

---

# Boundary data and discretization in gauge theories and gravity

with holographic perspectives

Simon Felix Langenscheidt

---



München 2025



---

**Boundary data and discretization in gauge  
theories and gravity**  
with holographic perspectives

Simon Felix Langenscheidt

---

Dissertation  
der Fakultät für Physik  
der Ludwig-Maximilians-Universität  
München

vorgelegt von  
Simon Felix Langenscheidt  
aus München

München, den 14. Mai 2025

Erstgutachter: Prof. Dr. Daniele Oriti  
Zweitgutachter: Prof. Dr. Ulrich Schollwöck  
Tag der mündlichen Prüfung: 30.06.2025

# Contents

<b>List of abbreviations</b>	<b>vii</b>
<b>Notations and conventions</b>	<b>ix</b>
<b>Zusammenfassung</b>	<b>xi</b>
<b>Declaration</b>	<b>xv</b>
<b>Preamble</b>	<b>3</b>
Two sides of the same coin . . . . .	3
A discrete subset of unbounded motivations . . . . .	7
Goals of this thesis . . . . .	10
<b>1 Preliminaries about boundaries, symmetries, dressings and glueing</b>	<b>13</b>
1.1 Lagrangians, phase spaces and symmetries . . . . .	14
1.2 Dressing fields and boundary conditions . . . . .	24
1.2.1 Internal symmetry dressings . . . . .	26
1.2.2 Naive boundary conditions and dressed Lagrangian . . . . .	29
1.3 Cutting and glueing . . . . .	33
1.3.1 The cutting and glueing problems, primo . . . . .	33
1.3.2 Algebraic cutting and the ‘corner algebra’ . . . . .	38
1.3.3 The cutting and glueing problems, secundo . . . . .	43
1.4 Discretization and sampling . . . . .	53
1.5 Case study of discretization . . . . .	57
1.5.1 BF theory: Generalities . . . . .	57
1.5.2 BF theory: Dressed phase space . . . . .	63
1.5.3 BF theory: Boundary dynamics . . . . .	66
1.5.4 BF theory: Discretization . . . . .	72

<b>2</b>	<b>Random tensor networks for discrete holography</b>	<b>77</b>
2.1	Introduction . . . . .	78
2.1.1	Questions and quantities of interest . . . . .	80
2.1.2	States under consideration . . . . .	80
2.1.3	A notion of holography . . . . .	86
2.2	Methodology . . . . .	90
2.2.1	Randomisation over vertex states . . . . .	90
2.2.2	Rewriting the Hamiltonian . . . . .	93
2.2.3	The puzzle of areas . . . . .	99
2.3	Isometry conditions on superposed spin networks . . . . .	103
2.4	Summary . . . . .	111
<b>3</b>	<b>Internal shift symmetries of 4D gravity</b>	<b>115</b>
3.1	Tetrad gravity . . . . .	116
3.2	The naive shifts . . . . .	122
3.2.1	Canonical derivation . . . . .	123
3.2.2	Contrast to symmetries of the Lagrangian . . . . .	124
3.3	The improved shifts . . . . .	127
3.3.1	The variation . . . . .	128
3.3.2	Algebra of charges . . . . .	130
3.3.3	Interpretation of corner charge . . . . .	136
3.4	Summary . . . . .	139
<b>4</b>	<b>Discussion</b>	<b>143</b>
4.1	Overview . . . . .	143
4.2	Outlook . . . . .	145
<b>A</b>	<b>Appendices to chapter 1</b>	<b>149</b>
A.1	Types of phase spaces . . . . .	149
A.2	Phase space ambiguities . . . . .	151
A.3	Charge brackets . . . . .	152
A.4	Cellular complexes: Language and notation . . . . .	154
A.5	Additional details on BF theory . . . . .	157

---

<b>B</b>	<b>Appendices to chapter 2</b>	<b>171</b>
B.1	Operator transport in the presence of nontrivial centers . . . . .	171
B.1.1	Trivial center . . . . .	173
B.1.2	Nontrivial center . . . . .	180
B.2	Derivation of random Ising model . . . . .	182
B.2.1	Bulk-to-boundary maps . . . . .	182
B.3	Bulk-to-boundary maps: Examples . . . . .	184
<b>C</b>	<b>Appendices to chapter 3</b>	<b>191</b>
C.1	Symplectic vector fields . . . . .	191
	<b>Acknowledgements</b>	<b>211</b>





# List of abbreviations

<b>QFT</b> Quantum Field Theory . . . . .	34
<b>LQG</b> Loop Quantum Gravity . . . . .	82
<b>YM</b> Yang-Mills . . . . .	60
<b>KR</b> Kalb-Ramond . . . . .	61
<b>AdS/CFT</b> Anti-deSitter/Conformal Field Theory . . . . .	79
<b>PEPS</b> Projected Entangled Pair States . . . . .	80
<b>ECH</b> Einstein-Cartan-Holst . . . . .	117
<b>BV</b> Batalin-Vilkovisky . . . . .	125
<b>BFV</b> Batalin-Fradkin-Vilkovisky . . . . .	125
<b>ADM</b> Arnowitt Deser Misner . . . . .	120
<b>NUT</b> Newman-Unti-Tamburino . . . . .	141



# Notations and conventions

Greek lowercase letters denote abstract or concrete spacetime indices. We work with signature  $(-, +, \dots, +)$ .

- $P \rightarrow M$ : principal  $G$ -bundles.
- $\Gamma(E)$ : sections of bundle  $E$ .
- Normalisation of  $p$ -forms:  $\omega = \frac{1}{p!} \omega_{\alpha \dots \beta} dx^\alpha \wedge \dots \wedge dx^\beta$ .
- Spacetime  $\rightarrow$  phase space notation:  $d \mapsto \delta, i_\xi \mapsto I_X, \mathcal{L}_\xi \mapsto L_X$
- Fundamental relation in symplectic geometry:  $\delta F_X + I_X \Omega = 0$ .
- Poisson bracket:  $\{F, G\} := X_F[G] = \Omega(X_F, X_G)$ .
- In chapter 3:  $V := P \times_\rho \mathbb{R}^{1,3}$ .  $I = 0, 1, 2, 3$  internal basis label for basis  $b_I$ .
- $\eta_{IJ}$ : Cartesian Minkowski metric in internal space.
- $\mathfrak{g} = \mathfrak{so}(1, 3) \cong \Lambda^2(\mathbb{R}^{1,3})$  used for Lie algebra-bivector conversions. Basis  $M_{IJ} = b_I \vee b_J$ .
- Lie inner product:  $\langle X, Y \rangle = \frac{1}{2} X^{IJ} Y_{IJ}$ .
- Lie algebra metric:  $\eta_{IJ, KL} = \eta_{IK} \eta_{JL} - \eta_{IL} \eta_{JK}$ .
- Internal Hodge dual:  $(\star M)_{IJ} = \frac{1}{2} \epsilon_{IJKL} M^{KL}$ .
- Lie algebra commutator on forms:  $[A, B]^{IJ} := A_K^I \wedge B^{KJ} - (-1)^{|A||B|} B_K^I \wedge A^{KJ}$
- $\star[A, B] = [\star A, B] = [A, \star B]$  holds generally.
- Matrix-vector product:  $(\alpha \cdot v)^I = \alpha_J^I v^J$ .
- Covariant derivatives of vectors:  $d_\omega v^I = dv^I + \omega_J^I \wedge v^J$ ,
- For bivector/Lie algebra valued objects:  $d_\omega A = dA + [\omega, A]$ .



# Zusammenfassung

In dieser Arbeit untersuchen wir nichttriviale Effekte in Eichtheorien und Gravitation, die mit Eichtransformationen an Rändern zusammenhängen. Diese zeigen sich in Rand-Symmetrien und randgestützten Freiheitsgraden in Form von Referenzsystemen, die als ‘Eich-Kantenmoden’ bekannt sind. Diese Kantenmoden sind notwendig für das korrekte Schneiden und Kleben von Eichtheorien, sowie für die Einbettung von Subsystemen in eine größere Theorie. Insbesondere bei der Konstruktion von Hilbert-Räumen und Pfadintegralen als Kontinuumslimiten von diskreten Theorien sind Kantenmoden von entscheidender Bedeutung für die Korrespondenz mit der Kontinuumsphysik.

Als unsere originären Beiträge stellen wir zwei Ergebnisse vor, die sich auf das Verhältnis zwischen diskreten und Kontinuumstheorien der Gravitation beziehen und Kantenmoden beinhalten.

Das erste dieser Ergebnisse analysiert den Grad der Holographie in der Gravitation, indem wir überprüfen, inwieweit die Geometrie eines Raumes aus seinem Rand rekonstruiert werden kann. Wir untersuchen dies im Kontext der Tensornetzwerkholographie, einem diskreten Paradigma, das konkret als ein Sektor der Quantengravitation verstanden werden kann. Wir zeigen, dass es strenge Beschränkungen darauf gibt, was aus Kantenmoden rekonstruiert werden kann. Um eine sinnvolle Relation zwischen der diskreten Theorie und dem Kontinuum zu erhalten, zeigen wir, dass zudem nichttriviale Einschränkungen auf die Verschränkung von Elementarbausteinen der diskreten Theorie und die Interpretation der Quantenzustände aufgestellt werden müssen.

Unser zweites Ergebnis betrifft die Symmetriestruktur der Gravitation und stellt eine Umformulierung dieser Struktur für Formulierungen in Tetradenform vor, die sich möglicherweise besser diskretisieren lässt. Wir zeigen, dass es eine produktive Umformung von Diffeomorphismen in interne Symmetrien namens ‘Shifts’ gibt, die im Gegensatz zu Diffeomorphismen den verbesserten Status haben, Generatoren zu besitzen, weshalb sie potentiell quantisiert werden können. Dies bringt die Struktur deutlich näher an die der Gravitation in 3D heran, welche bereits über diese Symmetrien quantisiert wurde, und ermöglicht auch einen direkten Zugriff auf den maximal möglichen Satz von Gravitationskantenmoden, so dass Schneiden und Kleben ohne Datenverluste möglich werden.



# Summary

In this thesis, we discuss nontrivial effects in gauge theories and gravity related to gauge transformations happening on boundaries. They show themselves in boundary-‘would-be-gauge’ symmetries and boundary-localised degrees of freedom in the form of reference frames, known as *gauge edge modes*.

Edge modes are a necessity for gauge theories in correctly cutting or glueing theories along spacetime boundaries, as well as for embedding small regions into the theory defined on a larger spacetime. When constructing Hilbert spaces and path integrals as continuum limits of discrete theories, edge modes play a crucial role in recovering continuum physics. As our original contribution, we present two results related to this context pertaining to the relation of discrete and continuum theories of gravity.

The first of these analyses the degree of holography in gravity, by verifying the extent to which the geometry of a space can be reconstructed from its boundary data. We study this in the context of tensor network holography, a discrete paradigm that can be concretely understood as a sector of quantum gravity. Our analysis reveals that there are strict limitations on what can be reconstructed from edge modes. Moreover, to establish a sensible correspondence between the discrete theory and the continuum, we must impose nontrivial constraints on the entanglement between building blocks of the discrete theory, and on the interpretation of the associated quantum states.

Our second result concerns the symmetry structure of gravity, and presents a reformulation of it for tetrad gravity in a way that is possibly more amenable to discretization. We show that there is a productive recasting of diffeomorphisms into internal symmetries called ‘shifts’, which enjoy the significant advantage of generically admitting generators that can be quantized, unlike diffeomorphisms. This puts the symmetry structure of 4D gravity much closer to that of the 3D theory, which was already quantized using these symmetries, and also gives direct access to the maximal set of gravitational edge modes, so that cutting and glueing can be performed without loss of information.





# Declaration

This dissertation is based on research done at the Ludwig Maximilian University of Munich (LMU) between July 2022 and July 2025. No part of it or anything substantially the same has been previously submitted for a degree or any other qualification at this or at any other University. This thesis is the result of the author's own work and of the scientific collaboration listed below. The work presented in this dissertation is based on the papers listed below, with annotation regarding their publication status.

Chapter 2 is based in large parts on

1. Eugenia Colafranceschi, Simon Langenscheidt, and Daniele Oriti. “Holographic properties of superposed quantum geometries”. Phys. Rev. D 110, 046024 (2024). arXiv:2207.07625.[1]
2. Simon Langenscheidt, Daniele Oriti, and Eugenia Colafranceschi. “Channel-State duality with centers” (2024). arXiv:2404.16004.[2], preprint.

Chapter 3 is based in part on an existing publication regarding the naive shifts,

3. Simon Langenscheidt and Daniele Oriti. “New edge modes and corner charges for first-order symmetries of 4D gravity”. Classical and Quantum Gravity 42, 075010 (2025)[3], but includes unpublished results about the improved shifts, as well,
4. Simon Langenscheidt. “Edge modes of tetrad gravity: Unlike diffeomorphisms, all shifts are integrable”. [4], in preparation.

Furthermore, much of the discussions and particularly the preamble is inspired by

5. Henrique Gomes, Simon Langenscheidt, and Daniele Oriti. “Boundaries, frames and the issue of physical covariance” (2024). arXiv:2412.00993 [5], to appear in “Open Systems: Physics, Metaphysics, and Methodology”, editors Michael Cuffaro, Stephan Hartmann, OUP.



"What today's nationalists and neosegregationists fail to understand," Kwame said, "is that the basis of every human culture is, and always has been, synthesis. No civilization is authentic, monolithic, pure; the exact opposite is true."

(Jonas Kyratzes)

I am interested in the way people relate to space. The best way to do this is to depict their interactions to the boundaries of these spaces.

(Francesca Woodman)

Really, the fundamental, ultimate mystery – the only thing you need to know to understand the deepest metaphysical secrets – is this: that for every outside there is an inside and for every inside there is an outside, and although they are different, they go together.

(Alan Watts)



# Preamble

## Two sides of the same coin

Most of modern physics is built on a set of founding pillars which concern our notions of space and time. They show up in many names and shapes - be it in the form of theories like that of special or general relativity, principles like general covariance or the equivalence principle, or in the form of objects, models and frameworks like the use of manifolds, continuous symmetries, field theories or the use of differential geometry as a whole.

What puts these frameworks into the same categories of thought is the common understanding of *smoothness*, *locality* and working at *arbitrary precision*. They form the platonic ideal of how we understand our physical world since the introduction of real numbers and differential calculus at the hands of Newton and Leibniz.

On quite a different trajectory, however, modern physics has been associated with a specific kind of enforced non-smoothness, non-continuity since the advent of quantum mechanics. The titular quantization of spectra of specific quantities of interest, particularly energy and angular momentum, has been an unavoidable deviation from the simple notions of ‘smooth everything’ that are so easy to intuit classically[6–8].

What is quite striking about the many examples of paradigmatic quantum systems is that, contrary to what the name claims, quantization of spectra only appears in specific cases, which are usually associated with bound systems or states. The classic example for this is the Schrödinger equation for the hydrogen atom, which admits quantised energy levels *exclusively* for wavefunctions with specific *boundary conditions*.

This should be of no surprise to anyone familiar with Fourier series expansions: A function defined on an interval, given no boundary conditions, will require a full Fourier transform with arbitrary frequency components, while with periodic boundary conditions (putting the function on a circle), we can only possibly see a (still infinite) set of discrete frequencies.

This may already spur the intuition, then, that a generic interplay of boundary conditions and our generic use of ‘continua’, that is, differential geometry, continuous spacetime and its symmetries, may be subject to deeper subtleties.

This is especially pressing when considering the ubiquitous need to regulate expressions in field theories, all due to the typically infinite amount of degrees of freedom localised in any region of space or spacetime of arbitrary size. Infrared divergences may be tamed by a restriction to finite regions, so by introducing a boundary. In the case of functions defined

on the real line, we know this has a ‘dual discreteness’ as its consequence, for example by decomposition into orthogonal polynomials[9].

However, ultraviolet divergences require a regulation of short-distance degrees of freedom, in many instances through a cutoff in frequencies, or a discretization in real space. It should be no surprise that these two are, again, dual statements: A boundary in frequency space corresponds to discretization in real space, and a boundary in real space corresponds to discretization in frequency space. Thus, a generically well-defined theory can always be understood to be *discrete and bounded*, or a family of such theories, related by transformations between different scales or cutoffs through coarse and fine graining.

A similarly fuzzy overturning of our intuitions of space and time has long been prophesied to come from the quantum pendence of our theories of gravity, particularly general relativity[10, 11], but may already come from classical gravity. Gravity is a peculiar case of a phenomenon - by its very nature, we cannot grasp it on a purely local level, as the equivalence principle tells us it looks indistinguishable from acceleration.

Gravity evades direct description.

This insight prompted mathematicians and physicists of the late 19th century to move to general coordinate systems, and even coordinate-free descriptions of physics to understand its actual content. It led Einstein and others onto the path of field theories, and into realising that gravity is an aspect of geometry, inextricably baked into the world we inhabit - unavoidable, all-affecting and all-encompassing.

To speak of gravity is to speak of the world (as a spacetime), and one cannot speak of the world without gravity.

What, then, is the content of gravity? Gravity comes with certain variables that pertain to geometry, yes - but it also comes with much less. Really, the Hole argument[12, 13] (a statement of active diffeomorphism invariance) shows Einstein’s original motivation for general relativity: physics is, or should be, independent of the local choice of reference frame, and simple covariance laws govern how phenomena, seen in different reference frames, relate to each other.

Nevertheless, the class of frames that do not affect physics is, in fact, *so* large that the content of a spacetime is actually massively underdetermined. This is not merely a statement of coordinate independence - the equations themselves exhibit invariance under active diffeomorphisms, allowing us to keep the spacetime fixed everywhere outside an arbitrarily large ball while freely relocating field values within it. Nothing in this can change the physically accessible information. Spacetime points and locations have no objective relevance in the equations of gravity, only relations between different data do[14]<sup>1</sup>.

Once again, Gravity evades direct description.

Yet, taking the perspective of an outside observer on a gravitational system changes the story: We can specify the locations of events, objects et cetera relative to ourselves, and therefore make sense of them, through *relational observables*[15–17]. But, what is an

---

<sup>1</sup>cf. p. 700, "Dem Gravitationsfeld *an einer Stelle* entspricht also noch nichts ‘physikalisch Reales’, wohl aber diesem Gravitationsfelde in Verbindung mit anderen Daten."

‘outside’ observer in a theory that encompasses all?

Whatever they are, they must be separated from the system in some capacity. Such a separation must inevitably show up as a figurative or concrete system-observer boundary. To be outside is to distinguish between outside and inside, to place a border in between. Of course, to be a dynamical entity in the theory, and not a background structure (which would simply reduce the diffeomorphism invariance to whatever preserves the structure), the physically meaningful observables are those defined as measured relative to another physical subsystem, with its own dynamics and coupling to the system of interest.[5] We can use this idea concretely: Given a boundary to a spacetime and some limited data on it, we can identify points in the bulk of the spacetime with physical measurements[18, 19]. Boundaries, in this context, are not a mere complication or formality for the act of describing a physical system realistically, but provide invaluable tools to even *formulate* physically interesting questions in a meaningful way. They form or encode a genuine physical subsystem that can be used as a reference - specifically whatever is on the other side of the boundary[20–22]. One may, in fact, think of the ‘outside’ observer as having a representative frame of reference on this boundary.

The resulting observables, though, differ between different reference systems and it takes care to know how to relate them[5, 23–25]. In particular, if we want to make use of the boundary as a reference system, we will need to be more sure about how different boundaries divide the system in different ways, and how they are related[26, 27].

So what should change about this in a quantum theory of gravity and spacetime? For one, comes the inevitable question of quantum uncertainty: If any of the properties of classical geometry are canonical conjugates, they cannot simultaneously be sharp in a quantum state. This, in fact, can lead to a new type of quantum discreteness of geometry: The standard case of this comes from canonical quantization of area operators[28, 29], understood as integrated versions of infinitesimal cross products: Given two small tangential vectors  $\vec{a}, \vec{b}$  of a surface, which locally approximate the area as a flat parallelogram, then a local measure of area is the cross product  $A\vec{n} = \vec{J} = \vec{a} \times \vec{b}$ . These are vectors in 3D, and just like angular momenta in quantum mechanics, they end up satisfying a vectorial commutator algebra

$$[\hat{J}^i, \hat{J}^j] = i\hbar\epsilon^{ijk}\hat{J}_k \quad (1)$$

which implies that the norm squared,  $\vec{J}^2 = A^2$ , becomes a Casimir with eigenvalues being a multiple of  $j(j+1)$ . Importantly, *the spectrum of the infinitesimal area is quantised*. There are other examples of this idea, generally under the umbrella name of *quantum geometry*[30–36], but we do not need any of their details to make our point: *quantization of geometry implies discreteness of geometry*. As stated before, a quantization of real-space properties like areas then entails a boundary in a dual space, like frequencies. Therefore, one can expect any quantum theory of geometry to bring with it an accompanying UV regulator.

Assuming, that this modifies our general picture of spacetime *geometry* as having a level of discreteness, this still leaves the same issues that the classical theory had with its

lack of observables: Once again, to productively work with gravity, one needs to investigate *relations* between objects of interest, but at the level of a quantum system. The discreteness of geometry does not lead by itself to a resolution of this, and we must once again provide the perspective of an outside observer, now with the added subtlety of whether we mean by this a quantum system or a classical one. Either way, boundaries are still an inevitably useful construct to provide observables and regulate the theory fully.

Our goal is thus clear: A well-regulated model of quantum spacetime must be capable of describing discrete spacetimes with boundaries. Nevertheless, as ideal as this outcome may seem, it lacks impact on its own, given that we experience a fundamentally continuous world. It is doubtful whether we will attain the experimental resolution necessary to observe discrete properties of spacetime in the near future. Therefore, we must proceed with due regard for the foundational continuum principles upon which our current understanding is built. The key question becomes how to inform our discrete model-building efforts so they remain as compatible as possible with continuum theories.



## A discrete subset of unbounded motivations

After this careful, general argument by appeal to stepwise reasoning, let us relax our heads a bit. *How many theoretical physicists does it take to change a lightbulb?*

*3+1: One to point out that the lightbulb is broken, one to protest it, and another one to point out the ramifications of replacing it. None of them end up with an implementation, so in addition a graduate student is needed at the end of the process.*

There is a nugget of truth to every bad joke. The road to quantum gravity is near boundless and full of obstacles, and we have as many beliefs about what it should be as we have doubts. Truly unshakeable common denominators are nigh nonexistent. We want to make this clear here - there are so many facets to quantum gravity as a field of research that we only need to name and discuss some of them to fill the entire thesis.

We therefore have to make a judicious choice of what to work on. As the title of this thesis suggests, we put the general relation of the continuum and the discrete into focus, and learn about the delicate ambiguities that show up in it. The reason this does not immediately emphasize gravity, or quantum theories, is that the majority of details we will show are not exclusive to them, and in fact best illustrated in more simple examples. Our choice rests on the fact that it is a broad enough topic to touch upon many quite interesting, nigh intersectional ideas in theoretical physics that may be fruitfully discussed in tandem, and therefore allows us to set the stage to build bridges between them.

Still, we do this with the explicit goal of informing research in the realm of gravity. We particularly have in mind to use the relations found in this way to better interpret the states and properties studied in existing quantum gravity formalisms, and how to best extend them. The projects we choose to present are part of the same realm of ideas that the general formalism connects with, and are instructive in understanding how certain – at first seemingly unrelated – questions link together.

Let us go through some of these ideas, and their relation to quantum gravity. Starting classically, since seminal work by Ströminger et al. [37], boundaries have been a place to look for new symmetries of field theories relevant to their low energy, infrared behaviour, in particular, with regards to scattering problems. This was first done for gravity[38–43], and has since expanded there in the form of entire research programs like Celestial[44–47] and Carrollian holography[48–51], but was quickly realised to be a more generic feature of *gauge theories* like Maxwell theory[52] and Yang-Mills theory[53]. New symmetries are continually being uncovered in less and less restrictive boundary conditions[54, 55].

The key feature in these situations is that the boundaries in question render certain bulk *gauge transformations* into *physical symmetries*. This is a generic phenomenon in theories with local symmetries, as we will expand upon in the technical parts of this thesis. What is promising about these symmetries is both that they give a refined control over states of field theories on spacetimes with fixed asymptotic behaviour (i.e. asymptotically flat or Anti-deSitter (AdS)), but also that they act on degrees of freedom of the bulk fields which are normally ‘inactive’, those which are usually considered ‘redundant’ due to gauge

invariance. Such ‘additional’ degrees of freedom are colloquially referred to as (gauge) ‘edge modes’[56, 57] since they live exclusively on boundaries of regions.

In particular, they very generically form a representation of the gauge group on the boundary, and therefore act as a sort of *reference frame* for the same group - one can tell all orientations relative to the edge modes, and therefore uniquely use them as a reference. One interpretation of their presence is that they incorporate data of an ‘outside observer’ in a way such that the whole system does not contain them explicitly, but rather implicitly as part of the system constituting the observer. This, of course, allows one to define many more observables which are ‘boundary-anchored’, compared to pure bulk observables. In this thesis, we will encounter a realisation of these edge modes in the form of *dressing fields*, which are symmetry group-valued fields acting as reference frames.

A similar, but distinct notion of edge modes predates this idea: In special condensed matter heterostructures such as HgTe/CdTe, one encounters something peculiar: Even though the material is itself an insulator, so does not support electronic low level excitations below its band gap, there nevertheless is a special kind of electronic conductivity on its surface. This is known as a *topological insulator*[58]. The key mechanism here is once again the presence of nontrivial boundaries: The field theories in the insulator and the surrounding material are not continuously deformable into each other (their topological invariants are distinct), which means that a nontrivial interface must compensate for this discontinuity. What is present here is a case of *nontrivial glueing* of two regions which each have trivial dynamics, but carry a nontrivial interface dynamics.

This is distinct from what we will consider in this thesis. In general, the edge modes we will discuss correspond to highly idealised interfaces which are not made of a real material. They are perfectly sharp walls, and the dressing fields inhabiting them idealise the reference frame aspect that a real edge mode carries with it. Still, we show in appendix A.5 that even on this idealised level, with certain boundary conditions we indeed find a sort of gapless dynamics for boundaries of a topological theory known as BF. Therefore, while one must be cautious about the direct links between the names, there are certain analogies that may be useful to consider.

Staying close to this example, we can see that topological properties due to cutting and glueing are quite relevant on boundaries. This is especially intriguing given the diffeomorphism invariance of gravity, which puts it close to topological field theories. Such theories have been at the forefront of much of mathematical physics research and have been the subject of highly rigorous study and axiomatisation. At their core lies the structure of cutting and glueing - which we will give a careful look (in section 1.3), as our goal will be to use the insights there to inform glueing in the discrete theory.

What makes topological theories with boundaries so intriguing is that they feature almost no bulk observables, and instead all their dynamics localises on boundaries (though, notably, not uniquely so). In this sense, they are the opposite of a standard scalar field theory. What makes gravity so striking is that it seemingly combines aspects of these polar opposites: It is diffeomorphism invariant, so it does not give meaning to any regions or observables outside of those that are specified relative to some physical data, but also it has local, propagating degrees of freedom in the form of gravitons.

This has spurred a direction of quantum gravity research known as *holography*. The central idea is to realise quantum gravity as a topological quantum field theory<sup>2</sup> which can be reconstructed in its entirety from boundary observables, states et cetera alone. There are cases in which this logic can be realised explicitly, most notably in asymptotically AdS spacetimes[59–62]. There are top-down approaches to this, but in low spacetime dimensions, it has also been fruitful to consider the same boundary symmetries we already mentioned, and the edge modes they act upon, as the consequential degrees of freedom that allow reconstruction[63–66].

Further evidence for this boundary-focused logic comes from ideas of spacetime thermodynamics, pioneered by Bekenstein’s[67] and Hawking’s[68] arguments on the entropy and lifecycle of black holes. If we interpret Bekenstein’s black hole entropy,

$$S_{Bk-Hk} = \frac{A_{BH}}{4\hbar G_N} \quad (2)$$

as counting the microstates of whatever system comprises the black hole apparent horizon from the viewpoint of an observer at asymptotic distance, then the system looks much more like a lower dimensional one, living on this boundary between ‘inside’ and ‘outside’ of the black hole. A similar formula arises in AdS-holography, by the name of Ryu-Takayanagi-Hubeny-Rangamani[69–71], where even more concretely, an *entangling entropy* is given as a geometric area.

In this context, ideas of building spacetime from boundary entanglement have flourished[72, 73]. As one typically deals with gauge theories there, the study of how to even quantify said entanglement in gauge theories[74, 75] has become ever more interesting and carries many connections to the field of, again, boundary symmetries and the specification of sub-systems for gauge theories[76–80].

Particularly these aspects of entanglement, but also holography and topological quantum field theories, are often investigated in the setting of discrete models like Lattice Gauge Theory[78, 81]. In these contexts, the intuitive notions of cutting and glueing are more easily examined, but the exact analogue in the continuum is left a bit unclear. A particular case that we will investigate is that of lattice ‘tensor network’ models of holography in a discrete quantum gravity context. Here, the notion of cutting and glueing, and the corresponding entanglement, is precisely what encodes connectivity between regions of space. It is therefore crucial to know for sure what happens at boundaries of these models, and how glueing in the discrete relates to glueing in the continuum.

Finally, discrete models of quantum gravity come with the challenge of requiring an implementation of diffeomorphism invariance or some analogue thereof. The continuum models all prominently feature it, but any generic lattice discretization or introduction of a cellulation reduces the invariance to a subgroup preserving the lattice. Furthermore, diffeomorphisms, even in the continuum, generically lack a generator that can be quantized unless specific conditions are met, making them technically involved<sup>3</sup>. It is therefore also

<sup>2</sup>By which we mean in this paragraph exclusively the property of being diffeomorphism invariant.

<sup>3</sup>We will expand on this point in 1.1.

important to consider closely how discrete notions of diffeomorphisms relate to continuum ones, in order for quantum gravity models to feature an appropriate continuum regime (be that realised as a parameter limit, a set of reasonable states, approximations, et cetera).

## Goals of this thesis

Before getting into the middle of things, let us give a rough overview of this work. Rather than focusing on a single specific question and providing a single answer, we instead present two projects within the larger, overarching context we established in the previous paragraphs. We chose this kind of presentation as it most accurately presents the way that these topics came to be - instead of a single thread, multiple lines of research loop back into the same core ideas and inform which direction to take next.

To begin, in 1 we first introduce mathematical and physical background about field theories in the presence of boundaries, in particular how their phase spaces may be constructed, which provides a foundation upon which discretization and quantization may be performed (1.1).

We will treat gauge transformations and how they become symmetries on boundaries (1.1), and introduce *dressing fields*, which provide implementation-agnostic reference frames which allow cleanly separating out the additional degrees of freedom living on boundaries (1.2.1). These are in particular relevant when discussing boundary conditions for gauge theories, where they allow one to impose boundary conditions only on the gauge-invariant parts of the field phase space (1.2.2).

Many of the subtler issues regarding boundaries in field theories are best understood through the question of how boundaries are inserted or removed, i.e. by cutting a space(time) or glueing along a common surface (1.3). We argue, from the example of a scalar field, that to glue algebras and Hilbert spaces together one needs a prescription which should always take the form of a modular tensor product (1.3.2). We also discuss in detail the issue of cutting and glueing gauge theories with the use of dressing fields (1.3.3).

We then discuss the central method of discretization that allows for a direct connection between the continuum and the discrete: Sampling of the continuum algebra on a special set of configurations (1.4). Then, we apply our foundations in an example that paves the way to applications in quantum gravity: BF theory, which is a particularly simple example of a topological theory (1.5). We present its symmetries and a discretization of it, which is useful as a preparation for what follows.

As a first major application, we work on verifying criteria for holography in discrete models of quantum gravity (2). As said already, holography is expected to be a feature of the phenomenology of quantum gravity and rely on quantum entanglement properties for boundary encoding of data. Whether this actually holds is a nontrivial check within nonperturbative quantum gravity. We extend previous work on discrete *toy models* of holography to include superpositions of geometries, with the much needed benefit of having

a proper geometric intuition for the quantum states in question. We calculate, in particular, entanglement entropy between spatial regions, and are forced to use a generalised notion of information transport due to the nontrivial factorisation properties of the algebra of observables. The algebra in question, and its representation on a Hilbert space, are clear examples of a modular tensor product

$$\mathcal{A} = \mathcal{A}_L \otimes_{\mathcal{A}_S} \mathcal{A}_R \quad (3)$$

and our notion of entanglement reflects this by using only bulk-supported operators as the bulk subsystem<sup>4</sup>. We find that holography requires low bulk dimensionality, but also prominently requires a *superselection rule* on the total boundary area of the region studied. This is, from the continuum perspective, just a perfectly expected superselection rule on total spin in the area and therefore totally natural.

This study shows directly that the continuum, and a careful analysis of entanglement in and discretization of gauge theories is crucial to properly understand holography in non-perturbative gravity.

Our second major application is the search and presentation of a reformulation of the symmetries of gravity and the study of their boundary implications (see chapter 3). The discrete model of the aforementioned study does not carry diffeomorphism invariance, which is generally quite difficult to implement. To tackle this issue, we take inspiration from the 3D gravity case and re-express diffeomorphisms as combinations of internal gauge transformations

$$\mathcal{L}_\xi = X_{\alpha_\xi}^{\text{Lorentz}} + Y_{\phi_\xi}^{\text{Shift}} \quad (4)$$

which show 2 major improvements over diffeomorphisms: First, they are internal symmetries, thus much easier to abstract away from the concrete setting of spacetime, and so fit much better on a lattice theory. Second, they are more amenable to quantization in the presence of boundaries, as they are agnostic about spacetime boundaries and leaking of information through them. This puts 4D gravity on the same footing as 3D gravity. While performing a discretization for this is out of the scope of this thesis, we can adapt results from the BF theory example to make guesses about the outcome of such a program. In this purely classical and continuum study, we see that a precise understanding of the continuum symmetry structure may have significant advantages in the process of discretization or in the presence of boundaries.

We then close the main text with an extended discussion and outlook (4), in which we present many natural follow-up directions of the present work, and issues we feel need to be addressed in future research.

The thesis also contains a number of supplementary sections.

---

<sup>4</sup>We already want to stress here that fixing the assignment of algebras to regions of space is a precursor to speaking about their factorization properties.

Appendix A contains complementary technical material that discusses issues that need consideration in going from the continuum to the discrete. A.1 explains the kind of phase spaces one can work with and how they are roughly related. A.2 discusses common ambiguities of the assignment of data we induce from Lagrangians. A.3 weighs different definitions of charge brackets for nonintegrable presymmetries against each other, demonstrating the degree of ambiguity present.

A.4 introduces notation and concepts relevant to discrete theories. This connects with the notions of sampling in 1.4, which are to be applied to the building blocks coming out of this section. Finally, we give more examples in this appendix: We discuss a topological field theory known as BF theory in A.5, in particular its boundary conditions and dynamics, where we show that a minimal ultralocal boundary condition that conserves all gauge corner charges induces the dynamics of a chiral scalar field onto the edge modes. We also give much more detail on the discretization procedure from the main text.

Appendix B includes relevant material for section 2, in particular an extended account of the appropriate notion of information/operator transport (B.1), a derivation of an auxiliary Ising model partition sum (B.2) and instructive examples (B.3 ).

Appendix C.1 includes a description of symplectic vector fields (i.e. allowed canonical transformations) on the phase space of tetrad gravity which is complementary to chapter 3.

# Chapter 1

## Preliminaries about boundaries, symmetries, dressings and glueing

For the main results of the thesis in chapters 2 and 3, we first need to provide sufficient technical and conceptual background. The main aspects in which we need to introduce topics are for phase spaces of field theories, including their symplectic structure (used to define Poisson brackets), which enable us to talk about generators of gauge transformations. We will also need to speak in detail about the notion of field-dependent transformations, and also touch upon the special case of diffeomorphisms (in particular, their nonintegrability). Afterwards, we introduce the notion of a dressing field, which functions as a reference frame for the gauge group of a theory of interest, and discuss how to use dressing fields to impose boundary conditions in a gauge invariant way. In this context, it will be that we see the emergence of ‘edge modes’.

As we want to deal with discrete theories, and also speak about entanglement between subsystems in 2, we will need to speak about cutting and glueing of field theories, in particular for gauge theories. We first give a general exposition of the problems involved, then speak about the most general way to glue algebras (as a stand-in for the full theory, which is useful for questions of entanglement), and specialise to gauge fields.

We then introduce our notions of discretization and sampling in full generality for the algebraic setting, and close with the relevant example of a field theory known as BF theory, which is topological but carries nontrivial boundary degrees of freedom.

This section will be in part expository and in part pedagogical; we believe that while it is possible to view the following chapters in a more isolated manner, this would lose track of the nontrivial ways in which their motivation, setting and physics are related. For this reason, we will emphasize the generality of gauge theory phenomena in many places.

We will make use of the facts and constructions discussed in this chapter in the chapters that follow thereafter. We do this by analysing specific instances of discrete Hilbert spaces, in particular the entanglement entailed by their glueing, and by carefully evaluating the gauge structure of gravity in order to have a grasp on the maximal set of edge modes present in the theory.

## 1.1 Lagrangians, phase spaces and symmetries

In this thesis, we will almost exclusively work with the *covariant-canonical phase space*. We do not attempt to be rigorous about the exposition on its technical details here and refer to existing literature[82–85], as its use is already quite widespread[20, 22, 86–105]. The content of the covariant-canonical phase space is essentially that, instead of constructing a phase space directly on a spatial Cauchy slice  $\Sigma$ , via the cotangent bundle, one first passes through the Lagrangian formalism.

The steps are as follows: One formulates a Lagrangian field theory and obtains, from its boundary piece in the variation of the action, a piece called the *presymplectic potential*. Then, choosing a Cauchy slice, one looks at the induced data on the slice coming from the covariant field theory. One then equips this slice data with a symplectic form built from the potential. This is often identical to the canonical phase space, particularly in first order theories, but often already imposes some identifications between momenta known as primary constraints. For the most part, this is practically identical to the canonical phase space, but written with fewer indices.<sup>1</sup>

### From Lagrangians to phase spaces

Much of what we present in this and the next subsection is a small adaptation of material presented in [83]. We will use bullet notation for 2-tuples indexed by codimension  $k$ , and consider a relative 2-tuple of manifolds<sup>2</sup>  $(M_\bullet) := (M_0, M_1)$  embedded in a  $D$ -dimensional manifold  $M$  which we treat as an ambient, complete spacetime in which all other ones we consider are contained:

$$M \supseteq M_0 \supseteq \partial M_0 \supseteq M_1 \quad (1.2)$$

Throughout the thesis, numerical indices  $k$  usually indicate the codimension of a manifold or differential form in  $M$ , i.e. a  $p$ -form gets an index  $D - p$  and is integrated naturally over  $p$ -manifolds, which are of codimension  $D - p$ .

We also allow (but do not formalise) the case where there are multiple  $M_k$  of the same codimension (which we index throughout the thesis with  $k$ ). A simple example of this is given by  $M = M_0$ ,  $M_1 = \partial M$ , all others empty. Our goal here is to

---

<sup>1</sup>There is also the actual ‘covariant phase space’, which refers instead to a (more or less formal) set of histories that fulfil the Lagrangian equations of motion. This is then equipped with a symplectic form that looks symbolically identical to the covariant-canonical one, but really is supposed to act on whole linearised field histories. Its Poisson bracket is then defined through Feynman propagators (‘causal Green’s functions’) of the linearized field theory. We do not employ this variant of the formalism here. Note, however, that in situations with well-posed initial value problems, the two phase spaces are generically identical.

<sup>2</sup>Such a relative tuple has that each  $M_k$  is contained in the boundary of some  $M_{k-1}$ . The appropriate notion of a boundary operator on these tuples is

$$\partial(M_\bullet) = \partial(M_0, M_1) := (\partial M_0 \setminus M_1, \partial M_1). \quad (1.1)$$



- Give an action principle determining an on-shell space  $P_M \subseteq \mathcal{C}_M$  as a set of solutions over  $M$ , or equivalently as collections of level sets of the action on a configuration space  $\mathcal{C}_M$ .
- Associate to each submanifold  $R$  of  $M$  a phase space  $\mathcal{C}_R$  of fields on  $R$  with fitting presymplectic potential.

The action principle is simply

$$S := \int_{M_0} L_0 + \int_{M_1} L_1 \quad \delta S = 0 \quad (1.3)$$

where each  $L_k \in \Omega^{D-k}$  is a  $(D-k)$ -differential form Lagrangian. One could naively view two tuples  $(L_\bullet)$  of Lagrangian forms as equivalent if they differ by transferring a total derivative into a higher codimension:

$$(L_0 + dl, L_1) \sim (L_0, L_1 + l). \quad (1.4)$$

We first need to fix a representative in this class in order to define the action principle unambiguously when boundaries are present. Therefore, in principle, the action, through a choice of boundaries, distinguishes the different tuples. Here, we will assume such a choice of representative has been made.

In what follows, we will use two notions of exterior derivatives. The one on spacetime,  $d$ , is simply the one of differential forms on  $M$ . The one on phase space,  $\delta$ , is instead more like a ‘variational’ derivative. The logic of variational calculus applies here throughout and may be made precise by use of jet bundles and variational bicomplexes. In particular, we will use different notations for the exterior derivatives, interior products and Lie derivatives: On spacetime,

$$d \quad i_\xi \quad \mathcal{L}_\xi \quad (1.5)$$

but on phase space,

$$\delta \quad I_X \quad L_X. \quad (1.6)$$

In particular, we will use Greek glyphs to denote spacetime vector fields ( $\xi \in \mathfrak{diff}(M) = \Gamma(TM)$ ), and capital Latin letters to denote phase space vector fields (infinitesimal canonical transformations  $X \in \Gamma(T\mathcal{C})$ ). We have, in particular, that all the spacetime and phase space operations commute; the only ‘exceptions’ appear when some quantities may be *field-dependent*, i.e. a  $\xi$  that depends on the phase space point. We will come back to this in 1.1.

By use of Anderson homotopy operators[93](which are essentially just the rule  $d \mapsto \delta$ ), we assign to each Lagrangian a symplectic potential (density)  $\theta_{k+1}$ , and via partial integration then have a split of the vertical/variational derivative  $\delta$  of each  $L_k$  into symplectic pieces and an equation of motion  $E_k$ :

$$\delta L_k = E_k + d\theta_{k+1} - i_{M_k}^* \theta_k \quad (1.7)$$

with starting condition  $\theta_0 = 0$  and  $i_{M_k}^*$  denotes pullback along the inclusion map of the submanifold  $M_k$ . This arrangement of symplectic potentials and equations of motion has amenable properties. First, see that the  $\theta$ -terms on the right hand side are simply the relative exterior derivative  $d(\theta_k, \theta_{k+1}) = (d\theta_k, d\theta_{k+1} - i_{M_k}^* \theta_k)$ . We can therefore write more compactly

$$\delta(L_\bullet) = (E_\bullet) + d(\theta_\bullet) \quad (1.8)$$

which looks more like the usual partial integration identity. The way the exterior derivative acts on a tuple is a straightforward generalization of the relative external derivative on pairs

$$d(\theta_\bullet) := (d\theta_1, d\theta_2 - i_{M_1}^* \theta_1) \quad (1.9)$$

whereas the variation distributes on each term  $\delta(L_\bullet) := (\delta L_\bullet)$ . In particular,

$$\delta S = \int_{M_0} E_0 + \int_{M_1} E_1 + \int_{N_1} \theta_1 + \int_{N_2} \theta_2 \quad (1.10)$$

where we define  $N_k := \partial M_{k-1} \setminus M_k$  as the parts of the codimension  $k$  boundary that do not have their own term in the action. On these surfaces, we need to prescribe boundary conditions by hand.<sup>3</sup>

If we then take the symplectic current  $\omega_k = \delta\theta_k$  to define symplectic forms, we find a closedness relation:

$$d\omega_{k+1} = \delta d\theta_{k+1} = \delta(\delta L_k - E_k + i_{M_k}^* \theta_k) = i_{M_k}^* \omega_k - \delta E_k. \quad (1.11)$$

or in other words

$$d\omega_{k+1} \approx i_{M_k}^* \omega_k \Leftrightarrow d(\omega_\bullet) + \delta(E_\bullet) = 0 \Leftrightarrow d\omega_1 \approx 0, d\omega_2 \approx i_{M_1}^* \omega_1 \quad (1.12)$$

where  $\approx$  denotes on-shell equality.

Let us now assume also a sequence of Cauchy slices for  $M_k$ ,

$$M_k \supset \Sigma_{k+1} \quad \partial\Sigma_k \supseteq \Sigma_{k+1} \quad (1.13)$$

which again form a relative 2-tuple  $(\Sigma_\bullet)$  that we use as an extended Cauchy slice. Then we assign to it the extended symplectic potential and form

$$\Theta_\Sigma = \int_{\Sigma_1} \theta_1 + \int_{\Sigma_2} \theta_2 \quad \Omega_\Sigma = \delta\Theta_\Sigma \quad (1.14)$$

This is a good assignment because it generically results in symplectic forms which are on-shell preserved in time - therefore the choice of slice does not matter. To see this, take two such slices  $(\Sigma_\bullet^\pm)$  connected by a cylinder spacetime and denote the relative tuple of

---

<sup>3</sup>This is just for completeness; in principle, we could have Lagrangians on all parts of all boundaries.

regions in  $(M_\bullet)$  connecting them as  $(\Delta_\bullet)^4$ . Then consider the symplectic form on  $\Delta_0$ , the part of  $M_0$  bounded by  $\Sigma_1^\pm \cup \Delta_1$ , which always vanishes because  $\omega_0 = \delta\theta_0 = 0$ :

$$\begin{aligned}
0 &= " \Omega_{\Delta_0} = \int_{\Delta_0} \omega_0 " = \int_{\Delta_0} d\omega_1 = \int_{\Sigma_1^+} \omega_1 - \int_{\Sigma_1^-} \omega_1 + \int_{\Delta_1} \omega_1 \\
&= \int_{\Sigma_1^\pm} \omega_1 + \int_{\Delta_1} d\omega_2 + \delta E_1 \approx \int_{\Sigma_1^\pm} \omega_1 + \int_{\Sigma_2^\pm} \omega_2 + \int_{\Delta_2} \omega_2 \\
&\approx \dots \\
&\approx \int_{\Sigma_1^\pm} \omega_2 + \int_{\Sigma_2^\pm} \omega_2 \approx \Omega_{\Sigma^+} - \Omega_{\Sigma^-}.
\end{aligned} \tag{1.15}$$

So in the end for these two slices,  $\Omega_{\Sigma^+} \approx \Omega_{\Sigma^-}$ .<sup>5</sup> This demonstrates that if we go on-shell of all of the equations of motion  $E_k$ , the symplectic form is conserved and the system is *closed*. This is, of course, nothing else than imposing boundary conditions in a coordinated, general fashion which is organised by a variational principle.

Given these forms, the algebra  $\mathcal{A}_R$  of a region  $R$  of the slice  $\Sigma$  in all this will be some subset of  $C^\infty(P_R)$  where  $P_R$  is the set of solutions to the  $E_\bullet$ , restricted to the submanifold  $R$ . So, the continuum classical algebra that is to be quantized or discretized is the Poisson algebra

$$\mathcal{A}_R = (C^\infty(P_R), \cdot, \{-, -\}_{\Omega_R}). \tag{1.17}$$

To give an easy example, consider Maxwell theory (in 3+1D), which is a theory of a 1-form  $A = A_\mu dx^\mu$ , and its field strength  $F = dA$ , written via the spacetime Hodge dual  $*$ ,

$$L_0 = \frac{1}{2} F \wedge *F. \tag{1.18}$$

The Anderson homotopy procedure replaces  $F = dA \mapsto \delta A$ , and so the symplectic potential is

$$\theta_1 = *F \wedge \delta A, \tag{1.19}$$

and associated equations of motion are

$$E_0 = -d * F \wedge \delta A. \tag{1.20}$$

We can give a boundary condition to it by setting for example  $*F \stackrel{M_1}{=} \pi$ , with  $\delta\pi = 0$ , i.e. it is a fixed value. This corresponds to the Lagrangian

$$L_1 = -\pi \wedge \delta A \implies \theta_2 = 0 \tag{1.21}$$

---

<sup>4</sup>So  $\partial\Delta_k = \Sigma_{k+1}^\pm \cup \Delta_{k+1}$ .

<sup>5</sup>If we make use of the relative Stokes's theorem, this is shortened to

$$0 \approx \int_{\Delta_{\bullet-1}} d(\omega_\bullet) = \int_{\partial(\Delta_{\bullet-1})} \omega_\bullet = \int_{\Sigma_\bullet^\pm} \omega_\bullet = \Omega_{\Sigma^+} - \Omega_{\Sigma^-} \tag{1.16}$$

and so

$$E_1 = (*F - \pi) \wedge \delta A \quad (1.22)$$

so by letting variations of  $A$  be arbitrary, we have the sought boundary condition as boundary equation of motion.

### Extended notion of symmetries and currents

We can then also define graded relative analogues of Noether currents. Let  $X$  denote a symplectic vector field on the covariant phase space. We call it a (pre)symmetry of the theory (in codimension  $k$ ) if

$$X[L_k] = d\lambda_{k+1} - i_{M_k}^* \lambda_k \quad (1.23)$$

for a sequence of  $(D - k)$  forms  $\lambda_k$  with  $\lambda_0 = 0$ . In the language of [83], this is called a d-symmetry of the Lagrangian tuple. Then we can do the same Noether derivation as usual: By contraction with the partial integration identity, we have

$$X[L_k] = I_X E_k + d(I_X \theta_{k+1}) - i_{M_k}^* I_X \theta_k. \quad (1.24)$$

If we define the codimension  $k$  Noether current  $j_k = I_X \theta_k - \lambda_k$ , we see that they are conserved on-shell

$$dj_{k+1} - i_{M_k}^* j_k = -I_X E_k \approx 0 \Leftrightarrow d(j_\bullet) + I_X(E_\bullet) = 0^6 \quad (1.25)$$

So, lower degree currents vanishing implies the conservation of higher degree currents. On-shell closedness implies that the Noether current

$$J_X = \sum_{k=1}^D \int_{\Sigma_k} j_k \quad (1.26)$$

is independent from the choice of the slice  $(\Sigma_\bullet)$ . Furthermore, if we have that the symplectic potential changes under the presymmetry by a specific variationally exact piece,

$$L_X \Theta_\Sigma = \delta \left( \sum_{k=1}^D \int_{\Sigma_k} \lambda_k \right), \quad (1.27)$$

then this Noether current is also the Hamiltonian generator of the vector field  $X$

$$\delta J_X = \delta I_X \Theta_\Sigma - \delta \left( \sum_{k=1}^D \int_{\Sigma_k} \lambda_k \right) = -I_X \delta \Theta_\Sigma = -I_X \Omega_\Sigma. \quad (1.28)$$

We call a presymmetry *Noether local* if, additionally, there exists *off-shell* a tuple of codimension  $k$  forms  $C_k$  with  $C_0 = 0$ , linear in the transformation parameters, that vanishes on-shell and, primarily, satisfies

$$dC_{k+1} - i_{M_k}^* C_k = -I_X E_k \Leftrightarrow d(C_\bullet) + I_X(E_\bullet) = 0. \quad (1.29)$$

---

<sup>6</sup>We note that there is a striking formal similarity between this equation and the fundamental relation of Hamiltonian vector fields  $\delta H + I_X \Omega = 0$ .

We call these the *local constraint forms*. In their presence, we can upgrade the conservation law to an off-shell relation

$$dj_{k+1} - i_{M_k}^* j_k = dC_{k+1} - i_{M_k}^* C_k \Leftrightarrow d(j_\bullet) = d(C_\bullet). \quad (1.30)$$

This in turn means that we can write, the Noether currents as constraints, which vanish on-shell, plus on-shell nonvanishing terms  $\rho_k$

$$j_k = C_k + \rho_k, \quad (1.31)$$

with

$$d\rho_{k+1} - i_{M_k}^* \rho_k = 0 \Leftrightarrow d(\rho_\bullet) = 0 \quad (1.32)$$

which are in the relative cohomology  $d\rho_\bullet = 0$  of the spacetime. These forms will often be boundary charges of interest for us. Note that this implies that, when pulled-back, all of them are closed  $d(i_{M_k}^* \rho_k) = d^2 \rho_{k+1} = 0$ .

The first two degrees are instructive:

$$d\rho_1 = 0, \quad j_1 = C_1 + \rho_1 \quad (1.33)$$

$$d\rho_2 = i_{M_1}^* \rho_1. \quad (1.34)$$

Then, if in fact  $\rho_1 = dq_2$  (an exact form usually called the *relative or corner charge*), we can elevate the chain to

$$d(\rho_2 - i_{M_1}^* q_2) = 0 \implies \rho_2 = i_{M_1}^* q_2 + \sigma_2, d\sigma_2 = 0 \quad (1.35)$$

$$j_2 = C_2 + q_2 + \sigma_2. \quad (1.36)$$

How do (pre)symmetries of the Lagrangians relate to (pre)symmetries of the action? Say a (pre)symmetry as above is given. Then,

$$X[S] = \int_{M_0} d\lambda_1 + \int_{M_1} d\lambda_2 - \lambda_1 = \int_{N_1} \lambda_1 + \int_{N_2} \lambda_2. \quad (1.37)$$

So, as usual, a (pre)symmetry of the action (for which  $X[S] = 0$ ) is a stronger statement, and depends on boundary conditions (so the prescribed state data on the  $N_k$ ).

We are now ready to differentiate among presymmetries based on whether they act trivially in a Hamiltonian sense. Say a presymmetry  $X$  has a Hamiltonian generator  $G_X$ , so  $I_X \Omega_\Sigma + \delta G_X = 0$ .

**If the generator is nonzero, we call  $X$  a *symmetry*. If it vanishes<sup>7</sup>, we call it a *redundancy*.**

We append the prefix "on-shell" as needed here, as many cases, as we will soon see, have vanishing bulk generators on-shell.

---

<sup>7</sup>or if it is a phase-space constant,

A typical scenario is that of a Noether-local presymmetries, known as a gauge transformations. For these, the generator (that is the Noether current) is typically

$$G_X = \int_{\Sigma_1} C_1 + dq_2 + \int_{\partial\Sigma} C_2 + \rho_2 \approx \int_{\partial\Sigma} q_2 - \rho_2, \quad (1.38)$$

and it has the special feature that it vanishes for transformations that are trivial on the corner  $\partial\Sigma$ , when the (pullbacks of the) equations of motion hold. Such transformations are then redundancies on-shell. *Boundary-supported transformations, on the other hand, have nonzero generator and are genuine symmetries of the existing theory.*

*In the Maxwell theory example*, we have as symmetry the  $U(1)$  gauge transformations with parameter  $\chi \in C^\infty(M)$

$$X_\chi[A] = d\chi \implies \lambda_0 = \lambda_1 = 0 \quad (1.39)$$

so it is a presymmetry in codimension 0, with Noether charge  $j_1^\chi = *F \wedge d\chi$ . This is also the corresponding canonical generator. In codimension 1,

$$X_\chi[L_1] = -\pi \wedge d\chi = -d(\pi\chi) + d\pi\chi \implies \lambda_2 = -\pi\chi \quad (1.40)$$

it is only a presymmetry if  $d\pi = 0$ , which however we must require for consistency with the equations of motion. So, in the presence of our boundary condition, the current

$$G_\chi = \int_{\Sigma_1} *F \wedge d\chi + \oint_{\Sigma_2} \pi\chi \quad (1.41)$$

is conserved. It is also a Noether-local symmetry: One can check that  $C_1 = -d * F$ ,  $C_2 = -( *F - \pi)$  a good constraint form, and in fact<sup>8</sup>

$$G_\chi = \int_\xi \chi C_1 + \oint_{\Sigma_2} \chi C_2 \approx 0 \quad (1.42)$$

shows the conserved charge is on-shell zero. Thus, the gauge transformations are redundancies. If we do not impose the boundary condition, then instead the canonical generator is

$$G_\chi \approx - \oint_{\Sigma_2} *F \chi \quad (1.43)$$

so the charge is nonzero, but conservation is meaningless unless we specify how the time evolution acts on the boundary. Still, the boundary-supported gauge transformations are *symmetries* if no boundary condition is imposed.

This split into bulk constraints and corner charges is the main distinguishing feature of *gauge theories*, which are the subject of the next subsection, and a majority of this thesis.

---

<sup>8</sup>Note that  $\Sigma_2$  carries an orientation from  $M_1$  which is opposite to the one induced from  $\Sigma_1$ .

### Field-dependence, finding gauge charges and integrability

Overall, a gauge theory comes equipped with a (concrete) group of presymmetries  $\mathcal{G}_\Sigma$  which acts on the kinematical phase space<sup>9</sup>, but which acts as a redundancy on-shell of certain constraints. In nice cases, this group can be understood as the image of an (abstract) group  $\hat{\mathcal{G}}$  which can be described without reference to the phase space. I.e. this is often the case for diffeomorphisms, where the abstract group is some subgroup of  $\text{Diff}(M)$  that acts as a Lie derivative. Whenever one uses phrasing such as ‘theories with  $SU(2)$  symmetry’, this refers to such abstract symmetry groups, which must always come with concrete actions on the kinematical phase space of the theory in question.

In many theories, there are multiple subgroups of  $\mathcal{G}_\Sigma$  of interest which can be cleanly distinguished, which are then colloquially regarded as ‘independent’ symmetries. This can be formalised by choosing a set of coordinates on  $\mathcal{G}_\Sigma$  or a generating set<sup>10</sup> in its Lie algebra  $\text{Lie}(\mathcal{G}_\Sigma)$ . Often, this shows up in the form of a split of  $\mathcal{G}_\Sigma$  into ‘external’ and ‘internal’ symmetries, where external ones are interpreted as diffeomorphisms. However, such a split is, like all coordinates, a priori arbitrary, and carries seemingly no physical information. However, we will see in a bit that this is not *entirely* true when it comes to corner transformations.

Now, if indeed we have chosen a set of generators on  $\mathcal{G}_\Sigma$ , a general presymmetry will be an exponentiation of an arbitrary combination of them. But how arbitrary? Say that we have encapsulated the split into two independent generators whose vector fields on phase space are  $X_\alpha, Y_\phi$ . Then, these are vector fields *on phase space*. The parameters  $\alpha, \phi$  belong to different model Lie algebras  $\hat{\mathfrak{g}}, \hat{\mathfrak{h}}$  which correspond to abstract groups which are subgroups of  $\hat{\mathcal{G}}$ , while the actual vector fields form the concrete (i.e. vector field) algebras  $\mathfrak{g}, \mathfrak{h}$ . So we have<sup>11</sup>

$$[X_\alpha, X_\beta]_{\mathfrak{g}} = X_{[\alpha, \beta]_{\hat{\mathfrak{g}}}} \in \mathfrak{g} \quad [Y_\phi, Y_\psi]_{\mathfrak{h}} = Y_{[\phi, \psi]_{\hat{\mathfrak{h}}}} \in \mathfrak{h} \quad (1.44)$$

Focusing on one of the two for the moment, there is a map  $X : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$  turning  $\alpha \mapsto X_\alpha$  which is a Lie algebra homomorphism. In writing this, we choose parameters only in the model algebra  $\hat{\mathfrak{g}}$  etc. However, the *concrete* algebra  $\text{Lie}(\mathcal{G}_\Sigma)$  is also closed much more generally. Indeed, suppose we have some mapping

$$\underline{\alpha} : \mathcal{C} \rightarrow \hat{\mathfrak{g}} \quad (1.45)$$

which associates to each configuration of the theory an abstract transformation parameter. Such a mapping is called a *field-dependent parameter*. Then, given two field-dependent parameters, we still have, as before,

$$[X_{\underline{\alpha}}, X_{\underline{\beta}}]_{\mathfrak{g}} = X_{[[\underline{\alpha}, \underline{\beta}]]_{\hat{\mathfrak{g}}}} \in \text{Lie}(\mathcal{G}_\Sigma) \quad (1.46)$$

<sup>9</sup>Much of this section is inspired by [106], but discussions of field-dependent symmetries are commonplace in the modern literature as well, i.e. see [99, 107, 108].

<sup>10</sup>Not technically a basis of the Lie algebra. Technically, a generating set is a basis of the kernel of the Hessian of the action, if an action is available. See 3.1.7 in [106].

<sup>11</sup>Note that all commutators of gauge transformations in this thesis will be modulo *trivial* gauge transformations which are vanishing on the on-shell pre-phase space. See section 3.1.5 of [106] for details.

but with the *field-dependent* or *modified Lie bracket*

$$[[\underline{\alpha}, \underline{\beta}]]_{\mathfrak{g}} := [\underline{\alpha}, \underline{\beta}]_{\hat{\mathfrak{g}}} - X_{\underline{\alpha}}[\underline{\beta}] + X_{\underline{\beta}}[\underline{\alpha}] \quad (1.47)$$

which features, in the first term, the abstract Lie bracket on the values of the field-dependent parameters, but also correction terms due to the fact that the fields inside the parameters also transform under the presymmetries.

We can therefore answer the question "how arbitrary" with "up to recombining them with phase space/field-dependent coefficients". So, in fact, *any* combination

$$X_{\underline{\alpha}} + Y_{\underline{\phi}} \in \mathcal{G} \quad (1.48)$$

is in the concrete gauge group of the theory. This means that, once we take advantage of this freedom, there are a plethora of reparametrizations we can do on the full gauge group for various goals. In particular, we might have some other abstract Lie algebra  $\mathfrak{k}$  (Fraktur k) and maps

$$\forall k \in \mathfrak{k} : \underline{\alpha}_k : \mathcal{C} \rightarrow \hat{\mathfrak{g}}, \quad \underline{\phi}_k : \mathcal{C} \rightarrow \hat{\mathfrak{h}} \quad (1.49)$$

and then define  $Z_k$  as the transformation

$$Z_k := X_{\underline{\alpha}_k} + Y_{\underline{\phi}_k} \quad (1.50)$$

and, if chosen correctly, we can see  $X, Z$  as independent generators of  $\mathcal{G}$  rather than  $X, Y$ . This is essentially a reparametrization of the concrete gauge group in terms of other abstract groups. A common example of this is in *generally covariant systems*, where it is possible to express the action of diffeomorphisms in terms of other gauge transformations, i.e.  $\mathfrak{k} = \mathfrak{diff}(M) \ni \xi$  and we have  $Z_{\xi} =: \hat{\xi}$

$$\hat{\xi} = X_{\underline{\alpha}_{\xi}} + Y_{\underline{\phi}_{\xi}} \quad (1.51)$$

usually onshell of some constraints. We will see examples of this in section 1.5, and it will feature centrally in chapter 3.

*As an example*, consider again Maxwell theory: It has two interesting classes of transformations, given by  $U(1)$  gauge transformations and diffeomorphisms (which are not all symmetries!), acting as  $\hat{\xi}[A] = \mathcal{L}_{\xi}A$ , so as a Lie derivative. These can play the role of our  $\mathfrak{g}, \mathfrak{h}$ , respectively. Then, it is perfectly sensible to define a new, *covariant diffeomorphism* as a field-dependent combination of the two. For this, see that one can always decompose

$$\mathcal{L}_{\xi}A = i_{\xi}F_A + d(i_{\xi}A) \quad (1.52)$$

and thus define the phase space vector field

$$Z_{\xi}[A] = -X_{i_{\xi}A} + \hat{\xi}[A] = i_{\xi}F_A \quad (1.53)$$

which is again parametrized by  $\mathfrak{diff}(M)$ , and features field-dependent parameter

$$\underline{\alpha}_{\xi}(A) = i_{\xi}A \in \mathfrak{u}(1)^M = \mathfrak{g} \quad (1.54)$$



These covariant diffeomorphisms are an interesting reparametrization because they do not include gauge transformation pieces on the face of it. One can therefore say perfectly well that covariant diffeomorphisms and gauge transformations are the full symmetry structure. Overall, we can see that the freedom to reparametrize our full concrete gauge group is enormous, and we must not be too attached to any one presentation of it in terms of abstract groups.

So, we understand under a gauge theory a theory whose ‘correct degree of redundancy’ is indicated by its nontrivial corner charges. If a corner charge is zero, the symmetry may be regarded as a pure artifact of parametrization. This interpretation of gauge theories as carrying extra data concerning corners or interfaces between systems has been argued for from multiple fronts [109]. We can understand even on a technical level that these data must be due to relations between a region and its complement (if present) as the charges of the joint region *always* vanish<sup>12</sup> Then, we can think of the data encoded about the system in the charges  $q^\alpha$  as telling us something about the region, but also its relation to the complementary region.

Let us now see how these corner charges can be found. As we stated before, the occurrence of gauge theories hinges on the presence of constraints coming from the equations of motion. In fact, given such a form of the equations of motion as above, we can *construct* canonical generators almost algorithmically. In essence, we just smear the equation  $\mathcal{E}_A$  by some appropriate parameters to create  $C_1^\alpha$ , and then look for a corner piece which makes this charge a Hamiltonian function on the kinematical phase space. One way to proceed is to work first on a closed slice  $\partial\Sigma = \emptyset$ , and find the corresponding transformation. Then, with the same transformation, but on an arbitrary slice, one finds the correct Hamiltonian generator. i.e.

$$G_\alpha^{(0)} = \int_\Sigma C_1^\alpha \quad I_{X_\alpha} \Omega + \delta G_\alpha^{(0)} = 0 \quad (1.55)$$

for a closed slice, and then one attempts to show that the *contraction*

$$\delta G_\alpha := -I_{X_\alpha} \Omega \quad (1.56)$$

is  $\delta$ -exact on the slice with boundary. If it is not exact, the transformation is termed *non-integrable on corners*.

Such behaviour, while it may appear pathological and strange, is actually commonplace. The most common example of this corner nonintegrability are diffeomorphisms, whose contractions for vector fields  $\xi$  and their phase space pendants  $\hat{\xi}$  typically feature terms such as

$$\delta D_\xi = \delta D_\xi - \oint_{\partial\Sigma_1} i_\xi \theta_1, \quad (1.57)$$

so consist of an exact piece and a *symmetry flux* piece  $\mathcal{F}_\xi = i_\xi \theta_1$ . Here, while vector fields tangential to  $\partial\Sigma_1$  give integrable transformations, those with nonvanishing normal

---

<sup>12</sup>Unless the two regions feature different theories, for example.

components do not. One can generally do this split

$$\oint G_\alpha = \delta G_\alpha + \mathcal{F}_\alpha, \quad (1.58)$$

but it is not unique in any capacity unless additional criteria are introduced. One choice[110] is the *Noetherian split*, valid for Lagrangian systems in which  $X_\alpha$  is a Lagrangian symmetry. Here, the charge is chosen to be the Noether one, which fixes the split uniquely. However, in more general contexts, the *charge-flux split is ambiguous*.

If a choice of charge-flux split is made, one then has the question of how to work with the nonintegrable transformation that is characteristic of the gauge theory. It acts in a fashion on the phase space which is not Hamiltonian, i.e. which will not be given by commutators in the quantum theory. Instead, the quantum action will have to be given by a *superoperator*

$$\hat{X}_\alpha[\hat{A}] = \frac{1}{i\hbar}[\hat{G}_\alpha, \hat{A}] + \hat{\mathcal{F}}_\alpha[\hat{A}] \quad (1.59)$$

which, in terms of algebra-centred language, will be a mixture of an *inner* (the commutator) and an *outer* (the flux term) automorphism of the algebra of operators. The most simple analogue of this is a time-dependent Hamiltonian in quantum mechanics, where the explicit time dependence gives us

$$\frac{\hat{d}}{dt}[\hat{A}] = \frac{1}{i\hbar}[\hat{H}, \hat{A}] + \frac{\partial}{\partial t}\hat{A} \quad (1.60)$$

and  $\frac{\partial}{\partial t}$  takes on the role of the flux term. Quite generically, the presence of a flux term indicates that the system on a given fixed slice  $\Sigma$  is insufficient to properly represent the action of the transformation on the phase space 'as-is'. This may be because the transformations take one out of the phase space, or because they shift the slice, or because they encode leaking of information out of the system. Overall, for the quantum theory, flux terms are difficult to deal with, as unlike with Hamiltonian functions, there is no standard way to implement them in the quantum theory. In fact, it is unclear what kind of form these superoperators should take, barring further information. Therefore, it is generally desirable to make gauge charges corner integrable.

For this, it is important to point out that given a generic contraction  $\oint G_\alpha$ , one may sometimes choose a field-dependent  $\alpha$  so that the transformation becomes integrable. We will see this in chapter 3.

## 1.2 Dressing fields and boundary conditions

In general, theories with gauge invariances in the classical sense share the phenomenon of having nontrivial generators for boundary-supported transformations  $G^{\partial M}$ . These *symmetries* have therefore a nontrivial action on boundary fields, therefore on boundary conditions.

It is here that the treatment of boundary conditions as equations of motion comes into

play: If we see boundary conditions as fixed and defining a theory, then these symmetries of the bulk theory are broken to the stabiliser group of the boundary conditions.<sup>13</sup> On the other hand, if instead the boundary conditions are dynamically determined, we may see the symmetries as truly exchanging physically distinguishable solutions of some equations of motion.

Also, here it enters that we worked with off-shell phase spaces; the symmetries on boundaries act specifically as gauge transformations on the fields, so if we had removed all this gauge freedom from the get-go, we would not be able to implement the symmetries at all.

So, there is in a sense a separate system with symmetry  $G^{\partial M}$  whose field values provide boundary conditions for the bulk, or which respond to the motions of the bulk. Configurations of the boundary theory can then be related by symmetries whose effects extend *trivially* into the bulk. Studying these boundary systems calls for a way to cleanly separate and couple the bulk from the boundary condition. This can be done with the help of auxiliary boundary fields. For a scalar field theory, this is easy to do. We use the Lagrangian system

$$L_0 = \pi \wedge d\phi - \mathcal{H}(\phi, \pi) \text{vol} \quad (1.61)$$

$$L_1 = -J \wedge (\phi - \psi) + \Lambda(\psi) \quad (1.62)$$

which identifies  $\phi|_{\partial M} = \psi$  and  $\pi|_{\partial M} = J$ , thus lifting the boundary condition to the question of prescribing values for  $\psi$  and  $J$ . This is the task of the boundary Lagrangian  $\Lambda$ . A choice for it should be made so that  $\psi, J$  effectively follow the pullback of the bulk equations of motion, for consistency.

In the case of gauge invariances, additionally there is a boundary condition ambiguity: Identifications of gauge fields or those transforming under gauge are defined only *up to a gauge transformation*. This means no simple boundary condition like  $A|_{\partial M} = a$  is possible, but we must allow for a gauge transformation  $\phi$  to act:

$$\phi \cdot A|_{\partial M} = A + d\phi = a \quad (1.63)$$

is a valid, gauge invariant boundary condition for  $\omega$ . We will in fact see  $\phi$  as a field on the boundary (more precisely, a section of  $\text{Conj}(P) \equiv \Omega^0(\partial M, G)$ ), with its own canonical pair and therefore a dynamical degree of freedom. We call  $\phi$  a *dressing field*[5, 21, 111], as it has the function of dressing  $A$  into a socially acceptable, gauge invariant object. It is these additional group-valued fields which embody the role of *edge modes*:

They are naturally present when boundaries are and are key when glueing together regions along codimension 2 cuts. They also correspond directly to gauge redundancies in the bulk, which indicate a ‘deactivated’ degree of freedom that may become active on boundaries. This (de)activation is analogous to that of electrons in an insulator, which, in certain materials and depending on boundary conditions, may become conducting on the boundary.

---

<sup>13</sup>Also, even if the boundary conditions are preserved by gauge transformations, they may still restrict the values of charges, thereby restricting to a subsector of the "full", open theory without boundary conditions.

It is *in this sense* that edges modes of condensed matter physics and the ones of classical field theory get the same name.

These fields have, in fact, another useful function: When extended into the bulk, they *compensate* any bulk gauge transformations in a transparent way. They provide, in a way, a stand-in for a physical reference frame with respect to which the gauge theory is defined.

<sup>14</sup>

### 1.2.1 Internal symmetry dressings

Let us first discuss the simpler case of an internal gauge group  $G$  acting on fields as an internal symmetry group, so it acts pointwise. We will think of matrix groups like  $SU(N)$  here, but many general cases can be treated, even diffeomorphisms. This is useful to us later on for the example of BF theory in section 1.5, but in principle also applies to the symmetries discussed in chapter 3. We require that the dressing field transforms as

$$\phi \mapsto g^{-1}\phi \quad (1.64)$$

under gauge transformations. If a field  $B$  transforms in the adjoint representation of the group, this means that

$$\phi \cdot B = \phi^{-1}B\phi \quad (1.65)$$

is in fact gauge invariant. This, of course, generalises to any representation. We will remark here that once we introduce this dressing, the notion of physical quantities becomes slightly different - a quantity on pre-configuration space is only a quantity on the actual configuration space (the symplectic manifold) if it does not depend on the bulk values of the dressing field. Corner values, instead, are perfectly allowed to appear. We will get back to this point at the end of the section.

Let us consider variations of this dressed field:

$$\delta(\phi \cdot B) = \delta\phi^{-1}B\phi + \phi^{-1}\delta B\phi + \phi^{-1}B\delta\phi = \phi^{-1}(\delta B + [B, \chi_R])\phi \quad (1.66)$$

Here we introduce the *right-invariant variational 1-form* or *field space connection*  $\chi_R := \delta\phi\phi^{-1}$ . So,  $\chi_R$  is  $Lie(G)$ -valued and its value serves as a parameter for an infinitesimal gauge transformation of the fields. By definition,  $\chi$  satisfies

$$I_{X_\alpha}\chi_R = (I_{X_\alpha}\delta\phi)\phi^{-1} = -\alpha\phi\phi^{-1} = -\alpha \quad (1.67)$$

and

$$\mathbf{L}_{X_\alpha}\chi_R = [\chi_R, \alpha] - \delta\alpha \quad (1.68)$$

---

<sup>14</sup>As stated beautifully in the beginning of *quantization of Gauge systems*, gauge theories are defined relative to an arbitrary reference frame. The dressing field here is then just the element of the covariance group that relates the physical reference frame to some standard one, e.g. some frame field on Minkowski space and the standard coordinate basis.

Notice that the left-invariant 1-form  $\chi_L = \phi^{-1}\delta\phi$  behaves very differently:

$$I_{X_\alpha}\chi_L = \phi^{-1}(I_{X_\alpha}\delta\phi) = -\phi^{-1}\alpha\phi \quad (1.69)$$

$$\mathbf{L}_{X_\alpha}\chi_L = -\phi^{-1}[\alpha, \delta\alpha]\phi \quad (1.70)$$

These two forms are related by  $\chi_L = \phi^{-1}\chi_R\phi$ .

Additionally, the curvature of this field space connection vanishes,

$$\mathcal{R}[\chi_R] = \delta\chi_R + \frac{1}{2}[\chi_R, \chi_R] = 0. \quad (1.71)$$

In general, it might be necessary to choose other field-space connections for which this curvature is nonzero, as exemplified in Yang-Mills theories through the Gribov problem. In our work, we choose to work *locally over field space* in order to have a simple implementation of field space connections as coming from dressing fields that we can interpret as edge modes.

In the Lagrangian picture, this introduces a few new techniques for dealing with gauge non-invariant systems. Assume that we work with a gauge non-invariant Lagrangian system  $[L_k]$ . Then, let  $[\phi \cdot L_k]$  be the invariant one where we replace the non-invariant terms by dressed versions. For example, a term in a gauge theory involving a background spin current  $\Sigma$  such as  $\Sigma \wedge \omega$  would be turned into  $\Sigma \wedge (\phi \cdot \omega)$ . Then the gauge transformation is *automatically* a symmetry in the bulk. However, the expression for the generator changes. If there are fields in the Lagrangians transforming with derivatives under gauge transformations, such as connection fields, these will lead to additional terms in the symplectic potential which change the phase space structure.

Instead of working out the general expressions, let us just *illustrate the point with a particular example* for Maxwell theory. Here, we need a dressing for the internal  $U(1)$  symmetry. For simplicity, however, we will treat the internal gauge group as  $\mathbb{R}$ , which makes the presentation a bit easier. Then, with a dressing field  $\phi \in C^\infty(M)$ , we take the bulk Lagrangian with electric source current 3-form  $\mathcal{J}$  and get a dressed Lagrangian

$$L_0 \mapsto \phi \cdot L_0 = \frac{1}{2}F \wedge *F + \mathcal{J} \wedge (A + d\phi). \quad (1.72)$$

If we vary  $\phi$ , we get the forced conservation law  $d\mathcal{J} = 0$ , which is required for consistent coupling:

$$E_0 = -d * F \wedge \delta A + d\mathcal{J}\delta\phi \quad (1.73)$$

In particular, the symplectic potential changes under this introduction of sources to

$$\theta_1 = *F \wedge \delta A - \mathcal{J}\delta\phi. \quad (1.74)$$

As long as we stay on the phase space level, we may treat  $\mathcal{J}$  like a canonical variable with nonzero variations. Then, we can see that sources of gauge constraints are conjugate to

the dressing fields. The role of the Maurer-Cartan form is here played by  $\delta\phi$ . If we include the boundary Lagrangian from before, its dressed version becomes

$$L_1 = -\pi \wedge (A + d\phi) \quad (1.75)$$

which now contributes a symplectic potential  $\theta_2 = -\pi\delta\phi$  (again, on the phase space level we can have  $\delta\pi \neq 0$ ) and has equation of motion

$$E_1 = (*F - \pi) \wedge \delta A + (d\pi - \mathcal{J})\delta\phi. \quad (1.76)$$

The gauge generator is also modified to include the source, but is still onshell zero.

$$G_\chi = \int_{\Sigma_1} *F \wedge d\chi + \mathcal{J}\chi + \oint_{\Sigma_2} \pi\chi \quad (1.77)$$

However, in this dressed phase space it is also clear that the boundary piece  $\pi\chi$  generates shifts of  $\phi$ ; one can therefore also consider a second set of transformations, in contrast to the bulk gauge transformations, which now also change the dressing field  $\phi$ ,

$$X_\alpha[A] = d\chi \quad X_\alpha[\phi] = -\chi : \quad (1.78)$$

We can consider *frame reorientations*[20, 112], which only change the dressing field by a right translation

$$\check{X}_\alpha[\phi] = \chi \quad \check{X}_\alpha[A] = 0; \quad (1.79)$$

This corresponds to a transformation that only changes how the frame  $\phi$  relates to its parent in any complement region. We can then ask when these are integrable transformations with respect to the modified ‘dressed’ symplectic form:

$$\delta Q_\chi + I_{\check{X}_\alpha} \Omega = 0 \quad Q_\chi = - \int_{\Sigma_1} \mathcal{J}\chi - \oint_{\Sigma_2} \pi\chi \quad (1.80)$$

This generator is nonzero, and counts the actual electric charge of the system. In particular, in the absence of sources,  $\pi$  is the radial electric field and  $Q$  tracks things like multipole moments of the bulk electromagnetic field. Thus, the frame reorientations are the actually physically relevant transformations.

### Some remarks on the uses of dressings

We need to stress the following important point about dressing fields, be they used on boundaries only or in the bulk as well: The *dressed* gauge transformations on the extended phase space for which the dressings are ‘bookkeepers’ are not physical operations. As was clear in the internal symmetry example, the only thing the addition of dressings does (kinematically, at least) is a reshuffling or separation of transformations. Any theory can be made invariant by such a screening process. We thereby ‘hide’ the actual symmetry content of the theory in the shifts of the dressing  $\phi$ .

Yet still, the technique is useful: It allows separating out neatly the corner symmetries

from redundancies, meaning that for example gauge fixings of bulk fields no longer need to take into account the presence of a boundary. With dressings, *all* joint gauge transformations are redundant throughout the bulk and can be uniformly treated. The nontrivial relation between bulk and boundary is then encoded in the redundancy of boundary gauge transformations, which are generated by the matching constraints.

*For example*, if we want to gauge fix Maxwell theory using some gauge fixing function  $f[A] = 0$ , then in the bulk this is no issue, but the transformations that would bring some arbitrary  $A$  into this form are not redundant on the corner. We would therefore need to choose gauge fixing functions that automatically vanish at the corner. The simple alternative with dressings is to just fix  $A$  as usual, and then the remaining corner degrees of freedom are encoded in the value of the dressing  $\phi$ .

Finally, we need to make a remark on the algebras extended by dressing fields. The phase space extension by a dressing does not necessarily give  $\phi$  a symplectic partner in the bulk, unless we make the sources come from dynamical fields. So, without a partner the phase space is at best a type of *contact manifold* equipped with an analogue of a symplectic potential, the contact form. To get a true symplectic or Poisson phase space, one needs to quotient out the parts of the dressing without partner, which often means all of the bulk values. I.e. while the procedure at the Lagrangian level extends the phase space by  $G^\Sigma$ , at the phase space level it often reduces to an extension by  $G^{\partial\Sigma}$ . This is perfectly analogous to the role Lagrange multipliers play in the covariant-canonical formalism.<sup>15</sup> This means in particular that the quantities on phase space cannot depend on the bulk values of the dressing.

This is of course different if we *manually* add a conjugate partner to the dressing in the bulk, which we saw can be interpreted as a *source*: Think back to the original example for a dressed Lagrangian, given by a source term for a gauge field. As we saw, such a term takes the form

$$\mathcal{J} \wedge A \mapsto \mathcal{J} \wedge (A + d\phi) \implies \theta_1 = \cdots - \mathcal{J} \delta\phi \quad (1.81)$$

and contributes to the symplectic potential. However, this does not work on the Lagrangian level as-is if  $\mathcal{J}$  is an unconstrained field, as it would constrain  $A + d\phi$  to be zero in the variation. Still, adding  $\mathcal{J}$  on the covariant phase space level on  $\Sigma$  is perfectly doable, and there are no inconsistencies from pretending  $\delta\mathcal{J} \neq 0$  on the phase space.

### 1.2.2 Naive boundary conditions and dressed Lagrangian

Now let us consider some naive boundary conditions, by going bottom-up as explained in appendix A.2. Suppose that the symplectic potential  $\theta_1$  is expressed in a Darboux coordinate system,

$$\theta_1 = \pi \wedge \delta\phi \quad (1.82)$$

---

<sup>15</sup>Generically, what they have in common is that certain variables appear in the Lagrangian that do not end up as phase space coordinates.

which means that the closure condition would be  $\delta\pi \wedge \delta\phi = \omega_1 \stackrel{E_0, E_1}{=} 0$ . If there are no gauge degrees of freedom, i.e. in a usual scalar field theory, we then essentially just have to choose a half-dimensional subspace of the phase space on  $M_1$  such that the symplectic form vanishes. This is technically called a Lagrangian subspace. There is a great freedom in choosing such a Lagrangian subspace, which practically can be highly nonlocal conditions on the fields. Given our finite lifespan and patience with this world, usually we restrict ourselves to local ones. There is then, in the above Darboux coordinates that make the symplectic current a local density, a special set of subspaces called *Robin boundary conditions*,

$$a\pi + b\phi = 0. \quad (1.83)$$

This equation should be read schematically; it really should be stated first for tangent vectors of  $P_{M_1}$  and its differentials, i.e.

$$a\delta\pi + b\delta\phi = 0. \quad (1.84)$$

Inserted into  $\omega_1$ , this gives

$$\omega_1 = -\frac{a}{b}\delta\pi\delta\pi = 0 \implies \theta_1 = -\frac{a}{2b}\delta(\pi^2) \quad (1.85)$$

so for phase space-independent  $a, b$  the symplectic potential can be absorbed into a Lagrangian  $L_1$ . Given the specific form above, involving  $\pi$  and  $\delta\phi$ , one then also distinguishes *Dirichlet*  $\delta\phi = 0$  and *Neumann*  $\pi = 0$  boundary conditions.<sup>16</sup> When a Robin boundary condition is given in terms of differentials, it must then be checked to actually give a defining equation for a Lagrangian *submanifold*. I.e. if again  $a, b$  are phase space constants, then the above condition is  $\delta(a\pi + b\phi) = 0$ , and therefore we get a *family* of boundary conditions labelled by the constant value of  $a\pi + b\phi$ , and the Lagrangian subspace is one of its level sets.

These boundary conditions are sufficient for many purposes, but must be amended at the very least for gauge theories. In those contexts, imposing them on the bare value of the off-shell fields leads to Lagrangian subspaces that are not invariant under the gauge transformations. This is problematic for a variety of reasons, i.e.

1. If one follows the viewpoint that gauge transformations at boundaries should be treated uniformly with the bulk, then the off-shell fields are not part of the observable algebra, and therefore cannot be restricted[113]. The only thing that can be meaningfully restricted is the gauge-invariant parts of the fields.
2. From the viewpoint that gauge transformations are symmetries at boundaries, the restriction of off-shell fields is overly strong and breaks the symmetries of the theory[22, 56, 114].

---

<sup>16</sup>These terms are a little more specific in their interpretation, i.e. Dirichlet fixes a value, while Neumann is usually reserved for fixing normal derivatives. However, if the variable  $\pi$  is interpreted as a normal derivative of  $\phi$ , then a Dirichlet condition on  $\pi$  is also a Neumann condition for  $\phi$ .



The latter argument holds also outside gauge theories: If there is a symmetry present in the theory, there are boundary conditions which break the symmetry. Geometrically, this means that the Lagrangian subspace chosen is not invariant under the symmetry, i.e. does not consist of orbits.

If one wants to impose gauge-invariant boundary conditions, one option is the *dressed Lagrangian*:

Say that one is working with a bulk Lagrangian  $L_0$  which carries a set of gauge invariances, collected into one collective group action. The central application of the dressed Lagrangian prescription for  $L_1$  is to provide gauge-invariant boundary conditions to the bulk fields. As a *simple example*, consider again the Maxwell field  $A$  with its gauge invariance  $A \mapsto A + d\chi$ . When it comes to boundary conditions, one may take two (seemingly conflicting) points of view:

1. Fixing a value of  $A \stackrel{M_1}{=} a$  is illegitimate because of cutting/glueing procedures: If a spacetime is split arbitrarily along a fictitious boundary  $M_1$ , then there is technically gauge invariance on that boundary. Therefore, if we fix boundary values for  $A$ , and wish to glue regions with these Dirichlet boundary conditions together, said gauge invariance must be reflected in an inexact matching:

$$A_L \stackrel{M_1}{=} A_R + d\chi_{LR} \quad (1.86)$$

This is also what one would expect from the principal bundle point of view - gauge potentials are only identifiable up to gauge transformations. So in fact, one can only give Dirichlet boundary conditions up to gauge transformations.

2. Fixing a Dirichlet value  $A \stackrel{M_1}{=} a$  is legitimate because gauge transformations on corners (and so their evolution  $M_1$ ) is not a redundancy, but a symmetry. In fact, one may see the ‘activation’ of gauge transformations at said corners as a reflection of the fact that they change the boundary condition - they change the state like a symmetry would. So, it is perfectly valid to specify a given representative of the gauge equivalence class on the boundary, fictitious or not.

*The resolution is, as is the case with any good paradox, that there is a false dichotomy at play.*

By adding dressing fields for the gauge invariance, at least to the boundary, we can cleanly separate all transformations into redundant gauge symmetry throughout bulk and corner, and into symmetries given by shifts of the dressing alone. This is simply a rewriting of the same data, but allows for a clean description of the process of setting boundary conditions and glueing: Under gauge transformations, the fields  $A, \phi$  transform as

$$A \mapsto A + d\chi, \phi \mapsto \phi - \chi. \quad (1.87)$$

This has as consequence that the combination  $A + d\phi$  on  $M_1$  is gauge invariant and may be unambiguously set a Dirichlet boundary condition. Then, when glueing with these gauge invariant conditions, we have

$$A_L + d\phi_L \stackrel{M_1}{=} a_L \stackrel{M_1}{=} a_R \stackrel{M_1}{=} A_R + d\phi_R \implies A_L \stackrel{M_1}{=} A_R + d(\phi_R - \phi_L). \quad (1.88)$$

So the ‘loose glueing’ expected from gauge invariance holds indeed. Furthermore, because shifts of  $\phi$  are considered symmetries, we can indeed change the boundary condition by changing  $\phi$ . If this is done in a synchronised manner for both  $\phi_L, \phi_R$ , then the glueing is preserved both at the gauge-invariant level and the level of the potentials  $A$ . In this way, by providing a gauge invariant boundary condition for the dressed field

$$\phi \cdot A = A + d\phi \stackrel{M_1}{=} a \quad (1.89)$$

we circumvent all ambiguities. This is useful to finding a Lagrangian because we can simply write the Lagrangian that fixes this boundary condition and happen to end up with a minimally constraining, gauge invariant system.

### Extended phase space

In practice, such a Lagrangian will take a simple enough form. Say the bulk potential takes the form

$$\theta_1 = *F \wedge \delta A. \quad (1.90)$$

Then a Lagrangian that fixes a Dirichlet value for  $A$  is

$$L_1 = -\pi \wedge (A - a) \rightarrow \phi \cdot L_1 = -\pi \wedge (A + d\phi - a) \quad (1.91)$$

and so it not only imposes the Dirichlet condition in a gauge-invariant fashion, but also induces two more equations of motion:

$$E_1 = (*F - \pi) \wedge \delta A - \delta\pi \wedge (A + d\phi - a) + d\pi \delta\phi. \quad (1.92)$$

Thus it enforces the *boundary continuity equation* and *boundary conservation law*

$$*F \approx \pi \quad d\pi \approx d * F \approx 0. \quad (1.93)$$

In Maxwell theory, the latter is just the pullback of one of the bulk equations of motion. It therefore is not a true boundary condition. Additionally, the continuity equation connects bulk and boundary objects that are generically involved in the gauge generators. One can see, then, this continuity equation as another piece  $C_2$  of a *gauge constraint*, and the full gauge generator turns out to be

$$G_\chi = \int_{\Sigma_1} C_1^\chi - \oint_{\Sigma_2} C_2^\chi \quad (1.94)$$

and is quite often differentiable as-is, but also vanishes on-shell *for any parameters*. This includes corner-supported parameters. Therefore, the *extended gauge transformations* given by this extended generator are pure redundancy, and the original corner symmetry is instead found in generators which involve only the boundary current  $\pi$ . This symmetry then

acts on the *extended phase space*[78] coordinatized by  $A, *F$  and the boundary variables  $\pi, \phi$ , with the extended presymplectic potential

$$\Theta_\Sigma = \int_{\Sigma_1} \theta_1 + \int_{\Sigma_2} \theta_2 \quad \theta_2 = -\pi \delta \phi. \quad (1.95)$$

In contrast to before, where we used Dirichlet boundary conditions, we can vary  $\pi$  also at the Lagrangian level. Like before, we can interpret  $\pi$  as the conjugate variable generating shifts in the dressing  $\phi$ . Therefore, on this extended phase space, an extended gauge transformation acts on the bulk fields as-is, but shifts the dressing as intended  $\phi \mapsto \phi - \chi$ . The corner *symmetries* instead act as *frame reorientations*  $\phi \mapsto \phi + \chi$  that don't impact the bulk fields at all. The phrase 'frame' here refers to the fact that the dressings are essentially group-valued, and so form a reference frame choice of the gauge symmetry group.

This procedure is simple and convenient as it disentangles gauge invariance cleanly from corner symmetries in a transparent fashion, and also allows in principle for implementation of less idealised scenarios: The dressing fields model a generic *ideal frame*<sup>17</sup>, but this can in principle be relaxed to model something more realistic, like using a material frame made from matter fields on which the corner symmetries act.

It is important to see, however, that the introduction of the dressing itself did *not* lead to this extension: instead, it was the demand that we impose a boundary condition in a gauge-invariant way, which required the introduction of the dressing in a derivative form. Therefore, we could have arrived at this symplectic potential contribution in a bottom-up way as well: Demanding that  $\delta A \stackrel{M_1}{=} -d\delta\phi$  for some  $\phi$ , we would have

$$\theta_1 \stackrel{M_1}{=} - * F \wedge d\delta\phi = d(- * F \delta\phi) + d * F \delta\phi \quad (1.96)$$

which, again, implies that if we use  $d * F \approx 0$ , which often is just a bulk equation of motion, we end up with a closed system, but with the additional symplectic contribution  $\theta_2 = - * F \delta\phi$ . So it really is just the boundary conditions that lead to this extension. If instead we impose gauge-variant boundary conditions, there is no such extension and the corner symmetry group is broken down to the stabiliser of the configuration chosen.

## 1.3 Cutting and glueing

### 1.3.1 The cutting and glueing problems, primo

We now wish to use our control of the formalism, including gauge invariant boundary conditions, to carefully examine the *glueing and cutting problems*. We will borrow terminology for these problems partially from existing literature[113] to state precisely what we mean. We will also assume that all the phase spaces we consider in the following, be they physical or auxiliary, are constructed in a presheaf-like manner: We consider them to be assignments

---

<sup>17</sup>See [24, 115] for the notion of ideal frames, and in particular their quantum extension as physical systems.

to local regions in a spacetime, with some restriction maps onto smaller subregions. On a very high level, what we want to discuss in this chapter are aspects of cutting (which is akin, but not necessarily identical to, restricting) and glueing (which is the inverse process, ideally). We will, in a way, be concerned with the criteria turning a presheaf into a sheaf. In particular, we will argue that usual local Quantum Field Theory (QFT)s, which are generated by smeared observables, are too singular to allow for simple glueing. We then give a prescription for how glueing *could* be realised and address how cutting and glueing works in gauge theories, where the obvious presheaf structure can be a bit misleading.

Let us first speak in terms of Cauchy slices. First, consider a given Cauchy slice  $\Sigma$ , divided into two adjacent ‘halves’ along a common, codimension 2<sup>18</sup> boundary  $S$ ,

$$\Sigma = \Sigma_L \cup_S \Sigma_R \quad (1.97)$$

which has opposing induced orientations from either side. We will usually use the orientation induced from  $\Sigma_L$  on  $S$ . Now, on this setup, we can ask whether there is a way, or multiple ways, of distributing the data of a field theory among the left (L) and right (R) sides. This gives rise to the glueing and cutting problems.

**Problem 1** (The first cutting problem).

*Given a phase space  $P_\Sigma$ , and a cutting surface  $S$ , is there a suitably unique (i.e. with well-characterised choices) cutting map to an appropriate codomain  $P_{\Sigma,LR} \subseteq P_{\Sigma_L} \times P_{\Sigma_R}$*

$$\begin{aligned} sp_S : P_\Sigma &\rightarrow P_{\Sigma,LR} \\ \Phi &\mapsto (\Phi_L, \Phi_R) \end{aligned} \quad (1.98)$$

*which respects useful criteria?*

and dual to that,

**Problem 2** (The first glueing problem).

*Given phase spaces  $P_{\Sigma_L}, P_{\Sigma_R}$  on regions with common glueing surface  $S$ , is there a suitably unique glueing map from an appropriate domain  $P'_{\Sigma,LR} \subseteq P_{\Sigma_L} \times P_{\Sigma_R}$*

$$\begin{aligned} gl_S : P'_{\Sigma,LR} &\rightarrow P_\Sigma \\ (\Phi_L, \Phi_R) &\mapsto gl(\Phi_L, \Phi_R) = \Phi_{LR} \end{aligned} \quad (1.99)$$

*which respects useful criteria?*

These two problems refer to the phase spaces as sets. We have phrased them, for now, deliberately as vague as possible, as we want to explore these questions with an open mind. Essentially, what these problems refer to is what definition of the regional phase spaces  $P_{\Sigma_L}$  allow for such maps to exist, and therefore for data of the full Cauchy slice to be reduced, through the cutting map, to smaller chunks of space. Dually, given such small chunks,

---

<sup>18</sup>with respect to spacetime

what kind of phase spaces do we need to be able to glue configurations on the chunks into a larger whole? What kind of matching of the configurations is necessary for appropriate glueing (as encoded in the choice of domain)? Also, although this may seem obvious, are the two maps  $\text{sp}_S, \text{gl}_S$  inverses of each other in a suitable sense?

On top of this set-level question, we have the further structure of Poisson bivectors and symplectic forms. It is not guaranteed that the structures split or glue trivially. Therefore, we also have the associated *second* cutting and glueing problems,

**Problem 3** (The second cutting/glueing problem).

*Given symplectic forms  $\Omega_\Sigma$  on the full phase space and  $\Omega_{\Sigma,L}, \Omega_{\Sigma,R}$ , what is the difference*

$$g\ell_S^*(\Omega_\Sigma) - (\Omega_{\Sigma,L} + \Omega_{\Sigma,R}) \quad (1.100)$$

*between the pullback of the full Poisson structure and the trivial combination of the structures on the pieces? Dually, what is*

$$\Omega_\Sigma - \text{sp}_S^*(\Omega_{\Sigma,L} + \Omega_{\Sigma,R})? \quad (1.101)$$

If the cutting and glueing maps are inverses, the two questions are equivalent. The main point here is that together with the question of what degrees of freedom are involved in glueing and cutting, their algebraic structure may also be nontrivial. For example, a reasonable scenario is that on regions with common boundary  $S$ , the symplectic forms might be shifted by a corner contribution

$$\Omega_{\Sigma,L} \mapsto \Omega_{\Sigma,L} + \Omega_S \quad \Omega_{\Sigma,R} \mapsto \Omega_{\Sigma,R} - \Omega_S \quad (1.102)$$

without changing the trivial combination  $\Omega_{\Sigma,L} + \Omega_{\Sigma,R}$ . So there is an ambiguity *per se* even if there is no nontrivial combination happening in the glueing process, which needs to be fixed. However, depending on the situation, there might also be a necessity for a nontrivial combination. This is the case for example in the on-shell (of constraints) symplectic forms of gauge theories, which typically split as

$$\text{gl}_S^* \Omega_\Sigma = \Omega_{\Sigma,L} + \Omega_{\Sigma,R} + \Omega_S \quad (1.103)$$

with an extra term  $\Omega_S$  that involves certain field values on the common boundary  $S$ . Often, these extra terms are interpreted as introducing new degrees of freedom to the phase space; however, this need not be the case[113]. Instead, the additional terms can merely be functionals of field values in the phase spaces  $P_{\Sigma,L}, P_{\Sigma,R}$ . Then, no phase space extension takes place, but the glueing is still nontrivial.

The mention of the on-shell phase space highlights an important aspect of these problems: We need to specify which phase spaces we want to split/glue. In the case of unconstrained phase spaces, this is usually not a hard question; one only needs to think about boundary conditions. When there are relevant constraints, however (so in gauge theories) one has in principle *separate* cutting/glueing problems for the off-shell and on-shell phase

spaces (assuming an off-shell phase space is available). One very desirable criterion for the choice of phase spaces is that these at first separate problems lead to *compatible* results. I.e. the reduced phase space of the full slice  $\Sigma$  should be the same as the glueing of the reduced phase spaces of the regions  $\Sigma_L, \Sigma_R$ . It is particularly crucial that these questions be asked on the level of *phase spaces*, so, say when all degeneracies of symplectic forms have been eliminated.

One can also envision that similarly to the second glueing problem, there is a *third glueing problem about observables*. One may see this as a separate problem, but once again it entwines with the other two. In particular, say one has fixed local bulk expressions for an observable such as a Hamiltonian  $H \sim \int_{\Sigma} h$ , then there is again an ambiguity in the corner contribution. However, one will usually want to coordinate said contribution to ensure differentiability under the structures found on the regions  $\Sigma_L, \Sigma_R$ , or that more specifically that the observable in question is the symplectic generator of a given transformation on the phase spaces. We will not further comment on this third glueing problem outside of more specific examples.

We also have to mention that, if one wishes to be precise, there is a principal arbitrariness in the way cutting works which does not appear so obviously when looking at glueing. This comes from the fact that, when we split a region into two, we introduce a new stratum, the cutting surface, on which in principle arbitrary fields, dynamics and structures may be placed. After all, as long as the corresponding glueing procedure erases them after, what is the issue? The issue is the arbitrariness itself - as long as we do not appeal to some principle for which data are allowed to appear on boundaries when cutting, the cutting and glueing problems are at best ambiguously solvable.

We therefore choose to codify the ‘common sense’ (which is really an implicit, imprecise assumption) notion that one should not create ‘extra data’ (although this should ring alarm bells already when thinking about boundary degrees of freedom). We will call this idea ‘minimal cutting/glueing’ and for the moment just point out that its use is more intuitive than it is precise or even rigorous. It is not even on the status of the usual minimal coupling procedure, as there seems to be no obvious way to codify what ‘minimality’ means.

Let us also turn to spacetimes. We have discussed the glueing and cutting of Cauchy slices with no further information about their embedding into a spacetime. However, we have different time developments for the slice depending on the bulk dynamics we are given and the boundary dynamics we choose. E.g. if we are in a relativistic theory and do not provide boundary field configurations along a causal boundary, we only generate the domain of causal dependence  $D(\Sigma)$  from the data on the Cauchy slice (i.e. a causal diamond). Therefore, with no further specification of boundary dynamics, and time evolution being trivial on the corner, the glueing of regions  $\Sigma_L, \Sigma_R$  simply gives the domain of dependence of their union (which is not the union of the individual domains). The glueing is then, on the spacetime level, *not* along any given codimension 1 surface, but exclusively along the connecting corner  $S = \partial\Sigma$ , and additional spacetime regions are generated by evolving

with a *different* boost diffeomorphism.

We can therefore see that the way a glueing of spatial regions looks in the spacetime picture depends on how we evolve the spatial slice. The choice of boost vector fields is simple from the canonical point of view as it vanishes on the corners; it therefore is more easily integrable. However, this notion of time evolution, due to this vanishing, does not compose nicely under cutting and glueing.

We distinguish this from the glueing along codimension 1 boundaries with specified boundary dynamics. In this case, we can work with cylindrical spacetimes, which are generated by diffeomorphisms which do not vanish on the corners. If integrability of these transformations is guaranteed by appropriate boundary conditions, we can glue the phase spaces and hope for the generators as well as the time evolution to behave nicely under cutting/glueing. Importantly, we can speak about this kind of glueing easily in a usual path integral picture, which allows us to transfer many statements between the two pictures.

Going back to phase space glueing on slices, let us specialise the discussion for a moment on the gauge-invariant degrees of freedom of a field theory, for example by focusing again on a scalar field. If we allow ourselves access to infinite resolution operations, we have no issues there: There is a straightforward way, classically speaking, of taking the field values, and we can use restrictions of fields to solve the first cutting problem: Assuming that the phase space  $P_\Sigma$  has a sensible notion of restriction maps (so if it is some kind of presheaf), we simply take

$$P_{\Sigma_L} = \{\phi : \Sigma_L \rightarrow \mathbb{R} \mid \text{suitable regularity}\} \quad (1.104)$$

(similar for  $\Sigma_L$ ) and as the codomain

$$P_{\Sigma,LR} := \{(\phi_L, \phi_R) \in P_{\Sigma_L} \times P_{\Sigma_R} \mid \phi_L|_S = \phi_R|_S\} \quad (1.105)$$

and the cutting map is just

$$\text{sp}_S(\phi) = (\phi|_{\Sigma_L}, \phi|_{\Sigma_R}). \quad (1.106)$$

The converse is true for glueing: Regularity issues aside, the first glueing problem is resolved by setting  $P'_{\Sigma,LR} = P_{\Sigma,LR}$  and glueing together the configurations through indicator or bump functions.

It therefore may appear that the glueing problem is restricted only to gauge degrees of freedom, or to nonlocally distributed gauge-invariant degrees of freedom where a simple restriction map will not be available. While classically, the discussion may be held on this level at the idealised level, this is not the case at all once quantum effects are considered. We will demonstrate this first in the path integral setting, where it will be apparent that strict glueing is not imposed for both field and conjugate momentum off-shell even in the naive glueing procedure. Then, going further, we demonstrate that the appropriate smearing of fields into local functionals will again lead to issues.

The problem at the heart of the matter lies with the notion of taking field values at the

corner. In a continuum field theory with local degrees of freedom at every point, this is an infinite energy operation (for free fields, essentially due to the uncertainty principle applied to canonical commutation relations)

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta(x, y). \quad (1.107)$$

If one were to force the state of the field into a given set of values  $\phi(x) = \Phi(x)$  at all points  $x \in S$ , one creates a state with zero uncertainty in the  $\phi$  variables, and therefore infinite uncertainty in the conjugate  $\pi$ . As this conjugate is quite directly related to energy, it is clear the state no longer lies in the Hilbert space proper.

Note that the restriction to finite energy is what is physically salient here: In the ‘larger Hilbert space’, which is just the product of Harmonic oscillators over all points,

$$\mathbb{H}_{\text{larger}} := \bigotimes_{x \in \Sigma} \mathbb{H}_x \quad \mathbb{H}_x = L^2(\mathbb{R}_x, d\lambda_x) \quad (1.108)$$

which has *no* continuity requirements on nearby field values, one clearly has no issues with factorization<sup>19</sup>. One can formally simply take any complementary subsets and the Hilbert space factorises with not a care in the world. Yet it is also clear that this is an unphysical Hilbert space. The reason for this is that the Hamiltonian contains a gradient term,

$$H = \int_{\Sigma} \frac{1}{2} |\nabla \phi|^2 + \dots \quad (1.109)$$

which shows that intuitively, discontinuous configurations will have an infinitely high energy. It is therefore the requirement of finite energy that brings us to the physical Fock space, so we really only care for those states that are ‘a finite energy away’ from the ground state. This intuitively introduces nontrivial regularity for the fields, which leads to the actual glueing issues.

Therefore, as soon as one considers states of finite energy only<sup>20</sup>, we have to face the fact that **glueing is a nontrivial problem even for free quantum scalar fields**.

We will not solve this problem here; Instead, we will highlight some intuitions for it, propose idealised ways of addressing it and then argue for some workarounds.

### 1.3.2 Algebraic cutting and the ‘corner algebra’

Let us get a better handle on these issues of cutting and glueing, and how they arise at the quantum level, *in an elementary example*. Consider the (canonical) algebra  $\mathcal{A}$  of a

<sup>19</sup>I would like to thank C. Rovelli, J. Kirklin and others present in the discussion for pointing this intuition out to me.

<sup>20</sup>There are, of course, technical caveats to this argument via energy such as whether we mean for example infinite *expectation values* or not, which are allowed if one considers the Hamiltonian only as a quadratic form. The main point here is that unphysical resources are required to distinguish field values exactly.



smearing scalar field  $\phi[\alpha]$  and its conjugate momentum  $D - 1$ -form  $\Pi[f]$  on a closed spatial slice  $\Sigma$  generated by the objects and relations

$$[\phi[\alpha], \pi[f]] = i\hbar \left( \int_{\Sigma} \alpha f \right) \mathbb{I} \quad (1.110)$$

with  $f \in \Omega^0(\Sigma), \alpha \in \Omega^{D-1}(\Sigma)$ . We wish to split the algebra in a way according to a cutting of  $\Sigma$  along a codimension 2 surface  $S$ . The subsystem algebras  $\mathcal{A}_{L|R}$  on the two sides, representing all operations we can perform on the subsystems, are easily found by restriction of the smearing parameters to  $\Sigma_{L|R}$ . However, these are not complementary subsystems<sup>21</sup>: Instead, their commutants are

$$(\mathcal{A}_{L|R})' = \mathcal{A}_{\dot{R}|\dot{L}}, \quad (1.111)$$

the algebras where smearing parameters are not allowed to have support on  $S$ . We call these the ‘interior algebras’. We can then define the corner algebras as quotients by the ideals  $\mathcal{I}_{L|R}$  generated from the interior algebras,

$$\mathcal{A}_{S,L|R} = \frac{\mathcal{A}_{L|R}}{\mathcal{I}_{L|R}} \quad (1.112)$$

which ideally are the same from both sides of the cut (or else we would not have an invertible cutting and glueing procedure). These are the closest equivalents to operators with support only on  $S$  that are available. We see that the only natural data we have is that of a *projection* onto the corner algebra, so we cannot see it as a subsystem of  $\mathcal{A}_{L|R}$ , which would usually be modelled as a *subalgebra*. If an inclusion exists, which renders the corner algebra as a true subsystem by itself, it must be prescribed *by hand*.

Let us compare this to the fully regularised case of a lattice scalar field, where the above smearings are unnecessary<sup>22</sup> and one can evaluate the field  $\phi$  and momentum  $\Pi$  directly on lattice sites  $x \in \Lambda$ . Here, if we divide the lattice into halves  $\Lambda_{L|R}$  meeting in a set of sites  $S$ , then there is a clear notion of a corner algebra: It is simply the algebra generated from all the  $\phi_x, \Pi_x$  on sites  $x \in S$ . This is, practically speaking, the intersection  $\mathcal{A}_L \cap \mathcal{A}_R$ , which is also a subalgebra and nonempty in this case. One can still, of course, do the same definitions as above, but in contrast to the continuum setting, the resulting quotient algebras admit simple injections into the full system. In this setting, it is also easy to see that glueing of two regions simply consists of identifying the corner operators (so the values of the fields) on both sides in a suitable way.

If indeed there is some injection of *some* ‘corner algebra’  $\mathcal{A}_S$  (e.g.  $\mathcal{A}_L \cap \mathcal{A}_R$  on the lattice)

$$\mathcal{A}_S \hookrightarrow \mathcal{A}_{L|R}, \quad (1.113)$$

<sup>21</sup>Complementary subsystems  $\mathcal{A}_L, \mathcal{A}_R$  are usually defined as obeying  $[\mathcal{A}_L, \mathcal{A}_R] = 0$ , and that they are each other’s commutants,  $(\mathcal{A}_{L|R})' = \mathcal{A}_{R|L}$ .

<sup>22</sup>Another way to phrase this would be that the resulting algebra is a *result* of smearing; an idea that we will get back to in 1.4 with the notion of sampling.

then we can construct the usual Dirichlet and Neumann boundary condition algebras easily, and glueing becomes clearer.<sup>23</sup> For example, one may then use the above inclusions to see the algebras  $\mathcal{A}_{L|R}$  as  $\mathcal{A}_S$ -modules (which, in this context, means vector spaces with an representation of  $\mathcal{A}_S$ ), and then imagine a glueing prescription for algebras via the spatial tensor product of  $\mathcal{A}_S$ -modules

$$\mathcal{A}_{\Sigma_L \cup_S \Sigma_R} \cong \mathcal{A}_{\Sigma_L} \otimes_{\mathcal{A}_S} \mathcal{A}_{\Sigma_R} \quad (1.114)$$

which generalises the usual glueing through the standard tensor product in finite dimensions. The meaning of the module structure should here just be seen as allowing some action of corner operators on the algebras of the regions. Whether this holds in concrete theories needs to be verified, and can therefore be seen as more of a tentative glueing prescription. This prescription, for the usual case of disjoint sets, together with  $\mathcal{A}_\emptyset = \mathbb{C}$ , reduces to the usual tensor product of algebras. This shows that the glueing prescription is subject to *at least* the same subtleties as the standard case of disjoint regions  $\Sigma_{L|R}$ , so it will only hold for certain, well-behaved theories.<sup>24</sup>

Represented on Hilbert spaces, such a glueing directly translates into a tensor product of *Hilbert modules*  $\mathbb{H}_{\Sigma_L}, \mathbb{H}_{\Sigma_R}$  which inherit the action of  $\mathcal{A}_S$  from their algebras of operators,

$$\mathbb{H}_{\Sigma_L \cup_S \Sigma_R} \cong \mathbb{H}_{\Sigma_L} \otimes_{\mathcal{A}_S} \mathbb{H}_{\Sigma_R}. \quad (1.115)$$

This means that the action of corner operators is *removed* from the joint Hilbert space, which implements the removal of the fictitious interface  $S$ . Such a tensor product can likely be realised in concrete cases through analogues of convolutions of wavefunctions.

How can such Hilbert modules be realised? In general, there is a simple way to get at least an  $\mathcal{A}_L$ -module, when the algebras  $\mathcal{A}_{\tilde{L}}$  and  $\mathcal{A}_L$  are known, together with a representation of  $\mathcal{A}_{\tilde{L}}$  on a 'bulk' Hilbert space  $\mathbb{H}_{\tilde{L}}$ . Then, the open-boundary-condition Hilbert space can be understood as the 'free-est' (no boundary relations/conditions) representation of the full algebra  $\mathcal{A}_L$  that restricts to the given  $\mathbb{H}_{\tilde{L}}$ . This is simply the induced representation of  $\mathcal{A}_L$ , which can be realised as the tensor product

$$\mathbb{H}_L = \mathbb{H}_{\tilde{L}} \otimes_{\mathcal{A}_{\tilde{L}}} \mathcal{A}_L \quad (1.116)$$

of the bulk space with the extended algebra. The action of the bulk operators is the same as before here, and the action of corner operators from  $\mathcal{A}_{S_L}$  is obvious, too. In principle, then, any boundary condition is imposed on this by quotienting by some ideal.

However, this is *not* an  $\mathcal{A}_{S_L}$ -module; instead to get one, one needs to quotient by the bulk space, yielding a kind of *boundary Hilbert space*

$$\mathbb{H}_{\partial\Sigma} := \frac{\mathbb{H}_\Sigma}{\mathbb{H}_{\tilde{\Sigma}}} \quad (1.117)$$

---

<sup>23</sup>Notice that this is simply a *section* of the short exact sequence  $0 \hookrightarrow \mathcal{I}_L \rightarrow \mathcal{A}_L \rightarrow \mathcal{A}_{S,L} \rightarrow 0$ , so requiring that the sequence *splits*.

<sup>24</sup>We also point to several technical challenges in realising this in the setting of local algebras assigned to *spacetime* regions, the likes of which are the real subject of study in algebraic QFT, as opposed to the canonical one we presented here.

which then inherits an action of the quotient algebra  $\mathcal{A}_{\partial\Sigma}$ . However, clearly this is missing all the bulk operators, so this can't quite be what we need. Indeed, unless there *are* regularised injections  $\mathcal{A}_{\partial\Sigma} \hookrightarrow \mathcal{A}_\Sigma$ , one cannot use the glueing prescription above. Instead, in the general case, one must take, for the algebras, a quotient by a more general corner ideal  $\mathcal{I}_S$

$$\mathcal{A}_{\Sigma_L \cup_S \Sigma_R} \cong \frac{\mathcal{A}_{\Sigma_L} \otimes \mathcal{A}_{\Sigma_R}}{\mathcal{I}_S}, \quad (1.118)$$

and for their representations, a projection map  $P_S$

$$\mathbb{H}_{\Sigma_L \cup_S \Sigma_R} \cong P_S(\mathbb{H}_{\Sigma_L} \otimes \mathbb{H}_{\Sigma_R}) \quad (1.119)$$

both of which in general cannot be understood as tensor products. In particular, they do not have immediately natural notions of separability. Therefore, the class of theories where a regularised inclusion of the corner algebra *does* exist has several (nontrivial) nice properties.

We also note that the picture of glueing presented here as modular tensor products has already found applications[56, 78]. This usually comes in cases where the corner algebra is taken to be a group algebra  $\mathcal{A}_S = \mathbb{C}[G]$ , in which case the glueing procedure amounts to matching up appropriate representations of the group  $G$  in the left and right tensor factors. This special case is then commonly known as the entangling product. Clearly, though, this can only apply to cases where the corner data is entirely captured by a group action, for example when the space of corner data forms a homogeneous space. If this is the case, then functions on said homogeneous space (as a stand-in for a more general corner algebra) can be lifted to functions on the symmetry group of the homogeneous space, giving elements of the group algebra. So while in certain cases, one may expect that the entangling product is sufficient to speak of glueing, we believe that the general glueing prescription is more transparent if phrased in terms of algebras of corner operators more generally.

We now wish to clarify how the direct-sum-of-boundary-conditions picture of glueing arises from the more abstract schematic we proposed here. For this, assume that the corner algebra here has the same general structure as the bulk one, generated e.g. not by smearings of  $\Pi$  but by those of  $i_{\hat{r}}\Pi$  on  $S$ , for  $\hat{r}$  the boundary radial vector field on  $S$ . Injections may be thought of as extending the smearing parameters of corner operators into a small neighbourhood of  $S$  in the interior, possibly parametrised by some arbitrary parameter  $\epsilon$ .<sup>25</sup> Then, we can get representations with fixed boundary conditions as follows: Select a commutative subalgebra  $C$  of  $\mathcal{A}_S$  corresponding to operators that should have definite values on the corner. In our example, this would be either the field operators or their momenta, again smeared. Then, define the ‘boundary conditioned’ algebra as its commutant

$$\mathcal{A}_L^C = (C)'_{\mathcal{A}_L} = \{A \in \mathcal{A}_L : [A, C] = 0\} \quad (1.120)$$

---

<sup>25</sup>This could lead to a family of injections whose regularity properties encode finer properties of the theory with respect to glueing. A similar process is already secretly taking place when we assume the algebra of quantities to be fully localised on a spatial slice  $\Sigma$ , more properly realised as a limit of some algebras on spacetime regions  $\Sigma \times (-\epsilon, \epsilon)$ .

which is a subalgebra of  $\mathcal{A}_L$ . For Dirichlet conditions,  $C$  will consist of smeared field operators, and  $\mathcal{A}_L^C$  will lack the corner momentum operators. This is completely analogous to the choice of ‘electric’ and ‘magnetic’ algebras for local regions in lattice gauge theories[116] - there is nothing special about gauge invariance. Rather, this ‘choice’ of subalgebras for local regions with boundary comes with an implicit choice of corner subalgebra or boundary condition.

Given this Dirichlet subalgebra, we can consider representations of it. Since  $C$  lies in the centre of it, a representation of the Dirichlet algebra has fixed, definite values for the corner field operators - a boundary condition. Name these representation spaces  $\mathbb{H}_{L,\psi}$ , indexed by their boundary conditions. Then, this does not give yet a representation of  $\mathcal{A}_L$  because the momentum operators on the corner are still missing. However, under favourable conditions like we have here, the ‘missing’ operators from  $\mathcal{A}_S$  can be recovered through an induced representation procedure. In our example, the momentum operators are generators of unitaries  $U(\chi)$  which affect a shift in field value:

$$e^{-i\Pi[\chi]} |\Phi\rangle = |\Phi + \chi\rangle. \quad (1.121)$$

With this, we can induce a representation of  $\mathcal{A}_S$  from one of  $C$ , and in turn one of  $\mathcal{A}_L$  from  $\mathcal{A}_L^C$ . Because it shifts between values of  $\psi$ , the representation space is the direct sum

$$\mathbb{H}_L = \bigoplus_{\psi} \mathbb{H}_{L,\psi}. \quad (1.122)$$

This is the ‘minimal’ extended Hilbert space we could have expected. With this new background, though, things are a lot more clear: This representation is an induced one of the full corner algebra. In principle, rather than doing this induction step, we could work with the full algebras directly and look for their representations.

It then also becomes clear that the standard matching prescription

$$\bigoplus_{\psi} \mathbb{H}_{L,\psi} \otimes \mathbb{H}_{R,\psi} \quad (1.123)$$

is a special case of this tensor product of modules. If, as above, we see the corner algebra as generated by operators  $\Phi, \Pi$ , then the modular tensor product can be realised as a subspace of the usual one where these corner operators act the same on both regions, i.e. it is the  $\mathbb{C}$ -linear span of pure tensors

$$\{\psi_L \otimes \psi_R \mid X\psi_L \otimes \psi_R = \psi_L \otimes X\psi_R \ \forall X \in \mathcal{A}_S\}. \quad (1.124)$$

In this expression, it is important to keep track of how the corner algebra acts on the two factors. The relevant choice for the example in question is that the field operators  $\Phi$  act the same on both sides, but the momenta  $\Pi$  act with *opposite sign*, meaning the associated shift operators are inverses of each other. Then, it follows that this subspace is precisely generated by states whose values for  $\Phi$  on  $S$  are matched up, therefore

$$\mathbb{H}_{\Sigma_L} \otimes_{\mathcal{A}_S} \mathbb{H}_{\Sigma_R} \cong \bigoplus_{\psi} \mathbb{H}_{L,\psi} \otimes \mathbb{H}_{R,\psi} \quad (1.125)$$

realising the usual intuition is reproduced by this more abstract construction.

### 1.3.3 The cutting and glueing problems, secundo

In the past few sections, we have expanded on the idea that a suitably lax notion of glueing on the algebraic level might be implemented through the use of a ‘corner algebra’ which could be used to define a generalised product.

We have pointed out that, when one takes the necessity to properly smear observables over space(time) regions seriously, it is not clear how to speak of values of fields at the corners where one wants to glue. This leads to a primary issue of the glueing and cutting problems in the continuum: The ‘corner algebra’ we propose is at the heart of properly identifying degrees of freedom across boundaries, is itself hard to identify because one needs infinite resolution to make statements about the values of fields at even codimension 2 surfaces.

We now return to the cutting problem. As stated before, if one has an algebra generated by smeared fields, then the simplest cutting operation is generated by restricting the support of the smearing functions. This is essentially the same as restricting the field configurations themselves. This produces algebras  $\mathcal{A}_{L|R}$  with support on the corner  $S$ , describing the open system (no boundary conditions). Then, once again, one can take their commutants to obtain the interior algebras. The cutting problem, from this algebraic level, can then be formally resolved as follows:

Given the algebra  $\mathcal{A}_L$  of the subregions  $\Sigma_L$ , we can find an equivalent to the ‘phase space’ by taking its set of algebraic states,  $\mathcal{S}(\mathcal{A}_L)$ [117]. In the classical case, this is precisely the set of evaluation maps  $ev_p : \mathcal{A}_L \rightarrow \mathbb{C}$  which correspond uniquely to points  $p \in P$  in phase space. In the quantum case, one has no such easy interpretation, but each element  $\omega \in \mathcal{S}(\mathcal{A}_L)$ , through GNS construction, gives a representation of the algebra as a Hilbert space, and by summing them over all such, we can construct a regional Hilbert space  $\mathbb{H}_L$ . The cutting problem, in its stated form, can then be expressed over  $\mathcal{S}(\mathcal{A})$  as inducing a state on  $\mathcal{A}_L$  from one on  $\mathcal{A}$ . This is of course just the restriction of such a state to the subalgebra  $\mathcal{A}_L$ , so our solution is formally

$$sp_S(\omega) = (\omega|_{\mathcal{A}_L}, \omega|_{\mathcal{A}_R}). \quad (1.126)$$

The codomain here is then just all pairs of states  $(\omega_L, \omega_R)$  which agree on the intersection  $\mathcal{A}_L \cap \mathcal{A}_R$ . This is also a subalgebra in principle, but it may be empty, as in the example of the smeared free scalar field algebra. This percolates to the level of Hilbert spaces by seeing rays  $|\psi\rangle \in \mathbb{H}$  as vector states in the algebra  $\mathcal{A}$ . I.e. the reduced state is the state  $\omega_L$  which agrees with  $|\psi\rangle$  in expectation value on all observables in  $\mathcal{A}_L$ . This, of course, will not in general be a vector state on  $\mathcal{A}_L$ , as the familiar example of reduced states generically being ‘density matrices’ shows. So we never get a cutting map  $\mathbb{H} \rightarrow \mathbb{H}_{LR} \subset \mathbb{H}_L \otimes \mathbb{H}_R$ , at best

$$sp_S : \mathcal{D}(\mathbb{H}) \rightarrow \mathcal{D}(\mathbb{H}_{LR}), \rho \mapsto \rho_L \otimes \rho_R. \quad (1.127)$$

This is just the usual statement that when reducing a pure state to a subsystem, it generically becomes a mixed state of the subsystem.

Of course, if the algebra is not generated by smearings of fields, then the cutting of the

algebra may not work this way and this is not valid.

The above procedure may be understood as a ‘minimal’ cutting, as far as usual intuition goes. What characterises this procedure as minimal is that it does not introduce any on-shell degrees of freedom on the glueing surface. In fact, it makes no reference to anything but the notion of subsystems by which we denote our regions on the level of observables. This means that as long as there is a ‘minimal’ or ‘universal’ way in which the subsystem algebras  $\mathcal{A}_{L|R}$  are subalgebras of  $\mathcal{A}$ , then there are no choices to be made and one can argue that this is a minimal way to split. However, this may not be the case. In particular, specifying what a subsystem *is* may, indeed, *require* the use of additional degrees of freedom. Particular cases are in gauge theories, and most strikingly gravity.<sup>26</sup> Because of this, we now investigate more closely what happens in gauge theories.

### Glueing and cutting problems in gauge theories

The most important class of situations where this is not the case is when constraints are present. This shows itself typically in gauge theories, when the above construction is valid on the kinematical phase space, but the physical phase space is not obviously generated by smearings of unconstrained fields. If gauge fixings are used, they also usually restrict the configurations in a non-ultralocal<sup>27</sup> way, i.e. the Lorenz or Coulomb gauges. In this case, the cutting and glueing problems acquire a new dimension: We need to ask whether projection to the physical/gauge-invariant phase space and cutting or glueing commute. Similarly, we can ask if there is any meaning to cutting and glueing on the kinematical phase space, when after all it contains physically irrelevant information.

We will see that working with the base space  $\Sigma$  only, instead of the total space (the principal bundle  $P$ ) that a gauge field is typically defined on, will lead to ambiguities and issues that are difficult to disentangle. In order to really sell this point, we will start our exposition on this level and demonstrate just how messy things can get, but ultimately resolve the tensions using geometric arguments.

We will take the common viewpoint that some off-shell cutting map  $\text{sp}_S$  is available on the off-shell configuration space  $\mathcal{C}_\Sigma$ , i.e. of the aforementioned type. Say we split an off-shell configuration  $A \in \mathcal{C}_\Sigma$  into its restrictions,  $(A_L, A_R)$ . Then, there is already an obvious problem: What is gauge on  $\Sigma$  is not gauge on  $\Sigma_{L|R}$ . This specifically holds for the degrees of freedom on the cut  $S$ . A procedure of defining the cutting on the gauge-invariant level therefore runs into the issue that using different gauge representatives for a cutting (passing through the off-shell phase space) yields *different* split gauge invariant

---

<sup>26</sup>In gravity, arguably specifying the location of a subregion is not viable without additional degrees of freedom - thus cutting is also not possible without those, and the notion of minimality is questionable.

<sup>27</sup>Ultralocal meaning acting, pertaining to, or being distributed per-point, as opposed to other notions of locality which may pertain to local regions with finite extent. In diffeomorphism invariant theories, this must be specified by way of physical reference frames. Note that operationally, this notion is also unphysical as one cannot resolve single points.

configurations:

$$\mathrm{sp}_S(A \in [A]) \neq \mathrm{sp}_S(A + df \in [A]) \quad \text{if } f|_S \neq 0. \quad (1.128)$$

There are multiple stances on this matter; the first is to take this as a sign that corner symmetries are, in fact, not symmetries but must be seen as a redundancy as well. This resolves the problem neatly, but leads to the fact that the cutting map is still not a simple restriction, even if gauge-fixings are used[113]. The second is to require that the cutting map on the off-shell space  $\mathcal{C}$  is actually gauge-invariant, so that the above *is* equal. However, this means that the cutting map is not injective on its domain, so that any glueing and cutting on the off-shell level will *not be inverses*. The third is to be more precise about how the surface  $S$  is specified. In particular, suppose that the surface is equipped *by definition* with a gauge frame  $\phi$  (i.e. for diffeomorphisms, this would be a set of coordinates defining the surface). This puts us into an extended phase space. Then, there is a preferred gauge representative  $(\underline{A}, \mathbb{I})$  of  $[(A, \phi)]$  in which  $\phi$  has been gauge fixed to be trivial. The residual gauge transformations are trivial on  $S$ . We can then define the cutting on the gauge invariant level by lifting to the representatives with this specific condition, cutting, and then quotienting again:

$$\mathrm{sp}_S([A]) := [\mathrm{sp}_S(\underline{A})] \quad (1.129)$$

More generally, all we need is some way of choosing a gauge fixing for the corner gauge transformations, which can be provided in many other ways, too.

We want to stress here that the use of some gauge fixing is *unavoidable* for the definition of the cutting map if one intends to use some variant of restriction maps. The general problem is that the equivalence classes  $[A]$  are not functions on spacetime, so do not admit restrictions; one then generically wants to solve this situation by choosing a representative that can be meaningfully restricted. Regardless of whether one sees corner transformations as redundant or not, this is always done in practice. The only difference is whether the choice of representative makes a difference or not. From the point of view of the full slice, it never does - but the subregion phase spaces might care, so if this is so, a particular choice must be provided.

This general fact reflects the *failure of subsystem assignment recursivity* that gauge theories generically feature<sup>28</sup>: It is not possible to resolve all three cutting problems in a neat fashion. Concretely, we mean by this means that one has to drop two of the three following properties (in our own nomenclature):

1. Symmetry recursivity: The symmetries of the theory on  $U \subset \Sigma$  are obtained ‘by restriction’ of symmetries on  $\Sigma$ .

---

<sup>28</sup>Note that our notion of subsystem recursivity is, a priori, different from the recent philosophy literature [118, 119]: We do not refer to isolation of subsystems, which is a dynamical notion, but rather to properties of the restriction process, compare [120]. In particular, we mean that in a subsystem recursive theory, the data assigned to any region is injectively contained in any superregion. A particular example where this holds is if the (co)sheaves used to assign the data are flabby.

2. Symplectic recursivity: The symplectic form/potential of the theory on  $U \subset \Sigma$  are obtained ‘by restriction’ of the form/potential on  $\Sigma$ .
3. Generator recursivity: The gauge generators of the theory on  $U \subset \Sigma$  are obtained ‘by restriction’ of the generators on  $\Sigma$ .

If one adopts either of the three aforementioned stances towards the cutting problem above, one ends up sacrificing some of these properties. First, if one chooses to see corner gauge transformations as symmetries, one does not have symmetry recursivity from the get-go. In the non-extended setting, then, potential and generators are recursive. In the extended phase space setting, this remedied by the dressings, and symmetry recursivity is restored at the cost of phase space/symplectic recursivity, as well as generator recursivity. If instead one sees corner gauge transformations as redundancies, one keeps symmetry recursivity, but needs to work with Poisson spaces, or perform superselection[113]. In the superselected setting, one then still loses symplectic recursivity, and arguably the phase space data is also not recursive due to superselection.

We can therefore see that while there may be technical problems to overcome in this matter, there is some amount of *physical content* to this problem: All of these refer implicitly to the idea that ‘the parts are induced from the whole’, which is, if put in the setting of the cutting and glueing problems, dual to ‘the whole is the sum of its parts’. Therefore, the failure of subsystem recursivity is, we would argue, a *reflection* of the failure of factorizability of gauge theories. This failure is well-known[116, 121, 122] and due to the presence of first class constraints, but of course equivalent to the presence of gauge invariances. Therefore, one cannot hope for a simple resolution that does not destroy some aspect that is quite fundamental to gauge theories.

We can already gleam from all this that the cutting problem tells us something crucial about the glueing problem: If we glue with representatives, we must be very sure *in which gauges* we are glueing together data. I.e. suppose we are given configurations on  $\Sigma_L, \Sigma_R$  which are in different gauge fixings each. How do we glue them? Intuitively, this seems meaningless as they belong, a priori, to different but isomorphic phase spaces. In practice, it may be possible to define a *global* gauge fixing function from the ones on the subregion, i.e. by combination with indicator or bump functions. Now, still, one might argue this does not matter because only physical degrees of freedom need to be glued. However, it is not clear from a practical viewpoint how this could be the case, as when a particular gauge is chosen for comparison, then depending on the type of continuity required, performing gauge transformations may or may not ruin the continuity in a different gauge. i.e.

$$A_L^{gf} \stackrel{S}{=} A_R^{gf} \stackrel{?}{\implies} A_L^{gf} + df_L \stackrel{S}{=} A_R^{gf} + df_R \quad (1.130)$$

If the gauge transformations in the subregions are restricted to vanish on the corner  $S$ , this will not be an issue. If not, then one needs to find a more elaborate glueing procedure, one which does not require continuity[113]. But even then, the question of the choice of gauge



fixing remains, and whether this is meaningful. Of course, if we use the extended phase space, then the gauge-fixed potential can be realised differently as the dressed potential  $A + d\phi$ , which then is gauge invariant throughout and can be made continuous.

Now, let us take the extended phase space point of view into focus: Here, we have full gauge invariance on the partial slices  $\Sigma_{L|R}$ , so one might think we avoid the issue of gauge fixing for cutting and glueing. This may be true, but reintroduces the ambiguity in a more obvious form: The phase space of the full slice,  $P_\Sigma$ , does not have the two copies of dressing degrees of freedom  $\phi_{L|R}$  that comes from the partial slices.

Therefore, when glueing, one of these pairs at least must be lost, and when cutting, at least one must be created (at the off-shell level, there are dressings throughout the region, so there is only one extraneous copy). Then, it is clear that the ambiguity in gauge fixing from before is just the choice of a value of  $\phi_L$  and  $\phi_R$  on the partial slices.<sup>29</sup> More precisely, the *relative orientation*

$$\Delta\phi := \phi_L - \phi_R \quad (1.131)$$

is a quantity that relates the two phase spaces on the partial slices that is not obviously given a value in the full theory. Recall our discussion of gauge invariant boundary conditions in 1.2.2: If we require continuity of gauge-invariant degrees of freedom  $a = A + d\phi$  across the cut,  $a_L = a_R$ , then these values are specified as usual on the partial slices by restricting  $a_L = a|_{\Sigma_L}$ . However, the full configuration also consists of  $\phi$  and the glueing is allowed to be *discontinuous* across the cut in  $A$ :

$$A_L - A_R = \Delta\phi \quad (1.132)$$

So in the extended phase space setting, it becomes clear that the cutting and glueing problem in gauge theories is due to the lack of required continuity in the gauge degrees of freedom.

### Resolving the first cutting/glueing problems in gauge theories

We have hopefully communicated in the previous section that the question of gauge fixings complicates the cutting and glueing process somewhat. We now hope to provide a simple resolution and then provide some geometric intuition.

First, we need a setup of phase spaces that includes gauge fixings. So, essentially, given the three off-shell (pre)configuration spaces  $\mathcal{C}_\Sigma, \mathcal{C}_{\Sigma_L}, \mathcal{C}_{\Sigma_R}$ , we choose arbitrary gauge fixing functions<sup>30</sup>  $f, f_L, f_R$ , which define the gauge-fixed subspaces  $\mathcal{C}_\Sigma^{gf}, \mathcal{C}_{\Sigma_L}^{gf}, \mathcal{C}_{\Sigma_R}^{gf}$ .

We then ask, instead of one overall cutting and glueing procedure, for an assignment of *individual* procedures

$$\text{sp}_S^{gf} : \mathcal{C}_\Sigma^{gf} \rightarrow \mathcal{C}_{\Sigma,LR}^{gf} \quad \text{gl}_S^{gf} : \mathcal{C}_{\Sigma,LR}^{gf} \rightarrow \mathcal{C}_\Sigma^{gf} \quad (1.133)$$

<sup>29</sup>Again, recall that these only live on the *prephase* spaces and must be quotiented out to get to the actual phase space.

<sup>30</sup>So, we require that they form a second-class pair with the gauge generators.

which we induce in a manner akin to the procedure around the Coulombic mismatch found in [113]: One glues and cuts the whole field in the given gauge fixings, and reconstructs the radiative pieces from there separately in each region. These maps have the property of making each individual global gauge-fixed potential continuous across the cut (conversely, the dressed potential in the extended phase space setting), but there may be discontinuities across the cut between the regional gauge-fixed/dressed potentials. Of course, it still makes intermediate use of a restriction map on  $\mathcal{C}_\Sigma$ .

Essentially, this resolution to the problem dispenses with the use of one mapping altogether, and instead says each gauge fixing must be treated with the appropriate care. Then, of course, this leads to a replacement problem<sup>31</sup>: If one now chooses a generic  $A \in \mathcal{C}_\Sigma$ , which mapping does one choose? A priori, there is no one particular gauge fixing condition that this  $A$  fulfils, in fact it fulfils infinitely many. Therefore, we argue that really the choice of gauge fixing, even here, *remains a choice* and at best one can find a family of gauge fixings such that each  $A$  lies in precisely one gauge-fixed configuration spaces. Such a family would *foliate* the configuration space. In the extended phase space setting *with bulk source*, such a family is easily given: It is simply

$$\{f = \phi - \phi_0 \mid \phi_0 \in \text{Conj}(P) \hat{=} \mathcal{G}\} \quad (1.134)$$

so just fixing a particular configuration of the dressing. So, one solution to the cutting and glueing problem on the classical level is given by a family of cutting and glueing maps parametrized by the configurations of the dressings at the global and subregion levels. As the corner values of the subregion dressings are not gauge in the individual regions, the ambiguity of the relative reorientation in cutting remains. Therefore, on the subregions, these families of dressing field configurations should be understood as *keeping the corner value fixed*.

We can also get a lot of insight from considering the geometry of principal bundles[81, 123], and how connections fit into it. In the geometric picture, we equip a spacetime  $\Sigma$  with a principal bundle  $P \rightarrow \Sigma$ , and a connection 1-form is first and foremost a 1-form  $\omega \in \Omega^1(P) \otimes \mathfrak{g}$  on  $P$  - globally defined, unlike on the base space.

Then, in this picture, it is relatively clear how to describe a restriction map: Given a submanifold  $R \hookrightarrow M_0$ , we can just pull back the bundle

$$P|_R \hookrightarrow P \quad (1.135)$$

and we can also just pull back the 1-form. So, *on the level of total spaces, connections are unambiguously restrictable*. Then, where do the problems come from? Essentially, one has two viewpoints: Either, one really wants to work with connection forms on the base spacetime, in which case one needs to choose local sections, whose role is very concretely played by the dressings  $\phi$ , on the individual regions. This brings us to the previous idea for glueing. The other viewpoint stays on the total space, and asks how two principal bundles

---

<sup>31</sup>Conservation of irritability holds.

may be glued, and furthermore connections on them.

Suppose we have a glueing of the base spaces  $\Sigma_L, \Sigma_R$  along their boundaries, so there is a diffeomorphism

$$f : \partial\Sigma_L \rightarrow \partial\Sigma_R. \quad (1.136)$$

Then, in order to glue the bundles atop them, we also need to specify how to glue the fibers, which each are isomorphic to the group  $G$ . So, we need in addition a bundle map

$$F : P|_{\partial\Sigma_L} \rightarrow P|_{\partial\Sigma_R} \quad (1.137)$$

which covers  $f$ . This additional data is essentially a gauge transformation supported on  $\partial\Sigma_L$ , and so comes with the same data as what is contained in the relative orientation  $\Delta\phi$ . Then, assuming continuity on the total space, we can match together the connection 1-forms on the total space to a glued global connection on  $P$ .

So, we can see that the presence of extra degrees of freedom on the boundary is necessary to ensure the right kind of glueing. The dressing fields in the bulk, instead, are only necessary for representing the connection forms on the base space.

As for minimal cutting and glueing, we can now see that the minimal required data for specifying a subsystem is, in fact, a gauge frame on the boundaries. *Therefore, if there is a notion of a minimal cutting, it introduces at most such a gauge frame, agnostic to what it is instantiated by.*

### Explicit cutting in extended phase space

Now let us be more explicit: Given a sample pre-configuration space parametrised by the bulk pair  $(A, *F)$  and the dressing pair  $(\phi, \pi)$ , how do we split? We first define a family of maps parametrized by  $\phi, \phi_L, \phi_R$  a group valued function on  $\Sigma$ , to the codomain

$$\begin{aligned} \mathcal{C}_{\Sigma, LR}^{\phi, \phi_L, \phi_R} &:= \{((A, *F, \pi, \phi)_L, (A, *F, \pi, \phi)_R) : \\ &\quad \phi_L \cdot (A_L, *F_L) \stackrel{S}{=} \phi_R \cdot (A_R, *F_R); \pi_L = \pi_R\} \\ &\subset \mathcal{C}_{\Sigma_L} \times \mathcal{C}_{\Sigma_R} \end{aligned} \quad (1.138)$$

And then define the cutting map on  $\mathcal{C}_\Sigma$  as

$$\text{sp}_S^{\phi, \phi_L, \phi_R}(A, *F, \pi, \phi) = ((A_L, *F_L, \pi_L, \phi_L), (A_R, *F_R, \pi_R, \phi_R)) \in \mathcal{C}_{\Sigma, LR}^{\phi, \phi_L, \phi_R} \quad (1.139)$$

with

$$\begin{aligned} a_L &= a|_L, a_R = a|_R, A_L = a_L - d\phi_L, A_R = a_R - d\phi_R \\ *F_L &= \phi_L^{-1} \cdot (\phi \cdot *F)|_L, *F_R = \phi_R^{-1} \cdot (\phi \cdot *F)|_R \end{aligned} \quad (1.140)$$

and  $\pi$  must be determined as a consequence of the boundary continuity equations  $C_2 = 0$ , which we can schematically rewrite as

$$\pi_L = *F_L|_S, \pi_R = *F_R|_S. \quad (1.141)$$

So, the principal freedom we have when cutting is twofold: The first is that  $\pi$  does not exist on the unsplit phase space, so we must *assign* a value to it that does not come from restrictions. The logical choice is to use the value it takes on-shell, but this is still a choice. The second is in the values of  $\phi_{L,R}$ , of which continuity is *not* required; off-shell, their value in the subsystems can in principle be completely arbitrarily assigned without affecting the gauge invariant content from the perspective of the full slice. Furthermore, on the actual phase space, they are quotiented away, meaning their values are really only important on the corner in any case.<sup>32</sup>

On-shell, the story becomes more interesting: The codomain we chose is such that the on-shell subset of  $\mathcal{C}_\Sigma$  is mapped into the on-shell subset of  $\mathcal{C}_{\Sigma,LR}^{\Delta\phi}$ . This ensures that we can take the quotient by redundancies on both sides. In addition, due to using the extended phase space, the gauge redundancy on the partial slices  $\Sigma_{LR}$  now includes *all* gauge transformations of the partial slices, so the choice of representatives ends up being irrelevant, and the cutting map descends to the physical phase spaces,

$$\text{sp}_S^{\phi,\phi_L,\phi_R} : P_\Sigma \rightarrow P_{\Sigma,LR}^{\phi,\phi_L,\phi_R} := \mathcal{C}_{\Sigma,LR}^{\phi,\phi_L,\phi_R} / (\mathcal{G}_L \times \mathcal{G}_R) \subset P_{\Sigma_L} \times P_{\Sigma_R} \quad (1.142)$$

where the right hand side is the quotient by the partial slice gauge transformations, which act independently on the phase spaces. To be precise, on all sides here one still needs to quotient out the bulk dressings to talk about the actual phase spaces. This is great, as it means that, when the ambiguity of how the dressings  $\phi, \phi_L, \phi_R$  are related is fixed, then cutting is made transparently gauge invariant. We can summarise this by the slogan: *Gauge theories and their cutting are defined relative to arbitrary reference frames.*

As for the second cutting problem, we must insert the expressions above into the (pre)symplectic potential on the full slice, then expand. As the expressions for the gauge fields involve the dressings, we get nontrivial corrections coming from their imprint on the cut. i.e. *for a simple example of Maxwell theory* where the corner charge is given by  $*F$ , the result is

$$\Theta_\Sigma = \Theta_{\Sigma_L} + \Theta_{\Sigma_R} + \oint_S *F|_S \delta(\phi_L - \phi_R). \quad (1.143)$$

In this expression, we mean by  $\Theta_{\Sigma_L}$  the symplectic form obtained by replacing in  $\Theta_\Sigma$  every field by its corresponding  $L$  or  $R$  version (not the restrictions!), and point out that the original gauge frame,  $\phi$ , does *not* appear in this rewriting at all, not even its value at  $S$ . In particular, we can read from this the usual

$$\Theta_{\Sigma_L} = \Theta_{\Sigma_L} + \oint_S \pi_L \delta\phi_L \quad (1.144)$$

and see that the symplectic potential  $\Theta_\Sigma$  splits into the two *extended* symplectic potentials without issue. In the treatment of the cutting problem where one quotients by corner symmetries, this has a direct analogue (see eq. (83) of [113]), where it is clear that on-shell

---

<sup>32</sup>Choosing a value for the  $\phi$  is choosing a particular representative of the physical configuration in the pre-phase space.

the ‘pair’  $(*F|_S, \phi_L - \phi_R)$  are functionals of the (mismatch of the) gauge-invariant data of the field theory<sup>33</sup>.

The third cutting problem is, however, not so easily resolved in this setting. The reason is that while there is a reasonable way that the symplectic potential splits into separate pieces with a clear corner contribution, this does not apply to the gauge generators. In fact, when one takes the (differentiable) gauge generator of  $\Sigma$  and restricts it to a subregion, i.e.  $\Sigma_L$ , there is generically no corner contribution at  $S$  like in the symplectic potential, and this makes it different from the *true* subregion gauge generator, which consists of a bulk constraint and the boundary continuity equation. Again, this due to a failure of subsystem recursivity, particularly obvious in this case. Of course, we might expect that gauge-invariant quantities restrict more predictably to the subregions. However, it is not obvious that what is gauge invariant on the full slice is also gauge invariant in the subregions. Instead, this is only true for operators which are ‘obviously’ gauge invariant. The prime example for this is the dressed potential  $a$ , which restricts nicely in our setting. Then, all its functionals are gauge invariant, as well. For typical objects like Wilson lines, this can then be reexpressed in terms of  $A, \phi$ : I.e. we can define a properly gauge invariant, dressed Wilson line on a curve  $\gamma$  as

$$W_\gamma[A, \phi] := \mathcal{P} \exp(i \int_\gamma a) = e^{i\phi(\gamma(1))} \mathcal{P} \exp(i \int_\gamma A) e^{-i\phi(\gamma(0))} \quad (1.145)$$

which is obviously gauge invariant, but also is the usual kind of dressed Wilson line[78] once reexpressed. In the bulk,  $\phi$  is pure redundancy even off-shell - this needs to be taken into account to connect with usual notions of observables and gauge invariance. In particular, one should see this as a Wilson line defined *relative* to the arbitrary frame defined by  $\phi$ . *The original gauge invariance notion we are familiar with is then fulfilled by quantities that do not depend on the bulk values of  $\phi$ .*

In particular, bulk Wilson lines are still not observable in this sense, but boundary-anchored ones are, as is usual. Cutting of Wilson loops, a typical example of non-factorisability in gauge theories, is then handled by cutting the above gauge invariant Wilson line: Say we have a loop  $\gamma$  based at  $x$ , and cut the slice  $\Sigma$  along  $S$ , which cuts  $\gamma$  into  $\gamma_L \cup_{p,q} \gamma_R$  meeting in the points  $p, q$ . Without loss of generality, we assume  $\gamma_R$  is based at  $x$  still, and decomposes into two lines  $\gamma_{R,\pm}$  with endpoints  $p, x$  and  $x, q$  each.  $\gamma_L$ , conversely, has

---

<sup>33</sup>In the treatment of [113], the gauge-variant  $A$  is taken continuous across the cut, while  $a$  is allowed to be discontinuous.

endpoints  $q, p$ . Then, we rewrite

$$\begin{aligned}
W_\gamma[A, \phi] &= \mathcal{P} \exp(i \int_{\gamma_L} a|_L) \mathcal{P} \exp(i \int_{\gamma_R} a|_R) \\
&= \mathcal{P} \exp(i \int_{\gamma_L} A_L + d\phi_L) \mathcal{P} \exp(i \int_{\gamma_R} A_R + d\phi_R) \\
&= e^{i\phi_R(x)} \mathcal{P} \exp(i \int_{\gamma_{R,+}} A_R) e^{-i\phi_R(p)} e^{i\phi_L(p)} \mathcal{P} \exp(i \int_{\gamma_L} A_L) e^{-i\phi_L(q)} e^{i\phi_R(q)} \mathcal{P} \exp(i \int_{\gamma_{R,-}} A_R) e^{-i\phi_R(x)} \\
&= W_{\gamma_{R,+}}[A_R, \phi_R] W_{\gamma_L}[A_L, \phi_L] W_{\gamma_{R,-}}[A_R, \phi_R]
\end{aligned} \tag{1.146}$$

So we *can* reasonably split the loops using the corner dressings, as long as we remember that we are working with path ordered quantities. In particular, while the presence of  $\phi_{L|R}(x)$  in the above is improper, it reminds us that without taking traces, the loop is not yet gauge invariant. In fact, when tracing, it drops out.

### Explicit glueing in extended phase space

Now, dually, how do we glue? Quite simply, we require

$$a_L = a_R, \pi_L = \pi_R \tag{1.147}$$

as the conditions defining the glueing map. We once again need to fix dressings  $\phi_L, \phi_R, \phi$  arbitrarily to work fully off-shell, and we have *off-shell glueing maps*

$$\text{gl}_S : \mathcal{C}_{\Sigma, LR}^{\phi_L, \phi_R} \rightarrow \mathcal{C}_\Sigma^\phi \tag{1.148}$$

with domain satisfying the above glueing conditions. We give its expression as

$$\text{gl}_S^{\phi_L, \phi_R, \phi}(A_{L,R}, *F_{L,R}, \pi_{L,R}, \phi_{L,R}) = (A, *F, \phi) \in \mathcal{C}_\Sigma^\phi \tag{1.149}$$

with

$$\begin{aligned}
A|_L &= A_L + d\phi_L - d\phi|_L, A|_R = A_R + d\phi_R - d\phi|_R \\
*F|_L &= (\phi_L \phi^{-1}) \cdot *F_L, *F|_R = (\phi_R \phi^{-1}) \cdot *F_R.
\end{aligned} \tag{1.150}$$

The second glueing problem, in turn, is immediate: one realises that the corner pieces, with opposite orientation, sum up to

$$\oint_S \pi_L \delta \phi_L - \pi_R \delta \phi_R = \oint_S \pi_L \delta (\phi_L - \phi_R) \tag{1.151}$$

and ends up with the same kind of expression as in the cutting problem. Therefore, the second glueing and cutting problem end up inverses to each other.

In the third glueing problem, we must of course use quantities that *can* be reasonably glued together. Such conditions are not obvious in general. For the special case of Wilson lines, it is clear that they must end in the same endpoints, upon which they join together

like in our cutting example, but reversed. With gauge generators, on the other hand, we have no issue, unlike with the cutting problem: We again have the corner pieces

$$G_{\chi_L}^L + G_{\chi_R}^R = \cdots + \oint_S \pi_L \delta \chi_L - \pi_R \delta \chi_R = \cdots + \oint_S \pi_L (\chi_L - \chi_R). \quad (1.152)$$

So, if we restrict to gauge parameters which are continuous across the glueing surface  $S$ , the corner piece vanishes and only the bulk contributions are present. The reverse was not possible: While we could have taken the global generator  $G_\chi$  and evaluated it separately on parameters  $\chi_L, \chi_R$  with supports in the partial slices, even with different values on the cut, there would have been no corner term from the point of view of the full slice, as the cut was entirely fictitious.

## 1.4 Discretization and sampling

Because ‘discretization’ is a bit of a vague term, let us first give a more precise version of what we try to achieve. Colloquially, a ‘discrete’ theory is one in which one has a *locally finite set of observables*. Suppose we are working with an algebra  $\mathcal{A}$  generated by smeared fields. Then, a discretization of this would colloquially be a restriction of the observables to such ones that are smeared only over discrete elements of a lattice. This can be made precise as the statement that, if we view the algebras as locally assigned, then the assignment

$$R \mapsto \mathcal{A}_R \quad (1.153)$$

should yield *finitely generated algebras*<sup>34</sup> for all compact regions  $R$  (be it over spacetime or on slices).<sup>35</sup>

Therefore, the task of discretization is to produce from a given local algebra assignment another one that is locally finitely generated. This can, generally speaking, come in two flavours, the *subalgebra* and *quotient* ones.

In the **subalgebra** flavour, one prescribes a systematic way to choose subalgebras  $\mathcal{A}_R^\Lambda \subset \mathcal{A}_R$  which are finitely generated. One instance of this would be to restrict the set of smearing parameters used to define observables, so that the set of all supports for observables is locally finite itself. The main instance of this will be the method of *sampling*, which we will explore in much more detail in a moment.

In the **quotient** flavour, one instead chooses an ideal  $\mathcal{I}$  of quantities in the algebra that is to be discarded, and regards the quotient  $\mathcal{A}_R^\mathcal{I} = \mathcal{A}_R / \mathcal{I}$  as the algebra of ‘relevant’ observables.

This process comes with a plethora of in principle arbitrary choices which are hard to characterise in full. A typical choice is to adapt a (subalgebra) discretization to a lattice,

<sup>34</sup>Note here that this means that the algebras need no regularisation, unlike in a continuum field theory. If the algebras are von Neumann, it implies they are (sums of factors) of type I.

<sup>35</sup>One might want to seek a more lax version where the ‘radius of finiteness’ is either bounded by some physical length scale or formally zero, so that only for infinitesimal regions one has this property.

although mesh-free methods are perfectly possible. Within such a setting, one still has a lot of ambiguity: While one could, of course, choose the cells of the cellulation  $\Lambda$  to house observables, it is equally viable to use instead any number and combination of cells from the dual cellulation  $\tilde{\Lambda}$  as well. The situation is somewhat analogous to a choice of topology, in that one has a maximal and minimal set of cells one can declare as ‘available for the fields’, but the interesting choices lie somewhere in between.

A typical requirement on a choice of supports is that the supports of conjugate variables should intersect transversely, i.e. in a single point in real space. This is so that Poisson brackets in the continuum yield simple, well-defined numbers. Furthermore, a choice of supports should be closed under the Poisson brackets of the corresponding observables.

It may seem like the logic of discretization is somewhat like losing or dropping information of the system, because we no longer have certain quantities of interest available. This intuition can be made more precise by the notion of **sampling**:

Consider the fact that in principle, even in a field theory with infinitely many degrees of freedom, we have only access to a limited set of facts we can verify with our limited resolution. We saw one reflection of this already in the splitting/glueing problem - in the regularised algebra, one does not have access to field values on the splitting surface  $S$ , which leads to issues in factorisation. In a similar vein, the infinite number of degrees of freedom near a splitting surface contributes in the form of a divergence to the entropy of a subregion. So if there is no limit to our resolution, we can glue and split perfectly, but the entropy diverges. Conversely, it seems that if we have finite resolution, the entropy should be finite, but glueing should become harder.

However, this does not need to be the case, quite on the contrary. Let us now consider that our finite resolution means we do not evaluate the fields on points, but on small balls of finite size, and we cannot distinguish the individual values on points inside those regions. We may then replace the field values by their average over the ball and not lose information from the point of view of our finite resolution measurements.

If this is so, we can always imagine that our measurement of the full state of the field is actually given by a family of spatially extended detectors that cover the region of interest, and which give us one value per ball. This then corresponds to a cellulation of space(time), and we approximate the field by the values found by our detectors. In this way, we have taken the continuum field configuration and replaced it by a *discrete field configuration* which agrees with the continuum one on a given set of quantities, here the average value of the field over the balls.

We then say that we **sampled the field configuration over the test regions**, or conversely that we **sampled the quantities of interest on the field**.

It should be clear that cellulations provide a systematic (though idealised) way to speak of sampled field configurations. This is also true for quantities of interest which are not smeared over dimension  $D$  or  $D - 1$ : E.g. if the field is a connection  $A$ , then its gauge-invariant content is often in line integrals or parallel transports

$$h_\gamma = \mathcal{P} \exp \left( \int_\gamma A \right) \quad (1.154)$$



which are over 1-dimensional supports. Such quantities can also be sampled over a cellulation by fixing a set of paths  $\gamma$  which are in the 1-skeleton of the cellulation (or its dual). One should generally have the image in mind that we select a number of quantities of interest from the continuum algebra and create an adapted cellulation/triangulation for them.

Then, as long as we only consider the quantities of interest, we can *replace* the continuum field configurations by *equivalence classes* which are distinguished by the values of quantities we care about. In the connection example, this means that given the values of  $h_\gamma$  on a single closed path  $\gamma$ , we cannot determine the field inside the loop in detail; instead, for all intents and purposes, we can replace the field in the vicinity of the loop by a *line defect*.<sup>36</sup> Now consider that we choose our sampling locations to also include the boundary (and in particular the splitting surface  $S$ ). Then, we have smeared our fields, but do not care about the fine structure near the surface anymore, so we might as well consider the values of detectors sitting adjacent to the boundary to **be** the boundary values. This means that we end up identifying a close approximation to a corner algebra  $\mathcal{A}_S$  by taking seriously the idea that we have only finite resolution anyway.

In this sense, **sampling is a solution to the glueing problem**.

What sampling therefore achieves is that, given operationally motivated restrictions on the algebra, we produce an adapted triangulation and a discretization of the continuum algebra which can approximate the continuum algebra as well as we want it to. This may be used both for off-shell and on-shell algebras and in principle applies to all field theories, the details being in the interpretation of the sampling through detectors or other mechanisms.

Crucially, the restriction to a smaller set of observables, as we said, also means we cannot distinguish many field configurations from each other. On the classical phase space level, this means we can work with equivalence classes of field configurations, or more easily, pick a simple field configuration of our liking that represents the data we can actually measure. In this way, we can simplify the phase space as well.

But conversely again, we can actually get the same result by *restricting* our field configurations to those simpler ones (e.g. the piecewise constant configurations for a scalar field), and then the algebra will instead collapse into a simpler one where most elements give the same values on all configurations as there is no substructure to probe. This can be more precisely expressed through the **quotient flavour** discretization:

We can field configurations as evaluation maps on the classical algebra, so in principle we can work more generally with algebraic states. To restrict our considered field configurations then means to restrict to a subset of states, and discard the rest. So, given the algebra  $\mathcal{A}$ , we could construct its states  $\mathcal{S}(\mathcal{A})$ , but now select a subset  $\mathcal{S}_d \subset \mathcal{S}(\mathcal{A})$ . We then ask: *Is there some algebra  $\mathcal{A}_d$  such that  $\mathcal{S}_d = \mathcal{S}(\mathcal{A}_d)$ ?*

---

<sup>36</sup>In cylindrical coordinates with the loop in the  $\phi$ -direction, the curvature of the connection would be  $F_A \sim \delta(\rho)$  for such a defect.

Such an algebra would be the analogue of the set of functions on the phase space of restricted field configurations.

We can answer this question in the classical case if the restriction on configurations we do is due to some *constraints*<sup>37</sup>  $f = 0$ , which are also functions on the phase space. Then, the machinery of symplectic/Poisson reduction kicks in and we can find the functions on  $P_d \subset P$  algebraically[106, 117] as

$$C^\infty(P_d) = \frac{C^\infty(P)}{\mathcal{I}_f} \quad (1.155)$$

where we quotient by the ideal  $\mathcal{I}_f$  generated from the constraints.

From this we can see by extension to the nonclassical case that, if we declare our restricted configuration space by the vanishing of certain observables  $f$ , then the equivalent of the algebra of that configuration space is effectively also this quotient. If there is no level-set description of  $\mathcal{S}_d$  available, however, then we can find an analogy as follows:

If one *does* have a description of  $\mathcal{S}_d$  through constraints  $f$ , then that means that for those algebra elements, and all  $\omega \in \mathcal{S}_d$ , we have  $\omega(f) = 0$ . So, we can take as a replacement of the constraints *the joint kernel of  $\mathcal{S}_d$* , so the intersection of all the kernels of the  $\omega$ . Then, we can again define the appropriate ideal

$$\mathcal{I}_{\mathcal{S}_d} := \langle \{f \in \mathcal{A} | \forall \omega \in \mathcal{S}_d, \omega(f) = 0\} \rangle \implies \mathcal{A}_d = \frac{\mathcal{A}}{\mathcal{I}_{\mathcal{S}_d}}. \quad (1.156)$$

Therefore, there is a sort of equivalence between the two flavours of discretization: Restricting attention to a few observables restricts the effectively distinguishable field configurations, and restricting the field configurations effectively restricts the algebra. The subalgebra flavour is usually nicer for considerations of the algebra itself, whereas the quotient flavour is better for studying the phase space or Hilbert space. A precise mapping between the two is possible if there is a way by which the algebra factors into observables to ‘keep’ and ‘lose’: If the algebra  $\mathcal{A}_d$  is both a quotient and a subalgebra<sup>38</sup>, it must be a *direct summand*

$$\mathcal{A} \cong \mathcal{A}_d \oplus \mathcal{A}_{\text{rest}} \quad (1.157)$$

which may for example be the case if one generates the subalgebra by dropping a number of *generators*.

We have now a good overview of the elements that go into discretization: One needs to sample a preferred set of observables at a finite resolution, or select a simple kind of field configurations to evaluate the algebra elements on, and thus produce a simpler algebra. In most examples, we can then choose a cellulation or triangulation adapted to the sampling we chose, and then split the theory into smaller pieces associated to the cells/simplices, on which the theory is not just locally finite, but totally finite(ly generated).

We stress that the choice of observables or sampling configurations is really the main ingredient. All other questions and ambiguities arise as a result of a chosen sampling.

<sup>37</sup>These are required to satisfy certain regularity conditions[106].

<sup>38</sup>Meaning it has canonical maps both from and into  $\mathcal{A}$

## 1.5 Case study of discretization

Now we present an important example, first of all to discuss the effects of gauge transformations in a context where we can isolate their special properties from the ‘usual’ local degrees of freedom. However, it also presents the same structures as the states of quantum gravity we are going to analyse in chapter 2.

The interesting thing in the purely topological case is that sampling and discretization do not present approximations to the theory. Instead, the highly limited number of degrees of freedom imply that we can perform any subdivision of space(time) and still capture the full physics as long as the sampling states are properly selected. Much of the material regarding algorithmic discretization was originally developed *for* topological field theories, and much of our material regarding the glueing and cutting problems was also inspired by the issues found in these theories.

As this is a very well-studied field[91, 124–132], we will mostly focus on exemplifying the specific boundary effects that exist in gauge theories, as these are nicely isolated in these theories and we have a clear separation of bulk (absent) degrees of freedom and those home to boundaries. For this purpose, we will employ *BF theory*, as it has the advantage of being remarkably rich yet very simple. A lot of our toolkit applies quite directly to it and it has many applications and relations to (quantum) gravity.

### 1.5.1 BF theory: Generalities

BF theory is a field theory of a connection  $\omega \in \Omega^1(M_0) \otimes \mathfrak{g}$  for some Lie algebra  $\mathfrak{g}$ , together with a field  $B \in \Omega^{D-2}(M_0) \otimes \mathfrak{g}$  valued in the Lie algebra. The role of this field is to impose that the curvature

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega] \quad (1.158)$$

vanishes. We choose a preferred inner product  $\langle -, - \rangle$  on the Lie algebra (we will always be thinking of the Killing form, so the trace in the matrix groups/algebras) and write the Lagrangian in the standard form

$$L_0 = \langle B, F_\omega \rangle. \quad (1.159)$$

This gives rise to the symplectic potential<sup>39</sup>

$$\theta_1 = (-1)^{D-2} \langle B, \delta\omega \rangle = \langle \delta\omega, B \rangle. \quad (1.160)$$

and equations of motion

$$E_0 = \langle \delta B, F_\omega \rangle + \langle \delta\omega, d_\omega B \rangle. \quad (1.161)$$

These equations impose the flatness of  $\omega$ ,  $F_\omega = 0$  as well as the *Gauss constraint*  $d_\omega B = 0$ . They are, respectively, a 2-form and a  $D - 1$ -form, a fact which will be relevant in constructing constraints.

---

<sup>39</sup>This is due to the relation  $\delta F_\omega = d_\omega \delta\omega$ .

BF theory carries Lagrangian presymmetries under three important kinds of transformations, which we will introduce in detail in a moment. They are the *Yang-Mills*, *Kalb-Ramond transformations and diffeomorphisms*. The fact that diffeomorphisms are a presymmetry will actually be seen to be a consequence of the invariance under the other two.

This theory is exactly solvable: Consider first the solution set on a contractible domain  $M_0$ , with principal  $G = \exp \mathfrak{g}$ -bundle  $P$  over it. Any flat connection  $\omega$  can locally be written as

$$\omega = g^{-1}dg \quad (1.162)$$

for some group-valued  $g \in \Omega^0(M_0, \text{Conj}(P))$ . This  $g$  can be understood as a *local holonomy* (defined up to global left translations by a group element)<sup>40</sup>, which can be defined for general connections  $\omega$ , but only on one path  $\gamma$  at a time, according to the formula

$$g^{-1} \frac{dg}{dt} = \omega(v_\gamma(t)) \quad (1.163)$$

where  $v_\gamma$  denotes the tangent vector velocity field along  $\gamma$ .

With a flat connection, we can actually fully solve the theory. This is done by use of the identity

$$d_\omega B = 0 \implies B = d_\omega b. \quad (1.164)$$

This is essentially an analogue of  $d\alpha = 0 \implies \alpha = d\beta$  from de Rham cohomology. Here, we have *twisted* the de Rham cohomology[124, 125, 133] by a flat connection which allows us to preserve the property  $d_\omega^2 = 0$ . So, our solutions are labelled by

$$(g, b) \in \Omega^0(M_0, \text{Conj}(P)) \times \Omega^{D-3}(M_0, \text{Ad}(P)). \quad (1.165)$$

We can then work out the fact that this theory is topological from evaluating the symplectic potential on the solutions:

$$\theta_1 = \langle d_\omega \chi_L(g), d_\omega b \rangle = d(\langle \chi_L(g), d_\omega b \rangle) \quad (1.166)$$

where we introduced  $\chi_L(g) = g^{-1}\delta g$ , which actually is a field-space connection<sup>41</sup>. So, the symplectic potential is actually a boundary term only, and so in the bulk of *any* slice, there are no local degrees of freedom on-shell. This is why we call BF theory a ‘properly’ topological field theory: It only has bulk degrees of freedom associated to nontrivial topology, which is why here, we had only a boundary term. Therefore, in any similar properly topological field theory, we have that the symplectic potential splits as

$$\theta_1 = E_1 + d\theta_2 - \delta L_1. \quad (1.168)$$

---

<sup>40</sup>This is slightly semantically incorrect, as holonomies actually refer to closed paths. In this thesis, we use the same name also for open paths.

<sup>41</sup>In particular, we have used

$$\delta(g^{-1}dg) = -g^{-1}\delta g g^{-1}dg + d(g^{-1}\delta g) + g^{-1}dg g^{-1}\delta g \quad (1.167)$$

but with  $E_1$  vanishing when  $E_0$  does. Therefore we can always close the system with an appropriate choice of Lagrangian  $L_1$  without restricting the set of configurations. This makes properly topological field theories very special.

Now, BF theory owes its topological nature to the fact that it is a gauge theory with as many redundancies as there are local off-shell degrees of freedom. Let us see how this comes about.

Again, it is useful to do a (D-1)+1 decomposition of the fields according to some normals<sup>42</sup>. As we do not have a metric in BF theory, one needs to be a bit more careful here. Assume that we have fixed an embedded slice  $\Sigma$ . Then, we have the set of normal 1-forms

$$N1F_\Sigma = \{\mathbf{n} \in \Omega^1(\Sigma \times (-\epsilon, \epsilon)) : i_\Sigma^* \mathbf{n} = 0\} \cong C_{\neq 0}^\infty(\Sigma \times (-\epsilon, \epsilon)) \quad (1.169)$$

on tubular neighbourhoods of the slice, unnormalized, but all a rescaling of some representative.

There are also the associated sets of vector field normals

$$NVF_\Sigma(\mathbf{n}) = \{\hat{n} \in \mathfrak{X}(\Sigma \times (-\epsilon, \epsilon)) : i_{\hat{n}} \mathbf{n} = \sigma\} \cong \Gamma(T\Sigma) \quad NVF_\Sigma = \prod_{\mathbf{n} \in N1F_\Sigma} NVF_\Sigma(\mathbf{n}) \quad (1.170)$$

which give the dual vector fields. As before, we choose a signature  $\sigma = \pm 1$ , i.e.  $-1$  for a timelike set of normals. Note that given a  $\mathbf{n}$ , there is a large set of dual vector fields - unlike when we have a metric, we do not get a unique one. This we do need to keep in mind.

Now given a choice  $(\mathbf{n}, \hat{n}) \in NVF_\Sigma$ , we can decompose the fields

$$\omega = \tilde{\omega} + \sigma \mathbf{n} \omega_n \quad B = \tilde{B} + \sigma \mathbf{n} \wedge B_n \quad (1.171)$$

where quantities with a tilde are always annihilated by contraction with  $\hat{n}$  and represent tangential pieces, while quantities with an index  $n$  refer to the normal components. Quite generically, then, the BF potential reduces to

$$\Theta_\Sigma = \int_\Sigma \langle \delta \tilde{\omega}, \tilde{B} \rangle. \quad (1.172)$$

which implies for the curvature<sup>43</sup>

$$F_\omega = F_{\tilde{\omega}} + \sigma \mathbf{n} \wedge (\mathcal{L}_{\hat{n}} \omega - d_\omega \omega_n) \quad (1.173)$$

We can further decompose

$$d_\omega \omega_n = \tilde{d}_\omega \omega_n + \sigma \mathbf{n}(\dots) \quad \mathcal{L}_{\hat{n}} \omega = \mathcal{L}_{\hat{n}} \tilde{\omega} + \sigma \omega_n \mathcal{L}_{\hat{n}} \mathbf{n} + \sigma \mathbf{n} \mathcal{L}_{\hat{n}} \omega_n \quad (1.174)$$

---

<sup>42</sup>We work here in analogy to [77, 134], but do not use any metric.

<sup>43</sup>Using the identity  $\mathcal{L}_\xi \omega = i_\xi F_\omega + d_\omega i_\xi \omega$

Usually, it is convenient to assume *hypersurface orthogonality*,  $\tilde{d}\mathbf{n} = 0$ , which we will do here. To lighten notation, let us assume that the acceleration  $\mathcal{L}_{\hat{n}}\mathbf{n}$  vanishes.

So  $B_n, \omega_n$  are more like Lagrange multipliers. The Lagrangian itself also splits into

$$L_0 = \sigma \mathbf{n} \wedge (\langle B_n, F_{\tilde{\omega}} \rangle + \langle \mathcal{L}_{\hat{n}}\tilde{\omega} - \tilde{d}_{\omega}\omega_n, \tilde{B} \rangle) \quad (1.175)$$

which shows that, in fact, there are no time derivatives of  $\omega_n, B_n$  involved at all indeed. By spatial partial integration, we turn the last term into

$$\sigma \mathbf{n} \wedge \langle \tilde{d}_{\omega}\tilde{B}, \omega_n \rangle - \sigma \mathbf{n} \wedge \tilde{d}(\langle \tilde{B}, \omega_n \rangle) \quad (1.176)$$

which clearly shows that the two Lagrange multipliers  $B_n, \omega_n$  impose tangential flatness and tangential Gauss constraint, respectively. This leads to a gauge redundancy. This is particularly easy to show because the theory is exactly solvable.

Note then that the Hamiltonian of BF theory is given by

$$H_{\Sigma} = - \int_{\Sigma} \langle B_n, F_{\tilde{\omega}} \rangle + \langle \tilde{d}_{\omega}\tilde{B}, \omega_n \rangle + \oint_{\partial\Sigma} \langle \tilde{B}, \omega_n \rangle \quad (1.177)$$

which has only constraints in the bulk - and therefore is trivial on on-shell configurations in the bulk. This is a feature of a class of theories commonly referred to as *generally covariant*, which includes generic properly topological field theories.

The fact that we have Lagrange multipliers shows us that we have *constraints*. However, here we cannot solve for the Lagrange multipliers, which is why we end up with a gauge redundancy. For seeing this, we construct the differentiable generators associated to these constraints. We start with the Gauss constraint. By smearing it, we find a non-differentiable expression, as

$$\delta(- \int_{\Sigma} \langle d_{\omega}B, \alpha \rangle) = - \int_{\Sigma} \langle d_{\omega}\delta B, \alpha \rangle + \langle [\delta\omega, B], \alpha \rangle. \quad (1.178)$$

Being (functionally) differentiable would mean that this has the form of a contraction with the symplectic form. This is not the case because of the  $d_{\omega}\delta B$ , in which the variation is wrapped in some object. But this can be remedied easily by adding a boundary term, giving the *Gauss generator*

$$J_{\alpha} := \int_{\Sigma} \langle d_{\omega}\alpha, B \rangle \quad (1.179)$$

which is differentiable and generates the transformation

$$X_{\alpha}[B] = [B, \alpha] \quad X_{\alpha}[\omega] = d_{\omega}\alpha. \quad (1.180)$$

We recognise that this generates internal gauge transformations of the bundle  $P$ , under which  $\omega$  transforms as a connection. We will call these the *Yang-Mills (YM) transformations*. On-shell, the bulk term vanishes - ergo, the bulk-supported gauge transformations are redundancies.

The other constraint, flatness, can be turned into the differentiable *Kalb-Ramond (KR) generator* parametrised by a Lie algebra valued  $D - 3$ -form  $\mu$

$$K_\mu := - \int_\Sigma \langle F_\omega, \mu \rangle + \oint_\Sigma \langle \omega, \mu \rangle \quad (1.181)$$

which generates the *KR shifts*

$$Y_\mu[B] = d_\omega \mu \quad (1.182)$$

under which  $B$  transforms as a connection. Again, all bulk-supported KR shifts are a redundancy. The full algebra of these is then

$$\Omega^0(\Sigma) \otimes \mathfrak{g} \ltimes \Omega^{D-3}(\Sigma) \otimes \mathfrak{g} \quad (1.183)$$

with the bracket

$$[(\alpha_1, \mu_1), (\alpha_2, \mu_2)] = ([\alpha_1, \alpha_2]_{\mathfrak{g}}, [\alpha_1, \mu_2]_{\mathfrak{g}} - [\alpha_2, \mu_1]_{\mathfrak{g}}). \quad (1.184)$$

Interestingly, then, we can see that the Hamiltonian can be expressed as

$$H_\Sigma = J_{\omega_n} + K_{B_n} - \oint_{\partial\Sigma} \langle \tilde{\omega}, B_n \rangle \quad (1.185)$$

and so, under the condition that  $B_n$  vanishes on the boundary of the slice, the Hamiltonian is a sum of the internal gauge transformations. This shows that with the right boundary conditions on the fields, the Hamiltonian is actually on-shell zero and given by a combination of gauge transformations. This holds more generally,

$$D_\xi = J_{i_\xi \omega} + K_{i_\xi B} - \oint_{\partial\Sigma} \langle \tilde{\omega}, i_\xi B \rangle \quad (1.186)$$

and means that *in properly topological field theories, internal gauge transformations and diffeomorphisms are expressions of the same thing*. This is crucial to understand their behaviour: It is through this relation that we can understand why they have no *propagating* degrees of freedom, but also the link to their topological invariance. That the diffeomorphism generators are all vanishing in the bulk is a sign that local deformations of the spacetime do not affect anything - there is nothing to move around, anyway. We need to sharply contrast this with the case of gravity later on, where most of these features are present, but with the stark exception of having propagating local degrees of freedom. Therefore, we cannot use the vanishing of the Hamiltonian as an indicator of the absence of degrees of freedom - it really is the dimension of the gauge groups that matters for this. Only when the internal gauge transformations reduce the degrees of freedom to zero can one speak of properly topological field theories.

At this stage we can tell that for a closed slice, all the on-shell  $B$  fields are of the form of a finite KR shift, and similarly for  $\omega$  through YM transformations. This means that any on-shell configuration is equivalent to the trivial configuration  $B = 0 = \omega$ . The physical phase space is therefore a point in the bulk. This is not so for the case with boundary,

as there we cannot gauge away boundary-supported configurations. Then, gauge fixing questions come in. We will come back to this in a moment in the dressed picture. Before moving on, though, we care about the case of nontrivial topology as well: In particular, we care about the twisted de Rham cohomology groups

$$H_{dR, \mathfrak{g}}^1(M_0) \times H_{dR, \mathfrak{g}}^{D-2}(M_0) \quad (1.187)$$

which represent the topological pieces of  $(\omega, B)$  which cannot be gauged away. These then form the physical phase space of BF theory on spaces without boundaries but with nontrivial topology. The decomposition that is relevant here is

$$\omega = g^{-1}\omega_0 g + g^{-1}dg \quad B = g^{-1}(d_{\omega_0}b + B_0)g \quad (1.188)$$

in which  $(\omega_0, B_0)$  contain the nontrivial pieces. Since the cohomology groups are finite dimensional, these pieces are always finite linear combinations of some generators of these groups. The bulk of BF theory is therefore a simple finite-dimensional mechanical system. In particular, all bulk degrees of freedom BF theory *has* are due to *topological defects*.

These defects are essentially just strata of the space(time) where either something topologically nontrivial happens, or where some fields live that give distributional sources to the equations of motion. I.e. consider the Gauss equation

$$d_{\omega}B = \mathcal{S} = S \mathbb{I}_{M_{D-1}} d^{D-1}x \quad (1.189)$$

so we suppose that  $\mathcal{S}$  is a  $D - 1$ -form distribution supported on a worldline  $\gamma = M_{D-1}$ . The Lagrangian with source,  $L_0 = \langle B, F_{\omega} \rangle - \langle \omega, \mathcal{S} \rangle$ , then gives rise to the action

$$S = \int_{M_0} \langle B, F_{\omega} \rangle - \int_{M_{D-1}} \langle \omega, S \rangle \quad (1.190)$$

so effectively we have a modification by an

$$L_{D-1} = -\langle \omega, S \rangle \quad S \in \Omega^0(M_{D-1}) \otimes \mathfrak{g} \quad (1.191)$$

This does not contribute to the symplectic potential as-is, but breaks gauge invariance. However in particular, it adds a nontrivial solution for the  $B$  field to the phase space, due to the fact that the (on spacelike slices pointlike) source gives rise to the same type of solutions as a space with nontrivial topology<sup>44</sup>. The only slight difference is that the configuration is uniquely *fixed* by the source's particular value, while a nontrivial topology gives a range of configurations in principle. This is easy to remedy as well by dressing the Lagrangian, as we will see shortly.

We just want to emphasize at this point that really, nontrivial topology should be understood as defects which *excite* the fields in BF. These can then be understood as static background data or given their own dynamics as well. BF theory is in a sense a prototypical example of this idea which applies to many gauge theories.

---

<sup>44</sup>This is due to the fact that a ball with a point removed is homotopy equivalent to a sphere, and BF theory's cohomology groups are homotopy invariant.



### 1.5.2 BF theory: Dressed phase space

We now have a good context to use the dressing field procedure. We will use the dressing fields

$$(\phi, q) \in \Omega^0(M_0) \otimes \mathfrak{g} \times \Omega^{D-3}(M_0) \otimes \mathfrak{g} \quad (1.192)$$

transforming as

$$X_\alpha[(\phi, q)] = (-\alpha\phi, [q, \alpha]) \quad Y_\mu[(\phi, q)] = (0, -\mu) \quad (1.193)$$

so  $\phi$  is a Yang-Mills dressing and  $q$  a Kalb-Ramond dressing, which also transforms as an adjoint representation object under Yang-Mills transformations. We apply the two in extending order (i.e. first the abelian part of the semidirect product in the gauge group) to the Lagrangian to get

$$L_0 = \langle (B + d_\omega q), F_\omega \rangle \quad (1.194)$$

which does not change the equation of motions - and get the bulk dressed potential

$$\begin{aligned} \theta_1 &= \langle \delta(\phi^{-1}\omega\phi + \phi^{-1}d\phi), \phi^{-1}(B + d_\omega q)\phi \rangle \\ &= \langle \phi^{-1}(\delta\omega + d_\omega\chi_R(\phi))\phi, \phi^{-1}(B + d_\omega q)\phi \rangle \\ &= \langle \delta\omega + d_\omega\chi_R(\phi), B + d_\omega q \rangle \end{aligned} \quad (1.195)$$

with  $\chi_R(\phi) = \delta\phi\phi^{-1}$ . This has four terms, the first being the undressed potential, then two relating to constraints, and a final fourth one which encodes mixing of the two types of gauge transformations. Expanding this, the dressed symplectic potential has the form

$$\theta_1 = \langle \delta\omega, B \rangle + J_{\chi_R(\phi)} - \delta K_q + K_{\delta q + [q, \chi_R(\phi)]} - d(\langle d\chi_R(\phi), q \rangle) \quad (1.196)$$

and we can see a structure clearly here: Apart from the exact piece  $\delta K$ , which we do not care so much about, the additional terms are either on the corner only or are related to the gauge structure. In particular, the additional terms take the form of gauge generators with variational 1-forms like  $\chi_R(\phi)$  as parameters. It is therefore clear that in the bulk, the additional data is trivial unless there are sources to the gauge constraints. Also important is the specific combination  $\delta q + [q, \chi_R(\phi)]$  for the KR shifts: This is the Yang-Mills-transformation invariant combination, so pertains only to variations of the dressing  $q$  which come from KR shifts.

The additional corner piece may appear surprising at first, but has an easy interpretation from the fact that the KR generator boundary term contains  $\omega$  by itself (without derivatives), and that  $\omega$  transforms non-tensorially under Yang-Mills transformations. This leads to the stray exterior derivative which is needed to account for the transformation behaviour of  $\omega$ .

It is now easy to evaluate the potential on-shell and see what we have as boundary data on slices which are not closed. Because of the dressing procedure, we can regard  $B, \omega$  *entirely* as gauge and set them to zero even when boundaries are present. The gauge-fixed potential is then

$$\begin{aligned} \theta_1^{gf} &\triangleq d\vartheta_2, \vartheta_2 = \langle \chi_R(\phi), B \rangle + \langle \omega, \delta q + [q, \chi_R(\phi)] \rangle - \langle d\chi_R(\phi), q \rangle - \delta \langle \omega, q \rangle \\ &= -\langle d\chi_R(\phi), q \rangle \end{aligned} \quad (1.197)$$

so carries only the dressing fields once the gauge fixing is imposed. We then also see immediately that the derivative of  $\phi$  is conjugate to the KR dressing  $q$ . So, we can now see that BF theory has two types of degrees of freedom:

1. In the bulk, there are only topological defect excitations, which may also be seen as being associated to the defects rather than BF theory, as we will see soon,
2. On the boundary, there is a canonical pair given by the dressing fields as gauge edge modes.

Therefore, without defects, the physical phase space is roughly speaking given by a Poisson structure on  $\mathfrak{g} \ltimes \mathfrak{g}$ . This is the conceptual starting point for many quantizations of BF theory[135–138].

Let us also consider the defects in more detail<sup>45</sup>. There are two equations of motion, and therefore two sources we can add. The general logic for defect actions is that we want to add sources for a  $p$ -form equation of motion through a codimension  $p$  Lagrangian, and that the sources are distributional  $p$ -forms supported on manifolds *transverse to the directions of the form*[127–129]. I.e. a volume form on the support, wedged with the source, will give a top-degree form in  $M_0$ . For the source  $\mathcal{S}$ , this meant that it was supported on 1D submanifolds, and the form components needed to be transverse to the worldline. I.e. for a timelike worldline, this means the source is a spacelike  $D - 1$  form. For the source of curvature,  $F_\omega = \mathcal{F}$ , which is a 2-form, this means

$$\mathcal{F} = F \mathbb{I}_{M_2} d^2x \quad F \in \Omega^0(M_2) \otimes \mathfrak{g} \quad (1.198)$$

So that the curvature defects are distributions supported on codimension 2 surfaces in spacetime, and the form directions are normal to that surface, infinitesimally speaking. We can then give *dressed* defect Lagrangians

$$\begin{aligned} L_2 &= -\langle \phi^{-1}(B + d_\omega q)\phi, F \rangle = -\langle B, \phi F \phi^{-1} \rangle - \langle d_\omega q, \phi F \phi^{-1} \rangle \\ L_{D-1} &= -\langle \phi^{-1}\omega\phi + \phi^{-1}d\phi, S \rangle = -\langle \omega, \phi S \phi^{-1} \rangle - \langle d\phi\phi^{-1}, \phi S \phi^{-1} \rangle \end{aligned} \quad (1.199)$$

which are now gauge invariant and contain derivatives of the dressing fields along the supports of the defects. This is a crucial point: If we do not have sources for constraints, then dressing fields are unphysical redundancy in the bulk. If instead we have sources, then they become conjugate to them. This means that generally speaking, dressing fields encode the *gauge group orientation of the sources*. We also can then derive the actual equations of motion: The symplectic potential  $\theta_1$  leaves no imprint on the defects as it only imprints on codimension 1 surfaces. The only place where this matters is on spacetime boundaries. Let us ignore this case for the moment; then, the equations of motion are

$$E_0 = \langle \delta B, F_\omega - \phi F \phi^{-1} \mathbb{I}_{M_2} d^2x \rangle + \langle \delta \omega, d_\omega B - \phi S \phi^{-1} \mathbb{I}_{M_{D-1}} d^{D-1}x \rangle \quad (1.200)$$

---

<sup>45</sup>We will only be discussing *vertical* defects here, so those which are supported on timelike submanifolds.

which is just the same bulk equations, but with distributional sources located on  $M_2, M_{D-1}$ ,

$$E_2 = \langle \delta\phi\phi^{-1}, [(B + d_\omega q), \phi F\phi^{-1}] \rangle + (-1)^{D-3} \langle \delta q, d_\omega(\phi F\phi^{-1}) \rangle \quad (1.201)$$

which encode the Bianchi identity  $d_\omega F_\omega = 0$  for the dressed curvature source  $\phi F\phi^{-1}$ , and the derivative of the Gauss law,  $d_\omega^2 B = [F_\omega, B] = 0$ , for the B-field on the defect (notably, not the Gauss law itself!), and

$$E_{D-1} = \langle \delta\phi\phi^{-1}, d_\omega(\phi S\phi^{-1}) \rangle \quad (1.202)$$

which encodes the Gauss law  $d_\omega B = 0$  on the worldline for the dressed Gauss source  $\phi S\phi^{-1}$ . Note that this one in particular can be understood as the source being covariantly conserved along the worldline - i.e. the particle moving along  $M = D - 1$  carries constant uniform Yang-Mills charge. For the other defects, this is the same for the dressed curvature. Thus, naturally, if we write our action principle for the background sources  $S, F$  included in BF theory, they must be constant Lie algebra elements associated to each defect<sup>46</sup>.

The defects also are equipped with their own symplectic potentials

$$\theta_2 = -\langle \delta q, \phi F\phi^{-1} \rangle \quad (1.203)$$

$$\theta_D = -\langle \delta\phi\phi^{-1}, \phi S\phi^{-1} \rangle = -\langle \phi^{-1}\delta\phi, S \rangle \quad (1.204)$$

which show very clearly that along the defects, the dressing fields are conjugated to the sources.

If any of the defects intersect the boundary  $M_1$ , then we also need to take care of the imprint of  $\theta_1$  on them. The boundary piece of the variation is then

$$\begin{aligned} & \int_{M_1} \theta_1 + \mathbb{I}_{M_2} \delta L_2 + \mathbb{I}_{M_{D-1}} \delta L_{D-1} \\ &= \int_{M_1} \langle \delta\omega, B \rangle - \int_{M_2} \langle \delta B, \phi F\phi^{-1} \rangle - \int_{M_{D-1}} \langle \delta\omega, \phi S\phi^{-1} \rangle \end{aligned} \quad (1.205)$$

which adds new pieces to the boundary variation that affect how we close the system. For the moment, see that if we want this to vanish for all  $\delta\omega$ , then on  $M_{D-1}$ , the component of  $B$  transverse to  $M_{D-1}$  is given by  $\phi S\phi^{-1}$ , and on  $M_2$ ,  $\phi F\phi^{-1} = 0$ . Of course, this will change if we introduce other boundary conditions. The main point is that defects intersecting the boundary will contribute to the question of boundary conditions, in principle.

We can therefore go and include arbitrary defects in BF theory as long as we consider the usual problem: At the Lagrangian level, the sources  $S, F$  must be nondynamical parameters, but on the phase space level we can easily take them to have nonzero, unconstrained variations without problem. However, we can also in principle resolve the issue by giving some constituent dynamics to them. Furthermore, one could even go further and specify some kind of creation/annihilation rules for the defects, which then situates BF theory as

---

<sup>46</sup>In principle, by doing some redefinitions, we may always assume these Lie algebra elements to lie in the commutative Cartan subalgebra of  $\mathfrak{g}$ .

a ‘vacuum’ of the theory thus built. The extended symplectic potential of BF theory with defects then looks like

$$\Theta_\Sigma = \int_{\Sigma_1} \langle \delta\omega, B \rangle - \sum_\alpha \int_{\Sigma_{3,\alpha}} \langle \delta q, \phi F_\alpha \phi^{-1} \rangle - \sum_\beta \int_{\Sigma_{D,\beta}} \langle \phi^{-1} \delta\phi, S_\beta \rangle \quad (1.206)$$

which contains a piece for each defect intersecting the slice  $\Sigma_1$  in some  $\Sigma_{3,\alpha}$  or a point  $\Sigma_{D,\beta}$ .

### 1.5.3 BF theory: Boundary dynamics

We are now going to look at the dynamics that the bulk  $M_0$  induces on the boundary  $M_1$ . We are going to solve the bulk constraints as to work only with proper physical degrees of freedom, and see that induces a dynamics on the boundary. As already stated, the bulk Hamiltonian is a sum of constraints and a boundary term, so on-shell on the dressed phase space it becomes the simple expression

$$H_\Sigma = \oint_{\partial\Sigma} \langle \tilde{d}_\omega \tilde{b} + \tilde{d}_\omega \tilde{q}, \omega_n \rangle. \quad (1.207)$$

This shows that  $b, q$  show up in the same way - which is not surprising as  $q$  is supposed to behave exactly like the KR components of  $B$ . Therefore, when we fix gauges throughout the bulk, i.e.  $\tilde{\omega} = 0 = \tilde{b}$ , the whole boundary dynamics is essentially that of  $q, g$ . The dressed Lagrangian itself also has the on-shell, gauge-fixed form<sup>47</sup>

$$\begin{aligned} L_0 &= \sigma \mathbf{n} \wedge (\langle \mathcal{L}_{\hat{n}}(\phi^{-1} \tilde{d}\phi), \tilde{d}\tilde{q} \rangle - \tilde{d}(\langle \tilde{d}\tilde{q}, \omega_n \rangle)) \\ &= -\sigma \mathbf{n} \wedge \tilde{d}(\langle \tilde{d}\tilde{q}, \omega_n + \phi^{-1} \mathcal{L}_{\hat{n}}\phi \rangle) \end{aligned} \quad (1.209)$$

which once again is a total boundary term living on  $M_1$ . It is easy to see that if we vary  $q$ , then there is a constancy condition on the relation between  $\omega_n$  and the time derivative of  $\phi$ :

$$\phi^{-1} \mathcal{L}_{\hat{n}}\phi = -\omega_n + \text{const} \quad (1.210)$$

In turn, the variation of  $\phi$  gives  $q$  its dynamics, i.e.

$$\mathcal{L}_{\hat{n}}(\phi \tilde{d}\tilde{q} \phi^{-1}) = 0 \quad (1.211)$$

Now suppose we were to treat the system as open, i.e. do not impose a condition on  $\omega_n$ . Then, its variation makes  $q$  constant, and the equations become indeterminate.  $\omega_n$  is then a Lagrange multiplier and the Lagrangian vanishes when it has been determined. The dynamics is therefore insufficiently determined. One *must* impose conditions on  $\omega_n$ .

Now instead suppose that we have

$$\omega_n = f[\phi, \tilde{q}]. \quad (1.212)$$

---

<sup>47</sup>We used

$$\mathcal{L}_{\hat{n}}(\phi^{-1} \tilde{d}\phi) = \tilde{d}_{\phi^{-1} d\phi}(\phi^{-1} \mathcal{L}_{\hat{n}}\phi) \quad (1.208)$$

to go to the second line.

Then this allows the equations to become nontrivial and determine  $\phi, \tilde{q}$ . So, we learn that even when the constraints reduce all data to the boundary, one does not get nontrivial equations of motion there *unless* one also imposes a boundary condition. In fact, one may think of the boundary conditions *as* choices of dynamics, since the bulk dynamics is essentially trivial.

We can therefore now ask what boundary dynamics have interesting effects. There are of course the usual naive boundary conditions,  $\delta\omega = 0$ , which we refer to as magnetic, and  $\delta B = 0$ , which we call magnetic. From the form of the Hamiltonian, we can immediately single out one particular class: If we set  $\delta\tilde{B} = 0$  or  $\delta\omega_n = 0$ , then the Hamiltonian vanishes completely on-shell, even the boundary piece. This makes the dynamics completely trivial, and the theory topological throughout. This is precisely the kind of condition one gets if one keeps the system open, so it confirms that BF theory is topological throughout the bulk at every fictitious interface.<sup>48</sup> Such boundary conditions may be said to be *topological boundary conditions*.

However, any other boundary condition will lead to a nontrivial Hamiltonian.

We need to stress this point because it can lead to confusion: *The bulk induces, on open or fictitious boundaries, the topological boundary conditions. If one expects or wants nontrivial boundary degrees of freedom, the boundary cannot be fictitious. It must be a physical interface with nontrivial boundary conditions.*

This really can be interpreted as follows: If one introduces a codimension 1 defect into the theory, acting as a physical wall, then this introduces a boundary condition because the degrees of freedom of the wall couple to the bulk. This coupling is the boundary condition. Further, the dynamics of the wall and its own degrees of freedom then determine the boundary condition's precise form. *Boundary conditions therefore just model different couplings to walls without specifying the constituent system of the wall.* The only thing that is required is some gauge-covariant degrees of freedom so that the bulk fields can couple to the wall.

Note also that these boundary conditions being topological or not has *nothing* to do with breaking gauge invariances: One could equally well impose the conditions in a gauge-invariant way and the conclusion would be the same.

*For an interesting example,* consider a dressed magnetic boundary condition, given by the boundary Lagrangian

$$L_1^M = -\langle \phi \cdot \omega - \omega_0, J \rangle \quad (1.213)$$

which gives rise to the symplectic potential

$$\theta_2 = -\langle \phi^{-1} \delta \phi, J \rangle \quad (1.214)$$

---

<sup>48</sup>I.e. if one subdivides the system fictitiously and looks at the induced dynamics on the boundary, it forces  $\tilde{B} = 0$ , which makes the boundary dynamics topological.

and has equations of motion on the boundary

$$E_1 = -\langle \phi \cdot \omega - \omega_0, \delta J \rangle + \langle \delta \omega, B - \phi J \phi^{-1} \rangle - \langle d_{\omega_0} J, \delta \phi \phi^{-1} \rangle. \quad (1.215)$$

This is quite simple: It fixes the value of the connection  $\omega$  in a gauge-invariant way, gives a continuity equation for  $B$  and requires that the Gauss constraint holds on the boundary. By our setup, the system is closed on-shell of these conditions, but we also artificially restricted the system into a single configuration of the local holonomy  $g$ . This models a wall which already carries a flat connection and we essentially just continue it into the bulk in a flat way. In terms of induced dynamics, this we work with the on-shell effective Lagrangian (which we get by eliminating  $\omega_n$ )

$$L_1^{eff} = \langle \tilde{d}\tilde{q}, \phi^{-1} \mathcal{L}_{\hat{n}} \phi + \omega_{0,n} \rangle + \langle \phi^{-1} \tilde{d}\phi - \tilde{\omega}_0, J_n \rangle \quad (1.216)$$

So we see that the profile of  $\phi$  is fixed up to constant terms. Meanwhile,  $q$  has the equation of motion

$$\mathcal{L}_{\hat{n}}(\phi \tilde{d}q \phi^{-1}) + \tilde{d}(\phi J_n \phi^{-1}) = 0 \quad (1.217)$$

which relates  $J_n$  and  $q$ . The Lagrangian also induces the symplectic potential

$$\theta_2 = (\tilde{d}q + J_n) \phi^{-1} \delta \phi \quad (1.218)$$

As we said, the profile of  $\phi$  is fixed, so regardless of the exact solution for  $q$ , the phase space is very small. So, while these boundary conditions *do* have nontrivial Hamiltonian as evidenced by the Lagrangian being nontrivial, the phase space is too small to make this meaningful. So, neither electric nor magnetic boundary conditions make for a particularly compelling phase space. However, they have the advantage of uniquely specifying a configuration in the bulk, which comes in handy in discretization.

There is a way to select a better boundary dynamics in BF theory: We do not have a notion of radiation or a need to drop it, as there is no on-shell flux. Instead, we can ask that the bulk symmetries given by YM and KR transformations are actual symmetries of the boundary dynamics. This is a *choice*, of course, but an interesting one. For this, consider the corner charges that are at play here:

$$J_\alpha \approx \oint_S \langle B, \alpha \rangle \quad K_\mu \approx \oint_S \langle \omega, \mu \rangle. \quad (1.219)$$

We will ask for conditions that leave the values of these charges unrestricted (so, no restrictions on the purely tangential components  $\tilde{B}, \tilde{\omega}$ ) and that leave their values conserved. This again falls in the class of conditions fixing  $B_n, \omega_n$ . Thus we check the change under a diffeomorphism:

$$\hat{\xi}[J_\alpha] = \oint_S \langle \mathcal{L}_\xi B, \alpha \rangle + \langle B, \hat{\xi}[\alpha] \rangle \quad (1.220)$$

In this, we also include a potential change of  $\alpha$ , assuming it could be field-dependent. More generally, we will suppose that a given dynamics will lead to a given class of time-dependent transformation parameters  $\alpha, \mu$  for which the charge is conserved. These classes

of parameters will be subject to some hyperbolic differential equations. We can use the equations of motion of  $B, \omega$ , to write the Lie derivative

$$\mathcal{L}_\xi B \hat{=} d_\omega i_\xi B + [B, i_\xi \omega], \quad \mathcal{L}_\xi \omega \hat{=} d_\omega i_\xi \omega \quad (1.221)$$

which once again encodes that on-shell, the diffeomorphisms are entirely gauge transformations. This means for the charges that we have

$$\hat{\xi}[J_\alpha] \hat{=} \oint_S \langle d_\omega \alpha, i_\xi B \rangle + \langle B, \mathcal{L}_\xi \alpha + [i_\xi \omega, \alpha] \rangle \quad (1.222)$$

$$\hat{\xi}[K_\mu] \hat{=} \oint_S -\langle i_\xi \omega, d_\omega \mu \rangle + \langle \omega, \mathcal{L}_\xi \mu \rangle \quad (1.223)$$

We can now specialise to  $\xi = \hat{n}$ , a timelike vector field. The key is now to choose  $B_n, \omega_n$  such that the conservation condition becomes a differential equation for  $\alpha, \mu$ . The easiest case for this is  $B_n = 0 = \omega_n$ , in which case all fields are constant in time, so obviously also the charges. Then, the charges are conserved for parameters satisfying  $\mathcal{L}_{\hat{n}} \alpha = 0 = \mathcal{L}_{\hat{n}} \mu$ , so the time-independent ones, naturally. However, more generally, any boundary condition implicitly defined by

$$\begin{aligned} \langle d_\omega \alpha, B_n \rangle &= \langle \tilde{B}, L_B[\alpha] \rangle \quad \forall \alpha \\ -\langle \omega_n, d_\omega \mu \rangle &= \langle \tilde{\omega}, L_\omega[\mu] \rangle \quad \forall \mu \end{aligned} \quad (1.224)$$

where  $L_B, L_\omega$  are spatial pseudodifferential operators on the corner  $S$ , also works and leads to conservation for charges with parameters

$$\begin{aligned} \mathcal{L}_{\hat{n}} \alpha + [\omega_n, \alpha] + L_B[\alpha] &= 0 \\ \mathcal{L}_{\hat{n}} \mu + L_\omega[\mu] &= 0 \end{aligned} \quad (1.225)$$

so we know that even with this requirement, there is a tremendous amount of freedom in choosing boundary dynamics, as *any*  $L_B, L_\omega$  will give rise to a consistent conservation law. In appendix A.5, we determine the largest off-shell class of boundary conditions. Here, let us restrict to  $D = 3$ , where our freedom is determined by three integral kernels  $a(x, y), b(x, y), c(x, y)$ , i.e. we use boundary conditions given in equation A.37. Then the condition for  $\omega_n$  gives

$$\begin{aligned} & - \oint_S d^{D-2} x \langle \omega_n(x), D_\theta^\omega \mu(x) \rangle \\ &= - \oint_S d^{D-2} x \int_{M_1} d^{D-1} y \langle a(x, y) B_\theta(y) + c(x, y) \omega_\theta(y), D_\theta^\omega \mu(x) \rangle \end{aligned} \quad (1.226)$$

so we should set  $a(x, y) = 0$  and require that  $c(x, y)$  is only nonzero when  $x, y$  are on the same slice, in which case we have

$$L_\omega[\mu_\theta](x) = - \oint_S d^{D-2} y c(y, x) D_\theta^\omega \mu(y) \quad (1.227)$$

and similarly, if we set  $b(x, y) = 0$ ,

$$L_\omega[\alpha](x) = \int_{M_1} d^{D-2} y c(y, x) D_\theta^\omega \alpha(y) \quad (1.228)$$

which eliminates some of the freedom of choosing dynamics, but still leaves open the choice of a spatial integral kernel  $c(x, y)$ . The easiest choice is of course ultralocal boundary conditions, but we have now seen that there is still much more compatible with even conservation of all of the charges. In fact, we should be surprised that the freedom is reduced so much. Notice, finally, that indeed the parameters generically can be field-dependent as they involve covariant derivatives.

In  $D = 4$ , we have largely the same problem but there are more restrictions. A similar set of boundary conditions is given by  $a = b = c = w = 0$ , leaving the boundary conditions

$$\omega_n[\omega_\theta, \omega_\phi], B_{n\theta}[B_{\theta\varphi}], B_{n\varphi}[B_{\theta\varphi}]. \quad (1.229)$$

There are two integral kernels  $u, v$  in this, but likely we can expect there to be some sensible sort of isotropy condition between the two, i.e.  $u = \cos\theta v$  or similar. We do not need the details for this at this point.

To see what happens in the dressed phase space with gauge-invariant boundary conditions, let us just *consider the example of a simple ultralocal integral kernel*  $c(x, y) = c\delta(x, y)$  in  $D = 3$ , where now  $c$  is a constant interpreted as a velocity. This we can implement through the undressed, non-covariant Lagrangian (employing a 2-form Lagrange multiplier  $\lambda = |\lambda|\mathbf{n} \wedge d\theta$ )

$$L_1 = -\langle\omega, J\rangle - \langle\lambda, J_n - cJ_\theta\rangle = (\langle J_n, \omega_\theta\rangle - \langle\tilde{J}_\theta, \omega_n\rangle - \langle|\lambda|, J_n - cJ_\theta\rangle)\mathbf{n} \wedge d\theta \quad (1.230)$$

which combines with the symplectic potential to give  $\theta_2 = 0$  and boundary equations of motion

$$\begin{aligned} E_1 = & (\langle\delta\tilde{J}_\theta, \omega_n - c|\lambda|\rangle - \langle\delta J_n, \omega_\theta - |\lambda|\rangle - \langle\delta|\lambda|, J_n - cJ_\theta\rangle \\ & + \langle\delta\omega_n, \tilde{J}_\theta - B_\theta\rangle - \langle\delta\omega_\theta, J_n - B_n\rangle)\mathbf{n} \wedge d\theta \end{aligned} \quad (1.231)$$

which allows us to infer  $B_n = cB_\theta, \omega_n = c\omega_\theta$ , as required. Quite clearly, this boundary condition breaks both diffeomorphism invariance and the internal gauge transformations, so we need to dress this and the bulk Lagrangian, choosing that  $J, \lambda$  are invariant under YM and KR transformations. This gives us<sup>49</sup>

$$L_0 = \langle B + d_\omega q, F_\omega \rangle \quad (1.232)$$

$$L_1 = -\langle\phi \cdot \omega, J\rangle - \langle\lambda, J_n - cJ_\theta\rangle \quad (1.233)$$

which modifies our equations of motion to the gauge-invariant

$$B + d_\omega q = \phi J \phi^{-1} \quad J_n = cJ_\theta \quad (\phi \cdot \omega)_n = c(\phi \cdot \omega)_\theta \quad d_\omega(\phi J \phi^{-1}) = 0. \quad (1.234)$$

In principle, then, these are boundary dynamics which close the system whether we are bulk on-shell or off-shell, they conserve all the corner charges satisfying

$$\begin{aligned} \mathcal{L}_{\hat{n}}\alpha + c[\omega_\theta, \alpha] + cD_\theta^\omega\alpha &= 0 \\ \mathcal{L}_{\hat{n}}\mu - cD_\theta^\omega\mu &= 0 \end{aligned} \quad (1.235)$$

---

<sup>49</sup> $\phi \cdot \omega = \phi^{-1}\omega\phi + \phi^{-1}d\phi$



and they are gauge invariant. They determine the gauge-invariant part of  $B, \omega$  such that all these properties hold, and the gauge part is then an equation of motion for  $\phi, q$  on the boundary. In particular, we can gauge fix throughout the bulk  $B = 0, \omega = 0$  and get the effective equations of motion

$$\phi^{-1}\partial_n\phi = c\phi^{-1}\partial_\theta\phi \quad \partial_n q = c\partial_\theta q = c\phi J_\theta\phi^{-1} \implies \partial_n J_\theta = c\partial_\theta J_\theta. \quad (1.236)$$

The operator  $\partial_n - c\partial_\theta$  describes a right/left moving chiral wave along the (cylindrical) boundary at speed  $c$ .

*Thus, the effective dynamics with minimally restrictive, conserving, ultralocal boundary conditions is given by chiral waves  $\phi, q, J$ , one of which is itself sourced by the others in a nonstandard way.* The parameter conditions also simplify to the same kind of equations,

$$\begin{aligned} \partial_n \alpha - c\partial_\theta \alpha &= 0 \\ \partial_n \mu - c\partial_\theta \mu &= 0 \end{aligned} \quad (1.237)$$

and the corner charges take on the expressions

$$J_\alpha \approx \oint_S \langle \phi^{-1}\partial_\theta q \phi, \alpha \rangle d\theta \quad K_\mu \approx \oint_S \langle \phi^{-1}\partial_\theta \phi, \mu \rangle d\theta \quad (1.238)$$

and evidently vanish for constant configurations of  $\phi, q$ , thereby marking those as vacua of the theory with maximal symmetry.

So there is a compelling picture here: the simplest conserving, off-shell closing boundary condition gives the dressing fields the dynamics of right/left moving waves, and charge conservation is due to their dynamics. The charges capture the full behaviour of the theory as decomposing the charges into Fourier modes will yield all the Fourier modes of the fields themselves (at least for  $J, q$ , but  $\phi$  can be understood as conjugate to them).

We need to stress once more that this is a *choice*, i.e. it models specific physical situations, but not others. In particular, this example is what one might find to be closest to the condensed matter phenomenon of *edge modes* of a topological material (i.e. an insulator which has no local degrees of freedom at low energies, therefore well-modelled by a properly topological theory).

What we have here is a quite abstract model of how surface waves may emerge on these materials due to some gauge-invariance related phenomena: In a real material, these are gapless (i.e. massless) electron waves concentrated in real space near the boundary. In our scenario, the role of these waves is played by the dressing fields, which model effective quantities built out of constituent boundary fields.

I.e. we could understand the above  $J$  to model the current density of a Lie algebra-valued fermion, which then fulfils a conservation law. Of course, the way we derived these equations makes *no* reference to constituents, and only used gauge invariance and a few limited properties of the bulk theory.

That is to say, the model of edge modes given from this is agnostic of microscopics and therefore cannot give any insight about which physical situations actually realise them.

Let us now continue forward by discretising BF theory.

### 1.5.4 BF theory: Discretization

Let us now apply the logic of sampling to the continuum BF phase space in order to arrive at a discretized theory whose quantization is directly relevant to the following chapters.

The main fact that enables the easy discretization of BF theory and other properly topological degrees of freedom is the simplicity of their phase space. In a generic field theory, we have infinitely many states to sample our observables on, but in BF theory there is a preferred physical bulk state given by the unique flat one modulo gauge redundancy. This allows discretizations to have much less ambiguity and be automatically closer to the continuum theory, as sampling is essentially the same as working with the on-shell theory, and so no data is lost in discretization.

There are numerous ways to perform the discretization, some manual and some by sampling. We prefer to give the version where one performs a quotient sampling with a set of constraints which are part of the physical ones[139–141]. Concretely, we select as our ideal  $\mathcal{I}$ , for one, the functions

$$\mathcal{I}_K = \{K_\mu \mid \mu|_{\partial\Sigma} = 0, \mu|_{\Lambda_\Sigma^{D-3}} = 0\} \quad (1.239)$$

which essentially imposes the flatness constraints everywhere except on the  $D - 3$ -skeleton  $\Lambda_\Sigma^{D-3}$  of some cellulation of  $\Sigma$  or lattice we care about (see A.4 for notation and explanations about cellulations). We also include the functions

$$\mathcal{I}_J = \{J_\alpha \mid \alpha|_{\partial\Sigma} = 0, \mu|_{\tilde{\Lambda}_\Sigma^0} = 0\} \quad (1.240)$$

which imposes the Gauss constraint everywhere except on the dual vertices. These are all part of the constraints that already exist in BF theory, so we are not even restricting the configurations artificially. This means that we go partially on-shell.

The result of this quotienting is that we no longer have observables to measure curvature anywhere except around the defects on  $\Lambda_\Sigma^{D-3}$  and  $\tilde{\Lambda}_\Sigma^0$ . The full ideal is then generated by  $\mathcal{I}_K, \mathcal{I}_J$ . That the result of quotienting this way is locally finite is guaranteed by local finiteness of the cellulation. Essentially, the remaining operators measure the field induced by *defects*, and so are only related to the degrees of freedom associated to them. In practice, this means that the thus discretised phase space is the same as the on-shell phase space of BF theory with defects. I.e, we can give its symplectic potential in the dressed phase space as<sup>50</sup>

$$\Theta_\Sigma = \int_{\Sigma_1} \langle \delta\omega, B \rangle - \sum_{e_\alpha^{D-3} \in \Lambda_\Sigma^{D-3}} \int_{e_\alpha^{D-3}} \langle \delta q, \phi F_\alpha \phi^{-1} \rangle - \sum_{\tilde{e}_\beta^0 \in \tilde{\Lambda}_\Sigma^0} \int_{\tilde{e}_\beta^0} \langle \phi^{-1} \delta\phi, S_\beta \rangle. \quad (1.241)$$

Now this is still before imposing the constraints. If we want to do so, we have the slight complication that the presence of defects changes the solution set. In particular, there are

---

<sup>50</sup>We use  $\alpha$  to label cells of the cellulation, i.e. tetrahedra in a triangulation of a spatial slice in  $D = 4$ , and  $\beta$  to label dual cells (centred at vertices), which can be more complicated polyhedra.

nontrivial flat connections around each curvature defect. I.e. in  $D = 3$  such a defect on a slice is a point, and if we choose a small radial coordinate chart around the point,  $(\rho, \varphi)$ , then such a configuration takes roughly the form

$$\omega = \frac{1}{2\pi} F_\alpha d\phi. \quad (1.242)$$

The point is that we can no longer use the same formulas for the on-shell symplectic potential as before, globally. However, *locally*, so within each cell  $e_\alpha^{D-1}$ , the connection is still flat - and within each dual cell  $\tilde{e}_\beta^{D-1}$ , there is no source for the Gauss constraint. We can use this to go on-shell locally, and then patch together the cells.

Let us now skip ahead and report the result of such a reduction of the phase space. All the data in the  $B, \omega$  fields is removed by the sampling, and all that is left is the dressing fields  $\phi, q$ . This is of course exactly the same as in the continuum, but absolutely crucial in understanding the interpretation of the final result. *All that is left after discretization is edge modes.*

These degrees of freedom now live on the boundaries of ‘wedges’

$$e_{\alpha\beta}^{D-1} := e_\alpha^{D-1} \cap \tilde{e}_\beta^{D-1} \quad (1.243)$$

which have boundary segments which are either cell/cell, or dual cell/dual cell boundaries. On each of these boundary segments, there is an imprint from its two sides, and the resulting net symplectic potential usually only involves constant quantities, in particular the *relative orientation*

$$G_{\alpha\alpha'} := \phi_\alpha \phi_{\alpha'}^{-1} \in G \quad (1.244)$$

between cells  $\alpha, \alpha'$ , and the *relative shifts*

$$X_{\alpha;\beta,\beta'} = \int_{e^{D-3}} q_{\alpha,\beta} - q_{\alpha,\beta'} \in \mathfrak{g} \quad (1.245)$$

which is smeared over a codimension 3 surface (i.e. a line in  $D = 4$ ) on the boundary of a surface between two wedges  $e_{\alpha\beta}^{D-1}, e_{\alpha\beta'}^{D-1}$  (but still inside a cell  $\alpha$ ).

Summarizing and simplifying the result, the symplectic potential terms can then be grouped together into parts associated to dual edges  $\tilde{e}_{\alpha,\alpha'}^1 \in \tilde{\Lambda}_\Sigma^1$ . Each part is of the form

$$\Theta_{\alpha,\alpha'} = -\langle \chi_R(G_{\alpha\alpha'}), \mathcal{Q}_{\alpha,\alpha'} \rangle \quad (1.246)$$

where  $\mathcal{Q}_{\alpha,\alpha'}$  is constructed from the  $X_{\alpha;\beta,\beta'}$ . The phase space, when discretized, is then a copy of  $T^*G$  associated to each dual edge, so

$$P_\Lambda = \prod_{\tilde{e} \in \tilde{\Lambda}_\Sigma^1} T^*G_{\tilde{e}}. \quad (1.247)$$

This is an incredibly simple structure that essentially just relies on the dual 1-skeleton, which can be represented as a graph. The Gauss constraint can then be represented efficiently on this phase space as

$$\mathcal{Q}_\alpha = \sum_{\alpha'} \mathcal{Q}_{\alpha,\alpha'} = 0 \quad (1.248)$$

which really follows from noticing that this sum is what appears as a counterpart to a YM charge defect at dual vertices. The absence of charges then implies this constraint. Similarly, the curvature constraint requires that one fixes the holonomy

$$\mathcal{H}_\mu = \prod_{(\alpha\alpha') \cap e_\mu^{D-3}} G_{\alpha\alpha'} = \mathbb{I} \quad (1.249)$$

around the  $(D-3)$ -surface  $e_\mu^{D-3}$ . Again, this appears as a counterpart to curvature defects, and when these are absent, the holonomy must be trivial.

In this phase space, it is also relatively easy to talk about glueing: If the cutting surface runs along cell/cell boundaries, the easiest sampling of the corner algebra given by  $\phi, q$  is the one that has the same kind of symplectic potential. The full symplectic potential with boundaries is then,

$$\Theta_\Lambda = - \sum_{(\alpha, \alpha') \in \tilde{\Lambda}_\Sigma^1} \langle \chi_R(G_{\alpha\alpha'}), \mathcal{Q}_{\alpha, \alpha'} \rangle - \sum_{(\alpha, \alpha') \in \tilde{\Lambda}_{\partial\Sigma}^0} \langle \chi_R(\phi_{\alpha\alpha'}), q_{\alpha, \alpha'} \rangle \quad (1.250)$$

so carries a copy of  $G, \mathcal{Q}$  on each internal edge of the dual graph, and a copy of  $\phi, q$  on each exposed boundary edge, so the ones sticking out through the boundary  $\partial\Sigma$ .

Glueing then consists of imposing the constraint

$$q_{\alpha, \alpha'} + q_{\alpha', \alpha} = 0 \quad (1.251)$$

on dual edges that need to be glued. This is just a matching or cancelling of corner gauge charges coming directly from the continuum, and leaves the combination

$$\phi_{\alpha\alpha'} \phi_{\alpha'\alpha}^{-1} \quad (1.252)$$

as invariant. Thus, we understand the glueing in this case fully if we identify as our corner algebra the algebra generated by the  $\phi_{\alpha\alpha'}, q_{\alpha\alpha'}$ , with the actions on the two sides of the cutting surface being orientation reversed.

We can then take this directly to the quantum realm, as  $T^*G$  with the given symplectic potential has a standard quantization on  $L^2(G)$ . In this representation,  $\mathcal{Q}, q$  act as right invariant vector fields (so, derivative operators) on the Lie group  $G$ , whereas  $G, \phi$  act as multiplication operators on wavefunctions

$$\Psi(\{G\}, \{\phi\}) \in \mathbb{H}_{\tilde{\Lambda}^1} = \bigotimes_{\tilde{e}} L^2(G). \quad (1.253)$$

The residual generators of YM gauge transformations now act per dual vertex, on the bulk  $G$  that touch a given dual vertex. If we look at a single vertex  $x$  with for example valence 4, then the YM transformations behave as

$$\Psi(\phi_1, \dots, \phi_4) \mapsto \Psi(g_x^{-1} \phi_1, \dots, g_x^{-1} \phi_4) \quad (1.254)$$

For KR transformations, instead, the relevant quantum operator becomes a holonomy around a loop in the dual graph.

The important point is then that the wavefunctions should carry invariance under these YM transformations, acting on bulk dual vertices. The physical gauge symmetries of the theory, instead, are the frame reorientations

$$\Psi(\phi_1, \dots, \phi_4) \mapsto \Psi(\phi_1 g_1, \dots, \phi_4 g_4) \quad (1.255)$$

which, in this picture, can easily be seen to be physical even when bulk gauge invariance holds. In fact, they exhaust the full set of states, once again showing that upon discretizing, we only have edge modes left.

For KR transformations, in principle the same holds: We require trivial holonomy around all loops, and a flat boundary connection, a stand-in for the KR charge, can be defined by taking combinations of the  $\phi$ : I.e. if  $\tilde{e}$ ,  $\tilde{e}'$  are boundary dual edges beginning at the same dual vertex, then

$$h_{\tilde{e}, \tilde{e}'} = \phi_{\tilde{e}'} \phi_{\tilde{e}}^{-1} \quad (1.256)$$

is a parallel transport on the boundary, and effectively a flat connection, so represents the relevant boundary initial data for BF theory.



## Chapter 2

# Random tensor networks for discrete holography

As we have seen in the previous chapter, many intricacies arise in the pursuit of understanding how discrete theories relate to the continuum, and in particular how this works in gauge theories. As we are primarily invested in these questions for the context of gravity, we will now present a specific case where this relation is subtle and understanding it is particularly meaningful.

The example in question concerns the study of the holographic principle and its validity in the context of 3+1D gravity, in a form known as tensor network holography. It arose historically from a need to better understand aspects of the continuum theory which were best studied in a setting with finite dimensional Hilbert spaces and simple spatial tensor product factorisations. This singled out discrete theories as a prime candidate, for all the reasons related to the glueing problem that we already expanded on in 1.3.

In essence, many of the questions relating to holography, so the idea that a field theory's bulk degrees of freedom may be encoded in a lower dimensional theory on the boundary<sup>1</sup>, are to do with entanglement. It is entanglement between regions of the boundary theory that allows faithful encoding of the bulk degrees of freedom, and therefore factorisation and non-factorisation and splitting into subregions lies at the heart of the mechanisms of the idea.

This chapter will present a particular setting of tensor network holography which is very close to the structures of discrete canonical gravity, and which allows a direct interpretation of the Hilbert spaces through quantum geometry. Originally, such interpretations were not available in tensor network models and have only become a relatively recent addition to the literature.

The primary goals here are the study of isometric encoding of bulk degrees of freedom into boundary Hilbert spaces, but implicitly the calculation of entropies as well. We will

---

<sup>1</sup>One may also phrase the story differently in terms of phase spaces (a la Aron Wall[142]): If the bulk  $D$ -dimensional theory reduces to the data of a  $(D-1)$ -Cauchy slice, then holography says that the phase space on it is equivalent to the phase space of the  $(D-1)$ -dimensional *spacetime boundary*, which is timelike in usual cases and therefore reduces itself to the data of its  $(D-2)$ -Cauchy slice on the corner.

see that the discrete theory, even though the glueing problem here reduces to the study of edge modes, still has subtleties in defining entanglement entropy and factorisation, which require an extended notion of *Channel-State duality for algebras with centers* (which we understand here as boundary-conditioned algebras). We introduce the necessary technical setup in B.1

Also, in order to achieve isometric mappings between the bulk and boundary, we find that corner effects rear their head: A necessary criterion for holographic reconstruction becomes that the *total area of the boundary is fixed*. This can be understood field-theoretically as a kind of gauge-fixing of the location of the boundary, but also from the perspective of corner charges: The full Hilbert space of the theory on the slice is superselected into sectors of constant total area, each of which separately allows for holographic reconstruction.

The overall picture is that the discrete theory here gives valuable insights about the general state of affairs that would be hard or impossible to get in the continuum theory with the same level of control or rigour. Given our previous work, we also understand the discrete-continuum correspondence better, which allows us to make inferences on the latter side. It also gives us better control over the elements that go into building the discrete model and notice its possible shortcomings and extensions.

The chapter is structured as follows. After a brief introduction in section 2.1, we detail the class of states under consideration in section 2.1.2, along with the criteria for holographic behavior. For this purpose, we need to adapt our notion of holographic isometry to the case of algebras with centers, which we work out in sections 2.1.3, B.1. Also within this section, we introduce the methodology for assessing holography: starting from a state of quantum geometry, we express the purity of a reduced boundary state - our "holography measure" - through a random Ising model defined on the graph underlying the quantum geometry. In section 2.3 we present our results: after analysing the properties of the aforementioned Ising model, we derive a precise criterion for the superposition of spin network states to feature isometry of the induced map between network subregions. We conclude with a summary and technical discussion of results in section 2.4. Some technical calculations are reported in the appendix B.

## 2.1 Introduction

Since the discovery of the Bekenstein-Hawking entropy formula for black holes [68, 143], holography has taken center stage in the exploration of the overlap of quantum, gravitational and thermodynamic properties of spacetime. Indeed, according to the conventional microstate counting interpretation of entropy, the Bekenstein-Hawking formula signifies that the number of degrees of freedom associated with a black hole does not scale with its *volume* (as for most systems), but instead scales with the *area* of the surface bounding that volume.



This implies a form of (informational) *holography*: information on the degrees of freedom of the system is encoded on the boundary of the region of space it occupies, from where it may be recovered. Although discovered in the context of semiclassical gravity, it is generally believed that holography calls for a quantum gravity explanation. In fact, this holographic behaviour is found for entanglement entropy in a number of quantum many-body systems and often (for local Hamiltonians) it characterizes ground states, distinguishing them from the vast majority of their quantum states [144]. The suggestion, therefore, is that it may also provide a similar characterizing role for quantum gravitational systems.

Following the discovery of the Bekenstein-Hawking area law for black hole entropy, other forms of holography have been related to gravity and used to gain access to its quantum properties. The most explored example is of course the Anti-deSitter/Conformal Field Theory (AdS/CFT) correspondence [59, 60], and its generalizations [62]. What all of these more recent examples have in common, and different from the original black hole case, is that they describe a relation between a *bulk* gravitational theory and degrees of freedom on an *asymptotic boundary*, apparently describing the same physics.

However, a different type of holography for finite regions of space/spacetime is suggested to exist in a variety of contexts, besides the original black hole one. This finite-distance holographic behaviour is signaled too by entropy bounds. For example, recent work in classical gravity suggests that corner charges of general relativity provide an encoding of bulk information [77, 99, 145], which applies to any finite region of space with boundary. In the already mentioned condensed matter context, holographic properties refer to finite regions; indeed, the ground states of local Hamiltonians on lattice systems are often found to be *short-range entangled* [146], so that the entropy of any finite region scales at most with the size of its boundary. States with such properties are highly desirable. They not only provide a more manageable subset of the state space for initiating searches for ground states, but also exhibit characteristics such as exponentially decaying correlations between regions, which mimic a local lightcone structure through Lieb-Robinson bounds [147, 148].

In the following, we seek this type of local holographic behavior, in a quantum gravity context, by identifying classes of quantum geometries that possess it. We focus for this on *spin network states*.

The goal of this chapter is to elaborate on criteria for holographic mappings to exist between patches of a finite spatial region. This is based on earlier work [149–154] where spin network states were seen as tensor networks, and analysed via random tensor network techniques, which have been well-explored in the quantum information literature in connection to holography (and AdS/CFT correspondence) [155–157].

These works established the Ryu-Takayanagi entropy formula [69, 71] for spin network states, as well as isometry conditions for bulk-to-boundary maps defined by the latter, with an important restriction: both combinatorial structures (the graph underlying the

spin network states) and the algebraic data labelling them (the eigenvalues of quantum geometric operators) were held fixed. Here, some of these restrictions are not imposed: the bulk region is modeled by a superposition of spin network states with fixed graph structure, but allow for general superpositions otherwise, i.e. we sum over the algebraic quantum numbers. Therefore, we are one step closer to studying local holography in full generality, which not only brings more realism to these constructions that were previously considered toy models, but also complements similar work generalising random tensor network holography [158–160].

### 2.1.1 Questions and quantities of interest

The goal here is to enhance the characterisations of spin network states in terms of holography that were put forward in [152]. This serves to select subsets of states with favorable typical properties, as well as to make connections with research done in the context of tensor network holography. The main questions we address are the following:

- Given a superposition of spin network states with fixed graph structure, consider the bulk-to-boundary map it defines; are these maps isometric?
- Using channel-state duality, properties of these maps can be traced back to properties of the corresponding states. In particular, isometry is related to maximised Rényi-2 entropy of the input region. We therefore ask: what is the Rényi-2 entropy of bulk and boundary subregions?

In doing so, we work with a vastly extended setting from that of previous works [156, 157, 159–161], in which only trivial superpositions were considered. In such a setting, many of the nontrivial subsystem and cutting/glueing issues in gauge theories do not appear, which we dealt with in sufficient generality here. Note that there is related work [158] that uses nontrivial link states, which are, in contrast to our setup, randomly chosen. As we will see in 2.2.3, this is not an option in our setup, as we need a specific kind of link state in order to match geometric area with entanglement entropy. However, their generalization, like ours, admits nontrivial entanglement spectra, and is therefore an improvement on previous classes of states.

### 2.1.2 States under consideration

Throughout this chapter, we will study a class of states made from superposing spin network states associated with the same (open) graph  $\gamma^2$  and different assignments of spins on its links. More specifically, we consider states constructed in analogy with Projected

---

<sup>2</sup>We distinguish between open graphs, which are allowed to have a subset of links, called ‘semilinks’ which do not end in another vertex (equivalently, one can see them as ending on 1-valent vertices), and closed ones, in which every link ends in 2 ( $D$ -valent) vertices (with  $D > 1$ ).

Entangled Pair States (PEPS), i.e. obtained by contracting tensors associated to open spin network vertices according to  $\gamma$ , and therefore denoted as *spin tensor network states*.

Even though most of the constructions in this chapter are generic in terms of the relevant Hilbert spaces, let us be concrete for illustration and introduce *spin network states*, which are a class of states in Hilbert spaces which are commonly found in canonical quantizations of gravity.

For the purposes of illustration, we ask the reader to think of a spatial slice  $\Sigma$ , and triangulate it into a cellular complex  $\Lambda$ . Then, the 1-skeleton of the dual cellulation,  $\tilde{\Lambda}^1$ , is a *graph* which we denote as  $\gamma$  and captures the connectivity of cells in the triangulation. We now think of special classes of states of the geometry of  $\Sigma$ , that is, its metric and parallel transports, that are associated with the given triangulation.

There are multiple ways to arrive at the same data, but we prefer the following: Consider the discretization procedure of BF theory we presented in section 1.5. This gave a piecewise flat geometry with curvature defects (if the curvature constraints are not imposed). This, while it does not have the variables of gravity, can equally be done in theories of gravity with similar variables, one continuum formulation of which we expand upon in the next chapter 3.1. If we sample the algebra of observables on geometries which are piecewise flat, we end up with very similar structures as in the BF theory example. The remaining degrees of freedom can then be quantised, and give rise to the data of spin networks based on the graph  $\gamma$ . This comes out as a Hilbert space associated to said graph, with group actions implementing gauge transformations for the isometry group of space, i.e.  $G = SU(2)$ .

For  $G$ -invariance, we further impose that the diagonal left action at each vertex  $x \in \gamma$  will be trivial. This corresponds to implementing the Gauss constraint (see eq. 1.248) as in the BF example. We will not implement flatness in our examples, and therefore not work with fully on-shell states.

We also note that these states are actually indistinguishable from those of BF theory on this level, and also the same as states of  $G$ -lattice gauge theory when considered on a fixed graph. The curvature restriction is special to BF theory, but the remainder of structures is *universal to flat-sampled theories of connections* with canonical pair structure  $(B, \omega)$ <sup>3</sup>.

To be more precise, given a (compact) Lie group  $G$ , let  $\mathbb{H}_x = L^2(G^D/G)$  be the Hilbert space of what we will refer to as an open,  $G$ -invariant spin network vertex  $x$  of valence  $D$ , pictured as a vertex of a graph with  $D$  semilinks sticking out. These semilinks are identified by an index  $\alpha = 1, \dots, D$ . For simplicity, we restrict the attention to the case in which all vertices of the graph possess the same valence (except the boundary vertices), but the analysis can be easily generalised.

A generic graph like  $\gamma$  can then be built from the individual open vertices by *glueing* the open semilinks<sup>4</sup>. The group elements  $\phi_\alpha \in G$  label faces of a cell, and correspond to the discretised  $\phi$  dressing field of the bulk. Then, the glueing consists of taking  $\phi_\alpha, \phi_{\alpha'}$  on

<sup>3</sup>A noteworthy exception to this is Chern-Simons theory, which has a different canonical pair structure.

<sup>4</sup>For details on how this comes about in the continuum, the reader may consult the appendix A.5.

two open links, and reducing the independent variables to the combination  $G_{\alpha\alpha'} = \phi_\alpha \phi_{\alpha'}^{-1}$ . Therefore, we implement glueing by a simple group-averaging procedure over the diagonal frame reorientations of the group elements  $\phi$ ,

$$\Psi(G_{\alpha\alpha'}) = \int dg \psi(G_{\alpha\alpha'} g, g) \quad (2.1)$$

which produces the invariant part of the wavefunction. See [152] for details.

A generic wavefunction  $\Psi$  in this glued Hilbert space can then be decomposed in a manner analogous to Fourier series, but with respect to a set of labels  $j$  associated to the irreducible representations of the group  $G$ . This is due to the Peter-Weyl theorem, which allows us to decompose generic  $L^2$ -functions on the group into analogues of plane waves. For the case of  $G = SU(2)$ , this reads

$$L^2(SU(2)) \cong \bigoplus_{j=0}^{\infty} V_j \otimes \bar{V}_j \quad (2.2)$$

where  $V_j$  is the irreducible unitary representation of the group of dimension  $d_j = 2j + 1$ .

Such  $SU(2)$  spin networks, whose representations are labeled by a *spin*  $j \in \frac{\mathbb{N}}{2}$ , describe piecewise-flat geometries whose cell-cell interface labels are interpreted as quantised microscopic area  $a_{\text{LoopQuantumGravity(LQG)}} = \sqrt{j(j+1)}$  and associated to the links of the spin network graph  $\gamma$ .

Glueing, in this decomposition, is dually expressed as the analogue of equation 1.251 from the BF example: We can interpret  $\mathcal{Q}_\alpha$  as the generator of right translations on the group, and imposing  $\mathcal{Q}_\alpha^L + \mathcal{Q}_\alpha^R = 0$  imposes the aforementioned right invariance. However,  $\mathcal{Q}_\alpha$  is also block diagonal in the Peter-Weyl decomposition, and the matching requirement translates into equality of spins  $j_L = j_R$  in the glued Hilbert space. This is just the matching of boundary conditions, here on the spins, that we already know from the naive picture of glueing of Hilbert spaces in 1.3.2. Here, we know it arises also from a more general glueing of algebras. We also know that this glueing involves a nontrivial corner algebra, generated by operators acting on boundary links.

This interpretation can be derived from a canonical quantization, in which these discrete sectors associated to triangulations naturally appear as subsectors of the continuum Hilbert space. A link<sup>5</sup> of  $\gamma$  is generically indicated by  $e \in \gamma$ , while vertices will be labeled by  $x \in \gamma$ . The boundary of  $\gamma$  (i.e. the set of boundary links) will be denoted by  $\partial\gamma$ . The fact that we get a specific notion of area from classical geometry which scales *linearly* in  $j$  is very important and we will return to this idea multiple times.

To recall, the single-vertex Hilbert space admits a decomposition

$$\mathbb{H}_x \cong \bigoplus_{j^x} \mathcal{I}_{j^x} \otimes V_{j^x} \quad (2.3)$$

---

<sup>5</sup>In this chapter, we avoid the edge/dual edge distinction we had to take care of in 1.5, and just refer to dual edges in  $\tilde{\Lambda}_\Sigma^1 = \gamma$ , which we call *links* of the graph  $\gamma$ .

into a direct sum over the representation labels  $\mathbf{j}^x := j_1^x, \dots, j_D^x$ , where each summand is the tensor product of the (boundary) space  $V^{\mathbf{j}^x} \cong \bigotimes_{\alpha=1}^D V^{j_\alpha^x}$ , given by the tensor product of the representation spaces of dimension  $d_j = 2j + 1$  associated to the open links of vertex  $x$ , and the (bulk) intertwiner space of vectors invariant under the diagonal  $G$ -action,  $\mathcal{I}^{\mathbf{j}^x} = \text{Inv}_G(\bigotimes_{\alpha} V^{j_\alpha^x})$ . These intertwiner spaces contain geometric data that is not fixed by the boundary spins, i.e. does not refer to a boundary distribution of areas. This includes, a priori, bulk volume information, but also bulk dihedral angles, which on-shell are the closest equivalent to gravitons.

The main way in which the choice of Hilbert spaces on links and vertices affects the results on isometry is through the dimensionalities, in particular the dependence of  $\mathcal{D}_{\mathbf{j}^x} := \dim(\mathcal{I}_{\mathbf{j}^x})$  on the adjacent link spaces. Additional structures such as group actions play less of a role in this context. In particular, the dimension of the bulk input space on a vertex,  $\dim(\mathcal{I}_{\mathbf{j}^x})$  depends on the dimensions of its adjacent link spaces,  $d_{j_e^x}$ . Additionally, we will consider only finite dimensional vertex Hilbert spaces - in the aforementioned example, we will implement this through imposing cutoffs  $\mathfrak{J} < j_e^x < J$  on the values of spin labels.

One may see this either as a technical restriction - needed for integrals to be well-defined - in which case one can in principle also switch to other methods of averaging such as Gaussians instead of the uniform distribution on states that will be used here. Alternatively, though, one may make analogies to the states of quantum geometry with nonzero cosmological constant, i.e. from the Turaev-Viro model[162] in 3D gravity, in which the cosmological constant acts as a natural cutoff to representation labels akin to the spins  $j$  here.

At this point, there is a need to stress a point of possible confusion for different audiences. While the interpretation of a link of the graph as dual to a surface is generic, there are multiple possible ‘area functions’ one can associate to it. The first obvious one is, as noted in our exposition, the LQG area spectrum  $a_{\text{LQG}} = \sqrt{j(j+1)}$ , obtained from quantizing the area function obtained from straightforward quantization of the surface area in classical general relativity or in simplicial geometry<sup>6</sup>. Another important measure of ‘area’ in our context is the ‘tensor network area’  $a_{TN} = \log(d_j)$ , motivated mostly from entanglement measures and their supposed matching to the Ryu-Takayanagi formula. In assigning a geometric interpretation to the states on the given graph, it is important not to conflate the two - a point to which we will return later in the discussion of the scaling of the entropy.

Importantly, the full Hilbert space of  $N$  distinguishable vertices decomposes into a direct sum of *spin sectors*, each having as a basis the spin network states with fixed link

---

<sup>6</sup>(the precise form of the spectrum depends on the chosen quantization map, an alternative being  $a'_{\text{LQG}} = j + 1/2$ )

spins:

$$\begin{aligned}\mathbb{H}^N &= \bigotimes_{x=1}^N \mathbb{H}_x \cong \bigoplus_{\vec{j}} \mathbb{H}_{\vec{j}} \\ \mathbb{H}_{\vec{j}} &\cong \mathcal{I}_{\vec{j}} \otimes V_{\vec{j}} = \bigotimes_{x=1}^N \mathcal{I}^{j^x} \otimes \bigotimes_{x=1}^N V^{j^x},\end{aligned}\tag{2.4}$$

where  $\vec{j} = j^1, \dots, j^N$  is the collection of spins over the whole set of semilinks. Our setup generalises immediately to the case where *any* single-vertex Hilbert space is used, as long as it admits a decomposition of the same type as 2.3. In particular, the dimension of the bulk space  $\mathcal{I}$  is allowed to depend on the representation labels in a nonlinear way. The crucial assumption is rather an initial factorisation of states over vertices, so that a PEPS-type construction of states is possible. This is not generally the case for states of quantum geometry in canonical or lattice quantum gravity (nor is the dependence on a single graph, even though it is more commonly assumed).

The first step for the construction of the spin tensor network states consists of picking a state in  $\mathbb{H}^N$  which factorizes per vertex:

$$|\Psi\rangle = \bigotimes_x |\Psi_x\rangle.\tag{2.5}$$

To turn this state into a spin network state with support on a graph  $\gamma$ , we apply a projection onto maximally entangled states of the spins living on the links (semilinks) forming the maximal closed subgraph  $\Gamma$  of  $\gamma$  consisting of all vertices with valence  $> 1$ , according to the prescription outlined in e.g. [150, 152]. More specifically, let

$$|e_j\rangle := \frac{1}{\sqrt{d_j}} \sum_m (-1)^{j+m} |j, m\rangle_1 |j, -m\rangle_2\tag{2.6}$$

be a normalised singlet state of two semilinks (labeled here by 1 and 2) carrying the spin  $j$ , and consider a normalised superposition

$$|e\rangle = \bigoplus_{j_e \in \frac{\mathbb{N}}{2}} g_{j_e} |e_{j_e}\rangle\tag{2.7}$$

with coefficients  $g_j$ . We will have more to say on the choice of these coefficients later, in 2.2.3. At this point, let us say that their choice can be motivated by kinematical, physical considerations. The gluing of the  $N$  vertices described by  $|\Psi\rangle$  into the graph  $\Gamma$  is then performed by projecting  $|\Psi\rangle$  onto  $|\Gamma\rangle = \bigotimes_{e \in \Gamma} |e\rangle$ , i.e. by applying to  $|\Psi\rangle$  the operator (up to normalisation)

$$\Pi_\Gamma = \bigotimes_{e \in \Gamma} \left( \bigoplus_{j_e} |g_{j_e}|^2 |e_{j_e}\rangle \langle e_{j_e}| \right)\tag{2.8}$$

The role of  $\Pi_\Gamma$  is precisely that of entangling, in *every spin sector*, the data of pairs of semilinks according to  $\Gamma$ . The result is a superposition of spin network states with support on  $\gamma$ , the set of which we denote by  $\mathbb{H}_\gamma = \Pi_\Gamma(\mathbb{H}^N)$ . Note that, when restricting the attention to vertex states living in a single spin sector, the setting reduces to that of previous work [153, 154]. We thus constrained the link spins to be in (a superposition of) singlet states, in order to be glued according to  $\Gamma$ .

In principle, we can proceed to constrain the intertwiner degrees of freedom as well, and focus on the resulting boundary state. To do so, first notice that the graph Hilbert space splits into a sum over boundary link labels  $E := \{j \in \partial\gamma\}$ :

$$\mathbb{H}_\gamma \cong \bigoplus_E \mathbb{H}_b^E \otimes \mathbb{H}_\partial^E. \quad (2.9)$$

where we introduced the Hilbert spaces for bulk ( $b$ ) and boundary ( $\partial$ ) degrees of freedom when the boundary links carry the spins  $E$ :

$$\mathbb{H}_b^E = \bigoplus_{\{j_e: e \in \Gamma\}} \bigotimes_{x \in \gamma} \mathcal{I}_x \quad \mathbb{H}_\partial^E = \bigotimes_{e \in \partial\gamma} V^{j_e} \quad (2.10)$$

in which the latter has a tensor product factorisation over boundary links, while the former retains a sum over bulk spins  $j_e$ . We can then choose pure states  $|\zeta_E\rangle \in \mathbb{H}_b^E = \bigoplus_{j_e: e \in \Gamma} \mathcal{I}_j$  and define another projector

$$\Pi_\zeta = \bigoplus_E \left( \frac{|\zeta_E\rangle \langle \zeta_E|}{\langle \zeta_E | \zeta_E \rangle} \otimes \mathbb{I}_{\mathbb{H}_\partial^E} \right) \quad (2.11)$$

which enforces the state to have certain intertwiner data in each sector.

Then, the projection

$$|\phi_{\partial\gamma}\rangle = \Pi_\zeta \Pi_\Gamma \bigotimes_x |\Psi_x\rangle \quad (2.12)$$

corresponds uniquely (by exact knowledge of the state of the bulk) to a state of the open (unglued) boundary semilinks, described by the Hilbert space

$$\mathbb{H}_{\partial\gamma} \cong \bigoplus_{j \in \partial\gamma} \bigotimes_{e \in \partial\gamma} V^{j_e} \cong \bigotimes_{e \in \partial\gamma} \bigoplus_{j_e} V^{j_e} \cong \bigotimes_{e \in \partial\gamma} V_e. \quad (2.13)$$

Due to this factorisation, it is straightforward to speak of entanglement and measures of it in the boundary-reduced case.

In contrast, the fixed-graph Hilbert space  $\mathbb{H}_\gamma$  with no restriction on the intertwiner data, thus with a generic superposition of them, has no obvious factorisation properties at all. In such a setting, the notion of entropy survives, but several subtleties arise in quantifying entanglement with it [116, 163, 164].

Nevertheless, holography can be characterised more directly than through entanglement scaling. We indeed use an alternative strategy, in which we can neglect this distinction and still make use of entropies as a computational tool to characterise holographic behavior; in particular, the latter will be traced back to isometric mappings between graph subregions. This is the task we tackle in the following.

### 2.1.3 A notion of holography

We introduce here a simple but effective notion of information transport that allows us to make assertions about a form of holography. Details may be found in the appendix B.1, in a more abstract context not focused on spin network states of quantum geometry.

Consider first the embedding of the graph Hilbert space  $\mathbb{H}_\gamma$  into the tensor product  $\mathbb{H}_b \otimes \mathbb{H}_\partial$ , with  $\mathbb{H}_b = \bigoplus_E \mathbb{H}_b^E$  and  $\mathbb{H}_\partial = \bigoplus_E \mathbb{H}_\partial^E$ . The main idea is then to introduce a map

$$\begin{aligned} i_b : B(\mathbb{H}_b) &\rightarrow B(\mathbb{H}_b \otimes \mathbb{H}_\partial) \\ X &\rightarrow X \otimes \mathbb{I}_\partial \end{aligned}$$

to (trivially) extend bulk operators into ‘bulk+boundary’ ones, and a partial-trace map

$$\begin{aligned} P\text{Tr}_\partial : B(\mathbb{H}_b \otimes \mathbb{H}_\partial) &\rightarrow B(\mathbb{H}_b) \\ Y &\rightarrow \text{Tr}_\partial[Y] \end{aligned}$$

which restricts ‘bulk+boundary’ operators to the boundary. We can then use these objects, suitably generalised, to define a mapping between algebras of operators associated to the bulk ( $\mathcal{A}_b$ ) or boundary ( $\mathcal{A}_\partial$ ) subsystems. The equivalence between the two subsystems is then the statement that the mapping between corresponding operator spaces is isometric. Such a statement may be translated into a calculable question about Rényi entropies of the bulk-reduced state  $\rho_b$ . This also connects the present framework to previous work.

The pieces are combined as follows: we suppose that holography can be expressed by turning operators (or operations)  $X_b$  on the bulk system into approximately equivalent operations on the boundary system. If the two are part of a larger system, we can extend bulk operators to the whole system by the operation  $i_b$ . We could then evaluate  $i_b(X_b)$  in the whole state  $\rho$  of the system (here, our graph state), to get  $\text{Tr}[i_b(X)\rho]$ , or instead reduce this to an effective operator on the boundary through the partial trace  $P\text{Tr}_\partial$ . The algebras  $\mathcal{A}_{b|\partial}$  serve in this as a restriction to sets of operators where this ‘operator transport’ suitably keeps all or most of the data of the system.

This operator-focused approach not only allows for immediate transport of operators on top of Hilbert space states, but it is also necessary for the direct-sum Hilbert spaces we consider here, which, as we remarked, do not have straightforward factorization properties. Indeed, if we were to apply a Hilbert-space mapping paradigm, mapping bulk states to boundary states, then we would have to confront the question of which Hilbert space would be the bulk one, and which one is the boundary one, and how we can see these as subsystems; this question is highly ambiguous, absent factorization.

To begin, we introduce the algebraic subsystems

$$\begin{aligned} \mathcal{B}_b &:= \bigoplus_{E \in \mathcal{W}_b} B(\mathbb{H}_b^E) \\ \mathcal{B}_\partial &:= \bigoplus_{E \in \mathcal{W}_\partial} B(\mathbb{H}_\partial^E) \end{aligned} \tag{2.14}$$



of the full algebra of operators on the graph Hilbert space

$$\mathcal{A} = B(\mathbb{H}_\gamma) = \bigoplus_{E, E'} B(\mathbb{H}_b^E \otimes \mathbb{H}_\partial^E, \mathbb{H}_b^{E'} \otimes \mathbb{H}_\partial^{E'}) \quad (2.15)$$

where we choose the index sets  $\mathcal{W}_{b|\partial}$  a posteriori to ensure isometric mapping between the two sets. The restriction can be motivated as follows. If the mapping of algebras is defined on all sectors, in general it will not be isometric.

By analysing the conditions for it to be so, one can identify necessary restrictions on the set of graphs colored by the labels  $E$ . In principle, then, the mapping is only isometric when restricted to the subalgebra fulfilling these ‘isometry conditions’. This may be seen as an analogue of the common practice of selecting a ‘code subspace’ which is properly reconstructible from the boundary data alone. This, of course, includes at a minimum the restriction that the dimension of the input algebra is not larger than that of the output algebra.

*Finding restrictions on the sets  $\mathcal{W}_b, \mathcal{W}_\partial$  will be the main formulation of the results of this chapter.*

We relate them to the full algebra through the associated extension and partial trace maps

$$\mathcal{B}_{b|\partial} \xrightarrow{i_{b|\partial}} \mathcal{A}_{b|\partial} \subset \mathcal{A} \xrightarrow{P\text{Tr}_{b|\partial}} \mathcal{B}_{b|\partial} \quad (2.16)$$

given for example for the input/bulk algebra by

$$\begin{aligned} i_b(X) &= \sum_E X_E \otimes \mathbb{I}_{\partial, E} \\ P\text{Tr}_b[X] &= \sum_E \text{Tr}_{\partial, E}[X_E] \end{aligned} \quad (2.17)$$

and whose images we name  $\mathcal{A}_{b|\partial} = \text{Im}(i_{b|\partial})$ .<sup>7</sup> These are the ‘naive’ partial trace and extension, and our choices of algebras can be motivated by this naive choice.

However, additionally, we must make this choice for bulk and boundary algebras where we only have sector-diagonal operators, because any non-diagonally acting operator will not have a clear notion of ‘leaving the complement invariant’. If we understand a bulk operator  $X = \sum_{E, F} X_{E, F}$  to leave the boundary  $\mathbb{H}_\partial^E$  invariant, its matrix elements with respect to a basis  $|E, i|j\rangle$  of the bulk  $\mathbb{H}_b^E$  labeled by  $i, j$ , and  $|E, m|n\rangle$  of the boundary  $\mathbb{H}_\partial^E$  labeled by  $m, n$ , should satisfy

$$\langle E, j | \langle E, n | X_{E, F} | F, i \rangle | F, m \rangle \sim \delta_{E, F} \delta_{m, n}, \quad (2.18)$$

but this is only a sensible equation if  $E = F$  holds on the right hand side, and therefore must be enforced on the left, giving a sector-diagonal operator.

On the other hand, if we instead defined bulk/boundary operators to be sector-mixing, such that

$$\mathcal{B}'_b = B\left(\bigoplus_E \mathbb{H}_b^E\right) \quad \mathcal{B}'_\partial = B\left(\bigoplus_E \mathbb{H}_\partial^E\right), \quad (2.19)$$

---

<sup>7</sup>These  $\mathcal{A}_{b|\partial}$  are, technically speaking, the actual subsystems, and  $i_{b|\partial}$  are the identifications allowing  $\mathcal{B}_{b|\partial}$  to be treated as subsystems.

then there would be no obvious way to extend or realise the non-sector-diagonal ones to/in the algebra  $\mathcal{A}$ . Said more succinctly, these sets are not (isomorphic to) subalgebras of  $B(\mathbb{H}_\gamma)$  and therefore do not function as algebraic subsystems.

We are therefore led to the choice of bulk and boundary algebras 2.14 as the largest sensible one, and notice that they are not realised as operators on a single Hilbert space, but only a certain subset of those. In fact, they are the subalgebras  $\mathcal{B}_{b|\partial} \subseteq \mathcal{B}'_{b|\partial}$  such that their *center* is given by the boundary spin Casimirs  $\{J_e^2 : e \in \partial\gamma\}$ . This is, in fact, what we get if we remove from the full algebra  $\mathcal{A}$  all *holonomy* operators  $h_e$  on boundary links  $e \in \partial\gamma$ . In this sense, because we are removing the boundary gauge field from the algebra (usually associated with magnetic degrees of freedom), we could interpret the bulk/boundary algebras 2.14 as *electric* algebras in the language of [116].

In our previous discussions on boundary-conditioned algebras in 1.3.2, we can also see it as the algebra of the graph conditioned on Dirichlet boundary conditions for the *fluxes*  $\vec{J}_e$  - but more precisely, *gauge invariant* boundary conditions that only fix the Casimirs. As we already discussed at length, in the continuum such a boundary condition would be implemented through addition of compensator fields. These are reflected here as the data of the *magnetic* indices  $m_e$  labelling a state in  $\mathbb{H}_\partial^E$ , which is a direct product of representations of  $SU(2)$ .

Now, with these preliminary choices at hand, we can define the superoperator mapping bulk operators into boundary operators via the Choi-Jamiołkowski isomorphism [165, 166]:

$$\begin{aligned}\mathcal{T}_\rho(X) &= K \text{PTr}_\partial[i_b(X)\rho] \\ &= \sum_E K c_E \text{Tr}_{b_E}[(X_E \otimes \mathbb{I}_{\partial E})\rho_{E,E}],\end{aligned}\tag{2.20}$$

where

$$\rho = \sum_{E,\tilde{E}} \sqrt{c_E c_{\tilde{E}}} \rho_{E,\tilde{E}} \in \mathcal{D}(\mathbb{H}_\gamma)\tag{2.21}$$

with  $\text{Tr}[\rho_{E,\tilde{E}}] = \delta_{E,\tilde{E}}$  and  $c_E = \text{Tr}_E[\rho] \geq 0$ ,  $\sum_E c_E = 1$ ,  $K > 0$ , and  $\mathcal{D}(\mathbb{H}_\gamma)$  the set of density matrices on  $\mathbb{H}_\gamma$ . The sum is again over boundary spin labels  $E = j_{\partial\gamma}$ .

In the following, we will work with proper subalgebras, so  $\mathcal{B}_{b|\partial} \subset \bigoplus_E \mathbf{B}(\mathbb{H}_{b|\partial,E})$ . The main reason is that it is generically impossible to make the mapping on the full bulk input algebra isometric, just for dimensional reasons alone. *We will make a choice of subalgebra by selecting a subindex set  $\mathcal{W}$  of sectors  $E$ .* The specific choice depends on the scenario of information transport we consider, and will be done a posteriori.

Assume now that the dimension of  $\mathcal{B}_b$  does not exceed that of  $\mathcal{B}_\partial$  so that isometry between spaces of operators is possible in principle. We find the condition for  $\mathcal{T}_\rho$  to be an *isometry in the Hilbert-Schmidt inner product on the operator algebras* by equating the two expressions on the input (bulk) and output (boundary) sides

$$\langle \mathcal{T}_\rho(A), \mathcal{T}_\rho(B) \rangle_\partial = \langle A, B \rangle_b.\tag{2.22}$$

As illustrated in [2], this is equivalent to the requirement on the state  $\rho$  that

$$c_E^2 K^2 \text{Tr}_{\partial_E^{\otimes 2}}[(\rho_{E,E} \otimes \rho_{E,E}) \mathcal{S}_{\partial_E}] = \mathcal{S}_{b_E} \quad \forall E \in \mathcal{W}. \quad (2.23)$$

in which we use the *swap operators*  $\mathcal{S}$  that exchange two identical copies of a Hilbert space. In the case where the states  $\rho_{E,E}$  are *pure*, the isometry condition above simplifies drastically to

$$(\rho_{E,E})_b = \frac{\mathbb{I}_{b_E}}{D_{b_E}} \quad c_E = \frac{D_{b_E}}{\sum_F D_{b_F}} \quad \forall E \in \mathcal{W}. \quad (2.24)$$

together with the condition on bulk dimensions  $D_{b_E} = \dim(\mathbb{H}_b^E)$  given by  $|K| = \sum_E D_{b_E} =: D_b$ . This, in fact, is just the requirement of trace preservation. Therefore, schematically,

$$\rho_{E,E} \text{ pure} : \mathcal{T}_\rho \text{ Quantum channel} \implies \mathcal{T}_\rho \text{ isometry} \quad (2.25)$$

We will restrict to pure states  $\rho_{E,E}$  in the following as results displayed in B.1 indicate that it is generically difficult to get isometry of  $\mathcal{T}$  from mixed states. *Therefore, in this work, we will establish when the transport superoperator is, typically, a quantum channel*, and this will be the sense in which we identify isometric and thus holographic states. Furthermore, trace preservation is easy to convert into a calculable statement about entropies, which also directly connects to previous work.

Let us consider entropies for those isometry-inducing states. For any normalised input state  $\sigma$ , hermiticity of the channel  $\mathcal{T}_\rho$  and isometry imply that the Rényi-2 entropy is left unchanged by the channel:

$$\begin{aligned} e^{-S_2(\mathcal{T}(\sigma))} &= \text{Tr}[\mathcal{T}(\sigma)^2] \stackrel{\mathcal{T} \text{ herm.}}{=} \text{Tr}[\mathcal{T}(\sigma)^\dagger \mathcal{T}(\sigma)] \\ &\stackrel{\mathcal{T} \text{ isom.}}{=} \text{Tr}[\sigma^\dagger \sigma] = \text{Tr}[\sigma^2] = e^{-S_2(\sigma)} \end{aligned} \quad (2.26)$$

The Rényi-2 entropy of the reduced state  $\rho_b$  itself is then also purely determined by the range of sectors and is maximal:

$$\begin{aligned} e^{-S_2(\rho_b)} &= \sum_E c_E^2 e^{-S_2((\rho_{E,E})_b)} \\ &= \sum_E \frac{\dim(\mathbb{H}_{b_E})^2}{D_b^2} \frac{1}{\dim(\mathbb{H}_{b_E})} = \frac{1}{D_b}. \end{aligned} \quad (2.27)$$

We now shift our focus towards determining which classes of spin network states satisfy these last requirements. The method we use is entirely analogous to the one used in previous works [153, 156, 167, 168] and relies on entropy calculations condition, written e.g. as  $S_2((\rho_{E,E})_b) = \log(D_{b_E})$ . Via the swap operator<sup>8</sup> we can rewrite the Rényi entropy as traces over two copies of the system:

$$\frac{\text{Tr}_{\mathbb{H}_b^2}[\rho_b^{\otimes 2} \mathcal{S}_b]}{\text{Tr}_{\mathbb{H}_b^2}[\rho_b^{\otimes 2}]} = \frac{\text{Tr}_{\mathbb{H}_b^2}[ (|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_b ]}{\text{Tr}_{\mathbb{H}_b^2}[ (|\phi\rangle \langle \phi|)^{\otimes 2} ]}. \quad (2.28)$$

---

<sup>8</sup>Letting  $\mathcal{S}$  be the operator swapping two copies of a Hilbert space  $\mathbb{H}$ , we have  $\text{Tr}_{\mathbb{H}^{\otimes 2}}[(A \otimes B) \mathcal{S}] = \text{Tr}_{\mathbb{H}}[AB]$  while  $\text{Tr}[(A \otimes B)] = \text{Tr}[A] \text{Tr}[B]$ .

We have introduced the state  $|\phi\rangle$  here as a stand-in for whichever pure state we assign to the full system. The specifics of bulk-to-boundary maps will be treated later in more detail.

## 2.2 Methodology

We start by explaining the issues we address and outlining the main features of our calculation. Several of these steps are analogous to previous results in the case of spin network states with fixed spins [167] (referred to as *single-sector case*), here generalised to include superposition of such states.

After giving examples for the types of quantum systems we would like to consider, we specify a class of tensor network states via their concrete construction. We then introduce an algebraic notion of holography, suitable for our context where the Hilbert space does not factorize over bulk and boundary regions. This notion of holography involves the bulk-to-boundary map defined by our states being an isometry, along the line of previous works on holographic tensor networks (see e.g. [156]), although the algebraic formulation represents a novel aspect of our analysis. We also inquire about the isometric character of the maps that our states define between boundary subregions.

As our work is statistical in nature, i.e. we ask questions about properties that may hold only ‘on average’ with respect to the geometric data of our states, we then specify a choice of randomisation.

Lastly, by adapting well-known techniques for random tensor networks to our general setting, we reframe the isometry condition in terms of the Rényi-2 entropy of the input region, and compute it via the analysis of a dual Ising model. A novel feature in this last step in order to calculate the Rényi entropy is the necessity to perform a cumulant expansion in terms of a statistical weight which is distinct from, but related to, the uniform distribution over states.

### 2.2.1 Randomisation over vertex states

Instead of calculating the entropy for a particular state, we will make a typicality statement about our class of spin tensor networks. So, colloquially, we will ask “*What is the average degree of isometry for states with specified graph structure?*”. The value of such a statement depends crucially on the deviation from the average.

However, as was shown in previous work on random tensor networks [156], the deviation is sufficiently small in a particular limit of bond dimensions. In our context, this limit must be taken on all bond dimension in the superposition. As we will discuss shortly, our construction requires the introduction of an upper cutoff  $J$  on dimensions, which may be arbitrarily large, but in addition we require a lower cutoff  $\mathbf{J}$ . By taking this lower cutoff to be large, all bond dimensions involved in the superposition are large enough to suppress deviations from the average.

In our class of states, then, we can write the average purity as a partition function of a randomised Ising model. To see this, we first average over a distribution of vertex states  $|\Psi_x\rangle = U_x |\Psi_{ref}\rangle$ , where we choose some arbitrary pure reference state  $|\Psi_{ref}\rangle$ . This distribution is chosen to be uniform over the unitary group relating different single-vertex-states<sup>9</sup>. More explicitly, we perform the integral

$$\begin{aligned} & \langle (|\Psi_x\rangle \langle \Psi_x|)^{\otimes 2} \rangle_{U_x} \\ &:= \int_{\mathcal{U}(\mathbb{H}_x)} d\mu(U_x) (U_x |\Psi_{ref}\rangle \langle \Psi_{ref}| (U_x)^\dagger)^{\otimes 2} \\ &=: R_x \left( (|\Psi_x\rangle \langle \Psi_x|)^{\otimes 2} \right) \end{aligned} \quad (2.29)$$

where  $d\mu(U_x)$  is the Haar measure on the unitary group  $U_x$ , for each vertex  $x$  separately; the last line defines the operator  $R_x$  implementing such an average. By linearity this average commutes with taking traces and we denote it by  $\langle - \rangle_U$  in the following.

If all participating spins in the state are sufficiently large, say larger than some lower cutoff  $\mathfrak{J}$ ,<sup>10</sup> we can suppress fluctuations in the quotient

$$\begin{aligned} \langle e^{-S_2(\rho_b)} \rangle_U &= \left\langle \frac{\text{Tr}_{\mathbb{H}^2} [ (|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_b ] }{\text{Tr}_{\mathbb{H}^2} [ (|\phi\rangle \langle \phi|)^{\otimes 2} ]} \right\rangle_U \\ &\approx \frac{\langle \text{Tr}_{\mathbb{H}^2} [ (|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_b ] \rangle_U}{\langle \text{Tr}_{\mathbb{H}^2} [ (|\phi\rangle \langle \phi|)^{\otimes 2} ] \rangle_U} =: \frac{Z_1}{Z_0} \end{aligned} \quad (2.30)$$

as has been shown in random tensor networks - the measure concentrates over the average if all spins are large.

A short note on this regime is in order. Operationally, we are simply making a statement about a class of states which, in Peter-Weyl decomposition, consist only of certain representations. Physically, we may inspect this (still relatively large) class of states from several angles.

For one, if no superposition is made and we deal with a single spin-network state, (LQG) area operators associated with individual links take sharp values. This is already not the case once we superpose. In this large- $\mathfrak{J}$  regime, the relative spacing in the spectrum of the area operator on any given link becomes arbitrarily small.

However, this phenomenon already occurs at lower spins for *sums* of area operators of different links, which are used for larger surfaces. Furthermore, while for a single sector the area values may be sharp, no sharpness is present ab initio for other quantities such as length operators or angles (spin networks are not eigenstates of those operators). Second, the states have a fixed entanglement pattern that is in no way erased by choosing large

<sup>9</sup>The choice of uniform probability distribution is of course not the only possible one. At this stage, interpretative viewpoints as well as dynamical considerations can play an important role and suggest different choices. For example, in [149, 151], this is where the group field theory dynamics of quantum geometry is inserted.

<sup>10</sup>What is required is that the lower cutoff scales in the number of vertices of the graph, as in  $\mathfrak{J} \gg N^k$  for some  $k > \frac{2}{\Delta E}$ , with  $\Delta E$  the spectral gap of the dual Ising model (see below); in fact, we need  $N^2 \ll \mathfrak{J}^{\ln(2\mathfrak{J}+1)}$ . This in turn implies  $N > 15$  for the argument to make sense.

representation labels from the beginning. These quantum information properties are our focus, so the area values are of secondary importance.

The operator  $R_x$  acting on two copies of the single-vertex Hilbert space has the property that it is invariant under unitary conjugation:

$$V^{\otimes 2} R (V^\dagger)^{\otimes 2} = R \quad (2.31)$$

by left-invariance of the Haar measure.

Crucially, this requires the group to be a finite-dimensional Lie group (this integral does not exist on the infinite unitary groups, so our Hilbert spaces must stay finite dimensional). Thus, we must require all spins of the state to be below some (arbitrarily large) upper cutoff  $J$ . *Therefore, all we can consider in our framework are subsets of the set of spin tensor network states which have only finitely many, sufficiently large spins in their superposition.* As stated before, one can see this as a technical requirement or as coming from the logic of the Turaev-Viro cutoff on spins due to the cosmological constant, with  $J \sim \frac{1}{\Lambda}$ . In this picture, the cutoff is naturally large for small cosmological constant.

With this property, we can easily find what  $R_x$  is - the only two operators invariant under this action are the identity and the swap operator, and are combined in the form

$$R_x = \frac{1}{\dim(\mathbb{H}_x)(\dim(\mathbb{H}_x) + 1)} (\mathbb{I}_x + \mathcal{S}_x). \quad (2.32)$$

However, the dimensions here of course need to be the ones of the truncated Hilbert spaces, as otherwise the right hand side would vanish. Since we average over each vertex separately, we really replace the initial random vertex states by

$$\frac{1}{\prod_x \dim(\mathbb{H}_x)(\dim(\mathbb{H}_x) + 1)} \bigotimes_x (\mathbb{I}_x + \mathcal{S}_x). \quad (2.33)$$

Then comes a crucial rewriting. To make working with the tensor product above tractable, we recognise that, when expanded as a sum, each term will have a number of swap operators, and identity operators do not matter. Each term can then be labeled by the set of vertices with swap operators on it, a  $-1$  indicating a swap.

The method introduced by Hayden *et al.* [156] assigns to each vertex a  $\pm 1$ -valued *Ising spin*  $\sigma_x$ , which indicates whether a swap is on that vertex or not. This means the product turns into the sum over Ising configurations

$$\prod_x \frac{1}{\dim(\mathbb{H}_x)(\dim(\mathbb{H}_x) + 1)} \sum_{\vec{\sigma}} \bigotimes_x \mathcal{S}_x^{\frac{1-\sigma_x}{2}} \quad (2.34)$$

To explain further this step, each term in the original sum is mapped to a unique Ising configuration such that the region of swap operators is the region of Ising spin-downs. Then,

every configuration must be summed over. This turns the numerator and denominator of the average purity into *Ising partition functions*:

$$Z_{1|0} = \sum_{\vec{\sigma}} \text{Tr}[\Pi^{\otimes 2} \bigotimes_x (\mathcal{S}_x^{\frac{1-\sigma_x}{2}}) \mathcal{S}_b^{1|0}] = \sum_{\vec{\sigma}} e^{-H_{1|0}(\vec{\sigma})} \quad (2.35)$$

and evaluation of the average purity is turned into a calculation of Ising-like partition sums. The projector  $\Pi$  will depend on the application. In the case of large bond dimensions, we can approximate the sums by their ground state values as the lowest bond dimension plays the role of inverse temperature. The result is that achieving minimal purity of the reduced state, corresponding to having an isometry, depends on the size of the local input and output legs, along with the underlying graph structure.

## 2.2.2 Rewriting the Hamiltonian

To calculate the partition functions, we need to find a usable expression for the Ising Hamiltonian. This is straightforward when the Hilbert space factorises over vertices or links, but in the case of superposed spin sectors, this is less immediate. Because the Hilbert space does not factorise, we first have to split the trace into a sum over the spin sectors, in which we can then easily determine the Hamiltonian.

More precisely, the total Hilbert space is the direct sum

$$\mathbb{H} = \bigotimes_x \mathbb{H}_x = \bigoplus_{\vec{j}} \mathbb{H}_{\vec{j}}.$$

We can thus decompose the trace over  $\mathbb{H}^{\otimes 2}$  into a sum over the spin sectors  $\mathbb{H}_{\vec{j}}$ , and rewrite the trace as follows:

$$\begin{aligned} Z_{1|0} &= \sum_{\vec{\sigma}} \text{Tr}[\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_x^{\frac{1-\sigma_x}{2}} (\mathcal{S}_b)^{1|0}] \\ &= \sum_{(\vec{j}, \vec{k}, \vec{\sigma})} \text{Tr}_{\mathbb{H}_{\vec{j}} \otimes \mathbb{H}_{\vec{k}}} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_x^{\frac{1-\sigma_x}{2}} (\mathcal{S}_b)^{1|0}]. \end{aligned} \quad (2.36)$$

The individual summands depend, through the choice of the Hilbert space traced over, on the spin sets  $\vec{j}, \vec{k}$ . The traces in each term are now over spaces that factorise over vertices and links, and accordingly the single-vertex swap operators do so, too:  $\mathcal{S}_x = \mathcal{S}_{\mathcal{I},x} \prod_{\alpha} \mathcal{S}_{\alpha,x}$ . The traces can then be evaluated over intertwiner, link and boundary parts separately. Then, the general form of decomposition we are looking for is as follows:

$$\text{Tr}_{j \times k} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_x^{\frac{1-\sigma_x}{2}} \dots] = \Delta(\vec{j}, \vec{k}, \vec{\sigma}) K_{\vec{j}} K_{\vec{k}} e^{-H(\vec{j}, \vec{k}, \vec{\sigma})}. \quad (2.37)$$

The three factors are non-unique, but fulfil specific functions:

- The  $\Delta$ -factor is boolean and indicates whether a term vanishes - depending on the combination of spin sectors and Ising configuration, the term might be zero. Constraints arising from this are to be incorporated here.

- The  $K$ -factor absorbs large contributions to the trace that depend only on the bond dimensions given through the spin sectors. They function as a normalising factor and will drop out of the calculation if one considers only a single factor. We can generally expect something of the form  $K_{\vec{j}} = \text{Tr}_{\mathbb{H}_{\vec{j}}}[\Pi]$ .
- The Hamiltonian  $H$  is the main quantity of interest and contains all dependence on the Ising configuration. It also depends, in a normalised way, on the spins of the spin sectors  $\vec{j}, \vec{k}$ . The function is designed such that it is nonvanishing only where the  $\Delta$ -constraints are satisfied. This means there is no ambiguity which couplings the Ising model is subject to.

Typically, in the following the decomposition for  $Z_0$  will be chosen such that the Hamiltonian  $H_0$  satisfies  $H_0(\vec{j}, \vec{k}, +1) = 0$ , and the resulting choice of  $K_{\vec{j}}$  will be applied for  $Z_1$  as well.

For example, for the case of bulk-to-boundary mappings, discussed in 2.3, we can find the following expressions, with details relegated to the appendix B.2.1:

$$\Delta_{1|0}(\vec{j}, \vec{k}; \vec{\sigma}) = \prod_x \delta_{\vec{j}^x, \vec{k}^x} \prod_{e \in \gamma} \delta_{j_e, k_e}^{\frac{1 - \sigma_{s(e)} \sigma_{t(e)}}{2}} \quad (2.38)$$

$$K_{\vec{j}} = \prod_x \mathcal{D}_{\vec{j}^x} \prod_{e \in \gamma} |g_{j_e}|^2 \prod_{e \in \partial \gamma} d_{j_e} = \text{Tr}_{\mathbb{H}_{\vec{j}}}[\Pi_{\Gamma, j_b}] \quad (2.39)$$

$$H_{1|0}(\vec{j}, \vec{k}, \vec{\sigma}) = \sum_{e \in \gamma} \lambda_e \frac{1 - \sigma_{s(e)} \sigma_{t(e)}}{2} + \sum_x \frac{1 - b\sigma_x}{2} \Lambda_x \quad (2.40)$$

with couplings  $\lambda_e = \log(d_{j_e})$ ,  $\Lambda_x = \log(\mathcal{D}_{\vec{j}^x})$ .

Let us also define, for reference, the quantities

$$Z_{1|0}^{(\vec{j}, \vec{k})} = \sum_{\vec{\sigma}} \Delta_{1|0}(\vec{j}, \vec{k}, \vec{\sigma}) e^{-H_{1|0}(\vec{j}, \vec{k}, \vec{\sigma})} \quad (2.41)$$

which enable us to phrase the discussion of the partition functions nicely. By defining the normalised distribution over spin sectors

$$P(\vec{j}, \vec{k}) = \frac{K_{\vec{j}} K_{\vec{k}}}{Z_0} Z_0^{(\vec{j}, \vec{k})} \quad (2.42)$$

we see our quantity of interest as a probability average over Ising models

$$\begin{aligned} \langle e^{-S_2(\rho_b)} \rangle_U &\approx \frac{Z_1}{Z_0} \\ &= \sum_{(\vec{j}, \vec{k})} P(\vec{j}, \vec{k}) \sum_{\vec{\sigma}} \frac{e^{-H_1(\vec{j}, \vec{k}, \vec{\sigma})}}{Z_0^{(\vec{j}, \vec{k})}} \Delta_1(\vec{j}, \vec{k}, \vec{\sigma}). \end{aligned} \quad (2.43)$$

If all spins in a given sector are large enough, we can perform a crucial approximation to



the partition sums. In the Ising model, we may approximate the partition function by its ground state contribution if the excited states have very low weight. This is the case if the couplings of the model (so the bond dimensions) are very large, as any spin flip will increase the energy by an amount proportional to that coupling constant. When the spins are all large, we have that

$$Z_0^{(\vec{j}, \vec{k})} \approx 1, \quad Z_1^{(\vec{j}, \vec{k})} \approx \exp(-H_1(\vec{j}, \vec{k}, \vec{\sigma}_{GS})) \quad (2.44)$$

where  $\vec{\sigma}_{GS}$  is the ground state configuration; this approximation massively simplifies the distribution  $P$  as well:

$$\begin{aligned} Z_0 &= \sum_{\vec{j}, \vec{k}} K_{\vec{j}} K_{\vec{k}} Z_0^{(\vec{j}, \vec{k})} \approx \left( \sum_{\vec{j}} K_{\vec{j}} \right)^2 \\ P(\vec{j}, \vec{k}) &\approx p_{\vec{j}} p_{\vec{k}} \quad p_{\vec{j}} = \frac{K_{\vec{j}}}{\sum_{\vec{k}} K_{\vec{j}}} \end{aligned} \quad (2.45)$$

In particular, given that  $Z_0 = \langle \text{Tr}[\rho]^2 \rangle_U$ , we can interpret the factorisation of the partition function as the statement  $\langle \text{Tr}[\rho]^2 \rangle_U = \langle \text{Tr}[\rho] \rangle_U^2$  in the high-spin regime. This, in turn, simply reflects that in the high-spin regime,  $\rho$  is on average a pure state, which is to be expected since we only work with such from the outset.

We can also derive general necessary conditions for the purity to be minimal. To be more precise, consider that the condition on purity and therefore the isometry condition may be written as

$$\langle p, Mp \rangle = \sum_{\vec{j}, \vec{k}} p_{\vec{j}} M_{\vec{j}\vec{k}} p_{\vec{k}} = 0 \quad M_{\vec{j}\vec{k}} = Z_1^{(\vec{j}, \vec{k})} - \frac{1}{\mathcal{D}_b} \quad (2.46)$$

where  $\mathcal{D}_b$  is the total bulk dimension. The special form of the matrix  $M$  allows us to get an idea of when this is the case. We may sketch the argument already at this stage, without referring to specific situations of interest. We can calculate the determinant using the matrix  $Z_1$  with entries  $Z_1^{(\vec{j}, \vec{k})}$ :

$$\begin{aligned} \det(M) &= \det\left(Z_1 - \frac{1}{\mathcal{D}_b} \vec{1}^T \otimes \vec{1}\right) \\ &= \det(Z_1) \left(1 - \frac{1}{\mathcal{D}_b} \vec{1}^T (Z_1)^{-1} \vec{1}\right) \\ &= \det(Z_1) \left(1 - \frac{1}{\mathcal{D}_b} \sum_{\vec{j}, \vec{k}} (Z_1)^{-1}_{\vec{j}\vec{k}}\right) \end{aligned} \quad (2.47)$$

So assuming  $\det(Z_1) \neq 0$ , we write it as a diagonal, invertible part  $W$  with entries  $W^{\vec{j}} = Z_1^{\vec{j}\vec{j}}$  plus a matrix with empty diagonal. Then, factoring out the diagonal we obtain the form  $Z_1 = W(\mathbb{I} + \alpha)$ . We can then solve for the inverse of  $Z_1$  and expand the above expression.

At first order,  $\det(\mathbb{I} + \alpha) \approx e^{\text{Tr}(\alpha)} = 1$  and  $(Z_1)^{-1} \approx (W)^{-1}(\mathbb{I} - \alpha)$ , meaning the above is

$$\begin{aligned} \det(M) &= \det(W) \left(1 - \frac{1}{\mathcal{D}_b} \sum_{\vec{j}, \vec{k}} (W^{\vec{j}})^{-1} (\delta_{\vec{j}, \vec{k}} - \alpha_{\vec{j}\vec{k}})\right) \\ &= \det(W) \left(1 - \frac{1}{\mathcal{D}_b} \sum_{\vec{j}} (W^{\vec{j}})^{-1}\right) \\ &\quad + \frac{\det(W)}{\mathcal{D}_b} \sum_{\vec{j}, \vec{k}} (W^{\vec{j}})^{-1} \alpha_{\vec{j}\vec{k}} \end{aligned} \quad (2.48)$$

Then, the necessary condition at zeroth order in  $\alpha$  is  $\sum_{\vec{j}} (W^{\vec{j}})^{-1} = \mathcal{D}_b$ . If, schematically,  $\mathcal{D}_b = \sum_{\vec{j}} \mathcal{D}_{\vec{j}}$ , then  $W^{\vec{j}} = \frac{1}{\mathcal{D}_{\vec{j}}}$  fulfils the condition and the vector  $p_{\vec{j}} = \frac{\mathcal{D}_{\vec{j}}}{\mathcal{D}_b}$  is a solution to Eq. (2.46). Therefore, for sector-diagonal  $Z_1$ , we know a necessary and sufficient criterion on the elements of the partition sum to fulfil the isometry condition:

$$p_{\vec{j}} = \frac{D_{\vec{j}}}{D} \stackrel{!}{=} \frac{\mathcal{D}_{\vec{j}}}{\mathcal{D}_b} \quad (2.49)$$

which requires that the *boundary dimensions*  $D_{\partial_{\vec{j}}}$  are constant across all sectors,  $D_{\partial_{\vec{j}}} \equiv D_{\partial}$  in order to factor out of the fraction. We will see this condition again later, as it persists to the general case.

It needs to be stressed that the nature of the averages  $\langle - \rangle_U$  and  $\langle - \rangle_P$  is quite different. The former is, effectively, a product of (uniform) averages over the Hilbert spaces  $\mathbb{H}_x$  and a classical one, in that whatever quantity  $\hat{X}$ <sup>11</sup> we compute in it (i.e. any average expectation value) corresponds to an ensemble average of complex numbers

$$\begin{aligned} \langle \text{Tr}[\hat{\rho} \hat{X}] \rangle_U &= \sum_{\psi} P_{\psi} X_{\psi} \\ X_{\psi} &= \langle \psi | \hat{X} | \psi \rangle = \text{Tr}[|\psi\rangle \langle \psi| \hat{X}] \end{aligned} \quad (2.50)$$

each of which is a quantum expectation value, and the probability  $P_{\psi}$  is the uniform distribution. As far as the meaning of the average  $\langle - \rangle_P$  is concerned, first note that we can write  $p$  as

$$p_{\vec{j}} = \frac{K_{\vec{j}}}{\sum_n K_{\vec{n}}} \quad K_{\vec{j}} = \text{Tr}_{\mathbb{H}_{\vec{j}}}[\hat{\Pi}] = \dim(\mathbb{H}_{\vec{j}}) \quad (2.51)$$

with the operator  $\hat{\Pi}$  that brings us from the full Hilbert space to the constrained one. For example, in the case of constraining onto states with definite graph pattern,  $\hat{\Pi}$  works as a projector that conditions the quantum probabilities on the given graph pattern. Then,  $p_{\vec{j}}$  can be understood as a kind of combinatorial probability for a given, uniform randomly

---

<sup>11</sup>For clarity of exposition we use hats to denote operators in this argument to distinguish them from c-numbers.

chosen vector  $|\psi\rangle \in \hat{\Pi}(\mathbb{H})$  to be in the subsector  $\mathbb{H}_{\vec{j}}$ : the larger the dimension of the sector, the larger the chance for a 1-dimensional subspace to be a part of it. The interpretation of  $\langle - \rangle_P$  is then the following. Given some operator  $\hat{X}$  on the system, we can write its average in the same way as above,

$$\begin{aligned} \langle \langle \hat{X} \rangle_\rho \rangle_U &= \frac{Z_1}{Z_0} \\ Z_1 &= \langle \text{Tr}_{\hat{\Pi}(\mathbb{H})}[\hat{\rho}\hat{X}] \rangle_U = \sum_{\vec{j}} \text{Tr}_{\mathbb{H}_{\vec{j}}}[\hat{\Pi}\hat{X}], \end{aligned} \quad (2.52)$$

with the *same*  $Z_0$  as above. Then approximately, this can again be written as

$$\begin{aligned} \langle \langle \hat{X} \rangle_\rho \rangle_U &= \sum_{\vec{j}} p_{\vec{j}} X_{\vec{j}} = \langle X \rangle_p \\ X_{\vec{j}} &= \frac{\text{Tr}_{\mathbb{H}_{\vec{j}}}[\hat{\Pi}\hat{X}]}{\text{Tr}_{\mathbb{H}_{\vec{j}}}[\hat{\Pi}]} = \langle \langle \hat{X} \rangle_{\rho_{\vec{j}}} \rangle_{U_{\vec{j}}} \end{aligned} \quad (2.53)$$

so to get the *average expectation value of  $\hat{X}$  in a pure state*, we take the probability of the pure state to be in sector  $\vec{j}$  and weigh by the average expectation value in that sector. For this interpretation, we see the object  $X$  as a random variable on the space of events given by the sectors  $\vec{j}$ , with values  $X_{\vec{j}}$ . To any quantum operator  $\hat{X}$ , we can associate such a classical random variable  $X$ .

The extension to operators on two copies of the system brings with it the modification that the weights  $X_{\vec{j},\vec{k}}$  are now dependent on a ground-state configuration  $\sigma_{GS}$  of the Ising model, but otherwise the interpretation is the same. We can therefore see that the average  $\langle - \rangle_\rho$  is quantum,  $\langle - \rangle_U$  is classical and statistical and  $\langle - \rangle_P$  is classical and combinatorial.

## Entropies

In the high-bond-dimension regime, we may make general statements about the behaviour of entropies, in particular of a bulk subregion  $R \subseteq \gamma$ . For this, we would reduce the full pure state of the graph to a subregion, and then in the Ising model calculate using a swap operator on each site of  $R$ . Assume that the Ising sums in Eq. (2.41) have been calculated, and denote their approximate value by

$$Z_1^{(\vec{j},\vec{k})} = e^{-X_{\vec{j},\vec{k}}}. \quad (2.54)$$

Then, the Rényi purity  $e^{-S_2(R)}$  takes the form of an expectation value in the probability density vectors  $P = p \otimes p$  over pairs of spin sectors:

$$\langle e^{-S_2} \rangle_U \approx \sum_{\vec{j},\vec{k}} p_{\vec{j}} p_{\vec{k}} e^{-X_{\vec{j},\vec{k}}} =: \langle e^{-X} \rangle_P. \quad (2.55)$$

We are therefore able to use a cumulant expansion for  $X$  and write

$$\langle e^{-X} \rangle_P = \exp \left( - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \kappa_n(X) \right) \quad (2.56)$$

with cumulants of the random variable  $X$ ,

$$\begin{aligned} \kappa_1(X) &= \langle X \rangle_P \\ \kappa_2(X) &= \langle X^2 \rangle_P - \langle X \rangle_P^2 \quad \text{etc.} \end{aligned} \quad (2.57)$$

Then, quite generally,

$$\begin{aligned} \langle S_2(\rho_R) \rangle_U &\approx \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \kappa_n(X) \\ &= \langle X \rangle_P - \frac{1}{2} (\langle X^2 \rangle_P - \langle X \rangle_P^2) + \dots \end{aligned} \quad (2.58)$$

which shows that the overall entropy will not be ‘sharp’ in the sectors, but be an average of the quantity  $X$  that depends on the contributing sectors nontrivially. In the particular case that the partition sums evaluate, individually, to an ‘area’ of a certain surface  $S_{\vec{j}, \vec{k}}$  which bounds a bulk region  $\Sigma_{\vec{j}, \vec{k}}$  in the graph, such as

$$X_{\vec{j}, \vec{k}} = \sum_{e \in S_{\vec{j}, \vec{k}}} \log(d_{j_e}) =: \frac{1}{4} A_{\vec{j}, \vec{k}} \quad (2.59)$$

then this gives a Ryu-Takayanagi-type formula for the entropy which takes the  $P$ -expectation value of the area operators  $\hat{A}_{\Sigma_{\vec{j}, \vec{k}}}$  associated to the set of minimal surfaces  $\Sigma_{\vec{j}, \vec{k}}$ :

$$\langle S_2(\rho_R) \rangle_U \approx \frac{1}{4} \langle A \rangle_P + \frac{1}{2 \cdot 4^2} (\langle A^2 \rangle_P - \langle A \rangle_P^2) + \dots \quad (2.60)$$

where remaining terms capture higher cumulants of  $A_{\Sigma}$ .

In the approximating case that the surfaces  $S_{\vec{j}, \vec{k}}$  are all the boundary of  $R$ , we then recover the area law of usual tensor networks, with the change that we take the average of the boundary area  $A_{\partial R}$  in the combinatorial distribution  $\langle - \rangle_P$ .

If we want to be even more proper, though, we need to think carefully about how we specify the subregion  $R$ , particularly in a diffeomorphism-invariant way. One typical way to do this in a maximally symmetric background [169] is to fix the boundary area of a spatial slice, which then selects the region  $R$  as the one with maximal volume in spacetime. This then generates a causal diamond spacetime. If we assume that  $R = \gamma$ , and assume the total boundary area is fixed, then the entropy becomes

$$\langle S_2(\rho_R) \rangle_U \approx A_{\partial R} \sum_{\vec{j}, \vec{k}: A_{\vec{j}, \vec{k}} = A_{\partial R}} p_{\vec{j}} p_{\vec{k}} \leq A_{\partial R} \quad (2.61)$$

which shows a typical area bound with a correcting factor that depends on the specifics of the state.

### 2.2.3 The puzzle of areas

A well-known fact about random tensor networks with fixed bond dimensions (equivalent to the fixed-spin case here) is that they feature a Ryu-Takayanagi formula with a *trivial area operator* [159, 170]. This means that the area operator  $\hat{A}_\Sigma$  of the minimal surface  $\Sigma$ , appearing as

$$S_{vN}(\rho_b) \approx \frac{1}{4} \langle \psi | \hat{A}_\Sigma | \psi \rangle, \quad (2.62)$$

is proportional to the identity operator on the graph Hilbert space,

$$\hat{A}_S = 4 \left( \sum_{e \in S} \log(d_{j_e}) \right) \mathbb{I}_{\mathbb{H}_\gamma}. \quad (2.63)$$

This is also the case in our setting, because for just a single sector, there is only one possible value for the spin on each link. The ‘tensor network area’  $a_{e,TN}(j_e) := \log(d_{j_e})$  is then just a function of those c-number labels on the Hilbert space. This changes with multiple sectors as seen above: we have instead that the area operator is evaluated on a *set* of minimal surfaces  $\Sigma_{\vec{j},\vec{k}}$ , and we can therefore write

$$\begin{aligned} \langle S_2(\rho_b) \rangle_U &\approx \frac{1}{4} \sum_{\vec{j},\vec{k}} P(\vec{j},\vec{k}) \langle \psi | \hat{A}_{\Sigma_{\vec{j},\vec{k}}} | \psi \rangle \\ \hat{A}_S &:= \sum_{e \in S} \hat{A}_e := \sum_{e \in S} \sum_{j_e} 4 \log(d_{j_e}) \mathbb{I}_{e,j_e}. \end{aligned} \quad (2.64)$$

The area operator on the right is not simply a multiple of the identity because it assigns different values to a surface  $S$  depending on the state. In this sense, our area operator is nontrivial in a very similar sense to that of recent studies [159, 160]. What is distinct is the possibility of multiple, distinct minimal surfaces which contribute to the entropy. However, this can be argued to be natural: if different sectors correspond to different sets of states with different metrics for spatial slices of a spacetime, then the condition of being the surface of minimal area depends on the metric in question, or more simply, on the sector. Therefore, to obtain the entropy, one does not evaluate the area operator on a single given surface in the bulk, but a number of potentially different minimal surfaces determined by the state, and average their areas according to  $P$ .

We note also that, while the area operator here is of the same form as the LQG area operator (it is block diagonal over the spin sectors)

$$\hat{A}_S^{\text{LQG}} := \sum_{e \in S} \hat{A}_e^{\text{LQG}} := \sum_{e \in S} \sum_{j_e} \sqrt{j_e(j_e + 1)} \mathbb{I}_{e,j_e}, \quad (2.65)$$

it differs in the area values themselves:  $a_{e,\text{LQG}}(s_e) = \sqrt{s_e(s_e + 1)}$  does not even match in terms of scaling. Therefore, we can not naively identify the Ryu-Takayanagi graph-area operator above with the discrete geometric LQG area operator. If we want to do such an

identification, there must be a difference between the ‘graph spins’  $j_e$  in our construction and the ‘LQG spins’  $s_e$  such that the two match

$$\sqrt{s_e(s_e + 1)} \approx \log(2j_e + 1) \quad (2.66)$$

at high values of the graph spins  $j_e$ . An identification like this is routine in tensor network models of holography - a priori, to interpret the logarithms of bond dimensions as areas, a relation between the two must be stipulated. This can for example be done by embedding the tensor network graph in an ambient metric space and matching bond dimensions to the areas of dual surfaces in the space. Here, we can instead stipulate the matching through the microscopic LQG area operator and even give it a preliminary interpretation. We also note that this is independent from the question of superpositions and already features in the same way on a single link with fixed spin.

This sort of matching makes sense if we see  $d_{j_e}$  as an ‘effective bond dimension’ of a system which subsumes many coarse-grained, or rather ‘reshuffled’ degrees of freedom, whereas the  $s_e$  describe the ‘microscopic’ geometry in terms of LQG’s areas. The area operator appearing on the right hand side of the Ryu-Takayanagi formula should then be understood as a ‘reshuffled’ one. In fact, from this point of view it seems most plausible to interpret the graph states themselves as a sort of ‘reshuffled’ spin networks. These kinds of reorganisations would not strictly be coarse-grainings in the usual sense of the word as the full dimensionality of the space of states is preserved. Instead, it may perhaps correspond to a type of ‘exact’ renormalisation procedure.

A particular example of this is easy to construct: Suppose that we see a link space  $\mathbb{H}_e$  as the space of states of a surface  $S$  dual to  $e$  as before, but now with a different substructure: Rather than being an indivisible quantum of area<sup>12</sup>, we see this surface as a conglomerate or *ensemble of  $M > 0$  indivisible areas*. Each of the quantal surfaces has its own area operator  $\hat{A}_{LQG}$  and spectrum labeled by the LQG area spins on a Hilbert space  $\mathbb{H}_e^{LQG}$ . Correspondingly, the conglomerate surface  $S$  instead will have the total area operator

$$\hat{A}_S := \sum_{i=1}^M \hat{A}_{i,LQG}. \quad (2.67)$$

If we now want to model only the macrostate of the surface  $S$  with a fixed area  $A$ , we can do so with the link Hilbert space of structure

$$\mathbb{H}_A = \bigoplus_A \mathbb{H}_{A,A} \quad \mathbb{H}_{S,A} := \bigoplus_{\sum_i a_{i,LQG}=A} \mathbb{H}_{a_i,i}^{LQG} \quad (2.68)$$

So the Hilbert space of  $S$  at fixed total area  $A$  consists of those of all constituent surfaces whose areas sum up to  $A$ . From counting arguments, it is then possible to show that (see section 3 of [172], and references therein)

$$\dim(\mathbb{H}_{e,A}) \sim e^A \quad (2.69)$$

---

<sup>12</sup>An *embadon*, as recently coined in [171].

which readily allows us to resolve the puzzle of areas: If we use the spaces  $\mathbb{H}_e$  as our link spaces in the tensor network construction, and in turn the bond dimensions  $d_j$  are now  $\dim(\mathbb{H}_{e,A})$  throughout, then clearly

$$a_{e,TN}(A_e) := \log(\dim(\mathbb{H}_{e,A_e})) \sim A_e \triangleq \sum_{i=1}^M A_{i,LQG} \quad (2.70)$$

so the tensor network area is the total area of the conglomerate surface, *as measured by the total LQG area operator*. Therefore, the puzzle is resolved cleanly if we think of the tensor networks we use as coarse grained states of geometry whose microscopic constituent quantal areas we do not resolve exactly.

We note that this is precisely the logic of ‘quantization’ in the original sense of Max Planck; In his case, the (continuous-looking) radiation of a black body needed to be a conglomerate of indivisible quanta of light (which in his case had a continuous spectrum of energy each, while here we have a discrete spectrum). This makes for a compelling picture of tensor network holography being a bridge between the continuum, semiclassical description of holography and the microscopic, but formally similar discrete language of spin networks.

Alternatively, we can try to think of this discrepancy in a totally different light, inspired by [56]. Recall that we chose our link states 2.7 with some arbitrary weights  $g_j$ . The tensor network area is a measure of entanglement across the link, and so the heuristic area law should really read

$$S_{vN}(\rho_R) = \sum_{e \in \partial R} S_{vN}(\rho_{e,+}) \quad \rho_{e,+} = \text{Tr}_{e,-}[|e\rangle\langle e|] \quad (2.71)$$

where  $\pm$  denote the two semilinks in a link.  $S_2(\rho_{e,+})$  therefore measures the entanglement across a single link of the graph. We can then think of the matching of area notions as a way to fix the weights  $g_j$ . We have that

$$\rho_{e,+} = \sum_s |g_s|^2 \rho_s \quad \rho_s := \text{Tr}_- [|e, s\rangle\langle e, s|] \quad (2.72)$$

As usual, the von Neumann entropy splits and we have

$$S_{vN}(\rho_{e,+}) = H(|g_s|^2) + \sum_s |g_s|^2 \log(d_s) \quad (2.73)$$

We can then think of different distributions  $w_s = |g_s|^2$  and look for one that matches the area notions. This is the case when the reduced state is to a first approximation the ‘thermal density matrix’ (with  $\kappa = 8\pi G_N$ )

$$\rho_R = \frac{1}{\text{Tr}[e^{-\frac{1}{2\kappa}\hat{A}_{\partial R}^{LQG}}]} e^{-\frac{1}{2\kappa}\hat{A}_{\partial R}^{LQG}} \quad (2.74)$$

using the block diagonality of  $\hat{A}^{LQG}$ , we can write this over a single link as the requirement

$$\rho_{e,+} = \frac{1}{\text{Tr}[e^{-\frac{1}{2\kappa}\hat{A}_e^{LQG}}]} e^{-\frac{1}{2\kappa}\hat{A}_e^{LQG}} = \frac{1}{\text{Tr}[e^{-\frac{1}{2\kappa}\hat{A}^{LQG}}]} \sum_s e^{-\frac{1}{2\kappa}a_{LQG}(s)} \mathbb{I}_s \quad (2.75)$$

By taking  $\mathbb{I}_s = d_s \rho_s$ , we convert this into

$$\rho_{e,+} = \sum_s \frac{d_s e^{-\frac{1}{2\kappa} a_{LQG}(s)}}{\text{Tr}[e^{-\frac{1}{2\kappa} \hat{A}_{LQG}}]} \rho_s \implies w_s = \frac{d_s e^{-\frac{1}{2\kappa} a_{LQG}(s)}}{\sum_{s'} d_{s'} e^{-\frac{1}{2\kappa} a_{LQG}(s)}} = \frac{1}{Z} d_s e^{-\frac{1}{2\kappa} a_{LQG}(s)} \quad (2.76)$$

So the requirement of the areas matching indeed does provide us with a set of coefficients. We calculate

$$\begin{aligned} S_{vN}(\rho_{e,+}) &= \frac{1}{Z} \sum_s d_s e^{-\frac{1}{2\kappa} a_{LQG}(s)} (\log(d_s) - \log(w_s)) \\ &= \log(Z) + \frac{1}{2\kappa} \langle a_{LQG} \rangle_w \end{aligned} \quad (2.77)$$

which gives the area law as expected up to a finite constant<sup>13</sup>. More importantly, though, this shows that there is merit to thinking about the issue in terms of link states: If we equip the graph with the link weights as above, then a generic bulk region, according to the tensor network area law, will have its entropy be proportional to the sum over link entropies. *In this way, if we use the link state*<sup>14</sup>

$$|e\rangle = \sum_s \frac{1}{\sqrt{Z}} e^{-\frac{1}{2\kappa} a_{LQG}(s)} |\tilde{e}, s\rangle \quad (2.78)$$

*we end up with a meaningful way in which the link entanglement, given by the tensor network area, is given by the geometric (LQG) area, and so to first approximation the entropy of a region is also given by the geometric area.*

It is also striking that the kind of state we end up with from this reasoning bears close resemblance to a thermofield double state where the role of the Hamiltonian is played by the geometric area, at fixed temperature  $\beta = \frac{2}{\kappa}$ .<sup>15</sup> This links this idea not only back to the previous alternative with the conglomerate surface, whose large scale statistics is well-described by such a canonical ensemble, but *also* to the literature on black hole and spacetime thermodynamics[173]. In that context, local regions in semiclassical spacetimes are shown to have properties of a thermal state of the fields. Here, we have an interpretation of such a thermal ensemble as one of microstates of quanta of area. The pictures we presented here are therefore compatible.

Finally, there is an independent motivation for the state  $|e\rangle$  as above: While we were working with  $SU(2)$  spin networks, there are in principle more symmetries and matching constraints on the data of general relativity. It has been argued [174] that the proper imposition of diffeomorphism constraints (akin to local translation invariance) forces a specific weight  $w_s$  in the link states. The specific form would need to involve the generator of spacetime boosts  $K$  on the surface dual to the link  $e$ , which in the LQG context is equal to the geometric area  $A$  (due to the simplicity constraints). Therefore, we may argue that

<sup>13</sup>In this particular example, it is given by  $\frac{1}{4} - 2 \log(\sqrt[3]{e} - 1) \approx 4.28258$ . The average area is  $\frac{1}{\sqrt[3]{e}-1} \approx 7.51041$  and so the total entropy is  $S_{vN}(\rho_{e,+}) \approx 6.16018$ .

<sup>14</sup>We redefined  $|\tilde{e}, s\rangle := \sqrt{d_s} |e, s\rangle$

<sup>15</sup>Alternatively, if we restore the angular Unruh temperature  $\beta_U = \frac{\hbar}{2\pi}$ , we have  $\frac{1}{\kappa} A = \frac{\hbar}{2\pi} \frac{A}{4G_N} = \beta_U S_{Bk-Hk}$ , with the Bekenstein-Hawking entropy seen by a Rindler observer.



the gravitational dynamics does, in fact, imply a matching of the two notions of area, and that this in turn allows an interpretation of the semiclassical surfaces to be made up of indivisible constituents.

## 2.3 Isometry conditions on superposed spin networks

We now turn our attention back to the conditions we must impose on our setup to get bulk-to-boundary isometry: By first determining some sufficient criteria, we get an idea of the restrictions that are involved in making holography possible. We then generalise the set of necessary criteria and find that they split into two classes:

1. A criterion per boundary-sector  $E$ , so per set of boundary spins in the input and output algebras under consideration
2. A global criterion on the full set of boundary spins, restricting the overall set by a single constraint.

The constraint in question turns out to be that the *total boundary dimension*

$$D_{\partial E} = \prod_{e \in \partial \gamma} d_{j_e} = D_{\partial} \quad \forall E \quad (2.79)$$

is a constant across the output boundary algebra. This means that the algebra may, in fact, only be acting on boundary spin sectors  $E$  which ‘sum’ to a given, nondynamical value, in the sense

$$\mathcal{B}_{\partial} := \bigoplus_{E \in \mathcal{W}_{\partial}} B(\mathbb{H}_{\partial}^E) \quad \mathcal{W}_{\partial} \subseteq \mathcal{W}_{\partial}(D_{\partial}) := \{E : D_{\partial E} = D_{\partial}\} \quad (2.80)$$

This is an interesting restriction for a number of reasons. First, it requires us to fix *a priori* a geometric property of the space we are modeling: Its total boundary area. This is because no matter which notion of area we are using, we end up fixing the total value of it by fixing the boundary dimension. This is a very natural criterion for the context of holographic or spacetime thermodynamic questions, in which usually, as we mentioned already, one fixes a slice through putting diffeomorphism invariant boundary conditions on the boundary of a slice. Therefore, in a given background spacetime, it is sensible to speak of ‘the slice with boundary area  $A$ ’. Here, we have no background, but the boundary area is still fixed in the same way to some finite number, as is required if we want to speak of something like area laws properly.

Second, it introduces a global notion of *scale* to the problem, which turns out to be crucial in making progress on finding necessary criteria. As in the single spin-sector case[153], one can most easily find them in a regime where the bond dimensions are ‘large’, which corresponds to a low energy limit in the dual Ising models. However, ‘low energy’ must be a consistent notion *across* the models with *different couplings* in order to universally

do such an approximation. The boundary area provides a necessary comparison scale in order for this to work.

The limit in which the approximations work, then, is when the bulk areas  $a_e$  on the individual links are all comparatively much smaller than the total boundary area,  $a_e \ll A$ . This speaks for a kind of refinement limit, in which the graph is ‘fine’ enough that it is made of many links of small (compared to the total system size) areas in the bulk. This does *not* conflict with our other limit of having large bond dimensions, as it is perfectly valid to have large bond dimensions  $d_e \sim e^{a_e}$  which are however still very small compared to another, even larger boundary dimension  $D_\partial \sim e^A$ .

Finally, we can also see this restriction as a kind of minimal gauge invariant boundary condition on the gravitational 2-form flux  $\theta_\beta^2$ : The Lorentz charges of tetrad gravity have a factor in the corner charges given by an ‘area density’  $w$ ,

$$\theta_\beta^2 = w (\epsilon_\parallel)_\beta d^2x \quad (2.81)$$

where  $\epsilon_\parallel$  is normalised so that the magnitude of the charge aspect is carried by  $w$ . Its 0th moment is precisely the total area,  $\oint_S w d^2x = A$ . As is typical, the lowest moments of corner charges correspond to bulk degrees of freedom in a very strict way. For comparison, the corresponding charge aspect in Maxwell theory would be the radial electric field  $E^r = \mathbf{q} d^2x$ , whose 0th moment gives the total electric charge of the region,  $\oint_S \mathbf{q} = Q$ . Therefore, by analogous superselection arguments, we could expect that just like how total charge is superselected, we should see total boundary area as superselected. This is then exactly what the holography criteria require of us, as well. This is further supported by the role the area density plays in corner symmetry algebra analyses, where it is a crucial Casimir element that controls the entire representation theory of the symmetry algebra[94, 175].

The other, sector-wise criteria, can be interpreted similarly to the single-sector case. It requires, roughly speaking, that the all-up Ising spin configuration is the minimum of the Ising model (and thus selects only sectors where this is the case). This, in turn, translates into a statement about the bond dimensions and site dimensions on any local region in the graph, and can be most easily satisfied if all the site dimensions are very small compared to the boundary dimension  $D_\partial$ , but also the bond dimensions surrounding the given site. In geometric terms, we can understand this as the values of possible volumes of polyhedra dual to sites being highly restricted. In turn, this means that given a set of bond dimensions, corresponding to some boundary areas  $a_e$  of the polyhedron, we get an almost unique function for the volume of the polyhedron,

$$\mathcal{V}_x(a_e, i_x) \approx \mathcal{V}_x(a_e) \quad (2.82)$$

so that we can reconstruct geometric properties of the bulk from boundary data only. This is the case for example in 3D gravity, where the usual 3-valent intertwiners are uniquely fixed through their surrounding spins, and therefore the area of a triangle is fixed uniquely through its surrounding edge lengths.

Therefore, geometrically speaking it is quite a natural criterion for reconstructibility. The crucial difference to the single-sector case is now that we *no longer* require this to hold in every subregion of the graph, but instead have a corresponding dimensionality argument for each boundary sector. This is slightly weaker and perhaps still allows for some bulk degrees of freedom (though this needs to be verified more explicitly in future work).

The new requirement is roughly that for a given set of boundary spins, the state  $|\Psi_\gamma\rangle$  is peaked on the geometries that are (almost uniquely) induced by the boundary geometry. This may be a unique one or may come with multiplicities. The lower the multiplicity, the better the reconstruction will be.

We now go into the details of deriving these criteria, and then proceed to check them in two very simple, yet already sufficiently involved, examples. However, their essence boils down to our previous comments.

### Bulk-to-boundary maps: General criteria

In seeking holographic behaviour, we are concerned with the equivalence of a bulk and a boundary space. The setting for us is to consider a fixed graph connectivity  $\gamma$  and a state in the Hilbert space

$$\mathbb{H}_\gamma \cong \bigoplus_E \mathbb{H}_b^E \otimes \mathbb{H}_\partial^E. \quad (2.83)$$

We have a nontrivial center given by sector-diagonal operators  $\sum_E \lambda_E \mathbb{I}_E$ . Our interest will be in determining which choices of connectivity result in quantum channels that are isometric between the spaces  $\bigoplus_E B(\mathbb{H}_b^E)$  and  $\bigoplus_E B(\mathbb{H}_\partial^E)$ . In this bulk-to-boundary mapping case, we will call such a channel, and by proxy the state from which it arises, *holographic*. The average purity of the reduced bulk state in the high spin regime is expressed as

$$\langle e^{-S_2(\rho_b)} \rangle_U = \sum_{\vec{\mathbf{j}}, \vec{\mathbf{k}}, \vec{\sigma}} p_{\vec{\mathbf{j}}} p_{\vec{\mathbf{k}}} \Delta_1(\vec{\mathbf{j}}, \vec{\mathbf{k}}; \vec{\sigma}) e^{-H_1(\vec{\mathbf{j}}, \vec{\sigma})} \quad (2.84)$$

with all quantities defined in detail in appendix B.2.1.

We will now present a sufficient condition on the graph data, and set of input sectors  $\mathcal{W}$ , such that the isometry condition is fulfilled. Unless a sector  $\vec{\mathbf{j}}$  is excluded by having  $\prod_{e \in \gamma} |g_{j_e}|^2 = 0$ ,

1.  $\forall \vec{\mathbf{j}} : \vec{\mathbf{j}}_{\partial\gamma} \in \mathcal{W}, \quad \vec{\sigma} = +1$  is the minimum of  $H_1(\vec{\mathbf{j}}, \vec{\sigma})$ .
2.  $\forall \vec{\mathbf{j}} : \vec{\mathbf{j}}_{\partial\gamma} \in \mathcal{W}, \quad \prod_{e \in \gamma} |g_{j_e}|^2 \prod_{e \in \partial\gamma} d_{j_e} = C$  with  $C$  independent of  $\vec{\mathbf{j}}$ .

That this condition is sufficient can be checked directly.

In general,  $\Delta_1$  allows only terms labeled by  $(\vec{\mathbf{j}}, \vec{\mathbf{k}}, \vec{\sigma})$  such that  $S_\uparrow \subseteq G_{\vec{\mathbf{j}}, \vec{\mathbf{k}}} := \{x \in \gamma | \mathbf{j}^x = \mathbf{k}^x\}$ . In particular, there are no restrictions when the sectors are equal. If  $\vec{\sigma} = +1$  is the ground state of  $H_1(\vec{\mathbf{j}}, \vec{\sigma})$ , then we approximate  $Z_1^{\vec{\mathbf{j}}, \vec{\mathbf{k}}}$  by the term corresponding to it. However, when  $\vec{\mathbf{j}} \neq \vec{\mathbf{k}}$ ,  $\Delta_1 = 0$  for this Ising configuration, implying that  $Z_1^{\vec{\mathbf{j}}, \vec{\mathbf{k}}} = Z_1^{\vec{\mathbf{j}}, \vec{\mathbf{j}}} \delta_{\vec{\mathbf{j}}, \vec{\mathbf{k}}}$ , with

$Z_1^{\vec{j}\vec{j}} = W^{\vec{j}} = \frac{1}{D_{b_{\vec{j}}}}$ . Additionally, the second condition implies for the probability weights  $p_{\vec{j}}$  that

$$\begin{aligned} p_{\vec{j}} &= \frac{D_{b_{\vec{j}}} \prod_{e \in \gamma} |g_{j_e}|^2 \prod_{e \in \partial \gamma} d_{j_e}}{\sum_{\vec{k}} D_{b_{\vec{k}}} \prod_{e \in \gamma} |g_{k_e}|^2 \prod_{e \in \partial \gamma} d_{k_e}} \\ &= \frac{D_{b_{\vec{j}}} C}{\sum_{\vec{k}} D_{b_{\vec{k}}} C} = \frac{D_{b_{\vec{j}}}}{\sum_{\vec{k}} D_{b_{\vec{k}}}} \end{aligned} \quad (2.85)$$

which is, as we discussed before, sufficient to reach minimal purity. This condition is in a sense the obvious one. It requires that all sectors which appear in the class of states must themselves be able to support holographic transport, and also constrains a little the weights by which they are superposed. However, this does not include the case where some sectors are not holographic by themselves, but their superposition is. But in fact, if the choice of data given by the cutoffs, the glueing pattern and the coefficients  $g$  are such that the matrix of partition sums  $Z_1^{\vec{j},\vec{k}}$  is approximately diagonal, then the general argument given before enforces this sufficient set of conditions on the sectors and makes it necessary. Therefore, to find more general sufficient conditions, we need to inquire how diagonal said matrix is.

Let us again use the language of nontrivial centers to investigate this issue. In the following, we denote by  $E$  the set of boundary spin values  $j_{\partial \gamma}$  which are values of the spectrum of the center of the algebra in our Hilbert space. We decompose our unnormalised states according to them:

$$\begin{aligned} |\phi\rangle &= \langle \Gamma | \Psi \rangle, \\ (\rho_{E,E})_I &= \frac{\text{Tr}_{\partial E} [|\phi\rangle \langle \phi|]}{\text{Tr}_E [|\phi\rangle \langle \phi|]}, \quad c_E = \frac{\text{Tr}_E [|\phi\rangle \langle \phi|]}{\text{Tr} [|\phi\rangle \langle \phi|]} \end{aligned} \quad (2.86)$$

We will study the impact of the trace preservation conditions 2.24 on these objects and the subsequent constraints on  $Z_1^{\vec{j},\vec{k}}$ . To begin, we first match the entropy calculations of the state  $\rho_b$  and its sector components in order to derive the right type of replica trick.

Using formula 2.27, we connect the following expressions:

$$\begin{aligned} e^{-S_2((\rho_{E,E})_b)} &= \frac{\text{Tr}_{\mathbb{H}_E^2} [(|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_{b_E}]}{\text{Tr}_{\mathbb{H}_E^2} [(|\phi\rangle \langle \phi|)^{\otimes 2}]} \\ c_E^2 &= \frac{\text{Tr}_{\mathbb{H}_E^2} [(|\phi\rangle \langle \phi|)^{\otimes 2}]}{\text{Tr}_{\mathbb{H}^2} [(|\phi\rangle \langle \phi|)^{\otimes 2}]} \\ e^{-S_2(\rho_b)} &= \frac{\text{Tr}_{\mathbb{H}^2} [(|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_b]}{\text{Tr}_{\mathbb{H}^2} [(|\phi\rangle \langle \phi|)^{\otimes 2}]} \\ &= \sum_{E, \bar{E}} \frac{\text{Tr}_{\mathbb{H}_E \otimes \mathbb{H}_{\bar{E}}} [(|\phi\rangle \langle \phi|)^{\otimes 2} \mathcal{S}_b]}{\text{Tr}_{\mathbb{H}^2} [(|\phi\rangle \langle \phi|)^{\otimes 2}]} \end{aligned} \quad (2.87)$$

which implies that we should use the sector-wise swap operator  $\mathcal{S}_b = \sum_E \mathcal{S}_{b_E}$  for this calculation.<sup>16</sup> While it is diagonal in boundary spins  $E$ , it is not so in the bulk ones. As we will now see, the same property holds for the matrix of partition sums. We can rewrite the purity  $e^{-S_2(\rho_b)}$  in the same Ising-oriented fashion as before,  $\langle e^{-S_2(\rho_b)} \rangle_U \approx \frac{Z_1}{Z_0}$ , but with sums over  $E$ :

$$Z_{1|0} = \sum_{E, \tilde{E}} \text{Tr}_{\mathbb{H}_E \otimes \mathbb{H}_{\tilde{E}}} [\langle (|\phi\rangle \langle \phi|)^{\otimes 2} \rangle_U \mathcal{S}_b^{1|0}], \quad (2.88)$$

which are *diagonal in  $E, \tilde{E}$*  due to the special swap operator here.

We may therefore write  $Z_{1|0} = \sum_E \bar{Z}_{1|0}^{E,E}$ , by defining the ‘boundary-fixed’ partition sums

$$\bar{Z}_{1|0}^{E, \tilde{E}} = \text{Tr}_{\mathbb{H}_E \otimes \mathbb{H}_{\tilde{E}}} [\langle (|\phi\rangle \langle \phi|)^{\otimes 2} \rangle_U \mathcal{S}_b^{1|0}] \quad (2.89)$$

which are related to the previous partition sums by

$$\bar{Z}_{1|0}^{E, \tilde{E}} = \sum_{j_B, k_B} K_{\vec{j}} K_{\vec{k}} Z_{1|0}^{(\vec{j}, \vec{k})} \quad (2.90)$$

where the full spin sectors  $\vec{j} = j_b \cup E$ ,  $\vec{k} = k_b \cup E$  are comprised of the bulk spin sets  $j_b, k_b$  and the boundary spins  $E$ , leading to

$$\langle e^{-S_2(\rho_b)} \rangle_U \approx \frac{Z_1}{Z_0} = \sum_E \left( \frac{\bar{Z}_0^{E,E}}{Z_0} \right) \left( \frac{\bar{Z}_1^{E,E}}{\bar{Z}_0^{E,E}} \right). \quad (2.91)$$

We can furthermore identify, through quick calculation, that

$$\langle e^{-S_2((\rho_{E,E})_b)} \rangle_U = \frac{\bar{Z}_1^{E,E}}{\bar{Z}_0^{E,E}} \quad \langle c_E^2 \rangle_U = \frac{\bar{Z}_0^{E,E}}{Z_0}. \quad (2.92)$$

The only assumption that goes into this is that the calculation of the two Rényi purity is the same if calculated over the full graph Hilbert space or over its individual sectors, which is true before averaging and which is true after averaging iff

$$\langle c_E^2 e^{-S_2((\rho_{E,E})_b)} \rangle_U \approx \langle c_E^2 \rangle_U \langle e^{-S_2((\rho_{E,E})_b)} \rangle_U. \quad (2.93)$$

So a certain type of localisation of the average is required, but we expect it to be naturally realised in the regime of large spins, in direct analogy to the arguments presented by Hayden et al[156]. In the case of a single vertex (see B.3 for an overview), one can verify this by direct but tedious calculation, leading to the result

$$\begin{aligned} \langle c_E^2 e^{-S_2((\rho_{E,E})_b)} \rangle_U &= \frac{D_{b_E} + D_{\partial_E}}{D^2} \left( 1 + \mathcal{O}\left(\frac{1}{D_E}, \frac{1}{D}\right) \right) \\ \langle c_E^2 \rangle_U &= \frac{D_E^2}{D^2} \left( 1 + \mathcal{O}\left(\frac{1}{D_E}, \frac{1}{D}\right) \right) \\ \langle e^{-S_2((\rho_{E,E})_b)} \rangle_U &= \frac{D_{b_E} + D_{\partial_E}}{D_E^2} \left( 1 + \mathcal{O}\left(\frac{1}{D_E}, \frac{1}{D}\right) \right). \end{aligned} \quad (2.94)$$

<sup>16</sup>Recall that 2.27 came about from the presence of a center, which in this case is given by boundary spins. Physical density matrices must then commute with these spin labels and therefore are diagonal with respect to them,  $\rho_{E, \tilde{E}} = \delta_{E, \tilde{E}} \rho_{E, E}$ .

This also directly demonstrates that the high-spin regime needs to apply to *all* sectors  $E$ . We stress that the partition sums need not be diagonal in the bulk spins. It is noteworthy that the average value of  $c_E$  can easily be computed:

$$\begin{aligned}\langle c_E \rangle_U &\approx \frac{\text{Tr}_E[\Pi_\Gamma \langle |\Psi\rangle \langle \Psi| \rangle_U]}{\text{Tr}[\Pi_\Gamma \langle |\Psi\rangle \langle \Psi| \rangle_U]} \\ &= \frac{\text{Tr}_E[\Pi_\Gamma]}{\text{Tr}[\Pi_\Gamma]} = \frac{D_E}{\sum_F D_F} = \sum_{j_b} p_{j_b \cup E}.\end{aligned}\quad (2.95)$$

Matching this with the required value necessitates that  $D_{\partial_E}$  is in fact independent of  $E$ . In our case, this means a restriction to boundary spins  $E = j_{\partial\gamma}$  such that

$$D_{\partial_E} = \prod_{e \in \partial\gamma} d_{j_e} = D_\partial \quad \forall E \quad (2.96)$$

So, we can only use certain sectors on the output side once we fix this value  $D_\partial$ . *In the following, we will assume such a value has been fixed once and for all.*

We have much more precise control over the isometry condition now. We can check it in every sector  $E$  separately and it amounts to

$$\frac{\bar{Z}_1^{E,E}}{\bar{Z}_0^{E,E}} = \frac{1}{\dim(\mathbb{H}_{b_E})} \quad \frac{\bar{Z}_0^{E,E}}{Z_0} = \frac{\dim(\mathbb{H}_{b_E})^2}{(\sum_{\tilde{E}} \dim(\mathbb{H}_{b_{\tilde{E}}}))^2} \quad \forall E \in \mathcal{W}. \quad (2.97)$$

Crucially, *these conditions are necessary and sufficient.* The calculation of these ‘boundary-fixed partition sums’  $\bar{Z}_{1|0}^{E,E}$  can again be done through dual Ising models. These conditions can be further reformulated to yield useful constraints.

Assuming the first condition, we can calculate what the second is:

$$\frac{\bar{Z}_0^{E,E}}{Z_0} = \frac{\bar{Z}_1^{E,E} D_{b_E}}{\sum_F \bar{Z}_1^{F,F} D_{b_F}} = D_{b_E}^2 \frac{1}{\sum_F D_{b_F} D_{b_E} \frac{\bar{Z}_1^{F,F}}{\bar{Z}_1^{E,E}}} \quad (2.98)$$

We can see from the expression in the denominator that we need  $\frac{D_{b_E}}{\bar{Z}_1^{E,E}} = \frac{1}{q}$ , with  $q$  constant, in order to achieve the second condition. This in turn means with the first condition

$$\begin{aligned}\bar{Z}_1^{E,E} &= q D_{b_E} = k \frac{D_E^2}{D_{b_E}} \\ \bar{Z}_0^{E,E} &= q D_{b_E}^2 = k D_E^2 \quad \forall E \in \mathcal{W}, \quad q = k D_\partial^2\end{aligned}\quad (2.99)$$

These are necessary conditions on all sectors which may be included in the bulk algebra  $\mathcal{A}_I$ , so the set  $\mathcal{W}$ . In fact, they are equivalent to the other set of conditions identified earlier.

With inspiration from the single-vertex case (see B.3 for details), we can make further clear the role of all these constraints. Define the new objects

$$Y_{1|0}^E = \frac{1}{D_E^2} \bar{Z}_{1|0}^{E,E} = \sum_{j_b, k_b} L_{j_b}^E L_{k_b}^E Z_{1|0}^{\vec{j}, \vec{k}} \quad K_{\vec{k}} = D_E L_{k_b}^E \quad (2.100)$$

for which the isometry conditions become (for some constant  $k$ )

$$Y_1^E = k \frac{1}{D_{b_E}} \quad Y_0^E = k \quad \forall E \in \mathcal{W}. \quad (2.101)$$

Let us summarise the constraints we have from isometry or trace preservation. On the partition sums  $Z_{1|0}^{\vec{j}, \vec{k}}$ , which represent the individual-sector data, there is the maximal entropy constraint for each Ising model. This can also pose constraints on the factors  $K_{\vec{j}}$ . For a single sector, this is all there is. On those factors as well as the dimensional data, there is the further restriction for multiple sectors that the output dimension must be fixed across all sectors, and possibly more subtle constraints.

Quite luckily, if the output dimension must be manually fixed to be constant across all sectors of the problem, this reintroduces a notion of *scale* into the discussion. What this means is that we can once again speak of low-energy limits for the Ising model in a sensible manner.

For this purpose, divide all the couplings in the ising model by  $\beta = \log(D_\partial)$ . Then we can perform a universal low-temperature approximation on  $Y_{1|0}^E$  by sending  $\beta \rightarrow \infty$ .

To be more precise, let us write out the full dependence of the partition sums in terms of these quantities:

$$\begin{aligned} Z_{1|0} = & \sum_E e^{2\beta} D_{b_E}^2 \sum_{j_b, k_b} L_{j_b}^E L_{k_b}^E \times \\ & \times \sum_{\vec{\sigma}} \Delta_{1|0}(\vec{j}, \vec{k}; \vec{\sigma}) e^{-\beta \tilde{H}_{1|0}(\vec{j}, \vec{k}; \vec{\sigma})} \end{aligned} \quad (2.102)$$

where we have rescaled  $\tilde{H}_{1|0} = \frac{H_{1|0}}{\beta}$  by rescaling the couplings  $\tilde{\lambda}_e = \frac{\lambda_e}{\beta}$ ,  $\tilde{\lambda}_x = \frac{\lambda_x}{\beta}$ . For the Ising models we consider for bulk-to-boundary and boundary-to-bulk, the  $L$ -factors are independent of  $\beta$ . Furthermore, there are subtle, indirect dependencies on  $\beta$  in  $D_{b_E}, \Delta$  which come from how many choices of  $E$  there are for a given value of  $\beta$ , but we neglect these here, assuming that in the low-temperature limit these do not matter so much.

Then, that limit is dominated by the lowest energy configurations of  $\tilde{H}_{1|0}$  and the combinations of  $j_B, k_B$  which minimise it furthest. We can find estimates for these quite easily. First, take into account the constraints from  $\Delta$ . Let

$$G_{\vec{j}, \vec{k}} = \{x \in \gamma : \mathbf{j}^x = \mathbf{k}^x\} \quad (2.103)$$

be the vertex set where the constraints do not change anything. Then for the different Hamiltonians, the constraints imply

$$\tilde{H}_1 : S_\uparrow \subseteq G_{\vec{j}, \vec{k}} \quad \tilde{H}_0 : S_\downarrow \subseteq G_{\vec{j}, \vec{k}} \quad (2.104)$$

or else the configuration does not contribute. We can then compare the values of the Hamiltonians in the all-up or all-down configurations (subject to the constraints) to see which corresponds more to a minimum:

$$\begin{aligned} \tilde{H}_1(+) &= \sum_{e \in \partial G_{\vec{j}, \vec{k}} \setminus \partial \gamma} \tilde{\lambda}_e + \sum_{x \in G_{\vec{j}, \vec{k}}} \tilde{\lambda}_x \\ \tilde{H}_1(-) &= 1 \quad \tilde{H}_0(+) = 0 \quad \tilde{H}_0(-) = 1 + s_{\vec{j}} \end{aligned} \quad (2.105)$$

where  $s_{\vec{j}} = \frac{\log(D_{\vec{j}})}{\log(D_{\partial})}$  and the reduced couplings are given as  $\tilde{\lambda}_e = \frac{\log(d_{j_e})}{\log(D_{\partial})}$ ,  $\tilde{\Lambda}_x = \frac{\log(D_{j^x})}{\log(D_O)}$ . We assumed here that the "all-up" configuration for  $\tilde{H}_1$  is up on  $G_{\vec{j},\vec{k}}$  and down elsewhere. This does not necessarily give the minimal energy configuration, but is a good approximation to it. In particular, when  $\vec{j} = \vec{k}$ , it reduces to  $\tilde{H}_1(+) = s_{\vec{j}}$ . We can think of the contribution from  $\partial G_{\vec{j},\vec{k}} \setminus \partial\gamma$  as being the analogue of a *bulk geometry-dependent minimal surface* - but we are calculating the entropy of the bulk here, so no such surface is attached to any boundary region and there is no meaning of RT formulas.

Notably, the all-up configuration is always the minimum of  $\tilde{H}_0$ , so the assumption  $Z_0^{\vec{j},\vec{k}} \approx 1$  holds well.

We can then approximate the partition functions as

$$Y_{1|0}^E = \sum_{j_b, k_b} L_{j_b}^E L_{k_b}^E e^{-\beta E_{1|0}(\vec{j}, \vec{k})} \quad (2.106)$$

which for  $Z_0$  gives the previous estimate and condition for isometry, and for  $Z_1$  selects a certain type of contribution of minimal energy  $E_1(j, k)$ . Which combinations of  $j_b, k_b$  give the lowest energy? This all depends a lot on the size, shape and values of spins in  $G_{\vec{j},\vec{k}}$ . However, we can think of two extreme cases for illustration. When the spin sets are equal,  $G_{\vec{j},\vec{k}}$  consists of all graph vertices and so  $E_1(j, j) = r_E = \frac{D_{bE}}{D_{\partial E}}$ . On the other hand, if the spin sets are nowhere equal,  $G_{\vec{j},\vec{k}} = \emptyset$  and no configurations with up-spins are allowed. Instead, consider spin sets which are equal at a *single* vertex  $\{z\} = G_{\vec{j},\vec{k}}$ , for which then

$$E_1(j, k) = \tilde{\Lambda}_z + \sum_{e \cap z, \notin \partial\gamma} \tilde{\lambda}_e \quad (2.107)$$

so it once again depends on the values of the spins at hand at any given vertex. However, it seems feasible that such an 'off-diagonal' contribution might be smaller than the 'diagonal' one, if the values of the spins are not too large. In the following, we will simply assume that there is a number  $g_E$  of combinations  $(j_b, k_b, \vec{\sigma})$  for which  $\tilde{H}_1$  is minimal at value

$$E_{min,E} = \sum_{e \in S_{min,E}} \tilde{\lambda}_e + \sum_{x \in \Sigma_{min,E}} \tilde{\Lambda}_x \quad (2.108)$$

for some bulk region  $\Sigma_{min,E}$  and a boundary segment  $S_{min,E}$  of it. Then

$$Y_1^E \approx g_E (2 - \delta_{j_b, min, k_b, min}) L_{j_b, min}^E L_{k_b, min}^E e^{-\beta E_{min,E}} \quad (2.109)$$

$$\begin{aligned} Y_0^E &\approx \left( \sum_{j_b} L_{j_b}^E \right)^2 \\ &= \left( \sum_{j_B} \prod_{e \in \gamma} |g_{j_e}|^2 \right)^2 = \left( \prod_{e \in \gamma} \sum_{j_e} |g_{j_e}|^2 \right)^2 = 1 \end{aligned} \quad (2.110)$$

and the next-to-leading term will be exponentially suppressed. As we can see, the condition on  $Y_0^E$  is generically fulfilled in the bulk-to-boundary model because  $|g_{j_e}| = 1$  for all boundary links, making the value independent of  $E$ .

*So we are now in a position to give general conditions for isometry to happen:*



1. First, the input and output algebras must be chosen such that the output dimension in each sector is a fixed  $D_\partial$ , which we take here to be quite large.
2. We assume the localisation of Eq. (2.93), which we conjecture to naturally happen in the regime of high spins.
3. The input Rényi-2 entropy in each *boundary* sector is maximal.

This means we can, in this setting of fixed output dimension, check the degree of isometry purely by finding the minimiser  $E_{\min,E}$  and its degeneracy  $g_E$ , meaning we are just looking at an Ising model with extended set of variables  $(j_b, k_b, \vec{\sigma})$  and finding its ground state. The problem is particularly simple because the minimal value is most likely the one of configurations where the bulk spins are close to the lower cutoff. This of course introduces a tension between the approximations made. We need a high  $D_\partial$  to perform the approximation, and require all spins to be large enough for the unitary average to localise, but still the minimal configuration will be the one with the smallest possible spins.

As the single-bulk-link example in appendix B.3 shows, the third of the three conditions implies restrictions on the weights  $g_j$  used to define the state  $\rho$ , of the form

$$\prod_{e \in \gamma} |g_j|^2 \sim \sqrt{D_{b_E}}. \quad (2.111)$$

Let us summarise the consequences of these results for the class of states themselves. In our setting, we consider PEPS-like spin network superpositions  $|\phi_\gamma\rangle$  whose spins may take values between  $\mathfrak{J}$  and  $J$ . If we further consider the (sufficient) restrictions

1. For *any* sector  $\vec{\mathbf{j}}$  that features in  $|\phi_\gamma\rangle$ , the boundary spins take the same *total* value, i.e.  $A_{\partial\gamma} = \sum_{e \in \partial\gamma} \log(d_{j_E})$  is independent of the sector.
2. Fix any set of boundary spins  $E = \{j_e\}$  in accordance to the above. There is a unique pair of spin sectors  $(\vec{\mathbf{j}}, \vec{\mathbf{k}})$  matching the boundary condition such that the Ising Hamiltonian  $H_1$  achieves its minimum, and the ground-state energy gives  $\log(D_{b_E})$ . In terms of spins, this requires as much inhomogeneity in the spins in  $\vec{\mathbf{j}}$  over the graph  $\gamma$  as possible. More generally, require that the state peaks on such sectors.

Then, provided we restrict the input and output algebras sufficiently, the state  $|\phi_\gamma\rangle$  induces an approximately holographic mapping  $\mathcal{T}_{|\phi_\gamma\rangle}$ .

In the appendix B.3, we illustrate the criteria investigated here in two examples: First of a single bulk vertex, then for a single bulk link, both of generic valence. The concrete model used is derived in B.2.1.

## 2.4 Summary

We have seen many different technical complications in this chapter which could give the impression that our main result is more complicated than it first appears. Let us therefore

give an elementary explanation at this point.

We are considering to which extent a given geometry, represented by a state, can be used to encode bulk geometric into its boundary only. This means that we are asking whether it is possible to provide a state of the boundary degrees of freedom only (the edge modes, so in this case the boundary spins), and reconstruct from it a state of the full bulk geometry to evaluate any observable in.

Instead of asking this for a single, special state, we considered a *typical* state, that is a randomly picked state, staying agnostic to the specifics of it.

Our result is that for the typical state, on average reconstruction *can* be done, assuming the state is picked from a certain class. So while the full graph Hilbert space is  $\mathbb{H}_\gamma$ , there is a subspace  $\mathbb{H}_{\gamma,\text{hol}}$  in which the typical state is holographic. Outside of this subspace, however, the typical state is not holographic. Our result characterises this holographic subspace.

The concrete characteristics involve only dimensions of certain Hilbert spaces, which can be interpreted geometrically. When equality between tensor network area and geometric area hold (as we argued, this is necessary for a good interpretation of the states and relation to continuum holography), i.e. with a specific choice of link states, then there is a simple interpretation of most of the criteria.

For one, there is the global criterion that  $\mathbb{H}_{\gamma,\text{hol}}$  is a superselection sector of the total boundary area: the spins  $j_e$  on the boundary of the graph, representing areas, may be different depending on the state, but they must at least give the same total area, asymptotically

$$\sum_e j_e = A_{\text{tot}} = a. \quad (2.112)$$

Thus we may label  $\mathbb{H}_{\gamma,\text{hol}}(a)$  by the total area. This is an interesting restriction because it automatically puts (very limited) restrictions on the boundary metric. Therefore, total area fluctuations are never reconstructible in the framework we work with. Note, in particular, that propagating metric fluctuations, so analogues of gravitons, *are* area preserving, so they still in principle fall within this class.

We already mentioned this, but emphasize again that if we take the analogy to Maxwell theory, then the global electric charge

$$Q = \oint_{\partial\Sigma} E_r = \oint_{\partial\Sigma} *F \approx G_1 \quad (2.113)$$

is just the  $l = 0$  spherical harmonic of the corner charge of the theory. It enjoys a special status among all the harmonics there, as it is superselected for - there can be no superpositions of different charges. In the gravitational case we studied here, the analogous statement is that the total area is the  $l = 0$  mode of several corner charges, in particular the area density  $\sqrt{g}$  on the boundary of the slice. This superselection is a statement about dynamical symmetries and holds in particular for time evolution in the form of a causal

diamond, so where no boundary conditions are placed on the system to close it. If some are placed and the  $l = 0$  harmonic is conserved, then once again we have a superselection rule. If instead we have some area-changing dynamics, like i.e. on expanding quasilocal horizons[176], then we have no such superselection rule. In this case, it is not entirely obvious whether reconstructibility can be achieved in our framework, but this may be subtle as it amounts to a choice of boundary dynamics, which we did not treat at all.

The other criterion is more closely related to local geometry. If  $\mathbb{H}_{\gamma,\text{hol}}(a)$  decomposes into

$$\mathbb{H}_{\gamma,\text{hol}}(a) = \bigoplus_{\sqrt{g}: A_{\text{tot}}=a} \mathbb{H}_{\gamma,\text{hol}}(\sqrt{g}), \quad (2.114)$$

then each of the sectors decomposes further into a bulk piece and edge modes,

$$\mathbb{H}_{\gamma,\text{hol}}(\sqrt{g}) = \mathbb{H}_b(\sqrt{g}) \otimes \mathbb{H}_\partial(\sqrt{g}) \quad (2.115)$$

and reconstructibility is possible if

$$\dim(\mathbb{H}_b(\sqrt{g})) \leq \dim(\mathbb{H}_\partial(\sqrt{g})) = D_\partial. \quad (2.116)$$

If this holds, then the typical state in  $\mathbb{H}_{\gamma,\text{hol}}(\sqrt{g})$  is very close to maximally entangled, so reconstruction is possible. Therefore, the holographic subspace consists of the direct sum of all subspaces of  $\mathbb{H}_{\gamma,\text{hol}}(a)$  such that 2.116 holds. This puts tight constraints on the possible bulk states: Say that we only consider bulk states that are all perturbations around some background geometry. Then assuming this background geometry is determined by the background boundary metric uniquely, all bulk states must again be 1-to-1 with boundary perturbations. In particular, *no* bulk state is reconstructible if it does not perturb the boundary metric.

This result becomes less obviously restrictive if we think about the continuum propagation of gravitons on a causal diamond: Say we remove, for the sake of the argument, all radiation that does not propagate inside the causal diamond. Then, any radiation that *starts* on the interior of the slice  $\Sigma$  will propagate on a null line to eventually meet the future null boundary of the causal diamond. It never touches the spatial boundary  $\partial\Sigma$ , so it does not have to perturb the spatial boundary. So, there will be no way to reconstruct the radiation on any of the spatial slices, and instead its data ends up on the future null boundary. In short, for full reconstruction of the bulk geometry, it may be necessary to not just work with spatial boundaries, but also null boundaries or timelike ones, depending on what needs to be reconstructed. It seems plausible that what *can* be reconstructed on spatial slices is the non-radiative, so Coulombic part of the geometry.

The study in this chapter tells us that many things enter into a concrete understanding of holography: One needs the right interpretation of states in relation to the continuum to make sensible interpretations. One needs a sensible notion of subsystems and cutting and of boundary degrees of freedom. One needs a handle on interpretations of entanglement, and even on the symmetries of the theory in order to make sure that typical quantum

information methods yield sensible results from the spacetime point of view.

All in all, the results here emphasize that it is unavoidable to engage with the symmetry structure of gauge theories if one wants to properly understand the phenomenology of gravity, in particular its degree of holography.

There is, however, an ingredient we did not get into in any amount of detail: The question of diffeomorphism invariance. As the action of those on an abstract lattice, as we presented things here, is not obvious at all, we now present another, closely related line of research that aims at clarifying this action, all the while carefully reevaluating the symmetry structure of gravity in the continuum.

# Chapter 3

## Internal shift symmetries of 4D gravity

With the work of the previous chapter, we have recognised that discrete theories carry many subtleties that we have now have better control over. However, the model we studied, based on the simplest form of spin networks, has also demonstrated a few issues which we have argued to be due to insufficient data of the states. As we already expanded on in section 1.5, we know that the link data of these states is heavily informed by the internal gauge symmetry content of the continuum theory - the link phase space is essentially that of a generic corner, with form  $T^*G$  - so that a precise accounting of the symmetries and corner data of the *continuum* theory is crucial for a correct interpretation of the discrete models in a common language. Therefore, it is desirable to have a common description of all gauge symmetries of gravity, including diffeomorphisms, in terms of internal gauge transformations (in the sense that their parameters are not vector fields, but functions). In this chapter, this is precisely what we study. We essentially perform a classical symmetry analysis of gravity in its tetrad formulation, with emphasis on the phase space of a partial Cauchy slice. This is because the spacetime formulation turns out to have slightly different symmetries which do not descend onto the phase space in a manageable form where they are generated by Hamiltonians. As the actual goal is to have extended data for discrete gravity, we have to pursue a formulation of the symmetries amenable to quantization, hence the focus on the phase space symmetries.

In pursuing this route, we ask more specifically:

- (1) What is the maximal set of internal gauge transformations (conversely, edge modes) in tetrad gravity?
- (2) Can we find a parametrization of symmetries such that the transformations have well-defined generators without restricting the phase space?

The shift symmetries we present in this chapter provide a positive answer to both of these questions. Their analogues are well-known in the 3D gravity literature and have received some limited attention also in the 4D case. In the 3D case, they are much more amenable to discretization, as well (though not identical due to relating degenerate and nondegenerate configurations), and are known to have a geometric interpretation on the lattice which

is tightly linked to the diffeomorphism or triangulation invariance of the resulting theory. We therefore believe that this set of symmetries in 4D may be useful in similar ways.

What we find concretely is that, in the correct parametrisation, the shift symmetries are fully integrable on corners for all parameters, which is much unlike diffeomorphisms. In this sense, they are a preferred parametrisation of the gauge group of tetrad gravity. Together with the internal Lorentz transformations, they form an algebra that bears resemblance to the Poincaré group, but deviates from it on corners.

We point out the following implications of the resulting symmetry structure:

- (1) The internal gauge group on links of spin networks may be understood to be not just the Lorentz group (or any  $SU(2)$  subgroup of it) but also the shifts.
- (2) The Hilbert space of a finite region must carry a representation of the algebra of the shift corner charges.
- (3) The action of the symmetry, once understood in the discrete, need not make reference to embeddings of lattices.
- (4) Implementation of invariance under the precise shift transformations is *equivalent* to the implementation of the dynamics of specifically GR.

We restrict ourselves here to the classical continuum analysis, which is involved enough.

### 3.1 Tetrad gravity

In this section, we detail technical elements of the phase space of tetrad GR: We specify the set of fields, its canonical structure and highlight that the symplectic form is degenerate, leading to the crucial requirement of fixing the so-called ‘structural gauge’. This is in stark contrast to the case in 3D gravity, where no such degeneracy is present. In contrast, in the 4D case the degeneracy forms an obstacle in relating the spacetime and phase space versions of the transformations we want to discuss, so we need to give a bit of intuition for the restriction we need to impose on the set of fields. Ultimately, it simply reflects the fact that the Hamiltonian evolution does not determine the connection field sufficiently, which must therefore be seen as having additional gauge freedom that is not apparent in the Lagrangian formulation.

In order to set the stage, let us properly introduce gravity in a tetrad-connection formulation. There is a particular class of actions we care about, which includes cosmological constant terms and the so-called ‘Holst term’[177], which is on-shell topological and related to the Nieh-Yan torsional invariant[178–180]. This term has been studied routinely in the context of canonical gravity and repeatedly been argued to be relevant[173, 181, 182] or irrelevant[183, 184] for physical quantizations of gravity.

While usual metric gravity is a theory of  $g_{\mu\nu}$ , which determines a full pseudo-Riemannian geometry independent of any observers, tetrad gravity goes a step further. It takes the logic of the equivalence principle seriously, stating that (very) locally, spacetime is equiva-

lent to Minkowski space. This can be implemented through the notions of tangent spaces, Riemann or geodesic normal coordinates, but in the tetrad framework it is furthermore realised through a set of ‘local Minkowski spaces’ which represent the viewpoints of local laboratories. These form the ‘fake tangent bundle’  $V$ , which is made isomorphic to the usual tangent bundle  $TM$  by use of a mapping called the *tetrad*,  $\theta : TM \rightarrow V$ . Each local Minkowski space is equipped with a flat metric  $\eta_{IJ}$ , and given any set of spacetime tangent vectors  $\xi, \zeta$ , we can use the tetrad to transport them into the frame of a local laboratory, where the scalar product is given by the flat one:

$$\xi \mapsto \theta(\xi) \quad g(\xi, \zeta) = \eta(\theta(\xi), \theta(\zeta)) = \eta_{IJ} \theta_\mu^I \theta_\nu^J \xi^\mu \zeta^\nu. \quad (3.1)$$

The local laboratories have their own, independent Lorentz frames, which can be rotated with complete freedom at every point. This is reflected in the fact that the metric  $g$  defined this way is independent of the Lorentz frame we express  $\theta, \eta$  in. In other words, the theory has a local  $SO(1, 3)$  invariance in addition to the data that the metric formulation of general relativity has.

As the laboratories are not yet related, there is still a need for a parallel transport. In principle, then, one also needs to work with a generic connection that relates the local Lorentz frames of the laboratories to each other through spacetime. Such a connection is valued in  $\mathfrak{so}(1, 3)$  and commonly referred to as the spin connection. A priori, there is no assumption made about the relation to the tetrad (or the metric, for that matter). This, instead, is a subject of the dynamics.

We will work with the dynamics presented by Einstein-Cartan-Holst (ECH) theory with cosmological constant<sup>1</sup>  $\Lambda$ :

$$S[\theta, \omega] = \int_M \frac{1}{2} \langle \theta_\beta^2, F_\omega \rangle - \frac{2\Lambda}{4!} \langle \theta^2, \star \theta^2 \rangle \quad (3.2)$$

in which  $\theta$  represents the tetrad,  $\omega$  the Lorentzian (Lie algebra  $\mathfrak{g}$ ) spin connection,  $\beta = \frac{1}{\gamma}$  the Immirzi parameter[177] and the  $\star$  indicates the internal Hodge dual (see the preamble page for notations for details). We use again the notation  $\langle, \rangle$  for the trace inner product on the Lie algebra. For simplicity, we will also use a shorthand for Lie algebra map appearing in the action,

$$(\theta^2)_\beta := (\star + \beta)(\theta^2), \quad \theta^2 := \theta \wedge \theta. \quad (3.3)$$

We will use the covariant phase space approach[83, 85, 86, 99, 111, 185, 186], although a canonical analysis along the same lines is equally possible. We will study the phase space of the theory on some fixed Cauchy slice  $\Sigma$ , and the off-shell (pre)phase space will be

$$\tilde{\mathcal{C}}_\Sigma = \Omega_{nd}^1(\Sigma, V) \times \mathcal{A}(\Sigma, \mathfrak{g}) \quad (3.4)$$

---

<sup>1</sup>We will generically refer to these theories, regardless of their values for  $\beta, \Lambda$ , simply as *tetrad gravity* throughout this work. This is meant to emphasize the *methods* presented here over the specific *theory* they are applied to, as well as evading issues of nomenclature.

the product of nondegenerate pullbacks of tetrads (equivalently, maps into the fake tangent bundle  $V$  whose image at all points of  $\Sigma$  spans a 3D subspace of  $V$ ) with the space of  $\mathfrak{g}$ -connections on the slice. Out of this phase space we will select then the physical (pre)phase space of solutions of the Einstein-Cartan equations

$$E_0 = \delta\theta_I \wedge (G_\omega - \Lambda \star \theta^3)^I - \langle d_\omega \theta_\beta^2, \delta\omega \rangle \quad G_\omega^I := (F_\omega)^{IJ}_\beta \wedge \theta_J \quad (3.5)$$

where we introduce the Einstein tensor  $G_\omega$ .

The presymplectic potential of the theory, in its integrated form, is

$$\Theta_\Sigma = \frac{1}{2} \int_\Sigma \langle \theta_\beta^2, \delta\omega \rangle = - \int_\Sigma \theta_I \wedge \delta\omega_\beta^{IJ} \wedge \theta_J. \quad (3.6)$$

The presymplectic form is produced by taking another variational derivative  $\delta$ , leading to the *degenerate* expression

$$\tilde{\Omega}_\Sigma = \frac{1}{2} \int_\Sigma \delta(\theta^2)_\beta \wedge \delta\omega = \int_\Sigma -\delta\theta_I \wedge \delta\omega_\beta^{IJ} \wedge \theta_J \quad (3.7)$$

which has nontrivial kernel given by the vector fields[186]

$$X_{\bar{\Delta}} = \bar{\Delta} \frac{\delta}{\delta\omega} \quad \bar{\Delta}_\beta^{IJ} \wedge \theta_J = 0 \quad \bar{\Delta} \in \Omega^1(\Sigma, \mathfrak{g}) \quad (3.8)$$

which we will refer to as kernel vector fields or more simply as  $\bar{\Delta}$ -vector fields. So, in fact, one needs to ‘fix a gauge’ even on this kinematical level<sup>2</sup>. The optimal way to do this has been found to be the so-called structural constraint on  $\omega$ [186, 187]. To state it, we first complete the (3-dimensional) image of the pullback of the tetrad to  $\Sigma$ ,  $\tilde{\theta}$ , to a 4-dimensional frame for the fake tangent bundle  $V$ . We do so with a fixed kinematical section  $\nu$ ,  $\delta\nu = 0$  (chosen normalised to  $\pm 1$  in the internal space metric) which can be chosen functionally independent of the fields as long as  $\tilde{\theta}$  induces a nondegenerate metric on  $\Sigma$ .<sup>3</sup> Then, the structural constraint can be stated as that  $\nu \wedge \tilde{d}_\omega \tilde{\theta}$  satisfies

$$\nu \wedge \tilde{d}_\omega \tilde{\theta} = \tau \wedge \tilde{\theta} \quad (3.9)$$

for some  $\tau \in \Omega^1(\Sigma, V)$ .<sup>4</sup> The left hand side has  $3 \cdot 6$  components acting as constraints, of which  $3 \cdot 4$  are alleviated by allowing for  $\tau$  to be arbitrary. Therefore, overall 6 constraints are imposed, fixing the kernel gauge precisely.

This yields a unique separation of any connection into a ‘reduced connection’ and  $\bar{\Delta}$ -part

$$\omega = \hat{\omega} + \bar{\Delta}. \quad (3.10)$$

<sup>2</sup>Usually, gauge fixing is only necessary on-shell of the constraints in order to have a nondegenerate Poisson structure. Here, however, we already need to do this off-shell.

<sup>3</sup>For notational simplicity, for the majority of the chapter we will work on the phase space level and drop the tilde as a restriction to the slice  $\Sigma$  is implicit.

<sup>4</sup> $A \wedge B$  includes the internal wedge product of (multi)vectors  $A^I, B^J$  in  $\Lambda^\bullet V$ .



Given a point  $(\omega, \theta)$ , one can then flow along the  $\bar{\Delta}$  orbits to a unique  $(\hat{\omega}, \theta)$ , where the specific reduced connection  $\hat{\omega}$  depends of course on the starting point.

With respect to the coordinate split  $(\theta, \omega) = (\theta, \hat{\omega}, \bar{\Delta})$ , the gauge fixing is done simply by restricting to  $\bar{\Delta} = 0$ . Then, any observable on the phase space will be a functional of  $\theta$  and  $\hat{\omega}$  only, and extended by constancy to the  $\bar{\Delta}$ -orbits. Naturally, the vector fields tangent to this gauge fixed phase space are then also of the form

$$X = X[\theta](\theta, \hat{\omega}) \frac{\delta}{\delta \theta} + X[\hat{\omega}](\theta, \hat{\omega}) \frac{\delta}{\delta \hat{\omega}}, \quad (3.11)$$

and the now nondegenerate symplectic form is

$$\Omega_\Sigma = - \int_\Sigma \delta \theta_I \wedge \delta \hat{\omega}_\beta^{IJ} \wedge \theta_J. \quad (3.12)$$

The phase space itself is then the total space of an affine vector bundle

$$\mathcal{C}_\Sigma \rightarrow \Omega_{nd}^1(\Sigma, V) \quad (3.13)$$

whose fibers are the spaces of reduced connections  $\mathcal{A}_{\text{red}}(\theta)$  satisfying the structural constraint.

An obvious basepoint for the affine fibers is the torsion-free Levi-Civita connection  $\gamma[\theta]$ , so we can write all points in the fibre as

$$\hat{\omega} = \gamma[\theta] + \kappa, \quad \kappa \in \Omega^1(\Sigma, \mathfrak{g}) \quad \nu \wedge q = \tau \wedge \tilde{\theta}, \quad q^I = \kappa^{IJ} \wedge \tilde{\theta}_J \quad (3.14)$$

We can interpret  $\kappa$  as a contorsion 1-form which, by virtue of the structural constraint, is kinematically restricted to be of a certain form. This restriction, however, is only relevant off-shell or in the presence of matter.

The general solution to the structural constraint can be given after decomposing  $\kappa$  and  $\tau$  with respect to the internal  $\nu$ :

$$\begin{aligned} \kappa &= \kappa_\perp \wedge \nu - \star(\kappa_\parallel \wedge \nu) & \kappa_\parallel^I &= V_J^I \theta_\parallel^J \\ \tau^I &= \tau_\parallel^I + \nu^I \tau_\perp & \theta^I &= \theta_\parallel^I + \nu^I \theta_\perp \end{aligned} \quad (3.15)$$

In this, the tangential  $\kappa_\parallel^I, \tau_\parallel^I$  and normal  $\kappa_\perp^I, \tau_\perp^I$  components to the spatial slice (seeing  $\nu$  as the time direction) are subject to relations under the structural constraint. By expressing  $\kappa_\parallel$  via some tensor  $V_J^I$  in the tetrad basis, we can solve it for  $\tau$

$$\tau_\perp = \epsilon_{ABCD} \theta_\parallel^A V^{BC} \nu^D \quad \tau_\parallel = -\kappa_\perp \quad (3.16)$$

under the condition that

$$V_{(IJ)} = 0. \quad (3.17)$$

In this,  $\kappa$  is constrained to only have degrees of freedom encoded in some antisymmetric tensor  $V_{IJ}$  orthogonal to  $\nu$  (which makes it effectively a spatial 3x3 matrix with 3 degrees of freedom). With this restriction, one can then solve for  $\tau$  in the form presented, meaning

the structural constraint is fulfilled.  
This selects contorsions of the form

$$\kappa = \kappa_\perp \wedge \nu + Q_n \wedge \theta_\parallel \quad Q_n^I \in \Omega^0(\Sigma, V), Q_n \cdot \nu = 0 \quad (3.18)$$

We can get an even better idea of the form of these contorsions when using not a kinematical normal, but the *adapted normal*  $u_\Sigma$  to the slice, which is field-dependent:

$$U_I = V_I^\mu \mathbf{n}_\mu \quad u_\Sigma = \frac{U}{\|U\|} \quad (3.19)$$

It is defined only from the normal 1-form  $\mathbf{n}$  to the slice  $\Sigma$  and the tetrad (and its inverse  $V$ ). In this decomposition, the solutions are all of the form

$$\kappa = i_{\hat{n}}(Q \wedge \theta) \quad Q \in \Omega^1(M, V) \quad (3.20)$$

where  $\hat{n}$  is the vector field associated to  $\mathbf{n}$  via the metric. Matching to the above, we have  $\tilde{Q} = -\kappa_\perp$ ,  $Q_n = Q_n$ . Ostensibly, this means that the solutions break spacetime covariance, but only on the slice  $\Sigma$ , which does so by itself. In terms of the torsion  $T^I = d_\omega \theta^I$ , we can understand the restriction as the *parallel part of the torsion, when expressed in the tetrad basis*,

$$T_\parallel^I \stackrel{\Sigma}{=} A_J^I \wedge \theta^J \quad (3.21)$$

being diagonal:

$$A_J^I = R \delta_J^I \quad R \in \Omega^1(\Sigma) \quad (3.22)$$

This is perhaps the simplest characterisation of the phase space of tetrad variable GR. We stress that this discussion shows an in-principle mismatch between the spacetime and phase space configuration variables of the theory: *Only equivalence classes of connections constitute physical data even off-shell*<sup>5</sup>. On-shell in vacuum, the solution sets are the same, but as we are concerned with off-shell symmetries of the system, this distinction is crucial. In fact, we will see that the Kalb-Ramond shifts we are seeking are sensitive to this mismatch.

It is extremely useful to pick an arbitrary reference connection  $\hat{\omega}_0$  and introduce the Arnowitt Deser Misner (ADM) momentum ‘aspect’[93] 2-form as

$$p^I := -(\hat{\omega} - \hat{\omega}_0)_{\beta}^{IJ} \wedge \theta_J \quad (3.23)$$

which, due to the structural constraint gauge fixing, is 1-to-1 with the reduced connections  $\hat{\omega}$  for given  $\omega_0$  as a basepoint.

With this, we can rewrite the symplectic form as simply

$$\Omega_\Sigma = \int_\Sigma \delta \theta_I \wedge \delta p^I \quad (3.24)$$

---

<sup>5</sup>This is directly related to the notion of primary constraints in canonical analysis, which, in the covariant phase space formalism, are encoded in *off-shell identities and gauge invariances on the configuration space*.

So upon a choice of (arbitrary!) basepoint section in the affine bundle  $\mathcal{C}_\Sigma \rightarrow \Omega_{nd}^1$ , we can realise the phase space as the cotangent bundle[186, 187]

$$\mathcal{C}_\Sigma \cong T^*\Omega_{nd}^1 \quad (3.25)$$

and the basepoint changes act on this cotangent bundle. Notice, however, that as  $\omega_0$  is not a phase space coordinate, it does not transform under Lorentz transformations. In turn,  $p$  does then not transform as a Lorentz vector.

Let us now come to the usual symmetries of tetrad gravity. The first one is the internal Lorentz transformations, generated in a Hamiltonian way by

$$J_\alpha = \frac{1}{2} \int_\Sigma \langle \theta_\beta^2, d_\omega \alpha \rangle \hat{=} \frac{1}{2} \oint_{\partial\Sigma} \langle \theta_\beta^2, \alpha \rangle. \quad (3.26)$$

with  $\alpha \in \Omega^0(\Sigma, \mathfrak{g})$ . This creates the transformation

$$X_\alpha[\theta]^I = -\alpha_J^I \theta^J \quad X_\alpha[\omega]^{IJ} = d_\omega \alpha^{IJ}. \quad (3.27)$$

By the equation of motion  $d_\omega \theta_\beta^2 = 0$ , the Gauss constraint, we see that this is a bulk redundancy on-shell. The Lagrangian is unchanged under these transformations. This is precisely like in BF theory.

Also similar to BF theory, diffeomorphisms are Lagrangian symmetries, and they are not integrable on boundaries. The fields and Lagrangian transform without diffeomorphism-anomaly,

$$\hat{\xi}[\theta] = \mathcal{L}_\xi \theta \quad \hat{\xi}[\omega] = \mathcal{L}_\xi \omega. \quad (3.28)$$

with Hamiltonian generator (if  $\xi|_{\partial\Sigma} \parallel \partial\Sigma$ )

$$D_\xi = \int_\Sigma \frac{1}{2} \langle \theta_\beta^2, d_\omega i_\xi \omega \rangle - \frac{1}{2} \langle i_\xi \theta_\beta^2, F_\omega \rangle \hat{=} \oint_{\partial\Sigma} \frac{1}{2} \langle \theta_\beta^2, i_\xi \omega \rangle \quad (3.29)$$

which is again a pure corner term on-shell. This means that also diffeomorphisms are redundancy in the bulk, just like in BF theory. In fact, the Hamiltonian generators also has the same structure.

Note that this is one of many different ways to let diffeomorphisms act on the fields, which we can refer to as the ‘naive lift’. There is overall a general family of transformations

$$X_\xi^\lambda[\theta] = \mathcal{L}_\xi \theta - \lambda_\xi \cdot \theta \quad X_\xi^\lambda[\omega] = \mathcal{L}_\xi \omega + d_\omega \lambda_\xi \quad (3.30)$$

parametrised by a (possibly field-dependent) map

$$\lambda : \mathfrak{diff}(M) \rightarrow \Omega^0(M, \mathfrak{g}) \quad (3.31)$$

which essentially just combines a naive lift and a Lorentz transformation. All of these transformations lead to the same behaviour on the metric  $g$ , so are, from the perspective of Einstein-Hilbert gravity, completely equivalent. The naive lift is particularly simple, but

different lifts have different amenable properties, i.e. the Kosmann lift preserving isometry behaviour.

Many of the properties we discussed, with the exception of the off-shell redundancy, are also found in BF theory. This can be understood by writing tetrad gravity *as* a BF theory with constraints:

$$L_0 = \langle B, F_\omega \rangle + t_I \wedge d_\omega \theta^I - \langle \lambda, B - \frac{1}{2} \theta_\beta^2 \rangle \quad (3.32)$$

This perspective allows many of the coming results to be understood in relation to BF theory's Kalb-Ramond shifts. The main difference is that BF theory's shifts are parametrised by a Lie algebra valued 1-form parameter  $\mu$ , whereas the tetrad shifts are parametrised by an internal 4-vector -form  $\phi$ . This has strong implications on how the discretization will change. However, we want to stress in this chapter that the properties of tetrad gravity can be almost entirely understood through internal 0-form symmetries, rather than making use of diffeomorphisms. This puts it even closer to topological theories, BF theory and 3D gravity in particular.

Let us now move on to the internal symmetry that improves upon diffeomorphisms.

## 3.2 The naive shifts

First, we present the 'obvious symmetry' as it appears when trying to use the Einstein constraint,  $G_\omega^I = (F_\omega)^{IJ}_\beta \wedge \theta_J$ , as the bulk piece of a generator of gauge transformations. With symbols to be introduced below, the vector field  $Y_\phi$  is parameterised by a 0-form internal vector  $\phi^I$  and acts as

$$Y_\phi[\theta] = d_\omega \phi + \mathbb{T}_\phi \quad Y_\phi[\hat{\omega}] = \mathbb{F}_\phi - \Lambda \mathbb{L}_\phi \quad (3.33)$$

which contains implicit expressions  $\mathbb{T}_\phi$ ,  $\mathbb{F}_\phi$  and  $\mathbb{L}_\phi$  that we purposefully wish to distinguish from other quantities and which stem from the implicit definitions

$$\begin{aligned} \phi \wedge d_\omega \theta &= \mathbb{T}_\phi \wedge \theta \\ \mathbb{F}_\phi \wedge \theta &= \phi \wedge F_\omega, \\ (\mathbb{L}_\phi)_\beta &= \star(\phi \wedge \theta). \end{aligned} \quad (3.34)$$

$\mathbb{L}$  can be easily solved for as  $\mathbb{L}_\phi = \frac{1+\beta\star}{1+\beta^2}(\phi \wedge \theta)$ , but we keep it implicit for the same reason as  $\mathbb{F}_\phi$  - it rarely appears on its own, rather in the form found in the implicit definitions. Similarly, given the contorsion  $\kappa$  from 3.14, we have

$$\mathbb{T}_\phi^I = -\frac{1}{3} \kappa^{IJ} \phi_J \quad (3.35)$$

whereas the expression for  $\mathbb{F}_\phi$  involves the inverse triad in principle.

In most calculations, one does not need the explicit forms of these objects. Instead, we can express their properties through the expressions

$$Y_\phi[\hat{\omega}]_\beta \wedge \theta = -((F_\omega)_\beta - \Lambda \star \theta^2) \cdot \phi, \quad (3.36)$$

and

$$Y_\phi[\theta \wedge \theta] = d_\omega(\phi \wedge \theta), \quad (3.37)$$

which are in fact sufficient to fully define  $Y_\phi$ .

We first present a simple derivation of this transformation from a canonical Ansatz charge. After this, we discuss the relation to symmetries of the Lagrangian and highlight that there is a discrepancy between the above phase space transformation and the appropriate analogous symmetry of the Lagrangian. This explains the need to consider the phase space.

### 3.2.1 Canonical derivation

By starting from the Einstein constraint as a function on phase space for a *closed* slice  $\Sigma$ , we will be able to find relevant symplectic vector fields on the phase space. We then try to extend the phase space transformation to any slice with non-empty boundary.

We have to first note that  $G_\omega - \Lambda \star \theta^3 = 0$ , unlike the Gauss constraint, is *not* constant across the kernel foliation (not left constant by the  $\bar{\Delta}$ -vector fields) in the presymplectic form of ECH theory. Therefore, the functions

$$G_\omega \neq G_{\hat{\omega}} \quad (3.38)$$

(where the latter is constant across orbits by definition) are not the same on the spacetime configuration space  $\mathcal{C}_M$  or the prephase space  $\tilde{\mathcal{C}}_\Sigma$ . This is not an accident: Together with the Gauss constraint,  $G_\omega - \Lambda \star \theta^3 = 0$  selects a *unique representative* of each orbit in the kernel foliation. Therefore, on the on-shell phase space itself, there is no redundancy in the symplectic form. In a way, the Einstein constraint comes ‘equipped’ with a gauge fixing for the  $\bar{\Delta}$ -transformations. However, when writing the offshell phase space using the gauge fixing of the structural constraint, the restriction to the onshell phase space is correctly performed by setting  $G_{\hat{\omega}} = \Lambda \star \theta^3$ . [186, 187]

We can take the variation of the Einstein tensor directly, keeping in mind that the variations are constrained to preserve the structural constraint.

$$\phi_I \delta G_{\hat{\omega}}^I = \langle (\phi \wedge \theta)_\beta, d_{\hat{\omega}} \delta \hat{\omega} \rangle - \delta \theta_I (F_{\hat{\omega}})^{IJ} \phi_J \quad (3.39)$$

For the cosmological constant contribution, we also have

$$\phi_I \delta (\star \theta^3)^I = \frac{1}{3} \delta \theta_I (\star \theta^2)^{IJ} \phi_J. \quad (3.40)$$

We can partially integrate this and for now neglect the boundary term to study the Hamiltonian vector field coming from the constraint. We therefore suppose  $\delta \phi = 0$  and do the contraction

$$\begin{aligned} -I_Y \Omega &= \int_\Sigma \delta(-\phi \cdot (G_{\hat{\omega}} - \Lambda \star \theta^3)) \\ &= - \int \langle (d_{\hat{\omega}} \phi \wedge \theta + \phi \wedge d_{\hat{\omega}} \theta), \delta \hat{\omega}_\beta \rangle - \delta \theta_I (F_{\hat{\omega}})^{IJ} \phi_J - \delta \theta_I \Lambda \star (\phi \wedge \theta)^{IJ} \theta_J \\ &= \int (d_{\hat{\omega}} \phi + \mathbb{T}_\phi)_I \wedge \delta \hat{\omega}_\beta^{IJ} \wedge \theta_J + \delta \theta_I ((F_{\hat{\omega}})^{IJ} \phi_J + \Lambda \star (\phi \wedge \theta)^{IJ} \theta_J) \end{aligned} \quad (3.41)$$

Using equation C.1 from appendix C.1, one can see that it generates the Hamiltonian vector field

$$Y_\phi[\theta] = d_{\hat{\omega}}\phi + \mathbb{T}_\phi \quad Y_\phi[\hat{\omega}] = \mathbb{F}_\phi - \Lambda \mathbb{L}_\phi. \quad (3.42)$$

So what is correctly shifted in 4-dimensional gravity by a derivative is not the tetrad, but the gravitational flux  $\theta^2$ , showing that this symmetry is a remnant from BF theory<sup>6</sup> after imposition of the simplicity constraints to arrive at gravity.

### 3.2.2 Contrast to symmetries of the Lagrangian

The presence of such constraints comes, by Noether's second theorem in a 1-to-1 way, with gauge symmetries of the theory. However, we need to be careful in invoking said theorem as we have not made use of Lagrangians so far. In fact, the vector field we presented is *not* the phase space pushforward of a transformation acting on the spacetime fields  $\theta, \omega$  appearing in the Lagrangian. There is still an analogue of it, which has been previously studied by Montesinos et. al[188], where by use of Noether identities, one obtains a symmetry of the Lagrangian that looks very similar. For this, we simply take derivatives of the Einstein tensor (this time in spacetime),

$$d_\omega G_\omega^I = (F_\omega)_{\beta}^{IJ} \wedge d_\omega \theta_J \quad (3.43)$$

and weigh it by an arbitrary internal 4-vector  $\phi^I$  to arrive at the Noether identity

$$d(\phi_I G_\omega^I) = d_\omega \phi_I \wedge G_\omega^I + \langle \phi \wedge d_\omega \theta, (F_\omega)_\beta \rangle. \quad (3.44)$$

For the cosmological constant part, we will also need

$$d(\phi_I \star (\theta^3)^I) = d_\omega \phi_I \wedge \star (\theta^3)^I + \langle (\phi \wedge d_\omega \theta), \star \theta^2 \rangle \quad (3.45)$$

This can be used to find vector fields which satisfy the characteristic relation of local symmetries (see section 1.1 for details),

$$I_X E_0 = dC_1. \quad (3.46)$$

Here,  $C$  is a codimension 1 form that vanishes on-shell of the equations of motion, and is referred to as the *constraint form*. For this, we need to rewrite the torsion term. There are essentially two options for this, which we can combine in a general way. First, note that the general contraction is

$$I_X E_0 = X_\phi[\theta]_I \wedge (G_\omega - \Lambda \star \theta^3)^I - \langle d_\omega \theta_\beta^2, X_\phi[\omega] \rangle. \quad (3.47)$$

The first of the two ways is to rewrite

$$\phi \wedge d_\omega \theta = \mathbb{T}_\phi \wedge \theta. \quad (3.48)$$

---

<sup>6</sup>In 4D BF theory, the 2-form Kalb-Ramond field  $B$  is shifted as  $B \mapsto B + d_\omega \mu$  with a Lie algebra-valued 1-form  $\mu$ .

The second option is to use  $\mathbb{F}_\phi$  such that

$$(\mathbb{F}_\phi)^{IJ}_\beta \wedge \theta_J = - (F_\omega)^{IJ}_\beta \phi_J, \quad (3.49)$$

which reproduces the torsion term in the contraction. Also useful is  $\mathbb{L}_\phi$  for the cosmological constant terms. We can then see by mixing and matching these ingredients that the family of vector fields for  $s \in \mathbb{R}$

$$Y'_{\phi,s}[\theta] = d_\omega \phi + s \mathbb{T}_\phi \quad Y'_{\phi,s}[\omega] = (1-s)\mathbb{F}_\phi + \Lambda(1 - \frac{s}{3})\mathbb{L}_\phi \quad (3.50)$$

are all local symmetries:

$$I_{Y'} E_0 = d \left( \phi_I (G_\omega - \Lambda \star \theta^3)^I \right) \quad (3.51)$$

$$Y'_{\phi,s}[L] = d \left( \phi_I (G_\omega - \Lambda \star \theta^3)^I + \theta_\beta^2 \wedge Y'_\phi[\omega] \right) \quad \forall s, \phi \quad (3.52)$$

These are associated to the Noether currents

$$j_s^\phi = \phi_I (G_\omega - \Lambda \star \theta^3)^I \quad (3.53)$$

which all vanish on-shell. However, we can already see that the pushforward of this vector field to the phase space  $\mathcal{C}_\Sigma$  will have issues: By analysing the symplectic form in detail, as we do in Appendix C.1, we can find criteria for a vector field to be symplectic. The result is that the vector fields here will not be symplectic in general.

This means, in particular, that the symmetry of the Lagrangian can not be canonically represented. In a potential quantization, this is unfortunate on a technical level and it is unclear how to realise such the action of a symmetry on the system through (unitary) operators. That the two transformations do not agree is perhaps more than an unlucky coincidence: It is known from careful Batalin-Vilkovisky (BV)-Batalin-Fradkin-Vilkovisky (BFV) studies[187, 189] that the spacetime formulation (BV) of tetrad gravity and the phase space (BFV) formulation, while separately equivalent to the Einstein-Hilbert formulations, are *not* equivalent in a more strict sense (BV-BFV). What this means is the following: The Hamiltonian evolution structure of ECH and its Lagrangian equations of motion are ‘incompatible’ in the sense that there is a discrepancy in configuration spaces:

$$\begin{aligned} \mathcal{C}_M^{Lagr} &= \Omega_{nd}^1(M, V) \times \mathcal{A}(M, \mathfrak{g}) \\ \mathcal{C}_M^{Ham} &= \Omega_{nd}^1(M, V) \times \mathcal{A}_{red}(M, \mathfrak{g}) \end{aligned} \quad (3.54)$$

The ‘reduced’ connections  $\mathcal{A}_{red}$  are simply the ones satisfying the structural constraint with respect to a given local foliation of the spacetime  $M$  into spatial slices. So, while the Lagrangian equations of motion are defined for all possible spin connections, the Hamiltonian evolution is only sensible for their equivalence class under the  $\bar{\Delta}$ -transformations. BV-BFV compatibility states now, in essence, that taking a covariant spacetime dynamics, restricting it to an initial slice  $\Sigma$ , and then evolving canonically on a cylinder  $\Sigma \times \mathbb{R}$  returns one to the same spacetime dynamics. Clearly, this is not the case in ECH.

This can in principle be remedied by restricting the spacetime configurations  $\omega$  to  $\hat{\omega}$ , but

this cannot be done in a covariant way. Overall, we can see that it is no surprise that the spacetime covariant symmetries  $X$  do not descend necessarily to the phase space. Therefore, as long as we want to perform phase space studies, we will want to stick to the vector fields  $Y$ , as these are the ones canonically represented.

We can actually make this point clear by studying the time evolution on the phase space: by splitting the equations of motion

$$G_\omega = 0 = d_\omega \theta \wedge \theta \quad (3.55)$$

into their horizontal (constraint) and vertical (evolution) parts, we can find the evolution equations of the canonical variables of the slice. We find, in particular, that they entail

$$\begin{aligned} \mathcal{L}_{\hat{n}} \tilde{\theta} &\approx \tilde{d}_\omega \theta_n + \tau_{\theta_n} - \omega_n \cdot \tilde{\theta} \\ \mathcal{L}_{\hat{n}} \tilde{\omega} &\approx \tilde{d}_\omega \omega_n + Y_{\theta_n}[\tilde{\omega}] + \bar{\Delta} \end{aligned} \quad (3.56)$$

where  $\omega_n = i_{\hat{n}} \omega$ ,  $\theta_n = i_{\hat{n}} \theta$  and the  $\bar{\Delta}$  piece is *arbitrary* and precisely of the type that is removed by the kernel quotient  $\tilde{\omega} \mapsto [\tilde{\omega}] \ni \hat{\omega}$  - its presence states that the time evolution of  $\tilde{\omega}$  under this equation is gauge invariant under the kernel foliation, and so, for the reduced connection, well defined. In principle, then, one must fix this gauge freedom in order to have well-posed time evolution. The  $Y$ -piece is precisely the vector field from the previous section, so in practice we have that

$$\mathcal{L}_{\hat{n}} \approx X_{\omega_n} + Y_{\theta_n} \quad (3.57)$$

or in other words, time evolution in tetrad gravity is pure gauge and a combination of Lorentz transformations and the new, nonlinear Kalb-Ramond transformations.

If, instead, we decompose the action of general diffeomorphisms on the spacetime fields in a similar way, by appealing to the split of diffeomorphisms into Lorentz transformations and covariant diffeomorphisms

$$\begin{aligned} \mathcal{L}_\xi \theta &= d_\omega i_\xi \theta - i_\xi \omega \cdot \theta + i_\xi d_\omega \theta \\ \mathcal{L}_\xi \omega &= i_\xi F_\omega + d_\omega(i_\xi \omega), \end{aligned} \quad (3.58)$$

then regular diffeos can then be rewritten as *yet different* field-dependent shifts

$$Y_p'' h i[\theta]^I = d_\omega \phi^I + \phi^J i_{\hat{V}_J} d_\omega \theta^I \quad Y_\phi''[\omega]^{IJ} = \phi^K i_{\hat{V}_K} F_\omega^{IJ} \quad (3.59)$$

together with Lorentz transformations  $X_\alpha$

$$\mathcal{L}_\xi = X_{i_\xi \omega} + Y_{i_\xi \theta}''. \quad (3.60)$$

While these other shifts  $Y_\phi''$  might superficially look similar to the  $Y_\phi$  ones, the latter are different due to the contraction pattern of  $\phi$ . They are also not symmetries of the Lagrangian like  $Y_{\phi,s}'$ .

We can therefore see that unlike in the case of 3D gravity, there are a multitude of shift



symmetries depending on the motivation. The presence of canonical generators and compatibility with time evolution singles out the transformation  $Y_\phi$  as the seemingly best choice for generators, leaving a schism between the Lagrangian and Hamiltonian formulation. Nevertheless, these transformations are genuine gauge symmetries of the Hamiltonian theory and can be used to rewrite diffeomorphisms as effective, field-dependent transformations like in 3D gravity.

Yet, there is one more freedom to exploit in our construction of the generator. As we said in the beginning, we only used the Einstein constraint  $G_\omega$  as the bulk piece. However, in principle our set of independent constraints,  $G_\omega, d_\omega \theta^2$ , can be linearly combined in any form we want and we still have an equivalent set of constraints defining the right on-shell phase space. As we will now see, if we redefine our shift constraint to be a combination of Einstein and Gauss, we get a different phase space transformation with better properties.

### 3.3 The improved shifts

If we are given the symplectic form of ECH gravity,

$$\Omega = \frac{1}{2} \int_\Sigma \delta \theta_\beta^2 \wedge \delta \hat{\omega}, \quad (3.61)$$

the shifts generated by the improved charge

$$P_\phi = - \int_\Sigma (\phi \wedge \theta)_\beta \wedge F_\omega + \frac{1}{2} d_\omega \theta_\beta^2 \cdot \alpha_\phi - \oint_{\partial \Sigma} p_I \phi^I \quad (3.62)$$

actually are integrable for field-independent  $\phi$ , if the Lorentz parameter  $\alpha_\phi^{IJ}$  satisfies

$$\alpha_\phi \wedge \theta = \hat{\omega} \wedge \phi. \quad (3.63)$$

This, in particular, implies that the corner charge

$$\oint \theta_\beta^2 \cdot \alpha_\phi = -\phi_I p^I \quad (3.64)$$

is the same Brown-York momentum that also is the conjugate variable to the tetrad in the bulk. We can also rewrite the generator in a completely bulk way, which then has a much more illuminating form:

$$P_\phi = - \int_\Sigma p_I \wedge d_\omega \phi^I + \phi_I (F_\omega - d_\omega \omega)^{IJ}_\beta \wedge \theta_J. \quad (3.65)$$

In this form, we can see that it consists of a momentum-derivative pair  $p d_\omega \phi$ , just like the Lorentz generator does, but also an additional piece. This one involves the peculiar non-tensorial combination  $F_\omega - d_\omega \omega = -\frac{1}{2}[\omega, \omega]$ , which vanishes for *reducible connections*,

so precisely those where the commutator vanishes. It is then clear that the corner charge piece comes from the first term, and the bulk constraint is

$$d_\omega p^I = (F_\omega - d_\omega \omega)_\beta^{IJ} \wedge \theta_J. \quad (3.66)$$

This has the form of a conservation law for the momentum  $p$ , with violations for non-reducible connections.

The charge generates the bulk transformation

$$Y_\phi[\theta] = d_\omega \phi - \alpha_\phi \cdot \theta \quad Y_\phi[\hat{\omega}] \wedge \theta = d_\omega(\alpha_\phi \wedge \theta) - F_\omega \wedge \phi \quad (3.67)$$

and in particular

$$Y_\phi[\hat{\omega}]_\beta \wedge \theta = -\hat{\omega}_\beta \wedge d_\omega \phi. \quad (3.68)$$

So that

$$Y_\phi[p] = \hat{\omega}_\beta \cdot \alpha_\phi \cdot \theta = -\alpha_\phi \cdot p + (d_\omega \alpha_\phi)_\beta \wedge \theta - (d\alpha_\phi)_\beta \wedge \theta. \quad (3.69)$$

This result has a few ramifications. As a first, we have

$$Y_{i_\xi \theta}[\theta] = \mathcal{L}_\xi \theta - i_\xi d_\omega \theta - (\alpha_{i_\xi \theta} - i_\xi \omega) \cdot \theta \quad (3.70)$$

$$Y[\hat{\omega}]_\beta \cdot \theta = (\mathcal{L}_\xi \hat{\omega} + d_\omega(\alpha_{i_\xi \theta} - i_\xi \omega))_\beta \cdot \theta - i_\xi(F_\omega)_\beta \cdot \theta + (\alpha_{i_\xi \theta})_\beta \cdot d_\omega \theta \quad (3.71)$$

so that on-shell of torsion and the Einstein constraint, an improved shift is equivalent to a diffeomorphism (as represented by the Lie derivative) and a Lorentz transformation with parameter  $\alpha_{i_\xi \theta} - i_\xi \omega$ . Conversely, standard diffeomorphisms are once again on-shell equivalent to field-dependent shifts and Lorentz transformations,

$$\mathcal{L}_\xi \approx X_{i_\xi \omega - \alpha_{i_\xi \theta}} + Y_{i_\xi \theta}. \quad (3.72)$$

In comparison to the naive shifts, we need to correct by the  $\alpha$ -term because the improved shifts contain (figuratively speaking) a Lorentz transformation. The main difference to diffeomorphisms is that the improved shifts are always integrable, as we will show now.

### 3.3.1 The variation

We vary the charge in pieces. We begin with the Einstein constraint bulk piece,

$$\delta(-(\phi \wedge \theta)_\beta \wedge F_\omega) = \delta\theta_I(F_\omega)_\beta^{IJ} \phi_J + d((\phi \wedge \theta)_\beta \wedge \delta\hat{\omega}) - d_\omega(\phi \wedge \theta)_\beta \wedge \delta\hat{\omega} \quad (3.73)$$

Next, we have the part of the Gauss constraint piece where we do not vary  $\alpha_\phi$ ,<sup>7</sup>

$$\begin{aligned} -\frac{1}{2}\delta(d_\omega \theta_\beta^2) \cdot \alpha_\phi &= -\frac{1}{2}[\theta_\beta^2, \alpha_\phi] \cdot \delta\hat{\omega} - \frac{1}{2}d_\omega \delta\theta_\beta^2 \cdot \alpha_\phi \\ &= -\frac{1}{2}[\theta^2, \alpha_\phi]_\beta \cdot \delta\hat{\omega} - d(\frac{1}{2}\delta\theta_\beta^2 \cdot \alpha_\phi) - \delta\theta_I d_\omega(\alpha_\phi)_\beta^{IJ} \theta_J \end{aligned} \quad (3.74)$$

---

<sup>7</sup>We make use of identities proved in [186].

Furthermore, the piece where we vary  $\alpha_\phi$ . For this, we need to use the identity

$$(\delta\alpha_\phi)_{\beta}^{IJ}\theta_J = \delta\hat{\omega}_{\beta}^{IJ}\phi_J - (\alpha_\phi)_{\beta}^{IJ}\delta\theta_J \quad (3.75)$$

which follows from varying the definition of  $\alpha_\phi$ . Then,

$$\begin{aligned} -\frac{1}{2}d\hat{\omega}\theta_{\beta}^2 \cdot \delta\alpha_{\phi} &= -d\hat{\omega}\theta_I\delta\alpha_{\phi}^{IJ}\theta_J \\ &= -d\hat{\omega}\theta_I\delta\hat{\omega}_{\beta}^{IJ}\phi_J + d\hat{\omega}\theta_I(\alpha_{\phi})_{\beta}^{IJ}\delta\theta_J \\ &= (\phi \wedge d\hat{\omega}\theta) \cdot \delta\hat{\omega}_{\beta} - \delta\theta_I(\alpha_{\phi})_{\beta}^{IJ}d\hat{\omega}\theta_J \end{aligned} \quad (3.76)$$

And we can combine the bulk pieces to get

$$\begin{aligned} &\delta\theta_I \left[ (F_{\hat{\omega}})_{\beta}^{IJ}\phi_J - d\hat{\omega}(\alpha_{\phi})_{\beta}^{IJ}\theta_J - (\alpha_{\phi})_{\beta}^{IJ}d\hat{\omega}\theta_J \right] \\ &- \left[ +d\hat{\omega}(\phi \wedge \theta) - (\phi \wedge d\hat{\omega}\theta) + \frac{1}{2}[\theta^2, \alpha_{\phi}] \right]_{\beta} \delta\hat{\omega}. \end{aligned} \quad (3.77)$$

We can see that the  $\phi \wedge d\hat{\omega}\theta$  term in the second row cancels the piece which would lead to the annoying torsion piece  $\mathbb{T}_{\phi}$  in the transformation of the tetrads. Furthermore, there is now instead a torsion piece in the first row, which modifies the transformation law of the connection, which would otherwise contain  $d\hat{\omega}\alpha_{\phi}$ , as usual for Lorentz transformations. We also see that the  $\mathbb{F}_{\phi}$  piece coming from the first term in the first line is unchanged.

We find that the charge generates the bulk vector field

$$Y_{\phi}[\theta] = d\hat{\omega}\phi - \alpha_{\phi} \cdot \theta \quad Y_{\phi}[\hat{\omega}] \wedge \theta = d_{\omega}(\alpha_{\phi} \wedge \theta) - F_{\omega} \wedge \phi \quad (3.78)$$

as we claimed.

We now look at the corner pieces. The full corner variation can be reformulated using the definition of  $\alpha_{\phi}$  again:

$$\begin{aligned} &(\phi \wedge \theta)_{\beta} \wedge \delta\hat{\omega} - \frac{1}{2}\delta\theta_{\beta}^2 \cdot \alpha_{\phi} - \delta p_I \phi^I \\ &= -\phi_I \delta\hat{\omega}_{\beta}^{IJ}\theta_J - \delta\theta_I(\alpha_{\phi})_{\beta}^{IJ}\theta_J + \phi_I \delta\hat{\omega}_{\beta}^{IJ}\theta_J + \phi_I \hat{\omega}_{\beta}^{IJ}\delta\theta_J \\ &= -\phi_I \delta\hat{\omega}_{\beta}^{IJ}\theta_J - \delta\theta_I \hat{\omega}_{\beta}^{IJ}\phi_J + \phi_I \delta\hat{\omega}_{\beta}^{IJ}\theta_J - \delta\theta_J \hat{\omega}_{\beta}^{IJ}\phi_I \\ &= 0 \end{aligned} \quad (3.79)$$

and so we have

$$I_{Y_{\phi}}\Omega + \delta P_{\phi} = 0 \quad (3.80)$$

which means the bulk vector field is perfectly integrable for all field-independent  $\phi$ . Their on-shell value is also nonzero for a generic configuration  $(\theta, \hat{\omega})$  for all values of  $\phi$ .

This is, notably, *not* the case for the naive shifts we studied before - those generically have a nonzero, non-exact term supported on  $\partial\Sigma$  when contracted with the symplectic form, so they do not have generators when corners are present. In contrast, we can use the improved shift charges and properly analyse their Poisson brackets, and in particular the on-shell algebra of corner charges.

We also note that this is *not* possible to do for the diffeomorphism charges: There is no analogue of the definition of  $\alpha_{\phi}$  that would lead to both bulk and boundary integrability for field-independent parameters  $\xi$  in the same way as here.

### 3.3.2 Algebra of charges

We can now present, through some involved calculations, the on-shell algebra of charges. For reference, the Lorentz generator is for field-independent  $\alpha$

$$J_\alpha = \frac{1}{2} \int_\Sigma \theta_\beta^2 \wedge d\hat{\omega} \alpha. \quad (3.81)$$

The algebra, once calculated and using the onshell relations  $G_\omega = 0 = d_\omega \theta_\beta^2$ , is

$$\begin{aligned} \{J_\alpha, J_\beta\} &= J_{-[\alpha, \beta]} \\ \{J_\alpha, P_\phi\} &= P_{\alpha \cdot \phi} + \oint_{\partial\Sigma} (\phi \wedge \theta)_\beta \wedge d\alpha \\ \{P_\phi, P_{\tilde{\phi}}\} &= J_{[\alpha_\phi, \alpha_{\tilde{\phi}}]} + \oint_{\partial\Sigma} \omega \cdot d(\phi \wedge \tilde{\phi})_\beta - (\phi \wedge \tilde{\phi})_\beta \cdot F_\omega \end{aligned} \quad (3.82)$$

The last equality in particular receives contributions from the bulk that must be carefully treated.<sup>8</sup> In addition to an expected Poincaré-structure and the term  $J_{[\alpha_\phi, \alpha_{\tilde{\phi}}]}$  coming from the included Lorentz transformation, we get extension terms which are in principle a little complicated due to field-dependence, meaning they are not central.

Note that we can rewrite this new term as

$$\oint_{\partial\Sigma} \omega \cdot d(\phi \wedge \tilde{\phi})_\beta - (\phi \wedge \tilde{\phi})_\beta \cdot F_\omega = - \oint_{\partial\Sigma} \frac{1}{2} [\omega, \omega]_\beta \cdot (\phi \wedge \tilde{\phi}) \quad (3.83)$$

so the term vanishes only for reducible connections.

First of all, the algebra stands as it is and only closes with structure functions in the  $P, P \rightarrow J$  part. The rest, however, naively speaking, also does not close, if we do not include charges linear in  $\theta$  and  $\omega$  supported on the corner, and a charge given by the smeared curvature on the corner. We do not need to see these additional objects as coming from constraints at this point, as they are the result of commutation relations of nonzero physical charges.

In particular, we can deduce once again from the Lorentz bracket a corner commutation relation of the rough form

$$\{\theta, \theta\} \sim 1 \quad (3.84)$$

from the logic

$$\{\theta^2, \theta^2\} \sim \theta^2 \{\theta, \theta\} \stackrel{!}{=} \theta^2. \quad (3.85)$$

The Lorentz-shift bracket tells us, in comparison

$$\{\theta^2, \theta\omega\} \sim \theta\omega + \theta^2 \{\theta, \omega\} \stackrel{!}{=} \theta\omega + \theta d(\dots) \quad (3.86)$$

where we use that  $p \sim \theta\omega$ . So, we can see that  $\{\theta, \omega\} \sim \frac{d(\dots)}{\theta}$ , and either inverse tetrads or implicit identities must be at play in the corner symplectic form.

---

<sup>8</sup>It involves expanding  $\tilde{\phi} \cdot d_\omega(\hat{\omega}_\beta \cdot \phi) - \phi \cdot d_\omega(\hat{\omega}_\beta \cdot \tilde{\phi}) + (\phi \wedge \tilde{\phi})_\beta \cdot F_\omega$ .

In a similar way, we can deduce from the shift-shift bracket

$$\begin{aligned}\{\theta\omega, \theta\omega\} &\sim \theta^2\{\omega, \omega\} + \theta\omega \frac{d(\dots)}{\theta} + \omega\omega \\ &\sim \theta^2\{\omega, \omega\} + \omega d(\dots) + \omega\omega \\ &\stackrel{!}{=} \theta^2[\alpha, \alpha] + \omega d(\dots) + F_\omega\end{aligned}\tag{3.87}$$

which informs us through the Lorentz charge contribution that  $Q_\phi^I := \omega_\beta^{IJ} \phi_J$  must satisfy something like

$$\{Q_{\phi,a}^I(x), Q_{\psi,b}^J(z)\} = \frac{1}{2} \epsilon_{ab} \delta(x, y) [\alpha_\phi, \alpha_\psi]_\beta^{IJ}(x)\tag{3.88}$$

So, we find that the appropriate corner symplectic form pairs tetrad with tetrad and connection with connection, but the latter in a nontrivial way with field-dependence. In particular, this motivates that the corner symplectic form of tetrad gravity is something which looks like the one of Chern-Simons theory for both  $\omega, \theta$ , but with a field-dependent pairing for  $\omega$ ,

$$\Omega_S = \int_S \epsilon^{ab} (\delta\theta_a^I N_{IJ} \delta\theta_b^J + \delta\omega_a^{IJ} M_{IJ,KL}(\theta, \omega) \delta\omega^{KL} + \dots)\tag{3.89}$$

These are mostly preliminary considerations, but hint towards the idea that the continuum *requires* the introduction of corner charges involving the tetrad and connection linearly, *in addition* to the shift charges. It also gives some constraints on the possible Poisson algebra of these charges, with the caveat being that gauge-invariant information in  $\theta, \omega$  is not affected by the symmetry charges.

We now derive the nontrivial Poisson brackets of the corner charges.

### Lorentz-shift algebra

We have for the bulk pieces

$$X_\alpha[(\phi \wedge \theta)_\beta F_\omega] = (\alpha \cdot \phi \wedge \theta)_\beta F_\omega\tag{3.90}$$

and

$$X_\alpha[\frac{1}{2} d_\omega \theta^2 \cdot \alpha_\phi] = \frac{1}{2} d_\omega \theta^2 \cdot [\alpha, \alpha_\phi] + \frac{1}{2} d_\omega \theta^2 \cdot X_\alpha[\alpha_\phi]\tag{3.91}$$

so we have in the bulk

$$\{J_\alpha, P_\phi\} = P_{\alpha \cdot \phi} + J_{A(\alpha, \phi)}\tag{3.92}$$

with

$$A(\alpha, \phi) := X_\alpha[\alpha_\phi] - \alpha_{\alpha \cdot \phi} - [\alpha_\phi, \alpha]\tag{3.93}$$

a piece that measures the Lorentz anomaly of  $\alpha_\phi$ : The action of Lorentz transformations should affect it at least through its argument, and as a Lie algebra element. However, it

carries an anomaly on top that transforms it with derivatives of  $\alpha$ . This is sensible from the definition of  $\alpha_\phi$ , as all pieces except  $\omega$  may be acted upon tensorially by the Lorentz transformations. If not for the total derivative term in  $d_\omega \alpha$ , we would have  $A = 0$ . So, we actually have

$$A(\alpha, \phi) \wedge \theta = d\alpha \wedge \phi \quad (3.94)$$

and the additional corner piece is a reflection of the fact that  $A \neq 0$ , or equivalently that  $\alpha$  transforms noncovariantly under Lorentz transformations.

With corners, we have

$$\{J_\alpha, P_\phi\} \approx - \oint (p \wedge \phi) \cdot \alpha + d\alpha_\beta \cdot (\phi \wedge \theta) \quad (3.95)$$

This is equal to the corner term of  $P_{\alpha \cdot \phi} + J_{A(\alpha, \phi)}$  if we identify

$$-\phi \cdot d\alpha_\beta = \theta \cdot A(\alpha, \phi)_\beta \Leftrightarrow A(\alpha, \phi) \wedge \theta = d\alpha \wedge \phi \quad (3.96)$$

So the Lorentz-anomaly of  $\alpha_o$  is indeed what gives rise to the extra term. In total, we have that off-shell,

$$\{J_\alpha, P_\phi\} = P_{\alpha \cdot \phi} + J_{A(\alpha, \phi)} \quad (3.97)$$

Let us compare this to the abstract model algebroid of Poincaré transformations; There, we would have the field-dependent Lorentz transformations  $\alpha, \tilde{\alpha}$  and field-independent translations  $\phi, \tilde{\phi}$  in the shape

$$[(\alpha, \phi), (\tilde{\alpha}, \tilde{\phi})] = ([\tilde{\alpha}, \alpha] - \delta_\alpha \tilde{\alpha} + \delta_{\tilde{\alpha}} \alpha, -\alpha \cdot \tilde{\phi} + \tilde{\alpha} \cdot \phi) \quad (3.98)$$

If we specialise this to  $\phi = 0, \tilde{\alpha} = \alpha_\phi$  and  $\alpha$  field-independent, then

$$[(\alpha, 0), (\alpha_\phi, \tilde{\phi})] = ([\alpha_\phi, \alpha] - X_\alpha \alpha_\phi, -\alpha \cdot \tilde{\phi}) = (-A(\alpha, \phi) + \alpha_{-\alpha \cdot \tilde{\phi}}, -\alpha \cdot \tilde{\phi}) \quad (3.99)$$

which is the parameters that give  $P_{\alpha \cdot \phi} + J_{A(\alpha, \phi)}$ <sup>9</sup>. So, we have a perfect representation of this algebroid both in bulk and corners. This is what we should expect from the fact that the transformations are integrable; It is also a crucial consistency check that is not available for the shift-shift bracket by itself, as things are much more involved there.

### Shift-shift algebra

The calculation for the shift Poisson brackets are much more involved, but still doable in practice, unlike the direct computation of the commutators of transformations, which contain too many implicit terms. We split the calculation into two pieces, one for the Einstein term, one for the Gauss term. We will use

$$Y_\phi[\theta^2] = 2d_\omega(\phi \wedge \theta) + [\theta^2, \alpha_\phi] - 2\phi \wedge d_\omega \theta \quad (3.100)$$

---

<sup>9</sup>Recall that  $P$  is a naive shift together with a Lorentz transformation, so corresponds to a parameter  $(\alpha_\phi, \phi)$ . Recall also that the charge parameters in the Poisson bracket come with the opposite sign to the ones we get in the abstract Lie bracket of parameters.

and

$$Y_\phi[(F_\omega)]_\beta \wedge \theta = d_\omega(Y_\phi[\omega]_\beta \wedge \theta) + Y_\phi[\omega]_\beta \wedge d_\omega \theta \quad (3.101)$$

$$= d_\omega(d_\omega \alpha_\beta \wedge \theta + \alpha_\beta \cdot d_\omega \theta) - d_\omega(F_\beta \cdot \phi) + Y_\phi[\omega]_\beta \wedge d_\omega \theta \quad (3.102)$$

$$= d_\omega^2 \alpha_\beta \wedge \theta + \alpha_\beta \cdot d_\omega^2 \theta - F_\beta \cdot d_\omega \phi + Y_\phi[\omega]_\beta \wedge d_\omega \theta \quad (3.103)$$

as well as

$$[\phi \wedge \theta, \alpha] = (-\alpha \cdot \phi) \wedge \theta + \phi \wedge (-\alpha \cdot \theta) \quad (3.104)$$

Let us look first at the Einstein term.

$$Y_\phi[(\tilde{\phi} \wedge \theta)_\beta F_\omega] = (\tilde{\phi} \wedge d_\omega \phi)_\beta F_\omega + [(\tilde{\phi} \wedge \theta)_\beta, \alpha_\phi] F_\omega \quad (3.105)$$

$$+ (\alpha_\phi \cdot \tilde{\phi} \wedge \theta)_\beta F_\omega + \tilde{\phi} \cdot Y_\phi[(F_\omega)]_\beta \cdot \theta \quad (3.106)$$

We examine the last term:

$$\tilde{\phi} \cdot Y_\phi[(F_\omega)]_\beta \cdot \theta \quad (3.107)$$

$$= (\tilde{\phi} \wedge \theta)_\beta \cdot [F_\omega, \alpha_\phi] - (\tilde{\phi} \wedge d_\omega \phi)_\beta F_\omega + (\tilde{\phi} \wedge d_\omega^2 \theta) \cdot \alpha_\beta + \tilde{\phi} \cdot Y_\phi[\omega]_\beta \cdot d_\omega \theta \quad (3.108)$$

The first two terms cancel with the first two in 3.105, so

$$Y_\phi[(\tilde{\phi} \wedge \theta)_\beta F_\omega] = (\alpha_\phi \cdot \tilde{\phi} \wedge \theta)_\beta F_\omega + (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi + (\tilde{\phi} \wedge d_\omega \theta) \cdot Y_\phi[\omega]_\beta \quad (3.109)$$

where we already see that in the first term, we have the action  $\tilde{\phi} \mapsto \alpha_\phi \cdot \tilde{\phi}$ . The other terms only involve torsion and are therefore functions of the Gauss constraint.

As for the Gauss piece itself, we can analyse it using its explicit form

$$\frac{1}{2} d_\omega \theta_\beta^2 \cdot \alpha_{\tilde{\phi}} = d_\omega \theta \cdot \omega_\beta \cdot \tilde{\phi} \quad (3.110)$$

which we can act on with  $Y_\phi$  directly to produce

$$Y_\phi[\frac{1}{2} d_\omega \theta_\beta^2 \cdot \alpha_{\tilde{\phi}}] = (d_\omega \theta \wedge \tilde{\phi})_\beta \cdot Y_\phi[\omega] + d_\omega Y_\phi[\theta] \cdot (\alpha_{\tilde{\phi}})_\beta \cdot \theta + (Y_\phi[\omega] \cdot \theta)_I (\alpha_{\tilde{\phi}})_\beta^{IJ} \cdot \theta_J. \quad (3.111)$$

We can rewrite the third term as

$$(Y_\phi[\omega] \cdot \theta)_I (\alpha_{\tilde{\phi}})_\beta^{IJ} \cdot \theta_J = \frac{1}{2} [Y_\phi[\omega], \theta^2]_\beta \cdot \alpha_{\tilde{\phi}}. \quad (3.112)$$

Now the first piece of 3.111 cancels with the last piece of 3.109, yielding

$$\begin{aligned} Y_\phi[(\tilde{\phi} \wedge \theta)_\beta F_\omega] + \frac{1}{2} d_\omega \theta^2 \cdot \alpha_{\tilde{\phi}} &= (\alpha_\phi \cdot \tilde{\phi} \wedge \theta)_\beta F_\omega + (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi \\ &\quad + \frac{1}{2} [Y_\phi[\omega], \theta^2]_\beta \cdot \alpha_{\tilde{\phi}} + d_\omega Y_\phi[\theta] \cdot (\alpha_{\tilde{\phi}})_\beta \cdot \theta. \end{aligned} \quad (3.113)$$

To write this in a manifestly antisymmetric form, we also need contributions coming from the boundary term. To ease notation, we introduce for the following the shorthand

$$Q_\phi^I = (\alpha_\phi)^{IJ} \theta_J = (\omega)^{IJ}_\beta \phi_J. \quad (3.114)$$

We have

$$\begin{aligned} Y_\phi[p_I \tilde{\phi}^I] &= -\tilde{\phi}_I (d_\omega Q_\phi^I - (F_\omega)^{IJ}_\beta \phi_J) - \tilde{\phi}_I \omega_\beta^{IJ} Y_\phi[\theta] \\ &= -(\phi \wedge \tilde{\phi}) \cdot (F_\omega)_\beta - \tilde{\phi}_I d_\omega Q_\phi^I + Q_{\tilde{\phi}}^I \wedge Y_\phi[\theta]_I \end{aligned} \quad (3.115)$$

of which we carry over the second and third term into the bulk,

$$\begin{aligned} d(-\tilde{\phi}_I d_\omega Q_\phi^I + Q_{\tilde{\phi}}^I \wedge Y_\phi[\theta]_I) &= d_\omega Q_\phi^I \wedge Y_\phi[\theta]_I - Q_{\tilde{\phi}}^I \wedge d_\omega Y_\phi[\theta]_I \\ &\quad - d_\omega \tilde{\phi}_I d_\omega Q_\phi^I - \tilde{\phi}_I d_\omega^2 Q_\phi^I \end{aligned} \quad (3.116)$$

and we rewrite

$$\begin{aligned} \frac{1}{2} [Y_\phi[\omega], \theta^2]_\beta \cdot \alpha_{\tilde{\phi}} &= (\phi \wedge \alpha_{\tilde{\phi}} \cdot \theta)_\beta \cdot F_\omega + d_\omega Q_\phi^I \wedge (\alpha_{\tilde{\phi}} \cdot \theta)_I \\ &= (\phi \wedge \theta)_\beta \cdot [F_\omega, \alpha_{\tilde{\phi}}] - (\alpha_{\tilde{\phi}} \cdot \phi \wedge \theta)_\beta \cdot F_\omega + d_\omega Q_\phi^I \wedge (\alpha_{\tilde{\phi}} \cdot \theta)_I \end{aligned} \quad (3.117)$$

and furthermore

$$\begin{aligned} (\phi \wedge \theta)_\beta \cdot [F_\omega, \alpha_{\tilde{\phi}}] &= [(\phi \wedge \theta), F_\omega]_\beta \cdot \alpha_{\tilde{\phi}} \\ &= (\theta \wedge (F_\omega \cdot \phi) - \phi \wedge (F_\omega \cdot \theta))_\beta \cdot \alpha_{\tilde{\phi}} \\ &= -Q_{\tilde{\phi}} \cdot d_\omega^2 \phi - (\phi \wedge d_\omega^2 \theta)_\beta \cdot \alpha_{\tilde{\phi}} \end{aligned} \quad (3.118)$$

in order to combine the pieces into

$$\begin{aligned} &Y_\phi[(\tilde{\phi} \wedge \theta)_\beta F_\omega + \frac{1}{2} d_\omega \theta^2 \cdot \alpha_{\tilde{\phi}}] \\ &= (\alpha_\phi \cdot \tilde{\phi} \wedge \theta)_\beta F_\omega + (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi + d_\omega Y_\phi[\theta] \cdot (\alpha_{\tilde{\phi}})_\beta \cdot \theta \\ &\quad + (\phi \wedge \theta)_\beta \cdot [F_\omega, \alpha_{\tilde{\phi}}] - (\alpha_{\tilde{\phi}} \cdot \phi \wedge \theta)_\beta \cdot F_\omega + d_\omega Q_\phi^I \wedge (\alpha_{\tilde{\phi}} \cdot \theta)_I \\ &\quad + d_\omega Q_{\tilde{\phi}}^I \wedge Y_\phi[\theta]_I - Q_{\tilde{\phi}}^I \wedge d_\omega Y_\phi[\theta]_I - d_\omega \tilde{\phi}_I d_\omega Q_\phi^I - \tilde{\phi}_I d_\omega^2 Q_\phi^I \\ &= ((\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi) \wedge \theta)_\beta F_\omega - d_\omega Q_\phi^I \wedge Y_{\tilde{\phi}}[\theta]_I + d_\omega Q_{\tilde{\phi}}^I \wedge Y_\phi[\theta]_I \\ &\quad - (\phi \wedge d_\omega^2 \theta)_\beta \cdot \alpha_{\tilde{\phi}} + (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi - Q_{\tilde{\phi}} \cdot d_\omega^2 \phi - \tilde{\phi}_I d_\omega^2 Q_\phi^I \end{aligned} \quad (3.119)$$

We can rewrite the last two terms as

$$-Q_{\tilde{\phi}} \cdot d_\omega^2 \phi - \tilde{\phi}_I d_\omega^2 Q_\phi^I = d(Q_{\tilde{\phi}} \cdot d_\omega \phi - \tilde{\phi}_I d_\omega Q_\phi^I) - d_\omega Q_{\tilde{\phi}} \cdot d_\omega \phi + d_\omega \tilde{\phi}_I d_\omega Q_\phi^I \quad (3.120)$$

and

$$Q_{\tilde{\phi}} \cdot d_\omega \phi - \tilde{\phi}_I d_\omega Q_\phi^I = -d(Q_{\tilde{\phi}}^I \phi_I) + \phi_I d_\omega Q_{\tilde{\phi}}^I - \tilde{\phi}_I d_\omega Q_\phi^I \quad (3.121)$$

So that everything is manifestly antisymmetric<sup>10</sup> in  $\phi, \tilde{\phi}$ . This is as expected since we are dealing with Hamiltonian charges, however it is, as demonstrated, a (very) nontrivial

<sup>10</sup>We do not need the codimension 3 term, but we can even write this one as  $\phi_I Q_{\tilde{\phi}}^I = \omega_\beta \cdot (\phi \wedge \tilde{\phi})$



consistency check. The bulk now consists of

$$\begin{aligned} & ((\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi) \wedge \theta)_\beta F_\omega - (\phi \wedge d_\omega^2 \theta)_\beta \cdot \alpha_{\tilde{\phi}} + (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi \\ & + d_\omega Q_\phi^I \wedge (\alpha_{\tilde{\phi}} \cdot \theta)_I - d_\omega Q_{\tilde{\phi}}^I \wedge (\alpha_\phi \cdot \theta)_I \end{aligned} \quad (3.122)$$

and the boundary piece of

$$-(\phi \wedge \tilde{\phi}) \cdot (F_\omega)_\beta + \phi_I d_\omega Q_{\tilde{\phi}}^I - \tilde{\phi}_I d_\omega Q_\phi^I \quad (3.123)$$

However, a meaningful bulk-boundary split only happens when we go on-shell of the bulk constraints. We therefore impose  $d_\omega \theta = 0 = (F_\omega)_\beta \cdot \theta$  which gives

$$d_\omega Q_\phi^I = (d_\omega \alpha_\phi)_\beta \cdot \theta \quad d_\omega^2 Q_\phi^I = [F_\omega, \alpha_\phi]_\beta \cdot \theta; \quad (3.124)$$

then in the bulk we are left with

$$\theta \cdot d_\omega (\alpha_\phi)_\beta \cdot \alpha_{\tilde{\phi}} \cdot \theta - \theta \cdot d_\omega (\alpha_{\tilde{\phi}})_\beta \cdot \alpha_\phi \cdot \theta \quad (3.125)$$

while on the boundary, we have

$$-(\phi \wedge \tilde{\phi}) \cdot (F_\omega)_\beta + \phi \cdot (d_\omega \alpha_{\tilde{\phi}})_\beta \cdot \theta - \tilde{\phi} \cdot (d_\omega \alpha_\phi)_\beta \cdot \theta. \quad (3.126)$$

We now study the bulk piece by comparing it to

$$d_\omega [\alpha_\phi, \alpha_{\tilde{\phi}}]_\beta. \quad (3.127)$$

To do so, we use a number of identities of traces and the homomorphism  $(-)_\beta$ . We focus in this first on the first term of the two. First, we rewrite it in the manifestly cyclic form<sup>11</sup>

$$\theta \cdot d_\omega (\alpha_\phi)_\beta \cdot \alpha_{\tilde{\phi}} \cdot \theta = (\theta^2)_J^I \wedge (d_\omega (\alpha_\phi)_\beta)_K^J (\alpha_{\tilde{\phi}})_I^K = \text{Tr}[\theta^2 d_\omega (\alpha_\phi)_\beta \alpha_{\tilde{\phi}}] \quad (3.128)$$

Then, we can rewrite this as

$$-\frac{1}{2} \theta^2 \cdot [d_\omega (\alpha_\phi)_\beta, \alpha_{\tilde{\phi}}] = -\frac{1}{2} \theta_\beta^2 \cdot [d_\omega \alpha_\phi, \alpha_{\tilde{\phi}}] \quad (3.129)$$

and together with the antisymmetric counterpart, we have in the bulk

$$-\frac{1}{2} \theta_\beta^2 \cdot d_\omega [\alpha_\phi, \alpha_{\tilde{\phi}}] \approx d(-\frac{1}{2} \theta_\beta^2 \cdot [\alpha_\phi, \alpha_{\tilde{\phi}}]) \quad (3.130)$$

so it becomes a pure boundary term. Therefore, we have on-shell:

$$\{P_\phi, P_{\tilde{\phi}}\} \approx (\phi \wedge \tilde{\phi})_\beta \cdot F_\omega - \phi \cdot d_\omega (\omega_\beta \cdot \tilde{\phi}) + \tilde{\phi} \cdot d_\omega (\omega_\beta \cdot \phi) + \frac{1}{2} \theta_\beta^2 \cdot [\alpha_\phi, \alpha_{\tilde{\phi}}] \quad (3.131)$$

---

<sup>11</sup>One can also just do the classic  $AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$  on  $A = (d_\omega \alpha_\phi)_\beta, B = \alpha_{\tilde{\phi}}$ .

Then, through carefully expanding out the second and third term, and using again the trace-commutator identities, we reach the final result

$$\{P_\phi, P_{\tilde{\phi}}\} \approx \oint_{\partial\Sigma} -(\phi \wedge \tilde{\phi})_\beta \cdot F_\omega + \frac{1}{2}\theta_\beta^2 \cdot [\alpha_\phi, \alpha_{\tilde{\phi}}] + \omega \cdot d(\phi \wedge \tilde{\phi})_\beta = \oint_{\partial\Sigma} q_{\phi, \tilde{\phi}} \quad (3.132)$$

which contain a Bianchi piece, a Lorentz piece and a nontrivial extension which vanishes for constant parameters.

Off-shell, the situation is more complicated, but by suitable further manipulations in the bulk we find

$$\begin{aligned} \{P_\phi, P_{\tilde{\phi}}\} = & \int_\Sigma -((\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi) \wedge \theta)_\beta \cdot F_\omega - \frac{1}{2}d_\omega \theta_\beta^2 \cdot 2[\alpha_\phi, \alpha_{\tilde{\phi}}] \\ & + \int_\Sigma (\phi \wedge d_\omega^2 \theta)_\beta \cdot \alpha_{\tilde{\phi}} - (\tilde{\phi} \wedge d_\omega^2 \theta)_\beta \cdot \alpha_\phi + \oint_{\partial\Sigma} q_{\phi, \tilde{\phi}} \end{aligned} \quad (3.133)$$

Now this is the kind of result we would expect; The shifts here contain Lorentz transformations, therefore it is reasonable that the translations can act nontrivially on each other and produce the parameter  $\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi$ . This is not just an artifact of the bulk parametrisation, but instead persists to the onshell phase space as seen above. Furthermore, while the exact way to express the bulk terms so that it looks like a combination of precisely  $P, J$  will involve some possibly quite complicated structure functions, it is obvious that the bulk pieces are combinations of the constraints. Therefore, as expected, for the constraint part of the charges (the bulk pieces), their Poisson bracket vanishes on-shell, as it should.

### 3.3.3 Interpretation of corner charge

We thus have charges which form an analogue of a Poincare algebra, and which are manifestly Lorentz covariant up to a transformation anomaly due to  $\omega$ .

Given that the charges and transformations we gave are intimately related to diffeomorphisms, an obvious question is how the corner piece relates to other ones found in the literature. There is an immediate way to do this for the charges corresponding to spatial translations: Adopting a 3+1 internal decomposition along the adapted internal normal  $u_\Sigma$  to a slice  $\Sigma$ , we perform decompositions as follows<sup>12</sup>: Any internal vector  $V^I$  has decomposition

$$V^I = V_\parallel^I + u_\Sigma^I V_\perp \quad V_\perp = V \cdot u_\Sigma \quad (3.134)$$

and Lorentz tensors  $M^{IJ}$  split into two vectors

$$\begin{aligned} M &= (M_\perp \wedge u_\Sigma) - \star(M_\parallel \wedge u_\Sigma) \\ M_\perp^I &= M^{IJ} u_{\Sigma J} \quad M_\parallel^I = (\star M)^{IJ} (u_\Sigma)_J \end{aligned} \quad (3.135)$$

Meanwhile, if, for the given (spacelike) slice  $\Sigma$ , an associated normal 1-form  $\mathbf{n}$  is available, together with some dual vector  $\hat{n}$  satisfying  $\mathbf{n}(\hat{n}) = 1$ , any differential form  $B$  can be

<sup>12</sup>We refer to [77] for notations, where our  $u_\sigma$  is called  $n^I$ , and  $\Gamma_\parallel^I = \tilde{\epsilon}^{IJK} \Gamma_{JK}$ .

decomposed into a horizontal piece (denoted by a tilde) and a vertical piece (denotes by an index  $n$  for normal to the slice):

$$B = \tilde{B} + \mathbf{n} \wedge B_n \quad B_n = i_{\hat{n}} B \quad (3.136)$$

Similarly, for (multi)vector fields  $\xi$ ,

$$\xi = \xi_{\perp} + \hat{n} \wedge \xi_n \quad \mathbf{n}(\xi) = \xi_n. \quad (3.137)$$

Now let us do this for tetrad gravity. The tetrad decomposes as

$$\theta = \tilde{e} + \mathbf{n} u_{\Sigma} \quad \tilde{e} \cdot u_{\Sigma} = 0, \quad (3.138)$$

into a spatial triad and the normals, the spin connection as

$$\begin{aligned} \omega &= (K \wedge u_{\Sigma}) + \Gamma \quad d_{\Gamma} u_{\Sigma} = 0, K = d_{\omega} u_{\Sigma} \\ &= ((K - du) \wedge u) - \star(\Gamma_{\parallel} \wedge u) \end{aligned} \quad (3.139)$$

and the curvature as

$$F_{\omega} = (d_{\Gamma} K \wedge u) + F_{\Gamma} - (K \wedge K). \quad (3.140)$$

we can expand the momentum  $p^I$  into<sup>13</sup>

$$\begin{aligned} p^I &= -(\omega)_{\beta}^{IJ} \wedge \theta_J \\ &= -\beta((\Gamma_{\parallel} - \gamma K) \times \theta)^I - (\Gamma_{\parallel} + \beta K)^I \wedge (u \cdot \theta) + u^I (\Gamma_{\parallel} + \beta K)_J \wedge \theta^J. \end{aligned} \quad (3.141)$$

In this expression, the combinations of the spatial spin connection  $\Gamma_{\parallel}$  and the extrinsic curvature  $K$

$$A = \Gamma_{\parallel} - \gamma K \quad C = \Gamma_{\parallel} + \beta K \quad (3.142)$$

are recognisable as the Ashtekar-Barbero-Sen connection and its counterpart which is needed to reconstruct the full spin connection. What's noteworthy is that because these objects are connections, neither of the terms above can be Lorentz vectors. However, as soon as we subtract a reference Lorentz connection offset  $\omega_0$ ,

$$p^I - p_0^I := -(\omega - \omega_0)_{\beta}^{IJ} \wedge \theta_J \quad (3.143)$$

this is no longer an issue. We will only be interested in the value on spatial slices and particular on corners. On a given spatial slice (with  $u_{\Sigma}^2 = 1$ ),  $\tilde{\theta} \stackrel{\Sigma}{=} \tilde{e}$  reduces to the spatial triad, and we have

$$\tilde{p}^I \stackrel{\Sigma}{=} -\beta((\tilde{\Gamma}_{\parallel} - \gamma \tilde{K}) \times \tilde{e})^I + u^I (\tilde{\Gamma}_{\parallel} + \beta \tilde{K})_J \wedge \tilde{e}^J. \quad (3.144)$$

Let us also further decompose the fields on a (closed, isolated) corner surface  $S \subset \Sigma$ , and decompose the triad there with respect to a spacelike 1-form  $\mathbf{s}$  of the surface when included

---

<sup>13</sup>Recall that  $\beta = \frac{1}{\gamma}$ , with  $\gamma$  the Barbero-Immirzi parameter.

in  $\Sigma$ . Let  $\hat{v}_I^i$  denote the inverse triad, and  $\varsigma_I = \hat{v}_I^i s_i$  the internal adapted radial normal, then we decompose

$$\tilde{e} \stackrel{S}{=} \bar{\tilde{e}} - \varsigma \mathbf{s} \quad (3.145)$$

into a zweibein  $\bar{\tilde{e}}$  and the radial piece. This implies through some calculations that the gravitational flux  $E = \frac{1}{2}(\tilde{e} \times \tilde{e})$  splits similarly as

$$\tilde{E}^I \stackrel{S}{=} \bar{E}^I - \mathbf{s} \wedge E_r \quad (3.146)$$

$$= -\mathbf{w} \varsigma^I - \mathbf{s}(\varsigma \times \bar{\tilde{e}})^I \quad (3.147)$$

where  $\mathbf{w}$  is the area density 2-form of the zweibein  $\bar{\tilde{e}}$  on  $S$ , defined as

$$\mathbf{w} = |\det(\bar{\tilde{e}}_a^I)| d^2x. \quad (3.148)$$

As  $S$  is 2-dimensional, any 2-dimensional 2-form must be related to  $\mathbf{w}$  through some prefactor. We can write this prefactor for  $p^I$  by first splitting

$$A^I \stackrel{S}{=} A_J^I e^J \quad C^I \stackrel{S}{=} C_J^I e^J, \quad (3.149)$$

then it is simply

$$\bar{p}^I \stackrel{S}{=} - \left[ \beta_{\varsigma J} (A^{JI} - A_C^J \tilde{\eta}^{IJ}) + u^I \star (u \wedge \varsigma)^{IJ} C_{IJ} \right] \mathbf{w}. \quad (3.150)$$

This is interesting as it suggests that the only nonzero components of  $p$  on the corner are the timelike  $u$  and radial  $\varsigma$  ones, unless  $A^{IJ}$  has off-diagonal elements as a 3x3 matrix. It is worth comparing this to the corner Lorentz charge of tetrad gravity, evaluated in the same way, which reads

$$\theta_\beta^2 \cdot \alpha \stackrel{S}{=} (u \wedge \varsigma) \cdot (1 - \beta \star) \alpha \mathbf{w} \quad (3.151)$$

and which also only picks up the internal directions normal to the surface  $S$ .

To get a better interpretation of this object, we go on-shell of the Gauss constraint, bringing us closer to metric gravity. In principle, this means that we replace  $\Gamma_{\parallel} = \gamma[\tilde{e}]$  by the 3D Levi-civita spin connection

$$\gamma_i^I[e] = \frac{1}{2} \hat{v}^{Ij} \tilde{e}_{iK} (\partial_t \tilde{e}_j^K - \partial_j u^K) \quad (3.152)$$

which one obtains via the Koszul formula[77] for the full 4D Levi-civita spin connection. On a slice  $\Sigma$ , it implies rather

$$\tilde{d}_{\Gamma_{\parallel}} \tilde{e} = 0 = \tilde{e}_I \wedge \tilde{K}^I. \quad (3.153)$$

By remembering that we should take differences of connections, we then can drop the  $\Gamma_{\parallel} \times e$  and  $K_I e^I$  terms in 3.144, and the expression simplifies to

$$\bar{p}^I \stackrel{S}{=} \left[ (\tilde{K}^{JI} - \tilde{\eta}^{IJ} \tilde{K}_C^C) \varsigma_J + u^I \star (u \wedge \varsigma)^{IJ} \gamma[\tilde{e}]_{IJ} \right] \mathbf{w} \quad (3.154)$$

and

$$\gamma[\tilde{e}]_{IJ} = \frac{1}{2} \hat{v}_I^j \tilde{\eta}_{JK} (\partial_t \tilde{e}_j^K - \partial_j u^K). \quad (3.155)$$

The object  $(K^{JI} - \tilde{\eta}^{IJ} K_C^C)$  is simply the usual gravitational ADM momentum built from the extrinsic curvature tensor, so the term with  $\varsigma$  is just the Brown-York charge [77, 190]. In particular, for timelike translations, this term will not contribute and only the term involving  $\gamma[\tilde{e}]$  is going to be relevant. Of course, because the original momentum is a Lorentz covariant object, we can equally do a 3+1 decomposition along a timelike slice in the boundary and find the same expression with the radial and timelike normals swapped. Therefore, in fact, the timelike translation generator is *also* just the Brown-York generator. Therefore, by swapping the roles of  $u, \varsigma$ , we can get the *equivalent* expression

$$\bar{p}^I \stackrel{S}{\cong} [(\bar{K}^{JI} - \bar{\eta}^{IJ} \bar{K}_C^C) u_J - \varsigma^I \star (u \wedge \varsigma)^{IJ} \gamma[\tilde{e}]_{IJ}] \mathbf{w} \quad (3.156)$$

in which we again identify  $\bar{K}^{JI} - \bar{\eta}^{IJ} \bar{K}_C^C$  as the main factor of the Brown-York stress energy tensor

$$\bar{T}_{BY}^{IJ} := |\det(\tilde{e}_a^I)| (\bar{K}^{IJ} - \bar{\eta}^{IJ} \bar{K}_C^C). \quad (3.157)$$

Therefore, we have for non-radial  $\phi$

$$\oint_{\partial\Sigma} \bar{p}^I \phi_I = \oint_{\partial\Sigma} \phi_I \bar{T}_{BY}^{IJ} u_J d^2x \quad (3.158)$$

and so we know that timelike shifts really are really generated by the Brown-York energy (where the offset of the Brown-York charge is the same offset we had to do in subtracting  $\omega_0$ ). It therefore automatically also asymptotes to the ADM energy at asymptotically flat boundaries.

*In summary, the charge  $p^I$  is really a first-order variant of the Brown-York energy-momentum.*

## 3.4 Summary

Let us give a less technical overview of the results of this chapter.

We showed that one can equally well understand the symmetries of 4D tetrad gravity to be shifts and Lorentz transformations. Diffeomorphisms are, in this picture, just combinations of shifts and Lorentz transformations. In the bulk, so in the absence of boundaries, this is just a mere reformulation and does not affect the on-shell phase space of physical configurations - but it does make a difference when boundaries are present.

When boundaries are present, there is a sharp difference between diffeomorphisms and shifts, in that diffeomorphisms which move the boundary are not representable as canonical transformations. As we said before, this means that they do not admit a simple quantization in terms of unitary operators - and there is no way to compute their Poisson bracket of generators, because there are no generators. It is therefore not clear what it would mean for the quantum theory to be symmetric under diffeomorphisms.

On the other hand, the right notion of shifts, as we presented it here, is a canonical transformation and can therefore in principle be quantized like any other symmetry. The only complication that arises is due to field-dependent structure constants (structure functions), which are however ubiquitous in generic field theories and can be handled, i.e. with the BV formalism of gauge fixing.

We also presented the algebra of the shifts and Lorentz transformations. While the Lorentz-Shift part is very comparable to a usual Poincaré algebra, the shift-shift part is really unlike it. One has to interpret it as follows: Let us think of a ‘naive’ shift charge  $\tilde{P}_\phi$  as the one that ‘should’ be commutative; then the improved shift charge is

$$P_\phi = \tilde{P}_\phi + J_{\alpha_\phi} \quad (3.159)$$

and so we have that the Poisson bracket of improved shifts splits formally into

$$\{P_\phi, P_{\tilde{\phi}}\} = \{\tilde{P}_\phi, \tilde{P}_{\tilde{\phi}}\} + \{J_{\alpha_\phi}, \tilde{P}_{\tilde{\phi}}\} + \{\tilde{P}_\phi, J_{\alpha_{\tilde{\phi}}}\} + \{J_{\alpha_\phi}, J_{\alpha_{\tilde{\phi}}}\} \quad (3.160)$$

The first term should then vanish, and the second and third term give something of the form

$$P_{\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi} \quad (3.161)$$

just from the commutator of Lorentz and translations. Similarly, the last term should give a term  $J_{[\alpha_\phi, \alpha_{\tilde{\phi}}]}$ . Indeed, this is almost the structure we find off-shell 3.133, but with deviations. Especially striking are the deviations due to  $d_\omega^2 \theta$  terms, which do not appear in any of the other gauge generators. Still, the logic is still here: when we go on-shell of the Gauss constraint, there is only the Einstein constraint piece with parameter of the form  $\alpha_\phi \cdot \tilde{\phi} - \alpha_{\tilde{\phi}} \cdot \phi$ .

It is worth noting that the constraint  $d_\omega^2 \theta = 0$ , in the continuum a consequence of the Gauss constraint, is not necessarily implied in the discrete[191]. Instead, it is sometimes an *independent* constraint, known as the *edge simplicity constraint*<sup>14</sup>

$$F_\omega^{IJ} \wedge \theta_J = 0. \quad (3.162)$$

In the discrete, this is the statement that the discretized tetrad, interpreted as an edge vector, is left fixed by holonomies around curvature defects.

We also studied the on-shell algebra on the corner, which generates the more interesting physical symmetries. We found that some pieces are analogous to 3D gravity, in particular a noncovariant term due to the transformation behaviours of  $\omega$ , but a curious contribution is due to the curvature,

$$\{P_\phi, P_{\tilde{\phi}}\} \approx - \oint_{\partial \Sigma} (\phi \wedge \tilde{\phi})_\beta \cdot F_\omega + \dots \quad (3.163)$$

which has been seen in related shift symmetry algebras as well[192]. This is, however, a covariant version in which the whole pullback of the spin connection curvature appears.

---

<sup>14</sup>Note the duality with the Einstein constraint,  $\star F_\omega^{IJ} \wedge \theta_J = 0$ .

This kind of curvature term is curious as a corner charge, because it is not one of the existing shift or Lorentz charges at all, and in fact mirrors another charge known from Maxwell theory. The charge in question is the *magnetic charge*

$$H_\chi = \oint_{\partial\Sigma} F\chi = \oint_{\partial\Sigma} B^r \chi d^2x \quad (3.164)$$

which has the *magnetic* instead of the electric radial field as a density. This means that its  $l = 0$  spherical harmonic is actually counting the presence of magnetic monopoles inside the region<sup>15</sup>.  $l = 1$  and higher then count magnetic dipoles and higher multipoles. Similarly, one can expect the corner charge here to be some sort of gravitational monopole charge as well, possibly related to Taub-Newman-Unti-Tamburino (NUT) charges and logarithmic supertranslations[193, 194].

Furthermore, the remaining terms in the charge algebra are *linear* in  $\theta, \omega$ . This again requires an extension by additional corner charges, further enriching the symmetry structure of the corner. In particular, It requires upon discretization that one amends the BF theory data we saw in 1.5,A.5 by additional variables sitting on interfaces, i.e. the terms linear in  $\omega$  become a boundary holonomy and those in  $\theta$  become edge vectors. This shows that for consistency, one needs to work with *both*  $\omega, \theta$ , rather than just with  $\omega$  and its conjugate  $\theta^2$ , or just with  $\theta$  and its conjugate  $p$ .

So, this parametrization of symmetries brings with it certain advantages that make it more natural to consider for quantization and discretization. Of course, no advantage comes for free, and we must use the full modern toolset of dealing with gauge theories in order to work with the set of structure functions. Still, the shifts put 4D gravity into the same perspective as 3D gravity, as now their symmetry structure is quite similar. Therefore, the difference between 3D and 4D gravity lies not in their different symmetry structure, but in the specifics of the dynamics, in particular the nontrivial local degrees of freedom of 4D. However, one might expect that 3D quantization techniques can then be applied at least to Coulombic sectors of gravity, where no radiation is present<sup>16</sup>.

---

<sup>15</sup>In particular, it is a multiple of the first Chern class over  $\partial\Sigma$ , so is a topological invariant of the principal bundle underlying the theory. Its vanishing implies the bundle is trivial over the boundary of the slice.

<sup>16</sup>This may not be restricted to the radiation-free sectors. What is essential is that the techniques from 3D gravity only refer to the quantization of Coulombic degrees of freedom, so radiative ones may require different treatment.





# Chapter 4

## Discussion

### 4.1 Overview

Let us take a moment to recapitulate what we have treated so far.

We dealt with several different aspects of boundary effects in gravity, both in the continuum and the discrete, most of them related to either the presence of boundary gauge symmetries or the effects of nontrivial cutting and glueing. In chapter 2, this was due to our interest in entanglement between spatial regions, which necessitates a clear notion of subsystem on the lattice gauge theory side. There, we saw that the entanglement we expect from the Ryu-Takayanagi-Rangamani formula in AdS/CFT only appears naturally if

1. we count the edge mode matching contribution to the entropy as a genuine entanglement entropy,
2. we adopt the correct link states or interpretation of link spaces so that the lattice notion of area matches with the geometric notion of area derived from the continuum.

Both of these are highly nontrivial from the pure lattice point of view. We also saw that the criteria for holography include a constraint on the total area of the region, which, in terms of continuum fields, would be expressed as

$$\delta\left(\oint_{\partial\Sigma} \sqrt{g} d^2x\right) = 0 \quad (4.1)$$

so the global or constant part of the area density  $\sqrt{g}$  is fixed. This corresponds to a global spin constraint in the gravitational theory, analogous to constancy of the total charge in Maxwell theory, and is therefore expected from the continuum side. *The continuum informs the discrete.*

In chapter 3, we instead reconsidered the symmetry structure of gravity from the outset and found that one may replace the previously common  $\text{Diff}(S) \ltimes SO(1,3)^S$  corner symmetry by a more complicated structure, with the underlying set

$$(SO(1,3) \times \mathbb{R}^{1,3})^S. \quad (4.2)$$

which features more complicated structure functions, but admits a conceptually more straightforward passage from continuum to discrete because it does not involve tangent vector fields. *Requirements made in the discrete inform the continuum.* We also saw that the algebra of corner charges requires us to include objects that are linear in the fields, so to extend the data we keep in discretizations compared to BF theory.

All of these ideas are contextualized by the ground work we did beforehand: Cutting and glueing are more complicated in gauge theories in the continuum, and one needs to choose the correct sampling of configurations to end up at the conclusions commonly made at the discrete level. In particular, we saw that we can attribute the nontrivial glueing of regions to two phenomena:

1. gauge invariant degrees of freedom, i.e. scalar field values, which require regularization,
2. gauge degrees of freedom in the form of dressing fields, which can be glued directly on the continuum level.

This distinction is essentially because the dressing fields only really live on the boundary interfaces anyway, and therefore their smearing over boundary surfaces is natural. This is not the case for bulk fields, which must be smeared over  $D$ -dimensional spacetime regions (or at least  $D - 1$  for free fields).

Let us also give a few remarks about the general context we explored with these topics. In all this, it was useful to know that discreteness and cutting/glueing were naturally linked with each other. When discretizing, we must take close care of cutting and glueing appropriately, particularly once we have sampled on certain special configurations. To solve the cutting and glueing problems, it turned out that discretization itself was a solution, but delivering different outcomes depending on the choice of sampling. So, the two ideas are interlocked in subtle ways that inform each other.

We also saw that the data used in the glueing of gauge theories was principally the dressing fields, which, as we already mentioned multiple times, have the status of extrinsic[20, 22, 112] reference frames for the gauge transformation group of a region. This shows that generally speaking, the nontrivial properties that gauge theories have can be encoded in the relations between different observers - i.e. those sitting on two sides of a boundary, related by a group element like the dressing field.

A subsequent discretization and *quantization* of the dressing fields, then, could correspond to a quantum version of such reference frames or relations between them. Such reference frames have received plenty of study (see e.g. [24, 26, 80, 104, 115, 195–203] and references therein) in recent years and gauge edge modes, in the guise of dressing fields, are a particularly rich instance of them.

In a sense, the study of these reference frames is what enables the specification of subregions in gauge theories; thus, to have a realistic description of physical systems - which, if operationally accessible, are of finite extent (even in cosmology) - one needs to have a good understanding of how subsystems are specified, which we now argue to have better control

over, both at the continuum as in the discrete.

## 4.2 Outlook

The concrete results presented in this thesis, but even more so the general context in which they are placed, are part of an ever expanding set of questions that informs and enriches quantum gravity model building, but also our general understanding of how gauge theories work both in the continuum and on the lattice.

Let us first start with direct consequences of the results we presented here, and then move to more broad ideas for the future of research in boundary effects in gauge theories and quantum gravity.

First, we can immediately see many interesting possible extensions of the work of chapter 2: The states we considered were kinematical, i.e. did not fulfil all the constraints needed to be considered proper initial data for gravity. The obvious logical step would be to consider (classes of) constraint-solving states<sup>1</sup> and their transport properties, i.e. how holographic they are. On this end, we can expect that the other constraints of gravity reduce the degrees of freedom on the sites, i.e. the intertwiners. By well-known arguments [204] via Lichnerowicz' equation, we can infer that at fixed spins  $j_\alpha$ , the remaining degrees of freedom after imposing the constraints do *not* include the volume, which is fixed by the constraints. Instead, the dihedral angles are still free, and the reduced intertwiner space is given by a specific fixed eigenspace of the volume operator, so it is much smaller. It is, however, nontrivial still, so the same kinds of criteria as in the main text are necessary to check for holography.

Another relatively obvious question is introducing a dynamical picture. Of course, in part the question is equivalent to working with on-shell states, due to the generally covariant nature of gravity. Thus, a choice of dynamics is really only relevant on boundaries, in particular in the form of boundary conditions. Still, there are first ideas on how to implement some kinds of dynamics in these tensor network models[205], which however should be expanded much further.

Also, the methodology in our holography analysis has been focused on a fixed graph structure, which, from the continuum point of view, corresponds to a fixed triangulation of a spatial slice. This, of course, is only a very restricted subset of all kinematical states (which include all possible graphs) and so, to make statements about the entire Hilbert space of the theory, one may need to find an appropriate extension of the tensor network holography setup that includes superpositions and equivalences of graphs.

A way in which this may be almost circumvented would be to work with a triangulation invariant model, as is the case in 3D gravity in the bulk. Then, the choice of graph is essentially irrelevant and one may always work with the minimal graph compatible with the

---

<sup>1</sup>Technically speaking, one would need to consider constraint solving functionals, i.e. bras, or equivalently states in higher degree BRST cohomology.

topology of the slice. Such triangulation invariance requires, in part[206–210], an analogue of the vertex translation symmetry in discrete 3D gravity[211] (though, this may be yet insufficient). So, if one were to establish an extended set of states, in which the 4D tetrad shifts act and one can find the invariant states, then one would not need to care about the bulk parts of the graph at all.

This naturally leads into the extensions of chapter 3: one of the main hopes with which we considered the reformulation of symmetries was to find a way to implement diffeomorphisms on the lattice without making reference to embeddings of the graph. In order to understand what is needed for such an implementation, let us compare to the case in 3D[211]. There, one needs a way to work with 'metric' data on the lattice, as opposed to the group-valued labels we originally ended up with in the case of BF theory in section 1.5. This is implemented through an analogue of a Fourier transform,

$$\Psi(g_1, g_2, g_3) \xleftrightarrow{\mathcal{F}} \hat{\Psi}(x_1, x_2, x_3) \quad (4.3)$$

where  $g_i \in SO(3)$ ,  $x_i \in \mathfrak{so}(3) \cong \mathbb{R}^3$ . In this representation of the state of a triangle, the variables  $x_i$  are the edge vectors of the triangle sides, so  $|\hat{\Psi}(x_1, x_2, x_3)|^2$  gives the likelihood in given state to find a triangle with the given edge vectors. In this 'metric' representation, the shifts correspond directly to vertex translations, which shift the edge vectors  $x_i$  as  $(x_1, x_2, x_3) \mapsto (x_1 + a, x_2 - a, x_3)$  et cetera. For the 4D gravity case, one would first need an appropriate analogue of the metric representation, and a way to see how the shift symmetry acts on a sampled configuration used for discretization. A reasonable choice for such a sampling is a piecewise flat, torsion free one - the geometric 'vacuum' of gravity is usually Minkowski space, so it makes sense to sample configurations near it if we want to describe physics comparable to the continuum.

However, the corner algebra (see equations 3.82) also shows that we need more than the analogy to the 3D case: We need additional, linear charges to make the algebra close, and these must be part of the discretization as well. The logic of simplicial geometry tells us that this should include discretizations of  $\theta$  on *edges* of the cellulation, whereas  $\omega$  must be discretized on bulk *and* boundary dual edges.

Such a set of data can be understood as coming from an extended, constrained BF theory formulation of tetrad gravity. Such a theory would have not 2, but instead (at least) 4 gauge generators

$$\begin{aligned} J_\alpha &\sim \oint \langle B, \alpha \rangle & P_\phi &\sim \oint \phi_A t^A \\ K_\mu &\sim \oint \langle \omega, \mu \rangle & \tau_v &\sim \oint \theta^A \wedge v_A \end{aligned} \quad (4.4)$$

and involve an  $\mathfrak{so}(1, 3)$ -BF pair  $B, \omega$  as well as one for  $\mathbb{R}^{1,3}$ ,  $t, \theta$ . These are the fields one would expect if one works with the gauge algebra  $\mathfrak{so}(1, 3) \oplus \mathbb{R}^{1,3}$ , but rather than seeing it as the usual Poincaré algebra, simplicial geometric arguments instead suggest that this should be seen as the algebra of the *Poincaré 2-group*[130, 191]: The same set, but treating the translations and rotations not on even footing. In the discrete, this difference is manifest

when placing the fields on a lattice: The usual  $B$  fields would be placed on faces,  $\omega$  or its holonomy on dual edges; but then the  $t$  and  $\theta$  variables can be discretized in different ways. One puts  $\theta$  on dual edges, leading to the Poincaré 1-group, whereas placing  $\theta$  on the edges leads to the Poincaré 2-group as the symmetry of the phase space. This difference is crucial for the simplicial geometric interpretation of the theory, as usual gravity is obtained from the BF fields through the simplicity constraint

$$B = \frac{1}{2}\theta_\beta^2 \quad (4.5)$$

which, when discretized, requires that  $\theta$  and  $B$  must live both on the cellulation, rather than one of them on the cellulation and one on the dual cellulation.

However, this would sell the implications of the shift symmetries short, still: They are just as interesting in the continuum, where their integrability promises that the asymptotic boundary symmetry behaviour might be analogous to that of other internal symmetries, i.e. the asymptotic symmetries of Yang-Mills theory[53, 114, 212–214]. In particular, the many ambiguities usually associated with nonintegrable gravitational diffeomorphism charges is largely absent for the shifts, with the notable exception of conservation, which still must be ensured by an appropriate choice of boundary condition and parameters (see [215] for an example in BF theory). It would be highly interesting to see how the resulting asymptotic symmetry algebra is related to existing cases derived from diffeomorphisms, i.e. the Weyl-Bondi-Metzner-Sachs group[216].

This is made especially intriguing by recent developments in asymptotic symmetries, in the study of logarithmic supertranslations[194, 217]. The additional terms appearing in the shift-shift commutator involve the boundary curvature, which is commonly associated with gravitational monopoles, also known as NUT charges. These NUT charges turn out[193] to be the lowest spherical harmonics in the generator of logarithmic supertranslations, leading one to guess that the shift algebra also requires us to keep track of a finite distance version of logarithmic supertranslations.

We can further speculate about many interesting future projects that make use of the shifts, but also the noted resolutions of the puzzle of areas in section 2.2.3. In particular, one could directly use the shift transformations in analogous ways to 3D gravity[218], where they allow a classical, continuum derivation of the quantum group symmetry known from the Turaev-Viro model[162, 219, 220]. The shifts have also been crucial in the derivation of refined spin network states known as Poincaré networks[221, 222], whose structure crucially relies on the corner commutators of the fields we saw appear in the algebra of charges.

On the other hand, the puzzle of areas suggests that there are ‘correct’ link states to use to entangle spatial cells in order to have a meaningful interpretation of entanglement entropies in terms of geometric areas. This should be puzzling, as it suggests that only for *certain types of entanglement*, one has a geometric interpretation, and as we already noted in the respective section, one can see this entanglement as being due to invariance under

certain symmetries related to gravitational dynamics. From the perspective of arguments that posit that gravity universally emerges from entanglement, this is unexpected, but seems to simply point to a few subtleties that require attention. It is enticing to investigate this relationship in more detail [223], especially because the semiclassical dynamics of gravity has long been known to be essentially *equivalent* to a certain type of spacetime thermodynamics[224–226], for example through the Clausius relation for processes near local Rindler horizons (i.e. energy-momentum conservation) as well as the Bekenstein-Hawking law for the thermodynamics observed by Rindler observers<sup>2</sup>.

In the realm of holography, it would be instructive to gain full control over 3D holography in the discrete, mainly because most pieces of the puzzle are already in place - 3D gravity is well understood in almost all aspects, the dual dynamics is known to be a Wess-Zumino-Witten model at finite boundaries, or Liouville theory on asymptotically AdS boundaries, and it is also known how to obtain the latter as the continuum limit of a matrix model, so a sum over cellulations of a discrete model. Spelling out this duality in full in the discrete setting would set a precedent for the 4D case, which is of course more interesting, but already very involved at the continuum level.

It has also been shown in recent research[169, 231–233] that regularization of entropies in QFT can be achieved by inclusion and dressing of observables by gravitational constraints - in effect, the entropy calculated with respect to gravitationally gauge-invariant observables, measured relative to a physical observer, is finite. This is not the case for the generic von Neumann entropy in QFT, which generically diverges with a leading area-law term. Such a phenomenon of regularization is known under the term ‘type reduction’. As the relevant ingredient in this process is gravity, together with a *reference frame*[234], we can speculate that boundary dressing fields might fulfil a similar function. As our discretization also leads to an effective type reduction (to type I algebras), this further begs the question: Can one reduce continuum algebras to discrete ones through a similar procedure?

Lastly, we note a missing piece in the literature: A formalism for the treatment of phase spaces and quantization of field theories with open boundary conditions. What we mean specifically is that when working with nonzero symplectic fluxes, i.e. when the system is not closed, one has to deal with nonintegrable diffeomorphisms. This leads to a number of more or less ad-hoc ways to deal with nonintegrable charges which however do not extend to the full phase space of the open systems. In fact, as of this writing, we are not aware of a formalism that extends the symmetry and phase space analysis we employ for closed systems, to the open case. We suspect that the setting of contact geometry may provide a fruitful generalization of our setup. This could eventually lead to a quantization prescription for field theories without regard to boundary conditions, which would in that setting be quantum states in a larger Hilbert space instead.

---

<sup>2</sup>Note that this refers to spacetime thermodynamics[227–229]. Such thermodynamics, and arguably all thermodynamics, depends on a notion of time and therefore observer. Rindler observers are associated to flow lines of boost vector fields in the vicinity of the horizon[230]

# Appendix A

## Appendices to chapter 1

### A.1 Types of phase spaces

We now need to speak about the choice of phase space and the associated structures on them. Ideally, we will package the choices into a few transparent, classical objects which we can motivate from a classical continuum field theory, and which allow for local assignments in such a way that glueing and splitting of regions is easy. We first need to clarify a preliminary question about the choice of phase space  $P$ , as there are two seemingly unrelated choices for this.

Describe a classical field theory through a number of fields  $\phi^A(x)$  of for now arbitrary spacetime transformation character. Fix also a Cauchy slice  $\Sigma$  of the spacetime  $M$  in question.

The first option is the so-called *canonical phase space*  $P_\Sigma^{\text{can}}$ . This is constructed from all the restrictions  $\phi^A|_\Sigma$  of spacetime fields to the slice  $\Sigma$ , collected into a *canonical configuration space*  $\mathcal{C}_\Sigma^{\text{can}}$ . The canonical phase space is then constructed as its cotangent bundle

$$P_\Sigma^{\text{can}} = T^*\mathcal{C}_\Sigma^{\text{can}} \quad (\text{A.1})$$

and equipped with the tautological 1-form of the cotangent bundle as its symplectic potential. Locally, we can express this potential as

$$\Theta_\Sigma^{\text{can}} = \int_\Sigma \pi_A \wedge \delta\phi^A \quad (\text{A.2})$$

where the conjugate momenta  $\pi_A$  must be of appropriate form degree.

The second option is the so-called *covariant phase space*  $P_M^{\text{cov}}$  and is the starting point for the BV formalism[106, 235]. This is the same as the other, but without the restrictions to the slice, so all spacetime fields being collected into a *covariant configuration space*  $\mathcal{C}_M^{\text{cov}}$ . The covariant phase space is then constructed as its cotangent bundle

$$P_M^{\text{cov}} = T^*\mathcal{C}_M^{\text{cov}} \quad (\text{A.3})$$

and equipped with the tautological 1-form of the cotangent bundle as its symplectic potential. Here, the symplectic partners of the fields are known as the *antifields*  $\phi^\mp$ , and we

write the symplectic potential as

$$\Theta_M^{\text{cov}} = \int_M \phi_A^\mp \wedge \delta \phi^A \quad (\text{A.4})$$

which, in contrast, is integrated over all points of spacetime  $M$ . In principle, these phase spaces are both ‘unphysical’ in that they have, respectively, too little and too much information relative to the actual physical configurations of the system. In the canonical phase space, one has only data on a given fixed slice, and must evolve it in time to recover a physical history. In the covariant phase space, instead, one has configurations which do not fulfil any equations of motion. In the former, one remedies this by including, by necessity, an evolution rule through a Hamiltonian function  $H$ . In the latter, one instead takes the subset of configuration space  $\mathcal{C}_M^{\text{cov}}$  which fulfils the given equations of motion, together with an appropriate symplectic reduction procedure to arrive at a proper phase space.<sup>1</sup>

There is also a hybrid of the two ideas which has been particularly popular in the literature on boundaries[20, 22, 76, 77, 83–105]. It is, for lack of a better term, *also* known as the “covariant phase space approach”, though one needs to be a little careful about the relation between them. It starts from the following idea: Given the action for a free point particle with position  $q$ , we can write its variation as

$$\delta S[q(t)] = \int_0^T dt \, m\ddot{q} \delta q + (m\dot{q} \delta q)|_0^T \quad (\text{A.5})$$

which consists of a bulk term given by the equations of motion,  $E = m\ddot{q} \delta q$ , and a boundary term at initial and final time, given by, effectively, the canonical symplectic potential  $p\delta q$ , once one of the Hamiltonian equations  $p = m\dot{q}$  has been used. The logic is then that one can use this boundary term as a sort of partially-on-shell version of the canonical symplectic potential, but derived from a spacetime description of the theory. Note already here that the ‘partially-on-shellness’ of this potential is just an artifact of using second order variables - if we rewrite the action in first-order form  $S[q, p] = \int_0^T dt \, p\dot{q} - \frac{p^2}{2m}$ , the variation is instead the canonical one

$$\delta S[q, p] = \int_0^T dt; \delta p(\dot{q} - \frac{p}{m}) - \dot{p}\delta q + (p\delta q)|_0^T \quad (\text{A.6})$$

and the potential is precisely the canonical one. Going on-shell of the equation of motion<sup>2</sup> of  $p$ , we recover the former expression. This suggests that in first-order formulations of theories, the boundary term in the variation can be used as-is as a symplectic potential. This gives rise to the following phase space, constructed in two steps:

Fix an action  $S$  and pick out the integrand of the boundary piece in the variation,  $\theta_1$ . Then, pick a slice  $\Sigma$  in the spacetime  $M$ . Then, we construct a *pre-symplectic phase space*  $\tilde{P}_\Sigma$  as the set of *spacetime configurations*  $\mathcal{C}_M^{\text{cov}}$ , equipped with the *pre-symplectic potential/form*

$$\tilde{\Theta}_\Sigma = \int_\Sigma \theta_1 \quad \tilde{\Omega}_\Sigma = \int_\Sigma \delta \theta_1 \quad (\text{A.7})$$

<sup>1</sup>This amounts to imposing the equations of motion as constraints, and then performing an appropriate gauge fixing on the antifields.

<sup>2</sup>We use the nomenclature that ‘the equation of motion of  $X$ ’ is the equation obtained by varying  $X$ .



which have a nontrivial kernel, so are vanishing on a number of nonzero vector fields on the pre-phase space. These vector fields are generically the infinitesimal changes of field configuration which affects points *not on the slice*  $\Sigma$ . They are not seen by  $\tilde{\Omega}_\Sigma$ , and therefore do not change the state of the system respective to it. In order to obtain a phase space, we must the quotient  $\tilde{P}_\Sigma$  by those vector fields  $X$  on it which have trivial action on the fields at the slice  $\Sigma$ . If this is possible, we obtain the phase space

$$(P_\Sigma = \tilde{P}_\Sigma / \sim, \Omega_\Sigma) \quad (\text{A.8})$$

which we could perhaps call the 'covariant-canonical phase space' due to its hybrid nature. Instead, we will follow the standard nomenclature in the literature on boundary effects and refer to the hybrid construction as the covariant phase space, and the phase space with antifields as the '*BV phase space*'.

## A.2 Phase space ambiguities

In our setup so far, we have chosen our procedure to be as algorithmic as possible. This, while absolutely an option, disguises some of the possible free choices one has in setting up the phase space. Let us quickly go over these options before we move on.

A priori, the only data that is really required for the phase space is the set itself, defined through the vanishing of a set of equations of motion and boundary conditions (the  $E_\bullet$ ), as well as the full symplectic form  $\Omega_\Sigma$ .

Therefore, we have a priori freedom to alter any element of the construction leading up to these objects. This applies to the Lagrangians  $L_\bullet$  and particularly the symplectic potentials  $\theta_\bullet$ , which have been known[236] to have a substantial amount of ambiguity in them from the phase space point of view. In the Lagrangians, we already spoke about the central ambiguity in terms of adding total derivative terms. This ambiguity also affects the symplectic potentials in a simple way, as, when we shift

$$L_k \mapsto L_k + dl_{k+1} - i_{M_k}^* l_k \Rightarrow \theta_k \mapsto \theta_k + \delta l_k \quad (\text{A.9})$$

then the fundamental relation of equations of motion and Lagrangians is unaffected. However, this is rather a subset of a larger class of ambiguities (or choices) for  $\theta_k$ ; one which can be related back to Lagrangians.

The full ambiguity comes from the fact that only the integrated symplectic form is fixed. This means first that we can add an arbitrary  $\delta$ -exact function to the symplectic potential. However, in addition we can add and remove total derivatives in the integrands without changing the full result, i.e.

$$\Omega = \int_{\Sigma_1} \omega_1 + \oint_{\partial\Sigma_1} \omega_2 \mapsto \int_{\Sigma_1} (\omega_1 + dY_1) + \oint_{\partial\Sigma_1} (\omega_2 - Y_1) = \Omega \quad (\text{A.10})$$

And so at the most general (local integrand) level, the ambiguities become

$$\theta_k \mapsto \theta_k + \delta l_k + dy_{k+1} - i_{M_k}^* y_k \quad (\text{A.11})$$

with  $y_0 = l_0 = 0$ . These ambiguities have been given multiple different names; We will refer to  $l_k$  as the *Lagrangian ambiguity* and  $y_k$  as the *symplectic ambiguity*.

Notice that in this setting of relative tuples, the ambiguity is restricted, i.e. when  $\Sigma_2$  is strictly less than the boundary of  $\Sigma_1$ . Then, adding a piece to  $\theta_1$ , even if compensated in  $\theta_2$ , does not lead to the same symplectic potential. If boundary conditions are imposed, this may again change; but in principle a slight reduction of ambiguity already takes place through this. We also already saw that any Lagrangian ambiguity can be reexpressed in terms of a different choice of Lagrangians  $L_k$  which does not change the equations of motion  $E_k$ .

We need to stress that our ambiguity resolution strategy is slightly different from those discussed in the literature: Usually, one considers only  $\theta_1$  subject to the ambiguities  $l_1, y_1$ . Then, *given a form of  $\theta_1$* , one appeals to the variational principle [236] and writes

$$i_{M_1}^* \theta_1 = -\delta L_1 + d\theta_2 + E_1. \quad (\text{A.12})$$

In this, one needs to fix once and for all the boundary condition/equation of motion  $E_1$  through the variational principle. Then,  $L_1$  and  $\theta_2$  are added to the full symplectic structure as in our procedure here. Any ambiguity in  $\theta_1$  is then translated into a change in  $L_1, \theta_2$  which cancels in the full symplectic form. Therefore, one can see this picture as 'building up bottom-up' the  $L_k, \theta_k$  piece by piece starting from  $\theta_1$  and a given set of equations of motion.

In contrast, in the presentation of the main text we take a 'top-down' approach to fixing the ambiguities; We fix the Lagrangians first, and uniquely assign them a symplectic potential. This resolves the ambiguities before they arise.

Overall, however, one needs to keep one thing in mind: No matter how hard one tries, *it is unavoidable to choose some  $E_1$  if one wants a  $\omega_2$* . Really, it is the act of choosing a boundary condition that allows one to do the split of  $\omega_k$  into a bulk piece and a boundary piece that one identifies as  $\omega_{k+1}$ . This is, in a sense, a more general version of the procedure above: One rewrites

$$i_{M_k}^* \omega_k = \delta E_k + d\omega_k. \quad (\text{A.13})$$

All that the  $E_k$  do is vanish, and essentially select a 'bulk' Lagrangian subspace of the symplectic form  $\omega_k$ , so where it vanishes (up to total derivative). This means that corner forms  $\omega_k$  are *induced* from these 'bulk-Lagrangian' subspaces of the phase space.

In general, however, if one does away with Lagrangians, as we do in the next section, one needs to keep in mind these ambiguities explicitly, as one needs a strategy to fix them.

### A.3 Charge brackets

Even in the classical theory, one has the complication that strictly speaking, one does not have a Poisson algebra of these charges as they are not Hamiltonian functions. There are, then, several inequivalent ways to define a *charge bracket* which replaces the Poisson bracket

[43, 99, 236]. We will give a non-exhaustive overview of some options. For reference, the usual Poisson bracket would be given as

$$\{G_\alpha, G_\beta\} = X_\alpha[G_\beta] = -X_\beta[G_\alpha] = \frac{1}{2}(X_\alpha[G_\beta] - X_\beta[G_\alpha]) = \Omega(X_\alpha, X_\beta) = \Pi(\delta G_\alpha, \delta G_\beta). \quad (\text{A.14})$$

Unlike the following alternatives, it enjoys a number of properties like off-shell antisymmetry and the Jacobi identity. It also realises the *charge algebra*

$$\{G_\alpha, G_\beta\} = G_{-[\alpha, \beta]} \quad (\text{A.15})$$

where the bracket of parameters is defined such that<sup>3</sup>

$$[X_\alpha, X_\beta] = X_{[\alpha, \beta]}. \quad (\text{A.16})$$

The first class of these is the Barnich-Troessaert bracket (or any its extensions), which relies on a given charge-flux split. For some charges  $G_\alpha$ , it is then defined as

$$\{G_\alpha, G_\beta\}_{BT, K} := X_\alpha[G_\beta] - I_{X_\beta} \mathcal{F}_\alpha + K(\alpha, \beta) \quad (\text{A.17})$$

with the use of a *cocycle*  $K$  which may depend on the phase space point. Such a cocycle must be chosen to ensure that the Jacobi identity is satisfied and that onshell, the bracket is antisymmetric. For the case of diffeomorphisms, it is understood how to do this[99] in such a way that the definition only depends on the choice of the Lagrangian  $L_0$ . Beyond this, it is less clear how to choose a split, and the choice of split generically affects the bracket.

The second large class is given by using different forms of the Poisson bracket. These have the general property of being independent of a charge-flux split, but are otherwise less contextualised. I.e. we have the left and right action charge brackets

$$\{G_\alpha, G_\beta\}_L := I_{X_\alpha} \delta G_\beta \quad \{G_\alpha, G_\beta\}_R := -I_{X_\beta} \delta G_\alpha \quad (\text{A.18})$$

as well as the antisymmetric bracket

$$\{G_\alpha, G_\beta\}_{as} := \frac{1}{2}(I_{X_\alpha} \delta G_\beta - I_{X_\beta} \delta G_\alpha). \quad (\text{A.19})$$

This bracket has the virtue of being off-shell antisymmetric. We also have the double contraction bracket

$$\{G_\alpha, G_\beta\}_{dc} := \Omega(X_\alpha, X_\beta) \quad (\text{A.20})$$

which in fact does not refer to charges at all and can be defined for *any* vector fields. On the converse, we have the Poisson contraction

$$\{G_\alpha, G_\beta\}_P := \Pi(\delta G_\alpha, \delta G_\beta). \quad (\text{A.21})$$

This can in principle be defined for any field-space 1-forms.

---

<sup>3</sup>This includes the modifications needed in the case of field-dependent parameters.

## A.4 Cellular complexes: Language and notation

Consider a  $D$ -dimensional relative tuple  $(M_\bullet) = (M_0, M_1, \dots)$  of smooth manifolds. In the spirit of topological field theories, we are going to decompose it into *cells*, each of which has the topology of an open ball  $B_d$  of variable dimension  $d$ . Our goal will be to introduce the right notation and context for the next section, in which, when an appropriate decomposition is chosen, we then define a discrete model from the continuum.

We will use the language of CW complexes[237–239] in order to have a general background, then specialise further onto triangulations and discuss relevant discrete operations on differential forms.

By a CW decomposition of a topological space  $M$ , we understand a partition of  $M$  into *cells*  $e_\alpha^d \subseteq M$  (pairwise disjoint) which are each homeomorphic to an open ball: For each cell, there is a *characteristic map*  $f_\alpha^d : B_d \mapsto M$  such that restricted to the interior of the ball,

we have  $\text{Int}(B_d) \xrightarrow{f_\alpha^d} e_\alpha^d$ . If we call the collection of all cells of dimension  $d$  in such a decomposition the *d-skeleton*  $sk_d(M)$  of  $M$ , then we require of the characteristic maps that the boundaries of the cells are themselves cells of lower dimension,  $\text{Im}(f_\alpha^d, \partial B_d) \subset sk_{d-1}(M)$ <sup>4</sup>. A topology is then fixed on this CW decomposition by requiring the characteristic maps to be continuous.

It is well-known that every *smooth manifold* admits a CW decomposition, though it may not have finitely many cells. An even stronger statement holds, though:

Let first  $\sigma_d$  be the standard *d-simplex* given by the convex hull of the basis vectors  $e_0, e_1, \dots, e_d$  in  $\mathbb{R}^{d+1}$ . Its boundary as a manifold decomposes into a number of lower dimensional d-simplices which are conventionally called *faces*. For different dimensionalities, the simplex is known under common names like the *vertex* ( $d = 0$ ), *edge* ( $d = 1$ ), *triangle* ( $d = 2$ ), *tetrahedron* ( $d = 3$ ) and *pentachoron* ( $d = 4$ ). As all of them are homeomorphic to balls, they can serve as cells for a CW decomposition, as well. There is, then, a specialisation of such a decomposition known as a *triangulation*.

We say that a topological space is triangulated if it is homeomorphic to a *simplicial complex*: A set  $S$  of simplices of variable dimension which satisfy two key notions of ‘proper connectedness’ and ‘non-overlap’:

1. If  $e_\alpha^d$  is a simplex in  $S$ , then all its faces are also simplices in  $S$
2. Two cells  $e_\alpha^d, e_\beta^{d'}$  are either *identical* or their interiors have no overlap.

The second condition in particular means that the intersection of two simplices, if non-empty, must be a *shared face* of the two simplices.

A typical class of simplicial complexes is those that are built out of glueing  $D$ -simplices, known as *homogeneous D-simplicial complexes*. These are such that every simplex in  $S$  of dimension less than  $D$  is a face of some  $D$ -simplex.

---

<sup>4</sup>More properly, the boundary is contained in a finite union of these cells. By abuse of notation, we refer by  $sk_d$  both to the set of cells as well as the set of finite unions of them.

We then have the stronger statement that *every smooth  $D$ -manifold admits a smooth triangulation by a homogeneous  $D$ -simplicial complex*.

While we will exclusively deal with such homogeneous triangulations for the remainder of this section, we note that there is a slightly larger but similarly nice class of spaces which are ‘almost’ smooth manifolds:

A  $D$ -dimensional *pseudomanifold* is a homogeneous  $D$ -simplicial complex where any two  $D$ -simplices are connected by a finite path of  $D$ -simplices through their shared  $(D - 1)$ -faces (strong connectedness) and where any  $(D - 1)$ -simplex is the face of precisely two  $D$ -simplices.

Many non-manifold spaces, such as complex algebraic varieties, are examples of pseudomanifolds. They are essentially manifolds with a number of codimension-2 singularities, so where an open set of dimension  $D$  has been collapsed into a set of dimension  $D - 2$ . Of course, this means that smooth manifolds are examples of pseudomanifolds, as well. Manifolds may be distinguished from pseudomanifolds by means of the shape of neighbourhoods of a point: In a smooth manifold of dimension  $D$ , a small neighbourhood of a point is always an open ball  $B_D$  with boundary  $S_{D-1}$ . This is not necessarily true in a pseudomanifold; manifolds are precisely those pseudomanifolds where this property holds.

As we will also need orientations on our manifolds, we work as follows: Because the standard  $d$ -simplex can be determined as the convex hull of  $(d + 1)$  points in  $\mathbb{R}^{d+1}$  (the end points of the standard basis vectors), one can fix an orientation on it by fixing an ordering of these points, written in a standard form as  $\sigma_d = [012 \dots d]$ . Then, any face of the simplex can be denoted schematically by which of these points it contains, e.g. an edge could be written  $[01]$ . This allows the standard orientation to induce orientations on all faces. Then, an orientation on the cells  $e_\alpha^d$  is induced through the characteristic maps, and we require that all the cells are consistently oriented. This means for example that if a simplex is a face of two other simplices, it inherits opposite orientations from either of them (making it a 2-sided face).

With the topology out of the way, let us proceed to field theory. Let us assume that our relative tuple  $(M_\bullet)$  has a triangulation in the sense that the triangulation of any  $M_k$ , when restricted to its boundary, induces a triangulation of all the  $M_{k+1}$  it intersects. If any  $M_{k+1}$  is in the boundary of multiple  $M_k$ , we require the triangulations to all match up, to give one global consistent one.

We will give a preferred **notation** for the sets of different cells in this for the rest of this work, given by  $\Lambda_{k,i}^{(\sim)(d)}$ . The meaning is as follows.

- The label  $(k, i)$  refers to a choice of manifold in the relative tuple,  $M_{k,i}$  of codimension  $k$  (and if the extra label is omitted, it refers to the union of all of those)
- The label  $d$  refers to the dimensionality of the cell.
- In absence of a tilde, it refers to the cells in the triangulation of  $M_{k,i}$ .

- In presence of a tilde, it refers to the cells in the *dual* cellulation of  $M_{k,i}$ .

The *dual* of a cellulation is an analogue of Poincare duality: One associates to each  $d$ -cell  $e_\alpha^d$  a  $(D - d)$ -cell  $\tilde{e}_\alpha^{D-d}$  in such a way that duals of  $D$ -cells form again a cellulation. Note that the dual of a triangulation is typically not a triangulation again. A typical instance of this is the *barycentric* dual of a simplicial complex[240]

For our case of multiple compatible triangulations, we stress the fact that even though a simplex  $\sigma_\alpha^d$  may be contained in multiple  $\Lambda_{k,i}^{(d)}$  (as it may be the face of many other simplices), it has an in principle *different* dual for each cellulation it is a part of. We must therefore speak, for any given element/simplex of our triangulation, of "*the dual of  $\sigma_\alpha^d$  in  $M_{k,i}$* " or similar.

As we will often deal with low-dimensional examples, it is useful to have specialised notation for the different cells. We will often denote 0-simplices by  $v$  (vertices), 1-simplices by  $e$  (edge), 2-simplices by  $t$  (triangle), 3-simplices by  $\tau$  (tetrahedron) and 4-simplices by  $\pi$  (pentachoron).

As an example, if we are working in  $D = 3$ , the statement  $e \in \Lambda_{1,i}^1$  means that the edge  $e$  is a face of a codimension 1, dimension 2 simplex  $t_i$  in  $M_1$ . We can also see the same edge as an element of  $\Lambda_{0,j}^1$ , where it is a face of a codimension 0, dimension 3 simplex  $\tau_j$  in  $M_0$ . Its duals in these cellulations are not the same: The dual  $\tilde{e}_{(t_i)} \in \tilde{\Lambda}_{1,i}^1$  in  $t_i$  is of dimension 1 and codimension 1 in  $t_i$ . The dual  $\tilde{e}_{(M_1)} \in \tilde{\Lambda}_1^1$  in  $M_1$  is of dimension 1 and codimension 1 in  $M_1$ , but is *not* identical to the first. Lastly, the dual  $\tilde{e}_{(M_0)} \in \tilde{\Lambda}_0^2$  in  $M_0$  is of dimension 2 and codimension 1 in  $M_0$ .

What this example shows, however, is that the dual of a cell in a high dimensional cellulation contains the duals of the same cell in lower dimensional cellulations: We have  $\tilde{e}_{(t_i)} \subset \tilde{e}_{(M_1)} \subset \partial \tilde{e}_{(M_0)}$ .

Now, let us consider more than just the manifold: We usually have classical fields as sections of bundles. Such bundles are typically trivial over contractible regions, such as the  $D$ -cells  $e_\alpha^D$ , and we will model them as such here. The only nontrivial bundle structures we can have are thus encoded in the transition functions between the cells. These can, for all intents and purposes, be approximated by constant transition group elements  $G_{\alpha,\beta}$  associated to the interfaces  $e_{\alpha,\beta}^{D-1}$ , in the structure group of the bundle. Cocycle conditions require that for any triple of cells labeled  $\alpha, \beta, \gamma$  intersecting in the codimension 2 cell  $e_{\alpha\beta\gamma}^{D-2}$ ,

$$G_{\alpha\beta} = G_{\alpha\gamma}G_{\gamma\beta}. \quad (\text{A.22})$$

This can equally be stated as the statement that the discrete parallel transport defined by the elements  $G$  has curvature/holonomy that stabilises/fixes the cell  $e_{\alpha\beta\gamma}^{D-2}$ , see i.e. [241] where this is remarked in the piecewise flat case. As such, we now need to keep in mind that if we consider any discrete fields, their continuity is defined up to the action of these group elements. If they can be trivialised by a change of section (here, we implicitly used the identity or zero section on the bundles of the  $D$ -cells), then the full bundle is trivial, and continuity is as usual.

## A.5 Additional details on BF theory

### BF theory: Boundary conditions

In this appendix, we consider boundary effects in BF theory and refer to the main text, section 1.5 for citations. For now, let us disregard defects intersecting the boundary and stay in the undressed framework. The main question, once again, is to find boundary conditions that close the system. For the former case, we need a class of Lagrangian subspaces for the symplectic form

$$\Omega_{M_1} = - \int_{M_1} \langle \delta\omega, \delta B \rangle. \quad (\text{A.23})$$

For this, introduce normal 1-form and vectors  $\mathbf{r}, \hat{r}$  for the boundary  $M_1$ , with  $\mathbf{r}(\hat{r}) = 1$  and do an analogous decomposition to the one on a spacelike slice. Here, we will denote the normal components by an index  $(-)_r$  and the tangential components by an overbar  $(-)$ . Then, the piece to be set to zero only involves the tangential components  $\bar{B}, \bar{\omega}$ . So unlike the scalar field case, we have to deal *neither with radial derivatives nor with radial components*. However, still, the ‘timelike’ pieces  $\omega_n, B_n$  enter the potential. Let us therefore for illustration purposes do a more refined decomposition: We fix the given radial normals, and choose further *timelike* normals (i.e. a foliation), though this can be done more generally. These normals  $\mathbf{n}, \hat{n}$  are required to satisfy

$$\mathbf{n}(\hat{n}) = -1 \quad \hat{\mathbf{r}} = 0 = \mathbf{r}(\hat{n}). \quad (\text{A.24})$$

So they are locally orthogonal. The decomposition is then

$$B = \bar{\bar{B}} + \mathbf{r} \wedge \tilde{B}_r - \mathbf{n} \wedge \bar{B}_n - \mathbf{n} \wedge \mathbf{r} \wedge B_{nr} \quad (\text{A.25})$$

$$\omega = \bar{\bar{\omega}} + \mathbf{r} \wedge \tilde{\omega}_r - \mathbf{n} \wedge \bar{\omega}_n - \mathbf{n} \wedge \mathbf{r} \wedge \omega_{nr} \quad (\text{A.26})$$

so that only  $\bar{\bar{B}}, \bar{B}_n, \bar{\bar{\omega}}, \bar{\omega}_n$  are in the symplectic potential, and we have the knowledge that the constraint equations on spatial slices tangent to  $\mathbf{n}$  must acquire boundary conditions. This may put restrictions on what Lagrangian subspaces we may choose. For this, we need to study the equations of motion in detail. With the covariant Lie derivative  $\mathcal{L}_\xi^\omega = d_\omega i_\xi + i_\xi d_\omega$ , we can get the timelike and radial projections of the Gauss equation: The Gauss constraint on a *spatial* slice has a single component, given by

$$\mathcal{L}_{\hat{r}}^\omega \bar{\bar{B}} = \tilde{d}_\omega \tilde{B}_r. \quad (\text{A.27})$$

Similarly, the flatness constraint on a slice gives

$$\mathcal{L}_{\hat{r}}^\omega \bar{\bar{\omega}} = \tilde{d}_\omega \tilde{\omega}_r. \quad (\text{A.28})$$

These may be understood as transport equations, or spatial conservation laws. We already know the local set of solutions for these equations. They are parametrised by  $(\tilde{g}, \tilde{b})$ , just like

at the spacetime level. Now, however, we must ask which data in the above fields we can freely prescribe and which boundary conditions we must fix *to arrive at these parameter functions*. As we have only *one* equation each, we must specify *all but one* components on some boundary, and specify the remaining one throughout the domain to be solved.

I.e. for the connection, we have the function  $\tilde{g}(r, x^a)$  and can derive its r-dependence as

$$\tilde{g}^{-1} \partial_r \tilde{g} = \tilde{\omega}_r. \quad (\text{A.29})$$

This must then be supplemented by an r-boundary condition, which then determines  $\tilde{\omega}$ , the non-radial part of the connection at the boundary. This means we must specify  $\tilde{\omega}$  on some codimension 2 surface as a boundary condition, and we can derive the radial dependence by integrating on the boundary

$$\tilde{g}^{-1} \tilde{d} \tilde{g} = \tilde{\omega} \quad (\text{A.30})$$

which, really, means one again has integrability conditions on these corners. We must therefore require *manually* that  $\tilde{\omega}(x^a)$  is flat, and can then always solve, from the  $\tilde{\omega}(x^a)$  on the boundary and  $\tilde{\omega}_r(r, x^a)$ , for  $\tilde{g}(r, x^a)$ , and therefore for  $\tilde{\omega}(r, x^a)$  throughout the region. Therefore, the free data of the connection is given by a *flat* boundary connection and the r-profile of  $\tilde{\omega}_r(r, x^a)$ .

For the  $B$ -field, it is similar - disregarding the dressing by  $\tilde{g}$ , we have that  $\tilde{b}$  is specified by a boundary configuration  $\tilde{B}(x^a)$  (not subject to further constraints) and the r-profile  $\tilde{B}_r(r, x^a)$ .

Therefore, it is *non-negotiable* for solving the equations of motion to give a whole time profile for  $\tilde{\omega}, \tilde{B}$  on  $M_1$ . In a variational principle, we want a single solution given an initial condition - This initial condition *must* then include the initial values on the boundary. As we also know,  $\tilde{\omega}_r, \tilde{B}_r$  must then be determined by a gauge fixing.

We now understand the constraints - and could assume it is fine to fix boundary conditions on  $\tilde{\omega}, \tilde{B}$  on  $M_1$  and move on. However, the dynamics on  $M_1$  is determined through the  $\tilde{B}_n, \tilde{\omega}_n$ , and nothing in the theory specifies them. Furthermore, while in the bulk these components are gauge, this is not the case on the boundary. Therefore, we know that it is much more reasonable to look instead for boundary conditions of the form

$$\tilde{\omega}_n \stackrel{M_1}{=} f[\tilde{\omega}, \tilde{B}] \quad \tilde{B}_n = G[\tilde{\omega}, \tilde{B}] \quad (\text{A.31})$$

which gives the variations

$$\begin{aligned} \delta \tilde{\omega}_n(x) &= \int_{M_1} d^{D-1} y \delta \tilde{\omega}(y) \frac{\delta f}{\delta \tilde{\omega}}(y) + \delta \tilde{B}(y) \frac{\delta f}{\delta \tilde{B}}(y) \\ \delta \tilde{B}_n(x) &= \int_{M_1} d^{D-1} y \delta \tilde{\omega}(y) \frac{\delta G}{\delta \tilde{\omega}}(y) + \delta \tilde{B}(y) \frac{\delta G}{\delta \tilde{B}}(y) \end{aligned} \quad (\text{A.32})$$

and so, in the expanded symplectic form,

$$\Omega_{M_1} = \int_{M_1} \mathbf{n} \wedge (\langle \delta \tilde{\omega}_n, \delta \tilde{B} \rangle - \langle \delta \tilde{\omega}, \delta \tilde{B}_n \rangle) \quad (\text{A.33})$$



vanishing turns into a number of conditions on  $f, G$ . These are best seen in local coordinates. We specialise here to the low dimensional cases of  $D = 2, 3, 4$ .

**D=2:** There are only timelike and radial components of  $\omega$  and a single component for  $B$  in this case. The second term in the expanded symplectic form vanishes identically, and a generic boundary condition is given by

$$\omega_n(x) = \int_{M_1} d^{D-1}y a(x, y) B(y) \quad (\text{A.34})$$

with  $a$  being a symmetric integral kernel.

**D=3:** There are tangential components, i.e.  $\omega_\theta, B_\theta$ . Apart from the dependence of  $\omega_n = f$  on  $B_\theta$  and  $B_n = G$  on  $\omega_\theta$ , which is like before allowed to be some symmetric integral kernel, we have another freedom coming from requiring the vanishing of

$$\int d^2x \int d^2y \frac{\delta \omega_n(x)}{\delta \omega_\theta(y)} \langle \delta \omega_\theta(y), \delta B_\theta(x) \rangle - \langle \delta \omega_\theta(x), \delta B_\theta(y) \rangle \frac{\delta B_n(x)}{\delta B_\theta(y)} \quad (\text{A.35})$$

which gives

$$\frac{\delta \omega_n(x)}{\delta \omega_\theta(y)} = \frac{\delta B_n(y)}{\delta B_\theta(x)} \quad (\text{A.36})$$

I.e. we can write the generic condition through three integral kernels  $a, b, c$  as

$$\omega_n(x) = \int_{M_1} d^{D-1}y a(x, y) B_\theta(y) + c(x, y) \omega_\theta(y) \quad (\text{A.37})$$

$$B_n(x) = \int_{M_1} d^{D-1}y c(y, x) B_\theta(y) + b(x, y) \omega_\theta(y) \quad (\text{A.38})$$

such that  $a, b$  are symmetric. Note that this has the structure of a symmetric matrix convolution again.

**D=4:** Here, we have once again the same structure; The only difference is that  $B_t$  now is a 1-form and has two components on  $M_1$ . We must therefore specify them both. Using spherical coordinates  $\theta, \phi$ , we then have the symplectic form

$$\langle \delta \bar{\omega}_n, \delta B_{\theta\phi} \rangle - \langle \delta \omega_\theta, \delta B_{n\phi} \rangle + \langle \delta \omega_\phi, \delta B_{n\theta} \rangle \quad (\text{A.39})$$

which gives the off-diagonal conditions

$$\frac{\delta \omega_n}{\delta \omega_\theta}(x, y) = \frac{\delta B_{n\phi}}{\delta B_{\theta\phi}}(y, x) \quad \frac{\delta \omega_n}{\delta \omega_\phi}(x, y) = -\frac{\delta B_{n\theta}}{\delta B_{\theta\phi}}(y, x) \quad \frac{\delta B_{n\phi}}{\delta \omega_\phi}(x, y) = -\frac{\delta B_{n\theta}}{\delta \omega_\theta}(y, x). \quad (\text{A.40})$$

Together with the diagonal integral kernels, there is a total freedom of a 6 integral kernels forming a block matrix that determine the boundary condition, i.e.

$$\begin{pmatrix} \omega_n \\ B_{n\theta} \\ B_{n\phi} \end{pmatrix} (x) = \int_{M_1} d^{D-1}y \begin{pmatrix} a & u & v \\ -u^T & b & w \\ v^T & -w^T & c \end{pmatrix} (x, y) \begin{pmatrix} B_{\theta\phi} \\ \omega_\phi \\ \omega_\theta \end{pmatrix} (y) \quad (\text{A.41})$$

where  $f^T(x, y) := f(y, x)$ . Overall, the structure is a lot more lax here, and in principle many boundary conditions are possible. This is the off-shell story. Now let us also consider the same symplectic form, on-shell of the bulk equations of motion.

The bulk constraints in particular tell us that  $\tilde{\omega} = g^{-1}\tilde{d}g$  and  $\tilde{B} = \tilde{d}_\omega\tilde{b}$  and this simplifies the symplectic form to

$$\begin{aligned}\langle \delta\tilde{\omega}_n, \delta\tilde{B} \rangle - \langle \delta\tilde{\omega}, \delta\tilde{B}_n \rangle &= \langle \delta\tilde{\omega}_n, \delta\tilde{d}_\omega\tilde{b} \rangle - \langle \tilde{d}_\omega(g^{-1}\delta g), \delta\tilde{B}_n \rangle \\ &= \langle \delta\tilde{\omega}_n, \tilde{d}_\omega\delta\tilde{b} \rangle - \langle \tilde{d}_\omega(g^{-1}\delta g), \delta\tilde{B}_n + [\delta\tilde{\omega}_n, \tilde{b}] \rangle \\ &= \langle g^{-1}\delta g, \tilde{d}_\omega(\delta\tilde{B}_n + [\delta\tilde{\omega}_n, \tilde{b}]) \rangle - \langle \tilde{d}_\omega\delta\tilde{\omega}_n, \delta\tilde{b} \rangle + \tilde{d}\vartheta_2\end{aligned}\tag{A.42}$$

with a (spatial) boundary term which usually drops out,

$$\vartheta_2 = \langle \delta\tilde{\omega}_n, \delta\tilde{b} \rangle - \langle g^{-1}\delta g, \delta\tilde{B}_n + [\delta\tilde{\omega}_n, \tilde{b}] \rangle.\tag{A.43}$$

This shows that on-shell of the bulk EoM, we have a different situation: To leave  $g, \tilde{b}$  arbitrary (so not restricting the phase space arbitrarily), we need to require that the variations of  $\omega_n, B_n$  satisfy a kind of covariant constancy:

$$\tilde{d}_\omega\delta\tilde{\omega}_n = 0 = \tilde{d}_\omega\delta\tilde{B}_n + [\delta\tilde{\omega}_n, \tilde{d}_\omega\tilde{b}]\tag{A.44}$$

This fits in with our previous argument that we need to fix data about the timelike components. This informs us of something interesting: As the dynamics of the theory is determined entirely by a choice of timelike components, this constrains the consistent choices of such dynamics. In fact, up to a small freedom, this *fixes* the boundary dynamics. If we go fully on-shell, then actually we know that  $\omega_n$  is the time derivative  $g^{-1}i_{\tilde{n}}dg = g^{-1}\dot{g}$ . This then tells us that

$$\tilde{d}_\omega i_{\tilde{n}}d_\omega(g^{-1}\delta g) = 0\tag{A.45}$$

which must be understood as a *linearised equation of motion* for  $g$ , akin to  $\partial_x\partial_t w = 0$ .

It is worth stressing that, while off-shell we might need to impose boundary conditions, these are actually superfluous on the on-shell set. *This means that we still need to choose some boundary condition when working off-shell* to make i.e. diffeomorphisms integrable, but on-shell one may keep all configurations induced from the bulk and no restrictions are necessary. *Therefore, there may be artificially restrictive off-shell boundary conditions that only keep some part of the on-shell phase space, and others which contain the full on-shell phase space.* The latter are, of course, preferable. The conditions we gave above are precisely of this type.

### BF theory: Details of sampling and discretization

Here, we will carefully put the main points of the discretization we sketched in the main text in section 1.5 into the perspective of our framework.

On a cell  $e_\alpha^{D-1}$ , we can write

$$\omega|_{e_\alpha^{D-1}} = g_\alpha^{-1} dg_\alpha. \quad (\text{A.46})$$

On dual cells, we have similarly

$$B|_{\tilde{e}_\beta^{D-1}} = d_\omega b_\beta, \quad (\text{A.47})$$

but we have the problem that  $\omega$  does not have a simple expression on the dual cells. Therefore, we subdivide the dual cells into pieces,

$$\tilde{e}_\beta^{D-1} = \cup_\alpha e_{\beta\alpha}^{D-1} := \cup_\alpha (e_\alpha^{D-1} \cap \tilde{e}_\beta^{D-1}) \quad (\text{A.48})$$

precisely into the intersections with the cells, which we refer to as *wedges*. In each of these, the solution is much more explicit,

$$B|_{e_{\beta\alpha}^{D-1}} = g_\alpha^{-1} (db_{\beta\alpha}) g_\alpha \quad (\text{A.49})$$

and so on each of the  $e_{\beta\alpha}^{D-1}$  we can again use the known on-shell potential

$$\langle \delta\omega, B \rangle|_{e_{\beta\alpha}^{D-1}} \hat{=} d(\langle \chi_R(g_\alpha), d_\omega b_{\alpha\beta} \rangle). \quad (\text{A.50})$$

The full potential is then constructed from these by summing up the contributions on all the interfaces between the  $e_{\beta\alpha}^{D-1}$ , together with the defects.

Let us step back for a moment and understand the big picture. We want to discretise a theory, and to do so we first *split* it into pieces which are easily solvable (or which keep only the data relevant for our preferred observables), and then *glue* the solved pieces back together. In the process, we must keep some nontrivial data so as to not arrive at a trivial phase space in the discretised theory. Therefore, what we have used here implicitly is the *splitting maps* along the interfaces, which we can refer to as  $e_{\beta\alpha,\gamma}^{D-25}$ . Therefore, starting from the global phase space  $\mathcal{C}_\Sigma$ , we apply the splitting maps

$$\text{sp}_{e_{\beta\alpha,\gamma}^{D-2}} : \mathcal{C}_\Sigma \mapsto \mathcal{C}_{\cup_\beta \cup_\alpha e_{\beta\alpha}^{D-1}} \subset \prod_\beta \prod_\alpha \mathcal{C}_{e_{\beta\alpha}^{D-1}} \quad (\text{A.51})$$

and then discretise the phase spaces  $\mathcal{C}_{e_{\beta\alpha}^{D-1}}$  by solving the constraints on them. This leaves us with only boundary data on each of them. The glueing maps then reassemble these boundary data into data on interfaces. It is crucial in this that the individual cells are treated as proper subsystems, i.e. the gauge transformations on their boundaries are *not* redundancies.

This is because the data on the interfaces is necessary to reconstruct all the global non-trivial configurations. In particular, the relative orientations of the dressing fields,  $\Delta\phi$ , actually determine the global connection  $\omega$  and are nontrivially related to the defects  $F$ .

---

<sup>5</sup> $\partial e_{\beta\alpha}^{D-2} = \cup_\gamma e_{\beta\alpha,\gamma}^{D-2}$ .

What this means is that to discretise the theory here, we really need the understanding we get from the general glueing problem. If the cells are trivialised, then the glueing must be nontrivial enough to ensure the theory does not become trivial.

Furthermore, if we were to consider this on a dynamical level, we would further need to think about the dynamics placed on the time evolution of the interfaces. As we have seen, the imprint of BF theory on its boundaries has topological boundary conditions  $\tilde{B} = 0$ , which gives us some idea about which glueing methods are sensible.

If we want to study also the time evolution of each of the cells *before glueing*, then we must be very sure that the boundary conditions we choose do not restrict any of the relevant features, i.e. are of the gauge-invariant, charge-preserving type we discussed before.

Let us now look at the concrete way these interface potentials look. We can closely follow [137, 242] for this. We first realise that on-shell in each cell, all the degrees of freedom localise on the boundaries of the cell, in the form of the dressing fields (if we choose the trivial gauge  $B = 0 = \omega$ ):

$$\theta_1 \hat{=} \langle d\chi_R(\phi), dq \rangle. \quad (\text{A.52})$$

Now, each cell boundary segment  $e_{\beta\alpha,\gamma}^{D-2}$  is also the boundary of another cell  $e_{\beta'\alpha',\gamma}^{D-2}$ , either in the same cell  $e_{\alpha}^{D-1}$  or the same dual cell  $\tilde{e}_{\beta}^{D-1}$ , so either  $\alpha = \alpha'$  or  $\beta = \beta'$ . Across all of these, we need to require, for consistent glueing, *continuity of the dressed fields*: I.e. across a cell interface, so  $\beta = \beta', \alpha \neq \alpha'$ , we must have

$$\phi_{\alpha}^{-1}\omega_{\alpha}\phi_{\alpha} + \phi_{\alpha}^{-1}d\phi_{\alpha} \stackrel{e_{\beta\alpha,\gamma}^{D-2}}{=} \phi_{\alpha'}^{-1}\omega_{\alpha'}\phi_{\alpha'} + \phi_{\alpha'}^{-1}d\phi_{\alpha'} \quad (\text{A.53})$$

Now, if we go into the trivial gauge on both sides of the interface, this reduces to the matching condition

$$\phi_{\alpha}^{-1}d\phi_{\alpha} \stackrel{e_{\beta\alpha,\gamma}^{D-2}}{=} \phi_{\alpha'}^{-1}d\phi_{\alpha'} \quad (\text{A.54})$$

which implies that the two dressing fields differ by a *constant* group element:

$$d(\phi_{\alpha}\phi_{\alpha'}^{-1}) = 0 \implies \phi_{\alpha}\phi_{\alpha'}^{-1} \equiv G_{\alpha,\alpha'} \quad (\text{A.55})$$

It is important that we choose this particular ordering, rather than  $\phi_{\alpha'}^{-1}\phi_{\alpha}$ , due to the transformation behaviour of the fields under gauge transformations versus frame reorientations (which act on the left, respectively the right, of the field  $\phi$ ). This combination is the one which is invariant under *diagonal frame reorientations*,

$$(\phi_{\alpha'}, \phi_{\alpha}) \mapsto (\phi_{\alpha'}g, \phi_{\alpha}g). \quad (\text{A.56})$$

This is the invariance one eventually associated with glueing of regions, and refers to the boundary symmetries, rather than redundancies. It is therefore a physical invariance associated to the combination of the subregions.

For the  $B$  field, instead we find

$$dq_{\beta\alpha'} = G_{\alpha,\alpha'}^{-1}dq_{\beta\alpha}G_{\alpha,\alpha'} = d(G_{\alpha,\alpha'}^{-1}q_{\beta\alpha}G_{\alpha,\alpha'}) \implies q_{\beta\alpha'} = G_{\alpha,\alpha'}^{-1}(q_{\beta\alpha} + Q_{\beta;\alpha,\alpha'})G_{\alpha,\alpha'} \quad (\text{A.57})$$

where  $dQ_{\beta;\alpha,\alpha'} = 0$ . Similarly, if we consider a dual cell interface  $\beta \neq \beta', \alpha = \alpha'$ , so staying within the same cell, then matching of  $B$  gives

$$\phi_\alpha^{-1}(B_{\beta\alpha} + d_{\omega_\alpha}q_{\beta\alpha})\phi_\alpha \stackrel{e_{\beta\alpha,\gamma}^{D-2}}{=} \phi_\alpha^{-1}(B_{\beta'\alpha} + d_{\omega_\alpha}q_{\beta'\alpha})\phi_\alpha \quad (\text{A.58})$$

and in the trivial gauge fixing, this gives again a constant Kalb-Ramond shift.

$$d(q_{\beta\alpha} - q_{\beta'\alpha}) = 0 \implies q_{\beta\alpha} - q_{\beta'\alpha} \equiv Q_{\beta,\beta';\alpha}. \quad (\text{A.59})$$

and similarly,  $\phi$  can in principle shift by a constant amount.

So when we cross from one (dual) cell to another, we change  $\phi$  by a constant group element  $G$  and rotate  $q$  by it as well. We also can add an arbitrary constant (rather, closed) shift  $Q$  to  $q$ . The elements  $G$  will encode nontrivial curvatures given by the defects  $F$ , roughly by the rule that the holonomy of a loop around  $e^{D-3}$  is given by  $F$ ,

$$\prod G_{\alpha,\alpha'} \sim F_{e^{D-3}}, \quad (\text{A.60})$$

while the shift elements  $Q$  will count the presence of charge defects  $S$ . Therefore, it will not be necessary to keep the group elements  $G$  inside the same cell labeled by  $\alpha$ , and the shifts  $Q$  associated to the same dual cell can be taken to be the same up to conjugation.

The logic is now that across an interface, we look at the difference of the imprints from the two sides and see what is left once we use the continuity equations.

For this, we need to make a *choice* on how to integrate  $\langle d\chi_R(\phi), dq \rangle$  into a boundary term. The reason this is a choice is that if we keep either of the derivatives, then the constant changes across the interface will not contribute. So, if we integrate to  $\langle \chi_R(\phi), dq \rangle$ , then  $Q$  will not contribute in the difference, and if we integrate to  $-\langle d\chi_R(\phi), q \rangle$ ,  $G$  will not contribute.

Because of the defects we chose, we want the elements  $Q$  to be present on *dual* cell boundaries *within the same cell*  $e_\alpha^{D-1}$ . Thus on these, we will integrate to  $-\langle d\chi_R(\phi), q \rangle$ , while on cell-to-cell boundaries, we will integrate to  $\langle \chi_R(\phi), dq \rangle$ , which lets us keep  $G$  on these interfaces.

First, let us look at a cell-cell interface  $e_{\beta,\alpha;\gamma}^{D-2}$ . There, we consider the difference (sign given by a preferred orientation)

$$\langle \chi_R(\phi_\alpha), dq_{\beta\alpha} \rangle - \langle \chi_R(\phi_{\alpha'}), dq_{\beta\alpha'} \rangle \quad (\text{A.61})$$

and use the continuity equations above, together with the identity

$$\chi_R(G_{\alpha\alpha'}) = \chi_R(\phi_\alpha) - G_{\alpha\alpha'}\chi_R(\phi_{\alpha'})G_{\alpha\alpha'}^{-1}, \quad (\text{A.62})$$

to rewrite it as

$$\langle \chi_R(G_{\alpha\alpha'}), dq_{\beta\alpha} \rangle. \quad (\text{A.63})$$

This piece sits then on all interfaces between cells. Importantly, all the  $\phi$  have dropped out and only a *constant* group element remains. This is absolutely crucial, as then, in the integral, this gives

$$\int_{e_{\beta,\alpha;\gamma}^{D-2}} \langle \chi_R(\phi_\alpha), dq_{\beta\alpha} \rangle - \langle \chi_R(\phi_{\alpha'}), dq_{\beta\alpha'} \rangle = \langle \chi_R(G_{\alpha\alpha'}), \oint_{\partial e_{\beta,\alpha;\gamma}^{D-2}} q_{\beta\alpha} \rangle \quad (\text{A.64})$$

which only requires the values of  $q_{\beta\alpha}$  on codimension 3 surfaces, on which it can be naturally smeared. These surfaces lie in the boundaries of the interfaces, so are part of  $\Lambda_\Sigma^{D-3}$ , but also of the dual cellulation of the *boundary* of  $e_\alpha^{D-1}$ ,  $\tilde{\Lambda}_{\partial e_\alpha^{D-1}}$ . Importantly, this happens on *all bulk cell-cell interfaces*. The combination of all these terms then contains information about the curvature defects  $F$ , when summing all contributions adjacent to a given defect.

On a dual cell-dual cell interface  $\tilde{e}_{\beta,\alpha;\gamma'}^{D-2}$  within the same cell labeled by  $\alpha$ , we instead consider

$$-\langle d\chi_R(\phi_\alpha), q_{\beta\alpha} \rangle + \langle d\chi_R(\phi_\alpha), q_{\beta'\alpha} \rangle \quad (\text{A.65})$$

which yields

$$-\langle d\chi_R(\phi_\alpha), Q_{\beta,\beta;\alpha} \rangle = -d\langle \chi_R(\phi_\alpha), Q_{\beta,\beta;\alpha} \rangle. \quad (\text{A.66})$$

This now is a term that lives on codimension 3 surfaces; these surfaces are part of both the dual cellulation  $\tilde{\Lambda}_{e_\alpha^{D-1}}$ , but also of the dual cellulation of the *boundary* of  $e_\alpha^{D-1}$ ,  $\tilde{\Lambda}_{\partial e_\alpha^{D-1}}$ . These serve different purposes: The former are contributions which must be summed up *within a cell*  $e_\alpha^{D-1}$ , and which are related to the spin defect  $S$  inside the cell. The latter combine with contributions from adjacent cells  $e_{\alpha'}^{D-1}$  and are not related to defects.

Let us now consider how these contributions fit together. The cell-cell interface potentials that sit in  $\Lambda_\Sigma^{D-3}$  are related to curvature defects, and sum up around one of them, say sitting on the surface  $e_\mu^{D-3}$  (but staying within the same dual cell  $\tilde{e}_\beta^{D-1}$ ), to

$$\sum_{e_\alpha^{D-1}, e_{\alpha'}^{D-1} \cap e_\mu^{D-3}} \epsilon_{\alpha,\alpha'} \langle \chi_R(G_{\alpha\alpha'}), \oint_{e_{\beta,\alpha;\gamma,\gamma'}^{D-3}} q_{\beta\alpha} \rangle \quad (\text{A.67})$$

where the sum goes over cell-cell interfaces  $(\alpha, \alpha')$  that touch  $e_\mu^{D-3}$ . Orientations induce some signs  $\epsilon_{\alpha,\alpha'}$  in the sum. As all the  $q_{\beta\alpha}$  within the same dual cell are just conjugates of each other by the  $G$  group elements, we can pick one of the cells  $\alpha$  and express all terms from that cell's frame of reference. The result is

$$\langle \chi_R(\mathcal{H}_{\mu,\beta}), \oint_{e_{\beta,\alpha;\gamma,\gamma'}^{D-3}} q_{\beta\alpha} \rangle \quad (\text{A.68})$$

where

$$\mathcal{H}_{\mu,\beta,\alpha} := \prod_{e_\alpha^{D-1}, e_{\alpha'}^{D-1} \cap e_\mu^{D-3}} G_{\alpha\alpha'}^{\epsilon_{\alpha,\alpha'}} \quad (\text{A.69})$$

is the holonomy around the defect  $\mu$ , within the dual cell  $\beta$ , anchored in the cell  $\alpha$ . There will of course also be other pieces associated to other dual cells, and the  $q$ s in those will be related by a shift. Since we want to assume that the nontrivial holonomy  $\mathcal{H}$  is created by the defect  $F$ , then the holonomies  $\mathcal{H}_{\mu,\beta,\alpha}$  *should be* independent of  $\beta$ <sup>6</sup>. Then, the total contribution of the interface potentials associated to the defect  $\mu$  is

$$\langle \chi_R(\mathcal{H}_{\mu,\alpha}), \sum_{\beta} \oint_{e_{\mu}^{D-3} \cap e_{\beta}^{D-1}} q_{\beta\alpha} \rangle \quad (\text{A.70})$$

which involves  $q$  only on the defect  $e_{\mu}^{D-3}$ . This is the part of discretised bulk BF theory that encodes the curvature defects. Compare this to the actual defect contribution

$$- \int_{e_{\mu}^{D-3}} \langle \phi F_{\mu} \phi^{-1}, \delta q \rangle \quad (\text{A.71})$$

which has a similar enough form. As we want the defect to actually *be* the source of the holonomy, we can identify an appropriate constraint

$$\chi_R(\mathcal{H}_{\mu,\alpha}) + \delta(\phi F_{\mu} \phi^{-1}) = 0 \quad (\text{A.72})$$

so the defect value  $F$ , dressed by  $\phi$ s values on the defect, should be opposite to the holonomy. In particular, if the defects *vanish*, so when there is no curvature present, then the holonomy must be trivial.

We have another defect contribution from the dual cell-dual cell interfaces, via pieces associated to the dual vertex  $\tilde{e}_{\alpha}^0$  of a cell  $e_{\alpha}^{D-1}$ . If we sum these all up, we get

$$- \sum_{\beta, \dots} \epsilon_{\beta, \beta', \dots} \int_{e_{\beta, \alpha; \dots}^{D-3}} \langle \chi_R(\phi_{\alpha}), Q_{\beta, \beta; \alpha} \rangle \quad (\text{A.73})$$

which counts the ‘total shift’ one has when going through all the wedges inside the cell  $e_{\alpha}^{D-1}$ , in an order which determines the signs  $\epsilon_{\beta, \dots}$ . The specifics of the summation labels do not matter so much. The main point is that these are integrals over full codimension 3 surfaces where *all* values of  $\phi$  contribute. The sampling therefore does *not* remove all values of the  $\phi$ s from inside the cells. In simplicial  $D = 4$ , this entails that one has integrals of  $Q, \phi$  over the dual edges of a tetrahedron.

While the sampling does not bring us further, it does seem reasonable to *enforce* instead that all these pieces should localise on the dual vertex  $\tilde{e}_{\alpha}^0$ . The continuum data must then be restricted, i.e. by assuming  $\phi_{\alpha}$  is constant along the union of the codimension 3 cells,

$$\phi_{\alpha}^{-1} d\phi_{\alpha} \stackrel{\cup e_{\beta, \alpha; \dots}^{D-3}}{=} 0. \quad (\text{A.74})$$

Then, the total is

$$- \langle \chi_R(\phi_{\alpha}(\tilde{e}_{\alpha}^0)), \mathcal{Q}_{\alpha} \rangle \quad (\text{A.75})$$

---

<sup>6</sup>We do not currently know of a good proof of this within this discretization procedure.

with

$$\mathcal{Q}_\alpha := \sum_{\beta, \dots} \epsilon_{\beta, \beta', \dots} \int_{e_{\beta, \alpha; \dots}^{D-3}} Q_{\beta, \beta'; \alpha} \quad (\text{A.76})$$

the 'Kalb-Ramond volume holonomy'. Comparing this again to the Yang-Mills charge defect term,

$$-\langle \chi_R(\phi)(\tilde{e}_\alpha^0), (\phi S_\alpha \phi^{-1})|_{\tilde{e}_\alpha^0} \rangle \quad (\text{A.77})$$

we see that the appropriate constraint is

$$\mathcal{Q}_\alpha + (\phi S_\beta \phi^{-1})|_{\tilde{e}_\alpha^0} = 0 \quad (\text{A.78})$$

which identifies the Kalb-Ramond volume holonomy with the charge defect. Again, if no charges are present, this means the KR holonomy must vanish.

What this teaches us is the following: By including sources and sampling, we can find discrete analogues of the continuum curvature and Gauss constraints by seeing which interface contributions coalesce onto a given defect. Thus, in the discrete model, we can see that our constraints of (sourceless) BF theory are

$$\mathcal{H}_{\mu, \alpha} = 0 = \mathcal{Q}_\alpha \quad (\text{A.79})$$

for all possible defects. We saw, however, that there are ever so slight non sequiturs in this, and one must be careful about how to precisely reduce the fields to make the interpretations simple.

There is one more set of contributions living on the dual codimension 3 surfaces  $\tilde{e}_{\alpha, \alpha', \gamma}^{D-3}$ . These are not a priori connected to any defects. By combining the interface potentials from the cell-cell and dual cell-dual cell interfaces that touch a given surface labeled by  $\gamma$ , one has a net contribution for each codimension 3 surface in  $\tilde{\Lambda}_{e_\alpha^{D-1} \cap e_{\alpha'}^{D-1}}$  which only includes constant objects. We anchor it on the  $\alpha, \beta$  wedge, whence it is expressed as

$$\langle \chi_R(G_{\alpha, \alpha'}), Q_{\beta \beta', \alpha} - Q_{\beta \beta', \alpha'} \rangle = \langle \chi_R(G_{\alpha, \alpha'}) + \chi_R(G_{\alpha, \alpha'}^{-1}), Q_{\beta \beta', \alpha} \rangle. \quad (\text{A.80})$$

For examples in triangulations, in  $D = 3$  this is a contribution which sits on the midpoints of edges between two triangles. In  $D = 4$ , instead, think of a triangle between two tetrahedra. This triangle itself has a dual triangulation. The contributions sit on the dual edges of that, so on lines connecting a central (dual) vertex to the midpoints of the three boundary edges.

This is quite curious, as the discretization thus does not yield a trivial phase space even without defects. If one wishes to, one may read this as a kind of discretization artifact, but we have little reason at this point to discard it. This set of contributions to the *bulk* symplectic potential of the discretised BF theory is of course still subject to the Gauss and flatness constraints, which reduce the degrees of freedom present here.

To summarise, we have sampled the continuum BF fields on the unique trivial configuration within each wedge and checked what remains of the bulk symplectic potential. We have found that the remaining data is entirely in terms of



1. the constant group elements  $G_{\alpha,\alpha'}$  associated to cell-cell interfaces, which encode parallel transport between frames associated to cells,
2. The ‘constant’ shift forms  $X_{\beta,\beta;\alpha;\gamma} := \int_{e^{D-3}} Q_{\beta,\beta;\alpha;\gamma}$  smeared over bulk codimension 3 dual surfaces, both inside cells and on their boundaries,
3. the smeared  $q$ -frames  $\mathbf{q}_{\mu,\alpha} := \sum_{\beta} \oint_{e_{\mu}^{D-3} \cap \tilde{e}_{\beta}^{D-1}} q_{\beta\alpha}$  over the defects, one per dual cell,
4. the bulk dual vertex value  $\phi_{\alpha}$  of  $\phi$  (strictly speaking, the frames  $\phi$  over dual edges of bulk cells, which we have eliminated here by requiring constancy).

The variables  $X_{\beta,\beta;\alpha;\gamma}$  are not free, as they come from closed forms  $Q$ . They satisfy the identity

$$\sum_{\gamma} X_{\beta,\beta;\alpha;\gamma} = 0 \quad (\text{A.81})$$

over each dual cell-dual cell interface  $\beta, \beta'$  within each cell  $\alpha$ .

Furthermore, the variables are all on-shell subject to the constraints  $\mathcal{H}_{\mu,\alpha} = 0 = \mathcal{Q}_{\alpha}$ . When these are imposed, then the 3rd and 4th degrees of freedom drop from the phase space. By analogy to the continuum, this means that they are the analogue of bulk dressing fields.

The full symplectic potential of this bulk set of variables is then

$$\begin{aligned} \Theta_{\Lambda} = & \sum_{\mu} (\langle \chi_R(\mathcal{H}_{\mu,\alpha}) + \delta(\phi_{\alpha} F_{\mu} \phi_{\alpha}^{-1}), \mathbf{q}_{\mu,\alpha} \rangle - \delta(\langle \phi_{\alpha} F_{\mu} \phi_{\alpha}^{-1}, \mathbf{q}_{\mu,\alpha} \rangle)) \\ & - \sum_{\alpha} \langle \chi_R(\phi_{\alpha}), \mathcal{Q}_{\alpha} + \phi_{\alpha} S_{\alpha} \phi_{\alpha}^{-1} \rangle \\ & + \sum_{(\alpha,\alpha'), (\beta,\beta')} \langle \chi_R(G_{\alpha,\alpha'}), Q_{\beta\beta',\alpha} - Q_{\beta\beta',\alpha'} \rangle \end{aligned} \quad (\text{A.82})$$

where we introduce the dressed, anchored curvature defect by making  $\phi$  constant along  $\mu$ , and setting it to be in the frame of the cell  $\alpha$ :

$$\phi F_{\mu} \phi^{-1} \mapsto \phi_{\alpha} F_{\mu} \phi_{\alpha}^{-1} \quad (\text{A.83})$$

Note that this is not quite proper: we have replaced the values of  $\phi$  on the defect with the ones on the dual vertex of  $\alpha$ . It is a sensible replacement, but still an ad-hoc one and should be kept in mind.

So, with a number of assumptions, we have arrived at the bulk symplectic potential we can expect from BF theory. We can see that there are additional contributions compared to the continuum, and may choose to keep or drop them. *In the main text, we drop these terms in order to connect with the standard set of states.* Let us now consider more closely what happens at the boundary of the slice  $\Sigma$ .

Once again, we have interface potentials imprinted on the boundary  $\partial\Sigma$  in its induced cellulation. The difference is now that there is no other side, so in addition to the existing

constant group elements  $G$ , there are still the full  $\phi, q$ . Then, on the boundary we have the contribution

$$\langle \chi_R(\phi_\alpha), dq_{\beta\alpha} \rangle \quad (\text{A.84})$$

and it does *not* reduce into a set of constant elements, as this only happens at bulk interfaces. We have already sampled on a configuration, and the discretization into constant elements happened automatically. *There is nothing in our sampling process that would tell us how to reduce the data on the boundary into a discrete set.* We thus arrive at the main insight about boundary dressing fields/gauge edge modes:

**Discretising a field theory in the bulk and on the boundary are separate processes.**

This should not be entirely surprising in the BF example, as the bulk is always topological, but its boundary dynamics can vary. There is no a priori relation between the two apart from some level of compatibility due to bulk constraints imprinting on the boundary. However, we may appeal to a kind of ‘minimal splitting and glueing procedure’, which tells us that all data created by splitting should be somehow already contained in the bulk in some fashion. The clearest way to do this in our current example is to note that every bulk cell-cell interface is equipped with one *constant* group element built from  $\phi$ s. Since now, on a boundary, we only have one of these  $\phi$ , a possible truncation is to keep only the constant part of  $\phi$  on each boundary surface  $e_{\alpha,\gamma}^{D-2}$ . Calling this  $\phi_{\alpha,\gamma}$ , we then see that the symplectic potential localises again:

$$\Theta_{\partial\Sigma} = \sum_{\alpha} \sum_{\gamma} \langle \chi_R(\phi_{\alpha,\gamma}), \sum_{\beta} \int_{e_{\alpha,\gamma}^{D-2} \cap \tilde{e}_{\beta}^{D-1}} dq_{\beta\alpha} \rangle \quad (\text{A.85})$$

one then again needs to combine the different contributions. The result is again a sum of terms for dual codimension 3 surfaces contained in the boundary dual triangulation, i.e. in  $D = 4$  the dual edges of boundary triangles. The form is

$$\Theta_{\partial\Sigma} = \sum_{\alpha} \sum_{\gamma} \langle \chi_R(\phi_{\alpha,\gamma}), \mathcal{Q}_{\alpha,\gamma} \rangle \quad \mathcal{Q}_{\alpha,\gamma} := \sum_{\beta} \epsilon_{\beta,\beta'} \int_{e_{\alpha,\gamma;\beta,\beta'}^{D-3}} Q_{\beta,\beta';\alpha;\gamma} \quad (\text{A.86})$$

so in fact it is just again a contribution for the Gauss constraint parts. In analogy to the continuum

$$\langle \chi_R(\phi), d_{\omega} q \rangle \quad (\text{A.87})$$

we can see that  $\mathcal{Q}_{\alpha,\gamma}$  is the discrete analogue of the  $B$  field on the boundary; in particular, we understand that we can write the corner YM charge as

$$J_{\alpha} = \oint_{\partial\Sigma} \langle J, \alpha \rangle = \oint_{\partial\Sigma} \langle \phi^{-1} d_{\omega} q \phi, \alpha \rangle \rightarrow \sum_{\alpha} \sum_{\gamma} \langle \phi_{\alpha,\gamma}^{-1} \mathcal{Q}_{\alpha,\gamma} \phi_{\alpha,\gamma}, \alpha_{\alpha,\gamma} \rangle \quad (\text{A.88})$$

In particular, we can see that the Gauss constraint essentially is requiring to sum all the discrete  $\mathcal{Q}_{\alpha,\gamma}$  around a dual vertex to zero. This is well-known as a formulation of the Gauss constraint in the discrete[28, 139, 243] and we have derived it here from sampling

and a constancy assumption on the field  $\phi_\alpha$ .

Similarly, the KR charges

$$K_\mu = \oint_{\partial\Sigma} \langle \phi^{-1} d\phi, \mu \rangle \quad (\text{A.89})$$

now localise on the interfaces  $e_{\alpha,\alpha'}^{D-3}$  between *boundary cells*, as this is where the derivative of  $\phi$  is still nonzero. Being heuristic, we find

$$K_\mu = \sum_{\alpha,\alpha'} \langle \phi_{\alpha,\gamma} \phi_{\alpha',\gamma}^{-1}, \mu_\gamma \rangle \quad \mu_\gamma = \int_{e_{\alpha,\alpha'}^{D-3}} \mu \quad (\text{A.90})$$

but we do not want to get into the details. The main point is that the KR charges are given by the parallel transport group elements from one boundary face to another, sitting on boundary codimension 3 surfaces, i.e., in  $D = 4$ , the parameters live on boundary edges. In this sense, the elements  $\phi_{\alpha,\gamma} \phi_{\alpha',\gamma}^{-1}$  represent a discrete boundary connection.

What this shows is that, once we commit to a *choice* of boundary discretization, we get also the expressions for charges in a mostly straightforward manner. What we meant to highlight is that

1. The bulk and boundary discretizations are *independent* of one another;
2. there are still intuitively minimal samplings of the boundary phase space that give rise to meaningful objects.

In particular, we know that given the right kind of dynamics, we do not need to restrict the boundary variables  $\mathcal{Q}_{\alpha,\gamma}, \phi_{\alpha,\gamma}$  apart from the induced flatness constraint on  $\phi_{\alpha,\gamma} \phi_{\alpha',\gamma}^{-1}$  coming from the bulk. Then, all the charges are conserved, and the fields  $\mathcal{Q}, \phi$  must satisfy some discrete analogue of the first order transport equations we saw before.

We can easily see now what kind of corner structure comes out of BF theory. The discretised theory has a clean corner algebra generated by the  $\phi_{\alpha,\gamma}, \mathcal{Q}_{\alpha,\gamma}$ , and we also know how glueing happens: The group elements must be combined into the  $G_{\alpha,\alpha'}$ . For demonstration, pick two cells labeled by  $\alpha, \alpha'$  with common interface labeled by  $\gamma$ . Then, we want to impose a redundancy

$$(\phi_{\alpha',\gamma}, \phi_{\alpha,\gamma}) \sim (\phi_{\alpha',\gamma} g, \phi_{\alpha,\gamma} g) \quad (\text{A.91})$$

which is a transformation generated by

$$(\tilde{\mathcal{Q}}_{\alpha',\gamma} + \tilde{\mathcal{Q}}_{\alpha,\gamma}), \quad \tilde{\mathcal{Q}}_{\alpha',\gamma} := \phi_{\alpha',\gamma}^{-1} \mathcal{Q}_{\alpha',\gamma} \phi_{\alpha',\gamma} \quad (\text{A.92})$$

This is a charge matching condition like in the continuum, for  $\tilde{\mathcal{Q}}$  which is an analogue of  $J$ . It is also the usual kind of lattice gauge theory matching condition that first spurred the recent interest in corner charges and symmetries.

Meanwhile, what about the KR charges? If there was no charge matching of some sort, then the group elements  $\phi_{\alpha,\gamma} \phi_{\alpha',\gamma}^{-1}$  and the  $G$ s, so both across the boundary and tangential

to it, would encode a holonomy around the codimension 3 surface labeled by  $\gamma$ . This holonomy, of course, will again need to fall into the form we saw in the bulk potential, for consistency. For more clarity, note that by imposing the charge matching on the sum of the symplectic boundary potentials, we arrive at

$$\langle \chi_L(G_{\alpha,\alpha'}), \mathcal{Q}_{\alpha',\gamma} \rangle \quad (\text{A.93})$$

which should appear familiar. Then, it is clear that once we check what the conjugate variable to the holonomy around  $\gamma$  is, it will be some function of the  $\mathcal{Q}_{\alpha',\gamma}$ s. So, imposing flatness will again lead to a KR shift invariance of these objects on the boundary. Inasmuch as we need to impose flatness either way, i.e. by asking the holonomy of bulk and boundary at  $\gamma$  to vanish already before glueing<sup>7</sup>, we thus know that imposing KR frame reorientation invariance at  $\gamma$  is given by matching the corner charges  $K$ . We should therefore expect it to be necessary, too:

$$K_{\alpha,\alpha';\gamma} + K_{\alpha'',\alpha''';\gamma} = 0 \quad (\text{A.94})$$

So we need to also glue the codimension 3 surfaces  $e_\gamma^{D-3} \in \Lambda_{\partial\Sigma}^{D-3}$  together.

These all show that BF is equipped, in this discretization, with a nontrivial, *noncommutative* corner algebra  $\mathcal{A}_{\partial\Sigma}$ , which gives rise to the glueing structure known from the entangling product

$$\mathcal{A}_L \otimes_{\mathcal{A}_{\partial\Sigma}} \mathcal{A}_R \quad (\text{A.95})$$

The splitting, in turn, comes from separating on a cut along cell-cell boundaries the variables  $G$  into constituents.

The quantisation of this discretised phase space, and its phase space, is then relatively easy by starting from a single cell and applying glueing laws. I.e. if we use a group polarisation, i.e. our wavefunctions look like

$$\Psi(\{G_{\alpha,\alpha'}\}; \{\phi_{\alpha,\gamma}\}) \quad (\text{A.96})$$

then we get YM transformations acting on the variables as shifts of argument (infinitesimally, as derivatives), and KR generators as multiplication operators. On the boundary, there is no invariance of these wavefunctions - instead, shifts of argument of  $\phi$  are changes of state, and similarly the boundary  $K$  charges referring to  $\phi$  are nontrivial multiplication operators. Of course, again, to have the correct phase space, one still needs to impose *boundary flatness*, so *boundary holonomies* must be trivial as well. We do not pursue this in more technical detail, as we think the logic is clear.

---

<sup>7</sup>The astute reader will notice that a priori, we could assign to a half-loop like this some nonzero group element instead of the trivial one; then a flatness constraint upon glueing would restrict their combination to be trivial.

# Appendix B

## Appendices to chapter 2

### B.1 Operator transport in the presence of nontrivial centers

This appendix is largely based on an existing preprint[2].

Let us start from the algebra  $\mathcal{A} = \mathbb{B}(\mathbb{H})$  of operators<sup>1</sup> acting on a Hilbert space  $\mathbb{H}$  of our choice. Then, select as subsystems two subalgebras  $\mathcal{A}_I, \mathcal{A}_O$ , to be later seen as inputs and outputs of the transport superoperator. These are understood as operations or observables of the subsystems in consideration. These two algebras are not necessarily a partition of  $\mathcal{A}$ , but rather we require the output to be the algebraic complement (the commutant) of the input,

$$(\mathcal{A}_I)' = \mathcal{A}_O \quad (\mathcal{A}_O)' = \mathcal{A}_I, \quad (\text{B.1})$$

so that the operators commuting with inputs are precisely the output operators. This structure, when present, captures the general properties we may expect of any operational definition of splitting of the system into two parts.<sup>2</sup> We have here labeled the two parts of the bipartition 'input' and 'output' in analogy to a quantum channel, but the labels could refer to any form of separation of subsystems (it could equally well be 'Left/Right', 'Inside/Outside', 'System/M Measurement apparatus' or 'Alice/Bob', etc). For more about this algebraic perspective on subsystems, we recommend the review[244].

In general, we have that  $\mathcal{A}_I \cup \mathcal{A}_O \neq \mathcal{A}$ , and, more importantly, we may have a nontrivial *center*  $\mathcal{Z} = \mathcal{A}_I \cap \mathcal{A}_O$ , which consists of operators which commute with all others. If the center is trivial, i.e. consisting only of multiples of the identity  $\lambda \mathbb{I}$ , we have that  $\mathcal{A}_I \cup \mathcal{A}_O$  factorises into  $\mathcal{A}_I \otimes \mathcal{A}_O$ . Its representations, then, also factorise into tensor products of Hilbert spaces. This is the simplest setting. In this special case, we can define subsystems as Hilbert subspaces in the tensor factorisation and extend subsystem operators uniquely:  $X_I \mapsto X_I \otimes \mathbb{I}_O$ <sup>3</sup>; moreover, the entanglement for pure states is well-defined and can be

---

<sup>1</sup>We will for simplicity work in this section with bounded operators on finite dimensional Hilbert spaces.

<sup>2</sup>Notice that, in particular, this is a pendent of properties in local QFT, as formalised by Haag duality, where subsystems are identified and distinguished by their localization on the spacetime manifold.

<sup>3</sup>We assume here that the algebras contain a unit, so an identity operator.

quantified, e.g. through von Neumann entropy. These properties do not generalise to the case with nontrivial center.[116, 164, 245]

In the case of nontrivial center, we have a *commutative subalgebra* of  $\mathcal{A}$ . In fact, because the input and output systems commute with this center, we may separate representations of the algebra into sectors labeled by eigenvalues of operators in  $\mathcal{Z}$ . This gives rise to the characteristic structure of the Hilbert space

$$\mathbb{H} = \bigoplus_E \mathbb{H}_E = \bigoplus_E \mathbb{H}_{I,E} \otimes \mathbb{H}_{O,E} \quad (\text{B.2})$$

in which the central operators have been diagonalised with eigenvalues given by  $E$ . A very concrete way to present this sort of algebra is through block matrices, where each block corresponds to a sector  $E$  in the above decomposition. Lacking a clear notion of separable states, the simple notion of entanglement as non-product states does not hold up, but, of course, this is not the end of the story, and for example entropies may still be calculated[164, 246, 247].

Still, the particular structure of the algebras and Hilbert spaces considered above indicates a natural way forward.

We still have a notion of subsystem in each sector due to factorisation, given by operators on input and output Hilbert spaces, in each of them. That is, there are associated subalgebras

$$\bigoplus_E \mathbb{B}(\mathbb{H}_{I,E}) \otimes \mathbb{I}_{O,E} \qquad \bigoplus_E \mathbb{I}_{I,E} \otimes \mathbb{B}(\mathbb{H}_{O,E}) \quad (\text{B.3})$$

of operators on the individual subsystems for each sector. These are a consistent definition of complementary subsystems in the case of the Hilbert space structure above - they form subalgebras, have the correct commutant relation and the right center given by sums of identity operators in each sector, which represent the diagonalised operators from  $\mathcal{Z}$ .

They also have unique extensions from the image of naive partial traces to the full algebra, and are the largest set to have this property. This means that, if we take the naive partial trace of an operator  $X = \sum_{E,F} X_{E,F} \in \mathbb{B}(\mathbb{H})$  (splitting into blocks over the different sectors), which is given by

$$\text{Tr}_O[X] = \sum_E \text{Tr}_{\mathbb{H}_{O,E}}[X] = \sum_E \text{Tr}_{\mathbb{H}_{O,E}}[X_{E,E}] \quad (\text{B.4})$$

then we only keep the diagonal blocks in  $X$ , because there is no notion of a trace on the non-diagonal blocks. The reason for this is that, while there is a natural 'evaluation' or trace map on  $V^* \otimes V$  for any vector space  $V$ , this is not true for

$$\mathbb{B}(\mathbb{H}_E, \mathbb{H}_F) \cong \mathbb{H}_E^* \otimes \mathbb{H}_F \quad (\text{B.5})$$

when  $E \neq F$  (or any vector spaces which are not equal). Without providing such maps by hand (which amounts to a different choice of partial trace map), we can only arrive at operators of the above form by reducing to subsystems. Similarly, if we want to extend

some abstract subsystem operator, for example given by  $U = \sum_{E,F} U_{E,F} \in \mathbb{B}(\bigoplus_E \mathbb{H}_{I,E})$  on the input subsystem, to the full system, then we would naively do so by extending it with 'identity operators'  $\mathbb{I}_{O;E,F}$

$$i_I(U) := \sum_{E,F} U_{E,F} \otimes \mathbb{I}_{O;E,F}, \quad (\text{B.6})$$

which however are only unambiguously defined, again, for  $E = F$ . This is simply the statement that the off-diagonal blocks will not have a clear notion of a diagonal, and certainly not of an 'identity', without prescribing it by hand. In this sense, the choice of subsystem we indicate here is the only unambiguous one - for others, we would need to prescribe by hand extra data for defining any extension and restriction. To summarize, the natural way forward is to use the unique unambiguous definition of subsystems in each sector and extend it to the sum over sectors, to obtain information channels and a generalised channel/state duality for the whole system. What we do in the following is to show that this way forward can indeed be pursued, that it leads to a well-defined result, and that the resulting construction is, in the sense we clarified, the only natural one.

In order to define a general notion of operator transport from system  $I$  to system  $O$ , we can follow the path indicated above, using a few ingredients:

1. Choices of input and output systems  $\mathcal{B}_{I|O}$ , e.g.  $\mathcal{B}_{I|O} = \bigoplus_E \mathbb{B}(\mathbb{H}_{I|O,E})$ ,
2. Identifications/Injections  $i_{I|O} : \mathcal{B}_{I|O} \hookrightarrow \mathcal{A}$ , whose images we identify as the complementary subsystems  $\mathcal{A}_{I|O}$ ,
3. Conjugate partial trace maps  $P\text{Tr}_{I|O} : \mathcal{A} \rightarrow \mathcal{B}_{I|O}$  that reduce an operator on the full system to a subsystem,
4. A mapping  $\Sigma : \mathcal{A} \rightarrow \mathcal{A}$ , usually related to a density matrix  $\rho$ , e.g.  $\Sigma(X) = X\rho^{t_I}$ .

We will now go into detail about this construction in the case of a *trivial center* at first, which corresponds to a system with simple tensor product factorisation in its Hilbert space. This will illustrate that the notion of transport operators is useful also in this simple case, and already shows the main behaviour of their properties, namely that there exists a 2-out-of-3 implication for purity of the state  $\rho$ , trace preservation and isometry of the mapping associated to it<sup>4</sup>. After that, we show the generalisation to the case with multiple blocks/sectors or nontrivial center, and find that the same thing holds, but the conditions split per sector.

### B.1.1 Trivial center

For starters, consider the  $C^*$ -algebra  $\mathcal{A} = \mathbb{B}(\mathbb{H})$  of bounded linear operators on a finite dimensional Hilbert space  $\mathbb{H}$ , equipped with the Hilbert-Schmidt inner product., together

<sup>4</sup>This 2-out-of-3 property appears to be due to the relatively rigid way entanglement shows itself in pure states, as manifested through there being a (mostly unique) measure of entanglement for pure states, which is not the case for mixed states.

with a tripartition of  $\mathbb{H} \cong \mathbb{H}_I \otimes \mathbb{H}_O \otimes \mathbb{H}_B$  into input, output and background spaces. In concrete cases, this induces subsystems  $\mathcal{A}_{I|O|B}$  and associated extension

$$i_{I|O|B} : \mathcal{B}_{I|O|B} \hookrightarrow \mathcal{A} \quad (\text{B.7})$$

and partial trace maps

$$P\text{Tr}_{I|O|B} : \mathcal{A} \rightarrow \mathcal{B}_{I|O|B} \quad (\text{B.8})$$

which are, respectively, injective and surjective and, up to normalisation, inverses of each other. As before, we identify  $\mathcal{A}_{I|O|B} = \text{Im}(i_{I|O|B})$ .

We prescribe usually the subsystems as the obvious choice of subalgebras of  $\mathbb{B}(\mathbb{H})$ :

$$\mathcal{B}_I = B(\mathbb{H}_I) \quad \mathcal{B}_O = B(\mathbb{H}_O) \quad \mathcal{B}_B = B(\mathbb{H}_B). \quad (\text{B.9})$$

In this concrete case, they are simply given by

$$i_I(X) = X \otimes \mathbb{I}_{BO} \quad P\text{Tr}_I(X) = \text{Tr}_{BO}[X] \quad (\text{B.10})$$

and will from now on refer to  $\mathcal{B}$  and  $\mathcal{A}$  interchangeably where there is no risk of confusion. In fact, the partial trace is best defined in terms of an adjoint to an extension. For example, for the bipartite case, it is the defining property of the partial trace that

$$\begin{aligned} \langle X_I \otimes \mathbb{I}_O, Y \rangle_{HS} &= \langle X_I, \text{Tr}_O[Y] \rangle_{HS,I} \\ \forall X_I \in \mathcal{B}_I, Y \in \mathcal{B}_O \end{aligned} \quad (\text{B.11})$$

Using this relation, we can define more general partial trace and extension maps which share the same behaviour.

These extension and partial trace maps may be used to create various kinds of transport maps from the input to the output system. In general, such a mapping will take the form

$$\mathcal{T}_\Sigma : \mathcal{B}_I \rightarrow \mathcal{B}_O \quad (\text{B.12})$$

$$X \mapsto P\text{Tr}_O[\Sigma(i_I(X))] \quad (\text{B.13})$$

where  $\Sigma : \mathcal{A} \rightarrow \mathcal{A}$  is some linear mapping that twists the trivial extension-restriction operation. For  $\Sigma = \text{id}_{\mathcal{A}}$ , this gives a completely depolarising channel up to normalisation. We are interested in a twisting by multiplication with a density operator  $\rho$  of the full system. These have the interpretation of first preparing the system in the state given by a density matrix  $\rho$ , acting on a subsystem  $I$  with some operator and then looking at the results of that action in subsystem  $O$ . This gives an effective induced operator in  $O$ , and therefore provides a notion of 'operator transport' similar to the 2-qubit case that we discussed in the introduction.

The first concrete case we are interested in is the choice  $\Sigma(X) = K \cdot X\rho$  with some positive constant  $K$ , which produces the Jamiolkowski-Pillis mapping

$$\mathcal{T}_\rho(X) = K P\text{Tr}_O[i_I(X)\rho]. \quad (\text{B.14})$$



For the purpose of generality, we do not assume the density matrix has been trace-normalised and keep its appearance explicit in the following. This mapping is characterised by the relation in Hilbert-Schmidt inner products

$$\begin{aligned}\langle \mathcal{T}_\rho(X), Y \rangle_O &= \langle K i_I(X) \rho, i_O(Y) \rangle \\ &= \langle X, K P \text{Tr}_I[\rho^\dagger i_O(Y)] \rangle_I.\end{aligned}\tag{B.15}$$

The middle form here gives a clear interpretation of the inner products: we extend both  $X$  from the input and  $Y$  from the output to the full system, then take their inner product with the density matrix in between. Due to the cyclicity of the full trace on  $\mathcal{A}$ , this is the same as the expression

$$\begin{aligned}K \text{Tr}_{\mathbb{H}}[i_O(Y) i_I(X)^\dagger \rho] &= K \langle \rho, i_O(Y) i_I(X)^\dagger \rangle \\ &= K \langle i_O(Y) i_I(X)^\dagger \rangle_\rho\end{aligned}\tag{B.16}$$

which is just, up to scaling, the expectation value of the operator given by  $X^\dagger$  on the input subsystem and by  $Y$  on the output subsystem, in the state  $\rho$ .

We can, of course, change the twisting map  $\Sigma$  to a different operation, but there is no uniquely compelling alternative. The perhaps most obvious alternative comes from a seemingly innocuous difference: the Choi mapping

$$\mathcal{T}_\rho(X) = K P \text{Tr}_O[i_I(X) \rho^{t_I}].\tag{B.17}$$

uses the *partial transpose* of the state with respect to the subspace  $\mathbb{H}_I$ . The Choi mapping has a number of more favorable properties compared to the Jamiolkowski-Pillis mapping. In particular, unlike the latter, the Choi mapping provides an isomorphism between the sets of completely positive maps  $\mathbb{H}_I \rightarrow \mathbb{H}_O$  and of (unnormalised) states on  $\mathbb{H}_I \otimes \mathbb{H}_O$ . Also, for bipartite systems, the Choi mapping for a pure state  $|\phi\rangle \langle\phi|$  can always be written as

$$\mathcal{T}_\rho(X) = \Phi X \Phi^\dagger\tag{B.18}$$

where the map  $\Phi : \mathbb{H}_I \rightarrow \mathbb{H}_O$  has components  $\langle o | \Phi | \iota \rangle = \langle \iota, o | \phi \rangle$ . This can be seen through

$$\langle o | \mathcal{T}_\rho(X) | \tilde{o} \rangle = \sum_{i, \tilde{i}} \langle i | X | \tilde{i} \rangle \langle \tilde{i}, o | (|\phi\rangle \langle\phi|)^{t_I} | i, \tilde{o} \rangle\tag{B.19}$$

$$= \sum_{i, \tilde{i}} \langle i | X | \tilde{i} \rangle \langle o | \Phi | i \rangle \langle \tilde{i} | \Phi^\dagger | \tilde{o} \rangle.\tag{B.20}$$

### Trace and isometry conditions

We will ask two important questions about this mapping: first, when it is a channel, and second, when it is an isometry (in the Hilbert-Schmidt sense).

The former is a standard question, but the latter has an interesting new aspect to it: if the mapping is isometric, we can see the system itself as providing an 'information funnel'

from input to output (this, in turn, has been used as a proxy of holographic behaviour in the literature).

First, if  $\rho \geq 0$ , it is clear that the mapping is completely positive. Trace preservation amounts to

$$\begin{aligned} K \text{PTr}_I[\rho] &= \mathbb{I}_I \\ \implies K &= \frac{\dim(\mathbb{H}_I)}{\text{Tr}[\rho]}, \text{PTr}_I\left[\frac{\rho}{\text{Tr}[\rho]}\right] = \frac{\mathbb{I}_I}{\dim(\mathbb{H}_I)}. \end{aligned} \quad (\text{B.21})$$

In other words, being a quantum channel fixes the overall normalisation of the mapping and also puts the requirement on the reduced input state that it must be flat.

Isometry may be expressed easily as well, using swap operators<sup>5</sup> on the Hilbert spaces:

$$\langle \mathcal{T}(X), \mathcal{T}(X) \rangle_O \quad (\text{B.22})$$

$$= \text{Tr}_O[\mathcal{T}(X)^\dagger, \mathcal{T}(Y)] \quad (\text{B.23})$$

$$= \text{Tr}_{O \otimes 2}[(\mathcal{T}(X)^\dagger \otimes \mathcal{T}(Y))\mathcal{S}_O] \quad (\text{B.24})$$

$$= K^2 \text{Tr}_{\mathbb{H} \otimes 2}[(X^\dagger \otimes Y)(\rho \otimes \rho)\mathcal{S}_O] \quad (\text{B.25})$$

$$= K^2 \text{Tr}_{I \otimes 2}[(X^\dagger \otimes Y)\text{Tr}_{OB \otimes 2}[(\rho \otimes \rho)\mathcal{S}_O]] \quad (\text{B.26})$$

which translates into the requirement

$$K^2 \text{Tr}_{O \otimes 2}[(\rho_{IO} \otimes \rho_{IO})\mathcal{S}_O] = \mathcal{S}_I. \quad (\text{B.27})$$

This in turn implies the two equalities (the second from multiplying the isometry condition by  $\mathcal{S}_I$ )

$$K^2 \text{Tr}[\rho]^2 e^{-S_2(\rho_O)} = D_I \quad (\text{B.28})$$

$$K^2 \text{Tr}[\rho]^2 e^{-S_2(\rho_{IO})} = D_I^2 \quad (\text{B.29})$$

expressed using the second Rényi entropy

$$e^{-S_2(\rho)} = \frac{\text{Tr}[\rho^2]}{\text{Tr}[\rho]^2} \quad (\text{B.30})$$

Combining these two leads to the general, normalisation-independent requirement

$$e^{-S_2(\rho_{IO}) + S_2(\rho_O)} = D_I \quad (\text{B.31})$$

The form of the exponent suggests looking for a subsystem inequality for Rényi entropies - however, it is known that such inequalities do not exist[248]<sup>6</sup>. Still, in general these conditions fix  $K^2 \text{Tr}[\rho]^2$  to be in the interval

$$[D_I, D_I D_O] \cap [D_I^2, D_I^3 D_O] = [D_I^2, D_I D_O]. \quad (\text{B.32})$$

<sup>5</sup>These simply take two factors in a tensor product and swap them,  $\mathcal{S}|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$ .

<sup>6</sup>However, we might still use *measured* Rényi entropies and mutual information[249]. These have a known expression and satisfy nice properties as an analogue of the von Neumann mutual information.

The minimum value  $K = \frac{D_I}{\text{Tr}[\rho]}$  is part of the trace condition above, while the maximum is  $K = \frac{D_I}{\text{Tr}[\rho]} \sqrt{\frac{D_O}{D_I}}$  is incompatible with being trace-preserving, in general. However, we should not preemptively choose the former value. Indeed, if we do, then the above conditions turn into

$$D_I^2 e^{-S_2(\rho_O)} = D_I \implies S_2(\rho_O) = \log(D_I) \quad (\text{B.33})$$

$$D_I^2 e^{-S_2(\rho_{IO})} = D_I^2 \implies S_2(\rho_{IO}) = 0 \quad (\text{B.34})$$

which means that the reduced state  $\rho_{IO}$  must be pure, and therefore the state must factorise  $\rho = \rho_{IO} \otimes \rho_B$ , and the reduced input (and output) state must be maximally mixed  $S_2(\rho_O) = S_2(\rho_I) = \log(D_I)$ .

In other words, isometry (ISOM) and trace preservation (TP) imply purity of the state  $\rho_{IO}$  (PURE).

$$\text{ISOM} \wedge \text{TP} \implies \text{PURE} \quad (\text{B.35})$$

However, this setup is too restrictive. On the other hand, the maximal value implies

$$e^{-S_2(\rho_O)} = \frac{1}{D_O} \implies S_2(\rho_O) = \log(D_O) \quad (\text{B.36})$$

$$e^{-S_2(\rho_{IO})} = \frac{D_I}{D_O} \implies S_2(\rho_{IO}) = \log(D_O) - \log(D_I), \quad (\text{B.37})$$

so once again the state reduced to the output system is maximally mixed. Now, however, the reduced state  $\rho_{IO}$  no longer needs to be pure. So quite intriguingly, the mapping we propose cannot be a quantum channel and an isometry in general, unless the state used factorises in a nice way.

Additionally, we may ask when the mapping we defined is unital. This gives an input-output swapped version of the trace preservation condition:

$$K \rho_O = \mathbb{I}_O \implies K = \frac{D_O}{\text{Tr}[\rho]}, \frac{\rho_O}{\text{Tr}[\rho]} = \frac{\mathbb{I}_O}{D_O} \quad (\text{B.38})$$

and we can again check when this is compatible with the mapping being isometric: we need  $D_O^2 \in [D_I^2, D_I D_O]$ , but when  $D_I \leq D_O$  this is only the case iff  $D_I = D_O$ . In that case, isometries are unitaries, and trace preservation and unitality are equivalent. Additionally,  $K$  is fixed uniquely to the value  $K = \frac{D_I}{\text{Tr}[\rho]} = \frac{D_O}{\text{Tr}[\rho]}$  and there is no other option than  $\rho_{IO}$  being pure.

We can frame this simple result as follows. If we fix a state  $\rho$ , then select manually input and output systems such that they are of equal size, then there is no way to have an isometry between the operator spaces from the Jamiolkowski-Pillis mapping if the state does not factorise into pure states.

Even in this simple setting, operator transport has clear limitations in the multipartite case. The 'environment' or 'bath'  $B$  generically makes it impossible for the mappings above to be isometric.

To specialise this discussion, let us assume that the state  $\rho_{IO} = |\phi\rangle\langle\phi|$  is pure and we use either the Choi or Jamiolkowski-Pillis mapping (the requirements, for both of them, turn out to be the same). Then the isometry condition is

$$|K|^2 \text{Tr}_{O^2}[\rho_{IO}^{\otimes 2} \mathcal{S}_O] = |K|^2 \text{Tr}_O[\rho_{IO}]^{\otimes 2} \mathcal{S}_I = \mathcal{S}_I \quad . \quad (\text{B.39})$$

This is simply the requirement of the reduced input state being flat:

$$\rho_I = \frac{\mathbb{I}_I}{|K|} = \frac{\mathbb{I}_I}{D_I}. \quad (\text{B.40})$$

This is precisely the condition we found before for trace preservation. So for pure states  $\rho_{IO}$ , trace preservation and isometry are in fact equivalent:

$$\begin{aligned} \text{PURE} \wedge \text{TP} &\implies \text{ISOM} \\ \text{PURE} \wedge \text{ISOM} &\implies \text{TP} \quad . \end{aligned} \quad (\text{B.41})$$

This, together with the implication we found before, shows a 2-out-of-3 property of the Jamiolkowski-Pillis or Choi mappings. Phrased in terms of entanglement properties, we may say that for pure states, isometry holds precisely when the induced transport is trace preserving, or equivalently when the two subsystems are maximally entangled.

We can also study the opposite case and ask what happens when the state  $\rho_{IO}$  is separable. In that scenario, we find (assuming a normalised  $\rho$ ) that

$$\frac{\langle \mathcal{T}_\rho(X), \mathcal{T}_\rho(Y) \rangle_O}{\langle X \rangle_{\rho_I} \langle Y \rangle_{\rho_I}} = C_{\text{separable}} = K^2 e^{-S_2(\rho_O)} \quad , \quad (\text{B.42})$$

which is, importantly, *independent of  $X$  and  $Y$* . In simple terms, this is just the situation in which the inner product in  $O$  factorises between  $X$  and  $Y$ . Of course, the same thing happens in the maximally mixed case - if  $\rho_{IO} = \frac{\mathbb{I}}{D_I D_O}$  in the standard setup, then the above formula holds for maximal entropy of  $\rho_O$ . This suggests that, while the isometry condition indeed seems to favour entangled states, it also disfavors mixed states, generally.

We note that already in [250], it had been shown that even for mixed states one can link separability with properties of the induced transport superoperator. It would be interesting to extend these considerations to our more general setting, but we leave this for future work.

### An example

We illustrate, for concreteness, the Choi mapping on the classic Werner states on 2 qubits

$$\rho = p \Psi^- + (1 - p) \frac{\mathbb{I}}{4} \quad (\text{B.43})$$

where the Bell state  $\Psi^- = |\psi^-\rangle\langle\psi^-|$  is maximally entangled, and so the mapping is expected to give isometry. The Choi map (here for  $K = 2$ ) is linear in the state  $\rho$ , and, for the Bell state alone, induces a conjugation by the 2nd Pauli matrix:

$$\mathcal{T}_{\Psi^-}(X) = \sigma_2 X \sigma_2 \quad (\text{B.44})$$

Therefore, the Werner states induce the superoperator

$$\mathcal{T}_\rho(X) = p\sigma_2 X \sigma_2 + (1-p)\frac{\mathbb{I}_O}{2} \quad . \quad (\text{B.45})$$

The isometry condition can therefore be checked directly:

$$\langle \mathcal{T}_\rho(X), \mathcal{T}_\rho(Y) \rangle = p^2 \langle X, Y \rangle_I + \frac{1-p^2}{2} \text{Tr}_I[X] \text{Tr}_I[Y] \quad (\text{B.46})$$

and we can see that isometry only holds for the pure case  $p = 1$ ; it is not a consequence of entanglement by itself, but rather of entanglement together with purity.

### Isometry degree of the average state

We can achieve a generic understanding of the tripartite case by employing Page-type averaging arguments[246, 251]. We can in principle just consider a random pure state  $\rho = |\psi\rangle\langle\psi|$  of the full system and compute quantities according to the unitary average  $\langle - \rangle_U$ , where states are given as  $U |\psi_{\text{ref}}\rangle$ . Then we can check the isometry condition in the average as well:

$$\begin{aligned} & \langle |K|^2 \text{Tr}_{O^{\otimes 2}}[(\rho_{IO} \otimes \rho_{IO}) \mathcal{S}_O] \rangle_U \\ &= |K|^2 \text{Tr}_{OB^{\otimes 2}}[\langle \rho^{\otimes 2} \rangle_U \mathcal{S}_O] \quad , \end{aligned} \quad (\text{B.47})$$

and use the result (found via Schur's theorem for the permutation group on the two copies of the system):

$$\begin{aligned} \langle \rho^{\otimes 2} \rangle_U &= \int_{\mathcal{U}(D)} d\mu_{\text{Haar}}(U) (U^\dagger \rho U)^{\otimes 2} \\ &= \frac{\mathbb{I}_{\mathbb{H} \otimes \mathbb{H}} + \mathcal{S}_{\mathbb{H} \otimes \mathbb{H}}}{D(D+1)} \quad . \end{aligned} \quad (\text{B.48})$$

This means that on average the left side of the isometry condition becomes

$$\frac{|K|^2 D_O^2 D_B}{D(D+1)} (\mathcal{S}_I + \frac{D_B}{D_O} \mathbb{I}_{\mathbb{H} \otimes \mathbb{H}}) \quad (\text{B.49})$$

which shows two conditions which must hold on average:

$$r = \frac{D_B}{D_O} \ll 1 \quad |K|^2 = \frac{D(D+1)}{D_O^2 D_B} \approx r D_I^2 D_O \quad . \quad (\text{B.50})$$

So we can see already that only small environments allow for the average state to still give rise to isometries. This is unsurprising: in that scenario, a typical reduced state  $\rho_{IO}$  is close to being pure. We can again take traces of this expression with  $\mathbb{I}_I$  and  $\mathcal{S}_I$  to find

$$D_I \stackrel{!}{=} 1 \quad . \quad (\text{B.51})$$

We interpret this as follows. In order to have a system whose average pure state gives rise to an isometric map, the system sizes must follow the above conditions. Of course, if we

restrict the average to a smaller class of states, we might find more lenient conditions. For example, we may only work with states of the form

$$\rho = \Pi^\dagger |\psi\rangle \langle \psi| \Pi$$

with some projector  $\Pi : \mathbb{H}_B \rightarrow P \subset \mathbb{H}_B$  to a subspace of the environment, suitably extended to the full system. This essentially restricts the environment into a class of states. Then, the above calculation goes through as before, but replacing  $D_B$  by  $\tilde{D}_B = \dim(P)$ . Such a projection can then make the first condition superfluous by choosing  $P$  to be small enough. So better knowledge of the state of the environment makes the effective Choi map more isometric. If we also want to check for trace preservation in this setting, we get the condition

$$\begin{aligned} \frac{K D_O \tilde{D}_B}{D} = 1 &\longleftrightarrow K_{TP} = D_I \\ K_{Isom} &= \sqrt{\tilde{D}_B} D_I \quad . \end{aligned} \tag{B.52}$$

So we have in fact that once again, choosing a small  $P$  makes trace preservation and isometry nearly equivalent. So by either making the environment small, or choosing its coupling to the system to be small, or by assuming strong knowledge of the system (for example assuming it to be in a pure state, making  $\tilde{D}_B = 1$ ), we can find isometries in the tripartite case.

### B.1.2 Nontrivial center

Let us first discuss the bipartite case. Consider an algebra  $\mathcal{A}$  with representation space  $\mathbb{H}$ , pre-selected subsystems  $\mathcal{A}_{I|O}$ , such that  $(\mathcal{A}_I)' = \mathcal{A}_O$ , but with nontrivial center<sup>7</sup>  $\mathcal{Z} = \mathcal{A}_I \cap \mathcal{A}_O$ . The case of interest to us is that of Hilbert spaces of the form

$$\mathbb{H} = \bigoplus_E \mathbb{H}_{I,E} \otimes \mathbb{H}_{O,E} \tag{B.53}$$

with the full algebra  $\mathcal{A} = \mathbb{B}(\mathbb{H})$ , and subsystem algebras  $\mathcal{B}_{I|O} = \bigoplus_E \mathbb{B}(\mathbb{H}_{I|O,E})$ . In this sector-split Hilbert space setting, extension and partial trace operations are defined sector-wise.

$$i_I(X) = \sum_E X_E \otimes \mathbb{I}_{O_E} \quad P\text{Tr}_I[X] = \sum_E \text{Tr}_{O_E}[X_E] \quad , \tag{B.54}$$

which are adjoints to each other under the Hilbert-Schmidt scalar products on the algebras. We also identify  $\mathcal{A}_{I|O}$ , the true subsystems, as the images of  $\mathcal{B}_{I|O}$  under the extension maps. In practical terms, any operator that may be reached by partial tracing needs to be in  $\mathcal{B}_{I|O}$ . Similarly, any operator that is obtained from extending one in  $\mathcal{B}_{I|O}$  must be in  $\mathcal{A}_{I|O}$ .

---

<sup>7</sup>This assumes an extension map and associated partial trace operation have been chosen.

We may once again define a Jamiolkowski-Pillis (or Choi with partial transpose) mapping via the property B.15 which is also fulfilled in the case of trivial center. We allow ourselves to rescale this mapping again by a constant  $K$ :

$$\begin{aligned}\mathcal{T}_\rho(X) &= K P\text{Tr}_O[i_I(X)\rho] \\ &= \sum_E K c_E \text{Tr}_{I_E}[(X_{E,E} \otimes \mathbb{I}_{O_E})\rho_{E,E}] \quad ,\end{aligned}\tag{B.55}$$

where we decompose the state as

$$\rho = \sum_{E, \tilde{E}} \sqrt{c_E c_{\tilde{E}}} \rho_{E, \tilde{E}} \quad ,\tag{B.56}$$

with  $\text{Tr}[\rho_{E, \tilde{E}}] = \delta_{E, \tilde{E}}$  and  $c_E = \text{Tr}_E[\rho] \geq 0$ ,  $\sum_E c_E = 1$ .

For the tripartite case, we can proceed analogously. We assume: 1) a Hilbert space structure

$$\mathbb{H} = \sum_E \mathbb{H}_{I,E} \otimes \mathbb{H}_{O,E} \otimes \mathbb{H}_{B,E} \quad ;\tag{B.57}$$

2) input/output algebras

$$\mathcal{B}_I = \bigoplus_E B(\mathbb{H}_{I,E}) \quad \mathcal{B}_O = \bigoplus_E B(\mathbb{H}_{O,E}) \quad ;\tag{B.58}$$

3) the mapping

$$\begin{aligned}\mathcal{T}_\rho(X) &= K P\text{Tr}_O[i_I(X)\rho] \\ &= \sum_E K c_E \text{Tr}_{I_E B_E}[(X_{E,E} \otimes \mathbb{I}_{O_E B_E})\rho_{E,E}] \quad ,\end{aligned}\tag{B.59}$$

and investigate about trace preservation and isometry.

Trace preservation is just the property

$$\text{Tr}_{B_E O_E}[\rho_{E,E}] = \frac{\mathbb{I}_{I_E}}{D_{I_E}} \quad c_E = \frac{D_{I_E}}{K} \quad .\tag{B.60}$$

Identifying isometry is made easier by the aforementioned relation B.15, which entails that the adjoint to  $\mathcal{T}_\rho$  is (unsurprisingly) given by

$$\mathcal{T}_\rho^*(X) = P\text{Tr}_I(\rho^\dagger i_O(X)) \quad .\tag{B.61}$$

Letting  $\sigma_E = \text{Tr}_{B_E}[\rho]$  and rescaling our definitions by  $K$ , we obtain the isometry condition

$$(\mathcal{T}^* \circ \mathcal{T})(X) \tag{B.62}$$

$$= \sum_E |K|^2 \text{Tr}_{O_E}[(\mathbb{I}_{I_E} \otimes \text{Tr}_{I_E}[(X_E \otimes \mathbb{I}_{O_E})\sigma_E])\sigma_E^\dagger] \tag{B.63}$$

$$= \sum_E |K|^2 \sum_{a,b,c,d} \langle b|X_{E,E}|a\rangle \cdot \tag{B.64}$$

$$\cdot |c\rangle \langle d| \cdot \text{Tr}_{O_E}[\langle a|\sigma_E|b\rangle \langle c|\sigma_E|d\rangle] \tag{B.65}$$

$$\stackrel{!}{=} X = \sum_E \sum_{a,b,c,d} \langle b|X_{E,E}|a\rangle \cdot |c\rangle \langle d| \cdot \delta_{a,d} \delta_{b,c} \quad , \tag{B.66}$$

where we choose some orthonormal basis of  $I_E$  labeled by  $a, b, c, d$  in the last line. This leads directly to the condition

$$|K|^2 \text{Tr}_{O_E^2} [\sigma_E^{\otimes 2} \mathcal{S}_{O_E}] = \mathcal{S}_{I_E}, \quad (\text{B.67})$$

as before. Notice however now that we use the same prefactor for *all* sectors, meaning the requirement hinges more on the properties of the reduced states  $\sigma_E$ . Therefore the sector-wise condition

$$e^{-S_2(\sigma_E) + S_2((\sigma_E)_O)} = D_{I_E} \quad (\text{B.68})$$

must hold as well as

$$\frac{K c_E}{D_{I_E}} e^{-S_2(\sigma_E)} = 1. \quad (\text{B.69})$$

Again, we can see that trace preservation together with isometry necessitates that the reduced state  $\sigma_E$  is pure.

Additionally, as before, if we assume the state  $\sigma_E = |\phi_E\rangle \langle \phi_E|$  to be pure, we reduce the isometry condition to

$$(\sigma_E)_I = \frac{\mathbb{I}_{I_E}}{\sum_F D_{I_F}} \quad c_E = \frac{D_{I_E}}{\sum_F D_{I_F}}, \quad (\text{B.70})$$

which comes from the condition  $|K| = \sum_E D_{I_E}$ . This is, again, just the trace preservation condition. So also in the setting with nontrivial center, demanding purity makes TP and isometry equivalent. So once again, the three conditions give a 2-out-of-3 implication.

## B.2 Derivation of random Ising model

This appendix is largely based on an existing publication[252].

### B.2.1 Bulk-to-boundary maps

Here we derive the random Ising model components necessary for the bulk-to-boundary mapping analysis. Crucially, we will need the projector

$$\Pi_\Gamma = \sum_{j_b} \Pi_{\Gamma, j_b} \quad \Pi_{\Gamma, j_b} = \bigotimes_{e \in \Gamma} |g_{j_e}|^2 |e_{j_e}\rangle \langle e_{j_e}| \quad (\text{B.71})$$

with the maximally entangled state 2.6 on each glued link, in each sector, with the weight  $|g_{j_e}|^2$ .



Then each term in the random Ising model becomes

$$\text{Tr}_{\mathbb{H}_{\vec{j}} \otimes \mathbb{H}_{\vec{k}}} [(\Pi_{\Gamma, \vec{j}_b} \otimes \Pi_{\Gamma, \vec{k}_b}) \mathcal{S}_{S_{\downarrow}} \mathcal{S}_b^{1|0}] \quad (\text{B.72})$$

$$= \text{Tr}_{\mathbb{H}_{\Gamma, \vec{j}_b} \otimes \mathbb{H}_{\Gamma, \vec{k}_b}} [(\Pi_{\Gamma, \vec{j}_b} \otimes \Pi_{\Gamma, \vec{k}_b}) \bigotimes_{e \in \Gamma} \mathcal{S}_e^{\frac{1-\sigma_{s(e)}\sigma_{t(e)}}{2}}] \times \quad (\text{B.73})$$

$$\times \text{Tr}_{\mathcal{I}_{\vec{j}} \otimes \mathcal{I}_{\vec{k}}} [\bigotimes_x \mathcal{S}_{b,x}^{\frac{1-b\sigma_x}{2}}] \text{Tr}_{\mathbb{H}_{\partial\gamma, \vec{j}_{\partial\gamma}} \otimes \mathbb{H}_{\partial\gamma, \vec{k}_{\partial\gamma}}} [\bigotimes_{e \in \partial\gamma} \mathcal{S}_e^{\frac{1-\sigma_{s(e)}}{2}}] \quad (\text{B.74})$$

where  $b = (-1)^{1|0}$  in  $Z_{1|0}$ . We can then evaluate each part separately. For the link factor, it factorises over each internal link:

$$\prod_{e \in \Gamma} |g_{j_e}|^2 |g_{k_e}|^2 \text{Tr}_{\mathbb{H}_{j_e} \otimes \mathbb{H}_{k_e}} [(|e_{j_e}\rangle \langle e_{j_e}| \otimes |e_{k_e}\rangle \langle e_{k_e}|) \mathcal{S}_e^{\frac{1-\sigma_{s(e)}\sigma_{t(e)}}{2}}] \quad (\text{B.75})$$

where we define  $\mathbb{H}_{j_e} := V_{s(e), j_e} \otimes V_{t(e), j_e}$ . This evaluates to

$$\prod_{e \in \Gamma} |g_{j_e}|^2 |g_{k_e}|^2 \left( \delta_{j_e, k_e} d_{j_e}^{-1} \right)^{\frac{1-\sigma_{s(e)}\sigma_{t(e)}}{2}}. \quad (\text{B.76})$$

Then, the intertwiner factor is similarly

$$\text{Tr}_{\mathcal{I}_{\vec{j}} \otimes \mathcal{I}_{\vec{k}}} [\bigotimes_x \mathcal{S}_{b,x}^{\frac{1-b\sigma_x}{2}}] = \prod_x \mathcal{D}_{j^x} \mathcal{D}_{k^x} \left( \delta_{j^x, k^x} \mathcal{D}_{j^x}^{-1} \right)^{\frac{1-b\sigma_x}{2}} \quad (\text{B.77})$$

where  $\mathcal{D}_{j^x} = \dim(\mathcal{I}_{j^x})$ . Lastly, the boundary factor is

$$\begin{aligned} & \text{Tr}_{\mathbb{H}_{\partial\gamma, \vec{j}_{\partial\gamma}} \otimes \mathbb{H}_{\partial\gamma, \vec{k}_{\partial\gamma}}} [\bigotimes_{e \in \partial\gamma} \mathcal{S}_e^{\frac{1-\sigma_{s(e)}}{2}}] \\ &= \prod_{e \in \partial\gamma} d_{j_e} d_{k_e} \left( \delta_{j_e, k_e} d_{j_e}^{-1} \right)^{\frac{1-\sigma_{s(e)}\sigma_{t(e)}}{2}} \end{aligned} \quad (\text{B.78})$$

where we set  $\sigma_{t(e)} = 1$  for all boundary links.

This means we have as our random Ising model data the choices:

$$\Delta_{1|0}(\vec{j}, \vec{k}; \vec{\sigma}) = \prod_x \delta_{j^x, k^x}^{\frac{1-b\sigma_x}{2}} \prod_{e \in \gamma} \delta_{j_e, k_e}^{\frac{1-\sigma_{s(e)}\sigma_{t(e)}}{2}} \quad (\text{B.79})$$

$$K_{\vec{j}} = \prod_x \mathcal{D}_{j^x} \prod_{e \in \gamma} |g_{j_e}|^2 \prod_{e \in \partial\gamma} d_{j_e} = \text{Tr}_{\mathbb{H}_{\vec{j}}} [\Pi_{\Gamma, j_b}] \quad (\text{B.80})$$

where we fix the convention  $g_{j_e} = 1$  for all  $e \in \partial\gamma$  to write this uniformly across all links of the graph. The Hamiltonian is

$$H_{1|0}(\vec{j}, \vec{k}, \vec{\sigma}) = \sum_{e \in \gamma} \lambda_e \frac{1 - \sigma_{s(e)}\sigma_{t(e)}}{2} + \sum_x \frac{1 - b\sigma_x}{2} \Lambda_x \quad (\text{B.81})$$

with couplings  $\lambda_e = \log(d_{j_e})$ ,  $\Lambda_x = \log(\mathcal{D}_{j_x})$ . It is clear that this Hamiltonian is bounded from above and below, with upper bound  $\sum_e \lambda_e + \sum_x \Lambda_x$ .

The diagonal partition sums  $Z_1^{\vec{j}\vec{j}}$ , can be reinterpreted through

$$\begin{aligned} Z_1^{\vec{j}\vec{j}} &= \langle \text{Tr}_{\mathbb{H}_{\vec{j}}^{\otimes 2}} \left[ \left( \frac{\Pi_{\Gamma, j_B}}{K_{\vec{j}}} \right)^{\otimes 2} (|\Psi\rangle \langle \Psi|)^{\otimes 2} \mathcal{S}_B \right] \rangle_U \\ &= \langle \text{Tr}_{\mathcal{I}_{\vec{j}}} [\text{Tr}_{\mathbb{H}_{\partial\gamma, j_{\partial\gamma}} \otimes \mathbb{H}_{\Gamma, j_B}} \left[ \frac{\Pi_{\Gamma, j_B}}{K_{\vec{j}}} |\Psi\rangle \langle \Psi| \right]^2] \rangle_U \end{aligned} \quad (\text{B.82})$$

as the reduced bulk entropy of a state in the given sector, so as

$$\begin{aligned} Z_1^{\vec{j}\vec{j}} &= \langle e^{-S_2(\rho_{b, \vec{j}})} \rangle_U \\ \rho_{b, \vec{j}} &= \text{Tr}_{\mathbb{H}_{\partial\gamma, j_{\partial\gamma}} \otimes \mathbb{H}_{\Gamma, j_B}} \left[ \frac{\Pi_{\Gamma, j_B}}{K_{\vec{j}}} |\Psi\rangle \langle \Psi| \right] \end{aligned} \quad (\text{B.83})$$

Therefore, the diagonal partition sums are in fact in the interval  $[\frac{1}{\dim(\mathcal{I}_{\vec{j}})}, 1]$ .

## B.3 Bulk-to-boundary maps: Examples

### Bulk-to-boundary: single vertex

We give a full analytical calculation of the partition sums for bulk-to-boundary mappings for a single spin network vertex. While this example may appear trivial, it illustrates the complexity of the calculations that already appear without tensor network contractions when superposition is allowed.

We begin with the full partition sums

$$Z_{1|0} = \sum_{\vec{\sigma}} \text{Tr}_{\mathbb{H}^{\otimes 2}} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_{\mathbb{H}_x}^{\frac{1-b\sigma_x}{2}}] = \sum_{E, \tilde{E}} Z_{1|0}^{E, \tilde{E}} \quad (\text{B.84})$$

with boundary-fixed partition sums

$$\begin{aligned} Z_{1|0}^{E, \tilde{E}} &= \sum_{\vec{\sigma}} \text{Tr}_{\mathbb{H}_E \otimes \mathbb{H}_{\tilde{E}}} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_{\mathbb{H}_x}^{\frac{1-b\sigma_x}{2}}] \\ &= \sum_{\vec{\sigma}} \sum_{j_b, k_b} \text{Tr}_{\mathbb{H}_{E \cup j_b} \otimes \mathbb{H}_{\tilde{E} \cup k_b}} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_{\mathbb{H}_x}^{\frac{1-b\sigma_x}{2}}] \\ &= \sum_{\vec{\sigma}} \sum_{j_b, k_b} K_{E \cup j_b} K_{\tilde{E} \cup k_b} Z_{1|0}^{E \cup j_b, \tilde{E} \cup k_b}. \end{aligned} \quad (\text{B.85})$$

Now each term in this is in general hard to compute already, but here it completely trivialises. Due to absence of internal links,  $\Pi$  and the sums over  $j_b, k_b$  disappear entirely. Furthermore, the sums are diagonal in boundary spins  $E$ , meaning we only need to consider

$$Z_{1|0}^{E, E} = K_E^2 Z_{1|0}^{E \cup \emptyset, E \cup \emptyset}. \quad (\text{B.86})$$

where the right partition sum is defined through the Ising model. Using B.2.1, this is

$$\begin{aligned} Z_{1|0}^{E\cup\emptyset, E\cup\emptyset} &= \sum_{\sigma_x} e^{-\frac{1-\sigma_x}{2} \sum_{e \in \partial\gamma} \lambda_e - \frac{1-b\sigma_x}{2} \Lambda_x} \\ &= D_{b_E}^{-(1|0)} + D_{\partial_E}^{-1} D_{b_E}^{-(0|1)} \end{aligned} \quad (\text{B.87})$$

and

$$K_E^2 = D_{b_E}^2 D_{\partial_E}^2 = D_E^2 \quad (\text{B.88})$$

and therefore the Ising sums are

$$\begin{aligned} Z_0 &= \sum_E (D_E^2 + D_E) \\ Z_1 &= \sum_E D_{b_E}^2 D_{\partial_E}^2 (D_{b_E}^{-1} + D_{\partial_E}^{-1}) \\ &= \sum_E D_{b_E} D_{\partial_E} (D_{b_E} + D_{\partial_E}). \end{aligned} \quad (\text{B.89})$$

Let us introduce  $r_E = \frac{D_{b_E}}{D_{\partial_E}}$ , which allows us to rewrite everything in terms of it as an expansion parameter  $r$ . Isometry requires  $r \leq 1$ , and this holds for any gauge invariant spin/tensor network vertex by definition of intertwiner spaces.

$$\begin{aligned} Z_0 &= \sum_E \left( \frac{D_{b_E}^2}{r_E} \left( \frac{D_{b_E}^2}{r_E} + 1 \right) \right) \\ Z_1 &= \sum_E D_{b_E}^3 \frac{1}{r_E} \left( 1 + \frac{1}{r_E} \right). \end{aligned} \quad (\text{B.90})$$

We can establish under which conditions we have holography by calculating the purity and weight separately in each sector. These are given, respectively, by 2.86, 2.87, 2.92:

$$\begin{aligned} \frac{Z_1^{E,E}}{Z_0^{E,E}} &= \frac{(D_{b_E} + D_{\partial_E}) D_E}{(1 + D_E) D_E} = D_{b_E} \frac{\frac{1}{r_E} (1 + \frac{1}{r_E})}{\frac{1}{r_E} (1 + \frac{D_{b_E}^2}{r_E})} \\ &= \frac{1}{D_{b_E}} \frac{1 + r_E}{1 + \frac{r_E}{D_{b_E}^2}} = \frac{1}{D_{b_E}} \left( 1 + \left( 1 - \frac{1}{D_{b_E}^2} \right) r_E + \mathcal{O}(r_E^2) \right) \end{aligned} \quad (\text{B.91})$$

which is generically close to the isometric value for small  $r_E$ , and in fact for 1-dimensional bulk spaces only differs at second order.

The cross-sector conditions can be understood as restrictions on which combinations of  $E$  we may have in the input algebra. We must demand for some constant  $q$  that

$$\begin{aligned} D_E (D_{b_E} + D_{\partial_E}) &= q D_{b_E} \\ D_E (D_E + 1) &= q D_{b_E}^2 \end{aligned} \quad (\text{B.92})$$

which already includes the condition from before that  $r_E$  is small. We can just solve the two conditions directly for some conditions by multiplying the first by  $D_{b_E}$  and solving for  $r_E$ :

$$D_{b_E}^2 + D_E = D_E + 1 \implies r_E = \frac{1}{D_E} \quad (\text{B.93})$$

Inserting this solution into the first equation yields

$$D_E + 1 = q = \text{const} \quad (\text{B.94})$$

so in fact *the dimensions must be independent of the sector label  $E$* . It also yields more trivially that

$$\begin{aligned} D_{I_E}^2 &= r_E D_E = 1 \\ \implies D_{b_E} &= 1, D_E = D_{\partial_E} = q - 1 = \text{const}. \end{aligned} \quad (\text{B.95})$$

So although we saw before that generically for small  $r_E$ , the entropy condition is fulfilled easily, this is not the case for the cross-sector condition: it requires that the boundary dimension may also not depend on the sector label. In fact, it also requires, far stronger, the restriction to 1-dimensional inputs in each sector. Therefore, we must restrict ourselves to a fixed boundary dimension  $D_{\partial}$  and select only those  $E$  such that  $D_{b_E} = 1, D_{\partial_E} = D_{\partial}$ . Then we can find holographic behaviour.

For the trivalent case, the input condition is always fulfilled; But in the 4-valent case already, we have to restrict ourselves: *either*, at least one of the dimensions on the boundary links is 1, *or* the largest dimension is

$$d_{max} = d_1 + d_2 + d_3 - 2. \quad (\text{B.96})$$

This means that there are overall 2 constraints on the 4 free variables we can choose. However, the overall message is clear: while the fixing of the output dimension is generically necessary, there are also strong restrictions on the input dimension. Incidentally, the restriction to dimension 1 inputs also makes corrections to the entropy vanish, simply because the minimal entropy is also the maximal entropy.

We note though that in the high- $D_{\partial}$  approximation, holography is generic:

$$\begin{aligned} Y_1^E &= \frac{1}{D_{\partial}} + \frac{1}{D_{b_E}} = e^{-\beta} + \frac{1}{D_{b_E}} \approx \frac{1}{D_{b_E}} \\ Y_0^E &= 1 + \frac{1}{D_E} = 1 + \frac{1}{D_{b_E}} e^{-\beta} \approx 1 \end{aligned} \quad (\text{B.97})$$

which trivially fulfils holography. We should therefore think of the restriction to bulk dimension 1 perhaps more as having ‘very low’ bulk dimension compared to the boundary  $D_{\partial}$ .

### Bulk-to-boundary: single internal link

In the case of a single link, we are also able to perform most of the calculations analytically. We label for convenience the endpoints by  $x$  and  $y$  and the set of boundary spins on  $x$  or

y as  $E_x$  and  $E_y$ , respectively. The spins of the single bulk link will be labeled by  $u$  or  $v$ , and we denote intertwiner dimensions on a vertex  $x$  depending on link spins by  $\mathcal{D}_{\{j_e^x\}}$ .

Once again we have

$$\begin{aligned} Z_{1|0}^{E,E} &= \text{Tr}_{\mathbb{H}_E \otimes \mathbb{H}_E} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_{\mathbb{H}_x}^{\frac{1-b\sigma_x}{2}}] \\ &= \sum_{u,v} \text{Tr}_{\mathbb{H}_{E \cup u} \otimes \mathbb{H}_{E \cup v}} [\Pi^{\otimes 2} \bigotimes_x \mathcal{S}_{\mathbb{H}_x}^{\frac{1-b\sigma_x}{2}}] \\ &= \sum_{u,v} K_{E \cup u} K_{E \cup v} Z_{1|0}^{E \cup u, E \cup v}. \end{aligned} \quad (\text{B.98})$$

and we can split

$$K_{E \cup u} = D_{\partial E} \mathcal{D}_{E_x \cup u} \mathcal{D}_{E_y \cup u} |g_u|^2 = L_{E \cup u} D_{\partial E} \quad (\text{B.99})$$

$$\begin{aligned} H_{1|0}^{E \cup u}(\vec{\sigma}) &= \frac{1-\sigma_x}{2} \sum_{j_e \in E_x} \log(2j_e + 1) \\ &\quad + \frac{1-\sigma_y}{2} \sum_{j_e \in E_y} \log(2j_e + 1) \\ &\quad + \frac{1-b\sigma_x}{2} \Lambda_x + \frac{1-b\sigma_y}{2} \Lambda_y \\ &\quad + \frac{1-\sigma_x \sigma_y}{2} \log(2u + 1) \end{aligned} \quad (\text{B.100})$$

into its bulk contribution  $L_{E \cup u}$ , which stays in the sum, and the pure boundary contribution  $D_{\partial E}$ . Next, we consider the effect of the constraints. The constraints, in general, state that for  $Z_1$  ( $Z_0$ ), the spin-down (spin-up) region must be contained in the region  $G_{\vec{j}, \vec{k}}$  of vertices where all the incident spin labelings agree. This restricts allowed Ising configurations and leads to a suppression of sector-off-diagonal partition sums, because the usual lowest energy configuration is typically disallowed.

In our case, for  $u \neq v$ ,  $G_{\vec{j}, \vec{k}} = \emptyset$  (not counting the boundary vertices), meaning that only the spin-down (spin-up) configuration can contribute to off-diagonal sums. Let

$$a_{x|y} = \mathcal{D}_{E_{x|y} \cup u}^{-1} \quad b_{x|y} = \prod_{j_e \in E_{x|y}} d_{j_e}^{-1}, \quad (\text{B.101})$$

then

$$Z_1^{E \cup u, E \cup v} = e^{-H_1^{E \cup u}(\vec{-1})} = b_x b_y \quad (\text{B.102})$$

$$Z_0^{E \cup u, E \cup v} = e^{-H_0^{E \cup u}(\vec{+1})} = 1. \quad (\text{B.103})$$

This means that the matrices  $Z_{1|0}^{E \cup u, E \cup v}$  (collected over  $u - v$ ) have the form of a constant contribution in each entry as well as a diagonal part. The diagonal terms are again

$$Z_1^{E \cup u, E \cup u} = a_x a_y + d_u^{-1} a_x b_y + d_u^{-1} a_y b_x + b_x b_y \quad (\text{B.104})$$

$$Z_0^{E \cup u, E \cup u} = 1 + d_u^{-1} a_x b_x + d_u^{-1} a_y b_y + a_x a_y b_x b_y \quad (\text{B.105})$$

and so we have obtained the full partition sums for fixed boundary spins.

We can use the other objects defined before,

$$L_{\vec{j}} = |g_u|^2 \quad Y_{1|0}^E = \sum_{u,v} |g_u|^2 |g_v|^2 Z_{1|0}^{E \cup u, E \cup v} \quad (\text{B.106})$$

and fix the output dimension

$$D_{\partial} = D_{\partial_E} = (b_x b_y)^{-1} \quad (\text{B.107})$$

and so we find:

$$Y_{1|0}^E = \sum_u |g_u|^4 Z_{1|0}^{E \cup u, E \cup v} + D_{\partial}^{(-1|0)} \left[ \left( \sum_u |g_u|^2 \right)^2 - \sum_u |g_u|^4 \right] \quad (\text{B.108})$$

Now by noting the normalisation condition  $\sum_u |g_u|^2 = 1$  for all internal links, the term in the brackets becomes a Rényi-2 entropy of the sequence of coefficients

$$\left( \sum_u |g_u|^2 \right)^2 - \sum_u |g_u|^4 = 1 - \exp(-S_2(|g_u|^2)_u).$$

We can evaluate for example  $Y_1^E$  in the special case where the two vertices are 4-valent and all the boundary spins are the same, at value  $j = \frac{n-1}{2}$ , meaning  $D_{\partial} = n^6$ . Then, we can explicitly calculate the intertwiner dimension in closed form, and find that  $D_{I_E} = n^2$ , and the sum splits into 3 parts: the constant part above, the part where  $m := 2u + 1$  is less than  $n$  and the part where  $m > n$ . The first part is as above, while the second and third are, respectively,

$$\sum_{m=1}^n |g_m|^4 \left( \frac{1}{n^6} + \frac{1}{m^2} \left( 1 + \frac{2}{n^3} \right) \right) \quad (\text{B.109})$$

$$\sum_{m=n+2k, k \in [1:n]} |g_{n+2k}|^4 \left[ \frac{2}{n^3(n-k)(2k+n)} + \frac{1}{(k-n)^2} + \frac{1}{n^6} \right] \quad (\text{B.110})$$

This is already quite complex, but because the sums are finite, they of course converge. In fact, for  $g = 1$  they even have a closed form each:

$$\frac{n^5 H_n^{(2)} + 2n^2 H_n^{(2)} + 1}{n^5} \\ \frac{-\pi^2 n^5 + 6n^5 \psi^{(1)}(1-n) + 4\gamma n - 4n\psi^{(0)}\left(\frac{n}{2} + 1\right) + 4n\psi^{(0)}\left(\frac{3n}{2} + 1\right) + 4n\psi^{(0)}(1-n) + 6}{6n^5}$$

where  $H_n^{(2)}$  gives the  $n$ th Harmonic number of order 2,  $\psi^{(k)}$  are the polygamma functions and  $\gamma$  is the Euler-Mascheroni constant. In the limit of large  $n$ , they have simple expansions

$$\frac{\pi^2}{6} - \frac{1}{n} + \mathcal{O}\left(\left(\frac{1}{n}\right)^2\right) \\ \frac{5\pi^2}{6} - \frac{1}{n} + \pi^2 \cot^2(\pi n) + \mathcal{O}\left(\left(\frac{1}{n}\right)^2\right).$$

In general, for varying  $n$  the sum shows oscillatory behaviour around any integer. As for  $Y_0^E$ , we find the same split with exact values

$$\begin{aligned} \frac{2n^3 H_n^{(2)} + H_n^{(2)} + n^7}{n^6} &= n + \mathcal{O}\left(\left(\frac{1}{n}\right)^2\right) \\ \frac{6n^7 + 4\gamma n^2 - 4n^2 \psi^{(0)}\left(\frac{n}{2} + 1\right) + 4n^2 \psi^{(0)}\left(\frac{3n}{2} + 1\right) + 4n^2 \psi^{(0)}(1 - n) + 6\psi^{(1)}(1 - n) - \pi^2}{6n^6} \\ &= n + \mathcal{O}\left(\left(\frac{1}{n}\right)^2\right) \end{aligned}$$

As a result, we see that indeed, with  $|g| = 1$  we cannot fulfil the isometry conditions, and that

$$\frac{Y_1^E}{Y_0^E} \approx \frac{\pi^2}{2n} \quad (\text{B.111})$$

which does not scale like  $\frac{1}{D_{I_E}} = \frac{1}{n^2}$ . This shows that the choice  $g = 1$  hinders holographic behaviour. Now, we can also find what choice for  $g$  we *must* make. In a high- $\beta$  approximation, the sums are generally ( $v_n = |g_{\frac{n-1}{2}}|^2$ ):

$$Y_1^E \approx \sum_n v_n^2 \frac{1}{D_{I_E}}, \quad Y_0^E \approx \left(\sum_n a_n\right)^2 \quad (\text{B.112})$$

So again, in this limit the only condition we need to check is the entropy one, which takes the form

$$\begin{aligned} \frac{Y_1^E}{Y_0^E} &\approx \sum_n r_n a_x a_y(n) \stackrel{!}{=} \frac{1}{\sum_n (a_x a_y(n))^{-1}} \\ r_n &= \frac{v_n^2}{(\sum_n v_n)^2} \end{aligned} \quad (\text{B.113})$$

which has as valid solutions for  $r_n$ , if there are  $M$  values for the spins,

$$\begin{aligned} r_n &= \frac{1}{a_x a_y(n)} \left( \frac{1}{M \sum_m (a_x a_y(m))^{-1}} + c_n \right) \\ \sum_n c_n &= 0. \end{aligned} \quad (\text{B.114})$$

Ultimately, solving for  $a_n$  here is irrelevant, but what matters is the scaling:

$$|g_u|^2 \sim \frac{1}{\sqrt{a_x a_y(u)}} \quad (\text{B.115})$$

So demanding isometry puts strong constraints on the scaling of the coefficients  $g$  we use to define the state  $\rho$  and map  $\mathcal{T}_\rho$ . This is interesting, for one because it is consistent with the assumption that large spins must dominate for the approximations to work, but also

because this is the only constraint we had to put on the problem to get isometry. This is because for the setting here, it was quite natural to assume that the all-up configuration is the ground state of  $\tilde{H}_1$ . Under this assumption, and the constancy of  $D_{\partial E}$ , though, we are however already almost at isometry, as given in the main section.

The most intriguing part of this scaling, though, is that the left hand side does not depend on  $E$ . This implies at least that

$$\frac{(a_x a_y(n))^{-1}}{\sum_m (a_x a_y(m))^{-1}} \quad (\text{B.116})$$

is independent of  $E$  for all  $n$ .

We can therefore understand that the full isometry condition boils down to 3 essential ingredients:

1. Constancy of  $D_{\partial E} = D_{\partial}$ .
2. Knowledge of the minima of  $\tilde{H}_1$  and their proximity to the all-up configuration.
3. Constraints on the  $g_{j_e}$  coefficients that relate them to the input dimensions.



# Appendix C

## Appendices to chapter 3

### C.1 Symplectic vector fields

Overall, with the gauge fixing from the structural constraint in place, the phase space of tetrad gravity is now 24-dimensional and has a nondegenerate symplectic form. We show here the general form of symplectic vector fields on this phase space. For this, first note the contraction

$$-I_X \Omega = \int_{\Sigma} X[\theta]_I \wedge \delta \hat{\omega}_{(\beta)}^{IJ} \wedge \theta_J - \delta \theta_I \wedge X[\hat{\omega}]_{(\beta)}^{IJ} \wedge \theta_J. \quad (\text{C.1})$$

$X$  will then be symplectic if  $\delta(I_X \Omega) = 0$ . Let us introduce the following shorthands:

$$\begin{aligned} A_{IJ} &= \frac{\delta X[\theta]_I}{\delta \theta^J} & B_{IRD} &= \frac{\delta X[\theta]_I}{\delta \hat{\omega}^{RD}} \\ U_{RSK} &= \frac{\delta X[\hat{\omega}]_{RS}}{\delta \theta^K} & V_{RSAB} &= \frac{\delta X[\hat{\omega}]_{RS}}{\delta \hat{\omega}^{AB}} \end{aligned} \quad (\text{C.2})$$

Then, we can fully expand the exterior derivative in differentials to arrive at the following set of conditions:<sup>1</sup>

$$(P_{(\beta)})_{KL}^{IJ} B_{IRD} \theta_J = 0 \quad \forall [RD] \neq [KL] \quad (\text{C.3})$$

$$(P_{(\beta)})^{IJ,RS} U_{RSK} \theta_J = 0 \quad \forall I \neq K \quad (\text{C.4})$$

$$\left[ A_{IK} (P_{(\beta)})_{AB}^{IJ} - \eta_{IK} (P_{(\beta)})^{IJ,RS} V_{RSAB} \right] \theta_J = 0 \quad \forall K, [AB] \quad (\text{C.5})$$

We note that  $(P_{(\beta)})^{IJ,RS} = (\star + \beta)^{IJ,RS}$  can be inverted to simplify these conditions. The third can be solved this way for  $A_{IK}$  as a function of  $V_{RSAB}$ . Therefore, there are stringent constraints on which transformations can be interpreted as canonical ones. However, as can be verified by a straightforward but tedious calculation, for example the Lorentz transformations

$$X_{\alpha} = -\alpha \cdot \theta \frac{\delta}{\delta \theta} + d_{\hat{\omega}} \alpha \frac{\delta}{\delta \hat{\omega}} \quad (\text{C.6})$$

---

<sup>1</sup>Here,  $(P_{(\beta)})_{KL}^{IJ}$  denotes the component expression of the map  $\star + \beta$  on the Lie algebra.

are symplectic, showing that some relevant vector fields can indeed satisfy these conditions. The vector fields 3.50 in the main text are not symplectic in general by this measure, as  $B_{IRD} \neq 0$ . An exception may exist though, as  $\mathbb{T}_\phi$  may contain pieces that cancel the relevant dependence on  $\hat{\omega}$ . In that situation, the tetrad's transformation would be

$$X_\phi[\theta] = d_{\gamma[\theta]}\phi \quad (\text{C.7})$$

which clearly does not depend on the connection. As it turns out, we can write

$$\mathbb{T}_\phi = -\frac{1}{3}\kappa \cdot \phi \quad (\text{C.8})$$

with the contorsion of a general connection, so indeed by choosing  $s = 3$ , we have a vector field with precisely this behaviour:

$$X_{\phi,3}[\theta] = d_{\gamma[\theta]}\phi \quad (X_{\phi,3}[\omega])_{(\beta)} \wedge \theta = 2(F_\omega)_{(\beta)} \cdot \phi \quad (\text{C.9})$$

This, at first, appears to be the only symmetry vector field among them that has a chance at being symplectic. In general, though, they all feature a *bulk* nonintegrability. This is similar to the case of timelike diffeomorphisms, as the bulk obstructions vanish on-shell (on-shell of the Gauss constraint for  $s = 0$ ). Unlike diffeomorphisms, however, there is no clear interpretation of these terms.

# Bibliography

- [1] Eugenia Colafranceschi, Simon Langenscheidt, and Daniele Oriti. “Holographic properties of superposed quantum geometries”. *Phys. Rev. D* **110**, 046024 (2024). arXiv:2207.07625.
- [2] Simon Langenscheidt, Daniele Oriti, and Eugenia Colafranceschi. “Channel-State duality with centers” (2024). arXiv:2404.16004.
- [3] Simon Langenscheidt and Daniele Oriti. “New edge modes and corner charges for first-order symmetries of 4D gravity”. *Classical and Quantum Gravity* **42**, 075010 (2025).
- [4] Simon Langenscheidt. “Edge modes of tetrad gravity: Unlike diffeomorphisms, all shifts are integrable”.
- [5] Henrique Gomes, Simon Langenscheidt, and Daniele Oriti. “Boundaries, frames and the issue of physical covariance” (2024). arXiv:2412.00993.
- [6] Theodore Lyman. “The Spectrum of Hydrogen in the Region of Extremely Short Wave-Length”. *Memoirs of the American Academy of Arts and Sciences* **13**, 125–146 (1906). arXiv:25058084.
- [7] N. Bohr. “I. *On the constitution of atoms and molecules*”. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **26**, 1–25 (1913).
- [8] Norbert Straumann. “Quantenmechanik: Ein Grundkurs über nichtrelativistische Quantentheorie”. Springer Berlin Heidelberg. Berlin, Heidelberg (2013).
- [9] Richard Courant and David Hilbert. “Methods of mathematical physics. Vol.1”. Wiley-VCH. Weinheim (2009).
- [10] Albert Einstein and Marcel Grossmann. “Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation: I. Physikalischer Teil von Albert Einstein. II. Mathematischer Teil von Marzel Großmann. Teubner, 1913”. *Monatshefte für Mathematik und Physik* **26**, A38–A38 (1915).
- [11] Albert Einstein. “Die Grundlage der allgemeinen Relativitätstheorie”. *Annalen der Physik* **354**, 769–822 (1916).

- [12] John Norton. “Einstein, The Hole Argument and the Reality of Space”. In John Forge, editor, *Measurement, Realism and Objectivity: Essays on Measurement in the Social and Physical Sciences*. Pages 153–188. Springer Netherlands, Dordrecht (1987).
- [13] Henrique Gomes and Jeremy Butterfield. “The Hole Argument and Beyond: Part I: The Story so Far”. *Journal of Physics: Conference Series* **2533**, 012002 (2023).
- [14] A. Einstein. “Dialog über Einwände gegen die Relativitätstheorie”. *The Science of Nature* **6**, 697–702 (1918).
- [15] Leonardo Chataignier. “Relational observables, reference frames, and conditional probabilities”. *Physical Review D* **103**, 026013 (2021).
- [16] Johannes Tambornino. “Relational Observables in Gravity: A Review”. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **8**, 017 (2012).
- [17] Emily Adlam. “How Do We Observe Relational Observables?”. *Journal for General Philosophy of Science* (2025).
- [18] Christophe Goeller, Philipp A. Hoehn, and Josh Kirklin. “Diffeomorphism-invariant observables and dynamical frames in gravity: Reconciling bulk locality with general covariance” (2022). [arXiv:2206.01193](#).
- [19] Alexandre Belin, Robert C. Myers, Shan-Ming Ruan, Gábor Sárosi, and Antony J. Speranza. “Does Complexity Equal Anything?”. *Phys. Rev. Lett.* **128**, 081602 (2022). [arXiv:2111.02429](#).
- [20] Sylvain Carrozza and Philipp A. Höhn. “Edge modes as reference frames and boundary actions from post-selection”. *Journal of High Energy Physics* **2022**, 172 (2022). [arXiv:2109.06184](#).
- [21] Sylvain Carrozza, Stefan Eccles, and Philipp A. Hoehn. “Edge modes as dynamical frames: Charges from post-selection in generally covariant theories”. *SciPost Physics* **17**, 048 (2024). [arXiv:2205.00913](#).
- [22] Goncalo Araujo-Regado, Philipp A. Hoehn, Francesco Sartini, and Bilyana Tomova. “Soft edges: The many links between soft and edge modes” (2025). [arXiv:2412.14548](#).
- [23] Augustin Vanrietvelde, Philipp A. Hoehn, Flaminia Giacomini, and Esteban Castro-Ruiz. “A change of perspective: Switching quantum reference frames via a perspective-neutral framework”. *Quantum* **4**, 225 (2020). [arXiv:1809.00556](#).
- [24] Philipp A. Hoehn, Alexander R. H. Smith, and Maximilian P. E. Lock. “The Trinity of Relational Quantum Dynamics”. *Physical Review D* **104**, 066001 (2021). [arXiv:1912.00033](#).

- [25] Esteban Castro-Ruiz and Ognyan Oreshkov. “Relative subsystems and quantum reference frame transformations” (2023). arXiv:2110.13199.
- [26] Viktoria Kabel, Časlav Brukner, and Wolfgang Wieland. “Quantum Reference Frames at the Boundary of Spacetime” (2023). arXiv:2302.11629.
- [27] Kristina Giesel, Viktoria Kabel, and Wolfgang Wieland. “Linking Edge Modes and Geometrical Clocks in Linearized Gravity” (2024). arXiv:2410.17339.
- [28] A. Barbieri. “Quantum tetrahedra and simplicial spin networks”. *Nuclear Physics B* **518**, 714–728 (1998). arXiv:gr-qc/9707010.
- [29] Thomas Thiemann. “Modern canonical quantum general relativity”. Cambridge Monographs on Mathematical Physics. Cambridge University Press. Cambridge (2007).
- [30] Hanno Sahlmann. “New insights in quantum geometry”. *Journal of Physics: Conference Series* **360**, 012007 (2012).
- [31] Abhay Ashtekar and Jerzy Lewandowski. “Quantum theory of geometry: I. Area operators”. *Class. Quant. Grav.* **14**, A55–A82 (1997). arXiv:gr-qc/9602046.
- [32] Benjamin Bahr, Bianca Dittrich, and Marc Geiller. “A new realization of quantum geometry”. *Classical and Quantum Gravity* **38**, 145021 (2021). arXiv:1506.08571.
- [33] Alberto S. Cattaneo and Alejandro Perez. “A note on the Poisson bracket of 2d smeared fluxes in loop quantum gravity”. *Classical and Quantum Gravity* **34**, 107001 (2017). arXiv:1611.08394.
- [34] Roukaya Dekhil, Matteo Laudonio, and Daniele Oriti. “A state sum for four-dimensional Lorentzian quantum geometry in terms of edge vectors” (2025). arXiv:2501.10115.
- [35] Laurent Freidel and Etera R. Livine. “Bubble Networks: Framed Discrete Geometry for Quantum Gravity”. *General Relativity and Gravitation* **51**, 9 (2019). arXiv:1810.09364.
- [36] E. Joung, J. Mourad, and K. Noui. “Three Dimensional Quantum Geometry and Deformed Poincare Symmetry”. *Journal of Mathematical Physics* **50**, 052503 (2009). arXiv:0806.4121.
- [37] Andrew Strominger. “On BMS Invariance of Gravitational Scattering”. *JHEP* **07**, 152 (2014). arXiv:1312.2229.
- [38] Steven Weinberg. “Infrared Photons and Gravitons”. *Physical Review* **140**, B516–B524 (1965).

- [39] Miguel Campiglia and Alok Laddha. “Asymptotic symmetries and subleading soft graviton theorem”. *Physical Review D* **90**, 124028 (2014).
- [40] Temple He, Vyacheslav Lysov, Prahar Mitra, and Andrew Strominger. “BMS supertranslations and Weinberg’s soft graviton theorem”. *Journal of High Energy Physics* **2015**, 151 (2015). arXiv:1401.7026.
- [41] Tullio Regge and Claudio Teitelboim. “Role of surface integrals in the Hamiltonian formulation of general relativity”. *Annals Phys.* **88**, 286 (1974).
- [42] Glenn Barnich and Cedric Troessaert. “BMS charge algebra”. *JHEP* **12**, 105 (2011). arXiv:1106.0213.
- [43] Glenn Barnich and Geoffrey Compere. “Surface charge algebra in gauge theories and thermodynamic integrability”. *Journal of Mathematical Physics* **49**, 042901 (2008). arXiv:0708.2378.
- [44] Daniel Kapec, Vyacheslav Lysov, Sabrina Pasterski, and Andrew Strominger. “Semiclassical Virasoro Symmetry of the Quantum Gravity S-Matrix”. *JHEP* **08**, 058 (2014). arXiv:1406.3312.
- [45] Sabrina Pasterski and Shu-Heng Shao. “A Conformal Basis for Flat Space Amplitudes”. *Phys. Rev. D* **96**, 065022 (2017). arXiv:1705.01027.
- [46] Sabrina Pasterski, Shu-Heng Shao, and Andrew Strominger. “Flat space amplitudes and conformal symmetry of the celestial sphere”. *Phys. Rev. D* **96**, 065026 (2017). arXiv:1701.00049.
- [47] Sabrina Pasterski, Monica Pate, and Ana-Maria Raclariu. “Celestial Holography” (2021). arXiv:2111.11392.
- [48] Arjun Bagchi, Rudranil Basu, Ashish Kakkar, and Aditya Mehra. “Flat Holography: Aspects of the dual field theory”. *JHEP* **12**, 147 (2016). arXiv:1609.06203.
- [49] Luca Ciambelli, Charles Marteau, Anastasios C. Petkou, P. Marios Petropoulos, and Konstantinos Siampos. “Flat holography and Carrollian fluids”. *JHEP* **07**, 165 (2018). arXiv:1802.06809.
- [50] Amartya Saha. “Carrollian approach to 1 + 3D flat holography”. *JHEP* **06**, 051 (2023). arXiv:2304.02696.
- [51] Laura Donnay, Adrien Fiorucci, Yannick Herfray, and Romain Ruzziconi. “Bridging Carrollian and Celestial Holography”. *Phys. Rev. D* **107**, 126027 (2023). arXiv:2212.12553.
- [52] Temple He, Prahar Mitra, Achilleas P. Porfyriadis, and Andrew Strominger. “New symmetries of massless QED”. *Journal of High Energy Physics* **2014**, 112 (2014).

- [53] Temple He, Prahar Mitra, and Andrew Strominger. “2D Kac-Moody symmetry of 4D Yang-Mills theory”. *Journal of High Energy Physics* **2016**, 137 (2016).
- [54] Miguel Campiglia and Adarsh Sudhakar. “Gravitational Poisson brackets at null infinity compatible with smooth superrotations” (2024). [arXiv:2408.13067](#).
- [55] Marc Geiller and Céline Zwikel. “The partial Bondi gauge: Gauge fixings and asymptotic charges” (2024).
- [56] Laurent Freidel, Marc Geiller, and Wolfgang Wieland. “Corner symmetry and quantum geometry” (2023). [arXiv:2302.12799](#).
- [57] Aldo Riello. “Quantum edge modes in 3d gravity and 2+1d topological phases of matter”. *Physical Review D* **98**, 106002 (2018). [arXiv:1802.02588](#).
- [58] B. Andrei Bernevig and Taylor L. Hughes. “Topological Insulators and Topological Superconductors”. Princeton University Press. (2013).
- [59] Edward Witten. “Anti De Sitter Space And Holography” (1998). [arXiv:hep-th/9802150](#).
- [60] Juan M. Maldacena. “The Large N Limit of Superconformal Field Theories and Supergravity”. *International Journal of Theoretical Physics* **38**, 1113–1133 (1999). [arXiv:hep-th/9711200](#).
- [61] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz. “Large N Field Theories, String Theory and Gravity”. *Phys. Rept.* **323**, 183–386 (2000). [arXiv:hep-th/9905111](#).
- [62] Goncalo Araujo-Regado, Rifath Khan, and Aron C. Wall. “Cauchy slice holography: A new AdS/CFT dictionary”. *Journal of High Energy Physics* **03**, 026 (2023). [arXiv:2204.00591](#).
- [63] S Carlip. “Dynamics of asymptotic diffeomorphisms in (2+1)-dimensional gravity”. *Classical and Quantum Gravity* **22**, 3055 (2005).
- [64] S. Carlip. “Conformal Field Theory, (2+1)-Dimensional Gravity, and the BTZ Black Hole”. *Classical and Quantum Gravity* **22**, R85–R123 (2005). [arXiv:gr-qc/0503022](#).
- [65] Rodrigo Andrade E Silva and Ted Jacobson. “Causal diamonds in  $(2 + 1)$ -dimensional quantum gravity”. *Physical Review D* **107**, 024033 (2023).
- [66] Andreas Blommaert, Thomas G. Mertens, and Shunyu Yao. “The q-Schwarzian and Liouville gravity”. *Journal of High Energy Physics* **2024**, 54 (2024).
- [67] Jacob D. Bekenstein. “Black Holes and Entropy”. *Phys. Rev. D* **7**, 2333–2346 (1973).

- [68] S. W. Hawking. “Particle creation by black holes”. *Communications In Mathematical Physics* **43**, 199–220 (1975).
- [69] Shinsei Ryu and Tadashi Takayanagi. “Holographic Derivation of Entanglement Entropy from the anti-de Sitter Space/Conformal Field Theory Correspondence”. *Physical Review Letters* **96**, 181602 (2006).
- [70] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. “A covariant holographic entanglement entropy proposal”. *Journal of High Energy Physics* **2007**, 062 (2007).
- [71] Mukund Rangamani and Tadashi Takayanagi. “Holographic Entanglement Entropy” (2017). [arXiv:1609.01287](#).
- [72] Brian Swingle and Mark Van Raamsdonk. “Universality of Gravity from Entanglement” (2014). [arXiv:1405.2933](#).
- [73] Mark Van Raamsdonk. “Building up spacetime with quantum entanglement II: It from BC-bit” (2018). [arXiv:1809.01197](#).
- [74] Aitor Lewkowycz and Juan Maldacena. “Generalized gravitational entropy”. *Journal of High Energy Physics* **2013**, 90 (2013). [arXiv:1304.4926](#).
- [75] Gabriel Wong. “A note on entanglement edge modes in Chern Simons theory”. *Journal of High Energy Physics* **08**, 020 (2018). [arXiv:1706.04666](#).
- [76] Marc Geiller, Christophe Goeller, and Nelson Merino. “Most general theory of 3d gravity: Covariant phase space, dual diffeomorphisms, and more”. *Journal of High Energy Physics* **2021**, 120 (2021).
- [77] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. “Edge modes of gravity. Part II. Corner metric and Lorentz charges”. *Journal of High Energy Physics* **2020**, 27 (2020).
- [78] William Donnelly and Laurent Freidel. “Local subsystems in gauge theory and gravity”. *Journal of High Energy Physics* (2016).
- [79] Shadi Ali Ahmad, Thomas D. Galley, Philipp A. Hoehn, Maximilian P. E. Lock, and Alexander R. H. Smith. “Quantum Relativity of Subsystems”. *Phys. Rev. Lett.* **128**, 170401 (2022). [arXiv:2103.01232](#).
- [80] Philipp A. Hoehn, Isha Kotecha, and Fabio M. Mele. “Quantum Frame Relativity of Subsystems, Correlations and Thermodynamics” (2023). [arXiv:2308.09131](#).
- [81] Daniel Freed. “Lectures on Field Theory and Topology”. Volume 133 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society. Providence, Rhode Island (2019).



- [82] C Crnkovic. “Symplectic geometry of the covariant phase space”. *Classical and Quantum Gravity* **5**, 1557–1575 (1988).
- [83] Juan Margalef-Bentabol and Eduardo J. S. Villaseñor. “Geometric formulation of the Covariant Phase Space methods with boundaries”. *Physical Review D* **103**, 025011 (2021). arXiv:2008.01842.
- [84] Igor Khavkine. “Covariant phase space, constraints, gauge and the Peierls formula”. *International Journal of Modern Physics A* **29**, 1430009 (2014). arXiv:1402.1282.
- [85] Daniel Harlow and Jie-qiang Wu. “Covariant phase space with boundaries”. *Journal of High Energy Physics* **2020**, 146 (2020). arXiv:1906.08616.
- [86] Adrien Fiorucci. “Leaky covariant phase spaces: Theory and application to  $\lambda$ -BMS symmetry” (2021). arXiv:2112.07666.
- [87] Wolfgang Wieland. “Barnich-Troessaert Bracket as a Dirac Bracket on the Covariant Phase Space”. *Classical and Quantum Gravity* **39**, 025016 (2022). arXiv:2104.08377.
- [88] Giulio Neri and Stefano Liberati. “Covariant phase space analysis of Lanczos-Lovelock gravity with boundaries” (2024). arXiv:2404.16981.
- [89] Marc Geiller. “Lorentz-diffeomorphism edge modes in 3d gravity”. *Journal of High Energy Physics* **2018**, 29 (2018).
- [90] Marc Geiller and Puttarak Jai-akson. “Extended actions, dynamics of edge modes, and entanglement entropy”. *Journal of High Energy Physics* **2020**, 134 (2020). arXiv:1912.06025.
- [91] Marc Geiller, Florian Girelli, Christophe Goeller, and Panagiotis Tsimiklis. “Diffeomorphisms as quadratic charges in 4d BF theory and related TQFTs” (2022). arXiv:2211.00068.
- [92] Adam Ball, Albert Law, and Gabriel Wong. “Dynamical Edge Modes and Entanglement in Maxwell Theory” (2024). arXiv:2403.14542.
- [93] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. “Edge modes of gravity. Part I. Corner potentials and charges”. *Journal of High Energy Physics* **2020**, 26 (2020).
- [94] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. “Edge modes of gravity. Part III. Corner simplicity constraints”. *Journal of High Energy Physics* **2021**, 1–64 (2021).
- [95] Geoffrey Compère and François Dehouck. “Relaxing the Parity Conditions of Asymptotically Flat Gravity”. *Class. Quant. Grav.* **28**, 245016 (2011). arXiv:1106.4045.
- [96] Geoffrey Compère, Samuel E. Gralla, and Hongji Wei. “An asymptotic framework for gravitational scattering”. *Class. Quant. Grav.* **40**, 205018 (2023). arXiv:2303.17124.

- [97] H. Adami, M. M. Sheikh-Jabbari, V. Taghiloo, H. Yavartanoo, and C. Zwikel. “Symmetries at Null Boundaries: Two and Three Dimensional Gravity Cases” (2020).
- [98] Robert McNees and Céline Zwikel. “The symplectic potential for leaky boundaries” (2024). arXiv:2408.13203.
- [99] Laurent Freidel, Roberto Oliveri, Daniele Pranzetti, and Simone Speziale. “Extended corner symmetry, charge bracket and Einstein’s equations”. *Journal of High Energy Physics* **2021**, 1–38 (2021).
- [100] Luca Ciambelli, Laurent Freidel, and Robert G. Leigh. “Quantum Null Geometry and Gravity”. *Journal of High Energy Physics* **2024**, 28 (2024). arXiv:2407.11132.
- [101] Luca Ciambelli, Laurent Freidel, and Robert G. Leigh. “Null Raychaudhuri: Canonical structure and the dressing time”. *JHEP* **01**, 166 (2024). arXiv:2309.03932.
- [102] Nicolas Cresto and Laurent Freidel. “Asymptotic Higher Spin Symmetries II: Noether Realization in Gravity” (2024). arXiv:2410.15219.
- [103] Nicolas Cresto and Laurent Freidel. “Asymptotic Higher Spin Symmetries I: Covariant Wedge Algebra in Gravity” (2024). arXiv:2409.12178.
- [104] Kristina Giesel, Viktoria Kabel, and Wolfgang Wieland. “Linking Edge Modes and Geometrical Clocks in Linearized Gravity” (2024). arXiv:2410.17339.
- [105] Wolfgang Wieland. “Evidence for Planck Luminosity Bound in Quantum Gravity” (2024). arXiv:2402.12578.
- [106] Marc Henneaux and Claudio Bunster. “Quantization of gauge systems”. *Physics*. Princeton University Press. Princeton, New Jersey (1992).
- [107] Glenn Barnich and Fabrizio Del Monte. “Introduction to Classical Gauge Field Theory and to Batalin-Vilkovisky Quantization” (2018). arXiv:1810.00442.
- [108] Glenn Barnich. “A note on gauge systems from the point of view of Lie algebroids” (2010). arXiv:1010.0899.
- [109] Carlo Rovelli. “Why Gauge?”. *Foundations of Physics* **44**, 91–104 (2014).
- [110] Laurent Freidel, Roberto Oliveri, Daniele Pranzetti, and Simone Speziale. “Extended corner symmetry, charge bracket and Einstein’s equations”. *JHEP* **09**, 083 (2021). arXiv:2104.12881.
- [111] Henrique Gomes, Florian Hopfmüller, and Aldo Riello. “A unified geometric framework for boundary charges and dressings: Non-Abelian theory and matter”. *Nuclear Physics B* **941**, 249–315 (2019). arXiv:1808.02074.

- [112] Sylvain Carrozza, Stefan Eccles, and Philipp A. Hoehn. “Edge modes as dynamical frames: Charges from post-selection in generally covariant theories” (2023). arXiv:2205.00913.
- [113] Aldo Riello. “Edge modes without edge modes” (2021). arXiv:2104.10182.
- [114] Adam Ball and Luca Ciambelli. “Dynamical Edge Modes in Yang-Mills Theory” (2024). arXiv:2412.06672.
- [115] Anne-Catherine de la Hamette, Thomas D. Galley, Philipp A. Hoehn, Leon Loveridge, and Markus P. Mueller. “Perspective-neutral approach to quantum frame covariance for general symmetry groups” (2021). arXiv:2110.13824.
- [116] Jennifer Lin and Djordje Radicevic. “Comments on Defining Entanglement Entropy” (2018). arXiv:1808.05939 [cond-mat, physics:hep-lat, physics:hep-th, physics:quant-ph].
- [117] Kehe Zhu. “An Introduction to Operator Algebras”. Number v.9 in Studies in Advanced Mathematics Ser. Chapman and Hall/CRC. Boca Raton (2018). 1st ed edition.
- [118] David Wallace. “Isolated systems and their symmetries, part II: Local and global symmetries of field theories”. *Studies in History and Philosophy of Science* **92**, 249–259 (2022).
- [119] Henrique Gomes. “The role of representational conventions in assessing the empirical significance of symmetries” (2021). arXiv:2110.14563.
- [120] David Wallace. “Isolated systems and their symmetries, part I: General framework and particle-mechanics examples”. *Studies in History and Philosophy of Science* **92**, 239–248 (2022).
- [121] William Donnelly. “Decomposition of entanglement entropy in lattice gauge theory”. *Physical Review D* **85**, 085004 (2012). arXiv:1109.0036.
- [122] Daniel Harlow and Daniel Jafferis. “The Factorization Problem in Jackiw-Teitelboim Gravity” (2019). arXiv:1804.01081.
- [123] Mark J.D. Hamilton. “Mathematical Gauge Theory”. Universitext. Springer International Publishing. Cham (2017).
- [124] Pavel Mnev. “Discrete BF theory” (2008). arXiv:0809.1160.
- [125] Alberto S. Cattaneo, Pavel Mnev, and Nicolai Reshetikhin. “A Cellular Topological Field Theory”. *Communications in Mathematical Physics* **374**, 1229–1320 (2020).

- [126] Giovanni Canepa and Michele Schiavina. “Fully extended BV-BFV description of General Relativity in three dimensions”. *Advances in Theoretical and Mathematical Physics* **26**, 595–642 (2022). arXiv:1905.09333.
- [127] John C. Baez and Alejandro Perez. “Quantization of strings and branes coupled to BF theory” (2007). arXiv:gr-qc/0605087.
- [128] L. Freidel, J. Kowalski–Glikman, and A. Starodubtsev. “Particles as Wilson lines of gravitational field”. *Physical Review D* **74**, 084002 (2006). arXiv:gr-qc/0607014.
- [129] Merced Montesinos and Alejandro Perez. “Two-dimensional topological field theories coupled to four-dimensional BF theory”. *Physical Review D* **77**, 104020 (2008).
- [130] Florian Girelli and Panagiotis Tsimiklis. “Discretization of 4d Poincare BF theory: From groups to 2-groups”. *Physical Review D* **106**, 046003 (2022). arXiv:2105.01817.
- [131] Viqar Husain and Seth Major. “Gravity and BF theory defined in bounded regions”. *Nuclear Physics B* **500**, 381–401 (1997). arXiv:gr-qc/9703043.
- [132] Mauricio Mondragon and Merced Montesinos. “Covariant canonical formalism for four-dimensional BF theory”. *J. Math. Phys.* **47**, 022301 (2006). arXiv:gr-qc/0402041.
- [133] Valentin Bonzom and Matteo Smerlak. “Bubble divergences: Sorting out topology from cell structure”. *Annales Henri Poincaré* **13**, 185–208 (2012). arXiv:1103.3961.
- [134] Roberto Oliveri and Simone Speziale. “Boundary effects in General Relativity with tetrad variables” (2020). arXiv:1912.01016.
- [135] Maïté Dupuis, Laurent Freidel, Florian Girelli, Abdulmajid Osumanu, and Julian Rennert. “On the origin of the quantum group symmetry in 3d quantum gravity” (2020). arXiv:2006.10105.
- [136] L. Freidel, F. Girelli, and Barak Shoshany. “2+1D loop quantum gravity on the edge”. *Physical Review D* (2019).
- [137] Maïté Dupuis, Laurent Freidel, and Florian Girelli. “Discretization of 3d gravity in different polarizations” (2017). arXiv:1701.02439.
- [138] Clement Delcamp, Laurent Freidel, and Florian Girelli. “Dual loop quantizations of 3d gravity” (2018). arXiv:1803.03246.
- [139] Laurent Freidel, Marc Geiller, and Jonathan Ziprick. “Continuous formulation of the Loop Quantum Gravity phase space” (2013). arXiv:1110.4833.
- [140] Barak Shoshany. “At the Corner of Space and Time” (2020). arXiv:1912.02922.
- [141] Eugenio Bianchi. “Loop Quantum Gravity a la Aharonov-Bohm”. *General Relativity and Gravitation* **46**, 1668 (2014). arXiv:0907.4388.

- [142] Goncalo Araujo-Regado, Rifath Khan, and Aron C. Wall. “Cauchy slice holography: a new ads/cft dictionary”. *Journal of High Energy Physics* **2023** (2023).
- [143] Jacob D. Bekenstein. “Black holes and entropy”. *Phys. Rev. D* **7**, 2333–2346 (1973).
- [144] J. Eisert, M. Cramer, and M. B. Plenio. “Area laws for the entanglement entropy - a review”. *Reviews of Modern Physics* **82**, 277–306 (2010). arXiv:0808.3773.
- [145] William Donnelly, Laurent Freidel, Seyed Moosavian, and Antony Speranza. “Gravitational edge modes, coadjoint orbits, and hydrodynamics”. *Journal of High Energy Physics* **2021**, 1–63 (2021).
- [146] J. I. Cirac and F. Verstraete. “Renormalization and tensor product states in spin chains and lattices”. *J. Phys. A: Math. Theor.* **42**, 504004 (2009). arXiv:0910.1130.
- [147] Bruno Nachtergaele and Robert Sims. “Lieb-Robinson Bounds in Quantum Many-Body Physics” (2010). arXiv:1004.2086.
- [148] S. Bravyi, M. B. Hastings, and F. Verstraete. “Lieb-Robinson bounds and the generation of correlations and topological quantum order”. *Physical Review Letters* **97**, 050401 (2006). arXiv:quant-ph/0603121.
- [149] Goffredo Chirco, Daniele Oriti, and Mingyi Zhang. “Group field theory and tensor networks: towards a Ryu–Takayanagi formula in full quantum gravity”. *Class. Quant. Grav.* **35**, 115011 (2018). arXiv:1701.01383.
- [150] Goffredo Chirco, Daniele Oriti, and Mingyi Zhang. “Ryu-Takayanagi Formula for Symmetric Random Tensor Networks”. *Phys. Rev. D* **97**, 126002 (2018).
- [151] Goffredo Chirco, Alex Goeßmann, Daniele Oriti, and Mingyi Zhang. “Group field theory and holographic tensor networks: dynamical corrections to the Ryu–Takayanagi formula”. *Class. Quant. Grav.* **37**, 095011 (2020). arXiv:1903.07344.
- [152] Eugenia Colafranceschi and Daniele Oriti. “Quantum gravity states, entanglement graphs and second-quantized tensor networks”. *Journal of High Energy Physics* **2021**, 52 (2021). arXiv:2012.12622.
- [153] Eugenia Colafranceschi, Goffredo Chirco, and Daniele Oriti. “Holographic maps from quantum gravity states as tensor networks” (2021). arXiv:2105.06454.
- [154] Goffredo Chirco, Eugenia Colafranceschi, and Daniele Oriti. “Bulk area law for boundary entanglement in spin network states: Entropy corrections and horizon-like regions from volume correlations”. *Physical Review D* **105**, 046018 (2022). arXiv:2110.15166.
- [155] Zhao Yang, Patrick Hayden, and Xiao-Liang Qi. “Bidirectional holographic codes and sub-AdS locality”. *Journal of High Energy Physics* **01**, 175 (2016). arXiv:1510.03784.

- [156] Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, and Zhao Yang. “Holographic duality from random tensor networks”. *J. High Energ. Phys.* **2016**, 9 (2016). arXiv:1601.01694.
- [157] Fernando Pastawski, Beni Yoshida, Daniel Harlow, and John Preskill. “Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence”. *J. High Energ. Phys.* **2015**, 149 (2015). arXiv:1503.06237.
- [158] Newton Cheng, Cécilia Lancien, Geoff Penington, Michael Walter, and Freek Witteveen. “Random tensor networks with nontrivial links” (2022). arXiv:2206.10482.
- [159] Xi Dong, Sean McBride, and Wayne W. Weng. “Holographic Tensor Networks with Bulk Gauge Symmetries” (2023). arXiv:2309.06436.
- [160] Chris Akers and Annie Y. Wei. “Background independent tensor networks” (2024). arXiv:2402.05910.
- [161] Harriet Apel, Tamara Kohler, and Toby Cubitt. “Holographic duality between local Hamiltonians from random tensor networks”. *Journal of High Energy Physics* **2022**, 52 (2022).
- [162] John W. Barrett, J. Manuel Garcia-Islas, and Joao Faria Martins. “Observables in the Turaev-Viro and Crane-Yetter models”. *Journal of Mathematical Physics* **48**, 093508 (2007). arXiv:math/0411281.
- [163] Stefan Hollands and Ko Sanders. “Entanglement measures and their properties in quantum field theory” (2018). arXiv:1702.04924 [gr-qc, physics:hep-th, physics:math-ph, physics:quant-ph].
- [164] Chen-Te Ma. “Entanglement with Centers”. *Journal of High Energy Physics* **2016**, 70 (2016).
- [165] Min Jiang, Shunlong Luo, and Shuangshuang Fu. “Channel-state duality”. *Physical Review A* **87**, 022310 (2013).
- [166] Władysław A. Majewski and Tomasz I. Tylec. “Comment on “Channel-state duality””. *Physical Review A* **88**, 026301 (2013).
- [167] Eugenia Colafranceschi and Gerardo Adesso. “Holographic entanglement in spin network states: A focused review”. *AVS Quantum Science* **4**, 025901 (2022). arXiv:2202.05116.
- [168] Xiao-Liang Qi, Zhao Yang, and Yi-Zhuang You. “Holographic coherent states from random tensor networks”. *Journal of High Energy Physics* **08**, 060 (2017). arXiv:1703.06533.

- [169] Kristan Jensen, Jonathan Sorce, and Antony Speranza. “Generalized entropy for general subregions in quantum gravity”. JHEP **12**, 020 (2023). arXiv:2306.01837.
- [170] Xi Dong, Daniel Harlow, and Donald Marolf. “Flat entanglement spectra in fixed-area states of quantum gravity”. Journal of High Energy Physics **2019**, 240 (2019). arXiv:1811.05382.
- [171] Luca Ciambelli, Laurent Freidel, and Robert G. Leigh. “Quantum Null Geometry and Gravity” (2024). arXiv:2407.11132.
- [172] J. Fernando Barbero G. and Daniele Pranzetti. “Black Hole Entropy in Loop Quantum Gravity” (2022). arXiv:2212.13469.
- [173] Jonathan Engle, Karim Noui, Alejandro Perez, and Daniele Pranzetti. “Black hole entropy from an  $SU(2)$ -invariant formulation of Type I isolated horizons”. Physical Review D **82**, 044050 (2010). arXiv:1006.0634.
- [174] Laurent Freidel, Marc Geiller, and Wolfgang Wieland. “Corner symmetry and quantum geometry” (2023). arXiv:2302.12799.
- [175] William Donnelly, Laurent Freidel, Seyed Farough Moosavian, and Antony J. Speranza. “Gravitational Edge Modes, Coadjoint Orbits, and Hydrodynamics”. JHEP **09**, 008 (2021). arXiv:2012.10367.
- [176] Abhay Ashtekar and Simone Speziale. “Null Infinity and Horizons: A New Approach to Fluxes and Charges”. Physical Review D **110**, 044049 (2024). arXiv:2407.03254.
- [177] Sören Holst. “Barbero’s Hamiltonian derived from a generalized Hilbert-Palatini action”. Phys. Rev. D **53**, 5966–5969 (1996). arXiv:gr-qc/9511026.
- [178] Simone Mercuri. “From the Einstein-Cartan to the Ashtekar-Barbero canonical constraints, passing through the Nieh-Yan functional”. Physical Review D **77**, 024036 (2008).
- [179] Osvaldo Chandia and Jorge Zanelli. “Torsional Topological Invariants (and their relevance for real life)” (1998). arXiv:hep-th/9708138.
- [180] G. de Berredo-Peixoto, L. Freidel, I. L. Shapiro, and C. A. de Souza. “Dirac fields, torsion and Barbero-Immirzi parameter in Cosmology”. Journal of Cosmology and Astroparticle Physics **2012**, 017–017 (2012). arXiv:1201.5423.
- [181] R. Capovilla, M. Montesinos, V. A. Prieto, and E. Rojas. “BF gravity and the Immirzi parameter”. Classical and Quantum Gravity **18**, L49–L52 (2001). arXiv:gr-qc/0102073.
- [182] Alejandro Perez and Carlo Rovelli. “Physical effects of the Immirzi parameter in loop quantum gravity”. Physical Review D **73**, 044013 (2006).

- [183] Sergei Alexandrov. “On choice of connection in loop quantum gravity”. *Physical Review D* **65**, 024011 (2002). arXiv:gr-qc/0107071.
- [184] Sergei Alexandrov. “SO(4,C)-covariant Ashtekar-Barbero gravity and the Immirzi parameter”. *Classical and Quantum Gravity* **17**, 4255–4268 (2000). arXiv:gr-qc/0005085.
- [185] Venkatesa Chandrasekaran, Éanna É. Flanagan, Ibrahim Shehzad, and Antony J. Speranza. “A general framework for gravitational charges and holographic renormalization”. *Int. J. Mod. Phys. A* **37**, 2250105 (2022). arXiv:2111.11974.
- [186] Alberto S. Cattaneo and Michele Schiavina. “The Reduced Phase Space of Palatini–Cartan–Holst Theory”. *Annales Henri Poincaré* **20**, 445–480 (2019).
- [187] Giovanni Canepa, Alberto S. Cattaneo, and Michele Schiavina. “Boundary structure of General Relativity in tetrad variables”. *Advances in Theoretical and Mathematical Physics* **25**, 327–377 (2021). arXiv:2001.11004.
- [188] Merced Montesinos, Diego González, Mariano Celada, and Bogar Díaz. “Reformulation of the symmetries of first-order general relativity”. *Classical and Quantum Gravity* **34**, 205002 (2017).
- [189] Giovanni Canepa, Alberto S. Cattaneo, and Michele Schiavina. “General Relativity and the AKSZ construction”. *Communications in Mathematical Physics* **385**, 1571–1614 (2021). arXiv:2006.13078.
- [190] J. David Brown and James W. York. “Quasilocal energy and conserved charges derived from the gravitational action”. *Phys. Rev. D* **47**, 1407–1419 (1993). arXiv:gr-qc/9209012.
- [191] Seth K. Asante, Bianca Dittrich, Florian Girelli, Aldo Riello, and Panagiotis Tsimiklis. “Quantum geometry from higher gauge theory”. *Classical and Quantum Gravity* **37**, 205001 (2020). arXiv:1908.05970.
- [192] Laurent Freidel, Etera R. Livine, and Daniele Pranzetti. “Kinematical Gravitational Charge Algebra”. *Physical Review D* **101**, 024012 (2020). arXiv:1910.05642.
- [193] Amitabh Virmani. “Asymptotic Flatness, Taub-NUT, and Variational Principle”. *Physical Review D* **84**, 064034 (2011). arXiv:1106.4372.
- [194] Oscar Fuentealba, Marc Henneaux, and Cédric Troessaert. “Logarithmic supertranslations and supertranslation-invariant Lorentz charges”. *JHEP* **02**, 248 (2023). arXiv:2211.10941.
- [195] Viktoria Kabel, Anne-Catherine de la Hamette, Luca Apadula, Carlo Cepollaro, Henrique Gomes, Jeremy Butterfield, and Časlav Brukner. “Identification is Pointless: Quantum Reference Frames, Localisation of Events, and the Quantum Hole Argument” (2024). arXiv:2402.10267.



- [196] Anne-Catherine de la Hamette and Thomas D. Galley. “Quantum reference frames for general symmetry groups”. *Quantum* **4**, 367 (2020). arXiv:2004.14292.
- [197] Flaminia Giacomini, Esteban Castro-Ruiz, and Časlav Brukner. “Relativistic quantum reference frames: The operational meaning of spin”. *Physical Review Letters* **123**, 090404 (2019). arXiv:1811.08228.
- [198] Flaminia Giacomini. “Spacetime Quantum Reference Frames and superpositions of proper times”. *Quantum* **5**, 508 (2021). arXiv:2101.11628.
- [199] Christopher J. Fewster, Daan W. Janssen, Leon Deryck Loveridge, Kasia Rejzner, and James Waldron. “Quantum reference frames, measurement schemes and the type of local algebras in quantum field theory”. *Communications in Mathematical Physics* **406**, 19 (2025). arXiv:2403.11973.
- [200] Sylvain Carrozza, Aidan Chatwin-Davies, Philipp A. Hoehn, and Fabio M. Mele. “A correspondence between quantum error correcting codes and quantum reference frames” (2024). arXiv:2412.15317.
- [201] Julian De Vuyst, Stefan Eccles, Philipp A. Hoehn, and Josh Kirklin. “Crossed products and quantum reference frames: On the observer-dependence of gravitational entropy” (2024). arXiv:2412.15502.
- [202] Philipp A. Hoehn, Alexander R. H. Smith, and Maximilian P. E. Lock. “Equivalence of approaches to relational quantum dynamics in relativistic settings”. *Front. in Phys.* **9**, 181 (2021). arXiv:2007.00580.
- [203] Leonardo Chataignier, Philipp A. Hoehn, Maximilian P. E. Lock, and Fabio M. Mele. “Relational Dynamics with Periodic Clocks” (2024). arXiv:2409.06479.
- [204] Niall O’Murchadha and James W. York. “Existence and uniqueness of solutions of the Hamiltonian constraint of general relativity on compact manifolds”. *Journal of Mathematical Physics* **14**, 1551–1557 (1973).
- [205] Tobias J. Osborne and Deniz E. Stiegemann. “Dynamics for holographic codes”. *JHEP* **04**, 154 (2020). arXiv:1706.08823.
- [206] Bianca Dittrich. “Diffeomorphism symmetry in quantum gravity models” (2008). arXiv:0810.3594.
- [207] B. Bahr and B. Dittrich. “Breaking and restoring of diffeomorphism symmetry in discrete gravity” (2009). arXiv:0909.5688.
- [208] Benjamin Bahr and Bianca Dittrich. “(Broken) Gauge symmetries and constraints in Regge calculus”. *Class. Quant. Grav.* **26**, 225011 (2009). arXiv:0905.1670.

- [209] Bianca Dittrich and Sebastian Steinhaus. “Path integral measure and triangulation independence in discrete gravity”. *Physical Review D* **85**, 044032 (2012). arXiv:1110.6866.
- [210] Bianca Dittrich, Wojciech Kamiński, and Sebastian Steinhaus. “Discretization independence implies non-locality in 4D discrete quantum gravity”. *Class. Quant. Grav.* **31**, 245009 (2014). arXiv:1404.5288.
- [211] Aristide Baratin, Florian Girelli, and Daniele Oriti. “Diffeomorphisms in group field theories”. *Physical Review D* **83**, 104051 (2011).
- [212] Roberto Tanzi and Domenico Giulini. “Asymptotic symmetries of Yang-Mills fields in Hamiltonian formulation”. *Journal of High Energy Physics* **2020**, 94 (2020). arXiv:2006.07268.
- [213] Aldo Riello and Michele Schiavina. “Null Hamiltonian Yang-Mills theory. Soft symmetries and memory as superselection” (2023). arXiv:2303.03531.
- [214] Silvia Nagy, Javier Peraza, and Giorgio Pizzolo. “Infinite-dimensional hierarchy of recursive extensions for all subn-leading soft effects in Yang-Mills”. *Journal of High Energy Physics* **2024**, 68 (2024).
- [215] Laurent Freidel, Christophe Goeller, and Etera R. Livine. “The Quantum Gravity Disk: Discrete Current Algebra”. *Journal of Mathematical Physics* **62**, 102303 (2021). arXiv:2103.13171.
- [216] Laurent Freidel, Roberto Oliveri, Daniele Pranzetti, and Simone Speziale. “The Weyl BMS group and Einstein’s equations”. *Journal of High Energy Physics* **2021**, 170 (2021).
- [217] Florian Girelli, Simon Langenscheidt, Giulio Neri, Christopher Andrew Pollack, and Céline Zwikel. “Logarithmic supertranslations in the covariant phase space”.
- [218] Christopher Andrew Pollack and Simon Langenscheidt. “Work in progress”.
- [219] Valentin Bonzom, Maité Dupuis, and Florian Girelli. “Towards the Turaev-Viro amplitudes from a Hamiltonian constraint”. *Physical Review D* **90**, 104038 (2014). arXiv:1403.7121.
- [220] Daniele Pranzetti. “Turaev-Viro amplitudes from 2+1 Loop Quantum Gravity”. *Physical Review D* **89**, 084058 (2014). arXiv:1402.2384.
- [221] Laurent Freidel, Alejandro Perez, and Daniele Pranzetti. “The loop gravity string”. *Physical Review D* **95**, 106002 (2017). arXiv:1611.03668.
- [222] Laurent Freidel, Etera Livine, and Daniele Pranzetti. “Gravitational edge modes: From Kac–Moody charges to Poincaré networks”. *Classical and Quantum Gravity* **36**, 195014 (2019).

- [223] Laurent Freidel, Fabio M. Mele, and Simon Langenscheidt. “Work in progress”.
- [224] Ted Jacobson. “Thermodynamics of Spacetime: The Einstein Equation of State”. *Physical Review Letters* **75**, 1260–1263 (1995).
- [225] Christopher Eling, Raf Guedens, and Ted Jacobson. “Nonequilibrium Thermodynamics of Spacetime”. *Physical Review Letters* **96**, 121301 (2006).
- [226] Ted Jacobson. “Entanglement Equilibrium and the Einstein Equation”. *Physical Review Letters* **116**, 201101 (2016).
- [227] T. Padmanabhan. “Gravity and the Thermodynamics of Horizons”. *Physics Reports* **406**, 49–125 (2005). [arXiv:gr-qc/0311036](https://arxiv.org/abs/gr-qc/0311036).
- [228] T. Padmanabhan. “General Relativity from a Thermodynamic Perspective”. *General Relativity and Gravitation* **46**, 1673 (2014). [arXiv:1312.3253](https://arxiv.org/abs/1312.3253).
- [229] T. Padmanabhan. “Gravity and its thermodynamics”. *Current Science* **109**, 2236–2242 (2015). [arXiv:24906663](https://arxiv.org/abs/24906663).
- [230] Theodore Jacobson and Manus R. Visser. “Gravitational thermodynamics of causal diamonds in (A)dS”. *SciPost Physics* **7**, 079 (2019).
- [231] Venkatesa Chandrasekaran, Roberto Longo, Geoff Penington, and Edward Witten. “An Algebra of Observables for de Sitter Space”. *JHEP* **02**, 082 (2023). [arXiv:2206.10780](https://arxiv.org/abs/2206.10780).
- [232] Venkatesa Chandrasekaran, Geoff Penington, and Edward Witten. “Large N algebras and generalized entropy”. *Journal of High Energy Physics* **2023**, 9 (2023). [arXiv:2209.10454](https://arxiv.org/abs/2209.10454).
- [233] “Completing the triangle: Local algebra type reduction from edge modes via quantum reference frames”. <https://ucrisportal.univie.ac.at/en/activities/completing-the-triangle-local-algebra-type-reduction-from-edge-mo>.
- [234] Julian De Vuyst, Stefan Eccles, Philipp A. Hoehn, and Josh Kirklin. “Gravitational entropy is observer-dependent” (2024). [arXiv:2405.00114](https://arxiv.org/abs/2405.00114).
- [235] Alberto S. Cattaneo and Nima Moshayedi. “Introduction to the BV-BFV formalism”. *Reviews in Mathematical Physics* **32**, 2030006 (2020). [arXiv:1905.08047](https://arxiv.org/abs/1905.08047).
- [236] Antony J. Speranza. “Ambiguity resolution for integrable gravitational charges”. *JHEP* **07**, 029 (2022). [arXiv:2202.00133](https://arxiv.org/abs/2202.00133).
- [237] Anatoly Fomenko and Dmitry Fuchs. “Homotopical Topology”. Volume 273 of *Graduate Texts in Mathematics*. Springer International Publishing. Cham (2016).

- [238] Razvan Gurau and James P. Ryan. “Colored Tensor Models - a Review”. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **8**, 020 (2012).
- [239] Daniele Oriti. “Spin Foam Models of Quantum Spacetime” (2003). [arXiv:gr-qc/0311066](#).
- [240] Fernando L. Teixeira. “Lattice Maxwell’s Equations”. *Progress In Electromagnetics Research* **148**, 113–129 (2014).
- [241] Rafael Sorkin. “Time-evolution problem in regge calculus”. *Phys. Rev. D* **12**, 385–396 (1975).
- [242] Maïté Dupuis, Laurent Freidel, Florian Girelli, Abdulmajid Osumanu, and Julian Rennert. “On the origin of the quantum group symmetry in 3d quantum gravity” (2020). [arXiv:2006.10105](#).
- [243] Laurent Freidel and Simone Speziale. “Twisted geometries: A geometric parametrisation of  $SU(2)$  phase space”. *Physical Review D* **82**, 084040 (2010). [arXiv:1001.2748](#).
- [244] Paolo Zanardi. “Virtual Quantum Subsystems”. *Physical Review Letters* **87**, 077901 (2001). [arXiv:quant-ph/0103030](#).
- [245] Horacio Casini, Marina Huerta, and Jose Alejandro Rosabal. “Remarks on entanglement entropy for gauge fields”. *Physical Review D* **89**, 085012 (2014). [arXiv:1312.1183](#).
- [246] Eugenio Bianchi and Pietro Dona. “Typical entanglement entropy in the presence of a center: Page curve and its variance”. *Physical Review D* **100**, 105010 (2019). [arXiv:1904.08370](#).
- [247] William Donnelly and Aron C. Wall. “Entanglement Entropy of Electromagnetic Edge Modes”. *Physical Review Letters* **114**, 111603 (2015).
- [248] Noah Linden, Milán Mosonyi, and Andreas Winter. “The structure of Renyi entropic inequalities”. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **469**, 20120737 (2013). [arXiv:1212.0248](#).
- [249] Samuel O. Scalet, Álvaro M. Alhambra, Georgios Styliaris, and J. Ignacio Cirac. “Computable Renyi mutual information: Area laws and correlations”. *Quantum* **5**, 541 (2021). [arXiv:2103.01709](#).
- [250] K. V. Antipin. “Channel-state duality and the separability problem” (2020). [arXiv:1909.13309](#).
- [251] Page. “Average entropy of a subsystem.”. *Physical review letters* (1993).
- [252] Eugenia Colafranceschi, Simon Langenscheidt, and Daniele Oriti. “Holographic properties of superposed spin networks” (2022). [arXiv:2207.07625](#).

# Acknowledgements

I have several people I am legally and morally required to express my gratitude to.

Then, there are my friends:

First of all, I am greatly indebted to my supervisor Daniele Oriti, who has not only taken me in when the group in Munich was about to dissolve, and pushed me to my absolute best form, but also taught me how to be a good researcher - not simply on a technical, but on a sociological and philosophical level; But he will never teach me how to use semicolons properly.

Second, my PhD time would not have been the same without Giulio Neri. Throughout much of this also quite complicated time, you stuck to me even when I wouldn't and were as close as I could've had to a PhD cohort colleague. And just as an aside, you also became one of my best friends in the process. I wish you only the best, forever.

I also would like to thank all the members of my PhD committee, foremost of them Uli Schollwöck, who also graciously took the function of my *de iure* supervisor at LMU. I further thank Ivo Sachs, Harald Weinfurtner and Michael Haack for taking the time to acquaint themselves with the thesis and testing me in the *rigorosum*.

This thesis would not exist without my many collaborators, especially Eugenia Colafranceschi, who co-authored some of the papers that went into the main text and has been a great mentor and friend throughout my time in academia so far. To my other current collaborators, as well - I thank Henrique Gomes, Christopher Pollack, Céline Zwikel and Florian Girelli.

In a similar vein, I thank all my collaborators from upcoming projects; Laurent Freidel, Fabio Mele, Bianca Dittrich, Aldo Riello, Abhishek Rajput, Tamara Kohler (and hopefully more! Maybe you??)

I furthermore would like to thank the people who have corrected the thesis and given me very necessary feedback to bring it from its natural form (completely unreadable) to its current state (a moderate size appendix). Thank you, Abhishek, Daniele, Fabio and especially you, Kaya - you're an angel.

Over the last three years, I have had many inspiring discussions contributing to the current state of the physics corner<sup>2</sup> of my mind - too many to count. *In lieu* of their number, I will list (in a diplomatically random fashion, and only those names not yet listed) the smart people I talked to instead:

I had many great chats with: Christophe Goeller, Oleksandra Hrytseniak, Luca Marchetti, Philipp Höhn, Francesco Sartini, Ann-Cathrin de la Hamette, Adrian Lopez, Viktoria Kabel, Wolfgang Wieland, Francesca Vidotto, Wojciech Kaminski, Renata Ferrero, Andrea de Biagio, Vanessa Brzić, Carlo Rovelli, Erik Curiel, Joshua Kirklin, Jacqueline Caminiti, Federico Capeccia, Simone Cepollaro, Ingmar Saberi, Rodrigo Andrade e Silva, Bruna Sahdo, Bilyana Tomova, Nicholas Ormrod, Yale Yauk, Ignacio Cirac, Toby Cubitt, Damiano Aliverti, Lluís Masenes, Christian Schilling, Sung-Sik Lee, Hank Chen, Marija Tomašević, Massimo Porrati, Ofek Bengyat, Goncalo Araujo Regado, Markus Dierigl, Simone Speziale, Luca Ciambelli, Daniele Pranzetti, Jochen Weller, Steffen Gielen, Ed Wilson-Ewing, Johanna Borissova, Folkert Kuipers, Nicholas Cresto, Goffredo Chirco, Eugenio Bianchi. I have learned from all of you more than I could give.

Beyond the scientific or professional, I made the acquaintance of too many lovely people in academia to not mention at least a representative amount of them.

First, the people at LMU Munich: Dusan, Da Yuan, Peng, Yonatan, Julia, Giovanni, Heloise, Elisa, William, Matteo, Luca, Italo, Christian, Elias, Antonia P., Carmine, Severin, Navid, Batuhan, Kim, Antonia W., George, Christoph, Zsófia, Lukas and so many more. Then, the people I met at Perimeter Institute: Raquel, Athanasios, Ifigenia, James, Pablo, Sasha, Kelly, Kiana, Eirini, Kallia, Mikka, Hassan, Cole, Dawit, Sercan, Robin, Antonia, Eivind and many I didn't have yet enough time to meet!

There are many many more people whom I would like to thank here or at least mention, as they have had my back throughout this entire time period of the PhD, and often for much, much, *much* longer. It goes without saying - my deepest gratitude to my parents, Beate and Siegfried, my siblings Matze, Fanny, Sofie and Jo, to my many nieces and nephews and all the rest of my family for educating me, bringing me back to earth and making me the person I am today. Ich danke insbesondere meinem Vater für intensive morgendliche Physikdiskussionen, typographische Einweisungen und den einen oder anderen Grappa ;)

I would furthermore like to thank my friends from my time at LMU: both Linus and Lukas, partners in crime since our first billard game at Schellingsalon, and Jonas, always pragmatic and Northern to the core. I also thank Robert, Pery and Miho for all the times they dragged me out of my monotropism over long, lovely coffee chats whose cardinality I better not start counting.

And, if any of you get to read this, feel seen you absolute nerds: Joshua, Håkan, Cereal, Noé, Ernie, Abi, Eyy, Loher, Arya; And also Nai, Em, Bubs, Proxy, Berke as well as all the others I omit for brevity. For as long I've been doing physics, you've been there whenever I needed a break from the world. Thank you.

---

<sup>2</sup>Haha, corner!

Now for the unceremonious finale:

I am very grateful to the Munich Center for Quantum Science and Technology for providing the funding for my PhD salary.

Furthermore, I am grateful to Dieter Lüst and his Mathematical Physics and String theory group for providing me with much coffee, banter and *Kost and Logis* throughout these 3 long years.

Similarly, I thank the Frey group for their endless hospitality and letting a foreign invader like me join for so many of their events (and especially drink such good coffee! A fountain of life). Finally, I would like to thank the Perimeter Institute for hospitality during a 3-month visit in 2024; but please replace the coffee blend in the bistro. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MEDT.

Finally finally, I would like to expressly thank in highest form *Thomas Waddleton* for graciously and meticulously pointing out not one, not two, but *three* expression mistakes in my thesis. Clearly this could never have been written without you, you handsome devil. Happy now?