A Functorial Approach To Orbifold Lifts And Flow Defects In Abelian GLSMs

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München 2025

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Mainz

München, den 05.03.2025

Erstgutachter: Prof. Dr. Ilka Brunner
Zweitgutachter: Prof. Dr. Ivo Sachs
Tag der mündlichen Prüfung: 15.05.2025

...und alles was man weiß, nicht bloß rauschen und brausen gehört hat, läßt sich in drei Worten sagen.

—Ferdinand Kürnberger, *attributed by Ludwig Wittgenstein in the Tractatus Logico-Philosophicus*

Zusammenfassung

Quanten Feld Theorien (QFTen) haben sich in der Hochenergiephysik als außerordentlich erfolgreich erwiesen, obgleich unser derzeitiges Verständnis dieser Theorien stark begrenzt ist. Während die perturbative Beschreibung von QFTen inzwischen auf einem rigorosen mathematischen Fundament steht, fehlt für nicht-perturbative QFTen bis zum heutigen Tage eine umfassende mathematische Beschreibung. Nichtsdestotrotz gibt es spezielle nicht-perturbative QFTen die eine solche mathematische Formulierung aufweisen. Es handelt sich hierbei um die topologischen und konformen Feldtheorien, welche mit Hilfe der Sprache der Kategorien in einer funktoriellen Weise axiomatisiert werden können.

Eine wichtige Quelle von Beispielen für diese funktoriellen Feldtheorien stellt die Stringtheorie dar. Die Stringtheorie, welche urspрüglich zur Beschreibung der starken Kernkraft dienen sollte, entwickelte sich später zu einem potentiellen Kandidaten für eine vereinheitliche Theorie der Grundkräfte, einschließlich der Gravitation. Über die phänomenologischen Implikationen hinaus erwies sich die Stringtheorie im Laufe ihrer Geschichte immer wieder als ein fruchtbarer Boden für neue Ideen in der theoretischen Physik und der Mathematik.

Die namensgebenden Strings der Stringtheorie können durch sogenannte geeichte lineare Sigma-Modelle (GLSMe) beschrieben werden. GLSMe besitzen topologische Untersektoren, welche durch sogenannte topologisch konforme Feldtheorien beschrieben werden. Bei niedrigen Energien verfügen GLSMe über eine reichhaltige Phasenstruktur, dieser Umstand macht sie zum idealen Versuchsumfeld zur Studie von Defekten und Phasenübergängen.

In dieser Dissertation präsentieren wir einen neuartigen Ansatz für die Konstruktion von Defekten, die Orbifaltigkeitsphasen in ihr entsprechendes GLSM funktoriell hochheben. Hierzu betrachten wir ausschließlich topologische Untersektoren von abelschen GLSMen. Insbesondere erlaubt unser Ansatz den Transport von Randbedingungen d.h. Branen von Orbifaltigkeitphasen in das GLSM. Wir präsentieren einen Überblick über topologische Feldtheorien, führen das topologische B-Modell für GLSMe ein und diskutieren unsere Konstruktion für Hochhebungsdefekte. Abschließend demonstrieren wir, dass unser Ansatz bekannte Ergebnisse für Branentransport und Flüsse in minimalen Modellen reproduziert.

Abstract

Quantum field theories (QFTs) have proven to be immensely successful in high energy physics, however, our present understanding of them is quite limited. While the perturbative approach to QFT is by now standing on a rigorous mathematical foundation, a comprehensive mathematical formulation of non-perturbative QFT is still missing to this date. Still, there are special kinds of non-perturbative QFTs that do admit such a mathematical formulation. These are topological and conformal field theories that can be axiomatized in a functorial manner, employing the language of categories.

An important source for examples of such functorial field theories is string theory. String theory, originally intended to describe the strong nuclear force, later on developed into a potential candidate for a unified QFT of the fundamental forces including gravity. Beyond its phenomenological implications, throughout its history string theory served as a fertile ground for new ideas in theoretical physics and mathematics.

The eponymous strings in string theory can be described by so called gauged linear sigma models (GLSMs). GLSMs admit topological subsectors which are captured by so called topological conformal field theories. At low energies GLSMs exhibit a rich phase structure making them an ideal testing ground to study defects and phase transitions.

In this thesis we present a novel approach for the construction of defects lifting so called orbifold phases to their respective GLSM in a functorial manner. To this end we restrict our attention to a topological subsector of GLSMs with Abelian gauge groups. Our construction in particular allows us to transport boundary conditions i.e. branes from orbifold phases to the GLSM. We present an overview of topological field theories, introduce the topological B-model for GLSMs and discuss our construction for lift defects. Finally we demonstrate that our approach reproduces known results for brane transport and for flows in minimal models.

Contents

1 Introduction

The framework of quantum field theories (QFTs) has been applied with great success to a wide range of problems in high energy physics and condensed matter theory. However, in particle physics only the perturbative approach to QFT is supported by a rigorous mathematical framework cf. [23]. Even though there is no general mathematical framework on the non-perturbative side, there are still some non-perturbative QFTs which admit a precise mathematical definition. Most notably conformal field theories (CFTs) and topological field theories (TQFTs) can be defined in a functorial manner [76] utilizing the powerful language of category theory. The former is scale invariant[1], while the latter's defining characteristic is metric independence. Other than in their metric dependence the functorial definition of CFTs and TQFTs are identical. For this reason on may think of CFTs as a refinement of TQFTs. Since their inception, TQFTs have been an active area of research in physics and mathematics.

TQFTs in Mathematics

Initially, mathematicians were interested in the classification and study of TQFTs, as they yielded new topological invariants which allowed for insights into knot theory, the categorification program and representation theory cf. [4, 5, 58, 79, 71]. Later on, even more branches of TQFT research formed in mathematics, for instance homological mirror symmetry and the geometric Langlands program [63, 57].

TQFTs in Physics

In physics, the applications of TQFTs are too plentiful to list them all, so we will just mention some notable examples. First off, TQFTs serve as testing ground to study general properties of QFTs and implement various physical ideas. TQFTs are prime candidates for such investigations, since they are mathematically well-defined, exactly solvable, and easily accessible. This approach dates back to early studies in the field e.g. in [74]. Later on, in condensed matter theory, it was realized that certain many body systems–subject to the fractional qunatum Hall-effect–can be described by TQFTs [83, 67]. In high energy physics, theorists were interested in Witten-type TQFTs which brought about new results

[1] Technically speaking conformal invariance is a slightly stronger notion than that of scale invariance cf. e.g. [73].

including two- and three-dimensional mirror symmetry and tests for non-abelian S-duality [15, 52, 81].

In recent years, new developments once again put the spotlight on TQFTs in the physics community. It was discovered that TQFTs allowed for a new conceptual understanding of and insights into structures of QFTs, namely symmetries and anomalies. This lead to the understanding of anomalies as invertible TQFTs and that of higher form symmetries as symmetry topological field theories (SymTFTs) [38, 39, 2].

Gauged Linear Sigma Models as TQFTs

In this thesis, we employ the techniques of functorial TQFTs to study defects in two-dimensional $\mathcal{N} = (2, 2)$ abelian gauge linear sigma models (GLSMs) [86]. More specifically, we consider the topological B-model of GLSMs which is a topological subsector of the full theory. These types of theories originated in string theory or, more specifically, in the study of topological strings.

String theory describes the dynamics of strings i.e. one-dimensional degrees of freedom. In string theory, open strings–as apposed to closed strings with periodic boundary conditions–can obey either Neumann or Dirichlet boundary condition cf. e.g. [46]. During the early days of string theory, Dirichlet boundary conditions were discarded, since they explicitly break supersymmetry. Later on, however, it was realized that they can be consistently implemented giving rise to D-branes–higher dimensional dynamical degrees of freedom on which the open strings can end–and, in turn, gauge symmetries [70]. Strings may be described by a GLSM. Mathematically speaking a GLSM describes how the world sheet of a propagating string, charged under some gauge symmetry, embeds into a target spacetime. By performing a topological B-twist we end up with the B-model of the GLSM meaning that we restrict our attention to the topological string. In this setting, it is possible to organize the B-type boundary conditions–the D-branes compatible with the B-twist–into a category. This category has as objects the D-branes and as morphisms the BRST-cohomology of string states between two D-branes [3]. The resulting category of B-type D-brane encodes the entire respective string theory and mathematically represents a so-called topological conformal field theory (TCFT).

At low energies, GLSMs admit a rich phase structure captured by the Fayet-Iliopoulos- and θ-parameters, which are part of the datum defining a GLSM. These parameters form a moduli space–which parametrize the theory space of low energy phases–in the non-anomalous case, but are subject to RG-flow in the anomalous case. The phases of a GLSM are characterized by breaking of the gauge symmetry. A discrete gauge symmetry remains unbroken in orbifold phases, while it is completely broken in geometric phases. Notably, on each phase boundary there is an unbroken $U(1)$. The moduli space of a non-anomalous model contains singular codimension-two-loci and different phases can be connected by homotopically different paths avoiding these singularities.

Mathematicians are interested in GLSMs for similar reasons as physicists. In toric geometry, they serve as a common framework to study various geometrical subspaces (the phases of the GLSM) and their relations to each other cf. [48].

Defects and Orbifold Lifts in GLSMs

The goal of this thesis is to use defects to devise a novel approach to constructing lifts from orbifold phases to the GLSM in a functorial manner. Here, by lift we mean a mapping in the opposite direction of the RG-flow. Thus, a lift is an identification of low energy degrees of freedom in the phase with the high energy ones in the GLSM. Flowing from the GLSM to a different phase subsequent to a lift then yields a defect between two different phases. The defects we will construct lie entirely within the sector protected by B-type supersymmetry and are, thus, applicable to both the non-anomalous and anomalous setting.

A defect is a codimension one subspace separating an ambient space into two subregions. They feature gluing conditions for the physical data attached to the two regions adjacent to the defect cf. e.g. [18]. In a TQFT, defects can be moved arbitrarily close to each other to fuse two separate defects into one. Defects are of particular interest, as they may implement various different concepts such as–by definition–operator insertions, boundaries, phase transitions and symmetry operations: Firstly, a boundary can be thought of as a defect between a theory and trivial theory which has no physical data attached to it. Secondly, phase transitions can be characterized by defects separating two adjacent phases. Lastly, symmetry operations can be represented by a defect whose gluing conditions implement the symmetry action on the fields. Defects can act on boundary conditions via fusion. Thus, defects describing phase transitions yield, in particular, brane transports.

We focus on defects to describe phase transitions. To be concise, we utilize the idea that defects can relate the physics at different points in moduli space cf. [13, 32, 59, 40, 41]. These defects are domain walls, separating two theories at different points in moduli space. One can think of them as originating from a trivial defect line in one of the theories. Upon perturbing the theory at one side of the defect, one obtains a defect between initial and perturbed theory. As this defect depends on the specific perturbation, one expects different defects for different paths in moduli space. Indeed, we have to make various choices when constructing a brane transport. Part of the data required to specify GLSM D-branes consists of a choice of representation on the Chan-Paton degrees of freedom, encoding the gauge sector. Consider the brane transport between a GLSM and a orbifold phase. Flowing from the GLSM with a continuous gauge symmetry $\mathrm{U}(1)^n$ to an orbifold phase, the representation of the initial gauge group simply restricts to a representation of the subgroup. On the other hand, a lift from an orbifold phase to the GLSM involves a choice of lift of the representation. Additionally, further restrictions on the possible lifts arise in the case that the lift of the D-brane to the GLSM can be completed by a flow to another phase, describing a path in moduli space from one phase to another [47].

In the approach we present in this thesis, lifts compatible with subsequent flows to other phases are implemented in a manifestly functorial way by defects. On general grounds, lift and deformation defects have some defining properties [59], one of which is semi-invertibility. This implies that one can associate, to any such defect, a projector that singles out a subcategory of the GLSM D-branes. We propose a concrete construction that yields defects that satisfy all expected properties. The starting point is the identity defect of the GLSM which is $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$ equivariant. Pushing the theory on one side of this defect into an orbifold phase breaks the symmetry to $\mathrm{U}(1)^n \times G$, where G is the remaining, unbroken gauge symmetry in the orbifold phase. The Chan-Paton representations on the defect transform to the respective induced representations of the subgroup $\mathrm{U}(1)^n \times G$. This however is not the whole story: we find that, in order to obtain consistent defects, we have to impose additional truncations of the representations. The truncation depends on the path of deformation in the GLSM parameter space. For each crossed phase boundary, we introduce an upper bound on the charges under the distinguished U(1) gauge symmetry preserved on this phase boundary. The choice of upper bounds, which we call cutoff parameters, characterizes our defect completely. It specifies to which subcategory of GLSM branes any brane of the phase can be lifted.

The question how D-branes are transported in GLSMs was previously addressed in the program initiated in [47]. Based on an analysis of the gauge sector, one of the proposals put forward in [47] is the band restriction rule for higher rank Abelian gauge theories. It states that for a smooth brane transport from an orbifold phase to a (partially) resolved phase, there is a restriction on the possible lifts from the phase to the GLSM, singling out subcategories of the GLSM. The possible choices compatible with the band restriction rule correspond to the different homotopy classes of paths in moduli space. In subsequent work, the result was confirmed by arguments using analyticity of the hemisphere partition function [50], extended to the non-Calabi-Yau case in [22] and to hybrid models in [62], see also [64, 61] for some results on the non-Abelian case. The grade restriction rule also inspired mathematicians to construct equivalences between D-brane categories, see [75, 31]. Our results are in complete agreement with the band restriction rule, where our choice of cutoff parameters precisely match with the bands appearing in [47].

Structure of this Dissertation

This thesis is structured as follows: The main part consists of two chapters. In Chapter 2 we review the theoretical framework we will be building on and introduce our construction of lifting defects. We then apply our construction to various examples in Chapter 3.

Chapter 2 begins in Section 2.1 with a review of TQFTs starting with oriented, closed TQFTs and ending with TCFTs, which are the TQFTs we will be studying in this thesis. We kick off this review with a brief retrospective of the field theoretic perspective on TQFTs followed by the axiomatic approach. Next, we introduce defects in two-dimensional TQFTs which formalize phase transitions in GLSMs. The topological subsectors of GLSMs we will look at are not captured by plain TQFTs, but by TCFTs which can be understood as a particular type of fully extended TQFT. We discuss the necessity of extended TQFTs for higher dimensional TQFTs which, in turn, leads to fully extended TQFTs as their generalization. By considering a homotopical version of CFTs we end up with TCFTs which can be understood as a special kind of fully extended TQFT. These are the formal frameworks describing the A- and B-model of a GLSM.

Following our discussion on TQFTs is an overview of the topological B-model for GLSMs in Section 2.2. Here, we discuss how B-branes in a GLSM can be described by a TCFT. We begin by giving a lightning review on the field theoretic perspective of the GLSM. In this discussion we will see how a GLSM serves as a common UV description of various IR phase. We follow up the discussion of the GLSM by introducing the topological A- and B-twists which yield the A- and B-models for GLSMs, respectively. Next, we present a class of GLSMs which individually encompass all minimal models up to a fixed level k. Later on, we use these GLSMs to construct all flows between minimal models. We then discuss B-type boundary conditions in GLSM and their phases and describe how they are captured by TCFTs.

Afterwards in Section 2.3, defects are introduced and we describe their general properties and how they are able to capture phase transitions in QFTs. We then explain the construction for defects lifting orbifold phases. They are obtained by pushing down the GLSM identity defect to the orbifold phase on its right and subsequently introducing a charge cut off. Concluding this chapter, we compare our construction to the grade- and band restriction rules from [47].

In Chapter 3, we apply our construction to a number of examples and compare our results to the ones from [47] and [13]. First, we discuss non-anomalous GLSMs in Section 3.1. We begin by investigating two GLSMs with gauge group $\mathrm{U}(1)^2$: the A_2 model and a model with $\mathbb{C}^5/\mathbb{Z}_8$-orbifold phase. We then study the general A_{N-1} model.

Finally, in Section 3.2 we turn our attention to the class of anomalous GLSMs which capture all minimal models. Applying our construction for lifting defects, we reproduce all flows between minimal models from [13].

Relevant Publications

Most of this thesis builds upon or is adopted with minor changes from [14] with permission of the coauthors. Parts of the Introduction are taken from [14]. In Chapter 2, parts of Sections 2.2 and 2.3 can be found in a similar form in [14]. Chapter 3 is mostly taken from [14] with minor changes to the text and presentation.

2 Topological Quantum Field Theories and Gauged Linear Sigma Models

In this chapter, the notion of a topological quantum field theory is introduced. We discuss the 'hands-on definition' of a TQFT originating from theoretical physics on the one hand, as well as the mathematical axiomatic definition of a TQFT on the other hand and sketch how the former motivates the latter. Subsequently, we discuss alterations and generalizations of TQFTs such as defect, (fully)-extended TQFTs and TCFTs.

Next, the concept of a gauged linear sigma model is introduced which is the main focus of this thesis. Performing a topological twist of a GLSM yields a topological theory whose low energy phases are realizations of a certain type of the aforementioned TQFTs, namely TCFTs. We discuss how minimal models–here, a certain type of superconformal field theory–can be described within the framework of GLSMs. Then, we describe the brane categories for GLSMs. Subsequently, we introduce defects in GLSMs and present the main result of this thesis: the construction of orbifold lifts in GLSMs. To finish this chapter, we contextualize our construction by comparing it to the grade- and band restriction rules introduced in [47].

2.1 Topological Quantum Field Theories

Before getting to the axiomatic definition of a topological quantum field theory we begin by rather informally discussing the 'hands-on definition'. In physics, a classical field theory describes the dynamics of classical fields in a fixed spacetime M[1]. A classical field is a smooth section of a bundle $E \xrightarrow{\pi} M$ i.e.

$$\phi \in \Gamma(M, E).$$

In case of a trivial bundle $E = X \times M$ for some space X the sections and, therefore, the fields can be identified with maps $\phi \in \mathrm{Hom}(M, X)$. Such a theory is called a *sigma model*. We will exclusively be concerned with sigma models in this thesis. In the Lagrangian description, the dynamics of the fields are determined by the choice of a density

$$L : J^\infty(E) \to \Omega^{\dim(M)} M,$$

[1]For simplicities sake, in the following we will omit giving precise definitions for spaces here and just take them to be some kind of smooth manifold.

where $J^\infty(E)$ denotes the jet bundle on E and L is called the *Lagrangian.* The equations of motion are then obtained via optimization of the *action functional*

$$S : \Gamma(M, E) \to \mathbb{R},\ \phi \mapsto \int_M L(\phi, \partial\phi, \partial^2\phi, \dots).$$

Quantum field theories are theories that arise from classical field theories through the process of quantization. Given a classical field theory we may quantize it by introducing the *path integral*

$$Z \coloneqq \int_{\Gamma(M,E)} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]},$$

which is called the *partition function* on M. The idea here is that when calculating expectation values in a quantum theory we need to account for all contributions of intermediate states i.e. field configurations by integrating the appropriate 'measure' $\mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}$ over all field configurations. This expression is merely symbolic and can in general only be made sense of in perturbation theory–as an asymptotic series in $\hbar \to 0$–or for particularly simple or constrained theories. However, whenever it is possible to assign meaning to this symbolic expression we can calculate correlation functions

$$\langle \mathcal{O}_1 \dots \mathcal{O}_m \rangle \coloneqq \frac{1}{Z} \int_{\Gamma(U, J^\infty(E|_U))} \mathcal{D}\phi \mathcal{O}_1 \dots \mathcal{O}_m e^{\frac{i}{\hbar}S[\phi]}$$

for dimension-n operators i.e. functionals on $\Gamma(U, J^\infty(E|_U))$ where $U \subseteq M$ is a n-dimensional subspace (e.g. $n = 1$ for line operators).

A quantum field theory is called *topological* iff all correlation functions are invariant under isomorphisms of M. Thus, in a TQFT correlation functions are topological invariants of M and we may think of a TQFT as a framework to either calculate correlation functions or to determine topological invariants.

2.1.1 Atiyah-Segal Quantum Field Theories

The ideas leading towards the modern axiomatic definition of a TQFT were first put forward in the seminal papers by Atiyah and Segal [5, 76]. We will proceed by giving a heuristic motivation of the axiomatic definition from the path integral approach to QFT. A more thorough and technical discussion of this matter can be found in [20].

Consider a QFT on a spacetime M with boundary i.e. $\partial M = B \neq \emptyset$. To calculate the path integral we need to specify the boundary conditions of the fields by fixing them to be some boundary field $\varphi \in \Gamma(B, E|_B)$. The path integral then reads

$$Z(M, \varphi) = \int_{\phi \in \Gamma(M,E),\ \phi|_B = \varphi} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}.$$

By identifying the space of states on the boundary with the Hilbert space $\mathcal{H}_B$ of (linear) functionals on the fields on B we find that the path integral actually defines a vector

$$Z(M, \bullet) \in \mathcal{H}_B.$$

Now, assume that the boundary can be decomposed into 'ingoing' and 'outgoing' boundary components $\partial M = B_{in} \sqcup B_{out}$. In this case we expect that $Z(M, \bullet) \in \mathcal{H}_{B_{in}} \otimes \mathcal{H}_{B_{out}}$ since the Hilbert spaces are ought to be independent for disjoint subspaces–in fact this holds true for local action functionals. Thus, we may identify the path integral with a (anti-)linear map[2]

$$Z(M, \bullet) : \mathcal{H}_{B_{in}} \to \mathcal{H}_{B_{out}}.$$

Suppose M is a closed manifold being composed of two manifolds M_1 and M_2 glued along a common codimension one manifold N, i.e. $M = M_1 \cup_N M_2$ we expect to get the following partition function

$$Z(M) = \int_{\varphi \in \Gamma(N, E|_N)} \mathcal{D}\varphi Z(M_1, \varphi) Z(M_2, \varphi).$$

Summarizing, given a space M a QFT assigns to each of its boundary components a Hilbert space of boundary states (by abuse of notation, to the empty boundary it assigns $\mathcal{H}_\emptyset := \mathbb{C}$), to the disjoint union of boundaries the tensor product of the respective state spaces and to M a linear map between the respective spaces of states. Additionally gluing of different spacetime patches amounts to composition of the corresponding linear maps.

These features can be repackaged into the *Atiyah-Segal type* axiomatic definition of a TQFT. We also refer to this as the *functorial* definition of a TQFT. Note that there are also other axiomatic approaches to defining QFTs such as *factorization algebras* coined by Costello and Gwilliam [26].

Definition 2.1.1 (Oriented Closed TQFT)
An n-dimensional oriented closed TQFT is a symmetric monoidal functor $Z : \mathrm{Bord}_n \to \mathrm{Vect}_k$ from the category of oriented n-bordisms to the category of k-vector spaces, for some field k.

To unpack the definition of an oriented closed TQFT we will now describe the categories Bord_n and Vect_k.

The category of k-vectors spaces Vect_k has as objects k-vector spaces and as morphisms k-linear maps. Composition of morphisms is given by composition of k-linear maps. The monoidal structure of Vect_k is given by the tensor product of vector spaces with monoidal unit k and the obvious associator, left and right unitors and symmetric structure.

The category of oriented n-bordisms Bord_n has as objects real smooth oriented closed $(n-1)$-manifolds. Morphisms in Bord_n are bordisms up to orientation-preserving diffeomorphisms of bordisms. Composition of a pair of morphisms $B_1 \to B_2$ and $B_2 \to B_3$ is given by gluing along the common boundary B_2. The monoidal structure is given by disjoint union with monoidal union given by the empty boundary $\emptyset$ and the obvious associator, left and right unitors and symmetric structure. For more details on these definitions cf. [20].

Note that by considering bordisms up to orientation-preserving diffeomorphisms we omit the geometrical structure of the bordisms as is required for the theory to be called

[2]The map is anti-linear due the opposite orientation of in- and outgoing boundaries.

topological. From now on we will be only considering oriented TQFTs and therefore drop the prefix 'oriented'.

With the functorial definition of a TQFT we essentially applied the idea of Feynman diagrams, which is to translate diagrams into mathematical expressions for correlation functions. In order to classify TQFTs we may now ask what algebraic properties the definition of a TQFT entails. We do so by investigating how $(n-1)$-dimensional manifolds are related and paired via bordisms i.e. Feynman 'graphs'.

It turns out that two-dimensional TQFTs are completely classified in terms of Frobenius algebras. This observation is captured by the following classification originally described by Dijkgraaf [30], more details on this can again be found in [20].

Theorem 2.1.1 (2D Closed TQFTs are Frobenius Algebras)
There is an equivalence of groupoids

$$\mathrm{Fun}_{\otimes,\mathrm{Sym}}(\mathrm{Bord}_2, \mathrm{Vect}_k) \to \mathrm{comFrob}_k$$

between the category of TQFTs and the category of commutative frobenius algebras.

We state this result here merely to familiarize us with classification results of functorial field theories in this simple setting before discussing generalizations which are relevant to this thesis.

There are different possibilities to generalize or alter the definition of a TQFT to account for various features that a QFT might exhibit. For instance one can equip the bordisms with additional structures such as a Riemannian structure or a conformal structure to describe euclidean QFTs or conformal field theories (CFTs) respectively cf. [76]. One can also consider higher categorical generalizations to probe homotopical or homological properties cf. [65]. We will explore examples for both these possibilities in the subsequent subsections.

2.1.2 Two-Dimensional Defect TQFTs

In the previous subsection we introduced the notion of a closed TQFT. However, for our purposes we need to familiarize ourselves with defects. To this end we will now introduce defect TQFTs. We will follow [17] but omit all technical details.

A defect is a codimension one subspace separating an ambient space into two subregions together with gluing conditions for the physical data attached to the two regions adjacent to the defect. Note that in general there can also be defects on defects which leads to a stratification of the ambient space. Defects conceptually unify various different concepts: operator insertion, symmetries, phase transitions and boundaries–which can be thought of as being defects separating spacetime from an space with no physical data attached to it. We will elaborate on some of these later on.

To incorporate defects into the definition of a TQFT we have to consider the category of stratified bordisms and decorate it according to the *defect data* $\mathbb{D}$. The defect data consists of labels for the strata encoding some physical data attached to it and information on how labeled strata are allowed to meet. More formally for two-dimensional TQFTs i.e. $n = 2$ we have that $\mathbb{D}$ is given by a tuple

$$\mathbb{D} = (D_1, D_2, s, t),$$

where D_1 and D_2 are sets labeling defect-lines and two-dimensional regions respectively. The *source-* and *target maps* $s, t : D_1 \to D_2$ signify which regions labeled by D_2 may border D_1 labeled defect-lines. Resulting from this procedure we get the category of two-dimensional defect bordisms $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D})$. For the precise definition of $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D})$ cf. [17]. We then have the following definition [28].

Definition 2.1.2 (2D Defect TQFT)
A two-dimensional defect TQFT is a symmetric monoidal functor $Z : \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_k$ from the category of two-dimensional defect bordisms to the category of k-vector spaces.

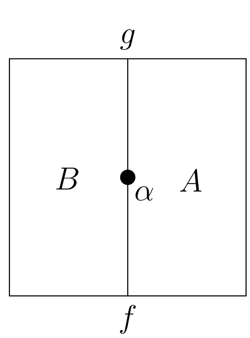
If we choose for the defect data $D_1 = \emptyset$ and for D_2 a singleton, we recover the definition of a two-dimensional closed TQFT. Similarly to the case of closed TQFTs we may again ask whether there is an algebraic structure of the state spaces to represent defect TQFTs. Indeed we have the classification.

Theorem 2.1.2 (2D Defect TQFTs are Pivotal 2-Categories)
For every two-dimensional defect TQFT $Z : \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_k$ one can construct a pivotal 2-category $\mathcal{B}_Z$.

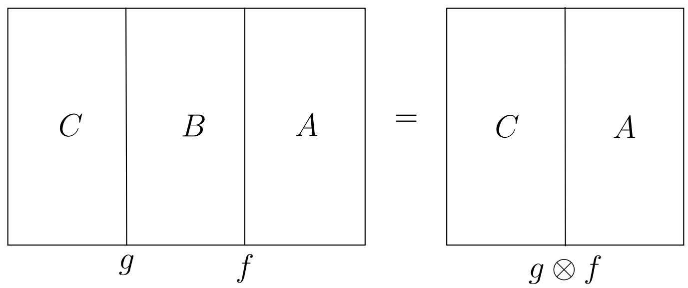
We will now introduce pivotal 2-category categories and *string diagrams* which will be used later on in this thesis to describe defects in gauged linear sigma models. By definition a (strict) 2-category $\mathcal{C}$ is a category enriched over the category of small categories. This means for any two objects A, B there is a category $\mathcal{C}(A, B)$ whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. Being enriched over the category of small categories, $\mathcal{C}$ exhibits functors

$$\otimes : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

which are called horizontal composition or composition along objects. The composition in $\mathcal{C}(A, B)$ is called vertical composition or composition along 1-morphims. There is a graphical calculus for morphisms in 2-categories given by string diagrams. A string diagram is a two-dimensional depiction in which regions represent objects of $\mathcal{C}$ edges represent 1-morphisms and nodes 2-morphisms. For instance



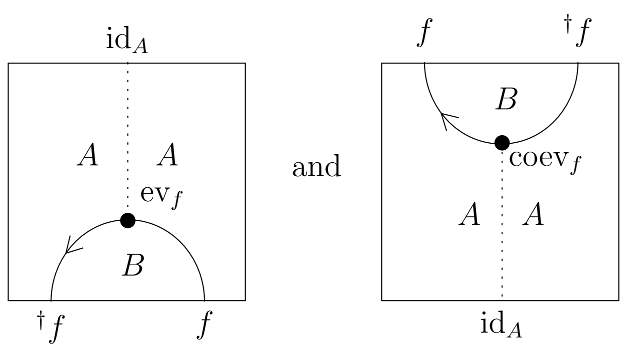
depicts the 2-morphism $\alpha : (f : A \to B) \Rightarrow (g : A \to B)$ and



depicts horizontal composition of 1-morphisms, where we adopt the convention to read from right to left and from bottom to top. Note that these diagrams already look like they describe defects–as well as their fusion–except that the edges do not have an orientation. This additional data of the string diagrams follows from the pivotality of $\mathcal{C}$. In a pivotal 2-category every 1-morphism f has a left and right adjoint ${}^{\dagger}f$ and $f^{\dagger}$ giving rise to the evaluation and coevaluation 2-morphisms

$$\mathrm{ev}_f : f \otimes {}^{\dagger}f \to \mathrm{id}_A \quad \text{and} \quad \mathrm{coev}_f : \mathrm{id}_A \to {}^{\dagger}f \otimes f$$

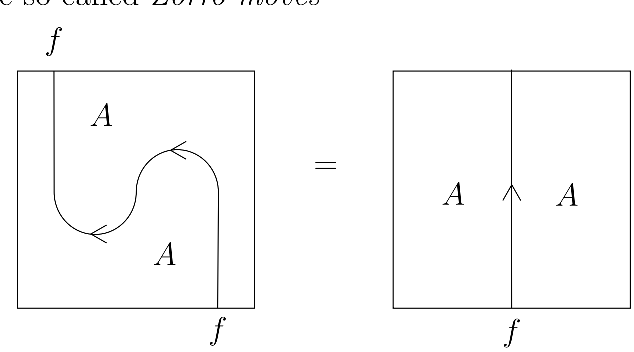
with corresponding string diagrams given by



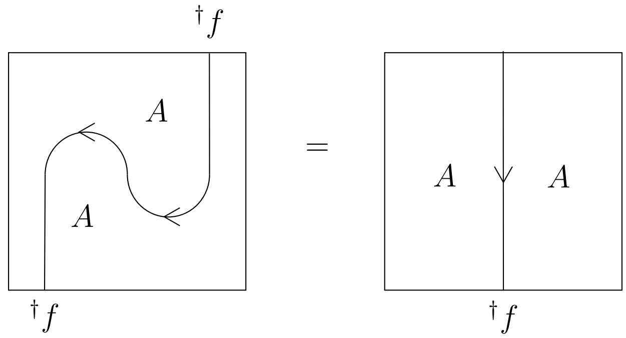
where dashed lines represent the identity 1-morphism, which by definition is the identity under fusion and is also part of the datum of a pivotal category. The evaluation and coevaluation are subject to the conditions

$$(\mathrm{id}_f \otimes \mathrm{ev}_f) \circ (\mathrm{coev}_f \otimes \mathrm{id}_f) = \mathrm{id}_f \quad \text{and} \quad (\mathrm{ev}_f \otimes \mathrm{id}_{{}^{\dagger}f}) \circ (\mathrm{id}_{{}^{\dagger}f} \otimes \mathrm{coev}_f) = \mathrm{id}_{{}^{\dagger}f}$$

which translate to the so called *Zorro moves*



and



in the diagrammatical language, where we neglected the identity 1-morphisms.

There are similar maps for the right adjoints, however in a pivotal category left and right adjoints are identical which reflects the fact that taking the adjoint merely corresponds to orientation reversal on the defect line.

We have stated that there is various ways of composing morphisms in a pivotal 2-category. From the perspective of a TQFT composition of these morphisms amounts to moving the corresponding defects arbitrarily close together, which can be done due the fact that the theory is topological. Merging two defects in this manner is called *fusion*.

2.1.3 (Fully)-Extended TQFTs

In this subsection we will briefly discuss extended TQFTs which will not be of immediate relevance for this thesis but serve as a bridge from TQFTs to TCFTs.

So far we have exemplified TQFTs only by two-dimensional theories. For the classification of these theories we asked for the algebraical structure on the state spaces. This analysis utilized that in two-dimensions the objects in Bord_2 are tensor-generated by just a single object, the circle S^1. Thus, the state spaces are as well generated by a single space $\mathcal{H} := Z(S^1)$. The algebraic structure on these spaces is then given by a finite set of conditions on these generators.

Considering 3D TQFTs for contrast we observe that there are infinitely many connected closed 2-manifolds. Accordingly the objects in Bord_3 are not tensor generated by a finite set of objects and thus we cannot hope to describe the algebraic structure on the state spaces by a finite amount of data[3]. This observation indeed remains true in any dimension ≥ 3.

To deal with this complication we may however extend the category Bord_n to a higher category. In three dimensions we do this by introducing the category $\mathrm{Bord}_{3,2,1}$, whose objects are oriented closed 1-manifolds, 1-morphisms are compact oriented bordisms between objects and 2-morphisms are compact oriented bordisms with corners between 1-morphisms up to diffeomorphisms.

[3]Note however that it is still possible to classify these TQFTs by more sophisticated algebraic structures–so called *J-algebras*–cf. [53].

For the target category there are several choices for a generalization of Vect_k, we simply call these *2-vector spaces* and denote them by $2\,\mathrm{Vect}_k$.

We have the following classification result due to Theorem 2 in [7], for more details see there.

Theorem 2.1.3 (Extended 3D TQFTs are MTCs)
Extended 3D TQFTs $Z : \mathrm{Bord}_{3,2,1} \to 2\,\mathrm{Vect}_k$ *are classified by non-anomalous modular tensor categories.*

Modular tensor categories can be constructed from 3D TQFTs by different means. The Turaev Viro method [80] relies on the fact that under mild conditions any n-manifolds admits a triangulation. It assigns a state space to each 1-simplex and infers their algebraic structure from homeomorphisms of the triangulations–the so called *Pachner moves*. In fact this procedure can in principle be performed in arbitrary dimensions, albeit being increasingly difficult with growing number of dimensions.

Generally speaking in higher dimensions the category of bordisms and the target category of Z becomes harder to define. It turns out that performing an additional extension of these categories actually leads to a setting which is better manageable and allows for a general classification conjecture. We perform this extension by considering the top-dimensional bordisms only up to homotopy i.e. passing to (∞, n)-categories.

The (∞, n)-category of bordisms $\mathrm{Bord}_{(\infty,n)}$ can be thought of as having disjoint unions of oriented points as objects, bordisms of $(m-1)$-manifolds as m-morphisms for $m \leq n$ and diffeomorphisms between n-bordisms, smooth homotopies of diffeomorphisms and so forth as m-morphisms for $m > n$. Note that $\mathrm{Bord}_{(\infty,n)}$ also admits a symmetric monoidal structure.

This is an informal definition but the advantage of working with (∞, n)-categories is that such definitions can be made precise by *models* of well controlled mathematical structures such as *n-fold complete Segal spaces*. A fully extended TQFT is then defined as follows.

Definition 2.1.3 (Fully Extended TQFT)
Let C be a symmetric monoidal (∞, n)-category, a fully extended TQFT is a symmetric monoidal functor $Z : \mathrm{Bord}_{(\infty,n)} \to C$.

A proper definition of $\mathrm{Bord}_{(\infty,n)}$ and Z can be found in [65]. The classification of fully extended TQFTs was conjectured by Baez and Dolan [6] and goes by the name cobordism hypothesis.

Theorem 2.1.4 (Cobordism Hypothesis)
Every fully extended TQFT $Z : \mathrm{Bord}_{(\infty,n)} \to C$ *is equivalent to a fully dulizable object in C given by evaluation of Z on a point.*

We will not be working with these results but only use them to contextualize the notion of TCFTs which will be introduced in the next subsection.

2.1.4 TCFTs

We will now introduce topological conformal field theories (TCFTs), we mainly follow [24]. In terms of physics TCFTs formalize A- and B-twisted superconformal field theories to be defined in Section 2.2.2. As their name suggest, TCFTs are topological field theories derived from CFTs. This is done by considering said CFTs 'up to homotopy'.

According to [76] we have following definition.

Definition 2.1.4 (Closed CFT)
A closed CFT is a symmetric monoidal functor $Z : \mathcal{R}_{\text{punct}} \to \mathrm{Vect}_k$ from the category of punctured Riemann surfaces to the category of k-vector spaces.

The objects of $\mathcal{R}_{\text{punct}}$ are finite sets. For a pair (I, J) of objects morphisms are punctured Riemann surfaces with 'ingoing' and 'outgoing' punctures being labeled by I and J respectively. We additionally impose that each connected component has at least one ingoing puncture. Composition in $\mathcal{R}_{\text{punct}}$ is given by gluing.

To pass to a homotopical version of a CFT first consider the symmetric monoidal functor $C_* : \mathrm{Top} \to \mathrm{Ch}_k$ from the category of topological spaces to the category of chain complexes of k-vector spaces. The functor C_* maps topological spaces to their homology chain complexes. Define the category $\mathcal{R}^{\text{Ch}}_{\text{punct}}$ to be the category whose objects are the objects of $\mathcal{R}_{\text{punct}}$ and whose hom-sets are given by

$$\mathrm{Hom}_{\mathcal{R}^{\text{Ch}}_{\text{punct}}}(I, J) = C_*(\mathrm{Hom}_{\mathcal{R}_{\text{punct}}}(I, J)).$$

We then have the following definition due to [44].

Definition 2.1.5 (Closed TCFT)
A closed TCFT is a symmetric monoidal functor $Z : \mathcal{R}^{\text{Ch}}_{\text{punct}} \to \mathrm{Ch}_k$.

For more details on the definition of a TCFT cf. [24]. This definition can also be modified by allowing intervals for boundaries of the Riemann surfaces to describe open TCFTs. One then introduces a set Λ of *D-branes* labeling these boundaries. In [24] following classification result was proven.

Theorem 2.1.5 (Open TCFTs are Calabi-Yau A_∞-Categories)
The category of open TCFTs with D-branes Λ is homotopy equivalent to the Calabi-Yau A_∞-category with objects Λ. Given a open TCFT Z this equivalence is given by evaluating $Z([0, 1])$.

We make two remarks to put this result into context. Firstly, being Calabi-Yau is a weaker notion than the one of fully dualizability, meaning that we cannot obtain this classification from the cobordism hypotheses mentioned above. It is however possible to formulate a generalized 'noncompact version' of the cobordism hypotheses to account for this circumstance and view TCFTs as a special type of fully extended TQFT [65].

Secondly, prototypical examples of Calabi-Yau A_∞-categories arise from the derived category of coherent sheaves on algebraic varieties. As we will describe in Section 2.2.4 derived categories naturally relate to D-branes in the B-model. Another class of examples is given by Fukaya categories associated to symplectic manifolds. These categories describe D-branes in the A-model.

2.2 The Topological B-Model for GLSMs

In this section we give a lightning review of $\mathcal{N} = (2,2)$ gauged linear sigma models in $1+1$ dimensions. We discuss the moduli spaces of GLSMs which provide a common framework for certain types of QFTs as their UV-completion [86], in particular this does also include minimal models.

Subsequently we review the topological A- and B-twists of $\mathcal{N} = (2,2)$ theories in $1+1$ dimensions. In case of the GLSMs these twists yield a description of topological strings.

We then introduce a class of GLSMs which encompasses all minimal model up to a fixed level k.

Lastly we will discuss the boundary conditions in the B-twisted GLSM and in the theories that constitute IR limits of the former.

2.2.1 Gauged Linear Sigma Models

For the subsequent discussion we mainly follow [51, 48], for a thorough introduction to supersymmetric field theories see [29]. Throughout this thesis we consider $1+1$ dimensional $\mathcal{N} = (2,2)$ *gauged linear sigma models* (GLSM for short). These theories can be considered as the worldsheet description of a superstring.

Since we are ultimately interested in a topological sector of the theory, the topology–and by extension the geometry–of the worldsheet does not enter the datum of the GLSM. For the discussion of general features of the theory we may therefore always restrict our attention to the flat Minkowski space $\mathbb{R}^{1,1}$. Having developed the general structure of the theory then allows for the computation of correlation functions on arbitrary surfaces.

The bulk theory of a GLSM is specified by the data

$$(G, V, (r,\theta), W),$$

where

- G is a compact Lie group
- $\rho : G \to \mathrm{GL}(V)$ is a faithful unitary representation of G
- $(r,\theta) \in \mathbb{R}^{\dim(G)} \times \mathbb{R}^{\dim(G)}$ are parameters such that $\exp(r_j + i\theta_j) \in \mathrm{Hom}(\pi_1(G), \mathbb{C}^*)^{\mathrm{Ad}_G}$
- $W \in \mathrm{Sym}(V^\vee)^G$ is a G-invariant polynomial.

We may identify $(r,\theta) \in \mathfrak{z}_{\mathbb{C}}^\vee \times \mathfrak{t}_{\mathbb{C}}^\vee$, where $\mathfrak{z}$ and $\mathfrak{t}$ are the Lie-algebras of the center Z_G and that of a maximal torus $T \subset G$ respectively. Define $t := r + i\theta$, this yields a linear twisted superpotential $\widetilde{W} := -t \in \mathrm{Sym}(\mathfrak{g}_{\mathbb{C}}^\vee)^{\mathrm{Ad}_G}$. In terms of physics, this data has the following interpretation

- G is the gauge group
- V is the chiral matter content
- (r, θ) are the Fayet-Iliopoulos- (FI) and θ-parameters, the former determining the IR-behavior of the theory the latter singularities of the moduli space
- W is the superpotential.

The definition of a Lagrangian also necessitates the choice of a G-invariant norm $\frac{1}{e^2}\langle .,.\rangle_{\mathrm{Ad}_G}$ on $\mathfrak{g}$ and a G-invariant hermitian inner product on V, where e is the coupling constant. The inner product on V induces a G-invariant smyplectic structure with momentum map $\mu : V \to \mathfrak{g}^\vee$. We assume that there is a vector R-symmetry $R : \mathrm{U}(1)_V \to \mathrm{GL}(V)$ such that the superpotential has R-charge 2.

The classical vacua of a GLSM are determined by the vanishing locus of the scalar potential

$$U(\sigma, X) = \frac{1}{8e^2}[\sigma, \bar{\sigma}]^2 + \frac{1}{2}\left(|\sigma X|^2 + |\bar{\sigma} X|^2\right) + \frac{e^2}{2}(\mu(X) - r)^2 + |dW(X)|^2$$

where σ is the scalar component of the vector multiplet–which we later on assume to be decoupled–and X is the scalar component of the chiral superfield. In this thesis we will be working exclusively with abelian gauge groups. Picking $G = \mathrm{U}(1)^n$ and $V = \mathbb{C}^k$ the scalar potential reads

$$U(\sigma, X) = \sum_{i=1}^{k}\left|\sum_{a=1}^{n} Q_{ai}\sigma_a X_i\right|^2 + \frac{e^2}{2}\sum_{a=1}^{n}\left(\sum_{i=1}^{k} Q_{ai}|X_i|^2 - r_a\right)^2 + \sum_{i=1}^{k}\left|\frac{\partial W}{\partial X_i}\right|^2,$$

where $Q_{ai} \in \mathfrak{u}(1)^\vee_{\mathbb{C}}$ are the gauge charges of X_i. Since each term in $U(\sigma, X)$ is positive they have to vanish independently. Vanishing of the first term in $U(\sigma, X)$ requires σ to lie in the stabilizer subgroup of X, whereas the vanishing locus of the last two terms defines the *vacuum manifold* given by

$$\mathrm{Crit}(W) \cap \{\mathbb{C}^k - \Delta_r\}/G_{\mathbb{C}}.$$

Here, $\mathrm{Crit}(W)$ is the vanishing locus of the F-term $\sum_{i=1}^{k}|\partial W/\partial X_i|^2 = 0$. The expression $\{\mathbb{C}^k - \Delta_r\}/G_{\mathbb{C}}$ is the the GIT quotient where $G_{\mathbb{C}}$ is the complexified gauge group and Δ_r is the set of $G_{\mathbb{C}}$-orbits that do not contain solutions to the D-term equation.

A phase of a GLSM is a domain in the FI-parameter space where the space of solutions to the D-term equation

$$\sum_{i=1}^{k} Q_{ai}|X_i|^2 - r_a = 0$$

is of maximal dimension n. Accordingly r lies on a phase boundary if the space of solutions to the D-term equation is $(n-1)$-dimensional. More generally, it is $n-m$-dimensional at the intersection of m phase boundaries.

The boundaries between different phases in parameter space can be described by positive cones of charge vectors: For any $I \subset \{1, \ldots, k\}$ we can associate the positive cone

$$\mathrm{Cone}_I = \left\{ \sum_{i \in I} \lambda_i Q_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \ \forall i \in I \right\} \subset \mathbb{R}^n \tag{2.1}$$

inside the FI parameter space. The cones associated to sets I of cardinality $|I| = n-1$ such that the Q_i, $i \in I$ are linearly independent (at least classically) describe the boundaries between different phases, see e.g. [22]. Along each such phase boundary the unbroken gauge group, which is the stabilizer of all the X_i, $i \in I$ contains a single $U(1)$.

Non-zero vaccum expectation values of the fields X_i give masses to the scalar component of the vector multiplet via the term

$$\sum_{i=1}^{k} \left| \sum_{a=1}^{n} Q_{ai} \sigma_a X_i \right|^2 .$$

As a consequence, in a phase the gauge group is broken to a discrete subgroup via the Higgs mechanism.

On the phase boundary there are Coulomb branches which render the phase boundaries singular. However on the quantum level these singularities get lifted and only discrete points on the phase boundary–whose locations are determined by the θ-parameter–remain singular [47]. We will elaborate further on this when discussing the grade and band restriction rules later on.

A *geometric phase* is characterized by a complete breaking of the gauge group and the fact that all modes transverse to the vacuum manifold $U(\sigma, X) = 0$ are massive. If the superpotential is non-vanishing some of the modes transverse to the vacuum manifold are massless. At sufficiently low energies the massive fields decouple and the theory reduces to a non-linear sigma model on the vacuum manifold. All non-vanishing masses are proportional to $e\sqrt{|r|}$, thus large e and $|r|$ can be considered to be describing the low energy regime.

For non-vanishing superpotentials $W \neq 0$ some modes transverse to the vacuum manifold are massive, a *Landau-Ginzburg phase* is a phase where the vacuum manifold is a single point and all transverse modes are massless. At low energies a Landau-Ginzburg phase is described by a Landau-Ginzburg theory or a Landau-Ginzburg orbifold theory in case there is a residual discrete gauge group.

Even though we so far only made classical considerations the statements regarding the phase structure asserted above remain valid in the quantum theory. However, the FI parameter may be subject to renormalization. To be more precise in case the *Calabi-Yau condition*

$$\sum_{i=1}^{k} Q_{ai} = 0 \ \forall a$$

is met the FI parameters are not affected by renormalization and are thus genuine parameters of the theory. In this 'non-anomalous' case the FI parameters are exactly marginal and capture honest deformations. The theory comes with a Kähler moduli space, and a phase transition corresponds to a marginal deformation. If the Calabi-Yau condition is not satisfied the FI parameters undergo a non-trivial renormalization group flow and the respective perturbations are relevant.

2.2.2 A- and B-Twists in (2,2) Models

As alluded to above we are ultimately interested in topological sectors of GLSMs and we may therefore restrict our attention to Minkowski space $\mathbb{R}^{1,1}$, which we will do in the following discussion. The topological sector is attained by the means of a so called topological twist, the procedure presented here was first introduced in [87]. In the following we will discuss this twist formally in the general case of $\mathcal{N} = (2, 2)$ theories in $1 + 1$ dimensions, we follow [51]. An introduction to this topic can be found in [82].

The super Poincaré algebra of $\mathbb{R}^{1,1}$ consists of the symmetry operators summarized in Table 2.1 below acting on the Hilbert space of states.

$\mathbb{Z}_2$-degree	0	1
	H, P, M, F_V, F_A	$Q_+, Q_-, \bar{Q}_+, \bar{Q}_-$

Table 2.1: Operators of the $\mathcal{N} = (2, 2)$ super Poincaré algebra of $\mathbb{R}^{1,1}$ and their $\mathbb{Z}_2$-degree.

The Hamiltonian H and momentum P form a vector with respect to Lorentz transformations M

$$[iM, H \pm P] = \mp 2(H \pm P),$$

where the imaginary unit enters due to Wick rotation and the supercharges $Q_\pm, \bar{Q}_\pm$ are spinors

$$\begin{aligned}[iM, Q_\pm] &= \mp Q_\pm \\ [iM, \bar{Q}\pm] &= \mp \bar{Q}_\pm.\end{aligned}$$

F_V and F_A are vector- and axial R-charges, at least one of which has to be a symmetry of the theory for the purpose of twisting as we will see subsequently. The supercharges satisfy

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P,$$

with all other brackets among them vanishing. The R-charges act on the supercharges via phase rotation

$$\begin{aligned}[F_V, Q_\pm] &= -Q_\pm \\ [F_V, \bar{Q}\pm] &= \bar{Q}_\pm \\ [F_A, Q_\pm] &= \mp Q_\pm \\ [F_A, \bar{Q}_\pm] &= \pm \bar{Q}_\pm.\end{aligned}$$

Now, define

$$Q_A \coloneqq \bar{Q}_+ + Q_- \quad \text{and} \quad Q_B \coloneqq \bar{Q}_+ + \bar{Q}_-.$$

We have that the pairs (Q_A, F_A) and (Q_B, F_V) satisfy

$$Q_{A/B}^2 = 0$$
$$[F_{A/V}, Q_{A/B}] = Q_{A/B}$$
$$[iM, Q_{A/B}] = -Q_{A/B}.$$

The nilpotency of $Q_{A/B}$ implies that this linear combination of supercharges defines a differential on the space of states and by extension on the space of local operators. Since the supercharges generate spacetime translations taking the cohomology with respect to this differential therefore renders the theory topological. There is however a caveat to this construction. On an arbitrary spacetime there does not necessarily exist covariantly constant spinors meaning that the supercharges would be subject to coordinate transformations. However, by modifying the Lorentz transformations according to

$$iM \mapsto iM + F_{A/V}$$

we find that the supercharges become Lorentz-scalars and thus the algebra is independent of the spacetime background. This procedure is called *topological twisting* and we call the twist of a GLSM by F_A the *topological A-model* and the twist by F_V the *topological B-model*.

Now, to describe spacetimes with boundaries consider a half-plane in Minkowski space $\mathbb{R}^{1,1}$. It turns out that the supercharges Q_A and Q_B together with their complex conjugates respectively comprise the maximal sets of supercharges that can be preserved by boundary conditions. Boundary conditions preserving Q_A and $Q_A^\dagger$ are called *A-branes* and *B-branes* if they preserve Q_B and $Q_B^\dagger$. The supercharges Q_A and Q_B act as differential on local operators on the boundary as well as on local operators insertions between two boundary components. We can then define the *categories of boundary conditions* $\mathcal{C}_{A/B}$ for A-branes and B-branes whose objects are boundary conditions and morphisms are cohomology classes of operator insertions between objects.

In fact the categories $\mathcal{C}_{A/B}$ encode the entire theory at hand and it turns out that they carry a Calabi-Yau A_∞-structure and thus represent TCFTs by Theorem 2.1.5. Conversely–on Riemann surfaces–TCFTs give rise to the A- and B-model. Namely in [84] it was argued that Chern-Simons theories are the effective background theories obtained from the A- and B-model and in [25] it was shown how to obtain Chern-Simons theories as effective background theories from a given TCFT.

The A- and the B-model are in one-to-one correspondence via a duality called *homological mirror symmetry* first conjectured by Kontsevich [63]. By now mirror symmetry is proven for various spaces cf. e.g. [45, 36, 1].

In this thesis we will restrict our attention to the B-model. We will discuss examples of categories of B-branes relevant for this thesis in Section 2.2.4.

2.2.3 Minimal Models

One notable observation about the structure of GLSMs is that it allows for the description of minimal models as low energy phases. In fact, as we will see, for every k there is a GLSM whose low energy phase structure encompasses all minimal models up to level k. We will exploit this fact later on when constructing all flow defects between minimal models.

The *minimal models* we are considering here are $1+1$-dimensional $\mathcal{N}=(2,2)$ theories which are minimal with respect to the superconformal algebra. This means they carry specific types of representations of the $\mathcal{N}=(2,2)$ superconformal algebra [54].

Minimal models admit a ADE classification cf. [16, 43, 42]. Here, we will restrict our attention to the A series of minimal models which are classified by a single number $k \in \mathbb{N}_0$, the *level* specifying the conformal weight. In [85] Witten demonstrated that A series minmal models can be obtained as the IR-fixed points of the RG flow of Landau-Ginzburg models $\mathcal{M}_k$ with a single chiral field X and superpotential $W = X^d$ for $d \in \mathbb{Z}_{\geq 2}$, where the level is given by $k = d-2$.

Minimal models exhibit relevant perturbations, which are captured by deformations of the superpotential of the respective Landau-Ginzburg model by lower degree monomials in X, i.e. they are given by

$$W = X^d + \lambda_1 X^{d-1} + \lambda_2 X^{d-2} + ... + \lambda_{d-2} X^2.$$

This implies that all minimal models $\mathcal{M}_k$ can be obtained as IR fixed points of relevant flows starting in minimal models $\mathcal{M}_{k_0}$ at levels $k_0 > k$. Along the flows, supersymmetric vacua become massive and decouple together with the A-type supersymmetric branes carrying the respective charges cf. [49, 66, 12]. Mirror symmetry interchanges A-type and B-type branes, so in the mirror model B-type branes decouple.

The mirror dual of a minimal model $\mathcal{M}_k$ is the $\mathbb{Z}_d$-orbifold of $\mathcal{M}_k$. Hence, mirror duals of minimal models can be obtained as IR fixed points of $\mathbb{Z}_d$-orbifolds of Landau-Ginzburg models with a single chiral field X and superpotential $W = X^d$. Here, the orbifold group acts on X by phase multiplication.

As we have seen the phases of a GLSM may be given by Landau-Ginzburg models or Landau-Ginuburg orbifolds. Thus, GLSMs can also capture minmal model. In fact for each k there exists a GLSM which contains the mirror duals to all minimal models up to level k as its phases [14]. This GLSM is given by

$$\mathrm{GLSM}_{\mathcal{M}_{d-2}} \coloneqq \left(\mathrm{U}(1)^{d-2}, \mathbb{C}^{d-1}, (r,\theta), W = \prod_{i=0}^{d-2} X_i^{d-i} \right),$$

where the representation $\mathbb{C}^{d-1}$ can be read of from the charge assignment of the chiral matter fields specified in Table 2.2 below.
Note that $U(1)_0$ is anomalous, whereas the other $U(1)_i$, $i > 0$ are non-anomalous. Hence the FI parameter of $U(1)_0$ has a non-trivial RG flow, whereas the other ones are honest parameters of the theory.

We observe that $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ exhibits $(d-1)$ phases. In each of these phases all but one of the chiral fields assume a non-trivial vacuum expectation value (VEV). We call the phase in which only the field X_i does not assume a VEV Phase$_i$. In the effective field theory describing this phase, only that field remains, the superpotential becomes $W = X_i^{d-i}$ and

	X_0	X_1	X_2	X_3	$\dots$	$\dots$	X_{d-3}	X_{d-2}
$U(1)_0$	$(d-1)$	$-d$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
$U(1)_1$	1	-2	1	0	$\dots$	$\dots$	$\dots$	0
$U(1)_2$	0	1	-2	1	0	$\dots$	$\dots$	0
$U(1)_3$	0	0	1	-2	1	0	$\dots$	0
$\vdots$	$\vdots$		$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$U(1)_{d-4}$	0	$\dots$	$\dots$	0	1	-2	1	0
$U(1)_{d-3}$	0	$\dots$	$\dots$	$\dots$	0	1	-2	1

Table 2.2: Charges of the chiral matter fields of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$.

the gauge group is broken to $\mathbb{Z}_{d-i}$. Thus, Phase_i is given by the Landau-Ginzburg orbifold whose IR fixed point is the mirror dual of the minimal model $\mathcal{M}_k$ at level $k = d - i - 2$. Therefore, $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ contains as phases all the minimal model orbifolds of levels up to $k = d - 2$.

Every such phase is separated from any other by a codimension-one phase boundary. The one separating phases i and j are located at the cones $\mathrm{Cone}_{\{k \notin \{i,j\}\}}$ consisting of the positive linear combinations of the charge vectors of all chiral fields X_k, $k \notin \{i, j\}$.

2.2.4 B-Branes in GLSMs

In this subsection we proceed by discussing the categories of B-branes for GLSMs and their phases. We begin by briefly discussing geometric phases i.e. non-linear sigma models followed up by a more detailed discussion of Landau-Ginzburg models and end up by describing the auxiliary definition of GLSM B-branes we employ in this thesis.

B-Branes in Non-Linear Sigma Models

For non-linear sigma models with target X the B-brane category is given by the (bounded) derived category of coherent sheaves

$$\mathcal{C}_B = \mathrm{D}^b(X).$$

The B-brane category for non-linear sigma models was first discussed in [63], even though at the time it was examined in the context of mirror symmetry and its connection to branes was not established. Later in [78] D-branes were described as objects in derived categories of coherent sheaves and in [27] the B-model was realized with derived categories of coherent sheaves at all genera.

The derived category of coherent sheaves being the prototypical example for a Calabi-Yau A_∞-category makes it apparent that $\mathcal{C}_B$ in fact represent a TCFT by Theorem 2.1.5.

From the physics perspective–to lowest order–a D-brane is given by a submanifold together with Chan-Paton data encoded in a vector bundle supported on that submanifold.

The idea of the derived category is to represent these vector bundles by–complexes of–coherent sheaves. This correspondence by itself is not one to one but localization with respect to quasi-isomorphisms identifies branes in the same universality class of the RG-flow. For a more detailed discussion of this point cf. [77].

B-Branes in Landau-Ginzburg Models

For Landau-Ginzburg models the category of B-branes was proposed by Kontsevich [55]. It is given by the homotopy category of matrix factorizations

$$\mathcal{C}_B = \mathrm{HMF}(R, W),$$

for a Landau-Ginzburg model with superpotential $W \in R$. For our purposes R will always be a polynomial ring over $\mathbb{C}$ but the statements discussed in this section generalize to regular rings of finite Krull dimension [34]. The category $\mathrm{HMF}(R, W)$ will be the main object of interest in this thesis and we will discuss its properties and features subsequently. The physics derivation of $\mathrm{HMF}(R, W)$ as the category of B-type boundary conditions in Landau-Ginzburg models is worked out in [9].

For our present review we mainly follow [34]. We start our discussion by giving the definition of a matrix factorization.

Definition 2.2.1 (Matrix Factorization)
Let R be a ring and $W \in R$. A matrix factorization of W is a $\mathbb{Z}_2$-graded R-module $P = P_0 \oplus P_1$ equipped with an odd endomorphism d such that $d^2 = W\,\mathrm{id}_P$.

Note that we may view matrix factorizations as two-periodic twisted complexes

$$\ldots P_0 \overset{d_0}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} P_1 \ldots =: P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$$

with differential

$$d = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix}.$$

Matrix factorizations admit a tensor product. Given two matrix factorizations P of $W \in R$ and P' of $W' \in R'$ their tensor product is given by the tensor product of the respective two-periodic twisted complexes

$$P \otimes_{\mathbb{C}} P' := \underbrace{(P_0 \otimes_{\mathbb{C}} P'_0) \oplus (P_1 \otimes_{\mathbb{C}} P'_1)}_{(P\otimes P')_0} \oplus \underbrace{(P_1 \otimes_{\mathbb{C}} P'_0) \oplus (P_0 \otimes_{\mathbb{C}} P'_1)}_{(P\otimes P')_1},$$

with the twisted differential

$$d := \begin{pmatrix} 0 & 0 & d_1 & -d'_1 \\ 0 & 0 & d'_0 & d_0 \\ d_0 & d'_1 & 0 & 0 \\ -d'_0 & d_1 & 0 & 0 \end{pmatrix}.$$

Note that the twists add under the tensor product and thus $P \otimes_{\mathbb{C}} P'$ is a matrix factorization of the sum $W + W'$.

Morphisms of matrix factorizations are given by the following definition.

Definition 2.2.2 (Morphisms of Matrix Factorizations)
Let P and P' be matrix factorizations of $W \in R$. A morphism between the matrix factorizations P and P' is a $\mathbb{Z}_2$-graded R-linear map.

The differential has an induced action on morphisms of matrix factorizations

$$d : \mathrm{Hom}(P, P') \to \mathrm{Hom}(P, P'), f \mapsto d_{P'} f - (-1)^{|f|} f d_P.$$

Having the definition of matrix factorizations and their morphisms–together with a differential acting on them–at hand we can define the homotopy category of matrix factorizations.

Definition 2.2.3 (Homotopy Category of Matrix Factorizations)
Let R be a ring and $W \in R$ the homotopy category of matrix factorizations $\mathrm{HMF}(R, W)$ *has as objects matrix factorizations of W and morphisms of matrix factorizations up to d-homotopy as morphisms.*

From the physics perspective, matrix factorizations correspond to B-branes and the hom-sets in $\mathrm{HMF}(W, R)$ correspond to state spaces of strings stretched between them.

Note that in the light of Theorem 2.1.5 we expect the category of Landau-Ginzburg B-branes to admit a Calabi-Yau A_∞-structure and indeed in [56] such a structure was found on the category of matrix factorizations $\mathrm{HMF}(R, W)$.

In [37] it was shown that for R a regular local ring matrix factorizations are equivalent to maximal Cohen-Macaulay Modules over the the hypersurface ring R/W. In fact there is a equivalence of categories furnished by the cokernel: For every matrix factorization of W

$$P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$$

we have that $\mathrm{coker}(d_1)$ is a maximal Cohen-Macaulay module over the the hypersurface ring R/W. Conversely, every Cohen-Macaulay R/W-module admits a resolution that turns two-periodic after the first step and this two periodic part of the resolution yields the associated matrix factorization. This leads to the equivalence first proven for regular local rings in [37].

Theorem 2.2.1 (Matrix Factorizations are Cohen-Macaulay Modules)
Let be a regular ring of finite Krull dimension and $W \in R$. There is an equivalence of categories

$$\mathrm{coker} : \mathrm{HMF}(R, W) \to \underline{\mathrm{MCM}}(R/W),$$

induced by the cokernel, where $\underline{\mathrm{MCM}}(R/W)$ is the stable category of maximal Cohen-Macaulay R/W-modules.

The stable category of maximal Cohen-Macaulay R/W-modules is obtained from the category of maximal Cohen-Macaulay R/W-modules $\mathrm{MCM}(R/W)$ by taking the quotient of the morphisms by the set of R/W-linear homomorphisms factoring through some free R/W-module.

So far we have discussed matrix factorizations which describe theories with trivial gauge group. In the case of a Landau-Ginzburg model with non-vanishing gauge group the ring of chiral fields R and the matrix factorization P carry representations $\rho_R : G \to \mathrm{GL}(R)$ and $\rho_P : G \to \mathrm{GL}(P)$ and we have to require compatibility of the matrix factorization with these representations. This means that

$$\rho_{P_i}(g)(rp) = \rho_R(g)(r)\rho_{P_i}(g)(p), \ \forall\ g \in G,\ r \in R,\ p \in P_i$$

and that the differential d has to be ρ-equivariant i.e.

$$\rho_{P_0}(g)d_1\rho_{P_1}^{-1}(g) = p_1, \quad \text{and} \quad \rho_{P_1}(g)d_0\rho_{P_0}^{-1}(g) = p_0,\ \forall\ g \in G.$$

A matrix factorization satisfying these properties is called *equivariant*. In the case $G = U(1)^n$ the representations ρ_i can be specified by means of the weights (charges) of the respective representations. Indeed, $U(1)^n$-equivariant matrix factorizations can be regarded as $\mathbb{Z}^n$-*graded matrix factorizations*. In [56], where it was shown that the category of matrix factorizations admit a Calabi-Yau A_∞-structure it was also discussed how to extend this result to the orbifold i.e. equivariant case. A proof for the existence of a Calabi-Yau structure on the category of equivariant matrix factorization can be found in [21]. This establishes that Landau-Ginzburg orbifold theories are also mathematically described by TCFTs via Theorem 2.1.5.

The correspondence between matrix factorizations and Cohen-Macaulay 2.2.1 modules can be extended to the equivariant, respectively graded case [37]. In particular, $\mathbb{Z}_n$-graded matrix factorizations are related to $\mathbb{Z}_n$-graded Cohen-Macaulay modules. This circumstance allows us to exploit the correspondence of matrix factorizations and Cohen-Macaulay modules also in the case of Landau-Ginzburg orbifold models.

We will use the fact that matrix factorizations can be represented by modules throughout this thesis. In particular we will also use this fact for the degenerate case of matrix factorizations of $R \ni W = 0$. By a matrix factorization of $W = 0$ we mean a $\mathbb{Z}_2$-graded module equipped with an odd differential. The category of B-branes can then essentially be thought of as something like the homotopy category of chain complexes of R-modules. To be more concise we may take the category of B-branes to be the *triangulated category of singularities*

$$\mathcal{C}_B = \mathrm{D}_{sg}(\mathrm{Mod}_R) \coloneqq \mathrm{D}^b(\mathrm{Mod}_R)/\mathrm{Perf}(\mathrm{Mod}_R),$$

where $\mathrm{D}^b(\mathrm{Mod}_R)$ denotes the bounded derived category of R-modules and $\mathrm{Perf}(\mathrm{Mod}_R)$ is the full subcategory comprised of *perfect complexes* i.e. complexes consisting of only projective modules. Just as in the non-degenerate case there is an equivalence between matrix factorizations and modules due to a theorem by Rickard [72].

Theorem 2.2.2 (Singularity Categories are Categories of Stabilized Modules) *Let A be a self-injective algebra then there is an equivalence of categories*

$$\mathrm{D}_{sg}(\mathrm{Mod}_A) \to \underline{\mathrm{Mod}}_A,$$

where $\underline{\mathrm{Mod}}_A$ is the stable category of A-modules.

Note that in the non-degenerate case Orlov showed in [69] that there is an equivalence between the category of matrix factorizations and the triangulated category of singularities. Thus we may identify

$$\mathcal{C}_B = \mathrm{D}_{sg}(\mathrm{Mod}_{R/W})$$

which covers both the non-degenerate and degenerate case.

B-Branes in GLSMs

As will be further elaborated on in the following sections, we are interested in constructing defects between various phases of GLSMs. To this end we will construct RG-type defects from the GLSM to its phases and defects lifting phases to the GLSM as an intermediate step. As such–from a mathematical point of view–these defects are ought to yield funtors between categories with the same kind of structures i.e. TCFTs. This necessitates a categorical description of the GLSM in terms of a TCFT which is at present not available to us.

As an auxiliary construction we will be treating the GLSM similar to the case of non-linear sigma models. We define the category of B-branes of a GLSM $(G = \mathrm{U}(1)^n, V, (r, \theta), W)$ as the (bounded) derived category of $\mathbb{Z}^n$-graded coherent sheaves on the subspace $W = 0$ of $\mathrm{Spec}(R)$ with $R = \mathrm{Sym}(V^\vee)$

$$\mathcal{C}_B := \mathrm{D}^b(R/W, G).$$

Since $R \cong \mathbb{C}[X_1, \ldots, X_{\dim(V)}]$ we have that $\mathrm{Spec}(R) = \mathbb{A}^n_{\mathbb{C}}$ and the category of B-branes is equivalent to the derived category of $\mathbb{Z}^n$-graded R/W-modules

$$\mathcal{C}_B \cong \mathrm{D}^b(\mathrm{Mod}^{\mathbb{Z}^n}_{R/W}).$$

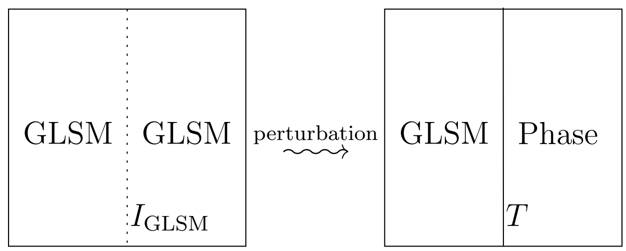
It can be shown that for any D-brane in an arbitrary phase a lift to this category does indeed always exist cf. section 8.2 in [47], justifying our definition as an auxiliary tool and a common framework for all the low energy phases of the GLSM.

2.3 Defects in GLSMs

We now have gathered all the ingredients needed to discuss our main objects of interest, defects in GLSM. As was briefly mentioned above we aim to construct RG-type defects R from the GLSM to its phases and defects T lifting phases to the GLSM. By fusing these types of defects we may construct transition defects between the phases of a GLSM. In this section we will be discussing the abstract approach of attaining RG-type and lifting defects and some of their properties and subsequently give the explicit constructions. We then compare our results to the grade- and band restriction rules from [47].

2.3.1 Generalities on GLSM Defects

We take the following approach. Starting out with a GLSM we choose a codimension one submanifold separating spacetime into two disjoint subspaces. Next we insert a invisible or trivial defect–the *identity defect*–gluing together both sides of the defect in a trivial manner, which leaves correlation functions unaffected. Subsequently we perform a deformation in the form of a relevant perturbation of the theory on one side of the defect by applying the RG flow. This is procedure is depicted in (2.2) below.



(2.2)

Lifting a phase to the GLSM via a defect T amounts to embedding it into the GLSM. This procedure is not unique as will be discussed at length below.

On the level of the B-brane categories, following the RG flow to a phase via a defect R amounts to either one of the following operations–assuming we are in the non-anomalous setting. Flowing to a Landau-Ginzburg phase corresponds to taking the quotient by perfect complexes to arrive at the triangulated category of singularities whereas for a geometric phase taking the quotient by appropriate torsion modules eliminates the deleted set Δ_r

$$\begin{tikzcd}
& \mathrm{D}^b(\mathrm{Mod}^{\mathbb{Z}^n}_{R/W}) \arrow[dl, "\text{perfect}"'] \arrow[dr, "\text{torsion}"] & \\
\mathrm{D}_{sg}\left(\mathrm{Mod}_{R/W}\right) \arrow[ur, hook] & & \mathrm{D}^b\left(\mathrm{Crit}(W) \cap \left\{\mathbb{C}^k - \Delta_r\right\}/G_{\mathbb{C}}\right) . \arrow[ul, hook]
\end{tikzcd}$$

Just as for T the defects R are not uniquely defined. Notably, due to a theorem by Orlov [68] we have the following equivalence of categories

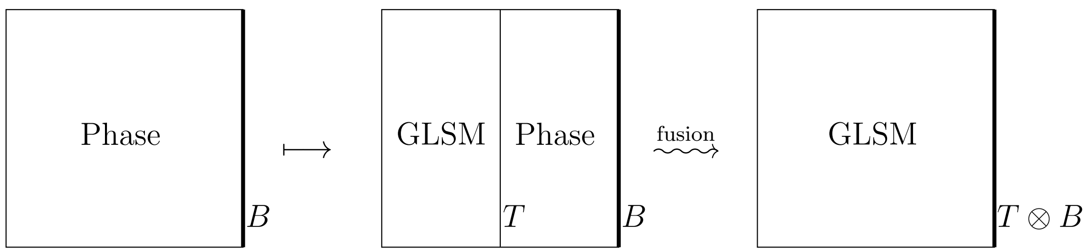
$$\mathrm{D}^b\left(\mathrm{Crit}(W) \cap \left\{\mathbb{C}^k - \Delta_r\right\}/G_{\mathbb{C}}\right) \cong \mathrm{D}_{sg}(\mathrm{Mod}_{R/W}),$$

for an accessible explanation of this theorem see also Section 10.6 in [47]. Indeed the categories of B-branes are equivalent for every pair of phases in the non-anomalous case.

In the anomalous setting certain branes decay along the RG flow and get projected out by the respective functor.

Since we are restricting our attention to a topological sector of the GLSM we may also fuse defects which in particular allows us to lift boundary conditions from a phase to the

GLSM as illustrated in (2.3). The thick lines in (2.3) represent boundary conditions i.e. defects with the empty theory to its right and the symbol $\otimes$ denotes the fusion which we will use from now on for the fusion product of defects. Fusion of defects is functorial and thus expressing bulk deformations via defects implies functoriality of the behavior of boundary conditions under bulk deformations.



(2.3)

Just as in the case of two-dimensional defect TQFTs, in the present setting the deformation defects are ought to constitute 1-morphisms in a 2-category where fusion is the horizontal composition, with the identity defect being the identity with respect to fusion. We expect the following properties for deformation defects. Every lift T_k admits a *adjoint* R_k such that

$$R_k \otimes T_k = I_{\text{Phase}},$$

where I_{Phase} is the identity defect of the phase and the index implies some choice which stems from finiteness conditions that we will impose. As a consequence, in the opposite direction we have that

$$(T_k \otimes R_k)^2 = T_k \otimes R_k \otimes T_k \otimes R_k = T_k \otimes R_k$$

so $P_k = T_k \otimes R_k$ is a idempotent defect in the GLSM. In fact P_k is supposed to be a defect projecting onto the IR degrees of freedom in the UV. More details on the properties of deformation defects and further applications can be found in [59].

We may also incorporate deformations in the form of perturbations that are marginal into our construction to attain transition defects between different phases of the GLSM. To do so we start out with a lift T^i_k of a phase Phase$_i$ and perform the deformation to flow to a different phase Phase$_j$ via R^j_l. The transition defect is then given by the fusion $R^j_l \otimes T^i_k$. This procedure is depicted in Figure 2.1 below, taken from [10].

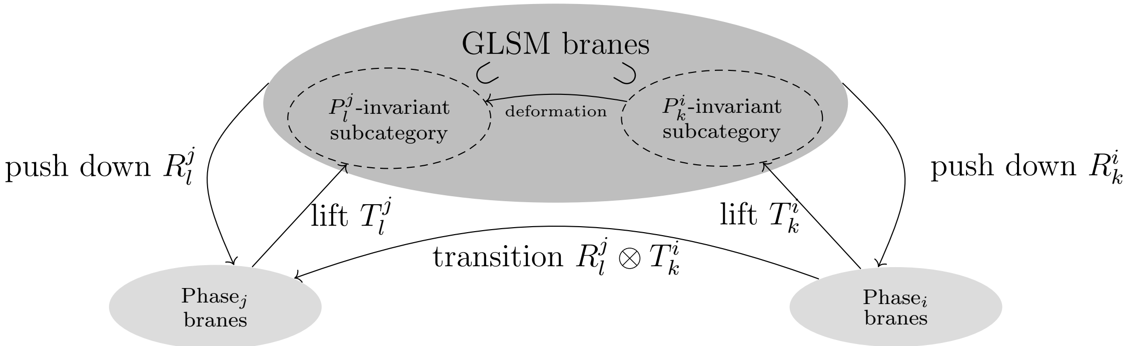


Figure 2.1: Lifts, RG-flows and phase transitions in a GLSM.

Two phases in a GLSM can generically be connected via different homotopy classes of paths in moduli space–in particular including monodromies–and we expect that the choices in picking T_k^i and R_l^j can be associated with the choices of homotopy classes. Indeed we will be able to make such an identification via matching our defects with the so called grade- and band restriction rules from [47] to be introduced in the next section.

2.3.2 Constructing GLSM Defects

When constructing transition defects in this thesis we will be exclusively concerned with Landau-Ginzburg phases to avoid technicalities[4]. As such it will be sufficient to study the subcategories of the GLSM given by the embeddings of matrix factorizations. We will thus take the category of boundary conditions for a GLSM $(G = \mathrm{U}(1)^n, V, (r, \theta), W)$ to be the category of G-equivariant matrix factorizations of W

$$\mathcal{C}_B^{\mathrm{GLSM}} := \mathrm{HMF}(\mathrm{Sym}(V^\vee), W)^G.$$

In case of Landau-Ginzburg models all physical data, including defect data, can be encoded in a *bicategory with adjunctions* $\mathcal{LG}_\mathbb{C}$ under mild assumptions on the superpotential cf. [19] for the precise statement. Note that the bicategory $\mathcal{LG}_\mathbb{C}$ is *graded pivotal* cf. [19]. This fits with our discussion in Section 2.1.2, where we have seen that plain two-dimensional defect TQFTs admit a pivotal structure. In contrast to the the pivotal case, in the graded pivotal case the left- and right adjoints may differ by a shift[5]. For Landau-Ginzburg orbifolds one would need to perform a orbifold construction as described in e.g. [8] to find the adjunction data on the respective bicategory cf. [20]. In this thesis we assume this structure to exist. We will not go into the technical details of this bicategory here and just proceed by describing its relevant aspects needed for our discussion of defects in GLSMs.

[4] Transitions between geometric phases have been considered in [11] for the case of a rank-1 abelian gauge group.

[5] That is, a shift with respect to the triangulated structure of the category of matrix factorizations, which we have not introduced in this thesis.

Given two GLSMs

$$(G_1, \mathbb{C}^{k^{(1)}}, (r^{(1)}, \theta^{(1)}), W^{(1)}) \quad \text{and} \quad (G_2, \mathbb{C}^{k^{(2)}}, (r^{(2)}, \theta^{(2)}), W^{(2)}),$$

a B-type defect between the two can be represented by a $G_1 \times G_2$-equivariant matrix factorization of the difference $W^{(1)} - W^{(2)}$ of the respective superpotentials over the ring of chiral bulk fields of the two models given by

$$R^{(1,2)} := R^{(1)} \otimes_{\mathbb{C}} R^{(2)} = \mathbb{C}[X_1^{(1)}, \dots, X_{k^{(1)}}^{(1)}, X_1^{(2)}, \dots, X_{k^{(2)}}^{(2)}],$$

where $R^{(i)} := \mathrm{Sym}(\mathbb{C}^{k^{(i)}\vee}) \cong \mathbb{C}[X_1^{(i)}, \dots, X_{k^{(i)}}^{(i)}]$ is the ring of chiral bulk fields for the model i. This means the category of 1-morphims of the bicategory of Landau-Ginzburg models is given by

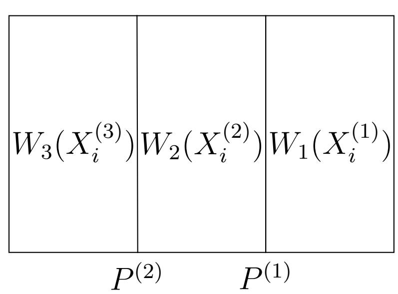
$$\mathcal{LG}_{\mathbb{C}}((\mathbb{C}^{k^{(1)}}, W^{(1)}), (\mathbb{C}^{k^{(2)}}, W^{(2)})) = \mathrm{HMF}(R^{(1,2)}, W^{(1)} - W^{(2)})^{G_1 \times G_2}.$$

Note that in the case that $(G_2, \mathbb{C}^{k^{(2)}}, (r^{(2)}, \theta^{(2)}), W^{(2)})$ is trivial i.e. $G_2 = \{e\}$, $k^{(2)} = 0$, $W = 0$ a defect between the two GLSM is a boundary condition of $(G_1, \mathbb{C}^{k^{(1)}}, (r^{(1)}, \theta^{(1)}), W^{(1)})$ as is expected.

Next we will have a look at the horizontal composition of 1-morphisms which amounts to fusion of the corresponding defects. We start by considering the non-equivariant case i.e. we restrict our attention to trivial gauge groups. Suppose we are given three models

$$(\{e\}, \mathbb{C}^{k^{(a)}}, (r^{(a)}, \theta^{(a)}), W^{(a)}), \ a = 1, 2, 3,$$

defects $P^{(1)}$ between model 2 and model 1 and $P^{(2)}$ between model 3 and model 2 as depicted in (2.4) below. Then $P^{(1)}$ is represented by a matrix factorization of $W_1(X_i^{(1)}) - W_2(X_i^{(2)})$ over $R^{(1,2)}$ and $P^{(2)}$ by a matrix factorization of $W_2(X_i^{(2)}) - W_3(X_i^{(3)})$ over $R^{(2,3)}$.



(2.4)

The fusion $P^{(1)} * P^{(2)}$ of $P^{(1)}$ and $P^{(2)}$ over the model 2 squeezed in between them is then given by the tensor product

$$P^{(1)} * P^{(2)} = P^{(1)} \otimes_{R^{(2)}} P^{(2)},$$

over the ring of bulk fields of model 2, regarded as matrix factorization over the ring $R^{(1,3)}$. Note that this matrix factorization still involves the chiral bulk fields $X_i^{(2)}$–now as defect degrees of freedom–of the model squeezed in between the defects. Thus, the resulting matrix factorization is a priori of infinite rank. The fusion $P^{(1)} * P^{(2)}$ can however be shown to be equivalent to a matrix factorization of finite rank cf. [12].

The equivariant setting requires a slight modification of the fusion product. Here the tensor product of the respective matrix factorization still carries a representation of the squeezed in model. The fusion is then given by the sub matrix factorization invariant under the intermediate gauge group. To be concise, now suppose that the model a has non-trivial gauge group G_a for $a = 1, 2, 3$. Then, the matrix factorizations $P^{(1)}$ and $P^{(2)}$ representing defects between the models are equivariant with respect to $G_1 \times G_2$ and $G_2 \times G_3$ respectively. The tensor product $P^{(1)} \otimes_{R^{(2)}} P^{(2)}$ then carries a representation of G_2, and the fusion of the respective defects is given by the G_2-invariant sub matrix factorization

$$P^{(1)} * P^{(2)} = \left[P^{(1)} \otimes_{R^{(2)}} P^{(2)}\right]^{G_2}.$$

More details on fusion in the equivariant setup can be found in [13].

On the level of Cohen-Macaulay modules, fusion is represented in a similar fashion, by taking the invariant part with respect of the intermediate gauge group of the tensor product of Cohen-Macaulay modules over the intermediate ring.

GLSM Identity Defects

To construct the identity defect consider two copies of the same GLSM

$$(G_1, \mathbb{C}^{k^{(1)}}, (r^{(1)}, \theta^{(1)}), W^{(1)}) = (G_2, \mathbb{C}^{k^{(2)}}, (r^{(2)}, \theta^{(2)}), W^{(2)}).$$

The identity defect in the GLSM acts as the identity with respect to fusion by enforcing trivial gluing conditions $X_i^{(1)} = X_i^{(2)}$ on the fields on either side. In the case of a trivial gauge group $G_1 = G_2 = \{e\}$, it is represented by a matrix factorization of *Koszul-type* with factors $(X_i^{(1)} - X_i^{(2)})$–i.e. it is a tensor product of rank-1 matrix factorizations with $d_1 = (X_i^{(1)} - X_i^{(2)})$ for each chiral bulk field X_i. In this case the Cohen-Macaulay module[6]

$$\mathcal{I}_{\mathrm{GLSM}} = \frac{S^{(1,2)}}{\langle (X_i^{(1)} - X_i^{(2)})_{i=1,\ldots,k} \rangle}, \tag{2.5}$$

acts as the identity on the GLSM, by identifying the fields $X_i^{(1)}$ with the fields $X_i^{(2)}$, where

$$S^{(1,2)} = R^{(1,2)} / \langle W(X_i^{(1)}) - W(X_i^{(2)}) \rangle.$$

Therefore, we can associate the functor $\mathcal{I}_{\mathrm{GLSM}}$ with the identity defect I_{GLSM}. From now on we will denote defects by italic letters and the corresponding 1-morphism in the bicategory of Landau-Ginzburg models with calligraphic letters. We will be somewhat sloppy when referring to the 1-morphism and just call them defects for the most part.

[6] Note that if the model has more than one chiral field, this module is not maximal Cohen-Macaulay–the resolution is not two-periodic from the start, but only after the number of chiral fields minus one steps.

When considering models with non-trivial gauge group G, the matrix factorization representing the identity defect has to be modified. This is due to the fact that the factors $(X_i^{(1)} - X_i^{(2)})$ are not equivariant with respect to the action of the product $G_1 \times G_2 = G \times G$ of the gauge groups on the left, respectively right side of the defect. The factors $(X_i^{(1)} - X_i^{(2)})$ facilitates the identification of the fields $X_i^{(1)}$ with the fields $X_i^{(2)}$ but does not map G_1 representations to G_2 representations. So to make them equivariant one has to translate the G_1 representation labels of $X_i^{(1)}$ to G_2 representations. To this end, we first tensor the Koszul factorization by the regular representation of the gauge group. For $G = U(1)$, the regular representation is given by

$$V_{\mathrm{reg}}^{U(1)} = \frac{\mathbb{C}[\alpha, \alpha^{-1}]}{\langle \alpha\alpha^{-1} - 1 \rangle}.$$

Here α corresponds to a defect field which has charges 1 and -1 under the $U(1)$ gauge groups on the left, respectively right of the defect. The field α^{-1} is a formal inverse of α i.e. its charges are inverse to those of α.

For $G = U(1)^n$, one such field α_i has to be introduced for every $U(1)$-factor, i.e. the regular representation is given by the module

$$V_{\mathrm{reg}}^{U(1)^n} = \frac{\mathbb{C}[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1}]}{\langle (\alpha_a \alpha_a^{-1} - 1)_{a=1,\ldots,n} \rangle}.$$

α_a has charges 1 and -1 under the left, respectively right ath $U(1)$ and is uncharged with respect to all the other $U(1)$s. The fields α_a^{-1} are formal inverses and thus have inverse charges to the ones of α_a. These fields now serve as intertwiners to render the Koszul factors in (2.5) equivariant by setting them to

$$\alpha_1^{-Q_{1\,i}} \ldots \alpha_n^{-Q_{n\,i}} X_i^{(1)} - X_i^{(2)},$$

here $Q_{a\,i}$ is the charge matrix specifying the gauge representations on the chiral matter fields.

The Cohen-Macaulay module associated to the identity defect in the GLSM

$$(U(1)^n, \mathbb{C}^k, (r, \theta), W)$$

is then represented by

$$\mathcal{I}_{\mathrm{GLSM}} := \frac{S^{(1,2)} \otimes_{\mathbb{C}} V_{\mathrm{reg}}^{U(1)^n}}{\langle (\alpha_1^{-Q_{1\,i}} \ldots \alpha_n^{-Q_{n\,i}} X_i^{(1)} - X_i^{(2)})_{i=1,\ldots,k}, \rangle}. \tag{2.6}$$

The construction of the identity defect in Landau-Ginzburg orbifold models is completely analogous. One only has to replace the regular representation of the gauge group $G = U(1)^n$ in (2.6) by the regular representation of the respective finite orbifold group.

For instance, in the important case of cyclic orbifold groups $\mathbb{Z}_d$, the regular representation is given by

$$V_{\mathrm{reg}}^{\mathbb{Z}_d} = \frac{\mathbb{C}[\alpha]}{\langle \alpha^d - 1 \rangle}.$$

For more details on the representation of the identity defect by means of matrix factorizations and Cohen-Macaulay modules see [12] for the non-equivariant case, [13] for the case of finite gauge groups and [10] for the case of gauge group $U(1)$.

Orbifold Lifts

With the GLSM identity defect at hand we now may proceed to construct orbifold lifts. To this end consider a GLSM

$$(G = U(1)^n, \mathbb{C}^k, (r, \theta), W)$$

which admits a Landau-Ginzburg orbifold phase. In such a phase the D-term equations require that some of the chiral fields of the GLSM obtain a non-trivial vacuum expectation value, which in turn breaks the gauge symmetry to a finite subgroup $H \subset G$.

We choose a labeling such that $X_{N+1}, \ldots, X_k$ are the fields acquiring a non-trivial VEV in the Landau-Ginzburg phase and denote the stabilizer of these fields by H. The Landau-Ginzburg model then has chiral fields given by $X_1, \ldots, X_N$ and superpotential W_{LG} obtained by inserting the vacuum-expectation values for the fields $X_{N+1}, \ldots, X_k$ into the superpotential W of the GLSM.

Our aim is to construct the defects T_γ, where γ labels the homotopy class of a path in the GLSM parameter space, which lift the Landau-Ginzburg phase to the GLSM. Such defects are B-type defects mediating between the Landau-Ginzburg model and the GLSM, and can therefore be represented by objects in

$$HMF(W - W_{\mathrm{LG}})^{G \times H}.$$

Let us first remark on the relationship between the identity defect of the GLSM and the one in the phase. Setting all the fields X_i, $N < i \leq k$ to their vevs (which w.l.o.g. we choose to be 1) on both sides of the GLSM identity defect yields the identity defect of the Landau-Ginzburg phase. Namely, setting $X_i^{(1)} = X_i^{(2)} = 1$ for $N < i \leq k$ in (2.6) changes the respective Koszul factors to

$$\alpha_1^{-Q_{1\,i}} \ldots \alpha_n^{-Q_{n\,i}} - 1,\ i \in \{N+1, \ldots k\}\ .$$

Dividing out $V_{\mathrm{ref}}^{U(1)^n}$ by this relation precisely yields the regular representation V_{reg}^H of H. Thus quotienting by $X_i^{(1)} = X_i^{(2)} = 1$ for $N < i \leq k$ in $\mathcal{I}_{\mathrm{GLSM}}$ produces the module associated to the identity defect in the Landau-Ginzburg phase $\mathcal{I}_{\mathrm{LG}}$.

Conversely, starting with the identity defect of the LG phase, one may ask how to lift to a defect of the GLSM. This can be done by re-introducing the variables X_i with $N < i \leq k$ on both sides of the defect and lifting the respective representations of H to representations of G. Indeed, there are many different such lifts which we refer to as "lifted identities", none of which are however GLSM identity defects.

We now turn to the construction of defects T^i_γ and R^{i}_γ for Landau-Ginzburg phases. These defects have to satisfy

$$R_\gamma \otimes T_\gamma = I_{\mathrm{LG}}.$$

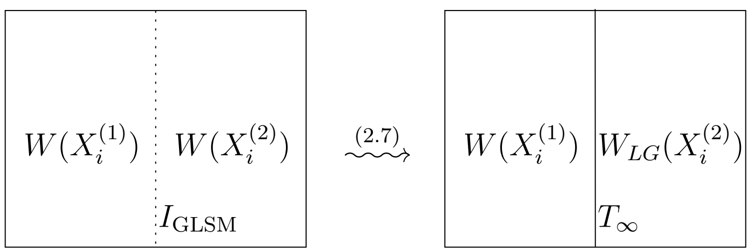
Our starting point is the module (2.6) representing the identity defect of the GLSM. Motivated by the observation that setting the fields to their expectation values on both sides yields the identity of the phase, we factorize the identity defect of the phase over the GLSM according to

$$R_\infty \otimes T_\infty = I_{\mathrm{LG}}.$$

The defects R_∞ and T_∞ are obtained by setting the fields X_i, $N < i \leq k$ to their VEVs, but only on one side of the GLSM identity defect. For example, to obtain T_∞ we set

$$X_i^{(2)} = 1,\ N < i \leq k \tag{2.7}$$

in the GLSM identity defect and arrive at the setting depicted in (2.8).



(2.8)

On the level of modules, imposing (2.7) in (2.6) yields a new module $\mathcal{T}_\infty$ with relations

$$\begin{aligned} \alpha_1^{-Q_{1\,i}} \dots \alpha_n^{-Q_{n\,i}} X_i^{(1)} &= X_i^{(2)}, \qquad && 1 \leq i \leq N \\ \alpha_1^{-Q_{1\,i}} \dots \alpha_n^{-Q_{n\,i}} X_i^{(1)} &= 1, \qquad && N < i \leq k. \end{aligned} \tag{2.9}$$

Thus, we have

$$\mathcal{T}_\infty = \frac{S^{(1,2)} \otimes V_{\mathrm{reg}}}{\langle (\alpha_1^{-Q_{1\,i}} \dots \alpha_n^{-Q_{n\,i}} X_i^{(1)} - X_i^{(2)})_{i=1,\dots,N} \rangle \langle (\alpha_1^{-Q_{1\,i}} \dots \alpha_n^{-Q_{n\,i}} X_i^{(1)} - 1)_{i=N+1,\dots,k)} \rangle}.$$

Similarly, we obtain a module $\mathcal{R}_\infty$ by setting fields on the left hand side of the defect to their expectation value i.e. by setting

$$X_i^{(1)} = 1,\ N < i \leq k.$$

The resulting module $\mathcal{T}_\infty$ however does not have the desired properties for lift defects. For one thing, $\mathcal{T}_\infty$ is not finitely generated and hence cannot be obtained by lifting the identity defect of the Landau-Ginzburg phase on the left side to the GLSM. What is more, $\mathcal{T}_\infty$ is unique and hence does not depend on a homotopy class of paths in moduli space.

For the case of $U(1)$ gauge groups, this problem was solved in [10]. There it was shown that the lift defects are obtained from $\mathcal{T}_\infty$ by a cutoff procedure which renders the module finitely generated. The choice involved in the cutoff procedure exactly corresponds to the choice of homotopy class of paths in parameter space. In the following, we will generalize this to higher rank abelian gauge groups and give a concrete construction of the desired transition defects.

Let γ be a path in parameter space such that it crosses a number of phase boundaries associated to cones Cone_{I_s}, $s = 1, \dots, m$ as defined in (2.1) but not intersect or encircle any singular point. On each phase boundary, a $U(1)$-subgroup $U(1)_{I_s}$ is preserved. To construct the lift defects, we now impose cutoffs

$$Q^L_{I_s} \leq N_{I_s} \tag{2.10}$$

in $\mathcal{T}_\infty$ for the respective charges $Q^L_{I_s}$ of the gauge group on the left of the defect for each such transition. More precisely, one considers the submodule $\mathcal{T}_{N_{I_1},\dots,N_{I_m}} \subset \mathcal{T}_\infty$ generated by all the generators whose charges satisfy (2.10). We claim that the associated defects $T_{N_{I_1},\dots,N_{I_m}}$ are the respective lift defects. While this does not offer an a priori assignment of defects to homotopy classes of paths, we observe that the choice of cutoff parameters $N_{I_1}, \dots, N_{I_m}$ are in one-to-one correspondence with the homotopy classes of paths with the chosen phase transitions. For each transition, the different paths must avoid the singular locus (2.12), and a homotopy class is specified by a connected component of $\mathbb{R}\backslash(2\pi\mathbb{Z} + \pi S_I)$. Thus, our construction provides a lift defect for every homotopy class of paths not encircling the singular points on the phase boundaries.

Indeed, the lift defects satisfy

$$R_\infty \otimes T_{N_{I_1},\dots,N_{I_m}} = \mathrm{id}_{\mathrm{LG}}$$

and hence are "lifted identities" from the perspective of the orbifold phase. Moreover, when the path crosses n phase boundaries (n being the rank of the gauge group), the respective cutoffs render the module finitely generated. Indeed, due to the relations (2.9) in $\mathcal{T}_\infty$, any cutoff (2.10) automatically leads to a lower limit on the respective charges of necessary generators of $\mathcal{T}_{N_{I_1},\dots,N_{I_m}}$ namely

$$N_{I_s} - M_{I_s} < Q^L_{I_s} \leq N_{I_s}. \tag{2.11}$$

Here M_{I_s} is an integer associated to I_s. This will be explained in more detail in the examples in the main body of this thesis. As will be outlined in the subsequent section, this exactly reproduces the band restriction rule for D-brane transport as put forward in [47].

From the lift defects $T_{N_{I_1},\dots,N_{I_m}}$ constructed in this way, one can then obtain the defects describing the transition between the LG phase and any other phase crossed by the path γ. This is done by pushing the GLSM to the respective phase on the left of the defect, which is the same as fusion with the respective defect R_∞. If the target phase is another Landau-Ginzburg phase, this just involves setting fields to their VEVs. For geometric phases, this is somewhat more complicated. It involves expanding the lift defect $T_{N_{I_1},\dots,N_{I_m}}$ into a complex of matrix factorizations and interpreting it as a hybrid between a matrix factorization and a complex of coherent sheaves on the target space. These steps were performed for one parameter models in [11]. We omit them here, focussing on the construction of the lift defects.

2.3.3 Grade- and Band Restriction Rules

As discussed earlier, on the classical level the emergence of Coulomb branches render the phase boundaries of a GLSM singular. On the quantum level, it turns out that the Coulomb branch only emerges at very specific values of θ on the classical phase boundaries defined by the cone Cone_I. This leads to codimension-two-loci of singular points in the Kähler moduli space (or rather the space parametrized by the complexified FI parameters $r_a+i\theta_a$). Indeed, on any such classical phase boundary, the singular loci are given by

$$\theta_I \in \{2\pi\mathbb{Z} + \pi S_I\}, \quad \text{with} \quad S_I = \sum_{Q_{I\,i}>0} Q_{I\,i}, \tag{2.12}$$

where θ_I is the θ-parameter in the direction of the $U(1)$-subgroup $U(1)_I$ unbroken on the phase boundary associated to Cone_I, and $Q_{I\,i}$ is the respective $U(1)_I$-charge of the chiral field X_i. For more details see [47].

To put our construction into perspective, we would like to compare it to results on D-brane transport on the Kähler moduli space of non-anomalous GLSMs in [47]. Indeed, fusion of transition defects with D-branes (boundary conditions) describes the behavior of the latter under the respective phase transitions. In this way, the defects constructed above can be used to describe D-brane transport between different phases in Kähler moduli space. Fusion of D-branes in the LG phase with the defects $T_{N_{I_1},\dots,N_{I_m}}$ lifts these D-branes to the GLSM in a way compatible with transport along a path associated to the choice of cutoff parameters. It turns out that this matches precisely with the *band restriction rule*–which is called *grade restriction rule* in special case of GLSMs with gauge group $U(1)$–proposed in [47].

The band restriction rule states the following: A path between two adjacent phases has to avoid the singular locus (2.12) in Kähler moduli space, i.e. it crosses the phase boundary at $\theta_I \in \mathbb{R}\backslash\{2\pi\mathbb{Z} + \pi S_I\}$. The choice of any such θ_I gives rise to a window

$$\mathbb{Z} \cap \left\{ -\frac{\theta_I}{2\pi} + \left(-\frac{S_I}{2}, \frac{S_I}{2} \right) \right\}$$

of consecutive integers, which only depends on the connected component of possible θ_I and hence on the homotopy class of paths from one phase to the adjacent one. According to the band restriction rule (see Section 7.3.2. of [47]), D-branes built from Wilson line branes $\mathcal{W}(q)$ can be transported straightforwardly along a chosen path in Kähler moduli if and only if their charges q under the $U(1)$s unbroken at all phase boundaries traversed lie in the respective windows. That is, for a path crossing the phase boundary given by Cone_I where $U(1)_I$ is unbroken the charge q_I of a Wilson line brane has to satisfy

$$-\frac{S_I}{2} < \frac{\theta_I}{2\pi} + q_I < \frac{S_I}{2}.$$

The matrix factorizations associated to the lift defects $T_{N_{I_1},\ldots,N_{I_m}}$ have the property that the generators of their underlying modules have charges (under the gauge group on the left of the defect) lying in the band (2.11). This means that fusing any boundary condition (D-brane) in the LG phase with $T_{N_{I_1},\ldots,N_{I_m}}$ produces only GLSM branes whose charges lie in this charge band. Thus, lifting LG-branes into the GLSM with $T_{N_{I_1},\ldots,N_{I_m}}$ produces GLSM branes in that charge band. As we will see concretely in the examples discussed in the next chapter, the charge bands singled out by the defect construction precisely match those of the band restriction rule in [47]. Note that while our construction requires the introduction of the upper bounds on the charges, the lower bounds automatically follow from it. In particular, the size of the bands is completely determined from the construction.

Note that a path from the small volume phase ($r_i << 0$ for all i) to the large volume phase ($r_i >> 0$ for all i) crosses at least r phase boundaries, where r is the rank of the gauge group. Hence, the corresponding band restriction rule restricts to a finite set of possible charges. In the language of defects and corresponding modules, this matches the fact that $\mathcal{T}_\infty$ gets truncated to a finite submodule in this case.

For the anomalous case, a generalization of the grade restriction rule has been discussed in [50, 35, 22]. Starting from the UV phase, D-branes are lifted into a 'large window', and D-branes with charges in a 'small window' that is a subset of the large window survive the flow to an IR phase. The location of the small window inside the large window can be shifted by symmetry and monodromy. Our defect construction applies to the anomalous case as well, the lift defects lift D-branes into the large window, and the small window arises automatically when pushing to the IR phase on the other side of the defect. We will confirm this for a class of examples, the Landau-Ginzburg models

$$\text{GLSM}_{\mathcal{M}_{d-2}} \coloneqq \left(\text{U}(1)^{d-2}, \mathbb{C}^{d-1}, (r,\theta), W = \prod_{i=0}^{d-2} X_i^{d-i}\right),$$

encompassing all minimal model orbifolds up to level $k = d-2$ as their respective phases.

3 Lift- and Flow Defects in Abelian GLSMs

In this chapter we discuss the application of the main result of this thesis–the prescription for the construction of defects $\mathcal{T}$ and $\mathcal{R}$–to various examples of GLSMs. The Results discussed here are the ones put forward in [14].

We begin by discussing orbifold lifts in three non-anomalous examples which where also considered in [47] and compare the results. First we consider the GLSM encoding the A_2 singularity and its resolution. Next, we study a Two Parameter Model with $\mathbb{C}^5/\mathbb{Z}_8$-orbifold phase before discussing the GLSM describing the A_{N-1} singularity for arbitrary N.

We then turn our attention to the construction of flow defects between minimal models via GLSMs. To this end we make use of the fact that there is a GLSM whose phases are precisely describing all minimal models up to a certain level. This allows us to exploit our construction of orbifold lifts and subsequently push down the defect to another phase on the left side of the defect. We demonstrate that this way we can reproduce the flow defects from [13].

3.1 Non-Anomalous Examples

In this section we will apply the construction of lift defects described in the previous chapter to two concrete examples of non-anomalous GLSMs with higher-rank abelian gauge groups. For simplicity we will choose examples with zero superpotential. The relevant defects are described by matrix factorizations of 0, which correspond to honest (as opposed to twisted) complexes as discussed in Section 2.2.4, and hence there is a direct connection with modules over polynomial rings.

The first example we will be studying is the A_{N-1}-model. It is a GLSM with $U(1)^{N-1}$ gauge group, whose orbifold phase is associated to the orbifold $\mathbb{C}^2/\mathbb{Z}_N$. Here $\mathbb{Z}_N$ acts by opposite phase multiplication on the two coordinates of $\mathbb{C}^2$ such that the Calabi-Yau condition is satisfied. The geometric phase of this model is the sigma model on the resolution of the corresponding A_{N-1}-singularity by a chain of $N-1$ Riemann spheres $\mathbb{P}^1$ intersecting according to the A_n Dynkin diagram. The volumes of the Riemann spheres are determined by the FI parameters.

The second example is a GLSM with gauge group $U(1)^2$ which has four phases, one of which is a $\mathbb{C}^5/\mathbb{Z}_8$-orbifold phase and one which is a non-linear sigma model on the total space of the line bundle $\mathcal{O}(-8)$ over the weighted projective space $\mathbb{P}_{(11222)}$.

These models are well studied. In particular, the D-brane transport between different phases in these models has been investigated in [47] (where the two models are referred to as example (D) and example (C), respectively). Now, D-brane transport is easy to describe in our framework, since it is just given by fusion of the D-branes with the transition defects between the respective phases. So after constructing the lift defects of the orbifold phases in these examples, we will show that fusion of D-branes with these defects reproduces the results on D-brane transport from [47]. The latter are formulated in terms of the grade restriction (for rank 1 gauge theories) or band restriction (higher rank) rule.

This section is organized as follows: We will start by studying two different two-parameter models, the A_2-model and the model with orbifold phase $\mathbb{C}^5/\mathbb{Z}_8$. In both cases we compare our results with the band restriction rule and find agreement. Subsequently, we will generalize the discussion of the A_2-model to the $N-1$-parameter A_{N-1}-model for arbitrary N.

Throughout this section, we will distinguish variables on the left and right of the defect by adding a prime, while the respective gauge groups are distinguished by a superscript L/R.

3.1.1 The A_2 Model

We start out by studying lifts of the orbifold phase of the A_2-GLSM. The latter is specified by the data

$$\mathrm{GLSM}_{A_2} \coloneqq (U(1)_1 \times U(1)_2,\ V,\ (r,\theta),\ W=0),$$

where the representation V, can be read off from the charge assignment of the chiral matter fields specified in Table 3.1 below.

	X_1	X_2	X_3	X_4
Q_1	1	-2	1	0
Q_2	0	1	-2	1

Table 3.1: Matter content of the A_2-model. Q_j denotes the respective $U(1)_j$-charge.

This model exhibits an orbifold phase for $r_1,\ r_2 \to -\infty$. In this phase the fields X_2 and X_3 acquire a vev and the low energy theory is the orbifold theory $\mathbb{C}^2/\mathbb{Z}_3$ with fields X_1 and X_4 whose $\mathbb{Z}_3$-charges are 1 and -1 respectively. The model also exhibits a large volume phase in the opposite limit $r_1,\ r_2 \to \infty$. This limit is described by a sigma model on the resolution of the $\mathbb{C}^2/\mathbb{Z}_3$ singularity, where both of the 2-spheres in the exceptional divisor are blown up. Apart from these there are two mixed phases related to partial resolutions of the singularity, where only one of the two 2-spheres is blown up.

The phase diagram of GLSM_{A_2} is depicted in Figure 3.1 below. Here (11) denotes the–unresolved–orbifold phase and (00) denotes the large volume phase, where both 2-spheres in the exceptional divisor are blown up. The intermediate phases in which only one of the two 2-spheres is blown up are denoted by (01) and (10), respectively.

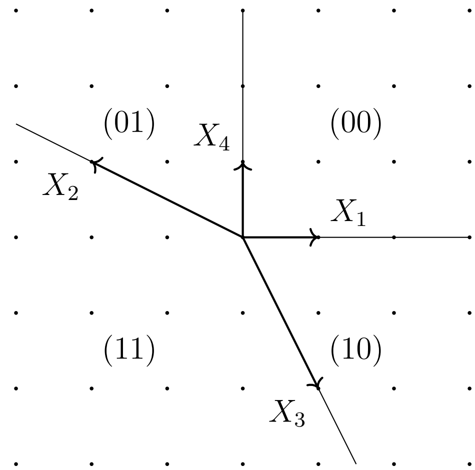


Figure 3.1: Phase diagram of GLSM_{A_2}. Phases are denoted by $(i_1 i_2)$ where $i_j = 0$ if the jth exceptional 2-sphere is blown up, and $i_j = 1$ if it is blown down. The phase boundaries denoted by X_i are located at $\mathrm{Cone}_{\{i\}}$.

The phase boundary $(11) \leftrightarrow (01)$ is located at

$$\mathrm{Cone}_{\{2\}} = \left\{ r = \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}_{>0} \right\}.$$

The D-term equation forces X_2 to acquire a vev on this phase boundary, and the isotropy group of the latter is given by

$$\{(g, g^2) | g \in U(1)\} \cong U(1).$$

This is the $U(1)$ unbroken on the entire phase boundary. Analogously one obtains the unbroken gauge groups at the other phase boundaries which can be read of from Table 3.2.

phase boundary	location	unbroken gauge group
$(11) \leftrightarrow (01)$	$\mathrm{Cone}_{\{2\}}$	$\{(g, g^2) \mid g \in U(1)\}$
$(11) \leftrightarrow (10)$	$\mathrm{Cone}_{\{3\}}$	$\{(g^2, g) \mid g \in U(1)\}$
$(00) \leftrightarrow (01)$	$\mathrm{Cone}_{\{4\}}$	$U(1)_1$
$(00) \leftrightarrow (10)$	$\mathrm{Cone}_{\{1\}}$	$U(1)_2$

Table 3.2: Unbroken subgroups of $U(1)_1 \times U(1)_2$ at the phase boundaries of GLSM_{A_2}.

A more detailed discussion on the A_2-model can be found in [47][1].

The GLSM Identity Defect

The identity defect of GLSM_{A_2} is associated to the $\mathbb{C}[X_1, \ldots, X_4, X_1', \ldots, X_4']$-module

$$\mathcal{I}_{\mathrm{GLSM}_{A_2}} \coloneqq \mathbb{C}[X_1, \ldots, X_4, X_1', \ldots, X_4', \alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}] / \langle (\alpha_1^{-Q_{1\,i}} \alpha_2^{-Q_{2\,i}} X_i - X_i'), (\alpha_i \alpha_i^{-1} - 1) \rangle.$$

[1]It is example (D) with $N = 3$.

Here the fields X_i and X'_i denote the chiral fields of the model on the left, respectively the right of the defect. We also introduced auxiliary defect fields α_i and α_i^{-1}, for each of the $U(1)$ gauge groups satisfying $\alpha_i\alpha_i^{-1} = 1$. The α_i are charged under both, the gauge groups on the left and the right of the defect. The gauge-charges of the various fields are given by Table 3.3 below.

	X_1	X_2	X_3	X_4	X'_1	X'_2	X'_3	X'_4	α_1	α_1^{-1}	α_2	α_2^{-1}
Q_1^L	1	-2	1	0	0	0	0	0	1	-1	0	0
Q_2^L	0	1	-2	1	0	0	0	0	0	0	1	-1
Q_1^R	0	0	0	0	1	-2	1	0	-1	1	0	0
Q_2^R	0	0	0	0	0	1	-2	1	0	0	-1	1

Table 3.3: Fields of the identity defect of the A_2-model. Q_j^L denotes the charges of the fields under the left gauge group $U(1)_j^L$ and Q_j^R denotes the charges of the fields under the right gauge group $U(1)_j^R$.

As an aside note that pushing the GLSM identity defect to the orbifold on both sides requires setting $X_2 = X'_2 = X_3 = X'_3 = 1$ in $\mathcal{I}_{\mathrm{GLSM}_{A_2}}$. This imposes the relations $\alpha_1^{-2}\alpha_2 = 1$ and $\alpha_2^{-2}\alpha_1 = 1$, thereby realizing the gauge symmetry breaking to the subgroup

$$\mathbb{Z}_3 \subset U(1)_1 \times U(1)_2$$

on the level of rings describing the regular representation. The result is the module

$$\mathcal{I}_{\mathrm{orb}} = \mathbb{C}[X_1, X_4, X'_1, X'_4, \alpha_1]/\langle(\alpha_1^3 = 1), (X_1\alpha_1^{-1} - X'_1), (X_4\alpha_1^{-2} - X'_4)\rangle$$

associated to the identity defect in the orbifold model $\mathbb{C}^2/\mathbb{Z}_3$. It is built on 3 generators $e_i = \alpha_1^{i-1}$, of $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$-charges[2] $([i-1]_3, [-i+1]_3)$ satisfying relations

$$X'_1 e_1 = X_1 e_2, \quad X'_1 e_2 = X_1 e_3, \quad X'_1 e_3 = X_1 e_1, \quad X'_4 e_1 = X_4 e_3, \quad X'_4 e_2 = X_4 e_1, \quad X'_4 e_3 = X_4 e_2.$$

Orbifold lift

In order to attain the defects that lift the orbifold phase to the GLSM we first have to push the GLSM on the right side of the GLSM identity defect to the orbifold phase. To do this, we set w.l.o.g. $X'_2 = X'_3 = 1$ in $\mathcal{I}_{\mathrm{GLSM}_{A_2}}$. This yields the $\mathbb{C}[X_1, \ldots, X_4, X'_1, X'_4]$-module

$$\mathcal{T}_\infty = \frac{\mathbb{C}[X_1, \ldots, X_4, X'_1, X'_4, \alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}]}{\langle(\alpha_1^{-Q_{1\,i}}\alpha_2^{-Q_{2\,i}}X_i - X'_i)_{i=1,4}, (X_2 - \alpha_1^{-2}\alpha_2), (X_3 - \alpha_1\alpha_2^{-2}), (\alpha_i\alpha_i^{-1} - 1)\rangle} . \tag{3.1}$$

The gauge symmetry of the theory on the right of the defect is broken from $U(1)_1^R \times U(1)_2^R$ to

$$\left\{\left(e^{\frac{2\pi in}{3}}, e^{-\frac{2\pi in}{3}}\right) | n \in \mathbb{Z}\right\} \cong \mathbb{Z}_3.$$

[2] Here, $[\bullet]_d$ denotes the $\mathbb{Z}_d$-reduction of the integer written in brackets.

This symmetry breaking is realized by the relations

$$X_2 = \alpha_1^{-2}\alpha_2 \quad \text{and} \quad X_3 = \alpha_1\alpha_2^{-2}$$

in (3.1). As expected, $\mathcal{T}_\infty$ is not a finitely generated $\mathbb{C}[X_1, \dots, X_4, X_1', X_4']$-module. To construct the lift defects, we now proceed according to our prescription presented in Section 2.3.2 by introducing cutoffs to charges of the $U(1)$s unbroken at all the phase boundaries traversed by the chosen path. For the transition $(11) \leftrightarrow (01) \leftrightarrow (00)$ we obtain the cutoff

$$Q_1^L + 2Q_2^L \leq N$$

from the phase boundary $(11) \leftrightarrow (01)$ and the cutoff

$$Q_1^L \leq M$$

from $(01) \leftrightarrow (00)$. Instead of the module $\mathcal{T}_\infty$ we consider the submodule $\mathcal{T}_{N,M}$ generated by only those generators whose charges satisfy these inequalities. The choice of cutoff parameters N and M corresponds to the choice of homotopy class of path between the respective phases.

Note that the relations

$$X_3\alpha_1^{-1}\alpha_2^2 = 1 \quad \text{and} \quad X_2\alpha_1^2\alpha_2^{-1} = 1,$$

in (3.1) can be used to write any generator e as

$$e = \left(X_3\alpha_1^{-1}\alpha_2^2\right)^r \left(X_2\alpha_1^2\alpha_2^{-1}\right)^s e = X_3^r X_2^s \left(\alpha_1^{-1}\alpha_2^2\right)^r \left(\alpha_1^2\alpha_2^{-1}\right)^s e =: X_3^r X_2^s\, e'. \tag{3.2}$$

Now, the charges under the unbroken $U(1)$s of $\alpha_1^{-1}\alpha_2^2$ and $\alpha_1^2\alpha_2^{-1}$ are given by Table 3.4 below.

	$\alpha_1^{-1}\alpha_2^2$	$\alpha_1^2\alpha_2^{-1}$
$Q_1^L + 2Q_2^L$	3	0
Q_1^L	-1	2

Table 3.4: Charges of $\alpha_1^{-1}\alpha_2^2$ and $\alpha_1^2\alpha_2^{-1}$ under the gauge groups $\{(g, g^2)\,|\, g \in U(1)\}$ and $U(1)_1$ unbroken at the phase boundaries $(11) \leftrightarrow (01)$ and $(00) \leftrightarrow (10)$ respectively.

Thus, any generator of $\mathcal{T}_{N,M}$ can–via (3.2)–be written as a generator whose charges lie in the band

$$\begin{aligned} N - 3 &< Q_1^L + 2Q_2^L \leq N \\ M - 2 &< Q_1^L \leq M. \end{aligned} \tag{3.3}$$

Thus, $\mathcal{T}_{N,M}$ is generated by generators with charges in this band. In particular it is finitely generated.

Indeed, this matches precisely with the analysis of D-brane transport in [47]. The charge bands above are the range of charges of D-branes lifted from the orbifold phase to the GLSM by fusion with the respective lift defects. They precisely match with all band restriction rules for this model from [47] (example (D)):

$$
\begin{aligned}
&-\frac{3}{2} < \frac{\theta_1 + 2\theta_2}{2\pi} + Q_1^L + 2Q_2^L < \frac{3}{2} \\
&-1 < \frac{\theta_1}{2\pi} + Q_1^L < 1,
\end{aligned}
$$

where the choice of cutoff parameters M and N corresponds to the choice of homotopy class of paths determined by θ_1 and θ_2.

It is very easy to determine the module $\mathcal{T}_{N,M}$ from the charge bands. For $N-M$ even the independent generators satisfying (3.3) are

$$
e_1 := \alpha_1^M \alpha_2^{\frac{N-M}{2}}, \quad e_2 := \alpha_1^M \alpha_2^{\frac{N-M}{2}} \alpha_1^{-1}, \quad e_3 := \alpha_1^M \alpha_2^{\frac{N-M}{2}} \alpha_2^{-1}.
$$

They are subject to the relations

$$
\begin{aligned}
X_1' e_1 &= X_1 e_2 \\
X_1' e_2 &= X_1 X_2 e_3 \\
X_1' e_3 &= X_1 X_2 X_3 e_1 \\
X_4' e_1 &= X_4 e_3 \\
X_4' e_3 &= X_3 X_4 e_2 \\
X_4' e_2 &= X_2 X_3 X_4 e_1.
\end{aligned}
$$

The $U(1)_1^L \times U(1)_2^L \times \mathbb{Z}_3^R$-charges of the generators are given by

$$
\begin{array}{ll}
e_1: & \left(M, \frac{N-M}{2}, \left[\frac{1}{2}(N-3M)\right]_3\right) \\
e_2: & \left(M-1, \frac{N-M}{2}, \left[\frac{1}{2}(N-3M)+1\right]_3\right) \\
e_3: & \left(M, \frac{N-M}{2}-1, \left[\frac{1}{2}(N-3M)+2\right]_3\right).
\end{array}
$$

For $N-M$ odd the independent generators satisfying (3.3) are

$$
e_1' := \alpha_1^{M-1} \alpha_2^{\frac{N-M-1}{2}}, \quad e_2' := \alpha_1^{M-1} \alpha_2^{\frac{N-M-1}{2}} \alpha_1, \quad e_3' := \alpha_1^{M-1} \alpha_2^{\frac{N-M-1}{2}} \alpha_2.
$$

They satisfy the relations

$$
\begin{aligned}
X_1' e_1' &= X_1 X_2 X_3 e_3' \\
X_1' e_3' &= X_1 X_2 e_2' \\
X_1' e_2' &= X_1 e_1' \\
X_4' e_1' &= X_2 X_3 X_4 e_2' \\
X_4' e_2' &= X_3 X_4 e_3' \\
X_4' e_3' &= X_4 e_1'
\end{aligned}
\tag{3.4}
$$

and their $U(1)^L_1 \times U(1)^L_2 \times \mathbb{Z}^R_3$-charges are given by

$$\begin{array}{ll} e'_1: & \left(M-1, \frac{N-M-1}{2}, \left[\frac{1}{2}(N+1-3M)\right]_3\right) \\ e'_2: & \left(M, \frac{N-M-1}{2}, \left[\frac{1}{2}(N+1-3M)+1\right]_3\right) \\ e'_3: & \left(M-1, \frac{N-M+1}{2}, \left[\frac{1}{2}(N+1-3M)+2\right]_3\right). \end{array} \tag{3.5}$$

Brane Lift to the GLSM

Lifting D-branes from the orbifold phase to the GLSM is described by fusing the D-branes with the respective lift defects.

In order to illustrate this, we will fuse the lift defects with the fractional D0 branes in the orbifold phase. The latter correspond to the $R' := \mathbb{C}[X'_1, X'_4]$-modules

$$\mathbb{C}[X'_1, X'_4]/\langle X'_1, X'_4\rangle\{[n]_3\}. \tag{3.6}$$

The lift of (3.6) is given by the fusion with $\mathcal{T}_{N,M}$

$$\mathcal{T}_{N,M} * R'/\langle X'_1, X'_4\rangle\{[n]_3\} = (\mathcal{T}_{N,M} \otimes_{R'} R'/\langle X'_1, X'_4\rangle\{[n]_3\})^{\mathbb{Z}_3},$$

For concreteness let us consider the case $N-M$ odd and $\frac{1}{2}(N+1-3M)+n = 0 \mod 3$. Then the $\mathbb{Z}_3$-invariant generator in $\mathcal{T}_{N,M} \otimes_{R'} R'/\langle X'_1, X'_4\rangle\{[n]_3\}$ is $e'_1 \otimes 1$. Therefore, replacing the variables X'_1 and X'_4 according to the relations (3.4) we find the lift

$$\mathcal{T}(N,M) * R'/\langle X'_1, X'_4\rangle\{[n]_3\} = \mathbb{C}[X_1, X_2, X_3, X_4]/\langle X_1X_2X_3, X_2X_3X_4\rangle \left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\}.$$

This module can be expressed equivalently by its Koszul resolution

$$R \otimes \bigwedge\nolimits^2 V^\vee \left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\} \xrightarrow{d_1} R \otimes \left(V_4^\vee \left\{\left(M-1, \tfrac{N-M+1}{2}\right)\right\} \oplus V_1^\vee \left\{\left(M, \tfrac{N-M-1}{2}\right)\right\}\right) \xrightarrow{d_2} R\left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\},$$

where $V := \operatorname{span}\{\pi_1, \pi_4\} =: V_1 \oplus V_4$ is a two-dimensional vector space, $R := \mathbb{C}[X_1, X_2, X_3, X_4]$ and

$$d_1 = \begin{pmatrix} X_1 \\ -X_4 \end{pmatrix}, \quad d_2 = \begin{pmatrix} X_2X_3X_4 & X_1X_2X_3 \end{pmatrix}.$$

Indeed, we may arrive at the same result by taking a slightly different route, first determining the resolution of the fractional brane (3.6) and then performing the fusion with $\mathcal{T}_{N,M}$. The fractional brane (3.6) can be represented by the Koszul resolution

$$A^\bullet := R' \otimes \bigwedge\nolimits^2 V^\vee\{[n]_3\} \to R' \otimes (V_4^\vee\{[n+1]_3\} \oplus V_1^\vee\{[n-1]_3\}) \to R'\{[n]_3\},$$

with differential $d := \sum_i X'_i \pi_i \lrcorner$. To compute the lift

$$\mathcal{T}_{N,M} * A^\bullet = (\mathcal{T}_{N,M} \otimes_{R'} A^\bullet)^{\mathbb{Z}_3}$$

of (3.6) we have to determine the $\mathbb{Z}_3$-invariant part of $\mathcal{T}_{N,M} \otimes_{R'} R'\{[n]_3\}$. This can be accomplished by using the charges (3.5) of the generators e'_i of $\mathcal{T}_{N,M}$. One obtains

$$R \otimes \bigwedge\nolimits^2 V^\vee \left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\} \xrightarrow{d_1} R \otimes \left(V_4^\vee \left\{\left(M-1, \tfrac{N-M+1}{2}\right)\right\} \oplus V_1^\vee \left\{\left(M, \tfrac{N-M-1}{2}\right)\right\}\right) \xrightarrow{d_2} R\left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\}.$$

Here

$$d_1 = \begin{pmatrix} A \\ -B \end{pmatrix}, \; d_2 = \begin{pmatrix} C & D \end{pmatrix}$$

with

$$A = X_1\,, \quad B = X_4\,, \quad C = X_2X_3X_4\,, \quad D = X_1X_2X_3\,,$$

which has been obtained by replacing X_i' variables with the X_i variables according to the relations (3.4).

Of course, one can perform the calculations also for the other fractional branes. The lifts for these cases are given by

$$R \otimes \bigwedge^2 V^\vee \left\{\left(M-1, \tfrac{N-M+1}{2}\right)\right\} \xrightarrow{d_1} R \otimes (V_4^\vee \left\{\left(M, \tfrac{N-M-1}{2}\right)\right\} \oplus V_1^\vee \left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\} \xrightarrow{d_2} R\left\{\left(M-1, \tfrac{N-M+1}{2}\right)\right\},$$

for $\frac{1}{2}(N+1-3M)+n = -1 \bmod 3$ where now

$$A = X_1X_2X_3 \quad B = X_3X_4 \quad C = X_4 \quad D = X_1X_2$$

and

$$R \otimes \bigwedge^2 V^\vee \left\{\left(M, \tfrac{N-M-1}{2}\right)\right\} \xrightarrow{d_1} R \otimes (V_4^\vee \left\{\left(M-1, \tfrac{N-M-1}{2}\right)\right\} \oplus V_1^\vee \left\{\left(M-1, \tfrac{N-M+1}{2}\right)\right\} \xrightarrow{d_2} R\left\{\left(M, \tfrac{N-M-1}{2}\right)\right\},$$

for $\frac{1}{2}(N+1-3M)+n = -2 \bmod 3$ with

$$A = X_1X_2\,, \quad B = X_2X_3X_4\,, \quad C = X_3X_4\,, \quad D = X_1\,.$$

This agrees with the results in 8.4.2 (D) of [47], where a lift of the fractional branes was determined for the homotopy class of paths given by $\theta_1 = -\pi,\ \theta_2 = -\frac{\pi}{2}$. This corresponds to the choice of our cutoff parameters $(N, M) = (2, 1)$.

3.1.2 A Two Parameter Model with $\mathbb{C}^5/\mathbb{Z}_8$-orbifold phase

Next we will apply our construction to another two parameter model, which is defined by the following data

$$\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8} = (U(1)_1 \times U(1)_2, V, (r, \theta), W = 0),$$

this is example (C) in [47]. Here the representation V, can be read off from the charge assignment of the chiral matter content given by table 3.5 below.

	X_0	X_1	X_2	X_3	X_4	X_5	X_6
Q_1	-4	0	0	1	1	1	1
Q_2	0	1	1	0	0	0	-2

Table 3.5: Matter content of the two parameter model. Q_j denotes the $U(1)_j$-charge of the field X_i.

The phase structure of this model is similar to the one of the A_2-model. It exhibits an orbifold phase for $r_1,\ r_2 \to -\infty$. In this phase the fields X_0 and X_6 aquire a vev and the low energy theory is the orbifold theory $\mathbb{C}_5/\mathbb{Z}_8$. We call this phase III. It also features a geometric, or large volume phase–labeled I–in the regime $r_1,\ r_2 \to \infty$. This phase has an effective description by a non-linear sigma model on the total space of the line bundle $\mathcal{O}(-8)$ over the weighted projective space $\mathbb{P}_{(11222)}$. Beyond these, there are two mixed phases labeled II and IV.

The phase diagram of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$ is depicted in Figure 3.2 below.

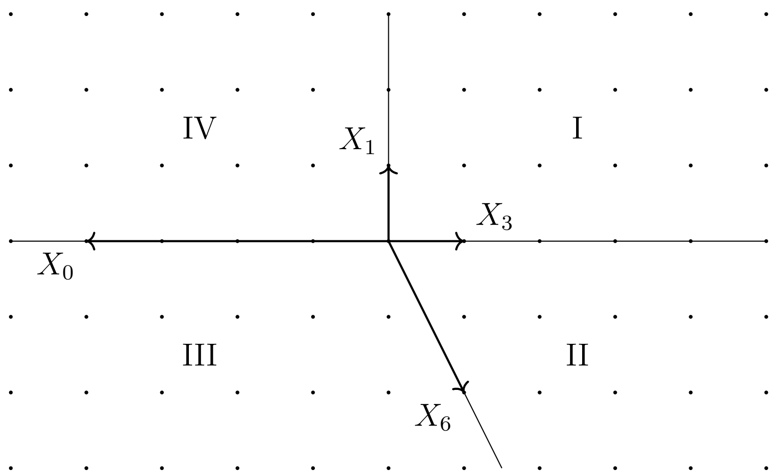


Figure 3.2: Phase diagram of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$. Here, I is the large volume phase, III is the orbifold phases and IV and II are mixed phases.

The locations of the phase boundaries and the respective unbroken gauge groups are given by Table 3.6 below.

phase boundary	location	unbroken gauge groups
(III) $\leftrightarrow$ (IV)	$\mathrm{Cone}_{\{0\}}$	$\mathbb{Z}_4 \times U(1)_2$
(IV) $\leftrightarrow$ (I)	$\mathrm{Cone}_{\{1\}}$	$U(1)_1$
(III) $\leftrightarrow$ (II)	$\mathrm{Cone}_{\{6\}}$	$\{(g^2, g) \mid g \in U(1)\}$
(II) $\leftrightarrow$ (I)	$\mathrm{Cone}_{\{3\}}$	$U(1)_2$

Table 3.6: Unbroken subgroups of $U(1)_1 \times U(1)_2$ at the phase boundaries of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$.

$\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$ Identity Defect

The identity defect of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$ is associated to the $\mathbb{C}[X_0,\ldots,X_6,X_0',\ldots,X_6']$-module

$$\mathcal{I}_{\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}} := \mathbb{C}[X_0,\ldots,X_6,X_0',\ldots,X_6',\alpha_1,\alpha_1^{-1},\alpha_2,\alpha_2^{-1}]/\langle(\alpha_1^{-Q_{1\,i}}\alpha_2^{-Q_{2\,i}}X_i - X_i'),(\alpha_i\alpha_i^{-1}-1)\rangle,$$

where an additional copy of the fields of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$ as well as auxiliary fields α_i and α_i^{-1} were introduced. The gauge-charges of the various fields are given by Table 3.7 below.

	X_0	$X_{1,2}$	$X_{3,4,5}$	X_6	X'_0	$X'_{1,2}$	$X'_{3,4,5}$	X'_6	α_1	α_1^{-1}	α_2	α_2^{-1}
Q_1^L	-4	0	1	1	0	0	0	0	1	-1	0	0
Q_2^L	0	1	0	-2	0	0	0	0	0	0	1	-1
Q_1^R	0	0	0	0	-4	0	1	1	-1	1	0	0
Q_2^R	0	0	0	0	0	1	0	-2	0	0	-1	1

Table 3.7: Fields of the identity defect of $\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}$. Q_j^L and Q_j^R denote the charge under the gauge groups $U(1)_j^L$, respectively $U(1)_j^R$ on the left and right of the defect.

Orbifold Lift

Starting from $\mathcal{I}_{\mathrm{GLSM}_{\mathbb{C}^5/\mathbb{Z}_8}}$ we push down the GLSM on the right side of the defect to the $\mathbb{C}^5/\mathbb{Z}_8$ orbifold phase by setting $X'_0 = X'_6 = 1$. This yields

$$\mathcal{T}_\infty \coloneqq \frac{\mathbb{C}[X_0, \ldots, X_6, X'_1, \ldots, X'_5, \alpha_i^{\pm 1}]}{\langle (X_0 - \alpha_1^{-4}), (\alpha_1^{-Q_{1\,i}}\alpha_2^{-Q_{2\,i}} X_i - X'_i), (X_6 - \alpha_1\alpha_2^{-2}), (\alpha_i \alpha_i^{-1} - 1)\rangle}. \tag{3.7}$$

The gauge symmetry on the right of the defect is broken according to

$$U(1)_1^R \times U(1)_2^R \to \{(g^2, g) | g^8 = 1\} \cong \mathbb{Z}_8.$$

This symmetry breaking is realized by the relations

$$X_0 = \alpha_1^{-4} \quad \text{and} \quad X_6 = \alpha_1 \alpha_2^{-2}.$$

Next, according to our prescription presented in Section 2.3.2 we have to introduce charge cutoffs in this module to obtain the modules associated to the lift defects. Let us first consider paths (III) $\leftrightarrow$ (IV) $\leftrightarrow$ (I) from the orbifold to the large volume phase transversing phase IV. The unbroken $U(1)$s on the phase boundaries traversed are $U(1)_2$ and $U(1)_1$ respectively. Thus, we have to impose charge cutoffs

$$Q_2^L \leq N$$

and

$$Q_1^L \leq M$$

for a choice of integers M and N. The submodule $\mathcal{T}_{N,M}^{\text{III-IV-I}} \subset \mathcal{T}_\infty$ with generators whose charges satisfy these inequalities corresponds to the desired lift defect. Here the choice of cutoff parameters corresponds to a choice of homotopy class of paths from phase III to phase I traversing phase IV.

As in the A_2-example discussed above, the relations in $\mathcal{T}_\infty$ render $\mathcal{T}_{N,M}^{\text{III-IV-I}}$ finitely generated. More precisely, we have the relations

$$X_6 \alpha_1^{-1} \alpha_2^2 = 1 \text{ and } X_0 \alpha_1^4 = 1. \tag{3.8}$$

The charges under the unbroken $U(1)$s of $\alpha_1^{-1}\alpha_2^2$ and α_1^4 are given by Table 3.8 below.

	$\alpha_1^{-1}\alpha_2^2$	α_1^4
Q_2^L	2	0
Q_1^L	-1	4

Table 3.8: Charges of $\alpha_1^{-1}\alpha_2^2$ and α_1^4 under the groups $U(1)_2$ and $U(1)_1$ unbroken at the phase boundaries (III) $\leftrightarrow$ (IV) and (IV) $\leftrightarrow$ (I) respectively.

Hence–by the same argument we made for the A_2 model–the charges of independent generators of the truncated module $\mathcal{T}_{N,M}^{\text{III-IV-I}}$ are also bounded from below and lie in the bands

$$\begin{aligned} N-2 &< Q_2^L \leq N \\ M-4 &< Q_1^L \leq M. \end{aligned} \tag{3.9}$$

It is not difficult to determine all the generators of $\mathcal{T}_{N,M}^{\text{III-IV-I}}$. The independent generators satisfying (3.9) are given by

$$\alpha_1^{M-3}\alpha_2^{N-1}\{e_1 \coloneqq 1,\ e_2 \coloneqq \alpha_1^3\alpha_2,\ e_3 \coloneqq \alpha_1^3,\ e_4 \coloneqq \alpha_1^2\alpha_2\ \ e_5 \coloneqq \alpha_1^2,\ e_6 \coloneqq \alpha_1\alpha_2,\ e_7 \coloneqq \alpha_1,\ e_8 \coloneqq \alpha_2\}.$$

As for the paths (III) $\leftrightarrow$ (II) $\leftrightarrow$ (I) from phase III to phase I traversing the other intermediate phase II, the respective unbroken $U(1)$s are given by $\{(g^2, g)\,|\, g \in U(1)\}$ and $U(1)_2$. Therefore, the corresponding cutoffs read

$$2Q_1^L + Q_2^L \leq N \text{ and } Q_2^L \leq M.$$

By virtue of the relations (3.8), one again finds that the truncated modules $\mathcal{T}_{N,M}^{\text{III-II-I}}$ are finitely generated. The charges of $\alpha_1^{-1}\alpha_2^2$ and α_1^4 under the unbroken $U(1)$s are given by Table 3.9 below.

	$\alpha_1^{-1}\alpha_2^2$	α_1^4
$2Q_1^L + Q_2^L$	0	8
Q_2^L	2	0

Table 3.9: Charges of $\alpha_1^{-1}\alpha_2^2$ and α_1^4 under the groups $\{(g^2, g)\,|\, g \in U(1)\}$ and $U(1)_2$ unbroken at the phase boundaries (III) $\leftrightarrow$ (II) and (II) $\leftrightarrow$ (I) respectively.

Therefore, the independent generators of $\mathcal{T}_{N,M}^{\text{III-II-I}}$ are the ones whose charges lie in the band

$$\begin{aligned} N-8 &< 2Q_1^L + Q_2^L \leq N \\ M-2 &< Q_2^L \leq M. \end{aligned} \tag{3.10}$$

The independent generators of $\mathcal{T}_{N,M}^{\text{III-II-I}}$ are given by

$$\alpha_1^{\frac{N-M-7}{2}}\alpha_2^{M-1}\{e'_1 \coloneqq \alpha_1^4,\ e'_2 \coloneqq \alpha_1^3\alpha_2,\ e'_3 \coloneqq \alpha_1^3,\ e'_4 \coloneqq \alpha_1^2\alpha_2\ \ e'_5 \coloneqq \alpha_1^2,\ e'_6 \coloneqq \alpha_1\alpha_2,\ e'_7 \coloneqq \alpha_1,\ e'_8 \coloneqq \alpha_2\}$$

for $N-M$ odd and

$$\alpha_1^{\frac{N-M-6}{2}}\alpha_2^{M-1}\{e_1,\ e_2,\ e_3,\ e_4\ e_5,\ e_6,\ e_7,\ e_8\}$$

for $N-M$ even.

As for the A_2-model we find complete agreement with [47]. The charge bands (3.9) and (3.10) of the lift defects associated to paths (III) $\leftrightarrow$ (IV) $\leftrightarrow$ (I), respectively (III) $\leftrightarrow$ (II) $\leftrightarrow$ (I) match with the respective band restriction rules. For (III) $\leftrightarrow$ (IV) $\leftrightarrow$ (I) the band restriction rules read

$$\begin{aligned} -1 &< \frac{\theta_2}{2\pi} + 2Q_2^L < 1 \\ -2 &< \frac{\theta_1}{2\pi} + Q_1^L < 2 \end{aligned}$$

and

$$\begin{aligned} -4 &< \frac{2\theta_1+\theta_2}{2\pi} + 2Q_1^L + Q_2^L < 4 \\ -1 &< \frac{\theta_2}{2\pi} + Q_2^L < 1. \end{aligned}$$

for (III) $\leftrightarrow$ (II) $\leftrightarrow$ (I).

To be more concrete, let us briefly list the charges of the modules for specific choices of cutoff parameters. The $U(1)_1^L\times U(1)_2^R\times\mathbb{Z}_8^R$ charges of the generators of $\mathcal{T}_{\text{III-IV-I}}(1,3)$ as well as $\mathcal{T}_{\text{III-II-I}}(7,1)$ are given by Table 3.10 below, whereas the generators of $\mathcal{T}_{\text{III-II-I}}(8,1)$ have charges given by Table 3.11 below.

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
$(0,0,[0]_8)$	$(3,1,[1]_8)$	$(3,0,[2]_8)$	$(2,1,[3]_8)$	$(2,0,[4]_8)$	$(1,1,[5]_8)$	$(1,0,[6]_8)$	$(0,1,[7]_8)$

Table 3.10: Charges of the generators of $\mathcal{T}_{\text{III-IV-I}}(1,3)$ and $\mathcal{T}_{\text{III-II-I}}(7,1)$.

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
$(4,0,[0]_8)$	$(3,1,[1]_8)$	$(3,0,[2]_8)$	$(2,1,[3]_8)$	$(2,0,[4]_8)$	$(1,1,[5]_8)$	$(1,0,[6]_8)$	$(0,1,[7]_8)$

Table 3.11: Charges of the generators of $\mathcal{T}_{\text{III-II-I}}(8,1)$.

This agrees with the band restrictions

$$\begin{aligned} \mathfrak{C}^{w_1}_{\text{III,II}}\cap\mathfrak{C}^{w'}_{\text{II,I}} = \mathfrak{C}^{w'}_{\text{III,IV}}\cap\mathfrak{C}^{w''}_{\text{IV,I}} &= \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}, \\ \mathfrak{C}^{w_0}_{\text{III,II}}\cap\mathfrak{C}^{w'}_{\text{II,I}} &= \{(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(4,0)\}, \end{aligned}$$

determined in [47] Section 8.4.2. (C), where the roman numerals label the phases and

$$\begin{aligned} w_0 &: \ -10\pi < 2\theta_1+\theta_2 < -8\pi \\ w_1 &: \ -8\pi < 2\theta_1+\theta_2 < -6\pi \\ w' &: \ -2\pi < \theta_2 < 0. \end{aligned}$$

Brane Lift to the GLSM

In [47] Section 8.4.2. (C) a lift of the fractional brane

$$\mathbb{C}[X_1', \dots, X_5']/\langle X_1', \dots, X_5' \rangle\{[0]_8\}$$

from the $\mathbb{C}^5/\mathbb{Z}_8$-phase to the GLSM is performed. For comparisons sake we briefly outline the lift of this brane via fusion with the transition defects constructed above. The fractional brane can be represented by the $R' \coloneqq \mathbb{C}[X_1', \dots, X_5']$-free resolution

$$\begin{tikzcd}
 & & & \Lambda^3 V^\vee \otimes R'\{[2]_8\} \arrow[dr, "Y"] & & & & \\
 & & \Lambda^4 V^\vee \otimes R'\{[1]_8\} \arrow[ur, "X"] \arrow[dr, "Y"] & & \Lambda^2 V^\vee \otimes R'\{[4]_8\} \arrow[dr, "Y"] & & & \\
0 \arrow[r] & \Lambda^5 V^\vee \otimes R'\{[0]_8\} \arrow[ur, "X"] \arrow[dr, "Y"] & & \Lambda^3 V^\vee \otimes R'\{[3]_8\} \arrow[ur, "X"] \arrow[dr, "Y"] & & V^\vee \otimes R'\{[6]_8\} \arrow[dr, "Y"] & & \\
 & & \Lambda^4 V^\vee \otimes R'\{[2]_8\} \arrow[ur, "X"] \arrow[dr, "Y"] & & \Lambda^2 V^\vee \otimes R'\{[5]_8\} \arrow[ur, "X"] \arrow[dr] & & R'\{[0]_8\} \arrow[r] & 0, \\
 & & & \Lambda^3 V^\vee \otimes R'\{[4]_8\} \arrow[ur, "X"] \arrow[dr, "Y"] & & V^\vee \otimes R'\{[7]_8\} \arrow[ur, "X"] & & \\
 & & & & \Lambda^2 V^\vee \otimes R'\{[6]_8\} \arrow[ur, "X"] & & &
\end{tikzcd}$$

where $V \coloneqq \operatorname{span}\{\pi_1, \dots, \pi_5\}$ is a five-dimensional vector space and

$$X = \sum_{i=1}^{2} X_i' \pi_i \lrcorner$$
$$Y = \sum_{i=3}^{5} X_i' \pi_i \lrcorner \,.$$

We lift this brane via fusion with $\mathcal{T}_{\text{III-IV-I}}(1,3)$, this yields the band restricted brane

$$\begin{tikzcd}
 & & & \Lambda^3 V^\vee \otimes R\{(1,0)\} \arrow[dr, "Y"] & & & & \\
 & & \Lambda^4 V^\vee \otimes R\{(0,1)\} \arrow[ur, "XX_6"] \arrow[dr, "Y"] & & \Lambda^2 V^\vee \otimes R\{(2,0)\} \arrow[dr, "Y"] & & & \\
0 \arrow[r] & \Lambda^5 V^\vee \otimes R\{(0,0)\} \arrow[ur, "X"] \arrow[dr, "Y"] & & \Lambda^3 V^\vee \otimes R\{(1,1)\} \arrow[ur, "XX_6"] \arrow[dr, "Y"] & & V^\vee \otimes R\{(3,0)\} \arrow[dr, "YX_0"] & & \\
 & & \Lambda^4 V^\vee \otimes R\{(1,0)\} \arrow[ur, "X"] \arrow[dr, "Y"] & & \Lambda^2 V^\vee \otimes R\{(2,1)\} \arrow[ur, "XX_6"] \arrow[dr] & & R\{(0,0)\} \arrow[r] & 0. \\
 & & & \Lambda^3 V^\vee \otimes R\{(2,0)\} \arrow[ur, "X"] \arrow[dr, "Y"] & & V^\vee \otimes R\{(3,1)\} \arrow[ur, "XX_0X_6"] & & \\
 & & & & \Lambda^2 V^\vee \otimes R\{(3,0)\} \arrow[ur, "X"] & & &
\end{tikzcd}$$

Just as for the A_2-model considered in the previous section this agrees with the respective lift in [47].

3.1.3 The A_{N-1}-Model

Next, we will generalize the treatment of the A_2-model from Section 3.1.1 to the GLSMs associated to A_{N-1}-singularities for general N. This GLSM is given by the data

$$\mathrm{GLSM}_{A_{N-1}} \coloneqq (U(1)^{N-1},\ V,\ (r,\theta),\ W=0),$$

where the representation V, can be read off from the charge assignment of the chiral matter content given by table 3.12 below.

	X_1	X_2	X_3	X_4	$\dots$	X_{N-1}	X_N	X_{N+1}
Q_1	1	-2	1	0	$\dots$ 0	0	0	0
Q_2	0	1	-2	1	$\dots$ 0	0	0	0
$\vdots$	$\vdots$							$\vdots$
Q_{N-2}	0	0	0	0	$\dots$ 1	-2	1	0
Q_{N-1}	0	0	0	0	$\dots$ 0	1	-2	1

Table 3.12: Matter content of the A_{N-1}-model. Q_j denotes the charge under $U(1)_j$.

Just as the A_2-model the A_{N-1}-model for general N also exhibits an orbifold phase for $r_1,\dots,r_{N-1}\to-\infty$. In this phase, the fields $X_2,\dots,X_N$ aquire a non-trivial vacuum expecation value, and the low energy theory is given by the orbifold model $\mathbb{C}^2/\mathbb{Z}_N$ with fields X_1 and X_{N+1} whose $\mathbb{Z}_N$-charges are 1 and -1 respectively. In the opposite limit $r_1,\dots,r_{N-1}\to\infty$ the theory is effectively described by a non-linear sigma model on the resolution of the A_{N-1}-singularity. This is the geometric or large volume phase. In total, the model has 2^{N-1} phases associated to the various partial resolutions of the singularity. The A_{N-1} singularity has an exceptional divisor of $N-1$ 2-spheres intersecting according to the A_{N-1}-Dynkin diagram, and a partial resolution is determined by which of these 2-spheres is blown up. We label the different phases by an $N-1$-tuple, whose ith entry is either 0 or 1 according to whether the ith 2-sphere in the exceptional divisor is blown up, or down. In particular $(00\dots0)$ corresponds to the full resolution of the singularity and therefore to the large volume phase, whereas $(11\dots1)$ denotes to the unresolved singularity and hence the orbifold phase. For more details on the A_{N-1}-model, see e.g. example (D) in [47].

$\mathrm{GLSM}_{A_{N-1}}$ Identity Defect

To write down the identity defect of $\mathrm{GLSM}_{A_{N-1}}$ we introduce a pair of auxiliary defect fields for each $U(1)$-factor of the gauge group denoted $\alpha_i,\ \alpha_i^{-1},\quad i=1,\dots,N-1$. The fields α_i carry $U(1)_i^L\times U(1)_i^R$-charge $(1,-1)$ and are uncharged under all other $U(1)$s whereas the fields α_i^{-1} carry $U(1)_i^L\times U(1)_i^R$-charge $(-1,1)$ and are likewise uncharged with respect to the other $U(1)$s. The $\mathrm{GLSM}_{A_{N-1}}$ identity defect is associated to the

$\mathbb{C}[X_1,\ldots,X_{N+1},X'_1,\ldots,X'_{N+1}]$-module

$$\mathcal{I}_{\mathrm{GLSM}_{A_{N-1}}} := \frac{\mathbb{C}[X_1,\ldots,X_{N+1},X'_1,\ldots,X'_{N+1},\alpha_1,\ldots,\alpha_{N-1},\alpha_1^{-1},\ldots,\alpha_{N-1}^{-1}]}{\langle(\alpha_1^{-Q_{1\,i}}\ldots\alpha_{N-1}^{-Q_{N-1\,i}}X_i - X'_i)_{i=1,\ldots,N+1},(\alpha_i\alpha_i^{-1}-1)_{i=1,\ldots,N-1}\rangle},$$

where the fields $X_1,\ldots,X_{N-1}$ and $X'_1,\ldots,X'_{N-1}$ are the bulk fields of the GLSM on the left, respectively right of the defect.

Orbifold Lift

To construct the defects lifting the orbifold phase to $\mathrm{GLSM}_{A_{N-1}}$ for arbitrary N we proceed in an analogous fashion to the case $N=3$ discussed in Section 3.1.1 above. We start out by pushing down the right side of the GLSM identity defect to the orbifold phase. This is accomplished by setting $X'_2=\cdots=X'_N=1$ in $\mathcal{I}_{\mathrm{GLSM}_{A_{N-1}}}$ and yields the $\mathbb{C}[X_1,\ldots,X_{N+1},X'_1,X'_{N+1}]$-module

$$\mathcal{T}_\infty := \frac{\mathbb{C}[X_1,\ldots,X_{N+1},X'_1,X'_{N+1},\alpha_1,\ldots,\alpha_{N-1},\alpha_1^{-1},\ldots,\alpha_{N-1}^{-1}]}{\langle(\alpha_1^{-1}X_1-X'_1),(\alpha_{N-1}^{-1}X_{N+1}-X'_{N+1}),(\alpha_1^{-Q_{1\,i}}\ldots\alpha_{N-1}^{-Q_{N-1\,i}}X_i-1)_{i=2,\ldots,N},(\alpha_i\alpha_i^{-1}-1)_{i=1,\ldots,N-1}\rangle}. \tag{3.11}$$

The gauge group on the right side of the defect is broken according to

$$U(1)_1^R\times\cdots\times U(1)_{N-1}^R \longrightarrow \{(g,g^2,\ldots,g^{N-1})|g\in\mathbb{Z}_N\}.$$

This symmetry breaking is realized by the relations

$$\alpha_1^{-Q_{1\,i}}\ldots\alpha_{N-1}^{-Q_{N-1\,i}}X_i=1,\ i=2,\ldots,N.$$

The Case $N=4$

We proceed by first discussing the A_3 case and then generalizing to arbitrary A_{N-1}. The phase structure of the A_3-model is depicted in Figure 3.3 below.

To exemplify our construction, we will derive the lift defects associated to paths from the orbifold phase (111) to the geometric phase (000) along two different classes of paths

$$(111)\leftrightarrow(011)\leftrightarrow(001)\leftrightarrow(000)$$

and

$$(111)\leftrightarrow(101)\leftrightarrow(001)\leftrightarrow(000).$$

We start out with the first one, $(111)\leftrightarrow(011)\leftrightarrow(001)\leftrightarrow(000)$. The relevant unbroken $U(1)$s at the traversed phase boundaries are given by Table 3.13 below.

phase boundary	location	unbroken $U(1)$
$(111)\leftrightarrow(011)$	$\mathrm{Cone}_{\{2,3\}}$	$\{(g,g^2,g^3)\,\|\,g\in U(1)\}$
$(011)\leftrightarrow(001)$	$\mathrm{Cone}_{\{2,5\}}$	$\{(g,g^2,1)\,\|\,g\in U(1)\}$
$(001)\leftrightarrow(000)$	$\mathrm{Cone}_{\{4,5\}}$	$\{(g,1,1)\,\|\,g\in U(1)\}$

Table 3.13: Unbroken subgroups of $U(1)^3$ at the phase boundaries $(111)\leftrightarrow(011)$, $(011)\leftrightarrow(001)$ and $(001)\leftrightarrow(000)$ of GLSM_{A_3}.

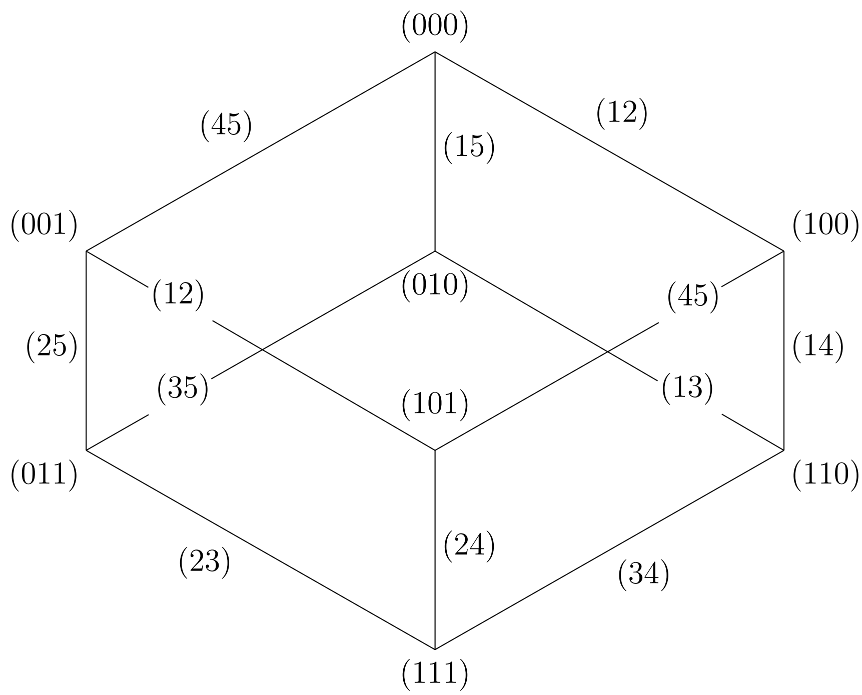


Figure 3.3: Phase diagram of GLSM_{A_3}. Here, phases are depicted by vertices, and phase boundaries by edges between them. The orbifold phase is labeled by (111) and the large volume phase where the singularity is fully resolved is labeled by (000). The phase boundaries (ij) are located at $\mathrm{Cone}_{\{i,j\}}$.

Thus, to obtain the modules describing the lift defects for this type of path, our construction requires to introduce charge cutoffs

$$\begin{aligned} &Q_1^L + 2Q_2^L + 3Q_3^L \leq N \\ &Q_1^L + 2Q_2^L \leq M \\ &Q_1^L \leq K. \end{aligned} \tag{3.12}$$

in the module $\mathcal{T}_\infty$. Let $\mathcal{T}_{N,M,K}^{(111)-(011)-(001)-(000)}$ be the submodule of $\mathcal{T}_\infty$ generated by all generators whose charges satisfy (3.12). Due to the relations

$$\begin{aligned} &X_2\alpha_1^2\alpha_2^{-1} = 1 \\ &X_3\alpha_1^{-1}\alpha_2^2\alpha_3^{-1} = 1 \\ &X_4\alpha_2^{-1}\alpha_3^2 = 1 \end{aligned} \tag{3.13}$$

in $\mathcal{T}_\infty$ the module $\mathcal{T}_{N,M,K}^{(111)-(011)-(001)-(000)}$ is in fact finitely generated. Namely, the charges under the unbroken $U(1)$s at the phase boundaries of $\alpha_1^2\alpha_2^{-1}$, $\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$ and $\alpha_2^{-1}\alpha_3^2$ are given by Table 3.14 below.

	$\alpha_1^2\alpha_2^{-1}$	$\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$	$\alpha_2^{-1}\alpha_3^2$
$Q_1^L+2Q_2^L+3Q_3^L$	0	0	4
$Q_1^L+2Q_2^L$	0	3	-2
Q_1^L	2	-1	0

Table 3.14: Charges of $\alpha_1^2\alpha_2^{-1}$, $\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$ and $\alpha_2^{-1}\alpha_3^2$ under the groups $\{(g,g^2,g^3)\,|\,g\in U(1)\}$, $\{(g,1,1)\,|\,g\in U(1)\}$ and $U(1)_1$ unbroken at the phase boundaries $(111)\leftrightarrow(011)$, $(011)\leftrightarrow(001)$ and $(001)\leftrightarrow(000)$ respectively.

Hence, $\mathcal{T}_{N,M,K}^{(111)-(011)-(001)-(000)}$ is in fact generated by all the generators of $\mathcal{T}_\infty$ whose charges lie in the finite band

$$\begin{aligned}&N-4<Q_1^L+2Q_2^L+3Q_3^L\leq N\\&M-3<Q_1^L+2Q_2^L\leq M\\&K-2<Q_1^L\leq K.\end{aligned}$$

This matches with the band restriction rules for all homotopy classes of paths along $(111)\leftrightarrow(011)\leftrightarrow(001)\leftrightarrow(000)$, from [47].

It is not difficult to describe $\mathcal{T}_{N,M,K}^{(111)-(011)-(001)-(000)}$ concretely. For instance for $K\in 2\mathbb{Z}$ and $M\in 3\mathbb{Z}$ a set of independent generators is given by

$$\alpha_1^K\alpha_2^{M-\frac{K}{2}}\alpha_3^{N-\frac{M}{3}}\{1,\alpha_1^{-1},\alpha_2^{-1},\alpha_3^{-1}\}.$$

For the second type of paths $(111)\leftrightarrow(101)\leftrightarrow(001)\leftrightarrow(000)$ we proceed in the same fashion. The relevant unbroken gauge groups at the traversed phase boundaries in that case are given by Table 3.15 below.

phase boundary	location	unbroken gauge group
$(111)\leftrightarrow(101)$	$\mathrm{Cone}_{\{2,4\}}$	$\{(g,g^2,ga)\,|\,g\in U(1),\,a\in\mathbb{Z}_2\}$
$(101)\leftrightarrow(001)$	$\mathrm{Cone}_{\{1,2\}}$	$\{(1,1,g)\,|\,g\in U(1)\}$
$(001)\leftrightarrow(000)$	$\mathrm{Cone}_{\{4,5\}}$	$\{(g,1,1)\,|\,g\in U(1)\}$

Table 3.15: Unbroken subgroups of $U(1)^3$ at the phase boundaries $(111)\leftrightarrow(011)$, $(101)\leftrightarrow(001)$ and $(001)\leftrightarrow(000)$ of GLSM_{A_3}.

Thus, the truncation of $\mathcal{T}_\infty$ describing the lift defects for this type of path is given by

$$\begin{aligned}&Q_1^L+2Q_2^L+Q_3^L\leq N\\&Q_3^L\leq M\\&Q_1^L\leq K.\end{aligned}\tag{3.14}$$

We denote the submodule of $\mathcal{T}_\infty$ generated by generators whose charges satisfy the inequalities (3.14) by $\mathcal{T}_{N,M,K}^{(111)-(101)-(001)-(000)}$. Using the relations (3.13) we again find that the latter module is generated by generators of charges in a finite charge band. Namely, the charges of $\alpha_1^2\alpha_2^{-1}$, $\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$ and $\alpha_2^{-1}\alpha_3^2$ under the unbroken $U(1)$s for this type of path are given by Tabel 3.16 below.

	$\alpha_1^2\alpha_2^{-1}$	$\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$	$\alpha_2^{-1}\alpha_3^2$
$Q_1^L+2Q_2^L+Q_3^L$	0	2	0
Q_3^L	0	-1	2
Q_1^L	2	-1	0

Table 3.16: Charges of $\alpha_1^2\alpha_2^{-1}$, $\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$ and $\alpha_2^{-1}\alpha_3^2$ under the groups $\{(g,g^2,g^3)\,|\,g\in U(1)\}$, $U(1)_3$ and $U(1)_1$ unbroken at the phase boundaries $(111)\leftrightarrow(011)$, $(101)\leftrightarrow(001)$ and $(001)\leftrightarrow(000)$ respectively.

and hence the charges of the generators of $\mathcal{T}_{N,M,K}^{(111)-(101)-(001)-(000)}$ lie in the band

$$\begin{aligned}
&N-2<Q_1^L+2Q_2^L+Q_3^L\leq N\\
&M-2<Q_3^L\leq M\\
&K-2<Q_1^L\leq K.
\end{aligned}$$

Again, this reproduces all band restriction rules for these type of paths from [47].

Also in this case the modules can be concretely described. For instance for $M+K\in 2\mathbb{Z}$ a set of independent generators is given by

$$\alpha_1^K\alpha_2^{N-\frac{M+K}{2}}\alpha_3^M\{1,\alpha_1^{-1},\alpha_3^{-1},\alpha_1^{-1}\alpha_2^{-1}\alpha_3\}.$$

General N

Since the phase structure of the A_{N-1} model becomes increasingly more complicated for higher values of N we will restrict our discussion of orbifold lifts to those associated to classes of paths

$$(00\ldots000)\leftrightarrow(00\ldots001)\leftrightarrow(00\ldots011)\leftrightarrow\cdots\leftrightarrow(01\ldots111)\leftrightarrow(11\ldots111).$$

The unbroken gauge group at the i-th phase boundary traversed by such a path is given by

$$\{(g,g^2,g^3,\ldots,g^i,1,1,\ldots,1)|g\in U(1)\}.$$

By our construction, the respective lift defects are associated to the truncated submodule $\mathcal{T}_{M_1,\ldots,M_{N-1}}$ which is generated by those generators in $\mathcal{T}_\infty$, whose charges satisfy the inequalities

$$
\begin{aligned}
&Q_1^L + 2Q_2^L + 3Q_3^L + \cdots + (N-3)Q_{N-3}^L + (N-2)Q_{N-2}^L + (N-1)Q_{N-1}^L \leq M_{N-1} \\
&Q_1^L + 2Q_2^L + 3Q_3^L + \cdots + (N-3)Q_{N-3}^L + (N-2)Q_{N-2}^L \leq M_{N-2} \\
&Q_1^L + 2Q_2^L + 3Q_3^L + \cdots + (N-3)Q_{N-3}^L \leq M_{N-3} \\
&\vdots \\
&Q_1^L + 2Q_2^L \leq M_2 \\
&Q_1^L \leq M_1.
\end{aligned}
$$

These implement the cut offs for the charges of generators with respect to the $U(1)$'s unbroken at the phase boundaries crossed by the paths in the homotopy class we are considering. As in the A_2 and A_3 cases, we can use the relations

$$
\begin{aligned}
&X_2\alpha_1^2\alpha_2^{-1} = 1 \\
&X_3\alpha_1^{-1}\alpha_2^2\alpha_3^{-1} = 1 \\
&X_4\alpha_2^{-1}\alpha_3^2\alpha_4^{-1} = 1 \\
&\vdots \\
&X_{N-1}\alpha_{N-3}^{-1}\alpha_{N-2}^2\alpha_{N-1}^{-1} = 1 \\
&X_N\alpha_{N-2}^{-1}\alpha_{N-1}^2 = 1
\end{aligned}
\tag{3.15}
$$

in $\mathcal{T}_\infty$ to show that the generators of $\mathcal{T}_{M_1,\ldots,M_{N-1}}$ have charges in a finite charge band. For this, we observe that the charges of the relevant monomials in the α_i are given by Table 3.17 below.

	$\alpha_1^2\alpha_2^{-1}$	$\alpha_1^{-1}\alpha_2^2\alpha_3^{-1}$	$\alpha_2^{-1}\alpha_3^2\alpha_4^{-1}$	$\ldots$	$\alpha_{N-2}^{-1}\alpha_{N-1}^2$
Q_1^L	2	-1	0	$\ldots$	0
$Q_1^L + 2Q_2^L$	0	3	-2	$\ldots$	$\vdots$
$Q_1^L + 2Q_2^L + 3Q_3^L$	0	0	4	0	0
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$-(N-2)$
$\sum_{i=1}^{N-1} iQ_i^L$	0	$\ldots$	0	0	N

Table 3.17: Charges of the relevant monomials under the $U(1)_1$ groups unbroken at the phase boundaries $(00\ldots000) \leftrightarrow (00\ldots001) \leftrightarrow (00\ldots011) \leftrightarrow \cdots \leftrightarrow (01\ldots111) \leftrightarrow (11\ldots111)$.

Therefore, $\mathcal{T}_{M_1,\ldots,M_{N-1}}$ is generated by generators of $\mathcal{T}_\infty$ whose charges lie in the band

$$\begin{aligned}
&M_{N-1} - (N-1) \leq Q_1^L + 2Q_2^L + \cdots + (N-1)Q_{N-1}^L \leq M_{N-1} \\
&M_{N-2} - (N-2) \leq Q_1^L + 2Q_2^L + \cdots + (N-2)Q_{N-2}^L \leq M_{N-2} \\
&M_{N-3} - (N-3) \leq Q_1^L + 2Q_2^L + \cdots + (N-3)Q_{N-3}^L \leq M_{N-3} \\
&\vdots \\
&M_2 - 2 \leq Q_1^L + 2Q_2^L \leq M_2 \\
&M_1 - 1 \leq Q_1^L \leq M_1.
\end{aligned} \tag{3.16}$$

Note that here, for convenience in solving the inequalities, we slightly changed the presentation of the charge bands as compared to the treatment of the A_2 and A_3 cases above, in that we added one to the lower bound and replaced the strict inequality with a $\leq$.

The inequalities (3.16) can be solved iteratively from bottom to top plugging in the solutions of the previous inequalities into the next. By induction we find that there are N solutions $(Q_1^L, Q_2^L \ldots, Q_{N-1}^L)$ to (3.16) and that $\sum_{j=1}^n jQ_j^L$ assumes $n+1$ consecutive integer values on the solutions for all $1 \leq n < N$. Starting from the last line we have that

$$Q_1^L = M_1 - 1, M_1.$$

Thus Q_1^L assumes two consecutive integer values on all solutions of the inequalities. The solutions for Q_1^L can then be plugged into the second to last line, which yields

$$\frac{M_2 - Q_1^L - 2}{2} \leq Q_2^L \leq \frac{M_2 - Q_1^L}{2}.$$

Thus, if 2 does not divide $(M_2 - Q_1^L)$ we have that Q_2^L has only one solution namely

$$Q_2^L = \frac{M_2 - Q_1^L - 1}{2}.$$

However, since Q_1^L runs through 2 consecutive integers, for one of them 2 does divide $M_2 - Q_1^L$. For this Q_1^L there are two solutions for Q_2^L:

$$Q_2^L = \frac{M_2 - Q_1^L - 2}{2}, \frac{M_2 - Q_1^L}{2}.$$

This yields 3 solutions, on which $Q_1^L + 2Q_2^L$ assumes 3 consecutive integer values. We can now go on inductively. For Q_n^L we have the inequality

$$\frac{M_n - n - \sum_{j=1}^{n-1} jQ_j^L}{n} \leq Q_n^L \leq \frac{M_n - \sum_{j=1}^{n-1} jQ_j^L}{n}.$$

Thus, if n does not divide $M_n - \sum_{j=1}^{n-1} jQ_j^L$ we find that Q_n^L has only one solution. However $\sum_{j=1}^{n-1} jQ_j^L$ assumes n consecutive integers. So for one of the solutions, $M_n - \sum_{j=1}^{n-1} jQ_j^L$ is divisible by n, in which case there are two solutions. So the last n inequalities have $n+1$ solutions, on which $\sum_{j=1}^{n} jQ_j^L$ take $n+1$ consecutive integer values. T herefore all the inequalities have N solutions.

For concreteness, let us pick $M_i = i$. Then the solutions to (3.16) are given by

$$(Q_1^L, Q_2^L \dots, Q_{N-1}^L) \in \{(0,\dots,0),(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}\,,$$

i.e. either all the Q_i^L are 0, or one of them is 1 and all the others are 0. The corresponding generators are

$$\{\alpha_0 \coloneqq 1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{N-1}\},$$

and they satisfy relations

$$X_1'\alpha_n = \begin{cases} X_1X_2\dots X_n\alpha_{n-1}, & n = 2,3,\dots,N \\ X_1\alpha_N, & n = 1 \end{cases}$$

and

$$X_{N+1}'\alpha_n = \begin{cases} X_{n+2}X_{n+3}\dots X_{N+1}\alpha_{n+1}, & n = 1,2,\dots,N-1 \\ X_2X_3\dots X_{N+1}\alpha_N, & n = N. \end{cases}$$

Also for this model we find agreement with [47]. The charge bands (3.16) of the lift defects match with the band restriction rules, and the fusion of fractional branes of the form $\mathbb{C}[X_1, X_{N+1}]/\langle X_1, X_{N+1}\rangle\{[n]_N\}$ with $\mathcal{T}_{1,2,\dots,N-1}$ agrees with the lifts of the respective D-branes derived in 8.4.2. in [47].

3.2 Flows Between Minimal Models from GLSMs

In this section, we will apply our construction introduced in 2.3.2 to attain flows between minimal models, and in particular re-derive the flow defects of [13]. This is possible, because we can model the mirror duals of minimal models in GLSMs. For single flow lines this was explained in [10], but indeed the entire parameter spaces of the mirror dual of $\mathcal{M}_{k=d-2}$ can be described in a single GLSM

$$\mathrm{GLSM}_{\mathcal{M}_{d-2}} \coloneqq \left(\mathrm{U}(1)^{d-2}, \mathbb{C}^{d-1}, (r,\theta), W = \prod_{i=0}^{d-2} X_i^{d-i}\right),$$

as discussed in Section 2.2.3. One can also consider the case with zero superpotential, where the respective GLSM can then be used to describes flows between $\mathbb{C}/\mathbb{Z}_d$-orbifold models. For convenience let us recall the charges of the chiral fields under the gauge group in Table 3.18 below. For a discussion of the phase structure of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ c.f. Section 2.2.3.

	X_0	X_1	X_2	X_3	$\dots$	$\dots$	X_{d-3}	X_{d-2}
$U(1)_0$	$(d-1)$	$-d$	0	$\dots$	$\dots$	$\dots$	$\dots$	0
$U(1)_1$	1	-2	1	0	$\dots$	$\dots$	$\dots$	0
$U(1)_2$	0	1	-2	1	0	$\dots$	$\dots$	0
$U(1)_3$	0	0	1	-2	1	0	$\dots$	0
$\vdots$	$\vdots$		$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$U(1)_{d-4}$	0	$\dots$	$\dots$	0	1	-2	1	0
$U(1)_{d-3}$	0	$\dots$	$\dots$	$\dots$	0	1	-2	1

Table 3.18: Charges of the chiral matter fields of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$.

Before proceeding with our discussion we would like to point out the apparent similarity between $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ and $\mathrm{GLSM}_{A_{d-2}}$ c.f. Section 3.1.3. This should not come as an surprise as $\mathrm{GLSM}_{A_{d-2}}$ is tailored to describe the physics of the singularity, and our model is in turn related to that one by mirror symmetry. Apart from the existence of the anomalous $U(1)_0$, the two models $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ and $\mathrm{GLSM}_{A_{d-2}}$ only differ by their superpotential. Setting the superpotential of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ to zero yields a GLSM describing the parameter space of the $\mathbb{C}/\mathbb{Z}_d$ orbifold model, which has very similar properties to the minimal model GLSM.

Having realized the entire parameter spaces of minimal models in an abelian GLSM, we can apply the strategy outlined in Section 2.3.2 to construct defects lifting the phases of the GLSM given by Landau-Ginzburg orbifolds to the GLSM. These can in particular be used to obtain the flow defects between different minimal model orbifolds. The procedure is exactly the same as in the non-anomalous case, namely we start out from the GLSM identity defect, then push the model down to a phase by setting fields to their VEVs on one side of the defect. Subsequently we introduce charge truncations for every phase boundary to be traversed by a chosen path, which then gives rise to a charge band. While the procedure is completely the same in anomalous and non-anomalous cases, there is a notable difference in the outcome however. In contrast to the non-anomalous case, one obtains different charge bands from lifting the UV and the IR phases[3] of a given flow. Indeed, lifting IR phases yields bands which are strictly smaller than the ones from the respective UV lifts. This is in qualitative agreement with the discussion of D-brane transport in other anomalous GLSMs in [50, 22], where it was found that the D-brane transport is goverend by a 'large' and a 'small' charge window. D-branes in the large window can be transported along the flow, but the ones which are not in the small window undergo some kind of decay.

Before giving the general construction, we will first spell out the details in a "truncated" example. RG flows of minimal models come in a hierarchy. The respective perturbations can be restricted to the i least relevant ones for $1 \leq i \leq d-2$. On the level of Landau-Ginzburg models this corresponds to restricting the deformation of the superpotential to $W = X^d + \lambda_1 X^{d-1} + \cdots + \lambda_i X^{d-i}$. The respective flow drives the minimal model at level $k = d-2$ at most to the one at level $k = d-2-i$, we call this the i-step perturbations. Indeed,

[3] Here, UV and IR refers to the energy hierarchy between the minimal models and not the one between GLSM and its phases. Given $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ The model of the highest level $d-2$ is the UV theory.

one can easily obtain GLSMs capturing these restricted parameter spaces by freezing the fields $X_j,\ j > i$ by setting them to their VEVs in the GLSMs describing the general perturbations. This procedure breaks the gauge groups $U(1)_j$ for $j > i-1$. The resulting GLSM has chiral fields $X_0, \ldots, X_i$, superpotential $W = X_0^d X_1^{d-1} \ldots X_i^{d-i}$ and gauge group $U(1)^i$. The charges of the fields X_i under the gauge group are given by Table 3.19 below.

	X_0	X_1	X_2	X_3	$\ldots$	$\ldots$	X_{i-1}	X_i
$U(1)_0$	$(d-1)$	$-d$	0	$\ldots$	$\ldots$	$\ldots$	$\ldots$	0
$U(1)_1$	1	-2	1	0	$\ldots$	$\ldots$	$\ldots$	0
$U(1)_2$	0	1	-2	1	0	$\ldots$	$\ldots$	0
$U(1)_3$	0	0	1	-2	1	0	$\ldots$	0
$\vdots$	$\vdots$		$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$U(1)_{i-2}$	0	$\ldots$	$\ldots$	0	1	-2	1	0
$U(1)_{i-1}$	0	$\ldots$	$\ldots$	$\ldots$	0	1	-2	1

Table 3.19: Charges of the chiral matter fields of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ truncated at $1 \leq i \leq d-2$.

Note that one can indeed construct GLSMs describing other subspaces of the minimal model parameter space by freezing arbitrary combinations of fields to VEVs in the GLSM containing the entire parameter space. Freezing for instance all but the fields X_0 and X_i, one arrives at a $U(1)$-GLSM describing a specific one-parameter flow from the the minimal model of level $k = d-2$ to the one at level $k = d-2-i$.[4]

Below we will give the concrete construction of the defects embedding the phases into the GLSM for two-step perturbations and derive the respective flow defects between the minimal models. Subsequently we will be discussing the lift- and flow defects for general perturbations.

3.2.1 The Two-Step Minimal Model Flows

As discussed above, the two-step flows starting in the minimal model of level $k = d-2$ can be modelled in the gauged linear sigma model with gauge group $U(1)^2$, chiral fields $X_0,\ X_1,\ X_2$ and superpotential $W = X_0^d X_1^{d-1} X_2^{d-2}$. The charges of the chiral fields can be read of from Table 3.20 below.

	X_0	X_1	X_2
$U(1)_0$	$(d-1)$	$-d$	0
$U(1)_1$	1	-2	1

Table 3.20: Charges of the chiral matter fields of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ truncated at $i = 2$.

This model exhibits three Landau-Ginzburg orbifold phases. In each of those two of the three chiral fields assume a non-trivial vaccum expectation value, and only one field remains

[4]These are the models studied in [10].

part of the effective low energy theory. The gauge group is broken to a finite subgroup. The phases are respectively described by the $\mathbb{Z}_{d-i}$-orbifolds of the Landau-Ginzburg theories with one chiral field X_i and superpotential $W = X_i^{d-i}$, where $i \in \{0, 1, 2\}$. The orbifold group acts by phase multiplication on the field X_i.

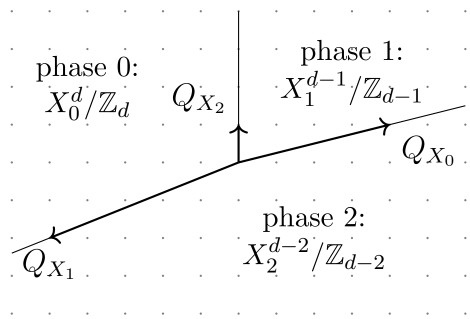


Figure 3.4: Phase diagram of the GLSM describing the two-step minimal model flows.

The phase boundary between phases i and j is located at the ray $\mathrm{Cone}_{\{k\}} = \mathbb{R}^{\geq 0} Q_k$ in the direction of the charge of the chiral field X_k, $k \notin \{i, j\}$ which assumes a non-trivial vev in both phases i and j. The phase diagram is given by Figure 3.4.

The preserved $U(1)$-gauge groups on the phase boundaries are the stabilizers of the respective fields having non-trivial VEV in both the adjacent phases and are given in Table 3.21 below.

phase boundary	preserved gauge group
(01)	$\{(g, 1) \mid g \in U(1)\} = U(1)_0$
(12)	$\{(g, g^{-(d-1)}) \mid g \in U(1)\}$
(02)	$\{(g^2, g^{-d}) \mid g \in U(1)\}$

Table 3.21: Unbroken subgroups of $U(1)_0 \times U(1)_1$ at the phase boundaries of $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ truncated at $i = 2$.

Lift Defects for the Two-Step Model

In the following we will construct the defects lifting the Landau-Ginzburg orbifold model ($W = X_0^d/\mathbb{Z}_d$) of phase 0 to the GLSM. Starting point is the GLSM identity defect, the respective module is given by

$$\mathcal{I}_{\mathrm{GLSM}} = \frac{S \otimes V}{\langle (X_0 - \alpha^{d-1}\beta X_0'), (X_1 - \alpha^{-d}\beta^{-2} X_1'), (X_2 - \beta X_2') \rangle}.$$

Here,

$$
\begin{aligned}
R &= \mathbb{C}[X_0, X_1, X_2, X_0', X_1', X_2'] \\
S &= \frac{R}{\langle X_0^d X_1^{d-1} X_2^{d-2} - (X_0')^d (X_1')^{d-1} (X_2')^{d-2} \rangle} \\
V &= \frac{\mathbb{C}[\alpha, \alpha^{-1}, \beta, \beta^{-1}]}{\langle (\alpha\alpha^{-1} - 1), (\beta\beta^{-1} - 1) \rangle}.
\end{aligned}
$$

The charges of the auxiliary fields $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ under the gauge groups of the left, respectively right of the defect are given by Table 3.22 below.

	α	α^{-1}	β	β^{-1}
Q_0^L	1	-1	0	0
Q_0^R	-1	1	0	0
Q_1^L	0	0	1	-1
Q_1^R	0	0	-1	1

Table 3.22: Charges of the auxiliary fields of the GLSM identity defect.

In phase 0 the fields X_1 and X_2 both have non-trivial vacuum expectation value. Hence, to obtain the defect embedding this phase into the GLSM, we have to set $X_1' = 1 = X_2'$ in $\mathcal{I}_{\text{GLSM}}$. This yields the module

$$
\mathcal{T}_\infty = \frac{\widetilde{S} \otimes V}{\langle (X_0 - \alpha^{d-1}\beta X_0'), (X_1 - \alpha^{-d}\beta^{-2}), (X_2 - \beta) \rangle}.
$$

with

$$
\begin{aligned}
\widetilde{R} &= \mathbb{C}[X_0, X_1, X_2, X_0'] \\
\widetilde{S} &= \frac{\widetilde{R}}{\langle X_0^d X_1^{d-1} X_2^{d-2} - (X_0')^d \rangle}.
\end{aligned}
$$

Next we have to impose the cutoffs they correspond to the chosen homotopy class of paths in parameter space. We would like to construct embeddings valid for flows from the UV phase (phase 0) all the way to the IR phase (phase 2). But there are two possibilities to go from phase 0 to phase 2. Either one can pass phase 1 on the way, or one can avoid it, crossing directly from phase 0 to phase 2. We will first discuss the case, in which phase 1 is passed, hence two phase boundaries, (01) and (12) are crossed. For each phase boundary, a cutoff is introduced.

On the phase boundary (01) $U(1)_0$ is preserved, implying a cutoff

$$
Q_{(01)}^L := Q_0^L \leq N_{(01)} \tag{3.17}
$$

for a choice of $N_{(01)} \in \mathbb{Z}$. On the phase boundary (12) the group $\{(g, g^{-(d-1)}) | g \in U(1)\}$ is preserved leading to the additional cutoff

$$Q^L_{(12)} := Q^L_0 - (d-1)Q^L_1 \leq N_{(12)} \tag{3.18}$$

for a choice of $N_{(12)} \in \mathbb{Z}$.

Introduction of the cutoffs means that one considers the submodule

$$\mathcal{T}_{N_{(01)},N_{(12)}} \subset \mathcal{T}_\infty$$

generated over the algebra $\widetilde{S}$ by only those generators whose charges satisfy the inequalities (3.17) and (3.18). Due to the relations

$$X_1\alpha^d\beta^2 = 1, \quad X_2\beta^{-1} = 1\,. \tag{3.19}$$

holding in the module $\mathcal{T}_\infty$ all generators in the truncated submodule can be obtained by applying the algebra $\widetilde{S}$ on generators whose charges lie in the band

$$\begin{array}{rcccl} N_{(01)} - d & < & Q^L_{(01)} := Q^L_0 & \leq & N_{(01)} \\ N_{(12)} - d + 1 & < & Q^L_{(12)} := Q^L_0 - (d-1)Q^L_1 & \leq & N_{(12)}. \end{array} \tag{3.20}$$

Namely, using the first relation in (3.19) any generator e of $\mathcal{T}_{N_{(01)},N_{(12)}}$ of charges $Q^L_{(01)}(e) \leq N_{(01)} - d$ can be written as

$$e = (X_1\alpha^d\beta^2)e =: X_1e'$$

where $e' \in \mathcal{T}_{N_{(01)},N_{(12)}}$ has charges

$$Q^L_{(01)}(e') = Q^L_{(01)}(e) + d \quad \text{and} \quad Q^L_{(12)}(e') = Q^L_{(12)}(e) - (d-2).$$

Hence, successively, $e = X_1^m e''$ with $e'' \in \mathcal{T}_{N_{(01)},N_{(12)}}$ and $N_{(01)} - d < Q^L_{(01)}(e'') \leq N_{(01)}$. One can now use the second relation in (3.19) in a similar fashion to write

$$e'' = (X_2\beta^{-1})^n e'' =: X_2^n e'''$$

for some $n \in \mathbb{N}_0$, such that

$$\begin{aligned} N_{(01)} - d &< Q^L_{(01)}(e''') \leq N_{(01)} \\ N_{(12)} - d + 1 &< Q^L_{(12)}(e''') \leq N_{(12)}. \end{aligned}$$

Hence, any generator of $\mathcal{T}_{N_{(01)},N_{(12)}}$ can be obtained by applying an element of $\widetilde{S}$ on a generator whose charges satisfy (3.20). Thus, the truncated module $\mathcal{T}_{N_{(01)},N_{(12)}}$ is generated by generators in this charge band.

In the following we will describe this submodule in more detail. It depends on the choice of the two integers $N_{(01)}$ and $N_{(12)}$. Define $z \in \mathbb{Z}$ and $n \in \{1, \ldots d-1\}$ such that

$$N_{(01)} - N_{(12)} = z(d-1) + n. \tag{3.21}$$

The first line of (3.20) implies that the charge $Q_0 \in \{1, \dots d-1\}$ of generators is given by

$$Q_0 \in \{1, \dots d-1\} = N_{(01)} - i, \quad i \in \{0, \dots d-1\}.$$

Substituting this into the second line in (3.20) and using the parametrization (3.21) one obtains

$$(z+1-Q_1^L)(d-1)+n > i \geq (z-Q_1^L)(d-1)+n.$$

The solutions to the inequalities (3.20) can now be read off as

$$\begin{array}{ll} i \in \{0, 1, \dots, n-1\} \quad \text{for} \quad Q_1^L = z+1 \\ i \in \{n, n+1, \dots, d-1\} \quad \text{for} \quad Q_1^L = z. \end{array}$$

The truncated module $\mathcal{T}_{N_{(01)},N_{(12)}}$ is therefore generated by

$$\alpha^{N_{(01)}}\beta^z \left\{\beta, \beta\alpha^{-1}, \beta\alpha^{-2}, \dots, \beta\alpha^{-(n-1)}, \alpha^{-n}, \alpha^{-n-1}, \dots, \alpha^{-(d-1)}\right\}$$

Denote the generators by

$$e_i := \alpha^{N_{(01)}}\beta^z\alpha^{-i} \cdot \begin{cases} \beta, & \text{if } 0 \leq i < n \\ 1, & \text{if } n \leq i < d. \end{cases}$$

Using the relations in the module $\mathcal{T}_\infty$ it is not difficult to find the relations in the truncated module $\mathcal{T}_{N_{(01)},N_{(12)}}$. For $i \in \{1, \dots, n-1, n+1, \dots, d-1\}$ one obtains

$$X_0' e_i = X_0\alpha^{-(d-1)}\beta^{-1} e_i = X_0X_1\alpha\beta e_i = X_0X_1X_2\alpha e_i = X_0X_1X_2 e_{i-1}\,,$$

and

$$\begin{array}{l} X_0' e_n = X_0\alpha^{-(d-1)}\beta^{-1} e_n = X_0X_1\alpha\beta e_n = X_0X_1 e_{n-1} \\ X_0' e_0 = X_0\alpha^{-(d-1)}\beta^{-1} e_0 = X_0 e_{d-1}. \end{array}$$

The module $\mathcal{T}_{N_{(01)},N_{(12)}}$ can be obtained as the cokernel of the matrix

$$p_1 = \begin{pmatrix} X_0' & & & & & & & & -X_0X_1X_2 \\ -X_0 & X_0' & & & & & & & \\ & -X_0X_1X_2 & \ddots & & & & & & \\ & & \ddots & X_0' & & & & & \\ & & & -X_0X_1X_2 & X_0' & & & & \\ & & & & -X_0X_1 & X_0' & & & \\ & & & & & -X_0X_1X_2 & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & -X_0X_1X_2 & X_0' \end{pmatrix}$$

on a free module of rank d whose generators have the same charges as the e_i. Note that in this matrix the elements on the secondary diagonal are $-X_0X_1X_2$ except in two places in one of which it is $-X_0$ and in the other one $-X_0 X_1$. The positions of these two entries is determined by the cutoff parameters $N_{(01)}$ and $N_{(12)}$.

This module corresponds to the defect lifting phase 0 into the GLSM in a way compatible with flows from phase 0 via phase 1 to phase 2. In fact, one can find a suitable $d \times d$-matrix p_0 which completes p_1 to a $U(1)^d \times \mathbb{Z}_d$-equivariant matrix factorization of $X_0^d X_1^{d-1} X_2^{d-2} - (X_0')^d$.

Indeed, these lift defects are consistent with what is known about flows between minimal models. Pushing the GLSM to phase 2 on the left of the defect should reproduce the flow defects between minimal models, which have been constructed in [13]. Indeed, this is accomplished by just setting $X_0 = 1 = X_1$ in the truncated module $\mathcal{T}_{N_{(01)},N_{(12)}}$. In this way one obtains the cokernel of the matrix

$$\begin{pmatrix}
X_0' & & & & & & & & -X_2 \\
-1 & X_0' & & & & & & & \\
 & -X_2 & \ddots & & & & & & \\
 & & \ddots & X_0' & & & & & \\
 & & & -X_2 & X_0' & & & & \\
 & & & & -1 & X_0' & & & \\
 & & & & & -X_2 & \ddots & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & -X_2 & X_0'
\end{pmatrix}. \tag{3.22}$$

Note that this matrix contains two entries -1 on the secondary diagonal. These can be used to reduce the presentation of the respective module. It can be written as a cokernel of a $(d-2)\times(d-2)$-matrix whose only non-zero entries are on the diagonal and the secondary diagonal. The entries on the diagonal are X_2 and the ones on the secondary diagonal are $-X_0$ except that depending on the choices of cutoff parameters either in two places we have $-X_0^2$ instead of $-X_0$, or in one place $-X_0^3$ instead of $-X_0$. These are exactly the defects describing the flows from the minimal model orbifold at level $k = d-2$ to the one at level $k = d-4$ [13]. Thus, the lift defects provide all the flow defects from phase 0 to phase 2.

As alluded to above, one can also cross over directly from phase 0 to phase 2 in the GLSM. Before constructing the lift defects associated to such paths, let us briefly step back and discuss the action of the lift defects on D-branes. To this end, let us determine the lifts of the elementary D-branes associated to the $S' := \mathbb{C}[X_0']/\langle (X_0')^d \rangle$-modules

$$\mathcal{M}_m := S'/X_0' S'\{[m]_d\}\,, \quad m = 0, \ldots d-1.$$

The lifts are given by the $\mathbb{Z}_d$-invariant parts of the tensor products $\mathcal{T}_{N_{(01)},N_{(12)}} \otimes_{S'} \mathcal{M}_m$. They are straight-forward to calculate, the $\mathbb{Z}_d$-projection singles out a single generator for each m. Let $i := N_{(01)} - m \bmod d \in \{0, \ldots, d-1\}$. Then, one obtains the $S'' =$

$\mathbb{C}[X_0, X_1, X_2]/\langle X_0^d X_1^{d-1} X_2^{d-2}\rangle$-module

$$\begin{array}{ll} S''/X_0X_1X_2S''\{(N_{(01)}-i, z+1)\} & \text{if } 0 \leq i < n-1 \\ S''/X_0X_1S''\{(N_{(01)}-i, z+1)\} & \text{if } i = n-1 \\ S''/X_0X_1X_2S''\{(N_{(01)}-i, z)\} & \text{if } n \leq i < d-1 \\ S''/X_0S''\{(N_{(01)}-i, z)\} & \text{if } i = d-1\,. \end{array} \tag{3.23}$$

Under the flow to the IR phase, which is implemented by setting $X_0 = 1 = X_1\ d-2$, the elementary D-branes in the UV phase are mapped to elementary D-branes in the IR, and two ($i = n-1, d-1$) are mapped to trivial D-branes and the respective D-branes decouple[5].

After this aside about D-branes, let us return to the construction of lift defects. The lift defects associated to paths directly crossing from phase 0 to phase 2 are obtained from $\mathcal{T}_\infty$ by only a single truncation in the direction of the $U(1)$ preserved along the phase boundary (02). The latter is given by $\{(g^2, g^{-d})|g \in U(1)\} \cong U(1)$ leading to the truncation

$$Q^L_{(02)} := 2Q^L_0 - dQ^L_1 \leq N_{(02)}. \tag{3.24}$$

As before, via the relations (3.19), all generators in the truncated module are obtained by action of $\widetilde{S}$ on generators whose charges lie in the band

$$N_{(02)} - d < Q^L_{(02)} := 2Q^L_0 - dQ^L_1 \leq N_{(02)}. \tag{3.25}$$

Since this is only a restriction in one direction on a rank-2 lattice of charges, there are still infinitely many generators. Writing

$$N_{(02)} = zd + n \quad \text{for} \quad z \in \mathbb{Z} \quad \text{and} \quad n \in \{0, \ldots, d-1\},$$

one can easily find the set of charges (Q^L_0, Q^L_1) satisfying (3.25), there are given by

$$\begin{array}{ll} & \{(-i-md, -2m-z)\,|\,0 \leq i < \frac{d-n}{2},\, m \in \mathbb{Z}\} \\ \cup & \{(-i-md, -2m-z-1)\,|\,\frac{d-n}{2} \leq i < \frac{2d-n}{2},\, m \in \mathbb{Z}\} \\ \cup & \{(-i-md, -2m-z-2)\,|\,\frac{2d-n}{2} \leq i < d,\, m \in \mathbb{Z}\}. \end{array}$$

Thus, the generators of the truncated module $\mathcal{T}_{N_{(02)}}$ are given by $e_{i,m}$ for $i \in \{0, \ldots, d-1\}$ and $m \in \mathbb{Z}$ with

$$e_{i,m} := \beta^{-z}(\alpha^d\beta^2)^{-m}\alpha^{-i} \cdot \begin{cases} 1, & \text{if } 0 \leq i < \frac{d-n}{2} \\ \beta^{-1}, & \text{if } \frac{d-n}{2} \leq i < \frac{2d-n}{2} \\ \beta^{-2}, & \text{if } \frac{2d-n}{2} \leq i < d. \end{cases}$$

The relations in $\mathcal{T}_{N_{(02)}}$ can be written as

$$X_1 e_{i,m} = e_{i,m+1}\,, \quad \text{and} \quad X'_0 e_{i,m} = \begin{cases} X_0X_1X_2\, e_{i-1,m} & \text{if } i \notin \{0, \lceil\frac{d-n}{2}\rceil, \lceil\frac{2d-n}{2}\rceil\} \\ X_0X_1\, e_{i-1,m} & \text{if } i \in \{\lceil\frac{d-n}{2}\rceil, \lceil\frac{2d-n}{2}\rceil\} \\ X_0X_2\, e_{i-1,m} & \text{if } i = 0, \end{cases}$$

[5]All the lifts of UV D-branes in (3.23) lie in the large window, but only the ones with $i \notin \{n-1, d-1\}$ lie in the small window.

where $\lceil r \rceil$ denotes the ceiling of r, i.e. the smalles integer $\geq r$. In contrast to the lift compatible with the flow from phase 0 via phase 1 to phase 2, this module corresponds to a matrix factorization of infinite rank. Pushing to phase 2 on the left side of the corresponding defect amounts to setting $X_0 = 1 = X_1$. This renders the rank of the module (resp. the matrix factorization) finite. Indeed, as for the case of flows passing phase 1, the resulting module can be written as a cokernel of a matrix of the form (3.22), which however is not as general as in the case of the flows passing through phase 1. Namely, the spacing of the positions of the scalars -1 on the secondary diagonal is fixed by d, whereas in the case of the flow passing through phase 1 it depends on the choice of cutoff parameter, and thereby on the chosen path in the parameter space. Thus, one only gets some flow defects between phases 0 and 2.

Indeed, there is another GLSM describing the direct flow between phases 0 and 2–the GLSM obtained from the two-step model by giving X_1 a vaccum expectation value. This GLSM has only the fields X_0, X_2, its gauge group is the stabilizer of X_1, and hence exactly the $U(1)$ in the two-step model which is preserved on the phase boundary between phases 0 and 2. The model has two phases, corresponding to phase 0 and 2 of the two-step model. We can now similarly lift phase 0 to this model, this requires the choice of a single cutoff. The resulting defects are of finite rank, and easy to construct. Fusing these lift defects with the defect between the GLSM with only fields X_0, X_2 and the two-step GLSM obtained by setting $X_1 = 1$ in the identity defect of the two-step GLSM, we obtain exactly the lift defects in the two-step model which are compatible with the direct crossover from phase 0 to phase 2.

3.2.2 Lift Defects for the Full Model

In an analogous fashion one can construct lift defects for the GLSM describing the full parameter space of minimal model orbifolds. We will briefly discuss the construction of defects lifting the UV phase $X^d/\mathbb{Z}_d$ into the full model, the field content of which is given in Table 3.18. Our construction works for all possible paths in parameter space, but we will restrict the discussion to paths crossing all the phase boundaries $(i, i+1)$, $i = 0, \ldots, d-3$. Those are paths which traverse all possible phases starting in the UV phase 0 and then step by step passing phases 1, 2, etc. going all the way down to the trivial phase $(d-2)$.

As before, the construction starts with the GLSM identity defect. The associated module is given by

$$\mathcal{I}_{\mathrm{GLSM}} = \frac{S \otimes V}{\left\langle \begin{array}{c} (X_0 - \alpha_0^{d-1}\alpha_1 X_0'), (X_1 - \alpha_0^{-d}\alpha_1^{-2}\alpha_2 X_1'), \\ (X_2 - \alpha_1\alpha_2^{-2}\alpha_3 X_2'), \ldots, (X_{d-4} - \alpha_{d-5}\alpha_{d-4}^{-2}\alpha_{d-3} X_{d-4}'), \\ (X_{d-3} - \alpha_{d-4}\alpha_{d-3}^{-2} X_{d-3}'), (X_{d-2} - \alpha_{d-3} X_{d-2}') \end{array} \right\rangle}.$$

Here

$$
\begin{aligned}
R &= \mathbb{C}[X_0, \ldots, X_{d-2}, X'_0, \ldots, X'_{d-2}] \\
S &= \frac{R}{\langle X_0^d X_1^{d-1} \cdot \ldots \cdot X_{d-2}^2 - (X'_0)^d (X'_1)^{d-1} \cdot \ldots \cdot (X'_{d-2})^2 \rangle} \\
V &= \frac{\mathbb{C}[\alpha_0, \alpha_0^{-1}, \ldots, \alpha_{d-3}, \alpha_{d-3}^{-1}]}{\langle (\alpha_0 \alpha_0^{-1} - 1), \ldots, (\alpha_{d-3}\alpha_{d-3}^{-1} - 1) \rangle}.
\end{aligned}
$$

The α_i have charges 1 and -1 with respect to $U(1)_i$ to the left, respectively right of the defect, and are not charged under the other $U(1)_j$, $j \neq i$.

Going down to phase 0 on the right of the defect amounts to setting $X'_i = 1$ for $i \neq 0$ in this module. This yields

$$
\mathcal{T}_\infty = \frac{\widetilde{S} \otimes V}{\left\langle \begin{array}{c} (X_0 - \alpha_0^{d-1}\alpha_1 X'_0), (X_1 - \alpha_0^{-d}\alpha_1^{-2}\alpha_2), \\ (X_2 - \alpha_1 \alpha_2^{-2}\alpha_3), \ldots, (X_{d-4} - \alpha_{d-5}\alpha_{d-4}^{-2}\alpha_{d-3}), \\ (X_{d-3} - \alpha_{d-4}\alpha_{d-3}^{-2}), (X_{d-2} - \alpha_{d-3}) \end{array} \right\rangle} \tag{3.26}
$$

with

$$
\begin{aligned}
\widetilde{R} &= \mathbb{C}[X_0, \ldots, X_{d-2}, X'_0] \\
\widetilde{S} &= \frac{\widetilde{R}}{\langle X_0^d X_1^{d-1} \cdot \ldots \cdot X_2^2 - (X'_0)^d \rangle}.
\end{aligned}
$$

Next, we have to impose a cutoff in this module, for every phase boundary $(i, i{+}1)$ traversed by the chosen path. The $U(1)$ gauge group preserved on this phase boundary is just the stabilizer of all the chiral fields X_j, $j \notin \{i, i+1\}$ obtaining a VEV in both phases i and $(i+1)$. It is given by

$$
U(1) \cong \{(g, g^{-(d-1)}, g^{-(d-2)}, \ldots, g^{-(d-i)}, 1, \ldots, 1) \,|\, g \in U(1)\} \subset U(1)^{d-2}.
$$

The respective cutoff reads

$$
Q^L_{(i\,i+1)} := Q^L_0 - \sum_{j=1}^{i} (d-j) Q^L_j \leq N_{(i\,i+1)}
$$

for a choice of cutoff parameter $N_{(i\,i+1)}$. We denote the submodule generated by all generators of $\mathcal{T}_\infty$ satisfying these bounds by $\mathcal{T}_{N_{(0\,1)}, \ldots, N_{(d-3\,d-2)}}$.

By virtue of the relations for X_i, $i > 0$ in (3.26) the charges of generators of the truncated module actually lie in a band. The size of the band can be read off from the charges of the X_i, $i > 1$ which can be read of from Table 3.23 below.

	X_1	X_2	X_3	$\ldots$	X_{d-3}	X_{d-2}
$Q^L_{(01)}$	$-d$	0	0	$\ldots$	$\ldots$	0
$Q^L_{(12)}$	$(d-2)$	$-(d-1)$	0	$\ldots$	$\ldots$	0
$Q^L_{(23)}$	0	$(d-3)$	$-(d-2)$	$\ddots$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$Q^L_{(d-4\,d-3)}$	0	$\ldots$	$\ldots$	0	-4	0
$Q^L_{(d-3\,d-2)}$	0	$\ldots$	$\ldots$	0	2	-3

Table 3.23: Charges of the chiral fields $X_i,\ i>0$ with respect to the $U(1)$ groups unbroken at the phase boundaries $(i,i+1),\ i=0,\ldots,d-3$.

Namely, one obtains that $\mathcal{T}_{N_{(01)},\ldots,N_{(d-3\,d-2)}}$ is generated by generators whose charges satisfy

$$\begin{array}{ccccc} N_{(01)}-d & < & Q^L_{(01)} & \leq & N_{(01)} \\ N_{(12)}-(d-1) & < & Q^L_{(12)} & \leq & N_{(12)} \\ & & \vdots & & \\ N_{(d-3\,d-2)}-3 & < & Q^L_{(d-3\,d-2)} & \leq & N_{(d-3\,d-2)}. \end{array}$$

Indeed, these inequalities can be solved successively. The inequality for $Q^L_{(01)}$ can be rewritten as

$$-d < Q^L_{(01)} - N_{(01)} = Q^L_0 - N_{(01)} \leq 0$$

which has solutions

$$Q^L_0 = N_{(01)} - a\,, \quad a \in \{0,\ldots,d-1\}.$$

The inequality for $Q^L_{(12)}$ can be brought into the form

$$(d-1)(Q^L_1 - 1) < Q^L_0 - N_{(12)} \leq (d-1)Q^L_1.$$

Hence Q^L_1 is determined by Q^L_0 according to

$$Q^L_1 = \left\lceil \frac{Q^L_0 - N_{(12)}}{d-1} \right\rceil.$$

In a similar way, the inequality for $Q^L_{(23)}$ determines Q^L_2 in terms of Q^L_0 and Q^L_1 according to

$$Q^L_2 = \left\lceil \frac{Q^L_0 - (d-1)Q^L_1 - N_{(23)}}{d-2} \right\rceil.$$

Going all the way we find d solutions for the inequalities given by

$$\begin{aligned} Q^L_0 &= N_{(01)} - a =: Q^L_0(a)\,, \quad a \in \{0,\ldots,d-1\} \\ Q_i &= \left\lceil \tfrac{Q^L_0 - \sum_{j=1}^{i-1}(d-j)Q^L_j - N_{(i\,i+1)}}{d-i} \right\rceil = \left\lceil \tfrac{N_{(01)} - N_{(i\,i+1)} - a - \sum_{j=1}^{i-1}(d-j)Q^L_j}{d-i} \right\rceil =: Q^L_i(a)\,,\ \text{for}\, i>0. \end{aligned}$$

Thus, the truncated module has d generators

$$e_a := \alpha_0^{Q_0^L(a)} \cdot \ldots \cdot \alpha_{d-3}^{Q_{d-3}^L(a)}, \ a \in \{0, \ldots, d-1\}.$$

These satisfy relations

$$X_0' e_a = X_0^{m_0^a} \cdot \ldots \cdot X_{d-2}^{m_{d-2}^a} e_{a-1},$$

where the m_i^a can take values 0 or 1 such that $\sum_a m_i^a = (d-i)$. In particular, $m_0^a = 1$ for all a, $m_1^a = 1$ for all but one a, $m_2^a = 1$ for all but two a's etc. Which of the m_i^a are zero is determined by the choice of $N_{(i\,i+1)}$ in a rather complicated manner depending on divisibility properties. In fact, $\mathcal{T}_{N_{(01)},\ldots,N_{(d-3\,d-2)}}$ is a free module divided by these relations.

For concreteness we will present the solution for the choice $N_{(i\,i+1)} = 0$ for all i. In this case

$$Q_i^L(a) = -\delta_{a,d-i}\,, \text{ for } i > 0.$$

In particular, the generators read

$$e_a = \begin{cases} \alpha_0^{-a}, & 0 \le a \le 2 \\ \alpha_0^{-a}\alpha_{d-a}^{-1}, & 3 \le a \le d-1. \end{cases}$$

They satisfy the relations

$$\begin{aligned} X_0' e_0 &= X_0 e_{d-1} \\ X_0' e_1 &= X_0 \cdot \ldots \cdot X_{d-2} e_0 \\ X_0' e_a &= X_0 \cdot \ldots \cdot X_{d-a} e_{a-1} \quad \text{for } 2 \le a \le d-1 \end{aligned}$$

and $\mathcal{T}_{N_{(01)},\ldots,N_{(d-3\,d-2)}}$ can be presented as the cokernel of the matrix

$$p_1 = \begin{pmatrix} X_0' & & & & & & & -X_0 \\ -X_0 \cdot \ldots X_{d-2} & X_0' & & & & & & \\ & -X_0 \cdot \ldots \cdot X_{d-2} & X_0' & & & & & \\ & & -X_0 \cdot \ldots \cdot X_{d-3} & X_0' & & & & \\ & & & -X_0 \cdot \ldots \cdot X_{d-4} & & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & X_0' & \\ & & & & & & -X_0 X_1 & X_0' \end{pmatrix}$$

which is one of the matrices of the matrix factorization representing the lift defect. Setting all but the $X_j = 1$ for $j \neq i$ one obtains the defect describing the transition between phase 0 and phase j along the chosen path. Again we find agreement with the flow defects constructed in [13].

4 Conclusion and Outlook

In this thesis we presented the findings of [14], where a novel functorial construction–presented in 2.3.2–for orbifold lift defects in GLSM was introduced. We demonstrated that our approach reproduces known results for brane transport from [47] and for flows in minimal models from [13]. The advantage of our construction is that it yields defects which completely describe the relation between two adjacent phases, rather than just their boundary conditions. Furthermore the rigid mathematical framework of TQFTs allows us to utilize mathematical properties of our defects–such as semi-invertibility–to constrain and refine our construction. There are various ways to build upon, explore and further deepen the understanding of the results of this thesis in greater detail.

We have only explored lifts in a limited class of examples, it would be interesting to apply our construction to a wider range of examples and in particular to different types of anomalous GLSMs. A good starting point would be the Hirzebruch-Jung models already discussed in [22] as this would allow for a direct comparison to their results concerning band restriction rules.

In Section 3.2 we noted the similarities between the models $\mathrm{GLSM}_{\mathcal{M}_{d-2}}$ encompassing all minimal models of levels $k \leq d-2$ as its phases and the models $\mathrm{GLSM}_{A_{d-2}}$ describing the A_{d-2} singularity. A more thorough investigation of these similarities may lead to interesting insights.

Our construction has the advantage of being functorial. This in particular allows for the application of defects not only to boundary conditions, as was discussed int his thesis, but also to morphisms between them. In particular this would also allow for the lift of bound states, which can be expressed as (chains) of maps between boundary conditions. To this end one may e.g. start by considering lifting the branes from the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold which appear in [33].

Other than in our discussion of minimal models we never pushed down our lift defects to another phase of the GLSM we are considering. We have avoided this purely to simplify matters, for geometric phases we could perform the push down by considering mixtures of matrix factorizations and complexes of coherent sheaves as was discussed in [11]. The generalization to mixed phases would then be the logical next step. By performing such push downs one would be able to construct flow defects between arbitrary pairs of orbifold phases on the one hand and geometric or mixed phases on the other. Furthermore one may get a better understanding of lifting geometric or mixed phases by attempting to replicate our approach of truncating the charge lattice of a naïve lift defect. Being able to construct lift- and RG type defects for arbitrary phases would then open the door to constructing arbitrary flow defects between phases including monodromies.

Lastly we note that the introduction of cut offs in our construction while leading to a consistent way of attaining lifting defects is ad hoc. It would be desirable to derive our approach purely from defect properties. From the conceptual point of view this is the most important hurdle to overcome in order to gain a complete understanding and contextualization of our construction. To be more concise we would like to better characterize the lifting- and RG defects T and R.

Consider the lifting defects T. We may construct them by either lifting the identity of a phase to the GLSM on its left or by pushing down the identity of a GLSM to a phase on its right. In this thesis we considered the latter, the ansatz for the former approach would entail lifting the representation of the gauge group H of the phase to the full gauge group $G \supset H$ of the GLSM. To this end one might have a look at the induced representation of G from the regular representation of H appearing in the module representing the identity defect of the phase. A consistent way of implementing this could necessitate additional choices which might coincide with the choices of cut offs.

Utilizing the properties of defects such as discussed in e.g. [60] may also impose certain constraints on the lifting defects which might match with the choices of cut offs. Namely one may attempt to understand the modules representing T and R as adjoints to define and characterize projection defects $P = T \otimes R$ as discussed in Section 2.3.1. In particular we would need to check certain properties of graded pivotal categories such as the existence of evaluations and coevaluations as discussed in Section 2.1.2. The fact that T and R are adjoint may imply some kind of finiteness condition potentially explaining our cut offs.

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Danksagung

An erster Stelle möchte ich meinen Dank an Ilka Brunner aussprechen, zum einen für die Möglichkeit unter ihrer Aufsicht in dem hervorragenden Umfeld des ASC promovieren zu dürfen, aber vor Allem für ihre exzellente Betreuung dieser Arbeit. Ihre Verfügbarkeit und fachliche Unterstützung hat es mir erlaubt stets Fortschritte mit meiner Arbeit zu machen und nie lange von Hindernissen ausgebremst zu werden.

Bei Daniel Roggenkamp möchte ich mich für eine produktive und angenehme Zusammenarbeit und für meinen Aufenthalt in Leipzig bedanken. Mein Dank gilt auch Dayuan Wang, Ed Segal, Gabriele Castellari, Ingmar Saberi und Johanna Knapp für hilfreiche Gespräche.

Ich danke meinen Eltern Eva-Maria und Paul Schneider sowie meinem Bruder Alexander Schneider für ihre uneingeschränkte Unterstützung, Rückhalt und Zuneigung. Meinen Freunden Frederic Wagner, Julian Schmittner, Mats Hansen und Mats Schero danke ich für ihre Unterstützung, dafür mir zu helfen als Person zu wachsen und dafür dass sie mir als Vorbilder in verschiedenen Bereichen des Lebens dienen. Einen besonderen Dank möchte ich Frederic Wagner für das Korrekturlesen von Teilen dieser Arbeit aussprechen.

Meine Stelle wurde durch das DFG-Projekt „Defekte und nicht perturbative Operatoren in superkonformen Theorien“ finanziert, wofür ich der Deutschen Forschungsgemeinschaft danken möchte.