Essays on Risk Modeling in Financial Econometrics

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Summary

The desire to understand financial markets has been unprecedented. This dissertation further contributes to it. We investigate the methods to reproduce the behavior of financial returns, examine their interrelationships within a portfolio during market stress, and introduce a new method to forecast correlation. The essays on the topics are embedded in this dissertation and presented as five coherent chapters.

The first chapter outlines the research goals of this dissertation. We motivate our research projects and provide a comprehensive literature review.

The second chapter presents a simulation framework that combines stochastic volatility models with various error distributions, conditional correlation models, and copulas. Empirical evidence has shown that the behavior of financial returns adheres to a set of distinct characteristics, commonly known as ”stylized facts”, such as heavy tails, volatility clustering, time-varying correlation, and non-linear dependence. The motivation for this project is that many established methods only cover one or a subset of these characteristics, thus producing inaccurate results. The simulation framework allows the combination of these models while maintaining traceability and is easily modified. One way to calibrate this model is to estimate the model components. Challenges such as optimal starting parameters for estimation and variance initialization are addressed during this process. Furthermore, sampling methods, in particular Filtered Historical Simulation (FHS) and Moving Block Bootstrap (MBB), are evaluated for deriving standard errors. The modular structure of the framework allows for future enhancements, e.g., incorporating more sophisticated variance models, error distributions, or advanced correlation dynamics.

The third chapter explores the dependence structures of financial returns, especially during market stress. A phenomenon often observed is the increase in correlation among financial assets during market downturns, where stock prices often drop jointly. However, the standard Pearson product-moment correlation cannot explicitly incorporate the effects of such tail events. We propose a methodology that constructs correlation matrices based on quantiles, particularly Value-at-Risk (VaR), and further modify the model to incorporate the Expected Shortfall (ES). The correlation matrix is conditioned on a specific tail area by selecting the quantile level associated with the respective risk measure and then implied by solving a linear system of equations formed by portfolio quantiles, which are determined by a set of specific weight vectors. This approach accommodates large portfolios and is suited for risk aggregation according to financial regulations. In this context, we investigate international stock indices across various quantile levels, return frequencies, and portfolio configurations in an empirical study. Furthermore, the study investigates systematic deviations between implied and Pearson correlation, revealing significant differences, particularly for extreme quantiles, thus justifying the necessity for tail-adjusted matrices.

The fourth chapter introduces a novel approach for forecasting correlation matrices based on the implied correlation methodology. The method combines quantile forecasts obtained from CAViaR models with implied correlation estimates to predict a correlation matrix.

Unlike traditional forecasting methods that require modeling the entire return distribution, the CAViaR model focuses on modeling the respective quantile dynamic directly. The empirical analysis uses six global stock indices, examining correlation forecasts under varying quantile levels and identification strategies. Comparisons are made with alternative forecasts, including Exponentially Weighted Moving Average (EWMA) and Dynamic Conditional Correlation (DCC) models. Since true correlation is not directly observable, we additionally evaluate the quality of the correlation forecast in terms of a Markowitz-styled Global Minimum Variance Portfolio (GMVP). Drawdown metrics such as the historical VaR, ES, and Maximum Drawdown are compared for different parameter settings. The empirical results are competitive but still need further research regarding alternative quantile dynamics, rebalancing strategies, and broader portfolio frameworks to fully realize the benefits of this approach.

We highlight our findings and discuss drawbacks and advantages in the last chapter. Finally, we present ideas for further potential research, which concludes the thesis.

Zusammenfassung

Der Wunsch, Finanzmärkte zu verstehen, ist ungebrochen. Diese Dissertation leistet dazu einen weiteren Beitrag. Wir erforschen Methoden, Renditezeitreihen von Finanzprodukten zu reproduzieren, beleuchten die deren Zusammenhänge vor allem in angespannten Marktumfeldern und stellen eine neue Methode zur Vorhersage von Korrelationen vor. Die Essays zu den einzelnen Themen werden in dieser Dissertation als fünf zusammenhängende Kapitel vorgestellt.

Das erste Kapitel stellt das Forschungsziel der Arbeit vor. Wir motivieren die Forschungsprojekte und geben zudem eine ausführliche Literaturübersicht.

Im zweiten Kapitel wird eine Simulationsmethode vorgestellt, die stochastische Volatilitätsmodelle mit verschiedenen Fehlerverteilungen, bedingten Korrelationsmodellen und Copulas kombiniert. Empirische Untersuchungen deuten darauf hin, dass Zeitreihen von Finanzrenditen eine Reihe von Eigenschaften haben, die üblicherweise als „stylized facts" bekannt sind. Beispiele hierfür sind schwere Ränder, Volatilitätscluster, zeitabhängige Korrelation und nicht lineare Abhängigkeitsstrukturen. Die Motivation für dieses Projekt besteht darin, dass viele etablierte Methoden nur einen Teil dieser Merkmale abdecken und daher suboptimale Ergebnisse liefern. Das Simulationsmodell kombiniert die einzelnen Modelle derart, dass die Ergebnisse nachvollziehbar sind. Die Schätzungen der einzelnen Modelle bieten eine Möglichkeit, wie die Simulationsparameter bestimmt werden können. Schwierigkeiten bei der Implementierung des Modells werden ebenfalls diskutiert, z.B. optimale Startparameter für die Likelihood-Optimierung und Startwerte für die Varianzrekursion. Darüber hinaus werden Samplingmethoden wie Filtered Historical Simulation (FHS) und Moving Block Bootstrap (MBB) zur Herleitung von Standardfehlern diskutiert. Die modulare Struktur des Frameworks ermöglicht Erweiterungen, z.B. Varianzmodelle mit komplexeren Dynamiken, andere Fehlerverteilungen oder Korrelationsdynamiken.

Im dritten Kapitel werden die Abhängigkeitsstrukturen von Finanzrenditen, vor allem bei angespannten Märkten, untersucht. Ein häufiges Phänomen bei fallenden Aktienmärkten ist der Anstieg der Korrelation zwischen Finanzanlagen, wodurch die Kurse gleichzeitig fallen. Die übliche Pearson Korrelation ist jedoch nicht in der Lage, die Auswirkungen von extremen Ereignissen zu berücksichtigen. Wir präsentieren eine Methode, die Korrelationsmatrizen basierend auf Quantilen konstruiert, insbesondere dem Value-at-Risk (VaR) und dem Expected Shortfall (ES). Die Korrelationsmatrix wird durch die Auswahl des Quantilniveaus auf einen bestimmten Bereich der Renditenverteilung bedingt und anschließend durch das Lösen eines Gleichungssystems impliziert, das von Portfolioquantile aufgespannt wird. Jedes Portfolio wird mit anderen Gewichten beschrieben. Der Ansatz über ein Gleichungssystem ist mit einer großen Anzahl an Anlageprodukten kompatibel und liefert eine Matrix, die sich für die Aggregierung von Risiken gemäß regulatorischen Vorschriften eignet. In einer empirischen Studie untersuchen wir die impliziten Korrelationen zwischen Aktienindizes für verschiedene Quantilniveaus, Renditefrequenzen und Portfoliokonfigurationen. Zudem werden systematische Abweichungen zwischen der impliziten und der Pearson-Korrelation untersucht, wobei es vor allem signifikante Unterschiede bei extremen

Quantilen gibt. Entsprechend wird damit die Notwendigkeit von angepassten Korrelationsmatrizen begründet.

Im vierten Kapitel wird ein neuer Ansatz zur Vorhersage von Korrelationsmatrizen vorgestellt, der auf der Methode der impliziten Korrelation beruht. Prognosen von Quantilen werden mittels CAViaR-Modellen erstellt und anschließend als Grundlage für implizite Korrelationen verwendet. Im Gegensatz zu traditionellen Prognosemethoden, die eine Modellierung der gesamten Renditeverteilung erfordern, modelliert das CAViaR-Modell auf direktem Wege das jeweilige Quantil, wodurch eine gezielte Spezifikation der Dynamik ermöglicht wird. Die empirische Analyse betrachtet sechs Aktienindizes und untersucht Korrelationsprognosen bei verschiedenen Quantilniveaus und Identifikationsstrategien. Die Ergebnisse werden mit Prognosen anderer Modelle, darunter das Exponentially Weighted Moving Average (EWMA) und Dynamic Conditional Correlation (DCC), verglichen. Da die tatsächliche Korrelation nicht direkt beobachtbar ist, bewerten wir die Qualität der Korrelationsprognose anhand eines Portfolios. Konkret verwenden wir das Global Minimum-Variance-Portfolio (GMVP) im Stile von Markowitz. Anschließend werden Drawdown-Metriken wie der historische VaR, ES und dem Maximum Drawdown für verschiedene Parametereinstellungen verglichen. Die empirischen Ergebnisse zeigen, dass weitere Forschung hinsichtlich alternativer Quantildynamiken, Rebalancing-Strategien und anderer Portfoliotypen notwendig ist, um das Potenzial dieser Prognosemethode voll auszuschöpfen.

Im letzten Kapitel werden sämtliche Ergebnisse vorgestellt. Wir präsentieren Vor- und Nachteile unserer Modelle und schließen die Arbeit zu möglichen weiterführenden Forschungsarbeiten ab.

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Chapter 1

Introduction

Understanding the distribution of financial returns is a fundamental aspect of financial econometrics. Accurate return modeling is essential to risk management, portfolio optimization, economic forecasting, and financial regulation, to name a few. Numerous empirical studies have identified specific statistical properties of financial time series, commonly referred to as ”stylized facts”, see McNeil et al. (2015). The most prominent include heavy tails, volatility clustering, time-varying correlation, and non-linear dependence structure, where each characteristic has different implications depending on the specific financial application. For example, heavy-tailed return distributions are particularly important to risk management in quantifying extreme losses. Modeling volatility clusters and time-varying correlation is essential for economic forecasting and portfolio optimization, while non-linear dependencies are vital for assessing risk diversification effects.

This thesis studies financial returns in a multivariate setting by introducing a modular simulation framework. The resulting simulation model finds application in various financial applications and sets the stage for studying the dependence structure of financial assets, especially during market stress. Specifically, we are interested in tail-correlation implied by quantiles and subsequently discover a method to predict correlation matrices. The essays on these research topics are embedded in this dissertation and presented as coherent chapters.

Regarding financial time series, despite extensive research on their unique characteristics, most models address only one or a subset of these properties, leaving the full range of financial return dynamics unexplored. We propose a unified framework that integrates existing models to account for the majority of the unique characteristics. Our contribution is to provide a simulation model that can be either calibrated according to the sample data through estimation or by setting the model parameters directly. The flexibility of this model allows the simulation of realistic financial scenarios while maintaining traceability. Furthermore, we discuss practical challenges encountered during the estimation process and the implementation of such a framework, which are often not mentioned further.

Building on the insights gained from studying financial returns, we motivate the necessity for an alternative measure of correlation during extreme events, which is based on another empirical observation that during financial distress, assets tend to decrease in synchrony. Figure 1 shows the historical prices of stock markets from different geographical regions and compares the Pearson correlation before and during events that had a significant impact on the financial markets, namely the global financial crisis caused by the American subprime mortgage crisis in 2008 and the outbreak of Covid-19 in 2020. Although the stock markets seem to be correlated in general, especially between 2000 and 2007, we observe a simultaneous sharp decline in prices during 2008 and 2020. The corresponding heatmaps of the Pearson correlation indicate that the correlation of the returns during these events is significantly higher compared to less turbulent times. However, the heatmaps presented in Figure 1 are ex-post correlation analyses and, therefore, have little practical use since efforts to maintain risk diversification based on correlation should be considered before the occurrence of such events.

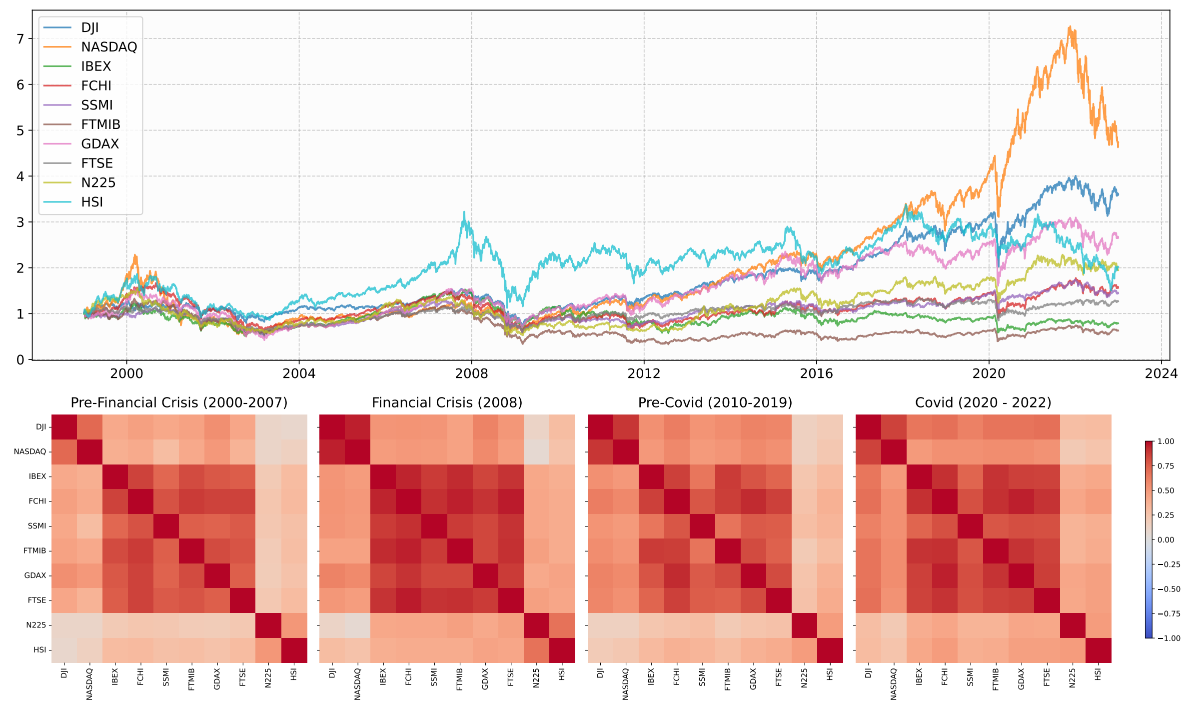


Figure 1: Historical prices & correlation of returns during different market periods.

The empirical fact that correlation increases during market stress has major consequences in risk management, portfolio strategies, and regulation. For example, the European Union introduced the Solvency II directive (European Union 2009) to establish a unified framework for European insurance practice. In principle, the directive requires insurance companies to determine a Solvency Capital Requirement (SCR) based on the Value-at-Risk (VaR), a monetary risk measure, for different risk categories in order to ensure solvency during market stress. The SCR is obtained by aggregating the individual risk estimates with a Pearson correlation matrix. However, the risk is only adequately aggregated in this way if the Pearson correlation fully describes the dependence structure among the financial assets. This is only true for elliptically distributed data, which assume symmetric tail behavior, thus ignoring the fact that extreme values in one variable increase the likelihood of extreme values in another. Based on these facts Mittnik (2011) point out that the methods proposed by the regulators do not account for these characteristics, thus rendering equity risk calibration inefficient, if not meaningless. In particular, a tail-adjusted correlation matrix should be used for risk aggregation instead of the standard Pearson correlation, which underestimates risk.

Our contribution is to construct a tail-adjusted correlation matrix suited for risk management practice by extending the methodology of Campbell et al. (2002), who proposed a correlation estimate implied by VaR although only for two asset portfolios. In particular, the tail-correlation matrix should fulfill the following requirements:

- Ability to handle large number of financial securities and portfolios
- Interpretability of the tail-correlation matrix given a pre-specified risk quantity described by Value-at-Risk or Expected Shortfall
- Suitability for (tail-) risk aggregation
- Minimal theoretical requirements

Regarding the first requirement, the literature discussed extreme correlations mostly in a bi-variate return setting. We imply the unique elements of a correlation in vectorized form by solving a system of linear equations. In doing so, we can obtain correlation matrices for a large asset universe while maintaining computational feasibility. Furthermore, we circumvent the definition of a threshold to classify extreme returns and rely on VaR or Expected Shortfall (ES) instead. This yields interpretable results, i.e., the correlation matrix describes the dependence structure of a specific tail area based on the choice of the quantile level associated with VaR or ES. If risk components are governed by a multivariate elliptical distribution, the obtained correlation coincides with their unconditional Pearson correlation counterpart, which offers a familiar way to interpret the results. If the underlying distribution is not elliptical, it can be viewed as a local elliptical approximation. A formal treatment of local ellipticity is provided in Tjøstheim & Hufthammer (2013). Finally, we can relax the requirement for the existence of moments since quantiles always exist.

A natural extension of the implied correlation methodology to forecasting correlation matrices follows directly from the possibility of forecasting quantiles. Traditionally, forecasting correlation is challenging due to several reasons. Starting with theoretical aspects, each correlation forecast must lie within $[-1, 1]$ while entire matrices must be positive-semidefinite and symmetric. In order to obtain those characteristics, numerical adjustments such as spectral correction for positive-semidefinitness and truncation for valid bounds are made when necessary. Furthermore, forecasting high-dimensional correlation matrices is particularly burdensome with the quadratic increase of correlation pairs. It is also well known that the volatility of asset returns exhibits clustering, which often increases correlation during turbulent market periods, thus adding another layer of complexity that needs to be considered. The main idea is to obtain a correlation matrix prediction based on quantile forecasting. Consequently, the first step is to acquire quantile forecasts of the corresponding financial assets, i.e., forecasting Value-at-Risk or Expected Shortfall. We will use the Conditional Autoregressive Value-at-Risk (CAViaR) method by Engle & Manganelli (2004) to forecast the necessary asset quantiles. Although ES provides more stable results, we will focus on VaR forecasts instead for simplicity. The predicted correlation matrix is given by the implied correlation matrix of the CAViaR forecasts, thus connecting dynamic quantiles with the implied correlation framework.

The thesis is structured as follows: The remainder of Chapter 1 provides a literature overview. Chapter 2 introduces a financial return simulation framework. Existing models and their roles to account for specific characteristics are presented. The empirical work demonstrates the simulation capabilities of the model based on three financial assets, followed by a thorough discussion of the potential drawbacks and challenges while implementing such a model. Chapter 3 introduces the concept of VaR- and ES-implied correlation. The bi-variate case is explained first and extended to an n-asset setting. We discuss different portfolio construction methods that are required to establish a linear system from which the correlation is obtained as a solution of the system. Finally, we analyze correlation patterns produced by different parameter settings in the first part and investigate asymmetries in the second part. Chapter 4 provides an overview of CAViaR models that are used to compute quantile forecasts. In the following step, dynamic quantiles by CAViaR are connected to the implied correlation from the previous chapter. Furthermore, evaluation strategies for correlation forecasts regarding potential benchmarks and portfolio performance are discussed. The viability of this method is assessed by an empirical study involving indices representing markets of different geographical regions. Chapter 5 summarizes all results and point out potential fields for further research.

1.1 Literature Review

The literature discussing models to approximate financial returns is vast. We provide an overview of this field with selected works and more recent literature. In an early study, Ibbotson & Sinquefield (1976) proposed a simulation model by combining forecasts of different economic factors such as government bond yields, interest rates, and inflation. Each factor is governed by a specific model that best describes the corresponding dynamics. For example, interest rates are assumed to follow an autoregressive process, while a random walk drives risk premium. A probability distribution of the returns is then constructed based on the forecast errors. The authors correctly predicted that the volatility of returns is not constant. However, these effects were not addressed due to the lack of statistical tools. After the introduction of the bootstrap by Efron (1982), Korajczyk (1985) applied it to financial returns. This approach was appealing due to its simplicity and the absence of required distributional assumptions. However, for the bootstrap to produce meaningful results, the sample needs to be independent and identically distributed (i.i.d.). Since financial returns exhibit serial correlation, especially in their volatility patterns, this criterion is violated, resulting in biased and inconsistent estimates. To address these issues, refined bootstrap methods were developed, such as the Moving Block Bootstrap (MBB), which samples blocks of data instead of just one data point, and the Filtered Historical Simulation (FHS) by Barone-Adesi et al. (1999) which uses conditional variance models to capture volatility clustering. Ruiz & Pascual (2002) provides an overview of Block Bootstraping methods specifically for financial time series. These sampling methods are a key component of the simulation effort.

Focusing on time series models, Engle (1982) introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model, which models volatility as a function of past innovations. Expanding on this concept, Bollerslev (1986) incorporated historical volatility as an additional component, resulting in the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model. Since then, the GARCH model has been continuously adapted to new findings on financial time series. For example, Nelson (1991) and Glosten et al. (1993) incorporated leverage effects where negative returns tend to increase volatility more than positive returns. Haas et al. (2004) presented Markov-Switching GARCH models that enable different volatility regimes while incorporating non-linearity by selecting normal mixture distributions to drive the innovation process. A mixture distribution is described as a weighted sum of multiple densities and is capable of producing a skewed distribution that is better suited to reflect the typical characteristics of financial returns. Extending these univariate conditional variance models to a multivariate framework has proven challenging due to the fast-increasing number of parameters. The development of conditional correlation models aimed to tackle the computationally burdensome estimation process. The core idea is to treat the volatility characteristics of the assets separately and join them through a correlation matrix based on the standardized residuals. The correlation matrix can be either static, as in the Constant Conditional Correlation model (CCC) by Bollerslev (1990), or dynamic, governed by its own law of motion producing the Dynamic Conditional Correlation model (DCC) by Engle (2002). Thus far, the presented models focus only on the first and second moments of the return distribution. However, higher moments such as skewness and kurtosis measure asymmetry and ”tailedness”, which are especially relevant for extreme market scenarios. In this context, Haas et al. (2004) proposed a mixed normal distribution as the error distribution for a GARCH model. The key idea is to combine several weighted component distributions with different parameters to reproduce the underlying return distributions. Another approach is to connect GARCH models with copulas, as was demonstrated in Jondeau & Rockinger (2006). Copulas provide additional modeling possibilities for non-linear relationships. Similar to conditional correlation models, the dependence structure is also derived from the standardized residuals. The correlation matrix is treated as a parameter of the copula where each element may follow a specific dynamic, e.g. Markov-switching model. The articles presented in this paragraph form the core literature of the first research project.

Regarding literature on correlation during declining markets, Erb et al. (1994) explored the correlation among financial markets concerning economic activity by constructing correlation matrices based on average returns over a time period and concluded that correlation is lowest when the economies expand and highest during joint recessions. Bae et al. (2003), focus on the joint occurrence of extreme returns rather than deriving the correlation. The authors suggest quantiles as a reference to classify extreme events instead of a certain return threshold level, which we will adapt in our research. The results of this study indicate that financial contagion, particularly for large negative returns expressed in joint exceedances, propagates differently in countries with no clear results. However, contagion can be predicted based on exchange rates, interest rates, and conditional stock volatility. Further studies by Longin & Solnik (1995), Butler & C. (2002), Ang & Bekaert (2002), Ang & Chen (2002) explored the behavior of correlations during market stress and the impact on portfolio diversification using GARCH and regime-switching models. Longin & Solnik (2001) investigate the conditional correlation with extreme value theory obtaining so-called exceedance correlation. Their approach is to derive the asymptotic distribution of extreme returns specified by a threshold and compare them with alternative distributions. Formally, let $\{r_{t,i}\}_{t\in I}$ and $\{r_{t,j}\}_{t\in I}$ denote the returns of two financial assets. The exceedance correlation is then defined by

$$\rho^{+} = \mathrm{Corr}(r_{t,i}, r_{t,j} | r_{t,i} \geq \theta_i^{+}, r_{t,j} \geq \theta_j^{+}) \qquad \theta_i^{+}, \theta_j^{+} \geq 0 \tag{1}$$

$$\rho^{-} = \mathrm{Corr}(r_{t,i}, r_{t,j} | r_{t,i} \leq \theta_i^{-}, r_{t,j} \leq \theta_j^{-}) \qquad \theta_i^{-}, \theta_j^{-} \leq 0 \tag{2}$$

where θ_i^{+}, θ_j^{+}, θ_i^{-} and θ_j^{-} are the corresponding thresholds. The exceedance correlation is differentiated between ρ^{+} representing correlation conditioned on significance gains and ρ^{-} for extreme losses. The results of their empirical study favor the perspective that the correlation of large negative returns increases while large positive returns do not. However, Forbes & Rigobon (2002) point out that cross-market correlations are inaccurate if heteroskedasticity of returns is not considered. The authors argue that correlation is conditional on market volatility. Hence, conditional correlation estimates in the style of exceedance correlation are upward biased, particularly during market stress. The central insight is that once heteroskedasticity effects are considered, financial contagion, vanishes to a certain degree, concluding that markets are already highly correlated beforehand and not only during crisis times. In addition, typical for methods focusing on extreme or rare events, the estimation also becomes increasingly unreliable due to data scarcity.

Finally, an early study on correlation forecast was done by Erb et al. (1994), who used the forecast results for portfolio optimization and obtained substantially different asset allocation weights compared to correlation matrices from historical data. The methodology is based on an instrumental variable regression approach proposed by Longin & Solnik (1995), which includes instruments that reflect the persistence of correlation and economic factors such as business cycles, recession, and expected returns. However, this approach could not account for correlation clustering and thus ignores one essential characteristic of financial time series.

Following the introduction of the GARCH model by Bollerslev (1986), its multivariate generalization produced several extensions to forecast covariance/correlation. The VEC-GARCH of Bollerslev et al. (1988) treats conditional variance and covariances as lagged squared returns and cross-products of returns. This model is flexible because each parameter can be modeled separately but has restrive conditions for the covariance matrix to be positive definite. The GARCH-BEKK (Engle & Kroner 1995) model circumvents this problem by replacing the constants with another parameter matrix that ensures positive definiteness. However, all three presented models are challenging to estimate due to the large amount of parameters and the requirement to invert several matrices. Silvennoinen & Teräsvirta (2009) provides an in-depth discussion of these models. The Dynamic Conditional Correlation model proposed by Engle (2002) amends the aforementioned drawbacks by modeling asset variance and portfolio correlation separately, which significantly reduces the number of parameters compared to VEC-GARCH or GARCH-BEKK, thus making DCC an efficient and parsimonious model. We refer to Section 2.2.3 in which the theoretical properties of the DCC models are discussed. Further content on DCC extensions is found in Engle (2009).

In a more recent study, Bollerslev et al. (2022) conducted correlation forecasting by combining a large set of features (lagged returns, high and low-frequency market data, sector information, etc.) into a linear model. Different machine learning-inspired estimation techniques (Ridge Regression, Elastic Net, or Neural Networks) are then explored with the LASSO-based estimates producing the most promising results regarding theoretical benchmarks and economic profitability. The Heterogeneous Autoregressive model (HAR) serves as a reference model. Similar to our study, the authors evaluate the viability of correlation forecasts by constructing a Global Minimum Variance Portfolio (GMVP) and evaluating its performance using various risk metrics.

The literature on forecasting VaR is vast, and providing an overview of all forecasting methods is out of the scope of this work. Classical methods like the historical, variance-covariance, or Monte Carlo approach are discussed in McNeil et al. (2015). A survey on advanced methods such as extreme value theory, time series with different error distributions, and quantile regression are provided in Kuester et al. (2005). Finally, although not related to risk management directly, Kelly & Xiu (2023) gives an overview of machine learning methods applied to topics in finance, including simulating returns in general. Deep quantile regression methods using neural networks are also intensively studied at the moment; see Chronopoulos et al. (2024) and the literature therein.

Chapter 2

Advances in Risk and Return Modeling: Estimation, Simulation, Application

2.1 Defining Financial Returns for Statistical Modeling

The main objects of interest are financial returns, which are more desirable for statistical modeling than prices due to their stationarity property. In addition, returns can be modified to account for taxes, dividends, and fees. They offer a scale-free quantity that is comparable across different asset classes.

Definition 2.1.1 (Net Return & Logarithmic Return)
Let P_t be the price of a financial security at time t. The one-period simple ***net return*** *R_t is given by*

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{3}$$

The ***logarithmic return*** *or continuously compounded return is defined as*

$$r_t = ln\left[\frac{P_t}{P_{t-1}}\right]. \tag{4}$$

Logarithmic returns often approximate a normal distribution more closely than simple returns, especially for daily or high-frequency intraday data. Furthermore, they can be aggregated linearly by a sum and behave symmetric in response to changes. The former property greatly increases computational efficiency if returns over multiple periods are considered.

We establish the concept of stationarity. Every observed time series can be viewed as a realization of a stochastic data-generating process. Understanding this data-generating process is the main goal of all modeling exercises. However, estimates are only reliable if certain stability assumptions are made. In particular, the marginal and joint distribution characterizing the time series must remain stable over time to ensure consistency and sound asymptotic behavior.

Throughout this thesis, we treat only real-valued random variables and time series models.

Definition 2.1.2 (Covariance Stationarity)
*A time series $\{Y_t\}_{t\in\mathbb{Z}}$ is **covariance stationary** or weakly stationary if*

$$\mathbb{E}[Y_t] = \mu \quad \textit{and} \quad \mathbb{V}[Y_t] = \sigma^2 \qquad \textit{for all } t \in \mathbb{Z}, \tag{5}$$

$$Cov[Y_t, Y_{t-k}] = \gamma_k \qquad \textit{for all } t, k \in \mathbb{Z}. \tag{6}$$

In other words, the mean, variance and autocovariance functions are finite and independent of time.

Assuming that a financial return series is stationary guarantees that its first and second moment are well-defined and suitable for modeling. We refer to Hansen (2022) and the literature therein for other stationarity concepts and focus on covariance stationary processes for the remainder of this work.

In a time series model, the randomness is usually described by a white noise process. It serves as an innovation process where realizations are also called (economic) shocks.

Definition 2.1.3 (White Noise Process)
*A process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a **white noise process** if*

$$\mathbb{E}[\varepsilon_t] = 0 \quad \textit{and} \quad \mathbb{V}[\varepsilon_t] = \sigma^2 < \infty \qquad \textit{for all } t \in \mathbb{Z}, \tag{7}$$

$$Cov[\varepsilon_t, \varepsilon_{t-k}] = 0 \qquad \textit{for } k \neq 0. \tag{8}$$

A white noise process is a serially uncorrelated zero mean process with finite variance.

In many econometric applications, white noise is assumed to follow a normal distribution. We will study alternative distributions that are more suited to modeling financial returns because a normal distribution cannot capture heavy tails or asymmetries.

2.2 Capturing Financial Time Series Dynamics

In this chapter, we provide an overview of the building blocks for the simulation framework and discuss the purpose of each model. Starting on a univariate level, stochastic volatility models capture asset-specific properties such as volatility clustering patterns, heavy tails, asymmetry, and leverage effects. This stage yields standardized residuals after filtering the original return series. Next, the linear dependence structure of the assets is captured by a correlation model. In particular, the law of motion is determined by the specification of the correlation model. Through this step, we obtain cross-filtered residuals. Finally, the remaining potential non-linear relationships or joint occurrences of extreme events are covered by a copula. Furthermore, we assume stationary return series with zero mean. Otherwise, the sample is centered by $r_t - \mu$ for all t. A constant μ is a direct consequence of stationarity. Figure 2 visualizes the role of each model component.

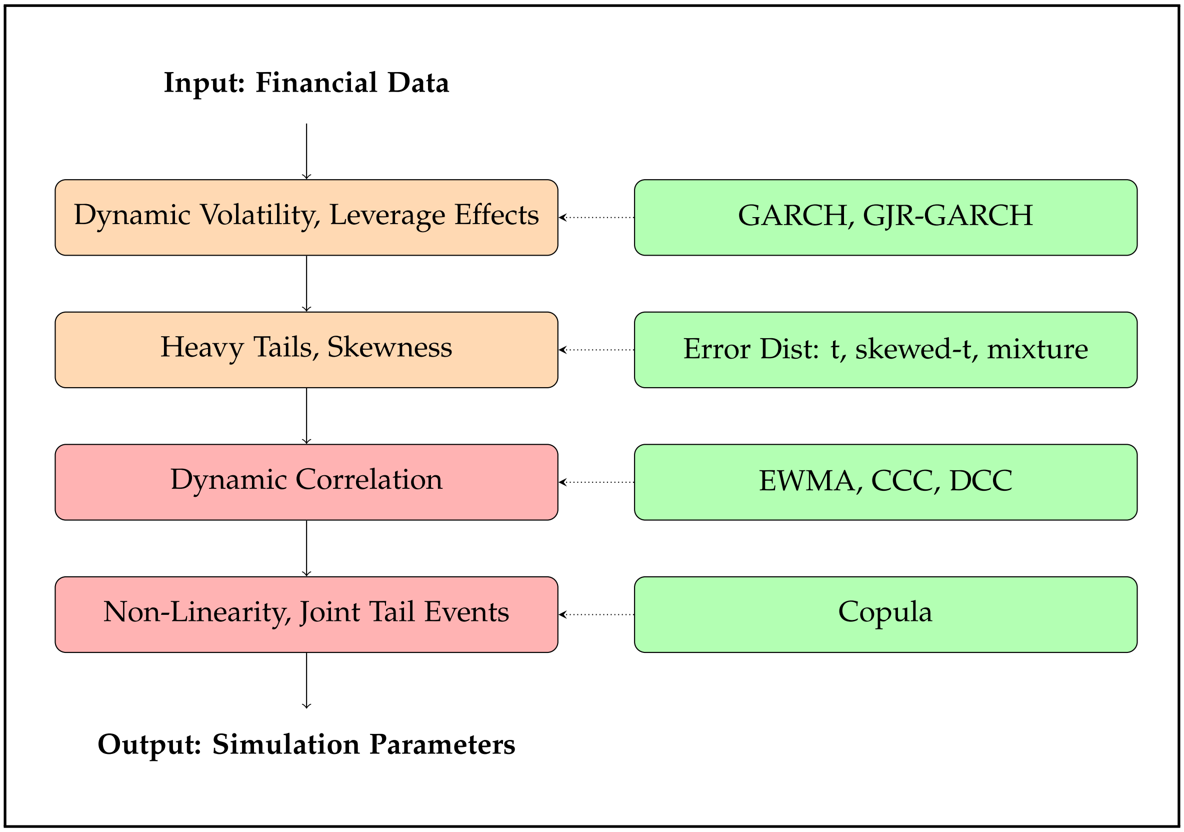


Figure 2: Workflow of addressing financial time series characteristics

2.2.1 Stochastic Volatility Models

The primary goal of stochastic volatility models is to address the dynamic nature of volatility, i.e., volatility clustering. Engle (1982) proposed the Autoregressive Conditional Heteroskedasticity (ARCH) model where the volatility is a function of squared historical shocks and provided stability criteria. Bollerslev (1986) generalized this model by including historical volatility additionally. The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model has become a standard model in both theory and pratices and serves as a benchmark for other volatility models, see Hansen & Lunde (2005). We present two variants of the GARCH model, namely the original GARCH and the GJR-GARCH which also incorporate leverage effects.

Definition 2.2.1 (GARCH)
*Let $\{r_t\}_{t\in\mathbb{Z}}$ denote a return series of a financial security. Define $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ as a white noise process of any distribution $\mathbb{P}$. A **GARCH(p,q)** process with $p, q \in \mathbb{N}$ is described by*

$$r_t = \sigma_t \varepsilon_t, \tag{9}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{10}$$

with $\omega, \alpha_i, \beta_j \in \mathbb{R}$, $0 < i \leq p$ and $0 < i \leq q$. The process is covariance stationary if

$$\omega > 0, \quad \alpha_i, \beta_j \geq 0 \quad \textit{for all } i, j \quad \textit{and} \tag{11}$$

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1. \tag{12}$$

If the process is stationary, the unconditional variance of the model is given by

$$\sigma^2 = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}. \tag{13}$$

The constant in the variance equation (10) contributes to the unconditional variance, while the coefficients describe the influence of squared past shocks and volatility. The sum of the coefficients in Equation (12) describes the overall persistence of shock on the model. It determines if the impact of any shocks decays over time and, thus, allows the model to revert to its unconditional long-term variance. Evidently, the unconditional variance grows significantly if the model exhibits strong persistence.

One drawback of the base GARCH model is that shocks are treated symmetrically. However, empirical works have shown that price declines tend to have a more significant impact on volatility than growth. We will also discuss the impact of this phenomenon on dependence structure in Chapter 3. On the univariate scale, Glosten et al. (1993) presents a modification that allows the GARCH model to account for such leverage effects.

Definition 2.2.2 (GJR-GARCH)
*Let $\{r_t\}_{t\in\mathbb{Z}}$ denote a return series of a financial security. Define$\{\varepsilon_t\}_{t\in\mathbb{Z}}$ as a white noise process of any distribution $\mathbb{P}$. A **GJR-GARCH(p,o,q)** process is defined by*

$$r_t = \sigma_t \varepsilon_t \tag{14}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{k=1}^{o} \gamma_k I_{t-k} r_{t-k}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{15}$$

with

$$I_t = \begin{cases} 1, & \text{if } r_t < 0 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

and $\omega, \alpha_i, \beta_k, \gamma_j \in \mathbb{R}$, $0 < i \leq p$, $0 < j \leq q$ and $0 < k \leq o$. The process is covariance stationary if

$$\omega > 0, \quad \alpha_i, \gamma_k, \beta_j, \geq 0 \quad \text{for all } i, k, j \quad \text{and} \tag{17}$$

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j + \frac{1}{2} \sum_{k=1}^{o} \gamma_k < 1. \tag{18}$$

Motivated by theoretical ambiguity and empirical conflicts regarding the trade-off between risk and returns, Glosten et al. (1993) reviewed works that state a positive relation between expected excess return and conditional variance. In contrast, others find a negative or no significant relation at all. It is argued that common models cannot account for seasonal effects and asymmetry of return innovations, which the GJR-GARCH model tries to overcome. Unlike the standard GARCH model, the GJR-GARCH model incorporates leverage effects. The leverage parameter γ_j captures the asymmetric response of volatility to positive and negative shocks in the previous returns, according to the sign of the last realizations. This model can be modified by including dummy variables to capture seasonal effects. We will focus on a simple GJR-GARCH model with $p = q = o = 1$. Zakoian (1994) extends the base model to a new family of threshold models to which the GJR-GARCH belongs.

2.2.2 Error Distribution

Selecting the error distribution for the white noise process plays an essential role in defining the behavior of the stochastic volatility model. While the variance equation only specifies the dynamic of the volatility, the error distribution determines the range and characteristics of each innovation such as tail events and asymmetry. We provide an overview of possible choices.

Definition 2.2.3 (Normal Distribution)
*Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable X follows a Gaussian or **normal distribution** with mean μ and variance σ^2, denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$, if $\mathbb{P}$ has a probability density function of the form*

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \text{with } x \in \mathbb{R} \tag{19}$$

where $exp(\cdot)$ represents the exponential function.

In many applications, normal white noise is assumed. However, Baillie & Bollerslev (1989) pointed out that although a GARCH model with normal white noise is able to generate a heavy-tailed return distribution up to a certain degree, the result still does not accurately reflect the underlying sample distribution. The authors propose the t-student distribution as an alternative.

Definition 2.2.4 (t-Student Distribution)
*Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable X follows a **t-distribution** with $\nu > 2$ degrees of freedom, denoted by $X \sim t(\nu)$, if $\mathbb{P}$ has a probability density function of the form:*

$$f_{\nu}(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{(\nu-2)\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \quad \text{with } x \in \mathbb{R} \tag{20}$$

where $\Gamma(\cdot)$ represents the Gamma function. An alternative parameterization of the t-distribution with a location parameter $\mu \in \mathbb{R}$, a scale parameter $\sigma > 0$ and a tail parameter $\nu > 2$ is given by:

$$f_{\mu,\sigma,\nu}(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sigma\sqrt{(\nu-2)\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{(x-\mu)^2}{\sigma^2(\nu-2)}\right)^{-\frac{\nu+1}{2}} \quad \text{with } x \in \mathbb{R}. \tag{21}$$

The additional degree of freedom parameter ν characterizes the tail behavior. For $\nu \to \infty$, the t-student distribution converges to the normal distribution. A small value of ν puts more probability mass on the tails of the conditional distribution, thus implying more extreme events.
However, Hansen (1994) points out that there is no reason to believe that the conditional distribution of returns is sufficiently described only by their first two moments. Indeed, numerous studies show that asymmetries exist in covariance and volatility, which is directly reflected in both frequency and severity of extreme losses during market stress, see Bekaert & Wu (2000). As a result, the requirements for a feasible distribution are a closed-form density function and a low-dimensional parameter vector. We present two generalizations of the t-student distribution that fulfill these two conditions and focus on the normalized version, i.e., zero mean and unit variance.

Definition 2.2.5 (Skewed t-Student Distribution, Hansen (1994))
Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable X follows a ***normalized skew t-distribution*** *with degrees of freedom ν and shape parameter λ, denoted by $X \sim st(\nu, \lambda)$, if $\mathbb{P}$ has a probability density function given by:*

$$f_{\nu,\lambda}(x) = \begin{cases} bc\left(1 + \frac{1}{\nu-2}\left(\frac{bx+a}{1-\lambda}\right)^2\right)^{-\frac{\nu+1}{2}} & \text{for } x < -a/b, \\ bc\left(1 + \frac{1}{\nu-2}\left(\frac{bx+a}{1+\lambda}\right)^2\right)^{-\frac{\nu+1}{2}} & \text{for } x \geq -a/b, \end{cases} \tag{22}$$

where $x \in \mathbb{R}$, $2 < \nu < \infty$, and $-1 < \lambda < 1$. The constants a, b, and c are given by:

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right), \tag{23}$$

$$b^2 = 1 + 3\lambda^2 - a^2, \tag{24}$$

$$c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}, \tag{25}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The parameter ν controls the tail behavior of the distribution. The parameter λ governs the skewness, with $\lambda = 0$ reducing the distribution to the standard t-distribution.

Theodossiou (1998) studies the natural extension of the skewed-t distribution towards the skew generalized t-distribution in which the kurtosis is also included. Further properties regarding variants of the t-distribution and ways to represent asymmetric elliptical distributions are studied in Azzalini & Capitanio (2003) and Azzalini & Genton (2008). In Gupta (2003), the skew t-distribution is extended to a multivariate framework.

2.2.3 Conditional Correlation Models

Univariate models are widely used in practice due to their simplicity and mathematical tractability. Nevertheless, univariate models, by nature, cannot capture the dependence structures with other assets. In a multivariate time series setting, the covariance matrix usually describes the dependence structure. The VEC-GARCH of Bollerslev et al. (1988) and GARCH-BEKK by Engle & Kroner (1995) are notable models for a multivariate GARCH extension. Although these models offer high flexibility, the estimation process becomes computationally cumbersome as the number of assets increase. In particular, the number of parameters increase quadratically. An additional challenge is to guarantee that the covariance matrix remains well-defined, i.e., positive semi-definite. A major advancement in treating volatility models in a multivariate framework while maintaining practical applicability was presented in Bollerslev (1990) and Engle (2002), where conditional variance and correlation are separately modeled.

Bollerslev (1990) developed the Constant Conditional Correlation (CCC) as an extension of the Seemingly Unrelated Regression (SUR) to understand the comovements of exchange rates. In particular, variance and covariance are dynamic, while the correlation is kept constant. In context of financial time series, denote $\{\boldsymbol{r}_t\}_{t\in I}$ as a series of $(n \times 1)$-dimensional vector of returns for an index set I and $\{\boldsymbol{z}_t\}_{t\in I}$ a multivariate white noise process. Assuming zero mean, the multivariate equivalent to the univariate conditional variance model can be expressed as

$$\boldsymbol{r}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{z}_t \tag{26}$$

where $\boldsymbol{H}_t$ denotes the conditional covariance matrix. The key idea is to decompose the covariance matrix into

$$\boldsymbol{H}_t = \boldsymbol{D}_t\boldsymbol{R}\boldsymbol{D}_t \tag{27}$$

where $\boldsymbol{D}_t$ denotes a diagonal matrix containing the asset specific stochastic volatilities governd by individual univariate models, i.e.

$$\boldsymbol{D}_t = \begin{bmatrix} \sigma_{1,t} & 0 & \cdots & 0 \\ 0 & \sigma_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_{n,t} \end{bmatrix}. \tag{28}$$

thus seperating volatility and dependence modeling which greatly increases estimation efficiency and stability. The model is called Constant Conditional Correlation because the correlation matrix $\boldsymbol{R}$ in Equation (27) is time-independent.

The natural time-dependent generalization was provided by Engle (2002). Three possible specifications for the correlation dynamics were investigated: A simple rolling correlation with a prespecified lag window, an exponential smoother with a decay factor, and GARCH-like dynamics. Regarding the first two methods, it is unclear what conditions are required for consistent estimates. The rolling correlation method requires selecting a lag window, while the exponential smoother has no criteria for choosing the decay factor. Using specific heuristic values is not backed by any theory but originates from practical experience. Regarding the GARCH dynamic, we assume normality, which provides a likelihood function that can be optimized. Even if the normality assumption does not hold, as long as the conditional mean and variance are correctly specified, the maximum likelihood estimate will be asymptotically normal and consistent, see Glosten et al. (1993). In this case, a Quasi-Maximum-Likelihood estimation is performed. We will introduce GARCH-like dynamics and refer to Engle (2002) for alternative dynamic correlation parameterizations.

Definition 2.2.6 (Dynamic Conditional Correlation)
Let $\{\mathbf{r}_t\}_{t\in\mathbb{Z}}$ be a d-dimensional vector of financial returns. Let $\{z\}_{t\in\mathbb{Z}}$ denote a vector-valued white noise process. The ***Dynamic Conditional Correlation*** *(DCC) model with GARCH-like dynamics is defined by the following equations:*

$$\boldsymbol{r}_t = \boldsymbol{C}_t\boldsymbol{x}_t + \boldsymbol{\varepsilon}_t \tag{29}$$
$$\boldsymbol{\varepsilon}_t = \boldsymbol{H}_t^{1/2}\mathbf{z}_t \tag{30}$$
$$\boldsymbol{H}_t = \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t \tag{31}$$
$$\boldsymbol{R}_t = diag(\boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t diag(\boldsymbol{Q}_t)^{-1/2} \tag{32}$$
$$\boldsymbol{Q}_t = (1-\lambda_1-\lambda_2)\boldsymbol{R} + \lambda_1\tilde{\boldsymbol{\varepsilon}}_{t-1}\tilde{\boldsymbol{\varepsilon}}_{t-1}^T + \lambda_2\boldsymbol{Q}_{t-1}. \tag{33}$$

where $\boldsymbol{C}_t$ is a matrix of parameters governing the mean process, $\boldsymbol{x}_t$ is a vector of independent variables which may contain lagged values of $\boldsymbol{r}_t$, $\boldsymbol{H}_t^{1/2}$ represents the Cholesky factor of the time-varying conditional covariance matrix $\boldsymbol{H}_t$ which corresponds to the multivariate volatility and $\tilde{\boldsymbol{\varepsilon}}_t = \boldsymbol{D}_t^{-1}\boldsymbol{\varepsilon}_t$ represents the vector of standardized residuals.

The DCC recursion in Equation (33) represents GARCH-like dynamics by incorporating historical squared residual vectors and correlation. For a stable DCC model, λ_1, λ_2 are nonnegative and satisfy $0 \leq \lambda_1 + \lambda_2 < 1$ similar to the stationarity condition. Equation (32) normalizes $\boldsymbol{Q}_t$ such that the dynamic correlation coefficients in $\boldsymbol{R}_t$ remain in [-1,1]. If $\boldsymbol{R}_t$ is assumed to be time-independent, then this model becomes the CCC model.

2.2.4 Copula

In this modeling framework, copula models constitute the last building block. They provide further flexibility regarding dependence modeling beyond linear relationships covered by the correlation models presented in the previous section. The correlation models yield standardized cross-correlation filtered residuals on which the copula is built. A key feature of copulas is their ability to model dependency structures independently of their margins. This separation enables additional options to reflect the unique characteristics of financial assets in terms of their interactions. At their core, copulas link multivariate distribution functions to their marginals, grounded in Sklar's theorem.

Theorem 2.2.1 (Sklar 1959)
Let H be a joint cumulative distribution function (cdf) with marginals $F_1(x_1), \ldots, F_d(x_d)$. Then there exists a copula $\mathrm{C} : [0,1]^d \rightarrow [0,1]$ such that for all $\mathbf{x} = (x_1, \ldots, x_d) \in \bar{\mathbb{R}}^d$:

$$H(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d)). \tag{34}$$

If the marginals $F_1(x_1), \ldots, F_d(x_d)$ are continuous, then C is unique. Otherwise, C is uniquely determined on the cartesian product of the ranges of the marginal cdfs $Ran(F_1) \times ... \times Ran(F_d)$.

Conversely, if $F_1(x_1), \ldots, F_n(x_d)$ are continuous marginal distribution functions and C a copula, then the function H is a joint distribution function with marginal distributions $F_1(x_1), \ldots, F_n(x_d)$. In other words, any multivariate joint distribution can be represented as a copula combining its marginals.

Remark 2.2.1
A copula density function is directly acquired by computing the derivative of C. Set $u_i \equiv F(x_i)$, then the copula density c is given by

$$c(\mathbf{u}) = \frac{\partial C(\mathbf{u})}{\partial u_1 \cdots \partial u_d} = \frac{h(\mathbf{x})}{\prod_{i=1}^{d} f_i(x_i)} \tag{35}$$

where $\mathbf{u} = (u_1, ..., u_d) \in [0,1]^d$. The existence of such a density function c is guaranteed by Sklar's theorem. The joint and marginal density is denoted by $h(\cdot)$ and $f_i(\cdot)$ respectively.

A Copula-GARCH model was proposed by Jondeau & Rockinger (2006). In their work, returns were modeled by a GARCH process with skewed-T distributed errors and joined by elliptical copulas, i.e., Gaussian and t-student. In particular, the parameters of the error distributions and the copula may vary over time. While an autoregressive process governs the parameters of the error distributions, the correlation parameters of the copula are characterized by either a time-varying correlation or Markov regime-switching models. Another approach to model multivariate dependence dynamically was presented by So & Yeung (2014), where the authors use vine copulas. The benefit of using vine copulas is the decomposition of a multidimensional density into a product of conditional bivariate copulas and marginal densities. As a result, this method also allows the study of high-dimensional portfolios. A survey on copula-based models can be found in Patton (2012).

We will focus on parametric copulas with continuous marginals because the dependence structure can be extracted directly from multivariate distributions with unique closed-form densities that enable estimation by the Maximum-Likelihood principle. A rigorous treatment of copula theory is found in Nelsen (2006) and Joe (1997). We present the Gaussian, t-student, and skewed-t copulas, and discuss their strengths and weaknesses in the context of financial return modeling.

Definition 2.2.7 (Gaussian Copula)
*Let Φ^{-1} be the inverse of the univariate cdf of the standard normal distribution Φ. Denote the multivariate normal cdf with correlation matrix $\boldsymbol{\mathcal{R}}$ as $\boldsymbol{\Phi}_{\boldsymbol{\mathcal{R}}}$. The d-dimensional Normal Copula or **Gaussian Copula** $C : [0,1]^d \to [0,1]$ is given by*

$$C(\mathbf{u}; \boldsymbol{\mathcal{R}}) = \Phi_{\boldsymbol{\mathcal{R}}}(\mathbf{z}) \tag{36}$$

with density

$$c(\mathbf{u}; \boldsymbol{\mathcal{R}}) = \frac{\phi_{\boldsymbol{\mathcal{R}}}(\mathbf{z})}{\prod_{i=1}^{n} \phi(\Phi^{-1}(u_i))} = \frac{1}{\sqrt{\det(\boldsymbol{\mathcal{R}})}} \exp(-\frac{1}{2}\mathbf{z}^\top(\boldsymbol{\mathcal{R}}^{-1} - \mathbf{I})\mathbf{z}) \tag{37}$$

where $\mathbf{z} = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))$ and $\mathbf{I}$ represents the identity matrix.

We use a different notation style for the correlation parmeter to avoid confusion with the conditional correlation models. The Gaussian copula is a member of the elliptical copula family and fully described by the correlation matrix $\boldsymbol{\mathcal{R}}$. However, it lacks the ability to account for tail dependence rendering this specific copula unsuitable for financial applications. The Gaussian copula gained attention in the aftermath of the financial crisis 2008 due to its (miss)usage in modeling credit derivatives, that is, modeling default probabilities. In this context, one crucial deficiency of the Gaussian copula is the inability to model extreme events such as default clustering. Extreme events are even treated independently as their size of default increases. In conjunction with reckless risk management and banking practices, it contributed to the American subprime mortgage crisis outbreak. Donnelly & Embrechts (2010) provide a detailed overview of this topic. Regarding market risk, the t-student copula stands out as a viable alternative, particularly in addressing the issue of tail dependence that the Gaussian copula fails to handle.

Definition 2.2.8 (t-Student Copula)
Let T_ν^{-1} be the inverse function of the univariate cdf of the t-student distribution T_ν. Denote the multivariate t-student cdf with correlation matrix $\boldsymbol{\mathcal{R}}$ and degree of freedom parameter $\nu > 2$ as $\mathbf{T}_{R,\nu}$. The d-dimensional t-student Copula $C : [0,1]^d \to [0,1]$ is given by

$$C(\mathbf{u}; \boldsymbol{\mathcal{R}}, \nu) = T_{\boldsymbol{\mathcal{R}},\nu}(T_\nu^{-1}(u_1), ..., T_\nu^{-1}(u_d)) \tag{38}$$

with density

$$c(\mathbf{u}; \boldsymbol{\mathcal{R}}, \nu) = \frac{t_{\boldsymbol{\mathcal{R}},\nu}(T_\nu^{-1}(u_1), ..., T_\nu^{-1}(u_d))}{\prod_{i=1}^{n} t_\nu(T_\nu^{-1}(u_i))} = \frac{1}{\sqrt{|\boldsymbol{\mathcal{R}}|}} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\nu\pi)^d}} \left(1 + \frac{\mathbf{z}^\top \boldsymbol{\mathcal{R}}^{-1}\mathbf{z}}{\nu}\right)^{-\frac{\nu+d}{2}}. \tag{39}$$

where $\mathbf{z} = (t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_d))$, Γ is the gamma function and ν is the degrees of freedom of the t-distribution.

The t-Student copula has an additional parameter that governs tail behavior allowing for a more realistic representation of joint extreme market movements. However, similar to the Gaussian copula the t-student copula assumes symmetric tail dependence, meaning it treats the likelihood of joint extreme losses the same as joint extreme gains.

A possible solution is to consider asymmetric copulas similar to adjusting the error distribution for the univariate conditional variance models. A candidate is the skewed-t copula proposed by Demarta & McNeil (2005). Note that this copula is not elliptical but rather an extension of the elliptical family. First, we introduce the multivariate skewed-t distribution as a mean-variance mixture. The dependence structure of the copula can be extracted in the same way as was done with the previous mentioned parametric copulas. We follow the notation of Demarta & McNeil (2005). A mean-variance mixture is of the form

$$\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{\gamma} g(W) + \sqrt{W}\boldsymbol{Z} \tag{40}$$

for some function $g : [0, \infty) \to [0, \infty)$ and a d-dimensional parameter vector $\boldsymbol{\gamma}$. With $\boldsymbol{\gamma} \neq 0$, setting g as the identity and $W \sim \text{Ig}(\nu/2, \nu/2)$ where Ig represents the inverse gamma distribution yields a skewed multivariate t distribution. The resulting density, which could be used to construct the copula, is defined by

$$f(\mathbf{x}) = c \frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu + (\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}))\boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}\right)\exp((\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma})}{\sqrt{(\nu + (\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}))\boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}^{\,\nu+d}\left(1+\frac{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{\frac{\nu+d}{d}}} \tag{41}$$

where

$$c = \frac{2^{\frac{2-(\nu+d)}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \tag{42}$$

is a normalizing constant, and K_λ denotes a modified Bessel function. An in-depth discussion on the properties of this copula can be found in Demarta & McNeil (2005). Investigating the effect of more sophisticated copula to capture non-linear and non-symmetric dependence is a potential topic for future research.

2.3 Simulation Parameters

2.3.1 Estimating the Model Components

Calibrating the simulation model amounts to estimating the model components starting with the univariate stochastic volatility models. Let $\{r_t\}_{t \in \mathbb{Z}}$ denote a return series of a specific asset.

For illustration, define a GARCH(1,1) model and a generic white noise process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$:

$$r_t = \mu + \sigma_t \varepsilon_t \tag{43}$$

$$\sigma_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{44}$$

with $\varepsilon_t \overset{iid}{\sim} \mathcal{WN}$. Assume that $\mu = 0$, otherwise, center the returns by substracting the mean. The likelihood function is determined by the choice of the error distribution. If normal white noise is assumed, i.e., $\mathcal{WN} \equiv \mathcal{N}(0,1)$, the logarithmic likelihood function for optimization with $\boldsymbol{\theta} = (\mu, \omega, \alpha, \beta)'$ is given by

$$l_r(\boldsymbol{\theta}) = \sum_{t=1}^{T} \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma_t^2) - \frac{(r_t - \mu_i)^2}{2\sigma_t^2} \right]. \tag{45}$$

The first term in the square brackets is usually discarded for the optimization process since they do not contain any parameters.

Similarly, if the error distribution follows a t-student distribution with degree of freedom $\nu > 2$, i.e., $\mathcal{WN} \equiv t(\nu)$ the target function with $\boldsymbol{\theta} = (\nu, \mu, \omega, \alpha, \beta)$ becomes

$$l_r(\boldsymbol{\theta}) = T\left[\ln\Gamma\left(\frac{v+1}{2}\right) - \ln\Gamma\left(\frac{v}{2}\right) - \frac{1}{2}\ln\left[(v-2)\pi\right]\right] \tag{46}$$

$$- \sum_{t=1}^{T} \left[\frac{v+1}{2} \ln\left(1 + \frac{(r_t - \mu)^2}{(v-2)\sigma_t^2}\right) + \ln(\sigma_t^2) \right]. \tag{47}$$

where Γ represents the gamma function. The degree of freedom parameter ν can be either jointly estimated with the remaining parameters or set as a predetermined value. If so, the common value is chosen between 4 and 8 according to Tsay (2010). In this case, the log-likelihood function is reduced to

$$l_r(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \left[\frac{v+1}{2} \ln\left(1 + \frac{(r_t - \mu)^2}{(v-2)\sigma_t^2}\right) + \ln(\sigma_t^2) \right]. \tag{48}$$

For the skewed-t distribution, we refer to Azzalini & Genton (2008) and the literature therein.

Estimating the DCC models also requires a distributional assumption for the data, here the standardized residuals. With it, the parameters of the conditional correlation model can be estimated with the Maximum Likelihood principle. Recall that the fundamental concept of the DCC model lies in the decomposition of the covariance matrix into a diagonal matrix $\boldsymbol{D}_t$ containing the conditional variance and a dynamic correlation matrix $\boldsymbol{R}_t$, formally

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t. \tag{49}$$

This is directly reflected in the likelihood function for a given distribution. Denote Θ as the vector containing all parameters of the correlation model. For normal distributed residuals, the likelihood function and its logarithmic counterpart are given by

$$L_{\varepsilon}(\boldsymbol{\Theta}) = \prod_{t=1}^{T} \frac{1}{(2\pi)^{\frac{n}{2}} \, |\boldsymbol{D_t}||\boldsymbol{R_t}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{\varepsilon}_t'(\boldsymbol{D_t}\boldsymbol{R_t}\boldsymbol{D_t})^{-1}\boldsymbol{\varepsilon}_t\right) \tag{50}$$

$$l_{\varepsilon}(\boldsymbol{\Theta}) = -\frac{1}{2}\sum_{t=1}^{T}\left(n\ln(2\pi) + 2\ln|\boldsymbol{D_t}| + r_t'\boldsymbol{D_t}^2 r_t + \ln|\boldsymbol{R_t}| + \boldsymbol{\varepsilon_t}'\boldsymbol{R_t}^{-1}\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon_t}'\boldsymbol{\varepsilon}_t\right). \tag{51}$$

Next, define $\boldsymbol{\theta}$ as the vector containing all parameters of the conditional variance models and $\boldsymbol{\psi}$ as the vector of the parameters for the correlation model, which contains the unique elements of the correlation matrix in addition to parameters describing its dynamics. The log-likelihood function in Equation (51) can be divided into a variance and correlation component. Formally,

$$l_{\varepsilon}(\boldsymbol{\Theta}) = l_{\boldsymbol{r}}(\boldsymbol{\theta}) + l_{\varepsilon}(\boldsymbol{\psi}) \tag{52}$$

with

$$l_{\boldsymbol{r}}(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{t=1}^{T}\left(n\ln(2\pi) + 2\ln|\boldsymbol{D_t}| + \boldsymbol{r}_t'\boldsymbol{D}_t^{-2}\boldsymbol{r}_t\right) \tag{53}$$

$$l_{\varepsilon}(\boldsymbol{\psi}) = -\frac{1}{2}\sum_{t=1}^{T}\left(\ln|\boldsymbol{R_t}| + \boldsymbol{\varepsilon}_t'\boldsymbol{R}_t^{-1}\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon_t}'\boldsymbol{\varepsilon}_t\right). \tag{54}$$

If the variance entries $\boldsymbol{D}_t$ are pre-specified by conditional variance models, estimating the parameters of the DCC model reduces to optimizing $l_{\varepsilon}(\boldsymbol{\psi})$ in Equation (54).

Recall that the dynamic of the correlation matrix is described by

$$\boldsymbol{R}_t = \text{diag}(\boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t\text{diag}(\boldsymbol{Q}_t)^{-1/2} \tag{55}$$

$$\boldsymbol{Q}_t = (1-\lambda_1-\lambda_2)\boldsymbol{R} + \lambda_1\tilde{\boldsymbol{\varepsilon}}_{t-1}\tilde{\boldsymbol{\varepsilon}}_{t-1}^T + \lambda_2\boldsymbol{Q}_{t-1} \tag{56}$$

In this setting, Equation (56) describes GARCH-like dynamics where $\boldsymbol{R}$ roughly corresponds to the constant. The matrix $\boldsymbol{R}$ is also called quasicorrelation, where each entry is treated as a parameter. Equation (55) scales the overall result $\boldsymbol{Q}$ back to a valid correlation matrix. However, the number of parameters still grows quadratically with increasing assets, rendering estimation cumbersome. In addition, $\boldsymbol{R}$ needs to remain positive definite, which is not guaranteed if $\lambda_1 + \lambda_2$ is close to one. Engle (2009) proposes an auxiliary estimator based on a moment condition. Formally, $\boldsymbol{R}$ is set to be the sample correlation matrix. This procedure drastically reduces computation time and is referred to as "Correlation Targeting". We will remain with the sample correlation. Further methods are discussed in Engle (2009) and the literature therein. Similar to the univariate stochastic volatility model, the DCC can retrieve residuals that are filterd for cross-correlation meaning that linear relationships among the standardized (cross-)filterd residuals should not be present anymore. The residuals are the data foundation for the last step.

The last component of the model aims to capture the remaining dependencen using copulas. As mentioned in Section 2.2.4, we consider parametric copulas with continuous marginals only. Given the unique copula density the estimation procedure amounts to maximizing the likelihood function

$$l_{\boldsymbol{\theta}}(u_1, \ldots, u_n) = \sum_{i=1}^{n} \log c_{\boldsymbol{\theta}}(u_i) \tag{57}$$

where $\boldsymbol{\theta}$ is a vector of parameters, $c_{\boldsymbol{\theta}}(\cdot)$ the corresponding copula density and $u_1, ..., u_n$ the marginals transformed into uniform scale using either a parametric or empirical cumulative distribution function.

If the DCC model is correctly specified, the correlation parameter of the copula should be approximately unity—the remaining parameters account for effects beyond linear relationships. For example, the parameters of a t-copula are given by a correlation matrix Σ and a degree of freedom parameter ν that specifies tail dependence. The traditional way to calibrate the copula is to estimate Σ and ν. In this framework, similar to correlation targeting, Σ can be set as a constant. The only remaining task is to estimate ν, which drastically simplifies the estimation process. Naturally, this holds for other elliptical copulas with additional parameters.

Overall, we can summarize the calibration process as a multistage estimation procedure. Let Θ_{var}, Θ_{DCC} and Θ_{cop} be compact subsets of finite Euclidian spaces representing the parameter space for the stochastic volatility models, DCC, and copula, respectively. Denote $n \in \mathbb{N}$ as the number of financial assets and $\Theta = \Theta_{\text{var}} \times \Theta_{\text{DCC}} \times \Theta_{\text{cop}}$. Then, the calibration process can be divided into three steps:

$$\hat{\theta}_{i,\text{var}} = \underset{\theta_{i,\text{var}} \in \Theta_{\text{var}}}{\text{argmax}} \; l_{i,\text{var}}(\boldsymbol{\theta}_{i,\text{var}}) \qquad \text{for } i = 1, ..., n \tag{58}$$

$$\hat{\theta}_{\text{DCC}} = \underset{\theta_{\text{DCC}} \in \Theta_{\text{DCC}}}{\text{argmax}} \; l_{\text{DCC}}(\boldsymbol{\theta}_{\text{DCC}}) \tag{59}$$

$$\hat{\theta}_{\text{cop}} = \underset{\theta_{\text{cop}} \in \Theta_{\text{cop}}}{\text{argmax}} \; l_{\text{cop}}(\boldsymbol{\theta}_{\text{cop}}) \tag{60}$$

where $l_{i,\text{var}}$, l_{DCC}, l_{cop} represent the log-likelihood functions of the corresponding model components. The final parameter vector is $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_{\text{var}}, \hat{\boldsymbol{\theta}}_{\text{DCC}}, \hat{\boldsymbol{\theta}}_{\text{cop}})$ with $\hat{\boldsymbol{\theta}}_{\text{var}} = (\hat{\boldsymbol{\theta}}_{1,\text{var}}, ..., \hat{\boldsymbol{\theta}}_{n,\text{var}})$. A possible simulation framework could be described by the following example:

Example 2.3.1

Given n financial assets, assume that their variance is described by a zero mean GARCH(1,1) model with normal innovations, i.e.

$$
\begin{aligned}
r_{i,t} &= \sigma_{i,t}\varepsilon_{i,t} \\
\sigma^2_{i,t} &= \omega_i + \alpha_{1,i} r^2_{i,t-1} + \beta_{1,i}\sigma^2_{i,t-1}
\end{aligned}
$$

with $\varepsilon_{i,t} \overset{iid}{\sim} \mathcal{N}(0,1)$ *and* $i = 1, ..., n$*. The univariate models are summarized in a DCC framework:*

$$
\begin{aligned}
\boldsymbol{r}_t &= \boldsymbol{H}_t^{1/2}\boldsymbol{z}_t \\
\boldsymbol{H}_t &= \boldsymbol{D}_t^{1/2}\boldsymbol{R}_t\boldsymbol{D}_t^{1/2} \\
\boldsymbol{R}_t &= diag(\boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t diag(\boldsymbol{Q}_t)^{-1/2} \\
\boldsymbol{Q}_t &= (1-\lambda_1-\lambda_2)\boldsymbol{R} + \lambda_1\tilde{\boldsymbol{\varepsilon}}_{t-1}\tilde{\boldsymbol{\varepsilon}}^T_{t-1} + \lambda_2\boldsymbol{Q}_{t-1}
\end{aligned}
$$

Assuming that $\{\boldsymbol{z}_t\}_{t\in\mathbb{Z}}$ *is multivarite normal white noise, we can apply the probability integral transform on the cross-correlation filtered residuals to transform the data into uniform scale. In this example, the t-student copula is used to model the remaining dependence in the tails of the distribution:*

$$C_{\nu,\boldsymbol{\mathcal{R}}}(u_1, ..., u_n) = T_{\nu,\boldsymbol{\mathcal{R}}}(T^{-1}_{\nu_1}(u_1), ..., T^{-1}_{\nu_n}(u_n)).$$

Overall, $n \times 3$ *parameters for the univariate models* $\hat{\boldsymbol{\theta}}_{i,var} = (\omega_i, \alpha_i, \beta_i)$*, two parameters* $\boldsymbol{\theta}_{DCC} = (\lambda_1, \lambda_2)$ *and one parameter for the copula* $\hat{\theta}_{cop} = \nu$*. If the correlation matrix* $\boldsymbol{\mathcal{R}}$ *is also estimated, additional* $n(n-1)$ *correlation parameters need to be estimated.*

2.4 Bootstrap Methods for Multistage Estimation

One challenge in multistage estimation is to evaluate the variability of the estimates for robustness. In particular, the standard error of each parameter at every stage must be considered for the subsequent one. In a recent article, Gonçalves et al. (2023) points out that the conventional way to derive errors includes the computation of numerous derivatives of the objective function where analytical derivatives cannot be easily obtained, which prompted the use of numerical derivatives. However, obtaining the derivative in both ways is prone to errors. The main finding in this article is that bootstrapping a two-stage quasi-maximum likelihood estimator still yields consistent results and can be generalized beyond two stages. As a result, using Bootstrap to derive standard errors and confidence intervals for time series data is justified under certain regularity conditions.

As part of the calibration effort for the simulation model, we present two approaches for sampling returns that are used to derive standard errors: The Moving Block Bootstrap (MBB) by Liu & Singh (1992), and a modified version of the Filtered Historical Simulation (FHS) by Adesi (2014). The latter has become a tool widely used in risk management practice. Both methods are specifically constructed to account for serial correlation in time series data.

2.4.1 Moving Block Bootstrap

The MBB method partitions the sample into a family of overlapping blocks given a specific block size. Indices of these blocks are then drawn with replacement until a bootstrap sample is formed. The desired statistics are computed over several iterations based on the bootstrap sample. The procedure is defined in the following way:

Define $\{r_t\}_{t=1,...,n}$ as a generic time series sample and $l \in \mathbb{N}$ as the bootstrap block length with $1 \leq l < n$. Define the blocks of l succesive observations beginning with r_t as

$$\boldsymbol{B}_{t,l} = \{r_t, r_{t+1}, ..., r_{t+l-1}\}. \tag{61}$$

If $l = 1$, the bootstrap reduces to the classical assumption of an i.i.d. sample. Given a partition of the sample by $n = kl$, where other specifications are also possible, the new bootstrap sample consists of $k = n/l$ draws with replacement from the set of overlapping blocks

$$\{\boldsymbol{B}_{1,j}, ..., \boldsymbol{B}_{n-1+l,l}\}. \tag{62}$$

The indices $I_1, ..., I_k$ are uniformly distributed on $\{0, ..., n-l\}$ and determine the choice of the blocks. Finally, we obtain a bootstrap sample

$$\{r_t^* = r_{\tau_t}, t = 1, ..., n\} \tag{63}$$

with τ_t defined as a set of indices according to a block

$$\{\tau_l\} \equiv \{I_1 + 1, ..., I_1 + l, ..., I_k + 1, ..., I_k + l\}. \tag{64}$$

More precisely, $\{I_k + 1, ..., I_k + l\}$ represents the k-th block with length l.

The block bootstrap samples several adjacent data points simultaneously instead of just one. However, a potential drawback is that block bootstrap can result in a non-stationary series while the original series is stationary. It needs to be clarified how to choose the size of the blocks, and no objective criterion so far exists for determining the appropriate length of the window. Gonçalves et al. (2023) state that the average block sizes in other literature is 3.90, however, this value is not backed by any theory. A detailed overview on bootstrapping financial time series is discussed in Ruiz & Pascual (2002).

2.4.2 Filtered Historical Simulation

The FHS method adapts a a semi-parametric approach to describe the sample. The underlying volatility dynamic is assumed to be driven by a stochastic volatility model. We describe this procedure directly in a multivariate setting. Assume that the conditional variance is governed by a multivariate GARCH model with zero mean. Without further specifiying the volatility dynamic, the scaled innovation process is given by

$$\boldsymbol{\varepsilon}_t = \boldsymbol{H}_t^{1/2} \mathbf{z}_t. \tag{65}$$

where $\boldsymbol{H}_t$ is the conditional covariance matrix and $\mathbf{z}_t$ a white noise vector. Consequently, if the model is correctly specified, filtering for heteroskedasticity should yield

$$\{\mathbf{z}_t : t = 1, ..., n\} = \{\boldsymbol{H}_t^{-1/2} \boldsymbol{\varepsilon}_t : t = 1, ..., n\} \tag{66}$$

which is a white noice process and thus should be independent and identically distributed. In this case, standard bootstrapping is possible by sampling $\{\mathbf{z}_t : t = 1, ..., n\}$ with replacement and thus creating a bootstrap sample $\{\mathbf{z}_t^* = \mathbf{z}_{\tau_t} : t = 1, ..., n\}$ where $\{\tau_t\}$ is defined as in (64) with $l = 1$.. The original purpose of Barone-Adesi et al. (1999) is to create a return distribution by scaling the bootstrap sample with a volatility forecast which is used to make a Value-At-Risk prediction. Instead, we obtain a synthetic return series by reintroducing heteroskedasticity where the bootstrapped residuals serve as a new innovation process. Formally, the return sample is obtained by

$$\{\boldsymbol{\varepsilon}_t^* : t = 1, ..., n\} = \{\boldsymbol{H}_t^{1/2} \mathbf{z}_t^* : t = 1, ..., n\} \equiv \{\boldsymbol{r}_t^* : t = 1, ..., n\}. \tag{67}$$

Reintroducing heteroskedasticity requires starting values to initiate the variance recursion. Common choices are historical or asymptotic values, however, they lack theoretical justification. In pratical appplication, the effects of initial values are smoothend by an initial phase or burn-in phase which discards the first $k > 0$ realization according to a predetermined value.

2.5 Empirical Application

In this section, we will calibrate the simulation model according to three financial assets. The focus of this study is to highlight the challenges of the entire process. Furthermore, we present two methods for obtaining standard errors, model diagnostics where appropriate and the limitations of each model component. Finally, we investigate possible areas of applications and discuss potential extensions of the simulation framework.

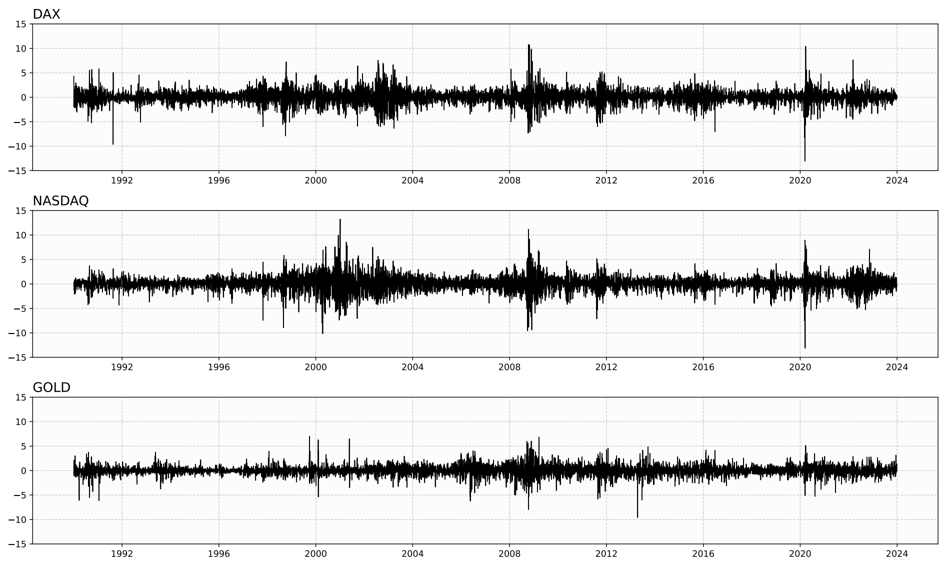


Figure 3: Chart of historical daily returns for DAX, NASDAQ and Gold from 1990 to 2024.

2.5.1 Data Description

The sample consists of daily returns for three assets: the German stock index DAX, the gold price as a commodity, and the NASDAQ Composite index for U.S. equities from January 1st, 1990, to December 29th, 2023, with $n = 8288$ data points. We choose this selection of assets to include two different asset classes as well as stock indices from two different geographic regions. Historical prices of the DAX and NASDAQ indices were retrieved from Yahoo! Finance, while gold prices were obtained from the World Gold Council website. The DAX consists of 40 major companies based in Germany, reflecting a broad range of sectors, including automotive, pharmaceutical, and technology industries.[1] Similarly, the NASDAQ Composite encompasses 100 companies, primarily from the technology sector. Since gold prices can vary across different regional markets, we use the gold price traded in the London Bullion Market (LBMA) for this analysis, as this price series is commonly referenced as a benchmark for global gold prices. Furthermore, the data is centered such that we can focus on variance and correlation while assuming a zero mean process. The analysis is built on centered logarithmic returns.

[1]The index consisted of 30 companies up to September 2021 and since then was extended to 40 companies

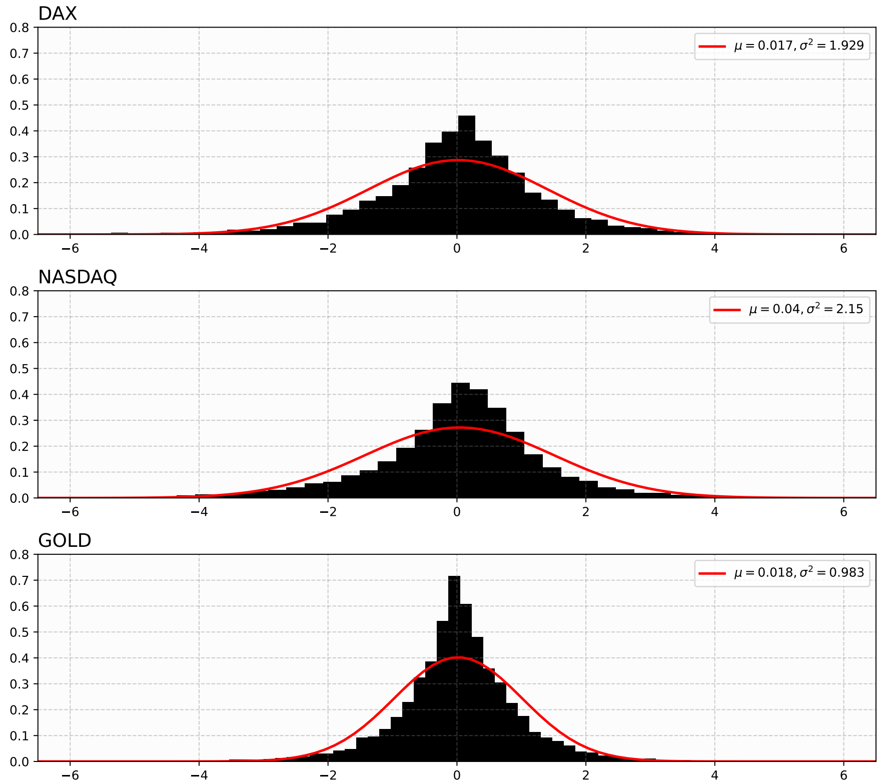


Figure 4: Histogram of daily returns with fitted normal density.

Figure 3 shows the return charts of the assets where events such as the dotcom crisis in 2000, the financial crisis in 2008, and the outbreak of the Covid-19 pandemic are reflected by significant changes visible, especially for the stock indices. Figure 4 displays the histogram of the sample data overlayed by the density of a normal distribution. The parameters are estimated by Maximum-Likelihood. Clearly, the densities are not able to approximate the sample distribution properly.

We compute descriptive statistics and provide additional tests for peakedness, heavy tails, and leptokurtosis proposed by Schmid & Trede (2003). The tests are based on quantiles which are more robust regarding outliers especially for higher moments. It is calculated as a ratio

$$\text{Test}_{\text{Schmid}} = \frac{x_{1-p} - x_p}{x_{1-q} - x_q}, \tag{68}$$

where x_p corresponds to the p-th quantile of the distribution. A specific choice of the quantiles determines the area of interest, i.e., heavy tails (T), peakedness (P) and leptokurtosis (L). The authors propose the following specifications:

$$T = \frac{x_{0.975} - x_{0.025}}{x_{0.875} - x_{0.125}}, \quad P = \frac{x_{0.875} - x_{0.125}}{x_{0.75} - x_{0.25}}, \quad L = \frac{x_{0.975} - x_{0.025}}{x_{0.75} - x_{0.25}}.$$

We refer to the article for critical values regarding hypothesis testing. The asymptotic distribution of the statistics is based on modified versions of the standard normal distribution. Tables of critical values are provided for finite samples ($n \leq 2000$) respectively.

Descriptive Statistics	**DAX**	**NASDAQ**	**Gold**
Mean	0.017	0.039	0.018
	(0.014)	(0.015)	(0.010)
Standard Error	1.388	1.466	0.991
	(0.028)	(0.035)	(0.019)
Excess Kurtosis	5.642***	6.670***	6.572***
	(1.000)	(1.116)	(1.030)
Skewness	-0.143***	-0.202***	-0.247***
	(0.159)	(0.160)	(0.175)
Schmid & Trede Statistics			
$\hat{T}_n$	2.112***	2.240***	2.227
	(0.013)	(0.014)	(0.015)
$\hat{P}_n$	1.955***	2.007***	1.958
	(0.009)	(0.010)	(0.010)
$\hat{L}_n$	4.132***	4.498***	4.361***
	(0.022)	(0.024)	(0.025)
Correlation Matrix			
DAX	1		
NASDAQ	0.507	1	
GOLD	-0.047	-0.010	1

*, **, and *** denote statistical significance levels at 10, 5, and 1 percent, respectively. The corresponding statistical tests evaluate whether the kurtosis and skewness are significantly different from those of a normal distribution, i.e., excess kurtosis and skewness of zero. Corresponding critical values regarding tailness, peakedness, and leptokurtosis are derived from a modified normal distribution. The tests evaluate whether the estimated values are statistically different from values derived from a normal distribution. Standard errors were obtained by block bootstrapping and displayed in brackets below the estimates. The correlation matrix is reported at the bottom, with only the lower triangular matrix entries.

Table 1: Descriptive statistics for daily logarithmic returns.

The descriptive statistics in Table 1 confirms that excess kurtosis and skewness are statistically different from their normal counterparts. This is further confirmed by the Schmid-Trede test, which also affirms that the estimates are highly significantly different from those by a reference normal distribution. This is inline with the empirical observations that returns are clearly not normal distributed.

2.5.2 Model Estimation

We estimate a simulation model consisting of univariate GJR-GARCH volatility model with skewed-t distributed errors, GARCH-like DCC dynamics and t-student copula. The standard errors are obtained by the modified FHS approach described in Section 2.4. Formally, the simulation framework with n=3 assets is described by

$$\begin{aligned} r_{i,t} &= \sigma_{i,t}\varepsilon_{i,t} \\ \sigma^2_{i,t} &= \omega_i + \alpha_{1,i} r^2_{i,t-1} + \gamma_{1,i} I_{t-1} r^2_{i,t-1} + \beta_{1,i}\sigma^2_{i,t-1} \end{aligned}$$

with $\varepsilon_{i,t} \overset{\text{iid}}{\sim}$ skewed-t$(0, 1, \nu_{i,\text{dist}}, \lambda_{i,\text{dist}})$, $i = 1, 2, 3$ and

$$I_{t-1} = \begin{cases} 1, & \text{if } r_{i,t} < 0 \\ 0, & \text{otherwise.} \end{cases}$$

The univariate models are summarized in a DCC framework:

$$\begin{aligned} \boldsymbol{r}_t &= \boldsymbol{H}_t^{1/2}\boldsymbol{\varepsilon}_t \\ \boldsymbol{H}_t &= \boldsymbol{D}_t^{1/2}\boldsymbol{R}_t\boldsymbol{D}_t^{1/2} \\ \boldsymbol{R}_t &= \text{diag}(\boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t\text{diag}(\boldsymbol{Q}_t)^{-1/2} \\ \boldsymbol{Q}_t &= (1-\lambda_1-\lambda_2)\boldsymbol{R} + \lambda_1\tilde{\boldsymbol{\varepsilon}}_{t-1}\tilde{\boldsymbol{\varepsilon}}^T_{t-1} + \lambda_2\boldsymbol{Q}_{t-1} \end{aligned}$$

with Copula

$$C_{\nu,\boldsymbol{\mathcal{R}}}(u_1, u_2, u_3) = T_{\nu,\boldsymbol{\mathcal{R}}}(T^{-1}_{\nu_1}(u_1), T^{-1}_{\nu_2}(u_2), T^{-1}_{\nu_n}(u_3)).$$

Overall, this simulation framework requires fifteen parameters for the univariate models $\hat{\boldsymbol{\theta}}_{i,\text{var}} = (\omega_i, \alpha_i, \beta_i, \nu_{i,\text{dist}}, \lambda_{i,\text{dist}})$, two parameters $\boldsymbol{\theta}_{\text{DCC}} = (\lambda_1, \lambda_2)$ and one parameter for the copula $\hat{\theta}_{\text{cop}} = \nu_{\text{cop}}$. If the correlation matrix $\boldsymbol{\mathcal{R}}$ is treated as a parameter, additional 3 correlation parameters need to be estimated. Estimation results are presented in Table 2.

The GJR-GARCH parameters of the marginal conditional variance models describe the corresponding volatility dynamics. The ω parameters of the equity indices DAX (0.034) and NASDAQ (0.027) exhibit higher baseline volatility compared to Gold's low ω (0.0003), which suggests minimal long-term volatility, affirming its role as a stable asset during crisis. In all three cases, the α_1 coefficient presents a rather small value, indicating that recent

	DAX	NASDAQ	Gold
GARCH(1,1) Parameters			
$\hat{\omega}$	0.034***	0.027***	0.0003***
SE filtered	(0.005)	(0.004)	(0.001)
t-stat	6.092	6.473	2.955
$\hat{\alpha}_1$	0.012*	0.029***	0.005****
SE filtered	(0.007)	(0.008)	(0.009)
t-stat	1.698	3.510	6.066
$\hat{\gamma}_1$	0.122***	0.125***	3.11×10^{-8}
SE filtered	(0.016)	(0.012)	(0.007)
t-stat	7.378	10.339	4.288
$\hat{\beta}_1$	0.905***	0.893***	0.941***
SE filtered	(0.009)	(0.008)	(0.008)
t-stat	96.697	108.436	109.623
Skewed t-Student Error Parameters			
$\hat{\nu}_{\text{dist}}$	844.065	414.255	790.148
SE filtered	204.662	130.803	161.011
$\hat{\lambda}_{\text{dist}}$	-0.121***	-0.190***	-0.215***
SE filtered	(0.022)	(0.015)	(0.032)
t-stat	5.373	12.140	6.576
DCC(1,1) Parameters			
$\hat{\lambda}_1$		0.011	
SE filtered		(0.007)	
t-stat		1.504	
$\hat{\lambda}_2$		0.983***	
SE filtered		(0.045)	
t-stat		21.733	
t-Copula Parameters			
$\hat{\nu}_{\text{cop}}$		3.331	
SE		(0.004)	
Σ_{cop}			
DAX	1		
NASDAQ	-0.003	1	
GOLD	0.005	-0.012	1

*,**, and *** denote statistical significance levels at 10, 5, and 1 percent, respectively. Corresponding critical values are derived from a standard normal distribution. The tests evaluate whether the estimated values are statistically different from zero. Standard errors were obtained using the filtered historical bootstrap method.

Table 2: Parameter estimates for GJR-GARCH-DCC-t-Copula model with univariate skewed-t student errors, GARCH-like correlation dynamics and t-Copula. Standard errors are obtained by filtered historical bootstrap.

shocks only have a minor contribution to the evolution of volatility. The β_1 values are all close to 0.9, thus demonstrating the strong persistence of past volatility impacts. Unlike a standard GARCH model, the GJR-GARCH model also considers the impact of returns on volatility depending on the sign. That is the effect of negative returns compared to positive returns. The parameter γ is highly significant for the equity indices, DAX (0.122) and NASDAQ (0.125), indicating a pronounced leverage effect, which is not true if gold is considered. This result again seems to agree with the assumption that gold as a commodity is an alternative investment instrument less volatile than equities and suited for portfolio diversification strategies.

Capturing leverage effects also have an impact on the estimates for the degrees of freedom parameter regarding the error distribution. The skewed-T distribution in the GJR-GARCH model captures heavy tails and skewness. Although not statistically significant on any level, controlling for leverage effects seems to account for heavy tails. The degrees of freedom parameters $\nu_{i,\text{dist}}$ are high, indicating tail behavior similar to a normal distribution. Negative skewness is reported, showing that negative returns increase future volatility more than positive returns. In all three cases, DAX (-0.121), NASDAQ (-0.190), and Gold (-0.215), the estimates are found to be significantly different from zero confirming the asymmetric nature of returns.

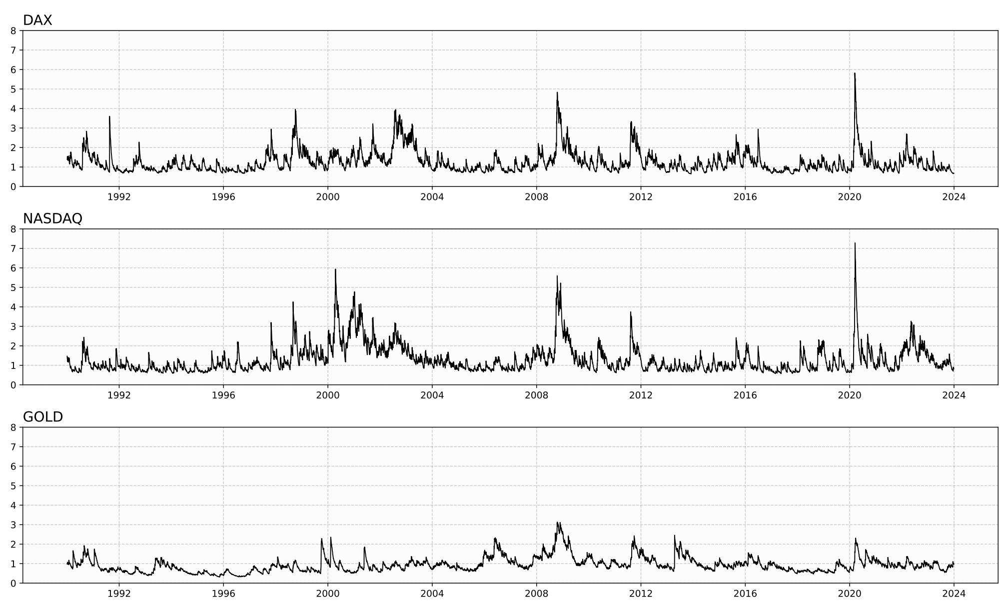


Figure 5: Fitted volatility by univariate GJR-GARCH models with normal errors.

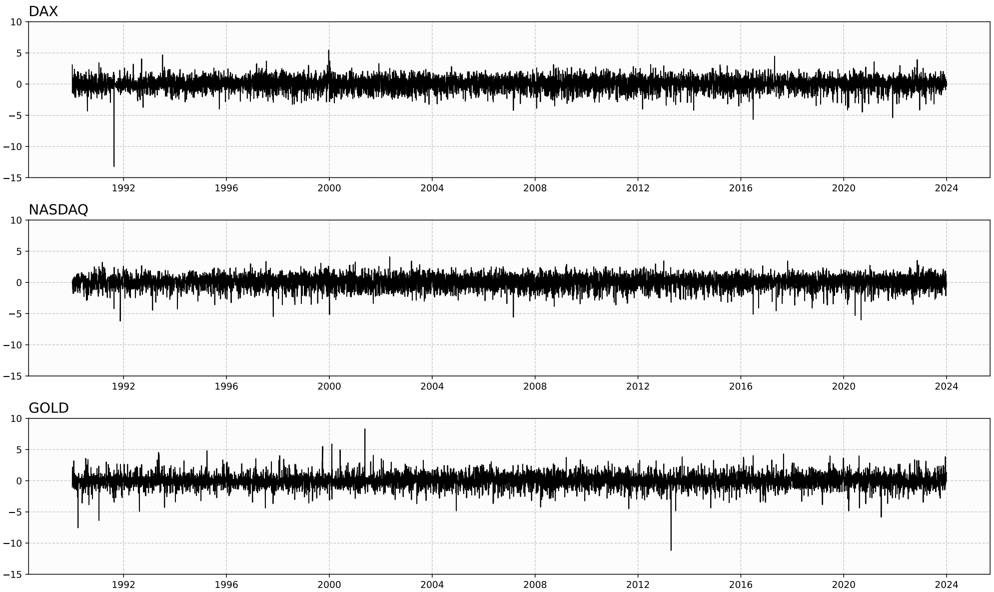


Figure 6: Standardized residuals obtained after filtering returns with a GJR-GARCH model.

Figure 5 displays the characteristic volatility spikes as a reaction to major financial market events. The early 2000s saw the dot-com bubble burst and the aftermath of the September 11 attacks, leading to economic recessions. The collapse of the housing market in the United States triggered the financial crisis in 2008. In more recent history, the COVID-19 pandemic in 2020 and the Russo-Ukrainian war in 2022 hit the financial markets once more. By the magnitude of the spikes, we can also observe that gold as a commodity reacts to these events but is not as pronounced as stock markets.

	DAX			Nasdaq			Gold		
	r_t	r_t^2	$\tilde{\varepsilon}_t$	r_t	r_t^2	$\tilde{\varepsilon}_t$	r_t	r_t^2	$\tilde{\varepsilon}_t$
Ljung-Box									
Test Stat.	20.58	3562.26	2.962	50.71	5665.83	20.070	15.47	1154.20	12.680
p-val	0.024	0.00	0.982	0.00	0.00	0.880	0.11	0.00	0.242
Jarque-Bera									
Test Stat.	11022.82		7706.54	15421.36		999.24	15000.95		8652.28
p-val	0.00		0.00	0.00		0.00	0.00		0.00

Table 3: Ljung-Box and Jarque-Bera test statistics.

Standardized residuals are obtained by filtering the original return series and depicted in Figure 6. Compared to Figure 3, the chart, apart from occasional spikes, does not display any clusterings. This is backed by the results of the Ljung-Box Test in Table 3, which tests for serial correlation. In every case, no evidence of serial correlation is reported for residuals but for the returns and squared returns. Finally, the Jarque-Bera test provides strong evidence against normality for both returns and standardized residuals. Consequently, selecting a normal white noise might not be appropriate for the innovation process.

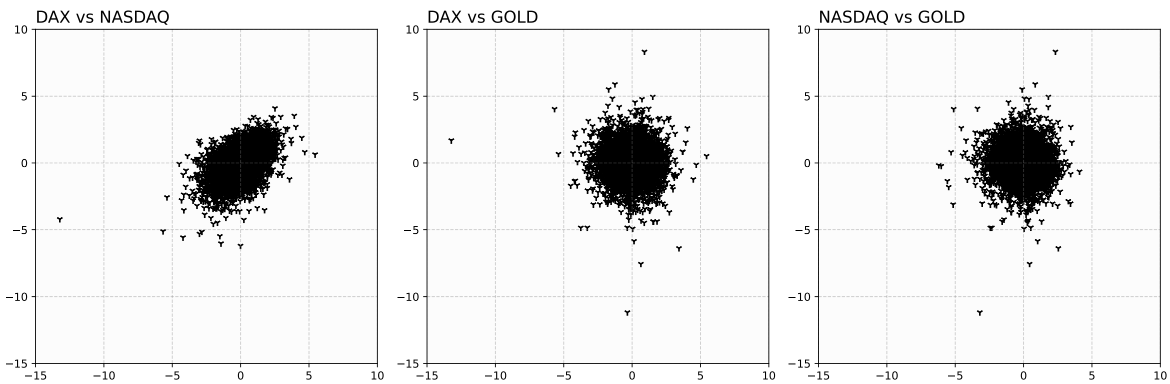


Figure 7: Scatterplot of standardized residuals - Visualization of cross-correlation between DAX, NASDAQ and Gold.

Although the univariate stochastic volatility models filtered serial correlation, correlation among the standardized residuals is still present. Figure 7 shows the scatterplot of the standardized residuals. The first scatterplot visualizes the relationship between the stock markets. The scatterplot hints at a positive correlation between DAX and NASDAQ due to its clockwise tilted oval shape, while the second and third plots display a more circular form, indicating a weaker correlation between the stock markets and gold.

Based on standardized residuals, the DCC model is estimated. Figure 8 visualizes the evolution of correlation coefficients over time compared to the static Pearson correlation. The correlation between DAX and NASDAQ remains positive. However, with varying magnitude, the correlation of the stock market indices with gold tends to turn negative during market distress. With only the λ_2 parameter (0.984) being statistically significant, the conditional correlation in the DCC setting is mainly determined by the previous realization and not the values of the residuals. Similar to volatility, correlation also persists over time.

Filtering the standardized results with the DCC model for cross-correlation should yield a multivariate normal white noise. However, Figure 9 still indicates some dependence in tail events. The scatterplot is segmented into 50 colored tiles where the color indicates the amount of points in a certain area. For DAX-NASDAQ and DAX-GOLD, potential tail-dependence is shown by darker colors in the corners.

Concluding the model calibration, we estimate a copula model based on the cross filtered

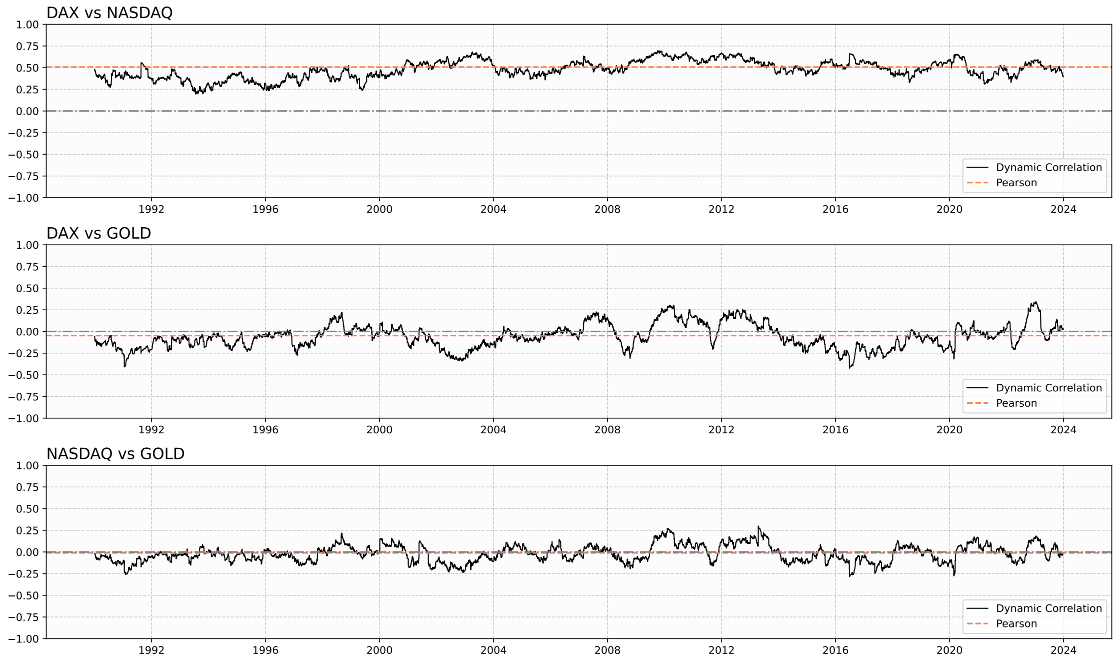


Figure 8: Chart of fitted correlation obtained bei a GARCH-like DCC model.

residuals. As expected, the correlation matrix Σ_{Cop} is close to the identity matrix since the linear dependence was accounted for by the DCC model. The degree of freedom parameter $\hat{\nu}_{\text{Cop}} = 3.836$ indicates heavier tails, meaning there is a higher probability of extreme co-movements.

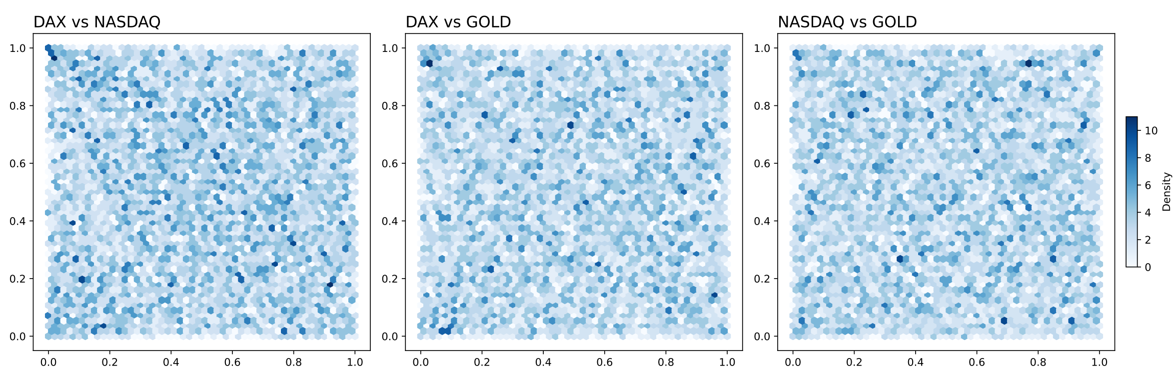


Figure 9: Scatterplot of cross-correlation filtered residuals after uniform transformation by the empirical cdf.

The parameters obtained in each step function as a potential default input for the simulation model. This entire process could be also described as a whitening process since the ultimate goal is to obtain i.i.d. residuals. Furthermore, additional parameters may be added to model the conditional mean by relaxing the zero-mean assumption. Overall, the framework allows a wide range of extensions to further enhance its simulation capabilities.

2.5.3 Simulating Returns

We calibrate the simulation framework according to the parameter estimates in Table 2 and simulate 5000 sample paths over 1500 days. Returns are then converted back to prices and visualized with an initial price of 100 seen in Figure 10. Note that the simulated paths of gold display a more narrow cone than the equities which is attributed to the lower volatility of the asset.

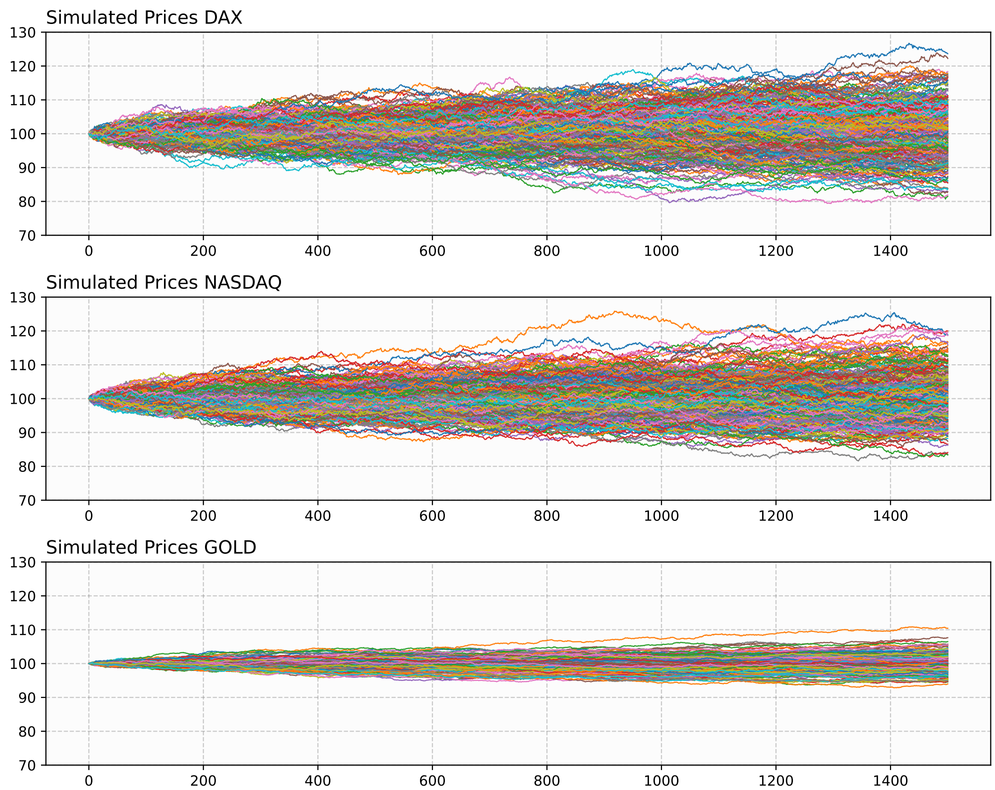


Figure 10: Simulated price paths according to a GJR-GARCH-DCC-T-Copula model with skewed-T student errors and GARCH like DCC dynamics.

Recall, that the simulation framework assumed zero mean indicated by the horizontal alignment of the cones. Incorporating a non-zero mean process tilts the cone in the respective angle as shown in Figure 11.

Our main goal is to present a simulation framework and its potential enhancements to existing financial econometric applications. The next step would involve integrating the results into an application of interest.

For portfolio analysis, simulated returns could provide insight into the behavior of a specific asset allocation, particularly in assessing the variability of the risk-return profile in the context of Markowitz-styled portfolios. The concept is to construct the set of efficient portfolios (efficient frontier) for every simulated scenario and thus acquire a distribution

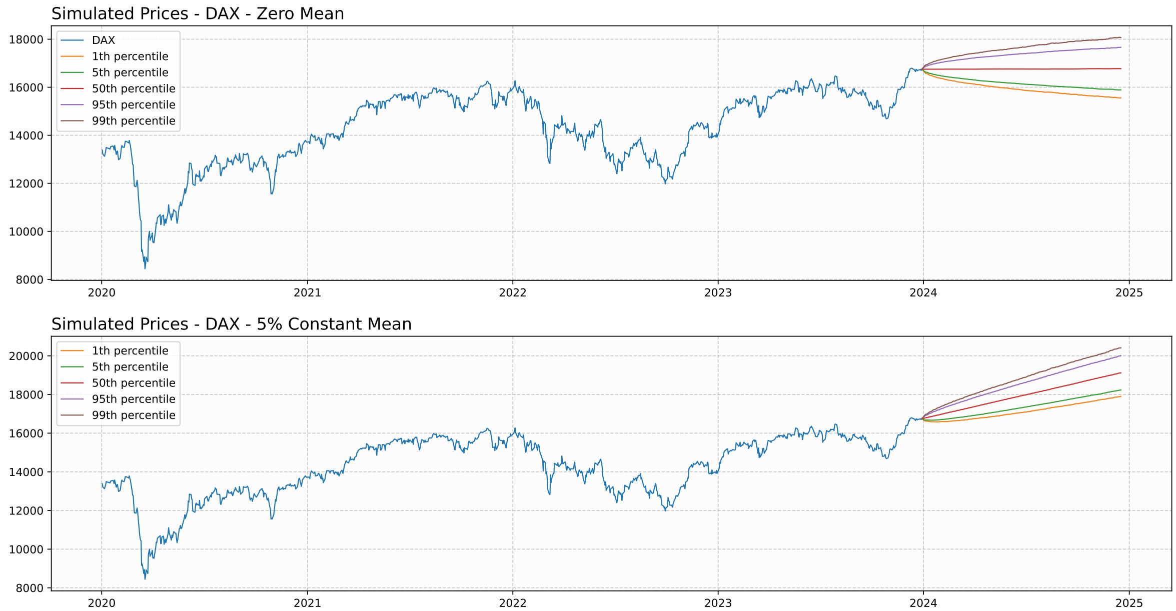


Figure 11: Price projection for DAX with zero (upper panel) and 5 percent (lower panel) constant mean.

of the set of efficient portfolios. The variance of this efficient frontier distribution offers an empirical measure of the variability for a specific asset allocation. As a result, portfolio optimization becomes more robust by averaging across multiple possible scenarios. This type of analysis is discussed in Michaud (2000) and Markowitz & Usmen (2005).

Similarly, simulated return scenarios can contribute to a better understanding of the risk associated with a financial asset since the model can mimic heavy tails, leptokurtosis, and asymmetries. In particular, different return scenarios generate a distribution of simulated returns from which Value-at-Risk or Expected Shortfall inferences can be obtained. This is a direct application of the Monte Carlo simulation method for risk modeling but with a more refined data-generating process. The parameters can also be adjusted for stress testing to encompass more extreme events during simulation.

Another possible application of this framework is further improving the FHS semi-parametric sampling methodology. In the original version by Barone-Adesi et al. (1999), stochastic volatility models filter serial correlation for each asset separately without incorporating the dependence structure. However, our model additionally allows us to filter for conditional correlation, which renders the resulting residuals approximately truly independent and identically distributed since each step of the model allows for control over at least one aspect of the ”stylized” properties. As a result, this sampling technique can be employed to evaluate standard errors of estimates where analytical or numerical derivatives are difficult to obtain.

2.6 Technical Aspects

2.6.1 Variance Initialization

A key challenge in estimating the proposed models is the optimization of the likelihood function in each step. The initial choice of the parameters carries significant weight, as the shape of the likelihood function can be complex with numerous local maxima. This makes the selection of starting values a crucial task, as it directly impacts the reliability of the parameter estimates. The implementation of the simulation framework draws from the *arch* python library of Sheppard (2015).

Starting with a simple GARCH(1,1) model,

$$r_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-j}^2$$

initial parameter for σ_1, ω, α_1 and β_1 are required to begin the optimization process. Potential choices for the initial variance are shown in table below:

Sample Variance	$\sigma_1^2 = \frac{1}{T}\sum_{t=1}^{T} r_t^2$
Unconditional Variance	$\sigma_1^2 = \frac{\omega}{1-\alpha_1-\beta_1}$
Backcasting with Exponential Smoothing	$\sigma_1^2 = \lambda^T \hat{\sigma}^2 + (1-\lambda)\sum_{j=0}^{T-1} \lambda^j r_{1+j}^2$

Table 4: Possible choices for the initial variance σ_1^2.

The sample variance reasonably approximates the unconditional variance, provided the time series is stationary. It is also proposed by Bollerslev (1986). However, the sample variance is a static measure and, therefore, cannot adapt to recent changes in volatility. Evidently, if the sample variance is calculated during a specific period of high volatility and vice versa, it might not represent the overall variance. Given a highly persistent GARCH process, the initial guess of σ_1 could be distant from its actual mean value.

The asymptotic unconditional variance presents an alternative way to initiate the variance. Although the unconditional variance is also a static measure like the sample variance, it is directly derived from the model itself and, therefore, less subjective. The unconditional variance leverages the theoretical properties of the GARCH model, ensuring that the initial value is consistent with the long-term behavior of the process. However, it requires accurate initial estimates of ω, α_1 and β_1. In addition, if the sum of α_1 and β_1 is close to 1, the optimization process can become highly unstable. For the process not to become a unit root process, the constraint of $\alpha_1 + \beta_1 < 1$ has to be met, which also ensures stationarity.

The exponential smoothing with backcast tries to overcome some of the aforementioned drawbacks. This method adjusts the initial variance based on recent data, which now accounts for changes in volatility patterns. It gives more weight to recent observations, making the initial variance responsive to recent market conditions. However, the effectiveness of this method depends on the choice of the smoothing parameter. An inappropriate choice can lead to either over-smoothing, which ignores recent changes, or under-smoothing, which makes the variance too reactive to short-term fluctuations. RiskMetrics (1996) proposed a decay factor of $\lambda = 0.94$ with a window length of $T = 74$ for trading and $\lambda = 0.97$ with $T = 151$ for investing.

Adding another routine to find an optimal initial value increases computational complexity. Regarding the exponential smoothing, the optimization routine could include both T and λ. This is reasonable if the parameter choice should be as objective as possible. However, the parameter increase may render the entire process unfeasible in extreme cases. In practice, heuristic values based on other empirical works or experience are often preferred over objectivity to reduce computation time.

Albeit cumbersome, a grid search significantly increases the accuracy by prescanning the geometry of the likelihood function. Subsequently, the space of possible parameters is narrowed down to a smaller area. Therefore, setting the grid resolution is a trade-off between estimation accuracy and computational resources. Assuming that the data of interest is well-studied, e.g., historical data of financial assets, one could justify the choice of the grid boundaries on the fact that the volatility of most financial time series are highly persistent. In this research project, the interval for the constant ω and ARCH parameter α_1 is set between $[0.001, 0.4]$. The GARCH parameter β_1 is set between $[0.5, 0.98]$. A grid search is performed once for each marginal variance model and held fixed during the bootstrap of the estimates for standard errors.

2.6.2 Likelihood Function Surface

We analyze the geometry of the multivariate negative log-likelihood function. A grid search greatly increases the quality of parameter estimates by providing refined starting values which increases the chance for the optimization algorithm to converge. Naturally, the numerical effort increases according to the grid resolution and number of model parameters.

For illustration, we analyze a GARCH(1,1) process with normal errors based on the same data (DAX, NASDAQ, Gold) studied in the empirical example, see Section 2.5. Let $\theta = (\omega, \alpha_1, \beta_1)$, then the likelihood function is of the form

$$l_{\mathbf{r}}(\theta) = \sum_{t=1}^{T} \left[-\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(\sigma_t^2) - \frac{r_t^2}{2\sigma_t^2} \right]. \tag{69}$$

For this specific task, we increased the resolution of the grid from 10 to 100, meaning that 100 equidistant points for $\omega, \alpha_1 \in [0.001, 0.4]$ and $\beta_1 \in [0.5, 0.98]$ are considered. Parameter sets are combined such that the stationarity constraint of $\alpha_1 + \beta_1 < 1$ is fulfilled. Overall, 10000 combinations are evaluated based on the corresponding likelihood function.

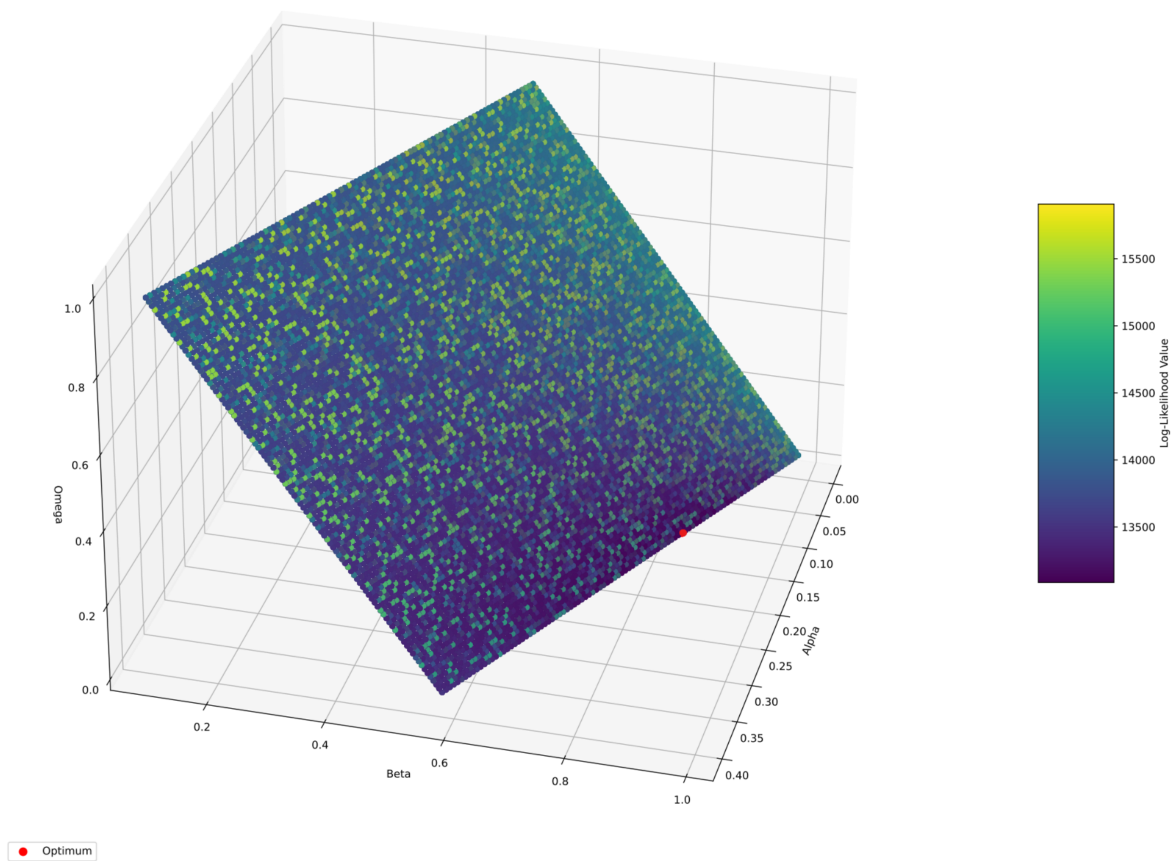


Figure 12: Negative logarithmic likelihood surface - DAX.

Figure 12 visualizes the surface structure of the negative log-likelihood function for the german stock index DAX. Each axis represents a parameter of the GARCH model while the color of each point indicate the value of the log-likelihood function. The range of the evaluated likelihood function is mapped to a color scheme where darker tones reflect lower values and vice versa. The red dot marks the optimal value which is a minimum since the negative log-likelihood function is evaluated due to the nature of the search algorithm.

The surface plot reveals that the area of interest is located in low ω, α_1 and relatively high β_1 parts (bottom right), consistent with the volatility persistency of financial time series. We can clearly see the irregular structure with numerous local extrema. For that reason, the choice of initial parameters has a crucial impact on the parameter outcome. In practical applications, however, i.e., software development, initial values are often chosen heuristically based on experience. Figures 23 and 24 in Appendix A.2 show the likelihood surface plots of the remaining assets which share a similar structure. So far, no clear strategy exists in terms of the best initial parameter values for optimization.

One particular observation is that the optimal parameters represent a solution on the boundary of the permissible set, which is problematic since boundary solutions render theoretical standard errors invalid. The asymptotic normality of the Maximum-Likelihood estimator only holds for interior points. As a result, the standard errors derived under the assumption of an interior solution do not accurately reflect the true variability of the estimator. This boundary behavior suggests that the model may not be a good fit for the data, as it points toward a potential near-non-stationary process or misspecification. Higher moments must likely be considered as well. Several attempts to obtain inner solutions by modeling the mean with an ARIMA process or increasing lags yield no better results, indicating that our model still cannot adequately represent the actual data-generating process. For estimation purposes, this phenomenon remains an obstacle. Nevertheless, using the model to simulate returns may still prove helpful.

A grid search was also conducted for the DCC model before the main optimization routine. Since the DCC recursion is motivated by the GARCH model, initial values of λ_1, which governs the impact of returns are searched in $[0.1, 0.4]$ while λ_2, which describes the impact of the previous correlation, is evaluated in the interval $[0.5, 0.9]$.

Naturally, the grid search becomes unfeasible for a large set of financial assets. In practice, it has become commom practice to select heuristic values based on past experience, i.e., parameters controlling the effects of past shocks are selected closer to zero, while parameters modeling the effects of past volatility or correlation is set close to one.

2.6.3 Comparison of Standard Errors by FHS and MBB

We estimate a simple GARCH-DCC-t-Copula model with normal distributed white noise to assess the difference of standard errors using MBB and FHS. Table 16 and Table 17 in Appendix A present the estimation results. Overall, the estimates for the model components are fairly close however the standard errors differ significantly. The MBB produces larger standard errors in almost almost all cases. Furthermore, it did not provide significant results for the constants ω of the univariate variance models and λ_1 for the DCC. It is not clear what causes these changes. Nevertheless, the ω and λ_1 are close to zero. One potential explanation for the differences is that standard errors derived using the MBB does not accuratly replicate the underlying data distribution. While block-bootstrapping preserves the dependence structure within blocks of data, it does not capture the time-varying nature of financial data as effectively especially for longer periods. The FHS approach is not reliant on specific data blocks. The choice of appropriate block size might affect the precision of the standard errors and add potential bias.

Chapter 3

Tail Correlation Matrices

Another empirical observation regarding financial returns is that they tend to decline jointly during bearish markets. Although the simulation model presented in the previous chapter accounts for dynamic correlation, the severity of correlation increase during such extreme market scenarios is not covered. In fact, the only parameters that account for joint occurrences of extreme events are partly determined by the parameters of the error distributions and the copulas, which become increasingly difficult to handle for more complex model choices. Consequently, for a risk management tool to remain viable for practical applications, particularly risk aggregation for high dimensional portfolios, we present an alternative strategy using tail-adjusted correlation matrices implied by quantiles instead.

3.1 Value-at-Risk Implied Correlation

Campbell et al. (2002) propose correlation estimates implied by VaR of a two-asset portfolio. First, we define the Value-at-Risk and follow the notation of McNeil et al. (2015):

Definition 3.1.1 (Value-at-Risk)
Given a confidence level $\alpha \in (0, 1)$. The VaR_α of a portfolio at the confidence level α is given by $l \in \mathbb{R}$ such that the probability of the loss L exceeding l is no larger than $1 - \alpha$. Formally,

$$VaR_\alpha = inf\{l \in \mathbb{R} : \mathbb{P}(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\} \tag{70}$$

The Value-at-Risk is a quantile associated with the loss distribution of an asset or portfolio.[2]

Using quantiles circumvents selecting a threshold for asset or portfolio returns, effectively collapsing a multivariate problem into a univariate setting. It is compatible with mean-variance portfolio optimization and directly applicable to risk aggregation. Ultimately, quantiles allow a direct economic interpretation depending on the choice of the quantile level. In this context, an event is classified as extreme if the returns falls below a predetermined bound, e.g., 5% quantile, if the losses are represented in the left/lower tail. First, we will outline the methodology for bi-variate portfolio following Campbell et al. (2002) and then present an extension of this method beyond two assets based on Mittnik (2014).

[2]Note that the loss distribution is a mirrored version of the return distribution such that losses are denoted by a positive sign.

Two Asset Scenario

Let r_1 and r_2 denote returns of two arbitrary financial assets. Define w_1, w_2 with $w_1 + w_2 = 1$ as the portfolio weights associated with the fraction of investment in each asset. The portfolio return and variance is then given by

$$r_p = w_1 r_1 + w_2 r_2, \tag{71}$$

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12}. \tag{72}$$

Assume that the returns follow an elliptical distribution. Then, any quantile can be represented as a transformed version of the standardized α-quantile of the marginal distribution with a location and scale parameter. Formally, given some α-level

$$q_{\alpha,i} = \mu_i + \sigma_i \Phi^{-1}(\alpha) \quad \text{with} \quad i = 1, 2. \tag{73}$$

For simplicity, set $\mu_i = 0$. We can rearrange (73) to $\sigma_i = q_{\alpha,i}/\Phi^{-1}(\alpha)$ and substitute the asset and portfolio variance in (72) resulting in

$$q_{\alpha,p}^2 = w_1^2 q_{\alpha,1}^2 + w_2^2 q_{\alpha,2}^2 + 2 w_1 w_2 q_{\alpha,1} q_{\alpha,2} \rho_{12}. \tag{74}$$

This step connects quantiles with portfolios. The choice of the $\alpha-$level determines how much we "move" into the tail of the return distribution. Finally, solving for ρ_{12} yields the VaR-implied correlation coefficient:

$$\rho_{\alpha,12} = \frac{q_{\alpha,p}^2 - w_1^2 q_{\alpha,1}^2 - w_2^2 q_{\alpha,2}^2}{2 w_1 w_2 q_{\alpha,1} q_{\alpha,2}}. \tag{75}$$

If normality holds, the implied correlation is constant and equals the Pearson correlation. Our analysis in Section 3.7.5 provide some evidence that this is not the case. Implied correlation indeed tends to deviate from the Pearson correlation depending on the return frequency and quantile level.

Multi Asset Scenario

The two-asset case relies only on one portfolio, which is solved for the correlation coefficient. Extending the portfolio to n assets introduces $n(n-1)/2$ unique correlation coefficients. Solving a linear system for the unique elements requires at least the same amount of equations. Here, the equations admit the form described in (74) where a different set of weights determines each portfolio.

For an n-asset portfolio, denote the return, variance and weight associated with each asset by r_i, σ_i^2 and w_i with $i = 1, ..., n$. Further, denote r_p, σ_p^2 as the portfolio return and portfolio variance respectively. Formally,

$$r_p = \sum_{i=1}^{n} w_i r_i \tag{76}$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij} \quad \text{with} \quad \rho_{ij} = 1 \quad \text{for} \quad i = j. \tag{77}$$

We establish a connection between variance and quantiles in the same way shown in (74) by substituting the variance with corresponding quantiles in (77), i.e.,

$$q_{\alpha,p}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j q_{\alpha,i} q_{\alpha,j} \rho_{\alpha,ij}. \tag{78}$$

As mentioned before, for n-assets, a total of at least $n(n-1)/2$ implied correlation coefficients are required, which is clearly not obtainable by one portfolio, i.e., by only one equation. Our idea is to jointly estimate the unique elements of the correlation matrix by constructing a linear system where each equation represents a portfolio quantile but with different weights. Hence, introducing $n(n-1)/2$ portfolios corresponds to an exact identified linear system, whereas going beyond this number results in an overidentified system solved by a least square calculation. The algebraic task is to derive an expression of the type $\boldsymbol{Ax} = \boldsymbol{\rho}$ where $\boldsymbol{\rho}$ represents the unique elements of a correlation matrix. We will derive the expressions for the components in the next step following Mittnik (2014).

Define $\mathcal{W} = \{\boldsymbol{w} = (w_1, ..., w_n) \in \mathbb{R}^n \,|\, \sum_{i=1}^{n} w_i = 1\}$ as the set of weight vectors where each element represents one specific wealth allocation. Define $\boldsymbol{q}_\alpha = (q_{\alpha_1,1}, ..., q_{\alpha_n,n})'$ as a vector of return quantiles and $\boldsymbol{R}_\alpha$ the quantile-implied correlation matrix. Technically, α_i can be different for each asset, but in our study, α is set constant for all quantiles. We discard the α-index for notational convenience. Possible choices for individual α levels will be discussed when investigating ES-implied correlation in Section 3.2.

Next, use the Schur-Product to rewrite (78) as a matrix product and factorize the quantiles that are associated with the variance:

$$\boldsymbol{q}_{\alpha_p,p}^2 = \boldsymbol{z}_{\boldsymbol{\alpha}}' \boldsymbol{R}_{\boldsymbol{\alpha}} \boldsymbol{z}_{\boldsymbol{\alpha}} \quad \text{with} \quad \boldsymbol{z}_{\boldsymbol{\alpha}} = \boldsymbol{q}_{\boldsymbol{\alpha}} \odot \boldsymbol{w} = \begin{bmatrix} w_1 q_{\alpha,1} & \dots & w_n q_{\alpha,n} \end{bmatrix} \tag{79}$$

$$\tilde{q}_{\alpha,p} = q_{\alpha,p}^2 - \sum_{i=1}^{n} q_{\boldsymbol{\alpha},i}^2 w_i^2 = \boldsymbol{z}_{\boldsymbol{\alpha}}'(\boldsymbol{R}_{\boldsymbol{\alpha}} - \boldsymbol{I})\boldsymbol{z}_{\boldsymbol{\alpha}}. \tag{80}$$

We derive an explicit expression of the unique elements in $\boldsymbol{R}_\alpha$ by applying the vectorization operator $\text{vec}(\cdot)$ on (80):

$$\tilde{q}_{\alpha,p} = \text{vec}(\boldsymbol{z}_{\boldsymbol{\alpha}}'(\boldsymbol{R}_{\boldsymbol{\alpha}} - \boldsymbol{I})\boldsymbol{z}_{\boldsymbol{\alpha}}) = (\boldsymbol{z}_{\boldsymbol{\alpha}} \otimes \boldsymbol{z}_{\boldsymbol{\alpha}})'\text{vec}(\boldsymbol{R}_{\boldsymbol{\alpha}} - \boldsymbol{I}) \tag{81}$$

where $\otimes$ is the Kronecker product. Since $\tilde{q}_{\alpha,p}$ is already a vector, it remains unchanged. The $\text{vec}(\cdot)$ operator creates a vector by vertically stacking the columns of a matrix. Furthermore, the operator admits an alternative representation with a duplication matrix $\boldsymbol{D}$ and the lower diagonal vectorization operator $\text{vecl}(\cdot)$, i.e. $\text{vec}(\cdot) = \boldsymbol{D}\text{vecl}(\cdot)$. The $\text{vecl}(\cdot)$ operator returns a vector by stacking the elements below the main diagonal column-wise of a matrix, effectively yielding the unique elements of the implied correlation matrix. Define $\text{vecl}(\boldsymbol{R}_{\boldsymbol{\alpha}}) := \boldsymbol{\rho}_{\boldsymbol{\alpha}}$ and we obtain

$$\tilde{q}_{\alpha,p} = (\boldsymbol{z}_{\boldsymbol{\alpha}} \otimes \boldsymbol{z}_{\boldsymbol{\alpha}})'\text{vec}(\boldsymbol{R}_{\boldsymbol{\alpha}} - \boldsymbol{I}) \tag{82}$$

$$= (\boldsymbol{z}_{\boldsymbol{\alpha}} \otimes \boldsymbol{z}_{\boldsymbol{\alpha}})'\boldsymbol{D}\text{vecl}(\boldsymbol{R}_{\boldsymbol{\alpha}}) \tag{83}$$

$$= (\boldsymbol{z}_{\boldsymbol{\alpha}} \otimes \boldsymbol{z}_{\boldsymbol{\alpha}})'\boldsymbol{D}\boldsymbol{\rho}_{\boldsymbol{\alpha}} \tag{84}$$

where $\boldsymbol{\rho}_{\boldsymbol{\alpha}}$ contains all unique elements of $\boldsymbol{R}_{\boldsymbol{\alpha}}$. Now, introduce $m \geq n(n-1)/2$ equations of the same type where the weight vector $\boldsymbol{w}_k$ of each equation is drawn from $\mathcal{W}$. Formally, define

$$\tilde{q}_{\alpha,p_k} = (\boldsymbol{z}_{\boldsymbol{\alpha},k} \otimes \boldsymbol{z}_{\boldsymbol{\alpha},k})'\boldsymbol{D}\boldsymbol{\rho}_{\alpha} \qquad \text{with} \quad k = 1, ..., m \tag{85}$$

In order to express all components by a linear system, we introduce quantities that allows us to express $(\boldsymbol{z}_{\boldsymbol{\alpha},k} \otimes \boldsymbol{z}_{\boldsymbol{\alpha},k})'\boldsymbol{D}$ for all k with a single matrix. Define

$$\tilde{\boldsymbol{q}}_p = (\tilde{q}_{\alpha,p_1}, \ldots, \tilde{q}_{\alpha,p_m})' \tag{86}$$

$$\boldsymbol{Z}_{\boldsymbol{\alpha}} = (\boldsymbol{z}_{\boldsymbol{\alpha},1}, \ldots, \boldsymbol{z}_{\boldsymbol{\alpha},m})' \tag{87}$$

$$\boldsymbol{X}_{\boldsymbol{\alpha}} = (\boldsymbol{Z}_{\boldsymbol{\alpha}} \otimes_r \boldsymbol{Z}_{\boldsymbol{\alpha}})\boldsymbol{D} \tag{88}$$

where $\otimes_r$ is the Khatri-Rao product or row-wise Kronecker product, see Appendix C definition C.2. The linear system that can be solved for $\rho_{\boldsymbol{\alpha}}$ has the form

$$\tilde{\boldsymbol{q}}_p = \boldsymbol{X}_{\boldsymbol{\alpha}}\boldsymbol{\rho}_{\boldsymbol{\alpha}}. \tag{89}$$

The implied correlation depending on the number of introduced equations is then given by

$$\boldsymbol{\rho}_{\boldsymbol{\alpha}} = \begin{cases} \boldsymbol{X}_{\boldsymbol{\alpha}}^{-1}\tilde{\boldsymbol{q}}_p, & \text{for } m = n(n-1)/2 \\ (\boldsymbol{X}_{\boldsymbol{\alpha}}'\boldsymbol{X}_{\boldsymbol{\alpha}})^{-1}\boldsymbol{X}_{\boldsymbol{\alpha}}'\tilde{\boldsymbol{q}}_p, & \text{for } m > n(n-1)/2. \end{cases} \tag{90}$$

Although our method produces tail correlation matrices, in some cases, they are not well-defined. To be precise, correlation coefficients may violate the theoretical range $[-1, 1]$ or not admit positive semi-definiteness (psd). The potential source for such occurrences may stem from the fact that VaR is not a coherent risk measure in the sense of Artzner et al. (1999) due to lack of subadditivity.

This deficit may propagate to the resulting correlation matrix, especially if local ellipticity does not hold. In addition, the VaR-implied correlation matrix might be inefficient due to lack of information beyond the specified quantile level. Recall that the VaR only controls the probability given a prespecified loss but does not account for the severity once a loss occurs (Föllmer & Schied 2004). This motivates ES-implied correlation to compensate for these drawbacks. In Section 3.3, we will discuss further methods to obtain well-defined correlation matrices.

3.2 Expected Shortfall Implied Correlation

Unlike VaR, the ES is a coherent risk measure. It is defined as an integral over the entire tail area. Thus, implying correlation from ES incorporates information about the severity of the loss beyond a single quantile. In our empirical study, using ES also contributes to numerical stability by reducing bound exceedances and violation of positive semi-definiteness. We define the Expected Shortfall first:

Definition 3.2.1 (Expected Shortfall)
Let L denote the loss of a financial instrument such that $\mathbb{E}[|L|] < \infty$. Given a confidence level $\alpha \in (0,1)$, the Expected Shortfall is defined as

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)du. \tag{91}$$

By integrating the VaR over the tail area beyond α, extreme events are now reflected in this risk measure. Depending on the study of interest, we can specify quantile regions by

$$\{\boldsymbol{q}_{\alpha_k} | \alpha_k \in \mathcal{A} \subset (0,1)^n\}. \tag{92}$$

For example, if only the lower and upper tail regions are of interest, $\mathcal{A}$ becomes

$$\mathcal{A}_{\text{lower}} = (0, \alpha_1] \times \cdots \times (0, \alpha_n] \tag{93}$$
$$\mathcal{A}_{\text{upper}} = [\alpha_1, 1) \times \cdots \times [\alpha_n, 1). \tag{94}$$

The choice of $\alpha_1, ..., \alpha_n$ consequently determines the range of the tail area with lower/higher levels representing more extreme regions. Integrating quantile regions into the implied correlation framework presented in Section 3.1 is straightforward. Given a quantile region $\mathcal{A}$, the natural representative quantile vector is given by the conditional expectation

$$\boldsymbol{q}_{\boldsymbol{\mathcal{A}}} = \mathbb{E}[\boldsymbol{q}_\alpha | \boldsymbol{\alpha} \in \boldsymbol{\mathcal{A}}]. \tag{95}$$

Let T_l denote the number of quantile vectors belonging to a specific tail region, e.g. $\mathcal{A}_{\text{lower}} = (0, \alpha]^n$. The empirical estimate corresponds to the discrete counterpart of the Expected Shortfall defined in 3.2.1. Formally,

$$\hat{\boldsymbol{q}}_{\mathcal{A}_l} = \frac{1}{T_l} \sum_{k=1}^{T_l} \boldsymbol{q}_{\boldsymbol{\alpha}_k} \qquad \text{with } \alpha_k \in \mathcal{A}_{\text{lower}}. \tag{96}$$

Technically, quantile regions enable a refined choice of confidence levels for each asset or portfolio. A possible application is to select the range of the region depending on the position of the corresponding asset. For example, given a long position, quantiles of the lower tail reflecting losses are of interest, while quantiles of the upper tails reflecting gains are considered for short positions. Selecting from both regions thus covers both position types in the portfolio. Another example is to introduce more estimation portfolios based on a neighborhood of the targeted confidence level $\alpha_k \in [\alpha_k \pm \delta]$ to incorporate more information on that area with different portfolio configurations. Studying all possible settings is not within the scope of this thesis. We will, therefore, focus on VaR and ES-based analysis with common tails and quantile levels.

3.3 Ensuring Well-Defined Correlation Matrices

Our proposed method to derive implied correlation does not guarantee well-defined matrices. Specifically, two requirements must be met: $\rho_{\alpha,i} \in [0, 1]$, where $\rho_{\alpha,i} \in \boldsymbol{\rho}_{\boldsymbol{\alpha}}$ are the unique elements of $\boldsymbol{R}_\alpha$ and $\boldsymbol{R}_\alpha$ is positive semi-definite. We present two strategies:

3.3.1 Quadratic Optimization

One strategy proposed by Mittnik (2014) is to treat the correlation estimation as a quadratic optimization problem:

$$\min_{\rho} \quad \frac{1}{2}\boldsymbol{\rho}_{\boldsymbol{\alpha}}' \boldsymbol{X}_{\boldsymbol{\alpha}}' \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{\rho}_{\boldsymbol{\alpha}} - 2\tilde{\boldsymbol{q}}_{\boldsymbol{p}}' \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{\rho}_{\boldsymbol{\alpha}} \tag{97}$$

subject to

$$|\boldsymbol{\rho}_{\boldsymbol{\alpha}}| \leq \mathbf{1}_{n(n-1)/2} \tag{98}$$

$$\lambda_i \geq 0 \quad \text{for} \quad i = 1, ..., n \tag{99}$$

where $\boldsymbol{X}_\alpha$ is defined as in (88) and $\boldsymbol{\rho}_\alpha$ in (90). The eigenvalues of $\boldsymbol{R}_{\boldsymbol{\alpha}}$ are defined λ_i with $i = 1, ..., n$ and $\mathbf{1}_{n(n-1)/2}$ represents a $n(n-1)/2$-dimensional vector of ones. The first constraint ensures that the implied correlation vector is properly bounded, while the second ensures positive semi-definiteness. However, similar to pairwise correlation estimation, this approach is unfeasible since the number of unknown parameters grows quadratically.

3.3.2 Spectral Correction

Spectral correction is a strategy that is often applied in portfolio optimization. The principle is to perform an eigenvalue decomposition of the covariance matrix and set all negative eigenvalues either to zero or a small positive value. The reconstructed covariance matrix is then positive semi-definite or definite. For correlation matrices, the reconstructed matrix must be scaled such that the diagonal elements admit unity.

Formally, define a matrix $\mathbf{R} \in \mathbb{R}^n$ that is not psd and perform an eigenvalue decomposition

$$\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top \tag{100}$$

where $\mathbf{Q}$ is the orthogonal matrix of eigenvectors, and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues λ_i for $i = 1, ..., n$. The spectral correction replaces all negative eigenvalues with zero, i.e.,

$$\tilde{\lambda}_i = \max(\lambda_i, 0). \tag{101}$$

Next, the adjusted correlation matrix $\tilde{\mathbf{R}}$ is reconstructed using the modified eigenvalues:

$$\tilde{\mathbf{R}} = \mathbf{Q}\tilde{\Lambda}\mathbf{Q}^\top \tag{102}$$

where $\tilde{\mathbf{\Lambda}}$ is the diagonal matrix of non-negative eigenvalues $\tilde{\lambda}_i$. According to Driessel (2007), by replacing negative eigenvalues, the resulting matrix is the approximation for the closest psd matrix w.r.t. the Frobenius norm and spectral norm. This first step ensures positive definiteness but does not guarantee unity on the diagonal. Rebonato & Jaeckel (2011) suggests rescaling the matrix for this property, i.e.,

$$\bar{\boldsymbol{R}} = \boldsymbol{S}\tilde{\boldsymbol{R}}\boldsymbol{S} \tag{103}$$

where $\boldsymbol{S}$ contains the reciprocal square roots of the diagonal elements of the matrix $\tilde{\boldsymbol{R}}$. The spectral correction will be applied to the implied correlation matrix if they are not positive semi-definite.

3.4 Portfolio Weights

The tail correlation estimate defined in (90) requires at least $n(n-1)/2$ equations to span a linear system from which $\boldsymbol{\rho}_\alpha$ is implied. A natural question is how to construct the portfolios, i.e., how to select portfolio weights. One minimal numerical requirement for $X_{\boldsymbol{\alpha}}$ is having full rank such that the inverse exists.

For an exact identified system Mittnik (2014) proposes equally weighted two-asset portfolios to maximize the degree of orthogonality. More precise, we choose weight vectors such that

$$\underset{w_i, w_j \in \mathcal{W}}{\arg\min} \quad \langle w_i, w_j \rangle \qquad \text{subject to} \quad w_i \neq w_j, \tag{104}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in a Euclidean space and $\mathcal{W}$ the space of portfolio weights. This procedure ensures orthogonality of the weight vectors and helps to avoid multicollinearity issues during estimation. Extension to an overidentified system in the same style is to include more equations by further adding equally weighted portfolios beyond two asset portfolios. For example by constructing a system consisting of equally weighted two-asset and three asset portfolios.

While maintaining the minimal numerical requirements, choosing the weights from an investment or risk management perspective revolves around allocation constraints. Such constraints may represent investment limits in certain asset classes or trading strategies. Accordingly, a set of portfolios that approximate the intended portfolio can be built. Variation of the weights provides the manager with additional insight into the tail dependence of the target portfolio. In this context, the choice of portfolio weights is motivated by understanding the tail correlation in a attainable set relevant to the investor rather than regions with unrealistic investments. Naturally, the weights can admit negative values if short positions are taken.

The choice of weight vectors is also subject to regulatory constraints such as the Solvency II directive (European Union 2009) or Basel III (Basel Committee on Banking Supervision 2013, 2017) accords. For example, the Basel III accords require banks to group financial instruments by similar risk characteristics. Correlation is then studied within and across the so-called risk buckets, e.g., interest rate, equity, or credit spread risk. Each risk bucket has a specific aggregation formula that includes risk weights and correlations. The final risk capital requirement, where diversification effects are considered, is then obtained by aggregating the different risk buckets using additional correlation parameters. The implied correlation matrix reflects these risk management practices if the weights are selected accordingly.

Finally, avoiding multicollinearity becomes cumbersome in high-dimensional portfolios since it is not feasible, if not impossible, to construct weight vectors accordingly in detail. In this case, a grid approach with randomly selected weights might provide a solution. A sufficiently large number of draws should provide a representative implied correlation of the subspace. The Dirichlet distribution is a candidate for the underlying distribution given random weights that ensure that portfolio weights sum up to one.

Definition 3.4.1 (Dirichlet Distribution)
Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ be a vector of positive real numbers, called the concentration parameters. The Dirichlet distribution with parameter α is a distribution over the k-dimensional probability simplex:

$$\Delta_{n-1} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}. \tag{105}$$

The probability density function of the Dirichlet distribution is given by:

$$f(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{B(\alpha)} \prod_{i=1}^{n} x_i^{\alpha_i - 1}, \tag{106}$$

for $(x_1, x_2, \ldots, x_n) \in \Delta_{n-1}$, where $B(\alpha)$ is the multivariate Beta function, defined as:

$$B(\alpha) = \frac{\prod_{i=1}^{n} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}, \tag{107}$$

and $\Gamma(\cdot)$ is the Gamma function.

If $\alpha_i = 1, \forall i = 1, ..., n$, the distribution equals a uniform distribution over the entire simplex corresponding to the number of assets. If the concentration parameters are larger than one, the resulting draw from the distribution is more constrained in smaller intervals.

3.5 Asymptotic Properties

Deriving the standard errors of correlation has been a challenge and subject of ongoing research; see Gnambs (2023) and the literature therein. There are several reasons: the underlying sampling distribution of a correlation coefficient in general is not normal, especially for small sample sizes and estimates near the boundary. Although Fisher's Z-transformation stabilizes the estimates to a certain degree, the distribution of the sample correlation remains skewed and asymmetric close to the boundaries which also persists for large samples. As a result, asymptotic normality only holds for moderate ρ. Furthermore, we lack knowledge of the true correlation distribution without further assumptions. In particular, this may pose a problem for financial returns since returns are not normal distributed. Nevertheless, this section derives the asymptotic distribution of the implied correlation vector $\boldsymbol{\rho}_{\boldsymbol{\alpha}}$ for the exact identified case. We will discuss alternative ways to obtain standard errors based on sampling methods in the subsequent section. Formally, we are interested in the limit distribution of

$$\sqrt{n}(\hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}} - \boldsymbol{\rho}_{\boldsymbol{\alpha}}). \tag{108}$$

Our strategy is based on two ideas. First, the correlation estimates can be interpreted as a function of asset and portfolio quantiles. In particular, the function is linear and differentiable. Second, sample quantiles in both univariate and multivariate settings are asymptotically normal distributed. As a result, the implied correlation estimates are also asymptotically normal distributed, and the covariance matrix can be obtained by the delta method. In summary, the asymptotic variance matrix of $\hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}}$ is derived in three steps:

1. Estimate the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{q}}$ of asset and portfolio quantiles.
2. Compute the gradient $\nabla\hat{\boldsymbol{\rho}}_{\alpha}$.
3. Scale $\boldsymbol{\Sigma}_{\mathbf{q}}$ with the gradient according to the delta method.

3.5.1 Asymptotic Distribution of Sample Quantiles

We present two key theorems that addresses the asymptotic normality of sample quantiles. Recall that for a distribution function F and $0 < \alpha < 1$ the α-quantile or the generalized inverse $F^{-1} : (0,1) \to \mathbb{R}$ of F is defined as

$$q_\alpha = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} | F(x) \geq \alpha\}. \tag{109}$$

For the sample equivalent, let $\{X_1, ..., X_n\}$ be a realization of F and $X_{(1)} \leq ... \leq X_{(n)}$ represent the order statistic, i.e., the sorted values of the sample. The empirical quantile is then defined as

$$\hat{q}_\alpha = cX_{(m_\alpha)} + (1-c)X_{(m_\alpha+1)} \tag{110}$$

where m_α is the integer part of $n\alpha$ and

$$c = \begin{cases} 1 & \text{if } n\alpha \in \mathbb{Z} \\ 0 & \text{else} \end{cases} \tag{111}$$

Hyndman & Fan (1996) discuss several sample quantile definitions and present computation methods. Interpolation techniques are an alternative way of determining quantiles if $n\alpha$ is not an integer. We will keep the intuitive definition using the integer part approach. In the next step, the asymptotic normality of sample quantiles is summarized in the two following theorems.

Theorem 3.5.1 (Asymptotic Distribution of a Sample Quantile)
Let F be a cumulative distribution function that is differentiable in the neighborhood of q_α with $0 < \alpha < 1$. Let the derivative $f(F^{-1}(\alpha)) > 0$ for any α. Then the empirical estimator $\hat{F}^{-1}(\alpha)$ of the theoretical quantile is asymptotically normal distributed with

$$\sqrt{n}(\hat{F}^{-1}(\alpha) - F^{-1}(\alpha)) \xrightarrow{d} \mathcal{N}\left(0, \frac{\alpha(1-\alpha)}{f^2(F^{-1}(\alpha))}\right). \tag{112}$$

Proof: See David & Nagaraja (2003, Theorem 10.3, p.288) and Walker (1968)

Babu & Rao (1988) showed that this result holds for quantile vectors. Let $\boldsymbol{X}_i = (x_{1i}, .., x_{ni})$ with $i = 1, ..., m$ be m independent copies of X. Define $q_i := F_i^{-1}(\alpha)$ as the marginal quantile function and $\hat{q}_i := \hat{F}_i^{-1}(\alpha)$ as the empirical counterpart. The theorem derives the asymptotic distribution of the vector $(\hat{q}_1, ..., \hat{q}_p)$ which represents the sample quantiles of the underlying data.

Theorem 3.5.2 (Joint Asymptotic Distribution of Quantile Functions)
Let F_i be continuously twice differentiable in a neighborhood of q_i and $\delta_i = f_i(q_i) > 0$, $i = 1, ..., m$ where f_i denotes the derivative of F_i. Then the asymptotic distribution of

$$\boldsymbol{y}_n := \sqrt{n}(\hat{q}_1 - q_1, ..., \hat{q}_m - q_m) \tag{113}$$

is m-variate normal with zero mean vector, and variance covariance matrix

$$\boldsymbol{\Sigma} = \begin{Bmatrix} \frac{\alpha_1(1-\alpha_1)}{\delta_1^2} & \frac{\sigma_{12}}{\delta_1\delta_2} & \cdots & \frac{\sigma_{1m}}{\delta_1\delta_m} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\sigma_{m1}}{\delta_m\delta_1} & \frac{\sigma_{m2}}{\delta_m\delta_2} & \cdots & \frac{\alpha_m(1-\alpha_m)}{\delta_m^2} \end{Bmatrix} \tag{114}$$

where $\sigma_{ij} = F_{ij}(q_i, q_j) - q_i q_j$.

Proof: See Babu & Rao (1988, p.17, Theorem 2.1)

This result ensures asymptotic normality and an explicit expression of the covariance matrix for sample quantiles which can be used to derive the limit distribution using the delta method.

3.5.2 The Limit Distribution

The Delta method presents a way to acquire the asymptotic distribution of desired statistics that can be represented as a function of another statistic with known asymptotic behavior. Assuming the function is continuous and differentiable, the limit distribution is derived directly from a Taylor expansion at a desired point. A linear transformation of the underlying known distribution then obtains the limit distribution.

Theorem 3.5.3 (Delta Method)
Let $\boldsymbol{\mu} \in \mathbb{R}^n$. Suppose that $\boldsymbol{Y}_n = (Y_{n1}, ..., Y_{nk})$ with $n \in \mathbb{N}$ is a sequence of random vectors such that

$$\sqrt{n}(\boldsymbol{Y}_n - \boldsymbol{\mu}) \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}) \tag{115}$$

Let $g : \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable function in the neighborhood of $\boldsymbol{\mu}$, $\nabla g(y)$ denote the gradient of g and $\nabla_\mu \equiv \nabla g(\mu) \neq 0$. Then

$$\sqrt{n}(g(\boldsymbol{Y_n}) - g(\boldsymbol{\mu})) \xrightarrow{d} \mathcal{N}(0, \nabla_\mu^T \boldsymbol{\Sigma} \nabla_\mu) \tag{116}$$

Proof: See Vaart (1998, p.26, Theorem 3.1)

We use the Delta method in order to derive the asymptotic distribution of the implied correlation vector. Note that the implied correlation vector can be interpreted as a linear map or a function of asset and portfolio quantiles. Let $\mathbf{q}$ be a vector containing asset and portfolio quantiles:

$$\mathbf{q} = \begin{bmatrix} q_{\alpha_1} & q_{\alpha_2} & \dots & q_{\alpha_n} & q_{p,1} & \dots & q_{p,m} \end{bmatrix}'. \tag{117}$$

Then, we can interpret (90) as a function of the quantile vector $\mathbf{q}$, formally

$$\boldsymbol{\rho}_{\boldsymbol{\alpha}}(\mathbf{q}) = \boldsymbol{X}_{\boldsymbol{\alpha}}^{-1}\tilde{\boldsymbol{q}}_p. \tag{118}$$

Following Theorem 3.5.2, the joint asymptotic distribution of sample quantiles follows a multivariate normal distribution

$$\sqrt{n}(\hat{\mathbf{q}} - \mathbf{q}) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{q}}) \tag{119}$$

where n is the sample size and $\hat{\mathbf{q}}$ is an estimate of the true population quantile $\mathbf{q}$. Babu & Rao (1988) point out that the covariance matrix $\Sigma_{\mathbf{q}}$ can be estimated by its sample equivalent with a kernel density estimator or bootstrap procedures. Both methods yield a consistent estimate of $\Sigma_{\mathbf{q}}$. Let $\nabla\boldsymbol{\rho}_{\alpha}$ denote the gradient of $\boldsymbol{\rho}_{\boldsymbol{\alpha}}(\cdot)$. Using the delta method in Theorem 3.5.3, the asymptotic distribution of $\hat{\boldsymbol{\rho}}$ is given by

$$\sqrt{n}(\boldsymbol{\rho}_{\boldsymbol{\alpha}}(\hat{\mathbf{q}}) - \boldsymbol{\rho}_{\boldsymbol{\alpha}}(\mathbf{q})) \sim \mathcal{N}(\mathbf{0}, \nabla'_{\rho_{\boldsymbol{\alpha}}}\mathbf{\Sigma}_{\mathrm{q}}\nabla_{\rho_{\boldsymbol{\alpha}}}) \tag{120}$$

from which standard errors can be obtained.

3.5.3 Computing the Gradient

We compute the gradient of $\boldsymbol{\rho}_{\boldsymbol{\alpha}}(\cdot)$ using tools from matrix algebra. A detailed description of how to manipulate matrices using matrix products such as the Kroncker product, Khatri-Rao product, and factorizing rules are found in Lütkepohl (2006). Matrix differentials are discussed in Abadir & Magnus (2005). Appendix C summarizes the most important features. Our approach is similar to Neudecker & Wesselman (1990), where the covariance matrix of a sample correlation matrix is derived. Recall that $\boldsymbol{\rho}_{\boldsymbol{\alpha}}$ is obtained as a solution of the linear system, assuming that $\boldsymbol{X}_{\boldsymbol{\alpha}}$ is of full rank. We suppress the α index for notational convenience.

Theorem 3.5.4
Let $\boldsymbol{\rho}(\cdot)$ be the estimator as defined in (118). *In particular $\boldsymbol{\rho} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ where n is the number of assets and $m = n(n-1)/2$ the number of portfolios. Further, let $\boldsymbol{A} \equiv \boldsymbol{A}_\alpha = \boldsymbol{Z}_\alpha \otimes \boldsymbol{Z}_\alpha$ where Z_α denotes the matrix of weighted quantiles defined in* (87). *Then the gradient $\nabla'_{\boldsymbol{\rho}}$ is given by*

$$\frac{d\boldsymbol{\rho}}{d\mathbf{q}'} = \boldsymbol{X}^{-1}\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p} - (\boldsymbol{\rho}' \otimes \boldsymbol{X}^{-1})(\boldsymbol{D}' \otimes \boldsymbol{S})\boldsymbol{\mathcal{D}}_A \tag{121}$$

where $\boldsymbol{D}$ and $\boldsymbol{S}$ denote a duplication matrix and a selection matrix, respectively. $\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p}$ and $\boldsymbol{\mathcal{D}}_A$ contain all partial derivatives with respect to the asset and portfolio quantiles.

Proof: First, apply the differential operator on (89). By the chain rule (165), the equation is of the form

$$d\tilde{\boldsymbol{q}}_p = (d\boldsymbol{X})\boldsymbol{\rho} + \boldsymbol{X}d\boldsymbol{\rho}$$

Next, solve for $d\boldsymbol{\rho}$ and use (169), (159):

$$d\boldsymbol{\rho} = \boldsymbol{X}^{-1}d\tilde{\boldsymbol{q}}_p - \boldsymbol{X}^{-1}(d\boldsymbol{X})\boldsymbol{\rho} \tag{122}$$

$$= \boldsymbol{X}^{-1}d\tilde{\boldsymbol{q}}_p - (\boldsymbol{\rho}' \otimes \boldsymbol{X}^{-1})d\text{vec}\boldsymbol{X} \tag{123}$$

We evaluate the expressions of the differentials $d\tilde{\boldsymbol{q}}_p$ and $d\text{vec}\boldsymbol{X}$ separately. Note that $\boldsymbol{X}^{-1}$ and $\tilde{\boldsymbol{q}}_p$ are functions of $\mathbf{q}$. Consider $d\tilde{\boldsymbol{q}}_p$, we have

$$d\tilde{\boldsymbol{q}}_p = \boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p}d\mathbf{q} \tag{124}$$

where $\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p}$ is the Jacobian containing all partial derivatives w.r.t. to $\mathbf{q}$. Technical notes on this step are presented in Appendix C.3.

For the second component, $\boldsymbol{X}$ was introduced in (88) as the Khatri-Rao product of $\boldsymbol{Z}_\alpha$ that contained the weighted quantiles given m different portfolios attached to a duplication matrix $\boldsymbol{D}$. Applying the differential on the vectorized matrix and reformulating the Khatri-Rao product to a Kronecker product with (161), we get:

$$\begin{aligned} d\text{vec}\boldsymbol{X} &= d\text{vec}((\boldsymbol{Z}_\alpha \otimes_r \boldsymbol{Z}_\alpha)\boldsymbol{D}) \\ &= d\text{vec}(\boldsymbol{S}(\boldsymbol{Z}_\alpha \otimes \boldsymbol{Z}_\alpha)\boldsymbol{D}) \\ &= (\boldsymbol{D}' \otimes \boldsymbol{S})d\text{vec}(\boldsymbol{Z}_\alpha \otimes \boldsymbol{Z}_\alpha) \end{aligned}$$

where $\boldsymbol{S}$ is another selection matrix. Further simplifying does not yield a more beneficial form of the derivative. Therefore, define $\boldsymbol{A} = \boldsymbol{Z}_\alpha \otimes \boldsymbol{Z}_\alpha$ and rewrite $d\text{vec}\boldsymbol{X}$ with differentials

$$d\text{vec}\boldsymbol{X} = (\boldsymbol{D}' \otimes \boldsymbol{S})d\text{vec}\boldsymbol{A} \tag{125}$$

$$= (\boldsymbol{D}' \otimes \boldsymbol{S})\boldsymbol{\mathcal{D}}_A d\mathbf{q} \tag{126}$$

Note that $\boldsymbol{\mathcal{D}}_A$ only depend on the asset quantiles, thus the Jacobian matrix has two blocks, the first block contains the derivatives with respect to the asset quantiles while the second block is a matrix of zeros. Now, we can state the derivative. Insert (124), (125) into (122) yields:

$$d\boldsymbol{\rho} = \boldsymbol{X}^{-1}\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p} d\mathbf{q} - (\boldsymbol{\rho}' \otimes \boldsymbol{X}^{-1})(\boldsymbol{D}' \otimes \boldsymbol{S})\boldsymbol{\mathcal{D}}_A d\mathbf{q} \tag{127}$$

$$= (\boldsymbol{X}^{-1}\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p} - (\boldsymbol{\rho}' \otimes \boldsymbol{X}^{-1})(\boldsymbol{D}' \otimes \boldsymbol{S})\boldsymbol{\mathcal{D}}_A) d\mathbf{q} \tag{128}$$

Finally, the gradient of the implied correlation estimator is given by

$$\nabla'_{\rho_\alpha} = \frac{d\boldsymbol{\rho}}{d\mathbf{q}'} = \boldsymbol{X}^{-1}\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{q}}_p} - (\boldsymbol{\rho}' \otimes \boldsymbol{X}^{-1})(\boldsymbol{D}' \otimes \boldsymbol{S})\boldsymbol{\mathcal{D}}_{A_\alpha} \tag{129}$$

q.e.d.

3.6 Semi-Parametric Estimation - Assessing Standard Errors

As mentioned in the previous section, the distribution of the sample correlation is asymmetric, especially for small sample sizes and estimates close to the theoretical bounds $[-1, 1]$, rendering theoretical standard errors unreliable. An alternative way to obtain standard errors is through bootstrapping. In particular, Horowitz (2001) points out that the Bootstrap is often more accurate in finite samples than first-order asymptotic approximations.

However, bootstrapping financial returns must be done with care. As was discussed in Chapter 2, sampling returns directly may render the bootstrapped statistics invalid due to the unique characteristics of financial returns, such as heteroscedasticity and serial correlation. Consequently, the sample is not independent and identically distributed as required for bootstrapping. Two bootstrap methods to account for these characteristics were discussed in Section 2.4, i.e., Moving Block Bootstrap (MBB) and Filtered Historical Simulation (FHS). Recall that although MBB can partially preserve dependence structure, it is ambiguous regarding block length and has the potential to induce structural breaks. For this specific task, continue with the filtering approach by estimating univariate GARCH(1,1) models with normal distributed errors for each asset. Ideally, the standardized residuals should be further filtered for cross-correlation by a multivariate model, e.g., Example 2.3.1, but this step was dropped to ensure numerical feasibility.

For standard errors of a statistic of interest, Efron & Tibshirani (1994) suggest repeating the sampling procedure at least 300 times. We follow this suggestion but add an initial stage (burn-in) of an additional 100 draws, thus sampling the standardized residuals 400 times. Heteroscedasticity is then reintroduced according to the model parameters (see Appendix B.2), and the first 100 values are discarded. This corrects for unrealistic initial values that may not represent the overall distribution. Although not pursued in this work, this procedure can also produce confidence intervals.

3.7 Empirical Study

The first part of the empirical study focuses on the impact of different parameter settings, e.g., different quantile levels, risk measures (VaR and ES), identification strategies for the portfolio system, variation in portfolio weights, and choice of return frequencies. In addition, we compare theoretical and bootstrapped standard errors for the exact identified case. The setup for this study allows a wide array of parameter combinations, which produces a substantial volume of results. To maintain clarity, we will present the most relevant findings to our research question, namely, whether correlations tend to increase during turbulent market periods and highlight surprising results. All remaining results are collected in Appendix B.

The second part of the study investigates the differences between implied correlation and Pearson correlation. Furthermore, we test whether the implied correlation is asymmetric conditional on the tail regions. In other words, does the correlation implied by losses differ from the correlation implied by gains?

3.7.1 Tail Correlation Patterns

We study international correlation patterns for stock indices of 10 developed countries across geographical regions. The dataset consists of $T_{\text{daily}} = 6604$ daily observations for each index starting from 1997-12-31 until 2022-12.31. For weekly and monthly returns the data set is reduced to $T_{\text{weekly}} = 1322$ and $T_{\text{monthly}} = 304$. This period features the impact of significant historical financial and political crises, including the Dot-com Bubble (2000-2002), the Global Financial Crisis precipitated by the US housing bubble (2007-2009), the Covid-19 pandemic (2020-2022), and the outbreak of the Russo-Ukrainian War. The data source is Thomson Reuters Datastream. Centered logarithmic returns computed are considered. A large data sample ensures numerical stability due to more datapoints in tail regions.

We present tail correlation estimates for lower and upper tails implied by Value-at-Risk and Expected Shortfall. The lower tail region addresses losses events, whereas the upper tail reflect gains. Given a risk measure, we cover different α-quantile levels, i.e., $\alpha \in \{0.001, 0.005, 0.01, 0.05, 0.1\}$ for daily and weekly returns. For monthly returns, we select $\alpha \in \{0.005, 0.01, 0.025, 0.05, 0.1\}$. The reason for a different set of quantiles while using monthly returns is the numerical instability that arises due to the potential lack of data. Our method relies heavily on sufficient data reflecting extreme scenarios which by nature are scarce.

No.	Symbol	Index Name (Country)
1	DJI	Dow Jones Industrial Average (United States)
2	NASDAQ	Nasdaq Composite (United States)
3	IBEX	IBEX 35 Index (Spain)
4	FCHI	CAC 40 Index (France)
5	SSMI	Swiss Market Index (Switzerland)
6	FTMIB	FTSE MIB Index (Italy)
7	GDAX	DAX Performance-Index (Germany)
8	FTSE	FTSE 100 Index (United Kingdom)
9	N225	Nikkei 225 Index (Japan)
10	HSI	Hang Seng Index (Hong Kong)

Table 5: Stock indices used for the empirical analysis across geographical regions.

3.7.2 Does Correlation Increase for Extreme Quantiles?

Figure 13 visualizes implied correlation for different quantile levels using heatmaps. The Pearson correlation is displayed on the bottom right. Each heatmap is divided by a diagonal of ones where the lower triangular matrix represents the correlation matrix implied by the lower tail and the upper triangular matrix by the upper tail. In doing so, we obtain a graphical comparison between two tails that might reveal potential asymmetries. Positively correlated indices are colored red, and negatively correlated pairs are colored blue. Darker colors signify stronger/weaker dependence. We will focus on correlation implied by the Expected Shortfall because it produce more stable results by incorporating information from the entire quantile region of interest. Correlation implied by VaR tends to behave erratic especially once extreme quantiles are considered.

We can observe that stock indices of countries that share a geographical vicinity tend to be stronger correlated resulting in distinct block patterns. For central European countries, this can be explained by the stronlgy intertwined economies. DJI and NASDAQ are both indices of the US, which is also plausible for pronounced interdependence. European markets also tend to exhibit stronger dependence towards US markets, which is not apparent for Asian indices. Asian markets overall seem to be loosely connected to their Western counterparts.

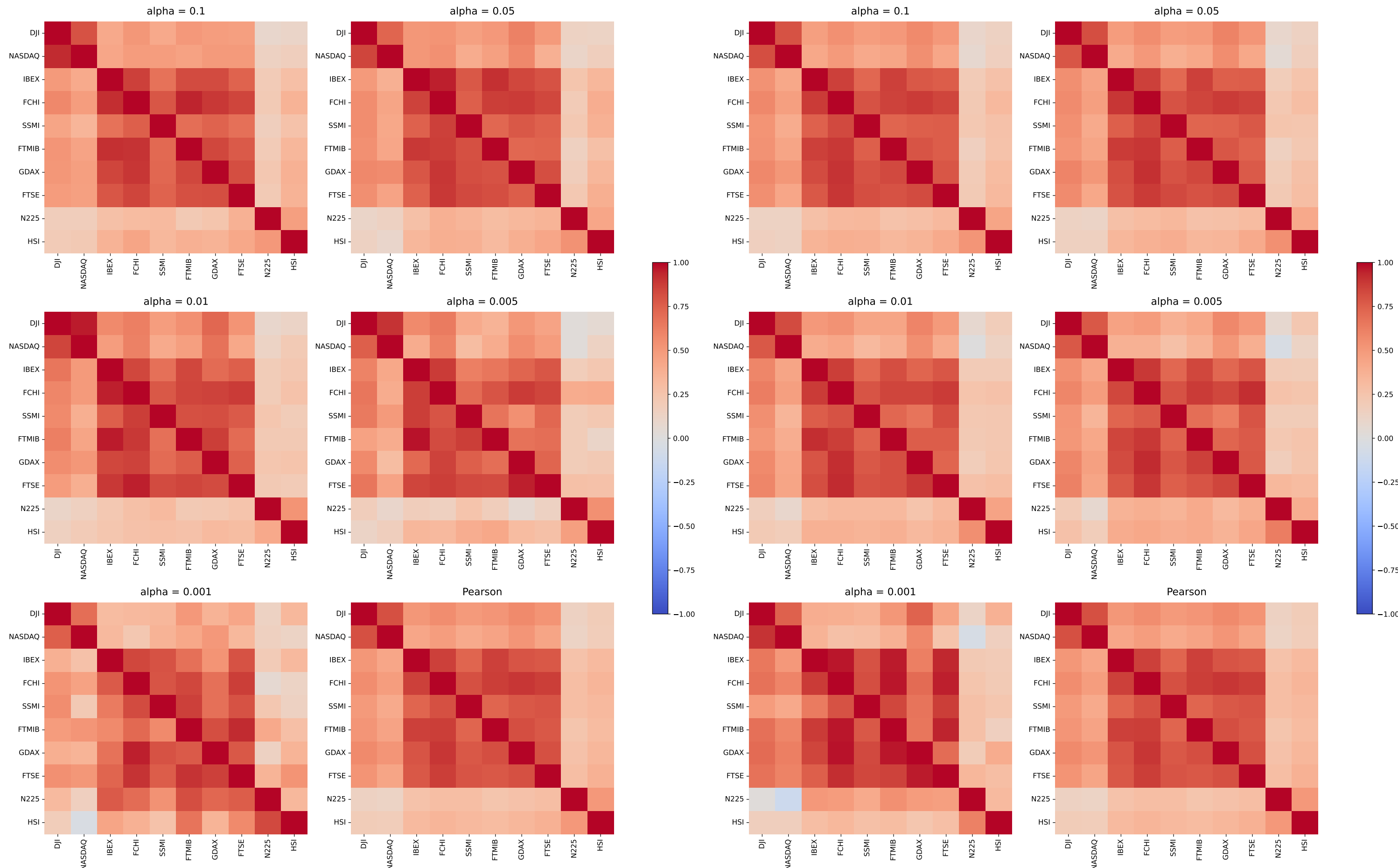
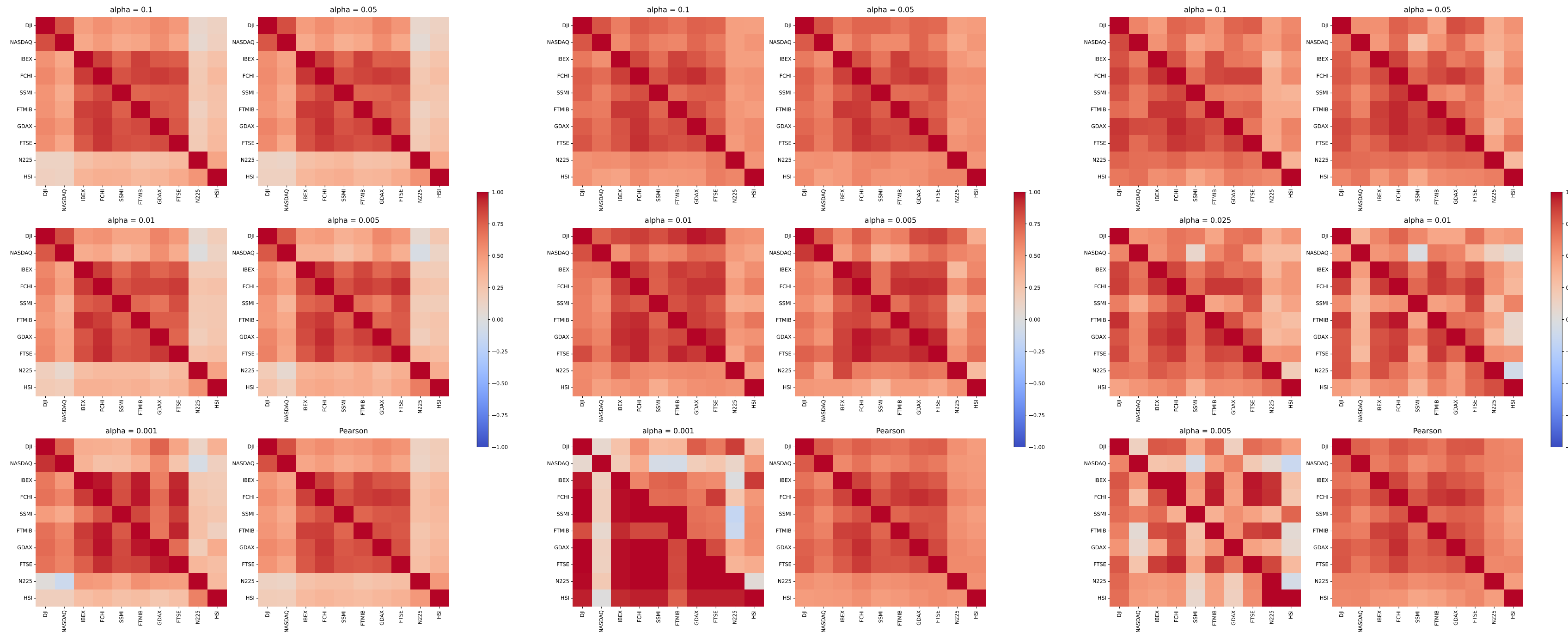


Figure 13: Heatmaps of ES-implied correlation for daily returns - The left panel features VaR- and the right panel shows ES-implied correlation from equal-weighted two-asset portfolios.



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

Figure 14: Heatmaps of implied correlation for daily, weekly and monthly returns - Implied correlation by equal-weighted two-asset portfolios. The lower triangular matrix shows correlation implied by the lower tail, the upper triangular matrix shows correlation implied by the upper tail.

Figure 14 displays ES-implied correlation for different return frequencies. While the correlation block patterns remain mostly distinguishable for daily and weekly returns, this block structure vanishes if monthly returns are considered, especially for extreme quantiles. One possible explanation is the reduced serial correlation for lower return frequencies than daily or weekly returns. Looking at different quantile levels, we can observe changes in implied correlation in both directions. Dependence tends to increase when moving into the tails, particularly for pairs already strongly correlated. On the other hand, pairs that exhibit weak correlation might even change signs. Drastic changes in correlation are only observed for lower tail implied correlation given weekly returns, especially for $\alpha = 0.001$. However, it is questionable if it makes sense that DJI and NASDAQ become less correlated in such an extreme scenario. Additional heatmaps for varying return frequencies and VaR-implied correlation are presented in Appendix B.5.

We provide another visualization of implied correlation using boxplots. Figure 15 plots implied correlation of all return frequencies according to a quantile level. More precisely, correlation coefficients over all return frequencies are collected and plotted against a quantile level. We distinguish between red boxes representing the lower tail while green boxes visualize the upper tail. Boxplots of both tails are more closely aligned for quantile regions closer to the center of the distribution. The horizontal line in the box represents the median, which is slightly higher for the upper tail regions. This trend shifts towards the lower tail implied correlation beginning with $\alpha = 0.25$. The difference is most extreme for $\alpha = 0.001$, which supports our hypothesis for increased correlation in extreme loss scenarios.

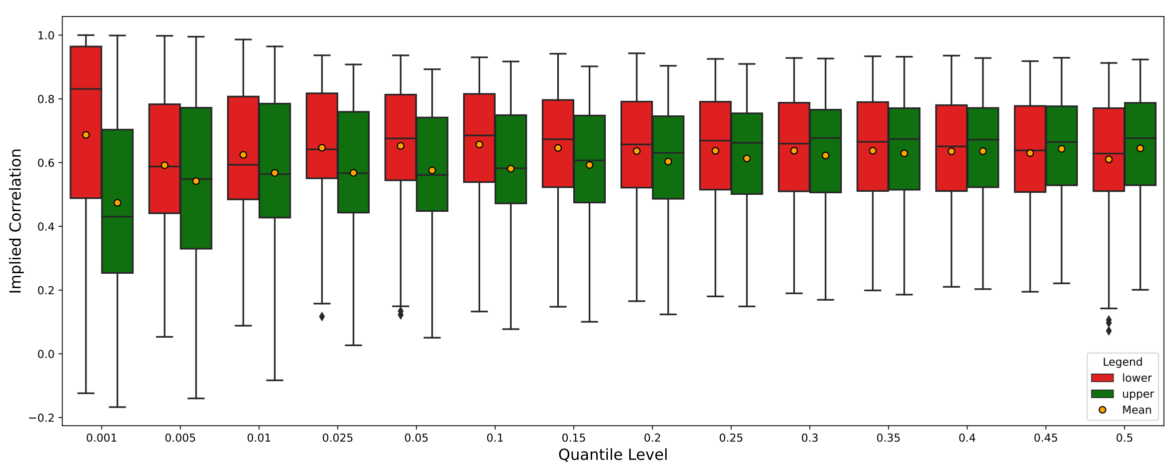


Figure 15: Boxplot of ES-implied correlation over all return frequencies.

We observe the same results also for different return frequencies, see Figure 27 in Appendix B.6. Asset-specific correlation are presented in Figure 28. One particular observation here is that the range of the boxplots is more narrow for pairs that are in closer proximity. Boxes with the largest range usually include indices of the Asian markets.

3.7.3 Theoretical and Bootstrapped Standard Errors

We investigate the efficiency of the correlation estimate in two settings. First, we compare theoretical standard errors with bootstrapped standard errors for correlation estimates implied by an exact identified portfolio system. In this case, equally weighted two-asset portfolios are considered. Second, bootstrapped standard errors of estimates implied by an exact and overidentified system are investigated. The overidentified portfolio system adds equally weighted two-asset portfolios and equally weighted three-asset portfolios. For $n = 10$ assets, the exact identified portfolio consists of 45 equations, while the overidentified portfolio system consists of 165 equations.

We observe the increase in standard errors when moving into tails because the amount of available data for modeling is scarce. This effect is more pronounced with decreasing return frequency. Regarding the variability of the implied correlation estimate, we further notice that theoretical standard errors differ significantly from bootstrapped standard errors for specific pairs with no clear patterns.

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.82 (0.044)	0.491 (0.047)	0.558 (0.047)	0.484 (0.043)	0.496 (0.045)	0.603 (0.144)	0.518 (0.298)	0.09 (0.046)	0.148 (0.041)
NASDAQ	0.784 (0.044)	1	0.409 (0.04)	0.501 (0.043)	0.389 (0.041)	0.429 (0.038)	0.561 (0.115)	0.424 (0.213)	0.051 (0.032)	0.169 (0.031)
IBEX	0.549 (0.049)	0.45 (0.043)	1	0.86 (0.07)	0.715 (0.064)	0.867 (0.074)	0.751 (0.151)	0.761 (0.335)	0.18 (0.041)	0.255 (0.037)
FCHI	0.569 (0.044)	0.469 (0.039)	0.893 (0.071)	1	0.802 (0.07)	0.851 (0.07)	0.882 (0.169)	0.852 (0.366)	0.221 (0.042)	0.29 (0.039)
SSMI	0.543 (0.043)	0.41 (0.04)	0.755 (0.066)	0.844 (0.065)	1	0.732 (0.064)	0.741 (0.165)	0.774 (0.367)	0.243 (0.048)	0.242 (0.041)
FTMIB	0.522 (0.047)	0.441 (0.04)	0.877 (0.073)	0.894 (0.066)	0.758 (0.063)	1	0.784 (0.153)	0.742 (0.325)	0.154 (0.038)	0.226 (0.035)
GDAX	0.595 (0.129)	0.51 (0.104)	0.818 (0.153)	0.909 (0.159)	0.801 (0.159)	0.836 (0.148)	1	0.767 (0.422)	0.188 (0.087)	0.28 (0.088)
FTSE	0.563 (0.294)	0.423 (0.212)	0.802 (0.347)	0.877 (0.37)	0.834 (0.38)	0.802 (0.341)	0.825 (0.406)	1	0.216 (0.192)	0.295 (0.181)
N225	0.134 (0.044)	0.122 (0.033)	0.279 (0.046)	0.307 (0.044)	0.329 (0.05)	0.27 (0.045)	0.28 (0.093)	0.309 (0.207)	1	0.419 (0.051)
HSI	0.154 (0.036)	0.149 (0.029)	0.339 (0.042)	0.373 (0.039)	0.391 (0.043)	0.339 (0.039)	0.345 (0.091)	0.41 (0.219)	0.544 (0.056)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.776 (0.14)	0.459 (0.155)	0.488 (0.152)	0.378 (0.129)	0.428 (0.156)	0.585 (0.489)	0.505 (0.2)	0.073 (0.124)	0.229 (0.134)
NASDAQ	0.778 (0.151)	1	0.373 (0.107)	0.371 (0.096)	0.28 (0.092)	0.362 (0.105)	0.514 (0.367)	0.39 (0.106)	-0.045 (0.072)	0.128 (0.084)
IBEX	0.543 (0.172)	0.431 (0.122)	1	0.884 (0.211)	0.726 (0.203)	0.837 (0.213)	0.722 (0.544)	0.791 (0.236)	0.201 (0.137)	0.188 (0.114)
FCHI	0.59 (0.159)	0.486 (0.113)	0.838 (0.21)	1	0.797 (0.194)	0.877 (0.193)	0.838 (0.575)	0.92 (0.226)	0.264 (0.132)	0.248 (0.109)
SSMI	0.516 (0.163)	0.351 (0.1)	0.73 (0.194)	0.767 (0.172)	1	0.692 (0.182)	0.622 (0.521)	0.796 (0.196)	0.194 (0.131)	0.193 (0.121)
FTMIB	0.51 (0.151)	0.423 (0.107)	0.848 (0.21)	0.887 (0.198)	0.74 (0.175)	1	0.728 (0.538)	0.766 (0.231)	0.221 (0.136)	0.253 (0.126)
GDAX	0.59 (0.396)	0.467 (0.292)	0.821 (0.464)	0.924 (0.509)	0.785 (0.461)	0.86 (0.47)	1	0.775 (0.804)	0.176 (0.327)	0.247 (0.282)
FTSE	0.616 (0.215)	0.438 (0.117)	0.781 (0.224)	0.897 (0.206)	0.755 (0.179)	0.796 (0.212)	0.851 (0.65)	1	0.329 (0.236)	0.312 (0.178)
N225	0.201 (0.151)	0.078 (0.09)	0.366 (0.171)	0.389 (0.16)	0.359 (0.154)	0.414 (0.157)	0.324 (0.308)	0.39 (0.23)	1	0.392 (0.156)
HSI	0.268 (0.116)	0.185 (0.076)	0.401 (0.134)	0.423 (0.116)	0.391 (0.113)	0.411 (0.119)	0.362 (0.277)	0.432 (0.14)	0.626 (0.18)	1

Table 6: ES-implied correlation estimates with theoretical standard errors - lower tail correlation estimates are on the lower triangle matrix while the upper triangular matrix represent implied correlation by upper quantiles.

Starting with Table 6 and 7: Bootstrapped SEs are smaller than their theoretical counterparts in most cases. This is visible for pairs involving the NASDAQ, DAX, or N225. Surprisingly, this is not the case for pairs involving DJI. Theoretical SEs involving DJI are overall smaller than bootstrapped SEs, especially for $\alpha = 0.05/0.95$, while bootstrapped SEs for $\alpha = 0.005/0.995$ show no significant differences. For weekly and monthly returns, theoretical SEs remain significantly larger than bootstrapped SEs, but SEs increase due to the reduced amount of underlying data in both cases.

The discrepancy between theoretical and bootstrapped standard errors may be attributed to finite sample properties since asymptotic theory often approximates properties for large

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.82 (0.103)	0.491 (0.079)	0.558 (0.079)	0.484 (0.073)	0.496 (0.08)	0.603 (0.147)	0.518 (0.211)	0.09 (0.048)	0.148 (0.049)
NASDAQ	0.784 (0.208)	1	0.409 (0.029)	0.501 (0.028)	0.389 (0.03)	0.429 (0.028)	0.561 (0.034)	0.424 (0.074)	0.051 (0.027)	0.169 (0.027)
IBEX	0.549 (0.217)	0.45 (0.028)	1	0.86 (0.018)	0.715 (0.029)	0.867 (0.019)	0.751 (0.035)	0.761 (0.09)	0.18 (0.03)	0.255 (0.031)
FCHI	0.569 (0.243)	0.469 (0.029)	0.893 (0.018)	1	0.802 (0.025)	0.851 (0.016)	0.882 (0.03)	0.852 (0.097)	0.221 (0.029)	0.29 (0.029)
SSMI	0.543 (0.231)	0.41 (0.028)	0.755 (0.026)	0.844 (0.025)	1	0.732 (0.027)	0.741 (0.034)	0.774 (0.093)	0.243 (0.029)	0.242 (0.027)
FTMIB	0.522 (0.213)	0.441 (0.029)	0.877 (0.019)	0.894 (0.02)	0.758 (0.027)	1	0.784 (0.032)	0.742 (0.091)	0.154 (0.028)	0.226 (0.028)
GDAX	0.595 (0.159)	0.51 (0.05)	0.818 (0.048)	0.909 (0.054)	0.801 (0.051)	0.836 (0.049)	1	0.767 (0.165)	0.188 (0.048)	0.28 (0.043)
FTSE	0.563 (0.175)	0.423 (0.127)	0.802 (0.129)	0.877 (0.143)	0.834 (0.126)	0.802 (0.129)	0.825 (0.114)	1	0.216 (0.059)	0.295 (0.059)
N225	0.134 (0.086)	0.122 (0.037)	0.279 (0.03)	0.307 (0.031)	0.329 (0.032)	0.27 (0.029)	0.28 (0.038)	0.309 (0.106)	1	0.419 (0.027)
HSI	0.154 (0.084)	0.149 (0.035)	0.339 (0.03)	0.373 (0.032)	0.391 (0.032)	0.339 (0.029)	0.345 (0.035)	0.41 (0.106)	0.544 (0.027)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.776 (0.11)	0.459 (0.084)	0.488 (0.085)	0.378 (0.075)	0.428 (0.086)	0.585 (0.082)	0.505 (0.217)	0.073 (0.066)	0.229 (0.068)
NASDAQ	0.778 (0.098)	1	0.373 (0.082)	0.371 (0.08)	0.28 (0.079)	0.362 (0.084)	0.514 (0.078)	0.39 (0.08)	-0.045 (0.055)	0.128 (0.066)
IBEX	0.543 (0.113)	0.431 (0.08)	1	0.884 (0.064)	0.726 (0.087)	0.837 (0.065)	0.722 (0.088)	0.791 (0.094)	0.201 (0.067)	0.188 (0.074)
FCHI	0.59 (0.118)	0.486 (0.076)	0.838 (0.065)	1	0.797 (0.085)	0.877 (0.061)	0.838 (0.079)	0.92 (0.093)	0.264 (0.069)	0.248 (0.075)
SSMI	0.516 (0.12)	0.351 (0.074)	0.73 (0.09)	0.767 (0.093)	1	0.692 (0.088)	0.622 (0.09)	0.796 (0.09)	0.194 (0.063)	0.193 (0.071)
FTMIB	0.51 (0.108)	0.423 (0.078)	0.848 (0.063)	0.887 (0.06)	0.74 (0.098)	1	0.728 (0.093)	0.766 (0.092)	0.221 (0.066)	0.253 (0.074)
GDAX	0.59 (0.129)	0.467 (0.08)	0.821 (0.107)	0.924 (0.101)	0.785 (0.105)	0.86 (0.104)	1	0.775 (0.097)	0.176 (0.071)	0.247 (0.077)
FTSE	0.616 (0.161)	0.438 (0.137)	0.781 (0.152)	0.897 (0.156)	0.755 (0.149)	0.796 (0.15)	0.851 (0.135)	1	0.329 (0.078)	0.312 (0.072)
N225	0.201 (0.124)	0.078 (0.055)	0.366 (0.07)	0.389 (0.073)	0.359 (0.071)	0.414 (0.071)	0.324 (0.07)	0.39 (0.122)	1	0.392 (0.08)
HSI	0.268 (0.122)	0.185 (0.054)	0.401 (0.064)	0.423 (0.07)	0.391 (0.065)	0.411 (0.064)	0.362 (0.067)	0.432 (0.123)	0.626 (0.077)	1

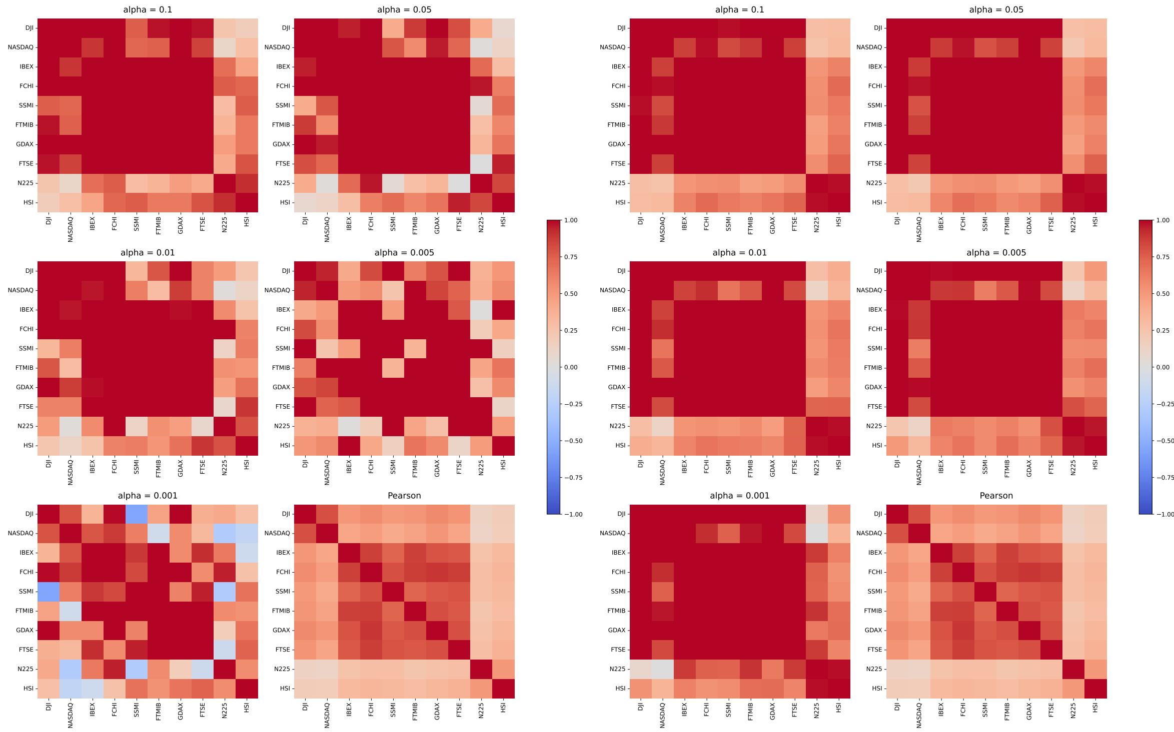
Table 7: ES-implied correlation estimates with bootstrapped standard errors - lower tail correlation estimates are on the lower triangle matrix while the upper triangular matrix represent implied correlation by upper quantiles.

samples. For extreme tail regions, asymptotic properties derived from theoretical SEs might not hold. In contrast, the bootstrap can provide a more accurate reflection of the actual sampling variability in smaller sample sizes, see Horowitz (2001). We find no evidence of an increase in efficiency for correlation implied by an overidentified system. The magnitude of SEs is in the same range of correlation implied by an exact identified system. Estimation results of all parameter sets are presented in Appendix B.3. In the subsequent analysis, only ES-implied correlations from exact identified systems are explored.

3.7.4 Portfolio Systems with Random Weights

Our study on correlation implied by portfolios with random weights yields no meaningful results. If weight is assigned to every asset, the random weights are drawn from a Dirichlet distribution that ensures that the sum of the weights equals one. The parameter of the distribution is selected such that the Dirichlet distribution reflects a uniform distribution over all its support. For two-asset portfolios, we draw a single weight w_1 from the uniform distribution and define the second weight as $w_2 = 1 - w_1$. Figure 16 presents heatmaps of two settings with random weights on all assets and random weights on two assets. In both cases, the portfolio system is exact identified.

While random weights on two asset portfolios still produce block patterns, although much more extreme, random weights on all assets produce erratic patterns. Further investigation reveals numerous boundary violations and frequent application of spectral corrections to produce well-defined correlation matrices. We did not continue further research in this area. However, this analysis highlights the critical role of portfolio weights when applying



(a) Random Weights on all assets

(b) Random weights on two assets

Figure 16: Heatmaps of lower ES-implied correlation - Exact identified systems with randomly weighted portfolios for daily returns are considered.

the implied correlation methodology. Future research could address the trade-off between selecting weights for numerical stability while maintaining an economic interpretation.

3.7.5 Difference between Implied Correlation and Pearson Correlation

We visualize the difference between implied correlation and Pearson correlation in two ways. Figure 17a shows the difference

$$\rho_{\text{Pearson}} - \rho_{\text{Implied}}$$

for different quantile levels and plots the sorted values in descending order. The x-axis represents the order of the corresponding pair, which differs across quantile levels. Pairs above the dotted line indicate a positive difference featuring cases where the Pearson correlation is larger than the implied correlation. The number of positive/negative pairs might provide insight into systematic behavior relative to each type of correlation.

We investigate pairs of the lower tail first. Implied correlation from moderate quantiles are close, with 0.005- and 0.001-quantile levels producing the largest deviations. Most of the differences are negative for daily and weekly returns, suggesting stronger tail dependence. Surprisingly for monthly returns, the Pearson correlation is larger than the correlation implied by the 0.005-ES and 0.01-ES in contrast to 0.1-ES which produces a larger tail correlation in all pairs. Regarding the upper tail, most Pearson correlations are slightly larger for daily returns and monthly returns. Similar to the lower tail, the largest differences are observed for extreme quantile levels. In this setting, we find no clear indications if the implied correlation deviates systematically from the Pearson correlation.

An alternative comparison is presented in Figure 17b where average implied and Pearson correlation are viewed instead of single pairs. The plots reveal that the ES-implied correlation behaves less volatile than the VaR-implied correlation, as was mentioned at the beginning of the empirical analysis. The average ES-implied correlation is close to its Pearson counterpart over all return frequencies for moderate quantile levels. For the lower tail, the average correlation tends to increase while moving into the tails. This effect is more observed for daily and weekly returns. In contrast, the average correlation decreases sharply for monthly returns given extreme quantiles. Regarding the upper tail, the average correlation tends to decrease for high quantile levels, suggesting a reduced correlation for gains. Overall, our results have similarities to Mittnik (2014), although no information on other return frequencies was provided. It was shown that correlation dependence behaves distinctly for high quantile levels and thus is not compatible with an elliptical data-generating process. We conclude with similar results.

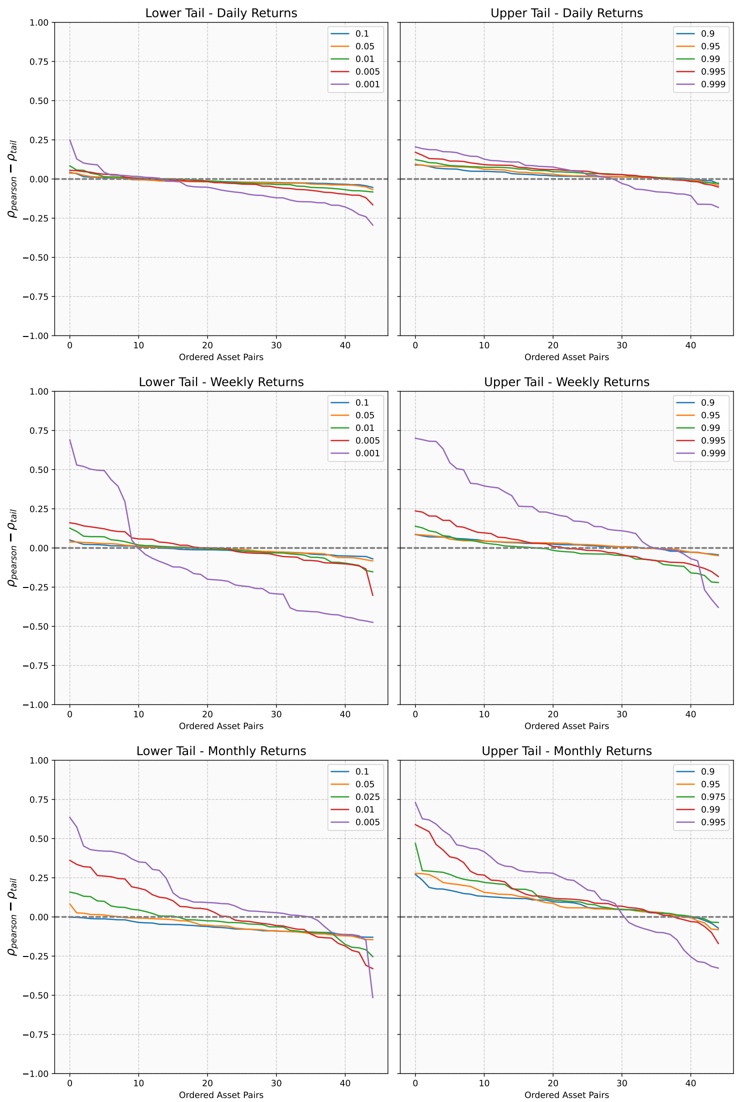
We conclude this section by comparing the ten strongest and weakest correlation pairs across different quantile levels with their Pearson counterpart. Table 8 displays the pairs for daily returns. Tables with other return frequencies are found in Appendix B.7. Considering lower tail correlation, graphical analysis already provided a hint that indices of countries that are geographically close to each other are strongly correlated. Indeed, nine out of ten pairs feature European indices such as France-Germany (FCHI-GDAX) or France-Italy (FCHI-FTSEMIB), with the former being the most substantial overall return frequency. The implied correlation constantly remains high for the ten strongly correlated pairs and significantly increases when selecting small quantile levels. In particular, ES implied correlations for $\alpha = 0.001$ are consistently higher than Pearson correlation for strongly correlated pairs, whereas only 6 out of 10 pairs behave similarly for the weakly correlated pairs. Note that nine out of ten weak correlation pairs always include an Asian index, HSI, or N225, further highlighting the geographical significance when studying dependence structures. Results for upper tail correlation are also mixed. The increasing correlation behavior for high quantile levels is inconsistent among the strong and weak correlated pairs. Unlike the lower tail where FCHI/GDAX are strongly correlated even for extreme quantiles, the correlation decreases from .878 to .704 when considering the 0.999 quantile on daily return frequency. In extreme cases, the sign of the correlation might even change, e.g., NASDAQ/N225, although it should be noted that the corresponding Pearson correlation is already close to zero. In summary, this analysis shows that pairs that are already strongly dependent in terms of Pearson correlation tend to increase if measured by implied correlation while moving into the tails. This behavior is more distinct for the lower tail.

	Lower Tail (Losses)					Pearson	Upper Tail (Gains)				
Correlation Pairs	**0.001**	**0.005**	**0.01**	**0.05**	**0.01**		**0.9**	**0.95**	**0.99**	**0.995**	**0.999**
FCHI vs GDAX	0.974	0.924	0.917	0.909	0.906	0.891	0.878	0.882	0.848	0.838	0.704
FCHI vs FTMIB	0.961	0.887	0.868	0.894	0.886	0.874	0.859	0.851	0.845	0.877	0.961
FCHI vs FTSE	0.921	0.897	0.922	0.877	0.891	0.869	0.851	0.852	0.881	0.920	0.951
IBEX vs FCHI	0.880	0.838	0.882	0.893	0.881	0.867	0.862	0.860	0.868	0.884	0.963
IBEX vs FTMIB	0.878	0.848	0.915	0.877	0.864	0.860	0.864	0.867	0.818	0.837	0.955
FTMIB vs GDAX	0.966	0.860	0.813	0.836	0.830	0.815	0.795	0.784	0.760	0.728	0.661
FCHI vs SSMI	0.801	0.767	0.803	0.844	0.834	0.810	0.797	0.802	0.796	0.797	0.815
DJI vs NASDAQ	0.906	0.778	0.779	0.784	0.814	0.809	0.804	0.820	0.827	0.776	0.749
GDAX vs FTSE	0.956	0.851	0.888	0.825	0.826	0.805	0.789	0.767	0.731	0.775	0.696
IBEX vs GDAX	0.846	0.821	0.796	0.818	0.822	0.794	0.776	0.751	0.731	0.722	0.619
NASDAQ vs N225	-0.124	0.078	0.088	0.122	0.134	0.125	0.077	0.051	0.002	-0.045	-0.048
DJI vs N225	0.019	0.201	0.169	0.134	0.133	0.148	0.098	0.090	0.078	0.073	0.123
NASDAQ vs HSI	0.170	0.185	0.186	0.149	0.144	0.187	0.164	0.169	0.140	0.128	0.160
DJI vs HSI	0.168	0.268	0.204	0.154	0.157	0.191	0.146	0.148	0.187	0.229	0.372
FTMIB vs N225	0.543	0.414	0.323	0.270	0.265	0.249	0.159	0.154	0.216	0.221	0.277
IBEX vs N225	0.504	0.366	0.295	0.279	0.274	0.263	0.200	0.180	0.199	0.201	0.213
GDAX vs N225	0.492	0.324	0.251	0.280	0.287	0.266	0.200	0.188	0.182	0.176	0.189
SSMI vs N225	0.413	0.359	0.324	0.329	0.335	0.292	0.222	0.243	0.219	0.194	0.279
FCHI vs N225	0.491	0.389	0.327	0.307	0.321	0.293	0.212	0.221	0.257	0.264	0.244
FTSE vs N225	0.475	0.390	0.323	0.309	0.321	0.297	0.207	0.216	0.267	0.329	0.338

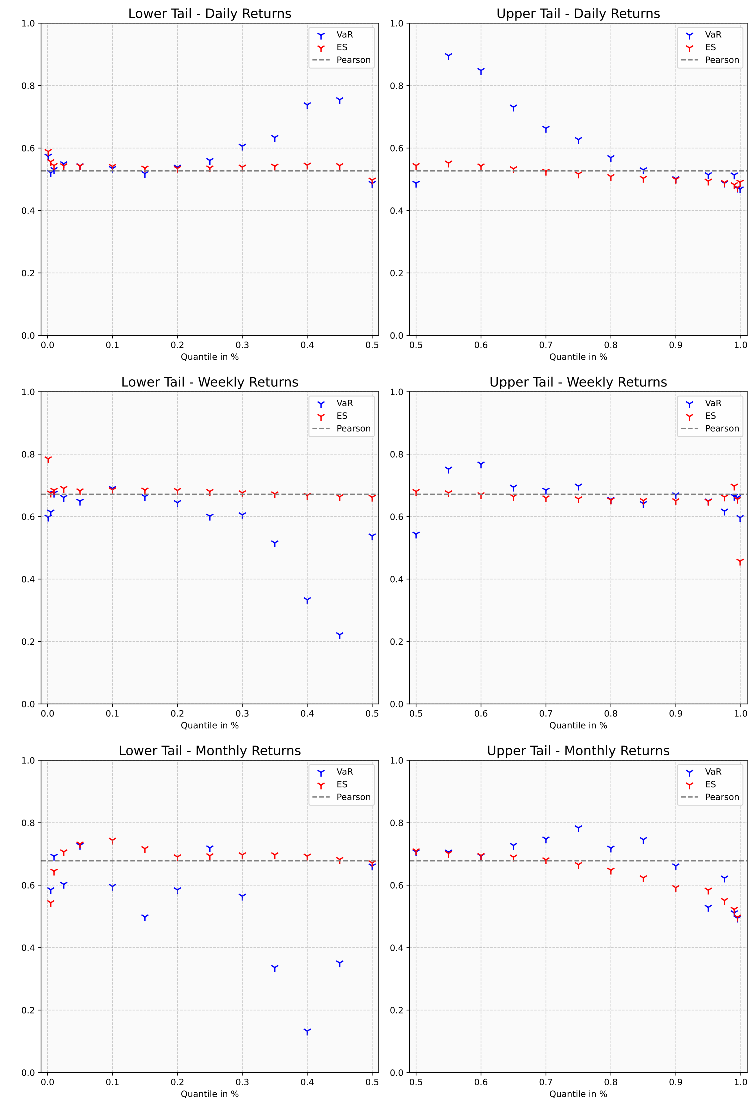
Table 8: Pearson and ES-implied correlation from daily returns - Presented are the ten strongest and weakest pairs according to Pearson correlation. We compare Pearson correlation with ES-implied correlation over different quantile levels.

In order to assess the statistical significance of the difference between Pearson and implied correlation, we construct confidence intervals based on the bootstrapped standard errors. In particular, the difference is likely significant if the confidence intervals are disjoint, i.e., do not overlap. We further apply Fisher's z-transformation to the sample correlations to stabilize the variance, enabling more robust statistical testing of the coefficients. As described in Section 3.5, this transformation yields approximately normally distributed variables, particularly for coefficients near the bounds and in cases of small sample sizes. The transformation is defined by

$$z = \frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right). \tag{130}$$



(a) Deviation from Pearson correlation sorted in descending order



(b) Average Implied Correlation vs Average Pearson Correlation

Figure 17: Differences between implied and Pearson correlation - Left Panel features the difference between implied and Pearson correlation - Right Panel shows the average correlation.

In the subsequent step, the confidence intervals are constructed by

$$\text{CI} = z \pm Z_{\alpha/2} \cdot \text{SE}_z \tag{131}$$

The results are brought back to original scale using the inverse fisher transform. Formally,

$$\rho = \frac{e^{2z} - 1}{e^{2z} + 1}. \tag{132}$$

Table 9 shows the confidence intervals of Pearson and ES-implied correlation for the lower tail ($\alpha = 0.01$) based on monthly returns. The five correlation pairs with the largest positive difference are shown, corresponding to the red line depicted in the bottom left plot in Figure 17a.

	Pearson		0.01 ES-Implied		
Correlation Pair	ρ_{Pearson}	CI	ρ_{ES}	CI	$\rho_{\text{Pearson}} - \rho_{\text{ES}}$
NASDAQ vs GDAX	0.734	[0.697, 0.767]	0.373	[0.021, 0.642]	0.361
NASDAQ vs FTSE	0.651	[0.599, 0.698]	0.316	[-0.059, 0.613]	0.335
NASDAQ vs FCHI	0.712	[0.671, 0.749]	0.391	[0.055, 0.648]	0.320
SSMI vs FTSE	0.741	[0.704, 0.773]	0.423	[0.220, 0.591]	0.317
NASDAQ vs FTMIB	0.627	[0.570, 0.677]	0.362	[0.032, 0.622]	0.264

Table 9: Implied correlation - 5% confidence intervals

We find three pairs (NASDAQ-GDAX, NASDAQ-FCHI, SSMI-FTSI) where the confidence intervals do not overlap, indicating statistical differences. However, evaluating the significance of the difference between Pearson and implied correlation in this way is less robust than a statistical test since no precise p-values or degree of significance are obtained. One criterion for existing statistical tests regarding the equality of correlation coefficients or correlation matrices is the independence of the underlying sample. In our case, both Pearson and implied correlations are obtained from the same sample. While Pearson correlation is computed over the entire sample, implied correlation only considers quantiles. It is unclear how statistical tests for the significance of differences are conducted under these circumstances. Nevertheless, we can conclude that the Pearson correlation matrix differs from the implied correlation matrix if only a single pair is already significantly different.

3.8 Tail Asymmetry

Literature on statistical tests specifically for tail correlations is scarce. Based on exceedance correlation according to Longin & Solnik (2001), see equation (1), Ang & Chen (2002) proposed a summary statistic that measures how much the observed (tail) correlation in the data deviates from those predicted by a reference model. Candidates for a reference model are, for example, normal distribution, GARCH, or regime-switching models. Therefore, the test evaluates whether the null hypothesis, which asserts that the reference model matches the empirical correlation, holds true. However, this approach relies on the assumption of a reference model and does not provide information on asymmetries in the return data itself. Hong et al. (2007) propose a non-parametric test that generally evaluates data asymmetry. As a result, if symmetry is rejected, the underlying data cannot be modeled by any symmetric distribution. In addition, the asymptotic behavior is known to follow a chi-square distribution. However, both studies suffer from the ambiguous choice of exceedance level. Different choices may yield varying degrees of evidence for asymmetry. In addition, the presented methods are only available for bi-variate portfolios.

We assess asymmetry or, more generally, distortion by adopting a regression approach. The idea is to regress the correlation implied by the upper tail against the lower tail. Formally,

$$\rho_{\text{lower}} = \beta_0 + \beta_1 \rho_{\text{upper}} + \epsilon. \tag{133}$$

The intercept β_0 relates to a systematic shift of implied correlation w.r.t. a reference correlation. For example, while $\beta_1 = 1$ holds, a positive intercept translates to a systematic positive deviation, stating a stronger correlation in general. The slope parameter β_1 addresses deviation conditional on the size of the correlation, i.e., whether correlation differs only for pairs that are stronger (or weaker) correlated. This translates to a rotation of the regression line. Geometrically, if the corresponding correlation pairs are identical, they should be situated on a 45-degree line. Consequently, several types of asymmetry are considered:

1. $\beta_0 \neq 0$ and $\beta_1 = 1$ shift distortion
2. $\beta_0 = 0$ and $\beta_1 \neq 1$ size distortion
3. $\beta_0 \neq 0$ and $\beta_1 \neq 1$ both

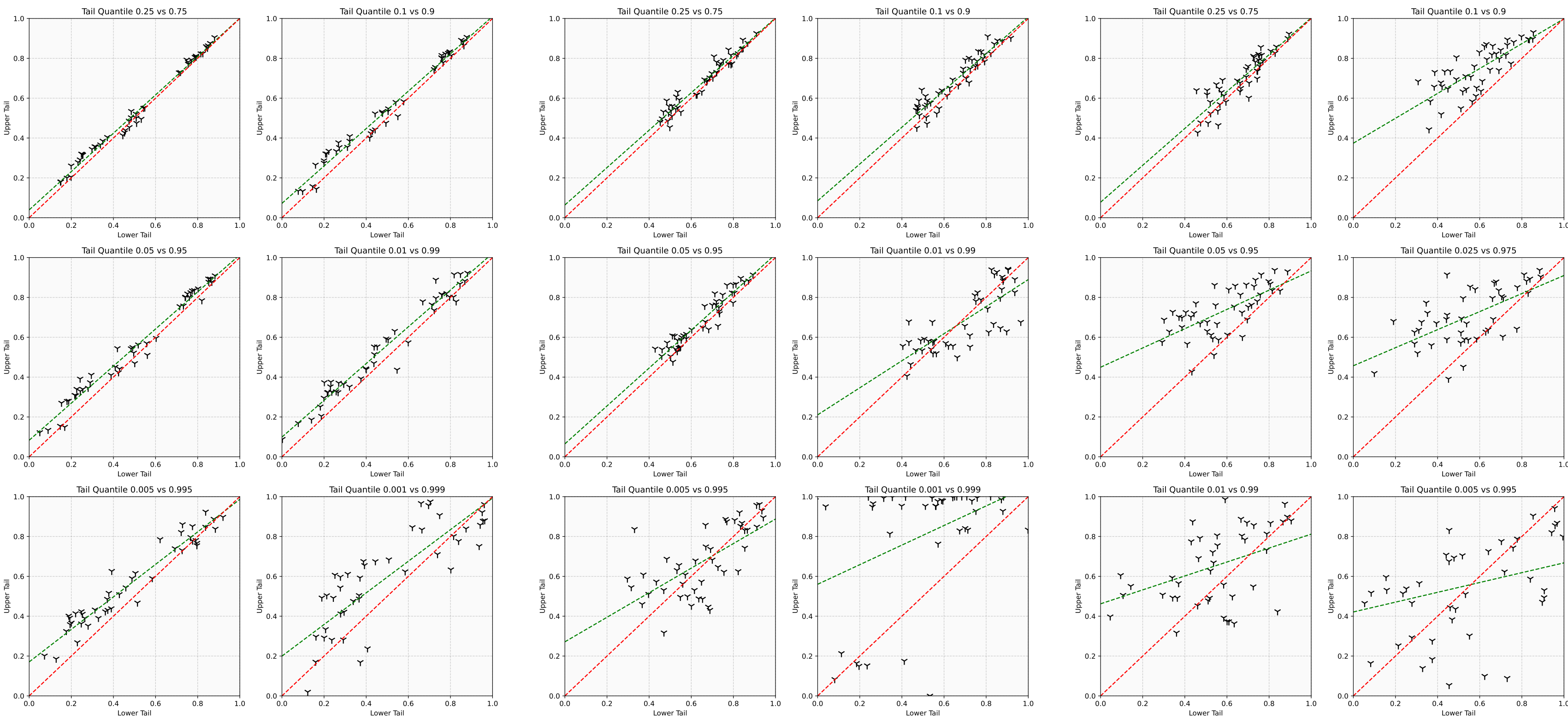
Similar to the previous section, we apply Fisher's Z-transformation for robustness. If the sets of correlation are similar, the intercept β_0 should be close to zero and the slope β_1 close to unity.

3.8.1 Distortion Analysis between upper and lower Tail Correlation

Regression results are reported in Table 10. ES-implied correlation by the lower tail is regressed on implied correlation by the upper tail of the same quantile level. The corresponding scatterplot is visualized in Figure 18. The intercept parameter is positive overall return frequencies and quantile levels. The intercept increases significantly once extreme quantile levels are considered, with 0.560 as the maximum for $\alpha = 0.001$ and weekly returns. The results indicate a systematic shift where the correlation implied by losses seems to be systematically higher than the correlation implied by gains. Concurrently, the slope parameter decreases down to 0.246 for monthly returns. We find the most considerable deviations for extreme quantile levels or lower return frequencies that coincide with our findings from previous analyses. For distortion in size, the clockwise rotation of the regression line suggests that large correlation coefficients implied by lower tails are larger relative to their counterpart in the upper tail. Apart from a few occurrences, the hypothesis "$\beta_0 = 0$" is always rejected on a high confidence level. This holds the same for "$\beta_1 = 1$". The F-statistics for the hypothesis "$\beta_0 = 0 \wedge \beta_1 = 1$" consistently rejects symmetry for all tail correlation pairs, including extreme quantiles.

Tail Region α	$\hat{\beta}_0$	$t_{\hat{\beta}_0=0}$	p-val	$\hat{\beta}_1$	$t_{\hat{\beta}_1=1}$	p-val	$F_{\beta_0=0,\beta_1=1}$	p-val
				Daily Returns				
0.250	0.039	4.343	0.000	0.962	60.341	0.000	16.517	0.000
0.100	0.072	6.222	0.000	0.939	45.404	0.000	35.652	0.000
0.050	0.083	6.281	0.000	0.931	39.206	0.000	36.444	0.000
0.010	0.099	6.267	0.000	0.922	31.901	0.000	38.019	0.000
0.005	0.171	9.814	0.000	0.816	25.354	0.000	65.843	0.000
0.001	0.200	4.430	0.000	0.793	9.958	0.000	12.979	0.000
				Weekly Returns				
0.250	0.064	2.577	0.013	0.939	25.402	0.000	13.034	0.000
0.100	0.085	2.624	0.012	0.923	18.951	0.000	15.913	0.000
0.050	0.065	2.050	0.046	0.952	19.899	0.000	15.426	0.000
0.010	0.211	3.433	0.001	0.679	7.955	0.000	7.493	0.002
0.005	0.271	3.691	0.001	0.616	5.701	0.000	6.811	0.003
0.001	0.560	6.729	0.000	0.492	3.224	0.002	31.468	0.000
				Monthly Returns				
0.250	0.078	1.569	0.124	0.926	12.633	0.000	6.115	0.005
0.100	0.374	7.818	0.000	0.624	7.951	0.000	101.842	0.000
0.050	0.449	8.989	0.000	0.484	5.863	0.000	80.775	0.000
0.025	0.457	8.940	0.000	0.453	5.167	0.000	62.797	0.000
0.010	0.462	7.363	0.000	0.350	3.215	0.002	28.451	0.000
0.005	0.421	6.367	0.000	0.246	2.187	0.034	23.228	0.000

Table 10: Relationship between lower and upper ES-implied correlations estimates - Reported are intercept and slope estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$, when regressing lower tail-correlation on upper tail-correlations. For a given tail region, specified by α, absence of shift and size symmetry is given vor "$\beta_0 = 0$" and "$\beta_1 = 1$". The corresponding t-statistics are reported in the next column. The F-statistic tests the joint absence of shift and size asymmetry, with p-values given in the last column.



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

Figure 18: ES-implied correlation: Lower tail vs Upper tail - Matching pairs will lie on the red line. The green line represents a linear approximation of the two correlation types. The corresponding regression results are reported in table 10.

3.8.2 Distortion Analysis between Implied and Pearson Correlation

The regression approach also enables us to study other correlation settings such as tail correlation with Pearsons correlation, i.e.,

$$\rho_{\text{tail}} = \beta_0 + \beta_1 \rho_{\text{Pearson}} + \epsilon, \tag{134}$$

where ρ_{tail} can either represent correlation implied from the upper or lower tail. Using the same approach from the previous section, we compare the differences by regressing implied correlation on Pearson correlation. Regression results are presented in Table 11 and 12. The corresponding scatterplot visualizations are found in Appendix B.8, Figure 29 and 30.

Regarding shift distortion for lower tails, the intercepts for different quantile levels are close or above zero with only one significant deviation of 0.401 for $\alpha = 0.001$ given weekly returns. Out of 18 configurations, only two cases reject the $\beta_0 = 0$ hypothesis on a 1% significance level. Therefore, we find little evidence for a systematic shift. Regarding size distortion, the $\hat{\beta}_1$ coefficient ranges from 0.572 to 1.119, with half of the cases above and below one. Except for one case, the Null hypothesis of $\beta_1 = 1$ is rejected on all levels, confirming a size distortion. However, we find no clear pattern concerning the direction for this type of asymmetry.

Considering the upper tail, we find mixed evidence for shift distortion with β_0 ranging between -0.543 and 0.084. Most negative estimates suggest that the upper implied correlations are smaller than their Pearson counterpart. However, we reject the hypothesis "$\beta_0 = 0$" on 1% significance level in 7 out of 18 cases. Slope estimates are close to 1, slightly varying between 0.952 and 1.379. Statistical tests uniformly reject "$\beta_1 = 1$", pointing to significant size distortion. We also note that in 12 out of 18 cases, the estimates are larger than 1. Geometrically, the regression line rotates counterclockwise, indicating that Pearson correlation tends to dominate their implied correlation counterpart for small sizes and vice versa. Similar to the results from investigating tail asymmetry, the F-statistics rejects symmetry in most of the cases.

Tail Region α	$\hat{\beta}_0$	$t_{\hat{\beta}_0=0}$	p-val	$\hat{\beta}_1$	$t_{\hat{\beta}_1=1}$	p-val	$F_{\beta_0=0,\beta_1=1}$	p-val
				Daily Returns				
0.500	-0.092	-6.069	0.000	1.119	42.470	0.000	21.518	0.000
0.100	0.007	1.030	0.309	1.013	86.268	0.000	13.585	0.000
0.050	0.004	0.500	0.620	1.023	76.904	0.000	14.527	0.000
0.010	0.012	0.831	0.410	1.011	42.166	0.000	4.701	0.014
0.005	0.076	4.535	0.000	0.911	31.275	0.000	13.857	0.000
0.001	0.040	1.026	0.311	1.041	15.357	0.000	7.686	0.001
				Weekly Returns				
0.500	-0.018	-0.846	0.402	1.012	32.427	0.000	2.977	0.062
0.100	0.029	1.383	0.174	0.977	31.540	0.000	6.390	0.004
0.050	0.037	1.466	0.150	0.962	26.314	0.000	3.173	0.052
0.010	-0.039	-0.820	0.417	1.077	15.316	0.000	1.537	0.227
0.005	-0.070	-0.952	0.346	1.109	10.382	0.000	0.553	0.579
0.001	0.401	1.535	0.132	0.572	1.499	0.141	3.251	0.048
				Monthly Returns				
0.500	-0.009	-0.293	0.771	1.004	22.079	0.000	0.810	0.452
0.100	0.104	3.137	0.003	0.944	19.586	0.000	68.903	0.000
0.050	0.121	2.563	0.014	0.901	13.164	0.000	23.750	0.000
0.025	0.109	1.293	0.203	0.881	7.162	0.000	2.445	0.099
0.010	0.120	0.777	0.441	0.774	3.455	0.001	1.314	0.279
0.005	-0.013	-0.065	0.949	0.821	2.820	0.007	7.969	0.001

Table 11: Relationship between Pearson and ES-implied correlations estimates for the lower tail region. Reported are intercept and slope estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$, when regressing lower tail correlation on Pearson correlations. For a given tail region, specified by α, absence of shift and size asymmetry is given for "$\beta_0 = 0$" and "$\beta_1 = 1$". The corresponding t-statistics are reported in the next column. The F-statistic tests the joint absence of shift and size asymmetry, with p-values given in the last column.

Tail Region α	$\hat{\beta}_0$	$t_{\hat{\beta}_0=0}$	p-val	$\hat{\beta}_1$	$t_{\hat{\beta}_1=1}$	p-val	$F_{\beta_0=0,\beta_1=1}$	p-val
				Daily Returns				
0.500	0.042	6.932	0.000	0.952	90.373	0.000	33.600	0.000
0.100	-0.063	-7.770	0.000	1.068	76.045	0.000	45.436	0.000
0.050	-0.076	-7.465	0.000	1.082	61.368	0.000	42.050	0.000
0.010	-0.085	-6.661	0.000	1.077	48.852	0.000	42.366	0.000
0.005	-0.099	-5.530	0.000	1.086	34.950	0.000	30.868	0.000
0.001	-0.089	-2.210	0.032	1.100	15.746	0.000	3.457	0.041
				Weekly Returns				
0.500	0.034	1.423	0.162	0.963	27.656	0.000	2.477	0.096
0.100	-0.014	-0.521	0.605	0.989	25.492	0.000	8.663	0.001
0.050	0.006	0.229	0.820	0.957	25.347	0.000	11.324	0.000
0.010	-0.092	-1.417	0.164	1.176	12.340	0.000	3.907	0.028
0.005	-0.114	-1.374	0.177	1.146	9.423	0.000	1.255	0.295
0.001	-0.318	-1.529	0.134	1.154	3.795	0.000	14.930	0.000
				Monthly Returns				
0.500	0.084	3.184	0.003	0.923	24.119	0.000	27.417	0.000
0.100	-0.134	-2.111	0.041	1.071	11.619	0.000	32.148	0.000
0.050	-0.202	-2.523	0.015	1.159	9.964	0.000	24.882	0.000
0.025	-0.384	-4.188	0.000	1.378	10.355	0.000	37.382	0.000
0.010	-0.543	-3.758	0.001	1.572	7.481	0.000	23.782	0.000
0.005	-0.291	-1.155	0.255	1.163	3.175	0.003	8.999	0.001

Table 12: Relationship between Pearson and ES-implied correlations estimates for the upper tail region. Reported are intercept and slope estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$, when regressing upper tail correlation on Pearson correlations. For a given tail region, specified by α, absence of shift and size asymmetry is given for "$\beta_0 = 0$" and "$\beta_1 = 1$". The corresponding t-statistics are reported in the next column. The F-statistic tests the joint absence of shift and size asymmetry, with p-values given in the last column.

Chapter 4

Forecasting Correlation Matrices

Based on the implied correlation methodology in the previous chapter, we propose a new method to forecast correlation matrices. The idea is to imply a correlation matrix based on quantile forecasts. Although Expected Shortfall implied correlation matrices provide more stable results, we will focus on Value-at-Risk for simplicity. Therefore, the initial step in acquiring correlation forecasts is to obtain Value-at-Risk predictions.

4.1 CAViaR - Conditional Autoregressive Value-at-Risk

Traditional methods for forecasting VaR rely on the entire return distribution. For example, Barone-Adesi et al. (1999) obtain VaR forecasts by bootstrapping filtered returns and scaling the bootstrap sample with a volatility forecast. The VaR is then obtained from the scaled bootstrap sample. In this case, bootstrapping filtered returns corresponds to modeling the entire distribution. In contrast, the key concept of CAViaR models, as introduced by Engle & Manganelli (2004), is to incorporate dynamics into quantiles using an autoregressive specification instead of building a return distribution. CAViaR models directly estimate VaR as a function of lagged variables from an information set, such as past VaR estimates, lagged returns, and other relevant factors. We outline the methodology in the following paragraph and adjust the notation for consistency.

Denote $\alpha \in \mathbb{R}$ as the level related to a quantile or probability associated with a VaR. Let $\{\mathbf{x}_t\}_{t\in\mathbb{Z}}$ denote a set of variables. Typical candidates for $\{\mathbf{x}_t\}_{t\in\mathbb{Z}}$ are financial returns. Define $\boldsymbol{\beta}_\alpha \in \mathbb{R}^p$ with $p = k + r + 1$ as a vector of undetermined parameters associated with quantile level α which will be estimated at a later stage. Further, define $q_{t,\boldsymbol{\beta}_\alpha} \equiv q_t(\boldsymbol{x}_t, \boldsymbol{\beta}_\alpha)$ as the α-quantile of the conditional return distribution derived at $t-1$. A generic CAViaR specification maybe described by

$$q_{t,\boldsymbol{\beta}_\alpha} = \beta_0 + \sum_{i=1}^{k} \beta_i q_{t-i,\boldsymbol{\beta}_\alpha} + \sum_{j=1}^{r} \beta_j l(\boldsymbol{x}_{t-j}). \tag{135}$$

where $l(\cdot)$ is a function of finitely lagged observed variables. The CAViaR representation in (135) allows a specific law of motion for a quantile according to any time series dynamic.

The authors propose four different CAViaR specifications:

$$\text{Adaptive:} \quad q_{t,\beta_\alpha} = q_{t-1,\beta_\alpha} + \beta_1\{[1+\exp(G[x_{t-1} - q_{t-1,\beta_\alpha}])]^{-1} - \theta\} \tag{136}$$

$$\text{Symmetric Absolute Value:} \quad q_{t,\beta_\alpha} = \beta_1 + \beta_2 q_{t-1,\beta_\alpha} + \beta_3|x_{t-1}| \tag{137}$$

$$\text{Indirect GARCH(1,1):} \quad q_{t,\beta_\alpha} = \left(\beta_1 + \beta_2 q^2_{t-1,\beta_\alpha} + \beta_3 x^2_{t-1}\right)^{1/2} \tag{138}$$

$$\text{Asymmetric Slope:} \quad q_{t,\beta_\alpha} = \beta_1 + \beta_2 q_{t-1,\beta_\alpha} + \beta_3(x_{t-1})^+ + \beta_4(x_{t-1})^-. \tag{139}$$

The adaptive model is a smoothed version of the step function given a finite value G, imposing the rule that the VaR increases when returns exceed it and decreases otherwise. However, this approach does not account for the extent to which returns exceed VaR, treating all exceedances equally. The symmetric absolute value and indirect GARCH specification treat gains and losses equally but incorporate the magnitude of the returns. Finally, the asymmetric slope not only incorporate past VaR estimates but additionally differentiates between positive and negative returns.

We provide a brief overview of the quantile regression framework by Koenker & Bassett (1978) which is used to estimate the presented CAViaR models. Unlike OLS regression where the object of interest is the conditional mean, quantile regression addresses the conditional quantile.

Definition 4.1.1 (Regression Quantile)
The conditional quantile of Y given $X = x$ is the value $q_\alpha(x)$ such that $\mathbb{P}(Y \leq q_\alpha(x)|X = x) = \alpha$. The regression model is of the form

$$y_t = q_\alpha(x_t) + \varepsilon_t \qquad \text{with} \quad \mathbb{Q}_\alpha(\varepsilon_t|x_t) = 0$$

$q_\alpha(\cdot)$ is also called the quantile regression function.

Note that the α-quantile of the error and not the error itself is centered around 0 zero. Hence, the error measures the deviation of the observed value from the conditional quantile as in contrast to the conditional mean. This setting splits the residuals into a non-negative and non-positive part proportionately according to the choice of α. Each residual is then evaluated by a function $\rho_\alpha(\cdot)$ that allows targeting a specific part of the destribution. Formally,

$$\rho_\alpha(u) = \begin{cases} \alpha \cdot u & \text{if } u \geq 0, \\ (\alpha - 1) \cdot u & \text{if } u < 0. \end{cases} \tag{140}$$

$$= u(\alpha - \mathbb{I}_{\{u<0\}}) \tag{141}$$

Note that for $\alpha = 0.5$ the loss function becomes symmetric and corresponds to regression by least absolute deviations where the median represents the optimum assuming linear dependency. Varying α tilts the function. Finally, the optimal parameters $\hat{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}$ are defined as a solution of

$$\min_{\beta} \frac{1}{T} \sum_{t=1}^{T} \rho_\alpha (y_t - q_{t,\beta_\alpha}). \tag{142}$$

One benefits of using quantile regression is that no assumptions regarding the return distributions are needed. Compared to traditional methods that impose at least an elliptical distributions on returns, the normal distribution being the most prominent among those, this method can cover any underlying distribution.

To put this method into context with CAViaR: The original quantile regression framework assumes only a linear relationship of the conditional quantile of the form $q_\alpha(x_t) = \boldsymbol{X}'\boldsymbol{\beta}$. Engle & Manganelli (2004) showed that quantile regression is also applicable for non-linear dynamic relationships, i.e., if the dynamic of $q_\alpha(x_t)$ is described by a CAViaR model, the parameter estimates remain consistent and asymptotically normal.

4.2 Dynamic Quantile Implied Correlation

4.2.1 Methodology

The Dynamic Quantile Implied Correlation (DQIC) or CAViaR-implied correlation is directly obtained by linking CAViaR to the asset and portfolio quantiles used to imply the correlation matrix. Recall the expression for the implied correlation as a function of quantiles for exact and overidentified systems defined in (90) and (118). We further assume that the asset and portfolio quantiles share the same α-level and drop the α subscript for better readability. The DQIC is directly obtained by substituting the static quantiles with dynamic quantiles:

$$\boldsymbol{\rho}_t \equiv \boldsymbol{\rho}(\mathbf{q}_t) = \begin{cases} \boldsymbol{X}_t^{-1}\tilde{\boldsymbol{q}}_{p,t} & \text{for } m = n(n-1)/2 \\ (\boldsymbol{X}_t'\boldsymbol{X}_t)^{-1}\boldsymbol{X}_t'\tilde{\boldsymbol{q}}_{p,t} & \text{for } m > n(n-1)/2 \end{cases} \tag{143}$$

with

$$\mathbf{q}_t = \begin{bmatrix} q_{1,t} & q_{2,t} & \dots & q_{n,t} & q_{p_1,t} & \dots & q_{p_m,t} \end{bmatrix}' \tag{144}$$

where $(q_{1,t}, ..., q_{n,t})$ and $(q_{p_1,t}, ..., q_{p_m,t})$ are separately modeled by a CAViaR model. In this study, we assume that the asset and portfolio quantiles are driven by an asymmetric slope CAViaR specification as defined in (139). The choice for this model is motivated by the nature of its ability to distinguish between the impact of positive and negative returns.

Thus, asset and portfolio quantiles as described by

$$q_{i,t} = \beta_{i,1} + \beta_{i,2} q_{i,t-1} + \beta_{i,3}(r_{i,t-1})^{+} + \beta_{i,4}(r_{i,t-1})^{-} \quad \text{for} \quad i = 1, ..., n \tag{145}$$

and

$$q_{p_j,t} = \beta_{p_j,1} + \beta_{j,2} q_{p_j,t-1} + \beta_{p_j,3}(r_{p_j,t-1})^{+} + \beta_{p_j,4}(r_{p_j,t-1})^{-} \quad \text{for} \quad j = 1, ..., m. \tag{146}$$

Note that the choice of the CAViaR α level corresponds directly to the desired tail area from which the correlation matrix is derived.

4.2.2 Evaluation of Correlation Forecast

Unlike return forecasts, where the true value is available for performance evaluation, correlation similar to volatility is not observable. Since both quantities are statistical measures, they are always subject to sampling error, measurement noise, or specific assumptions. We draw ideas from volatility forecasting. Usually, volatility forecast accuracy is valued ex-post, i.e., compared to a historical measure by a reference measure. Andersen & Bollerslev (1998) showed that using squared returns of high-frequency data as an approximation of the "true" volatility significantly improves the accuracy of volatility forecasts. Based on this finding, Skintzi & Xanthopoulos-Sisinis (2007) used realized correlation as a benchmark for the accuracy of the correlation forecast. The forecast performance is evaluated by a statistical loss function such as the mean absolute error (MAE) or the root mean squared error (RMSE).

Definition 4.2.1 (Realized Correlation)
Define $t = 0, ..., T$ as the index of a specific time interval, e.g. day, week or month. Further define $K \in \mathbb{N}$ as the amount of data points within this interval. Let $r_{i,k}$ define the return of the i-th asset at time $k \in \{1, ..., n\}$ within the t-th time interval. Then, the realized correlation is defined as

$$\rho_{ij,t} = \frac{\sum_{k=1}^{K} r_{i,k} r_{j,k}}{\sqrt{\sum_{k=1}^{K} r_{i,k}^2}\sqrt{\sum_{k=1}^{K} r_{j,k}^2}}. \tag{147}$$

The numerator represents realized covariance while the square root of the sum of squared returns corresponds to realized volatility.

Unfortunately, this approach requires high-frequency data, which were not available in this study. An attempt was made to construct weekly realized correlation based on daily returns, but the resulting series was highly volatile and exhibited unrealistic values. The results are presented in Appendix D.4.

An alternative benchmark is the Exponentially Weighted Moving Average (EWMA) model. The correlation matrix obtained in this way is a weighted sum of historical correlation matrices and squared returns. If the initial matrix is positive definite, the EWMA filter will preserve this property. We initialize the EWMA in our study with the sample covariance matrix, ensuring that the entire dataset is taken into account. Other initialization methods are discussed in Engle (2009). Regarding the smoothing parameter, RiskMetrics (1996) suggested $\lambda = 0.06$ based on empirical experience. The correlation matrix is then derived by normalizing the covariance matrix, with a diagonal matrix containing the conditional volatility similar to the Dynamic Conditional Correlation (DCC) model discussed in Section 2.2.3. Formally, assume that the mean of the return series is zero, then the exponential smoother is defined by

$$\boldsymbol{H}_t^{\text{exp}} = \lambda \boldsymbol{r}_{t-1}\boldsymbol{r}_{t-1}' + (1-\lambda)\boldsymbol{H}_{t-1}^{\text{exp}} \tag{148}$$

$$\boldsymbol{R}_t^{\text{exp}} = \boldsymbol{D}_t^{-1}\boldsymbol{H}_t^{\text{exp}}\boldsymbol{D}_t^{-1} \tag{149}$$

with $\boldsymbol{D}_t$ containing only the conditional volatilities on the diagonals.

However, evaluating correlation in this manner raises several concerns. As previously noted, true correlation is unknown and both realized correlation and EWMA are only approximations with no theoretical foundation. Additionally, both methods are extensions of Pearson correlation and lack the ability to capture tail behavior. Thus, comparing them with DQIC is questionable.

This motivates another approach that evaluates the quality of correlation forecast in terms of portfolio performance. The correlation forecasts are utilized to construct a global minimum variance portfolio (GMVP) without short-selling to avoid unrealistic portfolio weights. However, the correlation matrix does not incorporate the volatility of each asset, which is crucial for a meaningful asset allocation. Therefore, the implied correlation is re-scaled to a covariance matrix by historical volatility. Our choice for the GMVP is driven by the fact that this portfolio solely relies on covariance among assets and, by so, isolating covariance as the key driver of portfolio performance. Furthermore, estimating expected returns is not required, which significantly reduces the complexity of the portfolio construction process. Recall that the optimization problem for obtaining the weights of the GMVP can be formulated as follows:

Given n assets, let $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top$ denote the vector of portfolio weights, Σ the $n \times n$ covariance matrix of asset returns and $\mathbf{1}$ an n-dimensional vector of ones. The weights of the GMVP portfolio are obtained as the solution of the following optimization problem:

$$\begin{aligned} \min_{w} \quad & \mathbf{w}^\top \Sigma \mathbf{w} \\ \text{subject to} \quad & \mathbf{1}^\top \mathbf{w} = 1, \\ & \mathbf{w} \geq \mathbf{0}. \end{aligned}$$

The evolution of the portfolio value is then compared to the portfolios constructed in the same way using DCC forecasts and EWMA correlation instead. Finally, we evaluate the portfolio performance based on drawdown metrics, e.g. Maximum Drawdown (MDD), historical VaR and ES. The historical VaR for different alpha levels is obtained directly by the sample quantile.

Definition 4.2.2 (Maximum Drawdown)
*Let V_t represent the portfolio value at time t, where $t = 0, 1, 2, \dots, T$ over the evaluation period $[0, T]$. First, define the **cumulative peak** which tracks the portfolio highs up to time t as:*

$$P_t = \max_{0 \leq s \leq t} V_s \tag{150}$$

*A **drawdown** at time t is the relative decline from this cumulative peak:*

$$D_t = \frac{V_t - P_t}{P_t} \tag{151}$$

*The **Maximum Drawdown (MDD)** over the period $[0, T]$ is then the minimum of the drawdown values:*

$$MDD = \min_{0 \leq t \leq T} D_t \tag{152}$$

Note that the drawdown values are either zero or negative. Therefore, the minimum over all drawdown values corresponds to the largest deviation from zero hence representing the Maximum Drawdown.

We study the impact of different parameters on these drawdown metrics. More precise, effects of implied correlation obtained by solving exact and overidentified system, different CAViaR quantile levels and different rebalancing frequencies for daily and weekly returns. Technically, the portfolio can be evaluated by other measures such as the Sharpe ratio or

Sortino ratio. However, since the portfolio contains indices of different countries, it is not clear how to select the risk free rate. We will therefore remain with the drawdown metrics introduced in this section.

4.3 Empirical Study - Forecasting Correlation

For the empirical analysis, we investigate a subset of the global index portfolio from Section 3.7, selecting two indices per geographical location to represent the respective financial markets, i.e. DJI-NASDAQ (US-US), FCHI-GDAX (EU-EU), N225-HSI (ASIA-ASIA), DJI-GDAX (US-EU), DJI-N225 (US-ASIA), and GDAX-N225 (EU-ASIA). Our intention is to study implied correlation forecasts for markets in the same and across geographical regions. The dataset contains $T_{\text{daily}} = 6522$ data points from 1998-01-01 until 2022-12-30. We adopt an 80-20 split for the data, using 80% to estimate the CAViar/DCC models and 20% for out-of-sample testing. Naturally, the split does not affect the EWMA and realized correlation model. The empirical analysis is based on dynamic correlation estimates for daily and

No.	Symbol	Index Name (Country)
1	DJI	Dow Jones Industrial Average (US)
2	NASDAQ	Nasdaq Composite (US)
3	FCHI	CAC 40 Index (EU)
4	GDAX	DAX Performance-Index (EU)
5	N225	Nikkei 225 Index (ASIA)
6	HSI	Hang Seng Index (ASIA)

Table 13: List of six stocks indices from different geographical regions used in the empirical analysis.

weekly returns to assess short-term and longer-term dynamics in the correlation structure across different markets. Selected quantile levels are $\alpha \in \{0.25, 0.05, 0.01\}$. The realized correlation is only considered for weekly returns due to data requirements. More precisely, we only construct weekly realized correlation using daily data since daily realized correlation requires high-frequency intraday data. Our focus is to investigate the behavior of DQIC in general. We will, therefore, focus on the Asymmetric Slope CAViaR specification, see equation (139), and do not explore the effects of different specifications of quantile dynamics further. Finally, we employ two identification strategies for the DQIC and distinguish between correlation implied by an exact identified system and an overidentified system. Given n=6 assets, an exact identified system consists of 15 equations with equally weighted two-asset portfolios. For the overidentified system, we include additional equally weighted three-asset portfolios and, by so, raise the system to 35 equations.

4.3.1 Analysis of Correlation Forecasts using EWMA and DCC as Benchmarks

The DCC model produces correlation forecasts that are smoother and less erratic compared to those generated by EWMA and DQIC. This property is attributed to the inherent characteristics of the DCC model to incorporate volatility clustering by utilizing underlying conditional variance models that account for high and low volatility periods. In particular, the parameters of the DCC model are optimized for each asset time series such that resulting correlations are more adaptive to long-term trends rather than reactive to short-term fluctuations. As a result, the forecasts are less prone to noise and do not overreact to sudden spikes in volatility. Estimation results of the DCC model are presented in Appendix D.2.

In contrast, the EWMA model puts more emphasis on recent data by assigning exponentially greater weights to the latest observations. By construction, this property renders EWMA sensitive to recent changes in market conditions, making it more responsive to short-term shifts. However, unlike the DCC model, it lacks the ability to account for volatility clustering and, therefore, might overreact to temporary spikes without distinguishing between impermanent shocks and more persistent volatility patterns. As a result, the EWMA correlation estimates are more volatile due to their sensitivity to noise.

Correlation forecasts by the DQIC model are volatile, similar to the EWMA model, but their magnitudes vary considerably depending on the respective market. Markets that are traditionally stronger correlated, e.g., DJI and NASDAQ or GDAX and FCHI, exhibit less variation, whereas loosely connected markets, in particular western and asian markets, behave more noisy, see Figure 19.

The volatile behavior is further observed in both correlation forecasts implied by in-sample and out-of-sample CAViaR. Parameter estimates for the CAViaR model are shown in Appendix D.1. Although the dynamic of the conditional quantile in the CAViaR model does impose a certain law of motion, it does not incorporate the ability to specifically account for volatility clustering, resulting in the erratic nature of the correlation forecasts. Focusing on the effects of the CAViaR quantile level, we expect that a lower CAViaR level should result in stronger correlation coefficients, i.e., correlation forecasts implied by a 0.01 CAViaR model should exhibit a stronger tendency towards one compared to a 0.05 CAViaR specification. This effect is clearly visible for correlations beyond geographical regions and further holds for the two US markets. Surprisingly, we do not observe this behavior for European and Asian markets. Similar results are obtained for weekly return frequencies; see Appendix D.3.

We present the benchmark results in Table 14. Recall that the Mean Absolute Error and Root Mean Square Error are defined as follows:

$$\text{MAE} = \frac{1}{T}\sum_{t=1}^{T} |\hat{\rho}_{ij,t} - \rho_{ij,t}| \qquad \text{and} \qquad \text{RMSE} = \sqrt{\frac{1}{T}\sum_{t=1}^{T} (\hat{\rho}_{ij,t} - \rho_{ij,t})^2} \tag{153}$$

where $\hat{\rho}_{ij,t}$ represents the correlation forecast obtained by the DQIC and $\rho_{ij,t}$ the assumed true value. Comparing MAE and RMSE values across different index pairs, models, and DQIC levels suggests that forecasting based on daily returns yields more accurate predictions compared to weekly returns in general. Between models, DCC often performs better than EWMA at lower DQIC levels, while EWMA seems to perform better for less extreme quantiles. Among index pairs, DJI and NASDAQ have lower MAE and RMSE, particularly for 0.05 and 0.01, than all other pairs, indicating that DQIC is closest to DCC and EWMA for American markets. Regarding return frequency, daily returns yield better forecasts than weekly returns. By using weekly data, fluctuations within the week are discarded, leading to a loss of information that could improve the predictive powers of the model.

4.3.2 Analysis of Correlation Forecasts using Global Minimum Variance Portfolio

In the previous section, we assumed that EWMA or DCC approximates the true correlation and compared them against DQIC with different quantile levels. As mentioned before, the true correlation is not observable, and neither EWMA nor DCC are tail measures. In this section, we evaluate the quality of correlation forecasts by constructing global minimum variance portfolios and evaluating different drawdown metrics based on their performance. We will focus mainly on the Maximum Drawdown and historical Expected Shortfall.

Drawdown metrics for DQIC, DCC and EWMA portfolios are presented in Table 15. Starting with daily returns: If the portfolio is balanced on daily frequency, the exact identified DQIC 25% achieves the lowest MDD of 43.98%. This specific case also presents the largest ES compared to other settings. For overidentified systems, DQIC 25% yields the lowest ES for both 1% and 5%, although only slightly smaller than DCC and EWMA. Balancing the portfolio on monthly frequency yields roughly equal MDD values, with the exact identified DQIC 25% being slightly larger. Regarding ES, the portfolios constructed by DCC yield the lowest results. For weekly returns and weekly portfolio rebalancing, exact identified DQIC 5% and 25% yield the lowest MDD with 51.793% and 51.776%, respectively. Surprisingly, overidentified DQIC 1% presents a significantly larger MDD with 62%. The ES is approximately the same for all settings, with overidentified DQIC 5% (8.997) being slightly lower. Regarding monthly rebalancing, we find the exact identified DQIC 1% to have the lowest MDD of 50.995% while ES of all settings shows no notable patterns. A potential reason could be that incorporating information from the tails might lead to estimates that are more robust. DCC and EWMA generally yield lower drawdown estimates than DQIC portfolios on 5% and 1% levels. We also noticed that the rebalancing frequency only moderately im-

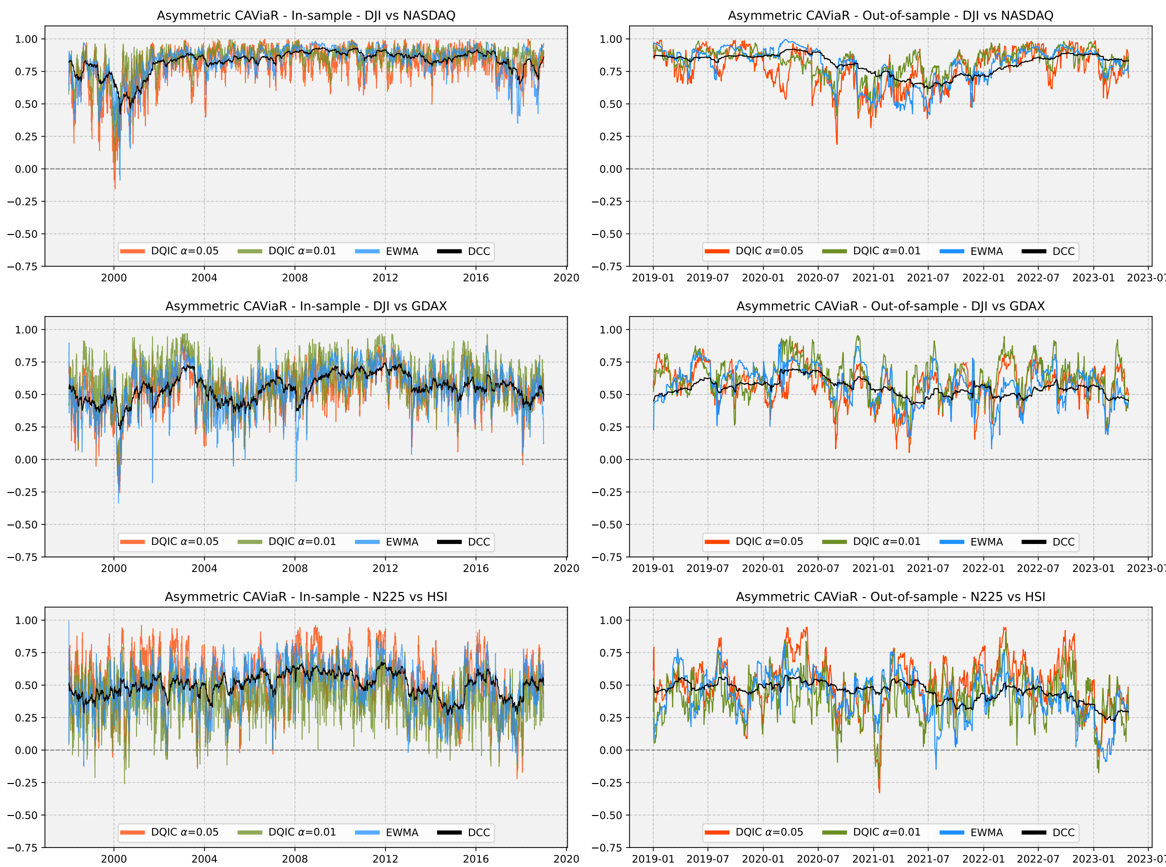


Figure 19: Dynamic Quantile Implied Correlation - Implied correlation from an exact identified portfolio system in comparison with DCC and EWMA correlation for daily returns.

EWMA		DJI vs. NASDAQ		DJI vs. GDAX		N225 vs. HSI	
	DQIC α	**MAE**	**RMSE**	**MAE**	**RMSE**	**MAE**	**RMSE**
Daily Returns							
Exact	0.25	0.349	0.451	0.464	0.570	0.334	0.418
	0.05	0.189	0.249	0.207	0.258	0.294	0.370
	0.01	0.152	0.210	0.218	0.282	0.290	0.363
Over	0.25	0.486	0.691	0.456	0.568	0.441	0.576
	0.05	0.193	0.254	0.199	0.250	0.313	0.400
	0.01	0.205	0.272	0.201	0.259	0.336	0.418
Weekly Returns							
Exact	0.25	0.356	0.524	0.327	0.429	0.567	0.684
	0.05	0.366	0.483	0.333	0.438	0.287	0.358
	0.01	0.397	0.566	0.347	0.462	0.439	0.538
Over	0.25	0.363	0.468	0.221	0.277	0.676	0.795
	0.05	0.363	0.497	0.309	0.395	0.324	0.400
	0.01	0.435	0.617	0.271	0.351	0.407	0.504

DCC		DJI vs. NASDAQ		DJI vs. GDAX		N225 vs. HSI	
	DQIC α	**MAE**	**RMSE**	**MAE**	**RMSE**	**MAE**	**RMSE**
Daily Returns							
Exact	0.25	0.388	0.517	0.476	0.572	0.279	0.344
	0.05	0.200	0.264	0.217	0.309	0.301	0.376
	0.01	0.144	0.191	0.251	0.309	0.296	0.368
Over	0.25	0.528	0.744	0.447	0.555	0.421	0.544
	0.05	0.195	0.251	0.215	0.265	0.332	0.412
	0.01	0.187	0.229	0.228	0.285	0.345	0.420
Weekly Returns							
Exact	0.25	0.376	0.542	0.324	0.422	0.612	0.721
	0.05	0.333	0.450	0.319	0.411	0.261	0.320
	0.01	0.355	0.522	0.318	0.424	0.367	0.436
Over	0.25	0.355	0.481	0.213	0.259	0.718	0.833
	0.05	0.333	0.461	0.286	0.361	0.295	0.370
	0.01	0.407	0.580	0.255	0.317	0.338	0.416

Table 14: Forecasting benchmarks - The DQIC α column indicates the CAViaR quantile levels from which the tail correlation matrix is implied. The quality of the implied correlation forecasts is evaluated by the MAE and RMSE using EWMA (top panel) and DCC (bottom panel) as reference models.

pacts the outcome. Furthermore, DCC mostly outperforms EWMA across all rebalancing frequencies and risk metrics, which could be related to the nature of DCC models to reflect financial returns better.

Although our analysis was mainly based on drawdown measures, we also visualized the portfolio performance with monthly rebalancing frequency in Figure 21 and 22. EWMA, DCC, and DQIC portfolios outperform the German stock index GDAX in all cases. However, the American tech index Nasdaq (NASDAQ) remains superior to all portfolios. For daily returns, the exact identified DQIC 1% portfolio performed surprisingly well, sharing the best performance with DQIC 5% and EWMA if overidentified DQICs are considered. One surprising result is the performance of weekly rebalanced DQIC 25% portfolios derived from an exact identified system that outperformed all other portfolios by a large margin. This effect is also observable for overidentified DQICs, although not as extreme. Portfolio performance with daily/weekly rebalancing frequency is visualized in Appendix D.5 in Figure 36 and 37 . One curious result is the portfolio performance of DQIC 25%, which surpasses even the NASDAQ by 1300 points. Although these results are appealing, the transaction costs render portfolio management with daily rebalancing unfeasible. In addition, stock markets from different geographical locations have different trading times. As a result, it is impossible to simultaneously trade all assets in a globally diversified portfolio during a single window.

We conclude this analysis by visualizing the portfolio weights and how they evolve for exact identified DQIC portfolios based on weekly returns in Figure 20. Portfolio weights produced by EWMA and DCC exhibit smooth and gradual changes in asset allocation. In contrast, DQIC portfolios are more reactive to extreme market conditions, especially for the 1% and 5% levels, featuring high variability in allocations. In this specific case, the DQIC at the 1% level also produces the lowest MDD and moderate ES compared to other portfolios. The portfolios are mainly composed of DJI, GDAX, and HSI, presenting one index per geographic location, which seems reasonable since including strongly correlated indices of the same area would not improve diversification benefits. While EWMA and DCC portfolios include the remaining indices occasionally and only for a short period, DQIC portfolios allocate wealth to these indices more often and abruptly.

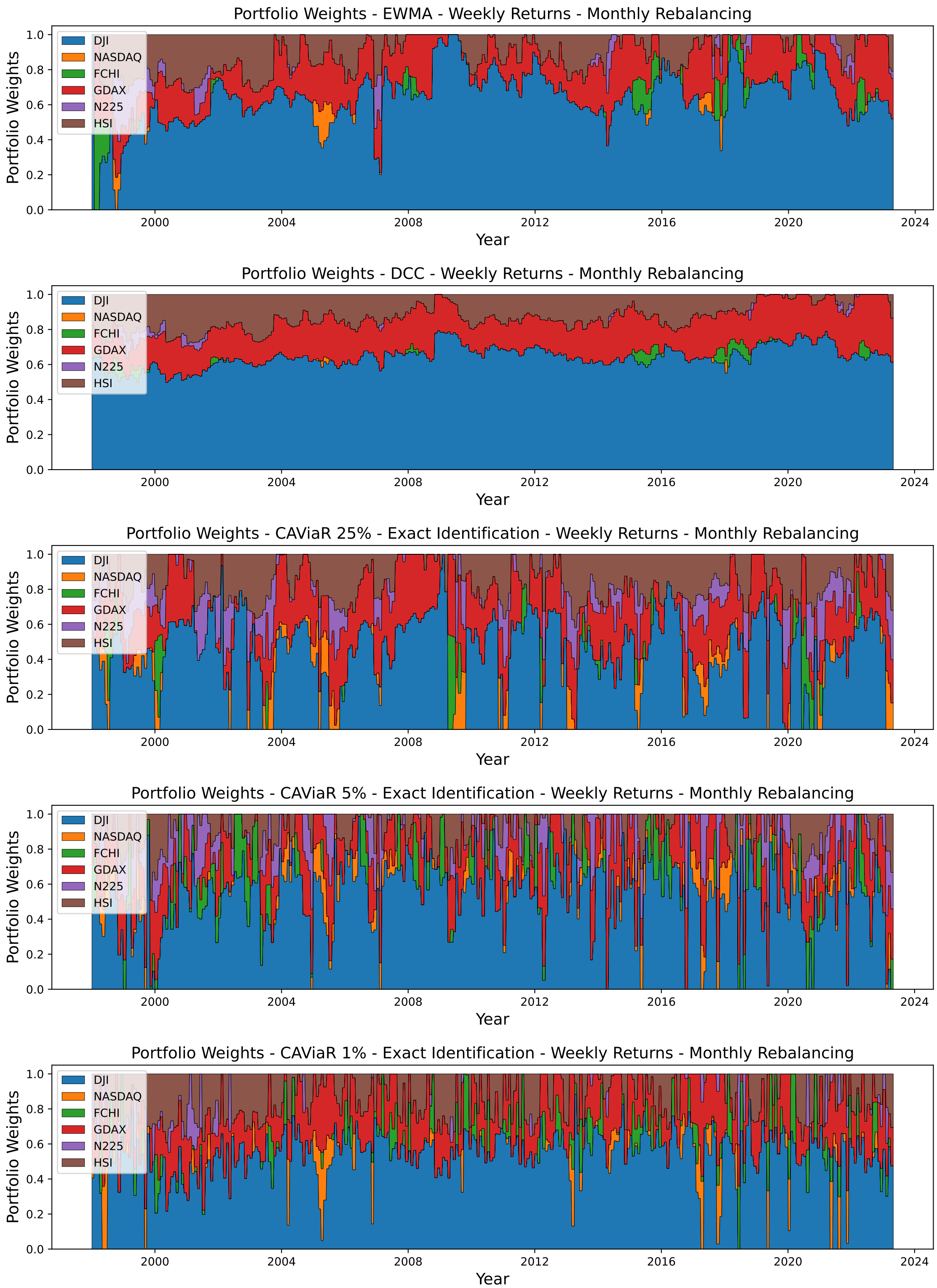


Figure 20: Evolution of portfolio weights - Presented are the portfolio weights with monthly rebalancing and weekly returns. DQIC from an exact identified system are considered.

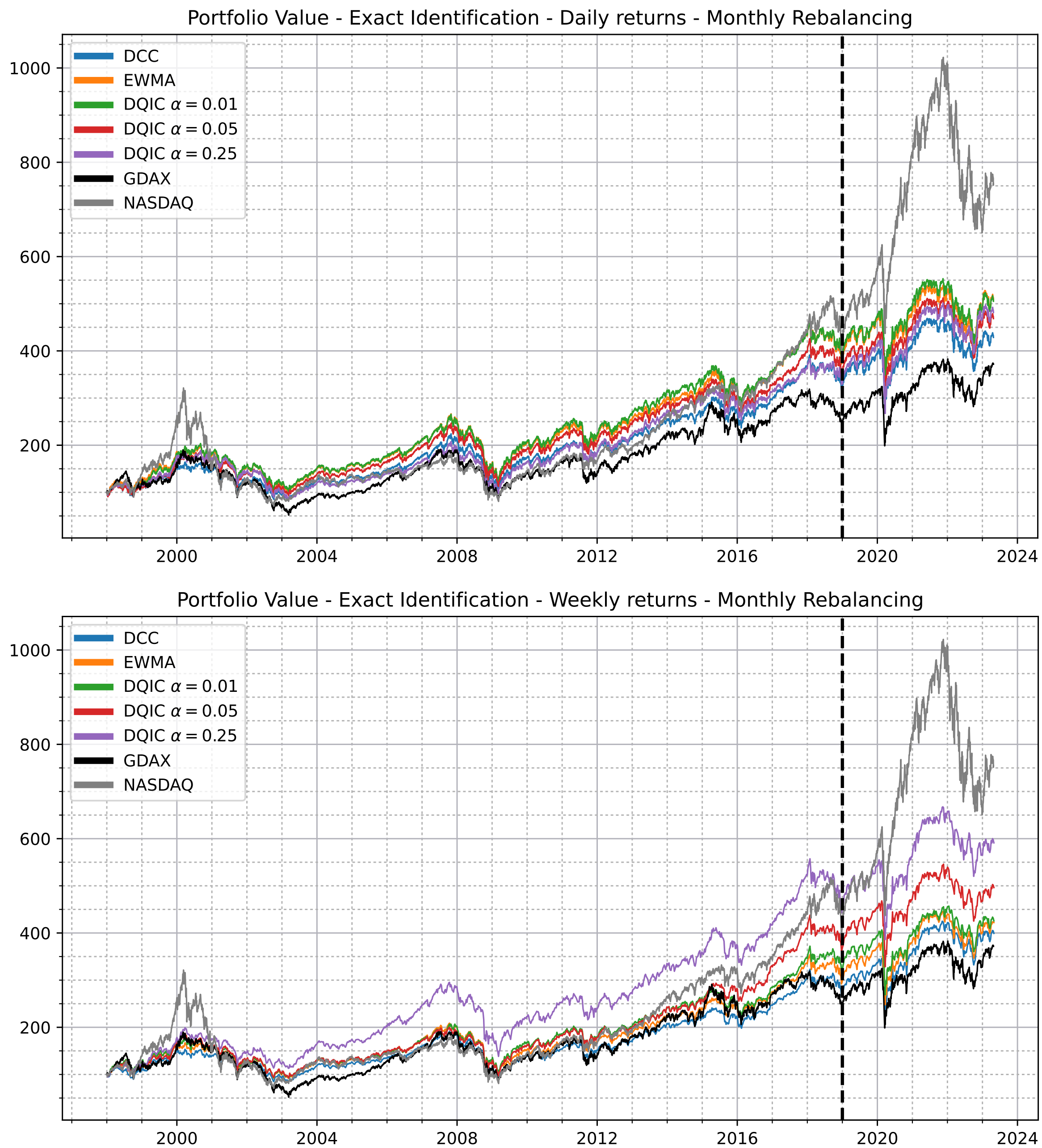


Figure 21: DQIC portfolio performance - Depicted are the evolution of portfolio values for daily (top) and weekly (bottom) returns. Correlation is implied by an exact identified system with monthly rebalancing frequency. The verticle line divides the chart into in-sample CAViaR/DCC (left) and out-of-sample CAViaR/DCC (right) predictions. Initial investment and indices were normed to 100 monetary units. Two indices (GDAX, NASDAQ) are included for comparison

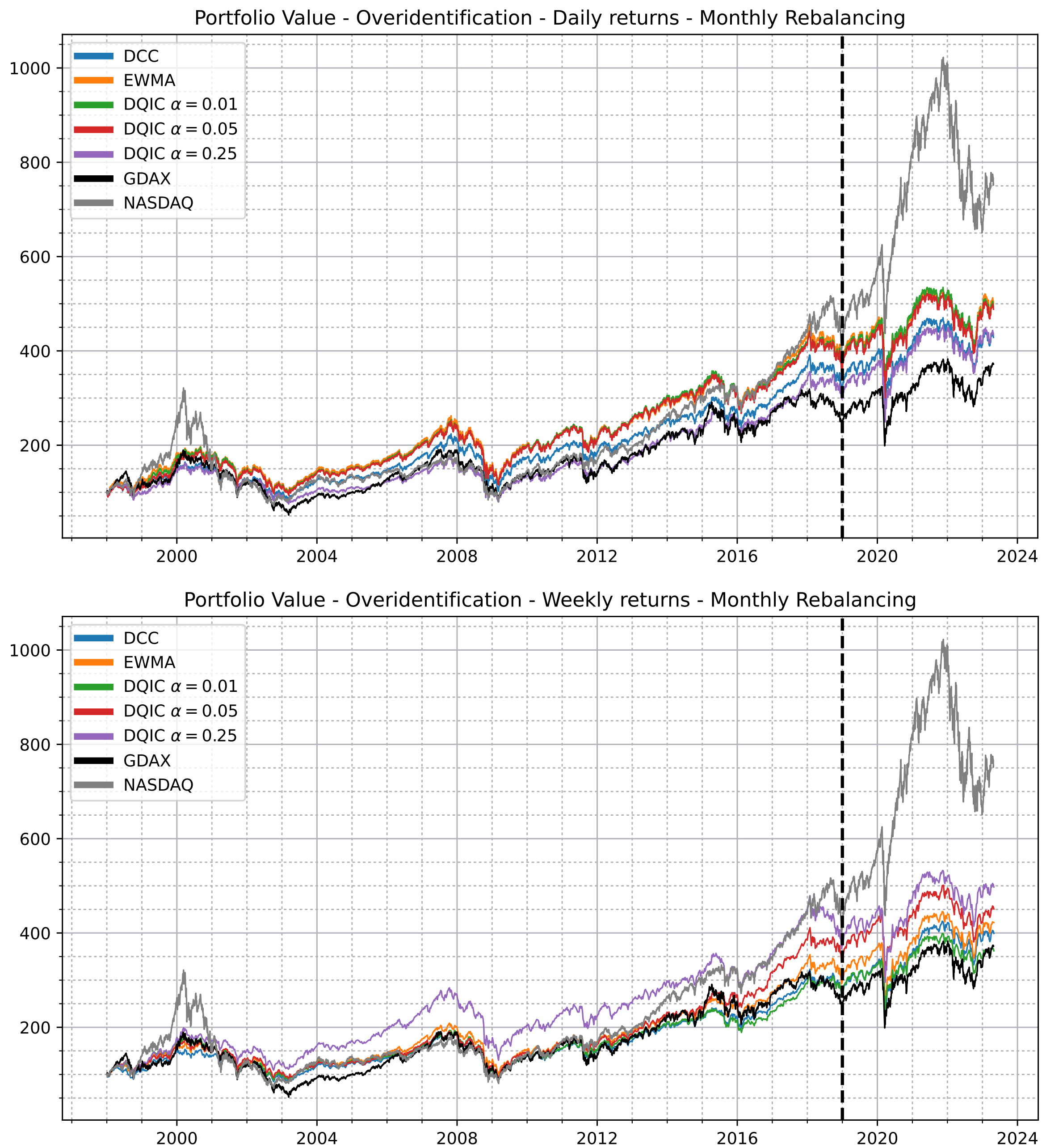


Figure 22: DQIC portfolio performance - Depicted are the evolution of portfolio values for daily (top) and weekly (bottom) returns. Correlation is implied by an overidentified system with monthly rebalancing frequency. The verticle line divides the chart into in-sample CAViaR/DCC (left) and out-of-sample CAViaR/DCC (right) predictions. Initial investment and indices were normed to 100 monetary units. Two indices (GDAX, NASDAQ) are included for comparison

Reported in %	Daily/Weekly Rebalancing					Monthly Rebalancing				
	MDD	**VaR 5%**	**VaR 1%**	**ES 5%**	**ES 1%**	**MDD**	**VaR 5%**	**VaR 1%**	**ES 5%**	**ES 1%**
Daily Returns										
EWMA	55.518	1.605	2.759	2.436	3.991	54.580	1.591	2.754	2.439	4.040
DCC	54.286	1.579	2.766	2.420	3.977	54.433	1.588	2.750	2.420	3.968
Exact Identification										
DQIC 25%	43.982	1.656	3.056	2.601	4.429	57.232	1.685	3.128	2.661	4.536
DQIC 5%	56.852	1.643	2.854	2.500	4.187	54.357	1.592	2.809	2.475	4.098
DQIC 1%	55.730	1.621	2.785	2.473	4.217	54.713	1.592	2.786	2.455	4.132
Overidentification										
DQIC 25%	49.940	1.550	2.815	2.394	3.860	55.204	1.655	3.011	2.585	4.286
DQIC 5%	55.130	1.633	2.819	2.490	4.207	54.059	1.585	2.792	2.477	4.110
DQIC 1%	56.247	1.616	2.872	2.464	4.122	54.573	1.582	2.753	2.448	4.079
Weekly Returns										
EWMA	53.895	3.720	6.246	5.629	9.192	53.447	3.745	6.301	5.623	9.182
DCC	53.640	3.662	6.476	5.674	9.335	54.452	3.663	6.513	5.680	9.344
Exact Identification										
DQIC 25%	51.793	3.700	6.191	5.439	8.986	53.178	3.667	6.456	5.533	9.038
DQIC 5%	51.776	3.673	6.272	5.527	9.023	51.574	3.620	6.541	5.573	9.000
DQIC 1%	55.686	3.892	6.665	5.747	9.631	50.955	3.773	6.345	5.590	9.384
Overidentification										
DQIC 25%	52.505	3.699	6.470	5.652	9.681	54.101	3.655	6.410	5.568	9.482
DQIC 5%	51.997	3.723	6.193	5.453	8.997	52.992	3.718	6.526	5.552	9.066
DQIC 1%	62.041	3.905	6.371	5.913	9.422	56.399	3.575	6.712	5.648	9.291

Table 15: Portfolio drawdown metrics - Drawdown metrics of portfolios for daily and weekly returns. Results of different identifcation and rebalancing strategies are presented. We report positive values that indicate losses.

Chapter 5

Conclusion

We present a simulation framework specifically designed to account for the characteristics of financial time series, also known as "stylized" facts. The building blocks of the framework include stochastic volatility models (GARCH, GJR-GARCH) with different error distributions (Normal, t-student) to account for asset-specific characteristics, while conditional correlation models and copulas address the dependence structure among the assets. We leveraged the abilities of each model and combined them into a streamlined, unified framework.

Obtaining the simulation parameters amounts to estimating the model components where the residuals of each stage are passed to the next until an independent and identically distributed sample is obtained. Financial returns are first filtered for heteroskedasticity and subsequently for cross-correlation. Copula models account for the remaining non-linear relationships in the data. Although the data is filtered in multiple stages, we find that the parameters describing the stochastic volatility model are boundary solutions to the likelihood optimization, implying that the solution does not necessarily represent a global optimum. In this case, the standard errors are meaningless, and the model is potentially misspecified, although it might prove sufficient for the tasks at hand. This reminds us that a statistical model remains an approximation of the true data-generating process.

Based on two stock indices (DAX, NASDAQ) and gold prices, we demonstrated an entire calibration process of the simulation model and provided an introductory example of how to obtain simulated return paths. Possible applications are discussed, such as portfolio optimization, risk projection, and Monte Carlo simulation. Parameters can be manually modified to produce more extreme events, especially for stress testing. One benefit of the modular structure regarding the simulation model is that the components can be easily exchanged. A better description of asset-specific characteristics can be acquired by selecting a more sophisticated error distribution. Alternative correlation dynamics beyond the GARCH-like law of motion and advanced copulas for more control over non-linear relationships are worth exploring.

Although, this model is able to account for many characteristics, it offers no dedicated way to address correlation for extreme market scenarios especially when returns jointly decline. In this framework only parameters of the error distributions governing the tail behavior cover this aspect. Modeling tail dependence structure also becomes increasingly difficult if more complex models are considered. Consequently, we seek an alternative solution by adjusting the correlation matrix to tail events.

Correlation remains essential for measuring dependence in financial applications. Although the drawbacks of this measure are well studied, it remains the primary building block in almost all financial modeling methods as soon as we depart from the univariate asset universe into multivariate return modeling, portfolio optimization, and risk management. Motivated by numerous empirical evidence in favor of increased correlation, especially during bearish markets, we propose a new method to measure correlation for different scenarios implied by a monetary quantile-based risk measure, namely, Value-at-Risk and Expected Shortfall. This enables us to condition correlation to extreme events that are economically interpretable by selecting a respective quantile instead of a static threshold. Correlation is implied by solving a linear system of equations, each representing a portfolio with different asset weights, which also allows for the consideration of large portfolios beyond two assets. The choice of portfolio weights depends on the financial constraints of an investor and must also be chosen such that numerical stability is guaranteed.

The empirical analysis studies implied correlation of 10 stock indices from different geographical location. We evaluate the effects of different parameter settings by varying quantile levels, return frequencies, risk measures, portfolio weights, and tail regions. In particular, extreme events are considered by choosing a low/high quantile level, which can be described as “moving into the tails” where lower quantiles represent losses and upper quantiles gains.

Our initial analysis showed that VaR-implied correlation tends to behave more volatile. ES-implied correlation produces stable results by its nature to incorporate additional information from the tail. Therefore, the remaining analysis is focused ES-implied correlation. We observe that indices that were already strongly correlated by Pearson standards tend to increase for implied correlation when extreme quantiles are considered. Pairs that were weakly correlated do not display consistent behavior; sometimes, they even change signs. This pattern is more pronounced for the lower tail quantiles, i.e., quantiles describing losses. A possible explanation is the scarcity of data for extreme quantiles and potential sampling errors. We further emphasize the geographical attribute of each index. European and US markets are traditionally closer aligned, while Asian markets seem less sensitive to changes in the Western economy. This is also reflected in the variability of the correlation estimates, where pairs involving an Asian index have significantly larger standard errors. If implied correlation estimates are viewed as an entire sample over different return frequencies, the median of lower implied correlation estimates is consistently higher than upper implied correlation once extreme quantiles are considered.

Regarding the system of portfolio equations from which correlation is implied, the choice of weights significantly impacts the estimation outcome. We mainly studied correlation implied by an exact identified system with deterministic two-asset portfolios where the weights are chosen such that multicollinearity is avoided. Our results in this setting differ slightly from an overidentified system with equally weighed two- and three-asset portfolios. However, randomly weighted portfolios induce significant instabilities such as boundary violations and non-positive-semidefinite matrices, which occur particularly often if weights are put on all assets. Randomly weighted two-asset portfolios produce similar patterns to the deterministic cases but remain unstable. Therefore, selecting weights is not straightforward and must be carefully done to guarantee numerical stability while maintaining economic interpretability.

The second part of our research investigates the differences between implied correlation and Pearson correlation. An initial descriptive analysis of the deviations between implied and Pearson correlation yields no apparent results. Differences between the two types of correlation vary for different parameter sets. However, the most considerable deviations are produced when extreme quantiles are considered. The average correlation tends to increase while moving into the lower tails for daily and weekly returns but not so for monthly returns. Further analysis of the ten strongest correlated pairs revealed a consistent increase for lower quantiles in the lower tail.

Finally, we investigate systematic deviations using a linear regression approach. Geometrically, if two correlation samples are equal, the corresponding scatterplot should lie on a diagonal line of 45 degrees. Analyzing distortions in a statistical context translates into testing whether the intercept differs from zero, the slope differs from one, or both. First, we compare lower and upper implied correlations. The results indicate a systematic shift where the correlation implied by losses seems to be systematically higher than that implied by gains, meaning that an intercept of zero is often rejected. Systematic decreasing slope coefficients suggest that large correlation coefficients implied by lower tails are larger relative to their counterpart in the upper tail. This finding is in line with our expectations. We find strong evidence for size distortion but no evidence for a systematic shift between lower implied and Pearson correlation. Similar results were also obtained when upper tails were explored instead. In almost all cases, symmetry was rejected.

Although the initial study produces mixed results, we believe that the principle of risk-implied correlation has the potential to be a valuable tool in addition to common dependence modeling. Further research can be conducted on the choice of risk measure. In theory, any quantile-based risk measure is viable for this approach. So far, only VaR and ES shortfall have been studied. Recent literature proposes the Range Value-at-Risk Cont et al. (2010) as a generalization that incorporates characteristics from both measures. Investigating quantile regions by a refined choice of quantile levels for each asset instead of one uniform quantile level might also be worth exploring.

Finally, we introduce a method to predict correlation matrices as an extension of the quantile implied correlation methodology. Our idea is first to obtain quantile forecasts using CAViaR models by Engle & Manganelli (2004) and subsequently imply correlation matrices based on these CAViaR predictions. One benefit of using CAViaR models is the possibility of modeling quantiles directly instead of the entire return distribution while specifying a particular law of motion. We selected the asymmetric slope dynamic for its ability to treat positive and negative returns differently. Naturally, other dynamics are possible but are not further explored in this initial study, which focuses on the general viability of this approach.

One obstacle in evaluating correlation forecasts is the lack of a valid reference value. Correlation is a statistical measure prone to bias, sampling errors, and false assumptions, e.g., linear relationships. As a result, realized correlation, Exponentially Weighted Moving Average, and Dynamic Conditional Correlation serve only as an approximation of the actual correlation. Realized correlation might provide an efficient benchmark similar to realized volatility but relies on high-frequency data for accuracy, which is not available in this study.

Nevertheless, using these quantities as benchmarks is questionable. However, we can deduce how DQICs behave compared to established methods while treating them as an approximation than a true benchmark. Unlike DCC, the DQIC and EWMA correlations exhibit more variability since neither model can control volatility clustering. The magnitude varies significantly depending on the markets. Pairs involving American and European markets exhibit moderate variability. In addition, DQICs with extreme quantiles, e.g., 1% CAViaR, are closer to one. The results involving Asian indices are less conclusive. A potential reason is the geographical distance and the less intertwined economic systems relative to Western markets.

An alternative approach is to construct portfolios based on the correlation forecasts and evaluate the portfolio value in terms of drawdown metrics, e.g., Maximum Drawdown, Value-at-Risk, or Expected Shortfall. Correlation forecasts were scaled with the historical volatility to obtain covariance matrices, subsequently used to construct Markowitz-style Global Minimum Variance Portfolios. DCC and EWMA portfolios generally produced similar results over all return and rebalancing frequencies. Regarding DQIC portfolios, rebalancing and identification strategies have the most effect on daily returns, while estimates based on weekly returns remain primarily unchanged. A possible reason is that DQIC models are more responsive to short-term changes or prone to noise. This is also reflected in the change in portfolio weights.

Although this initial study does not provide clear results, we encounter cases where tail-implied correlation portfolios do indeed yield better drawdown metrics and performance than portfolios based on Pearson correlation. Further studies on other parameter settings, e.g., different CAViaR quantile dynamics, identification, and rebalancing strategies, might provide additional insights. Finally, portfolio behavior for different quantile levels and a different type of portfolio in general beyond GMVP are worth exploring.

Appendix

A Advances in Risk and Return Modelling: Estimation, Simulation, Application

A.1 Benchmark Model with MBB Surface

	DAX	**NASDAQ**	**Gold**
GARCH(1,1) Parameters			
$\hat{\omega}$	0.032***	0.024***	0.003***
SE filtered	(0.007)	(0.004)	(0.001)
t-stat	4.270	4.963	3.439
$\hat{\alpha}_1$	0.086***	0.097***	0.058***
SE filtered	(0.009)	(0.008)	(0.007)
t-stat	8.875	11.042	8.208
$\hat{\beta}_1$	0.894***	0.889***	0.941***
SE filtered	(0.011)	(0.009)	(0.007)
t-stat	80.141	91.899	120.322
DCC(1,1) Parameters			
$\hat{\lambda}_1$		0.001	
SE filtered		(0.006)	
t-stat		1.643	
$\hat{\lambda}_2$		0.984***	
SE filtered		(0.054)	
t-stat		18.084	
t-Copula Parameters			
$\hat{\nu}_{\text{Cop}}$		3.836	
SE		(0.003)	
Σ_{cop}			
DAX	1		
NASDAQ	0.013	1	
GOLD	-0.004	-0.017	1

*,**, and *** denote statistical significance levels at 10, 5, and 1 percent, respectively. Corresponding critical values are derived from a standard normal distribution. The tests evaluate whether the estimated values are statistically different from zero. Standard errors were obtained using the modified filtered historical bootstrap method with $k = 300$ iterations.

Table 16: Parameter estimates for a GARCH-DCC-t-Copula model with univariate normal white noise, GARCH-like correlation dynamics and t-Copula. Standard errors are obtained by filtered historical bootstrap.

	DAX	NASDAQ	Gold
GARCH(1,1) Parameters			
$\hat{\omega}$	0.032	0.024	0.003
SE block	(0.063)	(0.061)	(0.052)
t-stat	0.504	0.390	0.068
$\hat{\alpha}_1$	0.086***	0.097***	0.058**
SE block	(0.022)	(0.024)	(0.025)
t-stat	3.775	3.923	2.287
$\hat{\beta}_1$	0.894***	0.889***	0.941***
SE block	(0.039)	(0.033)	(0.060)
t-stat	22.899	26.758	15.534
DCC(1,1) Parameters			
$\hat{\lambda}_1$		0.001	
SE block		(0.007)	
t-stat		1.405	
$\hat{\lambda}_2$		0.984***	
SE block		(0.245)	
t-stat		4.009	
t-Copula Parameters			
$\hat{\nu}_{\text{Cop}}$		3.836	
SE block		(0.002)	
Σ_{cop}			
DAX	1		
NASDAQ	0.013	1	
GOLD	-0.004	-0.017	1

*,**, and *** denote statistical significance levels at 10, 5, and 1 percent, respectively. Corresponding critical values are derived from a standard normal distribution. The tests evaluate whether the estimated values are statistically different from zero. Standard errors were obtained using the Moving Block Bootstrap with blocklength $l = 5$.

Table 17: Parameter estimates for a GARCH-DCC-t-Copula model with univariate gaussian errors, GARCH-like correlation dynamics and t-Copula. Standard errors are obtained by moving block bootstrap.

A.2 Likelihood Surface

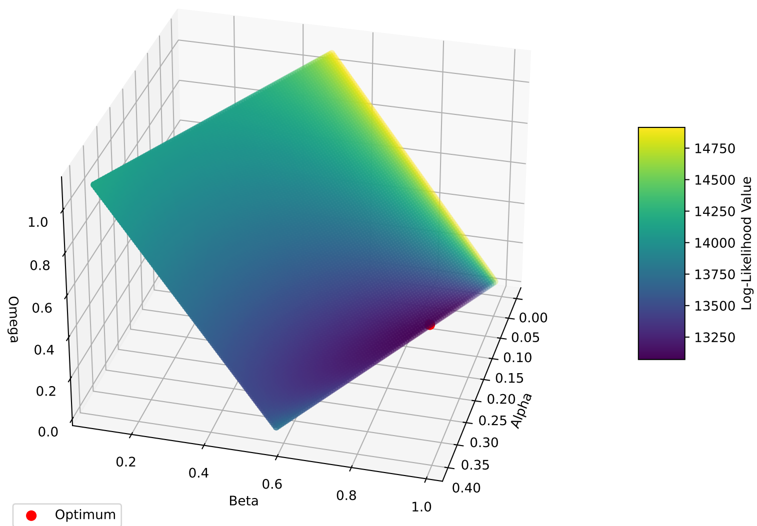


Figure 23: Negative logarithmic likelihood surface - NASDAQ

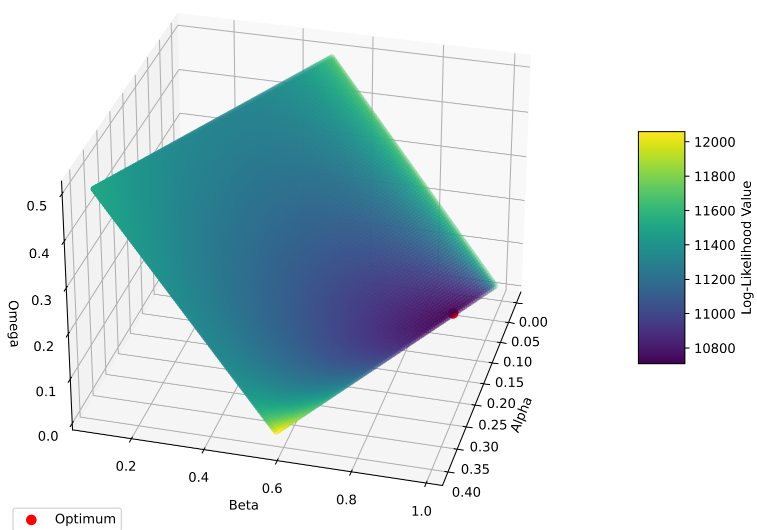


Figure 24: Negative logarithmic likelihood surface - GOLD

B Implied Correlation - Results

B.1 Pearson Correlation Estimates

Daily Returns	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1.000									
NASDAQ	0.809	1.000								
IBEX	0.510	0.433	1.000							
FCHI	0.556	0.483	0.867	1.000						
SSMI	0.492	0.407	0.728	0.810	1.000					
FTMIB	0.517	0.449	0.860	0.874	0.734	1.000				
GDAX	0.576	0.520	0.794	0.891	0.773	0.814	1.000			
FTSE	0.529	0.439	0.775	0.869	0.790	0.779	0.804	1.000		
N225	0.148	0.125	0.263	0.293	0.292	0.248	0.265	0.297	1.000	
HSI	0.199	0.187	0.317	0.351	0.324	0.307	0.339	0.372	0.506	1.000

Weekly Returns	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1.000									
NASDAQ	0.772	1.000								
IBEX	0.672	0.586	1.000							
FCHI	0.751	0.678	0.858	1.000						
SSMI	0.698	0.571	0.721	0.800	1.000					
FTMIB	0.677	0.612	0.861	0.883	0.732	1.000				
GDAX	0.747	0.684	0.818	0.915	0.787	0.838	1.000			
FTSE	0.760	0.648	0.771	0.878	0.797	0.787	0.831	1.000		
N225	0.545	0.509	0.534	0.599	0.533	0.550	0.581	0.572	1.000	
HSI	0.489	0.493	0.500	0.550	0.478	0.504	0.545	0.572	0.545	1.000

Monthly Returns	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1.000									
NASDAQ	0.748	1.000								
IBEX	0.678	0.633	1.000							
FCHI	0.777	0.712	0.849	1.000						
SSMI	0.730	0.568	0.794	0.794	1.000					
FTMIB	0.661	0.627	0.857	0.885	0.695	1.000				
GDAX	0.778	0.734	0.784	0.915	0.754	0.811	1.000			
FTSE	0.789	0.652	0.756	0.844	0.741	0.755	0.796	1.000		
N225	0.606	0.607	0.579	0.622	0.562	0.582	0.613	0.586	1.000	
HSI	0.580	0.590	0.537	0.524	0.439	0.470	0.535	0.592	0.483	1.000

B.2 GARCH Estimates for Filtering Heteroskedasticity

Presented below are the GARCH models that are used to filter the return series for bootstrapping. The standard error of an implied correlation estimate is derived in this way.

	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
GARCH(1,1) Parameters										
$\hat{\omega}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SE	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
t-stat	1.881e+04	5.221e+07	1.779e+05	3.369e+06	1.332e+06	4.827e+05	1.135e+05	2.109e+06	1.589e+04	5.366e+06
$\hat{\alpha}_1$	0.100	0.099	0.100	0.100	0.100	0.105	0.100	0.100	0.100	0.100
SE	(0.000)	(0.000)	(0.008)	(0.001)	(0.000)	(0.007)	(0.005)	(0.000)	(0.001)	(0.000)
t-stat	2.064e+02	2.973e+02	1.285e+01	8.209e+01	6.676e+03	1.509e+01	2.047e+01	6.231e+04	6.856e+01	3.329e+02
$\hat{\beta}_1$	0.880	0.871	0.879	0.880	0.880	0.884	0.880	0.880	0.880	0.880
SE	(0.003)	(0.003)	(0.007)	(0.003)	(0.003)	(0.003)	(0.006)	(0.002)	(0.003)	(0.003)
t-stat	3.325e+02	3.011e+02	1.178e+02	2.547e+02	3.421e+02	2.820e+02	1.537e+02	3.643e+02	2.522e+02	3.270e+02

Table 18: Estimates for a GARCH(1,1) model with normal errors for daily returns.

	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
GARCH(1,1) Parameters										
$\hat{\omega}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SE	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
t-stat	4.927	5.969	6.354	6.394	3.967	0.972	2.539	7.321	1.838	7.457
$\hat{\alpha}_1$	0.200	0.200	0.100	0.109	0.246	0.200	0.200	0.175	0.187	0.090
SE	(0.045)	(0.034)	(0.027)	(0.034)	(0.057)	(0.088)	(0.060)	(0.052)	(0.090)	(0.025)
t-stat	4.399	5.891	3.680	3.258	4.293	2.270	3.317	3.357	2.076	3.615
$\hat{\beta}_1$	0.700	0.780	0.880	0.871	0.678	0.700	0.700	0.782	0.588	0.892
SE	(0.041)	(0.029)	(0.027)	(0.021)	(0.047)	(0.181)	(0.079)	(0.028)	(0.183)	(0.020)
t-stat	16.918	26.686	32.521	41.763	14.383	3.865	8.852	27.649	3.222	44.466

Table 19: Estimates for a GARCH(1,1) model with normal errors for weekly returns.

	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
GARCH(1,1) Parameters										
$\hat{\omega}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.000
SE	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)
t-stat	2.018	1.640	1.351	0.819	1.608	1.348	2.005	1.626	3.265	2.311
$\hat{\alpha}_1$	0.100	0.151	0.041	0.192	0.082	0.055	0.149	0.152	0.161	0.157
SE	(0.042)	(0.062)	(0.058)	(0.156)	(0.059)	(0.046)	(0.080)	(0.070)	(0.093)	(0.054)
t-stat	2.400	2.445	0.712	1.234	1.392	1.188	1.860	2.181	1.729	2.903
$\hat{\beta}_1$	0.880	0.819	0.879	0.682	0.838	0.877	0.737	0.734	0.000	0.765
SE	(0.033)	(0.068)	(0.088)	(0.285)	(0.092)	(0.055)	(0.089)	(0.119)	(0.234)	(0.063)
t-stat	26.791	12.097	9.953	2.394	9.150	15.795	8.319	6.184	0.000	12.074

Table 20: Estimates for a GARCH(1,1) model with normal errors for monthly returns.

B.3 Estimates for Equally Weighted Two-Asset Portfolios - Exact Identification

Estimates on the lower triangular matrix represent correlation implied by losses and estimates from the upper triangular matrix represent correlation implied by gains.

Value-at-Risk - Daily Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.804 (0.077)	0.418 (0.065)	0.515 (0.07)	0.418 (0.066)	0.501 (0.068)	0.48 (0.17)	0.472 (1.607)	0.091 (0.064)	0.112 (0.058)
NASDAQ	0.925 (0.091)	1	0.432 (0.057)	0.492 (0.058)	0.48 (0.06)	0.454 (0.06)	0.495 (0.135)	0.497 (1.283)	0.148 (0.039)	0.168 (0.04)
IBEX	0.494 (0.07)	0.409 (0.051)	1	0.864 (0.1)	0.676 (0.086)	0.826 (0.098)	0.825 (0.204)	0.735 (1.829)	0.201 (0.048)	0.286 (0.049)
FCHI	0.584 (0.072)	0.483 (0.055)	0.916 (0.094)	1	0.783 (0.093)	0.944 (0.104)	0.89 (0.22)	0.85 (2.128)	0.207 (0.048)	0.364 (0.053)
SSMI	0.443 (0.068)	0.346 (0.052)	0.672 (0.084)	0.75 (0.086)	1	0.695 (0.085)	0.74 (0.215)	0.681 (2.02)	0.166 (0.05)	0.258 (0.051)
FTMIB	0.523 (0.07)	0.45 (0.052)	0.907 (0.097)	0.905 (0.093)	0.714 (0.084)	1	0.842 (0.203)	0.768 (1.888)	0.203 (0.047)	0.338 (0.051)
GDAX	0.512 (0.186)	0.474 (0.128)	0.846 (0.207)	0.891 (0.221)	0.723 (0.214)	0.836 (0.201)	1	0.815 (2.056)	0.227 (0.102)	0.373 (0.121)
FTSE	0.488 (1.146)	0.468 (0.828)	0.782 (1.384)	0.848 (1.501)	0.736 (1.492)	0.805 (1.381)	0.816 (1.512)	1	0.208 (0.784)	0.366 (1.059)
N225	0.175 (0.073)	0.175 (0.043)	0.276 (0.052)	0.309 (0.055)	0.316 (0.06)	0.211 (0.049)	0.248 (0.11)	0.37 (0.781)	1	0.474 (0.062)
HSI	0.197 (0.06)	0.218 (0.04)	0.365 (0.047)	0.441 (0.053)	0.332 (0.051)	0.375 (0.05)	0.367 (0.119)	0.423 (0.822)	0.506 (0.064)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.727 (0.072)	0.508 (0.074)	0.531 (0.082)	0.463 (0.067)	0.502 (0.083)	0.61 (0.234)	0.495 (0.432)	0.133 (0.076)	0.128 (0.059)
NASDAQ	0.848 (0.067)	1	0.513 (0.069)	0.545 (0.076)	0.403 (0.065)	0.465 (0.07)	0.58 (0.189)	0.382 (0.29)	0.116 (0.053)	0.165 (0.048)
IBEX	0.495 (0.066)	0.38 (0.054)	1	0.951 (0.125)	0.78 (0.108)	0.91 (0.123)	0.842 (0.255)	0.8 (0.491)	0.248 (0.066)	0.343 (0.064)
FCHI	0.559 (0.063)	0.443 (0.053)	0.857 (0.096)	1	0.75 (0.114)	0.868 (0.122)	0.877 (0.27)	0.837 (0.518)	0.2 (0.066)	0.394 (0.068)
SSMI	0.555 (0.066)	0.423 (0.059)	0.749 (0.093)	0.864 (0.1)	1	0.721 (0.107)	0.775 (0.274)	0.744 (0.528)	0.222 (0.073)	0.373 (0.068)
FTMIB	0.544 (0.07)	0.433 (0.056)	0.89 (0.101)	0.869 (0.096)	0.805 (0.093)	1	0.719 (0.229)	0.73 (0.456)	0.149 (0.058)	0.28 (0.062)
GDAX	0.583 (0.226)	0.563 (0.182)	0.786 (0.241)	0.892 (0.262)	0.814 (0.276)	0.804 (0.238)	1	0.814 (0.638)	0.175 (0.133)	0.339 (0.151)
FTSE	0.551 (0.587)	0.446 (0.378)	0.747 (0.606)	0.888 (0.694)	0.835 (0.71)	0.816 (0.63)	0.765 (0.701)	1	0.221 (0.271)	0.39 (0.302)
N225	0.111 (0.068)	0.141 (0.046)	0.275 (0.066)	0.376 (0.069)	0.347 (0.077)	0.3 (0.063)	0.33 (0.16)	0.355 (0.391)	1	0.434 (0.083)
HSI	0.141 (0.062)	0.094 (0.038)	0.338 (0.06)	0.387 (0.058)	0.382 (0.063)	0.314 (0.056)	0.387 (0.155)	0.431 (0.42)	0.533 (0.084)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.957 (0.149)	0.578 (0.148)	0.62 (0.152)	0.478 (0.134)	0.542 (0.137)	0.719 (0.703)	0.525 (0.171)	0.086 (0.149)	0.12 (0.097)
NASDAQ	0.851 (0.13)	1	0.481 (0.12)	0.617 (0.134)	0.41 (0.113)	0.474 (0.12)	0.678 (0.569)	0.427 (0.157)	0.132 (0.105)	0.208 (0.082)
IBEX	0.662 (0.155)	0.498 (0.117)	1	0.836 (0.199)	0.687 (0.178)	0.836 (0.189)	0.709 (0.635)	0.754 (0.245)	0.19 (0.128)	0.234 (0.098)
FCHI	0.588 (0.133)	0.499 (0.098)	0.952 (0.201)	1	0.774 (0.196)	0.846 (0.195)	0.859 (0.74)	0.878 (0.261)	0.195 (0.137)	0.258 (0.1)
SSMI	0.572 (0.138)	0.388 (0.093)	0.755 (0.173)	0.868 (0.165)	1	0.808 (0.199)	0.813 (0.81)	0.771 (0.238)	0.238 (0.152)	0.188 (0.104)
FTMIB	0.621 (0.142)	0.438 (0.1)	0.96 (0.201)	0.89 (0.173)	0.683 (0.148)	1	0.869 (0.735)	0.707 (0.229)	0.216 (0.136)	0.218 (0.093)
GDAX	0.56 (0.375)	0.514 (0.292)	0.841 (0.437)	0.859 (0.419)	0.708 (0.4)	0.76 (0.379)	1	0.748 (0.918)	0.238 (0.415)	0.254 (0.334)
FTSE	0.492 (0.167)	0.385 (0.129)	0.884 (0.25)	0.952 (0.219)	0.826 (0.205)	0.845 (0.223)	0.823 (0.631)	1	0.214 (0.233)	0.2 (0.15)
N225	0.116 (0.119)	0.156 (0.093)	0.22 (0.122)	0.277 (0.118)	0.315 (0.132)	0.212 (0.107)	0.221 (0.264)	0.25 (0.185)	1	0.525 (0.161)
HSI	0.149 (0.109)	0.199 (0.081)	0.239 (0.1)	0.272 (0.095)	0.287 (0.102)	0.266 (0.099)	0.319 (0.248)	0.3 (0.16)	0.421 (0.133)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.902 (0.231)	0.575 (0.254)	0.633 (0.224)	0.413 (0.195)	0.367 (0.201)	0.512 (0.487)	0.45 (0.307)	0.025 (0.164)	0.062 (0.123)
NASDAQ	0.757 (0.24)	1	0.402 (0.143)	0.604 (0.149)	0.298 (0.122)	0.399 (0.139)	0.557 (0.395)	0.487 (0.155)	0.027 (0.102)	0.137 (0.094)
IBEX	0.607 (0.26)	0.424 (0.146)	1	0.875 (0.264)	0.62 (0.228)	0.662 (0.251)	0.732 (0.594)	0.785 (0.299)	0.181 (0.178)	0.232 (0.136)
FCHI	0.658 (0.282)	0.404 (0.155)	0.873 (0.287)	1	0.706 (0.22)	0.794 (0.24)	0.881 (0.608)	0.846 (0.253)	0.405 (0.206)	0.407 (0.151)
SSMI	0.644 (0.31)	0.493 (0.201)	0.873 (0.339)	0.796 (0.331)	1	0.667 (0.222)	0.543 (0.504)	0.722 (0.238)	0.19 (0.16)	0.219 (0.141)
FTMIB	0.457 (0.219)	0.406 (0.127)	0.969 (0.274)	0.825 (0.26)	0.868 (0.319)	1	0.676 (0.539)	0.688 (0.267)	0.193 (0.17)	0.114 (0.114)
GDAX	0.576 (0.479)	0.301 (0.298)	0.715 (0.508)	0.858 (0.595)	0.752 (0.582)	0.691 (0.49)	1	0.731 (0.757)	0.188 (0.357)	0.211 (0.281)
FTSE	0.66 (0.461)	0.452 (0.143)	0.845 (0.304)	0.868 (0.36)	0.834 (0.402)	0.821 (0.286)	0.95 (0.807)	1	0.281 (0.241)	0.266 (0.171)
N225	0.179 (0.215)	0.098 (0.115)	0.173 (0.148)	0.155 (0.166)	0.252 (0.197)	0.178 (0.14)	0.063 (0.261)	0.146 (0.2)	1	0.541 (0.191)
HSI	0.122 (0.157)	0.17 (0.1)	0.336 (0.155)	0.323 (0.17)	0.398 (0.215)	0.422 (0.16)	0.31 (0.348)	0.28 (0.186)	0.475 (0.19)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.691 (0.57)	0.309 (0.423)	0.323 (0.418)	0.338 (0.496)	0.505 (0.495)	0.364 (0.659)	0.436 (0.951)	0.139 (0.462)	0.331 (0.425)
NASDAQ	0.755 (0.441)	1	0.326 (0.214)	0.233 (0.211)	0.36 (0.241)	0.428 (0.229)	0.505 (0.529)	0.334 (0.213)	0.15 (0.218)	0.128 (0.179)
IBEX	0.379 (0.411)	0.272 (0.212)	1	0.838 (0.407)	0.804 (0.453)	0.685 (0.369)	0.529 (0.636)	0.798 (0.412)	0.198 (0.288)	0.324 (0.264)
FCHI	0.53 (0.502)	0.46 (0.258)	0.773 (0.458)	1	0.794 (0.418)	0.839 (0.379)	0.682 (0.667)	0.868 (0.432)	0.063 (0.227)	0.128 (0.205)
SSMI	0.562 (0.533)	0.215 (0.208)	0.634 (0.404)	0.821 (0.49)	1	0.864 (0.437)	0.682 (0.746)	0.802 (0.424)	0.229 (0.351)	0.147 (0.238)
FTMIB	0.485 (0.466)	0.516 (0.279)	0.573 (0.403)	0.718 (0.449)	0.574 (0.392)	1	0.814 (0.786)	0.926 (0.439)	0.417 (0.372)	0.282 (0.237)
GDAX	0.383 (0.626)	0.359 (0.406)	0.673 (0.687)	0.946 (0.819)	0.797 (0.859)	0.772 (0.777)	1	0.78 (0.941)	0.143 (0.452)	0.359 (0.447)
FTSE	0.54 (0.68)	0.512 (0.254)	0.728 (0.456)	0.902 (0.473)	0.758 (0.506)	0.9 (0.452)	0.865 (0.937)	1	0.358 (0.478)	0.526 (0.374)
N225	0.316 (0.521)	0.158 (0.248)	0.768 (0.526)	0.702 (0.541)	0.534 (0.517)	0.813 (0.559)	0.725 (0.781)	0.761 (0.638)	1	0.329 (0.31)
HSI	0.183 (0.281)	-0.034 (0.131)	0.44 (0.3)	0.368 (0.277)	0.26 (0.273)	0.666 (0.375)	0.356 (0.444)	0.576 (0.373)	0.829 (0.47)	1

Value-at-Risk - Weekly Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.864 (0.215)	0.61 (0.179)	0.825 (0.158)	0.653 (0.131)	0.757 (0.132)	0.775 (0.131)	0.856 (0.141)	0.525 (0.095)	0.444 (0.132)
NASDAQ	0.685 (0.17)	1	0.667 (0.212)	0.77 (0.202)	0.638 (0.194)	0.677 (0.168)	0.776 (0.184)	0.853 (0.226)	0.53 (0.138)	0.624 (0.188)
IBEX	0.763 (0.203)	0.591 (0.182)	1	0.889 (0.245)	0.696 (0.218)	0.809 (0.199)	0.842 (0.21)	0.633 (0.198)	0.378 (0.124)	0.483 (0.174)
FCHI	0.873 (0.148)	0.624 (0.159)	0.891 (0.238)	1	0.86 (0.201)	0.88 (0.167)	0.896 (0.182)	0.861 (0.18)	0.459 (0.102)	0.497 (0.151)
SSMI	0.766 (0.116)	0.548 (0.156)	0.704 (0.201)	0.692 (0.15)	1	0.667 (0.141)	0.824 (0.163)	0.729 (0.144)	0.412 (0.099)	0.537 (0.154)
FTMIB	0.715 (0.108)	0.56 (0.131)	0.834 (0.187)	0.772 (0.13)	0.656 (0.112)	1	0.889 (0.148)	0.688 (0.134)	0.435 (0.09)	0.41 (0.117)
GDAX	0.891 (0.128)	0.692 (0.154)	0.834 (0.187)	0.879 (0.157)	0.681 (0.121)	0.916 (0.129)	1	0.751 (0.149)	0.565 (0.098)	0.61 (0.146)
FTSE	0.824 (0.116)	0.718 (0.176)	0.757 (0.208)	0.908 (0.164)	0.655 (0.106)	0.664 (0.109)	0.86 (0.13)	1	0.439 (0.091)	0.594 (0.154)
N225	0.576 (0.087)	0.602 (0.139)	0.786 (0.184)	0.759 (0.127)	0.489 (0.09)	0.591 (0.094)	0.634 (0.093)	0.761 (0.107)	1	0.561 (0.123)
HSI	0.488 (0.124)	0.572 (0.155)	0.413 (0.145)	0.613 (0.145)	0.483 (0.124)	0.479 (0.106)	0.536 (0.116)	0.693 (0.148)	0.621 (0.119)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.818 (0.338)	0.588 (0.287)	0.771 (0.201)	0.692 (0.178)	0.66 (0.142)	0.717 (0.151)	0.744 (0.171)	0.527 (0.118)	0.445 (0.158)
NASDAQ	0.83 (0.306)	1	0.616 (0.34)	0.69 (0.312)	0.67 (0.336)	0.613 (0.257)	0.723 (0.283)	0.688 (0.323)	0.556 (0.259)	0.529 (0.264)
IBEX	0.52 (0.217)	0.535 (0.288)	1	0.84 (0.372)	0.67 (0.33)	0.923 (0.353)	0.771 (0.302)	0.613 (0.316)	0.542 (0.272)	0.476 (0.265)
FCHI	0.66 (0.164)	0.815 (0.313)	0.801 (0.31)	1	0.702 (0.224)	0.848 (0.21)	0.871 (0.212)	0.688 (0.204)	0.534 (0.167)	0.556 (0.194)
SSMI	0.65 (0.153)	0.758 (0.324)	0.652 (0.268)	0.908 (0.242)	1	0.632 (0.178)	0.699 (0.194)	0.797 (0.217)	0.435 (0.138)	0.476 (0.198)
FTMIB	0.633 (0.131)	0.736 (0.268)	0.836 (0.266)	0.942 (0.214)	0.737 (0.173)	1	0.883 (0.175)	0.713 (0.165)	0.533 (0.125)	0.445 (0.148)
GDAX	0.723 (0.13)	0.759 (0.275)	0.753 (0.248)	0.844 (0.192)	0.798 (0.177)	0.806 (0.151)	1	0.746 (0.169)	0.518 (0.132)	0.603 (0.176)
FTSE	0.621 (0.129)	0.739 (0.307)	0.702 (0.272)	0.805 (0.21)	0.814 (0.194)	0.672 (0.142)	0.606 (0.129)	1	0.57 (0.143)	0.588 (0.193)
N225	0.414 (0.091)	0.678 (0.254)	0.479 (0.202)	0.634 (0.152)	0.557 (0.132)	0.627 (0.12)	0.621 (0.116)	0.656 (0.128)	1	0.535 (0.156)
HSI	0.323 (0.116)	0.45 (0.209)	0.257 (0.159)	0.524 (0.159)	0.386 (0.144)	0.531 (0.137)	0.475 (0.127)	0.443 (0.137)	0.509 (0.121)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.749 (1.095)	0.745 (0.728)	0.806 (0.593)	0.695 (0.543)	0.666 (0.363)	0.713 (0.448)	0.583 (0.36)	0.339 (0.229)	0.779 (0.363)
NASDAQ	0.725 (0.738)	1	0.567 (1.073)	0.705 (1.048)	0.655 (1.122)	0.616 (0.913)	0.608 (0.909)	0.528 (0.966)	0.345 (0.748)	0.537 (0.812)
IBEX	0.652 (0.769)	0.618 (0.898)	1	0.836 (0.973)	0.863 (1.045)	0.762 (0.724)	0.747 (0.824)	0.787 (0.809)	0.546 (0.581)	0.504 (0.543)
FCHI	0.52 (0.433)	0.441 (0.604)	0.724 (0.841)	1	0.778 (0.767)	0.867 (0.625)	0.939 (0.762)	0.774 (0.661)	0.561 (0.448)	0.585 (0.456)
SSMI	0.829 (0.505)	0.646 (0.696)	0.805 (0.887)	0.671 (0.606)	1	0.679 (0.57)	0.734 (0.678)	0.642 (0.625)	0.684 (0.517)	0.62 (0.467)
FTMIB	0.701 (0.315)	0.642 (0.593)	0.739 (0.697)	0.722 (0.462)	0.847 (0.508)	1	0.777 (0.517)	0.654 (0.407)	0.742 (0.352)	0.658 (0.357)
GDAX	0.803 (0.341)	0.588 (0.536)	0.728 (0.695)	0.581 (0.416)	0.676 (0.418)	0.824 (0.327)	1	0.858 (0.592)	0.606 (0.367)	0.607 (0.378)
FTSE	0.694 (0.315)	0.583 (0.629)	0.884 (0.849)	0.752 (0.5)	0.685 (0.458)	0.721 (0.297)	0.778 (0.324)	1	0.424 (0.283)	0.445 (0.308)
N225	0.506 (0.227)	0.523 (0.573)	0.551 (0.621)	0.483 (0.361)	0.78 (0.468)	0.831 (0.296)	0.453 (0.231)	0.546 (0.241)	1	0.602 (0.28)
HSI	0.631 (0.296)	0.605 (0.603)	0.668 (0.671)	0.746 (0.464)	0.562 (0.423)	0.711 (0.321)	0.69 (0.296)	0.914 (0.354)	0.59 (0.239)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.806 (0.879)	0.859 (0.851)	0.934 (0.665)	0.849 (0.666)	0.832 (0.55)	0.88 (0.521)	0.949 (0.537)	0.364 (0.291)	0.545 (0.334)
NASDAQ	0.819 (1.051)	1	0.603 (0.848)	0.746 (0.822)	0.824 (1.009)	0.77 (0.832)	0.613 (0.625)	0.617 (0.743)	0.462 (0.614)	0.246 (0.405)
IBEX	0.785 (0.799)	0.906 (1.167)	1	0.902 (1.06)	0.75 (1.027)	0.726 (0.916)	0.724 (0.77)	0.847 (0.943)	0.516 (0.629)	0.533 (0.58)
FCHI	0.621 (0.524)	0.592 (0.796)	0.807 (0.912)	1	0.684 (0.764)	0.743 (0.684)	0.825 (0.628)	0.836 (0.702)	0.423 (0.403)	0.623 (0.451)
SSMI	0.494 (0.457)	0.653 (0.783)	0.741 (0.84)	0.812 (0.632)	1	0.83 (0.817)	0.66 (0.627)	0.829 (0.773)	0.462 (0.497)	0.226 (0.361)
FTMIB	0.66 (0.372)	0.615 (0.674)	0.817 (0.678)	0.925 (0.54)	0.697 (0.434)	1	0.719 (0.495)	0.762 (0.575)	0.758 (0.463)	0.624 (0.404)
GDAX	0.544 (0.367)	0.631 (0.675)	0.831 (0.756)	0.695 (0.491)	0.789 (0.507)	0.654 (0.345)	1	0.876 (0.528)	0.259 (0.267)	0.439 (0.306)
FTSE	0.845 (0.435)	0.733 (0.936)	0.871 (0.932)	0.733 (0.557)	0.463 (0.459)	0.828 (0.388)	0.673 (0.386)	1	0.255 (0.284)	0.502 (0.345)
N225	0.542 (0.285)	0.662 (0.802)	0.518 (0.571)	0.245 (0.294)	0.351 (0.35)	0.322 (0.238)	0.286 (0.266)	0.277 (0.221)	1	0.515 (0.283)
HSI	0.775 (0.449)	0.34 (0.626)	0.414 (0.599)	0.469 (0.436)	0.327 (0.373)	0.52 (0.323)	0.42 (0.325)	0.581 (0.381)	0.376 (0.253)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.527 (1.006)	0.321 (0.692)	0.517 (0.751)	0.432 (0.773)	0.233 (0.421)	0.584 (0.728)	0.639 (0.676)	0.76 (0.63)	0.351 (0.397)
NASDAQ	0.68 (0.695)	1	0.43 (1.19)	0.716 (1.428)	0.358 (1.01)	0.342 (0.72)	0.708 (1.238)	0.6 (1.099)	0.629 (1.033)	0.455 (0.81)
IBEX	0.242 (0.422)	0.304 (0.711)	1	0.841 (1.358)	0.78 (1.363)	0.848 (0.973)	0.794 (1.201)	0.533 (0.94)	0.422 (0.721)	0.655 (0.814)
FCHI	0.258 (0.37)	0.36 (0.63)	0.996 (1.015)	1	0.819 (1.221)	0.682 (0.733)	0.859 (0.999)	0.696 (0.893)	0.564 (0.648)	0.682 (0.67)
SSMI	0.208 (0.415)	0.379 (0.797)	0.951 (1.224)	0.973 (1.09)	1	0.639 (0.841)	0.747 (1.067)	0.756 (0.992)	0.273 (0.687)	0.782 (0.885)
FTMIB	0.366 (0.309)	0.425 (0.598)	0.977 (0.804)	0.97 (0.721)	0.894 (0.787)	1	0.735 (0.729)	0.277 (0.463)	0.271 (0.382)	0.838 (0.521)
GDAX	0.262 (0.284)	0.366 (0.498)	0.995 (0.84)	1.0 (0.75)	0.973 (0.943)	0.971 (0.542)	1	0.806 (0.796)	0.735 (0.635)	0.791 (0.709)
FTSE	0.32 (0.301)	0.591 (0.65)	0.871 (0.778)	0.912 (0.738)	0.964 (0.901)	0.841 (0.482)	0.914 (0.558)	1	0.656 (0.578)	0.538 (0.507)
N225	0.179 (0.249)	0.305 (0.533)	0.995 (0.834)	0.996 (0.711)	0.968 (0.846)	0.962 (0.5)	0.996 (0.526)	0.894 (0.484)	1	0.239 (0.285)
HSI	-0.014 (0.172)	0.576 (0.591)	0.128 (0.353)	0.141 (0.321)	0.074 (0.315)	0.26 (0.268)	0.144 (0.245)	0.197 (0.229)	0.152 (0.179)	1

Value-at-Risk - Monthly Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.861 (0.526)	0.711 (0.286)	0.788 (0.359)	0.504 (0.262)	0.664 (0.324)	0.715 (0.324)	0.798 (0.358)	0.775 (0.273)	0.76 (0.33)
NASDAQ	0.844 (0.397)	1	0.811 (0.395)	0.837 (0.46)	0.677 (0.46)	0.688 (0.37)	0.763 (0.415)	0.851 (0.509)	0.602 (0.333)	0.807 (0.38)
IBEX	0.524 (0.218)	0.585 (0.251)	1	0.857 (0.261)	0.567 (0.216)	0.776 (0.243)	0.784 (0.288)	0.884 (0.287)	0.492 (0.168)	0.368 (0.184)
FCHI	0.658 (0.297)	0.747 (0.312)	0.78 (0.205)	1	0.739 (0.309)	0.907 (0.319)	0.773 (0.301)	0.868 (0.334)	0.515 (0.199)	0.486 (0.228)
SSMI	0.54 (0.226)	0.437 (0.267)	0.402 (0.145)	0.586 (0.217)	1	0.495 (0.24)	0.821 (0.319)	0.609 (0.252)	0.496 (0.197)	0.35 (0.229)
FTMIB	0.458 (0.25)	0.514 (0.255)	0.838 (0.228)	0.808 (0.26)	0.636 (0.226)	1	0.546 (0.261)	0.685 (0.31)	0.337 (0.183)	0.426 (0.225)
GDAX	0.824 (0.327)	0.855 (0.326)	0.792 (0.26)	0.786 (0.277)	0.627 (0.217)	0.662 (0.268)	1	0.691 (0.299)	0.826 (0.266)	0.424 (0.229)
FTSE	0.803 (0.303)	0.692 (0.306)	0.571 (0.178)	0.797 (0.262)	0.812 (0.218)	0.589 (0.237)	0.762 (0.262)	1	0.427 (0.198)	0.456 (0.265)
N225	0.388 (0.169)	0.642 (0.228)	0.567 (0.138)	0.441 (0.16)	0.452 (0.135)	0.611 (0.184)	0.546 (0.167)	0.451 (0.152)	1	0.561 (0.217)
HSI	0.47 (0.238)	0.303 (0.189)	0.537 (0.18)	0.399 (0.186)	0.295 (0.165)	0.384 (0.198)	0.329 (0.176)	0.629 (0.228)	0.451 (0.157)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.527 (0.424)	0.369 (0.288)	0.659 (0.391)	0.792 (0.434)	0.484 (0.312)	0.562 (0.368)	0.64 (0.418)	0.327 (0.223)	0.46 (0.309)
NASDAQ	0.914 (0.507)	1	0.615 (0.383)	0.671 (0.415)	0.525 (0.457)	0.456 (0.319)	0.516 (0.381)	0.42 (0.404)	0.464 (0.315)	0.612 (0.367)
IBEX	0.791 (0.381)	0.676 (0.343)	1	0.781 (0.34)	0.609 (0.291)	0.819 (0.302)	0.65 (0.336)	0.509 (0.287)	0.271 (0.182)	0.431 (0.251)
FCHI	0.914 (0.386)	0.734 (0.367)	0.847 (0.298)	1	0.498 (0.277)	0.596 (0.275)	0.643 (0.338)	0.597 (0.303)	0.263 (0.165)	0.571 (0.291)
SSMI	0.918 (0.378)	0.731 (0.387)	0.81 (0.285)	0.882 (0.27)	1	0.662 (0.245)	0.586 (0.335)	0.463 (0.238)	0.375 (0.18)	0.283 (0.283)
FTMIB	0.52 (0.264)	0.517 (0.259)	0.769 (0.25)	0.615 (0.224)	0.477 (0.176)	1	0.641 (0.309)	0.779 (0.313)	0.517 (0.195)	0.437 (0.236)
GDAX	0.886 (0.435)	0.713 (0.389)	0.749 (0.307)	0.901 (0.326)	0.842 (0.327)	0.667 (0.25)	1	0.425 (0.313)	0.38 (0.235)	0.867 (0.397)
FTSE	0.878 (0.401)	0.647 (0.379)	0.866 (0.319)	0.885 (0.318)	0.946 (0.268)	0.485 (0.196)	0.847 (0.361)	1	0.275 (0.15)	0.459 (0.3)
N225	0.77 (0.306)	0.702 (0.3)	0.565 (0.213)	0.858 (0.232)	0.6 (0.168)	0.456 (0.161)	0.705 (0.27)	0.577 (0.178)	1	0.316 (0.204)
HSI	0.801 (0.36)	0.755 (0.33)	0.508 (0.224)	0.726 (0.276)	0.611 (0.232)	0.53 (0.198)	0.89 (0.336)	0.592 (0.241)	0.665 (0.212)	1

$\alpha = 0.025/0.975$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.575 (0.717)	0.66 (0.436)	0.826 (0.694)	0.582 (0.52)	0.451 (0.394)	0.91 (0.72)	0.728 (0.584)	0.41 (0.368)	0.638 (0.444)
NASDAQ	0.537 (0.461)	1	0.386 (0.406)	0.7 (0.705)	0.47 (0.615)	0.498 (0.509)	0.749 (0.705)	0.563 (0.735)	0.468 (0.535)	0.738 (0.548)
IBEX	0.466 (0.255)	0.441 (0.307)	1	0.814 (0.385)	0.874 (0.329)	0.601 (0.256)	0.554 (0.321)	0.83 (0.372)	0.274 (0.195)	0.554 (0.238)
FCHI	0.343 (0.312)	0.566 (0.426)	0.501 (0.236)	1	0.798 (0.45)	0.805 (0.394)	0.856 (0.566)	0.852 (0.516)	0.339 (0.27)	0.601 (0.349)
SSMI	0.627 (0.341)	0.492 (0.415)	0.786 (0.263)	0.828 (0.352)	1	0.674 (0.315)	0.534 (0.422)	0.749 (0.353)	0.439 (0.229)	0.501 (0.292)
FTMIB	0.521 (0.264)	0.521 (0.342)	0.786 (0.236)	0.737 (0.295)	0.9 (0.254)	1	0.547 (0.321)	0.81 (0.39)	0.404 (0.213)	0.486 (0.241)
GDAX	0.451 (0.356)	0.743 (0.479)	0.645 (0.273)	0.923 (0.425)	0.797 (0.393)	0.772 (0.305)	1	0.681 (0.494)	0.317 (0.311)	0.633 (0.361)
FTSE	0.739 (0.364)	0.748 (0.462)	0.611 (0.218)	0.78 (0.362)	0.865 (0.264)	0.784 (0.25)	0.844 (0.36)	1	0.559 (0.289)	0.808 (0.363)
N225	0.649 (0.327)	0.496 (0.375)	0.571 (0.21)	0.514 (0.253)	0.576 (0.214)	0.677 (0.216)	0.673 (0.297)	0.619 (0.219)	1	0.768 (0.297)
HSI	0.386 (0.26)	0.783 (0.429)	0.296 (0.182)	0.294 (0.221)	0.253 (0.201)	0.411 (0.198)	0.444 (0.259)	0.348 (0.213)	0.378 (0.22)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.538 (0.732)	0.467 (0.463)	0.472 (0.543)	0.619 (0.518)	0.423 (0.425)	0.628 (0.613)	0.602 (0.6)	0.374 (0.412)	0.76 (0.57)
NASDAQ	0.716 (0.964)	1	0.602 (0.607)	0.748 (0.744)	0.16 (0.524)	0.874 (0.673)	0.798 (0.727)	0.444 (0.691)	0.44 (0.602)	0.342 (0.409)
IBEX	0.902 (0.634)	0.868 (0.769)	1	0.706 (0.499)	0.48 (0.403)	0.888 (0.418)	0.679 (0.514)	0.533 (0.434)	0.107 (0.264)	0.37 (0.315)
FCHI	0.77 (0.617)	0.821 (0.802)	0.877 (0.455)	1	0.471 (0.454)	0.796 (0.498)	0.915 (0.634)	0.819 (0.591)	0.414 (0.405)	0.378 (0.348)
SSMI	0.535 (0.566)	0.487 (0.702)	0.567 (0.327)	0.845 (0.448)	1	0.257 (0.339)	0.515 (0.509)	0.846 (0.473)	0.257 (0.271)	0.163 (0.364)
FTMIB	0.865 (0.582)	0.808 (0.718)	0.863 (0.35)	0.87 (0.459)	0.593 (0.355)	1	0.805 (0.493)	0.461 (0.386)	0.194 (0.264)	0.31 (0.273)
GDAX	0.762 (0.72)	0.797 (0.75)	0.798 (0.501)	0.889 (0.598)	0.597 (0.47)	0.884 (0.505)	1	0.746 (0.595)	0.2 (0.365)	0.453 (0.379)
FTSE	0.763 (0.599)	0.825 (0.861)	0.835 (0.35)	0.816 (0.461)	0.615 (0.3)	0.63 (0.311)	0.776 (0.517)	1	0.527 (0.411)	0.287 (0.391)
N225	0.411 (0.451)	0.656 (0.761)	0.496 (0.274)	0.731 (0.404)	0.6 (0.29)	0.513 (0.291)	0.821 (0.521)	0.779 (0.328)	1	0.145 (0.328)
HSI	0.373 (0.421)	0.523 (0.604)	0.686 (0.358)	0.717 (0.48)	0.585 (0.414)	0.423 (0.299)	0.483 (0.386)	0.584 (0.377)	0.415 (0.329)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.401 (0.971)	0.456 (0.442)	0.778 (0.846)	0.593 (0.491)	0.424 (0.386)	0.588 (0.673)	0.703 (0.62)	0.346 (0.343)	0.341 (0.361)
NASDAQ	0.411 (0.503)	1	0.681 (0.926)	0.677 (1.138)	0.098 (0.86)	0.626 (0.809)	0.452 (0.895)	0.374 (1.188)	0.086 (0.708)	0.065 (0.295)
IBEX	0.876 (0.474)	0.243 (0.323)	1	0.792 (0.669)	0.688 (0.456)	0.752 (0.372)	0.6 (0.532)	0.717 (0.515)	0.329 (0.326)	0.302 (0.294)
FCHI	0.725 (0.53)	0.257 (0.409)	0.475 (0.333)	1	0.646 (0.635)	0.85 (0.648)	0.84 (0.822)	0.896 (0.958)	0.307 (0.448)	0.333 (0.36)
SSMI	0.333 (0.304)	0.11 (0.405)	0.27 (0.26)	0.326 (0.301)	1	0.38 (0.384)	0.438 (0.585)	0.758 (0.437)	0.154 (0.238)	0.46 (0.393)
FTMIB	0.765 (0.5)	0.409 (0.446)	0.641 (0.303)	0.865 (0.503)	0.653 (0.382)	1	0.854 (0.637)	0.753 (0.497)	0.511 (0.353)	0.129 (0.225)
GDAX	0.716 (0.573)	0.413 (0.431)	0.683 (0.416)	0.627 (0.472)	0.755 (0.463)	0.911 (0.535)	1	0.819 (0.757)	0.39 (0.526)	0.104 (0.279)
FTSE	0.616 (0.428)	0.247 (0.468)	0.633 (0.334)	0.777 (0.443)	0.34 (0.22)	0.833 (0.428)	0.616 (0.446)	1	0.269 (0.294)	0.598 (0.481)
N225	0.75 (0.45)	0.498 (0.507)	0.752 (0.347)	0.542 (0.409)	0.67 (0.261)	0.831 (0.39)	0.809 (0.511)	0.765 (0.358)	1	-0.119 (0.245)
HSI	0.309 (0.361)	0.243 (0.407)	0.416 (0.3)	0.493 (0.4)	0.67 (0.379)	0.79 (0.436)	0.772 (0.461)	0.777 (0.387)	0.726 (0.36)	1

Expected Shortfall - Daily Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.804 (0.051)	0.475 (0.042)	0.54 (0.046)	0.477 (0.043)	0.504 (0.045)	0.578 (0.111)	0.505 (0.646)	0.098 (0.042)	0.146 (0.038)
NASDAQ	0.814 (0.044)	1	0.423 (0.033)	0.494 (0.037)	0.417 (0.036)	0.443 (0.035)	0.55 (0.087)	0.43 (0.427)	0.077 (0.026)	0.164 (0.025)
IBEX	0.532 (0.043)	0.423 (0.032)	1	0.862 (0.061)	0.722 (0.056)	0.864 (0.06)	0.776 (0.118)	0.765 (0.723)	0.2 (0.031)	0.272 (0.031)
FCHI	0.58 (0.042)	0.475 (0.032)	0.881 (0.055)	1	0.797 (0.062)	0.859 (0.061)	0.878 (0.133)	0.851 (0.805)	0.212 (0.033)	0.321 (0.034)
SSMI	0.525 (0.042)	0.401 (0.033)	0.745 (0.052)	0.834 (0.053)	1	0.728 (0.055)	0.757 (0.131)	0.758 (0.826)	0.222 (0.036)	0.268 (0.033)
FTMIB	0.535 (0.043)	0.442 (0.032)	0.864 (0.056)	0.886 (0.055)	0.755 (0.049)	1	0.795 (0.119)	0.761 (0.708)	0.159 (0.03)	0.258 (0.032)
GDAX	0.583 (0.106)	0.509 (0.078)	0.822 (0.122)	0.906 (0.131)	0.804 (0.131)	0.83 (0.12)	1	0.789 (0.789)	0.2 (0.064)	0.311 (0.067)
FTSE	0.548 (0.663)	0.436 (0.445)	0.778 (0.753)	0.891 (0.845)	0.815 (0.875)	0.8 (0.746)	0.826 (0.83)	1	0.207 (0.327)	0.322 (0.369)
N225	0.133 (0.039)	0.134 (0.023)	0.274 (0.034)	0.321 (0.034)	0.335 (0.036)	0.265 (0.032)	0.287 (0.07)	0.321 (0.418)	1	0.442 (0.04)
HSI	0.157 (0.034)	0.144 (0.021)	0.353 (0.032)	0.385 (0.031)	0.377 (0.032)	0.333 (0.03)	0.354 (0.07)	0.408 (0.467)	0.522 (0.04)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.82 (0.044)	0.491 (0.047)	0.558 (0.047)	0.484 (0.043)	0.496 (0.045)	0.603 (0.144)	0.518 (0.298)	0.09 (0.046)	0.148 (0.041)
NASDAQ	0.784 (0.044)	1	0.409 (0.04)	0.501 (0.043)	0.389 (0.041)	0.429 (0.038)	0.561 (0.115)	0.424 (0.213)	0.051 (0.032)	0.169 (0.031)
IBEX	0.549 (0.049)	0.45 (0.043)	1	0.86 (0.07)	0.715 (0.064)	0.867 (0.074)	0.751 (0.151)	0.761 (0.335)	0.18 (0.041)	0.255 (0.037)
FCHI	0.569 (0.044)	0.469 (0.039)	0.893 (0.071)	1	0.802 (0.07)	0.851 (0.07)	0.882 (0.169)	0.852 (0.366)	0.221 (0.042)	0.29 (0.039)
SSMI	0.543 (0.043)	0.41 (0.04)	0.755 (0.066)	0.844 (0.065)	1	0.732 (0.064)	0.741 (0.165)	0.774 (0.367)	0.243 (0.048)	0.242 (0.041)
FTMIB	0.522 (0.047)	0.441 (0.04)	0.877 (0.073)	0.894 (0.066)	0.758 (0.063)	1	0.784 (0.153)	0.742 (0.325)	0.154 (0.038)	0.226 (0.035)
GDAX	0.595 (0.129)	0.51 (0.104)	0.818 (0.153)	0.909 (0.159)	0.801 (0.159)	0.836 (0.148)	1	0.767 (0.422)	0.188 (0.087)	0.28 (0.088)
FTSE	0.563 (0.294)	0.423 (0.212)	0.802 (0.347)	0.877 (0.37)	0.834 (0.38)	0.802 (0.341)	0.825 (0.406)	1	0.216 (0.192)	0.295 (0.181)
N225	0.134 (0.044)	0.122 (0.033)	0.279 (0.046)	0.307 (0.044)	0.329 (0.05)	0.27 (0.045)	0.28 (0.093)	0.309 (0.207)	1	0.419 (0.051)
HSI	0.154 (0.036)	0.149 (0.029)	0.339 (0.042)	0.373 (0.039)	0.391 (0.043)	0.339 (0.039)	0.345 (0.091)	0.41 (0.219)	0.544 (0.056)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.827 (0.09)	0.504 (0.107)	0.536 (0.094)	0.438 (0.094)	0.439 (0.096)	0.6 (0.296)	0.495 (0.117)	0.078 (0.078)	0.187 (0.074)
NASDAQ	0.779 (0.098)	1	0.399 (0.079)	0.437 (0.071)	0.321 (0.07)	0.375 (0.078)	0.547 (0.243)	0.4 (0.099)	0.002 (0.054)	0.14 (0.055)
IBEX	0.588 (0.124)	0.442 (0.093)	1	0.868 (0.14)	0.712 (0.137)	0.818 (0.153)	0.731 (0.332)	0.782 (0.171)	0.199 (0.098)	0.201 (0.076)
FCHI	0.631 (0.108)	0.469 (0.079)	0.882 (0.168)	1	0.796 (0.126)	0.845 (0.139)	0.848 (0.356)	0.881 (0.162)	0.257 (0.093)	0.27 (0.074)
SSMI	0.552 (0.097)	0.351 (0.071)	0.763 (0.159)	0.803 (0.132)	1	0.723 (0.143)	0.669 (0.343)	0.814 (0.177)	0.219 (0.096)	0.23 (0.082)
FTMIB	0.514 (0.101)	0.392 (0.076)	0.915 (0.18)	0.868 (0.138)	0.735 (0.125)	1	0.76 (0.345)	0.758 (0.162)	0.216 (0.101)	0.232 (0.08)
GDAX	0.573 (0.248)	0.436 (0.189)	0.796 (0.306)	0.917 (0.329)	0.779 (0.304)	0.813 (0.296)	1	0.731 (0.471)	0.182 (0.204)	0.237 (0.179)
FTSE	0.592 (0.147)	0.438 (0.096)	0.82 (0.208)	0.922 (0.181)	0.802 (0.145)	0.816 (0.173)	0.888 (0.43)	1	0.267 (0.148)	0.293 (0.118)
N225	0.169 (0.091)	0.088 (0.06)	0.295 (0.103)	0.327 (0.099)	0.324 (0.093)	0.323 (0.095)	0.251 (0.174)	0.323 (0.143)	1	0.453 (0.109)
HSI	0.204 (0.085)	0.186 (0.057)	0.373 (0.095)	0.37 (0.084)	0.352 (0.078)	0.376 (0.088)	0.325 (0.183)	0.365 (0.115)	0.553 (0.111)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.776 (0.14)	0.459 (0.155)	0.488 (0.152)	0.378 (0.129)	0.428 (0.156)	0.585 (0.489)	0.505 (0.2)	0.073 (0.124)	0.229 (0.134)
NASDAQ	0.778 (0.151)	1	0.373 (0.107)	0.371 (0.096)	0.28 (0.092)	0.362 (0.105)	0.514 (0.367)	0.39 (0.106)	-0.045 (0.072)	0.128 (0.084)
IBEX	0.543 (0.172)	0.431 (0.122)	1	0.884 (0.211)	0.726 (0.203)	0.837 (0.213)	0.722 (0.544)	0.791 (0.236)	0.201 (0.137)	0.188 (0.114)
FCHI	0.59 (0.159)	0.486 (0.113)	0.838 (0.21)	1	0.797 (0.194)	0.877 (0.193)	0.838 (0.575)	0.92 (0.226)	0.264 (0.132)	0.248 (0.109)
SSMI	0.516 (0.163)	0.351 (0.1)	0.73 (0.194)	0.767 (0.172)	1	0.692 (0.182)	0.622 (0.521)	0.796 (0.196)	0.194 (0.131)	0.193 (0.121)
FTMIB	0.51 (0.151)	0.423 (0.107)	0.848 (0.21)	0.887 (0.198)	0.74 (0.175)	1	0.728 (0.538)	0.766 (0.231)	0.221 (0.136)	0.253 (0.126)
GDAX	0.59 (0.396)	0.467 (0.292)	0.821 (0.464)	0.924 (0.509)	0.785 (0.461)	0.86 (0.47)	1	0.775 (0.804)	0.176 (0.327)	0.247 (0.282)
FTSE	0.616 (0.215)	0.438 (0.117)	0.781 (0.224)	0.897 (0.206)	0.755 (0.179)	0.796 (0.212)	0.851 (0.65)	1	0.329 (0.236)	0.312 (0.178)
N225	0.201 (0.151)	0.078 (0.09)	0.366 (0.171)	0.389 (0.16)	0.359 (0.154)	0.414 (0.157)	0.324 (0.308)	0.39 (0.23)	1	0.392 (0.156)
HSI	0.268 (0.116)	0.185 (0.076)	0.401 (0.134)	0.423 (0.116)	0.391 (0.113)	0.411 (0.119)	0.362 (0.277)	0.432 (0.14)	0.626 (0.18)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.749 (0.468)	0.392 (0.42)	0.387 (0.379)	0.366 (0.439)	0.508 (0.447)	0.739 (0.768)	0.444 (0.573)	0.123 (0.341)	0.372 (0.323)
NASDAQ	0.906 (0.443)	1	0.366 (0.198)	0.278 (0.17)	0.295 (0.179)	0.371 (0.187)	0.585 (0.396)	0.252 (0.135)	-0.048 (0.133)	0.16 (0.134)
IBEX	0.655 (0.399)	0.505 (0.217)	1	0.963 (0.41)	0.802 (0.402)	0.955 (0.422)	0.619 (0.527)	0.936 (0.425)	0.213 (0.275)	0.2 (0.189)
FCHI	0.675 (0.415)	0.597 (0.228)	0.88 (0.371)	1	0.815 (0.411)	0.961 (0.399)	0.704 (0.548)	0.951 (0.335)	0.244 (0.276)	0.207 (0.181)
SSMI	0.488 (0.381)	0.421 (0.213)	0.634 (0.325)	0.801 (0.386)	1	0.839 (0.401)	0.665 (0.565)	0.874 (0.371)	0.279 (0.338)	0.237 (0.204)
FTMIB	0.684 (0.428)	0.593 (0.27)	0.878 (0.401)	0.961 (0.465)	0.776 (0.424)	1	0.661 (0.577)	0.941 (0.337)	0.277 (0.313)	0.162 (0.179)
GDAX	0.71 (0.644)	0.624 (0.444)	0.846 (0.579)	0.974 (0.681)	0.834 (0.644)	0.966 (0.698)	1	0.696 (0.586)	0.189 (0.373)	0.406 (0.365)
FTSE	0.675 (0.563)	0.607 (0.18)	0.751 (0.3)	0.921 (0.298)	0.839 (0.375)	0.858 (0.394)	0.956 (0.749)	1	0.338 (0.39)	0.291 (0.222)
N225	0.019 (0.213)	-0.124 (0.113)	0.504 (0.286)	0.491 (0.297)	0.413 (0.287)	0.543 (0.32)	0.492 (0.45)	0.475 (0.301)	1	0.313 (0.276)
HSI	0.168 (0.234)	0.17 (0.133)	0.292 (0.2)	0.334 (0.204)	0.281 (0.227)	0.297 (0.226)	0.237 (0.284)	0.282 (0.171)	0.611 (0.272)	1

Expected Shortfall - Weekly Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.793 (0.136)	0.627 (0.118)	0.76 (0.106)	0.724 (0.088)	0.668 (0.085)	0.749 (0.086)	0.733 (0.079)	0.474 (0.06)	0.467 (0.093)
NASDAQ	0.784 (0.11)	1	0.577 (0.131)	0.705 (0.131)	0.615 (0.119)	0.59 (0.106)	0.719 (0.117)	0.641 (0.119)	0.473 (0.086)	0.519 (0.12)
IBEX	0.651 (0.11)	0.548 (0.105)	1	0.839 (0.153)	0.691 (0.133)	0.854 (0.133)	0.785 (0.129)	0.702 (0.125)	0.476 (0.094)	0.471 (0.108)
FCHI	0.762 (0.09)	0.7 (0.104)	0.872 (0.14)	1	0.792 (0.121)	0.876 (0.118)	0.917 (0.124)	0.807 (0.114)	0.512 (0.079)	0.536 (0.106)
SSMI	0.748 (0.078)	0.613 (0.092)	0.748 (0.12)	0.819 (0.101)	1	0.69 (0.091)	0.757 (0.094)	0.763 (0.088)	0.481 (0.067)	0.516 (0.097)
FTMIB	0.663 (0.066)	0.639 (0.083)	0.889 (0.116)	0.886 (0.096)	0.727 (0.078)	1	0.823 (0.094)	0.718 (0.087)	0.51 (0.067)	0.483 (0.088)
GDAX	0.759 (0.076)	0.678 (0.089)	0.8 (0.113)	0.902 (0.098)	0.787 (0.081)	0.825 (0.076)	1	0.776 (0.093)	0.52 (0.069)	0.566 (0.093)
FTSE	0.791 (0.071)	0.693 (0.098)	0.792 (0.119)	0.91 (0.1)	0.835 (0.079)	0.802 (0.074)	0.834 (0.08)	1	0.494 (0.059)	0.574 (0.099)
N225	0.538 (0.055)	0.561 (0.079)	0.553 (0.09)	0.61 (0.07)	0.587 (0.06)	0.558 (0.058)	0.567 (0.057)	0.642 (0.056)	1	0.52 (0.087)
HSI	0.54 (0.08)	0.471 (0.087)	0.45 (0.098)	0.577 (0.092)	0.503 (0.078)	0.514 (0.071)	0.522 (0.072)	0.625 (0.079)	0.588 (0.071)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.799 (0.208)	0.656 (0.213)	0.731 (0.144)	0.733 (0.121)	0.667 (0.103)	0.734 (0.105)	0.717 (0.115)	0.462 (0.075)	0.485 (0.105)
NASDAQ	0.772 (0.178)	1	0.55 (0.23)	0.683 (0.202)	0.577 (0.184)	0.577 (0.162)	0.727 (0.179)	0.602 (0.188)	0.429 (0.15)	0.513 (0.159)
IBEX	0.648 (0.167)	0.552 (0.18)	1	0.812 (0.28)	0.664 (0.239)	0.872 (0.261)	0.747 (0.218)	0.72 (0.246)	0.502 (0.188)	0.46 (0.176)
FCHI	0.75 (0.122)	0.639 (0.169)	0.867 (0.218)	1	0.77 (0.171)	0.853 (0.172)	0.893 (0.161)	0.835 (0.175)	0.552 (0.119)	0.555 (0.134)
SSMI	0.73 (0.109)	0.593 (0.158)	0.756 (0.188)	0.861 (0.144)	1	0.7 (0.13)	0.712 (0.13)	0.799 (0.14)	0.511 (0.098)	0.527 (0.127)
FTMIB	0.677 (0.093)	0.616 (0.14)	0.883 (0.188)	0.879 (0.133)	0.761 (0.11)	1	0.805 (0.122)	0.749 (0.128)	0.541 (0.088)	0.534 (0.112)
GDAX	0.719 (0.093)	0.656 (0.146)	0.78 (0.173)	0.914 (0.131)	0.819 (0.119)	0.822 (0.101)	1	0.794 (0.119)	0.494 (0.08)	0.548 (0.104)
FTSE	0.763 (0.106)	0.638 (0.163)	0.781 (0.188)	0.897 (0.146)	0.864 (0.124)	0.813 (0.102)	0.824 (0.105)	1	0.535 (0.092)	0.563 (0.119)
N225	0.539 (0.068)	0.542 (0.134)	0.504 (0.139)	0.587 (0.094)	0.608 (0.085)	0.546 (0.071)	0.545 (0.069)	0.585 (0.076)	1	0.521 (0.101)
HSI	0.572 (0.107)	0.476 (0.141)	0.505 (0.148)	0.6 (0.125)	0.539 (0.109)	0.529 (0.096)	0.548 (0.093)	0.609 (0.114)	0.606 (0.1)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.746 (0.798)	0.835 (0.675)	0.868 (0.43)	0.812 (0.73)	0.897 (0.321)	0.964 (0.338)	0.936 (0.31)	0.547 (0.216)	0.526 (0.227)
NASDAQ	0.81 (0.571)	1	0.545 (0.825)	0.724 (0.82)	0.564 (0.763)	0.608 (0.654)	0.723 (0.71)	0.699 (0.797)	0.496 (0.58)	0.441 (0.49)
IBEX	0.666 (0.35)	0.677 (0.523)	1	0.882 (0.798)	0.759 (0.883)	0.877 (0.659)	0.839 (0.638)	0.881 (0.689)	0.433 (0.439)	0.548 (0.485)
FCHI	0.645 (0.271)	0.551 (0.434)	0.89 (0.467)	1	0.751 (0.694)	0.851 (0.411)	0.904 (0.445)	0.905 (0.453)	0.489 (0.277)	0.621 (0.353)
SSMI	0.625 (0.259)	0.519 (0.426)	0.825 (0.449)	0.781 (0.355)	1	0.807 (0.658)	0.867 (0.651)	0.775 (0.665)	0.405 (0.46)	0.424 (0.412)
FTMIB	0.629 (0.173)	0.57 (0.366)	0.902 (0.378)	0.927 (0.309)	0.743 (0.271)	1	0.874 (0.31)	0.826 (0.291)	0.553 (0.209)	0.663 (0.258)
GDAX	0.676 (0.202)	0.609 (0.417)	0.915 (0.399)	0.942 (0.331)	0.797 (0.312)	0.84 (0.215)	1	0.936 (0.313)	0.506 (0.214)	0.539 (0.249)
FTSE	0.825 (0.205)	0.657 (0.487)	0.882 (0.39)	0.938 (0.318)	0.788 (0.3)	0.94 (0.211)	0.89 (0.229)	1	0.433 (0.182)	0.643 (0.267)
N225	0.575 (0.156)	0.533 (0.421)	0.678 (0.329)	0.584 (0.225)	0.556 (0.235)	0.58 (0.167)	0.592 (0.183)	0.575 (0.154)	1	0.466 (0.192)
HSI	0.581 (0.239)	0.464 (0.44)	0.518 (0.339)	0.556 (0.281)	0.405 (0.249)	0.498 (0.214)	0.54 (0.232)	0.556 (0.236)	0.534 (0.208)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.763 (0.842)	0.572 (0.521)	0.755 (0.51)	0.56 (0.433)	0.62 (0.314)	0.823 (0.384)	0.854 (0.361)	0.727 (0.272)	0.397 (0.23)
NASDAQ	0.887 (0.802)	1	0.47 (0.679)	0.649 (0.789)	0.367 (0.566)	0.434 (0.541)	0.615 (0.648)	0.7 (0.778)	0.601 (0.668)	0.548 (0.574)
IBEX	0.608 (0.43)	0.53 (0.669)	1	0.942 (0.918)	0.669 (0.73)	0.866 (0.691)	0.834 (0.651)	0.841 (0.695)	0.331 (0.412)	0.582 (0.53)
FCHI	0.622 (0.342)	0.573 (0.661)	0.895 (0.684)	1	0.668 (0.635)	0.912 (0.592)	0.924 (0.611)	0.91 (0.619)	0.532 (0.403)	0.681 (0.448)
SSMI	0.562 (0.389)	0.46 (0.641)	0.749 (0.609)	0.857 (0.614)	1	0.692 (0.53)	0.829 (0.555)	0.769 (0.51)	0.297 (0.34)	0.47 (0.377)
FTMIB	0.678 (0.251)	0.573 (0.55)	0.833 (0.514)	0.848 (0.425)	0.736 (0.439)	1	0.853 (0.423)	0.806 (0.39)	0.373 (0.243)	0.654 (0.329)
GDAX	0.625 (0.224)	0.53 (0.534)	0.852 (0.521)	0.961 (0.465)	0.92 (0.503)	0.832 (0.282)	1	0.935 (0.444)	0.484 (0.288)	0.636 (0.331)
FTSE	0.743 (0.278)	0.683 (0.698)	0.867 (0.585)	0.957 (0.504)	0.875 (0.522)	0.882 (0.341)	0.93 (0.335)	1	0.542 (0.256)	0.689 (0.323)
N225	0.646 (0.179)	0.451 (0.501)	0.836 (0.511)	0.63 (0.333)	0.588 (0.369)	0.608 (0.212)	0.687 (0.22)	0.655 (0.241)	1	0.315 (0.216)
HSI	0.511 (0.192)	0.496 (0.565)	0.499 (0.431)	0.446 (0.314)	0.317 (0.31)	0.486 (0.213)	0.488 (0.204)	0.43 (0.221)	0.544 (0.178)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.081 (1.412)	0.263 (0.685)	0.543 (0.776)	0.314 (0.733)	0.343 (0.433)	0.758 (0.643)	0.642 (0.588)	0.872 (0.557)	0.26 (0.333)
NASDAQ	0.083 (0.54)	1	0.197 (1.485)	0.411 (1.886)	-0.062 (0.818)	-0.07 (0.534)	0.185 (1.248)	0.235 (1.604)	0.113 (1.055)	0.533 (1.791)
IBEX	0.965 (0.746)	0.149 (0.533)	1	0.594 (1.001)	0.733 (1.234)	0.751 (0.858)	0.588 (0.882)	0.57 (0.823)	-0.01 (0.317)	0.88 (0.939)
FCHI	0.993 (0.698)	0.175 (0.508)	0.979 (0.932)	1	0.708 (1.076)	0.713 (0.701)	0.65 (0.751)	0.882 (0.845)	0.241 (0.384)	0.512 (0.536)
SSMI	0.993 (0.675)	0.177 (0.519)	0.979 (0.895)	1.0 (0.806)	1	0.999 (1.061)	0.684 (0.916)	0.661 (0.928)	-0.167 (0.278)	0.56 (0.737)
FTMIB	0.813 (0.484)	0.081 (0.399)	0.927 (0.743)	0.833 (0.599)	0.833 (0.615)	1	0.7 (0.564)	0.674 (0.524)	-0.132 (0.212)	0.572 (0.402)
GDAX	0.994 (0.544)	0.164 (0.473)	0.982 (0.804)	1.0 (0.706)	1.0 (0.69)	0.841 (0.506)	1	0.821 (0.62)	0.417 (0.393)	0.562 (0.482)
FTSE	0.996 (0.484)	0.152 (0.457)	0.977 (0.712)	1.0 (0.642)	0.999 (0.63)	0.829 (0.436)	1.0 (0.543)	1	0.355 (0.331)	0.399 (0.351)
N225	0.986 (0.384)	0.213 (0.407)	0.981 (0.586)	0.999 (0.527)	0.999 (0.507)	0.837 (0.375)	0.998 (0.391)	0.997 (0.345)	1	0.038 (0.193)
HSI	0.949 (0.47)	-0.0 (0.535)	0.927 (0.705)	0.953 (0.633)	0.953 (0.619)	0.763 (0.433)	0.952 (0.482)	0.953 (0.425)	0.95 (0.285)	1

Expected Shortfall - Monthly Returns - Theoretical Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.599 (0.274)	0.49 (0.146)	0.731 (0.216)	0.693 (0.191)	0.533 (0.189)	0.732 (0.215)	0.761 (0.243)	0.461 (0.133)	0.585 (0.178)
NASDAQ	0.83 (0.276)	1	0.536 (0.189)	0.691 (0.243)	0.449 (0.251)	0.519 (0.2)	0.678 (0.224)	0.558 (0.28)	0.489 (0.216)	0.612 (0.204)
IBEX	0.804 (0.194)	0.645 (0.167)	1	0.798 (0.158)	0.575 (0.139)	0.835 (0.172)	0.658 (0.171)	0.634 (0.16)	0.308 (0.103)	0.511 (0.12)
FCHI	0.867 (0.236)	0.74 (0.213)	0.909 (0.166)	1	0.7 (0.187)	0.831 (0.208)	0.855 (0.213)	0.852 (0.237)	0.387 (0.118)	0.564 (0.154)
SSMI	0.797 (0.188)	0.647 (0.221)	0.759 (0.136)	0.841 (0.185)	1	0.649 (0.187)	0.622 (0.181)	0.632 (0.17)	0.384 (0.119)	0.359 (0.151)
FTMIB	0.71 (0.207)	0.633 (0.173)	0.895 (0.168)	0.896 (0.199)	0.742 (0.165)	1	0.724 (0.198)	0.748 (0.226)	0.421 (0.148)	0.417 (0.145)
GDAX	0.894 (0.221)	0.823 (0.213)	0.818 (0.177)	0.93 (0.204)	0.86 (0.176)	0.816 (0.192)	1	0.662 (0.214)	0.434 (0.13)	0.606 (0.16)
FTSE	0.881 (0.231)	0.707 (0.225)	0.794 (0.158)	0.897 (0.197)	0.87 (0.164)	0.773 (0.186)	0.862 (0.199)	1	0.416 (0.14)	0.582 (0.188)
N225	0.735 (0.162)	0.694 (0.184)	0.684 (0.114)	0.73 (0.144)	0.656 (0.12)	0.659 (0.138)	0.734 (0.14)	0.678 (0.146)	1	0.366 (0.124)
HSI	0.652 (0.198)	0.685 (0.209)	0.548 (0.129)	0.583 (0.152)	0.442 (0.13)	0.519 (0.152)	0.635 (0.151)	0.61 (0.168)	0.584 (0.138)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.553 (0.341)	0.546 (0.222)	0.744 (0.266)	0.672 (0.271)	0.453 (0.187)	0.815 (0.343)	0.758 (0.315)	0.405 (0.178)	0.533 (0.211)
NASDAQ	0.666 (0.445)	1	0.506 (0.246)	0.697 (0.33)	0.292 (0.279)	0.525 (0.226)	0.701 (0.341)	0.506 (0.359)	0.386 (0.233)	0.473 (0.221)
IBEX	0.76 (0.252)	0.63 (0.328)	1	0.852 (0.238)	0.634 (0.198)	0.806 (0.206)	0.639 (0.222)	0.717 (0.244)	0.302 (0.145)	0.539 (0.156)
FCHI	0.781 (0.262)	0.689 (0.365)	0.833 (0.234)	1	0.736 (0.233)	0.827 (0.234)	0.888 (0.305)	0.798 (0.275)	0.372 (0.142)	0.602 (0.179)
SSMI	0.724 (0.236)	0.576 (0.385)	0.753 (0.202)	0.887 (0.221)	1	0.609 (0.172)	0.541 (0.261)	0.692 (0.233)	0.387 (0.134)	0.434 (0.174)
FTMIB	0.77 (0.231)	0.612 (0.314)	0.868 (0.202)	0.937 (0.239)	0.839 (0.172)	1	0.763 (0.25)	0.664 (0.214)	0.429 (0.146)	0.412 (0.136)
GDAX	0.835 (0.285)	0.752 (0.369)	0.858 (0.241)	0.93 (0.274)	0.861 (0.24)	0.914 (0.233)	1	0.729 (0.327)	0.343 (0.192)	0.56 (0.206)
FTSE	0.814 (0.28)	0.676 (0.438)	0.762 (0.222)	0.88 (0.258)	0.863 (0.203)	0.813 (0.205)	0.855 (0.27)	1	0.444 (0.148)	0.673 (0.226)
N225	0.725 (0.203)	0.698 (0.38)	0.687 (0.169)	0.702 (0.169)	0.65 (0.129)	0.702 (0.144)	0.727 (0.207)	0.72 (0.147)	1	0.326 (0.131)
HSI	0.594 (0.216)	0.669 (0.343)	0.51 (0.162)	0.612 (0.187)	0.427 (0.149)	0.566 (0.157)	0.587 (0.2)	0.601 (0.189)	0.628 (0.15)	1

$\alpha = 0.025/0.975$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.528 (0.473)	0.555 (0.328)	0.669 (0.472)	0.628 (0.47)	0.445 (0.26)	0.661 (0.52)	0.691 (0.533)	0.395 (0.375)	0.522 (0.319)
NASDAQ	0.589 (0.38)	1	0.538 (0.313)	0.664 (0.398)	0.1 (0.263)	0.585 (0.309)	0.71 (0.438)	0.444 (0.404)	0.311 (0.303)	0.305 (0.218)
IBEX	0.853 (0.289)	0.669 (0.327)	1	0.838 (0.287)	0.64 (0.212)	0.779 (0.228)	0.678 (0.254)	0.705 (0.262)	0.346 (0.177)	0.51 (0.174)
FCHI	0.873 (0.42)	0.69 (0.457)	0.893 (0.327)	1	0.712 (0.274)	0.89 (0.285)	0.884 (0.402)	0.822 (0.356)	0.445 (0.221)	0.511 (0.217)
SSMI	0.627 (0.261)	0.419 (0.307)	0.638 (0.165)	0.798 (0.347)	1	0.442 (0.175)	0.513 (0.303)	0.776 (0.258)	0.291 (0.162)	0.452 (0.218)
FTMIB	0.914 (0.301)	0.592 (0.326)	0.851 (0.221)	0.904 (0.364)	0.69 (0.186)	1	0.811 (0.267)	0.579 (0.229)	0.352 (0.163)	0.291 (0.142)
GDAX	0.796 (0.368)	0.602 (0.356)	0.879 (0.284)	0.937 (0.472)	0.694 (0.293)	0.915 (0.317)	1	0.83 (0.412)	0.324 (0.239)	0.371 (0.191)
FTSE	0.835 (0.335)	0.589 (0.386)	0.805 (0.227)	0.882 (0.395)	0.642 (0.185)	0.841 (0.262)	0.822 (0.346)	1	0.521 (0.211)	0.546 (0.257)
N225	0.67 (0.253)	0.636 (0.341)	0.773 (0.186)	0.712 (0.284)	0.625 (0.142)	0.721 (0.192)	0.676 (0.258)	0.795 (0.202)	1	0.191 (0.166)
HSI	0.451 (0.303)	0.52 (0.368)	0.572 (0.243)	0.623 (0.399)	0.391 (0.244)	0.568 (0.264)	0.559 (0.318)	0.587 (0.293)	0.681 (0.291)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.364 (0.54)	0.592 (0.387)	0.727 (0.5)	0.584 (0.377)	0.437 (0.317)	0.43 (0.452)	0.686 (0.458)	0.472 (0.31)	0.511 (0.346)
NASDAQ	0.491 (0.469)	1	0.518 (0.492)	0.585 (0.571)	-0.02 (0.374)	0.634 (0.503)	0.601 (0.628)	0.361 (0.607)	0.144 (0.404)	0.046 (0.232)
IBEX	0.986 (0.394)	0.491 (0.356)	1	0.866 (0.467)	0.626 (0.359)	0.886 (0.369)	0.671 (0.466)	0.789 (0.42)	0.554 (0.289)	0.37 (0.241)
FCHI	0.856 (0.487)	0.392 (0.419)	0.875 (0.391)	1	0.725 (0.428)	0.875 (0.416)	0.807 (0.555)	0.905 (0.553)	0.536 (0.359)	0.34 (0.264)
SSMI	0.558 (0.277)	0.308 (0.343)	0.498 (0.224)	0.548 (0.282)	1	0.461 (0.313)	0.522 (0.423)	0.84 (0.403)	0.295 (0.234)	0.609 (0.36)
FTMIB	0.875 (0.374)	0.363 (0.335)	0.898 (0.294)	0.963 (0.406)	0.453 (0.216)	1	0.695 (0.431)	0.667 (0.385)	0.465 (0.27)	0.095 (0.173)
GDAX	0.775 (0.448)	0.373 (0.356)	0.803 (0.365)	0.867 (0.465)	0.628 (0.318)	0.869 (0.381)	1	0.788 (0.572)	0.342 (0.361)	0.107 (0.216)
FTSE	0.784 (0.396)	0.316 (0.426)	0.813 (0.295)	0.88 (0.397)	0.423 (0.185)	0.888 (0.3)	0.732 (0.378)	1	0.555 (0.327)	0.533 (0.385)
N225	0.791 (0.347)	0.551 (0.411)	0.805 (0.235)	0.668 (0.295)	0.507 (0.177)	0.69 (0.225)	0.493 (0.261)	0.755 (0.245)	1	-0.083 (0.213)
HSI	0.478 (0.308)	0.397 (0.367)	0.565 (0.262)	0.593 (0.319)	0.373 (0.224)	0.605 (0.272)	0.505 (0.286)	0.721 (0.331)	0.813 (0.292)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.155 (0.488)	0.778 (0.476)	0.759 (0.605)	0.441 (0.35)	0.718 (0.492)	0.159 (0.426)	0.703 (0.499)	0.641 (0.371)	0.478 (0.365)
NASDAQ	0.595 (0.439)	1	0.252 (0.389)	0.278 (0.477)	-0.058 (0.428)	0.455 (0.509)	0.624 (0.719)	0.214 (0.588)	0.085 (0.43)	-0.14 (0.178)
IBEX	0.788 (0.405)	0.538 (0.345)	1	0.995 (0.53)	0.518 (0.354)	0.942 (0.387)	0.485 (0.421)	0.967 (0.499)	0.906 (0.424)	0.278 (0.24)
FCHI	0.743 (0.508)	0.292 (0.354)	0.799 (0.41)	1	0.469 (0.419)	0.958 (0.546)	0.455 (0.503)	0.956 (0.652)	0.907 (0.533)	0.236 (0.263)
SSMI	0.707 (0.321)	0.63 (0.352)	0.704 (0.272)	0.383 (0.281)	1	0.375 (0.352)	0.551 (0.553)	0.46 (0.329)	0.329 (0.208)	0.73 (0.411)
FTMIB	0.623 (0.345)	0.053 (0.213)	0.82 (0.263)	0.856 (0.417)	0.276 (0.214)	1	0.532 (0.486)	0.854 (0.478)	0.897 (0.42)	0.054 (0.209)
GDAX	0.531 (0.407)	0.099 (0.241)	0.436 (0.296)	0.832 (0.513)	0.302 (0.273)	0.51 (0.31)	1	0.454 (0.526)	0.375 (0.448)	0.082 (0.252)
FTSE	0.776 (0.44)	0.252 (0.351)	0.868 (0.381)	0.942 (0.54)	0.444 (0.212)	0.904 (0.387)	0.675 (0.425)	1	0.84 (0.387)	0.313 (0.309)
N225	0.726 (0.276)	0.517 (0.272)	0.494 (0.191)	0.53 (0.243)	0.139 (0.106)	0.471 (0.185)	0.183 (0.179)	0.587 (0.209)	1	-0.068 (0.232)
HSI	0.693 (0.372)	0.495 (0.338)	0.466 (0.24)	0.514 (0.315)	0.089 (0.165)	0.465 (0.243)	0.164 (0.211)	0.565 (0.309)	0.998 (0.26)	1

Value-at-Risk - Daily Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.804 (0.143)	0.418 (0.136)	0.515 (0.15)	0.418 (0.136)	0.501 (0.14)	0.48 (0.191)	0.472 (0.177)	0.091 (0.084)	0.112 (0.087)
NASDAQ	0.925 (0.028)	1	0.432 (0.037)	0.492 (0.041)	0.48 (0.043)	0.454 (0.039)	0.495 (0.058)	0.497 (0.175)	0.148 (0.038)	0.168 (0.037)
IBEX	0.494 (0.033)	0.409 (0.039)	1	0.864 (0.036)	0.676 (0.045)	0.826 (0.037)	0.825 (0.054)	0.735 (0.228)	0.201 (0.042)	0.286 (0.04)
FCHI	0.584 (0.035)	0.483 (0.04)	0.916 (0.036)	1	0.783 (0.041)	0.944 (0.033)	0.89 (0.055)	0.85 (0.245)	0.207 (0.044)	0.364 (0.045)
SSMI	0.443 (0.033)	0.346 (0.039)	0.672 (0.04)	0.75 (0.04)	1	0.695 (0.04)	0.74 (0.057)	0.681 (0.256)	0.166 (0.04)	0.258 (0.045)
FTMIB	0.523 (0.032)	0.45 (0.042)	0.907 (0.035)	0.905 (0.034)	0.714 (0.039)	1	0.842 (0.053)	0.768 (0.226)	0.203 (0.039)	0.338 (0.043)
GDAX	0.512 (0.02)	0.474 (0.042)	0.846 (0.038)	0.891 (0.036)	0.723 (0.039)	0.836 (0.039)	1	0.815 (0.183)	0.227 (0.067)	0.373 (0.067)
FTSE	0.488 (0.098)	0.468 (0.161)	0.782 (0.204)	0.848 (0.212)	0.736 (0.184)	0.805 (0.206)	0.816 (0.162)	1	0.208 (0.176)	0.366 (0.176)
N225	0.175 (0.034)	0.175 (0.036)	0.276 (0.038)	0.309 (0.042)	0.316 (0.043)	0.211 (0.041)	0.248 (0.043)	0.37 (0.16)	1	0.474 (0.042)
HSI	0.197 (0.033)	0.218 (0.035)	0.365 (0.043)	0.441 (0.042)	0.332 (0.045)	0.375 (0.043)	0.367 (0.041)	0.423 (0.159)	0.506 (0.041)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.727 (0.14)	0.508 (0.123)	0.531 (0.134)	0.463 (0.121)	0.502 (0.128)	0.61 (0.213)	0.495 (0.194)	0.133 (0.086)	0.128 (0.084)
NASDAQ	0.848 (0.021)	1	0.513 (0.039)	0.545 (0.043)	0.403 (0.045)	0.465 (0.042)	0.58 (0.058)	0.382 (0.18)	0.116 (0.039)	0.165 (0.039)
IBEX	0.495 (0.024)	0.38 (0.045)	1	0.951 (0.039)	0.78 (0.044)	0.91 (0.04)	0.842 (0.058)	0.8 (0.213)	0.248 (0.045)	0.343 (0.041)
FCHI	0.559 (0.026)	0.443 (0.042)	0.857 (0.04)	1	0.75 (0.045)	0.868 (0.036)	0.877 (0.063)	0.837 (0.236)	0.2 (0.044)	0.394 (0.04)
SSMI	0.555 (0.025)	0.423 (0.043)	0.749 (0.048)	0.864 (0.045)	1	0.721 (0.043)	0.775 (0.056)	0.744 (0.238)	0.222 (0.047)	0.373 (0.047)
FTMIB	0.544 (0.023)	0.433 (0.046)	0.89 (0.042)	0.869 (0.041)	0.805 (0.046)	1	0.719 (0.061)	0.73 (0.216)	0.149 (0.041)	0.28 (0.039)
GDAX	0.583 (0.026)	0.563 (0.047)	0.786 (0.047)	0.892 (0.043)	0.814 (0.049)	0.804 (0.045)	1	0.814 (0.294)	0.175 (0.058)	0.339 (0.054)
FTSE	0.551 (0.097)	0.446 (0.132)	0.747 (0.171)	0.888 (0.18)	0.835 (0.156)	0.816 (0.171)	0.765 (0.128)	1	0.221 (0.151)	0.39 (0.145)
N225	0.111 (0.02)	0.141 (0.038)	0.275 (0.045)	0.376 (0.047)	0.347 (0.044)	0.3 (0.046)	0.33 (0.048)	0.355 (0.131)	1	0.434 (0.048)
HSI	0.141 (0.02)	0.094 (0.039)	0.338 (0.043)	0.387 (0.046)	0.382 (0.049)	0.314 (0.043)	0.387 (0.046)	0.431 (0.136)	0.533 (0.045)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.957 (0.136)	0.578 (0.111)	0.62 (0.114)	0.478 (0.125)	0.542 (0.11)	0.719 (0.169)	0.525 (0.223)	0.086 (0.08)	0.12 (0.087)
NASDAQ	0.851 (0.163)	1	0.481 (0.078)	0.617 (0.069)	0.41 (0.067)	0.474 (0.074)	0.678 (0.078)	0.427 (0.178)	0.132 (0.056)	0.208 (0.058)
IBEX	0.662 (0.199)	0.498 (0.064)	1	0.836 (0.069)	0.687 (0.079)	0.836 (0.07)	0.709 (0.085)	0.754 (0.172)	0.19 (0.065)	0.234 (0.066)
FCHI	0.588 (0.2)	0.499 (0.069)	0.952 (0.067)	1	0.774 (0.072)	0.846 (0.065)	0.859 (0.083)	0.878 (0.172)	0.195 (0.062)	0.258 (0.063)
SSMI	0.572 (0.189)	0.388 (0.07)	0.755 (0.075)	0.868 (0.072)	1	0.808 (0.073)	0.813 (0.08)	0.771 (0.182)	0.238 (0.067)	0.188 (0.064)
FTMIB	0.621 (0.193)	0.438 (0.069)	0.96 (0.067)	0.89 (0.066)	0.683 (0.072)	1	0.869 (0.084)	0.707 (0.171)	0.216 (0.063)	0.218 (0.062)
GDAX	0.56 (0.204)	0.514 (0.089)	0.841 (0.1)	0.859 (0.103)	0.708 (0.099)	0.76 (0.096)	1	0.748 (0.235)	0.238 (0.075)	0.254 (0.078)
FTSE	0.492 (0.213)	0.385 (0.096)	0.884 (0.086)	0.952 (0.103)	0.826 (0.093)	0.845 (0.078)	0.823 (0.061)	1	0.214 (0.126)	0.2 (0.138)
N225	0.116 (0.128)	0.156 (0.066)	0.22 (0.063)	0.277 (0.066)	0.315 (0.068)	0.212 (0.063)	0.221 (0.067)	0.25 (0.063)	1	0.525 (0.07)
HSI	0.149 (0.126)	0.199 (0.054)	0.239 (0.069)	0.272 (0.069)	0.287 (0.066)	0.266 (0.066)	0.319 (0.074)	0.3 (0.072)	0.421 (0.076)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.902 (0.146)	0.575 (0.111)	0.633 (0.117)	0.413 (0.133)	0.367 (0.109)	0.512 (0.146)	0.45 (0.242)	0.025 (0.083)	0.062 (0.087)
NASDAQ	0.757 (0.159)	1	0.402 (0.092)	0.604 (0.093)	0.298 (0.097)	0.399 (0.092)	0.557 (0.094)	0.487 (0.152)	0.027 (0.082)	0.137 (0.074)
IBEX	0.607 (0.177)	0.424 (0.088)	1	0.875 (0.101)	0.62 (0.108)	0.662 (0.095)	0.732 (0.119)	0.785 (0.151)	0.181 (0.077)	0.232 (0.083)
FCHI	0.658 (0.175)	0.404 (0.087)	0.873 (0.084)	1	0.706 (0.1)	0.794 (0.088)	0.881 (0.107)	0.846 (0.154)	0.405 (0.082)	0.407 (0.081)
SSMI	0.644 (0.164)	0.493 (0.083)	0.873 (0.099)	0.796 (0.094)	1	0.667 (0.098)	0.543 (0.104)	0.722 (0.167)	0.19 (0.084)	0.219 (0.082)
FTMIB	0.457 (0.156)	0.406 (0.086)	0.969 (0.086)	0.825 (0.091)	0.868 (0.09)	1	0.676 (0.111)	0.688 (0.141)	0.193 (0.076)	0.114 (0.074)
GDAX	0.576 (0.196)	0.301 (0.095)	0.715 (0.12)	0.858 (0.111)	0.752 (0.114)	0.691 (0.116)	1	0.731 (0.201)	0.188 (0.092)	0.211 (0.089)
FTSE	0.66 (0.253)	0.452 (0.07)	0.845 (0.062)	0.868 (0.063)	0.834 (0.065)	0.821 (0.059)	0.95 (0.068)	1	0.281 (0.134)	0.266 (0.137)
N225	0.179 (0.16)	0.098 (0.067)	0.173 (0.08)	0.155 (0.081)	0.252 (0.083)	0.178 (0.069)	0.063 (0.078)	0.146 (0.052)	1	0.541 (0.079)
HSI	0.122 (0.152)	0.17 (0.069)	0.336 (0.079)	0.323 (0.079)	0.398 (0.075)	0.422 (0.074)	0.31 (0.085)	0.28 (0.054)	0.475 (0.08)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.691 (0.174)	0.309 (0.155)	0.323 (0.149)	0.338 (0.168)	0.505 (0.146)	0.364 (0.168)	0.436 (0.247)	0.139 (0.141)	0.331 (0.137)
NASDAQ	0.755 (0.184)	1	0.326 (0.158)	0.233 (0.148)	0.36 (0.15)	0.428 (0.163)	0.505 (0.155)	0.334 (0.188)	0.15 (0.114)	0.128 (0.128)
IBEX	0.379 (0.184)	0.272 (0.163)	1	0.838 (0.158)	0.804 (0.181)	0.685 (0.146)	0.529 (0.203)	0.798 (0.15)	0.198 (0.129)	0.324 (0.142)
FCHI	0.53 (0.164)	0.46 (0.145)	0.773 (0.14)	1	0.794 (0.179)	0.839 (0.153)	0.682 (0.175)	0.868 (0.169)	0.063 (0.13)	0.128 (0.143)
SSMI	0.562 (0.161)	0.215 (0.156)	0.634 (0.177)	0.821 (0.162)	1	0.864 (0.174)	0.682 (0.181)	0.802 (0.162)	0.229 (0.137)	0.147 (0.132)
FTMIB	0.485 (0.185)	0.516 (0.158)	0.573 (0.147)	0.718 (0.133)	0.574 (0.16)	1	0.814 (0.201)	0.926 (0.154)	0.417 (0.125)	0.282 (0.145)
GDAX	0.383 (0.151)	0.359 (0.159)	0.673 (0.182)	0.946 (0.173)	0.797 (0.163)	0.772 (0.173)	1	0.78 (0.188)	0.143 (0.14)	0.359 (0.147)
FTSE	0.54 (0.103)	0.512 (0.065)	0.728 (0.076)	0.902 (0.064)	0.758 (0.071)	0.9 (0.072)	0.865 (0.075)	1	0.358 (0.155)	0.526 (0.168)
N225	0.316 (0.156)	0.158 (0.115)	0.768 (0.154)	0.702 (0.129)	0.534 (0.13)	0.813 (0.138)	0.725 (0.134)	0.761 (0.068)	1	0.329 (0.158)
HSI	0.183 (0.145)	-0.034 (0.099)	0.44 (0.129)	0.368 (0.12)	0.26 (0.122)	0.666 (0.129)	0.356 (0.119)	0.576 (0.063)	0.829 (0.165)	1

Value-at-Risk - Weekly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.864 (0.073)	0.61 (0.083)	0.825 (0.071)	0.653 (0.082)	0.757 (0.073)	0.775 (0.075)	0.856 (0.078)	0.525 (0.076)	0.444 (0.081)
NASDAQ	0.685 (0.073)	1	0.667 (0.092)	0.77 (0.078)	0.638 (0.09)	0.677 (0.081)	0.776 (0.077)	0.853 (0.085)	0.53 (0.082)	0.624 (0.083)
IBEX	0.763 (0.084)	0.591 (0.093)	1	0.889 (0.066)	0.696 (0.082)	0.809 (0.072)	0.842 (0.08)	0.633 (0.082)	0.378 (0.077)	0.483 (0.075)
FCHI	0.873 (0.07)	0.624 (0.077)	0.891 (0.069)	1	0.86 (0.071)	0.88 (0.058)	0.896 (0.052)	0.861 (0.064)	0.459 (0.075)	0.497 (0.078)
SSMI	0.766 (0.081)	0.548 (0.098)	0.704 (0.084)	0.692 (0.064)	1	0.667 (0.088)	0.824 (0.07)	0.729 (0.079)	0.412 (0.079)	0.537 (0.084)
FTMIB	0.715 (0.083)	0.56 (0.1)	0.834 (0.062)	0.772 (0.063)	0.656 (0.082)	1	0.889 (0.066)	0.688 (0.076)	0.435 (0.068)	0.41 (0.067)
GDAX	0.891 (0.068)	0.692 (0.079)	0.834 (0.066)	0.879 (0.045)	0.681 (0.071)	0.916 (0.062)	1	0.751 (0.074)	0.565 (0.077)	0.61 (0.075)
FTSE	0.824 (0.073)	0.718 (0.085)	0.757 (0.08)	0.908 (0.056)	0.655 (0.074)	0.664 (0.076)	0.86 (0.062)	1	0.439 (0.08)	0.594 (0.08)
N225	0.576 (0.092)	0.602 (0.098)	0.786 (0.086)	0.759 (0.084)	0.489 (0.091)	0.591 (0.087)	0.634 (0.085)	0.761 (0.082)	1	0.561 (0.071)
HSI	0.488 (0.088)	0.572 (0.084)	0.413 (0.088)	0.613 (0.074)	0.483 (0.094)	0.479 (0.081)	0.536 (0.079)	0.693 (0.082)	0.621 (0.085)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.818 (0.093)	0.588 (0.096)	0.771 (0.086)	0.692 (0.092)	0.66 (0.087)	0.717 (0.079)	0.744 (0.084)	0.527 (0.084)	0.445 (0.097)
NASDAQ	0.83 (0.085)	1	0.616 (0.103)	0.69 (0.098)	0.67 (0.112)	0.613 (0.093)	0.723 (0.092)	0.688 (0.108)	0.556 (0.095)	0.529 (0.095)
IBEX	0.52 (0.09)	0.535 (0.103)	1	0.84 (0.077)	0.67 (0.096)	0.923 (0.081)	0.771 (0.084)	0.613 (0.092)	0.542 (0.09)	0.476 (0.089)
FCHI	0.66 (0.088)	0.815 (0.096)	0.801 (0.075)	1	0.702 (0.083)	0.848 (0.068)	0.871 (0.059)	0.688 (0.075)	0.534 (0.091)	0.556 (0.09)
SSMI	0.65 (0.084)	0.758 (0.106)	0.652 (0.097)	0.908 (0.081)	1	0.632 (0.085)	0.699 (0.083)	0.797 (0.086)	0.435 (0.084)	0.476 (0.094)
FTMIB	0.633 (0.087)	0.736 (0.095)	0.836 (0.076)	0.942 (0.064)	0.737 (0.095)	1	0.883 (0.075)	0.713 (0.088)	0.533 (0.084)	0.445 (0.086)
GDAX	0.723 (0.085)	0.759 (0.09)	0.753 (0.078)	0.844 (0.063)	0.798 (0.081)	0.806 (0.076)	1	0.746 (0.082)	0.518 (0.077)	0.603 (0.084)
FTSE	0.621 (0.082)	0.739 (0.094)	0.702 (0.087)	0.805 (0.067)	0.814 (0.076)	0.672 (0.081)	0.606 (0.08)	1	0.57 (0.088)	0.588 (0.088)
N225	0.414 (0.092)	0.678 (0.109)	0.479 (0.093)	0.634 (0.09)	0.557 (0.091)	0.627 (0.092)	0.621 (0.081)	0.656 (0.088)	1	0.535 (0.079)
HSI	0.323 (0.094)	0.45 (0.104)	0.257 (0.091)	0.524 (0.091)	0.386 (0.094)	0.531 (0.086)	0.475 (0.083)	0.443 (0.091)	0.509 (0.092)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.749 (0.151)	0.745 (0.148)	0.806 (0.127)	0.695 (0.153)	0.666 (0.127)	0.713 (0.127)	0.583 (0.139)	0.339 (0.124)	0.779 (0.131)
NASDAQ	0.725 (0.153)	1	0.567 (0.162)	0.705 (0.162)	0.655 (0.171)	0.616 (0.147)	0.608 (0.162)	0.528 (0.162)	0.345 (0.135)	0.537 (0.148)
IBEX	0.652 (0.156)	0.618 (0.181)	1	0.836 (0.128)	0.863 (0.151)	0.762 (0.134)	0.747 (0.147)	0.787 (0.148)	0.546 (0.124)	0.504 (0.148)
FCHI	0.52 (0.142)	0.441 (0.167)	0.724 (0.123)	1	0.778 (0.142)	0.867 (0.113)	0.939 (0.104)	0.774 (0.137)	0.561 (0.125)	0.585 (0.146)
SSMI	0.829 (0.153)	0.646 (0.188)	0.805 (0.171)	0.671 (0.142)	1	0.679 (0.15)	0.734 (0.136)	0.642 (0.143)	0.684 (0.137)	0.62 (0.142)
FTMIB	0.701 (0.148)	0.642 (0.182)	0.739 (0.135)	0.722 (0.116)	0.847 (0.141)	1	0.777 (0.111)	0.654 (0.134)	0.742 (0.12)	0.658 (0.141)
GDAX	0.803 (0.151)	0.588 (0.164)	0.728 (0.147)	0.581 (0.114)	0.676 (0.138)	0.824 (0.13)	1	0.858 (0.124)	0.606 (0.113)	0.607 (0.142)
FTSE	0.694 (0.143)	0.583 (0.167)	0.884 (0.16)	0.752 (0.128)	0.685 (0.137)	0.721 (0.133)	0.778 (0.12)	1	0.424 (0.122)	0.445 (0.137)
N225	0.506 (0.145)	0.523 (0.159)	0.551 (0.153)	0.483 (0.14)	0.78 (0.162)	0.831 (0.14)	0.453 (0.139)	0.546 (0.151)	1	0.602 (0.129)
HSI	0.631 (0.141)	0.605 (0.16)	0.668 (0.169)	0.746 (0.155)	0.562 (0.174)	0.711 (0.138)	0.69 (0.138)	0.914 (0.138)	0.59 (0.133)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.806 (0.18)	0.859 (0.185)	0.934 (0.168)	0.849 (0.198)	0.832 (0.18)	0.88 (0.175)	0.949 (0.174)	0.364 (0.163)	0.545 (0.165)
NASDAQ	0.819 (0.179)	1	0.603 (0.188)	0.746 (0.192)	0.824 (0.21)	0.77 (0.189)	0.613 (0.198)	0.617 (0.179)	0.462 (0.18)	0.246 (0.171)
IBEX	0.785 (0.201)	0.906 (0.218)	1	0.902 (0.165)	0.75 (0.199)	0.726 (0.154)	0.724 (0.173)	0.847 (0.176)	0.516 (0.153)	0.533 (0.181)
FCHI	0.621 (0.189)	0.592 (0.188)	0.807 (0.136)	1	0.684 (0.165)	0.743 (0.138)	0.825 (0.136)	0.836 (0.159)	0.423 (0.156)	0.623 (0.175)
SSMI	0.494 (0.191)	0.653 (0.227)	0.741 (0.216)	0.812 (0.195)	1	0.83 (0.181)	0.66 (0.171)	0.829 (0.186)	0.462 (0.182)	0.226 (0.181)
FTMIB	0.66 (0.193)	0.615 (0.194)	0.817 (0.163)	0.925 (0.137)	0.697 (0.203)	1	0.719 (0.143)	0.762 (0.168)	0.758 (0.154)	0.624 (0.168)
GDAX	0.544 (0.176)	0.631 (0.191)	0.831 (0.165)	0.695 (0.144)	0.789 (0.19)	0.654 (0.15)	1	0.876 (0.177)	0.259 (0.15)	0.439 (0.161)
FTSE	0.845 (0.174)	0.733 (0.188)	0.871 (0.176)	0.733 (0.152)	0.463 (0.203)	0.828 (0.147)	0.673 (0.145)	1	0.255 (0.15)	0.502 (0.167)
N225	0.542 (0.191)	0.662 (0.186)	0.518 (0.201)	0.245 (0.197)	0.351 (0.212)	0.322 (0.188)	0.286 (0.183)	0.277 (0.208)	1	0.515 (0.154)
HSI	0.775 (0.174)	0.34 (0.195)	0.414 (0.186)	0.469 (0.183)	0.327 (0.218)	0.52 (0.171)	0.42 (0.19)	0.581 (0.168)	0.376 (0.161)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.527 (0.248)	0.321 (0.246)	0.517 (0.24)	0.432 (0.248)	0.233 (0.243)	0.584 (0.228)	0.639 (0.235)	0.76 (0.244)	0.351 (0.199)
NASDAQ	0.68 (0.25)	1	0.43 (0.265)	0.716 (0.249)	0.358 (0.261)	0.342 (0.241)	0.708 (0.246)	0.6 (0.271)	0.629 (0.219)	0.455 (0.211)
IBEX	0.242 (0.271)	0.304 (0.263)	1	0.841 (0.209)	0.78 (0.262)	0.848 (0.213)	0.794 (0.229)	0.533 (0.238)	0.422 (0.224)	0.655 (0.231)
FCHI	0.258 (0.263)	0.36 (0.262)	0.996 (0.173)	1	0.819 (0.243)	0.682 (0.198)	0.859 (0.186)	0.696 (0.224)	0.564 (0.253)	0.682 (0.229)
SSMI	0.208 (0.301)	0.379 (0.272)	0.951 (0.297)	0.973 (0.276)	1	0.639 (0.249)	0.747 (0.237)	0.756 (0.269)	0.273 (0.26)	0.782 (0.216)
FTMIB	0.366 (0.278)	0.425 (0.259)	0.977 (0.208)	0.97 (0.181)	0.894 (0.303)	1	0.735 (0.213)	0.277 (0.234)	0.271 (0.25)	0.838 (0.236)
GDAX	0.262 (0.267)	0.366 (0.267)	0.995 (0.24)	1.0 (0.198)	0.973 (0.285)	0.971 (0.231)	1	0.806 (0.232)	0.735 (0.23)	0.791 (0.221)
FTSE	0.32 (0.249)	0.591 (0.25)	0.871 (0.226)	0.912 (0.192)	0.964 (0.302)	0.841 (0.225)	0.914 (0.226)	1	0.656 (0.237)	0.538 (0.209)
N225	0.179 (0.288)	0.305 (0.252)	0.995 (0.286)	0.996 (0.293)	0.968 (0.318)	0.962 (0.32)	0.996 (0.308)	0.894 (0.304)	1	0.239 (0.195)
HSI	-0.014 (0.234)	0.576 (0.251)	0.128 (0.26)	0.141 (0.258)	0.074 (0.314)	0.26 (0.28)	0.144 (0.288)	0.197 (0.252)	0.152 (0.259)	1

Value-at-Risk - Monthly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.861 (0.125)	0.711 (0.137)	0.788 (0.117)	0.504 (0.126)	0.664 (0.133)	0.715 (0.117)	0.798 (0.121)	0.775 (0.136)	0.76 (0.15)
NASDAQ	0.844 (0.105)	1	0.811 (0.146)	0.837 (0.122)	0.677 (0.135)	0.688 (0.131)	0.763 (0.123)	0.851 (0.131)	0.602 (0.131)	0.807 (0.137)
IBEX	0.524 (0.163)	0.585 (0.157)	1	0.857 (0.128)	0.567 (0.137)	0.776 (0.121)	0.784 (0.109)	0.884 (0.117)	0.492 (0.129)	0.368 (0.172)
FCHI	0.658 (0.125)	0.747 (0.129)	0.78 (0.113)	1	0.739 (0.104)	0.907 (0.085)	0.773 (0.084)	0.868 (0.091)	0.515 (0.12)	0.486 (0.151)
SSMI	0.54 (0.147)	0.437 (0.14)	0.402 (0.131)	0.586 (0.127)	1	0.495 (0.118)	0.821 (0.102)	0.609 (0.105)	0.496 (0.105)	0.35 (0.144)
FTMIB	0.458 (0.174)	0.514 (0.164)	0.838 (0.103)	0.808 (0.09)	0.636 (0.127)	1	0.546 (0.117)	0.685 (0.129)	0.337 (0.149)	0.426 (0.142)
GDAX	0.824 (0.118)	0.855 (0.107)	0.792 (0.103)	0.786 (0.086)	0.627 (0.127)	0.662 (0.12)	1	0.691 (0.108)	0.826 (0.114)	0.424 (0.148)
FTSE	0.803 (0.117)	0.692 (0.145)	0.571 (0.129)	0.797 (0.122)	0.812 (0.162)	0.589 (0.154)	0.762 (0.109)	1	0.427 (0.119)	0.456 (0.159)
N225	0.388 (0.169)	0.642 (0.146)	0.567 (0.147)	0.441 (0.143)	0.452 (0.152)	0.611 (0.148)	0.546 (0.131)	0.451 (0.148)	1	0.561 (0.129)
HSI	0.47 (0.147)	0.303 (0.161)	0.537 (0.176)	0.399 (0.163)	0.295 (0.182)	0.384 (0.154)	0.329 (0.154)	0.629 (0.127)	0.451 (0.164)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.527 (0.139)	0.369 (0.15)	0.659 (0.127)	0.792 (0.146)	0.484 (0.147)	0.562 (0.139)	0.64 (0.127)	0.327 (0.114)	0.46 (0.165)
NASDAQ	0.914 (0.099)	1	0.615 (0.144)	0.671 (0.141)	0.525 (0.149)	0.456 (0.136)	0.516 (0.138)	0.42 (0.145)	0.464 (0.133)	0.612 (0.148)
IBEX	0.791 (0.144)	0.676 (0.163)	1	0.781 (0.123)	0.609 (0.141)	0.819 (0.109)	0.65 (0.132)	0.509 (0.132)	0.271 (0.13)	0.431 (0.153)
FCHI	0.914 (0.124)	0.734 (0.123)	0.847 (0.098)	1	0.498 (0.123)	0.596 (0.111)	0.643 (0.097)	0.597 (0.107)	0.263 (0.123)	0.571 (0.159)
SSMI	0.918 (0.115)	0.731 (0.13)	0.81 (0.13)	0.882 (0.124)	1	0.662 (0.144)	0.586 (0.127)	0.463 (0.123)	0.375 (0.11)	0.283 (0.162)
FTMIB	0.52 (0.164)	0.517 (0.162)	0.769 (0.094)	0.615 (0.1)	0.477 (0.139)	1	0.641 (0.143)	0.779 (0.111)	0.517 (0.139)	0.437 (0.167)
GDAX	0.886 (0.113)	0.713 (0.122)	0.749 (0.143)	0.901 (0.079)	0.842 (0.13)	0.667 (0.142)	1	0.425 (0.129)	0.38 (0.12)	0.867 (0.15)
FTSE	0.878 (0.111)	0.647 (0.14)	0.866 (0.128)	0.885 (0.101)	0.946 (0.12)	0.485 (0.133)	0.847 (0.121)	1	0.275 (0.136)	0.459 (0.176)
N225	0.77 (0.135)	0.702 (0.126)	0.565 (0.142)	0.858 (0.132)	0.6 (0.126)	0.456 (0.144)	0.705 (0.124)	0.577 (0.162)	1	0.316 (0.142)
HSI	0.801 (0.144)	0.755 (0.141)	0.508 (0.145)	0.726 (0.153)	0.611 (0.147)	0.53 (0.177)	0.89 (0.152)	0.592 (0.125)	0.665 (0.163)	1

$\alpha = 0.025/0.975$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.575 (0.162)	0.66 (0.163)	0.826 (0.159)	0.582 (0.151)	0.451 (0.194)	0.91 (0.156)	0.728 (0.157)	0.41 (0.143)	0.638 (0.185)
NASDAQ	0.537 (0.141)	1	0.386 (0.17)	0.7 (0.169)	0.47 (0.171)	0.498 (0.175)	0.749 (0.162)	0.563 (0.167)	0.468 (0.156)	0.738 (0.175)
IBEX	0.466 (0.146)	0.441 (0.181)	1	0.814 (0.151)	0.874 (0.153)	0.601 (0.128)	0.554 (0.147)	0.83 (0.15)	0.274 (0.145)	0.554 (0.188)
FCHI	0.343 (0.143)	0.566 (0.158)	0.501 (0.127)	1	0.798 (0.151)	0.805 (0.123)	0.856 (0.107)	0.852 (0.132)	0.339 (0.143)	0.601 (0.181)
SSMI	0.627 (0.136)	0.492 (0.174)	0.786 (0.157)	0.828 (0.134)	1	0.674 (0.178)	0.534 (0.158)	0.749 (0.159)	0.439 (0.166)	0.501 (0.185)
FTMIB	0.521 (0.142)	0.521 (0.167)	0.786 (0.114)	0.737 (0.116)	0.9 (0.126)	1	0.547 (0.148)	0.81 (0.179)	0.404 (0.167)	0.486 (0.178)
GDAX	0.451 (0.132)	0.743 (0.16)	0.645 (0.152)	0.923 (0.11)	0.797 (0.135)	0.772 (0.141)	1	0.681 (0.155)	0.317 (0.161)	0.633 (0.176)
FTSE	0.739 (0.136)	0.748 (0.168)	0.611 (0.153)	0.78 (0.116)	0.865 (0.126)	0.784 (0.127)	0.844 (0.134)	1	0.559 (0.146)	0.808 (0.173)
N225	0.649 (0.142)	0.496 (0.148)	0.571 (0.174)	0.514 (0.152)	0.576 (0.146)	0.677 (0.158)	0.673 (0.155)	0.619 (0.136)	1	0.768 (0.156)
HSI	0.386 (0.162)	0.783 (0.172)	0.296 (0.163)	0.294 (0.167)	0.253 (0.16)	0.411 (0.177)	0.444 (0.177)	0.348 (0.157)	0.378 (0.16)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.538 (0.204)	0.467 (0.18)	0.472 (0.203)	0.619 (0.183)	0.423 (0.191)	0.628 (0.199)	0.602 (0.169)	0.374 (0.183)	0.76 (0.219)
NASDAQ	0.716 (0.158)	1	0.602 (0.213)	0.748 (0.208)	0.16 (0.2)	0.874 (0.221)	0.798 (0.21)	0.444 (0.208)	0.44 (0.187)	0.342 (0.212)
IBEX	0.902 (0.163)	0.868 (0.199)	1	0.706 (0.176)	0.48 (0.174)	0.888 (0.157)	0.679 (0.189)	0.533 (0.174)	0.107 (0.186)	0.37 (0.208)
FCHI	0.77 (0.18)	0.821 (0.193)	0.877 (0.146)	1	0.471 (0.196)	0.796 (0.166)	0.915 (0.131)	0.819 (0.17)	0.414 (0.189)	0.378 (0.222)
SSMI	0.535 (0.16)	0.487 (0.199)	0.567 (0.162)	0.845 (0.162)	1	0.257 (0.175)	0.515 (0.205)	0.846 (0.196)	0.257 (0.192)	0.163 (0.209)
FTMIB	0.865 (0.153)	0.808 (0.182)	0.863 (0.132)	0.87 (0.14)	0.593 (0.144)	1	0.805 (0.167)	0.461 (0.187)	0.194 (0.19)	0.31 (0.187)
GDAX	0.762 (0.165)	0.797 (0.176)	0.798 (0.151)	0.889 (0.136)	0.597 (0.166)	0.884 (0.127)	1	0.746 (0.178)	0.2 (0.188)	0.453 (0.191)
FTSE	0.763 (0.165)	0.825 (0.198)	0.835 (0.157)	0.816 (0.158)	0.615 (0.131)	0.63 (0.141)	0.776 (0.131)	1	0.527 (0.183)	0.287 (0.194)
N225	0.411 (0.182)	0.656 (0.172)	0.496 (0.187)	0.731 (0.178)	0.6 (0.166)	0.513 (0.169)	0.821 (0.18)	0.779 (0.159)	1	0.145 (0.177)
HSI	0.373 (0.186)	0.523 (0.202)	0.686 (0.203)	0.717 (0.188)	0.585 (0.185)	0.423 (0.174)	0.483 (0.192)	0.584 (0.176)	0.415 (0.184)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.401 (0.229)	0.456 (0.232)	0.778 (0.24)	0.593 (0.215)	0.424 (0.243)	0.588 (0.21)	0.703 (0.223)	0.346 (0.239)	0.341 (0.235)
NASDAQ	0.411 (0.215)	1	0.681 (0.233)	0.677 (0.216)	0.098 (0.232)	0.626 (0.248)	0.452 (0.24)	0.374 (0.236)	0.086 (0.218)	0.065 (0.225)
IBEX	0.876 (0.175)	0.243 (0.238)	1	0.792 (0.249)	0.688 (0.287)	0.752 (0.199)	0.6 (0.227)	0.717 (0.268)	0.329 (0.359)	0.302 (0.231)
FCHI	0.725 (0.187)	0.257 (0.217)	0.475 (0.166)	1	0.646 (0.259)	0.85 (0.219)	0.84 (0.17)	0.896 (0.231)	0.307 (0.332)	0.333 (0.23)
SSMI	0.333 (0.196)	0.11 (0.25)	0.27 (0.215)	0.326 (0.183)	1	0.38 (0.279)	0.438 (0.238)	0.758 (0.21)	0.154 (0.301)	0.46 (0.211)
FTMIB	0.765 (0.185)	0.409 (0.217)	0.641 (0.157)	0.865 (0.152)	0.653 (0.184)	1	0.854 (0.207)	0.753 (0.26)	0.511 (0.357)	0.129 (0.191)
GDAX	0.716 (0.213)	0.413 (0.236)	0.683 (0.2)	0.627 (0.168)	0.755 (0.216)	0.911 (0.155)	1	0.819 (0.198)	0.39 (0.258)	0.104 (0.216)
FTSE	0.616 (0.19)	0.247 (0.233)	0.633 (0.181)	0.777 (0.163)	0.34 (0.141)	0.833 (0.158)	0.616 (0.174)	1	0.269 (0.298)	0.598 (0.204)
N225	0.75 (0.21)	0.498 (0.204)	0.752 (0.205)	0.542 (0.195)	0.67 (0.207)	0.831 (0.18)	0.809 (0.209)	0.765 (0.193)	1	-0.119 (0.195)
HSI	0.309 (0.216)	0.243 (0.238)	0.416 (0.224)	0.493 (0.228)	0.67 (0.209)	0.79 (0.204)	0.772 (0.216)	0.777 (0.202)	0.726 (0.214)	1

Expected Shortfall - Daily Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.804 (0.103)	0.475 (0.08)	0.54 (0.087)	0.477 (0.078)	0.504 (0.081)	0.578 (0.181)	0.505 (0.201)	0.098 (0.056)	0.146 (0.056)
NASDAQ	0.814 (0.393)	1	0.423 (0.022)	0.494 (0.021)	0.417 (0.023)	0.443 (0.021)	0.55 (0.034)	0.43 (0.079)	0.077 (0.023)	0.164 (0.023)
IBEX	0.532 (0.465)	0.423 (0.02)	1	0.862 (0.014)	0.722 (0.02)	0.864 (0.012)	0.776 (0.028)	0.765 (0.105)	0.2 (0.025)	0.272 (0.022)
FCHI	0.58 (0.486)	0.475 (0.019)	0.881 (0.011)	1	0.797 (0.017)	0.859 (0.012)	0.878 (0.033)	0.851 (0.116)	0.212 (0.025)	0.321 (0.019)
SSMI	0.525 (0.463)	0.401 (0.021)	0.745 (0.018)	0.834 (0.014)	1	0.728 (0.018)	0.757 (0.028)	0.758 (0.106)	0.222 (0.023)	0.268 (0.021)
FTMIB	0.535 (0.467)	0.442 (0.02)	0.864 (0.012)	0.886 (0.011)	0.755 (0.017)	1	0.795 (0.027)	0.761 (0.105)	0.159 (0.024)	0.258 (0.021)
GDAX	0.583 (0.477)	0.509 (0.029)	0.822 (0.029)	0.906 (0.034)	0.804 (0.029)	0.83 (0.03)	1	0.789 (0.204)	0.2 (0.044)	0.311 (0.043)
FTSE	0.548 (0.505)	0.436 (0.105)	0.778 (0.137)	0.891 (0.145)	0.815 (0.122)	0.8 (0.14)	0.826 (0.122)	1	0.207 (0.065)	0.322 (0.068)
N225	0.133 (0.414)	0.134 (0.023)	0.274 (0.024)	0.321 (0.022)	0.335 (0.021)	0.265 (0.023)	0.287 (0.028)	0.321 (0.115)	1	0.442 (0.021)
HSI	0.157 (0.424)	0.144 (0.021)	0.353 (0.021)	0.385 (0.022)	0.377 (0.021)	0.333 (0.021)	0.354 (0.026)	0.408 (0.109)	0.522 (0.019)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.82 (0.103)	0.491 (0.079)	0.558 (0.079)	0.484 (0.073)	0.496 (0.08)	0.603 (0.147)	0.518 (0.211)	0.09 (0.048)	0.148 (0.049)
NASDAQ	0.784 (0.208)	1	0.409 (0.029)	0.501 (0.028)	0.389 (0.03)	0.429 (0.028)	0.561 (0.034)	0.424 (0.074)	0.051 (0.027)	0.169 (0.027)
IBEX	0.549 (0.217)	0.45 (0.028)	1	0.86 (0.018)	0.715 (0.029)	0.867 (0.019)	0.751 (0.035)	0.761 (0.09)	0.18 (0.03)	0.255 (0.031)
FCHI	0.569 (0.243)	0.469 (0.029)	0.893 (0.018)	1	0.802 (0.025)	0.851 (0.016)	0.882 (0.03)	0.852 (0.097)	0.221 (0.029)	0.29 (0.029)
SSMI	0.543 (0.231)	0.41 (0.028)	0.755 (0.026)	0.844 (0.025)	1	0.732 (0.027)	0.741 (0.034)	0.774 (0.093)	0.243 (0.029)	0.242 (0.027)
FTMIB	0.522 (0.213)	0.441 (0.029)	0.877 (0.019)	0.894 (0.02)	0.758 (0.027)	1	0.784 (0.032)	0.742 (0.091)	0.154 (0.028)	0.226 (0.028)
GDAX	0.595 (0.159)	0.51 (0.05)	0.818 (0.048)	0.909 (0.054)	0.801 (0.051)	0.836 (0.049)	1	0.767 (0.165)	0.188 (0.048)	0.28 (0.043)
FTSE	0.563 (0.175)	0.423 (0.127)	0.802 (0.129)	0.877 (0.143)	0.834 (0.126)	0.802 (0.129)	0.825 (0.114)	1	0.216 (0.059)	0.295 (0.059)
N225	0.134 (0.086)	0.122 (0.037)	0.279 (0.03)	0.307 (0.031)	0.329 (0.032)	0.27 (0.029)	0.28 (0.038)	0.309 (0.106)	1	0.419 (0.027)
HSI	0.154 (0.084)	0.149 (0.035)	0.339 (0.03)	0.373 (0.032)	0.391 (0.032)	0.339 (0.029)	0.345 (0.035)	0.41 (0.106)	0.544 (0.027)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.827 (0.09)	0.504 (0.076)	0.536 (0.072)	0.438 (0.067)	0.439 (0.073)	0.6 (0.084)	0.495 (0.233)	0.078 (0.054)	0.187 (0.05)
NASDAQ	0.779 (0.093)	1	0.399 (0.063)	0.437 (0.062)	0.321 (0.062)	0.375 (0.058)	0.547 (0.063)	0.4 (0.084)	0.002 (0.045)	0.14 (0.048)
IBEX	0.588 (0.134)	0.442 (0.058)	1	0.868 (0.049)	0.712 (0.067)	0.818 (0.049)	0.731 (0.072)	0.782 (0.092)	0.199 (0.052)	0.201 (0.056)
FCHI	0.631 (0.133)	0.469 (0.063)	0.882 (0.044)	1	0.796 (0.063)	0.845 (0.046)	0.848 (0.055)	0.881 (0.098)	0.257 (0.053)	0.27 (0.054)
SSMI	0.552 (0.127)	0.351 (0.058)	0.763 (0.068)	0.803 (0.072)	1	0.723 (0.064)	0.669 (0.065)	0.814 (0.093)	0.219 (0.05)	0.23 (0.053)
FTMIB	0.514 (0.135)	0.392 (0.057)	0.915 (0.045)	0.868 (0.041)	0.735 (0.07)	1	0.76 (0.066)	0.758 (0.091)	0.216 (0.052)	0.232 (0.054)
GDAX	0.573 (0.17)	0.436 (0.074)	0.796 (0.09)	0.917 (0.085)	0.779 (0.094)	0.813 (0.087)	1	0.731 (0.115)	0.182 (0.06)	0.237 (0.059)
FTSE	0.592 (0.198)	0.438 (0.036)	0.82 (0.033)	0.922 (0.035)	0.802 (0.045)	0.816 (0.033)	0.888 (0.051)	1	0.267 (0.063)	0.293 (0.066)
N225	0.169 (0.159)	0.088 (0.047)	0.295 (0.05)	0.327 (0.057)	0.324 (0.059)	0.323 (0.052)	0.251 (0.062)	0.323 (0.038)	1	0.453 (0.059)
HSI	0.204 (0.151)	0.186 (0.041)	0.373 (0.051)	0.37 (0.053)	0.352 (0.055)	0.376 (0.053)	0.325 (0.059)	0.365 (0.034)	0.553 (0.059)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.776 (0.11)	0.459 (0.084)	0.488 (0.085)	0.378 (0.075)	0.428 (0.086)	0.585 (0.082)	0.505 (0.217)	0.073 (0.066)	0.229 (0.068)
NASDAQ	0.778 (0.098)	1	0.373 (0.082)	0.371 (0.08)	0.28 (0.079)	0.362 (0.084)	0.514 (0.078)	0.39 (0.08)	-0.045 (0.055)	0.128 (0.066)
IBEX	0.543 (0.113)	0.431 (0.08)	1	0.884 (0.064)	0.726 (0.087)	0.837 (0.065)	0.722 (0.088)	0.791 (0.094)	0.201 (0.067)	0.188 (0.074)
FCHI	0.59 (0.118)	0.486 (0.076)	0.838 (0.065)	1	0.797 (0.085)	0.877 (0.061)	0.838 (0.079)	0.92 (0.093)	0.264 (0.069)	0.248 (0.075)
SSMI	0.516 (0.12)	0.351 (0.074)	0.73 (0.09)	0.767 (0.093)	1	0.692 (0.088)	0.622 (0.09)	0.796 (0.09)	0.194 (0.063)	0.193 (0.071)
FTMIB	0.51 (0.108)	0.423 (0.078)	0.848 (0.063)	0.887 (0.06)	0.74 (0.098)	1	0.728 (0.093)	0.766 (0.092)	0.221 (0.066)	0.253 (0.074)
GDAX	0.59 (0.129)	0.467 (0.08)	0.821 (0.107)	0.924 (0.101)	0.785 (0.105)	0.86 (0.104)	1	0.775 (0.097)	0.176 (0.071)	0.247 (0.077)
FTSE	0.616 (0.161)	0.438 (0.137)	0.781 (0.152)	0.897 (0.156)	0.755 (0.149)	0.796 (0.15)	0.851 (0.135)	1	0.329 (0.078)	0.312 (0.072)
N225	0.201 (0.124)	0.078 (0.055)	0.366 (0.07)	0.389 (0.073)	0.359 (0.071)	0.414 (0.071)	0.324 (0.07)	0.39 (0.122)	1	0.392 (0.08)
HSI	0.268 (0.122)	0.185 (0.054)	0.401 (0.064)	0.423 (0.07)	0.391 (0.065)	0.411 (0.064)	0.362 (0.067)	0.432 (0.123)	0.626 (0.077)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.749 (0.142)	0.392 (0.134)	0.387 (0.133)	0.366 (0.12)	0.508 (0.14)	0.739 (0.135)	0.444 (0.227)	0.123 (0.094)	0.372 (0.116)
NASDAQ	0.906 (0.15)	1	0.366 (0.148)	0.278 (0.158)	0.295 (0.143)	0.371 (0.164)	0.585 (0.157)	0.252 (0.115)	-0.048 (0.107)	0.16 (0.119)
IBEX	0.655 (0.122)	0.505 (0.156)	1	0.963 (0.147)	0.802 (0.185)	0.955 (0.147)	0.619 (0.185)	0.936 (0.123)	0.213 (0.121)	0.2 (0.139)
FCHI	0.675 (0.139)	0.597 (0.164)	0.88 (0.128)	1	0.815 (0.182)	0.961 (0.137)	0.704 (0.154)	0.951 (0.114)	0.244 (0.128)	0.207 (0.141)
SSMI	0.488 (0.121)	0.421 (0.15)	0.634 (0.175)	0.801 (0.185)	1	0.839 (0.186)	0.665 (0.171)	0.874 (0.106)	0.279 (0.121)	0.237 (0.127)
FTMIB	0.684 (0.132)	0.593 (0.148)	0.878 (0.129)	0.961 (0.132)	0.776 (0.179)	1	0.661 (0.177)	0.941 (0.116)	0.277 (0.131)	0.162 (0.152)
GDAX	0.71 (0.13)	0.624 (0.156)	0.846 (0.183)	0.974 (0.181)	0.834 (0.173)	0.966 (0.182)	1	0.696 (0.125)	0.189 (0.122)	0.406 (0.142)
FTSE	0.675 (0.32)	0.607 (0.353)	0.751 (0.394)	0.921 (0.407)	0.839 (0.366)	0.858 (0.389)	0.956 (0.376)	1	0.338 (0.095)	0.291 (0.107)
N225	0.019 (0.127)	-0.124 (0.104)	0.504 (0.127)	0.491 (0.123)	0.413 (0.145)	0.543 (0.128)	0.492 (0.131)	0.475 (0.313)	1	0.313 (0.142)
HSI	0.168 (0.141)	0.17 (0.098)	0.292 (0.114)	0.334 (0.127)	0.281 (0.121)	0.297 (0.117)	0.237 (0.109)	0.282 (0.309)	0.611 (0.153)	1

Expected Shortfall - Weekly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.793 (0.036)	0.627 (0.046)	0.76 (0.038)	0.724 (0.041)	0.668 (0.04)	0.749 (0.033)	0.733 (0.037)	0.474 (0.042)	0.467 (0.045)
NASDAQ	0.784 (0.035)	1	0.577 (0.047)	0.705 (0.041)	0.615 (0.045)	0.59 (0.042)	0.719 (0.04)	0.641 (0.041)	0.473 (0.044)	0.519 (0.045)
IBEX	0.651 (0.039)	0.548 (0.048)	1	0.839 (0.032)	0.691 (0.043)	0.854 (0.035)	0.785 (0.037)	0.702 (0.04)	0.476 (0.047)	0.471 (0.048)
FCHI	0.762 (0.033)	0.7 (0.037)	0.872 (0.027)	1	0.792 (0.035)	0.876 (0.027)	0.917 (0.025)	0.807 (0.034)	0.512 (0.043)	0.536 (0.043)
SSMI	0.748 (0.037)	0.613 (0.042)	0.748 (0.037)	0.819 (0.033)	1	0.69 (0.042)	0.757 (0.036)	0.763 (0.036)	0.481 (0.049)	0.516 (0.048)
FTMIB	0.663 (0.038)	0.639 (0.039)	0.889 (0.031)	0.886 (0.025)	0.727 (0.04)	1	0.823 (0.031)	0.718 (0.039)	0.51 (0.045)	0.483 (0.044)
GDAX	0.759 (0.033)	0.678 (0.039)	0.8 (0.032)	0.902 (0.019)	0.787 (0.036)	0.825 (0.029)	1	0.776 (0.036)	0.52 (0.04)	0.566 (0.041)
FTSE	0.791 (0.03)	0.693 (0.035)	0.792 (0.034)	0.91 (0.023)	0.835 (0.029)	0.802 (0.033)	0.834 (0.028)	1	0.494 (0.044)	0.574 (0.041)
N225	0.538 (0.04)	0.561 (0.042)	0.553 (0.04)	0.61 (0.037)	0.587 (0.045)	0.558 (0.039)	0.567 (0.041)	0.642 (0.039)	1	0.52 (0.04)
HSI	0.54 (0.045)	0.471 (0.048)	0.45 (0.053)	0.577 (0.045)	0.503 (0.048)	0.514 (0.046)	0.522 (0.046)	0.625 (0.04)	0.588 (0.038)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.799 (0.06)	0.656 (0.063)	0.731 (0.052)	0.733 (0.057)	0.667 (0.05)	0.734 (0.054)	0.717 (0.051)	0.462 (0.053)	0.485 (0.063)
NASDAQ	0.772 (0.049)	1	0.55 (0.074)	0.683 (0.07)	0.577 (0.066)	0.577 (0.061)	0.727 (0.064)	0.602 (0.061)	0.429 (0.054)	0.513 (0.065)
IBEX	0.648 (0.056)	0.552 (0.071)	1	0.812 (0.049)	0.664 (0.06)	0.872 (0.054)	0.747 (0.052)	0.72 (0.062)	0.502 (0.059)	0.46 (0.068)
FCHI	0.75 (0.048)	0.639 (0.061)	0.867 (0.039)	1	0.77 (0.05)	0.853 (0.04)	0.893 (0.036)	0.835 (0.046)	0.552 (0.055)	0.555 (0.06)
SSMI	0.73 (0.051)	0.593 (0.062)	0.756 (0.063)	0.861 (0.054)	1	0.7 (0.053)	0.712 (0.052)	0.799 (0.048)	0.511 (0.061)	0.527 (0.06)
FTMIB	0.677 (0.049)	0.616 (0.059)	0.883 (0.043)	0.879 (0.034)	0.761 (0.054)	1	0.805 (0.04)	0.749 (0.047)	0.541 (0.055)	0.534 (0.059)
GDAX	0.719 (0.048)	0.656 (0.059)	0.78 (0.048)	0.914 (0.031)	0.819 (0.056)	0.822 (0.041)	1	0.794 (0.05)	0.494 (0.055)	0.548 (0.061)
FTSE	0.763 (0.047)	0.638 (0.059)	0.781 (0.048)	0.897 (0.033)	0.864 (0.045)	0.813 (0.04)	0.824 (0.039)	1	0.535 (0.055)	0.563 (0.059)
N225	0.539 (0.054)	0.542 (0.066)	0.504 (0.058)	0.587 (0.054)	0.608 (0.054)	0.546 (0.053)	0.545 (0.054)	0.585 (0.055)	1	0.521 (0.055)
HSI	0.572 (0.054)	0.476 (0.061)	0.505 (0.064)	0.6 (0.057)	0.539 (0.061)	0.529 (0.056)	0.548 (0.059)	0.609 (0.054)	0.606 (0.05)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.746 (0.125)	0.835 (0.124)	0.868 (0.122)	0.812 (0.13)	0.897 (0.12)	0.964 (0.115)	0.936 (0.112)	0.547 (0.106)	0.526 (0.108)
NASDAQ	0.81 (0.113)	1	0.545 (0.145)	0.724 (0.138)	0.564 (0.144)	0.608 (0.126)	0.723 (0.125)	0.699 (0.135)	0.496 (0.109)	0.441 (0.114)
IBEX	0.666 (0.129)	0.677 (0.146)	1	0.882 (0.108)	0.759 (0.132)	0.877 (0.119)	0.839 (0.124)	0.881 (0.126)	0.433 (0.1)	0.548 (0.126)
FCHI	0.645 (0.125)	0.551 (0.144)	0.89 (0.087)	1	0.751 (0.128)	0.851 (0.092)	0.904 (0.095)	0.905 (0.104)	0.489 (0.11)	0.621 (0.124)
SSMI	0.625 (0.129)	0.519 (0.147)	0.825 (0.146)	0.781 (0.145)	1	0.807 (0.126)	0.867 (0.119)	0.775 (0.119)	0.405 (0.117)	0.424 (0.111)
FTMIB	0.629 (0.119)	0.57 (0.132)	0.902 (0.094)	0.927 (0.074)	0.743 (0.14)	1	0.874 (0.099)	0.826 (0.112)	0.553 (0.103)	0.663 (0.118)
GDAX	0.676 (0.116)	0.609 (0.134)	0.915 (0.1)	0.942 (0.073)	0.797 (0.147)	0.84 (0.091)	1	0.936 (0.108)	0.506 (0.105)	0.539 (0.107)
FTSE	0.825 (0.114)	0.657 (0.129)	0.882 (0.114)	0.938 (0.083)	0.788 (0.136)	0.94 (0.087)	0.89 (0.081)	1	0.433 (0.098)	0.643 (0.118)
N225	0.575 (0.134)	0.533 (0.127)	0.678 (0.144)	0.584 (0.137)	0.556 (0.127)	0.58 (0.142)	0.592 (0.135)	0.575 (0.138)	1	0.466 (0.105)
HSI	0.581 (0.114)	0.464 (0.134)	0.518 (0.131)	0.556 (0.124)	0.405 (0.137)	0.498 (0.117)	0.54 (0.127)	0.556 (0.111)	0.534 (0.113)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.763 (0.151)	0.572 (0.18)	0.755 (0.173)	0.56 (0.179)	0.62 (0.166)	0.823 (0.157)	0.854 (0.169)	0.727 (0.14)	0.397 (0.14)
NASDAQ	0.887 (0.154)	1	0.47 (0.191)	0.649 (0.182)	0.367 (0.188)	0.434 (0.161)	0.615 (0.166)	0.7 (0.171)	0.601 (0.142)	0.548 (0.147)
IBEX	0.608 (0.179)	0.53 (0.188)	1	0.942 (0.147)	0.669 (0.192)	0.866 (0.151)	0.834 (0.167)	0.841 (0.178)	0.331 (0.142)	0.582 (0.17)
FCHI	0.622 (0.182)	0.573 (0.181)	0.895 (0.101)	1	0.668 (0.179)	0.912 (0.135)	0.924 (0.133)	0.91 (0.173)	0.532 (0.147)	0.681 (0.17)
SSMI	0.562 (0.195)	0.46 (0.196)	0.749 (0.194)	0.857 (0.2)	1	0.692 (0.163)	0.829 (0.168)	0.769 (0.167)	0.297 (0.156)	0.47 (0.157)
FTMIB	0.678 (0.179)	0.573 (0.168)	0.833 (0.122)	0.848 (0.103)	0.736 (0.206)	1	0.853 (0.142)	0.806 (0.163)	0.373 (0.147)	0.654 (0.171)
GDAX	0.625 (0.182)	0.53 (0.169)	0.852 (0.146)	0.961 (0.112)	0.92 (0.213)	0.832 (0.129)	1	0.935 (0.159)	0.484 (0.15)	0.636 (0.142)
FTSE	0.743 (0.17)	0.683 (0.176)	0.867 (0.137)	0.957 (0.111)	0.875 (0.197)	0.882 (0.12)	0.93 (0.118)	1	0.542 (0.137)	0.689 (0.149)
N225	0.646 (0.199)	0.451 (0.173)	0.836 (0.193)	0.63 (0.197)	0.588 (0.197)	0.608 (0.192)	0.687 (0.19)	0.655 (0.198)	1	0.315 (0.131)
HSI	0.511 (0.157)	0.496 (0.172)	0.499 (0.175)	0.446 (0.172)	0.317 (0.204)	0.486 (0.177)	0.488 (0.188)	0.43 (0.171)	0.544 (0.172)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.081 (0.293)	0.263 (0.304)	0.543 (0.284)	0.314 (0.309)	0.343 (0.291)	0.758 (0.299)	0.642 (0.312)	0.872 (0.251)	0.26 (0.242)
NASDAQ	0.083 (0.281)	1	0.197 (0.299)	0.411 (0.317)	-0.062 (0.326)	-0.07 (0.309)	0.185 (0.299)	0.235 (0.304)	0.113 (0.27)	0.533 (0.236)
IBEX	0.965 (0.33)	0.149 (0.327)	1	0.594 (0.28)	0.733 (0.333)	0.751 (0.276)	0.588 (0.315)	0.57 (0.307)	-0.01 (0.251)	0.88 (0.298)
FCHI	0.993 (0.332)	0.175 (0.324)	0.979 (0.199)	1	0.708 (0.319)	0.713 (0.27)	0.65 (0.249)	0.882 (0.283)	0.241 (0.28)	0.512 (0.284)
SSMI	0.993 (0.345)	0.177 (0.345)	0.979 (0.364)	1.0 (0.347)	1	0.999 (0.322)	0.684 (0.293)	0.661 (0.325)	-0.167 (0.319)	0.56 (0.254)
FTMIB	0.813 (0.327)	0.081 (0.306)	0.927 (0.231)	0.833 (0.207)	0.833 (0.348)	1	0.7 (0.289)	0.674 (0.293)	-0.132 (0.31)	0.572 (0.301)
GDAX	0.994 (0.317)	0.164 (0.302)	0.982 (0.282)	1.0 (0.227)	1.0 (0.338)	0.841 (0.258)	1	0.821 (0.287)	0.417 (0.276)	0.562 (0.26)
FTSE	0.996 (0.309)	0.152 (0.302)	0.977 (0.262)	1.0 (0.22)	0.999 (0.337)	0.829 (0.224)	1.0 (0.226)	1	0.355 (0.275)	0.399 (0.268)
N225	0.986 (0.337)	0.213 (0.318)	0.981 (0.32)	0.999 (0.346)	0.999 (0.378)	0.837 (0.367)	0.998 (0.336)	0.997 (0.344)	1	0.038 (0.244)
HSI	0.949 (0.308)	-0.0 (0.261)	0.927 (0.322)	0.953 (0.328)	0.953 (0.371)	0.763 (0.345)	0.952 (0.345)	0.953 (0.337)	0.95 (0.298)	1

Expected Shortfall - Monthly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.599 (0.089)	0.49 (0.095)	0.731 (0.082)	0.693 (0.076)	0.533 (0.088)	0.732 (0.082)	0.761 (0.077)	0.461 (0.074)	0.585 (0.093)
NASDAQ	0.83 (0.049)	1	0.536 (0.087)	0.691 (0.087)	0.449 (0.087)	0.519 (0.085)	0.678 (0.09)	0.558 (0.089)	0.489 (0.078)	0.612 (0.092)
IBEX	0.804 (0.067)	0.645 (0.071)	1	0.798 (0.066)	0.575 (0.08)	0.835 (0.053)	0.658 (0.072)	0.634 (0.08)	0.308 (0.094)	0.511 (0.088)
FCHI	0.867 (0.056)	0.74 (0.063)	0.909 (0.04)	1	0.7 (0.068)	0.831 (0.05)	0.855 (0.041)	0.852 (0.07)	0.387 (0.077)	0.564 (0.09)
SSMI	0.797 (0.063)	0.647 (0.082)	0.759 (0.064)	0.841 (0.066)	1	0.649 (0.068)	0.622 (0.072)	0.632 (0.068)	0.384 (0.078)	0.359 (0.086)
FTMIB	0.71 (0.077)	0.633 (0.071)	0.895 (0.033)	0.896 (0.041)	0.742 (0.066)	1	0.724 (0.07)	0.748 (0.069)	0.421 (0.087)	0.417 (0.089)
GDAX	0.894 (0.049)	0.823 (0.055)	0.818 (0.056)	0.93 (0.033)	0.86 (0.061)	0.816 (0.05)	1	0.662 (0.078)	0.434 (0.071)	0.606 (0.091)
FTSE	0.881 (0.049)	0.707 (0.073)	0.794 (0.057)	0.897 (0.051)	0.87 (0.057)	0.773 (0.066)	0.862 (0.045)	1	0.416 (0.079)	0.582 (0.096)
N225	0.735 (0.074)	0.694 (0.065)	0.684 (0.069)	0.73 (0.074)	0.656 (0.075)	0.659 (0.071)	0.734 (0.063)	0.678 (0.085)	1	0.366 (0.08)
HSI	0.652 (0.086)	0.685 (0.081)	0.548 (0.086)	0.583 (0.085)	0.442 (0.092)	0.519 (0.09)	0.635 (0.081)	0.61 (0.075)	0.584 (0.091)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.553 (0.118)	0.546 (0.122)	0.744 (0.114)	0.672 (0.102)	0.453 (0.115)	0.815 (0.108)	0.758 (0.102)	0.405 (0.099)	0.533 (0.118)
NASDAQ	0.666 (0.08)	1	0.506 (0.12)	0.697 (0.123)	0.292 (0.103)	0.525 (0.115)	0.701 (0.117)	0.506 (0.114)	0.386 (0.104)	0.473 (0.112)
IBEX	0.76 (0.087)	0.63 (0.108)	1	0.852 (0.09)	0.634 (0.101)	0.806 (0.057)	0.639 (0.105)	0.717 (0.109)	0.302 (0.112)	0.539 (0.113)
FCHI	0.781 (0.078)	0.689 (0.092)	0.833 (0.065)	1	0.736 (0.102)	0.827 (0.072)	0.888 (0.057)	0.798 (0.087)	0.372 (0.098)	0.602 (0.107)
SSMI	0.724 (0.072)	0.576 (0.105)	0.753 (0.075)	0.887 (0.079)	1	0.609 (0.089)	0.541 (0.111)	0.692 (0.107)	0.387 (0.109)	0.434 (0.098)
FTMIB	0.77 (0.085)	0.612 (0.101)	0.868 (0.05)	0.937 (0.058)	0.839 (0.071)	1	0.763 (0.087)	0.664 (0.1)	0.429 (0.107)	0.412 (0.114)
GDAX	0.835 (0.073)	0.752 (0.087)	0.858 (0.086)	0.93 (0.045)	0.861 (0.074)	0.914 (0.064)	1	0.729 (0.098)	0.343 (0.101)	0.56 (0.114)
FTSE	0.814 (0.074)	0.676 (0.105)	0.762 (0.081)	0.88 (0.072)	0.863 (0.067)	0.813 (0.076)	0.855 (0.064)	1	0.444 (0.109)	0.673 (0.1)
N225	0.725 (0.093)	0.698 (0.087)	0.687 (0.094)	0.702 (0.09)	0.65 (0.097)	0.702 (0.091)	0.727 (0.09)	0.72 (0.091)	1	0.326 (0.092)
HSI	0.594 (0.102)	0.669 (0.116)	0.51 (0.108)	0.612 (0.113)	0.427 (0.106)	0.566 (0.11)	0.587 (0.103)	0.601 (0.101)	0.628 (0.099)	1

$\alpha = 0.025/0.975$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.528 (0.146)	0.555 (0.155)	0.669 (0.148)	0.628 (0.14)	0.445 (0.151)	0.661 (0.144)	0.691 (0.142)	0.395 (0.16)	0.522 (0.156)
NASDAQ	0.589 (0.12)	1	0.538 (0.143)	0.664 (0.148)	0.1 (0.14)	0.585 (0.16)	0.71 (0.15)	0.444 (0.161)	0.311 (0.133)	0.305 (0.159)
IBEX	0.853 (0.104)	0.669 (0.148)	1	0.838 (0.127)	0.64 (0.149)	0.779 (0.097)	0.678 (0.142)	0.705 (0.136)	0.346 (0.173)	0.51 (0.149)
FCHI	0.873 (0.116)	0.69 (0.126)	0.893 (0.096)	1	0.712 (0.145)	0.89 (0.106)	0.884 (0.078)	0.822 (0.118)	0.445 (0.18)	0.511 (0.149)
SSMI	0.627 (0.101)	0.419 (0.146)	0.638 (0.105)	0.798 (0.115)	1	0.442 (0.139)	0.513 (0.149)	0.776 (0.138)	0.291 (0.164)	0.452 (0.141)
FTMIB	0.914 (0.105)	0.592 (0.132)	0.851 (0.072)	0.904 (0.082)	0.69 (0.101)	1	0.811 (0.11)	0.579 (0.145)	0.352 (0.167)	0.291 (0.147)
GDAX	0.796 (0.113)	0.602 (0.134)	0.879 (0.096)	0.937 (0.069)	0.694 (0.113)	0.915 (0.077)	1	0.83 (0.125)	0.324 (0.16)	0.371 (0.16)
FTSE	0.835 (0.108)	0.589 (0.141)	0.805 (0.113)	0.882 (0.094)	0.642 (0.084)	0.841 (0.106)	0.822 (0.089)	1	0.521 (0.168)	0.546 (0.14)
N225	0.67 (0.122)	0.636 (0.117)	0.773 (0.128)	0.712 (0.117)	0.625 (0.123)	0.721 (0.116)	0.676 (0.114)	0.795 (0.107)	1	0.191 (0.139)
HSI	0.451 (0.146)	0.52 (0.149)	0.572 (0.148)	0.623 (0.142)	0.391 (0.146)	0.568 (0.134)	0.559 (0.143)	0.587 (0.134)	0.681 (0.142)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.364 (0.219)	0.592 (0.196)	0.727 (0.189)	0.584 (0.186)	0.437 (0.198)	0.43 (0.175)	0.686 (0.195)	0.472 (0.216)	0.511 (0.206)
NASDAQ	0.491 (0.164)	1	0.518 (0.193)	0.585 (0.204)	-0.02 (0.181)	0.634 (0.212)	0.601 (0.209)	0.361 (0.196)	0.144 (0.191)	0.046 (0.215)
IBEX	0.986 (0.146)	0.491 (0.202)	1	0.866 (0.182)	0.626 (0.239)	0.886 (0.132)	0.671 (0.188)	0.789 (0.22)	0.554 (0.282)	0.37 (0.181)
FCHI	0.856 (0.166)	0.392 (0.183)	0.875 (0.128)	1	0.725 (0.198)	0.875 (0.155)	0.807 (0.133)	0.905 (0.172)	0.536 (0.255)	0.34 (0.195)
SSMI	0.558 (0.167)	0.308 (0.224)	0.498 (0.172)	0.548 (0.139)	1	0.461 (0.227)	0.522 (0.209)	0.84 (0.162)	0.295 (0.237)	0.609 (0.18)
FTMIB	0.875 (0.157)	0.363 (0.178)	0.898 (0.112)	0.963 (0.13)	0.453 (0.151)	1	0.695 (0.165)	0.667 (0.211)	0.465 (0.269)	0.095 (0.18)
GDAX	0.775 (0.164)	0.373 (0.189)	0.803 (0.144)	0.867 (0.111)	0.628 (0.157)	0.869 (0.125)	1	0.788 (0.167)	0.342 (0.222)	0.107 (0.197)
FTSE	0.784 (0.161)	0.316 (0.197)	0.813 (0.143)	0.88 (0.113)	0.423 (0.116)	0.888 (0.139)	0.732 (0.116)	1	0.555 (0.235)	0.533 (0.174)
N225	0.791 (0.166)	0.551 (0.18)	0.805 (0.151)	0.668 (0.159)	0.507 (0.144)	0.69 (0.133)	0.493 (0.157)	0.755 (0.157)	1	-0.083 (0.169)
HSI	0.478 (0.211)	0.397 (0.213)	0.565 (0.198)	0.593 (0.201)	0.373 (0.188)	0.605 (0.183)	0.505 (0.195)	0.721 (0.19)	0.813 (0.194)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.155 (0.256)	0.778 (0.31)	0.759 (0.293)	0.441 (0.256)	0.718 (0.295)	0.159 (0.252)	0.703 (0.281)	0.641 (0.31)	0.478 (0.27)
NASDAQ	0.595 (0.263)	1	0.252 (0.277)	0.278 (0.272)	-0.058 (0.267)	0.455 (0.269)	0.624 (0.293)	0.214 (0.288)	0.085 (0.274)	-0.14 (0.28)
IBEX	0.788 (0.223)	0.538 (0.289)	1	0.995 (0.254)	0.518 (0.331)	0.942 (0.192)	0.485 (0.27)	0.967 (0.315)	0.906 (0.392)	0.278 (0.244)
FCHI	0.743 (0.248)	0.292 (0.265)	0.799 (0.22)	1	0.469 (0.284)	0.958 (0.23)	0.455 (0.224)	0.956 (0.246)	0.907 (0.383)	0.236 (0.23)
SSMI	0.707 (0.24)	0.63 (0.33)	0.704 (0.258)	0.383 (0.237)	1	0.375 (0.333)	0.551 (0.287)	0.46 (0.227)	0.329 (0.333)	0.73 (0.259)
FTMIB	0.623 (0.228)	0.053 (0.245)	0.82 (0.154)	0.856 (0.209)	0.276 (0.188)	1	0.532 (0.239)	0.854 (0.3)	0.897 (0.394)	0.054 (0.218)
GDAX	0.531 (0.271)	0.099 (0.263)	0.436 (0.242)	0.832 (0.199)	0.302 (0.267)	0.51 (0.198)	1	0.454 (0.23)	0.375 (0.317)	0.082 (0.264)
FTSE	0.776 (0.236)	0.252 (0.292)	0.868 (0.204)	0.942 (0.19)	0.444 (0.2)	0.904 (0.201)	0.675 (0.197)	1	0.84 (0.334)	0.313 (0.216)
N225	0.726 (0.255)	0.517 (0.268)	0.494 (0.236)	0.53 (0.257)	0.139 (0.231)	0.471 (0.208)	0.183 (0.243)	0.587 (0.236)	1	-0.068 (0.215)
HSI	0.693 (0.281)	0.495 (0.304)	0.466 (0.249)	0.514 (0.285)	0.089 (0.257)	0.465 (0.243)	0.164 (0.263)	0.565 (0.259)	0.998 (0.273)	1

B.4 Estimates for Equally Weighted Two and Three-Asset Portfolios - Overidentification

Estimates on the lower triangular matrix represent correlation implied by losses and estimates from the upper triangular matrix represent correlation implied by gains.

Expected Shortfall - Daily Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.812 (0.096)	0.479 (0.082)	0.542 (0.087)	0.479 (0.079)	0.502 (0.083)	0.577 (0.173)	0.508 (0.187)	0.093 (0.054)	0.149 (0.052)
NASDAQ	0.822 (0.328)	1	0.421 (0.022)	0.494 (0.023)	0.417 (0.026)	0.441 (0.023)	0.55 (0.037)	0.434 (0.087)	0.079 (0.026)	0.17 (0.024)
IBEX	0.531 (0.398)	0.423 (0.02)	1	0.86 (0.021)	0.725 (0.024)	0.863 (0.022)	0.776 (0.035)	0.763 (0.11)	0.198 (0.026)	0.273 (0.024)
FCHI	0.577 (0.414)	0.471 (0.018)	0.88 (0.011)	1	0.795 (0.021)	0.857 (0.02)	0.875 (0.043)	0.848 (0.12)	0.206 (0.024)	0.319 (0.021)
SSMI	0.522 (0.397)	0.395 (0.02)	0.745 (0.017)	0.832 (0.015)	1	0.727 (0.022)	0.757 (0.038)	0.761 (0.112)	0.216 (0.024)	0.267 (0.026)
FTMIB	0.532 (0.399)	0.438 (0.019)	0.864 (0.012)	0.887 (0.011)	0.755 (0.017)	1	0.793 (0.035)	0.76 (0.11)	0.155 (0.025)	0.253 (0.023)
GDAX	0.585 (0.398)	0.505 (0.028)	0.82 (0.03)	0.906 (0.031)	0.803 (0.027)	0.831 (0.03)	1	0.785 (0.21)	0.192 (0.044)	0.308 (0.044)
FTSE	0.545 (0.447)	0.429 (0.094)	0.781 (0.132)	0.892 (0.145)	0.816 (0.117)	0.801 (0.135)	0.826 (0.1)	1	0.2 (0.061)	0.322 (0.069)
N225	0.138 (0.35)	0.129 (0.023)	0.284 (0.019)	0.324 (0.02)	0.335 (0.019)	0.269 (0.02)	0.293 (0.024)	0.323 (0.092)	1	0.449 (0.033)
HSI	0.157 (0.359)	0.148 (0.022)	0.351 (0.019)	0.387 (0.019)	0.378 (0.02)	0.336 (0.019)	0.355 (0.024)	0.41 (0.096)	0.516 (0.02)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.815 (0.098)	0.491 (0.08)	0.56 (0.083)	0.491 (0.075)	0.503 (0.081)	0.602 (0.146)	0.522 (0.218)	0.084 (0.045)	0.147 (0.047)
NASDAQ	0.799 (0.179)	1	0.416 (0.032)	0.504 (0.032)	0.396 (0.031)	0.434 (0.03)	0.559 (0.04)	0.434 (0.08)	0.056 (0.028)	0.165 (0.027)
IBEX	0.547 (0.202)	0.446 (0.032)	1	0.859 (0.026)	0.715 (0.03)	0.862 (0.027)	0.759 (0.038)	0.76 (0.101)	0.192 (0.029)	0.252 (0.03)
FCHI	0.571 (0.223)	0.464 (0.03)	0.892 (0.019)	1	0.801 (0.029)	0.85 (0.024)	0.876 (0.037)	0.851 (0.105)	0.217 (0.03)	0.283 (0.032)
SSMI	0.54 (0.218)	0.399 (0.032)	0.756 (0.027)	0.841 (0.022)	1	0.726 (0.027)	0.74 (0.039)	0.769 (0.099)	0.232 (0.029)	0.241 (0.03)
FTMIB	0.523 (0.197)	0.439 (0.029)	0.876 (0.018)	0.895 (0.016)	0.757 (0.025)	1	0.787 (0.039)	0.743 (0.1)	0.155 (0.03)	0.217 (0.029)
GDAX	0.591 (0.155)	0.496 (0.05)	0.822 (0.048)	0.909 (0.051)	0.802 (0.047)	0.835 (0.047)	1	0.767 (0.175)	0.189 (0.042)	0.279 (0.044)
FTSE	0.557 (0.163)	0.421 (0.086)	0.802 (0.078)	0.883 (0.087)	0.83 (0.079)	0.805 (0.075)	0.827 (0.059)	1	0.208 (0.056)	0.288 (0.064)
N225	0.132 (0.093)	0.117 (0.034)	0.29 (0.029)	0.322 (0.029)	0.335 (0.03)	0.284 (0.028)	0.289 (0.034)	0.315 (0.065)	1	0.425 (0.036)
HSI	0.153 (0.096)	0.139 (0.031)	0.34 (0.03)	0.37 (0.03)	0.378 (0.032)	0.34 (0.029)	0.341 (0.032)	0.404 (0.06)	0.542 (0.028)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.788 (0.097)	0.514 (0.086)	0.539 (0.088)	0.446 (0.074)	0.458 (0.085)	0.601 (0.102)	0.512 (0.225)	0.094 (0.052)	0.196 (0.054)
NASDAQ	0.807 (0.083)	1	0.403 (0.065)	0.429 (0.062)	0.319 (0.061)	0.38 (0.064)	0.551 (0.063)	0.391 (0.076)	0.003 (0.049)	0.141 (0.054)
IBEX	0.594 (0.137)	0.454 (0.058)	1	0.865 (0.049)	0.716 (0.069)	0.826 (0.05)	0.736 (0.064)	0.779 (0.089)	0.225 (0.052)	0.208 (0.049)
FCHI	0.624 (0.136)	0.488 (0.058)	0.886 (0.041)	1	0.797 (0.061)	0.847 (0.049)	0.846 (0.06)	0.881 (0.094)	0.28 (0.052)	0.26 (0.055)
SSMI	0.545 (0.133)	0.366 (0.055)	0.768 (0.061)	0.816 (0.059)	1	0.719 (0.067)	0.661 (0.067)	0.813 (0.091)	0.245 (0.046)	0.215 (0.056)
FTMIB	0.516 (0.139)	0.41 (0.055)	0.902 (0.042)	0.874 (0.039)	0.751 (0.059)	1	0.77 (0.063)	0.762 (0.082)	0.236 (0.053)	0.245 (0.053)
GDAX	0.582 (0.176)	0.457 (0.072)	0.809 (0.09)	0.922 (0.08)	0.786 (0.081)	0.821 (0.089)	1	0.741 (0.105)	0.201 (0.06)	0.24 (0.062)
FTSE	0.591 (0.242)	0.449 (0.082)	0.824 (0.084)	0.929 (0.085)	0.815 (0.082)	0.82 (0.083)	0.884 (0.089)	1	0.307 (0.057)	0.307 (0.061)
N225	0.185 (0.16)	0.113 (0.041)	0.306 (0.047)	0.354 (0.046)	0.345 (0.051)	0.346 (0.048)	0.268 (0.054)	0.349 (0.069)	1	0.467 (0.074)
HSI	0.184 (0.162)	0.182 (0.043)	0.372 (0.047)	0.381 (0.05)	0.367 (0.053)	0.382 (0.048)	0.333 (0.058)	0.382 (0.082)	0.548 (0.063)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.754 (0.108)	0.499 (0.086)	0.505 (0.085)	0.39 (0.081)	0.467 (0.084)	0.605 (0.091)	0.523 (0.226)	0.104 (0.065)	0.243 (0.066)
NASDAQ	0.808 (0.098)	1	0.389 (0.084)	0.371 (0.083)	0.294 (0.077)	0.367 (0.084)	0.523 (0.081)	0.378 (0.081)	-0.019 (0.056)	0.144 (0.068)
IBEX	0.556 (0.123)	0.443 (0.079)	1	0.874 (0.07)	0.712 (0.085)	0.835 (0.077)	0.728 (0.082)	0.787 (0.095)	0.226 (0.061)	0.191 (0.069)
FCHI	0.61 (0.122)	0.495 (0.074)	0.848 (0.056)	1	0.79 (0.086)	0.874 (0.062)	0.845 (0.079)	0.902 (0.095)	0.273 (0.069)	0.239 (0.075)
SSMI	0.527 (0.106)	0.353 (0.074)	0.745 (0.093)	0.783 (0.095)	1	0.692 (0.085)	0.631 (0.09)	0.8 (0.084)	0.235 (0.068)	0.19 (0.068)
FTMIB	0.521 (0.12)	0.42 (0.072)	0.851 (0.056)	0.881 (0.057)	0.748 (0.091)	1	0.749 (0.088)	0.773 (0.094)	0.24 (0.067)	0.248 (0.071)
GDAX	0.602 (0.132)	0.473 (0.079)	0.828 (0.106)	0.922 (0.1)	0.785 (0.106)	0.856 (0.104)	1	0.763 (0.098)	0.188 (0.076)	0.245 (0.077)
FTSE	0.614 (0.182)	0.452 (0.104)	0.786 (0.103)	0.897 (0.114)	0.775 (0.11)	0.798 (0.111)	0.859 (0.104)	1	0.348 (0.059)	0.329 (0.066)
N225	0.226 (0.128)	0.123 (0.051)	0.377 (0.065)	0.419 (0.071)	0.372 (0.072)	0.428 (0.066)	0.355 (0.07)	0.408 (0.092)	1	0.427 (0.09)
HSI	0.261 (0.124)	0.193 (0.052)	0.409 (0.063)	0.439 (0.064)	0.403 (0.065)	0.432 (0.064)	0.368 (0.065)	0.439 (0.091)	0.622 (0.076)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.723 (0.146)	0.395 (0.14)	0.434 (0.134)	0.427 (0.126)	0.455 (0.131)	0.731 (0.132)	0.47 (0.221)	0.091 (0.099)	0.399 (0.105)
NASDAQ	0.98 (0.147)	1	0.346 (0.135)	0.33 (0.134)	0.33 (0.13)	0.362 (0.134)	0.575 (0.138)	0.323 (0.099)	-0.053 (0.088)	0.212 (0.104)
IBEX	0.622 (0.131)	0.501 (0.146)	1	0.981 (0.139)	0.823 (0.175)	0.991 (0.129)	0.611 (0.173)	0.92 (0.124)	0.166 (0.099)	0.192 (0.128)
FCHI	0.641 (0.136)	0.566 (0.131)	0.872 (0.103)	1	0.836 (0.172)	0.973 (0.13)	0.723 (0.16)	0.945 (0.126)	0.219 (0.114)	0.235 (0.127)
SSMI	0.431 (0.122)	0.368 (0.136)	0.648 (0.158)	0.822 (0.158)	1	0.864 (0.177)	0.699 (0.165)	0.864 (0.109)	0.31 (0.102)	0.256 (0.126)
FTMIB	0.632 (0.132)	0.552 (0.133)	0.894 (0.117)	0.96 (0.114)	0.765 (0.163)	1	0.627 (0.169)	0.931 (0.124)	0.238 (0.097)	0.189 (0.148)
GDAX	0.667 (0.12)	0.593 (0.141)	0.851 (0.174)	0.979 (0.151)	0.831 (0.165)	0.954 (0.169)	1	0.717 (0.119)	0.176 (0.111)	0.434 (0.137)
FTSE	0.627 (0.306)	0.546 (0.369)	0.731 (0.409)	0.929 (0.434)	0.84 (0.389)	0.839 (0.414)	0.948 (0.397)	1	0.283 (0.097)	0.285 (0.108)
N225	0.009 (0.128)	-0.142 (0.105)	0.546 (0.125)	0.486 (0.125)	0.395 (0.116)	0.545 (0.128)	0.473 (0.118)	0.468 (0.358)	1	0.343 (0.149)
HSI	0.177 (0.134)	0.141 (0.085)	0.309 (0.112)	0.35 (0.11)	0.304 (0.1)	0.294 (0.114)	0.278 (0.103)	0.307 (0.346)	0.578 (0.143)	1

Expected Shortfall - Weekly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.799 (0.041)	0.631 (0.045)	0.76 (0.036)	0.726 (0.042)	0.662 (0.038)	0.746 (0.037)	0.729 (0.038)	0.475 (0.042)	0.451 (0.045)
NASDAQ	0.781 (0.032)	1	0.568 (0.051)	0.703 (0.044)	0.606 (0.047)	0.583 (0.044)	0.712 (0.042)	0.639 (0.046)	0.475 (0.045)	0.513 (0.047)
IBEX	0.663 (0.037)	0.561 (0.046)	1	0.836 (0.033)	0.688 (0.045)	0.855 (0.033)	0.772 (0.038)	0.698 (0.042)	0.469 (0.047)	0.46 (0.049)
FCHI	0.764 (0.034)	0.695 (0.04)	0.876 (0.026)	1	0.791 (0.039)	0.869 (0.028)	0.91 (0.024)	0.811 (0.034)	0.509 (0.042)	0.537 (0.045)
SSMI	0.731 (0.038)	0.598 (0.044)	0.741 (0.041)	0.812 (0.037)	1	0.688 (0.041)	0.754 (0.038)	0.76 (0.04)	0.488 (0.042)	0.517 (0.044)
FTMIB	0.67 (0.037)	0.631 (0.039)	0.882 (0.029)	0.884 (0.025)	0.723 (0.042)	1	0.813 (0.029)	0.711 (0.036)	0.512 (0.048)	0.485 (0.048)
GDAX	0.755 (0.033)	0.682 (0.035)	0.804 (0.031)	0.9 (0.019)	0.78 (0.04)	0.825 (0.027)	1	0.775 (0.037)	0.502 (0.045)	0.555 (0.044)
FTSE	0.795 (0.031)	0.695 (0.037)	0.793 (0.033)	0.91 (0.02)	0.828 (0.03)	0.807 (0.032)	0.833 (0.024)	1	0.484 (0.044)	0.564 (0.041)
N225	0.54 (0.04)	0.563 (0.044)	0.545 (0.041)	0.613 (0.039)	0.574 (0.039)	0.555 (0.04)	0.564 (0.04)	0.626 (0.038)	1	0.52 (0.037)
HSI	0.537 (0.045)	0.468 (0.047)	0.458 (0.044)	0.579 (0.045)	0.497 (0.047)	0.504 (0.043)	0.528 (0.041)	0.624 (0.039)	0.574 (0.039)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.801 (0.059)	0.657 (0.064)	0.733 (0.051)	0.719 (0.057)	0.67 (0.053)	0.738 (0.052)	0.716 (0.052)	0.464 (0.055)	0.478 (0.063)
NASDAQ	0.772 (0.055)	1	0.536 (0.07)	0.675 (0.062)	0.567 (0.065)	0.564 (0.061)	0.714 (0.061)	0.596 (0.063)	0.425 (0.06)	0.506 (0.066)
IBEX	0.655 (0.056)	0.551 (0.072)	1	0.812 (0.048)	0.676 (0.061)	0.865 (0.053)	0.752 (0.054)	0.728 (0.058)	0.504 (0.055)	0.444 (0.064)
FCHI	0.747 (0.051)	0.641 (0.065)	0.866 (0.044)	1	0.77 (0.052)	0.85 (0.034)	0.89 (0.035)	0.838 (0.046)	0.549 (0.054)	0.548 (0.056)
SSMI	0.73 (0.053)	0.573 (0.067)	0.757 (0.068)	0.857 (0.058)	1	0.688 (0.054)	0.716 (0.052)	0.796 (0.055)	0.509 (0.059)	0.531 (0.057)
FTMIB	0.68 (0.052)	0.612 (0.058)	0.859 (0.049)	0.895 (0.034)	0.76 (0.06)	1	0.796 (0.04)	0.741 (0.049)	0.555 (0.056)	0.523 (0.058)
GDAX	0.722 (0.046)	0.66 (0.057)	0.789 (0.049)	0.913 (0.031)	0.808 (0.064)	0.835 (0.042)	1	0.792 (0.048)	0.49 (0.053)	0.534 (0.055)
FTSE	0.77 (0.05)	0.643 (0.057)	0.785 (0.053)	0.896 (0.032)	0.858 (0.047)	0.83 (0.038)	0.836 (0.04)	1	0.522 (0.055)	0.579 (0.055)
N225	0.538 (0.055)	0.535 (0.062)	0.497 (0.059)	0.587 (0.056)	0.585 (0.059)	0.547 (0.055)	0.545 (0.054)	0.586 (0.055)	1	0.531 (0.055)
HSI	0.577 (0.057)	0.468 (0.072)	0.514 (0.068)	0.603 (0.064)	0.539 (0.063)	0.522 (0.061)	0.556 (0.062)	0.625 (0.057)	0.571 (0.057)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.758 (0.128)	0.824 (0.131)	0.872 (0.129)	0.81 (0.126)	0.901 (0.115)	0.969 (0.116)	0.931 (0.115)	0.586 (0.108)	0.524 (0.1)
NASDAQ	0.876 (0.126)	1	0.561 (0.149)	0.718 (0.149)	0.564 (0.141)	0.627 (0.135)	0.727 (0.128)	0.706 (0.134)	0.51 (0.113)	0.479 (0.114)
IBEX	0.644 (0.123)	0.653 (0.147)	1	0.894 (0.108)	0.751 (0.13)	0.878 (0.113)	0.843 (0.118)	0.891 (0.123)	0.468 (0.095)	0.525 (0.145)
FCHI	0.622 (0.123)	0.551 (0.136)	0.878 (0.083)	1	0.735 (0.123)	0.864 (0.086)	0.915 (0.09)	0.905 (0.103)	0.533 (0.107)	0.625 (0.122)
SSMI	0.612 (0.123)	0.515 (0.147)	0.833 (0.13)	0.785 (0.128)	1	0.8 (0.123)	0.854 (0.111)	0.773 (0.119)	0.426 (0.118)	0.409 (0.113)
FTMIB	0.63 (0.115)	0.579 (0.126)	0.904 (0.096)	0.921 (0.073)	0.763 (0.133)	1	0.876 (0.097)	0.835 (0.106)	0.573 (0.106)	0.656 (0.127)
GDAX	0.652 (0.118)	0.6 (0.124)	0.903 (0.112)	0.928 (0.072)	0.807 (0.124)	0.847 (0.095)	1	0.946 (0.105)	0.536 (0.104)	0.558 (0.107)
FTSE	0.787 (0.107)	0.686 (0.129)	0.89 (0.103)	0.943 (0.076)	0.801 (0.121)	0.943 (0.088)	0.889 (0.081)	1	0.483 (0.094)	0.621 (0.111)
N225	0.583 (0.118)	0.535 (0.127)	0.699 (0.13)	0.594 (0.126)	0.565 (0.121)	0.608 (0.133)	0.605 (0.124)	0.596 (0.128)	1	0.467 (0.093)
HSI	0.578 (0.114)	0.47 (0.125)	0.518 (0.128)	0.548 (0.118)	0.389 (0.128)	0.539 (0.114)	0.534 (0.123)	0.557 (0.119)	0.524 (0.11)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.728 (0.164)	0.587 (0.171)	0.75 (0.171)	0.563 (0.168)	0.616 (0.15)	0.807 (0.168)	0.864 (0.167)	0.786 (0.14)	0.437 (0.125)
NASDAQ	0.905 (0.164)	1	0.51 (0.182)	0.684 (0.196)	0.393 (0.189)	0.439 (0.167)	0.625 (0.179)	0.724 (0.181)	0.618 (0.15)	0.6 (0.144)
IBEX	0.58 (0.191)	0.502 (0.195)	1	0.933 (0.145)	0.68 (0.183)	0.876 (0.143)	0.859 (0.158)	0.844 (0.174)	0.342 (0.144)	0.641 (0.158)
FCHI	0.599 (0.192)	0.537 (0.189)	0.896 (0.101)	1	0.667 (0.178)	0.916 (0.131)	0.934 (0.127)	0.932 (0.164)	0.539 (0.148)	0.718 (0.144)
SSMI	0.56 (0.201)	0.428 (0.203)	0.757 (0.181)	0.874 (0.179)	1	0.695 (0.176)	0.817 (0.155)	0.742 (0.167)	0.305 (0.153)	0.5 (0.145)
FTMIB	0.685 (0.173)	0.554 (0.185)	0.843 (0.107)	0.866 (0.102)	0.769 (0.18)	1	0.863 (0.139)	0.819 (0.155)	0.38 (0.145)	0.699 (0.155)
GDAX	0.603 (0.179)	0.512 (0.184)	0.845 (0.139)	0.96 (0.099)	0.93 (0.187)	0.841 (0.121)	1	0.946 (0.151)	0.501 (0.145)	0.682 (0.137)
FTSE	0.719 (0.173)	0.666 (0.182)	0.863 (0.136)	0.962 (0.104)	0.885 (0.186)	0.889 (0.112)	0.939 (0.119)	1	0.576 (0.144)	0.685 (0.139)
N225	0.62 (0.186)	0.449 (0.17)	0.828 (0.178)	0.658 (0.182)	0.613 (0.188)	0.635 (0.186)	0.704 (0.186)	0.686 (0.198)	1	0.293 (0.132)
HSI	0.467 (0.167)	0.465 (0.183)	0.55 (0.17)	0.447 (0.166)	0.375 (0.193)	0.511 (0.171)	0.511 (0.181)	0.442 (0.167)	0.542 (0.156)	1

$\alpha = 0.001/0.999$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.202 (0.286)	0.373 (0.3)	0.585 (0.306)	0.35 (0.297)	0.339 (0.313)	0.748 (0.291)	0.679 (0.295)	0.861 (0.269)	0.359 (0.246)
NASDAQ	0.265 (0.296)	1	0.176 (0.303)	0.41 (0.309)	-0.019 (0.311)	-0.061 (0.294)	0.242 (0.305)	0.345 (0.312)	0.202 (0.246)	0.483 (0.208)
IBEX	0.989 (0.314)	0.307 (0.33)	1	0.711 (0.259)	0.757 (0.324)	0.797 (0.267)	0.636 (0.306)	0.63 (0.313)	0.023 (0.263)	0.875 (0.278)
FCHI	0.997 (0.318)	0.324 (0.346)	0.994 (0.191)	1	0.776 (0.3)	0.731 (0.236)	0.66 (0.241)	0.877 (0.289)	0.21 (0.284)	0.565 (0.268)
SSMI	0.997 (0.342)	0.326 (0.367)	0.994 (0.352)	1.0 (0.342)	1	0.988 (0.319)	0.681 (0.297)	0.677 (0.304)	-0.16 (0.315)	0.58 (0.273)
FTMIB	0.854 (0.301)	0.244 (0.309)	0.916 (0.215)	0.868 (0.193)	0.868 (0.357)	1	0.67 (0.275)	0.625 (0.314)	-0.169 (0.285)	0.631 (0.295)
GDAX	0.997 (0.306)	0.317 (0.335)	0.995 (0.277)	1.0 (0.212)	1.0 (0.336)	0.871 (0.251)	1	0.858 (0.302)	0.451 (0.287)	0.644 (0.262)
FTSE	0.998 (0.287)	0.31 (0.328)	0.994 (0.261)	1.0 (0.186)	1.0 (0.34)	0.865 (0.213)	1.0 (0.214)	1	0.398 (0.292)	0.499 (0.265)
N225	0.994 (0.322)	0.349 (0.319)	0.994 (0.311)	0.999 (0.333)	1.0 (0.367)	0.872 (0.331)	0.999 (0.35)	0.999 (0.341)	1	0.072 (0.231)
HSI	0.967 (0.294)	0.209 (0.29)	0.959 (0.313)	0.969 (0.305)	0.969 (0.364)	0.814 (0.322)	0.969 (0.334)	0.969 (0.304)	0.968 (0.308)	1

Expected Shortfall - Monthly Returns - Bootstrapped Standard Errors

$\alpha = 0.1/0.9$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.618 (0.081)	0.49 (0.098)	0.711 (0.078)	0.683 (0.074)	0.535 (0.096)	0.732 (0.075)	0.752 (0.07)	0.431 (0.077)	0.567 (0.092)
NASDAQ	0.852 (0.049)	1	0.55 (0.08)	0.693 (0.082)	0.447 (0.077)	0.525 (0.079)	0.684 (0.081)	0.568 (0.088)	0.465 (0.076)	0.61 (0.094)
IBEX	0.782 (0.071)	0.649 (0.083)	1	0.81 (0.064)	0.6 (0.079)	0.837 (0.049)	0.669 (0.073)	0.663 (0.073)	0.296 (0.092)	0.517 (0.086)
FCHI	0.873 (0.056)	0.734 (0.065)	0.896 (0.044)	1	0.711 (0.069)	0.833 (0.046)	0.855 (0.042)	0.834 (0.056)	0.388 (0.078)	0.564 (0.084)
SSMI	0.788 (0.063)	0.626 (0.08)	0.751 (0.068)	0.847 (0.061)	1	0.653 (0.069)	0.638 (0.076)	0.648 (0.069)	0.414 (0.084)	0.38 (0.087)
FTMIB	0.732 (0.078)	0.633 (0.074)	0.885 (0.034)	0.889 (0.04)	0.758 (0.058)	1	0.721 (0.066)	0.741 (0.069)	0.409 (0.081)	0.411 (0.092)
GDAX	0.886 (0.045)	0.822 (0.055)	0.816 (0.062)	0.929 (0.033)	0.845 (0.062)	0.826 (0.052)	1	0.689 (0.069)	0.417 (0.079)	0.616 (0.087)
FTSE	0.882 (0.045)	0.727 (0.074)	0.782 (0.065)	0.897 (0.053)	0.863 (0.055)	0.775 (0.071)	0.865 (0.048)	1	0.395 (0.081)	0.589 (0.089)
N225	0.744 (0.074)	0.71 (0.073)	0.69 (0.083)	0.744 (0.076)	0.659 (0.076)	0.681 (0.071)	0.75 (0.063)	0.676 (0.083)	1	0.384 (0.073)
HSI	0.679 (0.079)	0.71 (0.086)	0.55 (0.09)	0.607 (0.086)	0.456 (0.081)	0.544 (0.089)	0.644 (0.074)	0.647 (0.069)	0.595 (0.088)	1

$\alpha = 0.05/0.95$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.554 (0.124)	0.513 (0.114)	0.718 (0.096)	0.688 (0.095)	0.436 (0.111)	0.786 (0.096)	0.783 (0.1)	0.404 (0.105)	0.539 (0.123)
NASDAQ	0.649 (0.08)	1	0.522 (0.113)	0.697 (0.11)	0.334 (0.103)	0.541 (0.109)	0.706 (0.109)	0.529 (0.117)	0.361 (0.103)	0.473 (0.123)
IBEX	0.779 (0.08)	0.654 (0.11)	1	0.84 (0.083)	0.658 (0.1)	0.819 (0.061)	0.639 (0.096)	0.698 (0.101)	0.268 (0.1)	0.528 (0.111)
FCHI	0.789 (0.078)	0.71 (0.09)	0.834 (0.067)	1	0.74 (0.09)	0.851 (0.068)	0.901 (0.059)	0.791 (0.077)	0.358 (0.11)	0.598 (0.118)
SSMI	0.732 (0.073)	0.558 (0.11)	0.76 (0.078)	0.889 (0.077)	1	0.612 (0.088)	0.578 (0.098)	0.703 (0.09)	0.359 (0.11)	0.434 (0.109)
FTMIB	0.768 (0.08)	0.614 (0.103)	0.859 (0.056)	0.935 (0.059)	0.825 (0.073)	1	0.763 (0.083)	0.669 (0.096)	0.386 (0.104)	0.386 (0.112)
GDAX	0.821 (0.07)	0.754 (0.093)	0.851 (0.079)	0.948 (0.047)	0.87 (0.077)	0.924 (0.066)	1	0.745 (0.089)	0.326 (0.116)	0.574 (0.129)
FTSE	0.84 (0.068)	0.679 (0.101)	0.792 (0.084)	0.888 (0.072)	0.85 (0.066)	0.814 (0.081)	0.852 (0.069)	1	0.426 (0.111)	0.659 (0.108)
N225	0.733 (0.088)	0.65 (0.094)	0.717 (0.094)	0.72 (0.084)	0.632 (0.089)	0.701 (0.087)	0.726 (0.084)	0.733 (0.091)	1	0.329 (0.096)
HSI	0.616 (0.11)	0.686 (0.108)	0.551 (0.1)	0.615 (0.113)	0.435 (0.1)	0.576 (0.113)	0.602 (0.111)	0.615 (0.103)	0.635 (0.103)	1

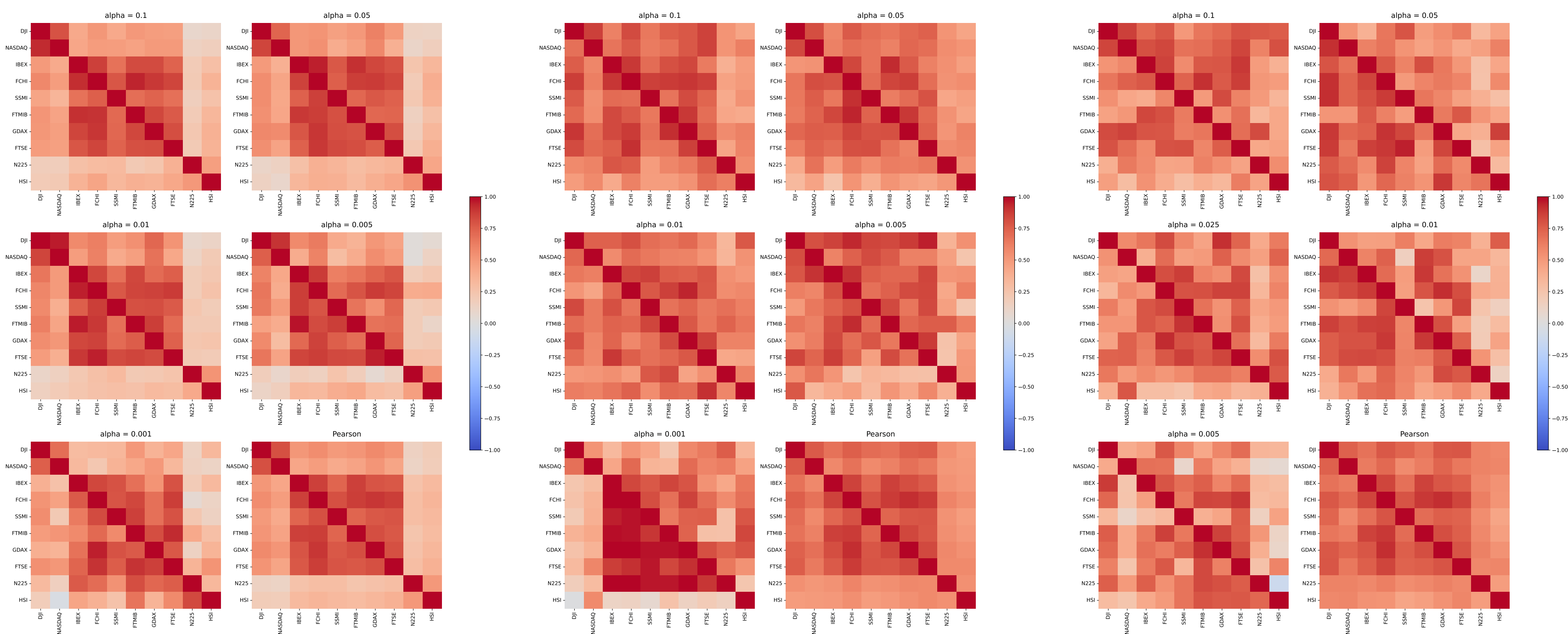
$\alpha = 0.025/0.975$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.455 (0.18)	0.513 (0.171)	0.697 (0.142)	0.641 (0.137)	0.41 (0.15)	0.672 (0.136)	0.745 (0.131)	0.409 (0.144)	0.544 (0.156)
NASDAQ	0.584 (0.136)	1	0.543 (0.14)	0.683 (0.153)	0.143 (0.147)	0.621 (0.16)	0.735 (0.152)	0.48 (0.159)	0.308 (0.137)	0.287 (0.162)
IBEX	0.853 (0.101)	0.649 (0.148)	1	0.835 (0.118)	0.653 (0.146)	0.791 (0.091)	0.672 (0.136)	0.696 (0.154)	0.318 (0.173)	0.514 (0.145)
FCHI	0.872 (0.111)	0.675 (0.131)	0.915 (0.099)	1	0.705 (0.133)	0.887 (0.097)	0.919 (0.083)	0.835 (0.115)	0.391 (0.17)	0.497 (0.152)
SSMI	0.643 (0.102)	0.407 (0.15)	0.655 (0.102)	0.782 (0.096)	1	0.455 (0.132)	0.524 (0.141)	0.803 (0.135)	0.315 (0.155)	0.493 (0.142)
FTMIB	0.913 (0.115)	0.595 (0.133)	0.867 (0.075)	0.919 (0.087)	0.682 (0.106)	1	0.813 (0.122)	0.568 (0.141)	0.341 (0.164)	0.277 (0.141)
GDAX	0.798 (0.117)	0.585 (0.136)	0.89 (0.095)	0.953 (0.076)	0.691 (0.109)	0.937 (0.08)	1	0.811 (0.127)	0.287 (0.154)	0.326 (0.154)
FTSE	0.825 (0.102)	0.592 (0.146)	0.821 (0.097)	0.895 (0.08)	0.67 (0.087)	0.848 (0.103)	0.822 (0.083)	1	0.476 (0.16)	0.586 (0.137)
N225	0.639 (0.127)	0.592 (0.121)	0.739 (0.133)	0.718 (0.119)	0.61 (0.118)	0.716 (0.117)	0.671 (0.137)	0.812 (0.112)	1	0.188 (0.136)
HSI	0.486 (0.143)	0.508 (0.149)	0.615 (0.135)	0.649 (0.143)	0.425 (0.143)	0.604 (0.137)	0.591 (0.142)	0.652 (0.131)	0.716 (0.134)	1

$\alpha = 0.01/0.99$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.366 (0.204)	0.618 (0.214)	0.723 (0.205)	0.554 (0.17)	0.449 (0.198)	0.476 (0.197)	0.729 (0.177)	0.486 (0.202)	0.48 (0.195)
NASDAQ	0.552 (0.176)	1	0.503 (0.186)	0.59 (0.209)	0.03 (0.185)	0.671 (0.205)	0.634 (0.22)	0.396 (0.21)	0.236 (0.187)	0.048 (0.196)
IBEX	0.995 (0.138)	0.519 (0.193)	1	0.886 (0.19)	0.671 (0.236)	0.896 (0.149)	0.678 (0.203)	0.824 (0.226)	0.566 (0.281)	0.418 (0.179)
FCHI	0.881 (0.164)	0.398 (0.183)	0.894 (0.147)	1	0.72 (0.21)	0.876 (0.165)	0.819 (0.132)	0.93 (0.171)	0.524 (0.263)	0.318 (0.188)
SSMI	0.569 (0.17)	0.385 (0.221)	0.547 (0.172)	0.548 (0.148)	1	0.435 (0.229)	0.531 (0.206)	0.829 (0.172)	0.271 (0.237)	0.587 (0.17)
FTMIB	0.919 (0.174)	0.413 (0.182)	0.929 (0.108)	0.976 (0.137)	0.475 (0.153)	1	0.716 (0.178)	0.706 (0.214)	0.526 (0.283)	0.12 (0.182)
GDAX	0.8 (0.176)	0.404 (0.185)	0.793 (0.142)	0.894 (0.115)	0.621 (0.165)	0.851 (0.125)	1	0.813 (0.172)	0.38 (0.225)	0.081 (0.188)
FTSE	0.807 (0.161)	0.353 (0.197)	0.849 (0.151)	0.868 (0.118)	0.472 (0.128)	0.894 (0.15)	0.745 (0.119)	1	0.523 (0.254)	0.48 (0.181)
N225	0.8 (0.171)	0.582 (0.177)	0.808 (0.147)	0.653 (0.172)	0.534 (0.164)	0.712 (0.138)	0.44 (0.174)	0.751 (0.168)	1	-0.024 (0.168)
HSI	0.544 (0.214)	0.418 (0.226)	0.56 (0.193)	0.618 (0.198)	0.383 (0.19)	0.626 (0.18)	0.484 (0.2)	0.697 (0.185)	0.784 (0.19)	1

$\alpha = 0.005/0.995$	DJI	NASDAQ	IBEX	FCHI	SSMI	FTMIB	GDAX	FTSE	N225	HSI
DJI	1	0.201 (0.263)	0.815 (0.301)	0.792 (0.293)	0.504 (0.24)	0.713 (0.295)	0.304 (0.267)	0.782 (0.26)	0.692 (0.296)	0.385 (0.268)
NASDAQ	0.658 (0.247)	1	0.299 (0.286)	0.335 (0.287)	-0.011 (0.283)	0.501 (0.285)	0.624 (0.299)	0.242 (0.288)	0.159 (0.298)	-0.07 (0.258)
IBEX	0.856 (0.211)	0.575 (0.299)	1	0.996 (0.264)	0.586 (0.319)	0.96 (0.207)	0.515 (0.301)	0.973 (0.299)	0.933 (0.375)	0.199 (0.225)
FCHI	0.793 (0.252)	0.333 (0.263)	0.79 (0.235)	1	0.549 (0.285)	0.973 (0.23)	0.524 (0.24)	0.965 (0.225)	0.94 (0.378)	0.151 (0.252)
SSMI	0.674 (0.253)	0.692 (0.31)	0.706 (0.273)	0.346 (0.241)	1	0.45 (0.329)	0.435 (0.31)	0.599 (0.223)	0.378 (0.338)	0.793 (0.239)
FTMIB	0.677 (0.239)	0.128 (0.253)	0.804 (0.161)	0.857 (0.214)	0.202 (0.203)	1	0.619 (0.26)	0.907 (0.285)	0.919 (0.385)	0.021 (0.23)
GDAX	0.558 (0.275)	0.212 (0.262)	0.455 (0.259)	0.823 (0.208)	0.332 (0.282)	0.466 (0.219)	1	0.54 (0.259)	0.449 (0.353)	0.034 (0.249)
FTSE	0.835 (0.231)	0.3 (0.271)	0.866 (0.229)	0.934 (0.17)	0.389 (0.193)	0.898 (0.212)	0.628 (0.204)	1	0.895 (0.345)	0.232 (0.223)
N225	0.714 (0.236)	0.587 (0.244)	0.447 (0.24)	0.487 (0.234)	0.124 (0.234)	0.437 (0.195)	0.153 (0.232)	0.551 (0.239)	1	-0.114 (0.23)
HSI	0.64 (0.257)	0.53 (0.291)	0.395 (0.256)	0.487 (0.271)	0.006 (0.223)	0.443 (0.243)	0.147 (0.263)	0.534 (0.248)	0.989 (0.265)	1

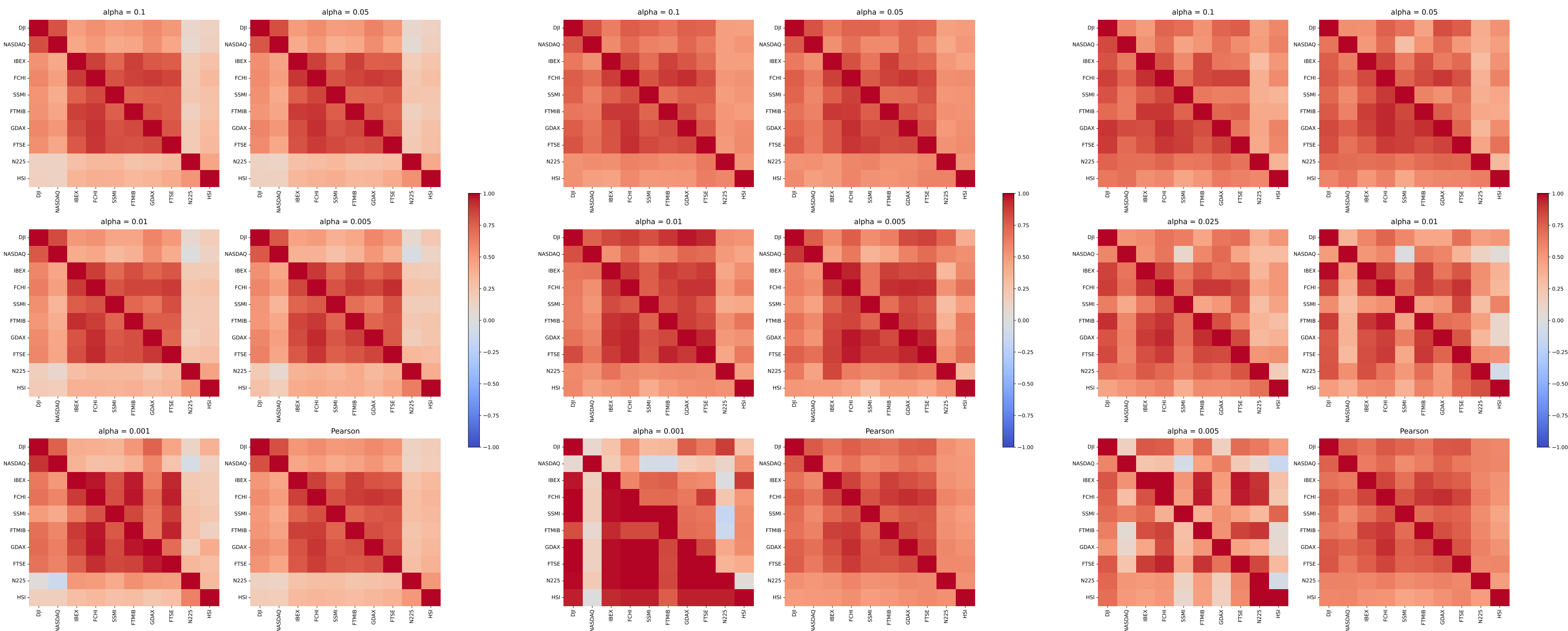
B.5 Heatmaps - Equally Weighted Two-Asset Portfolios

Value-at-Risk Implied Correlation - Equally Weighted Two Asset Portfolio - Exact Identification



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

Expected Shortfall Implied Correlation - Equally Weighted Two Asset Portfolio - Exact Identification



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

B.6 Boxplots - Expected Shortfall Implied Correlation

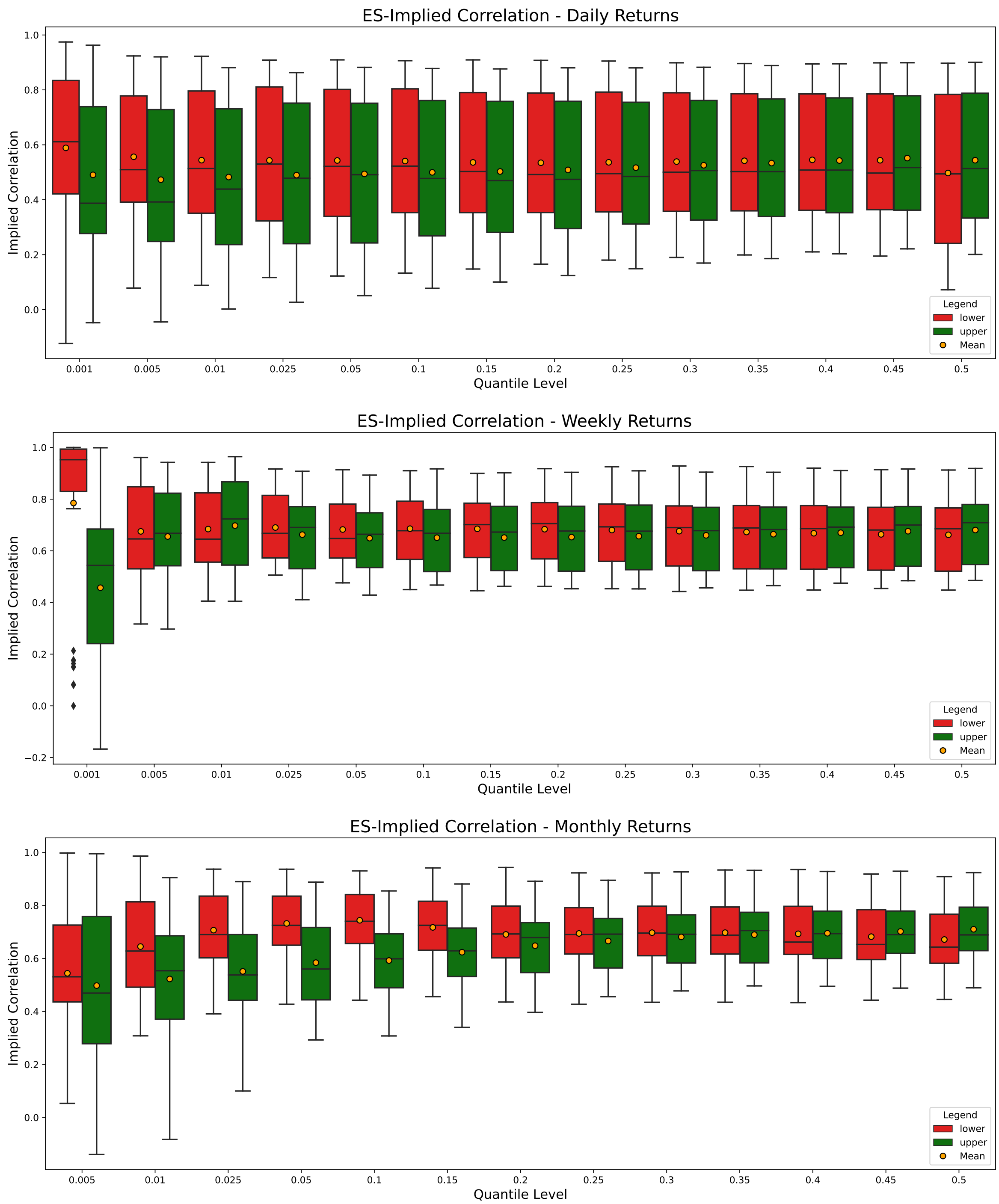


Figure 27: Average ES-implied correlation for daily, weekly and monthly returns.

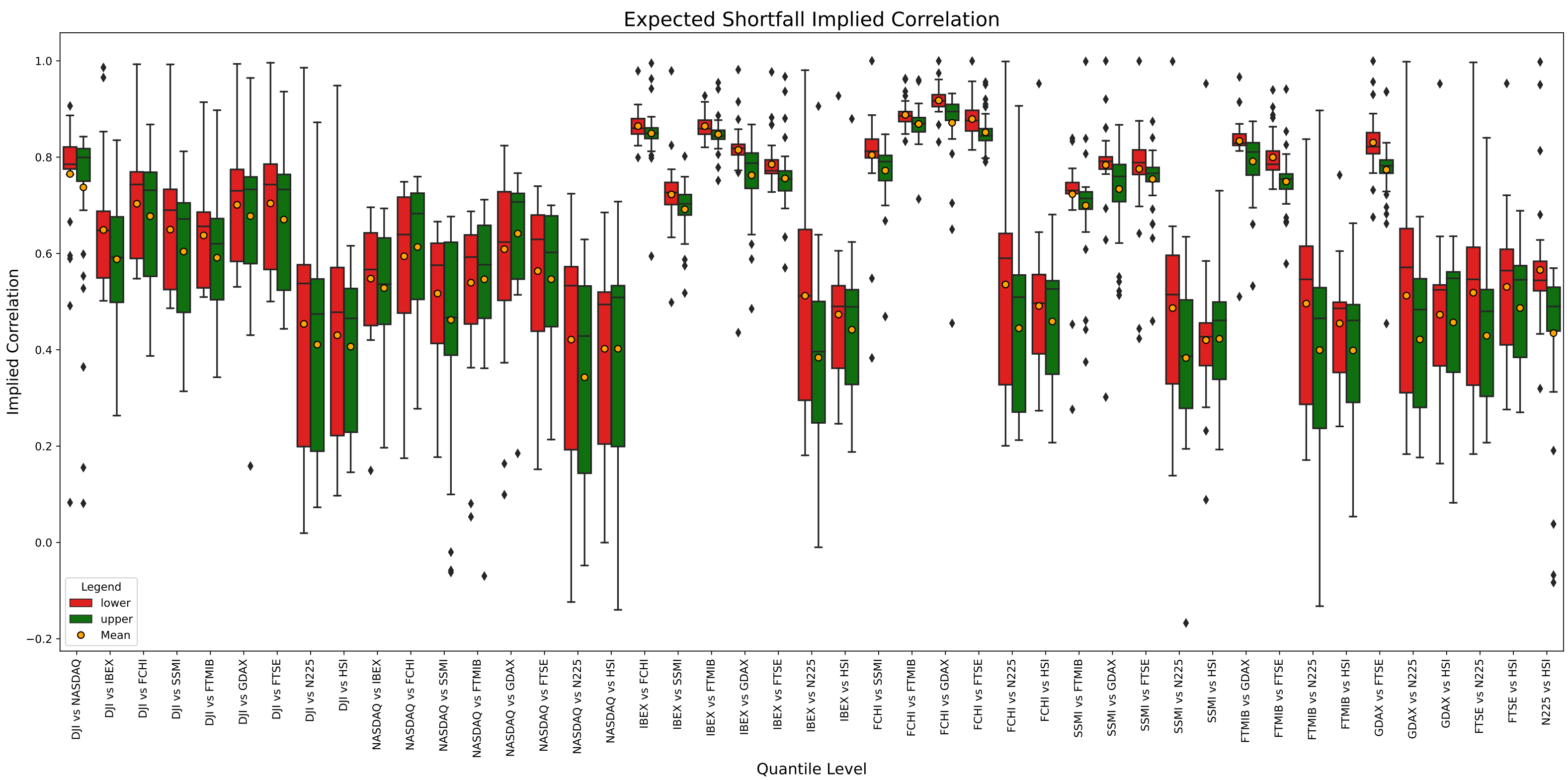


Figure 28: ES-implied correlation for asset pairs over all return frequencies.

B.7 Comparison of Strongest and Weakest Correlation Pairs

Correlation Pairs	Lower Tail (Losses) 0.001	0.005	0.01	0.05	0.01	Pearson	Upper Tail (Gains) 0.9	0.95	0.99	0.995	0.999
FCHI vs GDAX	0.999	0.961	0.942	0.914	0.902	0.915	0.917	0.893	0.904	0.924	0.650
FCHI vs FTMIB	0.833	0.848	0.927	0.879	0.886	0.883	0.876	0.853	0.851	0.912	0.713
FCHI vs FTSE	0.999	0.957	0.938	0.897	0.910	0.878	0.807	0.835	0.905	0.910	0.882
IBEX vs FTMIB	0.927	0.833	0.902	0.883	0.889	0.861	0.854	0.872	0.877	0.866	0.751
IBEX vs FCHI	0.979	0.895	0.890	0.867	0.872	0.858	0.839	0.812	0.882	0.942	0.594
FTMIB vs GDAX	0.841	0.832	0.840	0.822	0.825	0.838	0.823	0.805	0.874	0.853	0.700
GDAX vs FTSE	0.999	0.930	0.890	0.824	0.834	0.831	0.776	0.794	0.936	0.935	0.821
IBEX vs GDAX	0.982	0.852	0.915	0.780	0.800	0.818	0.785	0.747	0.839	0.834	0.588
FCHI vs SSMI	0.999	0.857	0.781	0.861	0.819	0.800	0.792	0.770	0.751	0.668	0.708
SSMI vs FTSE	0.999	0.875	0.788	0.864	0.835	0.797	0.763	0.799	0.775	0.769	0.661
SSMI vs HSI	0.953	0.317	0.405	0.539	0.503	0.478	0.516	0.527	0.424	0.470	0.560
DJI vs HSI	0.949	0.511	0.581	0.572	0.540	0.489	0.467	0.485	0.526	0.397	0.260
NASDAQ vs HSI	-0.000	0.496	0.464	0.476	0.471	0.494	0.519	0.513	0.441	0.548	0.533
IBEX vs HSI	0.927	0.499	0.518	0.505	0.450	0.500	0.471	0.460	0.548	0.582	0.880
FTMIB vs HSI	0.763	0.486	0.498	0.529	0.514	0.504	0.483	0.534	0.663	0.654	0.572
NASDAQ vs N225	0.213	0.451	0.533	0.542	0.561	0.509	0.473	0.429	0.496	0.601	0.113
SSMI vs N225	0.999	0.588	0.556	0.608	0.587	0.533	0.481	0.511	0.405	0.297	-0.167
IBEX vs N225	0.981	0.836	0.678	0.504	0.553	0.534	0.476	0.502	0.433	0.331	-0.010
N225 vs HSI	0.950	0.544	0.534	0.606	0.588	0.545	0.520	0.521	0.466	0.315	0.038
GDAX vs HSI	0.952	0.488	0.540	0.548	0.522	0.545	0.566	0.548	0.539	0.636	0.562

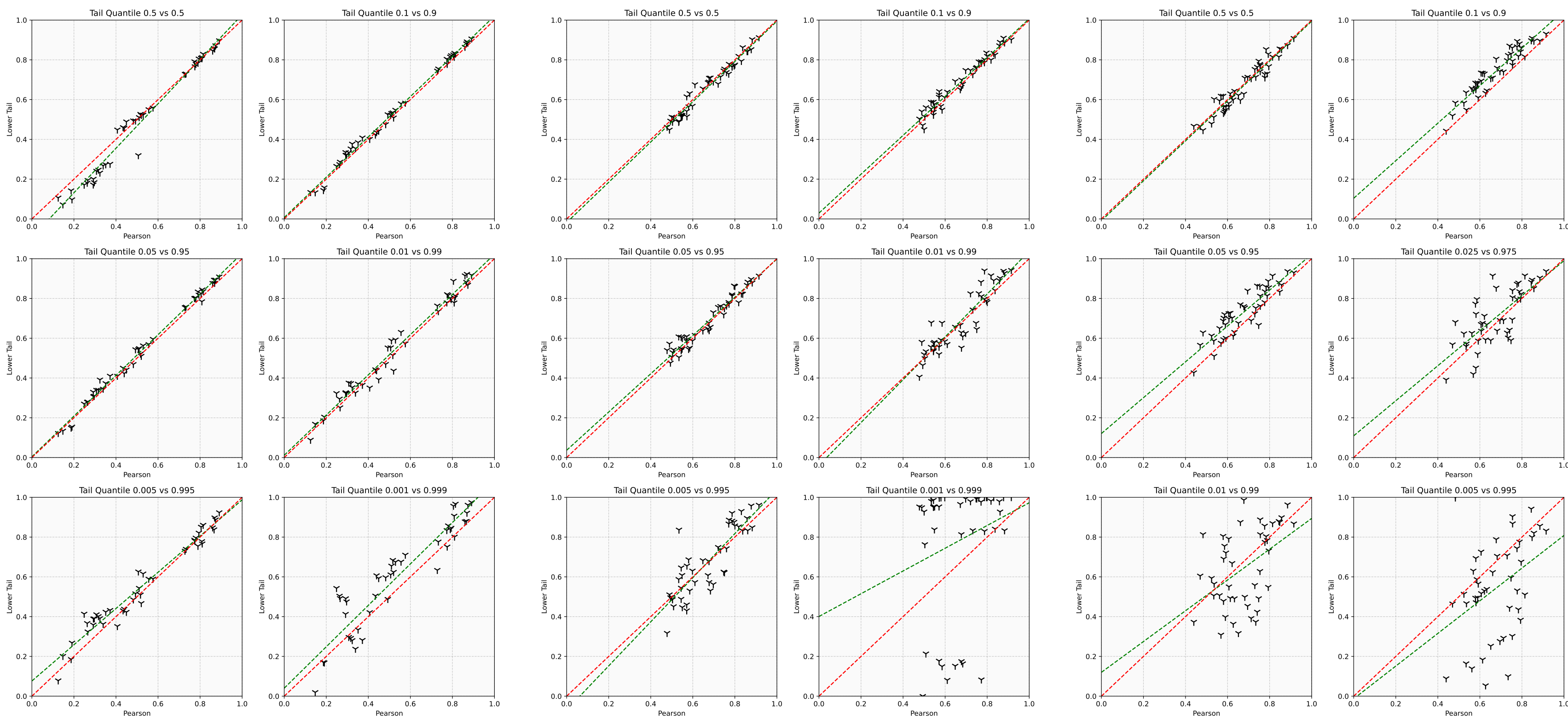
Table 21: ES-implied correlation - Top 10 strongest and weakest pairs for weekly returns

Correlation Pairs	Lower Tail (Losses) 0.001	0.005	0.01	0.05	0.01	Pearson	Upper Tail (Gains) 0.9	0.95	0.99	0.995	0.999
FCHI vs GDAX	0.832	0.867	0.937	0.930	0.930	0.915	0.855	0.888	0.884	0.807	0.455
FCHI vs FTMIB	0.856	0.963	0.904	0.937	0.896	0.885	0.831	0.827	0.890	0.875	0.958
IBEX vs FTMIB	0.820	0.898	0.851	0.868	0.895	0.857	0.835	0.806	0.779	0.886	0.942
IBEX vs FCHI	0.799	0.875	0.893	0.833	0.909	0.848	0.798	0.852	0.838	0.866	0.995
FCHI vs FTSE	0.942	0.880	0.882	0.880	0.897	0.844	0.852	0.798	0.822	0.905	0.956
FTMIB vs GDAX	0.510	0.869	0.915	0.914	0.816	0.814	0.724	0.763	0.811	0.695	0.532
GDAX vs FTSE	0.675	0.732	0.822	0.855	0.862	0.796	0.662	0.729	0.830	0.788	0.454
FCHI vs SSMI	0.383	0.548	0.798	0.887	0.841	0.794	0.700	0.736	0.712	0.725	0.469
DJI vs FTSE	0.776	0.784	0.835	0.814	0.881	0.789	0.761	0.758	0.691	0.686	0.703
IBEX vs GDAX	0.436	0.803	0.879	0.858	0.818	0.783	0.658	0.639	0.678	0.671	0.485
SSMI vs HSI	0.089	0.373	0.391	0.427	0.442	0.440	0.359	0.434	0.452	0.609	0.730
FTMIB vs HSI	0.465	0.605	0.568	0.566	0.519	0.470	0.417	0.412	0.291	0.095	0.054
N225 vs HSI	0.998	0.813	0.681	0.628	0.584	0.483	0.366	0.326	0.191	-0.083	-0.068
FCHI vs HSI	0.514	0.593	0.623	0.612	0.583	0.524	0.564	0.602	0.511	0.340	0.236
GDAX vs HSI	0.164	0.505	0.559	0.587	0.635	0.535	0.606	0.560	0.371	0.107	0.082
IBEX vs HSI	0.466	0.565	0.572	0.510	0.548	0.536	0.511	0.539	0.510	0.370	0.278
SSMI vs N225	0.139	0.507	0.625	0.650	0.656	0.562	0.384	0.387	0.291	0.295	0.329
NASDAQ vs SSMI	0.630	0.308	0.419	0.576	0.647	0.569	0.449	0.292	0.100	-0.020	-0.058
IBEX vs N225	0.494	0.805	0.773	0.687	0.684	0.579	0.308	0.302	0.346	0.554	0.906
DJI vs HSI	0.693	0.478	0.451	0.594	0.652	0.581	0.585	0.533	0.522	0.511	0.478

Table 22: ES-implied correlation - Top 10 strongest and weakest pairs for monthly returns

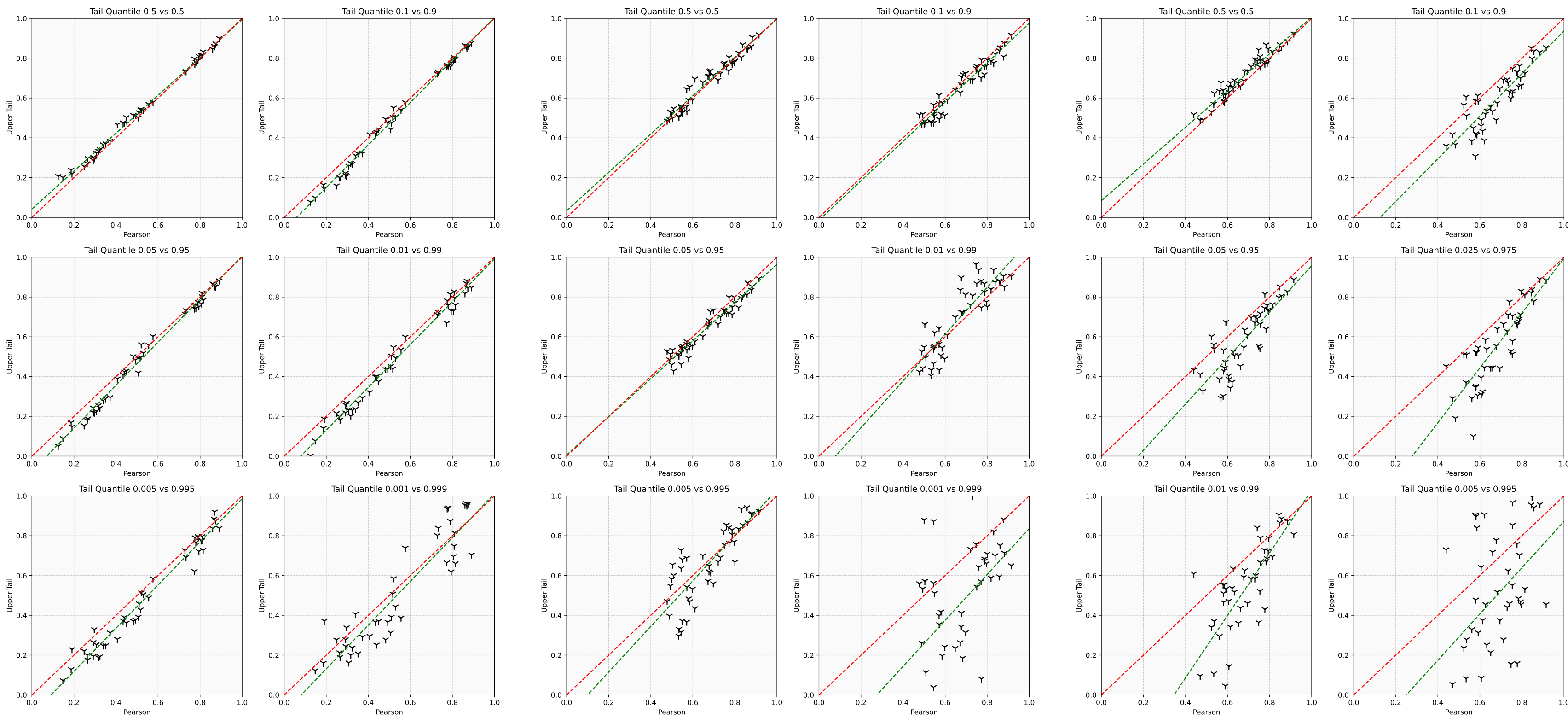
B.8 Tail Distortion

Distortion Analysis - Implied Correlation vs Pearson Correlation



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

Figure 29: ES implied correlation: Lower tail vs Pearson - Matching pairs will lie on the red line. The green line represents a linear approximation of the two correlation types. The corresponding regression results are reported in table 11.



(a) Daily Returns (b) Weekly Returns (c) Monthly Returns

Figure 30: ES implied correlation: Upper tail vs Pearson - Matching pairs will lie on the red line. The green line represents a linear approximation of the two correlation types. The corresponding regression results are reported in table 11.

C Notes on Matrix Algebra

C.1 Matrix Operators

Definition C.1 (Kronecker Product)
Let $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{p \times q}$. The Kronecker product or direct product of A and B is obtained by multiplying each entry in $\boldsymbol{A}$ by the entire matrix $\boldsymbol{B}$:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \tag{154}$$

The Kronecker product can be generalized into a block Kronecker product or Kathri-Rao product, see Khatri & Rao (1968). We are only interested in the row-wise Kroncker product which is a special case of the Kathri-Rao product. In our work this operation is required to summarize all portfolios into a linear system. Formally,

Definition C.2 (Kronecker Row-Product)
Let $\boldsymbol{A} = (a_{ij}), \boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{m \times n}$. The Kronecker row-product of A and B is obtained by multiplying each entry in $\boldsymbol{A}$ by the entire row of $\boldsymbol{B}$:

$$\boldsymbol{A} \otimes_r \boldsymbol{B} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \otimes_r \begin{pmatrix} B_1 \\ \hline \vdots \\ \hline B_m \end{pmatrix} = \begin{pmatrix} a_{11}B_1 & \dots & a_{1n}B_1 \\ \vdots & \vdots & \vdots \\ a_{m1}B_m & \dots & a_{mn}B_m \end{pmatrix} \tag{155}$$

Definition C.3 (Vectorization Operator)
The vectorization operator, denoted by vec, is a linear operator that transforms a matrix into a vector by stacking its columns into a single column vector. Let $A \in \mathbb{R}^{m \times n}$ such that

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \tag{156}$$

the vectorization of A, denoted by $vec(A)$, is defined as:

$$vec(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \in \mathbb{R}^{mn \times 1}. \tag{157}$$

Following Lütkepohl (2006)[3], the *vec* operator has the following properties:

Let A, B, C be two matricies with appropriate dimensions.

$$\text{vec}(\mathbf{A} + \mathbf{B}) = \text{vec}(\mathbf{A}) + \text{vec}(\mathbf{B}) \tag{158}$$

$$\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\text{vec}(\mathbf{B}) \tag{159}$$

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{p \times q}$

$$\text{vec}(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{I}_n \otimes \mathbf{K}_{qm} \otimes \mathbf{I}_p)(\text{vec}(\mathbf{G}) \otimes \text{vec}(\mathbf{F})) \tag{160}$$

We also need some results regarding the Khatri-Rao product from Lev-Ari (2005)[4]. We follow the same notation as the author

Let $A \in \mathbb{R}^{n \times L}$ and $B \in \mathbb{R}^{p \times L}$ be matrices that share the same amount of columns. Then

$$(\mathbf{A} \otimes_c \mathbf{B}) = (\mathbf{A} \otimes \mathbf{B})\mathbf{S}_L \tag{161}$$

where $\otimes_c$ denotes the column-wise Kronecker product, also known as special case of the Khatri-Rao product. Furthermore, $\mathbf{S}_L$ is a selection matrix of the form

$$S_L = \begin{bmatrix} e_1 & e_{L+2} & e_{2L+3} & \cdots & e_{L^2} \end{bmatrix} \tag{162}$$

and e_k is an $L^2 \times 1$ column vector with a unity element in the k-th position and zeros elsewhere. In our reserach, we use the row-wise Kronecker product hence

$$(\mathbf{A} \otimes_r \mathbf{B}) = \mathbf{S}_L'(\mathbf{A} \otimes \mathbf{B}). \tag{163}$$

[3] (158), (159) : p.622, (1),(2) ; (160): p.664, (27)

[4] (161) : p.127, (15),(16a),(16b)

C.2 Matrix Differentials

We revise some results and rules regarding matrix differentials from Abadir & Magnus (2005)[5].

For two matrix functions $\mathbf{F}$ and $\mathbf{G}$ of the same order, we have:

$$d(\mathbf{F}+\mathbf{G}) = d\mathbf{F} + d\mathbf{G} \tag{164}$$

Let $\mathbf{F}$ and $\mathbf{G}$ be two conformable matrix functions. For the chain rule and kronecker product respectively it holds:

$$d(\mathbf{FG}) = (d\mathbf{F})G + \mathbf{F}(d\mathbf{G}) \tag{165}$$

$$d(\mathbf{F}\otimes\mathbf{G}) = (d\mathbf{F})\otimes\mathbf{G} + \mathbf{F}\otimes(d\mathbf{G}) \tag{166}$$

For a vector function, the differential and derivative can be obtained by

$$d\mathbf{f}(\mathbf{x}) = \mathbf{A}d\mathbf{x} \qquad \text{with} \qquad D\mathbf{f}(\mathbf{x}) = A \tag{167}$$

If $\mathbf{A}$ is also a function of $\mathbf{x}$, say $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$, the differential is

$$d\mathbf{f}(\mathbf{x}) = d\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{A}(\mathbf{x})d\mathbf{x} \tag{168}$$

with

$$D\mathbf{f}(\mathbf{x}) = \frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}'} = (\mathbf{x}'\otimes\mathbf{I})\frac{\partial\ \mathrm{vec}(\mathbf{A})}{\partial\mathbf{x}'} + \mathbf{A}$$

Finally, on the interplay of the vec and differential operator:

$$d\mathrm{vec}(\mathbf{F}) = \mathrm{vec}(d\mathbf{F}) \tag{169}$$

[5](164): p.355, 13.1 ; (165) (166): p.355, 13.3 ; (167), (168): p.360, 13.13 ; (169): p.355, 13.12

C.3 Technical Notes on the Derivation of Theoretical Standard Errors

We present the exact identified case with $n(n-1)/2$ two asset equal weighted portfolios. First, consider $\mathcal{D}_{A_\alpha}$. Let $k, i = 1, \dots, n$ and $l = 1, \dots, m$. Note that Z_α only depends on the asset quantiles $\boldsymbol{q}_\alpha$.

$$A_\alpha = Z_\alpha \otimes Z_\alpha = \begin{bmatrix} \cdots & q_k w_k^{(1)} \gamma_1 & \cdots & q_k w_k^{(1)} \gamma_i & \cdots & q_k w_k^{(1)} \gamma_n & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & q_k w_k^{(l)} \gamma_1 & \cdots & q_k w_k^{(l)} \gamma_i & \cdots & q_k w_k^{(l)} \gamma_n & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & q_k w_k^{(m)} \gamma_1 & \cdots & q_k w_k^{(m)} \gamma_i & \cdots & q_k w_k^{(m)} \gamma_n & \cdots \end{bmatrix}$$

with $\gamma_i = \left[q_{\alpha_i} w_i^{(1)} \cdots q_{\alpha_i} w_i^{(m)} \right]^\top$ which is a column of Z_α. Vectorizing $Z_\alpha \otimes Z_\alpha$ yields

$$\text{vec}(Z_\alpha \otimes Z_\alpha) = \begin{bmatrix} \vdots \\ q_k w_k^{(1)} \gamma_i \\ \vdots \\ q_k w_k^{(l)} \gamma_i \\ \vdots \\ q_k w_k^{(m)} \gamma_i \\ \vdots \end{bmatrix}$$

Now, interpreting each entry as a function of $\mathfrak{q}$ we can compute the derivative. There are two cases to distinguish.

For $i = k$

$$q_k w_k^{(l)} \gamma_k = \begin{bmatrix} q_k w_k^{(1)} q_k w_k^{(1)} \\ \vdots \\ q_k w_k^{(l)} q_k w_k^{(l)} \\ \vdots \\ q_k w_k^{(m)} q_k w_k^{(m)} \end{bmatrix} \quad \overset{\frac{\partial}{\partial \mathfrak{q}}}{\Longrightarrow} \quad \begin{bmatrix} 0 & \cdots & 2 q_k w_k^{(1)} q_k w_k^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 2 q_k w_k^{(l)} q_k w_k^{(l)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 2 q_k w_k^{(m)} q_k w_k^{(m)} & \cdots & 0 \end{bmatrix}$$

The k-th column contains the derivatives and all other entries are zero.

For $i \neq k$: $i > k$

$$q_k w_k^{(l)} \gamma_i = \begin{bmatrix} q_k w_k^{(1)} q_i w_i^{(1)} \\ \vdots \\ q_k w_k^{(l)} q_i w_i^{(l)} \\ \vdots \\ q_k w_k^{(m)} q_i w_i^{(m)} \end{bmatrix} \xrightarrow{\frac{\partial}{\partial \mathsf{q}}} \begin{bmatrix} 0 & \cdots & w_k^{(l)} q_i w_i^{(1)} & \cdots & q_k w_k^{(1)} w_i^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & w_k^{(l)} q_i w_i^{(l)} & \cdots & q_k w_k^{(l)} w_i^{(l)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & w_k^{(l)} q_i w_i^{(m)} & \cdots & q_k w_k^{(m)} w_i^{(m)} & \cdots & 0 \end{bmatrix}$$

The k-th and i-th column contain the derivatives and all other entries are zero. If $i < k$ the columns in the last matrix is switched. Stacking the matrices after going through all indices delivers $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}_\alpha}$.

Now consider $\boldsymbol{\mathcal{D}}_{\tilde{\boldsymbol{q}}_p}$. Recall the definition of the squared excess quantile for one specific weight vector in (79),

$$\tilde{q}_p = q_p^2 - \sum_{i=1}^{n} q_{\alpha_i}^2 w_i^2$$

Interpreting the vector $\tilde{q}_p$ which reflects different squared excess portfolio quantiles, see (17), as a function of q, the derivative with respect to both asset and portfolio quantiles can be calculated directly.

$$\frac{\partial \tilde{\boldsymbol{q}}_p}{\partial \mathsf{q}'} = \begin{bmatrix} -2q_{\alpha_1} w_{1,(1)}^2 & \cdots & -2q_{\alpha_n} w_{n,(1)}^2 & | & 2q_{p,1} & \cdots & 0 \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ -2q_{\alpha_1} w_{1,(m1)}^2 & \cdots & -2q_{\alpha_n} w_{n,(m)}^2 & | & 0 & \cdots & 2q_{p,m} \end{bmatrix}$$

D Forecasting Correlation

D.1 CAViaR Estimation Results

	Asymmetric Slope					
	DJI	NASDAQ	FCHI	GDAX	NK225	HSI
1% **VaR**						
β_1	**-0.0728**	**-0.1184**	**-0.0657**	**-0.1060**	**-0.1681**	**-0.1307**
Standard Errors	0.0098	0.0187	0.0198	0.0323	0.0413	0.0441
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
β_2	**0.9199**	**0.9114**	**0.9337**	**0.9036**	**0.8661**	**0.9027**
Standard Errors	0.0165	0.0117	0.0112	0.0220	0.0225	0.0244
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
β_3	**-0.0100**	**-0.0651**	**-0.0777**	**-0.1472**	**-0.1274**	**-0.0582**
Standard Errors	0.0571	0.0396	0.0317	0.0654	0.0540	0.0444
p-values	0.4305	0.0503	0.0071	0.0122	0.0092	0.0952
β_4	**0.3582**	**0.2993**	**0.2193**	**0.2810**	**0.5152**	**0.3518**
Standard Errors	0.0830	0.0350	0.0224	0.0475	0.0782	0.0766
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5% **VaR**						
β_1	**-0.0457**	**-0.0441**	**-0.0327**	**-0.0488**	**-0.0656**	**-0.0414**
Standard Errors	0.0126	0.0071	0.0058	0.0065	0.0096	0.0111
p-values	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
β_2	**0.9013**	**0.9288**	**0.9233**	**0.9261**	**0.9032**	**0.9272**
Standard Errors	0.0212	0.0080	0.0160	0.0065	0.0109	0.0118
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
β_3	**-0.0213**	**-0.0527**	**-0.0256**	**-0.0042**	**-0.0380**	**-0.0442**
Standard Errors	0.0289	0.0204	0.0283	0.0171	0.0278	0.0234
p-values	0.2302	0.0049	0.1826	0.4028	0.0852	0.0297
β_4	**0.3054**	**0.1820**	**0.2500**	**0.2323**	**0.2736**	**0.2070**
Standard Errors	0.0667	0.0191	0.0471	0.0189	0.0319	0.0202
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
25% **VaR**						
β_1	**-0.0024**	**-0.0022**	**-0.0108**	**-0.0059**	**-0.0069**	**0.0000**
Standard Errors	0.0015	0.0023	0.0056	0.0031	0.0038	0.0017
p-values	0.0533	0.1679	0.0256	0.0292	0.0361	0.4993
β_2	**0.9579**	**0.9632**	**0.9427**	**0.9427**	**0.9610**	**0.9763**
Standard Errors	0.0060	0.0120	0.0248	0.0169	0.0080	0.0063
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
β_3	**0.0056**	**0.0174**	**0.0114**	**0.0089**	**0.0034**	**0.0037**
Standard Errors	0.0094	0.0098	0.0163	0.0134	0.0106	0.0068
p-values	0.2768	0.0379	0.2422	0.2546	0.3723	0.2914
β_4	**0.0624**	**0.0692**	**0.0793**	**0.0838**	**0.0471**	**0.0396**
Standard Errors	0.0043	0.0136	0.0226	0.0244	0.0071	0.0090
p-values	0.0000	0.0000	0.0002	0.0003	0.0000	0.0000

Table 23: Asymmetric CAViaR parameter estimates for daily returns.

	Asymmetric Slope					
	DJI	NASDAQ	FCHI	GDAX	NK225	HSI
1% **VaR**						
β_1	**-1.0213**	**-1.1511**	**-1.1403**	**-1.4296**	**-2.4247**	**-0.9387**
Standard Errors	1.1611	1.3076	0.3252	0.7127	0.8898	0.9416
p-values	0.1895	0.1893	0.0002	0.0224	0.0032	0.1594
β_2	**0.7244**	**0.7187**	**0.7091**	**0.7201**	**0.6341**	**0.7677**
Standard Errors	0.4038	0.2910	0.0800	0.1456	0.1598	0.2040
p-values	0.0364	0.0068	0.0000	0.0000	0.0000	0.0001
β_3	**0.0950**	**-0.1642**	**-0.0384**	**0.0195**	**0.3623**	**-0.1236**
Standard Errors	0.3243	0.2508	0.1203	0.1472	0.0958	0.2562
p-values	0.3848	0.2564	0.3748	0.4474	0.0001	0.3147
β_4	**1.0000**	**1.0000**	**0.8218**	**0.5699**	**0.7574**	**0.5954**
Standard Errors	2.0099	1.1189	0.2564	0.2509	0.6170	0.4982
p-values	0.3094	0.1857	0.0007	0.0116	0.1098	0.1161
5% **VaR**						
β_1	**-0.8565**	**-0.3214**	**-0.2688**	**-0.3384**	**-1.3337**	**-0.1385**
Standard Errors	0.6392	0.1589	0.0748	0.1021	0.4156	0.0461
p-values	0.0901	0.0216	0.0002	0.0005	0.0007	0.0013
β_2	**0.6070**	**0.8084**	**0.8886**	**0.8633**	**0.6892**	**0.9251**
Standard Errors	0.1982	0.0779	0.0334	0.0535	0.0899	0.0168
p-values	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000
β_3	**-0.1900**	**-0.1855**	**0.0273**	**0.0015**	**0.1652**	**-0.0062**
Standard Errors	0.0756	0.1142	0.0323	0.0663	0.0681	0.0206
p-values	0.0060	0.0522	0.1986	0.4907	0.0076	0.3823
β_4	**0.5354**	**0.3326**	**0.2440**	**0.2990**	**0.3686**	**0.1744**
Standard Errors	0.1624	0.1381	0.0743	0.1073	0.0756	0.0380
p-values	0.0005	0.0080	0.0005	0.0027	0.0000	0.0000
25% **VaR**						
β_1	**-0.0259**	**-0.0059**	**-0.0242**	**-0.0907**	**-0.0061**	**-0.0154**
Standard Errors	0.0248	0.0237	0.0165	0.0531	0.0238	0.0288
p-values	0.1484	0.4026	0.0714	0.0439	0.3992	0.2967
β_2	**0.9347**	**0.9513**	**0.9543**	**0.8803**	**0.9611**	**0.9368**
Standard Errors	0.0402	0.0119	0.0122	0.0420	0.0187	0.0331
p-values	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
β_3	**0.0188**	**0.0133**	**0.0185**	**0.0334**	**-0.0057**	**-0.0214**
Standard Errors	0.0397	0.0324	0.0134	0.0461	0.0246	0.0250
p-values	0.3177	0.3410	0.0832	0.2348	0.4083	0.1963
β_4	**0.0874**	**0.0822**	**0.0743**	**0.1455**	**0.0487**	**0.0765**
Standard Errors	0.0632	0.0216	0.0170	0.0382	0.0199	0.0235
p-values	0.0833	0.0001	0.0000	0.0001	0.0072	0.0006

Table 24: Asymmetric CAViaR parameter estimates for weekly returns.

D.2 DCC Estimation Results

	DJI	**NASDAQ**	**FCHI**	**GDAX**	**N225**	**HSI**
GARCH(1,1) Parameters						
$\hat{\omega}$	0.0142***	0.0170***	0.0205***	0.0205***	0.0205***	0.0802**
SE filtered	(0.0027)	(0.0044)	(0.0051)	(0.0051)	(0.0051)	(0.0096)
t-stat	5.0278	3.9110	3.7150	4.1807	8.0336	1.3360
$\hat{\alpha}_1$	0.0805***	0.0526***	0.0868***	0.0868***	0.0868***	0.0868***
SE filtered	(0.0122)	(0.0078)	(0.01)	(0.001)	(0.0094)	(0.0093)
t-stat	7.6356	10.2414	9.0514	8.5230	9.8952	5.9932
$\hat{\beta}_1$	0.9065***	0.9321***	0.9029***	0.901***	0.903***	0.9014***
SE filtered	(0.0115)	(0.0093)	(0.0091)	(0.0094)	(0.0099)	(0.0093)
t-stat	77.884	97.0858	100.1440	98.2194	99.94540	103.6271
DCC(1,1) Parameters						
$\hat{\lambda}_1$			0.0049			
SE filtered			(0.0071)			
t-stat			1.3951			
$\hat{\lambda}_2$			0.9947***			
SE filtered			(0.0147)			
t-stat			60.9139			

Table 25: Parameter estimates for a GARCH-DCC model with gaussian errors for daily returns.

	DJI	NASDAQ	FCHI	GDAX	N225	HSI
GARCH(1,1) Parameters						
$\hat{\omega}$	0.3770***	0.2258	0.1944	0.63021***	0.9831***	0.1204***
SE filtered	(0.1378)	(0.1690)	(0.1661)	(0.1661)	(0.1661)	(0.0317)
t-stat	2.7356	1.3357	1.1706	3.7949	5.9199	3.8038
$\hat{\alpha}_1$	0.2034***	0.1473***	0.1198***	0.1805***	0.1259***	0.0674
SE filtered	(0.0435)	(0.0315)	(0.0443)	(0.0443)	(0.0443)	(0.0443)
t-stat	4.6739	4.6742	2.7043	4.0745	2.8420	1.5214
$\hat{\beta}_1$	0.7413***	0.8381***	0.8647***	0.7680***	0.7666***	0.9193***
SE filtered	(0.0479)	(0.0465)	(0.0457)	(0.0457)	(0.0457)	(0.0457)
t-stat	15.4809	18.0412	18.9382	16.8204	16.7897	20.1341
DCC(1,1) Parameters						
$\hat{\lambda}_1$				0.01		
SE filtered				(0.0084)		
t-stat				1.1883		
$\hat{\lambda}_2$				0.88***		
SE filtered				(0.0313)		
t-stat				28.7305		

Table 26: Parameter estimates for a GARCH-DCC model with gaussian errors for weekly returns.

D.3 Charts - In-Sample and Out-of-Sample Correlation Forecasts

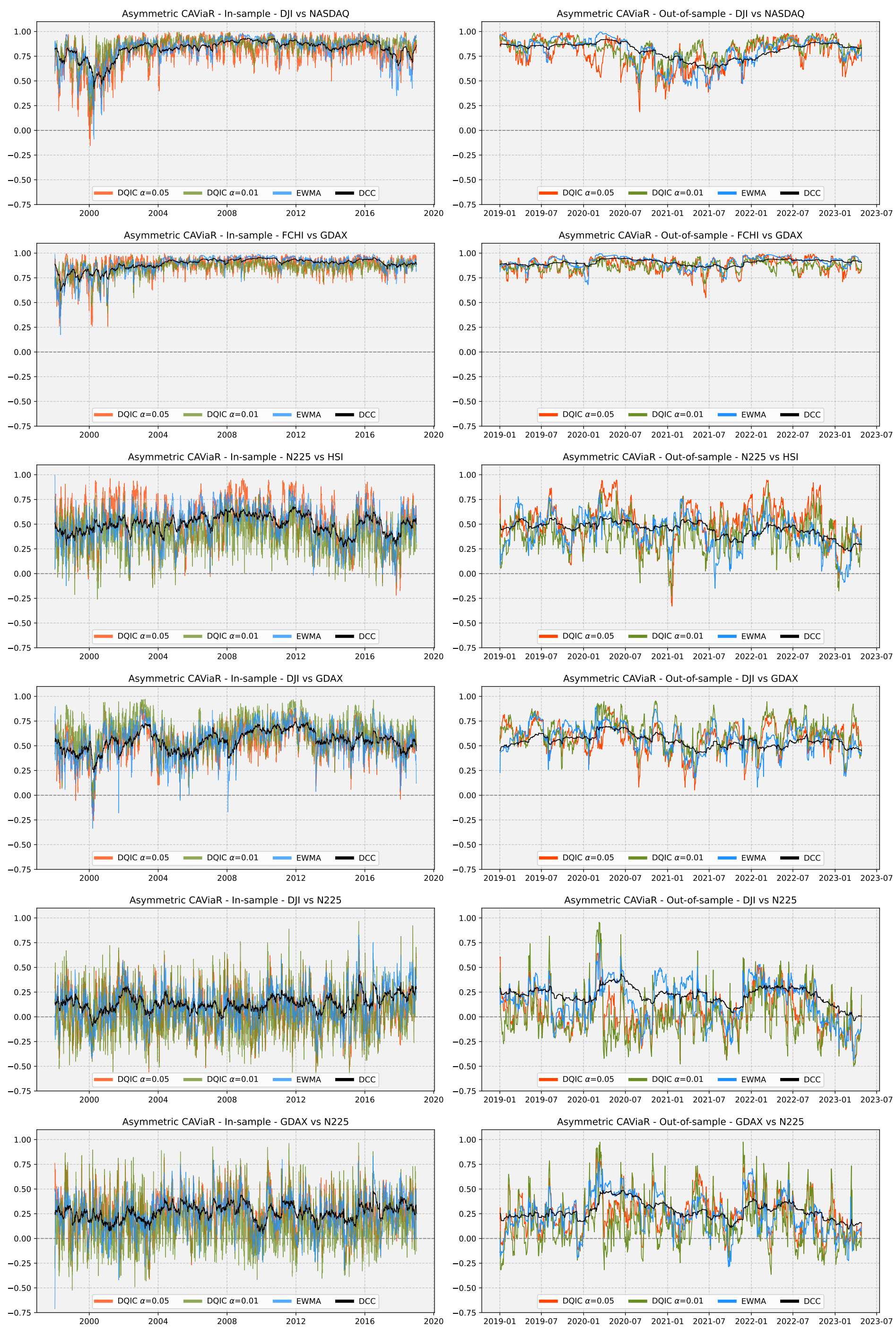


Figure 31: CAViaR implied correlation forecasts - Daily Returns - Exact Identification

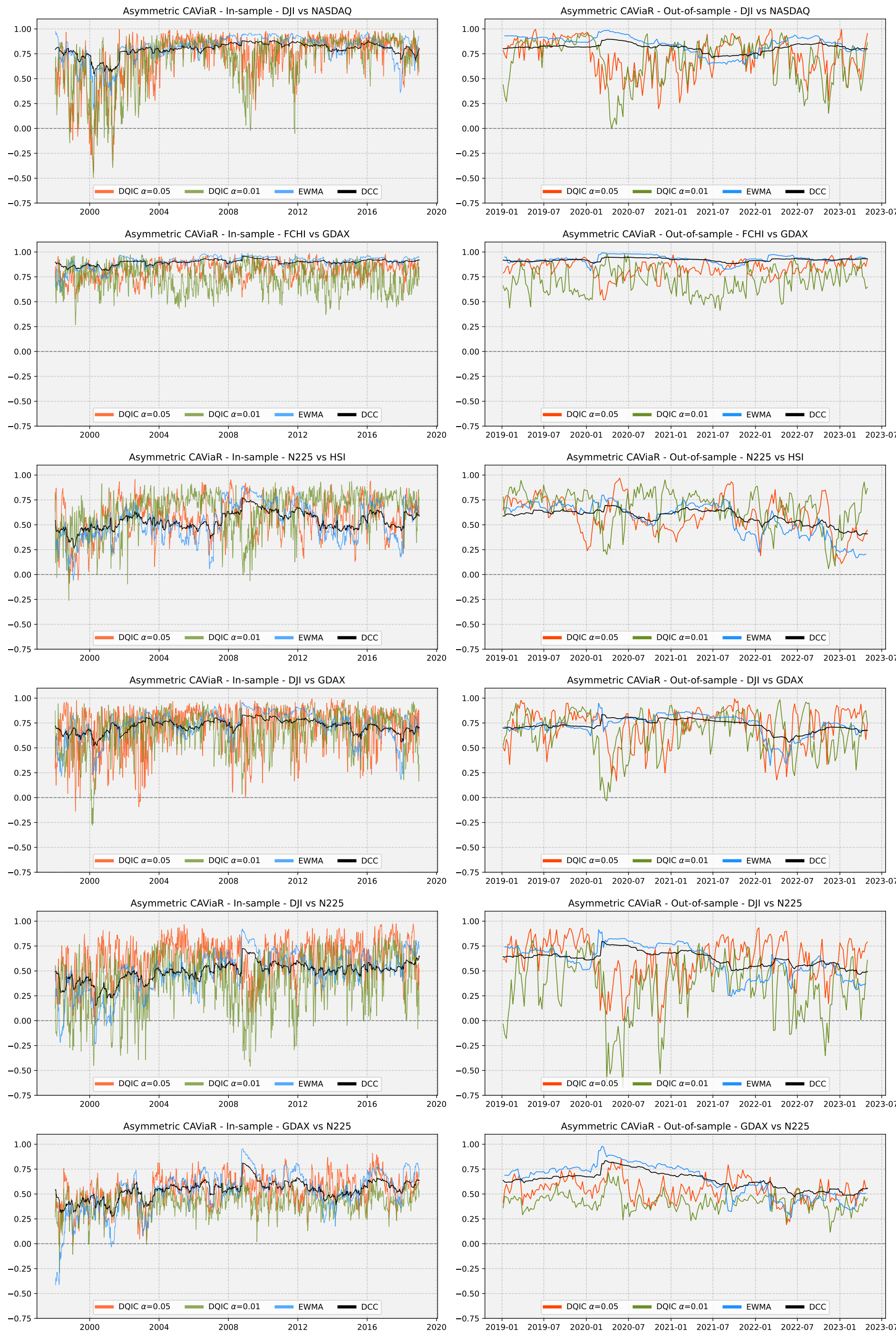


Figure 32: CAViaR implied correlation forecasts - Weekly Returns - Exact Identification

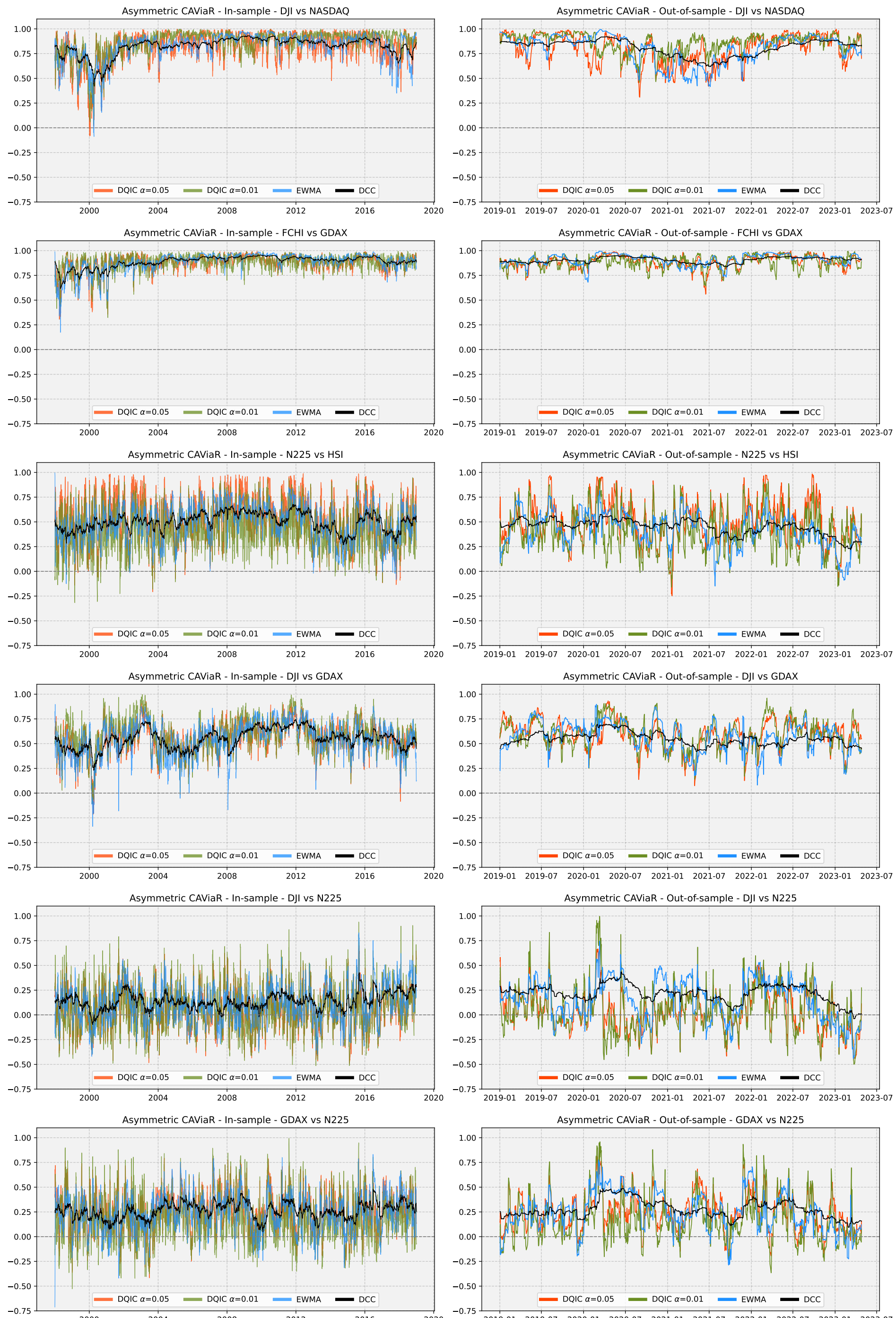


Figure 33: CAViaR implied correlation forecasts - Daily Returns - Overidentification

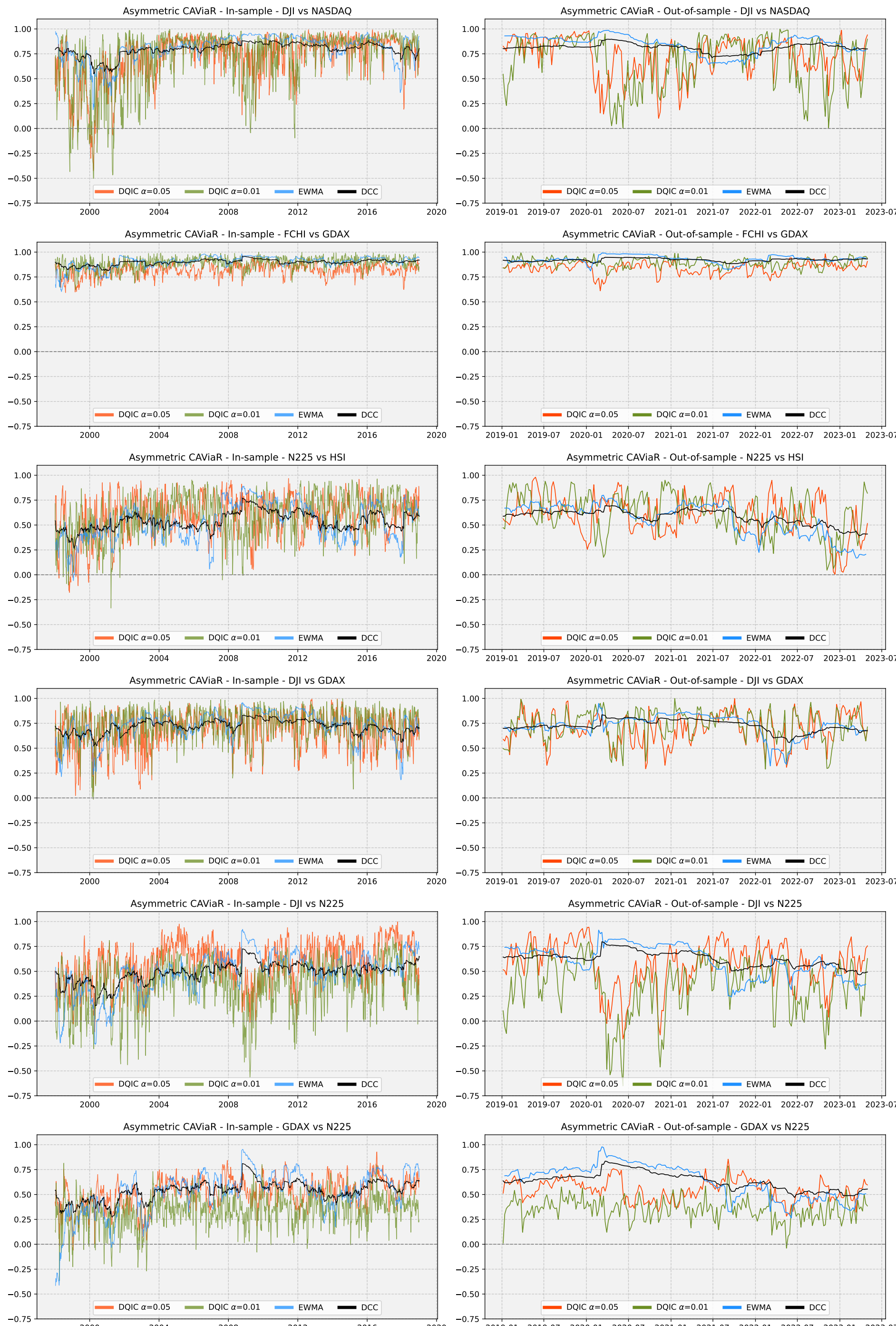


Figure 34: CAViaR implied correlation forecasts - Weekly Returns - Overidentification

D.4 Evaluation with Realized Correlation

Weekly realized correlation constructed from daily returns exhibits substantial variability. We observe erratic changes in figure 35 with large amplitudes where realized correlation admits unrealistic values. Andersen & Bollerslev (1998) emphasized that realized correlations are only accurate if sufficient data is available. In our case, high-frequency intra-day data would be required.

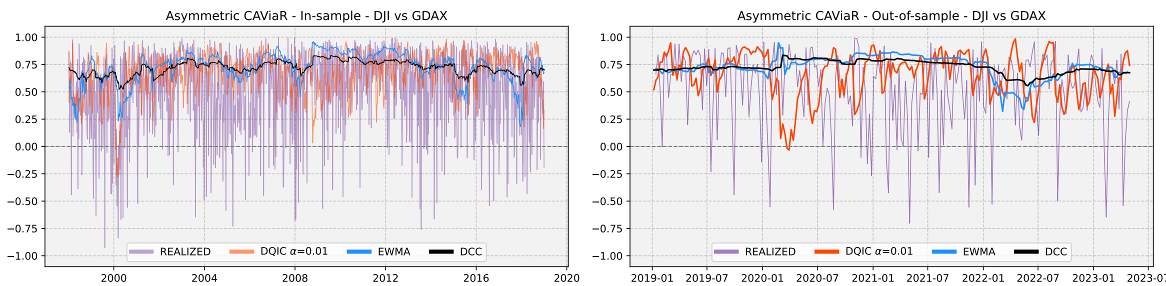


Figure 35: CAViaR implied correlation forecasts - Daily returns - Weekly realized correlation

D.5 Portfolio Evolution

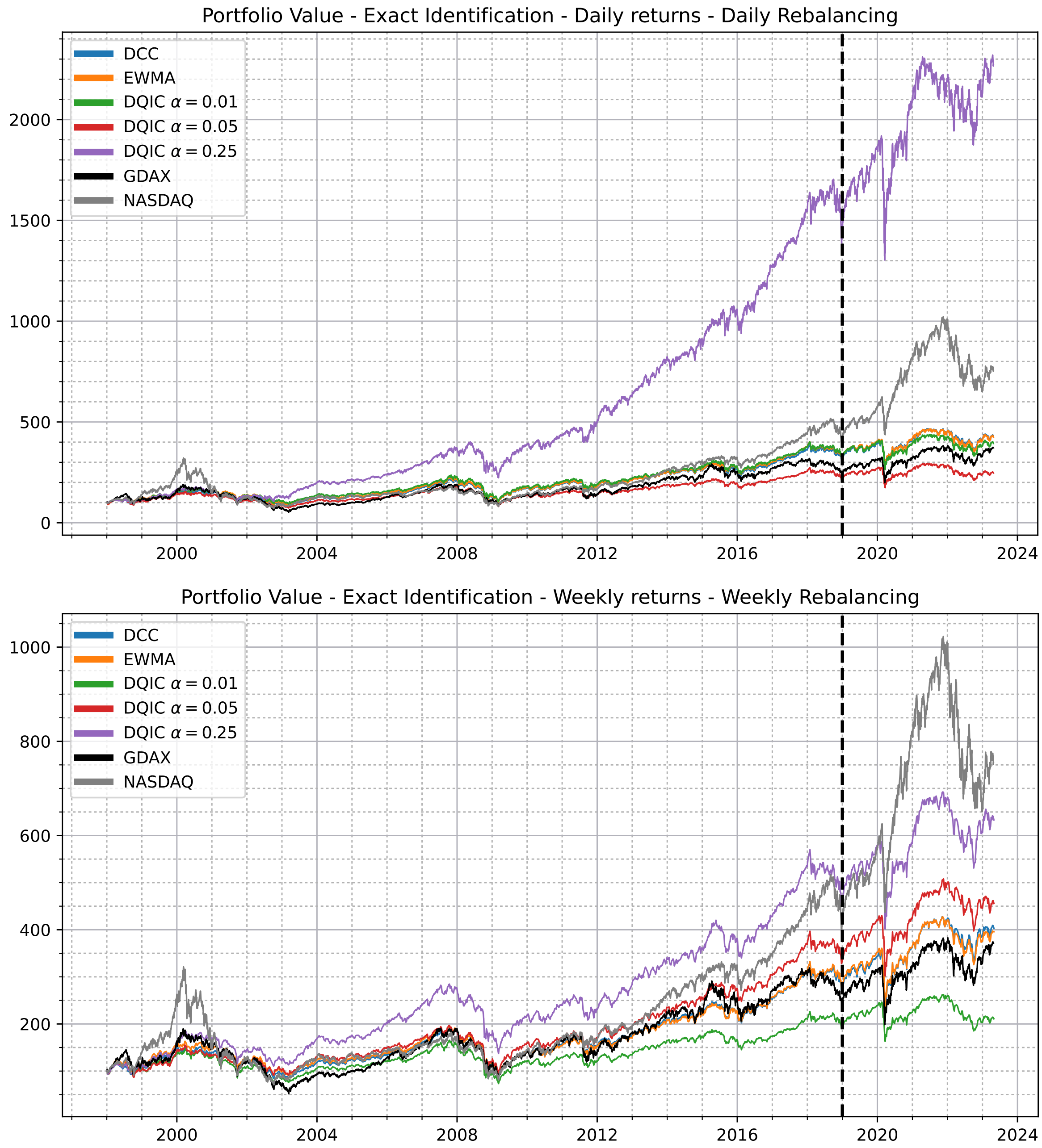
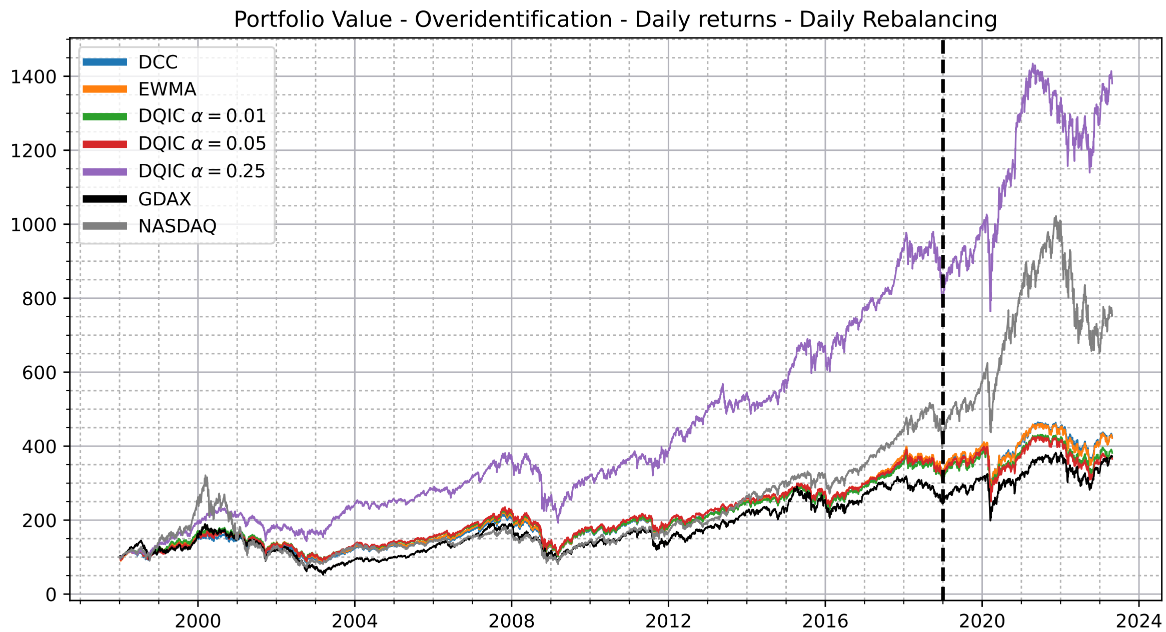


Figure 36: DQIC portfolio performance - Depicted are the evolution of portfolio values for daily (top) and weekly (bottom) returns. Correlation is implied by an exact identified system with daily/weekly rebalancing frequency. The verticle line divides the chart into in-sample CAViaR/DCC (left) and out-of-sample CAViaR/DCC (right) predictions. Initial investment and indices were normed to 100 monetary units. Two indices (GDAX, NASDAQ) are included for comparison



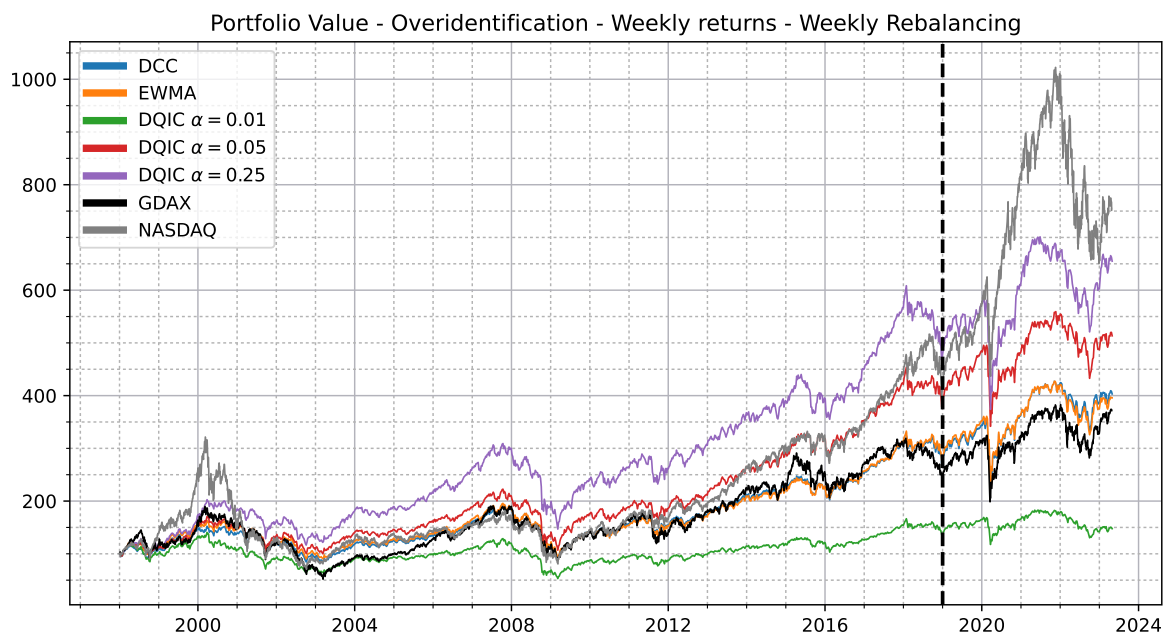


Figure 37: DQIC portfolio performance - Depicted are the evolution of portfolio values for daily (top) and weekly (bottom) returns. Correlation is implied by an overidentified system with daily/weekly rebalancing frequency. The verticle line divides the chart into in-sample CAViaR/DCC (left) and out-of-sample CAViaR/DCC (right) predictions. Initial investment and indices were normed to 100 monetary units. Two indices (GDAX, NASDAQ) are included for comparison

D.6 Stackplots - Portfolio Weights

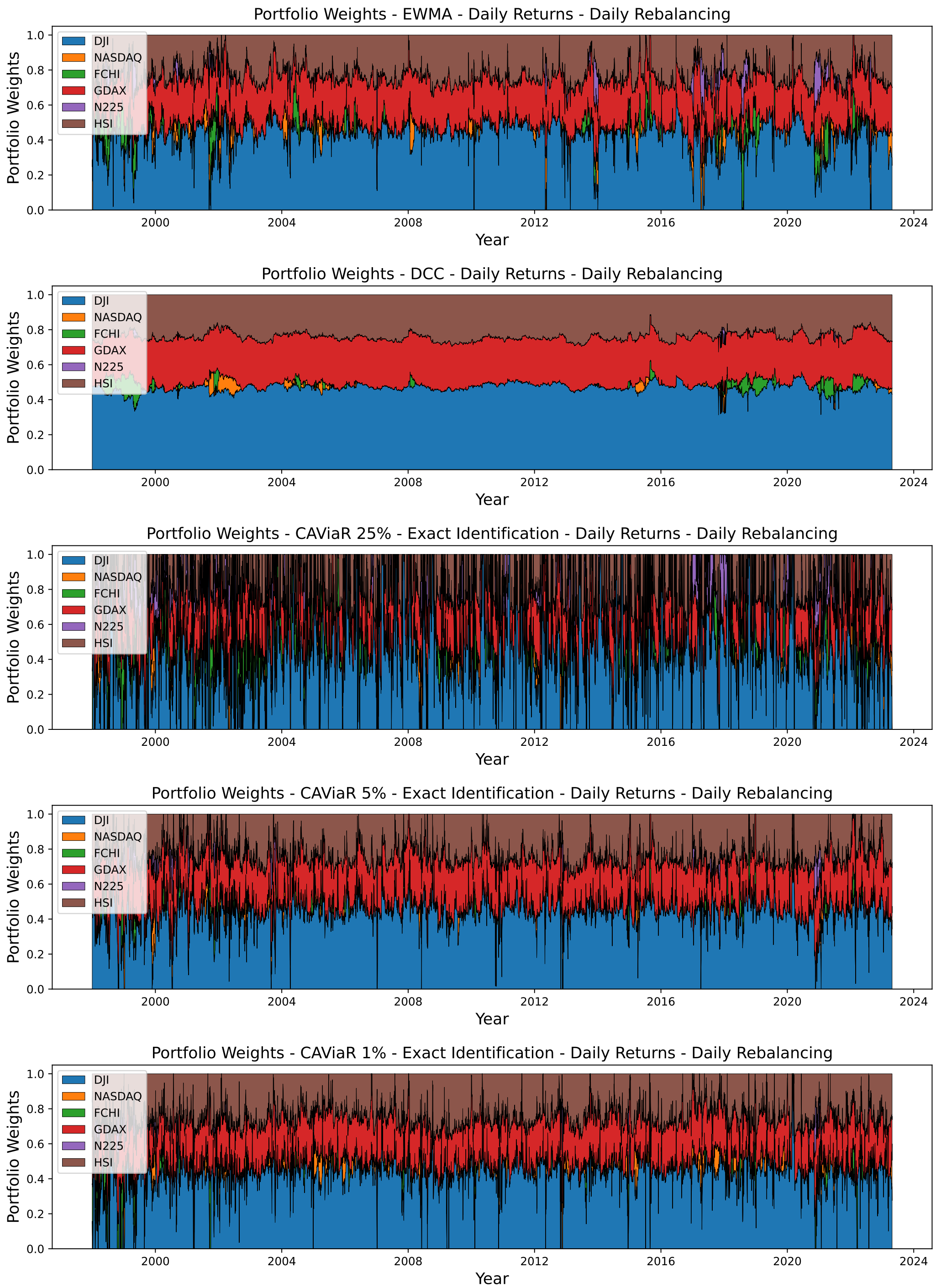


Figure 38: Evolution of portfolio weights - Presented are the portfolio weights with daily rebalancing and daily returns. DQIC from an exact identified system are considered.

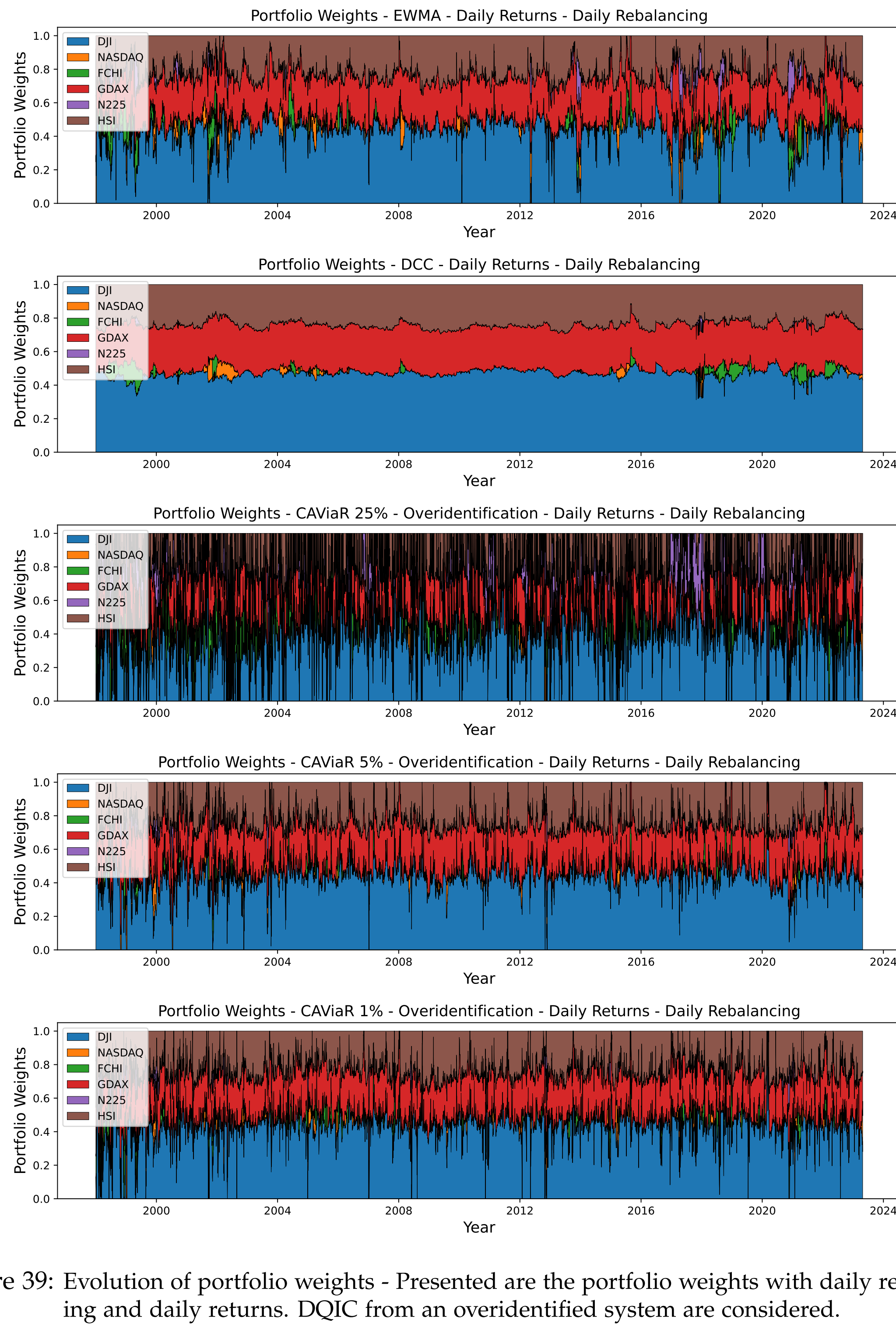


Figure 39: Evolution of portfolio weights - Presented are the portfolio weights with daily rebalancing and daily returns. DQIC from an overidentified system are considered.

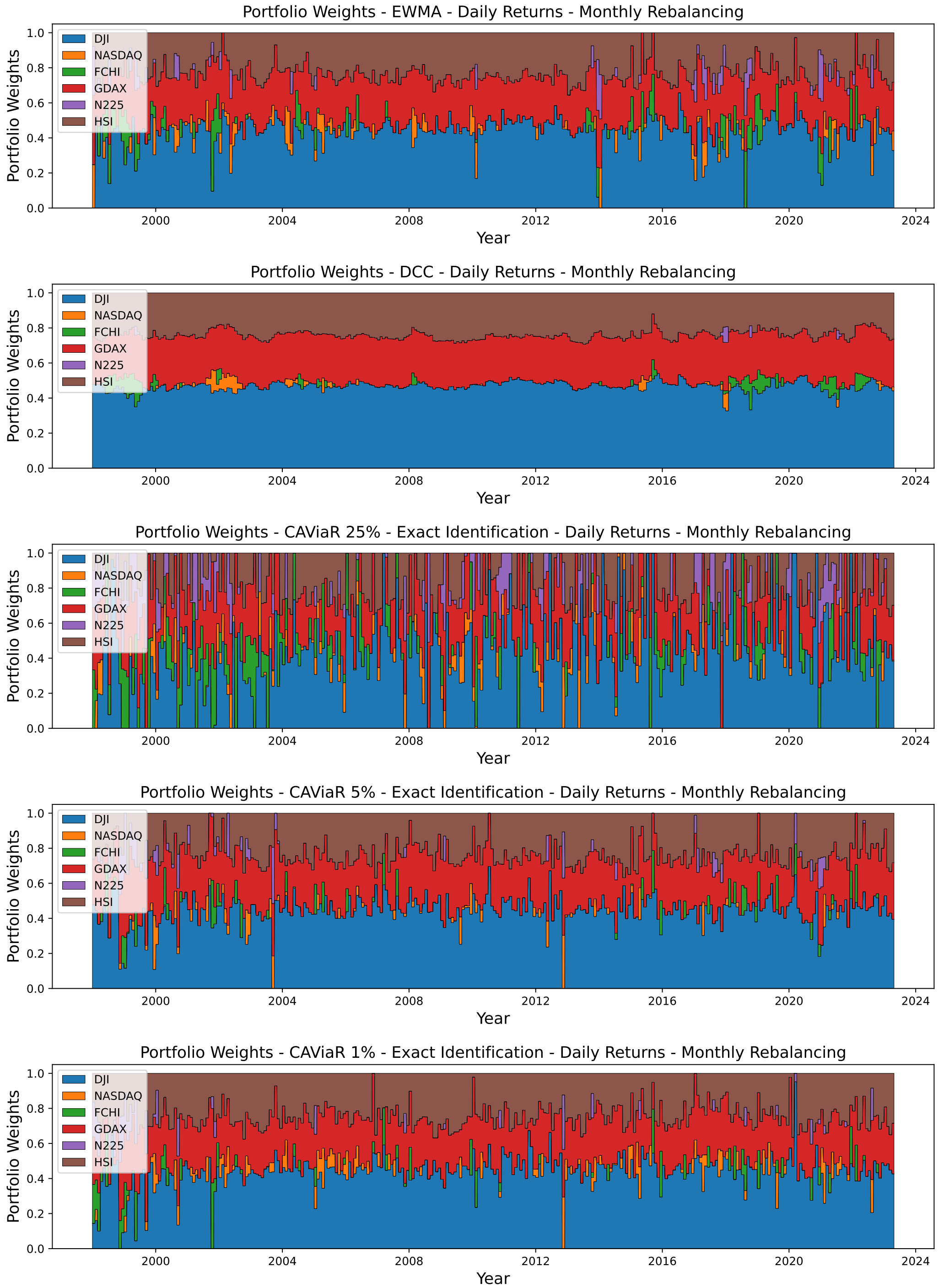


Figure 40: Evolution of portfolio weights - Presented are the portfolio weights with monthly rebalancing and daily returns. DQIC from an exact identified system are considered.

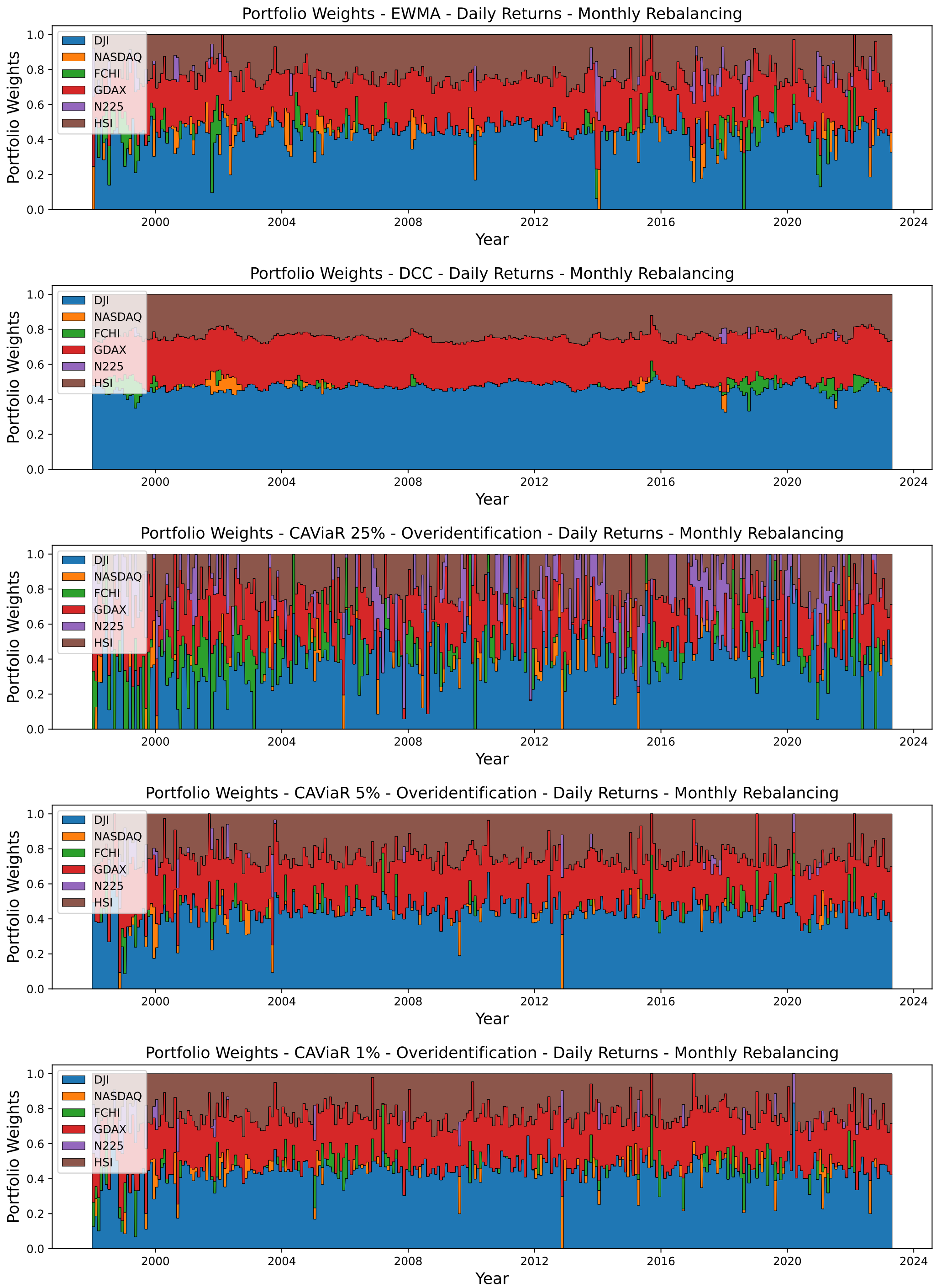


Figure 41: Evolution of portfolio weights - Presented are the portfolio weights with monthly rebalancing and daily returns. DQIC from an overidentified system are considered.

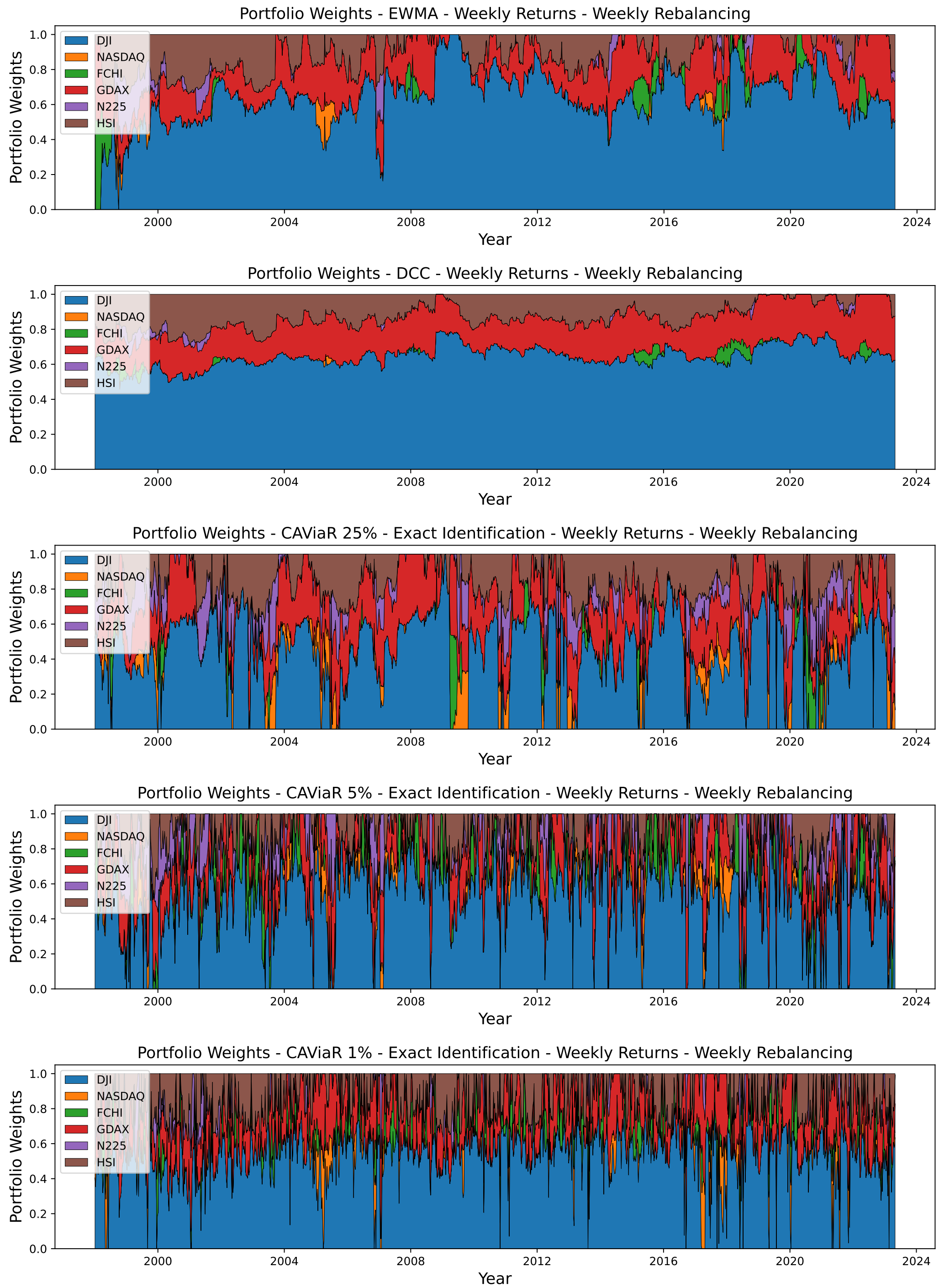


Figure 42: Evolution of portfolio weights - Presented are the portfolio weights with weekly rebalancing and weekly returns. DQIC from an exact system are considered.

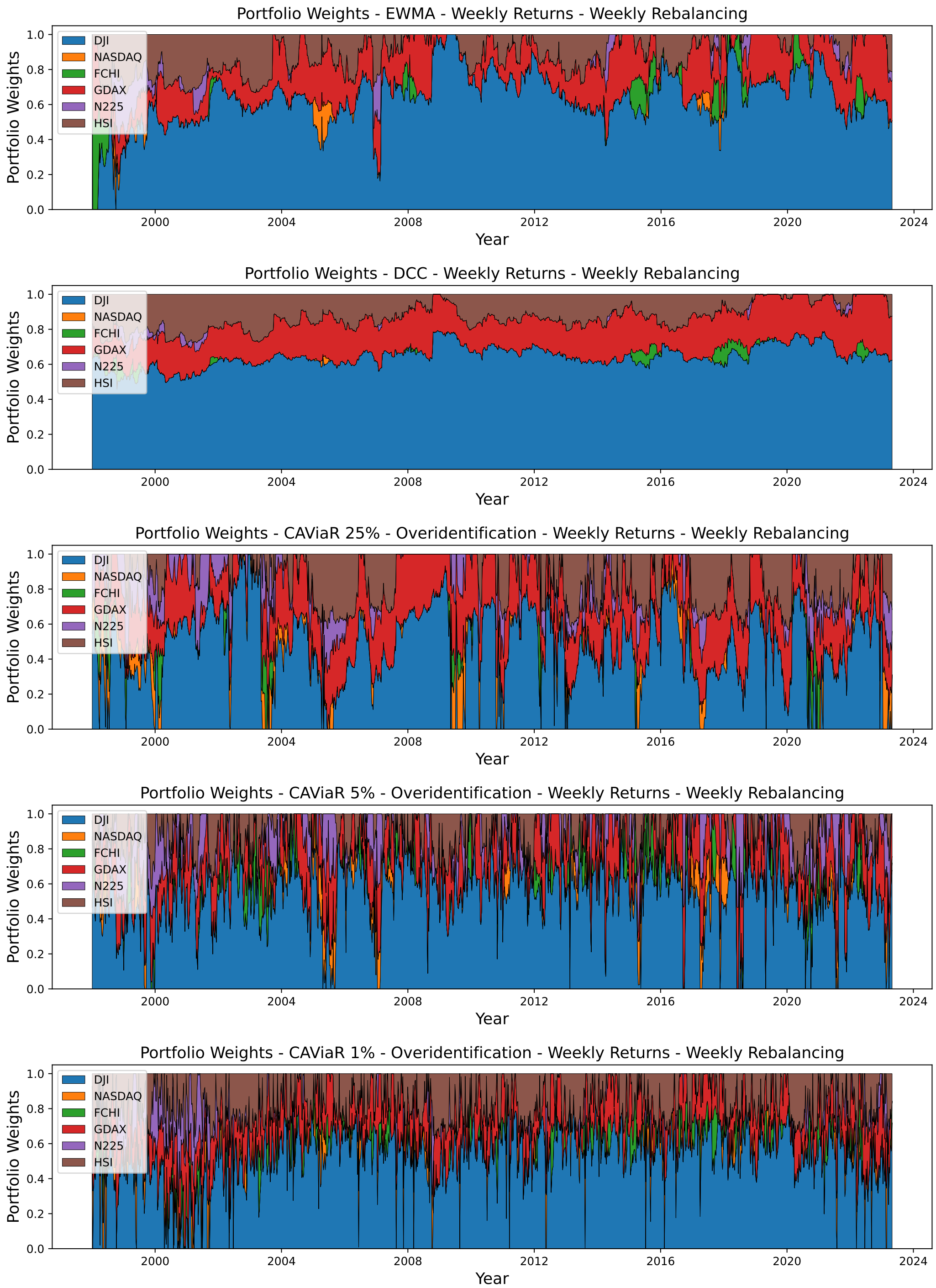


Figure 43: Evolution of portfolio weights - Presented are the portfolio weights with weekly rebalancing and weekly returns. DQIC from an overidentified system are considered.

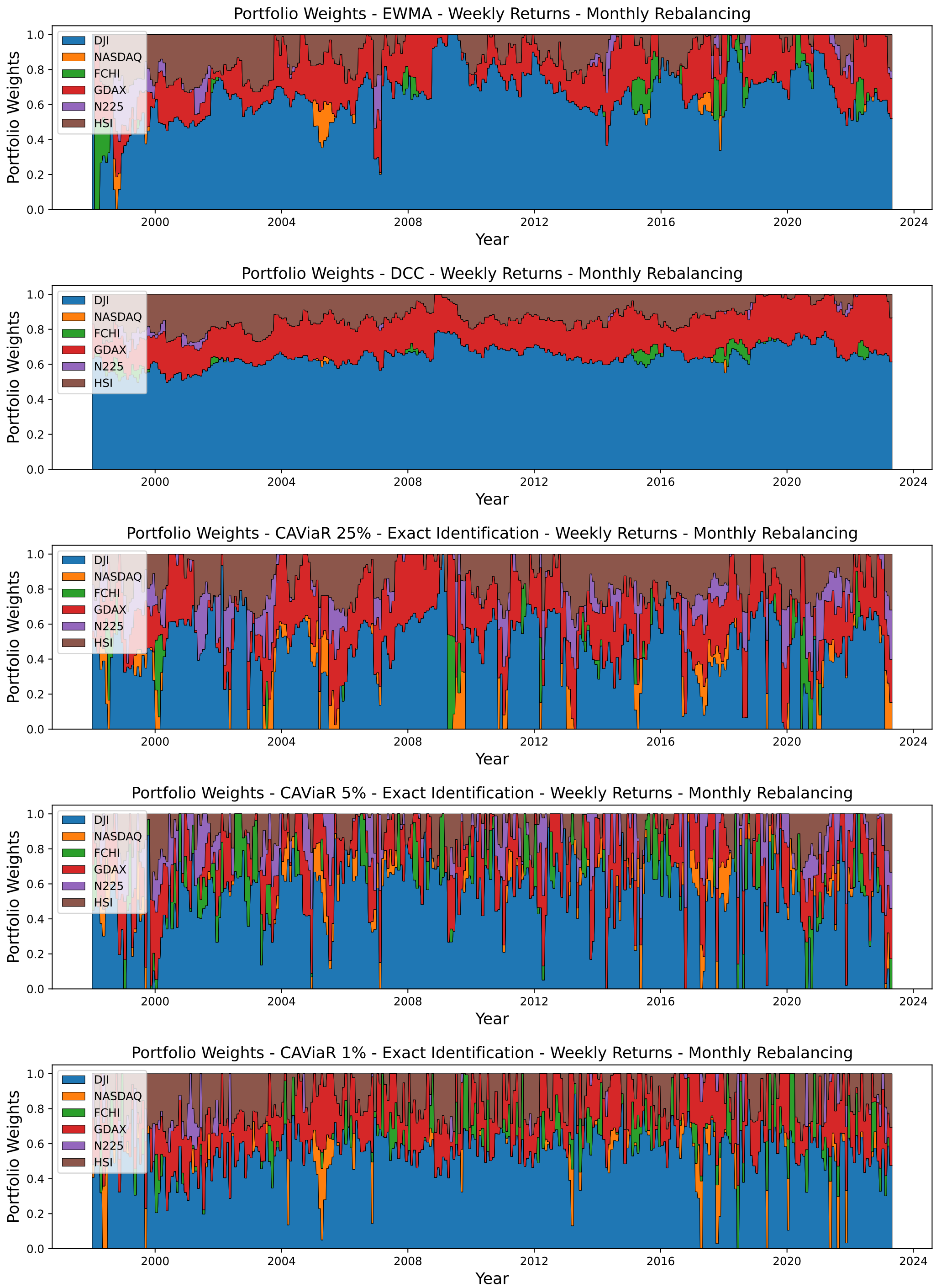


Figure 44: Evolution of portfolio weights - Presented are the portfolio weights with monthly rebalancing and weekly returns. DQIC from an exact identified system are considered.

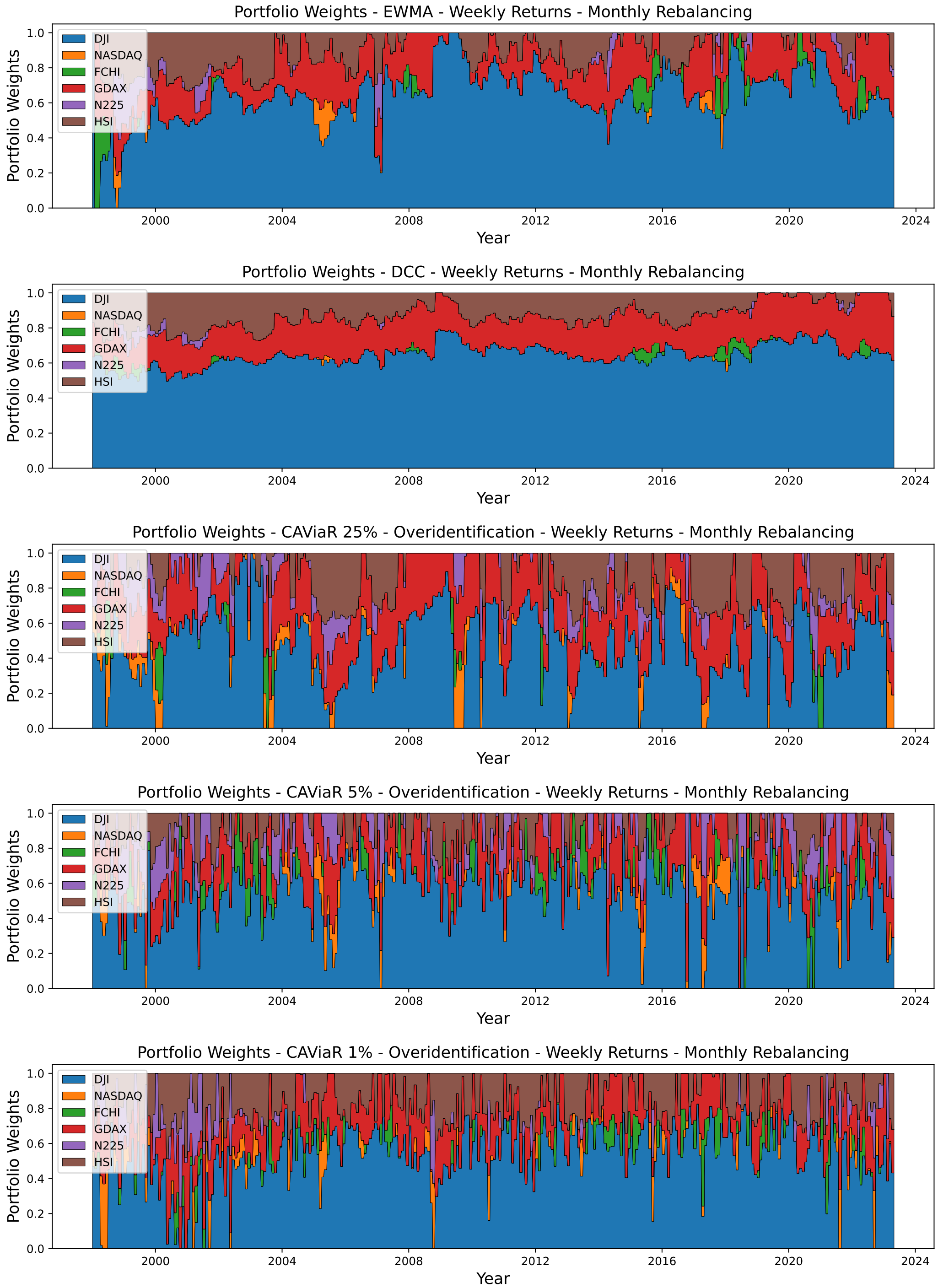


Figure 45: Evolution of portfolio weights - Presented are the portfolio weights with monthly rebalancing and weekly returns. DQIC from an overidentified system are considered.

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Eidesstattliche Versicherung (Affidavit)

(Siehe Promotionsordnung vom 12. Juli 2011, § 8 Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

München, 06.03.2025

Dennis Mao