Reference patterns of semantic paradoxes and the problem of their graph-theoretic characterization

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Abstract. This thesis is a contribution to the field of semantic paradoxes. It joins a tradition of works that investigate the questions of why certain referential structures of sentences or sentence systems lead to semantic paradoxes.

The approach presented in Chapter 2 is language-independent. The fundamental concept it is based on is that of a *Boolean network*. Other frameworks of the tradition that could be described as graph-theoretic approach to the semantic paradoxes can be embedded in ours straightforwardly.

As in various other accounts, reference patterns are formally conceived as directed graphs. A question every graph-theoretic account has to answer is what graphs should be conceived as being *potentially paradoxical*. The answer is not straightforward, since graphs that occur as reference graphs of paradoxical sentences usually do so not exclusively, but occur also as reference graphs of sentences that are not paradoxical.

I will suggest three candidates for the class of all potentially paradoxical directed graphs: *dangerous* digraphs, digraphs *of infinite character* and *not strongly kernel-perfect* digraphs. Each of them captures a different aspect of the intuition one might have about potentially paradoxical graphs and each gives rise to a *characterization problem*, i.e., the problem of finding a *graph-theoretic property* that is a necessary and sufficient condition for a graph to be a member to this class. To each of the three characterization problems I conjecture a solution. A directed graph is conjectured to be dangerous if and only if it contains a directed cycle or a *finitary inflation* of the Yablo-graph (i.e., the reference pattern of Yablo's paradox); it is conjectured to be of infinite character if and only if it contains a finitary inflation of the Yablo-graph; and it is conjectured to be not strongly kernel-perfect if and only if it contains an odd directed cycle or an *odd finitary inflation* of the Yablo-graph. It will be investigated how these conjectures are interrelated.

The goal of Chapter 3 is to show that any Boolean network (and the question of whether it has a fixed point in particular) can be analyzed in terms of an associated directed graph. Such a graph is called a *characteristic digraph* of the Boolean network and contains more information about it than a reference graph. This reduces the question of whether a sentence system is paradoxical to a purely graph theoretic one. In Chapter 4 it is shown that all three criteria conjectured as sufficient and necessary are sufficient indeed. In Chapter 5 it is shown that the criterion for dangerous digraphs is necessary under certain additional assumptions.

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A special thanks goes to Thomas Schindler, who is the co-author of two of my papers. The many inspiring discussions with him helped me a lot to develop my ideas. He has always been available to discuss any of my problems or to give me advice.

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Included material from co-authored publications. Section 1.1, which is presented here in a slightly modified form, was jointly written with Thomas Schindler and is part of our joint article [5]. Some of the material in Subsection 2.7.2 can also be seen as taken from this paper, though being presented here in a modified and more elaborate form. Conjecture 2.8.1 in Subsection 2.8.1 is also contained in [5] as Conjecture 4.24, as well as in [4] (another paper jointly written with Thomas Schindler) as Conjecture 1. Although Conjecture 2.8.1 is formulated for the more abstract domain of Boolean networks and is strictly speaking not equivalent to Conjecture 4.24 in [5], it seems to be fair to regard both as essentially the same.

Some theorems and definitions of Chapter 2 can be seen as more abstract counterparts of definitions and theorems in [5]. In most of these cases, however, the transfer is not a straightforward one. It is problably more appropriate to speak of *reconstructions* of concepts and theorems within a more abstract framework. This is the cases with some of the material in Section 2.3 (in particular Definition 2.3.1) and some of the material in Subsections 2.4.8, 2.4.10 and 2.4.11.

Finally, Definition 3.6.3 in Section 3.6 of Chapter 3 can be seen as a reconstruction of Definition 5.1 of [5].

Contents

Chapter 1

Introduction

1.1 Semantic paradoxes and reference patterns

'Why are some sentences paradoxical while others are not? Since Russell the universal answer has been: circularity, and more especially self-reference.' These are the opening lines of Stephen Yablo's paper 'Paradox without self-reference' [50] that he concludes with the assertion that self-reference is neither necessary nor sufficient for liar-like paradoxes, drawing on the now famous example of an infinite sequence of sentences each of which says that all the sentences appearing later in the sequence are not true.

In 1970, about two decades before Yablo's discovery, Hans Herzberger [24] already argued that there are referential patterns other than circularity that should be counted as pathological. According to his approach, any sentence has a *domain*, the *set of objects it is about*. Herzberger concedes that 'the general notion of a domain is more readily indicated than explicated'. However, he gives the following rules of thumb. A sentence of the form 'A is (not) true' is about A; a sentence of the form 'All φs are (not) true' is about all the φs. Of course, some objects in the domain of a sentence may be sentences themselves. Those sentences, too, have their own domain that may include sentences, and so forth. Let $D(\varphi)$ be the domain of the sentence φ and $D^2(\varphi)$ the union of the domains of all sentences in $D(\varphi)$. In this way, $D^k(\varphi)$ can be defined for all natural numbers k. (This hinges of course on the assumption that we have a definition of 'domain'.) Herzberger calls a sentence φ *groundless* iff for all k, $D^k(\varphi)$ is not empty. According to this picture, both the liar and Yablo's paradox are groundless; but while the liar is about itself, hence circular, no member of the Yablo sequence refers (directly or indirectly) to itself. Not all groundless sentences give rise to actual antinomies (the non-paradoxical truth-teller sentence is clearly groundless, for example), but they all suffer from 'vicious semantic regress', a form of 'semantic pathology' more general than merely involving a vicious circle, which, according to Herzberger, is responsible for the fact that groundless sentences 'lose their comprehensibility'. Thus, actual contradiction is 'but the extreme symptom of semantic pathology' ([24, pp. 149-150]).

Yablo did not answer the question 'Why are some sentences paradoxical while others are not?'; but the idea that each sentence has a domain invites the following crude answer:

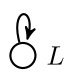
> *Some sentences are paradoxical because of their position in the reference graph of our language*, i.e., in the directed graph whose vertices are the sentences of the language, where two sentences φ, ψ are connected by an arc from φ to ψ iff φ is about (refers to, depends on) ψ.

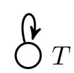
Let us have a look at some informal examples.

Example 1.1.1. The paradigms of a self-referential statement are the liar and the truth-teller and it is plausible to represent their reference patterns by simple loops.

L: (L) is false

T: (T) is true





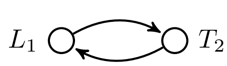
The sentence L is paradoxical in the following sense. Its truth value depends on its own truth value in a fashion that if we assign any truth value (true or false) to L and then evaluate L relative to this assigned truth value, then the value obtained by the evaluation turns out to be the opposite of the assigned value. In other words, whatever truth-value is assigned to L is rejected by the evaluation. The hypothesis that L is true implies that L is false and the hypothesis that L is false implies that L is true.

When it comes to the truth-teller sentence T we are confronted with a related but different phenomenon. Its truth value also depends on its own truth value but in a different way. Whatever truth-value is assigned to T is confirmed by the evaluation. The hypothesis that T is true implies that T is true and the hypothesis that T is false implies that T is false.

What both sentences share is the same *pattern of reference*, but they differ in their *mode of reference*.

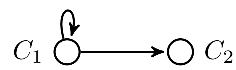
Example 1.1.2. We can also consider pairs of sentences that, even if they are not directly self-referential, still exhibit some kind of circularity:

L_1: (T_2) is false
T_2: (L_1) is true



Example 1.1.3. Similarly, for every natural number n, we can consider liar cycles of length n. A slightly different example is given by a version of Curry's paradox:

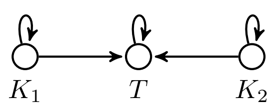
C_1: (C_1) is false or (C_2) is true
$C_2 : 1 + 1 = 3$



It is clear that self-reference or circularity is not a *sufficient* condition for paradox. But is self-reference or circularity a *necessary* condition for paradox? According to Yablo [50] that's not the case.

Example 1.1.4. Consider the following combination of Example 1.1.3 and Example 1.1.1:

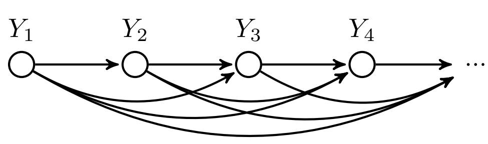
K_1: (K_1) is false or (T) is true
K_2: (K_2) is false or (T) is false
T: (T) is true



Example 1.1.5. Consider Yablo's paradox:

Y_1: (Y_n) is false for all $n > 1$
Y_2: (Y_n) is false for all $n > 2$
Y_3: (Y_n) is false for all $n > 3$
Y_4: (Y_n) is false for all $n > 4$
...

Informally still, we may represent the Yablo sequence by the following graph which does not contain any cycles.

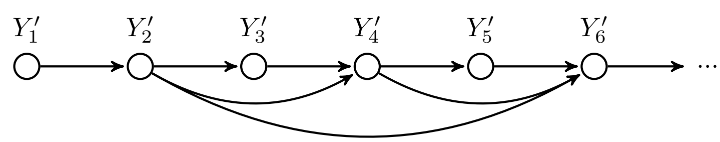


It is not possible to assign a truth value to every Y_n: in any such assignment, no more that two Y_n's could be declared true. Hence there exists some n_0 such that all Y_m with $m \geq n_0$ are declared false. This implies that Y_m is true while being declared false.

Yablo's paradox seems to have an *infinite character* in the sense that it is essential that there are infinitely many sentences involved.

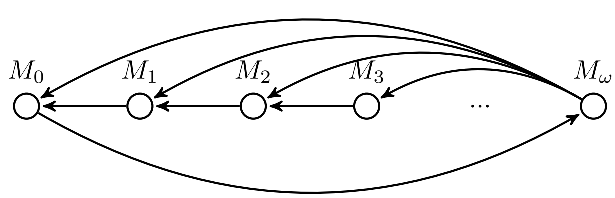
Example 1.1.6. Variations on the Yablo sequence deliver new paradoxes whose reference graphs do not contain any loops.

Y'_1: (Y'_2) is true
Y'_2: (Y'_3) is false and for all even $n > 2$, (Y'_n) is false
Y'_3: (Y'_4) is true
Y'_4: (Y'_5) is false and for all even $n > 4$, (Y'_n) are false
Y'_5: (Y'_6) is true
...



Example 1.1.7. Finally let us have a look at the following version of McGee's paradox,

M_0: (M_ω) is false
M_1: (M_0) is true
M_2: (M_1) is true
...
M_ω: (M_n) is true for all $n < \omega$.



In order to see the paradox in this construction, assume that M_ω is true. Hence for all $n < \omega$, M_n must be true. But then M_0 is true and thus claims truly that M_ω is false. Hence M_ω is false. This implies that M_0 is true. Which implies that M_1 is true. More generally, M_{n+1} is true, given that M_n is. Hence, by induction, we may conclude that M_ω is true.

McGee's paradox combines traits of both the liar-paradox and Yablo's paradox. While it has a cyclic structure, the fact that it involves infinitely many sentences seems to be essential. Is it a paradox of infinite character in the same sense as Yablo's paradox is?

These examples raise the question of how many types of paradox there are and what reference patterns underlie them. What are the 'paradoxical nodes' of the reference graph? And can they be characterized in graph-theoretic terms?

1.2 Graph-theoretic accounts of the semantic paradoxes

1.2.1 What is a graph-theoretic account?

Dating back to the early 1970s, there have been various attempts to explain why some sentences are paradoxical while others are not in terms of certain 'pathological' patterns of reference, which are most conveniently captured by directed graphs. All these accounts could be characterized as being rather 'naturalistic' (or 'naive' or 'empirical') in the sense that they seem to agree on Hans Herzberger's guideline that '[r]ather than try to eliminate those paradoxes [...] the idea is to stand back and let the paradoxes reveal their inner principles', (cf. [25]). Ideally, every such approach would involve, roughly speaking, at least the following four steps.

1. *Choice of language*: choose formal language $\mathcal{L}$.
2. *Notion of paradox*: define what sets of $\mathcal{L}$-sentences are paradoxical.
3. *Graph assignment*: assign to each set $\mathcal{S}$ of $\mathcal{L}$-sentences a directed graph $R(\mathcal{S})$ that describes a relation of reference/semantic dependence between the sentences of $\mathcal{S}$,
4. *Characterization problem*: classify the (non) paradoxical sets of sentences $\mathcal{S}$ in terms of $R(\mathcal{S})$!

1.2.2 The approach of Cook

As indicated in Section 1.1, the foundation for a graph-theoretic account of the semantic paradoxes has been laid by Herzberger in his article [24]. Another important and formally more rigorous contribution is Yablo's work, [49] which is thoroughly discussed in [5]. As the first rigorous and systematic attempt of a graph-theoretic account could probably be regarded Gaifman's approach in [21], which is, however, rather complicated.

In the following we shall discuss Cook's short and elegant essay [14] as paradigmatic of graph-theoretic accounts of the semantic paradoxes.[1]

[1] Other noteworthy contributions are: [9], [28], [29] and [46]. It is also worth mentioning that at the intersection of philosophical logic and theoretical computer science there exists a field of called *abstract argumentation frameworks* or *discourse logic*, which is quite flourishing

Cook considers (1) a formal language - denoted by $\mathcal{L}_F$ in this introduction, 'F' standing for 'false' - that consists of a set of sentence names, a falsity predicate and conjunction symbols of arbitrary arity. (Strictly speaking, $\mathcal{L}_F$ is not a language but a family of languages, parameterized by the set of sentence names. For the sake of simplicity we treat it as a language. This remark also applies to $\mathcal{L}_C$ considered in the next subsection.) A denotation function that maps sentence names to $\mathcal{L}_F$-sentences gives rise to what might be called a *negative sentence system*: A set of sentences $\mathcal{S}$ each of whose members refers to a subset of $\mathcal{S}$ and claims that all the members of this subset are false. From our above examples, $\{L\}$ and $\{Y_1, Y_2, \dots\}$ are sets of sentence names that can be thought of as such negative sentence systems (given the denotation function indicated in our examples by a colon), while our examples built on the sets $\{T\}$, $\{L_1, L_2\}$ and $\{Y_1', Y_2', \dots\}$ cannot.

Before discussing the notion of paradox (2), let us have a look how directed graphs are assigned to sentence systems: Cook considers a relation of *dependency* between sentences names that can be characterized as follows: a sentence name α depends on a sentence name β if and only β occurs in the sentence that is denoted by α.[2] The directed graph that corresponds to a given sentence system (let us call it the *reference graph* of this system) then consists of the sentence names (the vertices of the graph) and has an arc from α to β iff α depends on β. Clearly, the reference graphs of the sentence systems $\{L\}$ and $\{Y_1, Y_2, \dots\}$ are just as indicated by the drawings in the previous section: the loop and the Yablo-graph.

Now to the notion of paradox (2) that comes with Cook's account: a *truth-value assignment* is a function that maps sentences to truth values (t and f). A truth value assignment σ is *acceptable* if and only if a sentences name α receives the value t under σ iff every sentence-name that α depends on receives the value f under σ. A sentence system is said to be *paradoxical* iff it has no acceptable truth value assignment. Again, both $\{L\}$ and $\{Y_1, Y_2, \dots\}$ are paradoxical under this definition.

However, if we modify our example of the liar cycle $\{L_1, L_2\}$ in such a way that both L_1 and L_2 claim that the other is false (and only then this example can be expressed in $\mathcal{L}_F$), then the resulting sentence system comes out as not paradoxical: One sentence might be true and the other might be false.

Now to the characterization problem (4) that comes with this account. What can be said about the paradoxicality of a sentence system in terms of its reference graph? Cook emphasizes the (philosophical) importance of 'determining what patterns of dependency between sentences generate paradox', since in the absence of such a classification 'philosophical accounts of truth are constantly in danger of being overturned by the discovery of new sorts of paradox'.

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since the 1990s and which deals, at least from a formal point of view, with rather similar questions as graph-theoretic accounts of the semantic paradoxes do. Noteworthy papers are e.g. [18] and [20].

[2]In [4], [5] and in this thesis a dependency relation is understood as a relation between sentences and *sets* of sentences. This is done purely for technical convenience and both version are inter-translatable.

It turns out that the characterization problem can be tied straightforwardly[3] to a well-known and important concept in the theory of directed graphs, that of a kernel[4]: A set K of vertices of a digraph G is called *independent* iff there is no arc of G between any (not necessarily distinct) pair of (not necessarily distinct) vertices and K is called *absorbent* iff for all $x \notin K$ there exists $y \in K$ such that (x, y) is an arc of G. A *kernel* of G is a subset of the set of all vertices of G that is both independent and absorbent.

The crucial observation is that the characteristic function of any kernel of the reference graph that corresponds to a sentence system $\mathcal{S}$ defines an acceptable truth value assignment for $\mathcal{S}$ and, on the other hand, every acceptable truth value assignment for $\mathcal{S}$ can be read as the characteristic function of a kernel of its reference graph. Let us illustrate this point with the sentence system $\{Y_1, Y_2, \dots\}$ and its reference graph. Notice that a function $f : \omega \to \{0, 1\}$ is the characteristic function of a kernel of the Yablo-graph iff the following holds. For all $n > 1$: $f(n) = 1$ iff $f(k) = 0$, for all $k > n$, and $f(n) = 0$ iff there exists $k > n$ such that $f(k) = 1$. Replacing '$f(n) = 1$' by 'Y_n is true' and '$f(n) = 0$' by 'Y_n is false' generates the sentence system $\{Y_1, Y_2, \dots\}$ of our above example.[5]

As a consequence, the characterization problem is tantamount to the problem of classifying the digraphs that have no kernel. Promising as this correspondence might seem, it also brings to light the issue of complexity: the characterization problem could be much too difficult to have a satisfying solution. Cook himself concedes that 'elegant necessary and sufficient conditions for the existence of a kernel, even in the finite irreflexive case, have so far eluded discovery.' Indeed, this is not surprising. The problem of determining whether a finite digraph has a kernel or not is known to be NP-complete.[6] As shown in [8], for recursive infinite digraphs the problem is Σ^1_1-complete. This makes the corresponding versions of the characterization problem co-NP-complete and Π^1_1-complete, respectively.

This raises the question of whether there is any hope of ever formulating a simple graph-theoretic property that is a sufficient and necessary condition for a digraph having a kernel. Typically, graph-theoretic properties are formulated is in terms of excluded subgraph conditions. Examples are the characterization of the planar graphs or Hadwiger's conjecture. (Cf. [17]). That such a characterization is not possible for digraphs with (or without) kernel is due to the fact that for any digraph that has no kernel there is a superdigraph that has a kernel, and for any digraph that has a kernel, there is a superdigraph that has

[3]Cf. the appendix of [14]

[4]We will touch the notion of digraph kernel and its history a little more in Section1.4. A good survey is [10]. The very readable introduction to graph theory [3] dedicates an entire chapter to kernels.

[5]An important qualification must be made: Cook formulates and proves this correspondence only for *serial* digraphs - i.e. digraphs without *sinks*. The reason is that the language of his framework contains no logical constants. Even though the language of the framework of Rabern et al. (cf. Subsection 1.2.3 does contain logical constants, they restrict some of their results about kernel to graphs without sinks. Unnecessarily so, as will be shown in Proposition 2.3.8.

[6]For a discussion of complexity issues and for further references cf. [8].

no kernel.[7]

Theses considerations suggest that it is quite unlikely to ever find a satisfactory graph-theoretic property that is a necessary and sufficient condition for a digraph having a kernel.

1.2.3 The approach of Rabern et al.

Like Cook, Rabern et al. work in [41] with an infinitary propositional language, sentence names and a denotation function. The main difference, however, is that their language contains in addition to conjunction symbols the logical constants $\perp$ and $\top$ and, in particular, a negation symbol $\neg$. In this introduction we denote this language by $\mathcal{L}_C$ ('C' standing for 'complete'). Thus $\mathcal{L}_F$ can be considered as the negative fragment of $\mathcal{L}_C$. In the latter, all our above sentence systems - including the truth teller $\{T\}$ and the pair of liars $\{L_1, L_2\}$ - are expressible. Moreover, a dependency relation (called a reference relation) is defined between sentence names, completely analogously to [14], which gives rise to reference graphs for sentence systems.

Again, a sentence system is said to be paradoxical if and only if it has no acceptable truth value assignment σ, the definition of which could be seen as an adaption of Cook's to the context of more complex logical formulae: for every sentence name α, the value assigned to α by σ must must coincide with the semantic value relative to σ of the sentence denoted by α,

The price of the gain in expressive power is that it is no longer possible to characterize the paradoxicality of a sentence system in terms of its reference graph: e.g. both the paradoxical liar and the non-paradoxical truth-teller have a loop as their reference graph.

Probably for this reason, the authors shift the focus of the characterization problem, which makes no sense in the form as it is stated under (4) in Subsection 1.2.1. It makes no longer sense to speak of paradoxical reference patterns, only of *potentially paradoxical* ones: a directed graph is said to *dangerous* if and only if it is the reference graph of a paradoxical sentence system. The modified version of the characterization problem (4) becomes the following.

4′. Classify the dangerous directed graphs!

One advantage of this shift is a reduction in complexity: in contrast to (4), characterization problem (4′) seems to be a more tractable. Indeed, the dangerous finite directed graphs are exactly those that contain a directed cycle. Moreover, every directed graph that contains a dangerous graph is dangerous itself. This last point opens the possibility of an excluded subgraph or an excluded minor characterization of the dangerous directed graphs.

[7] In the first case just add a new vertex n to the digraph G and an arc from each vertex of G to n; a kernel of the new digraph consists just of n. In the second case proceed as in the first one but then add a further arc to the digraph from the n to itself. Then n cannot be in any kernel of the new digraph G'. But this is tantamount to the claim that there is no vertex v in any presumed kernel of of G' such that (n, v) is an arc of G'.

1.2.4 The approach of Beringer and Schindler

Indeed, the following conjecture was suggested by Beringer and Schindler (in [4] and in [5]).

Conjecture. A reference graph is dangerous if and only if it contains a directed cycle or the Yablo-graph as finitary minor[8].

However, it must be mentioned that in contrast to both of the previous accounts, [4] and [5] work with the language of first-order arithmetic augmented with a unary T-predicate. In this context, a syntactical notion of reference is no longer possible (or at least not obvious). Reference graphs are based on a notion of semantic dependence, first introduced by Leitgeb in [34]. For this reason the above conjecture does not pertain to directed graphs simpliciter but is relativized to reference graphs, a concept depending on the specific language of the framework. The same holds true also for the concept of *danger*. *Prima facie* it is not clear at all how the above conjecture relates logically to its translation into the framework of [41].

Conjecture*. A directed graph is dangerous* if and only if it contains a directed cycle or the Yablo-graph as finitary minor,

where dangerous* is meant to be defined just as in [41].

In order to tackle the problem that the question of whether a sentence is paradoxical cannot be answered in terms of its reference graph, a more fine grained version of reference graphs, *signed reference graphs*, are introduced. Signed reference graphs can distinguish the liar from the truth teller. In addition to the reference relation itself, expressed by an arc between sentences, the *mode of reference* is also taken into account and expressed by a label attached to the arcs. The loop of the liar graph turns out to be negative and the loop of the truth-teller positive. It is argued that the reference patterns that lead to paradox are all predominantly negative.

1.3 Guiding questions

The discussion of graph-theoretic accounts in the previous section raises a couple of questions that we partition into three groups.

1. Is there a language-independent graph-theoretic approach to the semantic paradoxes, integrating e.g. the frameworks of [14] and [41] on the one hand and that of [5] on the other hand? What would be the underlying dependency relation of such an abstract approach? Could it be in any sense meaningful or just an artificial construct? What is the exact relationship

[8]The concept of finitary minor will be introduced in Definition 2.7.10 and Definition 2.7.13

between the abstract framework and the particular accounts discussed in the previous section?

2. Is it possible to assign to sentence systems (or to their language-independent counterparts) directed graphs that capture the full information about their status of being paradoxical or not, just as in [14], but without any restriction of language? What would be the relation between these graphs, let us call them *characteristic graphs*, and dependency graphs? And what would be their relation to the signed reference graphs in [5]?

3. Is there some notion of *dangerous graph* in the abstract framework and how does it relate to the respective notions in the particular frameworks? What is a reasonable conjectured solution for the corresponding characterization problem? Is there an analogous characterization problem for characteristic graphs? What is a reasonable conjectured solution for this problem? How is the second related to the first? Can any of theses conjectured solutions be proven, at least partially?

These questions will serve as a guideline for this investigation, of which an outline shall be given in Section 1.5. Before doing so, it seems appropriate to provide the reader with some background in the theory of digraphs kernels, given the important role this concept plays throughout our investigation.

1.4 Richardson's theorem for infinite graphs

The notion of a *kernel* was first introduced by von Neumann and Morgenstern in their seminal work [45], in order to describe solutions for certain cooperative games. Kernels of directed graphs (at least of finite ones) have been studied extensively. For a survey of some classical results cf. [10] and [1].

A rare example of a book that treats kernels of infinite graphs to some extent is Berges *Théorie des graphes et ses applications* from 1958 [2].[9] With hindsight, it might seem a bit surprising that Yablo's paradox wasn't discovered by Berge back then - or, more precisely, its counterpart in kernel theory. In order to appreciate this remark, let us restate the definition of a kernel already given in Subsection 1.2.2 and then have a look at the pages 48 to 50 of the English translation.

A set K of vertices of a digraph G is called *independent* iff there is no arc of G between any (not necessarily distinct) pair of them, and K is called *absorbent* iff for all $x \notin K$ there exists $y \in K$ such that (x, y) is an arc of G. A *kernel* of G is a subset of the set of all vertices of G that is both independent and absorbent. After having stated Richardson's theorem (cf. [42]) Berge proceeds by proving two generalizations of it.

[9] English translation from 1962: *The theory of graphs and its applications*, ([3]).

Richardson's theorem. Every finite digraph possesses a kernel if it contains no cycles of odd length.

Generalization 1. If a totally inductive digraph contains no cycles of odd length, then it possesses a kernel.

Generalization 2. If a locally finite digraph contains no cycles of odd length, then it possesses a kernel.

Here a digraph G is *totally inductive* iff for every infinite directed walk $x_0, x_1, \ldots$ in G there exists a natural number n such that for all $n < k < l$, there is a directed walk from x_l to x_k, and G is *locally finite* iff every vertex of G has at most finitely many in-neighbors and finitely many out-neighbors.

It is hard to believe that at this point a mathematician like Berge hasn't given at least a moment's thought to the now apparent question of whether there is an infinite digraph that has no odd cycles and *no* kernel. Note that a negative answer would make the hypotheses that replace the condition of being finite in Richardson's theorem – i.e., being totally inductive and locally finite respectively – superfluous after all. On the other hand, if one suspects that such a digraph exists and one knows that it can neither be well-founded, nor totally inductive nor locally finite, the simplest example that comes to mind is the following.

Let $\mathbb{Y}$ be the digraph whose vertices are the natural number and whose arcs consist of all the ordered pairs (n, m) such that $n < m$. It is quite easy to see that $\mathbb{Y}$ cannot have a kernel. Every independent set contains at most one element n_0. But no $m > n_0$ has an arc leading to n_0, hence no independent set can be absorbent. As already argued in Subsection 1.2.2, this can be easily translated into the language of Example 1.1.5, which, in turn, is Yablo's infinite version of the liar paradox he discovered in 1985.

The correspondence between digraphs without a kernel and paradoxes of this kind is quite general. Every digraph can be interpreted as a sentence system by assigning to each vertex a sentence that claims that all its out-neighbors are false, and to every such Yablo-like sentence system, we can assign the digraph that describes its referential structure. This fact was first mentioned in the appendix of [14], a paper dealing with variations of Yablo's paradox, among others. So it contains implicitly the observation that $\mathbb{Y}$ is an acyclic infinite digraph without kernel – the answer to the question that Berge didn't ask in his book.

A generalization of Richardson's theorem was proved by Rabern et al. in [41]: a *sink-free* directed graph G has a kernel, if the *underlying undirected graph* of G is acyclic or if only finitely many vertices of G have infinite *out degree*.

Another generalization of Richardson's theorem (even though the name isn't mentioned) is proved in [5]. If a reference graph G contains neither an odd cycle nor an odd *double path*, then G has a kernel. As with all results in this paper, as we already mentioned in Subsection 1.2.4, the qualification must be added that strictly speaking they apply only to reference graphs and not to graphs in general. We will prove the general result in Subsection 2.6.3.

In 2019, Walicki [47] stated a conjecture about digraph kernel that is related to Conjecture 4.24 and that reads as follows: a directed graph has a kernel if it contains neither an odd cycle nor a ray that is *dominated* by infinitely many of its vertices. (This conjecture will be discussed in Section 3.4. We shall see that the second condition is equivalent to not containing the Yablo-digraph as a finitary minor in Section 4.4). This can be seen as conjectured generalization of Richardson's theorem. Walicki proved a special case of this conjecture: it holds if the digraph has only finitely many *ends* (cf. [47] and Section 3.4).

In Subsection 2.8.3 we shall state a conjecture that, if true, is a maximal strong version of Richardson's theorem in the sense that it provides a necessary and sufficient condition for a digraph to be strongly kernel-perfect (i.e. being such that every subdigraph has a kernel).

1.5 Outline of the thesis

Let me conclude this introduction with a rather detailed outline of each of the remaining chapters of this thesis. Aside from giving a first overview, this section is intended to help the reader navigate through these occasionally rather technical parts and can be consulted if and when the need arises.

1.5.1 Outline of Chapter 2

While Herzberger's paper [24], as has been indicated at the very beginning of this introduction, can be seen as perhaps the first systematic attempt to understand semantic paradoxes in terms of referential patterns, his works [25] and [26] could be seen as an essay on the dynamical aspect of semantics and their relevance for the understanding of semantic paradoxes. The opening passage of [25] reads as follows.

> 'One lesson I am inclined to draw from the recent history of philosophical struggles with semantic paradoxes, is that new techniques for suppressing them are unlikely to advance our understanding of the basic problems.[...]
>
> Rather than try to eliminate those paradoxes, I want to consider the experiment of positively encouraging them to arise and watching them work their own way out. This approach, which I call *naive semantics*, is a deliberately nondirective exercise. The idea is to stand back and let the paradoxes reveal their inner principles. Naive semantics is very much in the spirit of Charles Chihara's recent plea for a diagnosis of the paradoxes preceding their treatment.'

The phrase 'reveal their inner principles' almost certainly refers to the behavior a paradoxical sentence shows when subjected to a *revision process*. Such a process consists of the iterated semantic evaluation of a sentence system, starting from some first truth-value assignment. In [25] (as well as in [26] , [22] and [23]) such revision processes are iterated even transfinitely. We will discuss this point in Subsection 2.4.2, but for the time being let us leave the limit step aside.

Formally, a revision sequence is a *trajectory* of a point in a *discrete dynamical system*, which is simply a function $\Phi : U \to U$. (Cf. [43] or [48]). In our case the points, i.e., the elements of U, are truth-value assignment, i.e., elements of $U = \{0,1\}^X$, the set of all function of type $f : X \to \{0,1\}$. The elements of X are sentences, the functions $f \in \{0,1\}^X$ truth value assignments and Φ and operator that revises the truth-value assignments. The particular structure of the *state-space* U as set of all *Boolean functions* on a given set X makes the dynamical system Φ a *Boolean network*. The iterations $\Phi^1, \Phi^2, ..$ of Φ are of particular interest. Applied to an initial state $f \in \{0,1\}^X$ they produce a sequence of states that either reaches some *fixed point* after finitely many steps or goes on forever without stabilizing.

The concept of Boolean network was first introduced by Kaufman in [30] with an intended application in a completely different domain, namely biology. Since then the field has flourished and the concept has found its application also in theoretical computer science, e.g., [19]. Dedicated to research at the intersection of biology and computer science, there is the journal *Artificial Life*, many of whose articles are concerned with Boolean networks, explicitly or implicitly, e.g.[11].[10] However, the Boolean networks considered in the literature I am aware of are almost exclusively finite. (In [48] an infinite example is treated *en passant.*) For sure, no attempts have been made so far for a systematic and rigorous treatment of general (i.e., finite and infinite) Boolean networks.[11]

The goal of Chapter 2 is to undertake such an exposition and to develop a language-independent graph-theoretic account of the semantic paradoxes in terms of Boolean networks. In particular, almost all of the important concepts and theorems of [5] shall be reformulated in this more abstract setting. In Subsection 2.3.2 it is shown (or at least indicated how it can be shown) that the framework of Beringer and Schindler can be conceived of as a particular Boolean network and that their concepts and results are indeed instances of our abstract reconstructions of them.

In Section 2.2 we start out from scratch by expounding the basic concepts of a general theory of discrete dynamical systems that will be of importance for our endeavor like that of an *iteration graph* (Subsection 2.2.1), an *invariant subset* of the state space, or an *attractor* (Subsection 2.2.2). In Subsection 2.2.3 Boolean networks are formally introduced.[12]

[10]It should be mentioned that Boolean networks considered in biology are often (but not always) *non-deterministic*. I will only discuss *deterministic* Boolean networks in this thesis.

[11]At least not under the name of Boolean networks. As we have argued above, the works of Herzberger, Gupta and Belnap could be regarded as being about infinite Boolean networks. Another field that seems to be quite suitable for conceptualization in terms of Boolean networks (and which is formally related to revision theory) it that of *infinite time Turing machines* and *ordinal Turing machines*, cf. e.g. [35] and [13].

[12]Let me mention here that there is an unpublished paper by Landon Rabern and Brian Rabern, [40], which introduces a framework that seems to be equivalent to the present one taken at the stage of development it is in Subsection 2.3.1, although using a different termi-

Analogously to sentence systems, Boolean networks can be equipped with dependency graphs.[13] (Subsection 2.3.1) that capture the information on which members of the network the 'computation' any given *automaton* is about to perform depends on. Of course, the terms 'computation' and 'automaton' are strictly speaking not adequate for our general setting of infinite Boolean networks - Boolean functions involved are not necessarily computable. However, we shall stick to the term *automaton* when it comes to talking about a particular component of a Boolean network. Even if the corresponding Boolean function is not computable, one might think of a Boolean automaton as an oracle interacting with other members - other automata - of the network.

The notion of dependence that these graphs are based on can be seen as a straightforward generalization of the notion of semantic dependence introduced by Leitgeb in [34], which is a key concept of the framework of Beringer and Schindler. (Cf. [5], [4] and Subsection 2.3.2).

From a mathematical point of view, the guiding question of our investigation can now be formulated as follows. Given a dependency graph of a Boolean network, what can be said about its behavior as a dynamical system, in particular about its fixed points?

The main goal of Section 2.4 is to establish the aforementioned transfer of large parts of the conceptual apparatus of [5] to Boolean networks. In order to achieve a reconstruction of Kripke's theory of truth some pendant to three-valued logic is need. For this reason *general function networks* are introduced in Subsection 2.4.3. They are just like Boolean networks with the difference that each automaton can assume not just the states 0 and 1 but values from an arbitrary set - $\{0, \frac{1}{2}, 1\}$ for most of our purposes.

This allows us consider monotonic three-valued extension of a Boolean networks, called *Kripke extensions* in Subsection 2.4.8. In Subsection 2.4.7 the usual types of Kripke fixed points are defined and the notion of a *Kripke-paradoxical automaton* is defined in full analogy to that of a *paradoxical sentence* in [32].

After having defined the notion of a *subtype* of the state-space of a Boolean network we give in Subsection 2.4.9 an interpretation the Kripke fixed points, i.e., the fixed points of the Kripke-extension as subtype of the original Boolean network. This allows us to compare the concept of a Kripke-paradoxical automata to that of a *Herzberger-paradoxical* automata (already defined in Subsection 2.4.2) in terms of sets that are invariant with respect to the Boolean network Φ.

At the end of Section 2.4 a theorem from [5] whose translation states that every Boolean network has a fixed point if it has a dependency graph that is a tree is proven in Subsection 2.4.10. Finally, in subsection Subsection 2.4.11 we can reconstruct the notion of *periphery* and *core* of a Boolean network and show results that shall be of quite some use later in Chapter 5.

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nology leaving out the dynamical aspect. I have not been aware of it until at a rather late stage of my work on this thesis and I would like to thank Brian Rabern for the hint.

[13] These dependency graphs turn out to be the converse graphs of what is called a 'graphe de connexion' in [43], where the concept is, however, only defined for finite graphs.

The goal of Section 2.5 is to introduce notions of structure preserving transformations between dynamical systems in general and Boolean networks and constrained Boolean networks (cf. Definition 2.3.1 and the following remark) in particular. This has the purpose of preparing the conceptual ground for the notion of a *characteristic digraph* (cf. Definition 3.2.5) which shall play a crucial role throughout Chapter 3. Moreover, it allows us to formulate the concept of *dual paradox* (Cf. [14] and [41]) within our abstract framework, as well as criteria for the identity of paradoxes.

In in Subsection 2.5.2 we take a look at the concept of the *dual paradox* associated to a given paradox as formulated in [14] and generalized in [41].

In Section 2.6 we will introduce three classes of digraphs, each of which is associated to a different shade of the relation between paradox and reference pattern and each of which gives rise to a different characterization problem: *dangerous* (Subsection 2.6.1), *of infinite character* (Subsection 2.6.2) and *not strongly kernel-perfect* (Subsection 2.6.3). Showing that these characterization problems are interrelated and can at least be partially solved is the main goal of this thesis.

In Subsection 2.5.1, transformations of dynamical systems have been discussed. Some of them, like weak systems embeddings, preserve a lot less information about dynamical systems than system isomorphisms do. In Section 2.7 we introduce two types of operations on digraphs that preserve some but not all of the digraphs structure: subdivisions and inflations. Inflations in particular will play an important role throughout the rest of this thesis. They come in various flavors: finitary, regular and convergent. Moreover, the *parity* (the property of being even or odd) of a digraph inflation will be defined, a concept that is key in order to formulate the conjectured solutions for the characterization problem for strongly kernel-perfect digraphs (Problem 2.6.31).

In Section 2.8 solutions for the characterization problems given in the last section shall be conjectured in terms of finitary digraph inflations.

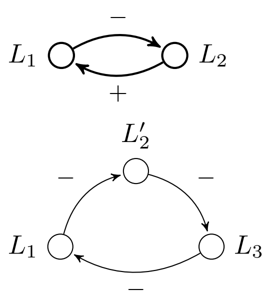
- Conjecture (A) (cf. Conjecture 2.8.1): If a directed graph contains neither a cycle nor a finitary inflation of the Yablo-graph, then it is safe.
- Conjecture (C) (cf. Conjecture 2.8.13): If an acyclic digraph contains no odd finitary inflation of the Yablo-graph, then is has a kernel.
- Conjecture (D) (cf. Conjecture 2.8.4): If a directed graph contains no finitary inflation of the Yablo-graph, then it is of finite character.

1.5.2 Outline of Chapter 3

When passing from the language $\mathcal{L}_F$ (cf. Subsection 1.2.2) to the broader language $\mathcal{L}_C$ (Subsection 1.2.3), the price one has apparently to pay for the gain of expressive power is the loss of the straightforward applicability of kernel theory to the analysis of sentence systems: It is no longer the case that a sentence system is potentially paradoxical if and only if its reference graph has no kernel. This correspondence remains valid solely for the fragment of $\mathcal{L}_C$ that can be thought of as an embedding of $\mathcal{L}_F$ in the more general language and that we may call the *negative fragment* $\mathcal{L}_C^-$ of $\mathcal{L}_C$: the reference graph of any such negative sentence system is *characteristic* for the sentence system in the sense that the reference graph has no kernel if and only if the sentence system is paradoxical.

This raises the question of whether there exists a translation from $\mathcal{L}_C$ into $\mathcal{L}_C^-$ such that a sentence system $\mathcal{S}$ is paradoxical if and only if its translation $\mathcal{S}'$ is paradoxical? A positive answer would extend the assignment of a characteristic graph to every sentence system of the complete language $\mathcal{L}_C$.

The existence of such a translation seems not implausible. E.g., take Example 1.1.2, the sentence system $\{L_1, L_2\}$, where L_1 claims that L_2 is false and L_2 claims that L_1 is true. On could add a sentence L_3 that claims that L_1 is false and replace L_2 with the sentence system $\{L_2', L_3\}$, while L_1 remains as it is.



Indeed, a procedure in this spirit has been suggested [8], at page 4, where a directed graph $G(T)$ is associated to an arbitrary theory T (formulated in a propositional language similar to $\mathcal{L}$) such T is satisfiable if and only if $G(T)$ has a kernel. This method is, however, not suitable for our purposes, which is to show that the behavior of any Boolean network and the question of whether it has a fixed point in particular can be analyzed in terms of an associated directed graph, called a *characteristic digraph* of the Boolean network, containing more information about it than a dependency graph.

So our list of desiderata is more comprehensive than those of [8]. We start (i) not from a theory of propositional logic but from a Boolean Φ network, which (ii) has a dependency graph G to whose structure the characteristic graph $G(\Phi)$ should bear some resemblance in addition to the fact that (iii) G has a kernel if and only if $G(\Phi)$ has a fixed point.

The points (i) and (ii) are obstacles for a direct application of the method of Bezem et al., in particular the second one. In light of the conjecture formulated in Section 2.8, the requirement that $G(\Phi)$ 'bear some resemblance to' G means in particular that (a) $G(\Phi)$ should be acyclic if and only if G is and (b) $G(\Phi)$ should contain a finitary inflation of the Yablo-graph if and only if G does. In particular (a) is not satisfied by the procedure of Bezem et al., since cycles would be inserted into acyclic graphs.

The first obstacle (i) can be dealt with by defining a *representation* of any Boolean network as a $\mathcal{L}_C$-sentence system, i.e., expressing the Boolean function associated to each automaton of the network by a $\mathcal{L}_C$-formula. This will be done in Subsection 3.5.1. A welcome byproduct of this step is the result that a digraph is dangerous in the sense of Rabern et al. if and only if it is dangerous in our senses. More generally, it even shows that the framework of Rabern et al. can be embedded in ours and vice versa.

In a second step, in Subsection 3.5.2, the the representation of a Boolean network is brought into a particular normal form to which then a directed graph is associated that satisfies our requirements of a characteristic graph $G(\Phi)$. In order to see it satisfies desideratum (ii), we introduce the concept of *regular inflation* in Section 3.1, show in Section 3.3 that all regular inflations satisfy the requirements (a) and (b). Furthermore, we show that $G(\Phi)$ is a regular inflation of G.

Strictly speaking, even more is achieved. In Section 3.2 characteristic digraphs will be defined in terms of *network inflations* (Definition 3.2.4) of *constrained Boolean networks*. A constrained Boolean network is an ordered pair consisting of a Boolean network Φ and a dependency graph G for Φ (which is thought of as a constraint on Φ in the sense that it puts a limitation on the set of all automata from which any automaton of the network can receive input.)

A network inflation of a constrained Boolean network consists of two components, a regular digraph inflation acting on the digraph part G by inflating it to a digraph $\mathcal{I}[G]$ and a *dense weak system embedding* (cf. Section 2.5) which transforms Φ into a Boolean network on $\mathcal{I}[G]$. This allows to describe not only the fixed points but the complete dynamical behavior of the Boolean network Φ in terms of its characteristic graph.

This method of describing a Boolean network in terms of its characteristic graph is epitomized by Theorem 3.2.6, the first main result of this thesis. Two applications of it are given in Chapter 3. One of them is a reconstruction of the concept of *signed reference graph* from [5] in Section 3.6.

The other and more important one is the exploitation of results from kernel theory for our theory of Boolean networks in Section 3.4. In particular, it allows us to use results from [47], already outlined in Section 1.4, in order to show that Conjecture (A) (already mentioned in Section 1.5.1) holds under the assumption that the directed graph in question has only finitely many ends. Moreover, we prove that Conjecture (A) follows from a conjecture by Walicki.

In Chapter 5 Theorem 3.2.6 will be used again in order to exploit another result from [47] for the purpose of Boolean network theory.

1.5.3 Outline of Chapter 4

The main goal of Chapter 4 is to prove the converse of each of the conjectures (A), (C) and (D). This shall be achieved with the theorems 4.3.4, 4.3.7 and 4.3.10 respectively. An important tool for this end is the concept of *convergent inflation*, introduced in Section 4.1.

The major part of the work will be done Section 4.2, culminating in Theorem 4.2.11 which states the a directed graph contains a finitary inflation of the Yablo-graph if and only if it contains a finitary convergent inflation of the Yablo-graph and that parity is respected by this equivalence. The point is that *odd* convergent inflation of the Yablo-graph cannot have kernels. The proof of this claim (Theorem 4.3.1) can be seen as a generalized form of the Yablo paradox. Sufficient conditions for digraphs without kernels have been investigated in [51], [14], [16] and [41]. (Cf. e.g. Lemma 31 in [41]). Theorem 4.3.1 provides a sharper result than any these.

Another application of convergent inflations is Theorem 4.4.2 in Section 4.4 which states the equivalence of a directed graphs containing a finitary inflation of the Yablo graph and the criterion Walicki uses in [47] for his conjecture about kernels.

1.5.4 Outline of Chapter 5

One goal of this final chapter is to show that Conjecture (A) holds under weaker assumptions than already established in Chapter 3. Theorem 5.4.6 states that any directed graph G is safe if and only if G is acyclic and contains no finitary inflation of the Yablo-graph, given that G contains only *normal* ends and at most countably many ends. (One direction of this equivalence has already been established in Chapter 4). In order to do this, a method is developed in Subsection 5.2 that seems to be more suitable for Conjecture (A) than Walicki's method from[47]. It takes into account the fact that for our purpose it suffices to focus on acyclic digraphs, and not on the larger class of digraphs without odd cycles.

A key method that is introduced in Section 5.1 and then used throughout this entire chapter is that of decomposing a Boolean network into subnetworks (cf. 2.4.6), finding a fixed point for each of them and the integrating these fixed points into a fixed point of the entire Boolean network. A precursor of this method (which has some applications in this chapter) has been the decomposition into periphery and core (Definition 2.4.89). We will primarily be concerned with Boolean networks – nevertheless, some results are formulated more generally for function networks. An important method of network decomposition is that of an *open exhaustion* (Subsection 5.1.1), which uses the topology of its dependency graph (defined in Subsection 2.4.6) in order produce a layer-wise decomposition of a Boolean network.

A particular useful type of an open exhaustion is discussed in Section 5.2. This procedure can be thought of as an inductive process, yielding a possibly transfinite sequence of larger and larger subnetworks, all of which have a fixed point. The bases case, the fact that the first subnetwork in this sequence has a fixed point, is due to an application of Theorem 3.2.6 (the characteristic graph method) to another result from [47] about digraph kernels. This inductive process can either exhaust the dependency graph completely, in which case it yields a fixed point for the entire Boolean network, or, stopping short of that, leave a non-empty subdigraph of the dependency graph as residuum. This residual digraph turns out to have a particular structure that is called *prolific*. Our strategy for proving Theorem 5.4.6 is to show that under certain circumstances every prolific directed graph contains a finitary inflation of the Yablo-graph.

The simplest case in which these circumstances are given is when the dependency graph is a *normal end* (Section 5.3). An end of a directed graph G is, roughly speaking, a subgraph H of G such that H contains a *ray* R (an infinite path) and all rays contained in H are *parallel* to R. This concept, introduced in this form by Walicki in [47], can be seen as an adaption of the usual concept of end for undirected graphs (cf. [17]), with the distinction that a certain anomaly can occur in the directed case which doesn't exist in the undirected case. An end of a directed graph is *normal* if it is free of this anomaly and behaves analogous to an end of an undirected graph.

In Section 5.4 this result is generalized to the case where a directed graph has only countably many ends and all ends are normal. An example of a directed graph that has *un*countable many ends is that of an infinite ever branching binary tree (each branch of which is an end that is simply a ray). This example, by the way, is crucial for the proof of Theorem 5.4.6.

Chapter 5 and thus our investigation concludes with Section 6.1, which is a short reflection on the methods used in order to prove special cases of the conjectures, and on the prospect of developing these methods further in order to achieve sharper results in future work.

Chapter 2

Boolean networks and semantic paradoxes

The goal of Chapter 2 is to develop a language-independent graph-theoretic account of the semantic paradoxes in terms of Boolean networks. In particular, almost all of the important concepts and theorems of [5] shall be reformulated in this more abstract setting. In Subsection 2.3.2 it is shown (or at least indicated how it can be shown) that the classical valuation scheme discussed in [5] can be conceived of as a particular Boolean network and that their concepts and results are indeed instances of our abstract reconstructions of them. For a more detailed outline of Chapter 2 the reader is referred to Subsection 1.5.1.

2.1 Basic terminology

In this section some rather basic (mostly graph-theoretic) notions are defined.

Notational conventions and general terminology

For all x, y we denote by (x, y) the *ordered pair* consisting of x and y. Given a function $f : A \to B$ and $X \subseteq A$, let $f[X] = \{y \in B \mid (\exists x \in A)(f(x) = y)\}$ be the *image of* X *under* f. Let $f \restriction X = \{(x, y) \mid x \in X \wedge f(x) = y\}$ be the *restriction of* f *to* X. Thus $f \restriction X : X \to B$. For all sets A, B let A^B be the set of all functions of type $B \to A$.

For all sets X, we denote the *identity map* on X by id_X, i.e., $\mathrm{id}_X(y) = y$, for all $y \in X$.

For functions $f : X \to Y$ and $g : Y \to Z$ let $f \circ g : X \to Z$ be defined by $(f \circ g)(x) = f(g(x))$, for all $x \in X$.

Digraphs, sub- and superdigraphs

A *directed graph* or *digraph* G is an ordered pair, consisting of a (possibly infinite) set $V(G)$, the *vertices* of G, and a set $A(G)$ of ordered pairs of elements of $V(G)$, called the *arcs* of G. This means that the set $A(G)$ is a *binary relation* with *field* $V(G)$. On the other hand, every binary relation can be conceived as a digraph. Note that in particular $(\varnothing, \varnothing)$ is a digraph.

We call the first vertex x of an arc (x, y) the *tail* of (x, y) (also denoted by $\mathrm{tail}(x, y)$) and the second vertex y the *head* of (x, y) (also denoted by $\mathrm{head}(x, y)$).

A digraph H is a *subdigraph* of G iff $V(H) \subseteq V(G)$ and $A(H) \subseteq A(G)$. In this case we also say that G *contains* H and write $H \subseteq G$. A digraph H is said to be a *superdigraph* of G iff G is a subdigraph of H. A subdigraph $H \subseteq G$ is said to be *induced* iff for all $x, y \in V(H)$, $(x, y) \in A(G)$ implies $(x, y) \in A(H)$. For $X \subseteq V(G)$ we denote by $G[X]$ the unique induced subdigraph $H \subseteq G$ such that $V(H) = X$. A subdigraph $H \subseteq G$ is said to be *spanning* iff $V(H) = V(G)$, and G is called a *spanning superdigraph* of H iff H is a spanning subdigraph of G. For digraphs G and H let us define the following operations: $G \cup H = (V(G) \cup V(H), A(G) \cup A(H))$, $G \cap H = (V(G) \cap V(H), A(G) \cap A(H))$. For $X \subseteq V(G)$ let $G \setminus X = G[V(G) \setminus X]$ and for $A \subseteq A(G)$ let $G \setminus A = (V(G), A(G) \setminus A)$.

Neighbors and degrees

For any vertex $x \in V(G)$, we call $y \in V(G)$ an *out-neighbor of x (in G)* iff $(x, y) \in A(G)$, and an *in-neighbor of x (in G)* iff $(y, x) \in A(G)$. Let $\mathrm{out}_G(x)$ be the set of all out-neighbors of x in G and $\mathrm{in}_G(x)$ be the set of all in-neighbors of x in G. We write $d^+_G(x)$ (the *out-degree of x in G*) for the cardinality of $\mathrm{out}_G(x)$ and $d^-_G(x)$ (the *in-degree of x in G*) for the cardinality of $\mathrm{in}_G(x)$.

A digraph G is said to be *finitely out-branching* iff $d^+_G(x)$ is finite for all $x \in V(G)$; it is said to be *finitely in-branching* iff $d^-_G(x)$ is finite for all $x \in V(G)$. A vertex $x \in V(G)$ is said to be a *sink* of G iff $\mathrm{out}_G(x) = \varnothing$, and a *source* of G iff $\mathrm{in}_G(x) = \varnothing$. We denote the set of all sinks of G by $\mathrm{snk}(G)$ and the set of all sources of G by $\mathrm{src}(G)$. A digraph G is said to be *sink-less* iff $\mathrm{snk}(G) = \varnothing$ and *source-less* iff $\mathrm{src}(G) = \varnothing$.

Paths, rays, walks, cycles and well-founded digraphs

A non-empty digraph P (i.e., a digraph with at least one vertex) is called a *(directed) path* (from x to y, of length n) iff there is an enumeration $(v_0, v_1, \ldots, v_n)$ of $V(P)$ such that for all $0 \le i, j \le n$, $(v_i, v_j) \in A(P)$ iff $j = i + 1$ with $x = v_0$ and $y = v_n$. Note that a digraph with one vertex and no arcs is a path of length 0. We call such a path *trivial.*

Let D be a digraph $x \neq y \in V(D)$. Then D is said to be a *double path* (from x to y) iff there are non-trivial paths $P_1 \neq P_2$ from x to y such that $V(P_1) \cap V(P_2) = \{x, y\}$ and $V(D) = V(P_1) \cup V(P_2)$ and $A(D) = A(P_1) \cup A(P_2)$. Note that it is necessary to stipulate that $P_1 \neq P_2$ in order to excluded an arc from being a double path.[1]

For any digraph G, call an infinite sequence of vertices $(v_0, v_1, \dots)$ of $V(G)$ an *infinite walk* in G iff for all $i \in \omega, (v_i, v_{i+1}) \in A(G)$. Analogously, *finite walk* is finite but non-empty sequence of vertices $(v_0, \dots, v_n)$ of $V(G)$ such that for all $0 \leq i < n, (v_i, v_{i+1}) \in A(G)$. We say that a walk (finite or infinite) in G *visits a vertex* x of G iff $x = v_i$, for all some index i of w. We say that a walk in G *visits an arc* $(x, y) \in A(G)$ iff $x = v_i$ and $y = v_{i+1}$ for some index i of w. We say that a walk (finite or infinite) is *straight* iff it visits no vertex twice.

The difference between a path and a finite walk is that a path is a digraph while a walk is a sequence of vertices in a digraph. Note that a digraph G is a path iff there is a finite walk in G that visits every vertex of G and every arc of G exactly once.

A digraph C is called a *cycle* (of length n) iff there is a finite walk $(v_0, \dots, v_n)$ in C that visits every arc of C exactly once and every vertex exactly exactly once, except for one, namely the vertex $v_0 = v_n$, which it visits twice. There may be many such walks, but all have the same length. A cycle of length 1 is called a *loop*. A cycle is said to be *odd* iff it has odd length and *even* iff it has even length. A digraph is said to be *acyclic* iff it contains no cycle.

A digraph R is said to be a *ray* iff there is an infinite walk in R that visits every vertex and every arc of R exactly once. Any subdigraph of a ray R that is itself a ray is called a *tail* of R. Hence every ray has infinitely many tails.

A digraph G is said to be *well-founded* iff there is no infinite walk in G. A digraph G is said to be *conversely well-founded* iff the *converse digraph* G^c of G is well-founded, where G^c is defined by $V(G^c) = V(G)$ and $(x, y) \in A(G^c)$ iff $(y, x) \in A(G)$.

The notion of a tail of a ray can be generalized as follows. A *tail* of a digraph G is a subdigraph of G that is induced by some $X \subseteq V(G)$ such that $G[V(G) \setminus X]$ contains no rays. Since an arc can be conceived as a digraph, danger of confusion between the notions *tail of an arc* and *tail of a digraph* could arise in principle . However, the meaning will be always clear from the context.[2]

Undirected graphs and connectivity

For any digraph G let G^{sym} be the digraph with $V(G^{\text{sym}}) = V(G)$ such that $(x, y) \in A(G^{\text{sym}})$ and $(y, x) \in A(G^{\text{sym}})$ iff $(x, y) \in A(G)$ or $(y, x) \in A(G)$. The digraph G^{sym} is always *symmetric*. We may identify symmetric digraphs with *undirected graphs*. (Cf. [3]).

A digraph G is said to be an *orientation* of an undirected graph H iff $H = G^{\text{sym}}$ and for all $x, y \in V(G) = V(H)$, either $(x, y) \in A(G)$ or $(y, x) \in A(G)$ but not both.

A digraph G is said to be an *undirected path* iff G^{sym} has an orientation that is a directed path.

[1] This condition is erroneously omitted in [5].

[2] Both ways to use the term are customary in the literature on digraphs, so we refrain from introducing new terminology.

We call G *weakly connected* if G^{sym} is connected (as an undirected graph), which is the case iff G^{sym} is *strongly connected* as a digraph, i.e., if for all $x, y \in V(G^{\mathrm{sym}})$ there is a path from x to y in G and a path from y to x in G.

A subdigraph H of G is said to be a *weak component* of G iff H is weakly connected and no proper superdigraph of H that is a subdigraph of G is weakly connected. Analogously a *strong component* can be defined. A digraph G is said to be *totally disconnected* iff $A(G) = \varnothing$.

Reachability, apgs and trees

A vertex $y \in V(G)$ is said to be *reachable (in G)* from $x \in V(G)$ iff there is a walk in G from x to y. For all digraphs G and all $x \in V(G)$ let $G\{x\}$ be the subdigraph of G that is induced by the set all vertices y of G such that y is reachable from x in G.

An *accessible pointed graph (apg)* is an ordered pair (G, x) where G is a digraph and $x \in V(G)$ such that every $y \in V(G)$ is reachable from x. We call x the *root* of the apg (G, x). By abuses of terminology we may sometimes call a digraph G an apg if there exists $x \in V(G)$ such that (G, x) is an apg. Notice that $G = G\{x\}$ whenever (G, x) is an apg.

For all digraphs G and all $X \subseteq V(G)$, let $\mathrm{Cl}_G(X)$ be the set of all $y \in V(G)$ such that there exists some $x \in X$ that is reachable in G from y. For $H \subseteq G$, let $\mathrm{Cl}_G(H) = \mathrm{Cl}_G(V(H))$. A set $X \subseteq V(G)$ is said to be *closed in G* iff $\mathrm{Cl}_G(X) = X$. In Subsection 2.4.6 we will see that the closed subsets of $V(G)$ are indeed closed with respect to a certain topology.

An *out-branching tree* is a digraph T such that there exists $r \in V(T)$ from which every $y \in V(T)$ is reachable in a unique walk. This makes (T, r) an apg and for no $r \neq y \in V(T)$, (T, y) is an apg, i.e, r is the unique root of T. A digraph G is said to be an *in-branching tree* iff its converse digraph T^c is an out-branching tree. Hence, a finite out-branching tree has a unique source (its root) and at least one but typically many sinks (its leaves), while a finite in-branching tree has a unique sink and at least one but typically many sources. For any out-branching tree (T, r) and any $x \in V(T)$, let $ht_T(x)$, the *height* of x in T, be the length of the unique path from r to x. For any in-branching tree (T, r) and any $x \in V(T)$, let $ht_T(x)$, the *height* of x in T, be the length of the unique path from x to r.

Note that every tree (out-branching or in-branching) is acyclic and contains no double-path.

A digraph G is said to be a(n) (in-branching, out-branching) *forest* iff every weak component of G is a(n) (in-branching, out-branching) tree.

Kernels

A set K of vertices of a digraph G is called *independent* iff there is no arc of G between any pair of (not necessarily distinct) elements of K; K is called *absorbent* iff for all $x \notin K$ there exists $y \in K$ such that (x, y) is an arc of G. A *kernel* of G is a subset of the set of all vertices of G that is both independent and absorbent. Note that $\varnothing$ is a kernel of the digraph $(\varnothing, \varnothing)$. -A digraph is said to be *strongly kernel-perfect* iff every of its subdigraphs has a kernel. It is said to be *kernel-perfect* iff every of its induced subdigraphs has a kernel.

Ends

The following definition from [47] is a (not totally straightforward) adaption of a concept well-known from the theory of infinite undirected graphs (cf. [17]) to infinite directed graphs. A digraph $H \subseteq G$ is said to be an *end in* G iff there exists some ray $R \subseteq G$ such that $V(H) = \mathrm{Cl}_G(R)$. A digraph G is said to be an *end* iff G is an end in G.

The Yablo-graph

As usual we denote the set of all finite ordinals by ω. Let $\mathbb{Y}$ be the digraph with $V(\mathbb{Y}) = \omega$ and $A(\mathbb{Y}) = \{(m, n) \mid m, n \in \omega \wedge m < n\}$. We call $\mathbb{Y}$ the *Yablo-graph*. It is clearly *isomorphic* (but not identical) to the digraph from Example 1.1.5. Digraph isomorphisms will be defined in Subsection 2.5.1. Of course the choice of ω as vertex set is arbitrary. At times we may identify $\mathbb{Y}$ with the graph from Example 1.1.5 (or any other isomorphic digraph) and call both, by abuse of terminology, *the* Yablo-graph.

2.2 Discrete dynamical systems

2.2.1 Dynamical systems and iteration graphs

Maybe one of the most ubiquitous type of problems in all mathematics (theoretical physics, theoretical computer science etc.) is the question of whether a given map $f : U \to U$ has a *fixed point*, i.e. some $x \in U$ such that $f(x) = x$. (Cf. e.g. [44]). A more general question is how any given point $x \in U$ behaves with respect to iterations on f.

Definition 2.2.1. For all $f : U \to U$ we define $f^0 = \mathrm{id}_U$ (the identity map on U) and $f^{n+1} = f \circ f^n$, for all $n \in \omega$. We call f^n the *n-th iteration* of f.

Definition 2.2.2. Let $U \neq \varnothing$ be a set.

1. A map $f : U \to U$ is called a *discrete dynamical system* or short a *dynamical system*.

2. For all $x \in U$, the *trajectory of* x (under f) is the sequence $\tau_f(x) = (f^n(x))_{n \in \omega}$.

3. We say that a trajectory $\tau_f(x) = (f^n(x))_{n\in\omega}$ *intersects* another trajectory $\tau_f(y) = (f^n(y))_{n\in\omega}$ (or some set X) iff $\{f^n(x)) \mid n \in \omega\} \cap \{f^n(y)) \mid n \in \omega\} \neq \varnothing$ (or $\{f^n(x)) \mid n \in \omega\} \cap X \neq \varnothing$ respectively).

4. For all $x, y \in U$, we say that x and y are *f-equivalent* (written $x \equiv_f y$) iff their trajectories have a non-empty intersection.

5. We call the equivalence classes of the $\equiv_f$-relation the *components* of the dynamical system $f : U \to U$. A system with exactly one component is said to be *connected*.

The fact that $\equiv_f$ is transitive (and thus an equivalence relation indeed) is implied by the following observation.

Proposition 2.2.3. *Let $f : U \to U$ and $x, y \in U$. If there are $m, n \in \omega$ such that $f^m(x) = f^n(y)$, then $f^{m+k}(x) = f^{n+k}(y)$ for all $k \in \omega$.*

An interesting question about a given trajectory is whether or not it shows periodic behavior.

Definition 2.2.4. We say that a sequence $(a_n)_{n\in\omega}$ of elements of a set A is

1. *periodic* iff there exists $k \in \omega$ such that for all $n \in \omega$, $a_n = a_{n+k}$,
2. *p-periodic* iff it is periodic and $p \in \omega$ is the least $k \geq 1$ such that $a_n = a_{n+k}$, for all $n \in \omega$,
3. *finally periodic* iff there exists $m \in \omega$ such that $(a_n)_{n\geq m}$ is periodic,
4. *finally p-periodic* iff there exists $m \in \omega$ such that $(a_n)_{n\geq m}$ is p-periodic,
5. *aperiodic* iff it is not finally periodic.

Dynamical systems can be visualized as *iteration graphs.* (Cf [43]).

Definition 2.2.5. The *iteration (di)graph* of the dynamical system $f : U \to U$ consists of U as the set of its vertices; moreover, for all $x, y \in U$, (x, y) is an arc of the iteration graph of f iff $y = f(x)$.

Notice that any trajectory $\tau_f(x)$ corresponds to a walk in the iteration graph of f and vice versa.

Proposition 2.2.6. *A digraph G is the iteration graph of some dynamical system iff every vertex of G has exactly one out-neighbor.*

Since this proposition establishes a canonical correspondence between dynamical systems and iteration digraphs of dynamical systems, we will call a digraph in which every vertex has a unique out-neighbor an iteration digraph (simpliciter). A weak component of an iteration digraph I is also called a *basin of I.* (Cf. [43]).

Call a digraph G *convergent* iff for all $x, y \in V(G)$ there exists $z \in V(G)$ and (maybe trivial) paths $P_x, P_y \subseteq G$ such that P_x leads from x to z and P_y leads from y to z.

Proposition 2.2.7. *Let I be an iteration digraph and B a basin of I. Then*

1. *B contains at most one cycle.*

2. *If $C \subseteq B$ is a cycle, then $B \setminus A(C)$ is either empty or consists a forest of in-branching trees, each of which has a vertex of C as its unique sink, i.e., its root.*

3. *If B contains no cycle, then B is a sink-less in-branching tree,*

4. *B is convergent.*

The following proposition states a useful criterion for the periodic behavior of a trajectory: if a trajectory repeats itself once, then it repeats itself forever.

Proposition 2.2.8. *Let $f : U \to U$ be a dynamical system and I its iteration graph. Let $x \in U$.*

1. *Then $I[\{f^k(x) \mid k \in \omega\}]$ is either a ray (if $\{f^k(x) \mid k \in \omega\}$ is infinite) or there exists some $n \in \omega$ such that $I[\{f^k(x) \mid k \geq n\}]$ is a cycle (if $\{f^k(x) \mid k \in \omega\}$ is finite). In the former case $\tau_f(x)$ is aperiodic while in the latter case $\tau_f(x)$ is finally p-periodic, where p is the length of the cycle $I[\{f^k(x) \mid k \geq n\}]$.*

2. *A trajectory $\tau_f(x)$ is*

   (a) *periodic iff there exists $k \geq 1$ such that $f^k(x) = x$,*

   (b) *finally periodic iff there exists $n \in \omega$ and $k \geq 1$ such that $f^{n+k}(x) = f^n(x)$.*

3. *If $x \equiv_f y$, then both $\tau_f(x)$ and $\tau_f(y)$ are either aperiodic or finally p-periodic for some $p \geq 1$.*

Definition 2.2.9. Let $f : U \to U$ be a dynamical system Let $\Pi_f : U \to (\omega + 1) \setminus \{0\}$ be defined by

$$\Pi_f(x) = \begin{cases} p, & \text{if } \tau_f(x) \text{ is finally } p\text{-periodic,} \\ \omega, & \text{if } \tau_f(x) \text{ is aperiodic.} \end{cases}$$

Definition 2.2.10. $f : U \to U$ be a dynamical system and $B \subseteq U$ a component if f. We say that B is *aperiodic* $\Pi_f(x) = \omega$ for some (and then all) $x \in B$ and that B is *p-periodic* iff $\Pi_f(x) = p$ or some (and then all) $x \in B$.

2.2.2 Invariant sets and attractors

This section is dedicated to a short discussion of the *f-invariant* subsets of the set of all states U a dynamical system $f : U \to U$ can assume. This concept will play a crucial role in Section 2.4 where notions of sentence-paradoxicality shall be considered.

Definition 2.2.11. Let $f : U \to U$ be a dynamical system. A set $X \subseteq U$ is said to be

1. *invariant* under f (or *f-invariant*) iff $f[X] \subseteq X$,

2. *strictly invariant* under f (or *strictly f-invariant*) iff $f[X] = X$.

Proposition 2.2.12. *For all $f : U \to U$ and $X \subseteq U$, if X is f-invariant, then $f[X]$ is f-invariant.*

Proof. Let $y \in f[X]$. Then $y = f(x)$ for some $x \in X$. Since X is f-invariant, $y \in X$ and so is $f(y)$. □

Definition 2.2.13. A set $A \subseteq U$ is said to be an *attractor* of $f : U \to U$ iff A is non-empty, strictly f-invariant and no proper subset of A is strictly f-invariant.

Definition 2.2.14. The *basin of the an attractor* A is the set of all $x \in U$ such that there exists $n \in \omega$ with $f^n(x) \in A$.

Note that in the iteration graph attractors correspond to cycles and *double-rays*. A double-ray is a digraph G such that for all $x \in V(G)$, $G\{x\}$ and the converse digraph $(G\{x\})^c$ are rays. This definition is in keeping with that of [43] and [48], where attractors are defined (only for finite dynamical systems) as a cycle of the iteration graph. Intuitively speaking, a dynamical system evolves by moving toward its attractors.

2.2.3 Boolean networks and Boolean automata

A *finite Boolean network* is usually defined as an n-tupel $(f_1, \dots, f_n)$ of functions $f_i : \{0,1\}^n \to \{0,1\}$ (cf. [43], [48]). For our purpose, we are interested in networks of arbitrary and in particular infinite cardinality.

Definition 2.2.15. For all $x \in X$ let $\pi_x : \{0,1\}^X \to \{0,1\}$, the *projection of the function space $\{0,1\}^X$ to the x-th coordinate*, be defined by $\pi_x(f) = f(x)$.

Definition 2.2.16. For all non-empty sets X,

1. a map $\Phi : \{0,1\}^X \to \{0,1\}^X$ is called a *Boolean network* on X.

2. For any Boolean network Φ on X and any $x \in X$, the *Boolean automaton at x induced by* Φ is the map $\Phi_x : \{0,1\}^X \to \{0,1\}$ given by $\pi_x \circ \Phi$.

Note that every Boolean network is a dynamical system.

Definition 2.2.17. Let X be a set.

1. Let $\Phi : \{0,1\}^X \to \{0,1\}^X$ be a Boolean network. Then we call $\{\pi_x \circ \Phi \mid x \in X\}$ the *family of Boolean automata associated with* Φ.

2. Let $\{\Phi_x \mid x \in X\}$ be a family of Boolean automata on X, i.e., a family of functions of type $\{0,1\}^X \to \{0,1\}$. Then we call $\Phi : \{0,1\}^X \to \{0,1\}^X$ defined by $\Phi(f) = \{(x, \Phi_x(f,x)) \mid x \in X\}$ the *Boolean network associated with* $\{\Phi_x \mid x \in X\}$.

Obviously, Boolean networks and families of Boolean automata are just different representations of the same concept. In particular, f is a fixed point of a Boolean network Φ iff $\pi_x \circ \Phi(f) = f(x)$ for all $x \in X$. We shall also conceive families of Boolean automata as functions of type $\{0,1\}^X \times X \to \{0,1\}$, with the convention that $\Phi(f,x) = \pi_x \circ \Phi(f)$, for all $x \in X$.

We also may make no terminological distinction between Boolean networks and families of Boolean automata and may call a function $\Phi : \{0,1\}^X \times X \to \{0,1\}$ a Boolean network (on X).

The additional structure that comes into play when considering Boolean networks instead of mere dynamical systems can, in a certain sense, be thought of as a ' 'coordinate system'. Note that the elements of the function space $\{0,1\}^X$ can be considered as vertices of a κ-dimensional cube, where $\kappa = |X|$. Focusing on a single Boolean automaton at some $x \in X$ is tantamount to projecting the network to the 'x-coordinate'. Shortly we shall analyze trajectories in terms of their projections to single coordinates. It would indeed make sense to call $|X|$ the *dimension* of a Boolean network on X. (Cf. Example 2.4.24).

Example 2.2.18. Let X be a set.

1. For all $f \in \{0,1\}^X$ and $x \in X$ let $\mathrm{Id}_X(f,x) = f(x)$. Then Id_X is a Boolean network on X. Every $f \in \{0,1\}^X$ is a fixed point of Id_X. Hence the iteration graph of Id_X has $2^{|X|}$ components each of which has cardinality 1 and consists entirely of its attractor, a loop.

2. Let $1_X(f,x) = 1$, for all $f \in \{0,1\}^X$ and $x \in X$. Then 1_X is also a Boolean network on X. A function $f \in \{0,1\}^X$ is a fixed point of 1_X iff $f(x) = 1$ for all $x \in X$, i.e., 1_X has a unique fixed point. On the other hand, each $f \in \{0,1\}^X$ is mapped by 1_X to the function that maps every $x \in X$ to 1. This means that 1_X has a single component whose attractor is a loop.

3. Likewise a Boolean network 0_X can be defined.

2.3 Dependency graphs

2.3.1 Automata dependency and dependency graphs

The first item of the following definition is an adaption from [34] and the second item is an adaption from [4] and [5].

Definition 2.3.1. Let X be a set and Φ be a Boolean network on X.

1. We say that $x \in X$ *depends on* $Y \subseteq X$ *with respect to* Φ iff for all $f, g \in \{0,1\}^X$, if $f \restriction Y = g \restriction Y$, then $\Phi(f,x) = \Phi(g,x)$.

2. Let G be a digraph with $V(G) = X$. Then Φ is said to be a *Boolean network on* G and G is said to be a *dependency graph for* Φ iff every $x \in V(G)$ depends on $\mathrm{out}_G(x)$ with respect to Φ.

3. An ordered pair (G, Φ) of a digraph G and a Boolean network Φ is said to be a *constrained Boolean network* iff Φ is a Boolean network on G.

Formally, this definition establishes a dependency relation between elements of the domain of the function space (in the above case the set X) and subsets of it. We may identify any $x \in X$ with the Boolean automaton Φ_x, i.e., with $\pi_x \circ \Phi_x$ or may think of x as a name for Φ_x.

In computational terms, the meaning of an automaton Φ_x depending on a set $Y \subseteq X$ can be explained as follows: given an *input* f, the output $\Phi_x(f)$ depends only on the restriction of f to Y. This means that Y can be thought of as the *input layer* (cf. [19]) of the automaton Φ_x, i.e., the space where the information to be processed by Φ_x is being stored. The *output layer* of Φ_x consists of the single *cell* x. In this picture, 'x depends on Y' means that Φ_x ignores everything that happens in the network outside of Y, i.e., it observes only changes in the cells $y \in Y$, each of which happens to be the output layer of an automaton Φ_y. Thus 'x depends on Y' can be read as 'the behavior of the automaton Φ_x depends only on the behavior of the automata family $\{\Phi_y \mid y \in Y\}$, i.e., the family of those automata whose output cell happens to lie in the input layer of Φ_x'.

By abuse of terminology we may sometimes call a constrained Boolean network (G, Φ) simply a Boolean network. But strictly speaking, the former belong to a different category than the latter. This difference becomes relevant in Section 2.5 where structure-preserving transformations (e.g. isomorphisms) are discussed, between Boolean networks on the one hand and between constrained Boolean networks on the other hand. For example, two constrained Boolean networks may not be isomorphic although their Boolean network parts are. (Cf. Example 2.5.23).

The following proposition is adopted from [34].

Proposition 2.3.2. *Let X be a set and Φ a Boolean network on X. Then*

1. *every $x \in X$ depends on X with respect to Φ,*

2. *if x depends on $Y \subseteq X$ with respect to Φ and if $Y \subseteq Z \subseteq X$, then x depends on Z with respect to Φ,*

3. *if x depends on $Y \subseteq X$ with respect to Φ and x depends on $Z \subseteq X$ with respect to Φ, then x depends on $Y \cap Z$,*

4. *x depends on $\varnothing$ with respect to Φ iff Φ is constant at x, i.e., $\Phi(f, x) = \Phi(g, x)$ for all $f, g \in \{0,1\}^X$.*

Consequently a Boolean network on G is also a Boolean network on every spanning superdigraph of G. Proposition 2.3.2 further implies that if X is finite, then there exists a digraph G with $V(G) = X$ such that G is a dependency graph for Φ and no proper spanning subdigraph of G is a dependency graph for Φ. We say then that G is a *minimal dependency graph* for Φ and that for all $x \in V(G)$, *x depends essentially on* $\mathrm{out}_G(x)$ *with respect to* Φ. (Cf. [5]).

There are Boolean networks that have no minimal dependency graph.

Example 2.3.3. Define $\Phi : \{0,1\}^\omega \times \omega \to \{0,1\}$ by

$$\Phi(f,n) = \begin{cases} 1, & \text{if } (\exists m \geq n)(\forall k > m)(f(k) = 0) \\ 0, & \text{else.} \end{cases}$$

Then $\mathbb{Y}$ is a dependency graph for Φ. However, it is not a minimal. Every digraph G with $V(G) = \omega$ is a dependency graph for Φ, as long as for all $n \in \omega$ there exists $\omega \ni m \geq n$ such that for all $k > m$, $(m,k) \in A(G)$. (Cf. the example in Section 2.3 of [5]).

However, it is not difficult to check that for any given digraph G (finite or infinite), the Examples of Boolean networks discussed in Subsection 2.3.4 below all have G as their minimal dependency graph.

The following will be the guiding question for the remainder of this thesis.

Question 2.3.4. *Given a dependency graph of a Boolean network, what can be said about its behavior as a dynamical system and what about its fixed points in particular?*

In other words our goal is to understand the dynamical properties of a Boolean network in terms of its structural properties, an endeavor that could be seen as an attempt to unify two strands of thought in Herzberger's work - epitomized by [24] and [25] respectively. (Cf. the remark at the beginning of Subsection 1.5.1). Question 2.3.4 will be investigated systematically from Section 2.6 onward.

2.3.2 Reconstruction of FOL frameworks

Consider the first-order language of Peano arithmetic augmented with a primitive unary predicate symbol T. We denote the set of its sentences by $\mathcal{L}_T$. We fix some coding of $\mathcal{L}_T$ into ω; for technical simplicity, we assume it is a bijection. The language of $\mathcal{L}_T$ contains a name for each sentence φ—i.e., the numeral of (the code of) φ—that we shall denote by $\ulcorner\varphi\urcorner$. Let $\mathbb{N}$ be the standard model of arithmetic and S the extension (interpretation) of the truth predicate T. We write $(\mathbb{N}, S) \vDash \varphi$ to indicate that φ is true in the model $(\mathbb{N}, S)$.

For $f \in \{0,1\}^\omega$, let $f^+ = \{n \in \omega \mid f(n) = 1\}$. Let $V : \{0,1\}^\omega \times \omega \to \{0,1\}$ be defined by

$$V(f,n) = \begin{cases} 1, & \text{if } n = \ulcorner\varphi\urcorner \wedge (\mathbb{N}, f^+) \vDash \varphi \\ 0, & \text{if } n = \ulcorner\varphi\urcorner \wedge (\mathbb{N}, f^+) \nvDash \varphi. \end{cases}$$

Then V is a Boolean network.

Then $n \in \omega$ depends on $X \subseteq \omega$ with respect to V iff for all $f, g \in \{0,1\}^\omega$, $f \restriction X = g \restriction X$ implies $V(f,n) = V(g,n)$. Given that $n = \ulcorner\varphi\urcorner$, this is equivalent to the following claim: for all $Y \subseteq \omega$, $(\mathbb{N}, Y) \vDash \varphi$ iff $(\mathbb{N}, Y \cap X) \vDash \varphi$. Hence $\ulcorner\varphi\urcorner$ depends on X with respect to V iff φ depends on X in the sense of [34].

Let G be a dependency graph for V. Let φ be an $\mathcal{L}_T$-sentence. Then $G\{\ulcorner\varphi\urcorner\}$ is *reference graph* in the sense of [4] and [5].

In the course of this thesis, we will encounter the abstract counterparts of various concepts and results that are discussed in [5], e.g. the theory of Kripke fixed points, (cf. Subsection 2.4.7) the notion of a *standard extension* of a Boolean network[3] (cf. Subsection 2.4.8) the concepts of *periphery and core* of a Boolean network (cf. Subsection 2.4.11) and the notion of a *signed dependency graph* (cf. Section 3.6.3). All these concepts have their counterparts in [5]. More precisely, their counterparts in [5] are instances of them. In this way the framework of [5] can be reconstructed from the more general Boolean network approach.

2.3.3 Constructing Boolean networks on digraphs

Instead of starting from a given Boolean network and trying to determine its dependency graph, we shall proceed the other way around in this subsection: take a given digraph G and consider some basic types of Boolean networks on G.

Definition 2.3.5. For $A \subseteq \{0,1\}$ define

- $\sup_2(A) = \begin{cases} 0, & \text{if } A = \varnothing \\ \max(A), & \text{else} \end{cases}$

- $\inf_2(A) = \begin{cases} 1, & \text{if } A = \varnothing \\ \min(A), & \text{else.} \end{cases}$

Definition 2.3.6. Let G be a digraph and $X = V(G)$. Let $x \in X$. We define several functions of type $\{0,1\}^X \to \{0,1\}$ as follows:

1. $\Phi^G_\top(f,x) = 1$,
2. $\Phi^G_\bot(f,x) = 0$,
3. $\Phi^G_\bigvee(f,x) = \sup_2\{f(x) \mid x \in \mathrm{out}_G(x)\}$,
4. $\Phi^G_\bigwedge(f,x) = \inf_2\{f(x) \mid x \in \mathrm{out}_G(x)\}$,
5. $\Phi^G_\downarrow(f,x) = 1 - \Phi^G_\bigvee(f,x)$,

[3] This corresponds to a *standard valuation scheme* in [5].

6. $\Phi^G_\uparrow(f,x) = 1 - \Phi^G_\bigwedge(f,x)$.[4]

Proposition 2.3.7. *Let G be a digraph. Then $\Phi^G_\top(\cdot,\cdot)$ $\Phi^G_\bot(\cdot,\cdot)$, $\Phi^G_\bigvee(\cdot,\cdot)$, $\Phi^G_\bigwedge(\cdot,\cdot)$, $\Phi^G_\downarrow(\cdot,\cdot)$ and $\Phi^G_\uparrow(\cdot,\cdot)$ are Boolean networks on G.*

Using Definition 2.3.6 as a toolbox, we can design on any given digraph G a Boolean network by assigning to each vertex a Boolean automaton of the above variety. We can do this in a homogeneous way, e.g. by considering $\Phi^G_\bigwedge : \{0,1\}^{V(G)} \times X \to \{0,1\}$, $(f,x) \mapsto \Phi^G_\bigwedge(f,x)$. Or, by mixing automata families up, e.g., by defining for a given partition $\{A,B\}$ of $V(G)$ a Boolean network $\Phi_{A,B} : \{0,1\}^{V(G)} \times X \to \{0,1\}$ on G by

$$\Phi_{A,B}(f,x) = \begin{cases} \Phi^G_\bigwedge(f,x), & \text{if } x \in A \\ \Phi^G_\bigvee(f,x), & \text{if } x \in B. \end{cases}$$

The following can be seen as a reconstruction of Theorem 5.15 from [5] and a generalization of Cook's result (cf. [14]) that has been discussed in Subsection 1.2.2.

Proposition 2.3.8. *For all digraphs G, there exists a bijection between the set of all kernels of G and the set of all fixed points of $\Phi^G_\downarrow$. In particular G has a kernel iff $\Phi^G_\downarrow$ has a fixed point.*

Notice that in Proposition 2.3.8, no conditions are stipulated for the digraph G, in particular not that G be sink-less - in contrast to [14], [16] and [41]. (Cf. also Subsection 1.2.2.)

Example 2.3.9. Let $V(G) = \mathbb{Z}$ and $(x,y) \in A(G)$ iff $y = x+1$ or $y = x-1$. Let $\Phi(f,x) = \mathrm{xor}(f \restriction out_G(x))$, where $\mathrm{xor}(b_1,b_2) = 1$ iff either $b_1 = 1$ or $b_2 = 1$ but not both. Then Φ is a special case of a Boolean network, a *cellular automaton* (of dimension 1) which is discussed in [7].

Example 2.3.10. Another case of a Boolean network that is cellular automaton, this time in dimension 2, is Conway's *Game of Life*. Instead of presenting it here, we refer the reader again to [7]. Game of Life is a quite interesting (family of) Boolean network(s), in in particular from a computational point of view. It is discussed thoroughly in [6].

2.3.4 Trajectories of automata

For the following recall Definition 2.2.2.

Definition 2.3.11. Let Φ be a Boolean network on X, $f \in \{0,1\}^X$ and $x \in X$. The *trajectory of f (under Φ)* at x (or the projection of $\tau_\Phi(f)$ to x) is the sequence $\tau_\Phi(f,x) = (\pi_x \circ \Phi^n(f))_{n\in\omega}$.

[4]The use of '$\downarrow$' alludes to its meaning as a logical connective and follows the convention that '$p \downarrow q$' stands for '$\neg(p \vee q)$'.The problem of representing Boolean networks in terms of Boolean expressions is investigated in Section 3.5.1

Note that the proposition that a trajectory repeats itself forever if it repeats itself once (cf. Proposition 2.2.8) doesn't hold necessarily for any of its projections. However, we have the following.

Proposition 2.3.12. *Let Φ be Boolean network on X, Then $f \in \{0,1\}^X$ is a fixed point of Φ iff $\tau_\Phi(f,x)$ is 1-periodic for all $x \in X$.*

The following observations illustrates that dependency graphs can be quite helpful when it comes to determining trajectories. The first claims that any trajectory can be reconstructed from the projections to its coordinates.

Proposition 2.3.13. *Let Φ be Boolean network on X and $f \in \{0,1\}^X$. Then $\tau_\Phi(f) = (\{(x, \pi_x \circ \Phi^n(f,x)) \mid x \in X\})_{n\in\omega}$.*

The second observation claims that, in order to determine the projection of a trajectory of f to x, we can focus only on the restriction of f to a set that x depends on.

Proposition 2.3.14. *Let Φ be Boolean network on X, $x \in X$ and $Y \subseteq X$. Then the following claims are equivalent.*

1. *x depends on Y with respect to Φ,*
2. *for all $f, g \in \{0,1\}^X$, if $f \restriction Y = g \restriction Y$, then $\tau_\Phi(f,x) = \tau_\Phi(g,x)$.*

Let us have a look how the examples from Section 1.1 can be modeled as Boolean networks. Since the examples are given in an informal manner we have to rely on our intuition. Later in Section 3.5.1 a formal language will be provided, so that process of assigning Boolean networks to sentence systems can be 'automatized'.

Example 2.3.15. Recall our informal presentation of the liar sentence from Example 1.1.1.

L: (L) is false.

Consider the following digraph: let $V(G_L) = \{L\}$ and $A(G_L) = \{(L,L)\}$. Define a Boolean automaton Φ_L on G_L by setting $\Phi_L(f,L) = \Phi^G_\downarrow(f,L)$, for all $f \in \{0,1\}^{V(G_L)}$. Then Φ_L is a Boolean network consisting of a single automaton. If we identify '0' with 'false' and '1' with 'true', then any $f \in \{0,1\}^{V(G_L)}$ can be looked at as a truth-value assignment for the sentence L. Moreover, $\Phi_L(f)$ corresponds to the intended classical evaluation of the liar sentence relative to the truth-value assignment f. As dynamical system the Boolean network Φ_L is connected and its unique component is a 2-periodic attractor.

In order to do the same analysis for the truth-teller sentence

T: (T) is true

we choose an isomorphic digraph G_T with $V(G_T) = \{T\}$ and $A(G_L) = \{(T,T)\}$. This time we define $\Phi_T(f,T) = \Phi^G_{\bigwedge}(f,T)$, for all $f \in \{0,1\}^{V(G_L)}$. The resulting dynamical system has two components, each of which is a 1-periodic attractor.

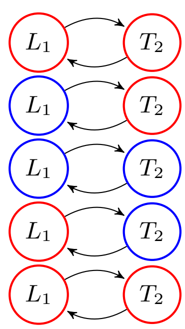
Example 2.3.16. Consider again Jourdain's paradox from Example 1.1.2.

L_1: (T_2) is false
T_2: (L_1) is true.

Let $V(G) = \{L_1, T_2\}$ and $A(G) = \{(L_2,T_1),(T_1,L_2)\}$. Define

$$\Phi(f,x) = \begin{cases} \Phi^G_{\downarrow}(f,x), & \text{if } x = L_1 \\ \Phi^G_{\bigwedge}(f,x), & \text{if } x = T_2. \end{cases}$$

The iteration-digraph of Φ is a cycle of length 4, a walk through which can be illustrated as follows, coding an assignment of 1 to a vertex by coloring it blue and an assignment of 0 by coloring it red.



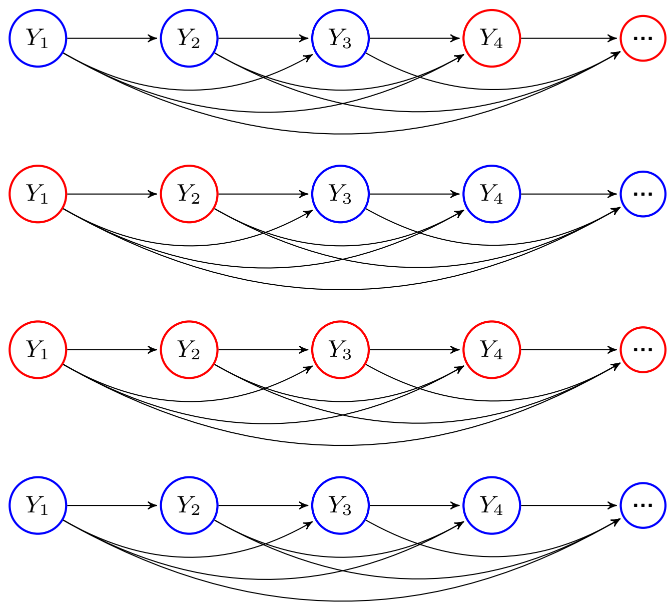
Hence Φ has a single component which is a 4-periodic attractor, i.e, every trajectory (and every $f \in \{0,1\}^{V(G)}$) is 4-periodic and the projections of it to L_1 and T_2 are both 2-periodic, but with a non-trivial *phase-shift* (by '180°', figuratively speaking).

Example 2.3.17. Yablo's paradox from Example 1.1.5:

Y_1: (Y_n) is false for all $n > 1$
Y_2: (Y_n) is false for all $n > 2$
Y_3: (Y_n) is false for all $n > 3$
Y_4: (Y_n) is false for all $n > 4$
...

Let $V(G) = \{Y_n \mid 1 \leq n \in \omega\}$ and $(Y_n, Y_m) \in A(G)$ iff $n < m$. Let $\Phi(f,x) = \Phi^G_{\downarrow}(f,x)$, for all $f \in \{0,1\}^{V(G_L)}$ and $x \in V(G)$. The following is an illustration of the trajectory of $f_3 \in \{0,1\}^{V(G)}$ defined by

$$f_3(x) = \begin{cases} 1, & \text{if } x = Y_n \wedge n \leq 3, \\ 0, & \text{if } x = Y_n \wedge n > 3. \end{cases}$$



The trajectory $\tau_\Phi(f_3)$ enters its attractor, which is a cycle of length 2 at step $\Phi^2(f_3)$. It is not difficult to see that every trajectory enters this cycle (after at most 2 or 3 steps). Hence Φ is connected and has a 2-periodic attractor. Every projection of every trajectory is also finally 2-periodic and they are all *synchronized* i.e., there is no phase-shift.

Let us conclude this subsection with an observation that relates the periodicity of trajectories to the periodicity of their projections.

Proposition 2.3.18. *Let Φ be Boolean network on X, $f \in \{0,1\}^X$ and $1 \leq p < \omega$. If $\tau_\Phi(f)$ is finally p-periodic, then for all $x \in X$ there exists $q_x \leq p$ such that $\tau_\Phi(f, x)$ is finally q_x periodic.*

The converse does not hold, as it will be illustrated by an Example 2.4.8 in the following section.

2.4 Paradoxical sentences and automata

Having discussed the periodic behavior of individual automata, a natural question is whether the fact that a Boolean network has a fixed point or not can be understood in terms of the behavior of all its single automata. In this context, an automaton whose behavior makes it impossible for a Boolean network to have a fixed point could be called *paradoxical*.

Since Boolean automata correspond to sentences (if a Boolean network is interpreted as a sentence system), this raises the question how existing accounts of sentence-paradoxicality can be reformulated for Boolean networks.

Well known formal accounts of sentence-paradoxicality are first and foremost those of Kripke ([32]) on the one hand (the *inductive* account) and of Herzberger ([26] and [25]) and Gupta ([22]) and Gupta and Belnap ([23]) on the other hand (the *semi-inductive* account).

In the existing graph-theoretic frameworks sentence-paradoxicality often does not play a central role. Neither Cook ([14],[16]) nor Rabern et al. ([41], [40]) give a definition of a paradoxical sentence, only of a paradoxical sentence system. In [46], the phenomenon of sentence-paradoxicality is mentioned but given a rather perfunctory treatment. Beringer and Schindler ([4], [5]) on the other hand put much emphasis on treating paradoxical sentences within their graph-theoretic framework and on investigating relations to Kripke's fixed point theory.

In this section, we will reformulate the inductive (Kripke), the semi-inductive (Herzberger) and the reference-based (Beringer and Schindler) accounts of sentence paradoxicality for Boolean networks. It will be shown that all these approaches can be understood and reformulated in terms of invariant subsets of the network's function space. In order to formulate Kripke's theory of fixed points and accommodate a three-valued logic, the notion of a *general function network* shall be introduced in Subsection 2.4.3.

2.4.1 Notions of paradoxical automata

Given a Boolean network Φ on a set X, what could a reasonable definition of x being paradoxical look like? The following first attempt could be seen as a conceptual precursor to Herzberger's approach which will be discussed in the next subsection.

Definition 2.4.1. Let Φ be a Boolean network on a set X. Then $x \in X$ is said to be

1. *trajectory-paradoxical* w.r.t. Φ iff there exists no $f \in \{0,1\}^X$ such that $(\Phi^n(f))(x) = (\Phi^m(f))(x)$ for all $n, m \in \omega$.
2. *trajectory-hypodoxical* w.r.t. Φ iff there exist $f_0, f_1 \in \{0,1\}^X$ such that $(\Phi^n(f_0))(x) = (\Phi^m(f_0))(x) = 0$ for all $n, m \in \omega$ and $(\Phi^n(f_1))(x) = (\Phi^m(f_1))(x) = 1$ for all $n, m \in \omega$.

E.g., the liar is trajectory-paradoxical and the truth-teller is trajectory-hypodoxical.

The next definition assigns to each Boolean network a family of potential notions of paradoxicality on automata level. In the following it will be shown that each notion of automata-paradoxicality of the inductive (Kripke) or semi-inductive (Herzberger) family can be expressed by an appropriate subset of $\mathfrak{S}_\Phi$, which, in turn, corresponds to the coarsest notion of paradoxicality, i.e., trajectory-paradoxicality.

Definition 2.4.2. Let Φ be a Boolean network on a set X.

1. Let $\mathfrak{S}_\Phi$ be the set of all $\varnothing \neq \Sigma \subseteq \{0,1\}^X$ s.t. Σ is invariant under Φ.
2. Let $\Sigma(x) = \{v \in \{0,1\} \mid \exists f \in \Sigma(f(x) = v)\}$, for all $\Sigma \in \mathfrak{S}_\Phi$, $x \in X$.
3. We say that Σ *determines a value for* x iff $\Sigma(x) = \{v\}$ (for $v \in \{0,1\}$) and also write $\Sigma(x) = v$ in this case.
4. Let $\varnothing \neq \mathfrak{M} \subseteq \mathfrak{S}_\Phi$ and $x \in X$. Then
   (a) x is said to be $\mathfrak{M}$-*paradoxical* iff there exists no $\Sigma \in \mathfrak{M}$ that determines a value for x,
   (b) x is said to be $\mathfrak{M}$-*hypodoxical* iff there are $\Sigma_0, \Sigma_1 \in \mathfrak{M}$ such that $\Sigma_0(x) = 0$ and $\Sigma_1(x) = 1$,
   (c) x is said to be $\mathfrak{M}$-*intrinsic* iff x is neither $\mathfrak{M}$-paradoxical nor $\mathfrak{M}$-hypodoxical.

The elements of $\mathfrak{S}_\Phi$ could be interpreted as *multi-decorations* of X in the sense of [5].

Proposition 2.4.3. *Let Φ be a Boolean network on X and $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{S}_\Phi$. Then for all $x \in X$,*

1. *if x is $\mathfrak{N}$-paradoxical, then x is $\mathfrak{M}$-paradoxical,*
2. *if x is $\mathfrak{M}$-hypodoxical, then x is $\mathfrak{N}$-hypodoxical.*

It could be an interesting investigation to take this definition as a basis for an *axiomatic account of sentence-paradoxicality* by stipulating further requirements on $\mathfrak{M} \subseteq \mathfrak{S}_\Phi$. For now, let us just note that the following property of $\mathfrak{M}$ should be certainly among the necessary requirements.

Proposition 2.4.4. *Let Φ be a Boolean network on X. Then there is a bijection between the set of all fixed points of Φ and the set $\{\Sigma \in \mathfrak{S}_\Phi \mid |\Sigma| = 1\}$.*

Definition 2.4.5. Let Φ be a Boolean network and $\varnothing \neq \mathfrak{M} \subseteq \mathfrak{S}_\Phi$. Then

1. $\Sigma \in \mathfrak{M}$ is said to be an *atom* of $\mathfrak{M}$ iff there exists no $\Sigma' \in \mathfrak{M}$ such that $\Sigma' \subsetneq \Sigma$.

2. $\mathfrak{M}$ is said to be *atomic* iff for all $\Sigma \in \mathfrak{M}$ there exists an atom Σ' of $\mathfrak{M}$ such that $\Sigma' \subseteq \Sigma$.

Proposition 2.4.6. *Let Φ be a Boolean network on X. Then $x \in X$ is trajectory-paradoxical w.r.t. Φ iff x is $\mathfrak{S}_\Phi$-paradoxical; and $x \in X$ is trajectory-hypodoxical w.r.t. Φ iff x is $\mathfrak{S}_\Phi$-hypodoxical.*

Proof. Both claims follow from the fact that the set $\{\Phi^n(f) \mid n \in \omega\}$ is invariant under Φ, for all $f \in \{0,1\}^X$. □

Proposition 2.4.7. *Let Φ be a Boolean network on X. Then $\Sigma \subseteq \{0,1\}^X$ is an atom of $\mathfrak{S}_\Phi$ iff Σ is an attractor of Φ.*

2.4.2 Transfinite trajectories

The following is an example of a Boolean network Φ such that $\mathfrak{S}_\Phi$ is not atomic. It can be seen as a motivation for prolonging trajectories transfinitely as it is done in [26].

Example 2.4.8. Consider the following version of Example 1.1.7:

M_0: (M_ω) is false
M_1: (M_0) is true
M_2: (M_0) and (M_1) are true
M_3: (M_0) and (M_1) and (M_2) are true
...
M_ω: (M_n) is true for all $n \in \omega$.

Let $V(G) = \{M_\alpha \mid \alpha \in \omega + 1\}$ and $(M_\alpha, M_\beta,) \in A(G)$ iff $\alpha > \beta$ or if $\alpha = 0$ and $\beta = \omega$. Let

$$\Phi(f,x) = \begin{cases} \Phi^G_\downarrow(f,x), & \text{if } x = M_0 \\ \Phi^G_\bigwedge(f,x), & \text{if } x = M_\alpha \wedge \alpha > 0. \end{cases}$$

Let $f_1 \in \{0,1\}^{V(G)}$ be the function that assumes constantly the value 1. Let us write functions in $\{0,1\}^{V(G)}$ as sequences of length $\omega + 1$, e.g.,

$f_1 = (1, 1, ...1)$.

The first position of such a sequence represents the value assigned to M_0 and the last position the value assigned to M_ω. With this convention, the trajectory of f_1 can be written as follows.

$(1, 1, ...1)$
$(0, 1, ...1)$
$(0, 0, ...0)$
$(1, 0, ...0)$

$(1, 1, ...0)$
$(1, 1, 1...0)$
$\cdots$

This sequence is clearly aperiodic while its projection to any M_n (for $n \in \omega$) is finally 1-periodic with the value 1 repeating itself. Its projection to M_ω is also finally 1-periodic, assuming the value 0. Let Σ be the set of all $f \in \{0,1\}^{V(G)}$ that occur in the trajectory of f_1. Then Σ is Φ-invariant but contains no non-empty subset that is strictly Φ-invariant. Hence $\mathfrak{S}_\Phi$ is not atomic.

It is not difficult to see that every trajectory intersects with the trajectory of f_1. Hence Φ has one component, which is aperiodic in the sense of Definition 2.2.10.

However, one might argue that this example shows a form of *transfinite* periodicity, based on the intuition that the trajectory *converges* to the function f_ω that assumes constantly the value 1 except at M_ω where it assumes the value 0. Moreover, $\Phi(f_\omega) = f_1$. This last step could be regarded as closing a cycle of length ω.

Let us conclude this discussion with the remark that the Boolean network Φ constitutes some kind of *anomaly* for the $\mathfrak{S}_\Phi$-account of sentence paradoxicality: No $x \in V(G)$ is $\mathfrak{S}_\Phi$-paradoxical (i.e., trajectory-paradoxical) but nevertheless Φ has no fixed point. This particular class of anomalies (stemming from infinite liar cycles) can be treated by further refinements of the notion of sentence paradoxicality, i.e., the accounts of Herzberger and Kripke. However, none of these accounts is free from anomalies. (Cf. Example 2.4.75 below.)

Transfinite trajectories (or *revision sequences*) were first studied by Herzberger in [25] and [26] and by Gupta in [22]. The crucial point is the definition of a limit rule. In the following we shall adapt Herzberger's framework to the context of Boolean networks.

Definition 2.4.9. Let Φ be a Boolean network on X. We define recursively for all ordinals α the *α-th iteration* Φ^α *of* Φ as follows.

- Let $\Phi^0 = \mathrm{Id}_X$ (cf. Example 2.2.18).
- For all ordinals α let $\Phi^{\alpha+1} = \Phi \circ \Phi^\alpha$.
- Let λ be a limit-ordinal. For all $f \in \{0,1\}^X$ and $x \in X$ define

$$(\Phi^\lambda(f))(x) = \begin{cases} 1, & \text{if } \exists \alpha < \lambda : \forall \beta(\alpha \leq \beta < \lambda : (\Phi^\beta(f))(x) = 1 \\ 0, & \text{if } \exists \alpha < \lambda : \forall \beta(\alpha \leq \beta < \lambda : (\Phi^\beta(f))(x) = 0 \\ 0, & \text{else.} \end{cases}$$

Note that Φ^λ is a well-defined function of type $\{0,1\}^X \to \{0,1\}^X$. For the following definition also cf. [26].

Definition 2.4.10. Let Φ be a Boolean network on X, $x \in X$ and $f \in \{0,1\}^X$.

- Say that x is *stable at* $\alpha \in On$ relative to f w.r.t. Φ iff for all $\beta \geq \alpha$: $(\Phi^{\beta}(f))(x) = (\Phi^{\alpha}(f))(x)$.
- Say that x *finally assumes the value* $v \in \{0,1\}$ relative to f w.r.t. Φ iff x is stable at some $\alpha \in On$ relative to f w.r.t. Φ and $(\Phi^{\alpha}(f))(x) = v$.
- Call x, *somewhere stable w.r.t.* Φ iff there is some $f \in \{0,1\}^X$ and $\alpha \in On$ such that x is stable at α relative to f w.r.t. Φ.
- Call x *nowhere stable* iff y is not somewhere stable w.r.t. Φ.

Definition 2.4.11. Let Φ be a Boolean network on X. A Boolean automaton $x \in X$ is said to be

1. *Herzberger-paradoxical w.r.t.* Φ iff x is nowhere stable w.r.t. Φ,
2. *Herzberger-hypodoxical w.r.t.* Φ iff there are $f_0, f_1 \in \{0,1\}^X$ such that x finally assumes the value 0 relative to f_0 w.r.t. Φ and x finally assumes the value 1 relative to f_1 w.r.t. Φ.

Definition 2.4.12. Let Φ be a Boolean network on X. We say that $\mathcal{F} \subseteq \{0,1\}^X$ is

1. λ-*invariant under* Φ iff $\Phi^{\alpha}[\mathcal{F}] \subseteq \mathcal{F}$, for all $\alpha \in On$.
2. *strictly* λ-*invariant under* Φ iff $\mathcal{F}$ is λ-invariant under Φ and for all $f \in \mathcal{F}$ there exist $g \in \mathcal{F}$ and $\alpha \in On$ such that $\Phi^{\alpha}(g) = f$,
3. a λ-*attractor* of Φ iff $\mathcal{F} \neq \varnothing$, $\mathcal{F}$ is strictly λ-invariant under Φ and no proper subset of $\mathcal{F}$ has these properties.

Note $\Phi^0[\mathcal{F}] = \mathcal{F} \subseteq \mathcal{F}$ and that by Proposition 2.2.12 $\Phi[\mathcal{F}] \subseteq \mathcal{F}$ implies that $\Phi^n[\mathcal{F}] \subseteq \mathcal{F}$ for all $n \in \omega$.

Clearly, the union of the elements of every *revision sequence* $(\Phi^{\alpha}(f))_{\alpha \in On}$ (or *transfinite trajectory*) is λ-invariant under Φ.

Now we can reformulate an important result from [26].

Proposition 2.4.13. *(Herzberger) Let* Φ *be a Boolean network on* X *and* $\mathcal{F} \subseteq \{0,1\}^X$.

1. *Then there exists some* $\xi(\mathcal{F}) \in On$ *such that* $\bigcup_{\xi(\mathcal{F}) \leq \alpha \in On} \Phi^{\alpha}[\mathcal{F}]$ *is strictly* λ-*invariant under* Φ.
2. *If* $\mathcal{F} \neq \varnothing$ *and* λ-*invariant under* Φ, *then* $\mathcal{F} \supseteq (\bigcup_{\xi(\mathcal{F}) \leq \alpha \in On} \Phi^{\alpha}[\mathcal{F}]) \neq \varnothing$.

Corollary 2.4.14. *Let* Φ *be a Boolean network on* X *and* $\varnothing \neq \mathcal{F} \subseteq \{0,1\}^X$ *be* λ-*invariant under* Φ. *Then there exists a unique* $\varnothing \neq \mathcal{F}' \subseteq \mathcal{F}$ *such that* $\mathcal{F}'$ *is strictly* λ-*invariant under* Φ *and no* $\mathcal{F}' \subsetneq \mathcal{G} \subseteq \mathcal{F}$ *is strictly* λ-*invariant under* Φ.

This corollary has a counterpart in Kripke's theory of fixed points that will be discussed later: every sound set can be extended to a maximal fixed point. (cf. Theorem 2.4.55).

Definition 2.4.15. Let Φ be a Boolean network on X. Let $\mathfrak{L}_\Phi$ be the set of all $\mathcal{F} \subseteq \{0,1\}^X$ such that $\mathcal{F}$ is λ-invariant under Φ.

Proposition 2.4.16. *For all Boolean networks Φ, $\mathfrak{L}_\Phi \subseteq \mathfrak{S}_\Phi$.*

Corollary 2.4.17. *Let Φ be a Boolean network on X and $x \in X$. Then*

1. *x is Herzberger-paradoxical w.r.t. Φ iff x is $\mathfrak{L}_\Phi$-paradoxical,*

2. *x is Herzberger-hypodoxical w.r.t. Φ iff x is $\mathfrak{L}_\Phi$-hypodoxical.*

Proposition 2.4.18. *Let Φ be a Boolean network on X. Then there is a bijection between the set of all fixed points of Φ and the set $\{\Sigma \in \mathfrak{L}_\Phi \mid |\Sigma| = 1\}$.*

Corollary 2.4.19. *For every Boolean network Φ on any set X,*

1. *$\mathfrak{L}_\Phi$ is atomic,*

2. *$\Sigma \in \{0,1\}^X$ is an atom of $\mathfrak{S}_\Phi$ iff Σ is a λ-attractor of Φ.*

This implies that anomalies of the type of Example 2.4.8 are avoided by Herzberger's approach. The price for this, in comparison with the $\mathfrak{S}_\Phi$-account, is a certain arbitrariness in the choice of the limit rule.

2.4.3 General function networks

This and the next subsections introduce concepts and tools that all shall play their role in later parts of this thesis, in particular in the formulation of Kripke's fixed point theory for Boolean networks in Subsection 2.4.7.

Definition 2.4.20. Let X be a set.

1. Let $(S_x)_{x \in X}$ be a family of non-empty classes. We call $\Sigma = \bigtimes_{x \in X} S_x$ a *type on X*, where $\bigtimes_{x \in X} S_x$ is the set of all functions f with $\mathrm{dom}(f) = X$ such that $f(x) \in S_x$. In other words, all $f \in \Sigma$ are functions of type $\Pi_{x:X} S_x$.

2. A *function network* of type Σ on X is a map $\Phi : \Sigma \to \Sigma$, where Σ is a type on X.

3. To every type $\Sigma = \bigtimes_{x \in X} S_x$ on X and every $x \in X$ we associate a map $\pi_x : \Sigma \to S_x$ defined by $\pi_x(f) = f(x)$ which we call the *projection* of Σ to x.

4. For all $x \in X$ define $\Phi_x : \Sigma \to S_x$ by $\Phi_x(f) = \pi_x \circ \Phi(f)$, the projection of Φ to x.

Clearly, every Boolean network is a function network.

Definition 2.4.21. Let X be a set and $\Sigma = \bigtimes_{x \in X} S_x$ a type on X.

1. A function network of type Σ is said to be a *Boolean (function) network* iff $\Sigma = \{0,1\}^X$.
2. A function network of type Σ is said to be a *finitary* iff S_x is finite for all $x \in X$.

Most of what has been said about Boolean networks so far also applies to function networks in general. In analogy to the correspondence between Boolean networks and families of Boolean automata in Subsection 2.2.3, function networks can be conceived as *families of functions* or as functions of type $\Pi_{((f,x):\Sigma \times X)} S_x$, i.e., as functions Φ that assign to each $(f,x) \in \Sigma \times X$ some $\Phi(f,x) \in S_x$.

Definition 2.4.22. Let X be a set and Φ a function network of type $\Sigma = \bigtimes_{x \in X} S_x$.

1. We say that x *depends on* $Y \subseteq X$ *with respect to* Φ iff for all $f, g \in \Sigma$, if $f \restriction Y = g \restriction Y$, then $\Phi(f,x) = \Phi(g,x)$.
2. Let G be a digraph with $V(G) = X$. Then Φ is said to be a *function network on* G and G is said to be a *dependency graph for* Φ iff and every $x \in V(G)$ depends on $\mathrm{out}_G(x)$ with respect to Φ.

Proposition 2.4.23. *Let X be a set and Φ be a function network of type $\Sigma = \bigtimes_{x \in X} S_x$. Then*

1. *every $x \in X$ depends on X with respect to Φ,*
2. *if x depends on Y with respect to Φ and if $Y \subseteq Z$, then x depends on Z with respect to Φ,*
3. *if x depends on Y with respect to Φ and x depends on Z with respect to Φ, then x depends on $Y \cap Z$,*
4. *x depends on $\varnothing$ with respect to Φ iff Φ is constant at x.*

Example 2.4.24. This is an example from calculus. Its main purpose is to illustrate the claim that the additional structure that a function network (or a Boolean network for that matter) has over mere dynamical systems can be looked at as 'coordinate system' indeed. Another purpose is to hint at how continuous dynamical systems can be approximated by discrete dynamical systems.

Let $B = \{x_1, x_2\}$ be a orthonormal basis of $\mathbb{R}^2$ and π_1, π_2 the projections of $\mathbb{R}^2$ to the subspaces of $\mathbb{R}^2$ spanned by x_1 and x_2 respectively. Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field and $v_1, v_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi_1 \circ v$ and $\pi_2 \circ v$ respectively. For all $k \in \omega$ define $\Phi_{v,k} : \mathbb{R}^B \times B \to \mathbb{R}$ by

$$\Phi_{v,k}(s,y) = \begin{cases} s(x_1) + \frac{1}{k} v_1(s(x_1), s(x_2)), & \text{if } y = x_1 \\ s(x_2) + \frac{1}{k} v_2(s(x_1), s(x_2)), & \text{if } y = x_2. \end{cases}$$

Clearly $\Phi_{v,k}$ is a function network on B. Note that elements of $\mathbb{R}^B$ can be conceived as points in $\mathbb{R}^2$ represented through the basis (or coordinate system) B. An interpretation of $\Phi_{v,k}$ and the significance of k will be discussed below.

First let us consider the question of dependency. If v is constant, then x_1 depends on $\varnothing$ and x_2 depends on $\varnothing$ with respect $\Phi_{v,k}$. If v is the identity map i.e., $v(x,y) = (x,y)$ then x_1 depends essentially on $\{x_1, x_2\}$ and x_2 depends essentially on $\{x_1, x_2\}$, i.e., the complete digraph on $\{x_1, x_2\}$ is the only dependency graph for $\Phi_{v,k}$. Can every digraph with vertices $\{x_1, x_2\}$ be realized by some v as essential dependency graph of $\Phi_{v,k}$?

Now to the interpretation. A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}^2$ is said to be a solution of initial value problem

$$z'(t) = v(x,y) \text{ (1a)}$$
$$z(0) = (x_0, y_0) \text{ (1b)}$$

(associated to the autonomous first order differential equation (1a)) if and only if $\varphi'(t) = v(x,y)$ and $\varphi(0) = (x_0, y_0)$. Physically φ could be interpreted as the trajectory of a particle that is at time $t = 0$ at (x_0, y_0) an moves through the plane with velocity $v(x,y)$ whenever it is at (x,y). In the discrete dynamical system $\Phi_{v,k}$ the particle doesn't move continuously but jumps from place to place. The value $\frac{1}{k}$ can be seen a the length of the time interval between two such jumps: the shorter this interval, the smaller is the displacement, given a fixed velocity field. The use of a coordinate system B helps to describe and analyze the problem in the usual manner. Clearly, the fixed points of $\Phi_{v,k}$ are exactly the points of $\mathbb{R}^2$ where v vanishes. (E.g. in the case where v is the identity $(0,0)$ is the unique fixed point). The trajectory $\tau_{v,k}(s_0)$ of s_0 under $\Phi_{v,k}$ can be seen as an infinite polygon chain $P_k^v(s_0)$ that converges (in a sense that can be made precise, at least under favorable circumstances - e.g., if v has only finitely many zeros) to a solution of (1) (at least locally), when $k \to \infty$.

Example 2.4.25. Function networks can serve as models for neural networks. For details the reader is referred to Chapter 12 of [43], where, among other things, a discussion of the famous Hopfield-model can be found. English-language sources on the Hopfield-model are e.g., [27] and [48].

2.4.4 Subtypes

Definition 2.4.26. Let X be a set, $\Sigma = \bigtimes_{x \in X} S_x$ and $\Sigma' = \bigtimes_{x \in X} S'_x$ be types on X. Then Σ' is said to be a *subtype* of Σ (written $\Sigma' \sqsubseteq \Sigma$) iff $S'_x \subseteq S_x$, for all $x \in X$.

Proposition 2.4.27. *Let $\Phi : \Sigma \to \Sigma$ be a function network on X and $\Sigma' \sqsubseteq \Sigma$. If Σ' is Φ-invariant i.e., $\Phi[\Sigma'] \subseteq \Sigma'$, then $\Phi \restriction \Sigma'$ is a function network on X of type Σ'.*

Definition 2.4.28. Let $\Phi : \Sigma \to \Sigma$ and $\Phi' : \Sigma' \to \Sigma'$ be function networks on a set X and $\Sigma' \sqsubseteq \Sigma$. We say that Φ is an *extension* of Φ' iff Σ' is Φ invariant and $\Phi' = \Phi \restriction \Sigma'$.

Definition 2.4.29. Let X be a set. A type $\Sigma = \bigtimes_{x \in X} S_x$ is said to be

1. *trivial iff* $|S_x| = 1$ for all $x \in X$,
2. *finitary iff* $|S_x| < \omega$ for all $x \in X$,
3. *sub-boolean iff* $S_x \subseteq \{0, 1\}$ for all $x \in X$.

Definition 2.4.30. Let Φ be a function network of type Σ. A subtype $\Gamma \sqsubseteq \Sigma$ is said to be Φ-*irreducible* iff there is no proper subtype $\Gamma' \sqsubseteq \Gamma$ that is invariant under Φ.

2.4.5 A topology for the state space

The goal of this subsection is to define a topology on the state space (or function space) of a finitary function network. For this aim, a recapitulation of some basic topological concepts is needed. (The reader can find this material also in any introductory text, e.g., in [31].)

Recall that for any set X a set $\mathcal{T}$ of subsets of X is said to be a *topology for* X iff (i) for all finite $\mathcal{I} \subseteq \mathcal{T}$, $\bigcap \mathcal{I} \in \mathcal{T}$, (ii) for all $\mathcal{I} \subseteq \mathcal{T}$, $\bigcup \mathcal{I} \in \mathcal{T}$ and (iii), $X, \varnothing \in \mathcal{T}$. If $\mathcal{T}$ is a topology for X, then the pair $(X, \mathcal{T})$ is said to be a *topological space*. A set $Y \subseteq X$ is said to be *open* (w.r.t. $\mathcal{T}$) iff $Y \in \mathcal{T}$; it is said to be *closed* (w.r.t. $\mathcal{T}$) iff $X \setminus Y \in \mathcal{T}$. A set $\mathcal{B} \subseteq \mathcal{T}$ is said to be a *basis for* $\mathcal{T}$ iff for all $Y \in \mathcal{T}$ there exists $\mathcal{I} \subseteq \mathcal{B}$ such that $Y = \bigcup \mathcal{I}$.

Given any set $\mathcal{B}$ of subsets of X that is closed under finite intersections, we say that the set $\{\bigcup \mathcal{I} \mid \mathcal{I} \subseteq \mathcal{B}\}$ is the *topology generated by* $\mathcal{B}$. In this case $\mathcal{B}$ is a basis for the topology generated by $\mathcal{B}$.

Example 2.4.31. Let X be a set.

- Then $\wp(X)$ is a topology for X, the so called *discrete topology*. The set of all singleton subsets of X is a basis for $\wp(X)$.
- For $S \subseteq X$, the set $\{Y \cap S \mid Y \in \mathcal{T}\}$ is a topology for S, the so called *subspace topology*.

Definition 2.4.32. Given any family $(S_x, \mathcal{T}_x)_{x \in X}$ of topological spaces, let the *product topology* for $\bigtimes_{x \in X} S_x$ be the topology generated by the set

$$\{\textstyle\bigcap_{x \in Y} \pi_x^{-1}(U_x) \mid U_x \in \mathcal{T}_x \wedge Y \subseteq X \wedge |Y| < \omega\},$$

where $\pi_x : \bigtimes_{y \in X} S_y \to S_x$ is given by $\pi_x(f) = f(x)$.

Now we define a topology on finitary types as follows.

Definition 2.4.33. For any finitary $\Sigma = \bigtimes_{x \in X} S_x$ let $\mathcal{T}(\Sigma)$ be the product topology for $\bigtimes_{x \in X} S_x$ given the family $(S_x, \wp(S_x))_{x \in X}$.

A topological space $(X, \mathcal{T})$ is said to be *compact* iff for all $\mathcal{I} \subseteq \mathcal{T}$, if $X = \bigcup \mathcal{I}$, then there exists some finite $\mathcal{F} \subseteq \mathcal{I}$ such that $X = \bigcup F$ - in other words iff every open cover has a finite subcover. The following is an immediate consequence of Tychonov's theorem (cf. [31]) which claims that a product space is compact given that each of its factors is compact.

Proposition 2.4.34. *Let $\Sigma = \bigtimes_{x \in X} S_x$ be finitary. Then*

1. *$(\Sigma, \mathcal{T}(\Sigma))$ is a compact topological space,*
2. *for all finite sets $Y \subseteq X$ and all $\mathcal{A} \subseteq \bigtimes_{x \in Y} S_x$, $\{f \in \Sigma \mid f \restriction Y \in \mathcal{A}\}$ is both open and closed in $\mathcal{T}(\Sigma)$.*

Proof. The first claim follows from Tychonoff's theorem, since every $(S_x, \wp(s_x))$ is compact (since it is finite). The second claim follows from the definition of the product topology (Definition 2.4.32) and the fact that every S_x is finite. □

For the following definition and proposition also cf. [31] p. 135.

Definition 2.4.35. A family $(X_i)_{i \in \mathcal{I}}$ of sets is said to have the *finite intersection property* iff $\varnothing \neq \bigcap_{i \in \mathcal{F}} X_i$, for all finite $\mathcal{F} \subseteq \mathcal{I}$.

Proposition 2.4.36. *A topological space is compact iff every family of closed sets that has the finite intersection property has a non-empty intersection.*

Proof. Cf. [31], Chapter 5, Theorem 1. □

This has the following consequence.

Proposition 2.4.37. *Let $(X, \mathcal{T})$ be a compact topological space. Let ξ be an ordinal and $(A_\alpha)_{\alpha < \xi}$ be a sequence of nonempty closed subsets of X that is $\subseteq$-descending. Then $\bigcap_{\alpha < \xi} A_\alpha$ is nonempty and closed.*

Proof. Let $A_{\alpha_0} \supseteq .... \supseteq A_{\alpha_n}$ be a finite subsequence of $(A_\alpha)_{\alpha < \xi}$. Then $A_{\alpha_0} \cap .... \cap A_{\alpha_n} = A_{\alpha_n} \neq \varnothing$ by premise. Hence $(A_\alpha)_{\alpha < \xi}$ has the finite intersection property. Hence, by the previous proposition $\bigcap_{\alpha < \xi} A_\alpha$ is nonempty. □

When it comes to finitary types we get the following application.

Proposition 2.4.38. *Let $\Sigma = \bigtimes_{x \in X} S_x$ be finitary and $(\Gamma_n)_{n \in \omega}$ a sequence of subsets of Σ such that $\Gamma_m \supseteq \Gamma_n$ whenever $m < n$. Then*

1. *$\bigcap_{n \in \omega} \Gamma_n$ is nonempty and closed, if every Γ_n is closed.*
2. *If for all $m < n$, $\Gamma_m \sqsupseteq \Gamma_n$ then $\bigcap_{n \in \omega} \Gamma_n = \bigtimes_{x \in X} (\bigcap_{k \in \omega} \pi_x[\Gamma_k])$.*

In particular, the intersection of a descending sequence of types is a type.

Let us conclude this subsection with a brief summary of how the account of Herzberger can be described in topological terminology. (Definitions of the topological notions used in the following can be found in any textbook, e.g. in [31]).

For every finitary type Σ the topological space $(\Sigma, \mathcal{T}(\Sigma))$ is a compact Hausdorff-space. (It is not necessarily sequence-compact). A sequence converges iff it converges pointwise. If a trajectory converges in $\mathcal{T}(\Sigma)$, then it converges to its Herzberger-limit (cf. Definition 2.4.9). A network contains no trajectory-paradoxical automaton iff the limit of every trajectory coincides with its Herzberger limit. If X is countable, then the Herzberger-limit of any trajectory is an accumulation point of this trajectory. Every trajectory has at least one accumulation point. It has more than one iff it does not converge.

2.4.6 Subnetworks and digraph topology

This subsection introduces a concept that shall be of crucial importance for the remainder of this thesis, in particular for the Chapters 3 and 5. While subsets of the state space of a function network have been the topic of Subsection 2.4.4, in the following we shall be concerned with subdigraphs of the dependency graph. First let us see how a digraph can be considered as a topological space.

Definition 2.4.39. Let G be a digraph. A set $X \subseteq V(G)$ is said to be *open* in G iff there are no $x \in X$ and $y \in V(G) \setminus X$ such that $(x, y) \in A(G)$. A set $X \subseteq V(G)$ is said to be *closed* in G iff $V(G) \setminus X$ is open in G.

Proposition 2.4.40. *Let G be a digraph and $X \subseteq V(G)$. Then X is open in G iff for all $x \in X$, $G\{x\} \subseteq G[X]$.*

The open sets of a digraph G form indeed a topology on $V(G)$, a rather special one, where even arbitrary intersections of open sets are open.

Proposition 2.4.41. *Let G be a digraph. Let $\mathcal{O}$ be the set of all subsets of $V(G)$ that are open in G. Then*

1. *$\varnothing \in \mathcal{O}$,*
2. *$V(G) \in \mathcal{O}$,*
3. *if $\mathcal{A} \subseteq \mathcal{O}$, then $\bigcup \mathcal{A} \in \mathcal{O}$,*
4. *if $\mathcal{A} \subseteq \mathcal{O}$, then $\bigcap \mathcal{A} \in \mathcal{O}$.*

Definition 2.4.42. Let G be a digraph and $Y \subseteq V(G)$. Let $\mathrm{Bd}^+_G(Y) = \bigcup\{\mathrm{out}_G(x) \mid x \in Y\} \setminus Y$ be the *outward boundary* of Y in G.

Proposition 2.4.43. *Let G be a digraph and $Y \subseteq V(G)$. Then the following are equivalent:*

1. *Y is open in G,*
2. *$\mathrm{Bd}^+_G(Y) = \varnothing$,*
3. *for all $x \in Y$, $V(G\{x\}) \subseteq Y$.*

Subnetworks

Definition 2.4.44. Let G be a digraph, $X = V(G)$, $\Sigma = \bigtimes_{x\in X} S_x$ a type on X and $\Phi : \Sigma \to \Sigma$ a function network on G. Let $Y \subseteq V(G)$ and $V(G) \setminus Y \supseteq Z \supseteq \mathrm{Bd}^+_G(Y)$. For any $h \in \bigtimes_{x\in Z} S_x$ define

$$\Phi^h[Y] : \bigtimes_{x\in Y} S_x \to \bigtimes_{x\in Y} S_x$$

by $(\Phi^h[Y])(f) = \Phi(g \cup h \cup f)$,

where $g \in \bigtimes_{x\in X\setminus(Y\cup Z)} S_x$ is arbitrary.

Since G is a dependency graph for Φ, the above definition does not depend on the choice of the function g.

Proposition 2.4.45. *Let G be a digraph, $X = V(G)$, $\Sigma = \bigtimes_{x\in X} S_x$ a type on X and $\Phi : \Sigma \to \Sigma$ a function network on G. Let $Y \subseteq V(G)$ and $V(G) \setminus Y \supseteq Z \supseteq \mathrm{Bd}^+_G(Y)$. Let $h \in \bigtimes_{x\in Z} S_x$. Then*

1. *$\Phi^h[Y]$ is a function network on $G[Y]$,*
2. *$\Phi^h[Y](f \restriction Y, y) = \Phi(f, y)$, for all $y \in Y$ and $f \in \Sigma$ such that $f \restriction Z = h$,*
3. *if f is a fixed point of Φ and $h = f \restriction Z$, then $f \restriction Y$ is a fixed point of $\Phi^h[Y]$.*

Following [19] we may call for any $X \subseteq V(G)$ and the set $\mathrm{Bd}^+_G(X)$ the *input layer* of the Boolean network $\Phi^{(\cdot)}[X]$. The idea is to imagine $\Phi^{(\cdot)}[X]$ as a (possibly nondeterministic and infinite) automaton, which, after being fed with an input $h \in \{0,1\}^{\mathrm{Bd}^+_G(X)}$ then comes up with a fixed point $f \in \{0,1\}^X$ (or not), which, if it exists, must satisfy $f \restriction Y = h$. If $G[X]$ is well-founded, then there is always a unique fixed point, i.e., $\Phi^h[X]$ terminates for each input and is deterministic.

The following definition will play a crucial role in Subsection 5.1.1, e.g. in Theorem 5.1.3.

Definition 2.4.46. Let G be a digraph, $X = V(G)$, $\Sigma = \bigtimes_{x\in X} S_x$ a type on X and $\Phi : \Sigma \to \Sigma$ a function network on G. Let $Y \subseteq V(G)$, $V(G) \setminus Y \supseteq Z \supseteq \mathrm{Bd}^+_G(Y)$ and $h \in \bigtimes_{x\in Z} S_x$.

1. Call $\Phi^h[Y]$ the *subnetwork of* Φ *induced by* X *relative to* h.
2. If Y is open in G, i.e., if $\mathrm{Bd}^+_G(Y) = \varnothing$, then $\mathrm{dom}\, h = \varnothing$ and we also write $\Phi[X]$ instead of $\Phi^h[X]$ and call it the *subnetwork of* Φ induced by X.
3. We say that Φ is
   (a) *absolutely solvable relative to* Y iff for all $h \in \bigtimes_{x\in Z} S_x$, $\Phi^h[Y]$ has a fixed point,

(b) *perfectly solvable relative to* Y iff for all $\varnothing \neq Z \subseteq Y$, Φ is absolutely solvable relative to Z,

(c) *relatively solvable relative to* Y iff there exists $h \in \bigtimes_{x \in Z} S_x$ such that $\Phi^h[Y]$ has a fixed point,

(d) *absolutely unsolvable relative to* Y iff it is not relatively solvable relative to Y.

The following two results are a first and rather trivial version of what could be called *decomposition theorems*. More sophisticated decomposition results shall be treated in Chapter 5.

Proposition 2.4.47. *Let G be a digraph and $\mathcal{P}$ a partition of $V(G)$ such that for all $X \in \mathcal{P}$, X is open in G. Let Φ be a function network on G. Then Φ has a fixed point if for all $X \in \mathcal{P}$, $\Phi[X]$ has a fixed point.*

Proof. The union of a set of fixed points of $\Phi[X]$ (for $X \in \mathcal{P}$) is a fixed point of Φ. □

Lemma 2.4.48. *Let G be a digraph, $X = V(G)$ and Φ a function network on G. Let $Y \subseteq X$ be open in G. If f is a fixed point of $\Phi[Y]$ and g is a fixed point of $\Phi^f[X \setminus Y]$, then $f \cup g$ is a fixed point of Φ.*

Proof. Straightforward by Proposition 2.4.45. □

2.4.7 Kripke fixed points

In the following we shall sketch how Kripke's work [32] can be adapted to the context of Boolean networks. The discussion of various standard valuation schemes in [5] will find its reflection in Subsection2.4.8.

Definition 2.4.49. For any set X and all $f, g \in \{0, \frac{1}{2}, 1\}^X$

1. define $f \subseteq g$ (and say that g is an *extension* of f) iff for all $x \in X$, $f(x) = 0$ implies $g(x) = 0$ and $f(x) = 1$ implies $g(x) = 1$.

2. For all $\varnothing \neq \mathcal{F} \subseteq \{0, \frac{1}{2}, 1\}^X$ let $\bigcap \mathcal{F} : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be defined by
$$(\bigcap \mathcal{F})(x) = \begin{cases} v, & \text{if } \ v \in \{0,1\} \wedge \forall f \in \mathcal{F} : f(x) = v \\ \frac{1}{2}, & \text{else.} \end{cases}$$

3. For all $\varnothing \neq \mathcal{F} \subseteq \{0, \frac{1}{2}, 1\}^X$ let $\bigcup \mathcal{F} : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be defined by
$$(\bigcup \mathcal{F})(x) = \begin{cases} v, & \text{if } \ v \in \{0,1\} \wedge \forall f \in \mathcal{F} : f(x) = v \\ v, & \text{if } \ v \in \{0,1\} \wedge \forall f \in \mathcal{F} : f(x) \in \{v, \frac{1}{2}\} \wedge \exists f \in \mathcal{F} : f(x) = v \\ \frac{1}{2}, & \text{else.} \end{cases}$$

   If $\mathcal{F} = \{f, g\}$ we also write $f \cup g$ for $\bigcup \mathcal{F}$.

4. A set $\varnothing \neq \mathcal{F} \subseteq \{0, \frac{1}{2}, 1\}^X$ is said to be *compatible* iff there are no $f, g \in \mathcal{F}$ such that $f(x) \neq g(x)$ and $f(x), g(x) \in \{0, 1\}$ for some $x \in X$.

5. Let $\mathrm{sdm}(f) = \{x \in X \mid f(x) \in \{0, 1\}\}$ be the *substantial domain* of f.

Proposition 2.4.50. *Let $\varnothing \neq \mathcal{F} \subseteq \{0, \frac{1}{2}, 1\}^X$. If $\mathcal{F}$ is compatible, then*

1. $\mathrm{sdm}(\bigcup \mathcal{F}) = \bigcup\{\mathrm{sdm}(f) \mid f \in \mathcal{F}\}$ *and*

2. *for all $f \in \mathcal{F}$ and all $x \in \mathrm{sdm}(f)$, $f(x) = (\bigcup F)(x)$.*

Definition 2.4.51. Let $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$.

1. Then Ψ is said to be *monotonic* iff $f \subseteq g$ implies $\Psi(f) \subseteq \Psi(g)$, for all $f, g \in \{0, \frac{1}{2}, 1\}^X$.

2. A function $f \in \{0, \frac{1}{2}, 1\}^X$ is said to be *sound* with respect to Ψ iff $f \subseteq \Psi(f)$,

3. and *maximally sound* with respect to Ψ iff f is sound with respect to Ψ and no proper extension of f is sound with respect to Ψ.

Definition 2.4.52. Let $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be monotonic. A fixed point f of Ψ is said to be *intrinsic* iff for all fixed points g of Ψ, $\{f, g\}$ is compatible.

Proposition 2.4.53. *Let $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be monotonic and $\varnothing \neq \mathcal{F} \subseteq \{0, \frac{1}{2}, 1\}^X$ be a set of fixed points of Ψ. Then*

1. *$\bigcap \mathcal{F}$ is a fixed point of Ψ. If $\mathcal{F}$ is the set of all fixed points of Ψ, we call $\bigcap \mathcal{F}$ the* least fixed point *of Ψ.*

2. *If $\mathcal{F}$ is compatible, then $\bigcup \mathcal{F}$ is a fixed point of Ψ. If $\mathcal{F}$ is the set of all intrinsic fixed points of Ψ, then $\mathcal{F}$ is compatible and we call $\bigcup \mathcal{F}$ the* largest intrinsic fixed point *of Ψ.*

Proposition 2.4.54. *Let $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be monotonic. Then for every sound $f \in \{0, \frac{1}{2}, 1\}^X$ there exists some maximally sound $g \supseteq f$.*

Proof. Consider the set $\mathcal{S}(f)$ of all sound $g \in \{0, \frac{1}{2}, 1\}^X$ extending f. Then $f \in \mathcal{S}(f)$ and since Ψ is monotonic, the union of every $\subseteq$- chain in $\mathcal{S}(f)$ is also an element of $\mathcal{S}(f)$. Hence the claim follows by Zorn's Lemma. □

Together with the observations that very function that assumes everywhere the value $\frac{1}{2}$ is sound and that soundness of f implies soundness of $\Phi(f)$, these last two propositions imply the following.

Theorem 2.4.55. *Let $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ be monotonic. Then*

1. *Ψ has a fixed point,*

2. *every sound $f \in \{0, \frac{1}{2}, 1\}^X$ can be extended to a maximal fixed point of Ψ,*

3. *there exists a unique fixed point f_0 of Ψ that is the* least *fixed point of Ψ in the senses that every fixed point of Ψ is an extension of f_0.*

2.4.8 Kripke-extensions of Boolean networks

Next we will study various ways in which a given Boolean network on X can be extended to a monotonic network of type $\{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$.

Definition 2.4.56. Let Φ be a Boolean network and G a digraph. Let $X = V(G)$. For all $f \in \{0, \frac{1}{2}, 1\}^X$ define $f^0_{\frac{1}{2}} \in \{0, 1\}^X$ by

$$f^0_{\frac{1}{2}}(y) = \begin{cases} f(y), & \text{if } f(y) \in \{0, 1\} \\ 0, & \text{else.} \end{cases}$$

The operation $f \mapsto f^0_{\frac{1}{2}}$ corresponds to *closing off* partial models.

Definition 2.4.57. Let Φ be a Boolean network and a digraph G. Let $X = V(G)$. We associate a three-valued function network $\Phi'_{WK}[G] : \{0, \frac{1}{2}, 1\}^X \times X \to \{0, \frac{1}{2}, 1\}$ to Φ as follows.

$$\Phi'_{WK}[G](f, x) = \begin{cases} \Phi(f^0_{\frac{1}{2}}, x), & \text{if } \mathrm{out}_G(x) \subseteq \mathrm{sdm}(f) \\ \frac{1}{2}, & \text{else.} \end{cases}$$

The index WK is intended to stand for 'Weak Kleene'. The relation of $\Phi'_{WK}[G]$ to the *weak Kleene valuation-scheme* (cf. [32] and also [5]) is a rather loose one when the vertices of G are non-linguistic entities. However, when the Boolean network Φ is *represented* as a sentence system formulated in an infinitary propositional language as in Subsection 3.5.1, then the vertices of G are sentence names and $(\alpha, \beta) \in A(G)$ means that β is a syntactic constituent of the sentence that is denoted by α in the sentence systems that represents the Boolean network. In this context, the relation to the weak Kleene valuation-scheme becomes apparent.

Analogously, it is possible to define a Strong-Kleene extension $\Phi'_{SK}[G]$ of a Boolean network Φ, given that each automaton of Φ behaves like the evaluation function of a sentence. E.g., if x computes the evaluation function of a conjunction, then $\Phi'_{SK}[G]$ would be defined at x as follows.

$$\Phi'_{SK}[G](f, x) = \begin{cases} \Phi(f^0_{\frac{1}{2}}, x), & \text{if } \mathrm{out}_G(x) \subseteq \mathrm{sdm}(f) \\ 0, & \text{if } \mathrm{out}_G(x) \not\subseteq \mathrm{sdm}(f) \wedge \exists y \in \mathrm{out}_G(x)(f(y) = 0) \\ \frac{1}{2}, & \text{else.} \end{cases}$$

Proposition 2.4.58. *Let Φ be a Boolean network and G a digraph. Then*

1. *$\Phi'_{WK}[G]$ is a function network on G,*
2. *$\Phi'_{WK}[G]$ is an extension of Φ and monotonic.*

Proposition 2.4.59. *Let Φ be a Boolean network and G a digraph. Then $x \in V(G)$ is in the substantial domain of the least fixed point of $\Phi'_{WK}[G]$ iff x is in the well-founded part of G.*

Proof. By induction of G. □

Definition 2.4.60. A function $f \in \{0, \frac{1}{2}, 1\}^X$ is said to be *complete* iff $\mathrm{sdm}(f) = X$.

Corollary 2.4.61. *Let Φ be a Boolean network on G. If G is well-founded, then the least fixed point of $\Phi'_{WK}[G]$ is complete and the unique fixed point of Φ.*

Definition 2.4.62. Let Φ be a Boolean network on X. We associate a three-valued function network $\Phi'_L : \{0, \frac{1}{2}, 1\}^X \times X \to \{0, \frac{1}{2}, 1\}$ to Φ as follows.

$$\Phi'_L(f, x) = \begin{cases} \Phi(f^0_{\frac{1}{2}}, x), & \text{if } x \text{ depends on } \mathrm{sdm}(f) \text{ w.r.t. } \Phi \\ \frac{1}{2}, & \text{else.} \end{cases}$$

Φ'_L is the direct counterpart of what is called the *Leitgeb valuation-scheme* V_L in [4] and [5]. Notice that the main difference between Φ'_L and $\Phi'_{WK}[G]$ is that $\Phi'_{WK}[G]$ requires a graph parameter.

Definition 2.4.63. For all $\Psi_1, \Psi_2 : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$, we say that

1. Ψ_2 is *stronger than* Ψ_1 and write $\Psi_2 \geq \Psi_1$ iff $\Phi_1(f) \subseteq \Phi_2(f)$, for all $f \in \{0, \frac{1}{2}, 1\}^X$,
2. Ψ_2 is *strictly stronger than* Ψ_1 and write $\Psi_2 > \Psi_1$ iff $\Psi_2 \geq \Psi_1$ and there exists $f \in \{0, \frac{1}{2}, 1\}^{V(G)}$ such that $\Phi_1(f) \subsetneq \Phi_2(f)$,
3. Ψ_2 is *equivalent to* Ψ_1 and write $\Psi_2 \equiv \Psi_1$ iff $\Psi_1 \leq \Psi_2$ and $\Psi_1 \geq \Psi_2$.

Proposition 2.4.64. *Let (G, Φ) be a Boolean network. Then*

1. *Φ'_L is a function network on G,*
2. *Φ'_L is an extension of Φ and monotonic,*
3. *$\Phi'_L \geq \Phi'_{WK}[G]$,*
4. *$\Phi'_L \equiv \Phi'_{WK}[G]$ iff G is a minimal dependency graph for Φ.*

The following notion of an *r-paradoxical* Boolean automaton is analogous to that of a *referentially paradoxical* (or *r-paradoxical*) sentence in [5]. Notice that there is no arbitrary choice involved in this notion – neither a particular Kripke-extension nor a particular limit rule.

Definition 2.4.65. Let Φ be a Boolean network on X. A Boolean automaton $x \in X$ is said to be *r-paradoxical relative to* Φ iff there is no dependency digraph G for Φ such that $\Phi[V(G\{x\})]$ has a fixed point. (Recall Definition 2.4.46 and observe that $V(G\{x\})$ is open in G.)

The following theorem is an adaptation from [5]. The proof carries over straightforwardly.

Theorem 2.4.66. *Let Φ be a Boolean network on X. Then $x \in V(G)$ is r-paradoxical relative to Φ iff it is Kripke-paradoxical w.r.t. Φ'_L.*

Definition 2.4.67. Let Φ be a Boolean network on X. We associate a three-valued function network $\Phi'_{FV} : \{0, \frac{1}{2}, 1\}^X \times X \to \{0, \frac{1}{2}, 1\}$ to Φ as follows:

$$\Phi'_{FV}(f, x) = \begin{cases} 1, & (\forall\, g \in \{0,1\}^X)(g \supseteq f \Rightarrow \Phi(g, x) = 1) \\ 0, & (\forall\, g \in \{0,1\}^X)(g \supseteq f \Rightarrow \Phi(g, x) = 0) \\ \frac{1}{2}, & \text{else.} \end{cases}$$

If the valuation scheme that is denoted in [5] by V_{FV} (which was introduced by Cantini in [12]) were to be transferred to the present context, it would be formulated as in the following proposition. However, for reasons of technical convenience, we stick to the formulation of Definition 2.4.67.

Proposition 2.4.68. *Let Φ be a Boolean network on X. Then for all $x \in X$ and all $f \in \{0, \frac{1}{2}, 1\}^X$*

$$\Phi'_{FV}(f, x) = \begin{cases} 1, & (\forall\, g \in \{0, \frac{1}{2}, 1\}^X)(g \supseteq f \Rightarrow \Phi(g^0_{\frac{1}{2}}, x) = 1) \\ 0, & (\forall\, g \in \{0, \frac{1}{2}, 1\}^X)(g \supseteq f \Rightarrow \Phi(g^0_{\frac{1}{2}}, x) = 0) \\ \frac{1}{2}, & \textit{else.} \end{cases}$$

Proposition 2.4.69. *Let Φ be a Boolean network on a digraph G. Then*

1. *Φ'_{FV} is a function network on G,*
2. *Φ'_{FV} is an extension of Φ and monotonic,*
3. *$\Phi'_{FV} \geq \Phi'_L$.*

Definition 2.4.70. Let (G, Φ) be a Boolean network and $X = V(G)$. A function network $\Psi : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ is said to be a *Kripke-extension* of Φ iff Ψ is a extension of Φ and monotonic.

Notice that in difference to [5] we do not require $\Psi \geq \Phi'_{WK}$, since the latter is only defined relative to a given graph.

Proposition 2.4.71. *Let Φ be a Boolean network on a digraph G. Then*

1. *every Kripke-extension of Φ is a function network on G,*
2. *Φ'_{FV} is the greatest Kripke-extension of Φ, i.e., $\Phi'_{FV} \geq \Psi$, for all Kripke-extensions Ψ of Φ.*

Proof. (1) is clear and (2) is analogous to the proof of Theorem 4.6 and Corollary 4.7. in [5]. □

Proposition 2.4.72. *Let Φ be a Boolean network on X and Ψ a Kripke-extension of Φ. Then*

1. *every fixed point of* Φ *is a complete fixed point of* Ψ *and*

2. *every complete fixed point of* Ψ *is a fixed point of* Φ*.*

Definition 2.4.73. Let (G, Φ) be a Boolean network and Ψ be a Kripke-extension of Φ. We say that $x \in V(G)$ is

1. *Kripke-paradoxical w.r.t.* Ψ iff there is no fixed point f of Ψ such that $f(x) \in \{0, 1\}$,

2. *Kripke-hypodoxical w.r.t.* Ψ iff there are fixed points f, g of Ψ such that $f(x) = 1$ and $g(x) = 0$.

3. *intrinsic w.r.t.* Ψ iff there is some intrinsic fixed point f of Ψ such that $f(x) \in \{0, 1\}$.

Proposition 2.4.74 (Kripke)**.** *Let* (G, Φ) *be a Boolean network and* Ψ *a Kripke-extension of* Φ*. Then* $x \in V(G)$ *is in the largest intrinsic fixed point of* Ψ *iff it is neither paradoxical nor hypodoxical w.r.t.* Ψ*.*

Example 2.4.75. The fact that a Boolean network Φ contains no paradoxical automata doesn't necessarily imply that it has a fixed point. (Cf. [32]). Recall Example 1.1.4:

K_1: (K_1) is false or (T) is true
K_2: (K_2) is false or (T) is false
T: (T) is true

The straightforward representation of these sentences as Boolean automata along the lines of the previous examples yields a Boolean network Φ on $X = \{K_1, K_2, T\}$ none of whose automata is Kripke paradoxical (or Herzberger-paradoxical) but that nevertheless has no fixed point: For any given $f \in \{0, 1\}^X$, T has period 1 while K_1 has period 1 iff $f(T) = 1$ and period 2 iff $f(T) = 0$. On the other hand, K_2 has period 1 iff $f(T) = 0$ and period 2 iff $f(T) = 1$.

Hence, Φ is an anomaly for any notion of paradoxicality of the Kripke or Herzberger family in the sense of the remark at the end of Example 2.4.8.

2.4.9 Kripke-paradoxicality in terms of subspace invariance

The goal of this subsection is to establish an interpretation of the fixed points of Φ'_{FV} as strictly Φ-invariant subtypes. In addition to Proposition 2.4.71 (cf. also the discussion in [5]) this result might be seen as further support for the claim that Φ'_{FV} is the most natural of all Kripke-extensions (valuation schemes).

Definition 2.4.76. Let X be a set. We denote $\wp(\{0, 1\}^X) \setminus \{\varnothing\}$ by $(\{0, 1\}^X)^*$. Let us define maps

1. $t : \{0, \frac{1}{2}, 1\}^X \to (\{0,1\}^X)^*$ by $t(f') = \{g \in \{0,1\}^X \mid g \supseteq f'\}$,
2. $r : (\{0,1\}^X)^* \to \{0, \frac{1}{2}, 1\}^X$ by $r(\mathcal{F}) = \bigcap \mathcal{F}$ (cf. Definition 2.4.49).

Lemma 2.4.77. *Let Φ be a Boolean network on X. Then for all $f', g' \in \{0, \frac{1}{2}, 1\}^X$, $f, g \in \{0,1\}^X$ and all $\mathcal{F}, \mathcal{G} \in (\{0,1\}^X)^*$ the following claims hold.*

1. $r \circ t = \mathrm{id}_{\{0,\frac{1}{2},1\}^X}$,
2. $t \circ r(\mathcal{F}) = \{g \in \{0,1\}^X \mid (\forall x \in X)(\exists f \in \mathcal{F})(g(x) = f(x))\}$,
3. $\mathcal{F} = (t \circ r)(\mathcal{F})$ *iff* $\mathcal{F}$ *is a type,*
4. $f \in \mathcal{F}$ *iff* $f \supseteq r(\mathcal{F})$,
5. $f \in t(f')$ *iff* $f \supseteq f'$,
6. $f' \subseteq g'$ *implies* $t(f') \supseteq t(g')$,
7. $\mathcal{F} \subseteq \mathcal{G}$ *implies* $r(\mathcal{F}) \supseteq r(\mathcal{G})$,
8. $g \supseteq \Phi'_{FV}(f') \Leftrightarrow (\forall x \in X)(\exists h \in \{0,1\}^X)(h \supseteq f' \wedge g(x) = \Phi(h,x))$.

Proof. All claims are clear except (8).

Ad (8): First recall that by Definition 2.4.67 we have

(a) $(\Phi'_{FV}(f'))(x) = 1 \Leftrightarrow (\forall h \in \{0,1\}^X \wedge h \supseteq f)(\Phi(h,x) = 1)$,

(b) $(\Phi'_{FV}(f'))(x) = 0 \Leftrightarrow (\forall h \in \{0,1\}^X \wedge h \supseteq f)(\Phi(h,x) = 0)$,

(c) $(\Phi'_{FV}(f'))(x) = \frac{1}{2} \Leftrightarrow (\exists h_1 \in \{0,1\}^X \wedge h_1 \supseteq f')\ (\Phi(h_1,x) = 1)$
$\wedge(\exists h_0 \in \{0,1\}^X \wedge h_0 \supseteq f')(\Phi(h_1,x) = 0)$.

$\Rightarrow$: Let $g \supseteq \Phi'_{FV}(f')$ and $x \in X$. Suppose $g(x) = 1$. Then $(\Phi'_{FV}(f'))(x) \in \{1, \frac{1}{2}\}$. In any case (either by (a) or by (c)) we get $(\exists h \in \{0,1\}^X)(h \supseteq f' \wedge g(x) = \Phi(h,x))$. If $g(x) = 0$ the argument goes analogously.

$\Leftarrow$: Suppose that $(\forall x \in X)(\exists h \in \{0,1\}^X)(h \supseteq f' \wedge g(x) = \Phi(h,x))$. We have to show that for all $x \in X$, $g(x) = (\Phi'_{FV}(f'))(x)$ or $(\Phi'_{FV}(f'))(x) = \frac{1}{2}$.

So let $x \in X$ and suppose that $(\Phi'_{FV}(f'))(x) \neq \frac{1}{2}$. Then $(\Phi'_{FV}(f'))(x) = 1$ or $(\Phi'_{FV}(f'))(x) = 0$. Suppose the first case. Then $(\forall h \in \{0,1\}^X \wedge h \supseteq f')(\Phi(h,x) = 1)$. This implies together with the hypothesis that $g(x) = 1$. Hence $g(x) = (\Phi'_{FV}(f'))(x)$. In the second case we get $g(x) = 0$ and $g(x) = (\Phi'_{FV}(f'))(x)$ analogously. □

Theorem 2.4.78. *Let Φ be a Boolean network on X. Then for all $f' \in \{0, \frac{1}{2}, 1\}^X$,*

1. $r(\Phi[t(f')]) = \Phi'_{FV}(f')$,

2. *f' is Φ'_{FV}-sound iff $t(f')$ is invariant under Φ,*

3. *f' is a Φ'_{FV}-fixed point iff $t(f')$ is strictly invariant under Φ,*

4. *f is a maximal fixed point of Φ'_{FV} iff $t(f)$ is Φ-irreducible.*

Proof. Ad 1: Let $f' \in \{0, \frac{1}{2}, 1\}^X$ and $g \in \{0,1\}^X$. Then

$$
\begin{aligned}
& g \supseteq \Phi'_{FV}(f')) \overset{(8)}{\Leftrightarrow} (\forall x \in X)\ (\exists h \in \{0,1\}^X)\ (h \supseteq f' \text{ and } g(x) = \Phi(h,x)) \\
& \overset{(5)}{\Leftrightarrow} (\forall x \in X)\ (\exists h \in t(f'))\ (g(x) = \Phi(h,x)) \\
& \overset{(2)}{\Leftrightarrow} g \in (t \circ r \circ \Phi)[t(f')] \\
& \overset{(4)}{\Leftrightarrow} g \supseteq (r \circ \Phi \circ t)(f')),
\end{aligned}
$$

where the numbers indicate the items of Lemma 2.4.77. Hence we can conclude by Lemma 2.4.77(1) that $(r \circ \Phi \circ t)(f') = \Phi'_{FV}(f')$.

Ad 2 and 3: Both claims follow straightforwardly from (1) and Lemma 2.4.77. □

Hence maximal fixed points of Φ'_{FV} correspond to Boolean subtypes that are strictly invariant under Φ and such that they have no proper subtype that is invariant under Φ. This makes them analogous to attractors in some sense (cf. Definition 2.2.13). But observe that they may have still a proper strictly invariant sub*set*.

In Section 2.5 we will interpret the relation between Φ and Φ'_{FV} from a more abstract point of view. (Cf. Theorem 2.5.13).

Corollary 2.4.79. *Let Φ be a Boolean network on X and Ψ a Kripke-extension of Φ. Let $f \in \{0, \frac{1}{2}, 1\}^X$. Then*

1. *$t(\Psi(f)) \supseteq t(\Phi'_{FV}(f))$,*

2. *if f is a fixed point of Ψ then $t(f) \supseteq t(g)$, for some g that is a fixed point of Φ'_{FV}.*

Hence fixed points of Kripke-extensions correspond to subtypes of the types that correspond to fixed points of Φ'_{FV}.

Definition 2.4.80. Let Φ be a Boolean network on X. Let $\mathfrak{T}_\Phi$ be the set of all $\Sigma \sqsubseteq \{0,1\}^X$ (i.e., all subtypes of $\{0,1\}^X$) that are invariant under Φ.

Corollary 2.4.81. *Let Φ be a Boolean network on X and $x \in X$. Then*

1. *x is $\mathfrak{T}_\Phi$-paradoxical iff x is Kripke-paradoxical w.r.t. Φ'_{FV},*

2. *x is $\mathfrak{T}_\Phi$-hypodoxical iff x is Kripke-hypodoxical w.r.t. Φ'_{FV},*

3. *x is $\mathfrak{T}_\Phi$-intrinsic iff x is Kripke-intrinsic w.r.t. Φ'_{FV}.*

This implies that Φ'_{FV} (in contrast to other Kripke-extensions of Φ) is *natural* in the sense that it involves no arbitrary choice, except for focussing on the rather natural concept of *subtype*.

For the following recall Definition 2.4.30.

Proposition 2.4.82. *Let Φ be a Boolean network on X and $\Sigma \subseteq \{0,1\}^X$. Then the following are equivalent.*

1. *Σ is an atom of $\mathfrak{T}_\Phi$,*
2. *$\Sigma \sqsubseteq \{0,1\}^X$ is Φ-irreducible,*
3. *$r(\Sigma)$ is a maximal fixed point of Φ'_{FV}.*

This means that in the context of $\mathfrak{T}_\Phi$, Φ-irreducible subtypes play the role that attractors play in $\mathfrak{S}_\Phi$ and λ-attractors play in $\mathfrak{L}_\Phi$.

Proposition 2.4.83. *For all Boolean networks Φ, every $\Sigma \in \mathfrak{T}_\Phi$ is λ-invariant with respect to Φ (cf. Definition 2.4.12).*

Corollary 2.4.84. *For all Boolean networks Φ, $\mathfrak{T}_\Phi \subseteq \mathfrak{L}_\Phi$.*

2.4.10 Fixed points by type reduction

Let us conclude this section with two applications of the Kripkean fixed point theory, demonstrating how fixed points of Φ'_{FV} (and thus of Φ) can actually be found.

Definition 2.4.85. Let Φ be a Boolean network on X and $\Sigma \sqsubseteq \{0,1\}^X$. Then

1. for $x \in X$, $v \in \Sigma(x)$ is said to be Φ-*realizable* in Σ iff there exists $f \in \Sigma$ with $\Phi(f,x) = v$,
2. Σ is said to be *pruned* with respect to Φ iff for all $x \in X$ and all $v \in \Sigma(x)$, v is Φ-realizable in Σ.

Proposition 2.4.86. *Let Φ be a Boolean network on X and $\Sigma \sqsubseteq \{0,1\}^X$. Then Σ is strictly Φ-invariant iff Σ is Φ-invariant and pruned with respect to Φ.*

The following results are related to Lemma 4.11. and Corollary 4.12. in [5].

Lemma 2.4.87. *Let T be a tree and Φ a Boolean network on T. Let $X = V(T)$. Let $\Sigma \sqsubseteq \{0,1\}^X$ be Φ-invariant. If Σ is pruned with respect to Φ and not trivial, then there exists a proper subtype Σ' of Σ that is Φ-invariant.*

Proof. Since Σ is not trivial there, exists $r \in X$ such that $\Sigma(r) = \{0,1\}$. Let $(y_\alpha)_{\alpha<\xi}$ be an enumeration of $V(T\{r\})$ such that $\alpha < \beta$ implies $ht_T(y_\alpha) \leq ht_T(y_\beta)$, for all $\alpha, \beta < \xi$. Then there is a sequence $(x_\alpha)_{\alpha\in(\xi\cap S_0)}$ (where S_0 is the class of all non-limit ordinals) that is also an enumeration of $V(T)$ (i.e., it induces a bijection between $\xi \cap S_0$ an $V(T\{r\})$) and such that $\alpha < \beta$ implies $ht_T(x_\alpha) \leq ht_T(x_\beta)$, for all $\alpha, \beta < \xi$. In particular $x_0 = r$.

We shall define recursively a sequence $(\Sigma_\alpha)_{\alpha \leq \xi}$ such that

1. $\Sigma_0 \sqsubseteq \Sigma$,
2. $\Sigma_\beta \sqsubseteq \Sigma_\alpha$, for all $\alpha \leq \beta \leq \xi$,
3. for all $v \in \Sigma_\alpha(x_\alpha)$ and all $f \in \Sigma_\alpha$, $v = \Phi(f, x_\alpha)$, for $\alpha \in S_0 \cap \xi$,
4. Σ_α is pruned for all $\alpha \leq \xi$,
5. $\Sigma_\lambda = \bigcap_{\alpha<\lambda} \Sigma_\alpha$, for all limits $\lambda \leq \xi$,
6. Σ_ξ is Φ-invariant and a proper subtype of Σ.

- Let $\alpha = 0$. Choose some $v_0 \in \Sigma(x_0)$. Since Σ is pruned, v_0 is Φ-realizable in Σ, i.e., there exists $f_0 \in \Sigma$ such that $v_0 = \Phi(f_0, x_0)$. Let $\Sigma_0 = \{g \in \Sigma \mid g(x_0) = v_0 \wedge (g \restriction \mathrm{out}_T(x_0)) = (f_0 \restriction \mathrm{out}_T(x_0))\}$. Then (1) holds, i.e., $\Sigma_0 \sqsubseteq \Sigma$. Moreover, since x_0 depends on $\mathrm{out}_T(x_0)$ w.r.t. Φ, we have (3), i.e., for all $f \in \Sigma_0$, $v_0 = \Phi(f, x_0)$.

  Finally, Σ_0 is pruned (4): Let $x \in V(T)$. If $x = x_0$, then (3) implies that v_0 is Φ-realizable in Σ_0. If $x \in V(T)\setminus\{x_0\}$, then $\mathrm{out}_T(x) \cap (\{x_0\} \cup \mathrm{out}_T(x_0)) = \varnothing$. Since x depends on $\mathrm{out}_T(x)$ w.r.t. Φ, the fact that Σ is pruned implies that for all $v \in \Sigma_0(x)$, v is Φ-realizable in Σ_0.

- Let $\alpha = \beta + 1$, for some $\beta < \xi$. Choose some $v_\alpha \in \Sigma(x_\alpha)$. Since Σ_β is pruned by induction hypothesis, there exists $f_\alpha \in \Sigma_\beta$ such that $v_\alpha = \Phi(f_\alpha, x_\alpha)$. Let $\Sigma_\alpha = \{g \in \Sigma \mid g(x_\alpha) = v_\alpha \wedge (g \restriction \mathrm{out}_T(x_\alpha)) = (f_\alpha \restriction \mathrm{out}_T(x_\alpha))\}$.

  Then (2) $\Sigma_\alpha \sqsubseteq \Sigma_\gamma$, for all $\gamma \leq \alpha$. Moreover, since x_α depends on $\mathrm{out}_T(x_\alpha)$ w.r.t. Φ it follows from induction hypothesis (3) that for all $f \in \Sigma_\gamma$, $v_\gamma = \Phi(f_\gamma, x_\gamma)$, for all $\gamma \in S_0 \cap \alpha + 1$.

  Furthermore, Σ_α is pruned (4): If $x = x_\beta$, for some $\beta \leq \alpha$, then the claim that x is Φ-realizable in Σ_α follows from (3).

  So let $x \in V(T) \setminus \{y \in V(T) \mid (\exists \beta \leq \alpha)(y = x_\beta)\}$. Then, since T is a tree and by definition of the enumeration $(x_\alpha)_{\alpha \in (\xi \cap S_0)}$ we obtain that $\mathrm{out}_T(x) \cap (\{y \in V(T) \mid (\exists \beta \leq \alpha)(y = x_\beta)\} \cup \bigcup \{\mathrm{out}_T(y) \mid y \in V(T) \wedge (\exists \beta \leq \alpha)(y = x_\beta)\}) = \varnothing$. Then, since x depends on $\mathrm{out}_T(x)$ w.r.t. Φ, the fact that Σ_β is pruned implies that for all $v \in \Sigma_\beta(x)$, v is Φ-realizable in Σ_β.

- Let α be a limit. Let $\Sigma_\alpha = \bigcap_{\beta<\alpha} \Sigma_\beta$, for all limits $\beta < \alpha$. Then by Proposition 2.4.38 $\Sigma_\alpha = \bigcap_{\beta<\alpha} \Sigma_\beta$ is a type. The rest of the claims follow straightforwardly.

It remains to be shown that Σ_ξ is Φ-invariant. (This, setting $\Sigma' = \Sigma_\xi$, yields the claim of the lemma.) So let $f \in \Sigma_\xi$ and $x \in V(T)$. We have to show that $\Phi(f, x) \in \Sigma_\xi(x)$. If $x \in V(T\{r\})$ then there exists $\alpha < \xi$ such that $x = x_\alpha$ and the claim follows from (3). Otherwise the claim follows from the hypothesis that Σ is Φ-invariant. □

This leads to the following result which has already been proven in [41] (in a very different manner) and in [5] (in a somewhat analogous manner).

Theorem 2.4.88. *If G is a tree, then every Boolean network on G has a fixed point.*

Proof. Let Φ be a Boolean network on $X = V(G)$. Let f be a maximal fixed point of $\Phi'(FV)$ and $\Sigma = t(f)$. Then $\Sigma \sqsubseteq \{0,1\}^X$ is Φ-invariant and pruned with respect to Φ. Moreover, Σ is Φ-irreducible. Hence it must be trivial by the previous lemma. □

2.4.11 Core, periphery and kernel-perfect digraphs

Definition 2.4.89. Let Φ be a Boolean network on X and f_l the least fixed point of Φ'_{FV}. Then

1. the set $\mathrm{sdm}(f_l) = \{x \in X \mid f_l(x) \in \{0,1\}\}$ is said to be the *periphery* of Φ,
2. the set $X \setminus \mathrm{sdm}(f_l)$ is said to be the *core* of Φ.

These concepts are analogous to those of Definition 4.1 in [5], although this might not be obvious at a first glance.

Proposition 2.4.90. *Let G a digraph, Φ a Boolean network on G and C the core of Φ. Then $G[C]$ is sink-less.*

Proposition 2.4.91. *Let (G, Φ) be a Boolean network. Then there exists a spanning subdigraph $G' \subseteq G$ such that G' is a dependency graph for Φ and the periphery of Φ is open in G'.*

Proof. The proof is straightforward by induction on the ordinal number of steps it takes Φ'_{FV} to reach f_l. □

Proposition 2.4.92. *Let (G, Φ) be a Boolean network and $P \subseteq V(G)$ the periphery of Φ. Suppose that P is open in G. Let f_l be the least fixed point of Φ'_{FV}. Then*

1. *$(f_l \restriction P) \in \{0,1\}^P$ is the unique fixed point of $\Phi[P]$,*
2. *for all $f \subseteq \{0,1\}^{V(G)}$, if f is a fixed point of Φ, then then set $\{f_l, f\}$ is compatible.*

Lemma 2.4.93. *Let G be a digraph and $\Phi = \Phi_{\downarrow}^{G}$. Let $X \subseteq V(G)$, $X^+ = \mathrm{Bd}_G^+(X)$ and $h \in \{0,1\}^{X^+}$. Let $C \subseteq X$ be the core of $\Phi^h[X]$, $P = X \setminus C$ the periphery of $\Phi^h[X]$ and $f_l \in \{0, \frac{1}{2}, 1\}^X$ the least fixed point of $(\Phi^h[X])'_{FV}$. Then for all $x \in C$ and all $y \in \mathrm{out}_G(x) \cap P$, $f_l(y) = 0$.*

Proof. Let $X^* = X \cup X^+$. Assume that there exists $x \in C$ and $y \in \mathrm{out}_G(x) \cap P$ such that $f_l(y) = 1$. Then for all $g \in \{0,1\}^X$ such that $g \supseteq f_l$, $(\Phi^h[X])(g,x) = 1 - \Phi_{\bigvee}^{G[X^*]}(h \cup g, x) = 1 - \sup_2\{g(z) \mid z \in \mathrm{out}_G(x)\} = 0$. Hence $(\Phi^h[X])'_{FV}(f_l, x) = 0$. But this contradicts the hypothesis that $x \in C$ which, since f_l is a fixed point of $(\Phi^h[X])'_{FV}$, implies that $\frac{1}{2} = f_l(x) = ((\Phi^h[X])'_{FV}(f_l))(x) = (\Phi^h[X])'_{FV}(f_l, x) \neq 0$. □

Recall that a digraph G is *kernel-perfect* iff every induced subdigraph of G has a kernel. Also recall Definition 2.4.46. The following theorem can be seen as a more elaborate version of Proposition 2.3.8.

Lemma 2.4.94. *Let G be a digraph and $\varnothing \neq X \subseteq V(G)$, $X^+ = \mathrm{Bd}_G^+(X)$ and $h \in \{0,1\}^{X^+}$. Let C be the core of $\Phi^h[X]$, where $\Phi = \Phi_{\downarrow}^{G}[X]$. Then $\Phi^h[X]$ has a fixed point, if C has a kernel.*

Proof. Let $P = X \setminus C$ be the periphery of $\Phi^h[X]$ and $f_l \in \{0, \frac{1}{2}, 1\}^X$ be the least fixed point of $(\Phi^h[X])'_{FV}$. If $C = \varnothing$, then $f_l \in \{0,1\}^X$ is a fixed point of $\Phi^h[X]$ and we are done. Let $C \neq \varnothing$ and g be the characteristic function of a kernel of $G[C]$. We will show that $f = f_l \cup g$ is a fixed point of $\Phi^h[X]$. So let $x \in X$.
Case 1: $x \in P$. Then it follows by Definition 2.4.67 that $\Phi^h[X](f,x) = \Phi^h[P](f \upharpoonright P, x) = f_l(x) = f(x)$.
Case 2: $x \in C$. Then $f(x) = g(x)$.

Case 2a: $f(x) = 0 = g(x)$. Since g is the characteristic function of a kernel of $G[C]$, there exits $y \in \mathrm{out}_G(x) \cap C$ such that $f(y) = g(y) = 1$. Observing that $(\Phi^h[X])(f,x) = 1 - \sup_2\{f(y) \mid y \in \mathrm{out}_G(x)\}$ and $f \upharpoonright C = g$, this implies that $\Phi^h[X](f,x) = 0$.

Case 2b: $f(x) = 1 = g(x)$. This implies that for all $y \in \mathrm{out}_G(x) \cap C$, $g(y) = 0$. By Lemma 2.4.93 we have that for all $y \in \mathrm{out}_G(x) \cap P$, $f_l(y) = 0$. Hence $\Phi^h[X](f,x) = 1$. □

Theorem 2.4.95. *A digraph G has a kernel iff the subdigraph of G induced by the core of $\Phi_{\downarrow}^{G}$ has a kernel.*

Proof. ⇒: Let K be a kernel of G and C the core of $\Phi_{\downarrow}^{G}$. It follows from Lemma 2.4.93 that $K \cap C$ is a kernel of $G[C]$.

⇐: Apply Lemma 2.4.94 to $X = V(G)$ and $\Phi = \Phi_{\downarrow}^{G}$. □

Theorem 2.4.96. *A digraph G is kernel-perfect iff for all $\varnothing \neq X \subseteq V(G)$, $\Phi_{\downarrow}^{G}$ is perfectly solvable relative to X.*

Proof. ⇒: Let $\Phi = \Phi_{\downarrow}^{G}$. Suppose that G is kernel-perfect. Let $\varnothing \neq X \subseteq V(G)$. Let $X^{+} = \mathrm{Bd}_{G}^{+}(X)$ and $h \in \{0,1\}^{X^{+}}$. We have to show that $\Phi^{h}[X]$ has a fixed point. (Observe that the claim that for all $\varnothing \neq X \subseteq V(G)$, $\Phi_{\downarrow}^{G}$ is absolutely solvable relative to X is equivalent to the claim that for all $\varnothing \neq X \subseteq V(G)$, $\Phi_{\downarrow}^{G}$ is perfectly solvable relative to X.) Let $C \subseteq X$ be the core of $\Phi^{h}[X]$ Then $G[C]$ has a kernel, since G is kernel-perfect. Then by Lemma 2.4.94 $\Phi^{h}[X]$ has a fixed point.
⇐: Let $\varnothing \neq X \subseteq V(G)$, $X^{+} = \mathrm{Bd}_{G}^{+}(X)$ and $h \in \{0,1\}^{X^{+}}$ be such that $h(x) = 0$, for all $x \in X^{+}$. Let f be a fixed point of $\Phi^{h}[X]$, which exists by hypothesis. It is straightforward to check that f is the characteristic function of a kernel. □

An application of Theorem 2.4.96 will be Corollary 5.1.11.

2.5 System- and network transformations

The goal of this section is to introduce notions of structure-preserving transformations between dynamical systems in general and Boolean networks and constrained Boolean networks (cf. Definition 2.3.1 and the remark following it) in particular. The purpose is to prepare the conceptual ground for the notion of a *characteristic digraph* (Definition 3.2.5) which shall play a crucial role in Chapter 3. Moreover, we can formulate the concept of *dual paradoxes* (cf. [14] and [41]) within our abstract framework, as well as criteria for the identity of paradoxes.

2.5.1 System transformations

The basic concepts of this section - *retraction*, *section* and *system map* are all well-know in the literature on category theory, cf. e.g., [33].

Definition 2.5.1. Let X and Y be sets, $i : X \to Y$ and $j : Y \to X$. We say that

1. j is a *retraction for* i iff $j \circ i = \mathrm{id}_X$,
2. j is a *section for* i iff $i \circ j = \mathrm{id}_Y$.

Proposition 2.5.2. *Let X and Y be sets, $i : X \to Y$ and $j : Y \to X$. Then j is a retraction for i iff i is a section for j.*

The following simple observation allows us to express the concept of *set isomorphism*, i.e., that of a bijective map in terms of retraction and section.

Proposition 2.5.3. *Let X and Y be sets. A map $i : X \to Y$ is*

1. *injective iff there exists a retraction $j : Y \to X$ for i,*
2. *surjective iff there exists a section $j : Y \to X$ for i,*

3. *bijective iff there exists a map $j : Y \to X$ that is a retraction and a section for i. If such a map j exists, then it is unique and $j = i^{-1}$.*

Further constraints lead to the following notions of dynamical system transformations.

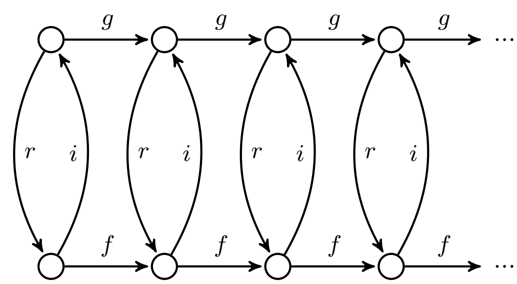
Definition 2.5.4. Let $f : U \to U$ and $g : V \to V$ be dynamical systems. Let $i : U \to V$ and $j : V \to U$. Then we say that

1. i is a *system map* iff $i \circ f = g \circ i$,
2. i is a *system isomorphism* iff i is a system map and there exists a system map $k : V \to U$ that is a retraction and a section for i,
3. i is a *system automorphism* iff it is a system isomorphisms and $f = g$,
4. i is a *strong system embedding (of f into g)* iff
   (a) i is a system map and
   (b) there exists a system map $r : V \to U$ that is a retraction for i,
5. i is a *system embedding (of f into g)* iff
   (a) i is a system map and
   (b) there exists a map $r : V \to U$ that is a retraction for i,
6. i is a *weak system embedding (of f into g)* iff there exists a map $r : V \to U$ that is a retraction for i that is such that
   (a) $f = r \circ g \circ i$ and
   (b) $f(x) = x$ implies $i(x) = (g \circ i)(x)$, for all $x \in U$,
7. i is a *dense (weak, strong) system embedding (of f into g)* iff i is a (weak, strong) system embedding and every fixed point of g is in $i[U]$.
8. a *closed (weak, strong) system embedding (of f into g)* iff i is a (weak, strong) system embedding and $i[U]$ is invariant under g.

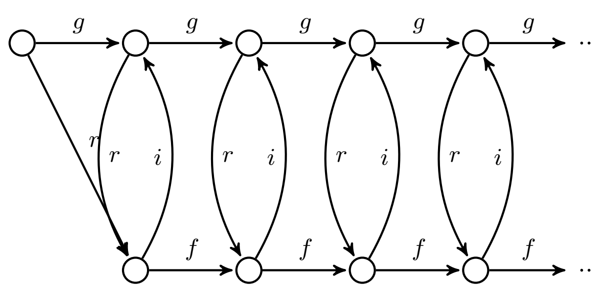
Proposition 2.5.5. *Every system embedding of f into g is a weak system system embedding of f into g.*

The concept of a dense weak system embedding shall play an important role in the definition of a *characteristic digraph* of a Boolean network (cf. Definition 3.2.5 and Chapter 3, Another application is Theorem 2.5.13 below.

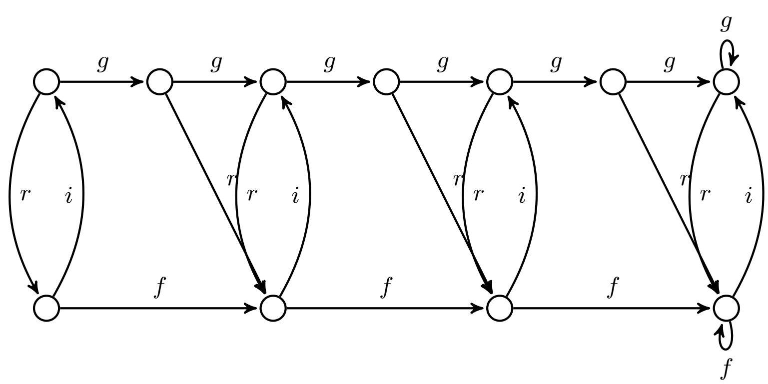
Let us illustrate the above definitions with some examples. Each of the following diagrams A, B, C and D represents the iteration graphs of two dynamical systems $f : U \to U$ (in the lower part of the diagram) and $g : V \to V$ (in the upper part of the diagram). From each vertex of U, (i.e., from each state of the dynamical system $f : U \to U$) an arrow labeled 'i' leads to exactly one vertex of U. Together, theses arrows represent a map $i : U \to V$. From each vertex of V, an arrow labeled 'r' leads to exactly one vertex of U. Together, these arrows represent a map $r : V \to U$. In all diagrams r is a retraction for i. But with respect to its status as system embedding the quality of i differs from example to example.



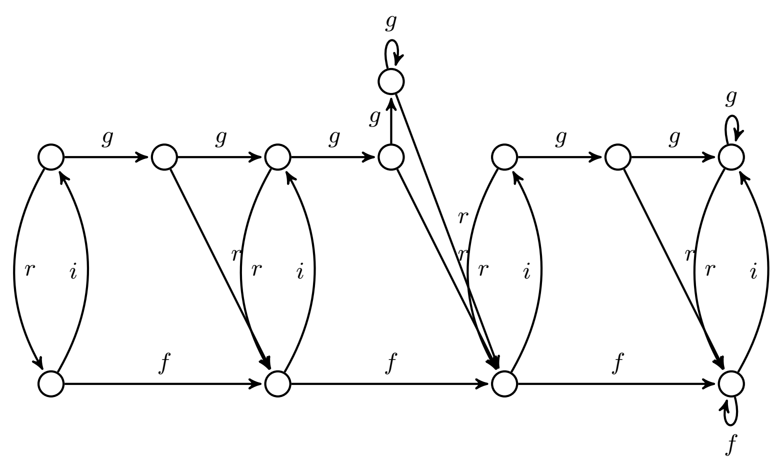
A: i is a strong system embedding of f into g with retraction r.



B: i is a system embedding of f into g, but not strong.



C: i is a dense weak system embedding of f into g, but no system embedding.



D: i is a weak system embedding of f into g, but not dense.

The following proposition ascertains that under a system embedding i of f into g trajectories of f are mapped to trajectories of g.

Proposition 2.5.6. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems and $i : U \to V$ a system embedding of f into g. Then for all $n \in \omega$, $i \circ f^n = g^n \circ i$.*

For the following corollary recall Definition 2.2.9. It expresses the fact that the period of trajectories remains unaltered under system embeddings.

Corollary 2.5.7. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems and $i : U \to V$ a system embedding of f into g. Then $\Pi_f(x) = \Pi_g(i(x))$, for all $x \in U$.*

Proposition 2.5.8. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems and $i : U \to V$ a system embedding of f into g. Let $r : V \to U$ be a system map that is a retraction for i. Let $x \in U$ and $y \in V$. Then*

1. *x is a fixed point of f iff $i(x)$ is a fixed point of g,*
2. *y is a fixed point of g iff $r(y)$ is a fixed point of f.*

In general, Proposition 2.5.6 does not hold for weak system embeddings. However, there is a weak counterpart of it that implies that every trajectory under f can be expressed in terms of g, i and r.

Proposition 2.5.9. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems and $i : U \to V$ a weak system embedding of f into g with retraction r. Then for all $n \in \omega$, $f^n = (r \circ g \circ i)^n$.*

Theorem 2.5.10. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems and $i : U \to V$ a dense weak system embedding of f into g. Then there exists a bijection between the fixed points of f and the fixed points of g.*

Power-systems and Kripke-extensions

Definition 2.5.11. Let $f : U \to U$ be a dynamical system. Define the *power system* $f^* : U^* \to U^*$ of f by

1. $U^* = \wp(U) \setminus \varnothing$,
2. $f^*(X) = f[X]$.

Proposition 2.5.12. *Let $f : U \to U$ be a dynamical system. Then $X \subseteq U$ is*

1. *a fixed point of f^* iff X is strictly invariant under f,*
2. *a minimal fixed point of f^* with respect to set inclusion iff X is an attractor of f.*

Notice that f^* is in general not monotonic with respect to set inclusion.

Recall that in Definition 2.4.76 we defined maps $t : \{0, \frac{1}{2}, 1\}^X \to (\{0,1\}^X)^*$ by $t(f') = \{g \in \{0,1\}^X \mid g \supseteq f'\}$ and $r : (\{0,1\}^X)^* \to \{0, \frac{1}{2}, 1\}^X$ by $r(\mathcal{F}) = \bigcap \mathcal{F}$. Clearly, r is a retraction for t. Moreover, Theorem 2.4.78 implies the following.

Theorem 2.5.13. *Let Φ be a Boolean network on some set X. Then t is a weak system embedding of $\Phi'_{FV} : \{0, \frac{1}{2}, 1\}^X \to \{0, \frac{1}{2}, 1\}^X$ into $\Phi^* : (\{0,1\}^X)^* \to (\{0,1\}^X)^*$ with retraction r.*

Transformations of iteration graphs

Just as iteration graphs (cf. Definition 2.2.5) provided a way of visualizing dynamical systems, transformations of dynamical systems can be captured by transformations of digraphs i.e., *digraph morphisms.*

Definition 2.5.14. Let G, H be directed graphs. A map $h : V(G) \to V(H)$ is said to be

1. a *digraph homomorphism* (from G to H) iff for all $x, y \in V(G)$, $(x, y) \in A(G)$ implies $(h(x), h(y)) \in A(H)$,
2. a *digraph isomorphism* iff it is a digraph homomorphism and there is a digraph homomorphism $g : V(H) \to V(G)$ that is a retraction and a section for h,
3. a *strong digraph embedding* iff it is a digraph homomorphism and has a retraction that is a digraph homomorphism.

Proposition 2.5.15. *Let G, H be digraphs. A map $h : V(G) \to V(H)$ is a digraph isomorphism iff the following conditions are satisfied.*

1. *h is a digraph homomorphism,*

2. *h is bijective and*
3. *h^{-1} is a digraph homomorphism.*

Proposition 2.5.16. *Let $f : U \to U$ and $g : V \to V$ be dynamical systems. Let I and J be their iteration graphs respectively. Let $k : U \to V$. Then*

1. *k is a system map from f to g iff k is a digraph homomorphism from I and J,*
2. *k is a system isomorphism from f to g iff k is a digraph isomorphism from I and J.*
3. *k is a strong system embedding from f to g iff k is a strong digraph embedding from I and J.*

2.5.2 Network isomorphisms

Recall that a constrained Boolean network (Definition 2.3.1) has a digraph component as well as a dynamical system component. After having formulated transformations for each of these categories in the previous subsection, we shall tackle in this subsection the compound category of constrained (Boolean) networks.

Definition 2.5.17. A Boolean network Φ is said to be a *representation* of a dynamical system $f : U \to U$ iff there exists a system isomorphism $i : U \to \{0,1\}^{V(G)}$ from f to Φ.

Of course the notion of representation could be defined analogously for arbitrary function networks.

Question 2.5.18. *Has every dynamical system a representation as a Boolean network?*

Given the following proposition a positive answer seems plausible.

Proposition 2.5.19. *Let $f : U \to U$ be a dynamical system. If there exists a cardinal κ such that $|U| = 2^\kappa$, then there exists a Boolean network that is a representation of f.*

Proof. Let $i : U \to 2^\kappa$ be bijective. Define $\Phi : \{0,1\}^\kappa \to \{0,1\}^\kappa$ by $\Phi = i \circ f \circ i^{-1}$. Then $\Phi \circ i = (i \circ f \circ i^{-1}) \circ i = i \circ f$. Hence i is a system map and so is i^{-1} by analogous reasoning. Since i is bijective, i^{-1} is both a retraction and a section for i. □

Although its proof is almost trivial, Proposition 2.5.19 may seem surprising from a certain point of view. In light of examples like Example 2.4.24 it may even seem intriguing. However, if a dynamical system has a representation, it usually has many of them and the dependency graphs of these Boolean networks may differ greatly.

Definition 2.5.20. Two Boolean networks Φ and Ψ are said to be *isomorphic* iff there exists a system isomorphism between them.

So any two representation of a dynamical system are isomorphic Boolean networks. If we want to preserve the dependency structure of a Boolean network, we need to formulate a more fine grained notion of isomorphism for the category of constrained Boolean networks. We call this kind of isomorphism *network* isomorphism in order to emphasize the dependency component.

Definition 2.5.21. Let (G, Φ) and (H, Ψ) be constrained Boolean networks. A *network isomorphism* from (G, Φ) to (H, Ψ) is an ordered pair (φ, i) such that

1. φ is a digraph isomorphism from G to H,
2. $i : \{0,1\}^X \to \{0,1\}^Y$ is a system isomorphism.

We write $(G, \Phi) \simeq (H, \Psi)$ and say that (G, Φ) and (H, Ψ) are *isomorphic* iff there exists a network isomorphism $(\varphi, i) : (G, \Phi) \to (H, \Psi)$.

Definition 2.5.22. A network isomorphism from (G, Φ) to (H, Ψ) is said to be a *network automorphism* iff $G = H$ and $\Phi = \Psi$ and a *network semi-automorphism* iff $G = H$.

Example 2.5.23. In order to illustrate the difference between network- and system isomorphisms, let us construct a Boolean network Φ that performs some very simple computation. (For models of computation based on Boolean networks cf. [19]). The task of Φ is to decide whether for a given number $0 \leq n \leq 3$ the claim that $n < 2$ is true or false. Our network Φ consists of three automata, I_0, I_1 and U. The set $\{I_0, I_1\}$ is thought of as the *input layer* of Φ, whereas the *output layer* consists of the single automaton U. Before we specify the functions of the automata we need a to choose a coding of the numbers $0 \leq n \leq 3$ in terms of of 0's and 1's. First let us stick to the standard coding $0 \mapsto 00, 1 \mapsto 01, 2 \mapsto 10, 3 \mapsto 11$. If $n < 2$, then the output automaton is expected to produce the value 1; if $n \geq 2$, the output value is expected to be 0.

Let us define the Boolean network Φ as follows. $\Phi(f, U) = f(I_0)$, $\Phi(f, I_0) = f(I_0)$ and $\Phi(f, I_1) = f(I_1)$, for all $f \in \{0,1\}^A$, where $A = \{I_0, I_1, U\}$. Then Φ performs the assigned task while leaving the input layer unchanged. The following set of arcs constitutes the minimal dependency graph for Φ:

$$A(G) = \{(I_0, I_0), (I_1, I_1), (U, I_0)\}.$$

Now let us change the coding as follows: $0 \mapsto 10, 1 \mapsto 01, 2 \mapsto 00, 3 \mapsto 11$. With respect to the new coding the network Φ does not solve the problem correctly anymore. For an amendment, the functions of the automata of the input layer can remain unchanged, but the function of the output-automaton has to be modified as follows.

$$\Phi'(f, U) = \begin{cases} 1, & \text{if } \ (f(I_0) = 1 \wedge f(I_1) = 0) \vee (f(I_1) = 0 \wedge f(I_1) = 1) \\ 0, & \text{else.} \end{cases}$$

Notice that G is not a dependency graph for Φ'. An arc must be added to G in order to come up with the minimal dependency graph G' for Φ':

$$A(G') = \{(I_0, I_0), (I_1, I_1), (U, I_0), (U, I_1)\}.$$

It is straightforward to find a system automorphism i from Φ to Φ'. Hence each of the Boolean networks Φ and Φ' can be expressed as a *simulation* of the other via i, e.g. $\Phi' = i \circ \Phi \circ i^{-1}$. But since G and G' are not isomorphic, there is no network semi-automorphism between (G, Φ) and (G', Φ').

In [14] Cook defines for a certain class of sentence systems their *dual* sentence system and then discusses dual versions of various paradoxes. Rabern et al. give a generalized version of Cook's definition (cf. [41], Definition 6). The following is an adaption of the latter for Boolean networks.

Definition 2.5.24. Let Φ be a Boolean network on some set X.

1. Let $\mathrm{iv}_X : \{0,1\}^X \to \{0,1\}^X$ be defined by $\mathrm{iv}_X(f) = \{(x, \bar{v}) \mid (x, v) \in f\}$, where $\bar{v} = 1$ if $v = 0$ and $\bar{v} = 0$ if $v = 1$, for $v \in \{0,1\}$.
2. Define $\bar{\Phi} : \{0,1\}^X \to \{0,1\}^X$ by $\bar{\Phi} = \mathrm{iv}_X \circ \Phi \circ \mathrm{iv}_X$. We call $\bar{\Phi}$ the *Boolean network dual* to Φ.

Proposition 2.5.25. *Let Φ be a Boolean network on G and $X = V(G)$. Then*

1. $\bar{\Phi}$ is a Boolean network on G,

2. $(\mathrm{id}_G, \mathrm{iv}_X)$ is a network semi-automorphism from (G, Φ) to $(G, \bar{\Phi})$ with $(\mathrm{id}_G, \mathrm{iv}_X)^{-1} = (\mathrm{id}_G, \mathrm{iv}_X)$,

3. For all $f \in \{0,1\}^X$, f is a fixed point of Φ iff $\mathrm{iv}_X(f)$ is a fixed point of $\bar{\Phi}$.

For the following proposition and examples recall Definition 2.3.6.

Proposition 2.5.26. *Let G be a digraph. Then*

1. $\bar{\Phi}^G_\top(f, x) = \Phi^G_\bot(f, x)$.

2. $\bar{\Phi}^G_\bigvee(f, x) = \Phi^G_\bigwedge(f, x)$,

3. $\bar{\Phi}^G_\downarrow(f, x) = \Phi^G_\uparrow(f, x)$.

Definition 2.5.27. Call a Boolean network Φ *self-dual* iff $\bar{\Phi} = \Phi$.

Example 2.5.28. The Boolean network from Example 2.3.15 that models the liar sentence is self-dual. Then same holds for the Boolean network from Example 2.3.15 that models the truth-teller sentence and for the Boolean network of Example 2.3.16 that models the sentence system of Jourdain's paradox.

Duality is not the only kind of *symmetry* a constrained Boolean network might show. (We would like to think of any non-trivial member of the semi-automorphism group of a constrained Boolean network (G, Φ) as a *symmetry* of (G, Φ)).

Definition 2.5.29. Let (G, Ψ) be a Boolean network and $\varphi : V(G) \to V(G)$ a digraph automorphism.

1. Let $i_\varphi : \{0,1\}^{V(G)} \to \{0,1\}^{V(G)}$ be defined by $i_\varphi(f) = \{(x, f(\varphi(x))) \mid x \in V(G)\}$,
2. and define $\Psi_\varphi : \{0,1\}^{V(G)} \to \{0,1\}^{V(G)}$ by $\Psi_\varphi = i_{\varphi^{-1}} \circ \ \Psi \circ i_\varphi$. We call (G, Ψ_φ) the *φ-shift* of (G, Ψ).

Proposition 2.5.30. *Let (G, Ψ) be a constrained Boolean network and $\varphi : V(G) \to V(G)$ a digraph automorphism. Then*

1. *(G, Ψ_φ) is a constrained Boolean network,*
2. *(φ, i_φ) is a network semi-automorphism from (G, Ψ) to (G, Ψ_φ) with $(\varphi, i_\varphi)^{-1} = (\varphi^{-1}, i_{\varphi^{-1}})$,*
3. *for all $f \in \{0,1\}^{V(G)}$, f is a fixed point of Ψ iff $i_\varphi(f)$ is a fixed point of Ψ_φ.*

Example 2.5.31. Let C be a directed cycle and $x, y \in V(C)$. Then there is a unique digraph automorphism $\varphi_{x,y}$ such that $\varphi_{x,y}(x) = y$ and every digraph automorphism on C can be represented in this way. If C is a cycle of liars, then we might think of the resulting Boolean network as the same cycle of liars. More precisely, (φ, i_φ) is an automorphism on $(C, \Phi^C_\downarrow)$. (It is non-trivial iff $x \neq y$).

On the other hand, consider Jourdain's paradox from Examples 1.1.2 and 2.3.16. Applying φ_{L_1,T_2} to Φ swaps the roles of the liar and the truth-teller and yields a Boolean network that is different from (but isomorphic to) Φ, which happens to be $\bar{\Phi}$, the dual network of Φ.

2.5.3 Function network products

The purpose of this short subsection is to have a quick look at what might happen to dependency graphs if function networks are iterated.

The following notion is well-known under the name of 'relation product'.

Definition 2.5.32. Let S be a set and G_1, G_2 digraphs with $V(G_1) = V(G_2) = S$. Define $V(G_1 \circ G_2) = S$ and $A(G_1 \circ G_2) = \{(x, z) \mid \exists y \in S, (x, y) \in A(G_1) \wedge (y, z) \in A(G_2)\}$.

Theorem 2.5.33. *Let G_1, G_2 be digraphs with $V(G_1) = V(G_2)$ and Φ_1, Φ_2 function networks on G_1 and G_2 respectively. Then $\Phi_1 \circ \Phi_2$ is a function network on $G_1 \circ G_2$.*

Definition 2.5.34. Let (G, Φ) a function network of type Σ.

- Let $\Phi^0 = \mathrm{Id}_\Sigma$ and $\Phi^n = \Phi \circ \Phi^{n-1}$, for all $n > 0$.
- Let $G^0 = \{(x, x) \mid x \in V(G)\}$ and $G^n = G \circ G^{n-1}$, for all $n > 0$.

Proposition 2.5.35. *Let G be a digraph and $x, y \in V(G)$ and $n \in \omega$. Then there is a non-trivial walk from x to y in G of length n iff $(x, y) \in A(G^n)$.*

Proposition 2.5.36. *Let G be an acyclic digraph such that there exists some $n \in \omega$ such that every path in G has length $\leq n$. Then G^{n+1} is totally disconnected.*

Example 2.5.37. Let G and G' be the digraphs from Example 2.5.23. Then $G^n = G$ and $(G')^n = G'$, for all $n \in \omega$.

2.5.4 What is a reference pattern?

Question 2.5.38. *How many network semi-automorphisms does any given constrained Boolean network have?*

This question is of interest in order to determine how many version of a given paradox there are. The semi-automorphisms of a constrained Boolean network form a group, each of whose members i gives rise to a constrained Boolean network $(G, i^{-1} \circ \Phi \circ i)$ which can be thought of as some kind of mirror image of (G, Φ). The semi-automorphism group contains always the identity-map and the involution iv_X (cf. Definition 2.5.24), i.e., it is at least of order two for every digraph whatsoever. A more specific version of the above question is under what circumstances the set of all φ-shifts together with the involution forms already a set of generators of the semi-automorphism group of (G, Φ)?

Question 2.5.38 leads to a deeper one.

Question 2.5.39. *Should two constrained Boolean networks be considered identical if they are isomorphic? Given that they have no fixed points and can be interpreted as sentence systems, is the paradox that arises from one of them identical to that arising from the other?*

There are well-known arguments that suggest a negative answer to the second part of Question 2.5.39, e.g.[15]. They imply that the concept of *reference* cannot be reduced to the concept *dependency* (in the sense of Definition 2.3.1) but that the way a reference is established is rather essential.

In any case, the framework suggested in this thesis is meant to capture only extensional aspects of semantic paradoxes and cannot help provide answers to questions like 2.5.39. On the other hand, it seems reasonable to assert that a reference-*pattern* is extensional, can be characterized in terms of dependency and is invariant under network isomorphisms.

But what exactly is a reference pattern? Supposing that a Boolean Φ describes some sentence systems (and in Section 3.5.1 we will show that every Boolean network can be thought of doing so) it would make sense say that a reference pattern of Φ is the isomorphism type (w.r.t. digraph isomorphisms, cf. Definition 2.5.14) of some dependency graph G of Φ. (The isomorphism type of a digraph G can be conceived as the class of all digraphs isomorphic to G).

This definition is too coarse in the sense that it cannot distinguish e.g., the liar sentence from the truth-teller. In [5] this problem is addressed by the introduction of so called *signed reference graphs*, i.e., labels are assigned to the arcs of a digraph in order to capture the *mode of reference*.

In Chapter 3 a more encompassing approach is chosen in order to model these more fine-grained reference patterns: that of a *characteristic digraph* of a constrained Boolean network. Indeed, it is shown in Section 3.6 that the concept of *signed dependency graph* can be reconstructed from that of a characteristic digraph.

2.6 Three related characterization problems

In this section we shall introduce three classes of digraphs, each of which is associated with a different aspect of the relation between paradoxes and reference patterns, and each of which gives rise to a different characterization problem: *dangerous* digraphs, digraphs *of infinite character* and *not strongly kernel-perfect* digraphs. To show that the corresponding characterization problems are interrelated and can, at least partially, be solved, is one of the main goals of this thesis.

2.6.1 Dangerous and safe digraphs

The notion of a dangerous digraph was originally introduced in [41] in a somewhat different setting. We will prove in Section 3.5.1 that a digraph is dangerous in our sense if and only if it is dangerous in the sense of Rabern et al. [41]. (Also cf. [40]). Observe that the empty digraph $(\varnothing, \varnothing)$ is trivially safe. All results in this subsections are already known from [41].

Definition 2.6.1. A digraph G is said to be *safe*[5] iff every Boolean network on G has a fixed point; it is said to be *dangerous* iff it is not safe.

Definition 2.6.2. G and H are *equi-dangerous* iff both G and H are dangerous or both are safe.

Proposition 2.6.3. *Every cycle is dangerous.*

Proposition 2.6.4. *A digraph is safe iff every subdigraph of it is safe.*

Proof. The proof follows that in [41]. Let H be a digraph and $G \subseteq H$. Suppose G is dangerous. Let Φ be a Boolean network on G such that Φ has no fixed point. Let $X = V(H) \setminus V(G)$. Let Φ_0 be defined on X such that $\Phi_0(f, x) = 0$ for all $f \in \{0,1\}^X$ and $x \in X$. Let $\Psi : \{0,1\}^{V(H)} \times V(H) \to \{0,1\}$ be defined

[5] Rabern et al. have no name for the digraphs that are not dangerous. Our choice seems to be the most suitable English word. However, notice that in [47] the term *safe* is used quite differently, in order to denote the class of digraphs that have a certain pattern that gives rise to Yablo-like paradoxes. This could be a particular awkward source of confusion, since we shall later state a conjecture that implies that an acyclic digraph is safe in Walicki's sense if and only if it is safe in our sense.

by $\Psi(f,x) = \Phi(f \upharpoonright V(G), x)$ if $x \in V(G)$ and $\Psi(f,x) = \Phi_0(f \upharpoonright X, x)$ if $x \in X$. Then Ψ is a Boolean network on $G \cup (X, \varnothing)$. Hence it is a Boolean network on H. Moreover, the restriction of every fixed point of Ψ to $V(G)$ is a fixed point of Φ. Hence H is dangerous. The other direction is trivial. □

Corollary 2.6.5. *Every digraph that contains a cycle is dangerous.*

Proposition 2.6.6. *Every well-founded digraph is safe.*

Proof. By Proposition 2.4.61. □

Proposition 2.6.7. *Every tree is safe.*

Proof. By theorem 2.4.88 □

Proposition 2.6.8. *A finite digraph is safe iff it is acyclic.*

Proof. Every finite and acyclic digraph is well-founded. □

Problem 2.6.9 (Rabern et al.)**.** *Characterize the dangerous digraphs!*

2.6.2 Compactness and digraphs of finite character

Corollary 2.6.5 implies that the concept of danger cannot distinguish between finitary (liar-like) and infinitary (Yablo-like or McGee-like) paradoxes: An infinite digraph may contain a cycle (and thus be dangerous) but nevertheless be not the reference pattern of any truly infinitary paradox. In order to fill this lacuna, we shall introduce the dichotomy of *digraphs of finite character* and *digraphs of infinite character*, which is based on the concept of a *compact Boolean network*.

Compact Boolean networks

The definitions and claims in this subsection can be formulated and proven for arbitrary finitary function networks, but for reasons of simplicity we present them only for Boolean networks.

The following definition and the next lemma are, *mutatis mutandis*, well-known from the literature. (Cf. e.g.,[17].) Recall the definitions in Subsection 2.4.6.

Definition 2.6.10. Let Φ be a Boolean network on a set X.

- For all $Y \subseteq X$, let $\mathcal{S}(Y)$ be the set of all $f \in \{0,1\}^{Y \cup \mathrm{Bd}^+_G(Y)}$ such that $f \upharpoonright Y$ is a fixed point of $\Phi^{f \upharpoonright \mathrm{Bd}^+_G(Y)}[Y]$.
- Call $\mathcal{F} \subseteq \wp(X)$ *compatible w.r.t.* Φ iff for all $Y \in \mathcal{F}$, there exists $f \in \{0,1\}^X$ such that for all $Y \in \mathcal{F}$, $f \upharpoonright (Y \cup \mathrm{Bd}^+_G(Y)) \in \mathcal{S}(Y)$.

Lemma 2.6.11 (Compactness principle)**.** *Let Φ be a Boolean network on a set X and $\mathcal{F}$ a set of finite subsets of X. Then $\mathcal{F}$ is compatible w.r.t. Φ if every finite $\mathcal{G} \subseteq \mathcal{F}$ is compatible w.r.t. Φ.*

Proof. Let $\Sigma = \{0,1\}^X$ and $\mathcal{T}(\Sigma)$ the product topology for Σ as in Definition 2.4.33. Then $(\Sigma, \mathcal{T}(\Sigma))$ is compact by Proposition 2.4.34 and for every finite $Y \subseteq X$, the set $S^*(Y) = \{f \in \Sigma \mid f \restriction (Y \cup \mathrm{Bd}_G^+(Y)) \in \mathcal{S}(Y)\}$ is closed in $\mathcal{T}(\Sigma)$.

Notice that for all $\mathcal{F} \subseteq \wp(X)$ the claim that $\mathcal{F}$ is compatible w.r.t. Φ is equivalent to the claims that the family $(S^*(Y))_{Y\in\mathcal{F}}$ has a non-empty intersection. Hence it follows from the hypothesis that for all finite $\mathcal{G} \subseteq \mathcal{F}$, $(S^*(Y))_{Y\in\mathcal{G}}$ has a non-empty intersection. Since Σ is compact, this implies by Proposition 2.4.36 that $(S^*(Y))_{Y\in\mathcal{F}}$ has a non-empty intersection and thus that $\mathcal{F}$ is compatible w.r.t. Φ. □

Definition 2.6.12. A Boolean network Φ on a set X is said to be

1. *compact* iff Φ has a fixed point, given that for all finite $\varnothing \neq Y \subseteq V(G)$, Φ is absolutely solvable relative to Y.

2. *strongly compact* iff Φ has a fixed point, given that for all finite $\varnothing \neq Y \subseteq V(G)$, Φ is relatively solvable relative to Y.

Clearly, every strongly compact Boolean network is compact.

Digraphs of finite- and infinite character

Definition 2.6.13. Let Φ be a Boolean network (on a set X) that has no fixed point. Then Φ is said to be

1. *paradoxical of finite character* iff there exits some finite $\varnothing \neq Y \subseteq X$ such that Φ is absolutely unsolvable relative to Y,

2. *paradoxical of quasi finite character* iff there exits some finite $\varnothing \neq Y \subseteq X$ such that Φ is relatively unsolvable relative to Y, but there exists no finite $\varnothing \neq Y \subseteq X$ such that Φ is absolutely unsolvable relative to Y,

3. *paradoxical of infinite character* iff every finite $\varnothing \neq Y \subseteq X$ is absolutely solvable relative to Y.

The idea of this trichotomy is that in the first two cases there exists some finite 'paradoxical' Boolean subnetwork (in a stronger sense in the first cases and in a weaker sense in the second) that is 'liable for' Φ not having a fixed point, whereas in the third case no finite paradoxical subnetwork is present. *Paradoxes of infinite character* are *paradoxes of non-compactness* in the sense that they are due to the non-applicability of the Compactness Principle 2.6.11, i.e., arguments as in the proof of Proposition 2.6.23 do not hold, because the digraph cannot be decomposed into suitable finite subdigraphs.

Proposition 2.6.14. *Let Φ be a Boolean network on a set X that has no fixed point. Then Φ is*

1. *paradoxical of finite character iff Φ is strongly compact,*

2. *paradoxical of quasi finite character iff is not strongly compact but compact,*

3. *paradoxical of infinite character iff not* Φ *compact.*

Definition 2.6.15. A digraph G is said to be *of finite character* iff every Boolean network on G is compact; it is said to be of *infinite character* iff it is not of finite character.

Example 2.6.16. Let us classify some previously discussed paradoxes.

1. Every liar cycle gives rise to a paradox of finite character. But there are also infinite Boolean networks that are paradoxical of finite character. By Proposition 2.6.23 (and Corollary 5.1.6) respectively), every Boolean network that has a finitely out-branching dependency graph but no fixed point is such a case. By Proposition 2.6.19, every dependency graph of such a network must contain a cycle.

2. McGee's paradox (cf. Example 2.4.8) is a paradox of quasi finite character. The subnetwork induced by $\{M_0, M_\omega\}$ is relatively unsolvable to the function h that assigns to each $0 < n < \omega$ the value 1: $\Phi^h[\{M_0, M_\omega\}]$ has no fixed point, it's iteration digraph is isomorphic to that of Jourdain's paradox (cf. Example 2.3.16). Hence Φ is compact. It is not strongly compact, however: let g be the function that assigns to each $0 < n < \omega$ the value 0. Then $\Phi^g[\{M_0, M_\omega\}]$ has the fixed point f, with $f(M_0) = 1$ and $f(M_\omega) = 0$. (Observe $G[X]$ is acyclic if X does not contain both M_0 and M_ω).

   Moreover, it can be show that the minimal dependency graph for Φ is of finite character.

3. Yablo's paradox is a paradox of infinite character. $\Phi^{\mathbb{Y}}_{\downarrow}$ is not compact. Hence $\mathbb{Y}$ is of infinite character.

This analysis seems to suggest that McGee's paradox is indeed closer to the paradoxes of finite character than to Yablo's paradox. In consideration of its dependency graph, this shouldn't come as a surprise.[6]

Proposition 2.6.17. *Every finite digraph is of finite character.*

Digraphs of finite character are closely related to safe digraphs.

Proposition 2.6.18. *Every safe digraph is of finite character.*

Proposition 2.6.19. *Every acyclic digraph of finite character is safe.*

Proof. By Proposition 2.6.8. □

[6]However, it may be surprising with regard to the fact that in frameworks that work with first order logic such as [5] [4] and [28] Yablo's paradox and McGee's paradox are similar in the sense that they both give rise to consistent first-order theories that are ω-inconsistent. This commonality can be explained be the fact that both Boolean networks are not strongly compact.

This leads to following theorem which, simple as it is, epitomizes the fact that there are two kinds of semantic paradoxes: finite paradoxes whose reference pattern is cyclic, and infinite paradoxes, the underlying mechanism of which is the phenomenon of *non-compactness*.

Theorem 2.6.20. *A digraph is dangerous iff it contains a cycle or if it is of infinite character.*

A consequence of this is a generalization of Proposition 2.6.8 to digraphs of finite character.

Corollary 2.6.21. *A digraph of finite character is safe iff it is acyclic.*

The question of how the reference patterns of infinitary paradoxes can be characterized is in the focus of this investigation.

Problem 2.6.22. *Characterize the digraphs that are of infinite character!*

The following proposition is essentially a well-known result - at least its version for safe digraphs (cf. [41] and [40]). Our proof employs the compactness principle and intends to exemplify how constraints on a given dependency graph can be used to obtain information about the dynamical behavior of a Boolean network.

Proposition 2.6.23. *Every finitely out-branching apg is of finite character.*

Proof. Let Φ be a Boolean network on G. Let r be the root of G. Let $X_0 = \{r\}$ and define $X_{n+1} = X_n \cup \mathrm{Bd}^+_G(X_n)$ for all $n \in \omega$. Then every X_n is finite. Let $\mathcal{F} = \{X_n \mid n \in \omega\}$. Then $V(G) = \bigcup \mathcal{F}$. Since unions of finite subsets of $\mathcal{F}$ are finite, it follows from the hypothesis that Φ is absolutely solvable relative to every non-empty finite subset of $V(G)$, that every finite subset of $\mathcal{F}$ is compatible w.r.t. Φ. Hence $\mathcal{F}$ is is compatible w.r.t. Φ by Lemma 2.6.11. Let $f \in \{0,1\}^{V(G)}$ be a witness to this claim. Then f is a fixed point of Φ. In order to see this, let $x \in V(G)$. Then $x \in X_n$ for some $n \in \omega$. Let $f_n = f \restriction X_n \cup \mathrm{Bd}^+_G(X_n)$. Then $f_n \in \mathcal{S}(X_n)$. Hence $\Phi(f_n, x) = f_n(x) = f(x)$. On the other hand, $\Phi(f_n, x) = \Phi(f, x)$, since x depends on $X_n \cup \mathrm{Bd}^+_G(X_n)$. Hence $\Phi(f, x) = f(x)$. □

The previous proposition shall later be generalized for finitely out-branching digraphs simpliciter (cf. Corollary 5.1.6). As a consequence, every acyclic and finitely out-branching digraph is safe.

Proposition 2.6.24. *A digraph is of finite character iff all of its subdigraphs are.*

Proof. If $G \subseteq H$ is not of finite character, then there exists a Boolean network Φ on G that is not compact, i.e., Φ has no fixed point but for all finite $\varnothing \neq Y \subseteq V(G)$, Φ is absolutely solvable relative to Y. Now we extend Φ to a Boolean network Ψ on H analogously to the proof of Proposition 2.6.4. The other direction is trivial. □

Skeletons

Let us conclude this subsection on digraphs of infinite character by having a quick look at a concept whose purpose it is to shed some more light on the distinction between cyclic and acyclic paradoxes.

Definition 2.6.25. Let G be a digraph. A *skeleton* of G is an acyclic spanning subdigraph of G that has no proper superdigraph that is an acyclic spanning subdigraph of G.

Clearly, every acyclic digraph is its own unique skeleton.

Proposition 2.6.26. *Every digraph has a skeleton.*

Proof. Let G be a digraph. Then G has an acyclic spanning subdigraph - the digraph $(V(G), \varnothing)$. Moreover, the union of every chain of acyclic spanning subdigraphs of G is an acyclic spanning subdigraph of G. Hence by Zorn's lemma there exists a maximal acyclic spanning subdigraph of G. This is a skeleton of G. □

Question 2.6.27. *Is there a digraph of infinite character that has a skeleton of finite character?*

A positive answer would imply the existence of some interesting kind of paradox. It would mean that, on the one hand, there is a Boolean network Φ on a digraph G (of infinite character) such that Φ has no fixed point, but for every non-empty finite $Y \subseteq V(G)$, Φ is absolutely solvable relative to Y. Hence the paradoxicality of Φ cannot be *localized* in any finite part of Φ, just as it is the case with Yablo's paradox. In particular, there is no specific finite structure of cycles that is responsible for Φ being paradoxical. On the other hand, there exists a skeleton H of G that is of finite character. Hence H is safe. This means that the cyclic structure of G is essential for Φ being paradoxical, but on a global rather than on a local scale. Φ would be a paradox that couldn't be explained in terms of danger. G would be dangerous because it contains a cycle, but that misses the point. Φ would be a cyclic paradox of truly infinite character and not just of quasi finite character as McGee's paradox is.

In Section 2.8 we shall briefly discuss which answer to Question 2.6.27 seems to be the plausible one and whether any result or conjecture could help to establish it.

2.6.3 Strongly kernel-perfect digraphs

The third class of digraphs to be considered is in some sense a more fine grained counterpart to dangerous digraphs. Proposition 2.6.28 can be seen as a first hint at this fact, Proposition 2.6.29 as another.

Recall that a digraph is said to *strongly kernel-perfect* iff every of its subdigraphs has a kernel.

Proposition 2.6.28 (Richardson)**.** *A finite digraph is strongly kernel-perfect iff it contains no odd cycle.*

Proof. Proofs can be found in [1], [3] or [42]. □

Proposition 2.6.29. *Every safe digraph is strongly kernel-perfect.*

Proof. By Propositions 2.6.4 and Proposition 2.3.8. □

A version of this proposition has already been formulated in [41]. [7]

In analogy to Corollary 2.6.21, Richardson's theorem can be generalized to digraphs of finite character.

Theorem 2.6.30. *A digraph of finite character is strongly kernel-perfect iff it contains no odd cycle.*

Proof. Let G be a digraph of finite character. For the non-trivial direction suppose that G contains no odd cycle. Let $H \subseteq G$. Then H is also of finite character. Hence $\Phi^H_\downarrow$ is compact. But then $\Phi^H_\downarrow$ must have a fixed point (which then corresponds to a kernel of H): Assuming otherwise implies by hypothesis the existence of a finite and non-empty set $Y \subseteq V(H)$ and $h \in \{0,1\}^{\mathrm{Bd}^+_H(Y)}$ such that $\Psi^h[Y]$ has no fixed point, where $\Psi = \Phi^H_\downarrow$. Let C be the core of $\Psi^h[Y]$ (cf. Definition 2.4.89). Then by Lemma 2.4.94 $G[C]$ has no kernel, which contradicts Richardson's theorem. □

Problem 2.6.31. *Characterize the strongly kernel-perfect digraphs!*

Bipartite digraphs

The goal of this subsection is to prove Theorem 2.6.41, which also has a counterpart in [5].

Definition 2.6.32. A digraph G is said to be *bipartite* iff there exists a partition of $V(G)$ into two components X and Y such that $A(G[X]) = A(G[Y]) = \varnothing$.

For the following proposition cf. also Theorem 2.2.1 in [1].

Proposition 2.6.33. *Every bipartite digraph is strongly kernel-perfect.*

Proof. Let G be a bipartite digraph and $H \subseteq G$. Let $X \subseteq V(H)$ be the core (cf. Definition 2.4.89) of $\Phi^H_\downarrow$. By Theorem 2.4.95 it suffices to show that $H[X]$ has a kernel. By Proposition 2.4.90 $H[X]$ is sink-less. Moreover, it is bipartite, since it is a subdigraph of a bipartite digraph. Let $\{X_0, X_1\}$ be a bipartition of $H[X]$. Then X_0 (as well as X_1) is a kernel of $H[X]$: X_0 is independent in H and for all $x \in X_0$, there must be some $y \in X_1$ such that $(x, y) \in A(H[X])$ (otherwise x would be a sink of $H[X]$). □

[7] Let me remark that in Appendix D of [41] the authors seem to be unaware of the fact that every safe digraph has a kernel: Corollary 32 and Corollary 33 are unnecessarily relativized to digraphs that have no sinks. Cf. also the remark in a footnote of Subsection 1.2.2.

Lemma 2.6.34. *Let G be a digraph and w a walk from x to y in G, for any $x, y \in V(G)$. Then there exists a straight walk from x to y in G.*

Proof. By avoiding every cycle of H, we find a straight walk w' from x to y in H. Notice that in the case of $x = y$, the trivial walk can be chosen. $\square$

Let us restate the definition of a double-path from Section 2.1.

Definition 2.6.35. Let D be a digraph and $x \neq y \in V(D)$. Then D is said to be a *double path* (from x to y) iff there are non-trivial paths $P_1 \neq P_2$ from x to x such that $V(P_1) \cap V(P_2) = \{x, y\}$ and $V(D) = V(P_1) \cup V(P_2)$ and $A(D) = A(P_1) \cup A(P_2)$.

Definition 2.6.36. Let G be a finite digraph. The *parity* of G is the parity of the set $A(G)$. In particular, we say that G is *even* iff $A(G)$ has even cardinality and that G is *odd* iff $A(G)$ has odd cardinality.

Proposition 2.6.37. *Every apg that contains neither an odd cycle nor an odd double path is bipartite.*

Proof. Let G be an apg with root r. Let X_0 be the set of all $x \in V(G)$ such that x is reachable from r in a straight walk of even length in G and X_1 the set of all $x \in V(G)$ such that x is reachable from r in a straight walk of odd length in G. Then, by Lemma 2.6.34, $V(G) = X_0 \cup X_1$.

Next we show that. $X_0 \cap X_1 = \varnothing$. Assume that there exists some $y \in X_0 \cap X_1$. Since G contains no odd cycle, $y \neq r$.

Let w_0 be an even straight walk from r to x in G and w_1 an odd straight walk from r to x in G. Since $y \neq r$, each w_i constitutes (in the obvious manner) a non-trivial path P_i. Then there exists some $y_1 \in (V(P_0) \cap V(P_1) \setminus \{r, y\})$. (Otherwise $P_0 \cup P_1$ would be an odd double path). Then there are paths Q_0 and Q_1 (both from y_1 to y) and R_0 and R_1 (both from r to y_1) such that $P_0 = Q_0 \circ R_0$ and $P_1 = Q_1 \circ R_1$. We may assume that we have chosen y_1 in such a way that $Q_0 \cup Q_1$ is a double path. Hence $V(Q_0 \cup Q_1)$ is even, i.e., Q_0 and Q_1 have the same parity. Then R_0 and R_1 must have different parities, otherwise P_0 and P_1 couldn't have different parities. Now we can apply the same argument as above and come up with some $y_2 \in (V(R_0) \cap V(R_1) \setminus \{r, y_1\})$. This procedure leads to the contradictory conclusion that the set $V(P_0 \cap P_1)$ is infinite. Hence $X_0 \cap X_1 = \varnothing$.

Now let us show that $\{X_0, X_1\}$ is bipartition of G. Assume that there exists $i \in \{0, 1\}$ and $x, y \in X_i$ such that $(x, y) \in A(G)$. Since G contains no odd cycle, $x \neq y$. There exists a straight walk $w_x = (r, \dots, x)$ of parity i from r to x and a straight walk $w_y = (r, \dots, y)$ of parity i from r to y.

Case 1: y does not occur in w_x. Then $(r, \dots, x, y)$ is a straight walk from r to y of different parity than i, which is a contradiction.

Case 2: y does occur in w_x, i.e, $w_x = (r, \dots y, \dots, x)$. Then the walk $(r, \dots, y)$ is straight and thus of parity i (since $X_0 \cap X_1 = \varnothing$). Hence the walk $(y, \dots, x)$ (which is also straight) is even. Since $x \neq y$, we obtain the contradiction that G contains an odd cycle. $\square$

For the following definition cf. [36].

Definition 2.6.38. Let G be a digraph. A set $X \subseteq G$ is said to be a *semi-kernel* iff the following conditions are satisfied.

1. X is independent in G,
2. for all $(v, w) \in A(G)$, if $v \in X$ and $w \in V(G) \setminus X$, then there exists some $v' \in X$ such that $(w, v') \in A(G)$.

Note that every kernel of a digraph G is a semi-kernel of G. Moreover, $\varnothing$ is always a semi-kernel of any digraph whatsoever, whereas $\varnothing$ is a kernel of a digraph G iff $G = (\varnothing, \varnothing)$.

Proposition 2.6.39. *Let G be a digraph and $x \in V(G)$. Then every kernel of $G\{x\}$ is a semi-kernel of G.*

Proof. The claim follows from the fact that $G\{x\}$ is open in G. □

The following is taken from [36].

Lemma 2.6.40. *Let G be a digraph such that every non-empty induced subdigraph of G possess a non-empty semi-kernel. Then G possess a kernel.*

Proof. Cf. Theorem 2 in [36]. □

Now we can prove a result from which Corollary 5.12. of [5] follows straightforwardly.

Theorem 2.6.41. *Every digraph that contains neither an odd cycle nor an odd double path is strongly kernel-perfect.*

Proof. Let G be a digraph that contains neither an odd cycle nor an odd double path. Let $H \subseteq G$. Assume H has no kernel. Then by Lemma 2.6.40 there exists some $\varnothing \neq X \subseteq V(H)$ such that $H[X]$ has no non-empty semi-kernel. Let $x \in X$. Then $H\{x\}$ has no kernel by Proposition 2.6.39. But $H\{x\} \subseteq G$, hence $H\{x\}$ contains neither an odd cycle nor an odd double path. Hence $H\{x\}$ is bipartite by Proposition 2.6.37 and thus is strongly kernel-perfect by Proposition 2.6.33. By Proposition 2.6.39 this leads to the contradiction that $H[X]$ has a non-empty semi-kernel. □

2.7 Digraph transformations

In Subsection 2.5.1 transformations of dynamical systems have been discussed. Some of them, like weak systems embeddings, preserve considerably less information than system isomorphisms do. In this section we introduce two types of operations on digraphs that preserve some but not all of the digraph's structure: subdivisions and inflations. Inflations in particular will play an important role throughout the remainder of this thesis. They come in various flavors: finitary, regular and convergent. In Chapter 3 regular inflations are used in order to formulate the notion of a *network inflation* which has a digraph component (a regular digraph inflation) and a dynamical systems component (a weak system embedding).

In Section 2.8 solutions for the characterization problems presented in the last section shall be conjectured in terms of finitary digraph inflations. The *parity* (the property of being even or odd) of a digraph inflation will be defined, a concept that is key in order to formulate the conjectured solution for the characterization problem for strongly kernel-perfect digraphs (Problem 2.6.31).

2.7.1 Subdivisions

Subdivisions and their relevance for the analysis of dangerous digraphs have been discussed in [41].

Definition 2.7.1 (Rabern et al.)**.** A *subdivision* of a digraph G is a digraph that is formed by replacing each arc $(x, y) \in A(G)$ by a path of length ≥ 1 from x to y.

Note that replacing an arc by a path of length $= 1$ means leaving the arc as it is. In particular, every digraph is a subdivision of itself.

Definition 2.7.2 (Rabern et al.)**.** A digraph G is said to be *homeomorphic* to a digraph H iff G has a subdivision that is isomorphic to some subdivision of H.

Proposition 2.7.3 (Rabern et al.)**.** *A digraph is dangerous iff every subdivision of it is dangerous.*

Corollary 2.7.4. *Homeomorphic digraphs are equi-dangerous.*

Let us note the following trivial observation for cycles, whose non-trivial counterpart for finitary inflations of $\mathbb{Y}$ we will shown below (Theorem 4.3.2).

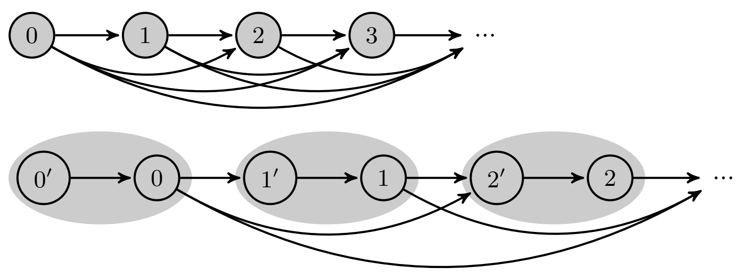
Proposition 2.7.5. *If C is a cycle, then some subdivision of C has no kernel.*

The following definition introduces a rather specific operation on digraphs that in general cannot be conceived as a subdivision.

Definition 2.7.6. Let G be a digraph. Define $G^{\rightarrow}$ as follows: for each $x \in V(G)$ let $x' = \{x\}$. Let $V(G^{\rightarrow}) = V(G) \cup \{x' \mid x \in V(G)\}$. For all $u, v \in V(G^{\rightarrow})$ let $(u, v) \in A(G^{\rightarrow})$ iff $u = x'$ and $v = x$ for some $x \in V(G)$ or if for $x, y \in V(G)$: $v = x$ and $u = y'$ and $(x, y) \in A(G)$.

Is every acyclic digraph that contains no subdivision of $\mathbb{Y}$ safe? The answer is negative as Rabern et al. show in Appendix B of [41]. They give a counterexample which can be formulated in terms of of Definition 2.7.6.

Example 2.7.7. There is a dangerous digraph G that is acyclic and contains no subdivision of $\mathbb{Y}$, namely $G = \mathbb{Y}^{\rightarrow}$.



Then no subdigraph of G is a subdivision of $\mathbb{Y}$, i.e., G is not homeomorphic to $\mathbb{Y}$. In order to see this, first observe that $d_G^-(n) = 1$ (since n' is the unique in-neighbor of n) for all $n \in \omega$. On the other hand, for every $n \in \omega$, $d_{\mathbb{Y}}^-(n) = n$. Since subdividing arcs does not alter the degree of any vertex of the original digraph, no subdivision of $\mathbb{Y}$ is contained in G.

Since G is clearly bipartite, G is strongly kernel-perfect by Proposition 2.6.33. However, it is dangerous. One can either apply Corollary 4.3.4 or find a Boolean network on G that has no fixed point - which is left to the reader.

2.7.2 Inflations and minors

There is a more general concept underlying the operation formulated in Definition 2.7.6, which shall be outlined in this subsection.

Definition 2.7.8. Let X, Y be sets. A *partition of Y with index set X* is a function $\mathcal{P} : X \to \wp(Y)$ such that for all $x, y \in X$,

1. $\mathcal{P}(x) \neq \varnothing$,
2. $\mathcal{P}(x) \cap \mathcal{P}(y) = \varnothing$,
3. $Y = \bigcup\{\mathcal{P}(x) \mid x \in X\}$.

$\mathcal{P}$ is said to be *finitary* iff $\mathcal{P}(x)$ is finite for all $x \in X$.

Notice that, as a function, every partition $\mathcal{P}$ is injective. As usually, we denote the inverse of $\mathcal{P}$ by $\mathcal{P}^{-1}$. We may conceive a partition $\mathcal{P}$ of Y with index-set X as a set of indexed subsets of Y and write $\mathcal{P} = \{\mathcal{P}(x) \mid x \in X\}$.

In the literature (e.g., in [17]), the *inflation* of an undirected graph H onto an undirected graph G is usually defined as a partition $\mathcal{P} = \{\mathcal{P}(x) \mid x \in V(H)\}$ of $V(G)$ such that (1) for all $x \neq y \in V(H)$, $\{x, y\} \in E(H)$ iff there is some $\{v_x, v_y\} \in E(G)$ such that $v_x \in \mathcal{P}(x)$ and $v_y \in \mathcal{P}(y)$ and (2) and for all $x \in V(H)$, $G[\mathcal{P}(x)]$ is connected. ($E(H)$) denotes the set of all *edges* od H.) An undirected graph H is said to be a *minor* of a graph G iff there exists an inflation of H onto some subgraph of G.

Now, let us generalize these notions for directed graphs. In order to do so, first consider the following notion of a *boundary*.

Definition 2.7.9. Let G, H be digraphs and $\mathcal{I}$ a partition of $V(G)$ with index set $V(H)$. For all $x \in V(H)$ let

1. the *inward-boundary* $\partial_G^- \mathcal{I}(x)$ of $\mathcal{I}(x)$ be
   (a) the set of all sources of $G[\mathcal{I}(x)]$, if $x \in \mathrm{src}(H)$,
   (b) the set of all $z \in \mathcal{I}(x)$ such that there exists $y \in V(G) \setminus \mathcal{I}(x)$ with $(y, z) \in A(G)$, if $x \notin \mathrm{src}(H)$,
2. the *outward-boundary* $\partial_G^+ \mathcal{I}(x)$ of $\mathcal{I}(x)$ be
   (a) the set of all sinks of $G[\mathcal{I}(x)]$, if $x \in \mathrm{snk}(H)$,
   (b) the set of all $z \in \mathcal{I}(x)$ such that there exists $y \in V(G) \setminus \mathcal{I}(x)$ with $(z, y) \in A(G)$, if $x \notin \mathrm{snk}(H)$.
3. The *boundary* $\partial_G \mathcal{I}(x)$ of $\mathcal{I}(x)$ be defined as $\partial_G \mathcal{I}(x) = \partial_G^- \mathcal{I}(x) \cup \partial_G^+ \mathcal{I}(x)$.

The following definition of an *inflation* is a modified version of the definition of *finitary inflation* given in the appendix of [5]. The differences are that we drop the requirement that $\mathcal{I}(x)$ is finite (we will treat this special case still under the name of *finitary inflation*); that we are more liberal when it comes to loops (here in clause (1) we just ignore them, whereas in [5] we didn't allow a digraph with loops to be inflated at all). Moreover, the notions of inward- and outward-boundary $\partial_G^-(\mathcal{I})$ and and $\partial_G^+(\mathcal{I})$ have been modified in order to accommodate sinks and sources. Finally, we added the last clause, that $G[\mathcal{I}(x)]$ is always weakly connected, because we want to exclude some degenerate cases and make sure that the notion of inflation for directed graphs coincides with that for undirected graphs, if a directed graph is symmetric.

Definition 2.7.10. Let G and H be digraphs and $\mathcal{I} : V(H) \to \wp(V(G))$ a partition of $V(G)$. We call $\mathcal{I}$ an *inflation of H onto G* iff the following conditions hold.

1. For all $x \neq y \in V(H)$, $(x, y) \in A(H)$ iff there is some $(v_x, v_y) \in A(G)$ with $v_x \in \mathcal{I}(x)$ and $v_y \in \mathcal{I}(y)$.
2. For all $x \in V(H)$ and all $y, z \in \mathcal{I}(x)$, if $y \in \partial_G^-(\mathcal{I}(x))$ and $z \in \partial_G^+(\mathcal{I}(x))$, then there is a path in $G[\mathcal{I}(x)]$ from y to z.
3. For all $x \in V(H)$, $G[\mathcal{I}(x)]$ is weakly connected.

Example 2.7.11. Recall Definition 2.7.6. Let G be any non-empty digraph. Then $\mathcal{I} = \{\{x, x'\} \mid x \in V(G)\}$ is an inflation of G onto $G^{\rightarrow}$.

Definition 2.7.12. Let G and H be digraphs.

1. A surjective function $f : V(G) \to V(H)$ is said to be a *contraction* of G onto H iff the partition $\mathcal{P} : V(H) \to \wp(V(G))$ given by $x \mapsto \mathcal{P}(x) = f^{-1}(x)$ is an inflation of H onto G.

2. By abuse of notation, we denote for any partition $\mathcal{P} : V(H) \to \wp(V(G))$ of $V(G)$ by $\mathcal{P}^{-1}$ the unique function $f : V(G) \to V(H)$ such that $\mathcal{P}(x) = f^{-1}(x)$, for all $x \in V(H)$.

Definition 2.7.13. Let G, H be digraphs.

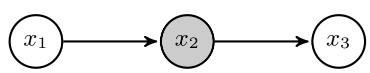
1. Then G is said to be *a(n) (finitary) inflation of* H iff there exists a (finitary) inflation of H onto G.

2. Likewise, we say that H is a *(finitary) contraction of* G iff there exists a finitary contraction of G onto H.

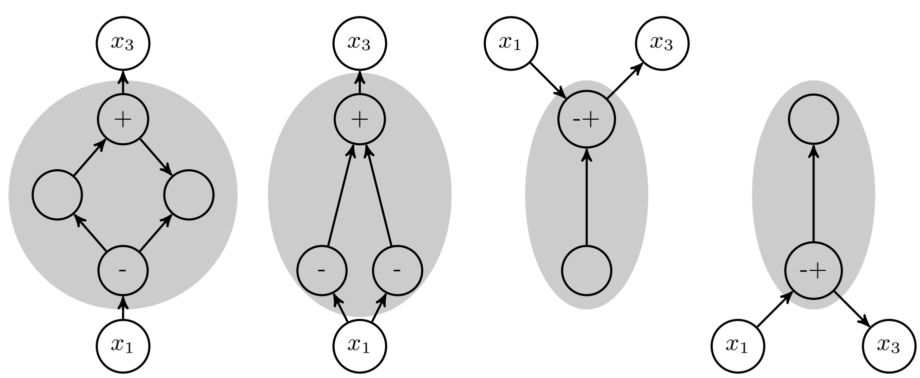
3. If G is an inflation of H we also say that G is an $\mathcal{I}[H]$; G is also called an an $\mathcal{I}_f[H]$ if G is a finitary inflation of H.

So, for a digraph H, the term 'an inflation of H' is ambiguous in the sense that it might denote a function (referring to Definition 2.7.10) as well as a digraph (referring to Definition 2.7.13). As long as we distinguish digraphs notationally from functions, the danger of conflation shouldn't be too high. Anyway, when talking about a digraph G, we will usually write 'G is an $\mathcal{I}[H]$' instead of 'G is an inflation of H'. This convention will be particularly convenient when in the next subsection the notion of parity is introduce for inflations as functions (Definition 2.7.18) on the one hand, and inflations as digraphs (Definition 2.7.19) on the other hand.

Let us illustrate Definition 2.7.10 with another example. Let H be a path of length 2.



Consider the following digraphs G_1, G_2, G_3 and G_4



and their partitions $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ whose respective non-trivial parts are indicated by the shaded areas, each of which corresponds to the vertex x_2

of H. All of them actually are inflations. Vertices belonging to the inward-boundary $\partial^-_{G_i}\mathcal{I}_i(x_2)$ of $\mathcal{I}_i(x_2)$ are labeled with '-', while those belonging to the outward-boundary $\partial^+_{G_i}\mathcal{I}_i(x_2)$ are labeled with '+'.

Observe that $\partial^+_{G_i}\mathcal{I}_i(x_2)$ does not necessarily coincide with $\mathrm{snk}(G_i[\mathcal{I}(x)])$. Whenever some sink y of $G_i[\mathcal{I}(x)]$ is not in the outward-boundary (as it is the case for $i = 1$), then it is the end of a *blind-alley*. The purpose of Condition 2 of Definition 2.7.10, which states that every vertex of the outward-boundary is reachable from every vertex of the inward-boundary, is to ensure that no walk entering a subdigraph $G[\mathcal{I}(x)] \subseteq G$ from the outside is ever forced to run into such a blind alley. Notice that loops of H are just ignored by Definition 2.7.10, which will be illustrated in the next examples.

Example 2.7.14.

1. Let G be a digraph and $1_G = \{\{x\} \mid x \in V(G)\}$ be the *trivial partition* of $V(G)$. Then 1_G is an inflation of G onto G.
2. Let $G = (\{0\}, \varnothing)$ and $H = (\{0\}, (0, 0))$. Consider again the trivial partition 1_G of $V(G) = V(H)$, i.e., the map $0 \mapsto \{0\}$. With respect to the digraphs G and H, 1_G can be interpreted in different ways: as an inflation of G onto G, an inflation of H onto H, an inflation of G onto H, and as an inflation of H onto G.
3. Let G and H be as in the previous example and let $F = (\{1, 2\}, \{(1, 2)\})$. Then the partition $\mathcal{I}(0) = \{1, 2\}$ is an inflation of H onto F, as well as an inflation of G onto F. Note that Condition (1) of Definition 2.7.10 is vacuously satisfied.

The previous examples show that an inflated digraph can be acyclic, even if the original digraph is not, given that the original digraph contains a loop. This is not the case if the original digraph is loop-free (cf. Lemma 3.3.1).

Definition 2.7.15. A digraph H is said to be a *(finitary) minor* of G iff there is some (finitary) inflation of H onto some subdigraph of G. We write $H \preceq G$ ($H \preceq_f G$), if H is a (finitary) minor of G, and $H \not\preceq G$ ($H \not\preceq_f G$), if H is not a (finitary) minor of G.

Proposition 2.7.16. *Let G, H and D be digraphs. Then the following claims hold.*

1. *If $H \subseteq G$ then $H \preceq_f G$.*
2. *$G \preceq_f G$.*
3. *If $D \preceq_f G$ and $G \preceq_f H$ then $D \preceq_f H$.*
4. *If G and H are symmetric then $H \preceq G$ iff H is a graph minor of G.*

Let us conclude this subsection with a notational convention that will be used throughout this investigation.

Definition 2.7.17. Let G, H be digraphs and $\mathcal{P}$ be a partition of G with index set $V(H)$.

1. For $X \subseteq V(H)$, let $\mathcal{P}[X] = \bigcup\{\mathcal{P}(x) \mid x \in X\}$.
2. For any subdigraph $H' \subseteq V(H)$, let $\mathcal{P}[H'] = G[\mathcal{P}[V(H')]]$.
3. For $Y \subseteq V(G)$, let $\mathcal{P}^{-1}[Y] = \{\mathcal{P}^{-1}(y) \mid y \in Y\}$.
4. For any subdigraph $G' \subseteq G$, let $\mathcal{P}^{-1}[G'] = H[\mathcal{P}^{-1}[V(G')]]$.

2.7.3 The parity of an inflated Yablo-graph

In Definition 2.6.36 we have defined the parity of a finite digraph. Is there such a thing as the parity of an infinite digraph? Such a notion seems to make no sense. The goal of this subsection is to show that in certain cases it does.

Definition 2.7.18. Let $\mathcal{I}$ be an inflation of H onto G. Then $\mathcal{I}$ is said to be

1. *even* iff for all $x \in V(H)$, $y \in \partial^-_G\mathcal{I}(x)$ and $z \in \partial^+_G\mathcal{I}(x)$, every path in $G[\mathcal{I}(x)]$ from y to z is even,
2. *odd* iff for all $x \in V(H)$, $y \in \partial^-_G\mathcal{I}(x)$ and $z \in \partial^+_G\mathcal{I}(x)$, every path in $G[\mathcal{I}(x)]$ from y to z is odd,
3. *determined* iff it is even or odd; and *undetermined* iff it is not determined.

Recall Definition 2.7.13 and Definition 2.6.36.

Definition 2.7.19. Let G and H be digraphs. Then G is said to be

1. an *even* $\mathcal{I}_f[H]$ iff there exists an odd finitary inflation of H onto G,
2. an *odd* $\mathcal{I}_f[H]$ iff there exists an even finitary inflation of H onto G,
3. a *determined* $\mathcal{I}_f[H]$ iff it is an even $\mathcal{I}_f[H]$ or an odd $\mathcal{I}_f[H]$.

In order to appreciate the unexpected inversion of 'even' and 'odd' in the above definition, consider a cycle C and its inflation $C^{\rightarrow}$ (cf. Definition 2.7.6). As an operation, the inflation $\mathcal{I}$ that maps C onto $C^{\rightarrow}$ is best conceived being odd, since each $C[\mathcal{I}(x)]$ is just an arc, i.e., a path of length $= 1$. But the result $C^{\rightarrow}$ is an even cycle, regardless of the parity of C. On the other hand, if one were to define $C^{2\rightarrow}$ analogously to $C^{\rightarrow}$ (i.e., inflating each vertex to a path of length $= 2$ instead of 1) this would be an even inflation (as an operation) but the result would be odd cycle $C^{2\rightarrow}$ iff C is odd and an even cycle iff C is even. More generally, every even cycle can be constructed as and odd finitary inflation of the loop $(\{x\}, \{(x, x)\})$ and every odd cycle as an even finitary inflation of it.

The following example shows that being an even $\mathcal{I}_f[G]$ doesn't prevent a digraph from being also and odd $\mathcal{I}_f[G]$ when it comes to infinite digraphs.

Example 2.7.20. Let Q and R be rays. Then Q is an even $\mathcal{I}_f[R]$ and an odd $\mathcal{I}_f[R]$.

The goal of the remainder of this section is to show that if a digraph G is a determined $\mathcal{I}_f[\mathbb{Y}]$, then it is not both, an even $\mathcal{I}_f[\mathbb{Y}]$ and an odd $\mathcal{I}_f[\mathbb{Y}]$. In contrast to the class of all cycles, not all members of the class of all digraphs that are finitary inflations of the Yablo-graph have a determined parity, but those member that do, have, in analogy to the class of all cycles, a unique one. This seems to be the most basic requirement for any meaningful *theory of parity*. Another reasonable requirement (stopping short of stipulating that every $\mathcal{I}_f[\mathbb{Y}]$ be determined) is that every $\mathcal{I}_f[\mathbb{Y}]$ contain a subdigraph that is a determined $\mathcal{I}_f[\mathbb{Y}]$. This will be ascertained by Theorem 4.2.13.

Definition 2.7.21. Let G be a digraph and $\mathcal{P}$ a partition of $V(G)$. Then G is said to be *convex* iff for all $X \in \mathcal{P}$, all $y, z \in X$ and all paths $P \subseteq G$: if P leads from x to y, then $P \subseteq G[X]$.

Proposition 2.7.22. *Let $\mathcal{I}$ be an inflation from H onto G. If H is acyclic, then $\mathcal{I}$ is convex.*

For the following recall Definition 2.7.17.

Definition 2.7.23. Let G, H be digraphs and $\mathcal{I}$ an inflation of H onto G.

1. Let $\mathcal{I}_\perp^+(G)$ be the set of all $x \in V(G)$ such that there exists no $y \in \partial_G^+\mathcal{I}(\mathcal{I}^{-1}(x))$ such that there exists a path from x to y in G.
2. Let $\mathcal{I}_\perp^-(G)$ be the set of all $x \in V(G)$ such that there exists no $z \in \partial_G^-\mathcal{I}(\mathcal{I}^{-1}(x))$ such that there exists a path P from z to x in G.
3. Let $\mathcal{I}_\perp(G) = \mathcal{I}_\perp^+(G) \cup \mathcal{I}_\perp^-(G)$.
4. Let $\mathcal{I}_\rightarrow(G) = V(G) \setminus \mathcal{I}_\perp(G)$.

Definition 2.7.24. Let G, H be digraphs and $\mathcal{I}$ an inflation of H onto G. Let $\mathcal{I}_\rightarrow : V(H) \rightarrow \wp(V(G))$ be defined by $\mathcal{I}_\rightarrow(y) = \mathcal{I}(y) \cap \mathcal{I}_\rightarrow(G)$.

Proposition 2.7.25. *Let G, H be digraphs and $\mathcal{I}$ a(n) (even, odd) finitary inflation of H onto G. Then $\mathcal{I}_\rightarrow$ is a(n) (even, odd) finitary inflation of H onto $G[\mathcal{I}_\rightarrow(G)]$.*

Proposition 2.7.26. *Let G, H be digraphs and $\mathcal{I}$ a finitary inflation of H onto G. Then*

1. *$\{\mathcal{I}_\rightarrow(G), \mathcal{I}_\perp^-(G), \mathcal{I}_\perp^+(G)\}$ is a partition of $V(G)$.*
2. *$\mathcal{I}_\perp^+(G)$ is open in G and each weak component of $G[\mathcal{I}_\perp^+(G)]$ is finite,*
3. *$\mathcal{I}_\perp^-(G)$ is closed in G, and each weak component of $G[\mathcal{I}_\perp^-(G)]$ is finite.*

Lemma 2.7.27. *Let G, H be digraphs and $\mathcal{I}$ an odd and convex inflation of H onto G, such that for all $x \in V(G)$, there exists $y^+ \in \partial_G^+\mathcal{I}(\mathcal{I}^{-1}(x))$ such that there is a path from x to y^+ in G and there exists $y^- \in \partial_G^-\mathcal{I}(\mathcal{I}^{-1}(x))$ such that there is a path from y^- to x in G. Then every double-path in G is even.*

Proof. Let D be a double-path in G, P_1 and P_2 be the two branches of D, x be the starting point of both P_1 and P_2 and y be their endpoint. Let $\mathcal{S} = \{\mathcal{I}(x) \mid x \in V(H) \wedge \mathcal{I}(x) \cap V(D) \neq \varnothing\}$. We call $(u, v) \in A(D)$ an *inner arc* if there exists $X \in \mathcal{S}$ such that $u, v \in X$; otherwise we call (u, v) an outer arc.

Suppose that $|\mathcal{S}| = 1$. Assume that D is odd. Let X be the single component of $\mathcal{S}$. Since there exists $y^+ \in \partial^+_G \mathcal{I}(X)$ and $x^- \in \partial^-_G \mathcal{I}(X)$, there must be two paths of different parity from $\partial^-_G \mathcal{I}(X)$ to $\partial^+_G \mathcal{I}(X)$, which contradicts the hypothesis that $\mathcal{I}$ is odd. Hence D is even.

Now suppose that $|\mathcal{S}| > 1$. Since $\mathcal{I}$ is convex $X = \mathcal{I}(x) \neq \mathcal{I}(y) = Y$. Let $\mathcal{S}_1$ be the set of all $U \in \mathcal{S} \setminus \{X, Y\}$ such that $G[U]$ contains only inner arcs of either P_1 or P_2, but not of both.

Clearly, $|\mathcal{S}_1| = N_o \mod 2$. (The argument is basically that the number of arcs of any double path equals the number of its vertices. So the parities of both are the same and that is not changed by subtracting two vertices.)

Now we show that D is even. First note that the number of inner arcs contained in $G[X]$ is even. Otherwise we could find two paths of different parity from $\partial^-_G \mathcal{I}(X)$ to $\partial^+_G \mathcal{I}(X)$, just as above, contradicting that $\mathcal{I}$ is odd. By the same argument, the the number of inner arcs contained in $G[Y]$ is also even.

Case 1: N_o is odd. Then total number of inner arcs contained in some $G[U]$, where $U \in \mathcal{S} \setminus (\mathcal{S}_1 \cup \{X, Y\})$ is even, because each such component contains an odd number of arcs of each path P_1 and P_2.

This leaves us with the outer arcs (an odd number by hypothesis) plus the inner arcs contained in $\mathcal{S}_1$. Since $|\mathcal{S}_1|$ is odd and $\mathcal{I}$ is an odd inflation, this last number is even. (Here we actually need the hypothesis that $\mathcal{I}$ is convex - so for all $U \in \mathcal{S}_1$, $G[U]$ contains an odd number of arcs of D, since neither P_1 nor P_2 can enter $G[U]$ multiple times.) Summing all arcs up, we conclude that D is even.

Case 2: N_o is even. The proof is analogous to the first case. The difference is that now we have an even number of the inner arcs contained in $\mathcal{S}_1$, because $|\mathcal{S}_1|$ even. Together with N_o being even, this yields an even number of arcs of D.

□

Theorem 2.7.28. *Let G be an even $\mathcal{I}_f[\mathbb{Y}]$ that contains no odd cycle. Then G possess as kernel.*

Proof. Let $\mathcal{I}$ be odd finitary inflation of $\mathbb{Y}$ onto G. Let $\mathcal{J} = \mathcal{I}_{\rightarrow}$. Then $\mathcal{J}$ is an odd inflation of $\mathbb{Y}$ onto $G[\mathcal{I}_{\rightarrow}(G)]$. By Proposition 2.7.22 $\mathcal{J}$ is convex. Notice that $G[\mathcal{I}^+_{\perp}(G)]$ has a partition into finite components, each of which is open in $G[\mathcal{I}^+_{\perp}(G)]$. Hence, by Proposition 2.6.28 and Proposition 2.4.47 the digraph $G[\mathcal{I}^+_{\perp}(G)]$ is strongly kernel-perfect. So let f_+ be a fixed point of $\Phi^G_{\downarrow}[\mathcal{I}^+_{\perp}(G)]]$.

Next we show that $G' = G[\mathcal{I}_{\rightarrow}(G)]$ is strongly kernel-perfect. By Theorem 2.6.41 it suffices to show that G' contains no odd double-path. Let $D \subseteq G'$ be a double-path. Since $\mathcal{J}$ satisfies the requirements of Lemma 2.7.27, we can conclude that D is even. Hence G' is strongly kernel-perfect.

Observe that the Boolean network $(\Phi^G_\downarrow)^{f_+}[\mathcal{I}_\rightarrow(G)]$ is well-defined since the set $\mathcal{I}_\rightarrow(G) \cup \mathcal{I}^+_+(G)$ is open in G by Proposition 2.7.26. Let C be the core of $(\Phi^G_\downarrow)^{f_+}[\mathcal{I}_\rightarrow(G)]$. Since C has a kernel, $(\Phi^G_\downarrow)^{f_+}[\mathcal{I}_\rightarrow(G)]$ has a fixed point $f_\rightarrow$ by Lemma 2.4.94. Since $\mathcal{I}^+_\perp(G)$ is open in $G[\mathcal{I}_\rightarrow(G) \cup I^+_\perp(G)]$, $f = f_\rightarrow \cup f_+$ is a fixed point $\Phi^G_\downarrow[V(G) \setminus \mathcal{I}^-_\perp(G)]$ by Lemma 2.4.48.

Analogous to $G[\mathcal{I}^+_\perp(G)]$, also $G[\mathcal{I}^-_\perp(G)]$ has a partition into finite, open components. Thus by Proposition 2.6.28 and Proposition 2.4.47 the digraph $G[\mathcal{I}^-_\perp(G)]$ is strongly kernel-perfect. Hence by Lemma 2.4.94 it follows that $(\Phi^G_\downarrow)^f[\mathcal{I}^-_\perp(G)]$ has a fixed point f_-. Since $\mathcal{I}_\rightarrow(G) \cup I^+_\perp(G)$ is open in G, $f \cup f_-$ is a fixed point $\Phi^G_\downarrow[V(G)] = \Phi^G_\downarrow$ by Lemma 2.4.48, and thus the characteristic function of a kernel of G. □

Proposition 2.7.29. *Let $X \subseteq \omega$. Then $\mathbb{Y}[X]$ is isomorphic to $\mathbb{Y}$ iff X is infinite.*

Corollary 2.7.30. *Let G be a determined $\mathcal{I}_f[\mathbb{Y}]$ that contains no odd cycles. Then G is strongly kernel-perfect iff G is an even $\mathcal{I}_f[\mathbb{Y}]$.*

Proof. If G is not an even $\mathcal{I}_f[\mathbb{Y}]$, then it is an odd $\mathcal{I}_f[\mathbb{Y}]$, since it is determined. Then by Theorem 4.3.7 G is not strongly kernel-perfect.

Now suppose that G is an even $\mathcal{I}_f[\mathbb{Y}]$. Let $H \subseteq G$. Let $X = \mathcal{I}^{-1}[V(H)]$. If X is finite, then H is finite and then strongly kernel-perfect by Proposition2.6.28. Hence we can assume that X is infinite. Then $\mathbb{Y}[X]$ is isomorphic to $\mathbb{Y}$ via some isomorphism φ. Hence $\mathcal{J}$ defined by $\mathcal{J}(x) = \mathcal{I}(\varphi(x)) \cap V(H))$ a finitary inflation of $\mathbb{Y}$ onto H. Moreover it is odd, since $\mathcal{I}$ is odd. Hence H is an even $\mathcal{I}_f[\mathbb{Y}]$ and has a kernel by Theorem 2.7.28. Hence G is strongly kernel-perfect. □

Corollary 2.7.31. *Let G be a determined $\mathcal{I}_f[\mathbb{Y}]$. Then G is either an even $\mathcal{I}_f[\mathbb{Y}]$ or and odd $\mathcal{I}_f[\mathbb{Y}]$ but not both.*

Proof. By Proposition 4.3.9 there exists some acyclic $H \subseteq G$ such that H is an even $\mathcal{I}_f[\mathbb{Y}]$ if G is and an odd $\mathcal{I}_f[\mathbb{Y}]$ if G is. Assuming that G is both an even and an odd $\mathcal{I}_f[\mathbb{Y}]$ implies that H is both. Since H is acyclic it follows by Corollary 2.7.30 that H is both strongly kernel-perfect and not strongly kernel-perfect. □

2.8 Conjectured solutions for the characterization problems

In this final section of Chapter 2 we will conjecture a solution for each of the three characterization problems formulated in Section 2.6.

2.8.1 The characterization problem for safe digraphs

The following is a reformulation of one direction of Conjecture 4.24 from [5] (cf. Conjecture 2.8.2 below), which, in turn, had already been formulated earlier in [4] as Conjecture 1.

Conjecture 2.8.1 (Conjecture A)**.** *If a digraph contains neither a cycle nor an $\mathcal{I}_f[\mathbb{Y}]$, then it is safe.*

In Chapter 4 we shall prove the converse of Conjecture (A)(Theorem 4.3.4), which implies one direction of the following.

Conjecture 2.8.2 (Beringer and Schindler 2015)**.** *A reference graph is dangerous iff it contains a subdivision of the liar-graph as a subgraph or the Yablo-graph as a finitary minor.*

There are two differences between Conjecture (A) and Conjecture 2.8.2. The first is that in [5] the definition of a digraph inflation is formulated differently. This difference, however, is of no significance, since it is not difficult to show that a digraph contains a finitary inflation of $\mathbb{Y}$ in one sense if and only if it contains a finitary inflation of $\mathbb{Y}$ in the other. The second and major difference is that Conjecture 2.8.2 is not formulated in terms of digraphs simpliciter but in terms of reference graphs (of sentences). Apparently, this restriction is simply due to the context in which the framework of [5] is formulated (first order logic and arithmetic) and not to any skepticism with respect to a broader validity of the conjecture. Thus Conjecture (A) can be regarded as a simple reformulation of one direction of Conjecture 2.8.2. Clearly, Conjecture (A) implies Conjecture 2.8.2. In Chapter 5 a special case of Conjecture (A) (Theorem 5.4.6) will be shown.

A natural question is why in Conjecture (A) an $\mathcal{I}_f[\mathbb{Y}]$ is required and not just an $\mathcal{I}[\mathbb{Y}]$? The answer is that otherwise the converse of Conjecture (A) would not hold. Consider the following example of a digraph that is an $\mathcal{I}[\mathbb{Y}]$ but safe nevertheless.

Example 2.8.3. For all $n \in \omega$ let S_n be the set of all sequences s such that $s = n$ or $s = n \circ t \circ 0$ or $s = n \circ t \circ 1$, where t is a sequence of arbitrary length each of whose members is the number 1.

Let $V(G) = \bigcup_{n \in \omega} S_n$ and let for all $x, y \in V(G)$, $(x, y) \in A(G)$ if one of the following conditions holds: (1) The sequences x and y have the same first member and the length of y is the length of x plus 1, or (2) if the last member of the sequence x is 0, the length of x is greater than 1 and $y = m > n$, where n is the first member of x.

Then G is safe by Proposition 2.6.23 and Corollary 2.6.21. However, $\{S_n \mid n \in \omega\}$ is clearly an inflation on $\mathbb{Y}$ onto G.

2.8.2 The characterization problem for digraphs of finite character

Conjecture 2.8.4 (Conjecture D)**.** *If a digraph contains no $\mathcal{I}_f[\mathbb{Y}]$, then it is of finite character.*

In Chapter 4 we shall prove the converse of Conjecture (D) (Theorem 4.3.10).

Proposition 2.8.5. *Conjecture (D) implies Conjecture (A).*

Proof. By Theorem 2.6.20. □

Now let us return to Question 2.6.27. Conjecture (D) takes a step towards an answer.

Proposition 2.8.6. *Conjecture (D) implies that every digraph of infinite character has a skeleton of infinite character.*

Proof. Let G be a digraph of infinite character, By Conjecture D, G contains some $\mathcal{I}_f[\mathbb{Y}]$. Then G contains some acyclic $\mathcal{I}_f[\mathbb{Y}]$ by Corollary 4.3.9, let us call it I. We can extend I to a skeleton H of G. By Theorem 4.3.10 I is of infinite character. Hence H is of infinite character. □

Can a digraph of infinite character have a skeleton of finite character and a skeleton of infinite character? This seems implausible, but how can it be ruled out?

Theorem 2.8.7. *If every digraph of infinite character has a skeleton of infinite character, then Conjecture (A) implies Conjecture (D).*

Proof. Assume Conjecture (D) fails. Then there exists a digraph of infinite character G that contains no $\mathcal{I}_f[\mathbb{Y}]$. Let H be a skeleton of G that is of infinite character. Then H is dangerous. But H contains no $I_f[\mathbb{Y}]$, since G does not. And H is acyclic since H is a skeleton. This contradicts Conjecture (A). □

2.8.3 The characterization problem for strongly kernel-perfect digraphs

The following conjecture describes how safety digraph safety can be expressed in terms of strong kernel-perfectness.

Conjecture 2.8.8 (Conjecture B)**.** *If every subdivision of a digraph G is strongly kernel-perfect, then G is safe.*

It is easy to see that the converse of Conjecture (B) holds.

Proposition 2.8.9. *If some subdivision of a digraph G has no kernel, then G is dangerous.*

In Chapter 4 we shall prove that Conjecture (A) implies Conjecture (B)(Proposition 4.3.6).

Theorem 2.8.10. *Suppose that Conjecture (A) holds. Then for all digraphs G the following claims are equivalent.*

1. *G is safe,*

2. *every subdivision of G is strongly kernel-perfect,*

3. *G contains neither a cycle nor an $\mathcal{I}_f[\mathbb{Y}]$.*

The following is a reformulation of a conjecture by Walicki (cf. [47]) which shall be discussed in Chapter 3 (Conjecture 3.4.1).

Conjecture 2.8.11 (Conjecture W)**.** *If a digraph contains no odd cycle and no $\mathcal{I}_f[\mathbb{Y}]$, then is has a kernel.*

The equivalence of Conjecture (W) and Conjecture 3.4.1 follows from Theorem 4.4.2.

Proposition 2.8.12. *Conjecture (D) implies Conjecture (W).*

Proof. Suppose G contains no $\mathcal{I}_f[\mathbb{Y}]$. Then it is of finite character by Conjecture (D). If it also contains no odd cycle, then it has a kernel by Theorem 2.6.30. □

A special case of Conjecture (W) was proved in [47].

In Chapter 3 we shall prove that Conjecture (W) implies Conjecture (A) (Theorem 3.4.2). The following seems to be a reasonable strengthening of Conjecture (W).

Conjecture 2.8.13 (Conjecture C)**.** *If a digraph contains no odd cycle and no odd $\mathcal{I}_f[\mathbb{Y}]$, then is has a kernel.*

The converse of Conjecture (C) does not hold. However, in Chapter 4 we shall prove Theorem 4.3.7.

Since Conjecture (C) implies Conjecture (W), we also get the following.

Proposition 2.8.14. *Conjecture (C) implies Conjecture (A).*

Theorem 2.8.15. *Conjecture (C) implies that a digraph G is strongly kernel-perfect iff G contains no odd cycle and no odd $\mathcal{I}_f[\mathbb{Y}]$.*

Proof. Suppose that G contains no odd cycle and no odd $\mathcal{I}_f[\mathbb{Y}]$. Let $H \subseteq G$. Then H contains no odd cycle and and no odd $\mathcal{I}_f[\mathbb{Y}]$. Hence H has a kernel by Conjecture (C). For the other direction suppose that G is strongly kernel-perfect. Clearly, G cannot contain an odd cycle. By Theorem 4.3.1, G cannot contain an odd $\mathcal{I}_f[\mathbb{Y}]$. □

Hence, Conjecture (C) can be seen as a generalization of Richardson's theorem that is *maximally strong* in the following sense: Just as the Richardson's theorem (Theorem 2.6.28) provides a necessary and sufficient condition for finite digraphs to be strongly kernel-perfect, Conjecture (C) does so for digraphs simpliciter.

Clearly, the converse of Conjecture (C) does not hold. However, in Chapter 4 we shall prove Theorem 4.3.7.

Chapter 3

Characteristic digraphs of constrained Boolean networks

The goal of Chapter 3 is to show that any Boolean network (and the question of whether it has a fixed point in particular) can be analyzed in terms of an associated directed graph. Such a graph is called a *characteristic digraph* of the Boolean network and contains more information about it than a reference graph. This reduces the question of whether a sentence system is paradoxical to a purely graph theoretic one. For a more detailed outline of Chapter 3 the reader is referred to Subsection 1.5.2.

3.1 Regular inflations

Definition 3.1.1. An inflation $\mathcal{I}$ of H onto G is said to be *regular* iff for all $x \in V(H)$,

1. $G[\mathcal{I}(x)]$ is well-founded,
2. there exists $r_x \in \mathcal{I}(x)$, called the *root of* $\mathcal{I}(x)$, such that
   (a) every $y \in \mathcal{I}(x)$ is reachable from r_x in $G[\mathcal{I}(x)]$,
   (b) $\partial^-_G \mathcal{I}(x) = \{r_x\}$.

Note that the r_x is unique, since $G[\mathcal{I}(x)]$ is acyclic.

Example 3.1.2. Recall Definition 2.7.6. For all digraphs G, there exists a regular inflation of G onto $G^{\rightarrow}$.

Definition 3.1.3. Let G and H be digraphs. If there is a regular inflation of H onto G, we say that G is an $\mathcal{I}_r[H]$.

Regular inflations behave in many respects like finitary inflations. E.g., there is a counterpart to Theorem 2.7.28 for regular inflations.

Theorem 3.1.4. *Let G be an even $\mathcal{I}_r[\mathbb{Y}]$ that contains no odd cycle. Then G possess as kernel.*

Proof. By Proposition 2.6.18 every well-founded digraph is of finite character. So the proof can proceed in analogy to that Theorem 2.7.28, invoking Theorem 2.6.30 instead of Richardson's original theorem. □

Analogous to Corollary 2.7.30 we obtain the following.

Corollary 3.1.5. *Let G be a determined $\mathcal{I}_r[\mathbb{Y}]$ that contains no odd cycles. Then G is strongly kernel-perfect iff it is an even $\mathcal{I}_r[\mathbb{Y}]$.*

3.2 Characteristic digraphs

We have previously discussed the ambiguous status of loops with respect to inflations. Now a method shall be introduced how loops can be replaced by cycles of length 3 (an odd number in order to preserve the parity) without altering the dynamical behavior of the system in a significant way. The technical reason for eliminating loops is to ensure that a digraph can always be inflated regularly (cf. Definition 3.2.4).

Definition 3.2.1. Let (G, Φ) be a constrained Boolean network. Define the *loop-cleansed form* (G', Φ') of (G, Φ) as follows.

- Let G' be the result of subdividing each loop of G twice.
- For all x in $V(G')$, let
  - $x^* = x$, if $x \in V(G)$ and such that $(x, x) \notin A(G)$,
  - x^* be the unique out-neighbor of x in $V(G') \setminus V(G)$, if $x \in V(G)$ and $(x, x) \in A(G)$,
  - x^* be the unique out-neighbor of x, if $x \in V(G') \setminus V(G)$.
- For all $f' \in \{0,1\}^{V(G')}$ define $r(f') \in \{0,1\}^{V(G)}$ by $r(f') = f' \restriction V(G)$.
- Define $\Phi' : \{0,1\}^{V(G')} \times V(G') \to \{0,1\}$ by

$$\Phi'(f', x) = \begin{cases} \Phi(r(f'), x), & \text{if } x \in V(G) \\ f'(x^*), & \text{if } x \in V(G') \setminus V(G). \end{cases}$$

Notice that $(G', \Phi') = (G, \Phi)$, if G is loop-free.

For the following recall Definition 2.5.4.

Proposition 3.2.2. *Let (G, Φ) be a constrained Boolean network and (G', Φ') its loop-cleansed form. Then there exists a dense weak system-embedding of Φ into Φ'.*

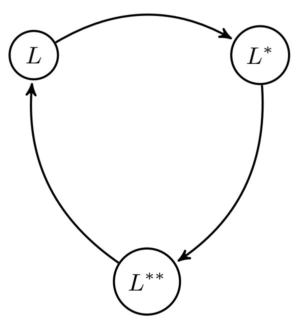
Proof. Define $i : \{0,1\}^{V(G)} \to \{0,1\}^{V(G')}$ as follows:

$$(i(f))(x) = \begin{cases} f(x), & \text{if } x \in V(G) \\ f(x^*), & \text{if } x \notin V(G) \wedge x^* \in V(G) \\ f(x^{**}), & \text{if } x \notin V(G') \wedge x^{**} \in V(G). \end{cases}$$

Then i is a dense weak system embedding of Φ into Φ' with retraction r. $\square$

Example 3.2.3. The dependency graph of the liar sentence

and its loop-cleansed form.



A further example for a loop-cleansed form of a digraph is discussed in Example 3.5.24 below.

Definition 3.2.4. Let (G, Φ) and (H, Ψ) be constrained Boolean networks. A *network inflation* of (H, Ψ) onto (G, Φ) is an ordered pair $(\mathcal{I}, i)$ such that

1. $\mathcal{I}$ is a regular inflation of H onto G,
2. $i : \{0,1\}^{V(H)} \to \{0,1\}^{V(G)}$ is a dense weak system embedding of Ψ into Φ.

Definition 3.2.5. Let Ψ be a Boolean network on H. Then a digraph G is said to be a *characteristic digraph* of the constrained Boolean network (H, Ψ) iff there exists a network inflation of (H', Ψ') into $(G, \Phi^G_{\downarrow})$, where (H', Ψ') is the loop-cleansed form of (H, Ψ).

The following theorem is the main result of this Chapter. We shall prove it at the end of Subsection 3.5.2.

Theorem 3.2.6. *Every constrained Boolean network has a characteristic digraph.*

Proposition 3.2.7. *Let (H, Ψ) be a constrained Boolean network and G a characteristic digraph of (H, Ψ). Then there exists a bijection between the fixed points of (H, Ψ) and the kernels of G.*

Proof. By Theorem 2.5.10. □

Definition 3.2.8. A class $\mathcal{C}$ of digraphs is said to *closed under regular inflation and under twofold subdivision of loops* iff for all $G \in \mathcal{C}$, $G' \in \mathcal{C}$, given that G' is a regular inflations of G or arises from G by subdividing each loop of G twice.

Corollary 3.2.9. *Let $\mathcal{C}$ be a class of digraphs that is closed under regular inflation and under twofold subdivision of loops. If every element of $\mathcal{C}$ has a kernel, then every element of $\mathcal{C}$ is safe.*

Proof. Let $H \in \mathcal{C}$, Ψ a Boolean network on H and G a characteristic digraph of (H, Ψ). Then $G \in \mathcal{C}$, since $\mathcal{C}$ is closed under regular inflation and under twofold subdivision of loops. Hence G has a kernel. Hence Ψ has a fixed point by Proposition 3.2.7. Hence H is safe. □

Proposition 3.2.10. *Let $\mathcal{E}$ be the class of digraphs that contain no odd cycles and $\mathcal{O}$ the class of all digraphs that contain no even cycles. Then both $\mathcal{E}$ and $\mathcal{O}$ are closed under regular inflation and under twofold subdivision of loops.*

3.3 Danger preserving digraph transformations

This section deals with subdigraphs and minors that are preserved by contractions and provides tools for the following sections. The main result is Theorem 3.3.10 which states that finitary $\mathbb{Y}$-minors are preserved when loop-free digraphs undergo a finitary or a regular contraction.

For the following Lemma and the rest of this section recall Definition 2.7.17.

Lemma 3.3.1. *Let G and H be digraphs and let $\mathcal{I}$ an inflation of H onto G. Then the following claims hold.*

1. *If H is loop-free and G is acyclic, then H is acyclic.*
2. *Let $P \subseteq H$ be a path from x to y. Then there exists a path $P' \subseteq \mathcal{I}[P] \subseteq G$ from some $x' \in \mathcal{I}(x)$ to some $y' \in \mathcal{I}(y)$. If $x \neq y$ and $\partial^+_G \mathcal{I}(x) \neq \varnothing$ then we can assume that $y' \in \partial^+_G \mathcal{I}(y)$. Moreover, if $\partial^-_G \mathcal{I}(x) \neq \varnothing$, then we can assume that $x' \in \partial^-_G \mathcal{I}(x)$.*
3. *Let $R \subseteq H$ be a ray and $x = src(R)$. Then there exists a ray $R' \subseteq \mathcal{I}[R] \subseteq G$ such that $R = \mathcal{I}^{-1}[R']$. Moreover, if $\partial^-_G \mathcal{I}(x) \neq \varnothing$, then we can assume that $src(R') \in \partial^-_G \mathcal{I}(x)$.*
4. *Let $P \subseteq G$ be a path from x to y. Then $\mathcal{I}^{-1}[P]$ is such that there exists a walk from $\mathcal{I}^{-1}(x)$ to $\mathcal{I}^{-1}(y)$ in $\mathcal{I}^{-1}[P]$.*

Proof. Ad 1: Suppose that H is loop-free and assume that C is a cycle in H. Then C must have length ≥ 2. Let $x_0, \ldots, x_n$ be an enumeration of the vertices of C such that $(x_i, x_{i+1}) \in A(C)$, for all $0 \leq i \leq n$.

Let $X_i = \mathcal{I}(x_i)$, for all $0 \leq i \leq n$. By Definition 2.7.10(1) there exist $a_0, \ldots, a_n \in A(G)$ such that $tail(a_i) \in X_i$ and $head(a_i) \in X_{i+1}$. Hence by Definition 2.7.10(2) there are paths P_i from $head(a_i)$ to $tail(a_{i+1})$. Thus a cycle $a_0 \circ P_0, \ldots, a_n \circ P_{n+1}$ can be constructed in G, which is a contradiction.

Ad 2: Let $x_0, \ldots, x_n$ be the order-preserving enumeration of $V(P)$, where $x_0 = x$ and $x_n = y$. Then for all $0 \leq k < n$, there exists $a_k \in A(G)$ from some $x'_k \in \mathcal{I}(x_k)$ to some $x^*_k \in \mathcal{I}(x_{k+1})$, by Definition 2.7.10(1). Then for all $0 \leq k < n$, $x^*_k \in \partial^-_G(\mathcal{I}(x_{k+1}))$ and $x'_{k+1} \in \partial^+_G(\mathcal{I}(x_{k+1}))$. Hence, by Definition 2.7.10(2) there exists a path $P_{k+1} \subseteq G[(\mathcal{I}(x_{k+1})]$ from x^*_k to x'_{k+1} (which is trivial iff $x^*_k = x'_{k+1}$). Let $P' = a_0 \circ P_1 \circ \cdots \circ P_{n-1} \circ a_{n-1}$. Then $P' \subseteq \mathcal{I}[P]$ is a path with $src(P') = tail(a_0) = x'_0 \in \mathcal{I}(x_0)$ and $snk(P') = head(a_{n-1}) = x^*_{n-1} \in \mathcal{I}(x_n)$. By setting $x' = x'_0$ and $y' = x^*_{n-1}$, this proves the first part of the claim.

If $x' \in \partial^-_G(\mathcal{I}(x))$, then there is by Definition 2.7.10(2) a path Q_x from x' to x'_0 (x'_0 defined as above). So $Q_x \circ P'$ is as desired. Likewise we obtain a path $P' \circ Q_y$ with an appropriate sink.

Ad 3: Analogous to (2).

Ad 4: By By 2.7.10(1). □

Lemma 3.3.2. *Let G and H be digraphs and $\mathcal{I}$ an inflation of H onto G. Let $R \subseteq G$ be a ray.*

1. *If $\mathcal{I}(x') \cap V(R)$ is infinite for some $x' \in V(H)$, then one of the following is the case.*
   (a) *For some $x' \in V(H)$, $\mathcal{I}[x']$ contains a tail of R,*
   (b) *for some $x' \in V(H)$, $\partial^-_G\mathcal{I}(x')$ is infinite and $\mathcal{I}^{-1}[R]$ contains a cycle.*
2. *If $\mathcal{I}(x') \cap V(R)$ is finite for all $x' \in V(H)$, then $\mathcal{I}^{-1}[R]$ contains a ray.*

Proof. Let $(x_n)_{n\in\omega}$ be the order-preserving enumeration of the vertices of R.

Ad 1: Suppose that for no $x' \in V(H)$, $\mathcal{I}[x']$ contains a tail of R. Let $R' = \mathcal{I}^{-1}[R]$.

Then clearly for some $x' \in V(H)$, $\partial^-_G\mathcal{I}(x')$ is infinite. Moreover, there exist $x' \in V(R')$ and $i < j < k$ such that $x_i, x_k \in \mathcal{I}(x') \cap V(R)$ and $x_j \notin \mathcal{I}(x') \cap V(R)$. Since $x_j \notin \mathcal{I}(x') \cap V(R)$, then $x_j \in V(R) \setminus \mathcal{I}(x')$ and thus $x_j \in \mathcal{I}(y')$ for some $x' \neq y' \in V(R')$. Hence, by Lemma 3.3.1, there is a walk from $\mathcal{I}(x')$ to $\mathcal{I}(y')$ in R' and a walk from $\mathcal{I}(y')$ to $\mathcal{I}(x')$ in R'. Hence R' contains a cycle.

Ad 2: Let $X_n = \mathcal{I}(\mathcal{I}^{-1}(x_n))$, for all $n \in \omega$. Since all X_n are finite, it is straightforward to construct a sequence $(n_i)_{i\in\omega}$ such that $(X_{n_i})_{i\in\omega}$ is an order-preserving enumeration of the vertices of a ray in H. □

The following definition is due to [47].

Definition 3.3.3. Let G be a digraph, $x \in V(G)$ and $R \subseteq G$ a ray. We say that x *dominates* R in G iff there are infinitely many paths from x to R in G that are pairwise disjoint except at x.

Definition 3.3.4. Call a contraction $\mathcal{I}^{-1}$ from G onto H

1. *ray-preserving* iff for all rays $R \subseteq G$, $\mathcal{I}^{-1}[R]$ contains a ray,
2. *cycle-preserving* iff for all cycles $C \subseteq G$, $\mathcal{I}^{-1}[C]$ contains a cycle,
3. *dominance-preserving* iff it is ray-preserving and for all rays $R \subseteq G$ and all $x \in V(G)$: if x dominates R in G, then $\mathcal{I}^{-1}(x)$ dominates every ray $R' \subseteq \mathcal{I}^{-1}[R]$ in H.

Lemma 3.3.5. *Let G and H be digraphs and $\mathcal{I}$ an inflation of H onto G such that $\partial_G^- \mathcal{I}(x)$ is finite for all $x \in V(H)$. Let $R \subseteq G$ be a ray and $v \in V(G)$ be such that v dominates R in G. If $R' \subseteq \mathcal{I}^{-1}[R]$ is a ray, then $\mathcal{I}^{-1}(v)$ dominates R' in H.*

Proof. Let $(P_n)_{n\in\omega}$ be a family of paths in G, joining v to R, being pairwise disjoint except at v.

First, assume that for all $x \in V(H)$ there are only finitely many $n \in \omega$ such that $P_n \cap (\mathcal{I}(x) \setminus \{v\}) \neq \varnothing$. Then there exists an infinite subfamily $(P_{n_i})_{i\in\omega}$ of $(P_n)_{n\in\omega}$ such that $\mathcal{I}^{-1}[P_{n_i}]$ and $\mathcal{I}^{-1}[P_{n_j}]$ are disjoint for $i \neq j$, except at $\mathcal{I}^{-1}(v)$. Let $P'_{n_i} = \mathcal{I}^{-1}[P_{n_j}]$ if $\mathcal{I}^{-1}[P_{n_j}]$ is a path and otherwise let it be a maximal subdigraph of $\mathcal{I}^{-1}[P_{n_j}]$ that is a path and starts at $\mathcal{I}^{-1}(v)$. Hence the collection of all P'_{n_i} witnesses to $\mathcal{I}^{-1}(v)$ dominating R' in H.

Now, assume that there exists $x \in V(H)$ such that $P_{n_i} \cap (\mathcal{I}(x) \setminus \{v\}) \neq \varnothing$ for infinitely many i. Then for each $i \in \omega$ there exists $p_i \in V(P_{n_i})$ such that $p_i \in \partial_G^- \mathcal{I}(x)$ or $p_i = v$. Since $\partial_G^- \mathcal{I}(x)$ is finite, this means that $v \in \mathcal{I}(x)$. On the other hand, $\partial_G^+ \mathcal{I}(x)$ must contain infinitely many q_i with $q_i \in V(P_{n_i})$ such that all q_i are pairwise disjoint. Let Q_i be the maximal subpath of P_{n_i} with initial vertex q_i. Then for all $x \neq y \in V(H)$: $Q_i \cap \mathcal{I}(y) \neq \varnothing$ for only finitely many $i \in \omega$ (otherwise we would have $v \in \mathcal{I}(y)$ by the above argument). Hence there exists a subfamily $(Q_{i_k})_{k\in\omega}$ of $(Q_i)_{i\in\omega}$ such that $\mathcal{I}[Q_{i_k}]$ and $\mathcal{I}[Q_{i_l}]$ are disjoint for all $k \neq l \in \omega$ (with the possible exception of $\mathcal{I}^{-1}(v)$). As above, we may choose a path $Q' \subseteq \mathcal{I}[Q_{i_k}]$ if necessary. Thus we obtain that $\mathcal{I}^{-1}(v)$ dominates R'. □

Proposition 3.3.6. *Let $\mathcal{I}^{-1}$ be a dominance-preserving contraction of G onto H, where H is acyclic. Then $\mathbb{Y} \preceq_f H$, if $\mathbb{Y} \preceq_f G$.*

Proof. Suppose $\mathbb{Y} \preceq_f G$. Then by Theorem 4.4.2 G contains a ray R such that there are infinitely many vertices in $V(R)$ dominating R in G. Then there exists a ray $R' \subseteq \mathcal{I}^{-1}[R]$, because $\mathcal{I}^{-1}$ being dominance-preserving implies that $\mathcal{I}^{-1}$ is ray-preserving. Let $(x_n)_{n\in\omega}$ be the order-preserving enumeration of $V(R)$ and let $(x'_n)_{n\in\omega}$ the order-preserving enumeration of $V(R')$.

Then $\mathcal{I}(x'_n) \cap V(R)$ is finite for all $n \in \omega$.(The assumption that $\mathcal{I}(x'_n) \cap V(R)$ is infinite would imply that every infinite walk in $\mathcal{I}^{-1}[R]$ visits $\mathcal{I}(x'_n)$ infinitely many times. Hence $\mathcal{I}^{-1}[R]$ cannot contain a ray.) This implies that there are infinitely many $n \in \omega$ such that $\mathcal{I}(x'_n)$ contains some $x \in V(R)$ such that x dominates R in G. Since $\mathcal{I}^{-1}$ is dominance-preserving, R' contains infinitely many vertices dominating R' in H. Since H is acyclic, Theorem 4.4.2 implies that $\mathbb{Y} \preceq_f H$. □

Proposition 3.3.7. *Let G be acyclic and $\mathcal{I}$ an inflation of H onto G. Then $\mathbb{Y} \preceq_f G$, if $\mathbb{Y} \preceq_f H$.*

Proof. Suppose $\mathbb{Y} \preceq_f H$. Then, by Theorem 4.4.2, there is a ray $R \subseteq H$ such that there are infinitely many vertices in $V(R)$ dominating R in H. Let $r = src(R)$. We can assume w.l.o.g. that $\partial_G^- \mathcal{I}(r) \neq \varnothing$. Let $r' \in \partial_G^- \mathcal{I}(r)$ and let $R' \subseteq \mathcal{I}[R]$ be a ray with $src(R') = r'$ by Lemma 3.3.1(3).

If a vertex $x \in V(R)$ dominates R in H, then there exists $x' \in \mathcal{I}(x) \cap V(R')$ such that x' dominates R' in G.

In order to see this, let $x' \in \partial_G^- \mathcal{I}(x) \cap V(R')$. (Such an x' exists, because $\partial_G^- \mathcal{I}(x) \neq \varnothing$, for all $x \in V(R)$). Let $\mathcal{P}$ be an infinite set of paths from x to R that are pairwise disjoint except at x. For each $P \in \mathcal{P}$ (leading from x to some $y \in V(R)$) let $P' \subseteq \mathcal{I}[P]$ be a path from x' to some $y' \in \partial_G^- \mathcal{I}(y)$, according to Lemma 3.3.1(2). Since there is some $z' \in V(R') \cap \partial_G^+(\mathcal{I}(y))$ – note that $y \in V(R)$ and that R' must leave $\mathcal{I}(y)$– we can, by Definition 2.7.10(2), choose a path from y' to z' and thus obtain a path P^*, leading from x' to $z' \in V(R')$. Then the set $\mathcal{P}^*$ of all path constructed in this way (they are disjoint except at x', since those of $\mathcal{P}$ are disjoint except at x) witnesses the claim that x' dominates R' in G.

Hence, there are infinitely many vertices in $V(R')$ dominating R' in G. Since G is acyclic we can apply Theorem 4.4.2 and obtain $\mathbb{Y} \preceq_f G$. □

Proposition 3.3.8. *Let $\mathcal{I}$ be an inflation of H onto G. If $\mathcal{I}$ is regular, then $\mathcal{I}^{-1}$ is cycle-preserving.*

Proof. Let $C \subseteq G$ be a cycle and and let $x_0, \ldots, x_n$ be an order-preserving enumeration of its vertices. Let $C' = \mathcal{I}^{-1}[C]$. Since $\mathcal{I}$ is regular, there exists no $x' \in V(H)$ such that $V(C') \subseteq \mathcal{I}(x')$. Hence $\mathcal{I}^{-1}(V(C))$ contains at least two elements.

If there exist $x' \in V(C')$ and $i < j < k$ such that $x_i, x_k \in \mathcal{I}(x') \cap V(C)$ and $x_j \notin \mathcal{I}(x') \cap V(C)$, then we obtain a cycle $C'' \subseteq C'$ by the same argument as in Lemma 3.3.2. So assume that this is not the case. Then we have an order-preserving enumeration $x'_0, \ldots, x'_m$ of the vertices of C' such that $\{x_{i_0}, \ldots, x_{j_0}\} \subseteq \mathcal{I}(x'_0), \ldots, \{x_{i_m}, \ldots, x_{j_m}\} \subseteq \mathcal{I}(x'_m)$, where $0 = i_0 \leq j_0 < \cdots < i_m \leq j_m = n$. Hence C' is a cycle. □

Proposition 3.3.9. *Let H be acyclic and $\mathcal{I}^{-1}$ a contraction of G onto H that is finitary or regular. Then $\mathcal{I}^{-1}$ is dominance preserving. In particular, $\mathcal{I}^{-1}$ is ray-preserving.*

Proof. In any case, we have that $\partial_G^- \mathcal{I}(x)$ is finite for all $x \in V(H)$. First we show that $\mathcal{I}^{-1}$ is ray-preserving. Let $R \subseteq G$ be a ray. Then for all $x \in V(H)$, $\mathcal{I}(x) \cap V(R)$ must be finite by Lemma 3.3.2(1). We conclude by Lemma 3.3.2(2) that $\mathcal{I}[R]$ contains a ray.

In order to show that $\mathcal{I}^{-1}$ is dominance-preserving, let $x \in V(G)$ such that x dominates $R \subseteq G$. We apply Lemma 3.3.5 and obtain that $\mathcal{I}^{-1}[x]$ dominates R'. □

Theorem 3.3.10. *Let H be loop-free, G acyclic and $\mathcal{I}$ an inflation of H onto G that is finitary or regular. Then $\mathbb{Y} \preceq_f H$ iff $\mathbb{Y} \preceq_f G$.*

Proof. $\Rightarrow$: Since G is acyclic Proposition 3.3.7 yields the claim.

$\Leftarrow$: By Lemma 3.3.1(1) we conclude that H is acyclic. Then we apply Proposition 3.3.9 and Proposition 3.3.6. □

3.4 Walicki's conjecture

In this section we shall use Corollary 3.2.9 in order to transfer some results from [47], concerning sufficient conditions for a digraph having a kernel, to results on sufficient conditions for a digraph being safe. In particular, we show that Conjecture (W) (2.8.11) implies Conjecture (A) (2.8.1.)

Recall Definition 3.3.3. The following was conjectured in [47].

Conjecture 3.4.1 (Walicki's Conjecture). *A digraph G has a kernel if it contains neither an odd cycle nor a ray that is dominated by infinitely many of its vertices.*

We will later show (Theorem 4.4.2) that $\mathbb{Y} \preceq_f G$ iff G contains a ray R such that there are infinitely many vertices in $V(R)$ dominating R in G. Hence the above conjecture is indeed equivalent to our reformulation as Conjecture (W) (2.8.11).

Theorem 3.4.2. *Conjecture (W) implies Conjecture (A).*

Proof. Let $\mathcal{C}$ be the class of all acyclic digraphs G such that $\mathbb{Y} \not\preceq_f G$. By Theorem 3.3.10 $\mathcal{C}$ is closed under regular inflation. Since every element of $\mathcal{C}$ is acyclic, $\mathcal{C}$ is also closed under twofold subdivision of loops. Then Conjecture (W) implies that that every element of $\mathcal{C}$ is has a kernel. It follows by Theorem 3.2.9 that every element of $\mathcal{C}$ is safe. □

Let us restate the following definition of an *end*. A subdigraph $H \subseteq G$ is called an *end* of G iff there is a ray $R \subseteq G$ such that for all $x \in V(H)$ there exists a path from x to some vertex of R in G. Ends of digraphs will be discussed in Section 5.3.

The following was proved in [47].

Theorem 3.4.3 (Walicki). *A digraph G has a kernel if it has only finitely many ends and contains neither an odd a cycle nor a ray that is dominated by infinitely many of its vertices.*

It is not difficult to prove the following proposition, employing various results from Section 3.3.

Proposition 3.4.4. *If a digraph has only finitely many ends, then every regular inflation of it has only finitely many ends.*

Corollary 3.4.5. *Let G be an acyclic digraph that has only finitely many ends and such that $\mathbb{Y} \not\preceq_f G$. Then G is safe.*

Proof. Let $\mathcal{C}$ be the class of all acyclic digraphs G such that $\mathbb{Y} \not\preceq_f G$ and G has only finitely many ends. Then by Theorem 3.3.10 and Proposition 3.4.4, $\mathcal{C}$ is closed under regular inflation. It is trivially closed under twofold subdivision of loops. By Theorem 3.4.3, every element of $\mathcal{C}$ has a kernel. Hence every element of $\mathcal{C}$ is safe by Corollary 3.2.9. $\square$

3.5 Existence of characteristic digraphs

Now that we have discussed some of its applications, we shall proceed to the proof of Theorem 3.2.6. A first step towards this aim is a description of Boolean networks in terms of an infinitary propositional logic.

3.5.1 Representation of Boolean networks as sentence systems

The goal of this subsection, aside from an exposition of the framework of Rabern et al., is to show that every Boolean network can be represented as a sentence system. Many of the definitions and results of this subsection are from [41]. Cf. also [40].

Given a set (of arbitrary cardinality) $\mathcal{S}$, the elements of which are called *sentence names*, the language $\mathcal{L}_\mathcal{S}$ consists of the elements of $\mathcal{S}$, the nullary operators $\top$ and $\bot$, the unary operator $\neg$, the binary operator $\wedge$ and the infinitary operator $\bigwedge$.

Definition 3.5.1. Let $\mathcal{S}$ be a set of sentence names. Define the set of sentences $\mathcal{S}^+$ over $\mathcal{S}$ recursively as follows.

1. $\alpha \in \mathcal{S}^+$, for all $\alpha \in \mathcal{S}$,
2. $\top, \bot \in \mathcal{S}^+$,
3. if $\varphi \in \mathcal{S}^+$ then $\neg\varphi \in \mathcal{S}^+$,
4. if $\varphi \in \mathcal{S}^+$ and $\psi \in \mathcal{S}^+$, then $\varphi \wedge \psi \in \mathcal{S}^+$,
5. if $\{\varphi_i \mid i \in I\}$ is a non-empty set of cardinality less than or equal the cardinality of $\mathcal{S}$,[1] and if $\{\varphi_i \mid i \in I\} \subseteq S^+$, then $\bigwedge_{i\in I} \varphi_i \in \mathcal{S}^+$.

We call sentences that are formed according to rule (1) *primitive* $\mathcal{L}_\mathcal{S}$-sentences and sentences that are formed according to the rules (2) - (5) *complex* $\mathcal{L}_\mathcal{S}$-sentences (even thought $\top$ and $\bot$ might not be considered complex in the ordinary sense). In the following we always assume that the set of all primitive

[1] Without this cardinality restriction we would obtain a proper class of sentences. Note that the cardinality of the set of all $\mathcal{L}_\mathcal{S}$-sentences is strictly greater than that of $\mathcal{S}$.

$\mathcal{L}_\mathcal{S}$-sentences and the set of all complex $\mathcal{L}_\mathcal{S}$-sentences are disjoint, i.e., we exclude *degenerate* sets of sentences names like e.g., $\mathcal{S} = \{\alpha, \neg\alpha\}$, where $\neg\alpha$ would be both, primitive and complex. If $\mathcal{S}$ is a *non-degenerate* sets of sentences, then the set of all complex $\mathcal{L}_\mathcal{S}$-sentences is $\mathcal{S}^+ \setminus \mathcal{S}$.

Definition 3.5.2. We say that a sentence name α *occurs* in $\psi \in \mathcal{S}^+$ iff α is a syntactic constituent of ψ.

Definition 3.5.3. Let v be a *truth-value assignment*, i.e., a function from $\mathcal{S}$ to $\{0,1\}$. Given v, we recursively define the *evaluation function* on $\mathcal{S}$, $\|\cdot\|(\cdot) : \mathcal{S}^+ \times \{0,1\}^\mathcal{S} \to \{0,1\}$ as follows.

1. $\|\top\|(v) = 1$,
2. $\|\bot\|(v) = 0$,
3. $\|\alpha\|(v) = v(\alpha)$,
4. $\|\neg\varphi\|(v) = 1 - \|\varphi\|(v)$,
5. $\|\varphi \wedge \psi\|(v) = \min\{\|\varphi\|(v), \|\psi\|(v)\}$,
6. $\|\bigwedge_{i\in I} \varphi_i\|(v) = \min\{\|\varphi_i\|(v) \mid i \in I\}$.

A *sentence system* is an ordered pair $(\mathcal{S}, d)$, where $\mathcal{S}$ is a set and d is a function from $\mathcal{S}$ to $\mathcal{S}^+$. A truth-value assignment $v : \mathcal{S} \to \{0,1\}$ is an *interpretation* of $(\mathcal{S}, d)$ iff for all $\alpha \in S$, $\|\alpha\|(v) = \|\mathrm{d}(\alpha)\|(v)$; and v is an *interpretation* of a set of $\mathcal{L}_S$-sentences $\mathcal{T}$ iff for all $\varphi \in \mathcal{T}$: $\|\varphi\|(v) = 1$. Furthermore, we say that $\mathcal{T}$ is *satisfiable* iff there exists an interpretation of $\mathcal{T}$. For any sentence system $(\mathcal{S}, d)$ we call the set $\mathcal{T}(\mathcal{S}, d) = \{\alpha \leftrightarrow d(\alpha) \mid \alpha \in \mathcal{S}\} \subseteq \mathcal{S}^+$ the *set of T-schemes associated with* $(\mathcal{S}, d)$. A *paradoxical* sentence system is one that has no interpretation. Clearly, $v : \mathcal{S} \to \{0,1\}$ is an interpretation of $(\mathcal{S}, d)$ iff it is an interpretation of $\mathcal{T}(\mathcal{S}, d)$.

Definition 3.5.4. The *reference graph* $\mathcal{G}_{\mathcal{S},d}$ *of a sentence system* $(\mathcal{S}, d)$ is the digraph G with $V(G) = \mathcal{S}$ and such that $A(G)$ is the set of all (α, β) such that β occurs in $d(\alpha)$.

Note that, in contrast to a dependency graph of a Boolean network, there is only one reference graph of a sentence system.

For example, if $\mathcal{S}_0 = \{\alpha_0\}$ and $d_0(\alpha_0) = \bot$, then $(\mathcal{S}_0, d_0)$ is a sentence system whose reference graph consists of a single vertex α_0 with no arcs. For $\mathcal{S}_\lambda = \{\lambda\}$ and $d_\lambda(\lambda) = \neg\lambda$, the corresponding reference graph consists of a single vertex λ and an arc (a loop) from λ to λ. The reference graph of $(\{\tau\}, d_\tau(\tau) = \tau)$ is isomorphic to that of $(\mathcal{S}_\lambda, d_\lambda)$, even though $(\mathcal{S}_\lambda, d_\lambda)$ is paradoxical while $(\mathcal{S}_\tau, d_\tau)$ is satisfiable. Finally, for $n \in \omega$, let $d_\mathbb{Y}(n) = \bigwedge_{m>n} \neg m$. Then $(\omega, d_\mathbb{Y})$ is a paradoxical sentence system whose reference graph is $\mathbb{Y}$.

The following construction up to Corollary 3.5.12 is, in some coarser form, already indicated in Appendix A of [41].

Definition 3.5.5. Let $(\mathcal{S}, d)$ be a sentence system. For all $x \in V(\mathcal{G}_{S,d}) = \mathcal{S}$ and all $f : V(\mathcal{G}_{S,d}) \to \{0,1\}$ define $\mathcal{V}_d(f, x) = \|d(x)\|(f)$.

Proposition 3.5.6. *Let G be the reference graph of a sentence system $(\mathcal{S}, d)$. Then $\mathcal{V}_d : \{0,1\}^{V(G)} \times V(G) \to \{0,1\}$ is a Boolean network on G.*

Proof. Let $X = V(G)$. Let $f, g \in \{0,1\}^X$. Let $x \in X$ and suppose $f \restriction out_G(x) = g \restriction out_G(x)$. Then $\mathcal{V}_d(f, x) = \|d(x)\|(f)$, where $d(x) \in \mathcal{S}^+$ is such that $out_G(x)$ is the set of all sentence names occurring in $d(x)$. By induction on the complexity of x one shows that $\|d(x)\|(f) = \|d(x)\|(g)$. Hence $\mathcal{V}_d(f, x) = \mathcal{V}_d(g, x)$. □

Definition 3.5.7. Let (G, Φ) be a constrained Boolean network. A sentence system $(\mathcal{S}, d)$ is said to be a *representation* of (G, Φ) iff $G = \mathcal{G}_{\mathcal{S},d}$ and $\Phi = \mathcal{V}_d$.

Observe that in particular $V(G) = \mathcal{S}$, if $(\mathcal{S}, d)$ is a representation of (G, Φ). The vertices of G are simply used as sentence names.

Proposition 3.5.8. *Let Φ be a Boolean network and $(\mathcal{S}, d)$ a representation of Φ. Then $f : \mathcal{S} \to \{0,1\}$ is a fixed point of Φ iff f is an interpretation of $(\mathcal{S}, d)$.*

Every Boolean network on G has a representation. In order to show this, we need the following lemma from [41].

Lemma 3.5.9. *For all sets $\mathcal{S}$ of sentence names and all $f : \{0,1\}^{\mathcal{S}} \to \{0,1\}$, there exists a sentence $\zeta_f \in \mathcal{S}^+$ such that $\|\zeta_f\| = f$.*

Proof. Then sentence ζ_f will be the *representation of f in disjunctive normal form* (DNF), generalized to infinitary propositional logic. In the following, read $\bigvee_{i\in I} \varphi_i$ as an abbreviation of $\neg \bigwedge_{i\in I} \neg\varphi_i$.

If $f(v) = 0$ for all $v \in \{0,1\}^{\mathcal{S}}$, then let $\zeta_f = \bot$. Otherwise let

$$\zeta_f = \bigvee_{v\in C} \bigwedge_{\alpha\in\mathcal{S}} h(v, \alpha),$$

where $C = \{v \in \{0,1\}^{\mathcal{S}} \mid f(v) = 1\}$ and $h(v, \alpha) = \begin{cases} \alpha, & \text{if } v(\alpha) = 1 \\ \neg\alpha, & \text{if } v(\alpha) = 0. \end{cases}$ □

Theorem 3.5.10. *For all digraphs G, there is a map $\Phi \mapsto d_\Phi$ from the set of all Boolean networks on G into the set of all sentence systems over $\mathcal{S} = V(G)$ such that $\mathcal{G}_{\mathcal{S},d_\Phi} = G$ and $\mathcal{V}_{d_\Phi} = \Phi$.*

Proof. Let Φ be a Boolean networks on G. For every $x \in \mathcal{S}$ let $\mathcal{S}_x = \mathcal{S} \cap \mathrm{out}_G(x)$. For $x \in \mathcal{S}$ and $f : \mathcal{S} \to \{0,1\}$ let $\Phi_x(f) = \Phi(f, x)$. Thus $\Phi_x : \{0,1\}^{\mathcal{S}} \to \{0,1\}$. By Lemma 3.5.9 there exists a sentence $\zeta_{\Phi_x} \in \mathcal{S}_x^+$ such that $\|\zeta_{\Phi_x}\| = \Phi_x$. Since $\mathcal{S}_x \subseteq \mathcal{S}$, $\zeta_{\Phi_x} \in \mathcal{S}^+$. Let $d_\Phi(x) = \zeta_{\Phi_x}$. Then $(\mathcal{S}, d_\Phi)$ is a sentence system and $G = \mathcal{G}_{\mathcal{S},d_\Phi}$. Moreover, for all $f : V(X) \to \{0,1\}$ and $x \in V(G)$: $\mathcal{V}_{d_\Phi}(f, x) = \|d_\Phi(x)\|(f) = \Phi(f, x)$ by definition of d_Φ. □

Corollary 3.5.11. *Every constrained Boolean network has a representation.*

Corollary 3.5.12. *A digraph G is dangerous iff there exists a paradoxical sentence system $(\mathcal{S}, d)$ such that $G = \mathcal{G}_{\mathcal{S},d}$.*

In other words, a digraph is dangerous in the sense of [41] iff it is dangerous in the sense of Definition 2.6.1.

Observe that representations are only defined for constrained Boolean networks and not for Boolean networks simpliciter. The reason for this can be found in the crucial role the dependency graph plays in defining the denotation function of the sentence system in the proof of Theorem 3.5.10. This point becomes particularly apparent when Boolean networks with no minimal dependency graphs as in Example 2.3.3 are considered. Such Boolean networks cannot be fully captured by any sentence system without prior interpretation of their dependency structure.

Definition 3.5.13. Let (G, Ψ) be a constrained Boolean network. We call the sentence system assigned to (G, Ψ) in the proof of Theorem 3.5.10 the *standard representation of* (G, Ψ) *in DNF.*

3.5.2 Construction of a characteristic digraph

The goal of this subsection is to prove Theorem 3.2.6.

Definition 3.5.14. Let $\mathcal{S}$ be a set of sentence names. An $\mathcal{S}$-sentence φ is said to be in *F-normal form (FNF)* iff one of the following holds.

1. $\varphi = \top$,
2. $\varphi = \neg\alpha_0$, for some $\alpha_0 \in \mathcal{S}$,
3. $\varphi = \nu_1 \wedge \nu_2$, where $\nu_i = \top$ or $\nu_i = \neg\alpha_i$, for $i \in \{0, 1\}$ and $\alpha_i \in \mathcal{S}$,
4. $\varphi = \bigwedge_{i\in I} \nu_i$, where $\nu_i = \top$ or $\nu_i = \neg\alpha_i$, for $i \in I$ and $\alpha_i \in \mathcal{S}$.

A sentence system $(\mathcal{S}, d)$ is said to be in *F-normal form* iff for all $\alpha \in \mathcal{S}$, $d(\alpha)$ is in F-normal form.

The 'F' in 'F-normal form' stands for 'false'. Also cf. '$\mathcal{F}$-system' in [41].

Proposition 3.5.15. *Let $(\mathcal{S}, d)$ be a sentence system in F-normal form. Then $\|d(x)\|(v) = \Phi_{\downarrow}^{\mathcal{G}_{\mathcal{S},d}}(v, x)$, for all $x \in \mathcal{S}$, $v \in \{0,1\}^{\mathcal{S}}$. In particular, $\mathcal{G}_{\mathcal{S},d}$ has a kernel iff $(\mathcal{S}, d)$ is satisfiable.*

In a certain sense, the goal of the remainder of this section is to establish a more elaborate version of a result from [8]. In the following, we shall define for every sentence system $(\mathcal{S}, d)$ an equi-satisfiable sentence system $(\mathcal{S}_F, F(d))$ in F-normal form. One advantage that this procedure has over that suggested in [8] is that no artificial loops are created when passing from the original reference graph to the new one, which, given that the original one is loop-free, will be a regular inflation of it.

For this purpose, let $\mathcal{S}$ be a non-degenerate set of sentence names that shall be fixed in the following. For every sentence φ (of every language whatsoever, i.e., for every object in the set-theoretic universe) and every $\beta \in \mathcal{S}$, let

$$\ulcorner\varphi\urcorner^{\beta} = \begin{cases} \alpha, & \text{if } \varphi = \alpha, \text{ for some } \alpha \in \mathcal{S} \\ (\varphi, \beta), & \text{if } \varphi \notin \mathcal{S}. \end{cases}$$

Definition 3.5.16. We define by simultaneous recursion a set of sentence names $\mathcal{S}_F$ and a set of sentences $\mathcal{S}_F^+$ over $\mathcal{S}_F$ as follows.

1. If $\alpha \in \mathcal{S}$, then $\alpha \in \mathcal{S}_F$,
2. if $\varphi \in (S_F)^+$, then $\varphi \in \mathcal{S}_F^+$,
3. if $\varphi \in \mathcal{S}_F^+$ and $\beta \in \mathcal{S}$, then $\ulcorner\varphi\urcorner^{\beta} \in \mathcal{S}_F$.

Then $\mathcal{S}_F$ is a non-degenerate set of sentence names, $\mathcal{S} \subseteq \mathcal{S}_F$, and $\mathcal{S}^+ \subseteq \mathcal{S}_F^+ = (\mathcal{S}_F^+)^+$. Note that $\ulcorner\varphi\urcorner^{\beta} \in \mathcal{S}_F^+$, for all $\varphi \in \mathcal{S}_F^+$ and $\beta \in \mathcal{S}$.

The basic idea behind the following translation of $\mathcal{L}_{\mathcal{S}}$-sentences into $\mathcal{L}_{\mathcal{S}_F}$-sentences is to replace unnegated sentences by doubly negated sentences and, if necessary, substituting sentences names from $\mathcal{S}_F^+$ for sub-sentences in order to obtain sentences in F-normal form. The role of the set $\mathcal{S}_F$ is to ensure that occurrences of identical sub-sentences in different contexts (i.e., in $d(\alpha)$ and $d(\beta)$, where d is some denotation function for $\mathcal{S}$ and $\alpha \neq \beta$) are replaced by different sentence names. This is necessary if we want the reference graph of the resulting sentence system to be a regular inflation of the reference graph of the original one.

Definition 3.5.17. Define a maps $F, F^1 : \mathcal{S} \times \mathcal{S}^+ \to \mathcal{S}_F^+$ simultaneously by recursion on the syntactic complexity of $\mathcal{L}_{\mathcal{S}}$-sentences as follows.

1. $F_\beta(\top) = F_\beta^1(\top) = \top$,
2. $F_\beta(\bot) = F_\beta^1(\bot) = \neg\ulcorner\top\urcorner^{\beta}$,
3. $F_\beta(\alpha) = \neg\ulcorner\neg\alpha\urcorner^{\beta}$, where $\alpha \in \mathcal{S}$,
4. $F_\beta(\neg\alpha) = F_\beta^1(\neg\alpha) = \neg\alpha$, where $\alpha \in \mathcal{S}$,
5. $F_\beta(\neg\varphi) = F_\beta^1(\neg\varphi) = \neg\ulcorner F_\beta(\varphi)\urcorner^{\beta}$, where $\varphi \in \mathcal{S}^+ \setminus \mathcal{S}$,
6. for all $\varphi, \psi \in \mathcal{S}^+$ let
   (a) $F_\beta(\varphi \wedge \psi) = F_\beta^1(\varphi) \wedge F_\beta^1(\psi)$,
   (b) $F_\beta^1(\varphi \wedge \psi) = \neg\ulcorner\neg(F_\beta^1(\varphi) \wedge F_\beta^1(\psi))\urcorner^{\beta}$,
7. for all $\varphi_i \in \mathcal{S}^+$ let
   (a) $F_\beta(\bigwedge_{i\in I} \varphi_i) = \bigwedge_{i\in I} F_\beta^1(\varphi_i)$,
   (b) $F_\beta^1(\bigwedge_{i\in I} \varphi_i) = \neg\ulcorner\neg(\bigwedge_{i\in I} F_\beta^1(\varphi_i))\urcorner^{\beta}$.

The reason for this nested recursion is to distinguish between occurrences within conjunctions and outside of conjunctions.

It will be shown below (cf. Lemma3.5.20) that these functions are indeed well-defined and behave as intended. For an illustration cf. Example 3.5.24.

In order to associate a truth-value assignment v_F on $\mathcal{S}_F$ to any given truth-value assignment v on $\mathcal{S}$, let us first extend the evaluation function $\|\cdot\|(v) : S^+ \to \{0,1\}$ to the extended language $\mathcal{S}_F^+$.

Definition 3.5.18. For $v : \mathcal{S} \to \{0,1\}$ define $\|\cdot\|^*(v) : \mathcal{S}_F^+ \to \{0,1\}$ recursively as follows.

1. $\|\top\|^*(v) = 1$,
2. $\|\bot\|^*(v) = 0$,
3. $\|\alpha\|^*(v) = v(\alpha)$, if $\alpha \in \mathcal{S}$,
4. $\|\ulcorner\varphi\urcorner^\beta\|^*(v) = \|\varphi\|^*(v)$, if $\varphi \in \mathcal{S}_F^+ \setminus \mathcal{S}$ and $\beta \in \mathcal{S}$,
5. $\|\neg\varphi\|^*(v) = 1 - \|\varphi\|^*(v)$,
6. $\|\varphi \wedge \psi\|^*(v) = \min\{\|\varphi\|^*(v), \|\psi\|^*(v)\}$,
7. $\|\bigwedge_{i\in I}\varphi_i\|^*(v) = \min\{\|\varphi_i\|^*(v) \mid i \in I\}$.

The crucial clause of Definition 3.5.18 is, of course, (4).

Definition 3.5.19. Let $v : \mathcal{S} \to \{0,1\}$. Define $v_F : S_F \to \{0,1\}$ by $v_F(\nu) = \|\nu\|^*(v)$, for all $\nu \in S_F$.

Lemma 3.5.20. *Let $(\mathcal{S}, d)$ be a sentence system, $\varphi \in S^+$, $\beta \in \mathcal{S}$ and $v \in \{0,1\}^{\mathcal{S}}$. Then*

1. *$v = v_F \restriction \mathcal{S}$,*
2. *$F_\beta(\varphi)$ is an $\mathcal{S}_F$-sentence in FNF,*
3. *$\|F_\beta(\varphi)\|(v_F) = \|F_\beta(\varphi)\|^*(v) = \|F^1_\beta(\varphi)\|^*(v) = \|\varphi\|(v)$.*
4. *Let $\alpha \in \mathcal{S}$. If α occurs in $F_\beta(\varphi)$, then α occurs in φ.*
5. *If φ is in F-normal form, then $F_\beta(\varphi) = \varphi$.*

Proof. Ad 1: $v_F(\alpha) = \|\alpha\|^*(v) = v(\alpha)$ by Definitions 3.5.19 and 3.5.18.
Ad 2 and 3: We prove the claims simultaneous by induction on the syntactic complexity of φ.

- Suppose $\varphi \in \{\top, \bot, \alpha, \neg\alpha\}$, $(\alpha \in \mathcal{S})$. Then $F_\beta(\varphi) \in \{\top, \neg\ulcorner\top\urcorner^\beta, \neg\ulcorner\neg\alpha\urcorner^\beta\, \neg\alpha\}$. This proves (2). Claim (3) is straightforward by computation, e.g., $\|\neg\ulcorner\neg\alpha\urcorner^\beta\|(v_F) = \|\neg\ulcorner\neg\alpha\urcorner^\beta\|^*(v) = 1 - (1 - v(\alpha)) = \|\alpha\|(v)$.

- Now let $\varphi = \neg\psi$, for $\psi \in \mathcal{S}^+ \setminus \mathcal{S}$. Then $F_\beta(\varphi) = \neg\ulcorner F_\beta(\psi)\urcorner^\beta$, where $F_\beta(\psi) \in \mathcal{S}_F^+$ by induction hypothesis (2). Hence $\ulcorner F_\beta(\psi)\urcorner^\beta \in \mathcal{S}_F$ by Definition 3.5.16, which proves claim (2).

  Ad 3: $\|\neg\ulcorner F_\beta(\psi)\urcorner^\beta\|(v_F) \overset{\text{i.h.}}{=} 1 - \|\ulcorner F_\beta(\psi)\urcorner^\beta\|^*(v) = 1 - \|F_\beta(\psi)\|^*(v) \overset{\text{i.h.}}{=} 1 - \|\psi\|(v) = \|\neg\psi\|(v)$.

- Now let $\psi_1, \psi_2 \in \mathcal{S}^+$ and $\varphi = \psi_1 \wedge \psi_2$. Then $F_\beta(\varphi) = F_\beta^1(\psi_1) \wedge F_\beta^1(\psi_2)$. Ad (2): We have to show that $F_\beta^1(\psi_1) \in \{\top, \neg\nu_1\}$ and $F_\beta^1(\psi_2) \in \{\top, \neg\nu_2\}$, for $\nu_1, \nu_2 \in \mathcal{S}_F$. This is done by side induction on the syntactic complexities of ψ_1 and ψ_2. We only show the computation for ν_1, that for ν_2 is completely analogous.

  - Suppose $\psi_1 \in \{\top, \bot, \alpha, \neg\alpha\}$, $(\alpha \in \mathcal{S})$. Then $F_\beta^1(\psi_1) \in \{\top, \neg\ulcorner\top\urcorner^\beta, \neg\ulcorner\neg\alpha\urcorner^\beta\, \neg\alpha\}$, whicb proves the claim.
  - Now let $\psi_1 = \neg\chi$, for $\chi \in \mathcal{S}^+ \setminus \mathcal{S}$. Then $F_\beta^1(\psi_1) = \neg\ulcorner F_\beta(\chi)\urcorner^\beta$, where $F_\beta(\chi) \in \mathcal{S}_F^+$ by induction hypothesis (2). Hence $\nu_1 = \ulcorner F_\beta(\chi)\urcorner^\beta \in \mathcal{S}_F$.
  - Now let $\psi_1 = \chi \wedge \xi$, for $\chi, \xi \in \mathcal{S}^+$. Then $F_\beta^1(\psi_1) = F_\beta^1(\chi \wedge \xi) = \neg\ulcorner\neg(F_\beta^1(\chi) \wedge F_\beta^1(\xi))\urcorner^\beta$, where $F_\beta^1(\chi)$, $F_\beta^1(\xi) \in \mathcal{S}_F^+$ by side induction hypothesis (2). Hence $\nu_1 = \ulcorner\neg(F_\beta^1(\chi) \wedge F_\beta^1(\xi))\urcorner^\beta \in \mathcal{S}_F$.
  - Suppose $\psi_1 = \bigwedge_{i \in I} \chi_i$. The argument is completely analogous to the previous case.

  Ad 3: By induction hypothesis (3) we have $\|F_\beta(\varphi)\|(v_F) = \|F_\beta^1(\psi_1) \wedge F_\beta^1(\psi_2)\|(v_F) \overset{\text{i.h.}}{=} \min\{\|F_\beta^1(\psi_1)\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = \|F_\beta^1(\psi_1) \wedge F_\beta^1(\psi_2)\|^*(v) = \|F_\beta(\varphi)\|^*(v)$.
  Moreover, $\|F_\beta^1(\psi_1)\|^*(v) = \|F_\beta(\psi_1)\|^*(v) = \|\psi_1\|(v)$ and $\|F_\beta^1(\psi_2)\|^*(v) = \|F_\beta(\psi_2)\|^*(v) = \|\psi_2\|(v)$ by induction hypothesis (3).

  - Suppose $\psi_1 = \top$. Then $F_\beta^1(\psi_1) = \top$. Hence $\|F_\beta(\varphi)\|(v_F) = \|F_\beta^1(\psi_2)\|^*(v) = \|F_\beta(\psi_2)\|^*(v) = \|\psi_2\|(v) = \|\psi_1 \wedge \psi_2\|(v) = \|\varphi\|(v)$.
  - Suppose $\psi_1 = \bot$. Then $F_\beta^1(\psi_1) = \neg\ulcorner\top\urcorner^\beta$. Hence $\|F_\beta(\varphi)\|(v_F) = \min\{\|F_\beta^1(\psi_1)\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = 0 = \|\psi_1 \wedge \psi_2\|(v) = \|\varphi\|(v)$.
  - Suppose $\psi_1 = \alpha$, for $\alpha \in \mathcal{S}$. Then $F_\beta^1(\psi_1) = \neg\ulcorner\neg\alpha\urcorner^\beta$. Hence $\|F_\beta(\varphi)\|(v_F) = \min\{\|F_\beta^1(\psi_1)\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = \min\{v(\alpha), \|F_\beta^1(\psi_2)\|^*(v)\} = \min\{v(\alpha), \|\psi_2)\|(v)\} = \|\psi_1 \wedge \psi_2\|(v) = \|\varphi\|(v)$.
  - Suppose $\psi_1 = \neg\chi$. Then $F_\beta^1(\psi_1) = \neg\ulcorner F_\beta(\chi)\urcorner^\beta$. Hence $\|F_\beta(\varphi)\|(v_F) = \min\{\|F_\beta^1(\psi_1)\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = \min\{\|\neg\ulcorner F_\beta(\chi)\urcorner^\beta\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = \min\{1 - \|F_\beta(\chi)\|(v), \|\psi_2)\|(v)\} = \|\psi_1 \wedge \psi_2\|(v) = \|\varphi\|(v)$.
  - Suppose $\psi_1 = \chi \wedge \xi$. Then $F_\beta^1(\psi_1) = F_\beta^1(\chi \wedge \xi) = \neg\ulcorner\neg(F_\beta^1(\chi) \wedge F_\beta^1(\xi))\urcorner^\beta$. Hence $\|F_\beta(\varphi)\|(v_F) =$
    $\min\{\|\neg\ulcorner\neg(F_\beta^1(\chi) \wedge F_\beta^1(\xi))\urcorner^\beta\|^*(v), \|F_\beta^1(\psi_2)\|^*(v)\} = \min\{b_1, \|\psi_2)\|(v)\} = \|\psi_1 \wedge \psi_2\|(v) = \|\varphi\|(v)$, where $b_1 = 1 - (1 - \min\{\|\chi\|^*(v), \|\xi\|^*(v)\})$.

– Suppose $\psi_1 = \bigwedge_{i\in I} \chi_i$. The argument is completely analogous to the previous case.

- The $\bigwedge$-case is analogous to the $\wedge$-case.

Ad 4 and 5: Straightforward by induction on the syntactic complexity of φ. $\square$

Definition 3.5.21. For all $\varphi \in \mathcal{S}_F^+$ define

1. the set $D^*(\varphi)$ of all *direct proper transparent subsentences* of φ as follows.
   (a) If $\varphi = \bot$ or $\varphi = \top$ or $\varphi = \alpha$, for some $\alpha \in \mathcal{S}$, then $D^*(\varphi) = \varnothing$,
   (b) if $\varphi = \ulcorner\psi\urcorner^\beta$, where $\psi \in \mathcal{S}_F^+ \setminus \mathcal{S}$ and $\beta \in \mathcal{S}$, then $D^*(\varphi) = \{\psi\}$, (observe that the case that $\psi \in \mathcal{S}$ cannot occur),
   (c) if $\varphi = \neg\psi$, then $D^*(\varphi) = \{\psi\}$,
   (d) if $\varphi = \psi_1 \wedge \psi_2$, then $D^*(\varphi) = \{\psi_1, \psi_2\}$,
   (e) if $\varphi = \bigwedge_{i\in I} \psi_i$, for some non-empty set I, then $D^*(\varphi) = \{\psi_i \mid i \in I\}$,
2. the set $P^*(\varphi)$ of all *proper transparent subsentences* of φ recursively by $P^*(\varphi) = D^*(\varphi) \cup \bigcup\{P^*(\psi) \mid \psi \in D^*(\varphi)\}$.
3. We say that $\psi \in \mathcal{S}_F^+$ *occurs transparently* in φ iff $\psi \in P^*(\varphi)$ or $\psi = \varphi$.

Definition 3.5.22. Let $(\mathcal{S}, d)$ be a sentence system.

1. Define $F(d) : \mathcal{S}_F \to \mathcal{S}_F^+$ by
$$(F(d))(\nu) = \begin{cases} F_\alpha(d(\alpha)), & \text{if } \nu = \alpha, \text{ for some } \alpha \in \mathcal{S} \\ \varphi, & \text{if } \nu = \ulcorner\varphi\urcorner^\beta, \text{ for } \varphi \in \mathcal{S}_F^+ \setminus \mathcal{S} \text{ and } \beta \in \mathcal{S}. \end{cases}$$
2. Let $\mathcal{F}_d(\alpha)$ (for $\alpha \in \mathcal{S}$) be the union of $\{\alpha\}$ and the set of all $\nu \in \mathcal{S}_F \setminus \mathcal{S}$ such that ν occurs transparently in $(F(d))(\alpha)$.
3. Let $\mathcal{S}_f = \bigcup_{\alpha\in\mathcal{S}} F_d(\alpha)$ and $d_f = (F(d)) \restriction \mathcal{S}_f$.
4. For all $v : \mathcal{S} \to \{0,1\}$ define $v_f = v_F \restriction \mathcal{S}_f$. (Cf. Definition 3.5.19).

Lemma 3.5.23. *Let $(\mathcal{S}, d)$ be a sentence system and $\alpha \in \mathcal{S}$. Then the following hold.*

1. *$(\mathcal{S}_f, d_f)$ is a sentence system in F-normal form.*
2. *$\mathcal{F}_d = \{\mathcal{F}_d(\alpha) \mid \alpha \in \mathcal{S}\}$ is a partition of $\mathcal{S}_f$.*
3. *α is the root of $\mathcal{F}_d(\alpha)$, i.e., every $\nu \in \mathcal{F}_d(\alpha)$ is reachable from α in $G_{\mathcal{S}_f,d_f}[\mathcal{F}_d(\alpha)]$.*
4. *If $\mathcal{G}_{\mathcal{S},d}$ is loop-free, then $\mathcal{G}_{\mathcal{S}_f,d_f}[\mathcal{F}_d(\alpha)]$ is a well-founded digraph.*
5. *$\partial^-_{G_{\mathcal{S}_f,d_f}} \mathcal{F}_d(\alpha) = \{\alpha\}$.*
6. *If v is an interpretation of $(\mathcal{S}, d)$, then v_f is an interpretation of $(\mathcal{S}_f, d_f)$.*
7. *For every interpretation u of $(\mathcal{S}_f, d_f)$, there exists an interpretation v of $(\mathcal{S}, d)$ such that $u = v_f$.*

8. *$\mathcal{F}_d$ is a regular inflation of $\mathcal{G}_{\mathcal{S},d}$ onto $\mathcal{G}_{\mathcal{S}_f,d_f}$, if $\mathcal{G}_{\mathcal{S},d}$ is loop-free.*

Proof. For better readability, let $G = \mathcal{G}_{\mathcal{S},d}$ and $G' = \mathcal{G}_{\mathcal{S}_f,d_f}$ throughout the proof.

Ad 1: This is a consequence of Lemma 3.5.20(2).

Ad 2: It suffices to show that $\mathcal{F}_d(\alpha) \cap \mathcal{F}_d(\beta) = \varnothing$, for $\alpha \neq \beta \in \mathcal{S}$. But this is a consequence of the definitions of $\ulcorner\varphi\urcorner^\alpha$ and $\mathcal{F}_d(\alpha)$, since for all $\gamma \in \mathcal{S}$, every $\nu \in F_d(\gamma)$ is of the form $\ulcorner\varphi\urcorner^\gamma$ for some $\varphi \in \mathcal{S}_F^+$.

Ad 3: Observe that for all $\alpha \neq \nu \in \mathcal{F}_d(\alpha)$, either ν is a direct proper subsentence of $d_f(\alpha)$ or there exists a unique $\nu' \in \mathcal{F}_d(\alpha)$ such that ν is a proper direct subsentence of ν'. Hence there is a sequence $\nu_0, \cdots, \nu_n$ in $\mathcal{F}_d(\alpha)$ with $\nu_0 = \alpha$ and $\nu_n = \nu$ such that ν_{i+1} occurs in $d_f(\nu_i)$.

Ad 4: Assuming otherwise implies the contradiction that the realtion $\{(\varphi, \psi) \mid \psi \in D^*(\varphi)\}$ is not well-founded.

Ad 5: Let $\nu \in \partial^-_{G'} F_d(\alpha)$. If $\nu \in \mathcal{S}$, then $\nu = \alpha$. Hence we can assume that $\nu = \ulcorner\varphi\urcorner^\alpha$, for some $\varphi \in \mathcal{S}_F^+ \setminus \mathcal{S}$. Clearly $\ulcorner\varphi\urcorner^\alpha$ does not occur transparently in any $(F(d))(\beta)$, for $\beta \neq \alpha$. This leads to the contradiction that $\nu \notin \partial^-_{G'} F_d(\alpha)$.

Ad 6 and 7: By induction on the syntactic complexity.

Ad 8: By (2) - (5). □

Proof of Theorem 3.2.6. Let (H, Ψ) be a constrained Boolean network. We have to show that (H, Ψ) has a characteristic digraph, i.e., a digraph G such that there exists a network inflation (cf. Definition 3.2.4) $(\mathcal{I}, i)$ of (H', Ψ') onto $(G, \Phi^G_\downarrow)$, where (H', Ψ') is the loop-cleansed form of (H, Ψ) (cf. Definition 3.2.1). Let us assume for simplicity of notation that $(H, \Psi) = (H', \Psi')$. (Because of Proposition 3.2.2 we can assume this without loss of generality). So, we must find a digraph G and a regular inflation $\mathcal{I}$ of H onto G and a dense weak system-embedding i of Ψ into $\Phi^G_\downarrow$.

- Let $(\mathcal{S}, d)$ be the standard representation of (H, Ψ) in DNF. (cf. Definition 3.5.13).
- Let $\mathcal{S}_f$ be as in Definition 3.5.22.
- Let $G = \mathcal{G}_{\mathcal{S}_f,d_f}$.
- Let $\mathcal{I} = \mathcal{F}_d$ be the partition of $\mathcal{S}_f$ from Definition 3.5.22.
- Let $i : \{0,1\}^{\mathcal{S}} \to \{0,1\}^{\mathcal{S}_f}$ be defined by $i(v) = v_f$ as in Definition 3.5.22.
- Let $r : \{0,1\}^{\mathcal{S}_f} \to \{0,1\}^{\mathcal{S}}$ be defined by $r(v) = v \restriction \mathcal{S}$.

Then

1. $V(H) = \mathcal{S}$ (by definition),
2. $V(G) = S_f$ (by definition),
3. $\mathcal{S}_f \supseteq \mathcal{S}$ (by Definition 3.5.22),

4. $(\mathcal{S}_f, d_f)$ is in FNF (by Lemma 3.5.23),

5. $\mathcal{V}_d = \Psi$ (by Theorem 3.5.10),

6. $\mathcal{V}_{d_f} = \Phi^G_\downarrow$ (by Proposition 3.5.15),

7. $\mathcal{F}_d$ is a regular inflation of $\mathcal{G}_{\mathcal{S},d} = H$ onto $\mathcal{G}_{\mathcal{S}_f,d_f} = G$ (by Lemma 3.5.23(8)).

Hence $\mathcal{I}$ is a regular inflation of H onto G by (7). It remains to be shown that i is a dense weak system embedding of Ψ into $\Phi^G_\downarrow$ with retraction r. By Defnition 2.5.4 this means that we have to show that

i. $\Psi = r \circ \Phi^G_\downarrow \circ i$,

ii. $\Psi(v) = v$ implies $i(v) = \Phi^G_\downarrow \circ i(v)$, for all $v \in \{0,1\}^{\mathcal{S}}$,

iii. $v = r(i(v))$, for all $v \in \{0,1\}^{\mathcal{S}}$,

iv. if $\Phi^G_\downarrow(v) = v$, then $v = i(r(v))$, for all $v \in \{0,1\}^{\mathcal{S}_f}$.

In order to prove (i), recall that

$$(F(d))(\nu) = \begin{cases} F_\alpha(d(\alpha)), & \text{if } \nu = \alpha, \text{ for some } \alpha \in \mathcal{S} \\ \varphi, & \text{if } \nu = \ulcorner\varphi\urcorner^\beta, \text{ for } \varphi \in \mathcal{S}^+_F \setminus \mathcal{S} \text{ and } \beta \in \mathcal{S}. \end{cases}$$

Moreover, $d_f = (F(d)) \restriction \mathcal{S}_f$.

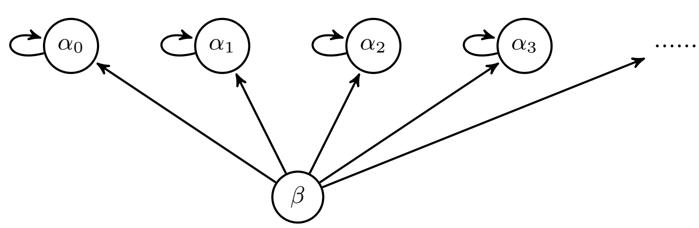
We have (cf. Definition 3.5.5) for all $\alpha \in \mathcal{S}$, $\Psi(v,\alpha) = \mathcal{V}_d(v,\alpha) = \|d(\alpha)\|(v)$. On the other hand, $\Phi^G_\downarrow(i(v),\alpha) = \mathcal{V}_{d_f}(i(v),\alpha) = \|d_f(\alpha)\|(i(v)) = \|d_f(\alpha)\|(v_f) = \|(F(d))(\alpha)\|(v_F) = \|F_\alpha(d(\alpha))\|(v_F) = \|F_\alpha(d(\alpha))\|^*(v) = \|d(\alpha)\|(v) = \Psi(v,\alpha)$. (The last three identifies follow by Lemma 3.5.20(3)). Hence $\Psi = (\Phi^G_\downarrow \circ i(v)) \restriction \mathcal{S}$, and thus $\Psi = r \circ \Phi^G_\downarrow \circ i$.

Moreover, (ii) follows from Lemma 3.5.20 (3), (iii) from Lemma 3.5.20 (1), and (iv) from Lemma 3.5.23(7)). This proves Theorem3.2.6.

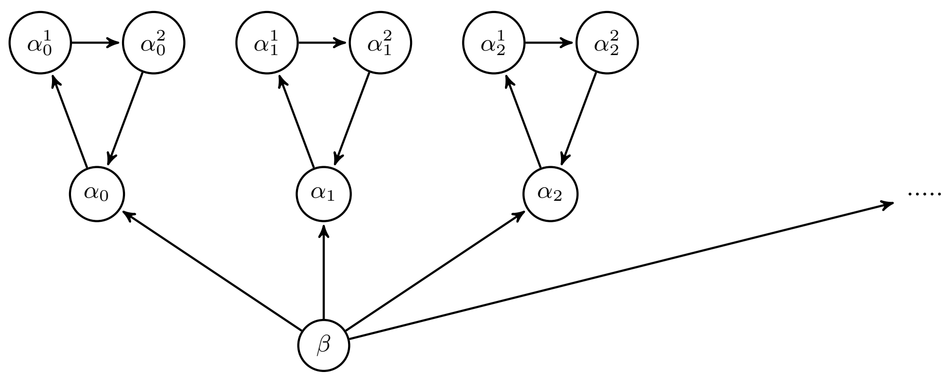
Let us conclude this chapter with an example.

Example 3.5.24. Let $(\mathcal{S}, d)$ be the sentence system such that $\mathcal{S} = \{\beta\} \cup \{\alpha_i \mid i \in \omega\}$, $d(\beta) = \varphi = \bigwedge_{i\in\omega} \alpha_i$, $d(\alpha_i) = \neg\alpha_i$.

Note that $(\mathcal{S}, d)$ is not in FNF. The reference graphs $\mathcal{G}_{\mathcal{S},d}$ of $(\mathcal{S}, d)$ looks as follows.



The loop-cleansed form $\mathcal{G}'_{\mathcal{S},d}$ of $\mathcal{G}_{\mathcal{S},d}$ (cf. Definition 3.2.1) looks as follows.



The corresponding Boolean network can be described by the following sentence system $(\mathcal{S}', d')$. This sentence system arises from $(\mathcal{S}, d)$ by adding new sentences names α_i^0 and α_i^1, for all $i \in \omega$ and changing d to d' as follows. For all $i \in \omega$ let

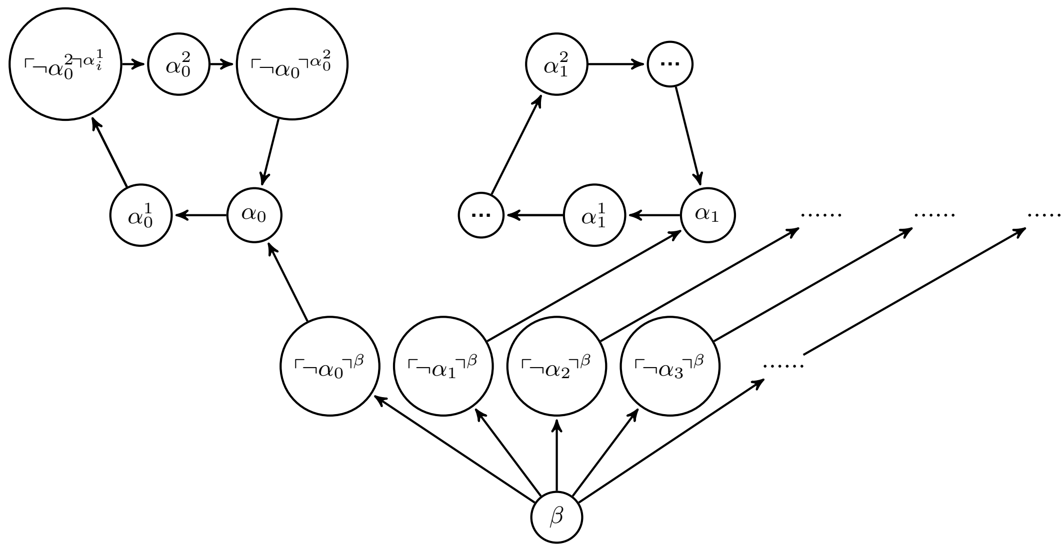
1. $d'(\alpha_i) = \neg\alpha_i^1$,
2. $d'(\alpha_i^1) = \alpha_i^2$,
3. $d'(\alpha_i^2) = \alpha_i$.

For all all other sentence names ν, let $d'(\nu) = d(\nu)$.

Finally, we transform $(\mathcal{S}', d')$ into FNF, i.e., into the sentence system $(\mathcal{S}'_f, d'_f)$. This is done on the basis of the following computations.

- $F_\beta(d'(\beta)) = F_\beta(\varphi) = \bigwedge_{i\in\omega} F^1_\beta(\alpha_i) = \bigwedge_{i\in\omega} \neg\ulcorner\neg\alpha_i\urcorner^\beta$.
- $F_{\alpha_i}(d'(\alpha_i)) = \neg\alpha_i^1$,
- $F_{\alpha_i^1}(d'(\alpha_i^1)) = \neg\ulcorner\neg\alpha_i^2\urcorner^{\alpha_i^1}$
- $F_{\alpha_i^2}(d'(\alpha_i^2)) = \neg\ulcorner\neg\alpha_i\urcorner^{\alpha_i^2}$.

The reference graph $\mathcal{G}_{\mathcal{S}'_f,d'_f}$ of $(\mathcal{S}'_f, d'_f)$ looks as follows.



3.6 Signed dependency graphs

Definition 3.6.1. Let $\mathcal{S}$ be a set of sentence names. We call an $\mathcal{L}_\mathcal{S}$-sentence φ

1. *positive* iff it is logical equivalent to an $\mathcal{L}_\mathcal{S}$-sentence ψ such that the only operator symbols that occur in ψ are $\top$, $\wedge$, $\vee$, $\bigwedge$ and $\bigvee$.

2. *negative* iff it is logical equivalent to an $\mathcal{L}_\mathcal{S}$-sentence ψ such that the only operator symbols that occur in ψ are $\bot$, $\wedge$, $\vee$, $\bigwedge$ and $\bigvee$ and $\neg$ and, ,moreover, $\neg$ occurs only in front of sentence names and in front of every sentence name.

Definition 3.6.2. A sentence system $(\mathcal{S}, d)$ is said to be *positive* iff $d(\alpha)$ is positive for all $\alpha \in \mathcal{S}$; it is said to be *negative* iff $d(\alpha)$ is negative for all $\alpha \in \mathcal{S}$. A Boolean network is said to be *positive* iff its standard representation in DNF is positive; it is said to be *negative* iff its standard representation in DNF is negative.

The following is a (more general) reconstruction of Definition 5.1 of [5].

Definition 3.6.3. Let (H, Ψ) be a constrained Boolean network and $(\mathcal{S}, d)$ the standard representation of (H, Ψ) in DNF. For all $(x, y) \in A(H)$ define $S_\Psi(x, y) \in \{+, -, \bot\}$ as follows.

1. $S_\Psi(x, y) = +$ iff $d(x)$ is positive,

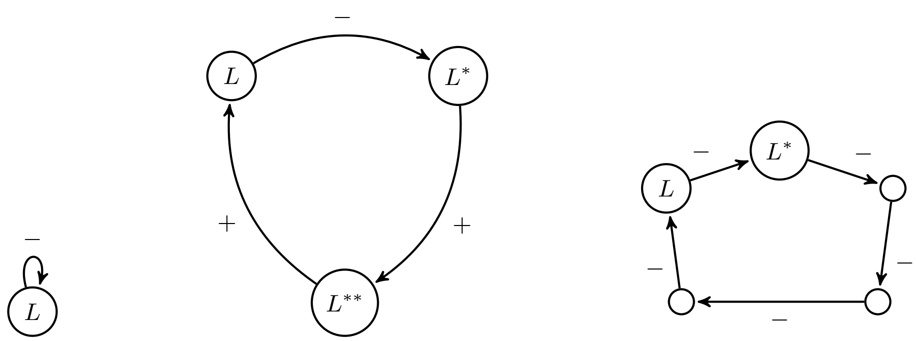
2. $S_\Psi(x, y) = -$ iff $d(x)$ is negative,

3. $S_\Psi(x, y) = \bot$ iff $d(x)$ is neither positive nor negative.

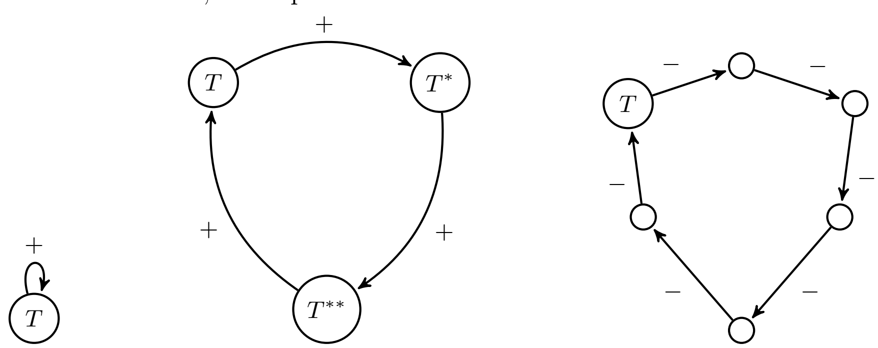
We call $(V(H), \{((a, S_\Psi(a))) \mid a \in A(H))\})$ the *signed dependency graph* of the constrained Boolean network (H, Ψ).

Let us illustrate this definition with two examples.

Example 3.6.4. The following digraphs are signed dependency graphs of the liar sentence, its loop-cleansed form and its F-normal form.



Example 3.6.5. The following digraphs are signed dependency graphs of the truth-teller sentence, its loop-cleansed form and its F-normal form.



There is, of course, a deeper reason behind the fact that the characteristic digraph of the liar is an odd cycle while that of the truth-teller is an even cycle - and it can be understood in terms of arc-signatures. Recall Definition 2.7.18.

Theorem 3.6.6. *Let (H, Ψ) be a constrained Boolean network and $(\mathcal{S}, d)$ the standard representation of (H, Ψ) in DNF. Let $\mathcal{I} = \mathcal{F}_d$ (cf. Definition 3.5.22) be the regular inflation of H onto the characteristic digraph G of (H, Ψ) as in in the proof of Theorem 3.2.6. Then or all $(x, y) \in A(H)$ the following claims hold.*

1. *$S_\Psi(x, y) = +$ iff every path $P \subseteq G$ that leads from x to y and is such that $(V(P) \setminus \{y\}) \subseteq \mathcal{I}(x)$ is even,*
2. *$S_\Psi(x, y) = -$ iff every path $P \subseteq G$ that leads from x to y and is such that $(V(P) \setminus \{y\}) \subseteq \mathcal{I}(x)$ is odd,*

3. *$S_\Psi(x, y) = \perp$ iff there exists an even $P_0 \subseteq G$ that leads from x to y and is such that $(V(P_0) \setminus \{y\}) \subseteq \mathcal{I}(x)$ and there exists an odd $P_1 \subseteq G$ that leads from x to y and is such that $(V(P_1) \setminus \{y\}) \subseteq \mathcal{I}(x)$.*

Notice that for $(x, y) \in A(H)$, $y \notin \mathcal{I}(x)$. For this reason the path P from Definition 3.6.3 is always one arc longer than the respective path figuring in Definition 2.7.18, that lies completely inside of $\mathcal{I}(x)$. In particular, for all $(x, y) \in A(H)$, $S_\Psi(x, y) = -$, if $\mathcal{I}$ is the trivial inflation. More generally, the following holds.

Corollary 3.6.7. *Let (H, Ψ) be a constrained Boolean network and $\mathcal{I}$ the regular inflation of H onto the characteristic digraph G of (H, Ψ) as in the proof of Theorem 3.2.6. Then*

1. *$\mathcal{I}$ is odd iff for all $(x, y) \in A(H)$, $S_\Psi(x, y) = +$,*
2. *$\mathcal{I}$ is even for all $(x, y) \in A(H)$, $S_\Psi(x, y) = -$.*

Now we can derive Corollary 5.11 of [5].

Corollary 3.6.8. *Every positive Boolean network has a fixed point.*

Proof. By Theorem 3.1.5. □

Example 3.6.9. Let us conclude this chapter by revisiting Example 3.5.24. The given Boolean network is neither positive nor negative: $d(\beta)$ is a positive formula while all $d(\alpha_i)$ are negative. Consequently, the inflation of the loop-cleansed form onto the characteristic digraph is neither even nor odd. Every positive arc is expanded to a path of length 2 (corresponding to an odd inflation of a vertex), while the negative arcs (leading from α_i to α_i^1) are not expanded at all ((corresponding to the trivial and thus even inflation of a vertex).

Chapter 4

The parity of an inflation and strong kernel-perfectness

The main goal of Chapter 4 is to prove the converse of each of the conjectures (A), (C) and (D). This is done by proving Theorems 4.3.4, 4.3.7 and 4.3.10 respectively. An important tool for this end is the concept of *convergent inflation*, introduced in Section 4.1. For a more detailed outline of Chapter 4 the reader is referred to Subsection 1.5.3.

4.1 Convergent inflations

Recall the definition of an in-branching tree from Section 2.1.

Definition 4.1.1. An inflation $\mathcal{I}$ of H onto G is said to be *convergent* iff for all $x \in V(H)$,

1. $G[\mathcal{I}(x)]$ is an in-branching tree,
2. $\partial_G^+ \mathcal{I}(x) \subseteq \mathrm{snk}(G[\mathcal{I}(x)])$.

Convention. Let $\mathcal{I}$ be an inflation of H onto G and $x \in V(H)$. For better readability we write also $\mathcal{I}[x]$ for $G[\mathcal{I}(x)]$.

Notice that this convention is, if not entirely in the spirit of, then at least not in conflict with Definition 2.7.17.

Proposition 4.1.2. *Let $\mathcal{I}$ be a convergent inflation of H onto G. Then $\mathcal{I}$ is*

1. even *iff for all $x \in V(H)$ and all $y \in \partial_G^- \mathcal{I}(x)$, $ht_{\mathcal{I}[x]}(y)$ is even,*
2. odd *iff for all $x \in V(H)$ and all $y \in \partial_G^- \mathcal{I}(x)$, $ht_{\mathcal{I}[x]}(y)$ is odd,*

where $ht_{\mathcal{I}[x]}(y)$ is the height of the vertex y in the in-branching tree $\mathcal{I}[x] = G[\mathcal{I}(x)]$.

The following definition is in full analogy to Definition 2.7.19.

Definition 4.1.3. For any digraph H, a digraph G is said to be

1. an *even* $\mathcal{I}_{fc}[H]$ iff there exists an odd finitary convergent inflation of H onto G,
2. an *odd* $\mathcal{I}_{fc}[H]$ iff there exists an even finitary convergent inflation of H onto G.

Lemma 4.1.4. *Let G be an $\mathcal{I}_{fc}[H]$. Then there exists a subdivision G' of G that is an even (odd) $\mathcal{I}_{fc}[H]$.*

Lemma 4.1.5. *Let $\mathcal{I}$ be an even convergent inflation of H onto G and let K be a kernel of G. Then for all $x \in V(H)$, all $y \in \partial^+_G \mathcal{I}(x)$ and all $z \in \partial^-_G \mathcal{I}(x)$, $y \in K$ iff $z \in K$.*

Proof. Let $x \in V(H)$ and $T = \mathcal{I}[x]$. Then for all $v \in V(T) \setminus \mathrm{snk}(T)$, $\mathrm{out}_G(v) \subseteq \mathrm{out}_T(v)$ and there exists a unique $v' \in V(T)$, such that $v' \in \mathrm{out}_T(v)$. Hence

$(*)$ $v \in K$ iff $v' \notin K$, for all $v \in V(T) \setminus \mathrm{snk}(T)$.

Let $y \in \partial^+_G \mathcal{I}(x)$ and $z \in \partial^-_G \mathcal{I}(x)$. Then y is the unique sink of T and $ht_T(z)$ is even by Proposition 4.1.2. Hence the unique path P in T from z to y has even length. We can assume that $y \neq z$, i.e., z is not the sink of T. Then the claim follows from $(*)$, by which the vertices of P are alternately in K and not in K or vice versa. □

4.2 Construction of convergent inflations of the Yablo-graph

Definition 4.2.1. Let G_0, G_1 and G_2 be digraphs, $\mathcal{I}$ an inflation of G_0 onto G_1 and $\mathcal{J}$ an inflation of G_1 onto G_2. Let $\mathcal{J} \circ \mathcal{I} : V(G_0) \to \wp(V(G_2))$ be defined by $(\mathcal{J} \circ \mathcal{I})(x) = \bigcup\{\mathcal{J}(y) \mid y \in \mathcal{I}(x)\}$.

Proposition 4.2.2. *Let $\mathcal{I}$ be an inflation of G_0 onto G_1 and $\mathcal{J}$ an inflation of G_1 onto G_2. Then*

1. *$\mathcal{J} \circ \mathcal{I}$ is an inflation of G_0 onto G_2,*
2. *if $\mathcal{I}$ and $\mathcal{J}$ are finitary, then $\mathcal{J} \circ \mathcal{I}$ is finitary,*
3. *if both $\mathcal{I}$ and $\mathcal{J}$ are determined, then $\mathcal{J} \circ \mathcal{I}$ is determined,*

Proposition 4.2.3. *Let $\mathcal{I}$ be a determined inflation of G_0 onto G_1 and $\mathcal{J}$ a determined inflation of G_1 onto G_2. Then*

1. *if $\mathcal{J} \circ \mathcal{I}$ is even, iff $\mathcal{I}$ and $\mathcal{J}$ have the same parity,*
2. *$\mathcal{J} \circ \mathcal{I}$ is odd iff $\mathcal{I}$ and $\mathcal{J}$ have different parities.*

We write $H \preceq_{fc} G$ ($H \preceq_{fsc} G$) iff there exists some $G' \subseteq G$ such that G' is a finitary and convergent inflation of H.

Proposition 4.2.4. *If $D \preceq_{fc} G$ and $G \preceq_{fc} H$, then $D \preceq_{fc} H$.*

Proof. Let $\mathcal{I}_0$ be a finitary and convergent inflation of D onto $G' \subseteq G$ and $\mathcal{I}_1$ be a finitary and convergent inflation of G onto $H' \subseteq H$. For all $x \in V(D)$ let $\mathcal{I}$ be defined by $\mathcal{I}(x) = \bigcup\{\mathcal{I}_1(y) \mid y \in \mathcal{I}_0(x)\}$. By Proposition 4.2.2, $\{\mathcal{I}(x) \mid x \in V(D)\}$ is inflation of D onto $H' \subseteq H$. However, $H'_x = H'[\mathcal{I}(x)]$ might not be an in-branching tree, for some $x \in V(D)$. On the other hand, clearly $\partial^+_{H'}\mathcal{I}(x) \subseteq \text{snk}(H'_x)$, for all $x \in V(D)$. The problem can be fixed by choosing for each $x \in V(D)$ an in-branching spanning tree T_x of H'_x and considering the digraph H^* with $V(H^*) = V(H')$ such that $(u, v) \in A(H^*)$ iff there exists $x \in V(D)$ such that $(u, v) \in A(T_x)$ or there exists $y \neq z \in V(D)$ such that $u \in \mathcal{I}(y)$ and $v \in \mathcal{I}(z)$ and $(u, v) \in A(H)$. Then $H^* \subseteq H'$ is spanning, and $\mathcal{I}$ is an inflation of D onto H^*, with $\mathcal{I}[x] = T_x$. Since $\partial^+_{H'}\mathcal{I}(x) = \partial^+_{H^*}\mathcal{I}(x)$ and $\partial^+(H'_x) = \text{snk}(T_x)$, it follows that $\mathcal{I}$ is a finitary convergent inflation of D onto $H^* \subseteq H$. □

Proposition 4.2.5. *For all $X \subseteq \omega$, $\mathbb{Y}[X]$ is isomorphic to $\mathbb{Y}$ iff X is infinite.*

Proof. Suppose that X is infinite. Let $(x_n)_{n\in\omega}$ be the order-preserving enumeration of X. Then $n \mapsto x_n$ is a digraph isomorphism from $\mathbb{Y}$ to $\mathbb{Y}[X]$. The other direction is trivial. □

Definition 4.2.6. A spanning subdigraph $G \subseteq \mathbb{Y}$ is said to be *Yablo-like* iff $n + 1 \in out_G(n)$ and there are infinitely many $n \in \omega$ such that $\text{out}_G(n)$ is infinite.

Definition 4.2.7. Let G be a Yablo-like digraph. We say that G is *evenly spaced* iff for all $n \in \omega$, all $k \in \text{out}_G(n) \setminus \{n + 1\}$ are even, and that G is *oddly spaced* iff for all $n \in \omega$, all $k \in \text{out}_G(n) \setminus \{n + 1\}$ are odd.

Example 4.2.8. The digraph $\mathbb{Y}^{\rightarrow}$ from Example 2.7.7 is isomorphic to an evenly spaced Yablo-like digraph.

Lemma 4.2.9. *Every Yablo-like digraph contains*

1. *a subdigraph that is isomorphic to a Yablo-like digraph and is evenly spaced and*
2. *a subdigraph that is isomorphic to a Yablo-like digraph and is oddly spaced.*

Proof. Let G be a Yablo-like digraph. Let $n \in \omega$ be such that $\text{out}_G(n)$ is infinite. Then $\text{out}_G(n)$ has an infinite subset of even numbers or an infinite subset of odd numbers. Moreover, G has infinitely many even vertices with infinitely many out-neighbors or infinitely many odd vertices with infinitely many out-neighbors. Combining both observations yields an infinite set $X \subseteq \omega$ such that either every $n \in X$ has infinitely many even out-neighbors, or every $n \in X$ has infinitely many odd out-neighbors. Thinning out G accordingly (by deleting arcs) yields a Yablo-like subdigraph H of G that is evenly spaced or oddly spaced.

Let $H' = H[\{n \in \omega \mid n > 0\}]$. Then H' is isomorphic to the Yablo-like digraph H^* that arises from H' by renumbering every vertex of H' according to the rule $n \mapsto n-1$. Hence H is oddly spaced iff H^* is evenly spaced. This proves both claims (1) ans (2). □

Lemma 4.2.10. *Let G be isomorphic to a Yablo-like digraph. Then*

1. *there exists a subdigraph of G that is an $\mathcal{I}_{fc}[\mathbb{Y}]$,*
2. *if G is evenly spaced, there exists a subdigraph of G that is an odd $\mathcal{I}_{fc}[\mathbb{Y}]$,*
3. *if G is oddly spaced, there exists a subdigraph of G that is an even $\mathcal{I}_{fc}[\mathbb{Y}]$.*

Proof. We can assume that G is a spanning subdigraph of $\mathbb{Y}$. We shall define recursively a sequence $(k_n)_{n\in\omega}$ of natural numbers and a sequence of spanning subdigraphs $(G_n)_{n\in\omega}$ of G such that

i. $G_0 = G$ and for all $n \in \omega$,
   (a) if $n > 0$ then G_n is a spanning subdigraph of G_{n-1},
   (b) $\text{out}_{G_n}(l) = \text{out}_G(l)$, for all $l \geq k_n$,
   (c) $\text{out}_{G_n}(i) = \text{out}_{G_m}(i)$ and $\text{in}\, G_n(i) = \text{in}_{G_m}(i)$, for all $m \leq n$ and $i \leq k_m$,

ii. $k_{n-1} < k_n$ and $k_n - k_{n-1}$ is odd, for all $n > 0$,

iii. for all $n > 0$ and all $m < n$ there exists some $k_{n-1} < i_m \leq k_n$ such that $(k_m, i_m) \in A(G_n)$,

iv. $\text{out}_{G_n}(k_n)$ is infinite, for all $n \in \omega$,

v. for all $n \in \omega$ and all $i < k_n$, if $i \neq k_l$ for all $l \in \omega$, then $\text{out}_{G_m}(i) = \{i+1\}$.

- Let $G_0 = G$ and $k_0 = 0$.
- Suppose $n > 0$.
  - Let k_n be the least $l > k_{n-1}$ that has the same parity as n and is such that for all $m \leq n-1$ there is some $k_{n-1} < i_m \leq l$ such that $(k_m, i_m) \in A(G_{n-1})$.
    (Such a number exists, since by induction hypotheses (iv), for every $m \leq n$, k_m has infinitely many out-neighbors in G_n, and because of induction hypothesis (i), $\text{out}_{G_n}(l)$ is infinite for all $l \geq k_n$.)
  - Let $V(G_{n+1}) = V(G_n)$ and $A(G_{n+1}) = A(G_n) \setminus \{(x,y) \in A(G_n) \mid k_n < x < k_{n+1} \wedge y > x+1\}$.

- Let $G_\omega \bigcap_{n\in\omega} G_n$.

We prove the claims (i)-(v) simultaneously by induction on n. For $n = 0$, they are all trivial or clear.

Let $n > 0$. All claims (i)(a), (b) and (c) hold, because by definition of G_n only arcs between k_{n-1} and k_n are affected by the removal that takes place when passing form G_{n-1} to G_n. Claims (ii) and (iii) follow immediately from the definition of k_n (for (ii) note that the k_n's have alternating parities). Claim (iv) follows from the fact that G is Yablo-like and from (i)(a). Claim (v) follows from the definition of G_n.

Notice that (v) implies that for all n, $G_\omega[\{i \mid k_n < i \leq k_{n+1}\}]$ is a path. This fact together with (iii) for all $n \in \omega$ and all $m > n$, k_m is reachable from k_n in G_ω. Furthermore $k_1 = 1$. This means we can define an infinite walk in G_ω as follows. We start at 2 and follow the path that leads to k_2 Then we take the step from k_2 to the least $m \in \mathrm{out}_{G_\omega}(k_2)$ such that $k_3 < m \leq k_4$. Then we walk from m to k_4, from where we proceed in an analogous manner as from k_2. This procedure yields a ray $R \subseteq G_\omega$, such that for all $n > 0$, $k_n \in V(R)$ iff n is even.

Let $H = G_\omega[V(R)]$. For all $n > 0$, let $\mathcal{I}(n) = \{i \in V(R) \mid k_{2n+1} < i \leq k_{2n+2}\}$. We show that $\mathcal{I}$ is a finitary convergent inflation of $\mathbb{Y}$ onto H. First, observe that $\mathcal{I}$ is a partition of $V(R)$ - in particular $\mathcal{I}(0) = \{2, \ldots, k_2\}$. By construction of H and definition of $\mathcal{I}$, for all $n \in \omega$, $H[\mathcal{I}(n)]$ is a path which terminates in k_{2n+2}. By (v), $\partial^+_H \mathcal{I}(n) = \{k_{2n+2}\}$. Hence $\mathcal{I}$ is a finitary convergent inflation (Cf. Definition 4.1.1), provided that it is an inflation at all.

This observation clearly implies that for all $n \in \omega$ and all $i, j \in \mathcal{I}(n)$, if $i \in \partial^-_H(\mathcal{I}(n))$ and $j \in \partial^+_G(\mathcal{I}(n))$, then there is a path in $G[\mathcal{I}(n)]$ from i to j. Hence 2.7.10(2) is satisfied. Moreover,2.7.10(3) is clearly satisfied.

Next, let us show that 2.7.10(1) is satisfied, i.e., that for all $m \neq n \in \omega$, $m < n$ iff there is some $(v_m, v_n) \in A(H)$ with $v_m \in \mathcal{I}(m)$ and $v_n \in \mathcal{I}(n)$. Since $H \subseteq \mathbb{Y}$, the direction from left to right holds. The other direction follows from (iii). Hence $\mathcal{I}$ is an inflation of $\mathbb{Y}$ onto H.

It remains to be shown that $\mathcal{I}$ is an even inflation, provided that G is evenly spaced. Let $i \in \partial^-_H(\mathcal{I}(n))$. Since k_{2n+2} is even, every (i.e., the only) path from i to k_{2n+2} is even iff i is even. But $i \in \mathrm{out}_H(k_{2n})$, because $\mathcal{I}$ is an inflation of $\mathbb{Y}$ onto H. On the other hand, $i \neq k_{2n} + 1 \notin V(H)$. Since k_{2n} is even this implies that i is even, provided that G is evenly spaced. Hence the only path from i to k_{2n+2} is even. Hence $\mathcal{I}$ is an even finitary convergent inflation (Cf. Definition 4.1.3). Hence $\mathcal{I}[\mathbb{Y}]$ is an odd convergent inflation of $\mathbb{Y}$. If G is oddly spaced, we show that $\mathcal{I}$ is an odd inflation completely analogously. This proves all claims of the lemma. $\square$

Theorem 4.2.11. *Let G be a digraph. Then*

1. *G contains an $\mathcal{I}_f[\mathbb{Y}]$ iff G contains an $\mathcal{I}_{fc}[\mathbb{Y}]$,*

2. *G contains an $\mathcal{I}_f[\mathbb{Y}]$ of parity p iff G contains an $\mathcal{I}_{fc}[\mathbb{Y}]$ of parity p.*

Proof. Recall that $V(\mathbb{Y}) = \omega$. Let $\mathcal{I} = \{\mathcal{I}(n) \mid n \in \omega\}$ be a finitary inflation of $\mathbb{Y}$ onto $H \subseteq G$. For better readability let $V_n = \mathcal{I}(n)$. Since $\mathcal{I}$ is finitary, for all $n \in \omega$ there exists $x_n \in V_n$ such that $d^+_H(x_n) = \omega$ and $x_n \in \partial^+_H(V_n)$. Let $T_n \subseteq H[V_n]$ be an in-branching spanning tree of the subdigraph of $H[V_n]$ that consists of the union of all paths in $H[V_n]$ from some $y \in \partial^-_H(V_n)$ to x_n such that $\partial^+(T_n) = \{x_n\}$. Observe that this subdigraph is non-empty because of Definition 2.7.10.(2). Notice that if $|\mathcal{I}(n)| = 1$, then T_n is simply the point $(x_n, \varnothing)$.

Let H_0 be the subdigraph of H induced by $\bigcup_{n\in\omega} V(T_n)$. Let H_1 be the spanning subdigraph of H_0 that arises from H_0 by deleting all the arcs $a \in A(H_0)$ such that there is no $n \in \omega$ with $a \in A(T_n)$ or $tail(a) = x_n$.

Let $R \subseteq H_1$ be an ray such that for all $n \in \omega$, if $x_n \in V(R)$ then $(x_n, y) \in A(R)$, where $y \in \mathrm{out}_{H_1}(x_n) \cap V(T_k)$ for some $k \in \omega$ such that for all $l \in \omega$ and all $z \in \mathrm{out}_{H_1}(x_n)$, if $z \in V(T_l)$ then $k \leq l$. Let $(k_n)_{n\in\omega}$ be the sequence of natural numbers such that x_{k_n} is an enumeration (preserving the natural order of R) of all vertices of R that have infinite out-degree in H. Let H_2 be the subdigraph of H_1 induced by $\bigcup_{n\in\omega} V(T_{k_n})$.

For all $n \in \omega$, let $\mathcal{J}(n) = V(T_{k_n})$. Then $\mathcal{J}$ is a partition of H_2. Let $Y = \mathcal{J}^{-1}[H_2]$. Then Y is isomorphic to a Yablo-like digraph by construction and $\mathcal{J}$ is a finitary convergent inflation of Y onto $H_2 \subseteq G$. Hence $Y \preceq_{fc} G$.

By Lemma 4.2.10(1) we obtain $\mathbb{Y} \preceq_{fc} Y$ and thus $\mathbb{Y} \preceq_{fc} G$ by Proposition 4.2.4. This proves the first claim. (The converse direction is trivial.)

When it comes to the second claim, notice that $\mathcal{J}$ is an inflation of parity $\bar{p}$, where $\bar{p}$ is even if p is odd and vice versa. In any case, let $Y' \subseteq Y$ be an evenly spaced Yablo-like digraph by Lemma4.2.9. By Lemma 4.2.10(2) we obtain an even inflation $\mathcal{J}'$ of $\mathbb{Y}$ onto some $Y^* \subseteq Y' \subseteq Y$. Then $\mathcal{J} \circ \mathcal{J}'$ is an inflation of $\mathbb{Y}$ onto some subdigraph of H_2. By Proposition 4.2.3, $\mathcal{J} \circ \mathcal{J}'$ is an inflation of parity $\bar{p}$, Hence G contains an $\mathcal{I}_{fc}[\mathbb{Y}]$ of parity p. The converse direction is, again, trivial. □

Lemma 4.2.12. *If G is an $\mathcal{I}_c[\mathbb{Y}]$, then G has a subdigraph that is an even $\mathcal{I}_c[\mathbb{Y}]$ or it has a subdigraph that is an odd $\mathcal{I}_c[\mathbb{Y}]$.*

Proof. Let $\mathcal{I}$ be a convergent inflation of $\mathbb{Y}$ onto G. We say that $(x, y) \in A(G)$ is an *inner arc* of G (wiht respect to $\mathcal{I}$) iff $\mathcal{I}^{-1}(x) = \mathcal{I}^{-1}(y)$, and an *outer arc* of G (wiht respect to $\mathcal{I}$) iff $\mathcal{I}^{-1}(x) \neq \mathcal{I}^{-1}(y)$. Let $H \subseteq G$ be spanning such that for all outer arcs $(v, w) \neq (x, y) \in A(H)$, $\mathcal{I}^{-1}(v) = \mathcal{I}^{-1}(x)$ implies $\mathcal{I}^{-1}(w) \neq \mathcal{I}^{-1}(y)$. (Such an H can be constructed by choosing one arcs from each set of outer arcs of G that is contracted by $\mathcal{I}^{-1}$ onto the same arc of $\mathbb{Y}$). Then $\mathcal{I}$ is a convergent inflation of $\mathbb{Y}$ onto H.

An outer arc $(x, y) \in A(H)$ is said to be *even* iff $ht_{T_y}(y)$ is odd and *odd* iff $ht_{T_y}(y)$ is even, where $T_y = \mathcal{I}[\mathcal{I}^{-1}(y)]$. Let E be the set of all even outer arcs of H and O the set of all odd outer arcs of H. Let $E' = \{(n, m) \mid n = \mathcal{I}^{-1}(x) \wedge m = \mathcal{I}^{-1}(y), (x, y) \in E\}$, and $O' = \{(n, m) \mid n = \mathcal{I}^{-1}(x) \wedge m = \mathcal{I}^{-1}(y), (x, y) \in O\}$. Then $\{E', O'\}$ is a partition of $A(\mathbb{Y})$. Hence it induces a partition $\{E'_U, O'_U\}$ of $E(K^\omega)$, because K^ω is the underlying graph of $\mathbb{Y}$. By Ramsey's theorem, there exists an infinite $M \subseteq \omega$, such that $E(K^\omega[M]) \subseteq E'_U$ or $E(K^\omega[M]) \subseteq O'_U$. Hence

$(*) A(\mathbb{Y}[M]) \subseteq E'$, or $A(\mathbb{Y}[M]) \subseteq O'$.

Since $\mathbb{Y}[M]$ is isomorphic to $\mathbb{Y}$ by Proposition 4.2.5, $\mathcal{I}[\mathbb{Y}[M]]$ is an $\mathcal{I}_c[\mathbb{Y}]$. Because of $(*)$, $\mathcal{I}[\mathbb{Y}[M]]$ is an even $\mathcal{I}_c[\mathbb{Y}]$ or an odd $\mathcal{I}_c[\mathbb{Y}]$. □

Theorem 4.2.13. *Every $\mathcal{I}_f[\mathbb{Y}]$ contains an even $\mathcal{I}_f[\mathbb{Y}]$ or an odd $\mathcal{I}_f[\mathbb{Y}]$.*

4.3 Necessary conditions for the characterization problems

Theorem 4.3.1. *No odd $\mathcal{I}_c[\mathbb{Y}]$ has a kernel.*

Proof. Let G be a digraph and $\mathcal{I}$ be an even convergent inflation from $\mathbb{Y}$ onto G. Assume that K is a kernel of G. Let $\partial^+_G(\mathcal{I}) = \bigcup_{n\in\omega} \partial^+_G \mathcal{I}(n)$. Then $K \cap \partial^+_G(\mathcal{I})$ has at most one element: Assume that $x \neq y \in K \cap \partial^+_G(\mathcal{I})$. Then $x \in \partial^+_G \mathcal{I}(n)$ and $y \in \partial^+_G \mathcal{I}(m)$ for $n < m$ (since $\mathcal{I}$ is convergent). Then there exists an arc in G from x to some $y' \in \partial^-_G \mathcal{I}(m)$. Hence $y' \notin K$. But then Lemma 4.1.5 implies that $y \notin K$, which is a contradiction.

Now assume that there exists $x \in \partial^+_G(\mathcal{I}) \cap K$. Let $y \in \mathrm{out}_G(x)$. Then $y \notin K$ and $y \in \partial^-_G \mathcal{I}(n)$ for some $n \in \omega$. Hence, by Lemma 4.1.5 , the unique $y' \in \partial^+_G \mathcal{I}(n)$ is not in K either. Hence there exists $z \in \mathrm{out}_G(y')$ such that $z \in K \cap \partial^-_G \mathcal{I}(m)$ for some $m > n$. Thus, again by Lemma 4.1.5, the unique $z' \in \partial^+_G \mathcal{I}(m)$ is an element of K, which is a contradiction because we have already shown that there can br at most one such vertex.

Hence $\partial^+_G(\mathcal{I}) \cap K = \varnothing$. But this implies by Lemma 4.1.5 that $\partial^-_G(\mathcal{I}) \cap K = \varnothing$, where $\partial^-_G(\mathcal{I}) = \bigcup_{n\in\omega} \partial^-_G \mathcal{I}(n)$. This is impossible, because for $x \in \partial^+_G(\mathcal{I})$ we have $\mathrm{out}_G(x) \subseteq \partial^-_G(\mathcal{I})$. □

Corollary 4.3.2. *If $\mathbb{Y} \preceq_f G$, then there exists some $H \subseteq G$ such that some subdivision of H has no kernel.*

Proof. By Theorem4.2.11 $H = \mathcal{I}_{fc}[\mathbb{Y}] \subseteq G$. By Lemma 4.1.4 there exists a subdivision H' of H such that H' is an odd $\mathcal{I}_{fc}[\mathbb{Y}]$. Then by Theorem 4.3.1 H' has no kernel. □

Corollary 4.3.3. *If G contains a cycle or an $\mathcal{I}_f[\mathbb{Y}]$, then there is a subdivision of G that is not strongly kernel-perfect.*

Corollary 4.3.4. *If a digraph contains a cycle or an $\mathcal{I}_f[\mathbb{Y}]$, then it is dangerous.*

Proof. Let G be a digraph. If G contains a cycle, then it is dangerous by Corollary 2.6.5. So, suppose $\mathbb{Y} \preceq_f G$. Let G' be a subdivision of G that is not strongly kernel-perfect by Corollary 4.3.3. Then, by Proposition 2.6.29, G' is dangerous. Hence G is dangerous by Proposition 2.7.3. □

Recall Conjecture (B)2.8.8.

Proposition 4.3.5. *Conjecture (B) implies that a digraph is safe iff every subdivision of it is strongly kernel-perfect.*

Proof. If some subdivision G' of a digraph G is not strongly kernel-perfect, then G' is dangerous. Hence G is dangerous by Proposition 2.7.3. The other direction follows by Conjecture (B). □

Proposition 4.3.6. *Conjecture (A) implies Conjecture (B).*

Proof. Let G be a digraph. Suppose that G is dangerous. Then, by Conjecture 2.8.1, G contains a cycle or $\mathbb{Y} \preceq_f G$. Hence, by Corollary 4.3.3 there exists a subdivision of G and that is not strongly kernel-perfect. □

Theorem 4.3.7. *A digraph that contains an odd cycle or an odd $\mathcal{I}_f[\mathbb{Y}]$ is not strongly kernel-perfect.*

Recall the discussion of digraphs of (in)finite character from Subsection 2.6.2. Moreover, recall Conjecture (D) (2.8.4), which states that if a digraph contains no $\mathcal{I}_f[\mathbb{Y}]$, then it is of finite character. We shall now prove the converse statement.

Proposition 4.3.8. *Every acyclic $\mathcal{I}_f[\mathbb{Y}]$ is of infinite character.*

Proof. By Corollary 2.6.21 and Corollary 4.3.4. □

Proposition 4.3.9. *Every $\mathcal{I}_f[\mathbb{Y}]$ has an acyclic subdigraph that is an $\mathcal{I}_f[\mathbb{Y}]$.*

Proof. By Theorem 4.2.11. □

Theorem 4.3.10. *A digraph that contains an $\mathcal{I}_f[\mathbb{Y}]$ is of infinite character.*

Proof. By Proposition 4.3.9, Proposition 4.3.8 and Proposition 2.6.24. □

4.4 Finitary inflations of the Yablo-graph

Definition 4.4.1. Let G be a digraph and R a ray in G.

1. Let $x \in V(G)$. An *x-R-fan* is an infinite set $\mathcal{F}_x$ of x-R paths such that any two distinct members of $\mathcal{F}_x$ have exactly the vertex x in common. The set $\mathcal{F}_x$ is called an *x-R-fan in G* iff $P \subseteq G$ for all $P \in \mathcal{F}_x$.

2. A vertex $x \in V(G)$ is said to be a *focal point* of R in G iff exists x-R-fan in G.

Theorem 4.4.2. *For all digraphs G the following are equivalent:*

1. $\mathbb{Y} \preceq_f G$,

2. *there exists some ray $R \subseteq G$ such that infinitely many vertices of R are focal points of R in G.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (1): It is not difficult to see that G contains an finitary inflation of a Yablo-like digraph (cf. Definition 4.2.6). The claim follows then by Lemma 4.2.10. $\square$

Hence for all digraphs G, G contains a ray R such that there are infinitely many vertices in $V(R)$ dominating R in G (in the sense of Section 3.4) if and only if $\mathbb{Y} \preceq_f G$.

Chapter 5

Safe digraphs

One goal of this final chapter is to show that Conjecture (A) holds under weaker assumptions than already established in Chapter 3. Theorem 5.4.6 states that any directed graph G is safe if and only if G is acyclic and contains no finitary inflation of the Yablo-graph, given that G contains only *normal* ends and at most countably many ends. (One direction of this equivalence has already been established in Chapter 4).

In order to do this, a method is developed in Subsection 5.2 that seems to be more suitable for Conjecture (A) than Walicki's method from[47]. It takes into account the fact that for our purpose it suffices to focus on acyclic digraphs, and not on the larger class of digraphs without odd cycles. For a more detailed outline of Chapter 5 the reader is referred to Subsection 1.5.4.

5.1 Partitions of function networks

A key method that shall be developed and used throughout this chapter is that of decomposing a Boolean network into subnetworks (cf. Subsection 2.4.6), finding a fixed point for each of them and the integrating these fixed points into a fixed point of the entire Boolean network. A very rudimentary precursor of this method is Proposition 2.4.47, a more sophisticated one the decomposition into periphery and core (cf. Definition 2.4.89). We will primarily be concerned with Boolean networks – nevertheless, some results are formulated more generally for function networks.

5.1.1 Open exhaustions

For the following recall Definition 2.4.39.

Definition 5.1.1. Let G be a digraph, ξ an ordinal and $(X_\alpha)_{\alpha<\xi}$ a sequence of subsets of $V(G)$. We say that $(X_\alpha)_{\alpha<\xi}$ is an *open exhaustion* of G iff it satisfies the following conditions.

1. X_α is open in G, for all $\alpha < \xi$.
2. $X_\alpha \subsetneq X_\beta$, for all $\alpha < \beta < \xi$,
3. $X_\xi = V(G)$,
4. $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$, for all limits $\lambda \leq \xi$.

Definition 5.1.2. Let (G, Φ) be a constrained function network, ξ an ordinal and $(X_\alpha)_{\alpha<\xi}$ an open exhaustion of G. We say that Φ is *absolutely solvable relative to* $(X_\alpha)_{\alpha<\xi}$ iff

1. $\Phi[X_0]$ has a fixed point,
2. Φ is absolutely solvable relative to $X_{\alpha+1} \setminus X_\alpha$ for all $\alpha < \xi$.

Theorem 5.1.3. *Let (G, Φ) be a function network, ξ an ordinal and $(X_\alpha)_{\alpha<\xi}$ an open exhaustion of G such that Φ is absolutely solvable relative to $(X_\alpha)_{\alpha<\xi}$. Then Φ has a fixed point.*

Proof. Let $X_\xi = \bigcup_{\alpha<\xi} X_\alpha$. We shall define recursively a sequence of functions $(f)_{\alpha\leq\xi}$ with $f_\alpha : X_\alpha \to \{0, 1\}$ such that for all $\alpha \leq \xi$ the following holds.

1. f_α is a fixed point of $\Phi[X_\alpha]$,
2. $f_\beta \subseteq f_\alpha$, for all $\beta \leq \alpha$.

- Let $\alpha = 0$. Then Φ is absolutely solvable relative to X_0 by hypothesis. Moreover $\mathrm{Bd}^+_G(X_0) = \varnothing$, since X_0 is open in G. Hence there exists some $f_0 \in \{0, 1\}^{X_0}$ that is a fixed point of $\Phi[X_0]$.
- For the successor step, we use the fact that Φ is absolutely solvable relative to $X_{\alpha+1} \setminus X_\alpha$, which implies that $\Phi^{f_\alpha}[X_{\alpha+1} \setminus X_\alpha]$ has a fixed point $f'_{\alpha+1}$. Then by induction hypothesis and Lemma 2.4.48 $f_{\alpha+1} = f_\alpha \cup f'_{\alpha+1}$ is a fixed point of $\Phi[X_{\alpha+1}]$. Moreover, $f_\alpha \subseteq f_{\alpha+1}$. Hence $f_\beta \subseteq f_{\alpha+1}$, for all $\beta < \alpha + 1$.
- If α is a limit, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then (2) is satisfied. In order to prove (1), let $x \in X_\alpha$. Then there exists some $\beta < \alpha$ such that $x \in X_\beta$. By induction hypothesis, f_β is a fixed point of $\Phi[X_\beta]$. Hence $\Phi[X_\beta](f_\beta, x) = f_\beta(x)$. Since X_β is open in G, X_β is also open in $G[X_\alpha]$ (because $X_\beta \subseteq X_\alpha$). Hence $\mathrm{out}_{G[X_\alpha]}(x) \subseteq X_\beta$. Together with $f_\beta \subseteq f_\alpha$, this implies that $\Phi[X_\alpha](f_\alpha, x) = f_\alpha(x)$. Hence f_α is a fixed point of $\Phi[X_\alpha]$.

From (2) follows that f_ξ is a fixed point of Φ. □

Corollary 5.1.4. *A digraph G is safe iff $G\{x\}$ is safe, for all $x \in V(G)$.*

Proof. For the non-trivial direction let $(x_\alpha)_{\alpha<\xi}$ be an enumeration of the vertices of G. For all $\alpha < \xi$, let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta \cup V(G\{x_\alpha\})$. Then $(X_\alpha)_{\alpha<\xi}$ is an open exhaustion of G. Since $G\{x_\alpha\}$ is safe for all $\alpha < \xi$, it follows that every Boolean network on G is absolutely solvable relative to $(X_\alpha)_{\alpha<\xi}$. Hence Theorem 5.1.3 implies that G is safe. □

Corollary 5.1.5. *A digraph G is of finite character iff $G\{x\}$ is of finite character, for all $x \in V(G)$.*

Corollary 5.1.6. *Every finitely out-branching digraph is of finite character.*

Proof. By Proposition 2.6.23. □

As a further corollary we obtain a result from [41] (Corollary 23).

Corollary 5.1.7. *If a digraph G is acyclic and has only finitely many vertices with infinite out-degree, then G is safe.*

5.1.2 Well-founded partitions

Definition 5.1.8. Let G be a digraph and $\mathcal{P}$ a partition of $V(G)$. Then $\mathcal{P}$ is said to a *well-founded partition of* G iff for all infinite walks $(x_n)_{n\in\omega}$ in G there exist $k \in \omega$ and $X \in \mathcal{P}$ such that $x_n \in X$, for all $n \geq k$.

In other words, a partition of G is well-founded iff every ray of G has a tail that is contained in some component of the partition and every cycle of G is contained in some component of the partition.

For the following theorem recall Definition 2.4.46.

Theorem 5.1.9. *Let (G, Φ) be a constrained function network, and $\mathcal{P}$ a well-founded partition of G. If Φ is perfectly solvable relative every $X \in \mathcal{P}$, then Φ has a fixed point.*

Proof. For all $x \in V(G)$ let $\pi(x)$ be the unique $X \in \mathcal{P}$ such that $x \in X$. We define recursively an open exhaustion $(X_\alpha)_{\alpha\leq\xi}$ of G such that Φ is absolutely solvable relative to $(X_\alpha)_{\alpha\leq\xi}$. Then the claim follows from Theorem 5.1.3.

- Let X_0 be the set of all $x \in V(G)$ such that $G\{x\} \subseteq \pi(x)$. Then $X_0 \neq \varnothing$. (Assume otherwise. Then for all $x \in V(G)$ there exists a path that leads from x to some $y \in V(G) \setminus \pi(x)$. Hence we can construct an infinite walk that finally leaves every $Y \in \mathcal{P}$ after having entered it - or having started within it. This contradicts the assumption that $\mathcal{P}$ is well-founded.)
- Let α be an ordinal such that X_α is already defined.
  - If $X_\alpha = V(G)$, then let $\xi = \alpha$ and stop the procedure.
  - We can assume that $X_\alpha \neq V(G)$. Then let $X'_{\alpha+1}$ be the set of all $x \in V(G) \setminus X_\alpha$ such that $G\{x\} \setminus X_\alpha \subseteq \pi(x)$. Then $X'_{\alpha+1} \neq \varnothing$ by the same argument as in the base case. Let $X_{\alpha+1} = X'_{\alpha+1} \cup X_\alpha$.
- Let λ be a limit such that X_α is already defined for all $\alpha < \lambda$. Then let $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$.

Note that the procedure must terminate: $V(G)$ is a set and not a proper class. We show that $(X_\alpha)_{\alpha\leq\xi}$ is an open exhaustion of G and that Φ is absolutely solvable relative to $(X_\alpha)_{\alpha\leq\xi}$. Clauses (2), (3) and (4) of Definition 5.1.1 are clearly satisfied.

Ad 5.1.1 (1): We show by induction on α that X_α is open in G for all $\alpha < \xi$. (The cases $\alpha = \xi$ is trivial.) Let $0 < \alpha < \xi$ and $x \in X_\alpha$. We have to show that $V(G\{x\}) \subseteq X_\alpha$. So let $y \in G\{x\}$. Then $G\{y\} \subseteq G\{x\}$. We have to show $y \in X_\alpha$ and consider the following three cases.

- $\alpha = 0$. Then $G\{x\} \subseteq \pi(x)$ by definition. Hence $G\{y\} \subseteq \pi(x)$. But $G\{y\} \subseteq G\{x\} \subseteq \pi(x)$ also implies that $\pi(x) = \pi(y)$. Hence $G\{y\} \subseteq \pi(y)$ and thus $y \in X_\alpha$.
- $\alpha = \beta + 1$ for some ordinal β. Because of the induction hypothesis we can assume that $x \in X_\alpha \setminus X_\beta$. By definition we have $G\{x\} \setminus X_\beta \subseteq \pi(x)$. If $y \in X_\beta$ we are done, because $X_\beta \subsetneq X_\alpha$. So we can assume $y \notin X_\beta$. Hence $y \in \pi(x)$, i.e., $\pi(y) = \pi(x)$. Moreover, $G\{y\} \setminus X_\beta \subseteq \pi(x) = \pi(y)$. Hence $y \in X_\alpha$.
- α is a limit ordinal. Because of Definition 5.1.1(4) there exists some $\beta < \alpha$ such that $x \in X_\beta$. Then the claims follows by induction hypothesis.

Hence $(X_\alpha)_{\alpha\leq\xi}$ is an open exhaustion of G It remains to be shown that (1) and (2) of Definition 5.1.2 are satisfied.

Ad (1): We have to show that $\Phi[X_0]$ has a fixed point. Let $\mathcal{P}_0 = \{Y \cap X_0 \mid Y \in \mathcal{P}\}$. Then $\mathcal{P}_0$ is a partition of X_0. Moreover, for every $Z \in \mathcal{P}_0$, Z is open in $G[X_0]$. (Assume otherwise: then there is $Y \neq Z \in \mathcal{P}$ such that there exists an arc $(v, w) \in A(G[X_0])$ with $v \in Z \cap X_0$ and $w \in Y \cap X_0$. Hence $G\{v\} \not\subseteq \pi(v)$ which leads to the contradiction that $v \notin X_0$.) Let $Y \in \mathcal{P}$. Since Φ is perfectly solvable relative to Y by hypothesis and Y is open in X_0, $\Phi[Y \cap X_0]$ has a fixed point. By applying Lemma 2.4.47, we can paste together the solutions of these open components of X_0. Hence $\Phi[X_0]$ has a fixed point.

Ad (2): Let $\alpha < \xi$. Let $X'_{\alpha+1} = X_{\alpha+1} \setminus X_\alpha$ We have to show that Φ is absolutely solvable relative to $X'_{\alpha+1}$. Let $\mathcal{P}_{\alpha+1} = \{Y \cap X'_{\alpha+1} \mid Y \in \mathcal{P}\}$. Then $\mathcal{P}_{\alpha+1}$ is a partition of $X'_{\alpha+1}$. Analogously to (1) one shows that each of its components is open in $G[X'_{\alpha+1}]$. We have to show that Φ is absolutely solvable relative to $X'_{\alpha+1}$. Let $\Sigma = \bigtimes_{x\in V(G)} S_x$ be the type of Φ. Let $Z = \mathrm{Bd}^+_G(X'_{\alpha+1})$ and $h \in \bigtimes_{x\in Z} S_x$. Let $Y \in \mathcal{P}$ such that $Y \cap X'_{\alpha+1} \neq \varnothing$. By hypothesis Φ is absolutely solvable relative to $Y \cap X'_{\alpha+1}$. Hence $\Phi^h[Y \cap X'_{\alpha+1}]$ has a fixed point. ($\Phi^h[Y \cap X'_{\alpha+1}]$ is indeed a function network by Proposition 2.4.45: This is so because $(Y \cap X_{\alpha+1}) = \varnothing$ and $\mathrm{Bd}^+_G(Y \cap X'_{\alpha+1}) \subseteq Z$.) Since each $Y \in \mathcal{P}$ is open in $G[X'_{\alpha+1}]$, Lemma 2.4.47 yields a fixed point of $\Phi^h[X'_{\alpha+1}]$. □

Corollary 5.1.10. *Let G be a digraph an $\mathcal{P}$ a well-founded partition of G. Then G is of finite character, if $G[X]$ is of finite character, for all $X \in \mathcal{P}$.*

Proof. Let Φ be a Boolean network on G. We have to show that Φ is compact, i.e, that either there exists some finite $\varnothing \neq Y \subseteq V(G)$ such that Φ is not absolutely solvable relative to Y or Φ has a fixed point. Suppose that Φ has no fixed point. Then Theorem 5.1.9 implies that there exists some $X \in \mathcal{P}$ such that Φ is not perfectly solvable relative to X. In other words, there exists $\varnothing \neq Z \subseteq X$ such that Φ is not absolutely solvable relative to Z. Hence there exists $h \in \{0,1\}^{\mathrm{Bd}_G^+(Z)}$ such that $\Phi^h[Z]$ has no fixed point. Since $\Phi^h[Z]$ is a Boolean network on $G[Z]$ and $G[Z]$ is of finite character (by hypothesis and Proposition 2.6.24), there exists some finite $\varnothing \neq Y \subseteq Z$ such that $\Phi^h[Y]$ is not absolutely solvable relative to Y. This means that there exists $g \in \{0,1\}^{\mathrm{Bd}_G^+(Y)}$ that is compatible with h and such that $(\Phi^h[Z])^g[Y]$ has no fixed point. But $Y \subseteq Z$ implies that $(\Phi^h[Z])^g[Y] = \Phi^{h \cup g}[Y]$. Hence Φ is not absolutely solvable relative to Y. □

Corollary 5.1.11. *Let G be a digraph and $\mathcal{P}$ a well-founded partition of G. Then G has a kernel, if for all $X \in \mathcal{P}$, $G[X]$ is kernel-perfect.*

Proof. By Theorem 2.4.96 and Theorem 5.1.9. □

Corollary 5.1.12. *Let G be a digraph an $\mathcal{P}$ be a well-founded partition of G. Then G is safe, if for all $X \in \mathcal{P}$, $G[X]$ is safe.*

5.2 Prolific digraphs

5.2.1 Quasi-finitely out-branching digraphs

Definition 5.2.1. Let G be a digraph, $R \subseteq G$ a ray and $x \in V(G)$. We say that a set $X \subseteq V(G)$ *separates* x from R in G iff $x \notin X$ and for all tails R' of R one of the following is the case.

1. X has a non-empty intersection with $V(R')$,
2. every x-R'-path in G has a non-empty intersection with X.

Definition 5.2.2. Let G be a digraph and $x \in V(G)$. Let $x \in V(G)$ and $\mathcal{R}$ be a set of rays in G that all have x as their root. We say that a set $X \subseteq V(G)$ *separates* x from $\mathcal{R}$ iff X separates r from R, for all $R \in \mathcal{R}$.

Definition 5.2.3. Let G be a digraph.

- A vertex $x \in V(G)$ is said to be *quasi-grounded* (in G) iff there exists some finite $Y \subseteq V(G)$ such that $x \notin Y$ and every ray in G with root x has a non-empty intersection with Y.
- A digraph G is said to be *quasi-finitely out-branching* iff every $x \in V(G)$ is quasi-grounded in G.

Proposition 5.2.4. *Let G be a digraph. A vertex $x \in V(G)$ is quasi-grounded in G iff there exists some finite $X \subseteq V(G)$ that separates x from the set of all rays of G.*

Proof. ⇒: Let $X \subseteq V(G)$ be finite and such that $x \notin X$ and every ray in G has some non-empty intersection with X. Let $R \subseteq G$ be a ray and R' a tail of R. Suppose that $V(R') \cap X = \varnothing$. Then every x-R' path must have a non-empty intersection with X. Otherwise we could construct a ray in G that has an empty intersection with X.

⇐: Let $X \subseteq V(G)$ be finite and such that X separates x from the set of all rays of G. Let $R \subseteq G$ be a ray. Assume that $V(R) \cap X = \varnothing$. Let x_1 be the unique out-neighbor of x in R. Then $R\{x_1\}$ is a tail of R with $V(R\{x_1\}) \cap X = \varnothing$. Moreover (x, x_1) is an x-$R\{x_1\}$-path in G that has a non-empty intersection with X, which is a contradiction. □

Proposition 5.2.5. *Every ray-less digraph is quasi-finitely out-branching.*

Proof. Let G be a ray-less digraph. Then the empty set separates every from the set of all rays of G. □

Hence a well-founded digraph is quasi-finitely out-branching. Any finitely out-branching digraph is, of course, also quasi-finitely out-branching.

The following is just a rephrasing of a theorem by Walicki (Corollary 3.18 in [47]) for the present context.

Proposition 5.2.6 (Walicki)**.** *Every quasi finitely out-branching digraph is kernel-perfect if it contains no odd cycle.*

This result is obtained at the end [47] as a corollary of the main result. We shall use it to initiated an inductive hierarchy of classes of digraphs that will turn out to be safe.

Proposition 5.2.7. *Let G be a quasi-finitely out-branching digraph. Then every regular inflation of G is quasi-finitely out-branching.*

Theorem 5.2.8. *Every acyclic quasi-finitely out-branching digraph is safe.*

Proof. Let $\mathcal{C}$ be the class of all acyclic quasi-finitely out-branching digraphs. Then $\mathcal{C}$ is closed under regular inflation by Proposition 5.2.7. By Proposition 5.2.6 every element of $\mathcal{C}$ has a kernel. Hence every element of $\mathcal{C}$ is safe by Corollary 3.2.9. □

Corollary 5.2.9. *No quasi-finitely out-branching digraph contains an $\mathcal{I}_f[\mathbb{Y}]$.*

5.2.2 Prolific digraphs

Definition 5.2.10. For any digraph G and $X \subseteq V(G)$, let $\mathbf{G}(X)$ be the set of all $x \in X$ such that $G[X]\{x\}$ is quasi-finitely out-branching.

Definition 5.2.11. For all digraphs G and all $\alpha \in On$, define recursively $\mathbf{G}'_\alpha = \mathbf{G}(V(G) \setminus \bigcup_{\beta<\alpha} \mathbf{G}'_\beta)$.

Clearly, for any digraph G there exists an ordinal ξ such that for all $\alpha \geq \xi$, $\mathbf{G}'_\alpha = \varnothing$. Let ξ_G be the least ordinal ξ such that $\mathbf{G}'_\alpha = \varnothing$, for all $\alpha \geq \xi$.

Definition 5.2.12. For all digraphs G and all $\alpha < \xi_G$ define

1. $\mathbf{G}_\alpha = \bigcup_{\beta \leq \alpha} \mathbf{G}'_\beta$.
2. $\mathbf{G}_{\xi_G} = \bigcup_{\alpha < \xi_G} \mathbf{G}_\alpha$,
3. $\mathbf{G}_* = \mathbf{G}_{\xi_G}$.

Proposition 5.2.13. *Let G be a digraph. Then $G[\boldsymbol{G}_*]$ contains no $\mathcal{I}_f[\mathbb{Y}]$.*

Proposition 5.2.14. *Let G be a digraph and $H \subseteq G$. Then $\boldsymbol{H}_* \supseteq \boldsymbol{G}_* \cap V(H)$.*

Proposition 5.2.15. *Let G be a digraph G. Then*

1. *$(\boldsymbol{G}_\alpha)_{\alpha \leq \xi_G}$ is an open exhaustion of $G[\boldsymbol{G}_*]$.*
2. *$\boldsymbol{G}_*$ is open in G.*
3. *Let Φ be a Boolean network on G that is absolutely solvable relative to all $X \subseteq V(G)$ such that $G[X]$ is quasi-finitely out-branching. Then Φ is absolutely solvable relative to $(\boldsymbol{G}_\alpha)_{\alpha \leq \xi_G}$.*

Proof. Ad (1): We shall check clauses (1)-(4) of Definition 5.1.1.

Clause 1: We prove that $\mathbf{G}_\alpha$ is open in G for all $\alpha \leq \xi_G$ by induction on α. We have $\mathbf{G}_\alpha = \mathbf{G}'_\alpha \cup X_\alpha$, where $X_\alpha = \bigcup_{\beta<\alpha} \mathbf{G}'_\beta$. Assume that $\mathbf{G}_\alpha$ is not open in G. Then there exists $(x, y) \in A(G)$ such that $x \in \mathbf{G}_\alpha$ and $y \in V(G) \setminus \mathbf{G}_\alpha$ Since X_α is open in G by induction hypothesis (as a union of open sets), $x \in \mathbf{G}'_\alpha$. Hence $G[V(G) \setminus X_\alpha]\{x\}$ is quasi-finitely out-branching. Since $(x, y) \in A(G[V(G) \setminus X_\alpha])$ we have $(G[V(G) \setminus X_\alpha])\{y\} \subseteq (G[V(G) \setminus X_\alpha])\{x\}$. Hence $(G[V(G) \setminus X_\alpha])\{y\}$ is quasi finitely out-branching, which yields the contradiction that $y \in \mathbf{G}'_\alpha \subseteq \mathbf{G}_\alpha$.
Clauses 2,3 and 4 are clear. Ad (2): By (1), since the union of open sets is open.

Ad (3): We have to check both conditions of Definition 5.1.2. The first i.e., the claim that $\Phi[\mathbf{G}_0]$ has a fixed point holds by hypothesis. As for the second, we have to show that Φ is absolutely solvable relative to $\mathbf{G}'_{\alpha+1}$, for all $\alpha < \xi_G$. Since $\mathbf{G}'_{\alpha+1}$, is quasi-finitely out-branching the claim follows analogously also from the hypothesis. □

Theorem 5.2.16. *Let G be a digraph.*

1. *If $G[\boldsymbol{G}_*]$ contains no odd cycles, then $G[\boldsymbol{G}_*]$ has a kernel.*
2. *If $G[\boldsymbol{G}_*]$ is acyclic, then $G[\boldsymbol{G}_*]$ is safe.*

Proof. Ad 1: By Proposition 5.2.6, Proposition 5.2.15 and Theorem 5.1.3.
Ad 2: By Theorem 5.2.8, Proposition 5.2.15 and Theorem 5.1.3. □

Definition 5.2.17. A digraph G is said to be *prolifically out-branching* (or simply *prolific*) iff $V(G) \neq \varnothing$ and for all $x \in V(G)$ there is a walk in G from x to some vertex that is not quasi-grounded in G.

Proposition 5.2.18. *Let G be a digraph.*

1. *If $V(G) \setminus \boldsymbol{G}_* \neq \varnothing$, then $G[V(G) \setminus \boldsymbol{G}_*]$ is prolific and contains a ray.*

2. *For all $x \in \boldsymbol{G}_*, G\{x\}$ is not prolific.*

Proof. Ad 1: Suppose that $X = V(G) \setminus \mathbf{G}_* \neq \varnothing$. Assume that $G[X]$ is not prolific. Then there exists $x \in X$ such that $G[X]\{x\}$ is quasi-finitely out-branching. But this implies the contradiction that $x \in \mathbf{G}_*$. Now assume that $G[V(G) \setminus \mathbf{G}_*]$ contains no ray. Then for all $x \in X$, $G[X]\{x\}$ is quasi-finitely out-branching by Proposition 5.2.5, yielding the same contradiction.

Ad 2: By induction on ξ_G.

□

Theorem 5.2.19. *Let G be an acyclic digraph. Then*

1. *G is equi-dangerous to $G[V(G) \setminus \boldsymbol{G}_*]$.*

2. *G is kernel-perfect iff $G[V(G) \setminus \boldsymbol{G}_*]$ is kernel-perfect.*

Proof. Ad 1: For the non-trivial direction, it follows from Proposition 5.2.15 that $\mathbf{G}_*$ is open in G. Hence the claim follows from Lemma 2.4.48.

Ad 2: Applying Theorem 2.4.96, the argument is analogous to (1). □

5.3 Ends

5.3.1 Ends and end covers

Let us restate the following definitions.

Definition 5.3.1. A *tail* of a digraph G is a subdigraph of G that is induced by some $X \subseteq V(G)$ such that $G[V(G) \setminus X]$ contains no rays.

Definition 5.3.2. Let G be a digraph.

- A digraph $H \subseteq G$ is said to be an *end in* G iff there exists some ray $R \subseteq G$ such that $V(H) = \mathrm{Cl}_G(R)$. We call such a ray R a *principal ray* of H.

- G is said to be an *end* iff G is an end in G.

Proposition 5.3.3. *Let G be a digraph and $H \subseteq G$ an $\mathcal{I}_f[\mathbb{Y}]$. Then H is contained in some end of G.*

Definition 5.3.4. Let G be a digraph and Q, R rays in G. We define the following relations between Q and R.

1. $Q \preceq_G R$ iff for every tail Q' of Q and every tail R' of R there exists a path from Q' to R' in G.

2. $Q \preceq_G^\infty R$ iff there exists an infinite set $\mathcal{P}$ of pairwise disjoint (possibly trivial) paths such that for every tail Q' of Q and every tail R' of R there exists some $P \in \mathcal{P}$ that joins Q' to R' in G.

3. $Q \preceq_G^f R$ iff $Q \preceq_G R$ and there exists a finite set $X \subseteq V(G)$ such that $Q' \not\preceq_{G \setminus X} R'$, for all tails Q' of Q and R' of R such that $Q', R' \subseteq G \setminus X$.

4. $Q \prec_G R$ iff $Q \preceq_G R$ and $R \not\preceq_G Q$.

5. $Q \prec_G^\infty R$ iff $Q \preceq_G^\infty R$ and $R \not\preceq_G^\infty Q$.

6. We say that Q and R are *incomparable* iff $R \not\preceq_G Q$ and $Q \not\preceq_G R$.

7. We say that Q and R are *parallel* iff $R \preceq_G Q$ and $Q \preceq_G R$.

The following two definitions are adopted from [47].

Definition 5.3.5. A digraph G is said to be *flat* iff for all $Q, R \subseteq G$ either Q and R are incomparable or parallel.

Definition 5.3.6. Let G be a digraph and $Q, R \subseteq G$ rays. We say that

- a set $X \subseteq V(G)$ *separates* Q from R iff every Q-R path has a non-empty intersection with X,

- Q is *finitely separable from* R iff there exists a finite set that separates Q from R.

Proposition 5.3.7. *Let G be a digraph and $Q, R \subseteq G$ be rays.*

1. *If $Q \preceq_G^\infty R$ or $Q \preceq_G^f R$, then $Q \preceq_G R$.*

2. *If $Q \preceq_G R$, then either $Q \preceq_G^f R$ or $Q \preceq_G^\infty R$, but not both.*

3. *If $Q \preceq_G R$, then $Q \preceq_G^f R$ iff then Q is finitely separable from R.*

4. *If G is acyclic and $Q \preceq_G R$ and $R \preceq_G Q$, then $Q \preceq_G^\infty R$ and $R \preceq_G^\infty Q$.*

5. *If G is acyclic, then there exists no ray $Q \subseteq \mathrm{Cl}_G[R]$ such that $R \preceq_G^f Q$.*

Lemma 5.3.8. *Let G a digraph, $X \subseteq V(G)$ closed in G and $R \subseteq G$ a ray. Then $V(R) \cap X$ is finite or $V(R) \cap (V(G) \setminus X)$ is finite.*

Proof. Assume otherwise. Let $y \in V(R) \cap (V(G) \setminus X)$. Then it follows from the assumption there exists some $z \in V(R) \cap X$ that is reachable from y via R. But since X is closed in G, it follows that $y \in X$, which is a contradiction. □

Definition 5.3.9. Let G be a digraph. An *end cover* of G is a set $\mathcal{H}$ of ends in G such that for all ends F in G there exists some $H \in \mathcal{H}$ with $F \subseteq H$.

Theorem 5.3.10. *Let G be an acyclic digraph and $\mathcal{H}$ a countable end cover of G. Then*

1. *G is (strongly) kernel-perfect iff all $H \in \mathcal{H}$ are (strongly) kernel-perfect,*
2. *G is safe is safe iff all $H \in \mathcal{H}$ are safe,*
3. *G is of finite character iff all $H \in \mathcal{H}$ are of finite character iff G is of finite character.*

Proof. First we construct a well-founded partition of G. Let $(H_n)_{n\in\omega}$ be an enumeration of the elements of $\mathcal{H}$. We define recursively a sequence $(X_n)_{n\in\omega}$ of subsets of $V(G)$ as follows. Let $X_0 = V(H_0)$. Now let $n > 0$ and suppose we have already defined X_i, for all $i \leq n$. Let $X_{n+1} = V(H_{n+1}) \setminus \bigcup_{i\leq n} X_i$. An an end in G, every $V(H_i)$ is closed in G. Hence $\bigcup_{i\leq n} X_i$ is closed in G for all $n \in \omega$. Let $\mathcal{P} = \{X_n \mid n \in \omega \wedge X_n \neq \varnothing\}$. Then $\mathcal{P}$ is a partition of $V(G)$.

Let us show that for every ray $R \subseteq G$ there exists some $X \in \mathcal{P}$ such R has a tail in X. Let $R \subseteq G$ be a ray. Then $\mathrm{Cl}_G[R]$ is an end of G. Hence there exists some $n \in \omega$ such that $\mathrm{Cl}_G[R] \subseteq H_n$. Hence $\mathrm{Cl}_G(R) \subseteq \bigcup_{i\leq n} X_i$. Let Q be a tail of R. Then there exists some $k \leq n$ such that $X_k \cap V(Q)$ is infinite. Then for all $k \neq i \leq n$, $X_i \cap V(Q)$ is finite: assume otherwise and let $k \neq i \leq n$ be such that $X_i \cap V(Q)$ is infinite. We may assume w.l.o.g. that $i < k$. Since $Y = \bigcup_{m\leq i} X_i$ is closed in G and since $V(G)\setminus Y$ contains infinitely many vertices of Q, X_i contains not infinitely many vertices of Q by Lemma 5.3.8, yielding a contradiction.

Hence X_k contains a tail of R. It follows that, given that G is acyclic, $\mathcal{P}$ is a well-founded partition of G.

Now the direction from right to left of the claims follows by Corollaries 5.1.11, 5.1.12 and 5.1.10 respectively. The other directions are clear. □

5.3.2 Normal ends

The last theorem raises the question under what circumstances ends themselves are kernel-perfect, safe or of finite character.

Definition 5.3.11. An end G with principal ray R is said to be *normal* iff for all rays $Q \subseteq G$, $Q \preceq^{\infty}_{G} R$.

If all ends of a digraph are normal and the digraph is symmetric i.e., undirected, then they coincide just with the *undirected ends* i.e., those in the sense of the usual definition of end in the theory of undirected graphs, cf. [17].

Proposition 5.3.12. *Every acyclic and flat end is normal.*

Proof. By Proposition 5.3.7 (5). □

Proposition 5.3.13. *Every end that is acyclic, conversely well-founded and finitely in-branching is normal.*

Proof. Let G be an end and R a principal ray of G. Let $Q \subseteq G$ be a ray. Then $Q \preceq_G R$, i.e., for all $y \in V(Q)$ there is a path from y to some vertex of R. We have to show that $Q \preceq_G^\infty R$. By Proposition 5.3.7(2,3) it suffices to prove that there exists no finite $X \subseteq V(G)$ that separates Q from R.

Assume that X is such a set. For each $y \in V(Q)$, let P_y be a path from y to R. Then for all $y \in V(Q)$, there exists some $x \in X$ such that $x \in V(P_y)$. By cutting the P_y's off at x, we get for all $x \in X$ a set $\mathcal{P}_x$ of Q-x paths. For all $x \in X$, let T_x be an in-branching spanning tree of the digraph $G[\bigcup \mathcal{P}_x]$ (each of whose vertices is conversely reachable from x, hence such a spanning tree exists). Since $V(Q)$ is infinite, one T_x must be infinite. By König's lemma T_x contains a vertex of infinite in-degree or a converse ray, neither of which is possible by hypothesis. □

For the following recall Definition 4.4.1.

Proposition 5.3.14. *Let G be a digraph and $Q, R \subseteq G$ rays. If $x \in V(G)$ is a focal point of Q and $Q \preceq_G^\infty R$, then x is a focal point of R.*

Recall Definition 5.2.3.

Lemma 5.3.15. *Let G be digraph, $r \in V(G)$ and $R \subseteq G$ a ray with root r. If there is no finite set that separates r from R, then r is a focal point of R.*

Proof. Let $R \subseteq G$ be a ray with root r. We construct recursively a sequence $(\mathcal{P}_n)_{n\in\omega}$ such that for all $n \in \omega$, (i) every $P \in \mathcal{P}_n$ is an r-R path, (ii) the members of $\mathcal{P}_n$ are pairwise disjoint except at r, and (iii) $\mathcal{P}_n \subsetneq \mathcal{P}_{n+1}$.

Let $x_0, x_1, \ldots$ be the R-order preserving enumeration of the set $V(R)$. Since $\{x_1\}$ does not separate r from R by hypothesis, there exists an r-$R\{x_2\}$-path P_0 in G such that $V(P_0) \cap \{x_1\} = \varnothing$. Let $\mathcal{P}_0 = \{P_0\}$. Let $k_0 \in \omega$ be such that x_{k_0} is the head of P_0. Let $X_0 = (V(P_0) \cup \mathrm{Cl}_R(x_{k_0}) \setminus \{r\}$.

Now let $n > 0$ and suppose that we have already constructed $\mathcal{P}_n$, k_n and X_n. By induction hypothesis X_n is finite. Hence it does not separate r from R. In particular, there exists an r-$R\{x_{k_n+1}\}$-path P in G such that $X_n \cap V(P) = \varnothing$. Let $\mathcal{P}_{n+1} = \mathcal{P}_n \cup P$. Then (i), (ii) and (iii) hold. Let k_{n+1} be such that $x_{k_{n+1}}$ is the head of P. Let $X_{n+1} = (\bigcup\{V(P) \mid P \in \mathcal{P}_{n+1}\} \cup \mathrm{Cl}_R(x_{k_{n+1}})) \setminus \{r\}$.

Finally, set $\mathcal{F} = \bigcup_{n<\omega} \mathcal{P}_n$. Then it follows from (i)-(iii) that $\mathcal{F}$ is an r-R-fan. Hence r is a focal point of R. □

Proposition 5.3.16. *Let G be digraph, $r \in V(G)$ and $\mathcal{R} = \{R_0, \ldots, R_n\}$ a finite set of rays in G with root r. If there is no finite set that separates r from $\mathcal{R}$, then r is a focal point of R_i, for some $0 \leq i \leq n$.*

Proof. We construct recursively a sequence $(\mathcal{P}_n)_{n\in\omega}$ such that for all $n \in \omega$, (i) every $P \in \mathcal{P}_n$ is an r-R_i path, for some $0 \leq i \leq n$, (ii) the members of $\mathcal{P}_n$ are pairwise disjoint except at r, and (iii) $\mathcal{P}_n \subsetneq \mathcal{P}_{n+1}$.

This can be done analogously to the procedure in the proof of Lemma 5.3.15. Then let $\mathcal{F} = \bigcup_{n<\omega} \mathcal{P}_n$. Since $\mathcal{F}$ is infinite and $\mathcal{R}$ finite, there must exist some $0 \leq i \leq n$ such that r is a focal point of R_i. □

Lemma 5.3.17. *Let G be an end with principal ray $R \subseteq G$. Let $r \in V(G)$. If G is normal and r is not quasi-grounded in G, then r is a focal point of R.*

Proof. Before proceeding with the proof, let us state some definitions. A *star* in G with center r is a non-empty set $\mathcal{S}$ of rays in G such that $\bigcap_{R\in\mathcal{S}} V(R) = \{r\}$. A star $\mathcal{S}$ is said to be *infinite* iff the set $\mathcal{S}$ is infinite; it is said to be *finite* iff the set $\mathcal{S}$ is finite. A star $\mathcal{S}$ is said to be *maximal* (in G) iff there exists no star $\mathcal{T}$ (in G) such that $\mathcal{S} \subsetneq \mathcal{T}$.

Case 1: There exists an infinite star $\mathcal{S}$ with center r in G. We shall construct recursively a sequence $(\mathcal{P}_n)_{n\in\omega}$ of r-R paths such that for all $n \in \omega$, (i) $\mathcal{P}_n \subseteq \mathcal{P}$, (ii) the members of $\mathcal{P}_n$ are pairwise disjoint except at r, and (iii) $\mathcal{P}_n \subsetneq \mathcal{P}_{n+1}$.

Let $(R_i)_{n\in\omega}$ be an enumeration of a countable subset of $\mathcal{S}$. Since G is normal, we have $R_i \preceq_G^\infty R$, for all $n \in \omega$. Picking some R_0-R path and prolonging it backwards to r yields some path P_0 from r to R. Let $\mathcal{P}_0 = \{P_0\}$.

Now let $n > 0$ and suppose that we have already defined $\mathcal{P}_n$. Since $X_n = V(\bigcup \mathcal{P}_n) \subseteq V(G) \setminus \{r\}$ is finite and $\mathcal{S}$ is an infinite star, there exists some $k_{n+1} \in \omega$ such that $V(R_{k_{n+1}}) \cap X_n = \varnothing$. Since $R_{k_{n+1}} \preceq_G^\infty R$ and X_n is finite, there exists some $R_{k_{n+1}}$-R path P with $V(P) \cap X_n = \varnothing$. Prolonging P backwards to r yields a path $P_{k_{n+1}}$ from r to R such that $V(P_{k_{n+1}}) \cap V(Q) = \{r\}$, for all $Q \in \mathcal{P}_n$. Set $\mathcal{P}_{n+1} = \mathcal{P}_n \cup P_{k_{n+1}}$. This shows that r is a focal point of R.

Case 2: There exists no infinite star with center r in G. Since G is an end with principal ray R, here exists at least one star with center r: the one that has R as its single ray. Moreover, the union of every ascending $\subseteq$-chain of stars with center r is a star with with center r. Hence, by Zorn's lemma, there exists a finite maximal star $\mathcal{S}$ with center r in G. Let $\mathcal{S} = \{R_0, \ldots, R_n\}$.

Assume that for all $0 \le i \le n$, there exists some finite $X_i \subseteq V(G) \setminus \{r\}$ that separates r from R_i. We show that this leads to a contradiction. Let $X_i' = \mathrm{Cl}_{R_i}(X_i) \setminus \{r\}$, for all $0 \le i \le n$. Then $X = X_0' \cup \cdots \cup X_n'$ is finite and X separates r from $\{R_i \mid 0 \le i \le n\}$. We claim that every ray $Q \subseteq G$ with root r has a non-empty intersection with X. (This claim yields a contradiction to the assumption that r is not quasi-grounded in G.) Assume $V(Q) \cap X = \varnothing$. Then there exists $r \ne y \in V(Q) \cap (\bigcup\{V(R_i) \mid 0 \le i \le n\})$. (Otherwise $\mathcal{S} \cup Q$ would be a star, contradicting the maximality of $\mathcal{S}$.) Let k be the unique $0 \le i \le n$, such that $y \in V(R_i)$. Since $y \notin X$, $V(R_k\{y\}) \cap X = \varnothing$. On the other hand $\mathrm{Cl}_Q[y]$ is an r-R_k-path that has an non-empty intersection with X. This yields the contradiction that X does not separates r from $\{R_i \mid 0 \le i \le n\}$.

Hence there is some $0 \le i \le n$ such that no finite $X \subseteq V(G) \setminus \{r\}$ separates r from R_i. Hence r is a focal point of R_i by Lemma 5.3.15 and, since G is normal, also a focal point of R by Proposition 5.3.14. □

Definition 5.3.18. A digraph G is said to contain *only finitely many ends* iff the set of all subdigraphs H of G such that H is an end in G is finite.

Theorem 5.3.19. *Let G be an acyclic end.*

1. *If G is normal and prolific, then G contains an $\mathcal{I}_f[\mathbb{Y}]$.*

2. *If G contains only finitely many ends and is prolific, then G contains an $\mathcal{I}_f[\mathbb{Y}]$.*

Proof. Ad 1: Let R be a principal ray of G. We define recursively a sequence $(x_n)_{n\in\omega}$ of vertices of G such that for all $n \in \omega$,

1. if n is even, then x_n is a focal point of R,
2. if n is odd, then $x_n \in V(R)$,
3. there is a path in G from x_n to x_{n+1},

as follows.

- Let $x_0 \in V(G)$ be such that x_0 is not quasi-grounded in G. Then by Lemma 5.3.17, x_0 is a focal point of R.
- Let $n > 0$.
  - If n is even, let x_{n+1} be some vertex of R that is reachable from x_n. Since x_n is a focal point of R by induction hypothesis, such a vertex must exist.
  - If n is odd, let x_{n+1} be some vertex that is not quasi-grounded in G and that is reachable from x_n. Since G is prolific, such a vertex must exist. Then by Lemma 5.3.17, x_{n+1} is a focal point of R.

Now let Q be the concatenation of the all paths from x_n to x_{n+1} that exist by (3). Since G is acyclic, Q is a ray. Since Q and R have infinitely many vertices in common, $R \preceq_G^\infty Q$. Hence by Proposition 5.3.14, for all even $n \in \omega$, x_n is a focal point of Q. Thus Q is a ray in G such that infinitely many vertices of Q are focal points of Q. Hence G contains an $\mathcal{I}_f[\mathbb{Y}]$ by Proposition 4.4.2.

Ad 2: Let $r \in V(G)$ and $\{H_0, \ldots, H_n\}$ be the set of ends of G that are reachable from r. Let $\mathcal{R} = \{R_0, \ldots, R_n\}$ be a set of principal rays of $\{H_0, \ldots, H_n\}$. We may assume that r is the root of each R_i, for $0 \leq i \leq n$.

Since every ray in $G\{r\}$ is parallel to some ray in $\mathcal{R}$, Proposition 5.2.4 implies that $x \in V(G)$ is quasi-grounded in $G\{r\}$ iff there exists a finite set that separates x from $\mathcal{R}$. Hence we can define recursively a sequence $(x_n)_{n\in\omega}$ of vertices of G such that for all $n \in \omega$,

i. if n is even, then x_n is a focal point of some element of $\mathcal{R}$,

ii. if n is odd, then x_n is a vertex of some element of $\mathcal{R}$,

iii. there is a path in G from x_n to x_{n+1}.

This is done analogously to the proof of (1), but by invoking Proposition 5.3.16 instead of Lemma 5.3.17. Then a ray Q is constructed analogously to the proof of (1). Proposition 5.3.14 is not needed: since $\mathcal{R}$ is finite, Q has infinitely many vertices in common with some member of $\mathcal{R}$. Hence Q contains infinitely many of its focal points. $\square$

Question 5.3.20. *Is there an acyclic and prolific end that contains no finitary inflation of $\mathbb{Y}$?*

The answer seems to be positive.

5.4 Digraphs with countably many ends

Definition 5.4.1. Let G be a digraph and $R \subseteq G$ a ray. A vertex $x \in V(R)$ is said to be an *exit point* of R in G iff there exists $y \in V(G) \setminus \mathrm{Cl}_G(R)$ such that y is reachable from x in G.

Lemma 5.4.2. *Let G be a digraph such that $\boldsymbol{G}_* = \varnothing$ and such that no end in G is prolific. Then every ray in G has an exit point in G.*

Proof. Assume that there is a ray $R \subseteq G$ that has no exit point in G. Since G is prolific, $H = \mathrm{Cl}_G(R)$ is a prolific end: for each $x \in V(R)$ there exists $y \in V(G)$ that is reachable from x and is not quasi-grounded in G. Since $G\{y\} \subseteq H$ (otherwise x would be an exit point), y is not quasi-grounded in H. Since for all $z \in V(H)$ there exists $x \in V(R)$ such that x is reachable from z, H is prolific. □

Proposition 5.4.3. *Let G be a digraph such that $\boldsymbol{G}_* = \varnothing$ and such that no end in G is prolific. Then G contains uncountably many ends.*

Proof. Let $R \subseteq G$ be a ray. Then there exists an exit point x of R and a ray $R_1 \subseteq G$ that originates from x and is otherwise disjoint from R. (This is so because ends are closed.) Let $R_0 = R\{x\}$. Analogously, there are exit points x_0 and x_1 for R_0 and R_1 respectively, both above x, giving rise to rays R_{00}, R_{01}, R_{10}, R_{11}, each of which has an exist point x_{00}, x_{01}, x_{10}, x_{11}. Proceeding in this way, we define an injection of 2^ω into the set of all ends of G. □

Definition 5.4.4. Call a digraph G *normal* iff every end of G, is normal or contains only finitely many ends.

Theorem 5.4.5. *Let G be an acyclic and normal digraph. If G contains no $\mathcal{I}_f[\mathbb{Y}]$, then $\boldsymbol{G}_* = V(G)$ or $G[V(G) \setminus \boldsymbol{G}_*]$ contains uncountably many ends.*

Proof. Suppose $\mathbf{G}_* \neq V(G)$. Assume that $H = G[V(G) \setminus \mathbf{G}_*]$ contains only countably many ends. Since $\mathbf{H}_* = \varnothing$ it follows from Proposition 5.4.3 that H has some prolific end H'. Since H' is normal or contains only finitely many ends, it follows from Theorem 5.3.19 that H' contains an $\mathcal{I}_f[\mathbb{Y}]$. □

Theorem 5.4.6. *Let G be a normal digraph with only countably many ends. Then G is safe iff G is acyclic and contains no $\mathcal{I}_f[\mathbb{Y}]$.*

Proof. $\Rightarrow$: By Corollary4.3.4. $\Leftarrow$: Let G be acyclic and containing no $\mathcal{I}_f[\mathbb{Y}]$. Then $\mathbf{G}_* = V(G)$ by Theorem 5.4.5. Hence G is safe by Theorem 5.2.16. □

Chapter 6

Final remarks

Let us conclude this thesis with a discussion of a couple of technical and philosophical questions regarding the most important results and methods, as well as their potential for further applications.

6.1 How to make further progress?

Theorem 5.4.5 has a striking resemblance to Cantor-Bendixson's theorem in the sense that it claims that a closed set is either empty or very large. It could be interesting to think about stronger versions of Theorem 5.4.5 that work with modified definitions of $\mathbf{G}_*$. Such modifications could be realized by initializing the hierarchy of Definition 5.2.12 with a more comprehensive class of digraphs than the quasi-finitely out-branching ones. A prerequisite for this would be to establish a counterpart to Theorem 5.2.16.

Another even more interesting line of thought arises from the observation that an end cover of a digraph (cf. Definition 5.3.9) tends to have a tree-like structure. This phenomenon is illustrated in the proof of Proposition 5.4.3, which can be read as a recipe for constructing an uncountable end cover of a digraph G, given that $\mathbf{G}_* = \varnothing$ and no end in G is prolific.

Of course, systematizing this construction also requires a treatment of infinite chains of ends, i.e., infinite $\preceq_G$-chains of their principal rays $R_0 \preceq_G R_1 \preceq_G R_2 \cdots$ (cf. Definition 5.3.4). It seems plausible, however, (at least under favorable circumstances) that for every such infinite chain some diagonal ray $R_\omega \subseteq G$ can be found such that $R_0 \preceq_G R_1 \preceq_G R_2 \cdots \preceq_G R_\omega$.

This observation suggests that in order to establish stronger versions of Theorem 5.4.6, a better understanding is needed of tree-like uncountable end covers, and, in particular, of the role they play in the construction of global fixed points of Boolean networks relative to local fixed points that are restricted to ends. (Note that there are countable digraphs that have no countable end cover - e.g. infinitely out-branching binary trees. Hence we cannot eliminate in Theorem 5.4.6 the requirement that G have only countably many ends, simply by focusing on countable digraphs.)

Let us recall that, with regard to the structure of a digraph G, there are basically three classes of techniques for constructing fixed points of a Boolean network on G: techniques based on the well-foundedness of G (treated in Subsection 2.4.7 and Subsection 2.4.8), techniques based on the compactness of G (treated in Subsection 2.6.2) and techniques based on G being a tree (treated in Subsection 2.4.10 and resulting in Theorem 2.4.88). (A combination of the methods of well-foundedness and compactness is used in Definition 5.2.12).

Of these three methods, it seems that the one based on trees is still the least understood - and at the same time the one with the largest potential. In contrast with the other two, we haven't been able to apply it in Chapter 5. Instead, we used a different structural property of a digraph, that of having a countable end cover - but this method is rather limited in scope.

So, progress towards a proof of Conjecture (A) seems to require a generalization of Theorem 2.4.88 to digraphs that are not just trees but rather are *tree-like*, or more specifically, have a tree-like end cover. Or, formulated negatively, it seems that any counterexample to Conjecture (A) must be essentially a tree. And this makes it hard to imagine that such a counterexample exists.

6.2 Are there only two semantic paradoxes?

Does any of the three main conjectures, Conjecture (A), (C) or (D) support the claim that the Liar paradox and Yablo's paradox are *essentially* the only semantic paradoxes?

Conjecture (A) (Conjecture 2.8.1) states that the loop and the Yablo-graph are the building blocks from which every dangerous digraph can be generated via the process of first inflating any of these two digraphs and then adding arbitrary vertices and arcs to the result. In this sense, (1) the loop and the Yablo-graph are essentially the only dangerous digraphs. If one concedes that (2) a semantic paradox is essentially captured by its dependency graph, then it is plausible to conclude that Conjecture (A) implies that there are essentially only two paradoxes.

Premise (2), however, is highly problematic. It runs into at least three objections: (i) there are Boolean networks that have no canonical (i.e., minimal) dependency graph (cf. Example 2.3.3); (ii) two isomorphic digraphs might be dependency graphs of two different Boolean networks, each of which might be paradoxical in a different way or not paradoxical at all; (iii) even two isomorphic constrained Boolean networks (cf. Definition 2.5.21) can describe two different paradoxes, since the way one automaton of a network refers to other automata does matter - in short, reference is an intensional phenomenon and essential for the identity of a paradox (cf. Subsection 2.5.4).

But even Premise (1) is not as evident as it may seem. As indicated at the end of Subsection 2.6.2, a positive answer to Question 2.6.27 ('Is there a digraph of infinite character that has a skeleton of finite character?') would imply the existence of a paradox of infinite character whose paradoxicality nevertheless depends on its cyclic structure. It seems fair to say that its dependency graph would constitute a reference pattern of hybrid character that cannot be fully captured by the either the loop or the Yablo-graph alone. (In Subsection 2.8.2 we remarked that it is not obvious how a negative answer to Question 2.6.27 can be derived from any of the conjectures - but, of course, we could be mistaken.)

What about the other conjectures? Conjecture (C) (Conjecture 2.8.13) is more specific than Conjecture (A) in the sense that it asserts the existence of fixed points for the Boolean network $\Phi^G_\downarrow$ associated to a digraph G, given that G contains neither an even finitary inflation of the loop nor of the Yablo-graph. Since there is a unique correspondence between Boolean networks of the class $\Phi^G_\downarrow$ and their dependency graph G, this makes the association between semantic paradox and reference pattern much closer than in the previous case. This invalidates Objections (i) and (ii) against Premise (2) - all adapted to the context of Conjecture (C) - while Objection (iii) is just as applicable as in the case of Conjecture (A). (And so is also the objection against Premise (1)).

However, one should keep in mind that Conjecture (C) in itself does not make an assertion about all Boolean networks, but only about a certain subclass of them and has to rely on a translation process in order to claim universality. On the other hand, given the results of Chapter 3 and Theorem 3.2.6 in particular, one could argue that this translation process via regular inflation (cf. Definition3.1.1) of the dependency graph and dense weak system embedding (cf. Definition2.5.4) of Boolean networks into another preserves all the essential information about the original constrained Boolean network. I would contend that that Conjecture (C) is indeed a better candidate to support the claim that there are essentially only two semantic paradoxes than Conjecture (A).

And a better one than Conjecture (D) (Conjecture 2.8.4), which deals with the concept of digraphs of infinite character - which is at least as abstract as that of danger and thus again susceptible to Objection (i) and (ii).

6.3 Application to axiomatic theories of truth

In a series of papers ([37], [38] and [39]) Picollo introduces a notions of *alethic reference* for sentences of the language of first-order Peano arithmetic extended with a truth predicate, based upon which she formulates various axiomatic theories of truth which then are proved to be (ω-)consistent. The basic underlying idea is to restrict the T-scheme to sentences that exhibit a benign alethic reference pattern, the paradigm of which is, as might be expected, well-foundedness. But Picollo also considers a theory that allows for non-well founded reference patterns, as long as they satisfy a propery that corresponds to our notion of a finitely out-branching digraph (cf. Section 2.1).

It would be interesting to investigate whether even proof-theoretically stronger axiomatic theories of truth than those in [39] can be formulated and proved consistent by using results and methods from this thesis. The first idea that comes to mind is to work with the intertwined hierarchy of finitely out-branching and well-founded digraphs from Definition 5.2.12 and apply Theorem 5.2.19 to the Boolean network discussed in Subsection 2.3.2.

More advanced question then could be, for example, whether positive and negative reference (in the sense of Section 3.6) can also be formulated in the vein of Picollo's approach. And, whether an axiomatic theory of truth can be formulated that avoids (odd) cycles and reference patterns corresponding to (odd) finitary inflations of the Yablo-graph, and whether it can be shown to be ω-consistent relative to Conjecture (A) and Conjecture (C), respectively.

6.4 Application to the logical paradoxes

Finally, it seems worthwhile to consider briefly the question of whether our abstract approach to the semantic paradoxes via Boolean networks is also applicable the logical paradoxes, i.e., the class - and property-theoretic paradoxes. Can class-theoretic paradoxes be analyzed in terms of dependency graphs?

There are probably various ways to do this. The most straightforward one is to look at Boolean networks that have a Boolean automaton for every ordered pair (x,y) of classes (or even class terms) that is supposed to 'compute' whether or not x is an element of y, given the results of the entire network and assuming the validity of the comprehension scheme. Without going into details, it seems clear that the automaton that 'computes' whether the Russell class R is an element of itself has a (signed) dependency graph isomorphic to that of the liar sentence. This is so because the truth value of the proposition 'R is an element of R' depends on the truth value of the proposition 'R is an element of R' in the same negative way the truth value of the liar sentence depends on itself, producing an endless alternating sequence of output values of their automata. On the other hand, the picture for the universal class U is quite different: For every class x whatsoever, the proposition 'x is an element of U' is true by definition of U. The computation process stabilizes at step one and the dependency graph is the trivial one, consisting of an isolated point, just as that for a sentence like 'snow is white'.

Investigations along these lines could provide insights into how the semantic- and the logical paradoxes are interrelated. Are the paradoxical dependency patterns the same as for the semantic paradoxes? In particular, is there a class-theoretical counterpart to the Yablo-paradox? (The answer is probably language-dependent). And, more ambitiously, can results about fixed point existence for Boolean networks be used in order construct models for set theories with a universal set?

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