DAMTP Cambridge **10.8** (2018).

[104] H. J. Rothe, *Lattice gauge theories: an introduction* World Scientific Publishing Company (2012).

[105] A. Starobinsky, *A new type of isotropic cosmological models without singularity.* Physics Letters **B91** (1980) 99.

[106] Chamseddine, A. H., Mukhanov, V. . *On unification of gravity and gauge interactions.* Journal of High Energy Physics, **3** (2016) 1-13.