# *Spectral estimates for singular systems*

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# Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.2011, §8, Abs. 2, Pkt. 5.)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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# Zusammenfassung

Diese Doktorarbeit besteht aus vier Artikeln, die sich mit Eigenwertabschätzungen und Eigenwertasymptotiken, isoperimetrischen Ungleichungen und Funktionalungleichungen befassen.

Wir beweisen eine universelle Abschätzung für die Anzahl der negativen Eigenwerte von Schrödingeroperatoren auf Hölder-Gebieten mit Neumann-Randbedingungen. Anschließend benutzen wir diese Abschätzung, um eine semiklassische Asymptotik für die jeweiligen Schrödingeroperatoren zu zeigen. Wir diskutieren außerdem Fälle, in denen diese Abschätzung und Asymptotik nicht gelten.

In einem anderen Artikel zeigen wir verschiedene Weyl-Asymptotiken für den Laplace-Beltrami-Operator auf singulären Riemannschen Mannigfaltigkeiten. Die Motivation für das betrachtete Modell stammt von der Untersuchung der Ausbreitung von Schallwellen in Gasplaneten.

Zudem zeigen wir eine isoperimetrische Ungleichung für das Massenträgheitsmoment von der Menge aller Punkte in einem Gebiet mit festem Abstand zum Rand. Mithilfe dieser Ungleichung kann man eine isoperimetrische Ungleichung für den ersten Eigenwert von magnetischen Robin-Laplace-Operatoren zeigen.

Außerdem beweisen wir eine stärkere Version der Hardy-Ungleichung. Alle Terme in unserer Ungleichung haben dasselbe Skalierungsverhalten.

# Abstract

This thesis consists of four works that deal with estimates and asymptotics for eigenvalues, isoperimetric and functional inequalities.

We prove a universal bound for the number of negative eigenvalues of Schrödinger operators on Hölder domains with Neumann boundary conditions. This bound is used to deduce semiclassical asymptotics for those operators. We also discuss a class of examples where those bounds and asymptotics fail.

In another work, we prove Weyl asymptotics for the Laplace-Beltrami operator on a class of singular Riemannian manifolds. This model is motivated by the study of the propagation of sound waves in gas planets.

We also show an isoperimetric inequality for moments of inertia of inner parallel curves. This inequality can be used to prove an isoperimetric inequality for the first eigenvalue of the magnetic Robin Laplacian.

Furthermore, we prove an improved version of Hardy's inequality. Our inequality only contains terms that scale in the same way.

# Declaration of own contribution to the publications

This thesis consists of the following four research papers:

- [14] Semiclassical estimates for Schrödinger operators with Neumann boundary conditions on Hölder domains. Preprint 2023, arXiv:2304.01587
- [9] (with Yves Colin de Verdière, Maarten de Hoop and Emmanuel Trélat) Weyl formulae for some singular metrics with application to acoustic modes in gas giants. Preprint 2024, arXiv:2406.19734
- [16] (with Ayman Kachmar and Vladimir Lotoreichik) Isoperimetric inequalities for inner parallel curves. J. Spectr. Theory 14.4 (2024), 1537–1562, DOI: 10.4171/JST/534
- [17] (with Phan Thành Nam) Hardy-Sobolev interpolation inequalities. Calc. Var. 63, 184 (2024), DOI: `https://doi.org/10.1007/s00526-024-02800-x`

I will explain my own contribution to those papers in the following.

- [14] Semiclassical estimates for Schrödinger operators with Neumann boundary conditions on Hölder domains. Preprint 2023, arXiv:2304.01587

  Phan Thành Nam suggested this project to me and we discussed the project in great detail together.

- [9] (with Yves Colin de Verdière, Maarten de Hoop and Emmanuel Trélat) Weyl formulae for some singular metrics with application to acoustic modes in gas giants. Preprint 2024, arXiv:2406.19734

  For the subcritical case, I initially had a proof idea that used an argument similar to Rozenblum's proof of the CLR inequality by covering the domain by boxes, each of which corresponds to at most one eigenvalue. One of my collaborators found the change of functions, which was the key ingredient to make the analysis more convenient so we could avoid weighted measures. This was later particularly helpful for the proofs in the critical and supercritical case.

The proofs presented in this paper rely on heat kernel estimates, which mostly uses ideas from my collaborators. However, in parallel to the development of the proofs using heat kernels, I found alternative proofs involving Dirichlet-Neumann bracketing, which we mentioned in a remark in the paper. I explain these proofs in Section 3 below. My collaborators took care of large parts of the editing, and I carefully checked everything.

- [16] (with Ayman Kachmar and Vladimir Lotoreichik) Isoperimetric inequalities for inner parallel curves. J. Spectr. Theory 14.4 (2024), 1537–1562, DOI: 10.4171/JST/534

  The idea for this project came from a talk from Ayman Kachmar, where he mentioned our result as an open problem. After many helpful discussions, I had the idea for the construction of the closed curve covering the inner parallel curve, which is the key ingredient of the paper. Vladimir Lotoreichik found a very important reference, which allowed us to use a certain result on the structure of inner parallel curves, which is intuitive but has a very lengthy proof. I wrote very detailed handwritten drafts, and my collaborators did large parts of the editing, which I then checked.

- [17] (with Phan Thành Nam) Hardy-Sobolev interpolation inequalities. Calc. Var. 63, 184 (2024), DOI: `https://doi.org/10.1007/s00526-024-02800-x`

  Many ideas of this work were found during discussions with Phan Thành Nam. He had the idea of using monotonicity for expressions of the form $A(B-A)$, which was a key idea of the proof. I found two important references that greatly helped us with the proof and I also found the counterexamples detailed in the paper. Furthermore, I had the idea of how to extend the first result to Lorentz spaces. While I took care of writing detailed handwritten drafts for our results, Phan Thành Nam took care of most of the editing of the paper and afterwards, I checked everything.

# Contents

# Part I

# Introduction

# Chapter 1

# General introduction

One of the great early successes of quantum mechanics was its rigorous proof of atomic stability, a phenomenon that classical physics could not explain. In the simplest case of the hydrogen atom, the question of stability asks whether there exists a constant $C > 0$ such that for all $u \in H^1(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx \geqslant -C. \tag{1.1}$$

Here the left-hand side of (1.1) represents the energy of the wave function describing an electron in a hydrogen atom. Put differently, we ask if the energy for $L^2$-normalised wave functions $u$ is bounded from below.

In the case of the hydrogen atom, the optimal constant in (1.1) is known to be $C = 1/4$ using explicit computations. In the following, I would like to explain an alternative approach of proving (1.1) for some (non-optimal) $C > 0$ that relies on Hardy's inequality, which can be seen as an uncertainty principle. Hardy's inequality states that for every function $u \in \dot{H}^1(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} |\nabla u|^2 \geqslant \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx. \tag{1.2}$$

Now note that

$$0 \leqslant \left(1 - \frac{1}{2|x|}\right)^2 = 1 - \frac{1}{|x|} + \frac{1}{4|x|^2}, \quad \text{so} \quad \frac{1}{|x|} \leqslant \frac{1}{4|x|^2} + 1. \tag{1.3}$$

We obtain using (1.3) and (1.2) that for all $u \in H^(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} dx \geqslant \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} - \int_{\mathbb{R}^3} |u|^2 \geqslant -1. \tag{1.4}$$

This shows that the hydrogen atom is stable in the sense of (1.1).

The quantity

$$\inf_{u\in H^1(\mathbb{R}^3),\|u\|_{L^2}=1}\left(\int_{\mathbb{R}^3}|\nabla u|^2-\int_{\mathbb{R}^3}\frac{|u(x)|^2}{|x|}dx\right) \tag{1.5}$$

is equal to the lowest eigenvalue (which is usually called the ground state) of the Schrödinger operator[1] $-\Delta - 1/|x|$. The eigenfunctions of $-\Delta - 1/|x|$ correspond to the bound states of the hydrogen atom. The eigenvalues of $-\Delta - 1/|x|$ can be computed using the min-max principle, see Section 2.3 below.

More generally, one may consider Schrödinger operators $-\Delta + V$ on domains $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, where $V : \Omega \to \mathbb{R}$ is usually called a potential. The corresponding quadratic form describing the energy of $L^2$-normalised wave functions $u$ is

$$Q(u) := \int_\Omega |\nabla u|^2 + \int_\Omega V(x)|u(x)|^2\,dx\,. \tag{1.6}$$

If $\Omega \not\subseteq \mathbb{R}^d$ is a domain and we consider $Q(u)$ for all $u \in H^1(\Omega)$, we say that we take Neumann boundary conditions[2]. If we only consider all $u \in H_0^1(\Omega)$ (where $H_0^1(\Omega)$ denotes the closure of all smooth compactly supported functions in $\Omega$ with respect to the $H^1(\Omega)$ norm), we refer to this as Dirichlet boundary conditions.

Similarly to (1.5), the lowest eigenvalue[3] of $-\Delta + V$ is characterised by the infimum of $Q(u)$ taken over all $u$ in the quadratic form domain (here $H^1(\Omega)$ or $H_0^1(\Omega)$) with $\|u\|_{L^2} = 1$. Thus, the lowest eigenvalue of Schrödinger operators with Neumann boundary conditions is less than or equal to the lowest eigenvalue of the corresponding Schrödinger operator with Dirichlet boundary conditions. This is also true for higher eigenvalues. Thus, the number of eigenvalues below a certain threshold, say $\lambda$, of a Schrödinger operator with Neumann boundary conditions is larger than or equal to this number for the corresponding Schrödinger operator with Dirichlet boundary conditions.

In a seminal work, Weyl proved in 1911 [50] that for any bounded domain $\Omega \subset \mathbb{R}^d$, the number of eigenvalues below $\lambda$ of the Dirichlet Laplacian $-\Delta$ on $\Omega$, which we

[1] For simplicity, we ignore domain issues and the question of self-adjointness here.

[2] Strictly speaking, we would like to assume that $V$ is such that $\int_\Omega V|u|^2 < \infty$ for all $u \in H^1(\Omega)$ here in order to ensure that $Q(u) < \infty$ for all $u \in H^1(\Omega)$.

[3] Since we do not know in general if the operator has discrete spectrum, we should rather speak of the lowest min-max value.

denote by $N(\lambda)$, satisfies the following asymptotics

$$N(\lambda) = C_d |\Omega| \lambda^{d/2} + o\left(\lambda^{d/2}\right) \text{ as } \lambda \to \infty\,, \tag{1.7}$$

where $C_d > 0$ is a constant that only depends on the dimension $d$.

Since then, deducing similar asymptotics has remained an active field of research. For instance, one can ask if similar asymptotics hold in the case of the Neumann Laplacian. There the situation is more complicated, and in fact, there are bounded domains such that the corresponding Neumann Laplacian has zero in its essential spectrum [29]. Furthermore, instead of considering a Laplace operator $-\Delta$, one can ask for asymptotics for the number of negative eigenvalues of Schrödinger operators $-\Delta + \lambda V$ as $\lambda \to \infty$. The result is well-known for Lipschitz domains, see for example [22]. My work [14] described in Section 2 below studies this question for the Neumann Laplacian on Hölder domains.

Another direction is to prove Weyl asymptotics for the Laplace-Beltrami operator on Riemannian manifolds. In [9] with Yves Colin de Verdière, Maarten de Hoop and Emmanuel Trélat, see Section 3 below, we prove Weyl asymptotics for the Laplace-Beltrami operator on a singular Riemannian manifold. Our model comes from the study of the propagation of soundwaves in gas planets and has links to sub-Riemannian geometry.

In certain cases, geometric inequalities can be useful to deduce properties of the spectrum. Together with Ayman Kachmar and Vladimir Lotoreichik [16], we proved weighted isoperimetric inequalities for smooth, bounded, and simply connected domains. More precisely, we show that the moment of inertia of inner parallel curves for domains with fixed perimeter attains its maximum for a disk. This inequality, which was previously only known for convex domains, allows us to extend an isoperimetric inequality for the first eigenvalue of the magnetic Robin Laplacian.

One can also investigate other functional inequalities and their properties. In a joint work with Phan Thành Nam [17], we derive a family of interpolation estimates which improve Hardy's inequality (1.2) and are critical in some sense. We also determine all optimisers among radial functions in the corresponding radial problem and discuss open questions.

This thesis contains the works [14, 9, 16, 17]. In this introduction, we give a brief overview of each of the papers. We also explain the context, mention the most relevant known results and references, and describe how they are linked to the paper.

Furthermore, for each paper, we choose to explain or emphasise certain aspects, which we find useful to simplify the reading or to give additional background information. In particular, we will explain the concept of min-max values and Dirichlet-Neumann bracketing in Section 2. In Section 3, we give an alternative proof sketch of the main result of [9] using Dirichlet-Neumann bracketing. In Section 4, we explain the intuition and key steps of the elementary geometric construction we used in [16]. In Section 5, we focus on different counterexamples and give an outlook on the remaining open question of existence of optimisers for our functional inequality.

# Chapter 2

# Semiclassical estimates for Schrödinger operators on Hölder domains

## 2.1 Overview of [14]

In a work of mine [14], I considered Schrödinger operators $-\Delta + V$ on bounded Hölder domains $\Omega \subset \mathbb{R}^d$ (that is, the boundary of $\Omega$ is locally the graph of a Hölder-continuous function) with Neumann boundary conditions, where the potential $V$ is a non-positive function on $\Omega$. My work [14] was motivated by the detailed study [41] of Weyl asymptotics for the Laplacian $-\Delta$ on bounded Hölder domains with Neumann boundary conditions.

I proved the validity of the same leading order eigenvalue asymptotics as for Schrödinger operators on Lipschitz domains, under certain assumptions on the potential $V$ in the optimal range of Hölder exponents. More precisely, if $N(-\Delta + \lambda V)$ denotes the number of negative eigenvalues of the Schrödinger operator $-\Delta + \lambda V$, then

$$N(-\Delta + \lambda V) = C_d \lambda^{d/2} \int_\Omega |V|^{d/2} + o\left(\lambda^{d/2}\right) \text{ as } \lambda \to \infty, \tag{2.1}$$

where $C_d > 0$ is a universal constant only depending on the dimension $d$. My result is valid in all dimensions $d \geqslant 2$, but for simplicity, let us assume $d \geqslant 3$ in the following explanations.

Furthermore, I provided an example, which shows that the assumptions on the potential $V$ in the setting of Hölder domains indeed need to be stronger than in the case of Lipschitz domains (where $V \in L^{d/2}(\Omega)$ is sufficient). This is a surprising result since the condition $V \in L^{d/2}(\Omega)$ seems very natural in view of (2.1).

The proof of (2.1) relies on the new universal bound for the number $N(-\Delta+V)$ of negative eigenvalues of Schrödinger operators $-\Delta + V$

$$N(-\Delta + V) \leqslant C|||V|||^{d/2}, \tag{2.2}$$

where $|||\cdot|||$ is a weighted $L^p$-norm for $p > d/2$ with a weight that grows near the boundary $\partial\Omega$. The proof of (2.2) is inspired by the proof of Rozenblum of the "classical" Cwikel-Lieb-Rozenblum inequality [45], but I made substantial modifications and adaptations.

I covered the domain $\Omega$ by smaller domains, which are carefully chosen rectangles intersected with $\Omega$ such that each of the smaller domains supports at most one negative eigenvalue of the Schrödinger operator $-\Delta + V$. To this end, I proved a new Poincaré-Sobolev inequality and a new Besicovitch-type covering theorem for those smaller domains. The proof of the Poincaré-Sobolev inequality uses a Sobolev embedding for Hölder domains [34]. The key ingredient for the Besicovitch-type covering theorem is that the smaller domains are chosen such that the ratio of the side-lengths of each rectangle are comparable for all chosen rectangles intersecting each other. The number of negative eigenvalues $N(-\Delta+V)$ can then be estimated by the number of smaller domains chosen using the min-max principle, see Section 2.3 below.

One of the main challenges in this problem was to get an exponent of $d/2$ on the right-hand side of (2.2). This is essential for deducing (2.1). Previous results, see for example [23], could only get an exponent that is strictly larger than $d/2$. On the technical side, the careful choice of the smaller domains, an application of Hölder's inequality for sums of real numbers and the use of the weighted $L^p$-norm $|||\cdot|||$ made it possible to obtain (2.2) with the exponent $d/2$.

## 2.2 Context for [14]

For the Dirichlet Laplacian on domains $\Omega \subset \mathbb{R}^d$ with finite measure, we always have the leading-order Weyl asymptotics (1.7) [50, 44]. For the Neumann Laplacian, the situation is much more delicate. Intuitively speaking, the eigenfunctions of the Neumann Laplacian can grow near the boundary and if the boundary is very rough, in particular with many outward pointing cusps, then many eigenfunctions can accumulate near the boundary leading to a larger number of eigenvalues than what one would expect from (1.7). There are bounded domains that have zero in the essential spectrum of the Neumann Laplacian, and one can even construct for any closed $S \subset [0, \infty)$ bounded domains such that the essential spectrum of the Neumann Laplacian is given by $S$ [29].

If the domain $\Omega$ is an $H^1$-extension domain, that is, we can extend functions in $H^1(\Omega)$ to $H^1$-functions on the entire space $\mathbb{R}^d$ in such a way that their $H^1(\mathbb{R}^d)$-norm is comparable with their $H^1(\Omega)$-norm, and similarly for the $L^2$-norms, then a classical argument shows that the Neumann Laplacian has discrete spectrum and the leading-order Weyl asymptotics (1.7) hold, see for example [22, Theorem 3.20]. Lipschitz domains are extension domains, but Hölder domains for a Hölder exponent $\gamma < 1$ are in general not extension domains. Thus, it is a priori unclear if we have (1.7) for Hölder domains.

In [41], the authors considered the Neumann Laplacian on bounded Hölder domains $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$. Denoting the Hölder exponent of the function locally describing the boundary by $\gamma$, they showed that if $\gamma \in ((d-1)/d, 1)$, then the usual Weyl asymptotics (1.7) as in the case of Lipschitz domains hold. Note that a Hölder exponent $\gamma = 1$ means that $\Omega$ is a Lipschitz domain. Furthermore, if $\gamma \in (0, (d-1)/d]$, then they provided an explicit counterexample for a domain $\Omega \subset \mathbb{R}^d$ with Hölder exponent $\gamma$ and such that (1.7) fails. It is remarkable that they could identify a critical value $(d-1)/d$, where the asymptotic behaviour changes drastically.

My work [14] treats similar questions in the case of Schrödinger operators $-\Delta + V$ on bounded Hölder domains $\Omega \subset \mathbb{R}^d$ with Neumann boundary conditions.

In the following, I will explain the min-max principle and Dirichlet-Neumann bracketing, which have been important tools for my work [14].

## 2.3 The min-max principle

This section is devoted to an explanation of the min-max principle, which goes back to [43, 20, 13]. It is a very powerful technique in spectral theory, see [37, Theorem 12.1] for an overview.

Let us first introduce the min-max principle in a general setting [37, Theorem 12.1, Version 3]. If $\mathcal{H}$ is a Hilbert space and $A : \mathcal{D}(A) \to \mathcal{H}$ is a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$ and with $A \geqslant -C$ for some $C > 0$, then we define by

$$\mu_n(A) := \inf_{M \subset \mathcal{D}(A),\, \dim(M) = n} \ \sup_{u \in M,\, \|u\| = 1} \langle u, Au \rangle \tag{2.3}$$

the $n$-th min-max value for all $n \in \mathbb{N}$. The $M$ in the infimum in (2.3) is a subspace of $\mathcal{D}(A)$ of dimension $n$. If $A$ has at least $n$ discrete eigenvalues below its essential spectrum, then $\mu_n(A)$ agrees with the $n$-th lowest eigenvalue of $A$, counted with multiplicity.

In particular, if $A$ has compact resolvent, then (2.3) gives a variational characterisation of the eigenvalues of $A$. More generally, instead of considering $M$ as an $n$-dimensional subspace of $\mathcal{D}(A)$ in (2.3), we can also take $M$ as an $n$-dimensional subspace of a form core of the corresponding quadratic form $Q(\cdot)$, and replace $\langle u, Au\rangle$ by $Q(u)$. This is very convenient as we need not know the domain of $A$ explicitly, but it suffices to work on the level of quadratic forms.

While we can refer to the $\mu_n(A)$ from (2.3) as min-max values (think of the infimum as a minimum and the supremum as a maximum), there is also a version of max-min values [37, Theorem 12.1, Version 2]

$$\nu_n(A) := \sup_{L\subset\mathcal{D}(A),\,\dim(L)=n-1}\ \inf_{u\in L^\perp,\,\|u\|=1} \langle u, Au\rangle. \tag{2.4}$$

Again, one can alternatively work with the corresponding quadratic form and a quadratic form core, and furthermore, by the proof of [37, Theorem 12.1, Version 2], it suffices to assume that $L$ is an $n-1$-dimensional subspace of our Hilbert space $\mathcal{H}$, which is sometimes referred to as Glazman's lemma [22, Theorem 1.26]. One can show that for all $n \in \mathbb{N}$

$$\mu_n(A) = \nu_n(A)\,, \tag{2.5}$$

see [37, Theorem 12.1, Version 2, 3]. Thus, we have two very different variational characterisations of the $n$-th eigenvalue below the essential spectrum of an operator. Both of them are very helpful in applications (see also Section 2.4).

For instance, the second characterisation can be useful when estimating the value of the second eigenvalue for a Neumann Laplacian: Suppose that we consider a box $\Omega := [0,1]^d$ in dimension $d \geqslant 3$ and we consider the Schrödinger operator $-\Delta + V$ with Neumann boundary conditions and $V \in L^{d/2}(\Omega)$. Assume that we have shown that

$$\int_\Omega |\nabla u|^2 + \int_\Omega V|u|^2 \geqslant 0 \quad \text{for all } u \in H^1(\Omega) \text{ with } \int_\Omega u = 0. \tag{2.6}$$

Note that the corresponding quadratic form $Q(u)$ is the left-hand side of (2.6) with quadratic form domain $H^1(\Omega)$ (since we consider the Neumann Laplacian). Then defining $L$ as the subspace of $H^1(\Omega)$ that is spanned by the constant function on $\Omega$, the condition $\int_\Omega u = 0$ is equivalent to saying $u \in L^\perp$. Thus, (2.6) states that

$Q(u) \geqslant 0$ for all $u \in L^\perp$. We deduce that $\nu_2(-\Delta + V) \geqslant 0$, or put differently, the number of negative eigenvalues of $-\Delta + V$ is at most one. The argument we presented here is classical [45] and we also use it in different variations in [14], for instance in the proof of [14, Lemma 1.8]. In practice, if $\Omega$ is a box as in our example, a sufficient condition for (2.6) is a sufficiently small negative part of $V$ in the $L^{d/2}(\Omega)$-norm.

## 2.4 Dirichlet-Neumann bracketing

In this section, we explain Dirichlet-Neumann bracketing. A historically particularly relevant work is due to Weyl [50]. We would like to refer to [47, Theorem 7.5.28] for a more recent source that also includes historical remarks.

I will explain Dirichlet-Neumann bracketing for a domain $\Omega$ in euclidean space that is decomposed into two smaller domains $\Omega_1$ and $\Omega_2$. I will allow for mixed boundary conditions for $\Omega_1$ and $\Omega_2$, which appear naturally when imposing Dirichlet or Neumann boundary conditions on $\Omega$ and performing the Dirichlet-Neumann bracketing. More generally, this technique can also be used for Schrödinger operators or for the Laplace-Beltrami operator on a Riemannian manifold. We used Dirichlet-Neumann bracketing in both [14] and [9].

Let us consider an open set $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. Recall that

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; u \text{ weakly differentiable and } \nabla u \in L^2(\Omega) \right\} . \tag{2.7}$$

Then the Neumann Laplacian on $\Omega$ is the unique self-adjoint operator that corresponds to the quadratic form

$$\int_\Omega |\nabla u|^2 \quad \text{for } u \in H^1(\Omega). \tag{2.8}$$

Let us now define for every $E \subset \mathbb{R}^d$ the space $A_E(\Omega)$ of $H^1(\Omega)$-functions that vanish near $E$ as follows: $u \in A_E(\Omega)$ if and only if $u \in H^1(\Omega)$ and there exists an open neighbourhood $V \subset \mathbb{R}^d$ of $E$ such that $u(x) = 0$ for almost every $x \in V \cap \Omega$. Note that for any $E \subset \mathbb{R}^d$, we have $A_E(\Omega) = A_{E\cap\overline{\Omega}}(\Omega)$. A similar definition to the spaces $A_E(\Omega)$ can be found for instance in [26].

Then the Dirichlet Laplacian on $\Omega$ can be defined as the unique self-adjoint operator that corresponds to the (closeable) quadratic form

$$\int_\Omega |\nabla u|^2 \quad \text{for } u \in A_{\partial\Omega}(\Omega). \tag{2.9}$$

Note that the closure of $A_{\partial\Omega}(\Omega)$ with respect to the $H^1(\Omega)$-norm (which is in this case also the quadratic form norm) is equal to the closure of $C_c^\infty(\Omega)$ functions with respect to the $H^1(\Omega)$-norm, which is usually denoted by $H_0^1(\Omega)$.

Using a quadratic form with form domain $A_E(\Omega)$ for different sets $E \subset \partial\Omega$ allows us to define Laplace operators with mixed boundary conditions. The closeable quadratic form $\int_\Omega |\nabla u|^2$ with $u \in A_E(\Omega)$ gives rise to a unique self-adjoint operator, which we refer to as having Dirichlet boundary conditions on $E$ and Neumann boundary conditions on the rest of the boundary.

Denote for $\lambda \in \mathbb{R}$ and measurable $E \subset \mathbb{R}^d$ by $N_\Omega^E(\lambda)$ the number of eigenvalues (counted with multiplicity) below $\lambda$ of the operator that corresponds to the quadratic form

$$\int_\Omega |\nabla u|^2 \quad \text{for } u \in A_E(\Omega), \tag{2.10}$$

and let $N_\Omega^E(\lambda) = \infty$ if $\lambda$ is greater than or equal to the infimum of the essential spectrum of that operator.

**Proposition 2.1** (Dirichlet-Neumann bracketing)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open subset and let $\Omega_1, \Omega_2 \subset \Omega$ be open and disjoint such that $\Sigma = \Omega \setminus (\Omega_1 \cup \Omega_2)$ has zero Lebesgue measure. Then for any $\lambda \in \mathbb{R}$, we have a two-sided estimate for the Dirichlet Laplacian on $\Omega$*

$$N_{\Omega_1}^{\partial\Omega_1}(\lambda) + N_{\Omega_2}^{\partial\Omega_2}(\lambda) \leqslant N_\Omega^{\partial\Omega}(\lambda) \leqslant N_{\Omega_1}^{\partial\Omega}(\lambda) + N_{\Omega_2}^{\partial\Omega}(\lambda) \tag{2.11}$$

*and a two-sided estimate for the Neumann Laplacian on $\Omega$*

$$N_{\Omega_1}^{\Sigma}(\lambda) + N_{\Omega_2}^{\Sigma}(\lambda) \leqslant N_\Omega^{\varnothing}(\lambda) \leqslant N_{\Omega_1}^{\varnothing}(\lambda) + N_{\Omega_2}^{\varnothing}(\lambda)\,. \tag{2.12}$$

*More generally, if $S \subset \partial\Omega$ denotes a measurable set, then we have a two-sided estimate for the Laplacian on $\Omega$ with Dirichlet boundary conditions on $S$ and Neumann boundary conditions elsewhere*

$$N_{\Omega_1}^{S\cup\Sigma}(\lambda) + N_{\Omega_2}^{S\cup\Sigma}(\lambda) \leqslant N_\Omega^{S}(\lambda) \leqslant N_{\Omega_1}^{S}(\lambda) + N_{\Omega_2}^{S}(\lambda). \tag{2.13}$$

(2.11) and (2.12) show that one can estimate the eigenvalues for a Laplacian with Dirichlet or Neumann boundary conditions on $\partial\Omega$ by the sums of the number of the corresponding eigenvalues on the two subdomains $\Omega_1$, $\Omega_2$ after putting additional Dirichlet and Neumann boundary conditions on the interface $\Sigma$.

*Proof.* For concreteness, we stated the two most commonly used versions where we have Dirichlet or Neumann boundary conditions on the entire boundary of $\Omega$ separately, but of course, it suffices to show (2.13) and then set $S = \partial\Omega$ or $S = \varnothing$ to obtain (2.11) or (2.12).

First note that

$$\Sigma = \Omega \cap (\partial\Omega_1 \cup \partial\Omega_2). \tag{2.14}$$

The proof of this is a contradiction argument that involves the construction of small open balls completely contained in $\Sigma$, which is a contradiction to $\Sigma$ having zero Lebesgue measure. We omit the details here.

**Estimate with extra Dirichlet conditions on $\Sigma$.** Let us first show that

$$N^{S\cup\Sigma}_{\Omega_1}(\lambda) + N^{S\cup\Sigma}_{\Omega_2}(\lambda) \leqslant N^S_\Omega(\lambda). \tag{2.15}$$

To this end, we will use the min-max characterisation of eigenvalues (2.3). For $k \in \{1, 2\}$, we denote $K_k := N^{S\cup\Sigma}_{\Omega_k}(\lambda)$ and note that by the version of (2.3) with a quadratic form core, there exists a subspace $M_k \subset A_{S\cup\Sigma}(\Omega_k)$ with

$$\int_{\Omega_k} |\nabla u_k|^2 < \lambda \int_{\Omega_k} |u_k|^2 \quad \text{for all } 0 \not\equiv u_k \in M_k. \tag{2.16}$$

By extending the functions $u_k$ by zero on $\Omega \setminus \Omega_k$, we can think of them as functions in $A_S(\Omega)$. Here we used that the $u_k$ vanish in a neighbourhood of $S \cup \Sigma$. Define

$$M := \{u_1 + u_2 \mid u_1 \in M_1,\, u_2 \in M_2\}, \tag{2.17}$$

and note that $K := \dim(M) = K_1 + K_2$. Writing every $0 \not\equiv u \in M$ as $u = u_1 + u_2$ with $u_k \in M_k$, $k = 1, 2$ and $0 \not\equiv u_1$ or $0 \not\equiv u_2$, it follows that

$$\int_\Omega |\nabla u|^2 = \int_{\Omega_1} |\nabla u_1|^2 + \int_{\Omega_2} |\nabla u_2|^2 < \lambda \int_{\Omega_1} |u_1|^2 + \lambda \int_{\Omega_2} |u_2|^2 = \lambda \int_\Omega |u|^2, \tag{2.18}$$

so $N^S_\Omega(\lambda) \geqslant K = K_1 + K_2$. Using the definition of $K_1$ and $K_2$, we deduce (2.15).

**Estimate with extra Neumann conditions on $\Sigma$.** Next, let us show that

$$N^S_\Omega(\lambda) \leqslant N^S_{\Omega_1}(\lambda) + N^S_{\Omega_2}(\lambda). \tag{2.19}$$

In order to see this, we will use the second characterisation of min-max values, which we referred to as max-min values above. For $k \in \{1, 2\}$, we define $K_k := N^S_{\Omega_k}(\lambda)$. By (2.4), or more precisely its version with a quadratic form core and $L$ a subspace of the corresponding Hilbert space, see for example [22, Theorem 1.26], there exists a subspace $L_k \subset L^2(\Omega_k)$ of dimension $K_k$ such that for every $u_k \in A_S(\Omega_k)$ with $u_k \in L_k^\perp$, we have

$$\int_{\Omega_k} |\nabla u_k|^2 \geqslant \lambda \int_{\Omega_k} |u_k|^2. \tag{2.20}$$

Next, we identify functions $l_k \in L_k$ with functions in $L^2(\Omega)$ by extending them by zero on $\Omega \setminus \Omega_k$. Define

$$L := \{l = l_1 + l_2 \mid l_1 \in L_1, \, l_2 \in L_2\} \tag{2.21}$$

and note that $L$ is a subspace of $L^2(\Omega)$ of dimension $K := K_1 + K_2$.

Let $u \in A_S(\Omega)$ with $u \in L^\perp$. In particular, for any $k \in \{1, 2\}$ and any $l_k \in L_k$, we have

$$\int_{\Omega_k} u l_k = \int_{\Omega} u l_k = 0\,, \tag{2.22}$$

where we used that we extended the $l_k$ by zero on $\Omega \setminus \Omega_k$. Thus, the restriction of $u$ to $\Omega_k$, which we denote by $u_k \in A_S(\Omega_k)$ satisfies $u_k \in L_k^\perp$. By (2.20), it follows that

$$\int_{\Omega_k} |\nabla u_k|^2 \geqslant \lambda \int_{\Omega_k} |u_k|^2. \tag{2.23}$$

Putting everything together and using that $\Sigma$ has zero Lebesgue measure, we obtain

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega_1} |\nabla u_1|^2 + \int_{\Omega_2} |\nabla u_2|^2 \geqslant \lambda \int_{\Omega_1} |u_1|^2 + \lambda \int_{\Omega_2} |u_2|^2 = \lambda \int_{\Omega} |u|^2 \tag{2.24}$$

Thus, $N^S_\Omega(\lambda) \leqslant K$, which completes the proof of (2.19). □

# Chapter 3

# Weyl formulae for some singular metrics with application to acoustic modes in gas giants

## 3.1 Overview of [9]

Seismology for gas planets plays an important role in understanding their interior. Astrophysicists use for instance ring seismology for Saturn when measuring eigenfrequencies of soundwaves inside the planet [38]. A particularity of gas planets is that the speed of sound goes to zero near the boundary of the planet. Physicists use a model for this behaviour that includes a parameter $\alpha$ depending on the density profile of the planet near the boundary, which depends on the chemical decomposition of the planet.

The propagation of sound waves in gas planets can be modelled by the Laplace-Beltrami operator on a compact Riemannian manifold $X$ with boundary with a certain class of Riemannian metrics that become singular at the boundary. Near the boundary $\partial X$, these Riemannian metrics are of the form

$$g = u^{-\alpha}\bar{g}, \tag{3.1}$$

where $u$ is a suitable coordinate transverse to the boundary with $u = 0$ at the boundary, $\bar{g}$ is a smooth Riemannian metric on our manifold up until its boundary and $0 < \alpha < 2$. It has been shown in [30] that the spectrum of the corresponding Laplace-Beltrami operator is discrete.

Our work with Yves Colin de Verdière, Maarten de Hoop and Emmanuel Trélat

[9] provides a proof of the asymptotics of the eigenfrequencies of gas planets for any $0 < \alpha < 2$, which are the physically relevant cases [9, Theorem 1]. Furthermore, we show how most of the eigenfunctions are distributed inside the planet [9, Theorem 2].

When $\alpha$ is larger than a certain critical parameter, then the Hausdorff dimension of $(X, g)$ is strictly larger than its topological dimension and we proved in [9] that the Weyl asymptotics are determined by $\alpha$ and the metric $\bar{g}$ at the boundary. In this case, the Weyl measure, which describes the limiting averaged distribution of the eigenfunctions, is the uniform distribution on the boundary. Below this critical parameter, the Weyl asymptotics agree with the usual Weyl asymptotics on smooth Riemannian manifolds and the Weyl measure is a uniform distribution on $X$. We also covered the case of the critical value of $\alpha$.

We have two different proofs; one of them uses heat kernel asymptotics and the other one uses Dirichlet-Neumann bracketing. Both in the critical and supercritical parameter range of $\alpha$, it is crucial that our metric near the boundary is quasiisometric to a metric that has nice scaling properties.

## 3.2 Context for [9]

It is a classical result that for smooth compact Riemannian manifolds $X$ of dimension $d \in \mathbb{N}$ with a smooth non-degenerate Riemannan metric $g$, the associated Laplace-Beltrami operator $\Delta_g$ has discrete spectrum and the eigenvalues satisfy the leading-order Weyl asymptotics

$$N(\lambda) = C_d \mathrm{vol}(X, g)\lambda^{d/2} + o\left(\lambda^{d/2}\right) \quad \text{as } \lambda \to \infty, \tag{3.2}$$

where $N(\lambda)$ denotes the number of eigenvalues of $\Delta_g$ that are less than $\lambda$ counted with multiplicity, and $\mathrm{vol}(X, g)$ the corresponding Riemannian volume.

In [40], the authors gave a full asymptotic expansion of the corresponding heat kernel, which can be used to deduce (3.2) by a Karamata theorem [19, Chapter XIII, Theorem 2]. More generally, we still have (3.2) if $X$ has a smooth boundary, for instance with Dirichlet boundary conditions, see for example [25] and references therein for a review of different techniques for heat kernel asymptotics in the case with and without boundary.

More generally, one can ask for eigenvalue asymptotics in the case of singular metrics, where the singularity could be both on the boundary of the manifold or in its interior.

In the case of domains in euclidean space with a certain power-type singularity at the boundary, Métivier [39] gave an extensive description and derived Weyl's law in different settings. In his proofs, he worked with weighted Sobolev spaces. Using his results, one can also derive the Weyl law in our setting (3.1) after suitably localising near the boundary and using local coordinates, though the constants in [39] are not explicit. We learnt about [39] after the completion of our work [9] from Bernard Helffer, who we would like to thank for this very helpful comment. We would also like to thank an anonymous referee for pointing out the reference [49], which also includes the Weyl law in our setting (3.1) in suitable coordinates.

More recently, in [4], the authors considered the example of a Grushin cylinder and a Grushin sphere where they computed the eigenvalues and corresponding eigenfunctions explicitly, thereby also establishing a Weyl law. The Weyl law in their example is a special case for our setting (3.1) with a critical value of $\alpha$.

We would also like to mention the recent work [8], which considered singular Riemannian manifolds with a singularity at the boundary under different assumptions on the metric. The authors identify subcritical, critical and supercritical cases depending on the behaviour of the Riemannian volume of the set of points with distance at most $1/\sqrt{\lambda}$ for $\lambda \to \infty$ and derive a corresponding Weyl law in the subcritical and critical case. In the supercritical case, they derive corresponding bounds.

We would also like to refer to [12, 10, 11] for various results on asymptotic expansions of heat kernels in sub-Riemannian geometry and local Weyl laws. In particular, these papers treat the Grushin case, which corresponds to $\alpha = 1$ in (3.1).

In [9, Theorem 2], we showed that in the supercritical regime, most of the eigenfunctions accumulate at the boundary. In [18], I showed with Larry Read that this happens at scale $\lambda^{-1/(2-\alpha)}$, and we also identified a profile that emerges after zooming in at that scale.

In the following, I would like to give more details on the statement and the proof of [9, Theorem 1] using Dirichlet-Neumann bracketing [9, Remark 2]. Some of the proof ideas we explain here were rewritten in [18, Section 2] in the critical case for sums of eigenvalues and with a rescaled test function. After the completion of this work, we realised that very similar ideas were also used in [49].

## 3.3 Normal form and change of variables

We let $n \in \mathbb{N}$ such that the Riemannian manifold $X$ is of dimension $n+1$. The part of $X$ near the boundary is diffeomorphic to $[0,1] \times M$, where $M$ is a smooth $n$-dimensional Riemannian manifold that is diffeomorphic to the boundary; we identify $\{0\} \times M$ with $\partial X$. Near the boundary $\partial X$, we can choose our coordinates such that $g$ can be expressed as

$$g = \frac{1}{u^\alpha}(du^2 + g_0(u)) \quad \text{for } (u, y) \in [0,1] \times M, \tag{3.3}$$

which is referred to as a normal form, see [30, Lemma 5.2]. Here $(g_0(u))_{u \in [0,1]}$ is a family of smooth non-degenerate Riemannian metrics on $M$ that depend continuously on $u$.

We can then perform a change of variables $x = 1/(1-\alpha/2)u^{1-\alpha/2}$ and obtain

$$g = dx^2 + x^{-\beta} g_1(x) \quad \text{for } (x, y) \in [0,1] \times M, \quad \text{where } \beta := \frac{2\alpha}{2-\alpha}. \tag{3.4}$$

Again, $(g_1(x))_{x \in [0,1]}$ is a family of smooth non-degenerate Riemannian metrics on $M$ that depend continuously on $x$.

## 3.4 Statement of [9, Theorem 1] in the subcritical, critical and supercritical case

Denote by $\Delta_g$ the Laplace-Beltrami operator on $X$ with Dirichlet boundary conditions on $\partial X$ (consider the corresponding quadratic form for smooth compactly supported functions). We would like to understand the asymptotics for the number $N(\lambda)$ of eigenvalues of $\Delta_g$ that are less than $\lambda$ for $\lambda \to \infty$. For the parameter $0 < \alpha < 2$ in (3.1), there is a subcritical regime $\alpha < 2/(n+1)$, a critical value $\alpha = 2/(n+1)$, and a supercritical regime $\alpha > 2/(n+1)$.

In the **subcritical case** $\alpha < 2/(n+1)$, we show in [9, Theorem 1] that we have the asymptotics

$$N(\lambda) = C_{n+1} \mathrm{vol}(X, g) \lambda^{(n+1)/2} + o\left(\lambda^{(n+1)/2}\right) \quad \text{as } \lambda \to \infty. \tag{3.5}$$

Here $C_{n+1}$ denotes a universal semiclassical constant depending on the dimension $n+1$, and $\mathrm{vol}(X, g)$ is the Riemannian volume of $X$ with respect to $g$. These asymptotics (3.5) agree with the leading order Weyl asymptotics for non-singular smooth

Riemannian metrics on compact Riemannian manifolds, possibly with boundary.

In the **critical case** $\alpha = 2/(n+1)$, we show in [9, Theorem 1] that

$$N(\lambda) = K_n \mathrm{vol}(M, g_1(0))\lambda^{(n+1)/2}\log(\lambda) + o\left(\lambda^{(n+1)/2}\log(\lambda)\right) \quad \text{as } \lambda \to \infty \tag{3.6}$$

for some explicit constant $K_n$ only depending on $n$. Here $\mathrm{vol}(M, g_1(0))$ denotes the Riemannian volume of $M$ with respect to $g_1(0)$.

In the **supercritical case** $\alpha > 2/(n+1)$, [9, Theorem 1] states that

$$N(\lambda) = K_{n,\alpha}\mathrm{vol}(M, g_1(0))\lambda^{n/(2-\alpha)} + o\left(\lambda^{n/(2-\alpha)}\right) \quad \text{as } \lambda \to \infty, \tag{3.7}$$

where $K_{n,\alpha}$ is an explicit constant only depending on $n$ and $\alpha$.

## 3.5 Hausdorff dimension

The exponents appearing in (3.5), (3.6) and (3.7) are linked to the Hausdorff dimension of $(X, g)$. For $\alpha \leqslant 2/(n+1)$, the Hausdorff dimension of $(X, g)$ agrees with its topological dimension and is equal to $n+1$. For $\alpha > 2/(n+1)$, the Hausdorff dimension of $(X, g)$ is given by $d := 2n/(2-\alpha)$. Hence, up to the logarithmic factor in the critical case, we always have a behaviour like $\lambda^{d_H/2}$ if $d_H$ denotes the Hausdorff dimension of $(X, g)$. This is not surprising in view of the modified Weyl-Berry conjecture [35] (which is however false in general [36]) and that in our case, the Hausdorff dimension and the Minkowski dimension agree.

## 3.6 Discreteness of the spectrum

As we have mentioned previously, the discreteness of the spectrum of the Laplace-Beltrami operator $\Delta_g$ for $g$ given in (3.1) with $0 < \alpha < 2$ was shown in [30, Proposition 29]. For critical and supercritical $\alpha$, that is, $2 > \alpha \geqslant 2/(n+1)$, the Riemannian volume $\mathrm{vol}(X, g)$ is infinite. It might at first sight be a bit surprising that we nevertheless have discrete spectrum in this case. Morally speaking, I would say that the discrete spectrum for $\alpha < 2$ seems more linked to the fact that for $\alpha < 2$, every point in $X$ has a finite distance to the boundary measured with respect to the metric $g$. By contrast, for $\alpha \geqslant 2$, every point in the interior has an infinite distance to the boundary. This infinite distance is helpful in the construction of Weyl sequences that can show the existence of essential spectrum. Note that $\alpha = 2$ corresponds to the case of hyperbolic geometry.

## 3.7 Quasi-isometric metric

Let us use the notation $X_{\tilde{\varepsilon}} = [0, \tilde{\varepsilon}]_x \times M_y$ for any $\tilde{\varepsilon} > 0$. Here the indices $x$ and $y$ indicate that we work with coordinates $(x, y) \in [0, \tilde{\varepsilon}] \times M$ as in (3.4), and we identify this set with the corresponding part of $X$. Since the family of metrics $g_1(x)$ on $M$ is continuous in $x$ (see (3.4)), we can find for any $\delta > 0$ an $\varepsilon > 0$ (which we fix in the following) and a smooth metric $g_\varepsilon$ on $X$ such that

$$g_\varepsilon = dx^2 + x^{-\beta} g_1(x = 0) \quad \text{on } X_{3\varepsilon}$$

and

$$(1 + \delta)^{-1} g \leqslant g_\varepsilon \leqslant (1 + \delta) g \quad \text{on } X. \tag{3.8}$$

From (3.8), we obtain

$$(1 + c_\delta)^{-1} \Delta_g \leqslant \Delta_{g_\varepsilon} \leqslant (1 + c_\delta) \Delta_g$$

in the sense of quadratic forms for some $c_\delta > 0$ with $c_\delta \to 0$ as $\delta \to 0$. In particular, the leading order Weyl asymptotics for $\Delta_g$ and $\Delta_{g_\varepsilon}$ will only differ by a factor of at most $(1 + c_\delta)$. Since we can choose $\delta$ arbitrarily small, it suffices to prove the desired asymptotics for $\Delta_{g_\varepsilon}$. For simplicity of notation, we assume in the following without loss of generality that $g = g_\varepsilon$.

## 3.8 Change of function and change of measure

The change of functions consists in replacing any function $\tilde{f} \in L^2(X, dv_g)$ by a function $f$ which is $x^{-\beta n/4} \tilde{f}$ on $X_{2\varepsilon}$ and is equal to $\tilde{f}$ on $X \setminus X_{3\varepsilon}$, and $f$ is some interpolation of the two behaviours on $X_{3\varepsilon} \setminus X_{2\varepsilon}$. We also make a change of measure such that this transformation becomes unitary. In particular, on $X_{2\varepsilon}$, the Riemannian volume measure $dv_g(x, y) = x^{-n\beta/2} dx dv_G(y)$ is replaced by $dx dv_G(y)$, where $G = g_1(x = 0)$ is a metric on $M$. In particular, for any $\tilde{f}, \tilde{h} \in L^2(X_{2\varepsilon}, dv_g)$, we have

$$\int_{X_{2\varepsilon}} \tilde{f} \tilde{h} \, dv_g(x, y) = \int_{X_{2\varepsilon}} f g \, dx dv_G(y). \tag{3.9}$$

On $X_{2\varepsilon}$ in local coordinates, the operator $\Delta_{g_\varepsilon}$ (which we from now on refer to as $\Delta_g$ since we assume without loss of generality $g = g_\varepsilon$) is of the form

$$-\partial_x^2 + \frac{C_\beta}{x^2} + x^\beta \Delta_M \quad \text{with } C_\beta = \frac{\beta n}{4} \left( 1 + \frac{\beta n}{4} \right). \tag{3.10}$$

Recall that $\beta$ was defined in (3.4). $\Delta_M$ denotes the Laplace-Beltrami operator on $M$ with respect to the metric $G = g_1(0)$. Notice that the operator in (3.10) allows to separate the variables $x$ and $y$. This will be used in Section 3.9 below.

### 3.8.1 Splitting of the Riemannian manifold

The next step in the proof consists in splitting the Riemannian manifold $X$ into a part close to the boundary $X_\varepsilon$ and the rest $X \setminus X_\varepsilon$.

The number $N(\lambda) := N_X(\lambda)$ of negative eigenvalues of the Laplace-Beltrami operator $\Delta_g$ (which is a nonnegative operator for us) on $X$ less than $\lambda > 0$ can then be estimated using Dirichlet-Neumann bracketing by the sum of the corresponding numbers of eigenvalues of the Laplace-Beltrami operator on the two parts:

$$N^D_{X_\varepsilon}(\lambda) + N^D_{X\setminus X_\varepsilon}(\lambda) \leqslant N(\lambda) \leqslant N^N_{X_\varepsilon}(\lambda) + N^N_{X\setminus X_\varepsilon}(\lambda) \tag{3.11}$$

Here the $D$ or $N$ stand for Dirichlet or Neumann boundary conditions at the boundary of $X_\varepsilon$ in the interior of $X$, namely at $\{\varepsilon\} \times M$. Strictly speaking, we will first perform the change of function and change of measure from Section 3.8 and then put the Dirichlet or Neumann boundary conditions.

We then examine the asymptotics of the terms in (3.11) separately. The parts corresponding to the interior of $X$, namely $N^{D/N}_{X\setminus X_\varepsilon}(\lambda)$ can be treated using classical results since the metric $g$ is non-singular there. We have

$$N^{D/N}_{X\setminus X_\varepsilon}(\lambda) = C_{n+1}\mathrm{vol}(X \setminus X_\varepsilon, g)\lambda^{(n+1)/2} + o\left(\lambda^{(n+1)/2}\right) \quad \text{as } \lambda \to \infty. \tag{3.12}$$

The main contribution is a precise understanding of the asymptotics for the term near the boundary $N^{D/N}_{X_\varepsilon}(\lambda)$.

## 3.9 Separation of variables and the one-dimensional operator $P_\mu$

Note that when considering $\Delta_g$ on $X_\varepsilon$ in local coordinates (both with Dirichlet or Neumann boundary conditions at $\{\varepsilon\} \times M$), we can find an orthonormal basis of eigenfunctions of $\Delta_g$ with eigenvalue $\eta \geqslant 0$ that are of the form

$$\Phi(x, y) = \varphi(x)\psi(y), \quad (x, y) \in [0, \varepsilon] \times M \tag{3.13}$$

where $\varphi$, $\psi$ are $L^2$-normalised, $\psi$ is an eigenfunction of $\Delta_M$ with eigenvalue $\mu \geqslant 0$, and $\varphi$ is an eigenfunction of

$$P_\mu := -\partial_x^2 + \frac{C_\beta}{x^2} + \mu x^\beta \tag{3.14}$$

on $[0, \varepsilon]$ with eigenvalue $\eta$. Here we used (3.10).

Using the notation $N^{D/N}_{\mu,[0,\varepsilon]}(\lambda)$ for the number of eigenvalues of $P_\mu$ less than $\lambda$ with Dirichlet or Neumann boundary conditions at $x = \varepsilon$ and always Dirichlet boundary conditions at $x = 0$ (since we have Dirichlet boundary conditions for $\Delta_g$ at $\partial X$), we obtain

$$N^{D/N}_{X_\varepsilon}(\lambda) = \sum_{j=0}^{\infty} N^{D/N}_{\mu_j,[0,\varepsilon]}(\lambda). \tag{3.15}$$

Here we denote by $(\mu_j)_{j\in\mathbb{N}_0}$ the eigenvalues of $\Delta_M$ counted with multiplicity.

## 3.10 Application of the Weyl asymptotics for $M$

Note that for any $\mu \geqslant 0$, $P_\mu \geqslant -\partial_x^2$, and therefore, from the explicit theory for the one-dimensional Laplacian on an interval,

$$N^{D/N}_{\mu,[0,\varepsilon]}(\lambda) \leqslant 1 + \frac{1}{\pi}\varepsilon\sqrt{\lambda} = o(\lambda^{(n+1)/2}) \text{ as } \lambda \to \infty. \tag{3.16}$$

Thus, any of the terms in (3.15) individually is of subleading order.

At the same time, from the Weyl asymptotics on $M$, we know that

$$\mu_j = c_M j^{2/n}(1 + o(1)) \text{ as } j \to \infty,$$

where $c_M = (|B_1^n|(2\pi)^{-n} v_G(M))^{-2/n}$ and $|B_1^n|$ denotes the volume of the unit ball in dimension $n$. For any $\delta > 0$ we can find $K \in \mathbb{N}$ large enough such that

$$(1+\delta)^{-1} c_M j^{2/n} \leqslant \mu_j \leqslant (1+\delta) c_M j^{2/n} \tag{3.17}$$

for all $j \geqslant K$. Replacing $\mu_j$ by the estimates in (3.17) only changes the leading order asymptotics for $N^{D/N}_{X_\varepsilon}(\lambda)$ by at most a factor that converges to 1 as $\delta \to 0$. Since we can take $\delta$ arbitrarily small, we can without loss of generality assume that $\mu_j = c_M j^{2/n}$ for all $j \geqslant K$.

Combining these two aspects, we see that for the leading order asymptotics of

$N^{D/N}_{X_\varepsilon}(\lambda)$, it suffices to find the leading-order asymptotics of

$$\sum_{j=K}^{\infty} N^{D/N}_{c_M j^{2/n},[0,\varepsilon]}(\lambda). \tag{3.18}$$

## 3.11 Unitary equivalence of $P_\mu$ and scaling

For any $\mu > 0$ and $a > 0$, the operator $P_\mu$ on $[0,a]$ is unitarily equivalent to $\mu^{\frac{2}{2+\beta}} P_1$ on $[0, \mu^{1/(2+\beta)}a]$, see [9, Proposition 5]. This can be seen through the unitary transformation

$$(Uf)(x) \mapsto \mu^{\frac{1}{2(2+\beta)}} f\left(\mu^{\frac{1}{2+\beta}} x\right). \tag{3.19}$$

It follows from this and (3.18) that we need to understand the leading-order asymptotics of

$$\sum_{j=K}^{\infty} N^{D/N}_{c_M j^{2/n},[0,\varepsilon]}(\lambda) = \sum_{j=K}^{\infty} N^{D/N}_{1,[0,\varepsilon(c_M j^{2/n})^{1/(2+\beta)}]}\left((c_M j^{2/n})^{-2/(2+\beta)}\lambda\right). \tag{3.20}$$

The scaling we use here is one of the key steps in the proof. In particular, it is essential for the proof in the critical and supercritical case to obtain asymptotics instead of leading-order bounds with non-matching constants.

In the subcritical case, we will show that (3.20) is an arbitrarily small factor times the leading order term. In the critical and supercritical case, it will be the main order term and it does not depend on $\varepsilon$. By replacing $\varepsilon$ by $\varepsilon(c_M)^{-1/(2+\beta)}$ and replacing $\lambda$ by $(c_M)^{-2/(2+\beta)}\lambda$, on can check in those cases that the sum (3.20) is proportional to $v_G(M)$, as in (3.6) and (3.7), and one can also keep track of the constants. Therefore, it suffices to understand the asymptotics of

$$\sum_{j=K}^{\infty} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right), \tag{3.21}$$

where we use the notation $d := n(1+\beta/2)$. Note that we have $d > n$ for all $\beta > 0$.

## 3.12 Removing zero and subleading contributions

Note that there exists some $c > 0$ (depending on $\beta$) such that

$$\frac{C_\beta}{x^2} + x^\beta \geqslant c \quad \text{for all } x > 0. \tag{3.22}$$

This expression appears in the definition of $P_1$, see (3.14), and shows that $P_1 \geqslant c$, independently of the interval on which we consider $P_1$ or which boundary conditions we pick. Hence, there exists $C_0 > 0$ such that for all $j \geqslant C_0 \lambda^{d/2}$, we have

$$N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) = 0. \tag{3.23}$$

Thus, by (3.23), we can express (3.21) as

$$\sum_{j=K}^{\infty} N^{D/N}_{j^{n/2},[0,\varepsilon]}(\lambda) = \sum_{j=K}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right). \tag{3.24}$$

Furthermore, for any $L > 0$ we have by (3.16)

$$\sum_{j=K}^{L\lambda^{n/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \leqslant L\lambda^{n/2}\left(1 + \frac{1}{\pi}\varepsilon\sqrt{\lambda}\right) = \frac{1}{\pi}\varepsilon L \lambda^{(n+1)/2} + o(\lambda^{(n+1)/2}) \tag{3.25}$$

as $\lambda \to \infty$.

In the subcritical case, we will first choose $L > 0$ very large and then we only need to choose $\varepsilon = \varepsilon(L) > 0$ small enough (namely we want $\varepsilon L$ to be small). Then we get a term of leading order, but with arbitrarily small prefactor. In the critical and supercritical case, this term will be of subleading order, and we may choose $L = L(\varepsilon) > 0$ large depending on $\varepsilon > 0$.

Thus, for $L > 0$, we will need to understand the asymptotics of

$$\sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right), \tag{3.26}$$

which is a sum of counting functions for the operator $P_1$, which was defined in (3.14), considered on intervals of different sizes. Continuing from this point, we give a separate end of proof in the subcritical, critical and supercritical case.

## 3.13 End of the proof in the critical case

In the critical case, we have $\beta = 2/n$. For some $E > 0$ small and $L > 0$ to be chosen, we split the sum (3.26) as follows:

$$\begin{aligned} &\sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \\ &= \sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) + \sum_{j=E\lambda^{d/2}+1}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) =: T + S, \end{aligned} \tag{3.27}$$

where we assume without loss of generality that $E\lambda^{d/2}, C_0\lambda^{d/2} \in \mathbb{N}$.

Let us first estimate $T$. We use the notation $\omega := j^{-2/d}\lambda$ and note that for $L\lambda^{n/2} \leqslant j \leqslant E\lambda^{d/2}$, we have

$$L^{-2/d}\lambda^{\beta/(2+\beta)} \geqslant \omega \geqslant E^{-2/d}. \tag{3.28}$$

If $L = L(\varepsilon) > 0$ is chosen large enough depending on $\varepsilon$, then we have $\varepsilon\sqrt{\lambda/\omega} > \omega^{1/\beta}$ for all $j$ with $L\lambda^{n/2} \leqslant j \leqslant E\lambda^{d/2}$ and thus, by Dirichlet-Neumann bracketing,

$$N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \leqslant N^{N}_{1,[0,\omega^{1/\beta}]}(\omega) + N^{N}_{1,[\omega^{1/\beta},\varepsilon\sqrt{\lambda/\omega}]}(\omega) = N^{N}_{1,[0,\omega^{1/\beta}]}(\omega)\ , \tag{3.29}$$

where $N$ stands for additional Neumann boundary conditions at $x = \omega^{1/\beta}$. Here we used that $P_1 \geqslant \omega$ on $[\omega^{1/\beta}, \varepsilon\sqrt{\lambda/\omega}]$ in the last step. Similarly, we get

$$N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \geqslant N^{D}_{1,[0,\omega^{1/\beta}]}(\omega)\,. \tag{3.30}$$

Next, we claim that for any $\gamma > 0$, we can find $E > 0$ small enough such that for all $\omega := j^{-2/d}\lambda$ with $L\lambda^{n/2} \leqslant j \leqslant E\lambda^{d/2}$ and $L = L(\varepsilon) > 0$ chosen as above, we have

$$N^{D/N}_{1,[0,\omega^{1/\beta}]}(\omega) \in \left[(1+\gamma)^{-1}A\omega^{1/2+1/\beta}, (1+\gamma)A\omega^{1/2+1/\beta}\right], \tag{3.31}$$

where

$$A := \frac{1}{\pi}\int_0^1 \sqrt{1 - z^\beta}\,dz. \tag{3.32}$$

The proof of (3.31) follows along the lines of the proof of [9, Proposition 4]. Using Dirichlet-Neumann bracketing, we can split the interval $[0, \omega^{1/\beta}]$ into smaller intervals. On each of those smaller intervals, we estimate the potential $C_\beta x^{-2} + x^\beta$ in the definition of $P_1$ by a constant from above and below. Then we use estimates for eigenvalues for the Dirichlet and Neumann Laplacian on intervals. We omit further details here.

By (3.31), we get

$$T \in \left[ (1+\gamma)^{-1} A \sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} \left(j^{-2/d}\lambda\right)^{1/2+1/\beta}, (1+\gamma) A \sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} \left(j^{-2/d}\lambda\right)^{1/2+1/\beta} \right]. \quad (3.33)$$

Since $\beta = 2/n$, we have $(1/2+1/\beta) = (n+1)/2$ and $2/d(1/2+1/\beta) = 1$. Thus,

$$\sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} \left(j^{-2/d}\lambda\right)^{1/2+1/\beta} = \lambda^{(n+1)/2} \sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} \frac{1}{j}. \quad (3.34)$$

Recall that

$$\sum_{j=1}^{E\lambda^{d/2}} \frac{1}{j} = \log(E\lambda^{d/2}) + \mathcal{O}(1) = \frac{d}{2}\log(\lambda) + \mathcal{O}(1) \quad \text{as } \lambda \to \infty \quad (3.35)$$

and similarly,

$$\sum_{j=1}^{L\lambda^{n/2}} \frac{1}{j} = \frac{n}{2}\log(\lambda) + \mathcal{O}(1) \quad \text{as } \lambda \to \infty. \quad (3.36)$$

Using $d = n+1$, so $d/2 - n/2 = 1/2$, it follows from (3.35) and (3.36) that

$$\sum_{j=L\lambda^{n/2}}^{E\lambda^{d/2}} \frac{1}{j} = \frac{1}{2}\log(\lambda) + \mathcal{O}(1) \quad \text{as } \lambda \to \infty. \quad (3.37)$$

Combining (3.33), (3.34) and (3.37), we get

$$T \in \left[(1+\gamma)^{-1}, (1+\gamma)\right] \frac{A}{2}\lambda^{(n+1)/2}\log(\lambda) + \mathcal{O}\left(\lambda^{(n+1)/2}\right) \quad \text{as } \lambda \to \infty. \quad (3.38)$$

Recall that $\gamma > 0$ can be chosen arbitrarily small, and this will give us the leading-order asymptotics for (3.26) in the critical case.

In order to complete the proof of (3.6) in the critical case, it remains to show that $S$ is of subleading order. To this end, note that if $j \geqslant E\lambda^{d/2}$ and $\lambda > 0$ is large enough (depending on $\varepsilon$ and $E$), then we always have $E^{-2/(d\beta)} \leqslant \varepsilon j^{1/d}$ and by

Dirichlet-Neumann bracketing, we have

$$\begin{aligned} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) &\leqslant N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(E^{-2/d}\right) \\ &\leqslant N^{N}_{1,[0,E^{-2/(d\beta)}]}\left(E^{-2/d}\right) + N^{N}_{1,[E^{-2/(d\beta)},\varepsilon j^{1/d}]}\left(E^{-2/d}\right) \\ &= N^{N}_{1,[0,E^{-2/(d\beta)}]}\left(E^{-2/d}\right) =: C(E), \end{aligned}$$

where we used that $P_1 \geqslant E^{-2/d}$ on $[E^{-2/(d\beta)}, \varepsilon j^{1/d}]$. It follows using $d = n+1$ that

$$S = \sum_{j=E\lambda^{d/2}+1}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \leqslant C_0\lambda^{d/2}C(E) = C_0 C(E)\lambda^{(n+1)2}\,, \tag{3.39}$$

and thus,

$$S = o\left(\lambda^{(n+1)/2}\log(\lambda)\right) \qquad \text{as } \lambda \to \infty\,. \tag{3.40}$$

This completes the proof of (3.6) in the critical case.

## 3.14 End of the proof in the supercritical case

Recall that by the unitary equivalence of $P_\mu$ on $[0,a]$ and $\mu^{\frac{2}{2+\beta}}P_1$ on $[0,\mu^{1/(2+\beta)}a]$ via (3.19),

$$N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) = N^{D/N}_{j^{2/n},[0,\varepsilon]}\left(\lambda\right), \tag{3.41}$$

which is monotone decreasing in $j$. Using this monotonicity and that each of those terms is of subleading order, see (3.16), we can interpret our sum as a Riemann sum and obtain

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{-d/2} \sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \\ &= \lim_{\lambda\to\infty} \lambda^{-d/2} \sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon\left(j\lambda^{-d/2}\right)^{1/d}\sqrt{\lambda}]}\left(\left(j\lambda^{-d/2}\right)^{-2/d}\right) \\ &= \lim_{\lambda\to\infty} \int_{L\lambda^{n/2-d/2}}^{C_0} N^{D/N}_{1,[0,\varepsilon s^{1/d}\sqrt{\lambda}]}\left(s^{-2/d}\right)\,ds. \end{aligned} \tag{3.42}$$

In a similar way to the proof of (3.31), one can also show that there exists a (non-optimal) constant $C$ depending only on $C_0$ such that for any $\lambda > 0$, $\varepsilon > 0$ and for any $\omega \geqslant {C_0}^{-2/d}$, we have

$$N^{D/N}_{1,[0,\varepsilon\sqrt{\lambda/\omega}]}\left(\omega\right) \leqslant C\omega^{1/2+1/\beta}. \tag{3.43}$$

Applying (3.43) with $\omega = s^{-2/d}$ for $s \leqslant C_0$ and using $\beta > 2/n$, we obtain

$$N^{D/N}_{1,[0,\varepsilon s^{1/d}\sqrt{\lambda}]}\left(s^{-2/d}\right) \leqslant Cs^{-2/d(1/2+1/\beta)} = Cs^{-2/(n\beta)} \in L^1([0,C_0]). \tag{3.44}$$

Thus, the limit in (3.42) is finite and we have found a dominating function, which we will later use to apply the dominated convergence theorem in (3.42).

We claim that

$$\lim_{\lambda\to\infty} N^{D/N}_{1,[0,\varepsilon s^{1/d}\sqrt{\lambda}]}\left(s^{-2/d}\right) = N_{1,[0,\infty)}\left(s^{-2/d}\right) \quad \text{for almost every } s > 0. \tag{3.45}$$

Note that we cannot expect pointwise convergence since $N^{D/N}_{1,[0,\varepsilon s^{1/d}\sqrt{\lambda}]}\left(s^{-2/d}\right) \in \mathbb{N}_0$. For (3.45), it suffices to show that for every $k \in \mathbb{N}$, the $k$-th eigenvalue of $P_1$ on $[0,a]$ converges to the $k$-th eigenvalue of $P_1$ on $[0,\infty)$ as $a \to \infty$. In the case of Dirichlet boundary conditions, this can be seen using the min-max principle. In the case of Neumann boundary conditions, one can show an Agmon-type estimate to get the convergence. We will omit the details here.

Using (3.44), (3.45) and the dominated convergence theorem, we obtain

$$\lim_{\lambda\to\infty}\int_{L\lambda^{n/2-d/2}}^{C_0} N^{D/N}_{1,[0,\varepsilon s^{1/d}\sqrt{\lambda}]}\left(s^{-2/d}\right) = \int_0^{C_0} N_{1,[0,\infty)}\left(s^{-2/d}\right) \in (0,\infty), \tag{3.46}$$

and by (3.42), this completes the proof of (3.7).

## 3.15 End of the proof in the subcritical case

Using (3.44) and multiplying everything in (3.42) by $\lambda^{d/2}$, we get

$$\sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \leqslant \lambda^{d/2}\int_{L\lambda^{n/2-d/2}}^{C_0} Cs^{-2/(n\beta)}\,ds + o(\lambda^{d/2}) \text{ as } \lambda\to\infty. \tag{3.47}$$

Note that

$$\lambda^{d/2}\int_{L\lambda^{n/2-d/2}}^{C_0} s^{-2/(n\beta)}\,ds \leqslant C\lambda^{d/2}\left((L\lambda^{n/2-d/2}\right)^{1-2/(n\beta)} \leqslant CL^{1-2/(n\beta)}\lambda^{(n+1)/2} \tag{3.48}$$

and since $\beta < 2/n$, the factor $L^{1-2/(n\beta)}$ can be arbitrarily small if we choose $L > 0$ large enough. Since we have $d < n+1$, we find that

$$\sum_{j=L\lambda^{n/2}}^{C_0\lambda^{d/2}} N^{D/N}_{1,[0,\varepsilon j^{1/d}]}\left(j^{-2/d}\lambda\right) \leqslant CL^{1-2/(n\beta)}\lambda^{(n+1)/2} + o(\lambda^{(n+1)/2}) \quad \text{as } \lambda \to \infty. \tag{3.49}$$

Combining this with (3.25) and (3.12), we obtain (3.5).

# Chapter 4

# Isoperimetric inequalities for inner parallel curves

## 4.1 Overview of [16]

Together with Ayman Kachmar and Vladimir Lotoreichik [16], we studied an isoperimetric problem that was motivated by a previous paper of theirs [32]. We proved the following statement: Let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain. Let $B \subset \mathbb{R}^2$ be the ball centred at the origin with $|\partial B| = |\partial\Omega|$. Denote for all $t \geqslant 0$

$$S_t(\Omega) := \{x \in \overline{\Omega} \mid \text{ dist } (x, \partial\Omega) = t\}$$

and similarly for $S_t(B)$. Define the centroid of $S_t(\Omega)$ by

$$c(t) := |S_t(\Omega)|^{-1} \int_{S_t(\Omega)} x d\mathcal{H}^1(x) \,. \tag{4.1}$$

Then

$$\int_{S_t(\Omega)} |x - c(t)|^2 d\mathcal{H}^1(x) \leqslant \int_{S_t(B)} |x|^2 d\mathcal{H}^1(x) \text{ for almost every } t \geqslant 0, \tag{4.2}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Note that $\int_{S_t(B)} |x|^2 d\mathcal{H}^1(x)$ is *not* the variance of $x$ for $S_t(B)$ since we do not divide by $|S_t(B)|$.

(4.2) was the missing ingredient to extend an isoperimetric inequality for the lowest eigenvalue of the magnetic Robin Laplacian with negative boundary parameter for convex domains in [32] to all smooth simply connected domains in $\mathbb{R}^2$ with finite perimeter.

The case $t = 0$ of (4.2) is a classical result due to Hurwitz [31] relying on Fourier techniques. In order to extend this result to $t > 0$, we used in [16] an explicit construction of a closed curve $\Sigma_t$ with $S_t(\Omega) \subset \Sigma_t$ and $|\Sigma_t| \leqslant |S_t(B)|$ for almost every $t > 0$.

We also studied a related question: Fix $p > 0$. Let $\Gamma \subset \mathbb{R}^2$ be a closed Lipschitz curve with the origin as its centroid, and let $B \subset \mathbb{R}^2$ be a ball centred at the origin with $|\Gamma| = |\partial B|$. Then do we always have

$$\int_\Gamma |x|^p d\mathcal{H}^1(x) \leqslant \int_{\partial B} |x|^p d\mathcal{H}^1(x)\,?$$

We could give a partial answer to this question, namely yes for all $p \leqslant 2$, and no for all $p > 3$. Using a Fuglede-type argument, we could show that if $2 < p \leqslant 3$ in a suitable sense, for nearly circular domains $\Omega$ locally near a ball, the answer is yes. The case $2 < p \leqslant 3$ in its full generality remains an interesting open problem.

In the following, we focus on (4.2). We first consider the example of a dumbbell in Section 4.3. In Section 4.5 and Section 4.6, we explain the strategy of how to define the closed curve $\Sigma_t$ with $S_t(\Omega) \subset \Sigma_t$ and $|\Sigma_t| \leqslant |S_t(B)|$ for almost every $t > 0$. In Section 4.4, we explain why this is enough to conclude (4.2).

## 4.2 Context for [16]

Using Fourier analysis, Hurwitz [31, pp. 396-397] showed that for any closed curve $\Gamma$ of fixed length with the origin as its centroid, one has

$$\int_\Gamma |x|^2 \, d\mathcal{H}^1(x) \leqslant \frac{|\Gamma|^3}{(2\pi)^2} \tag{4.3}$$

with equality if and only if the curve is a circle around the origin. Recently, generalisations of (4.3) to higher dimensions and corresponding quantitative versions were shown in [33].

For inner parallel curves, Hartman [28, Corollary 6.1] showed that $|S_t(\Omega)| \leqslant |S_t(B)|$ for almost every $t > 0$.

Combining the two results [31, 28], one can show that if $S_t(\Omega)$ is a closed curve, then (4.2) holds, see for instance the proof of [32, Proposition 4.4] or our explanation in Section 4.4 below. For instance, $S_t(\Omega)$ is a closed curve for all $t > 0$ if $\Omega$ is convex, see [32, Proposition 4.4]. Our main contribution in [16, Theorem 1.1] is to generalise this result to general smooth, bounded, simply connected domains (the smoothness assumption is technical and can probably be relaxed).

Our work [16] was motivated by [32]. Assuming (4.2), the authors of [32] proved an isoperimetric inequality for the first eigenvalue of the magnetic Robin Laplacian [32, Theorem 4.8]. They left as an open question for which domains $\Omega$ one has (4.2). Using (4.2) for all smooth, bounded simply connected domains with finite perimeter, we could straightforwardly extend this isoperimetric inequality for the first eigenvalue of the magnetic Robin Laplacian [16, Theorem 1.5].

There are many results in the literature on weighted isoperimetric inequalities. In [3], the authors consider weighted integrals over the boundary of the domain for different domains with the same Lebesgue measure. In [1] a similar setting is considered, but the Lebesgue measure is replaced by some weighted Lebesgue measure with a power-type weight. Another result on weighted isoperimetric inequalities we would like to mention is [6].

## 4.3 The dumbbell example

Let us consider the example where $\Omega$ is a dumbbell, such as in Figure 4.1. Ayman Kachmar mentioned this example to me when we started working on the project. By symmetry of the dumbbell, we have $c(t) = 0$ for all $t \geqslant 0$. Let us without loss of generality assume that $|\partial\Omega| = 2\pi$, so we need to compare $\Omega$ with a ball of radius 1 centred at the origin.

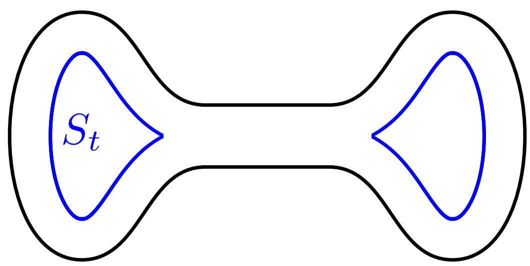


Figure 4.1: This picture shows an example where $\Omega$ is a dumbbell-shaped region. The set $S_t(\Omega) =: S_t$ for some $t > 0$ is drawn in blue. Note that it consists of two connected components.

For $t$ in a certain range, $S_t(\Omega)$ consists of two connected components that are far away from each other. We know that $|S_t(\Omega)| \leqslant |S_t(B)|$ for almost every $t > 0$ by [28, Corollary 6.1]. However, only using this information is not enough to conclude since the two parts of $S_t(\Omega)$ are very far away from the origin, so the $|x|^2$ for $x \in S_t(\Omega)$

in (4.2) can be larger than 1. Note that the two connected components of $S_t(\Omega)$ cannot be arbitrarily far away from the origin since they are a subset of $\Omega$ and we have $|\partial\Omega| = 2\pi$, so we always have $|x| \leqslant \pi$ for $x \in S_t(\Omega)$ (using the symmetry of the dumbbell around the origin). Also note that in the case of a dumbbell with a very long middle axis, $|S_t(\Omega)|$ is much smaller than $|S_t(B)|$, see Figure 4.1.

## 4.4 General proof strategy

In this section, we explain why for the proof of (4.2), it is sufficient to construct a closed curve $\Sigma_t$ with $S_t(\Omega) \subset \Sigma_t$ and of length $|\Sigma_t| \leqslant |S_t(B)|$ for almost every $t > 0$. This result can be deduced from [31, pp. 397-397]. The following explanation can also be found in [16, Section 4].

Fix $t > 0$ and suppose we have already constructed such a curve $\Sigma_t$ of length $L := |\Sigma_t| \leqslant |S_t(B)|$. Let $\sigma : [0, L] \to \mathbb{R}^2$ be a parametrisation of that curve by arc-length. Assume without loss of generality that the centroid of $\Sigma_t$ is the origin, that is,

$$\int_0^L \sigma(s)\, ds = 0 \tag{4.4}$$

(otherwise shift the whole coordinate system). Then

$$\int_{S_t(\Omega)} |x-c(t)|^2 d\mathcal{H}^1(x) \leqslant \int_{S_t(\Omega)} |x|^2 d\mathcal{H}^1(x) \leqslant \int_{\Sigma_t} |x|^2 d\mathcal{H}^1(x) = \int_0^L |\sigma(s)|^2\, ds. \tag{4.5}$$

Note that the first step can be shown by a short computation that uses the definition of $c(t)$ as a centroid and the fact that the power is equal to 2. Next, we apply the Poincaré inequality using (4.4), see for example [27, Section 7.7], to get

$$\int_0^L |\sigma(s)|^2\, ds \leqslant \frac{L^2}{(2\pi)^2} \int_0^L |\sigma'(s)|^2\, ds = \frac{L^3}{(2\pi)^2}\,, \tag{4.6}$$

where we used the arc-length parametrisation of $\sigma$, that is, $|\sigma'(s)| = 1$ for all $s$, in the last step.

By scaling in (4.2), we may without loss of generality assume that $|\partial\Omega| = |\partial B| = 2\pi$, that is, $B$ is a ball of radius 1 around the origin. In particular, we have $|S_t(B)| = 2\pi(1-t)$. Then

$$\int_{S_t(B)} |x|^2 d\mathcal{H}^1(x) = (1-t)^3 2\pi. \tag{4.7}$$

Thus, if

$$L \leqslant (1-t)2\pi = |S_t(B)|\,, \tag{4.8}$$

then by (4.5), (4.6), (4.7) and (4.8), we obtain (4.2).

This shows that in order to complete the proof, it suffices to find a closed curve $\Sigma_t$ with $S_t(\Omega) \subset \Sigma_t$ and of length $|\Sigma_t| \leqslant |S_t(B)|$.

## 4.5 A first naive try for the construction

Since $\Omega$ is assumed to be simply connected and we assume without loss of generality that $|\partial\Omega| = 2\pi$, we can parametrise its boundary by a curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ that is parametrised by arc-length. Moreover, we may assume that $\gamma$ has positive orientation. We denote by $n : [0, 2\pi] \to \mathbb{R}^2$ the corresponding inward pointing normal vector

$$n(s) = (-\gamma_2'(s), \gamma_1'(s))^T = J\gamma'(s), \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.9}$$

and by $\kappa : [0, 2\pi] \to \mathbb{R}$ the curvature defined by

$$\gamma''(s) = \kappa(s)n(s). \tag{4.10}$$

Since $\gamma$ is a positively oriented non-self-intersecting closed curve, we have by [2, Satz 2.2.10]

$$\int_0^{2\pi} \kappa(s)\, ds = 2\pi. \tag{4.11}$$

Using the above notation, we have [16, Lemma 2.3]

$$S_t(\Omega) \subset \{\gamma(s) + tn(s) \mid x \in [0, 2\pi]\}. \tag{4.12}$$

In view of (4.12), a natural guess for a closed curve containing $S_t(\Omega)$ and with length less than or equal to $|S_t(B)|$ is given by the curve $A_t$ parametrised by

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \quad \alpha(s) = \gamma(s) + tn(s). \tag{4.13}$$

The curve $A_t$ for the dumbbell-shaped region that was shown in Figure 4.1 is illustrated in Figure 4.2.

The following results and explanations for $A_t$ can be found in [16, Lemma 2.3].

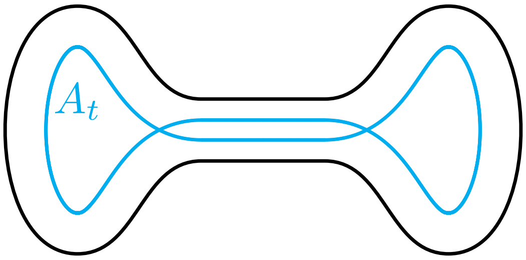


Figure 4.2: This image shows an example where $\Omega$ is a dumbbell-shaped region. The curve $A_t$ for some $t > 0$, which is parametrised by (4.13) is drawn in cyan. Note that we have $S_t(\Omega) \subset A_t(\Omega)$, compare with Figure 4.1. In this example we have $\kappa(s) \leqslant 1/t$ for all $s \in [0, 2\pi]$ and thus, by (4.16), we have $|A_t| = |S_t(B)|$.

We have

$$\begin{aligned} \alpha'(s) &= \gamma'(s) + tn'(s) = \gamma'(s) + t\frac{\mathrm{d}}{\mathrm{d}s}(J\gamma'(s)) = \gamma'(s) + tJ\gamma''(s) \\ &= \gamma'(s) + tJ\kappa(s)n(s) = \gamma'(s) + tJ\kappa(s)J\gamma'(s) = (1 - t\kappa(s))\gamma'(s) \end{aligned} \tag{4.14}$$

Here we used (4.9) in the second step, (4.10) in the fourth step, and $J^2 = -Id$ in the last step. Thus, the length of the curve $A_t$ is

$$|A_t| = \int_0^{2\pi} |\alpha'(s)|\, ds = \int_0^{2\pi} |1 - t\kappa(s)|\, ds\,, \tag{4.15}$$

where we used (4.14) and that $\gamma$ is parametrised by arc-length.

Note that using (4.11), we have

$$\int_0^{2\pi} (1 - t\kappa(s))\, ds = 2\pi(1 - t)\,, \tag{4.16}$$

so if $1 - t\kappa(s) > 0$ for all $s \in [0, 2\pi]$, then $|A_t| = 2\pi(1 - t) = |S_t(B)|$, which is the desired length, see (4.8). Thus, for $t$ small enough, the construction of the curve $A_t$ and the considerations from Section 4.4 yield (4.2).

By contrast, for $t$ such that $|\{s \in [0, 2\pi] \mid 1 - t\kappa(s) < 0\}| > 0$, we see from (4.15) and (4.16) that $|A_t| > 2\pi(1 - t) = |S_t(B)|$, which is not enough to conclude. Such an example is shown in Figure 4.3.

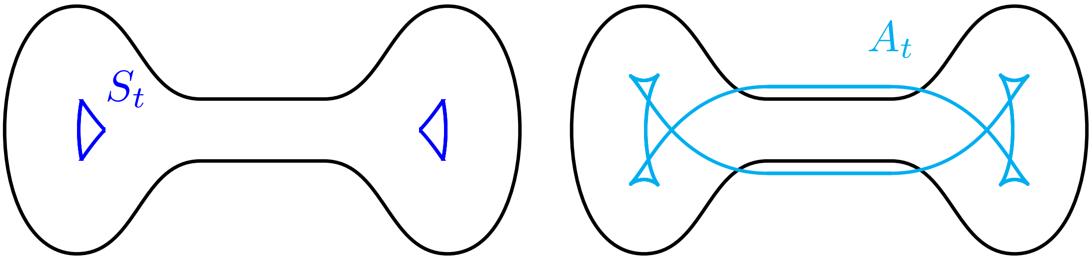


Figure 4.3: These images show the same example of a dumbbell $\Omega$ as in Figure 4.1 and Figure 4.2. Here we consider a larger value of $t$ and these images show the corresponding inner parallel curve $S_t(\Omega) =: S_t$ on the left and the curve $A_t$ on the right. In this example we have $\kappa(s) > 1/t$ for some $s \in [0, 2\pi]$ and thus, by (4.16) and (4.15), we deduce $|A_t| > |S_t(B)|$.

## 4.6 General ideas on the modification of the closed curve

In the case where $|\{s \in [0, 2\pi] \mid 1 - t\kappa(s) < 0\}| > 0$, that is, where $\kappa(s) > 1/t$ on a set of positive measure, a natural idea is to try to modify the curve $A_t$ in such a way that its length decreases to at most $2\pi(1-t) = |S_t(B)|$ while still covering $S_t(\Omega)$.

Recall that the circle of radius $1/\kappa(s)$ that is tangent to the curve at $\gamma(s)$ is the circle that approximates the curve best at this point, namely up to second order in the Taylor expansion, see for example [16, Lemma 2.2]. If $\kappa(s) \leqslant \kappa_0$ for all $s \in [0, 2\pi]$, then for any $s \in [0, 2\pi]$ the ball of radius $1/\kappa_0$ that is tangent to the curve at $\gamma(s)$ is contained in $\Omega$. On the other hand, if for some $s \in [0, 2\pi]$, we have $\kappa(s) > 1/t$, then we do *not* have $\gamma(s) + tn(s) \in S_t(\Omega)$ in general.

Thus, we would like to discard all points from the construction of our curve $A_t$, where $\kappa(s) > 1/t$. There are two restrictions we should keep in mind here: First, the resulting curve $\Sigma_t$ should be a closed continuous curve, so in particular, we cannot allow for any jumps. Second, we need to make sure that the new curve $\Sigma_t$ has a length of at most $2\pi(1-t) = |S_t(B)|$. For the second point, note that if $|\{s \in [0, 2\pi] \mid 1 - t\kappa(s) < 0\}| > 0$, then

$$\int_{\{s\in[0,2\pi]\,|\,\kappa(s)\leqslant 1/t\}} |1 - t\kappa(s)|\, ds > 2\pi(1-t) \tag{4.17}$$

by (4.16). Thus, just removing those points $\alpha(s)$ in the curve for which $\kappa(s) > 1/t$ is *not* enough.

## 4.7 Construction of the closed curve $\Sigma_t$

In order to deal with the two restrictions we explained in Section 4.6, we proceed as follows.

We will first remove all points $\alpha(s)$ from our curve that satisfy $\alpha(s) \notin S_t(\Omega)$. For almost every $t > 0$, we will obtain a finite number of curve segments, see [16, Proposition 2.1] and the references therein. More precisely, there is $m \in \mathbb{N}$ and there are $0 \leqslant a_1 < b_1 < a_2 < b_2 < \ldots < a_m < b_m \leqslant 2\pi$ such that

$$S_t(\Omega) = \{\gamma(s) + tn(s) \mid s \in [a_k, b_k] \text{ for some } 1 \leqslant k \leqslant m\}. \tag{4.18}$$

The result from [46, Theorem 4.4.1] we use here assumes smoothness of the curve, which could probably be relaxed.

Then, we will add some straight line segments to connect those curve segments $\gamma(s)+tn(s)$, $s \in [a_k, b_k]$ and obtain a closed curve $\Sigma_t$. More precisely, we add straight lines connecting $\gamma(b_k) + tn(b_k)$ with $\gamma(a_{k+1}) + tn(a_{k+1})$, where we use the notation $a_{m+1} := a_1$. The details on the construction of $\Sigma_t$ can be found in the proof of [16, Theorem 3.1]. An illustration of $\Sigma_t$ is shown in Figure 4.4.

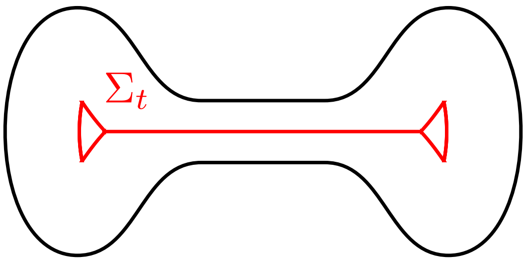


Figure 4.4: This picture shows the same setting as in Figure 4.3. The closed curve $\Sigma_t$ is drawn in red. Note that the straight line in the middle is doubly covered by $\Sigma_t$. Four straight line segments from the construction of $\Sigma_t$ just consist of a single point.

Finally, we need to show that $|\Sigma_t| \leqslant |S_t(B)|$, see [16, Theorem 3.1]. To this end, we use a lemma [16, Proposition 2.7] that estimates the length of a curve outside two disks starting and ending tangently to those by the distance of the centres of the two disks and the integral over the curvature. More precisely, [16, Proposition 2.7] states the following:

Suppose that we have two disks with radius $t$, possibly intersecting each other, with distance $R$ of the two centres, and a smooth curve $\Gamma$ parametrised by arc-length by $\gamma : [a, b] \to \mathbb{R}^2$ that always stays in the complement of the two disks and starts at $\gamma(a)$ tangently in a positively oriented way to the first disk and ends at $\gamma(b)$ tangently in a positively oriented way to the second disk, then

$$|\Gamma| \geqslant R + t \int_a^b \kappa(s)\, ds, \tag{4.19}$$

where $\kappa(s)$ denotes the curvature of $\Gamma$ at $\gamma(s)$. (4.19) is very intuitive, but until now, we only have proofs using relatively technical and powerful results from the literature [7, Theorem 3.3] (see the proof of [16, Proposition 2.7]) or tedious elementary proofs (see for example [15, Appendix A.2]).

With $|\Sigma_t| \leqslant |S_t(B)|$, we can conclude (4.2) by our explanations in Section 4.4, see [16, Section 4].

# Chapter 5

# Hardy-Sobolev interpolation inequalities

## 5.1 Overview of [17]

Recall that the classical Hardy inequality states that in any dimension $d \geqslant 3$, we have for all $u \in \dot{H}^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geqslant \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx. \tag{5.1}$$

Using the ground state representation, see for example [24, Eq. (2.14)], and radial decreasing arrangement, one can show that for $u \not\equiv 0$, the inequality is strict.

One might thus ask for improvements of (5.1) that involve an error term. Note that by translation, from (5.1), one also gets for all $u \in \dot{H}^1(\mathbb{R}^d)$ with $u \not\equiv 0$

$$\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2} dx > 0. \tag{5.2}$$

Previous results estimated the left-hand side in (5.2) by a subcritical $L^p$-norm [5], see (5.6) below, or a subcritical Sobolev norm [21], while fixing the $L^2$-norm of $u$.

In a joint work with Phan Thành Nam [17] we prove an inequality, see (5.3) below, that involves the left-hand side in (5.2) and is critical in the sense that all terms scale in the same way. More precisely, we show [17, Theorem 1] for $d = 3$ and

$\theta = 1/3$ the following Hardy-Sobolev inequality: For all $u \in \dot{H}^1(\mathbb{R}^d)$, we have

$$\left( \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2} dx \right)^{\theta} \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2} dx \right)^{1-\theta} \geqslant C \|u\|_{L^{2^*}}^2, \tag{5.3}$$

where $C > 0$ only depends on $d, \theta$ and $2^* = 2d/(d-2)$. We also show that (5.3) does not hold if $d \geqslant 4$ or if $\theta \neq 1/3$. Furthermore, we show a more general analogue of (5.3) involving $L^p$-spaces for $p \in [2, d)$ and Lorentz spaces [17, Theorem 2]. The new point of (5.3) with respect to previous estimates, see for example [5] or (5.6) below, is the appearance of the critical exponent $2^*$ on the right-hand side.

Our proof relies on a Morrey-type inequality going back to Palatucci–Pisante [42], the ground state representation for the Hardy inequality, symmetric decreasing rearrangement and an optimal estimate in the radial case. In fact, we can determine all the optimisers of the corresponding radial problem explicitly [17, Theorem 3(i)], which are given by the family

$$u^{\eta}(x) = \frac{1}{(|x|^{1-\eta}(1+|x|^{2\eta}))^{(d-2)/2}}, \quad \eta \in (0, \infty), \tag{5.4}$$

up to dilation and multiplication by a constant. It remains an open problem to prove whether there exist minimisers for the full non-radial inequality (5.3).

## 5.2 Context of [17]

Brezis and Vázquez [5, Theorem 4.1 and Extension 4.3] showed that if

$$d \geqslant 3, \quad 2 < q < 2^* = \frac{2d}{d-2}, \quad \theta = d\left(\frac{1}{2} - \frac{1}{q}\right), \tag{5.5}$$

then there exists a constant $C > 0$ such that for all $u \in H^1(\mathbb{R}^d)$

$$\left( \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2} dx \right)^{\theta} \|u\|_{L^2}^{2(1-\theta)} \geqslant C \|u\|_{L^q}^2, \tag{5.6}$$

Note that the $q$ in (5.5) is subcritical with respect to the Sobolev critical exponent $2^*$. More recently, a corresponding version involving the corresponding subcritical Sobolev norms was shown in [21, Theorem 1.2].

As we show below in Section 5.3, a corresponding version of (5.6) does *not* hold in the critical case. One of the main points of [17] is that we *do* have a corresponding inequality only involving quantities with critical scaling behaviour if we replace the $L^2$-norm in (5.6) by a suitable Morrey-type norm.

A related Morrey-type inequality, which we also use in our proof, is the following inequality for all $d \geqslant 3$ and $1-2/d \leqslant \theta \leqslant 1$ that goes back to Palatucci and Pisante [42, Theorem 1]

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\theta} \left(\sup_{R>0, x\in\mathbb{R}^d} R^{-2} \int_{B_R(x)} |u|^2\right)^{1-\theta} \geqslant C\|u\|^2_{L^{2^*}}, \quad \text{for all } u \in \dot{H}^1(\mathbb{R}^d). \tag{5.7}$$

We would also like to refer to [48] for a recent simplified proof of (5.7) involving the use of maximal functions.

Below we explain a radial example from [17, Section 2] in Section 5.3 and a bubbling example from [17, Section 3.1] in Section 5.5 in detail. Both examples combined explain why (5.3) does *not* hold if $d \geqslant 4$ or if $\theta \neq 1/3$, see Section 5.4 and Section 5.5. Furthermore, the radial example is important for understanding why (5.6) can only work for subcritical norms. The bubbling example exhibits a key feature that makes it challenging to answer the question of existence of optimisers for (5.3) with $d = 3$ and $\theta = 1/3$, see Section 5.6.

## 5.3 Radial example

This section is devoted to an explanation of why the inequality estimating the first factor in (5.3) by a subcritical $L^p$-norm, see (5.6) [5, Theorem 4.1 and Extension 4.3], does *not* hold for critical $L^p$-norms. The understanding of the counterexample explained below was the starting point our work in [17].

For our counterexample, we will consider radial functions $u \in H^1(\mathbb{R}^d)$ and write them as

$$u(x) = \frac{f(|x|)}{|x|^{(d-2)/2}}, \tag{5.8}$$

which corresponds to the ground state representation for Hardy's inequality, see for example [24, Eq. (2.14)]. With this notation, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |u|^{2^*} = |\mathbb{S}^{d-1}| \int_0^\infty \frac{|f(r)|^{2^*}}{r}\, dr \\ &\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx = \int_{\mathbb{R}^d} \frac{|\nabla f(x)|^2}{|x|^{d-2}}\, dx = |\mathbb{S}^{d-1}| \int_0^\infty |f'(r)|^2 r\, dr, \end{aligned} \tag{5.9}$$

where $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere in $\mathbb{R}^d$.

For our counterexample, we choose $f = f_\varepsilon$ depending on a parameter $\varepsilon > 0$ (which we will later send to zero) as follows:

$$f_\varepsilon(r) = \begin{cases} r^\varepsilon, & r \in (0,1], \\ r^{-\varepsilon}, & r \in [1,\infty). \end{cases} \tag{5.10}$$

The idea is to consider very flat profiles $f_\varepsilon$, which correspond to $\varepsilon$ small. If $0 < \varepsilon \leqslant (d-2)/2$, then the function

$$u_\varepsilon(x) = f_\varepsilon(|x|)|x|^{-\frac{d-2}{2}} = \begin{cases} |x|^{\varepsilon - \frac{d-2}{2}} & , |x| \leqslant 1, \\ |x|^{-\varepsilon - \frac{d-2}{2}} & , |x| \geqslant 1 \end{cases} \tag{5.11}$$

is radially symmetric decreasing, and hence,

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_\varepsilon(x)|^2}{|x-y|^2}\, dx = \int_{\mathbb{R}^d} \frac{|u_\varepsilon(x)|^2}{|x|^2}\, dx. \tag{5.12}$$

With this choice of $f = f_\varepsilon$, we have for any $p > 0$

$$\int_0^\infty r|f_\varepsilon'(r)|^2\, dr = \varepsilon, \qquad \int_0^\infty \frac{|f_\varepsilon(r)|^p}{r}\, dr = \frac{2}{p\varepsilon}. \tag{5.13}$$

By (5.9) and (5.13), we get for any $\varepsilon > 0$ for the corresponding function $u_\varepsilon$ defined by (5.11)

$$\int_{\mathbb{R}^d} |\nabla u_\varepsilon|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_\varepsilon(x)|^2}{|x-y|^2}\, dx = |\mathbb{S}^{d-1}|\varepsilon \tag{5.14}$$

and

$$\int_{\mathbb{R}^d} |u_\varepsilon|^{2^*} = |\mathbb{S}^{d-1}| \frac{2}{2^*\varepsilon}. \tag{5.15}$$

Sending $\varepsilon \to 0$, we find that (5.6) cannot hold with a constant $C > 0$ independent of $u$ if $d \geqslant 3$, $q = 2^*$, $\theta = 1$.

## 5.4 Application of the radial example to (5.3)

Note that in our example defined by (5.11), we also have for $0 \leqslant \varepsilon \leqslant (d-2)/2$

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_\varepsilon(x)|^2}{|x-y|^2} = \int_{\mathbb{R}^d} \frac{|u_\varepsilon(x)|^2}{|x|^2}\, dx = |\mathbb{S}^{d-1}| \int_0^\infty \frac{|f_\varepsilon(r)|^2}{r}\, dr = |\mathbb{S}^{d-1}| \frac{1}{\varepsilon}, \tag{5.16}$$

where we used (5.12) in the first step and (5.13) in the last step. Having noted this, it seems natural to ask if (5.3) holds for a suitable range of $\theta$. Plugging in (5.14), (5.15) and (5.16) into (5.3), we get (for a different constant $C$)

$$\varepsilon^{\theta}\varepsilon^{-(1-\theta)} \geqslant C\varepsilon^{-2/2^*}, \tag{5.17}$$

and since we may send $\varepsilon \to 0$, (5.3) can only hold if

$$\theta \leqslant \frac{1}{2}\left(1 - \frac{2}{2^*}\right) = \frac{1}{d}. \tag{5.18}$$

## 5.5 Bubbling example

In this section, I would like to explain a bubbling example from [17, Section 3.1]. Combined with $\theta \leqslant 1/d$, see (5.18), this example shows that (5.3) can only hold for $d = 3$, $\theta = 1/3$. Furthermore, this bubbling example seems to be the main obstruction in showing existence of an optimising function $u \not\equiv 0$ for (5.3) with the optimal constant $C > 0$ for $d = 3$, $\theta = 1/3$, see Section 5.6 below.

We take $N \in \mathbb{N}$ identical bubbles travelling away from each other. Later we will let $N \to \infty$. Fix any $0 \neq \varphi \in C_c^\infty(B_1(0))$, $0 \neq z \in \mathbb{R}^d$ and define

$$u_N(x) = \sum_{n=1}^{N} \varphi(x + nNz) \tag{5.19}$$

for every $N \in \mathbb{N}$. Note that for $N$ large enough, the individual bubbles $\{\varphi(\cdot + nNz)\}_{n=1}^N$ have disjoint support and are far away from each other.

Thus, for $N$ large enough, we have

$$\int_{\mathbb{R}^d} |u_N|^{2^*} = N \int_{\mathbb{R}^d} |\varphi|^{2^*}, \tag{5.20}$$

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_N(x)|^2}{|x-y|^2}\,dx = \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\varphi(x)|^2}{|x-y|^2}\,dx + o(1) \quad \text{as} \quad N \to \infty \tag{5.21}$$

and

$$\int_{\mathbb{R}^d} |\nabla u_N|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_N(x)|^2}{|x-y|^2}\,dx = N \int_{\mathbb{R}^d} |\nabla\varphi|^2 + \mathcal{O}(1) \quad \text{as} \quad N \to \infty\,. \tag{5.22}$$

The estimate in (5.21) indicates that a single term, which corresponds to one bubble, dominates the full integral, while the term in (5.22) is well approximated by the sum of the corresponding terms for all the bubbles.

Thus, (5.3) can only hold if (for a different constant $C$), we have for all $N \in \mathbb{N}$ large enough

$$N^\theta \geqslant C N^{2/2^*}. \tag{5.23}$$

It follows that we need

$$\theta \geqslant \frac{2}{2^*} = 1 - \frac{2}{d}. \tag{5.24}$$

Combining (5.24) with (5.18) for $d \geqslant 3$, we find that (5.3) can only possibly hold for $d = 3$, $\theta = 1/3$.

## 5.6 Outlook: Existence of optimisers?

One may ask if (5.3) for $d = 3$ and $\theta = 1/3$ admits nonzero-optimisers: Namely if $C > 0$ is the optimal constant in

$$\left( \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|^2} dx \right)^{1/3} \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|^2} dx \right)^{2/3} \geqslant C \|u\|_{L^6}^2, \tag{5.25}$$

then is there a function $u \in \dot{H}^1(\mathbb{R}^3)$ with $u \not\equiv 0$ such that we have equality in (5.25)?

We solved the corresponding radial problem in [17, Theorem 3(i)]. We showed that for the inequality

$$\left( \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx \right)^{1/3} \left( \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx \right)^{2/3} \geqslant C \|u\|_{L^6}^2, \tag{5.26}$$

for radial $u \in \dot{H}^1(\mathbb{R}^3)$ with the optimal constant $C > 0$ (possibly a different one than in (5.25)), all optimisers (up to scaling and multiplication by constants) are given explicitly by (5.4) above. By contrast, the full non-radial problem (5.25) remains open.

A standard strategy for showing the existence of an optimiser is to take an optimising sequence $(u_n)_{n \in \mathbb{N}} \subset \dot{H}^1(\mathbb{R}^3)$ that is normalised in a suitable sense, for example $\int_{\mathbb{R}^3} |\nabla u|^2 = 1$, and to show that up to a subsequence, translation and scaling, it converges weakly to a candidate for an optimiser $\tilde{u} \not\equiv 0$, and then to show that $\tilde{u}$ is indeed an optimiser. A key difficulty in our case is to show that $\tilde{u}$ is non-zero, or put differently, to exclude the vanishing case in the concentration-compactness method.

Let us elaborate a bit more on this problem in a related setting, namely the inequality

$$\int_{\mathbb{R}^3} |\nabla u|^2 \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|^2} dx \right)^2 \geqslant C \int_{\mathbb{R}^3} |u|^6 \tag{5.27}$$

for all $u \in \dot{H}^1(\mathbb{R}^3)$, which can be deduced from (5.7). Note that all three terms in (5.27) scale in the same way when replacing $u(x)$ by $au(b(x-c))$ for $a, b > 0$ and $c \in \mathbb{R}^3$. Therefore, using a rescaling of $u$, it is only possible to fix the value of one of these three terms.

Moreover, if $\varphi \in C_c^\infty(\mathbb{R}^3)$ is an almost-optimiser, then taking an arbitrarily large number of copies of $\varphi$ and sending them far away from each other by translation is also an almost-optimiser of (5.27). This bubbling phenomenon is the key challenge to overcome when trying to understand if (5.27) admits non-zero optimisers.

More precisely, for any given $\delta > 0$, there is $\delta \geqslant \varepsilon > 0$ and $0 \not\equiv \varphi \in C_c^\infty(\mathbb{R}^3)$ depending on $\varepsilon$ such that

$$\int_{\mathbb{R}^3} |\nabla \varphi|^2 \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|^2} dx \right)^2 = (C-\varepsilon) \int_{\mathbb{R}^3} |\varphi|^6. \tag{5.28}$$

Furthermore, if we take $N \in \mathbb{N}$ copies of $\varphi$ and send them far away from each other, namely to define $u_N$ as in (5.19), then

$$\lim_{N \to \infty} \frac{\int_{\mathbb{R}^3} |\nabla u_N|^2 \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_N(x)|^2}{|x-y|^2} dx \right)^2}{\int_{\mathbb{R}^3} |u_N|^6} = C - \varepsilon. \tag{5.29}$$

If we define $\tilde{u}_N := N^{-1/2} u_N$ in order to fix the $\dot{H}^1$-norm, then $\int_{\mathbb{R}^3} |\nabla \tilde{u}_N|^2 = \int_{\mathbb{R}^3} |\nabla \varphi|^2$ for all $n \in \mathbb{N}$ large enough. At the same time,

$$\int_{\mathbb{R}^3} |\tilde{u}_N|^6 = N^{-2} \int_{\mathbb{R}^3} |\varphi|^6 \to 0 \quad \text{as} \quad N \to 0\,. \tag{5.30}$$

Thus, any translated and rescaled subsequence of $(\tilde{u}_n)_{n \in \mathbb{N}}$ that is weakly convergent in $\dot{H}^1(\mathbb{R}^3)$ has zero as its weak limit.

Using the above example with different $\varphi$ and a diagonal sequence argument, one can construct different minimising sequences of (5.27) that consist of an arbitrary number $N \in \mathbb{N}$ of bubbles that are smooth and compactly supported. Furthermore, these sequences, which are normalised in the $\dot{H}^1(\mathbb{R}^3)$-norm, vanish in any $L^p(\mathbb{R}^3)$ norm for $p > 2$.

The main reason why this bubbling phenomenon occurs is a lack of strict convexity in the problem. By contrast, for a strictly convex problem, such as the Sobolev inequality

$$\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^3 \geqslant C \int_{\mathbb{R}^3} |u|^6, \tag{5.31}$$

taking two or more copies of a compactly supported function are always far away from being an optimiser. In (5.31), the ratio of the left-hand side divided by the right-hand side will be at least $N^2$ if we take $N$ disjoint bubbles. This shows that minimising sequences for (5.31) cannot exhibit a bubbling phenomenon as described above.

In our inequality (5.25), one can expect that the term

$$-\frac{1}{4} \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_N(x)|^2}{|x-y|^2} dx \tag{5.32}$$

in the first factor could prevent the bubbling phenomenon. However, it is less clear than in the case of strict convexity how to make this idea work in a proof. This remains a problem for future works.

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# Part II

# Articles

In this part, the articles [14, 9, 16, 17] are printed in the version in which they appeared in the journal or on the preprint server arXiv.

# SEMICLASSICAL ESTIMATES FOR SCHRÖDINGER OPERATORS WITH NEUMANN BOUNDARY CONDITIONS ON HÖLDER DOMAINS

CHARLOTTE DIETZE



Abstract. We prove a universal bound for the number of negative eigenvalues of Schrödinger operators with Neumann boundary conditions on bounded Hölder domains, under suitable assumptions on the Hölder exponent and the external potential. Our bound yields the same semiclassical behaviour as the Weyl asymptotics for smooth domains. We also discuss different cases where Weyl's law holds and fails.

## 1. Introduction

The celebrated correspondence principle, which goes back to Niels Bohr in the early days of quantum mechanics, states that quantum systems exhibit classical behaviour in the limit of large quantum numbers. In the context of spectral analysis of Schrödinger operators, this leads to the semiclassical approximation, which suggests that any bound state can be related to a volume of size $(2\pi)^d$ in the phase space $\mathbb{R}^d \times \mathbb{R}^d$ [1, Section 4.1.1]. In particular, the number of negative eigenvalues $N(-\Delta_\Omega - \lambda)$ of $-\Delta_\Omega - \lambda$ on a domain $\Omega \subset \mathbb{R}^d$ with suitable boundary conditions can be approximated by its semiclassical analogue

$$N(-\Delta_\Omega - \lambda) \approx \frac{1}{(2\pi)^d} \left| \left\{ (p,x) \in \mathbb{R}^d \times \Omega \;\middle|\; |p|^2 - \lambda < 0 \right\} \right| = \frac{|B_1(0)|}{(2\pi)^d} |\Omega| \lambda^{\frac{d}{2}} \tag{1}$$

in the large coupling limit $\lambda \to \infty$, with $|B_1(0)|$ the volume of the unit ball in $\mathbb{R}^d$. More generally, for a general potential $V : \Omega \to (-\infty, 0]$ one might expect that

$$N(-\Delta_\Omega + \lambda V) \approx \frac{1}{(2\pi)^d} \left| \left\{ (p,x) \in \mathbb{R}^d \times \Omega \;\middle|\; |p|^2 + \lambda V(x) < 0 \right\} \right| = \frac{|B_1(0)|}{(2\pi)^d} \int_\Omega |\lambda V|^{\frac{d}{2}}. \tag{2}$$

Rigorous justifications of (1) and (2) have been shown for a large class of smooth domains $\Omega$ and potentials $V$. On the other hand, in general, implementing the semiclassical approximation for rough domains and potentials is difficult. In this paper, we will discuss the validity of (2) for Hölder domains $\Omega$ and suitable $L^p$-integrable potentials $V$. We will focus on Neumann boundary conditions as Dirichlet boundary conditions have been well understood.

1.1. **Main results.** Let $d \in \mathbb{N}$, $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a domain, that is, an open bounded and connected subset of $\mathbb{R}^d$. For $\gamma \in (0,1]$, a $\gamma$-Hölder domain is a domain $\Omega \subset \mathbb{R}^d$ which is locally the subgraph of a $\gamma$-Hölder continuous function $f$, that is, there exists a constant $c > 0$ such that

$$|f(x) - f(y)| \leq c|x-y|^\gamma \tag{3}$$

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for all $x, y$ in the domain of $f$, which is an open subset of $\mathbb{R}^{d-1}$. In the case $\gamma = 1$, we call $\Omega$ a Lipschitz domain. See Section 2.1 for details.

Denote the Dirichlet Laplacian on $\Omega$ by $-\Delta_\Omega^D$ and the number of negative eigenvalues of $-\Delta_\Omega^D - \lambda$ by $N\left(-\Delta_\Omega^D - \lambda\right)$, where $\lambda > 0$. One of the fundamental results of spectral theory is Weyl's law [2, 3, 4, 5], which states that

$$N\left(-\Delta_\Omega^D - \lambda\right) = \frac{|B_1(0)|}{(2\pi)^d}|\Omega|\lambda^{\frac{d}{2}} + o\left(\lambda^{\frac{d}{2}}\right) \text{ as } \lambda \to \infty, \tag{4}$$

thus rigorously justifying (1), see also [6] for a second order result. This asymptotics also holds for the Neumann Laplacian[1] $-\Delta_\Omega^N$ for Lipschitz domains $\Omega$, or more generally for extension domains, see [7] and also [8, Theorem 3.20].

In general, the Weyl asymptotics for the Neumann Laplacian does *not* hold for arbitrary domains $\Omega$. It is well-known, see for example [9], that there are domains $\Omega$ such that zero is contained in the essential spectrum of $-\Delta_\Omega^N$. Interestingly, Netrusov and Safarov showed that the Weyl asymptotics (4) holds for the Neumann Laplacian $-\Delta_\Omega^N$, with any $\gamma$-Hölder domain $\Omega$, if and only if $\gamma > (d-1)/d$ (see [10, Corollary 1.6 and Theorem 1.10]).

In the present paper, we are interested in $N\left(-\Delta_\Omega^N + V\right)$ with a potential $V : \Omega \to (-\infty, 0]$ on a Hölder domain $\Omega$. Unlike the case of constant potentials, the problem with a general potential $V$ is more subtle as the following theorem shows.

**Theorem 1.1** (Example with non-semiclassical behaviour). *Let $d \geq 2$. For every $\gamma \in \left(\frac{d-1}{d}, 1\right)$ there exists a $\gamma$-Hölder domain $\Omega \subset \mathbb{R}^d$ and $V : \Omega \to (-\infty, 0]$ with $V \in L^{\frac{d}{2}}(\Omega)$ such that*

$$\limsup_{\lambda \to \infty} \lambda^{-d/2} N\left(-\Delta_\Omega^N + \lambda V\right) = \infty. \tag{5}$$

We prove Theorem 1.1 by constructing a $\gamma$-Hölder domain $\Omega$ in the same way as Netrusov and Safarov in [10, Theorem 1.10], and we choose the potential $V$ to be growing to infinity near the boundary in such a way that $-\Delta_\Omega^N + \lambda V$ can support significantly more than $\lambda^{\frac{d}{2}}$ bound states, see Section 7.1 for more details.

Our next result is a universal bound on the number of negative eigenvalues of the Schrödinger operator $-\Delta_\Omega^N + V$ on $L^2(\Omega)$ with a suitable potential $V$ on a Hölder domain $\Omega$. The Cwikel-Lieb-Rozenblum inequality [11, 12, 13] states that for any open set $\Omega \subset \mathbb{R}^d$ for $d \geq 3$, there exists a constant $C = C(d) > 0$ such that for every $V : \Omega \to (-\infty, 0]$, we have

$$N\left(-\Delta_\Omega^D + V\right) \leq C \int_{\mathbb{R}^d} |V|^{\frac{d}{2}}. \tag{6}$$

Here $\|V\|_p$ is the $L^p$ norm of $V$. In the case of the Neumann Laplacian on a Lipschitz domain $\Omega$, if $d \geq 3$, then

$$N\left(-\Delta_\Omega^N + V\right) \leq C_\Omega \left(1 + \int_{\mathbb{R}^d} |V|^{\frac{d}{2}}\right) \tag{7}$$

for a finite constant $C_\Omega > 0$ independent of the potential $V$, see for example [8, Corollary 4.37]. In dimension $d = 2$, (6) and (7) still hold, provided that $\|V\|_{\frac{d}{2}}$ is replaced by the Orlicz norm $\|V\|_{\mathcal{B}}$ [14, 15].

[1]Here $-\Delta_\Omega^N$ is the self-adjoint operator generated by the quadratic form $\int_\Omega |\nabla u|^2$ for all $u \in H^1(\Omega)$.

On the other hand, Theorem 1.1 implies that (7) cannot be extended to Hölder domains. Our main result is a replacement of (7) for Hölder domains under suitable assumptions on the potential $V$. We need to use a weighted $L^p$-norm with a weight that grows to infinity as one approaches $\partial\Omega$. A simplified version of our result reads as follows.

**Theorem 1.2** (Cwikel-Lieb-Rozenblum type bound)**.** *Let $d \geq 2$. Let $\gamma \in \left[\frac{d-1}{d}, 1\right)$ and let $\Omega$ be a $\gamma$-Hölder domain. Then there exists a constant $C_\Omega = C_\Omega(d, \gamma, \Omega) > 0$ such that for every $V : \Omega \to (-\infty, 0]$ with $|||V||| < \infty$, we have*

$$N\left(-\Delta_\Omega^N + V\right) \leq C_\Omega \left(1 + |||V|||^{\frac{d}{2}}\right). \tag{8}$$

*Here the norm $|||V||| = \|V\|_{\tilde{p},\beta}$ is given in Definition 2.4, with $\beta$ and $\tilde{p}$ chosen as in* (22)*. Moreover, if $\gamma \in \left[\frac{2(d-1)}{2d-1}, 1\right)$, then $|||V|||$ can be replaced by $\|V\|_p$ where $p = p_{d,\gamma} > \frac{d}{2}$ is a constant depending only on $d$ and $\gamma$ satisfying*

$$\lim_{\gamma \to 1} p_{d,\gamma} = \frac{d}{2}. \tag{9}$$

A more precise statement is given in Theorem 1.4 below. The norm $|||V|||$ is stronger than $\|V\|_{\frac{d}{2}}$ (in particular, the potential $V$ in Theorem 1.1 satisfies $|||V||| = \infty$). Nevertheless, by (8), we still get the correct semiclassical behaviour as soon as $|||V||| < \infty$, namely

$$N\left(-\Delta_\Omega^N + \lambda V\right) = \mathcal{O}\left(\lambda^{\frac{d}{2}}\right) \text{ as } \lambda \to \infty. \tag{10}$$

On the technical level, our norm $|||V|||$ is chosen carefully to capture the correct leading order behavior of the number of bound states close to the boundary. By following Rozenblum's method [13], it is possible to obtain the following bound

$$N\left(-\Delta_\Omega^N + V\right) \lesssim 1 + \int_\Omega |V|^{\frac{d}{2}} + \int_{\text{close to } \partial\Omega} |V|^{\tilde{p}} \tag{11}$$

for some $\tilde{p} > \frac{d}{2}$ (this bound could also be obtained from the analysis in [16] and [17]). However, (11) is insufficient to deduce (10).

As a consequence of Theorem 1.2, we are able to come back to sharp semiclassics.

**Theorem 1.3** (Weyl's law for Schrödinger operators on Hölder domains)**.** *Let $d \geq 2$. Let $\gamma \in \left[\frac{d-1}{d}, 1\right)$ and let $\Omega \subset \mathbb{R}^d$ be a $\gamma$-Hölder domain. Let $V : \Omega \to (-\infty, 0]$ be measurable and such that $|||V||| < \infty$, where the norm $|||\cdot|||$ is the same as in Theorem 1.2. Then*

$$N\left(-\Delta_\Omega^N + \lambda V\right) = (2\pi)^{-d} |B_1(0)| \lambda^{\frac{d}{2}} \int_\Omega |V|^{\frac{d}{2}} + o\left(\lambda^{\frac{d}{2}}\right) \text{ as } \lambda \to \infty. \tag{12}$$

Let us remark that Weyl's law for constant potentials fails for $\gamma = \frac{d-1}{d}$ [10, Theorem 1.10]. This does not contradict Theorem 1.3 since $|||-1_\Omega||| = \infty$ for $\gamma = \frac{d-1}{d}$.

1.2. **Main ingredients of Theorem 1.2.** Now let us explain the proof strategy of our main result Theorem 1.2. Let us focus on the case $d \geq 3$.

Our general approach is inspired by the method of Rozenblum [13] where the number of bound states is bounded using techniques from microlocal analysis. The main idea is to first localize the Schrödinger operator $-\Delta + V$ in small domains such that it has at most one bound state in each domain, and then put these local bounds together by a covering lemma. In the present paper, since we have to deal with Hölder domains with Neumann boundary conditions, we need to deal with "oscillatory domains" when working close to the boundary, and in particular we need to introduce a new Poincaré-Sobolev inequality and a new Besicovitch-type covering lemma for those domains.

To be precise, while Rozenblum [13] works with cubes $Q \subset \mathbb{R}^d$, we will work with "oscillatory domains" $D \subset \Omega$ which are given either by cubes if they are far enough away from $\partial\Omega$, or rectangles intersected with $\Omega$ if they are close to the boundary. Since $\Omega$ is a $\gamma$-Hölder domain, the classical Poincaré-Sobolev inequality fails in general for oscillatory domains $D$ with $\overline{D} \cap \partial\Omega \neq \emptyset$. Therefore, we will develop a Poincaré-Sobolev inequality for those domains (see Corollary 3.3), which involves the ratio of the two side-lengths of the rectangle and a $L^{p^*}$-norm for some $p^* = p^*(d,\gamma) > \frac{d}{2}$. Consequently, we show that for oscillatory domains $D$ that are small enough in a suitable sense, we have

$$N\left(-\Delta_D^N + KV 1_D\right) \leq 1. \tag{13}$$

Here, $K = K(d,\gamma) \in \mathbb{N}$ is the constant from our Besicovitch-type covering theorem for oscillatory domains (see Lemma 4.1). Then the total number of bound states is controlled by a counting argument, eventually leading to the weighted norm

$$|||V||| := \|V\|_{\tilde{p},\beta,D} := \|V\|_{\frac{d}{2},D} + |V|_{\tilde{p},\beta,D}, \tag{14}$$

where $\tilde{p} = \tilde{p}(d,\gamma) > \frac{d}{2}$, $\beta = \beta(d,\gamma) > 0$ and $|V|_{\tilde{p},\beta}$ is a weighted $L^{\tilde{p}}(\Omega)$-seminorm with a weight supported near the boundary of $\Omega$ that grows at a rate determined by $\beta$ as one approaches the boundary, see (22) and Definition 2.4 for the precise definitions.

In (14) we need a $\tilde{p} > p^*$ for the following reason. Due to the Hölder-regularity of $\partial\Omega$, our oscillatory domains close to the boundary will in many cases look like very narrow rectangles intersected with $\Omega$, so the ratio of the two side-lengths of these rectangles is very far away from one. This influences the constant in the Poincaré-Sobolev inequality (Corollary 3.3). Using Hölder's inequality, we get

$$\|V\|_{p^*,D} \leq \|V\|_{\tilde{p},D} \|1\|_{\tilde{r},D} = \|V\|_{\tilde{p},D} |D|^{\frac{1}{\tilde{r}}} \tag{15}$$

for $\frac{1}{p^*} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{r}}$. The quantity $|D|^{\frac{1}{\tilde{r}}}$, which is relatively small for these narrow rectangles, cancels the effect of a growing constant in the Poincaré-Sobolev inequality as the rectangle gets more narrow.

At this point, it seems natural to measure the "size" of oscillatory domains close to the boundary by $\|V\|_{\tilde{p},D}$. Following [13], we would get (11) but it does not capture the correct semiclassical behaviour. In order to get the desired semiclassical behaviour for the parts close to the boundary as well, the key idea is to count the number of oscillatory domains $\{D_j\}_{j\in J_3}$, with some index set $J_3$, which are narrow rectangles intersected with $\Omega$ close to the boundary, by using a convexity argument. More precisely, we introduce coefficients $A_j > 0$ depending only on the larger side-length of the corresponding rectangle and the distance of the centre of the oscillatory domain $D_j$ to the boundary measured in a suitable sense. Now for suitably chosen $s, s' \in (1,\infty)$ with $\frac{1}{s} + \frac{1}{s'} = 1$, we apply Hölder's inequality for sums of products of real numbers to get

$$|J_3| = \sum_{j\in J_3} A_j^{-1} A_j \leq \Big(\sum_{j\in J_3} A_j^{-s'}\Big)^{\frac{1}{s'}} \Big(\sum_{j\in J_3} A_j^{s}\Big)^{\frac{1}{s}}. \tag{16}$$

The coefficients $A_j$ will be chosen in such a way that

$$A_j^s \lesssim |V|_{\tilde{p},\beta,D_j}^{\tilde{p}} \tag{17}$$

for all $j \in J_3$. Therefore,

$$\sum_{j \in J_3} A_j^s \lesssim \sum_{j \in J_3} |V|_{\tilde{p},\beta,D_j}^{\tilde{p}} \lesssim |V|_{\tilde{p},\beta}^{\tilde{p}} \lesssim \|V\|_{\tilde{p},\beta}^{\tilde{p}} . \tag{18}$$

In the proof of Rozenblum, the cubes $Q$ were chosen such that $\|V\|_{\frac{d}{2},Q} \le 1$. By contrast, here the oscillatory domains $\{D_j\}_{j \in J_3}$ are chosen such that $\|V\|_{\tilde{p},D_j}$ is significantly larger than one if the distance of the centre of the oscillatory domain $D_j$ to the boundary is significantly larger than the largest side-length of the corresponding rectangle. This is possible, even though we need to ensure $N\left(-\Delta_{D_j}^N + KV1_{D_j}\right) \le 1$ since in fact, we gain something in the Poincaré-Sobolev inequality for oscillatory domains because we consider an $L^{\tilde{p}}$-norm instead of an $L^{p^*}$-norm. The fact that $\|V\|_{\tilde{p},D_j}$ can be significantly larger than one in certain circumstances combined with the weight that grows near the boundary in the definition of $|V|_{\tilde{p},\beta}$ allow us to choose $A_j$ so large that

$$\sum_{j \in J_3} A_j^{-s'} \lesssim \|V\|_{\tilde{p},\beta}^{-\frac{1}{2}} \tag{19}$$

while (17) holds. We choose all parameters in such a way that

$$\frac{\tilde{p}}{s} - \frac{1}{2s'} = \frac{d}{2}, \tag{20}$$

so by (18) and (19),

$$|J_3| \le \left(\sum_{j \in J_3} A_j^{-s'}\right)^{\frac{1}{s'}} \left(\sum_{j \in J_3} A_j^{s}\right)^{\frac{1}{s}} \lesssim \|V\|_{\tilde{p},\beta}^{-\frac{1}{2s'}} \|V\|_{\tilde{p},\beta}^{\frac{\tilde{p}}{s}} = \|V\|_{\tilde{p},\beta}^{\frac{d}{2}} , \tag{21}$$

which has the desired semiclassical behaviour.

Combining these computations with the computations for oscillatory domains far enough away from the boundary, we obtain the following.

**Theorem 1.4** (Precise version of Theorem 1.2)**.** *Let $d \in \mathbb{N}$ with $d \ge 2$, $\gamma \in \left[\frac{d-1}{d}, 1\right)$, $c > 0$, $0 < h_\Omega < 1$, $L \in \mathbb{N}$ and let $\emptyset \ne \Omega \subset (0,1)^d$ be a $\gamma$-Hölder domain with parameters $c$, $h_\Omega, L$. Define*

$$\beta := \frac{1}{d+1}\left(\frac{d-1}{\gamma}+1\right)\left[\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 - d\right] > 0, \quad \tilde{p} := \frac{1}{2d}\left(\frac{d-1}{\gamma}+1\right)^2 > \frac{d}{2}. \tag{22}$$

*Let $V : \Omega \to (-\infty, 0]$ be measurable and such that $\|V\|_{\tilde{p},\beta} < \infty$. Define*

$$\delta_0 := \delta_0(V) := \min\left(\frac{h_\Omega}{\sqrt{d}}, \|V\|_{\tilde{p},\beta}^{-\frac{1}{2}}\right) \le 1. \tag{23}$$

*Then*

$$N\left(-\Delta_\Omega^N + V\right) \lesssim \delta_0^{-d} , \tag{24}$$

*where the constant in the inequality may depend on $d$, $\gamma$, $c$, $h_\Omega$, $L$.*

The definition of a "$\gamma$-Hölder domain with parameters $c$, $h_\Omega, L$" will be given in Definition 2.1. The condition $\Omega \subset (0,1)^d$ is not a restriction since we can recover the same result for all (bounded) $\gamma$-Hölder domains by scaling.

Note that from (24) in Theorem 1.4, we can deduce the first claim (8) of Theorem 1.2. Moreover, the second claim of Theorem 1.2 follows from the following corollary.

**Corollary 1.5.** *If* $\gamma \in \left[\frac{2(d-1)}{2d-1}, 1\right)$ *and* $p > \frac{\tilde{p}}{1-\beta}$*, where* $\tilde{p}$ *and* $\beta$ *were defined in Theorem 1.4, then*

$$N\left(-\Delta_\Omega^N + V\right) \lesssim 1 + \|V\|_p^{\frac{d}{2}}\,, \tag{25}$$

*where the constant in the inequality may depend on* $d$, $\gamma$, $c$, $h_\Omega$, $L$, $p$.

The details of the proofs of Theorem 1.4 and Corollary 1.5 can be found in Section 5. Let us give the key ingredients of the proof of Theorem 1.4 below. First, we have

**Lemma 1.6.** *There exists* $K = K(d, \gamma, L) \in \mathbb{N}$ *such that for any measurable* $V : \Omega \to (-\infty, 0]$ *with* $\|V\|_{\tilde{p},\beta} < \infty$ *there are families* $\mathcal{F}_1, \ldots, \mathcal{F}_K$ *of oscillatory domains* $D \subset \Omega$ *such that the following properties are satisfied:*

*(a) The oscillatory domains* $D$ *in every* $\mathcal{F}_k$ *with* $k \in \{1, \ldots, K\}$ *are disjoint and*

$$\Omega = \bigcup_{k=1}^{K} \dot{\bigcup_{D\in\mathcal{F}_k}} D\,. \tag{26}$$

*(b) For every* $k \in \{1, \ldots, K\}$ *and every* $D \in \mathcal{F}_k$ *we have*

$$N\left(-\Delta_D^N + V\right) \leq 1\,. \tag{27}$$

*(c) Moreover, for an implicit constant depending on* $d$, $\gamma$, $c$, $h_\Omega$, $L$,

$$\sum_{k=1}^{K} |\mathcal{F}_k| \lesssim \delta_0^{-d}. \tag{28}$$

To prove Lemma 1.6, we need a new covering lemma (see Lemma 4.1 for details). From Lemma 1.6, we can deduce the following two lemmata.

**Lemma 1.7** (Selfadjointness of the operator $-\Delta_\Omega^N + V$)**.** *Let* $V : \Omega \to (-\infty, 0]$ *be measurable and such that* $\|V\|_{\tilde{p},\beta} < \infty$*. Then the operator* $-\Delta_\Omega^N + V$ *is a selfadjoint operator, which is bounded from below and has the* $H^1(\Omega)$ *norm as quadratic form norm.*

**Lemma 1.8.** *Let* $V : \Omega \to (-\infty, 0]$ *be measurable and such that* $\|V\|_{\tilde{p},\beta} < \infty$*. Let* $K = K(d, \gamma, L) \in \mathbb{N}$ *and the families* $\mathcal{F}_1, \ldots, \mathcal{F}_K$ *of oscillatory domains be chosen as in Lemma 1.6. Then*

$$N\left(-\Delta_\Omega^N + \frac{1}{K}V\right) \lesssim \delta_0^{-d}\,. \tag{29}$$

The proofs of Lemmas 1.6, 1.7 and 1.8 can be found in Section 5.1. We are now ready to prove Theorem 1.4 assuming those lemmata.

*Proof of Theorem 1.4.* Let $K = K(d, \gamma, L) \in \mathbb{N}$ be chosen as in Lemma 1.6. Now apply Lemma 1.8 to $KV$ to get

$$N\left(-\Delta_\Omega^N + V\right) = N\left(-\Delta_\Omega^N + \frac{1}{K}KV\right) \lesssim \delta_0(KV)^{-d} \lesssim \delta_0(V)^{-d}\,, \tag{30}$$

where we used that $K$ only depends on $d$, $\gamma$, $L$ in the second last step. □

**Structure of the paper.** Sections 2 to 5 are devoted to proving the Cwickel-Lieb-Rozenblum type bound (Theorem 1.2). In Section 2, we introduce some preliminaries, including the definition and some basic properties of oscillatory domains. We prove several estimates for oscillatory domains in Section 3. In Section 4, we prove a covering lemma for oscillatory domains. In Section 5, we combine the results from Sections 2 to 4 and prove Theorem 1.2. In Section 6, we prove Weyl's law for Schrödinger operators on Hölder domains (Theorem 1.3). In Section 7, we construct an example with non-semiclassical behaviour and therby prove Theorem 1.1.

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**Notation.** For two numbers $A, B \geq 0$, we write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$ which may depend on $d$, $\gamma$, $c$, $h_\Omega$, $L$ (see Definition 2.1). Similarly we write $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

## 2. Preliminaries

In this section we collect some technical definitions and preliminary results.

2.1. **Hölder domains.** We often write elements in $\mathbb{R}^d$ as $x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. We denote the infinity norm on $\mathbb{R}^{d-1}$ by $|\cdot|_\infty$. We have the following technical definition which agrees with the definition of Hölder domains given around (3).

**Definition 2.1** ($\gamma$-Hölder domain with parameters $c, h_\Omega, L$)**.** *Let $d \in \mathbb{N}$ with $d \geq 2$, $\gamma \in (0,1]$, $c > 0$, $h_\Omega > 0$, $L \in \mathbb{N}$ and let $\emptyset \neq \Omega \subset \mathbb{R}^d$ and be a bounded open set. We call $\Omega$ a* $\gamma$-Hölder domain with parameters $c$, $h_\Omega, L$ *if there exists a collection $\{O_l\}_{l=1}^L \subset \mathbb{R}^d$ of open sets covering $\partial\Omega$ with the following properties:*

*(i) For every $l \in \{1, \dots, L\}$ there exists an orthogonal map $R_l$ and a translation map $T_l$ such that for $\Omega_l := O_l \cap \Omega$ we have*

$$\Omega_l = T_l R_l \left\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x \in \Omega_l^{(d-1)},\, 0 < x_d < f_l(x') \right\} \tag{31}$$

*and*

$$O_l \setminus \Omega_l \subset T_l R_l \left\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x \in \Omega_l^{(d-1)},\, f_l(x') \leq x_d \right\} \tag{32}$$

*for some open sets $\Omega_l^{(d-1)} \subset \mathbb{R}^{d-1}$ and for a function $f_l : \Omega_l^{(d-1)} \to (3h_\Omega, \infty)$ with*

$$|f_l(x') - f_l(y')| \leq c\, |x' - y'|_\infty^\gamma \text{ for all } x', y' \in \Omega_l^{(d-1)} . \tag{33}$$

*(ii) For every $l \in \{1, \dots, L\}$ define*

$$\hat{\Omega}_l^{(d-1)} := \left\{ x' \in \Omega_l^{(d-1)} \;\middle|\; \operatorname{dist}\left(x', \partial\Omega_l^{(d-1)}\right) > 2h_\Omega \right\}, \tag{34}$$

*where $\operatorname{dist}\left(x', \partial\Omega_l^{(d-1)}\right)$ denotes the distance from $x'$ to $\partial\Omega_l^{(d-1)}$ with respect to the Euclidean norm in $\mathbb{R}^{d-1}$, and*

$$\hat{\Omega}_l := T_l R_l \left\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x' \in \hat{\Omega}_l^{(d-1)},\, 2h_\Omega < x_d < f_l(x') \right\} . \tag{35}$$

*Then*

$$\partial\Omega \subset \bigcup_{l=1}^{L} \overline{\hat{\Omega}_l} . \tag{36}$$

**Definition 2.2** ($\tilde{\Omega}_l^{(d-1)}$ and $\tilde{\Omega}_l$)**.** *For every $l \in \{1, \dots, L\}$ define*

$$\tilde{\Omega}_l^{(d-1)} := \left\{ x' \in \Omega_l^{(d-1)} \,\middle|\, \mathrm{dist}\left(x', \partial\Omega_l^{(d-1)}\right) > h_\Omega \right\}, \tag{37}$$

$$\tilde{\Omega}_l := T_l R_l \left\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x' \in \tilde{\Omega}_l^{(d-1)},\, h_\Omega < x_d < f_l(x') \right\}, \tag{38}$$

$$\Omega_\varepsilon^b := \left\{ x \in \Omega \mid \mathrm{dist}\,(x, \partial\Omega) < \varepsilon \right\}, \tag{39}$$

*where* $\mathrm{dist}\,(x, \partial\Omega)$ *denotes the distance with respect to the Euclidean norm in* $\mathbb{R}^d$.

**Lemma 2.3.** *Let $\tilde{\Omega}_l$ and $\Omega_\varepsilon^b$ be defined as in Definition 2.2. Then*

$$\bigcup_{l=1}^{L} \tilde{\Omega}_l \supset \Omega_{h_\Omega}^b. \tag{40}$$

*Proof.* Let $x \in \Omega$ with $\mathrm{dist}\,(x, \partial\Omega) < h_\Omega$. Then there exists $y \in \partial\Omega$ with $|x - y| = \mathrm{dist}\,(x, \partial\Omega)$. By in Definition 2.1(ii), we know (36), so there exists $l \in \{1, \dots, L\}$ with $y \in \partial\hat{\Omega}_l$. Without loss of generality, we may assume that $T_l$, $R_l$ are the identity map. Hence, we have $y = (y', f_l(y'))$ with $y' \in \hat{\Omega}_l^{(d-1)}$. It follows that $\mathrm{dist}\left(y', \partial\Omega_l^{(d-1)}\right) > 2h_\Omega$ by (34). By the triangle inequality, we obtain $\mathrm{dist}\left(x', \partial\Omega_l^{(d-1)}\right) > h_\Omega$, so $x' \in \Omega_l^{(d-1)}$. Furthermore, by the definition of $f_l$, we have $f_l(y') > 3h_\Omega$. Again, by the triangle inequality, this implies $x_d > 2h_\Omega > h_\Omega$. It follows that $x \in \tilde{\Omega}_l$. □

**Definition 2.4** ($h_{x,l}$, $|\cdot|_{p,\beta}$ and $\|\cdot\|_{p,\beta}$)**.** *Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a $\gamma$-Hölder domain with constant $c > 0$ and parameters $h_\Omega > 0$ and $L \in \mathbb{N}$ for some $\gamma \in (0, 1]$. In the following, we use the notation from Definition 2.1.*

*(i) For any $l \in \{1, \dots, L\}$ and $x = T_l R_l(x', x_d) \in \Omega_l$, define*

$$h_x := h_{x,l} := f_l(x') - x_d > 0\,. \tag{41}$$

*Moreover, for any $x \in \bigcup_{l=1}^{L} \Omega_l$, we let*

$$h_{x,\min} := \min_{l \in \{1,\dots,L\}\, with\, x \in \Omega_l} h_{x,l}\,. \tag{42}$$

*(ii) For $\beta > 0$ and $p \in [1, \infty)$, define the seminorm $|\cdot|_{p,\beta}$ by*

$$|f|_{p,\beta}^p := \int_{\bigcup_{l=1}^{L} \Omega_l} \mathrm{d}x\, h_{x,\min}^{-\beta} |f(x)|^p \tag{43}$$

*for all measurable functions $f : \Omega \to \mathbb{C}$.*

*(iii) Define the norm $\|\cdot\|_{p,\beta}$ by*

$$\|f\|_{p,\beta} := \|f\|_{\frac{d}{2},\Omega} + |f|_{p,\beta} \quad \text{if } d \geq 3\,, \quad \|f\|_{p,\beta} := \|f\|_{\mathcal{B},\Omega} + |f|_{p,\beta} \quad \text{if } d = 2 \tag{44}$$

*for all measurable functions $f : \Omega \to \mathbb{C}$. See [14, p. 1] for the definition of $\|f\|_{\mathcal{B},\Omega}$.*

2.2. **Oscillatory domains.** In this subsection we define oscillatory domains and prove several properties of oscillatory domains.

**Definition 2.5** ($c_0$, $c_1$ and $c_2$)**.** *Let $c > 0$ be as in Definition 2.4. We define*

$$c_0 := \left[ \min\left( \frac{1}{c_1}, \frac{2^\gamma}{64c}, \frac{1}{2^{\gamma+3}c} \right) \right]^{\frac{1}{\gamma}}, \quad c_1 := 16, \quad c_2 := c_0 c_1^{\frac{1}{\gamma}}. \tag{45}$$

**Definition 2.6** (Oscillatory domain $D$)**.** *For every $l \in \{1,\dots,L\}$, $x \in \tilde{\Omega}_l$ and $\delta \in (0,\delta_0]$ define*

$$a := a_x := a_x(\delta) := a_{x,l}(\delta) := \min\left(\delta, c_0 \max\left(h_{x,l}, c_1\delta\right)^{\frac{1}{\gamma}}\right), \tag{46}$$

*where $c_0$, $c_1$ were defined in Definition 2.5. Define $D := D_x := D_x(\delta) := D_{x,l}(\delta)$ by*

$$D := T_l R_l \left\{ (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; |y' - x'| < \frac{1}{2}a,\ |y_d - x_d| < \frac{1}{2}\delta,\ f_l(y') > y_d \right\}. \tag{47}$$

**Lemma 2.7.** *Let $l \in \{1,\dots,L\}$, $x \in \tilde{\Omega}_l$ and $\delta \in (0,\delta_0]$. Then*

*(i) $D = D_x$ is well-defined and $D \subset \Omega_l$.*

*(ii) For all $y = T_l R_l(y', y_d) \in D$, we have*

$$f_l(y') \geq x_d - \frac{\delta}{4}. \tag{48}$$

*Proof.* **Proof of (i).** Let $y \in D$. Then $y = T_l R_l(y', y_d)$ for some $(y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with $|y' - x'| < \frac{1}{2}a \leq \frac{1}{2}h_\Omega$, $|y_d - x_d| < \frac{1}{2}\delta \leq \frac{1}{2}h_\Omega$ and $f_l(y') > y_d$. Here we used that $a \leq \delta \leq \delta_0 \leq \frac{h_\Omega}{\sqrt{d}} \leq h_\Omega$. Since $x' \in \tilde{\Omega}_l^{(d-1)}$ and $|y' - x'| < \frac{1}{2}h_\Omega$, we get $y' \in \Omega_l^{(d-1)}$. Thus, $f_l(y')$ is well-defined and therefore, $D$ is well-defined. Since $x \in \tilde{\Omega}_l$, we have $x_d > h_\Omega$, so using $|y_d - x_d| < \frac{1}{2}h_\Omega$, we get $y_d > 0$. To sum up, we have shown that $y' \in \Omega_l^{(d-1)}$ and $f_l(y') > y_d > 0$, so $y \in \Omega_l$ by (31).

**Proof of (ii).** Let $y = T_l R_l(y', y_d) \in D$. Then

$$|f_l(x') - f_l(y')| \leq c\,|x' - y'|_\infty^\gamma < c\left(\frac{1}{2}a\right)^\gamma = \frac{c}{2^\gamma}a^\gamma, \tag{49}$$

so

$$f_l(y') \geq f_l(x') - |f_l(x') - f_l(y')| \geq f_l(x') - \frac{c}{2^\gamma}a^\gamma = h + x_d - \frac{c}{2^\gamma}a^\gamma, \tag{50}$$

where we used (41). Thus, in order to show (48), it suffices to show that

$$\frac{\delta}{4} + h - \frac{c}{2^\gamma}a^\gamma \geq 0, \tag{51}$$

If $a = c_0 h^{\frac{1}{\gamma}}$ or $a = c_0(c_1\delta)^{\frac{1}{\gamma}}$, then (51) holds by

$$\max\left(2^{\gamma+1}cc_0^\gamma, \frac{cc_0^\gamma c_1}{2^\gamma}\right) \leq \frac{1}{4}. \tag{52}$$

If $a = \delta$, then $\delta \leq c_0 h^{\frac{1}{\gamma}}$ by the definition of $a$, and it reduces to the case $a = c_0 h^{\frac{1}{\gamma}}$. □

**Lemma 2.8.** *Let $l \in \{1,\dots,L\}$, $x \in \tilde{\Omega}_l$ and $\delta_x \in (0,\delta_0]$. Then the following holds true:*

*(i) If $a_x = \delta_x$ or $a_x = c_0 h_x^{\frac{1}{\gamma}}$, then $D_x$ is a cuboid, namely*

$$D_x := T_l R_l \left\{ (w', w_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |w' - x'| < \frac{1}{2}a_x,\ |w_d - x_d| < \frac{1}{2}\delta_x \right\}. \tag{53}$$

*(ii) In addition, let $y, z \in \tilde{\Omega}_l$ with $\delta_y, \delta_z \in (0,\delta_0]$ and assume that*

$$a_w = c_0 h_w^{\frac{1}{\gamma}} \text{ for all } w \in \{x, y, z\}. \tag{54}$$

*Furthermore, assume that $\delta_x \leq 2\delta_y$, $\delta_z \leq 2\delta_y$, $D_x \cap D_y \neq \emptyset$ and $D_x \cap D_z \neq \emptyset$. Then*

$$\frac{1}{2}h_y \leq h_w \leq 2h_y \text{ for all } w \in \{x, z\}. \tag{55}$$

*(iii) If* $a_x = c_0 h_x^{\frac{1}{\gamma}}$*, then*

$$\frac{1}{2}h_x \leq h_w \leq 2h_x \text{ for all } w \in D_x\,. \tag{56}$$

*(iv) If* $a_x = c_2 \delta_x^{\frac{1}{\gamma}}$*, then*

$$|h_w - h_x| \leq \delta_x \text{ for all } w \in D_x\,. \tag{57}$$

*(v) If* $a_x = c_0 \max(h_x, c_1\delta)^{\frac{1}{\gamma}}$ *and* $f : \Omega \to \mathbb{C}$ *is measurable, then*

$$|f|_{\tilde{p},\beta,D_x}^{\tilde{p}} \gtrsim \max(h_x, c_1\delta_x)^{-\beta} \|f\|_{\tilde{p},D_x}^{\tilde{p}}\,. \tag{58}$$

*Proof.* For simplicity of notation, we write $a := a_x$, $\delta := \delta_x$, $h := h_x$, $D := D_x$ in the proof.

**Proof of (i).** Since $c_0 c_1^{\frac{1}{\gamma}} \leq 1$ and $\delta_0 \leq 1$, we know by the definition of $a$ and by $a = \delta$ or $a = c_0 h^{\frac{1}{\gamma}}$ that $h \geq c_1\delta$. For all $w = T_l R_l(w', w_d)$ with $|w' - x'| < \frac{1}{2}a$ and $|w_d - x_d| < \frac{1}{2}\delta$, we have

$$\begin{aligned} f_l(w') &\geq h + x_d - \frac{c}{2^\gamma}a^\gamma \geq h + x_d - \frac{c}{2^\gamma}\left(c_0 h^{\frac{1}{\gamma}}\right)^\gamma \\ &= x_d + h\left(1 - \frac{c c_0^\gamma}{2^\gamma}\right) \geq x_d + \frac{1}{2}h \geq x_d + \frac{c_1}{2}\delta = x_d + 8\delta \geq w_d + 7\delta\,, \end{aligned} \tag{59}$$

where we used (50) in the first step, $a = \delta$ or $a = c_0 h^{\frac{1}{\gamma}}$ in the second step, (52) in the fourth step, and $c_1 = 16$ in the second last step. By (47), we get $w \in D$.

**Proof of (ii).** First note that by $a_w = c_0 h_w^{\frac{1}{\gamma}}$, we have $h_w \geq c_1\delta_w$ for all $w \in \{x, y, z\}$. Also note that in order to show (55), it suffices to show that

$$|h_w - h_y| \leq \frac{1}{2}\max(h_w, h_y)\,. \tag{60}$$

Let us begin by showing (60) for $w = x$. Since $D_x \cap D_y \neq \emptyset$, we have

$$|x_d - y_d| < \frac{1}{2}\delta_x + \frac{1}{2}\delta_y \leq \frac{1}{2}2\delta_y + \frac{1}{2}\delta_y \leq 2\delta_y \tag{61}$$

and

$$|x' - y'| < \frac{1}{2}a_x + \frac{1}{2}a_y = \frac{c_0}{2}\left(h_x^{\frac{1}{\gamma}} + h_y^{\frac{1}{\gamma}}\right) \leq c_0\left[\max(h_x, h_y)\right]^{\frac{1}{\gamma}}\,. \tag{62}$$

We obtain

$$\begin{aligned} |h_x - h_y| &= |f_l(x') - x_d - \big(f_l(y') - y_d\big)| \leq |f_l(x') - f_l(y')| + |x_d - y_d| \leq c|x' - y'|^\gamma + 2\delta_y \\ &\leq c c_0^\gamma \max(h_x, h_y) + \frac{2}{c_1}h_y \leq \left(c c_0^\gamma + \frac{2}{c_1}\right)\max(h_x, h_y) \leq \frac{1}{2}\max(h_x, h_y)\,. \end{aligned}$$

Here we used $h_y \geq c_1\delta_y$ in the second last step and we used (52) and $c_1 = 16$ in the last step. This shows (60) for $w = x$.

Next, let us show (60) for $w = z$. Since $D_x \cap D_y \neq \emptyset$ and $D_x \cap D_z \neq \emptyset$, we have

$$|z_d - y_d| \leq |z_d - x_d| + |x_d - y_d| < \frac{1}{2}\delta_z + \delta_x + \frac{1}{2}\delta_y \leq \frac{1}{2}2\delta_y + 2\delta_y + \frac{1}{2}\delta_y \leq 4\delta_y \tag{63}$$

and

$$\begin{aligned} |z' - y'| &\leq |z' - x'| + |x' - y'| < \frac{1}{2}a_z + a_x + \frac{1}{2}a_y \leq 2\max(a_x, a_y, a_z) \\ &= 2c_0 \max\left(h_x^{\frac{1}{\gamma}}, h_y^{\frac{1}{\gamma}}, h_z^{\frac{1}{\gamma}}\right) \leq 2c_0\max\left((2h_y)^{\frac{1}{\gamma}}, h_y^{\frac{1}{\gamma}}, h_z^{\frac{1}{\gamma}}\right) \leq 2^{1+\frac{1}{\gamma}} c_0 \max\left(h_y^{\frac{1}{\gamma}}, h_z^{\frac{1}{\gamma}}\right)\,, \end{aligned}$$

where we used $h_x \le 2h_y$ in the second last step. We obtain

$$|h_z - h_y| = |f_l(z') - z_d - \big(f_l(y') - y_d\big)| \le |f_l(z') - f_l(y')| + |z_d - y_d|$$
$$\le c|z' - y'|^\gamma + 4\delta_y \le c\left(2^{1+\frac{1}{\gamma}} c_0 \left[\max(h_x, h_y)\right]^{\frac{1}{\gamma}}\right)^\gamma + 4\delta_y$$
$$\le 2^{\gamma+1} c c_0^\gamma \max(h_x, h_y) + \frac{4}{c_1} h_y \le \left(2^{\gamma+1} c c_0^\gamma + \frac{4}{c_1}\right) \max(h_x, h_y) \le \frac{1}{2}\max(h_x, h_y)\ .$$

where we used (52) and $c_1 = 16$ in the last step. This shows (60) for $w = z$.

**Proof of (iii).** Let $w \in D_x$. By $a = c_0 h^{\frac{1}{\gamma}}$, we have $h \ge c_1\delta$. As in the proof of (ii), we get

$$|h_w - h| = |f_l(w') - w_d - \big(f_l(x') - x_d\big)| \le |f_l(w') - f_l(x')| + |w_d - x_d|$$
$$\le c|w' - x'|^\gamma + \frac{1}{2}\delta \le c\left(\frac{1}{2}a\right)^\gamma + \frac{1}{2c_1}h = c\left(\frac{1}{2}c_0 h^{\frac{1}{\gamma}}\right)^\gamma + \frac{1}{2c_1}h = \frac{c c_0^\gamma}{2^\gamma}h + \frac{1}{2c_1}h \le \frac{1}{2}h\,,$$

where we used (52) and $c_1 = 16$ in the last step. This shows (55).

**Proof of (iv).** Let $w \in D_x$. By $a = c_2\delta^{\frac{1}{\gamma}}$, we have $h \le c_1\delta$. As in the proof of (iii), we get

$$|h_w - h| = |f_l(w') - w_d - \big(f_l(x') - x_d\big)| \le |f_l(w') - f_l(x')| + |w_d - x_d|$$
$$\le c|w' - x'|^\gamma + \frac{1}{2}\delta \le c\left(\frac{1}{2}a\right)^\gamma + \frac{1}{2}\delta = \frac{c c_0^\gamma c_1}{2^\gamma}\delta + \frac{1}{2}\delta \le \delta\,,$$

where we used (52) and $c_1 = 16$ in the last step.

**Proof of (v).** Let $w \in D_x$. Then

$$h_w = |f_l(w') - w_d| \le h + |f_l(w') - w_d - \big(f_l(x') - x_d\big)| \le h + |f_l(w') - f_l(x')| + |w_d - x_d|$$
$$\le h + c|w' - x'|^\gamma + \frac{1}{2}\delta \le h + c\left(\frac{1}{2}c_0 \max(h_x, c_1\delta)^{\frac{1}{\gamma}}\right)^\gamma + \frac{1}{2}\delta \lesssim \max(h, c_1\delta)\ ,$$

By the definition of $|f|^{\tilde p}_{\tilde p,\beta,D}$, we have

$$|f|^{\tilde p}_{\tilde p,\beta,D} = \int_D \mathrm{d}y\, h_{y,\min}^{-\beta} |f(y)|^{\tilde p} \gtrsim \max(h, c_1\delta)^{-\beta} \|f\|^{\tilde p}_{\tilde p,D}\ . \tag{64}$$

□

## 3. Estimates for oscillatory domains

3.1. **Poincaré-Sobolev inequality for oscillatory domains.** In this subsection we prove a Poincaré-Sobolev inequality for oscillatory domains. The following Lemma and its proof is a version of [17] for oscillatory domains (Corollary 3.3).

**Lemma 3.1** (Sobolev inequality for oscillatory domains). *Let $c > 0$, $\gamma \in (0,1)$ and $\delta \in (0,1)$. Let $\tilde a > 0$ and let $f : [-\tilde a/2, \tilde a/2]^{d-1} \to (0,\delta)$ be such that*

$$|f(x') - f(y')| \le c\,|x' - y'|_\infty^\gamma \ \text{ for all } x', y' \in \left[-\tfrac{\tilde a}{2}, \tfrac{\tilde a}{2}\right]^{d-1}\ . \tag{65}$$

*Define $q^* \in (2,\infty)$ by*

$$\frac{1}{q^*} := \frac{1}{2} - \frac{1}{\frac{d-1}{\gamma} + 1} \tag{66}$$

*and define*

$$\tilde{D} := \left\{ (x', x_d) \in \left[-\tfrac{\tilde{a}}{2}, \tfrac{\tilde{a}}{2}\right]^{d-1} \times (0,\delta) \,\middle|\, f(x') > x_d \right\} . \tag{67}$$

*Define the "straight part I'm nots of the boundary of $\tilde{D}$" by*

$$\tilde{B} := \partial\tilde{D} \setminus \left\{ (x', x_d) \in \left[-\tfrac{\tilde{a}}{2}, \tfrac{\tilde{a}}{2}\right]^{d-1} \times (0,\delta) \,\middle|\, f(x') = x_d \right\} . \tag{68}$$

*Then there exists a constant $C_S = C_S(d, c, \gamma) > 0$ such that for all $u \in H^1(\tilde{D})$ with $u \restriction_{\tilde{B}} \equiv 0$ in the trace sense, we have*

$$\|u\|_{q^*,\tilde{D}}^2 \leq C_S \|\nabla u\|_{2,\tilde{D}}^2 . \tag{69}$$

*Proof.* We first consider the case of smooth functions $u$ and later deduce (69) for all $u \in H^1(\tilde{D})$ with $u \restriction_{\tilde{B}} \equiv 0$ in the trace sense. Let $u \in C^\infty(\tilde{D}) \cap H^1(\tilde{D})$ with $u \restriction_{\tilde{B}} \equiv 0$. Let

$$\psi : (0,\infty) \to \mathbb{R} , \quad \psi(s) = c^{-\frac{1}{\gamma}} s^{\frac{1}{\gamma}} . \tag{70}$$

Fix $\tilde{x} \in \tilde{D}$. For simplicity of notation, we shift the coordinate system and reflect the last coordinate by replacing every $x \in \mathbb{R}^d$ by $(x_1 - \tilde{x}_1, \dots, x_{d-1} - \tilde{x}_{d-1}, -x_d + \tilde{x}_d) \in \mathbb{R}^d$, so without loss of generality, we may assume $\tilde{x} = 0$. Define

$$K := \left\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,\middle|\, 0 < x_d ,\ 0 \leq |x'| < \psi(x_d) \right\} . \tag{71}$$

and note that $u$ it is well-defined on $K$ since we can extend it by zero on $K \setminus \tilde{D}$ due to $u \restriction_{\tilde{B}} \equiv 0$ in the trace sense. Let $y' \in \mathbb{R}^{d-1}$ with $|y'| < 1$ and note that $(y'\psi(x_d), x_d) \in K$ for all $x_d \in (0,\delta)$. Since $u$ is smooth, we can apply Newton's theorem and $u((y'\psi(\delta), \delta)) = 0$ to obtain

$$\begin{aligned} -u(0) &= \int_0^\delta \mathrm{d}x_d \, \frac{\partial}{\partial x_d} u(y'\psi(x_d), x_d) \\ &= \int_0^\delta \mathrm{d}x_d \left( \sum_{j=1}^{d-1} (\partial_j u)(y'\psi(x_d), x_d) y'_j \psi'(x_d) + (\partial_d u)(y'\psi(x_d), x_d) \right) . \end{aligned}$$

Here $\psi'$ denotes the derivative of $\psi$ while $y' \in \mathbb{R}^{d-1}$. Integrating over $y' \in B_1^{(d-1)}(0)$ and using the notation $\omega_{d-1} = |B_1^{(d-1)}(0)|$, we get

$$\begin{aligned} &-\omega_{d-1} u(0) \\ &= \int_{B_1^{(d-1)}(0)} \mathrm{d}y' \int_0^\delta \mathrm{d}x_d \left( \sum_{j=1}^{d-1} (\partial_j u)(y'\psi(x_d), x_d) \frac{y'_j \psi(x_d)}{\psi(x_d)} \psi'(x_d) + (\partial_d u)(y'\psi(x_d), x_d) \right) \\ &= \int_K \mathrm{d}x \, \frac{1}{\psi(x_d)^{d-1}} \left( \sum_{j=1}^{d-1} (\partial_j u)(x) \frac{x'_j}{\psi(x_d)} \psi'(x_d) + (\partial_d u)(x) \right) , \end{aligned}$$

where we used the change of variables $x = (y'\psi(x_d), x_d)$. Note that

$$\psi'(s) = \frac{1}{c^{\frac{1}{\gamma}}} \frac{1}{\gamma} s^{\frac{1}{\gamma}-1} . \tag{72}$$

Thus, if $x = (x', x_d) \in K$, then using $x_d < \delta \in (0,1)$, we get

$$\left| \frac{x'_j}{\psi(x_d)} \psi'(x_d) \right| \leq |\psi'(x_d)| = \frac{1}{c^{\frac{1}{\gamma}}} \frac{1}{\gamma} x_d^{\frac{1}{\gamma}-1} \leq \frac{1}{c^{\frac{1}{\gamma}}} \frac{1}{\gamma} \delta^{\frac{1}{\gamma}-1} \lesssim 1 . \tag{73}$$

Therefore, by (73) and the Cauchy-Schwarz inequality on $\mathbb{R}^d$,

$$\begin{aligned}\omega_{d-1}|u(0)| &\leq \int_K \mathrm{d}x\, \frac{1}{\psi(x_d)^{d-1}} \sum_{j=1}^{d} |(\partial_j u)(x)| \lesssim \int_K \mathrm{d}x\, \frac{1}{\psi(x_d)^{d-1}} \sqrt{\sum_{j=1}^{d} |(\partial_j u)(x)|^2}\sqrt{d} \\ &\lesssim \int_K \mathrm{d}x\, \frac{1}{\psi(x_d)^{d-1}} |\nabla u(x)|\,.\end{aligned}$$

We get

$$|u(0)| \lesssim \int_K \mathrm{d}x\, \frac{1}{\psi(x_d)^{d-1}} |\nabla u(x)|\,, \tag{74}$$

where the constant in the inequality only depends on $d$, $\gamma$, $c$. Let us now undo the change of variables, which was convenient for the above computation. In the old coordinate system (74) reads

$$|u(\tilde{x})| \lesssim \int_{\tilde{D}} \mathrm{d}y\, 1_K(\tilde{x}-y) \frac{1}{\psi(\tilde{x}_d - y_d)^{d-1}} |\nabla u(y)| = \left(\left(1_K \frac{1}{\psi(\cdot_d)^{d-1}}\right) * |\nabla u|\right)(\tilde{x}) \tag{75}$$

for all $\tilde{x} \in \tilde{D}$. Here we used the rotational symmetry of $K$ with respect to the first $d-1$ variables. We define $r \in (1,\infty)$ by

$$1 + \frac{1}{q^*} = 1 + \frac{1}{2} - \frac{1}{\frac{d-1}{\gamma} + 1} =: \frac{1}{2} + \frac{1}{r}\,. \tag{76}$$

By the weak Young inequality and (74), we get

$$\|u\|_{q^*} \lesssim \|\nabla u\|_2 \left\| 1_K \frac{1}{\psi(\cdot_d)^{d-1}} \right\|_{r,w}\,. \tag{77}$$

Here $\|\cdot\|_{r,w}$ denotes the weak $L^r$-norm. From the definitions of $\psi$ and $K$ in (70) and (71), we have

$$\left\| 1_K \frac{1}{\psi(\cdot_d)^{d-1}} \right\|_{r,w} \lesssim 1, \tag{78}$$

which completes the proof for smooth $u$. The claim for general $u$ follows a standard density argument. □

**Lemma 3.2** (Poincaré inequality for oscillatory domains with Neumann boundary conditions)**.** *Let $\delta > 0$, let $\hat{a} \in \left(\frac{\delta}{3}, \delta\right)$ and let $f : [-\hat{a}/2, \hat{a}/2]^{d-1} \to [\delta/4, \delta)$ be continuous. Define*

$$\hat{D} := \left\{ (x', x_d) \in \left[-\tfrac{\hat{a}}{2}, \tfrac{\hat{a}}{2}\right]^{d-1} \times \left(-\tfrac{\delta}{4}, \delta\right) \,\middle|\, f(x') > x_d \right\}\,. \tag{79}$$

*Then there exists a constant $C_P = C_P(d) > 0$ such that for all $u \in H^1(\hat{D})$ with $\int_{\hat{D}} u = 0$, we have*

$$C_P \frac{1}{\delta^2} \|u\|_{2,\hat{D}}^2 \leq \|\nabla u\|_{2,\hat{D}}^2\,. \tag{80}$$

*Proof.* The proof can be found in [10, Lemma 2.6(2)] for slightly different side lengths of the domain. □

**Corollary 3.3** (Poincaré-Sobolev inequality for oscillatory domains)**.** *Let $M \geq 1$, $\delta \in (0,1)$ and let $f : [-\delta/(2M), \delta/(2M)]^{d-1} \to [\delta/4, \delta)$ be such that*

$$|f(x') - f(y')| \leq c\,|x'-y'|_\infty^\gamma \text{ for all } x', y' \in \left[-\tfrac{\delta}{2M}, \tfrac{\delta}{2M}\right]^{d-1} \tag{81}$$

*for some $c > 0$, $\gamma \in (0,1)$. Define $q^* \in (2,\infty)$ by*

$$\frac{1}{q^*} = \frac{1}{2} - \frac{1}{\frac{d-1}{\gamma} + 1}\,. \tag{82}$$

*and define*

$$D := \left\{(x', x_d) \in \left(-\tfrac{\delta}{2M}, \tfrac{\delta}{2M}\right)^{d-1} \times (0,\delta) \,\middle|\, f(x') > x_d\right\} . \tag{83}$$

*Then there exists a constant* $C_{PS} = C_{PS}(d,c,\gamma) > 0$ *such that for all* $u \in H^1(D)$ *with* $\int_D u = 0$*, we have*

$$\|u\|_{q^*,D}^2 \leq C_{PS} M^{(d-1)\left(1-\frac{2}{q^*}\right)} \|\nabla u\|_{2,D}^2 . \tag{84}$$

*Proof.* We combine Lemma 3.1 and Lemma 3.2. Define $\tilde{M}$ as the largest odd number such that $\tilde{M} \leq M$ and note that $\tilde{M} \leq M \leq 3\tilde{M}$, so

$$\hat{a} := \tfrac{\tilde{M}\delta}{M} \in \left(\tfrac{\delta}{3}, \delta\right) . \tag{85}$$

We define

$$\tilde{f} : \left[-\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M}\right]^{d-1} \to \left[\tfrac{\delta}{4}, \delta\right) \tag{86}$$

by reflecting $f$: We can write every $x' \in \left[-\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M}\right]^{d-1}$ as

$$x' = \tfrac{\delta}{M} z' + w' \tag{87}$$

with $z' = (z_1, \ldots, z_{d-1}) \in \mathbb{Z}^{d-1}$ and $w' = (w_1, \ldots, w_{d-1}) \in \left[-\tfrac{\delta}{2M}, \tfrac{\delta}{2M}\right]^{d-1}$. Define $\tilde{f}$ by

$$\tilde{f}(x') := f\left(((-1)^{z_1} w_1, \ldots, (-1)^{z_{d-1}} w_{d-1})\right) . \tag{88}$$

Note that $\tilde{f}$ is well defined, continuous and

$$|\tilde{f}(x') - \tilde{f}(y')| \leq c \left|x' - y'\right|_\infty^\gamma \text{ for all } x', y' \in \left[-\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M}\right]^{d-1} . \tag{89}$$

Define

$$\tilde{D} := \left\{(x', x_d) \in \left(-\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M}\right)^{d-1} \times \left(-\tfrac{\delta}{4}, \delta\right) \,\middle|\, \tilde{f}(x') > x_d\right\} \tag{90}$$

and

$$\hat{D} := \left\{(x', x_d) \in \left(-\tfrac{\tilde{M}\delta}{2M}, \tfrac{\tilde{M}\delta}{2M}\right)^{d-1} \times (0,\delta) \,\middle|\, \tilde{f}(x') > x_d\right\} . \tag{91}$$

Note that $\hat{D}$ consists of $\tilde{M}^{d-1}$ reflected copies of $D$. Moreover, $\tilde{D}$ consists of less than $2(3\tilde{M})^{d-1}$ reflected copies of $D$ in the sense that $\tilde{D} \cap \{x_d \geq 0\}$ consists of $(3\tilde{M})^{d-1}$ reflected copies of $D$ but $\tilde{D} \cap \{x_d < 0\}$ is only contained in $(3\tilde{M})^{d-1}$ reflected copies of $D$. Let

$$\varphi \in C_c^\infty\left(\left(-\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M}\right)^{d-1} \times \left(-\tfrac{\delta}{4}, \tfrac{5\delta}{4}\right)\right)$$
$$\text{with } 0 \leq \varphi \leq 1, \; \varphi \restriction_{\left[-\frac{\tilde{M}\delta}{2M}, \frac{\tilde{M}\delta}{2M}\right]^{d-1} \times [0,\delta]} \equiv 1 \text{ and } \|\nabla\varphi\|_\infty \lesssim \frac{1}{\delta} .$$

It is possible to choose such a $\varphi$ by scaling. Note that $\varphi \restriction_{\hat{D}} \equiv 1$ by the definition of $\hat{D}$. Let $u \in H^1(D)$ with $\int_D u = 0$ and define the corresponding reflected version $\tilde{u} \in H^1(\tilde{D})$ by

$$\tilde{u}(x) := u\left(((-1)^{z_1} w_1, \ldots, (-1)^{z_{d-1}} w_{d-1}, |w_d|)\right) . \tag{92}$$

for every $x = \frac{\delta}{M} z + w \in \tilde{D}$ with $z = (z_1, \ldots, z_{d-1}, 0) \in \mathbb{Z}^d$ and $w = (w_1, \ldots, w_d) \in \left[-\frac{\delta}{2M}, \frac{\delta}{2M}\right]^{d-1} \times \left(-\frac{\delta}{4}, \infty\right)$. Note that $\tilde{u}$ is well-defined because $H^1$ functions are defined up to almost everywhere equality and since reflections of $H^1$ functions are again $H^1$ functions. Moreover, by $\tilde{u} \restriction_D \equiv u$, we know that $\tilde{u} \restriction_{\tilde{D}}$ consists of less than $2(3\tilde{M})^{d-1}$ reflected copies of

$u$, and $\tilde{u} \restriction_{\hat{D}}$ consists of $\tilde{M}^{d-1}$ reflected copies of $u$. Also note that $\int_{\hat{D}} \tilde{u} = 0$. Furthermore, $\varphi\tilde{u} \in H^1(\tilde{D})$, $\varphi\tilde{u} \restriction_{\hat{D}} = \tilde{u}$ and $\varphi\tilde{u} \restriction_{\tilde{B}} \equiv 0$, where

$$\tilde{B} := \partial\tilde{D} \setminus \left\{ (x', x_d) \in \left[ -\tfrac{3\tilde{M}\delta}{2M}, \tfrac{3\tilde{M}\delta}{2M} \right]^{d-1} \times \left( -\tfrac{\delta}{4}, \delta \right] \,\middle|\, \tilde{f}(x') = x_d \right\} . \tag{93}$$

Since $\int_{\tilde{D}} \tilde{u} = 0$, $\hat{a} \in \left(\frac{\delta}{3}, \delta\right)$ and $\tilde{D}$ satisfies the assumptions of Lemma 3.2, we can apply Lemma 3.2 to get

$$\begin{aligned} \|\nabla(\varphi\tilde{u})\|_{2,\tilde{D}}^2 &\lesssim \int_{\tilde{D}} |\varphi|^2 |\nabla\tilde{u}|^2 + \int_{\tilde{D}} |\tilde{u}|^2 |\nabla\varphi|^2 \lesssim \int_{\tilde{D}} |\nabla\tilde{u}|^2 + \frac{1}{\delta^2} \int_{\tilde{D}} |\tilde{u}|^2 \\ &\leq 2 \cdot 3^{d-1} \left( \int_{\hat{D}} |\nabla\tilde{u}|^2 + \frac{1}{\delta^2} \int_{\hat{D}} |\tilde{u}|^2 \right) \lesssim \int_{\hat{D}} |\nabla\tilde{u}|^2 = \tilde{M}^{d-1} \int_D |\nabla u|^2 . \end{aligned}$$

In the third step we used that $0 \leq \varphi \leq 1$ and $\|\nabla\varphi\|_\infty \lesssim \frac{1}{\delta}$, and in the second last step we used that $\tilde{u} \restriction_{\hat{D}}$ consists of $\tilde{M}^{d-1}$ reflected copies of $u$. On the other hand, $\varphi\tilde{u} \restriction_{\tilde{B}} \equiv 0$ and $\tilde{D}$ satisfies the assumptions of Lemma 3.1, so by Lemma 3.1, we obtain

$$\begin{aligned} \|\nabla(\varphi\tilde{u})\|_{2,\tilde{D}}^2 &\gtrsim \|\varphi\tilde{u}\|_{q^*,\tilde{D}}^2 \geq \|\tilde{u}\|_{q^*,\hat{D}}^2 = \left( \int_{\hat{D}} |\tilde{u}|^{q^*} \right)^{\frac{2}{q^*}} = \left( \tilde{M}^{d-1} \int_D |\tilde{u}|^{q^*} \right)^{\frac{2}{q^*}} \\ &= \tilde{M}^{(d-1)\frac{2}{q^*}} \left( \int_D |\tilde{u}|^{q^*} \right)^{\frac{2}{q^*}} = \tilde{M}^{(d-1)\frac{2}{q^*}} \|u\|_{q^*,D}^2 . \end{aligned}$$

To sum up, we get

$$\tilde{M}^{d-1} \int_D |\nabla u|^2 \gtrsim \|\nabla(\varphi\tilde{u})\|_{2,\tilde{D}}^2 \gtrsim \tilde{M}^{(d-1)\frac{2}{q^*}} \|u\|_{q^*,D}^2 , \tag{94}$$

so

$$\tilde{M}^{(d-1)\left(1 - \frac{2}{q^*}\right)} \int_D |\nabla u|^2 \gtrsim \|u\|_{q^*,D}^2 . \tag{95}$$

Now recall that $M \sim 3\tilde{M}$, so we obtain the desired result. □

3.2. **Choice of the oscillatory domains.** In this subsection, we choose depending on $V$ for every $x \in \Omega$ close to $\partial\Omega$ an oscillatory domain $D_x$ with centre $x$ such that

$$N\left(-\Delta_{D_x}^N + V\right) \leq 1 . \tag{96}$$

For the proof, we use the Poincaré-Sobolev inequality for oscillatory domains (Corollary 3.3). At the same time, we choose the oscillatory domains $D_x$ such that a certain norm of $V$ on $D_x$ is not too small. This will be needed in the following subsection.

**Lemma 3.4.** *Let $l \in \{1, \ldots, L\}$ and let $x \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$. Let $\delta \in (0, \delta_0]$ and let $D := D_x(\delta)$.*

*(i) Suppose $a_x(\delta) = \delta$ and $\|V\|_{\frac{d}{2},D} \lesssim 1$ if $d \geq 3$ and $\|V\|_{\mathcal{B},D} \lesssim 1$ if $d \geq 2$. Then*

$$N\left(-\Delta_D^N + V\right) \leq 1 . \tag{97}$$

*(ii) Suppose $a_x(\delta) = c_0 \max\left(h_x, c_1\delta\right)^{\frac{1}{\gamma}}$ and*

$$\|V\|_{\tilde{p},D}^{\tilde{p}} \lesssim \max\left( \frac{h_x}{c_1\delta}, 1 \right)^{\frac{d-1}{\gamma}} . \tag{98}$$

*Then*

$$N\left(-\Delta_D^N + V\right) \leq 1 . \tag{99}$$

*All the constants in $\lesssim$ in this Lemma only depend on $d$, $\gamma$, $c$.*

*Proof.* **Proof of (i).** Let $0 \not\equiv u \in H^1(D)$ with $\int_D u = 0$. Then using Hölder's inequality and the Poincaré-Sobolev inequality for cubes, see [18, Theorem 8.12], we get for $d \geq 3$

$$\int_D |\nabla u|^2 + \int_D V|u|^2 \geq \int_D |\nabla u|^2 - \|V\|_{\frac{d}{2},D} \|u\|^2_{\frac{2d}{d-2},D} \geq \|\nabla u\|^2_{2,D} \left(1 - C_{PS} \|V\|_{\frac{d}{2},D}\right).$$

Hence, since $\nabla u \not\equiv 0$, we get $\int_D |\nabla u|^2 + \int_D V|u|^2 > 0$ if $\|V\|_{\frac{d}{2},D} < \frac{1}{C_{PS}}$. If $d = 2$, we use the Poincaré-Sobolev inequality for Orlicz norms, see [14, Proposition 2.1].

**Proof of (ii).** Choose $M := \delta/a$ and note $M \geq 1$. By Lemma 2.7(ii), we know that $D$ is an oscillatory domain as in Corollary 3.3. In order to show $N\left(-\Delta_D^N + V\right) \leq 1$, it suffices to show that for all $0 \not\equiv u \in H^1(D)$ with $\int_D u = 0$, we have

$$\int_D |\nabla u|^2 + \int_D V|u|^2 > 0\,. \tag{100}$$

To this end, let $0 \not\equiv u \in H^1(D)$ with $\int_D u = 0$. Define $p^*$ by

$$1 = \frac{1}{p^*} + \frac{1}{\frac{q^*}{2}}, \quad \text{namely} \quad \frac{1}{p^*} := 1 - \frac{2}{q^*} = \frac{2}{\frac{d-1}{\gamma} + 1}. \tag{101}$$

By Hölder's inequality and the Poincaré-Sobolev inequality for oscillatory domains, see Corollary 3.3, we get

$$\int_D |\nabla u|^2 + \int_D V|u|^2 \geq \int_D |\nabla u|^2 - \|V\|_{p^*,D} \|u\|^2_{q^*,D} \geq \|\nabla u\|^2_{2,D} \left(1 - C_{PS} M^{\frac{d-1}{p^*}} \|V\|_{p^*,D}\right).$$

Hence, since $\nabla u \not\equiv 0$, the left-hand side is strictly positive if

$$M^{\frac{d-1}{p^*}} \|V\|_{p^*,D} < \frac{1}{C_{PS}}\,. \tag{102}$$

We define $\tilde{r} \in (1, \infty)$ by

$$\frac{1}{\tilde{r}} := \frac{1}{p^*} - \frac{1}{\tilde{p}} = \frac{2(d-1)}{\left(\frac{d-1}{\gamma} + 1\right)^2} \left(\frac{1}{\gamma} - 1\right), \tag{103}$$

where $p^*$ and $\tilde{p}$ are given in (101) and (22). Using Hölder's inequality with (103), $|D| \sim \delta^d M^{-(d-1)}$ and $M = \delta/a$, it follows that

$$\begin{aligned} M^{\frac{d-1}{p^*}} \|V\|_{p^*,D} &\lesssim M^{\frac{d-1}{p^*}} \|V\|_{\tilde{p},D} |D|^{\frac{1}{\tilde{r}}} \lesssim M^{\frac{d-1}{p^*}} \|V\|_{\tilde{p},D} \left(\delta^d M^{-(d-1)}\right)^{\frac{1}{\tilde{r}}} \\ &\sim \|V\|_{\tilde{p},D}\, \delta^{\frac{d}{\tilde{r}}} \left(\delta^{1-\frac{1}{\gamma}} \max\left(\frac{h}{c_1\delta}, 1\right)^{-\frac{1}{\gamma}}\right)^{\frac{d-1}{\tilde{p}}} = \left(\|V\|^{\tilde{p}}_{\tilde{p},D} \max\left(\frac{h}{c_1\delta}, 1\right)^{-\frac{d-1}{\gamma}}\right)^{\frac{1}{\tilde{p}}} \lesssim 1\,, \end{aligned}$$

where we used that

$$\frac{d}{\tilde{r}} + \left(1 - \frac{1}{\gamma}\right) \frac{d-1}{\tilde{p}} = 0 \tag{104}$$

in the third step and the assumption (98) in the last step. Hence, if the constant in (98) is chosen small enough, we can deduce (102), which is what we wanted to show. ☐

**Lemma 3.5** (Choice of the oscillatory domains)**.** *Let $l \in \{1, \dots, L\}$ and let $x \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$. Then there exists $\delta_x \in (0, \delta_0]$ such that for $D := D_x := D_x(\delta_x)$ we have*

$$N\left(-\Delta_D^N + V\right) \leq 1 \tag{105}$$

*and at least one of the following properties is satisfied:*

*(1) $\delta_x = \delta_0$.*

*(2)* $a_x = \delta_x$ *and*

$$\|V\|^{\frac{d}{2}}_{\frac{d}{2},D} \gtrsim 1 \text{ if } d \geq 3 \quad \text{and} \quad \|V\|_{\mathcal{B},D} \gtrsim 1 \text{ if } d = 2\,. \tag{106}$$

*(3)* $a_x = c_0 \max(h_x, c_1\delta_x)^{\frac{1}{\gamma}}$ *and*

$$\|V\|^{\tilde{p}}_{\tilde{p},D} \gtrsim \max\left(\frac{h_x}{c_1\delta_x}, 1\right)^{\frac{d-1}{\gamma}}\,. \tag{107}$$

*All the constants in $\gtrsim$ in this Lemma only depend on $d$, $\gamma$, $c$.*

*Proof.* Let us first explain how to choose $\delta$ if $c_0 \max(h, c_1\delta_0)^{\frac{1}{\gamma}} > \delta_0$. It follows that $a(\delta) = \delta$ for all $\delta \leq \delta_0$. Pick $\delta \leq \delta_0$ such that

$$\|V\|^{\frac{d}{2}}_{\frac{d}{2},D(\delta)} \sim 1 \text{ if } d \geq 3 \quad \text{and} \quad \|V\|_{\mathcal{B},D(\delta)} \sim 1 \text{ if } d = 2\,. \tag{108}$$

holds, so (2) is satisfied. By (108) and Lemma 3.4(i), we get $N\left(-\Delta_D^N + V\right) \leq 1$.

Let us now assume that $c_0 \max(h, c_1\delta_0)^{\frac{1}{\gamma}} \leq \delta_0$. Define $\delta_c := c_0 h^{\frac{1}{\gamma}} \leq \delta_0$, so $c_0 \max(h, c_1\delta_c)^{\frac{1}{\gamma}} = \delta_c$. We have $a(\delta) = c_0 \max(h, c_1\delta)^{\frac{1}{\gamma}}$ for all $\delta \in [\delta_c, \delta_0]$ and $a(\delta) = \delta$ is for all $\delta \leq \delta_c$. If

$$\|V\|^{\tilde{p}}_{\tilde{p},D_x(\delta_0)} \lesssim \max\left(\frac{h}{c_1\delta_0}, 1\right)^{\frac{d-1}{\gamma}}\,, \tag{109}$$

pick $\delta := \delta_0$ and note that (1) is satisfied. Furthermore, by Lemma 3.4(ii), we have $N\left(-\Delta_D^N + V\right) \leq 1$. Else, if there exists $\delta \in [\delta_c, \delta_0]$ such that

$$\|V\|^{\tilde{p}}_{\tilde{p},D(\delta)} \sim \max\left(\frac{h}{c_1\delta}, 1\right)^{\frac{d-1}{\gamma}}\,, \tag{110}$$

where the constant in $\sim$ only depends on $d$, $\gamma$, $c$, pick this $\delta$. By (110), (3) is satisfied and moreover, we have $N\left(-\Delta_D^N + V\right) \leq 1$ by Lemma 3.4(ii). Else, since the left hand side of (110) is increasing in $\delta$ and the right hand side of (110) is decreasing in $\delta$, we have (107) for $\delta = \delta_c$. Now if $d \geq 3$ and $\|V\|_{\frac{d}{2},D_x(\delta_c)} \lesssim 1$ or if $d = 2$ and $\|V\|_{\mathcal{B},D_x(\delta_c)} \lesssim 1$, pick $\delta := \delta_c$ and note that (3) is satisfied and $N\left(-\Delta_D^N + V\right) \leq 1$ by Lemma 3.4(i). Else, pick $\delta < \delta_c$ such that (108) holds. Thus, (2) is satisfied and $N\left(-\Delta_D^N + V\right) \leq 1$ by Lemma 3.4(i). □

## 4. Covering lemmas

4.1. **Covering of the part close to the boundary by oscillatory domains.** In this subsection, we prove a Besicovitch type covering lemma for oscillatory domains. It is one of the key ingredients of the proof of Theorem 1.2 since it allows us to choose a family of oscillatory domains as in the previous subsection, which cover the part of $\Omega$ close to $\partial\Omega$, but which do not overlap too much. Using this result, we show that the number of oscillatory domains we choose is bounded by a constant times $\delta_0^{-d}$.

**Lemma 4.1** (Covering lemma). *Let $l \in \{1, \dots, L\}$ and suppose that for every $x \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ we are given a $\delta_x \in (0, \delta_0]$. We define the oscillatory domains $D_x := D_x(\delta_x)$ as in Definition 2.6. Then there exists $K_l = K_l(d, \gamma) \in \mathbb{N}$ and subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{K_l}$ of oscillatory domains $D_x := D_x(\delta_x) \subset \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}$ such that*

*(i) For every $k \in \{1, \dots, K_l\}$ all oscillatory domains in $\mathcal{F}_k$ are disjoint.*

*(ii)*

$$\bigcup_{k=1}^{K_l} \dot{\bigcup_{D\in\mathcal{F}_k}} D \supset \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0} \,. \tag{111}$$

*Proof.* Recall that for every $x \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ we are given a $\delta_x \in (0, \delta_0]$ and the corresponding $a_x$ is given by

$$a_x = \min\left(\delta_x, c_0 \max\left(h_{x,l}, c_1\delta_x\right)^{\frac{1}{\gamma}}\right) \,. \tag{112}$$

Furthermore, the oscillatory domain $D_x = D_x(\delta_x)$ is given by

$$D_x := T_l R_l \left\{ (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; |y' - x'| < \frac{1}{2}a,\; |y_d - x_d| < \frac{1}{2}\delta_x,\; f_l(y') > y_d \right\} \,. \tag{113}$$

Without loss of generality, let us assume that $T_l R_l$ is the identity map. To begin with, we decompose $\tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ into three parts

$$\tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0} = A_1 \cup A_2 \cup A_3 \tag{114}$$

such that

$$A_1 := \left\{ a_x = c_0 h_x^{\frac{1}{\gamma}} \right\}, \quad A_2 := \left\{ a_x = c_2 \delta_x^{\frac{1}{\gamma}} \right\}, \quad A_3 := \left\{ a_x = \delta_x \right\} \,.$$

For each of these sets, we will prove a corresponding covering theorem with a constant $K_1^l$, $K_2^l$, $K_3^l$ depending on $d$, $\gamma$. Combining these results, we will get the desired result with $K^l := K_1^l + K_2^l + K_3^l \in \mathbb{N}$.

The beginning of the proof for $A_b$ with $b \in \{1, 2\}$ will be similar to the proof of Besicovich's covering theorem for cubes, see for example. Let $b \in \{1, 2\}$. We will denote families of oscillatory domains $D_x$ with $x \in A_b$ by $(\mathcal{F}_k)_{k\in\mathbb{N}}$. At the beginning of our construction, the $\mathcal{F}_k$ are all assumed to be empty. We put oscillatory domains inside those $\mathcal{F}_k$ according to the following procedure:

First, choose $\tilde{x}_1 \in A_b$ with

$$\tilde{\delta}_1 := \delta_{\tilde{x}_1} \geq \frac{1}{2} \sup_{x\in A_b} \delta_x \tag{115}$$

and denote the corresponding domain by $\tilde{D}_1 := D_{\tilde{x}_1}$. Put $\tilde{D}_1$ in $\mathcal{F}_1$. Then, if possible, choose $\tilde{x}_2 \in A_b \setminus \tilde{D}_1$ such that

$$\tilde{\delta}_2 := \delta_{\tilde{x}_2} \geq \frac{1}{2} \sup_{x\in A_b\setminus\tilde{D}_1} \delta_x \tag{116}$$

and denote the corresponding domain by $\tilde{D}_2 := D_{\tilde{x}_2}$. If $\tilde{D}_2 \cap \tilde{D}_1 = \emptyset$, put $\tilde{D}_2$ in $\mathcal{F}_1$. Otherwise, put $\tilde{D}_2$ in $\mathcal{F}_2$. More generally, if $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ have already been chosen for some $n \in \mathbb{N}$, we proceed as follows: If

$$\bigcup_{m=1}^{n-1} \tilde{D}_m \supset A_b \,, \tag{117}$$

then stop. Else, we can choose $\tilde{x}_n \in A_b \setminus \bigcup_{m=1}^{n-1} \tilde{D}_m$ such that

$$\tilde{\delta}_n := \delta_{\tilde{x}_n} \geq \frac{1}{2} \sup_{x\in A_b\setminus\bigcup_{m=1}^{n-1}\tilde{D}_m} \delta_x \tag{118}$$

and denote the corresponding domain by $\tilde{D}_n := D_{\tilde{x}_n}$. Put $\tilde{D}_n$ in $\mathcal{F}_k$, where $k$ is the lowest natural number such that $\tilde{D}_n \cap D = \emptyset$ for all $D \in \mathcal{F}_k$. Note that by construction, we know that all oscillatory domains in each $\mathcal{F}_k$ are disjoint.

We are done if we can show that there exists $K_b^l = K_b^l(d, \gamma) \in \mathbb{N}$ such that $\mathcal{F}_k = \emptyset$ for all $k \geq K_b^l + 1$. For the moment, let us assume we had already shown that.

The oscillatory domains, which we have chosen in the above construction, cover $A_b$, namely,

$$\bigcup_{n\in\mathbb{N}} \tilde{D}_n \supset A_b\,, \tag{119}$$

where we use the convention that $\tilde{D}_n = \emptyset$ if we have to stop before the $n^{\text{th}}$ step in the procedure above. In order to see this, note that (119) is clear from the construction above if we have to stop after a finite number of steps. If the number of steps is infinite, we claim

$$\lim_{n\to\infty} \tilde{\delta}_n = 0\,. \tag{120}$$

By construction, we know that $\tilde{\delta}_n \leq 2\tilde{\delta}_m$ for all $n \geq m$. Thus, in order to show (120), it suffices to show that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $\tilde{\delta}_n < \varepsilon$. Suppose this was wrong, that is, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, we have $\tilde{\delta}_n \geq \varepsilon$. It follows that

$$\sum_{n\in\mathbb{N}} |\tilde{D}_n| = \infty\,. \tag{121}$$

On the other hand, we have $\tilde{D}_n \subset \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}$ for all $n \in \mathbb{N}$, $|\Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}| \leq |\Omega| \leq 1$ and by assumption, all $\mathcal{F}_k$ with $k \geq K_b^l + 1$ are empty. Since the oscillatory domains in each $\mathcal{F}_k$ are disjoint, we get

$$\sum_{n\in\mathbb{N}} |\tilde{D}_n| \leq \sum_{k=1}^{K_b^l} |\Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}| \leq K_b^l\,, \tag{122}$$

which contradicts (121). This proves (120). Now suppose (119) was wrong, that is, there exists $\tilde{x} \in A_b \setminus \bigcup_{n\in N} \tilde{D}_n$. By (120), there exists $n \in \mathbb{N}$ with $\tilde{\delta}_n \leq \frac{1}{4}\delta_{\tilde{x}}$, which contradicts (118), and thereby proves (119).

It remains to show that there exists $K_b^l = K_b^l(d, \gamma) \in \mathbb{N}$ such that $\mathcal{F}_k = \emptyset$ for all $k \geq K_b^l + 1$. Let $m \in \mathbb{N}$ with $\mathcal{F}_m = \emptyset$. Let $D_m \in \mathcal{F}_m$. By the above construction, for every $k \in \{1, \ldots, m-1\}$ there exists $D_k \in \mathcal{F}_k$ with $D_k \cap D_m \neq \emptyset$ and such that $D_1, \ldots, D_{m-1}$ were all chosen before $D_m$ in the construction. By relabelling the families $\mathcal{F}_k$ and the corresponding domains $D_k$, we may without loss of generality assume that $D_k$ was chosen before $D_n$ for all $k, n \in \{1, \ldots, m\}$ with $k \leq n$. We denote the centres of $D_1, \ldots, D_m$ by $x_1, \ldots, x_m \in A_b$, the corresponding $\delta$ by $\delta_1, \ldots, \delta_m$ and the corresponding $h$ by $h_1, \ldots, h_m$. Note that by construction, we have

$$\delta_n \leq 2\delta_k \text{ for all } k, n \in \{1, \ldots, m\} \text{ with } k \leq n\,. \tag{123}$$

Note that the $D_k$ do *not* agree with the $\tilde{D}_k$ from the construction above but we have $D_k \in \bigcup_{n\in\mathbb{N}} \tilde{D}_n$ for all $k \in \{1, \ldots, m\}$.

Our goal is to show that $m \leq K_b^l$ for some $K_b^l = K_b^l(d, \gamma) \in \mathbb{N}$, and we will show this for $b = 1$ and $b = 2$ separately.

**Proof of $m \leq K_1^l < \infty$ for $A_1$.** By Lemma 2.8(i), we note that the $D_k$, $k \in \{1, \ldots, m\}$

are all cuboids contained in $\Omega$. Applying Lemma 2.8(ii) with $x = x_m$, $y = x_1$ and $z = x_k$, we get for all $k \in \{1, \dots, m-1\}$

$$\left|x_k' - x_m'\right|_\infty < \frac{1}{2}c_0 h_k^{\frac{1}{\gamma}} + \frac{1}{2}c_0 h_m^{\frac{1}{\gamma}} \le \frac{1}{2}c_0 \left(4h_m\right)^{\frac{1}{\gamma}} + \frac{1}{2}c_0 h_m^{\frac{1}{\gamma}} \le \frac{\alpha}{2}c_0 h_m^{\frac{1}{\gamma}} \tag{124}$$

with $\alpha := \lceil 4^{\frac{1}{\gamma}} + 1\rceil \in \mathbb{N}$. Now define the lattice $\mathcal{G}$ by

$$\mathcal{G} := \left\{ (g', g_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; \begin{array}{l} g_d - (x_m)_d \in \left\{-\frac{\delta_m}{2}, -\frac{\delta_m}{4}, 0, \frac{\delta_m}{4}, \frac{\delta_m}{2}\right\} \text{ and} \\ g' - x_m' = \frac{j}{4\alpha}c_0 h_m^{\frac{1}{\gamma}} \text{ for some } j \in \mathbb{Z}^{d-1} \text{ with } |j|_\infty \le 2\alpha^2 \end{array} \right\}.$$

Here $(x_m)_d$ is the $d^{\text{th}}$ coordinate of $x_m$. Note that

$$|\mathcal{G}| = 5 \cdot \left(2 \cdot 2\alpha^2 + 1\right)^{d-1}, \tag{125}$$

which only depends on $d$, $\gamma$. For every $k \in \{1, \dots, m-1\}$, we associate a $g \in \mathcal{G}$ to $x_k$, which is chosen such that

$$\left|g' - x_k'\right|_\infty + \left|g_d - (x_k)_d\right| \tag{126}$$

is minimal among all $g \in \mathcal{G}$. Note that by the choice of $\mathcal{G}$, if $g \in \mathcal{G}$ is associated to $x_k$, then by (124), $h_m \le 4h_k$ and the definition of $\alpha$, we obtain

$$\left|x_k' - g'\right|_\infty \le \frac{1}{8\alpha}c_0 \left(4h_k\right)^{\frac{1}{\gamma}} \le \frac{1}{8}c_0 h_k^{\frac{1}{\gamma}}. \tag{127}$$

Furthermore, we claim that

$$\left|g_d - (x_k)_d\right| < \frac{1}{2}\delta_k. \tag{128}$$

If $\left|(x_m)_d - (x_k)_d\right| \ge \frac{1}{2}\delta_m$, then by the triangle inequality and $D_m \cap D_k \neq \emptyset$, we get (128). If $\left|(x_m)_d - (x_k)_d\right| < \frac{1}{2}\delta_m$, we use the definition of $\mathcal{G}$ and $2\delta_k \ge \delta_m$ to get (128). Combining (127) and (128), we obtain that if $g \in \mathcal{G}$ is associated to $x_k$, then $g \in D_k$.

Now, if $g \in \mathcal{G}$ is associated to both $x_k$ and $x_n$ for $k, n \in \{1, \dots, m-1\}$ with $k < n$, then by (127) and $h_m \le 4h_k$,

$$\left|x_k' - x_n'\right|_\infty \le \left|x_k' - g'\right|_\infty + \left|g' - x_n'\right|_\infty \le \frac{1}{8\alpha}c_0 h_m^{\frac{1}{\gamma}} + \frac{1}{8\alpha}c_0 h_m^{\frac{1}{\gamma}} \le \frac{1}{4}c_0 h_k^{\frac{1}{\gamma}} \tag{129}$$

where we used $\alpha := \lceil 4^{\frac{1}{\gamma}} + 1\rceil$ in the last step. Since $k < n$, we have $x_n \notin D_k$ by construction. Using (129) and $x_n \notin D_k$, we deduce that

$$\left|(x_n)_d - (x_k)_d\right| \ge \frac{1}{2}\delta_k. \tag{130}$$

**Claim.** Fix $g \in \mathcal{G}$. Then there are at most two indices $k, n \in \{1, \dots, m-1\}$ with $k \neq n$ such that $(x_k)_d \ge g_d$ and $(x_n)_d \ge g_d$ and such that $g$ is associated to both $x_k$ and $x_n$.

**Proof of the claim.** Suppose there were $k, n, j \in \{1, \dots, m-1\}$ with $k < n < j$ such that $g$ is associated to $x_k$, $x_n$ and $x_j$ and such that $(x_k)_d \ge g_d$, $(x_n)_d \ge g_d$ and $(x_j)_d \ge g_d$. Without loss of generality, let us assume that $g_d = 0$. By $g \in D_i$ for all $i \in \{k, n, j\}$, we have

$$0 \le (x_i)_d < \frac{1}{2}\delta_i \tag{131}$$

By (130) and $k < n$, we have

$$(x_n)_d \ge (x_k)_d + \frac{1}{2}\delta_k. \tag{132}$$

Note that $0 \le (x_n)_d \le (x_k)_d$ is not possible since this would imply $x_n \in D_k$ by (129), $g \in D_k$ and the fact that $D_k$ is a cuboid. Similarly, we find that

$$(x_j)_d \ge (x_n)_d + \frac{1}{2}\delta_n \,. \tag{133}$$

Combining (131), (132) and (133), we obtain

$$\frac{1}{2}\delta_j > (x_j)_d \ge (x_n)_d + \frac{1}{2}\delta_n \ge (x_k)_d + \frac{1}{2}\delta_k + \frac{1}{2}\delta_n \ge \frac{1}{2}\delta_j \,, \tag{134}$$

which is a contradiction. In the last step we used that $2\delta_k \ge \delta_j$ and $2\delta_n \ge \delta_j$ since $k, n < j$. This finishes the proof of the claim.

Similarly, we can show that there are at most two indices $k, n \in \{1, \dots, m-1\}$ with $k \ne n$ such that $(x_k)_d \le g_d$ and $(x_n)_d \le g_d$ and such that $g$ is associated to both $x_k$ and $x_n$. Hence, for every $g \in \mathcal{G}$ there exist at most four different indices $j_1, j_2, j_3, j_4 \in \{1, \dots, m-1\}$ such that $g$ is associated to $x_{j_i}$ for all $i = 1, 2, 3, 4$. This shows that $m - 1 \le 4|\mathcal{G}|$, where the right hand side only depends on $d$, $\gamma$. It follows that a possible choice is

$$K_1^l := 4|\mathcal{G}| + 1. \tag{135}$$

**Proof of $m \le K_2^l < \infty$ for $A_2$.** Define the lattice $\mathcal{G}$ by

$$\mathcal{G} := \left\{ (g', g_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; \begin{array}{l} g_d - (x_m)_d \in \left\{-\frac{\delta_m}{2}, -\frac{\delta_m}{4}, 0, \frac{\delta_m}{4}, \frac{\delta_m}{2}\right\} \text{ and} \\ g' - x'_m = \frac{j}{2\alpha} c_2 \delta_m^{\frac{1}{\gamma}} \text{ for some } j \in \mathbb{Z}^{d-1} \text{ with } |j|_\infty \le \alpha \end{array} \right\},$$

where $\alpha := \lceil 2^{\frac{1}{\gamma}} \rceil \in \mathbb{N}$. Note that $\mathcal{G} \subset \overline{D_m}$ and

$$|\mathcal{G}| = 5 \cdot (2\alpha + 1)^{d-1} \,, \tag{136}$$

which only depends on $d$, $\gamma$. For every $k \in \{1, \dots, m-1\}$, we associate a $g \in \mathcal{G}$ to $x_k$, which is chosen such that

$$\left|g' - x'_k\right|_\infty + |g_d - (x_k)_d| \tag{137}$$

is minimal among all $g \in \mathcal{G}$. We may also say that $x_k$ is associated to $g \in \mathcal{G}$. Note that if $g \in \mathcal{G}$ is associated to $x_k$, then

$$|g_d - (x_k)_d| < \frac{1}{2}\delta_k \,. \tag{138}$$

If $x_k \notin D_m$, this follows from $D_m \cap D_k \ne \emptyset$ and $\mathcal{G} \subset \overline{D_m}$. If $x_k \in D_m$, this follows from $\delta_m \le 2\delta_k$ and the choice of $\mathcal{G}$. Furthermore, we have

$$\left|x'_k - g'\right|_\infty \le \frac{1}{2} c_2 \delta_k^{\frac{1}{\gamma}} \,. \tag{139}$$

If $x_k \notin D_m$, this follows from $D_m \cap D_k \ne \emptyset$ and $\mathcal{G} \subset \overline{D_m}$. If $x_k \in D_m$, this follows from

$$\frac{1}{4\alpha} c_2 \delta_m^{\frac{1}{\gamma}} \le \frac{1}{4} c_2 \left(\frac{\delta_m}{2}\right)^{\frac{1}{\gamma}} \le \frac{1}{4} c_2 \delta_k^{\frac{1}{\gamma}} \,. \tag{140}$$

Combining (138) and (139), we get $g \in D_k$.

Fix $g \in \mathcal{G}$ and consider all $k \in \{1, \dots, m-1\}$ such that $g$ is associated to $x_k$. Without loss of generality, we may assume that we chose our coordinate system in such a way that $g = (0, \dots, 0)$. For each coordinate $i \in \{1, \dots, d\}$, we distinguish between two cases:

$$(0) \;\; (x_k)_i \ge 0 \quad \text{and} \quad (1) \;\; (x_k)_i < 0,$$

and associate a $\sigma = (\sigma_1, \dots \sigma_d) \in \{0, 1\}^d$ to $x_k$ such that condition $(\sigma_i)$ is satisfied for $(x_k)_i$ for all $i \in \{1, \dots, d\}$. Fix $\sigma \in \{0, 1\}^d$ and assume without loss of generality that

$\sigma = (0, \dots, 0)$. Otherwise, rotate and reflect the coordinate system accordingly. In the following, we would like to count the number of $x_k$ such that $x_k$ is associated to $g$ and to $\sigma = (0, \dots, 0)$.

Let $k_1 \in \{1, \dots, m-1\}$ be the smallest number such that $x_{k_1}$ is associated to $g$ and to $\sigma = (0, \dots, 0)$. For $n \in \mathbb{N}$, if it exists, let $k_n \in \{1, \dots, m-1\}$ be the $n^{\text{th}}$ smallest number such that $x_{k_n}$ is associated to $g$ and to $\sigma = (0, \dots, 0)$. By construction, we have $\delta_{k_n} \leq 2\delta_{k_1}$ for all $n \geq 1$, and moreover, we also have $g \in D_{k_n}$ as we have noticed before. Hence, with the notation $S_\sigma := [0, \infty)^d$, we have by $\delta_{k_n} \leq 2\delta_{k_1}$

$$|D_{k_n} \cap S_\sigma| \leq \left| \left[0, c_2 \delta_{k_n}^{\frac{1}{\gamma}}\right]^{d-1} \times [0, \delta_{k_n}] \right| \leq 2^{\frac{d-1}{\gamma}+1} \left( c_2 \delta_{k_1}^{\frac{1}{\gamma}} \right)^{d-1} \delta_{k_1} \tag{141}$$

for all $n \geq 1$. Let $n \geq 1$. We claim that $\delta_{k_n} \geq \delta_{k_1}$. To see this, recall that $x_{k_n} \in S_\sigma$ but $x_{k_n} \notin D_{k_1}$ since $k_n > k_1$. Therefore, there exists $i_0 \in \{1, \dots, d\}$ with

$$(x_{k_n})_{i_0} \geq (x_{k_1})_{i_0} + \frac{1}{2} c_2 \delta_{k_1}^{\frac{1}{\gamma}} \text{ if } i_0 \in \{1, \dots, d-1\} \tag{142}$$

and

$$(x_{k_n})_d \geq (x_{k_1})_d + \frac{1}{2} \delta_{k_1} \text{ if } i_0 = d\,. \tag{143}$$

At the same time, since $g \in D_{k_n}$, we have

$$(x_{k_n})_i \leq \frac{1}{2} c_2 \delta_{k_n}^{\frac{1}{\gamma}} \text{ for all } i \in \{1, \dots, d-1\} \tag{144}$$

and $(x_{k_n})_d \leq \frac{1}{2}\delta_{k_n}$. We deduce that $\delta_{k_n} \geq \delta_{k_1}$. For all $n \geq 1$, we have by $\delta_{k_n} \geq \delta_{k_1}$ and $x_n \notin D_j$ for $j < n$,

$$\left| (D_{k_n} \cap S_\sigma) \setminus \bigcup_{j=1}^{n-1} D_{k_j} \right| \geq \frac{1}{2^d} \left( c_2 \delta_{k_1}^{\frac{1}{\gamma}} \right)^{d-1} \delta_{k_1}\,. \tag{145}$$

Now assume that $x_{k_1}, \dots, x_{k_1}$ are all associated to $g$ and to $\sigma = (0, \dots, 0)$. By (145),

$$\left| S_\sigma \cap \bigcup_{n=1}^{N} D_{k_n} \right| = \sum_{n=1}^{N} \left| (D_{k_n} \cap S_\sigma) \setminus \bigcup_{j=1}^{n-1} D_{k_j} \right| \geq N \frac{1}{2^d} \left( c_2 \delta_{k_1}^{\frac{1}{\gamma}} \right)^{d-1} \delta_{k_1}\,. \tag{146}$$

On the other hand, we have (141) and thus,

$$N \leq 2^{\frac{d-1}{\gamma}+1+d}\,. \tag{147}$$

Repeating the same argument for every $g \in \mathcal{G}$ and every $\sigma \in \{0,1\}^d$, we find by (136) that

$$K_2^l := 5 \cdot (2\alpha+1)^{d-1} \cdot 2^{\frac{d-1}{\gamma}+1+2d} + 1 \in \mathbb{N} \tag{148}$$

is a possible choice. Note that the right-hand side only depends on $d$, $\gamma$.

**Covering theorem for $A_3$.** This is simply the Besicovitch covering theorem for cubes and here $K_3^l = K_3^l(d) \in \mathbb{N}$. □

**Definition 4.2** (Choice of a subfamily of oscillatory domains)**.** *Let $l \in \{1, \dots, L\}$ and for every $x \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ let $\delta_x \in (0, \delta_0]$, and hence also $D_x := D_x(\delta_x)$, be chosen as* *in Lemma 3.5. Now apply Lemma 4.1 to this collection of oscillatory domains to get subfamilies* $\mathcal{F}_1, \ldots, \mathcal{F}_{K_l}$ *of oscillatory domains, where* $K_l = K_l(d, \gamma) \in \mathbb{N}$. *Write*

$$\bigcup_{k=1}^{K_l} \mathcal{F}_k =: \{D_j\}_{j \in J} . \tag{149}$$

*where* $J = J_1 \cup J_2 \cup J_3$ *is an index set such that for every* $m \in \{1,2,3\}$ *and all* $j \in J_m$ *the oscillatory domain* $D_j$ *satisfies condition* $(m)$ *in Lemma 3.5. In the following, for every* $m \in \{1,2,3\}$ *and every* $j \in J_m$ *the point* $x_j \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ *is such that* $D_j = D_{x_j}$.

**Definition 4.3.** *Let* $\gamma \in \left[\frac{d-1}{d}, 1\right)$. *Define* $s, s', \omega, \zeta$ *by*

$$\frac{1}{s'} := \frac{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 - d}{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 + 1}, \qquad \frac{1}{s} := \frac{d+1}{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 + 1}, \tag{150}$$

*and*

$$\omega := \left(\frac{d-1}{\gamma}+1\right)\frac{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 - d}{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 + 1}, \quad \zeta := \frac{1}{s}\left(-\beta + \frac{d-1}{\gamma}\right). \tag{151}$$

We are now ready to prove that the number of oscillatory domains we choose is bounded by a constant times $\delta_0^{-d}$.

**Lemma 4.4.** *Let* $l \in \{1, \ldots, L\}$ *and let* $\{D_j\}_{j \in J}$ *be chosen as in Definition 4.2. Then*

$$|J| \lesssim \delta_0^{-d} . \tag{152}$$

*Proof.* Recall that $J = J_1 \cup J_2 \cup J_3$, where each $J_m$, $m \in \{1,2,3\}$ is chosen such that all $D_j$ with $j \in J_m$ satisfy condition $(m)$ in Lemma 3.5. We will show for all $m \in \{1,2,3\}$ that $|J_m| \lesssim \delta_0^{-d}$.

**Notation.** If $j \in J$ and $D_j = D_{x_j}(\delta_{x_j})$ for some $x_j \in \tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$, write

$$\delta_j := \delta_{x_j}, \quad h_j := h_{x_j}, \quad a_j := a_{x_j} = \min\left(\delta_j, c_0 \max\left(h_j, c_1\delta_j\right)^{\frac{1}{\gamma}}\right), \quad K := K_l . \tag{153}$$

**Estimate for $J_1$.** Recall that if $j \in J_1$, then $\delta_j = \delta_0$. We have

$$\begin{aligned} |J_1| &\le |\{j \in J_1 \mid a_j = \delta_0\}| + \left|\left\{j \in J_1 \mid a_j = c_0 h_j^{\frac{1}{\gamma}}\right\}\right| + \left|\left\{j \in J_1 \mid a_j = c_2 \delta_0^{\frac{1}{\gamma}}\right\}\right| \\ &=: |J_{1,1}| + |J_{1,2}| + |J_{1,3}| . \end{aligned}$$

**Estimate for $J_{1,1}$.** Note that if $a_j = \delta_0$, then $|D_j| = \delta_0^d$ since by Lemma 2.7(i) and Lemma 2.8(i), $D_j$ is a cuboid contained in $\Omega_l$. Thus,

$$|J_{1,1}| = |\{j \in J_1 \mid a_j = \delta_0\}| = \delta_0^{-d} \sum_{j \in J_{1,1}} |D_j| \lesssim \delta_0^{-d}\left|\Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}\right| \lesssim \delta_0^{-d} , \tag{154}$$

where we used the covering lemma (Lemma 4.1) and $D_j \subset \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}$ for all $j \in J$ in the second last step . In the last step, we used that $\left|\Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}\right| \le |\Omega| \le 1$.

**Estimate for $J_{1,2}$.** Recall that if $a_j = c_0 h_j^{\frac{1}{\gamma}}$, then $h_j \geq c_1\delta_0$ and $|D_j| \sim \delta_0 h_j^{\frac{d-1}{\gamma}}$ by Lemma 2.8(i). Using these properties, Lemma 2.8(iii) and Lemma 4.1, we get

$$|J_{1,2}| = \left|\left\{ j \in J_1 \mid a_j = c_0 h_j^{\frac{1}{\gamma}} \right\}\right| \sim \delta_0^{-1} \sum_{j\in J_{1,2}} |D_j| h_j^{-\frac{d-1}{\gamma}} \sim \delta_0^{-1} \sum_{j\in J_{1,2}} \int_{D_j} \mathrm{d}w\, h_w^{-\frac{d-1}{\gamma}}$$
$$\lesssim \delta_0^{-1} \int_{\left\{ w\in\Omega_l\cap\Omega^b_{\sqrt{d}\delta_0} \,\middle|\, h_w \geq \frac{1}{2}c_1\delta_0 \right\}} \mathrm{d}w\, h_w^{-\frac{d-1}{\gamma}} .$$

Note that $\frac{d-1}{\gamma} > 1$ since $d \geq 2$ and $\gamma < 1$. We estimate using $\frac{d-1}{\gamma} \leq d$ and $\delta_0 \leq 1$

$$|J_{1,2}| \lesssim \delta_0^{-1} \int_{\left\{ w\in\Omega_l\cap\Omega^b_{\sqrt{d}\delta_0} \,\middle|\, h_w \geq \frac{1}{2}c_1\delta_0 \right\}} \mathrm{d}w\, h_w^{-\frac{d-1}{\gamma}} \lesssim \delta_0^{-1} \int_{\frac{1}{2}c_1\delta_0}^{\infty} \mathrm{d}h\, h^{-\frac{d-1}{\gamma}} \sim \delta_0^{-\frac{d-1}{\gamma}} \leq \delta_0^{-d} .$$

**Estimate for $J_{1,3}$.** By $a_j = c_2\delta_0^{\frac{1}{\gamma}}$, we have $h_j \leq c_1\delta_0$ for all $j \in J_{1,3}$. Furthermore,

$$|D_j| \sim \delta_0 a_j^{d-1} \sim \delta_0^{\frac{d-1}{\gamma}+1} . \tag{155}$$

By Lemma 2.8(iv), we have $|h_w - h_j| \leq \delta_0$ for all $j \in J_{1,3}$ and all $w \in D_j$. Since $h_j \leq c_1\delta_0$, we get

$$h_w \leq h_j + |h_w - h_j| \leq c_1\delta_0 + \delta_0 = (c_1+1)\delta_0 \text{ for all } w \in D_j . \tag{156}$$

We obtain

$$|J_{1,3}| = \left|\left\{ j \in J_1 \mid a_j = c_2\delta_0^{\frac{1}{\gamma}} \right\}\right| = \delta_0^{-\left(\frac{d-1}{\gamma}+1\right)} \sum_{j\in J_{1,3}} |D_j|$$
$$\lesssim \delta_0^{-\left(\frac{d-1}{\gamma}+1\right)} \left|\left\{ w \in \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0} \,\middle|\, h_w \leq (c_1+1)\delta_0 \right\}\right| \sim \delta_0^{-\frac{d-1}{\gamma}} \leq \delta_0^{-d} ,$$

where were used (155) in the second step, the covering lemma (Lemma 4.1) and (156) in the third step, and $\frac{d-1}{\gamma} \leq d$ and $\delta_0 \leq 1$ in the last step.

**Estimate for $J_2$.** Recall that if $j \in J_2$, then $\|V\|_{\frac{d}{2},D_j}^{\frac{d}{2}} \gtrsim 1$ if $d \geq 3$ and $\|V\|_{\mathcal{B},D_j} \gtrsim 1$ if $d = 2$. Using the covering lemma (Lemma 4.1), we obtain for $d \geq 3$

$$|J_2| = \sum_{j\in J_2} 1 \lesssim \sum_{j\in J_2} \|V\|_{\frac{d}{2},D_j}^{\frac{d}{2}} \lesssim \|V\|_{\frac{d}{2},\Omega_l\cap\Omega^b_{\sqrt{d}\delta_0}}^{\frac{d}{2}} \lesssim \|V\|_{\frac{d}{2},\Omega}^{\frac{d}{2}} \lesssim \delta_0^{-d}. \tag{157}$$

For $d = 2$, using [14, Lemma A.1] we get

$$|J_2| = \sum_{j\in J_2} 1 \lesssim \sum_{j\in J_2} \|V\|_{\mathcal{B},D_j} \lesssim \|V\|_{\mathcal{B},\Omega_l\cap\Omega^b_{\sqrt{d}\delta_0}} \lesssim \|V\|_{\mathcal{B},\Omega} \lesssim \delta_0^{-d}, \tag{158}$$

**Estimate for $J_3$.** Recall that if $j \in J_3$, then $a_j = c_0 \max(h_j, c_1\delta_j)^{\frac{1}{\gamma}}$ and

$$\|V\|_{\tilde{p},D_j}^{\tilde{p}} \gtrsim \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{\frac{d-1}{\gamma}} . \tag{159}$$

In the following, we will use $s'$, $s$, $\omega$ and $\zeta$ as defined in Definition 4.3. By

$$\frac{1}{s} + \frac{1}{s'} = 1 \tag{160}$$

and Hölder's inequality, we have

$$|J_3| = \sum_{j\in J_3} \delta_j^{\omega} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\zeta} \delta_j^{-\omega} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{\zeta}$$

$$\leq \left(\sum_{j\in J_3} \delta_j^{\frac{d-1}{\gamma}+1} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\zeta s'}\right)^{\frac{1}{s'}} \left(\sum_{j\in J_3} \delta_j^{-\beta} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\beta+\frac{d-1}{\gamma}}\right)^{\frac{1}{s}}, \quad (161)$$

where we used

$$\omega s' = 1 + \frac{d-1}{\gamma} \quad \text{and} \quad \omega s = \beta \quad (162)$$

and the definition of $\zeta$, see Definition 4.3, in the last step. By (159) and Lemma 2.8(v), we have for every $j \in J_3$

$$\delta_j^{-\beta} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\beta+\frac{d-1}{\gamma}} \lesssim \max(h_j, c_1\delta_j)^{-\beta} \|V\|_{\tilde p, D_j}^{\tilde p} \lesssim |V|_{\tilde p,\beta,D_j}^{\tilde p}. \quad (163)$$

Using the covering lemma (Lemma 4.1), we get by (23)

$$\sum_{j\in J_3} \delta_j^{-\beta} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\beta+\frac{d-1}{\gamma}} \lesssim \sum_{j\in J_3} |V|_{\tilde p,\beta,D_j}^{\tilde p} \lesssim \|V\|_{\tilde p,\beta}^{-\tilde p} \lesssim |V|_{\tilde p,\beta,\Omega}^{\tilde p} \lesssim \delta_0^{-2\tilde p}. \quad (164)$$

If we can show that

$$\sum_{j\in J_3} \delta_j^{\frac{d-1}{\gamma}+1} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\zeta s'} \lesssim \delta_0, \quad (165)$$

then we get by (161) and (164) that

$$|J_3| \lesssim \delta_0^{\frac{1}{s'}} \delta_0^{-2\tilde p \frac{1}{s}} = \delta_0^{-d},$$

where we used

$$-2\tilde p \frac{1}{s} + \frac{1}{s'} = -d \quad (166)$$

in the last step. Hence, it remains to show (165). First note that by $a_j = c_0 \max(h_j, c_1\delta_j)^{\frac{1}{\gamma}}$ and by Lemma 2.7(ii), we have

$$|D_j| \sim \delta_0 a_j^{d-1} = \delta_0 \left(c_0 \max(h_j, c_1\delta_j)^{\frac{1}{\gamma}}\right)^{d-1} \sim \delta_0^{\frac{d-1}{\gamma}+1} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{\frac{d-1}{\gamma}} \quad (167)$$

for all $j \in J_3$. Thus,

$$\sum_{j\in J_3} \delta_j^{\frac{d-1}{\gamma}+1} \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\zeta s'} \sim \sum_{j\in J_3} |D_j| \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)}$$

$$= \sum_{j\in J_3,\, h_j\geq c_1\delta_0} |D_j| \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} + \sum_{j\in J_3,\, h_j< c_1\delta_0} |D_j| \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)}$$

$$=: S_1 + S_2.$$

**Estimate for $S_1$.** If $j \in J_3$ with $h_j \geq c_1\delta_0$, then we have $h_j \geq c_1\delta_0 \geq c_1\delta_j$ since $\delta_0 \geq \delta_j$. Thus,

$$a_j = c_0 \max(h_j, c_1\delta_j)^{\frac{1}{\gamma}} = c_0 h_j^{\frac{1}{\gamma}}. \quad (168)$$

By Lemma 2.8(iii), it follows that

$$\begin{aligned}
S_1 &\leq \sum_{j\in J_3,\, h_j\geq c_1\delta_0} |D_j| \left(\frac{h_j}{c_1\delta_0}\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \sim \sum_{j\in J_3,\, h_j\geq c_1\delta_0} \int_{D_j} \mathrm{d}w \left(\frac{h_w}{c_1\delta_0}\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \\
&\lesssim \int_{\left\{w\in\Omega_l\cap\Omega^b_{\sqrt{d}\delta_0}\,\middle|\, h_w\geq\frac{1}{2}c_1\delta_0\right\}} \mathrm{d}w \left(\frac{h_w}{c_1\delta_0}\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \lesssim \int_{\frac{1}{2}c_1\delta_0}^{\infty} \mathrm{d}h \left(\frac{h}{c_1\delta_0}\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \\
&= c_1\delta_0 \int_{\frac{1}{2}}^{\infty} \mathrm{d}t\, t^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \lesssim \delta_0\,.
\end{aligned}$$

Here we used $h_j \geq c_1\delta_0$, $\delta_0 \geq \delta_j$ and

$$\zeta s' + \frac{d-1}{\gamma} > 1 \tag{169}$$

in the second step. We explain (169) below. In the fourth step, we used Lemma 2.8(iii) to get $h_w \geq \frac{1}{2}h_j \geq \frac{1}{2}c_1\delta_0$ and moreover we applied the covering lemma (Lemma 4.1). In the second last step we used the change of variables $t = \frac{h_w}{c_1\delta_0}$ and in the last step we used (169) to deduce that the integral is finite.

In order to show (169), note that we have

$$\zeta s' + \frac{d-1}{\gamma} = \frac{1}{s}\left(-\beta + \frac{d-1}{\gamma}\right)s' + \frac{d-1}{\gamma} = 1 - 2 + \frac{d-1}{\gamma}\frac{d+1}{\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 - d}\,. \tag{170}$$

Therefore, (169) is equivalent to

$$\frac{d-1}{\gamma} > \frac{2}{d+1}\left[\frac{1}{d}\left(\frac{d-1}{\gamma}+1\right)^2 - d\right] = \frac{2d}{d+1}\left[\frac{1}{d^2}\left(\frac{d-1}{\gamma}+1\right)^2 - 1\right]. \tag{171}$$

Define

$$Y := \frac{1}{d}\left(\frac{d-1}{\gamma}+1\right) \tag{172}$$

and note that since $\gamma \in \left[\frac{d-1}{d}, 1\right)$, we have $\frac{d-1}{\gamma} \in (d-1, d]$ and therefore, $Y \in \left(1, \frac{d+1}{d}\right]$. The inequality (171) now reads

$$dY - 1 > \frac{2d}{d+1}\left[Y^2 - 1\right]. \tag{173}$$

Using $\frac{d}{d+1} \leq \frac{1}{Y}$, $\frac{2d}{d+1} > 1$ and $d \geq 2$, we have

$$\frac{2d}{d+1}\left[Y^2 - 1\right] \leq \frac{2}{Y}Y^2 - \frac{2d}{d+1} < 2Y - 1 \leq dY - 1\,, \tag{174}$$

which shows (173) and hence, (169) holds.

**Estimate for $S_2$.** By (169), we get

$$S_2 = \sum_{j\in J_3,\, h_j<c_1\delta_0} |D_j| \max\left(\frac{h_j}{c_1\delta_j}, 1\right)^{-\left(\zeta s'+\frac{d-1}{\gamma}\right)} \leq \sum_{j\in J_3,\, h_j<c_1\delta_0} |D_j|\,. \tag{175}$$

Let $j \in J_3$ with $h_j < c_1\delta_0$. By the definition of $J_3$, we have $a_j = c_0 h_j^{\frac{1}{\gamma}}$ or $a_j = c_2\delta_j^{\frac{1}{\gamma}}$. If $a_j = c_0 h_j^{\frac{1}{\gamma}}$, then by assumption $h_j \le c_1\delta_0$, so we get by Lemma 2.8(iii)

$$h_w \le 2h_j \le 2c_1\delta_0 \quad \text{for all} \quad w \in D_j\,. \tag{176}$$

If $a_j = c_2\delta_j^{\frac{1}{\gamma}}$, we get by Lemma 2.8(iv)

$$h_w \le h_j + |h_w - h_j| \le c_1\delta_0 + \delta_j \le c_1\delta_0 + \delta_0 \le 2c_1\delta_0 \quad \text{for all} \quad w \in D_j\,. \tag{177}$$

In both cases we get $h_w \le 2c_1\delta_0$ for all $w \in D_j$. Using this fact, (175) and the covering lemma (Lemma 4.1), we obtain

$$S_2 \le \sum_{j\in J_3,\, h_j<c_1\delta_0} |D_j| \lesssim \left|\left\{w \in \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0} \,\middle|\, h_w \le 2c_1\delta_0\right\}\right| \lesssim \delta_0\,. \tag{178}$$

□

4.2. **Covering of the interior by cubes.** In this subsection, we consider the part of $\Omega$ far enough away from $\partial\Omega$ and show that we can choose a family of cubes $D_x$ with centre $x \in \Omega$ far enough away from $\partial\Omega$ such that

$$N\left(-\Delta^N_{D_x} + V\right) \le 1 \tag{179}$$

and such that the number of cubes we choose is bounded by a constant times $\delta_0^{-d}$. This part of the proof mimics the proof strategy of Rozenblum [13], [8, Section 4.5.1].

**Definition 4.5.** *For all $x \in \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$, define $\delta_x \in (0, \delta_0]$ by*

$$\delta_x := \sup\left\{\tilde\delta \in (0,\delta_0] \,\middle|\, \|V\|^{\frac{d}{2}}_{\frac{d}{2},D_x(\tilde\delta)} \lesssim 1\right\} \ \text{if } d \ge 3 \tag{180}$$

*and*

$$\delta_x := \sup\left\{\tilde\delta \in (0,\delta_0] \,\middle|\, \|V\|_{\mathcal{B},D_x(\tilde\delta)} \lesssim 1\right\} \ \text{if } d = 2 \tag{181}$$

*where*

$$D_x(\tilde\delta) := \left\{y \in \mathbb{R}^d \,\middle|\, |y - x|_\infty < \frac{1}{2}\tilde\delta\right\} \tag{182}$$

*and where $|\cdot|_\infty$ denotes the $\infty$-norm on $\mathbb{R}^d$. Here the constants in $\lesssim$ have to be chosen small enough depending on $d$.*

**Lemma 4.6.** *Let $x \in \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ and let $\delta_x \in (0, \delta_0]$ be as in Definition 4.5. Then $D := D_x := D_x(\delta_x) \subset \Omega$ and*

$$N\left(-\Delta^N_D + V\right) \le 1\,. \tag{183}$$

*Proof.* Since $x \in \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ and $\delta_x \le \delta_0$, we have $\operatorname{dist}(x, \partial\Omega) \ge \frac{1}{2}\sqrt{d}\delta_0$. By the definition of $D$, we obtain $D \subset \Omega$. The bound (183) can be proved as in Lemma 3.4(i). □

**Lemma 4.7** (Covering lemma for the interior of $\Omega$)**.** *For every $x \in \Omega\setminus\Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$ let $\delta_x \in (0, \delta_0]$ and $D_x := D_x(\delta_x)$ be as in Definition 4.5.*

*(i) Then there exists $K_0 = K_0(d, \gamma) \in \mathbb{N}$ and subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{K_0}$ of oscillatory domains $D_x = D_x(\delta_x) \subset \Omega_l \cap \Omega^b_{\sqrt{d}\delta_0}$ such that for every $k \in \{1, \dots, K_0\}$ all oscillatory domains in $\mathcal{F}_k$ are disjoint, and moreover,*

$$\bigcup_{k=1}^{K_0} \dot{\bigcup_{D\in\mathcal{F}_k}} D \supset \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}\,. \tag{184}$$

*(ii) Let $J_0$ be an index set and denote*

$$\bigcup_{k=1}^{K_0} \mathcal{F}_k =: \{D_j\}_{j\in J_0} \,. \tag{185}$$

*Then*

$$|J_0| \lesssim \delta_0^{-d} \,. \tag{186}$$

*Proof.* **Proof of (i).** In order to get the desired result, it suffices to apply the Besicovitch covering lemma for cubes to the family $\{D_x\}_{x\in\Omega\setminus\Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}}$.

**Proof of (ii).** For every $j \in J_0$, let $\delta_j$ be such that $D_j := D_{x_j}(\delta_j)$ for some $x_j \in \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$. Write $J_0 = J_{0,0} \cup J_{0,1}$, where $J_{0,0}$ and $J_{0,1}$ are chosen such that $\delta_j = \delta_0$ for all $j \in J_{0,0}$, $\|V\|_{\frac{d}{2},D_j} \sim 1$ for all $j \in J_{0,1}$ if $d \geq 3$, and $\|V\|_{\mathcal{B},D_j} \sim 1$ for all $j \in J_{0,1}$ if $d = 2$. This is possible by the definition of $\delta_x$ for $x \in \Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$. By (i), $|D_j| = \delta_0^d$ and $|\Omega| \leq 1$ for all $j \in J_{0,0}$, so we get $|J_{0,0}| \lesssim \delta_0^{-d}$. Using (i), we obtain

$$|J_{0,1}| = \sum_{j\in J_{0,1}} 1 \lesssim \sum_{j\in J_{0,1}} \|V\|^{\frac{d}{2}}_{\frac{d}{2},D_j} \lesssim \|V\|^{\frac{d}{2}}_{\frac{d}{2},\Omega\setminus\Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}} \lesssim \|V\|^{\frac{d}{2}}_{\frac{d}{2},\Omega} \lesssim \delta_0^{-d} \tag{187}$$

if $d \geq 3$ and

$$|J_{0,1}| = \sum_{j\in J_{0,1}} 1 \lesssim \sum_{j\in J_{0,1}} \|V\|_{\mathcal{B},D_j} \lesssim \|V\|_{\mathcal{B},\Omega\setminus\Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}} \lesssim \|V\|_{\mathcal{B},\Omega} \lesssim \delta_0^{-d} \tag{188}$$

if $d = 2$. Thus, we get (186). □

## 5. Conclusion of Theorem 1.2 and Theorem 1.4

In this section, we conclude Theorem 1.4 and we also prove Corollary 1.5. We remarked in Section 1.2 that these two results imply Theorem 1.2.

5.1. **Proof of Theorem 1.4.** In this subsection, we combine the results from the previous subsections to prove Lemma 1.6. From this, we deduce Lemma 1.7 and Lemma 1.8. As we have already shown in Section 1.2, we obtain Theorem 1.4 from these lemmata.

*Proof of Lemma 1.6.* Define $K := K_0 + K_1 + \ldots K_L$, where $K_0$ was defined in Lemma 4.7(i) and $K_l$ for $l \in \{1, \ldots, L\}$ was defined in Lemma 4.1. Note that $K$ only depends on $d$, $\gamma$, $L$. Denote by $\mathcal{F}^l_k$ with $l \in \{0, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$ the corresponding families of oscillatory domains. By Lemma 4.7(i) and Lemma 4.1(i), the oscillatory domains in each $\mathcal{F}^l_k$ are disjoint and moreover,

$$\Omega \supset \bigcup_{l=0}^{L} \bigcup_{k=1}^{K_l} \dot{\bigcup_{D\in\mathcal{F}^l_k}} D \supset \left(\Omega \setminus \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}\right) \cup \left(\bigcup_{l=1}^{L} \left(\tilde{\Omega}_l \cap \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}\right)\right) \supset \Omega \,, \tag{189}$$

where we used in the last step that by $h_\Omega \geq \sqrt{d}\delta_0$ and Lemma 2.3, we have $\bigcup_{l=1}^{L} \tilde{\Omega}_l \supset \Omega^b_{\frac{1}{2}\sqrt{d}\delta_0}$. This shows (a). For (b), note that by Lemma 4.6 and Lemma 3.4, we have

$$N\left(-\Delta^N_D + V\right) \leq 1 \text{ for all } l \in \{0, \ldots, L\}\,,\ k \in \{1, \ldots, K_l\} \text{ and } D \in \mathcal{F}^l_k \,. \tag{190}$$

For (c), we obtain by Lemma 4.4 and Lemma 4.7(ii)

$$\sum_{l=0}^{L} \sum_{k=1}^{K_l} |\mathcal{F}_k| \lesssim \delta_0^{-d} \,. \tag{191}$$

□

*Proof of Lemma 1.7.* By Friedrich's extension, it suffices to prove that $-\Delta_\Omega^N + V$ is bounded from below with the quadratic form domain $H^1(\Omega)$. Let $K = K(d, \gamma, L) \in \mathbb{N}$ be as in Lemma 1.6. Denote $\tilde{V} := 4KV$ and recall $\|V\|_{\tilde{p},\beta} < \infty$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_K$ be the families of oscillatory domains $D \subset \Omega$ which we got from Lemma 1.6 applied to $\tilde{V}$. We compute

$$\begin{aligned} -\Delta_\Omega^N + V &\geq -\frac{1}{2}\Delta_\Omega^N + \frac{1}{2}\left(\frac{1}{K}\sum_{k=1}^{K}(-\Delta_\Omega^N) + \sum_{k=1}^{K}\sum_{D\in\mathcal{F}_k} 2V1_D\right) \\ &\geq -\frac{1}{2}\Delta_\Omega^N + \frac{1}{2K}\sum_{k=1}^{K}\sum_{D\in\mathcal{F}_k}\left(-\Delta_D^N + \frac{1}{2}\tilde{V}1_D\right). \end{aligned} \tag{192}$$

Let $k \in \{1, \ldots, K\}$, $D \in \mathcal{F}_k$, $u \in H^1(\Omega)$ and $u_D := \frac{1}{|D|}\int_D u \in \mathbb{R}$. Note that for $v := u - u_D$, we have $v \in H^1(\Omega)$, $\int_D v = 0$ and $\int_D |\nabla u|^2 = \int_D |\nabla v|^2$. We get using $\tilde{V} \leq 0$

$$\begin{aligned} &\int_D |\nabla u|^2 + \int_D \frac{1}{2}\tilde{V}|u|^2 \geq \int_D |\nabla v|^2 + \int_D \frac{1}{2}\tilde{V}\left(2|v|^2 + 2|u_D|^2\right) \\ &= \int_D |\nabla v|^2 + \int_D \tilde{V}|v|^2 + \left|\frac{1}{|D|}\int_D u\right|^2 \int_D \tilde{V} \geq \frac{1}{|D|}\int_D \tilde{V}\int_D |u|^2. \end{aligned}$$

In the fourth step we used that $\int_D |\nabla v|^2 + \int_D \tilde{V}|v|^2 \geq 0$ by the choice of the family $\mathcal{F}_k$ and $\int_D v = 0$ for the first two summands, and we used Jensen's inequality and $\tilde{V} \leq 0$ for the last summand. We deduce that in the sense of quadratic forms,

$$-\Delta_D^N + \frac{1}{2}1_D\tilde{V}1_D \geq -\left(\frac{1}{|D|}\int_D |\tilde{V}|\right)1_D. \tag{193}$$

We obtain by (192) and (193),

$$-\Delta_\Omega^N + V \geq \frac{1}{2}\left(-\Delta_\Omega^N\right) - \left(\frac{1}{2K}\sum_{k=1}^{K}\sum_{D\in\mathcal{F}_k}\frac{1}{|D|}\int_D |\tilde{V}|\right)1_\Omega, \tag{194}$$

where the constant in the last part is finite since $\sum_{k=1}^{K} |\mathcal{F}_k| \lesssim \delta_0(V)^{-d} < \infty$. It follows that the quadratic form for $-\Delta_\Omega^N + V$ with domain $H^1(\Omega)$ is well-defined, bounded from below and its form norm is given by the $H^1(\Omega)$-norm. This finishes the proof of the self-adjointness of $-\Delta_\Omega^N + V$. □

*Proof of Lemma 1.8.* As at the beginning of the proof of Lemma 1.7, in the sense of quadratic forms, we have

$$-\Delta_\Omega^N + \frac{1}{K}V \geq \frac{1}{K}\sum_{k=1}^{K}\sum_{D\in\mathcal{F}_k}\left(-\Delta_D^N + V1_D\right), \tag{195}$$

where we used $V \leq 0$ and $\Omega = \bigcup_{k=1}^{K}\bigcup_{D\in\mathcal{F}_k} D$ in the first step. In the second step, we used that for every $k \in \{1, \ldots, K\}$ the oscillatory domains $D \in \mathcal{F}_k$ are disjoint and therefore, $\int_\Omega |\nabla u|^2 \geq \sum_{D\in\mathcal{F}_k}\int_D |\nabla u|^2$ for all $u \in H^1(\Omega)$. By Lemma 1.6(b), we have

$$N\left(-\Delta_D^N + V\right) \leq 1 \tag{196}$$

for every $k \in \{1, \dots, K\}$ and every $D \in \mathcal{F}_k$. Hence, by the min-max principle [18, Theorem 12.1, version 2][2], for every $k \in \{1, \dots, K\}$ and $D \in \mathcal{F}_k$ there exists a function $u^D \in H^1(D) \subset L^2(\Omega)$ such that

$$\int_D |\nabla u|^2 + \int_D V|u|^2 \geq 0 \text{ for all } u \in H^1(\Omega) \text{ with } \int_\Omega \overline{u^D} u = 0 \,. \tag{197}$$

It follows that if $u \in H^1(\Omega)$ is in the orthogonal complement in the $L^2(\Omega)$ sense of $\operatorname{span}\left\{u^D \mid D \in \mathcal{F}_k,\, k = 1, \dots K\right\}$, then

$$\int_\Omega |\nabla u|^2 + \int_\Omega \frac{1}{K} V|u|^2 \geq \frac{1}{K} \sum_{k=1}^K \sum_{D \in \mathcal{F}_k} \int_D \left( |\nabla u|^2 + \int_D V|u|^2 \right) \geq 0 \,. \tag{198}$$

Since the orthogonal complement of $\operatorname{span}\left\{u^D \mid D \in \mathcal{F}_k,\, k = 1, \dots K\right\}$ is a subspace of $L^2(\Omega)$ of dimension at most $\sum_{k=1}^K |\mathcal{F}_k|$, we obtain by the min-max principle [18, Theorem 12.1, version 2] and (195)

$$N\left( -\Delta_\Omega^N + \frac{1}{K} V \right) \leq \sum_{k=1}^K |\mathcal{F}_k| \lesssim \delta_0^{-d} \,, \tag{199}$$

where we used Lemma 1.6(c) in the last step. □

5.2. **Proof of Corollary 1.5.** For the proof of Corollary 1.5, we need the following lemma.

**Lemma 5.1** (A subset of the $\gamma$ with $\beta < 1$)**.** *Let $d \geq 2$.*

*(i) If $\gamma \in \left[\frac{2(d-1)}{2d-1}, 1\right]$, then*

$$\beta = \beta(d, \gamma) = \frac{1}{d+1} \left( \frac{d-1}{\gamma} + 1 \right) \left[ \frac{1}{d} \left( \frac{d-1}{\gamma} + 1 \right)^2 - d \right] < 1. \tag{200}$$

*(ii) If $\gamma \in (0, 1)$ is such that $\beta < 1$, then $\|\cdot\|_{\tilde{p},\beta} \lesssim \|\cdot\|_p$ for all $p > \frac{\tilde{p}}{1-\beta}$ .*

*Proof.* **Proof of (i).** Let $\gamma \in \left[\frac{2(d-1)}{2d-1}, 1\right]$. Note that

$$Y := \frac{1}{d} \left( \frac{d-1}{\gamma} + 1 \right) \leq \frac{1}{d} \left( \frac{2d-1}{2} + 1 \right) = 1 + \frac{1}{2d}, \tag{201}$$

so $Y^2 - 1 \leq \frac{1}{d}\left(1 + \frac{1}{4d}\right)$. Hence, we have

$$\beta = \frac{d^2}{d+1} Y \left[ Y^2 - 1 \right] \leq \frac{d^2}{d+1} \left( 1 + \frac{1}{2d} \right) \frac{1}{d} \left( 1 + \frac{1}{4d} \right) < 1 \,.$$

**Proof of (ii).** Let $q := \frac{p}{\tilde{p}} > \frac{1}{1-\beta} > 1$ and note that $\frac{1}{q'} = 1 - \frac{1}{q} > 1 - (1-\beta) = \beta$, so $\beta q' < 1$. By Hölder's inequality, we get

$$\begin{aligned} |V|_{\tilde{p},\beta}^{\tilde{p}} &= \int_{\bigcup_{l=1}^L \Omega_l} \mathrm{d}x\, h_{x,\min}^{-\beta} |V(x)|^{\tilde{p}} \leq \left( \int_{\bigcup_{l=1}^L \Omega_l} \mathrm{d}x\, h_{x,\min}^{-\beta q'} \right)^{\frac{1}{q'}} \left( \int_{\bigcup_{l=1}^L \Omega_l} \mathrm{d}x\, |V(x)|^{\tilde{p} q} \right)^{\frac{1}{q}} \\ &\lesssim \|V\|_p^{\tilde{p}} \,, \end{aligned}$$

[2] As the proof [18, Theorem 12.1, version 2] shows, in fact, the subspace $M$ need not be a subset of $H^1(\Omega)$ but it suffices to take $M \subset L^2(\Omega)$.

where we used $\beta q' < 1$ and $\tilde{p}q = p$ in the last step. Furthermore, since $|\Omega| \leq 1$ and $p > \tilde{p} \geq \frac{d}{2}$, we can apply Jensen's inequality to get

$$\|V\|_p^p = \int_\Omega |V|^p = |\Omega| \frac{1}{|\Omega|} \int_\Omega \left(|V|^{\frac{d}{2}}\right)^{\frac{2p}{d}} \geq |\Omega| \left(\frac{1}{|\Omega|} \int_\Omega |V|^{\frac{d}{2}}\right)^{\frac{2p}{d}} = |\Omega|^{1-\frac{2p}{d}} \|V\|_{\frac{d}{2}}^p \geq \|V\|_{\frac{d}{2}}^p .$$

For $d = 2$, we have $\|V\|_p \gtrsim \|V\|_{\mathcal{B},\Omega}$ by [19, Chapter 5.1, Theorem 3, p. 155] and $p > \tilde{p} \geq \frac{d}{2} = 1$. Using Definition 2.4(iii), we obtain $\|V\|_{\tilde{p},\beta} \lesssim \|V\|_p$ for any $d \geq 2$. □

*Proof of Corollary 1.5.* By Lemma 5.1, we have $\|\cdot\|_{\tilde{p},\beta} \lesssim \|\cdot\|_p$. Therefore, by Theorem 1.4,

$$\begin{aligned} N\left(-\Delta_\Omega^N + V\right) &\lesssim \left[\min\left(\frac{h_\Omega}{\sqrt{d}}, \|V\|_{\tilde{p},\beta}^{-\frac{1}{2}}\right)\right]^{-d} = \max\left(\left(\frac{h_\Omega}{\sqrt{d}}\right)^{-d}, \|V\|_{\tilde{p},\beta}^{\frac{d}{2}}\right) \\ &\lesssim 1 + \|V\|_{\tilde{p},\beta}^{\frac{d}{2}} \lesssim 1 + \|V\|_p^{\frac{d}{2}} . \end{aligned}$$

□

## 6. Weyl's law for Schrödinger operators (Theorem 1.3)

In this section, we deduce Theorem 1.3 using Theorem 1.2. The main idea is to first reduce to Weyl's law for continuous compactly supported potentials, namely

$$N\left(-\Delta_\Omega^N + \lambda W\right) = (2\pi)^{-d} |B_1(0)| \lambda^{\frac{d}{2}} \int_\Omega |W|^{\frac{d}{2}} + o\left(\lambda^{\frac{d}{2}}\right) \text{ as } \lambda \to \infty. \tag{202}$$

for all $W \in C_c(\Omega)$ with $W \leq 0$. Using (202) combined with the Cwikel-Lieb-Rozenblum type bound (Theorem 1.2), we can then deduce Theorem 1.3.

6.1. **Reduction to compactly supported potentials.** Let $V$ be as in Theorem 1.3. Assume that we have (202) for $0 \geq W \in C_c(\Omega)$. Since $|||\cdot||| := \|\cdot\|_{\tilde{p},\beta}$ is a weighted $L^{\tilde{p}}$-norm on $\Omega$ with $\tilde{p} < \infty$, there exists a sequence $(V_n)_{n\in\mathbb{N}} \subset C_c(\Omega)$ with $V_n \leq 0$ such that

$$\|V_n - V\|_{\frac{d}{2}} \lesssim |||V - V_n||| \to 0 \text{ as } n \to \infty. \tag{203}$$

Let $\delta \in (0, 1)$. Recall that from the min-max principle [18, Theorem 12.1, version 2], one can deduce that

$$N(A + B) \leq N(A) + N(B) \tag{204}$$

for any two self-adjoint operators $A$ and $B$ defined on the same Hilbert space with the same quadratic form domain. We have for every $n \in \mathbb{N}$ and $\lambda > 0$

$$\begin{aligned} N\left(-\Delta_\Omega^N + \lambda V\right) &\leq N\left((1-\delta)\left(-\Delta_\Omega^N\right) + \lambda V_n\right) + N\left(\delta\left(-\Delta_\Omega^N\right) + \lambda\left(V - V_n\right)\right) \\ &= N\left(-\Delta_\Omega^N + \lambda \frac{V_n}{1-\delta}\right) + N\left(-\Delta_\Omega^N + \lambda \frac{V - V_n}{\delta}\right) \\ &\leq N\left(-\Delta_\Omega^N + \lambda \frac{V_n}{1-\delta}\right) + C_\Omega\left(1 + \delta^{-\frac{d}{2}} \lambda^{\frac{d}{2}} |||V - V_n|||^{\frac{d}{2}}\right), \end{aligned} \tag{205}$$

where we used Theorem 1.2 in the last step. Using (203), (205), (202) for $W = \frac{V_n}{1-\delta}$ and the definition of $|||\cdot|||$, we get

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_\Omega^N + \lambda V\right) \\
&\leq \limsup_{\delta\to 0}\limsup_{n\to\infty}\limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \left(N\left(-\Delta_\Omega^N + \lambda\frac{V_n}{1-\delta}\right) + C_\Omega\left(1+\delta^{-\frac{d}{2}}\lambda^{\frac{d}{2}}|||V-V_n|||^{\frac{d}{2}}\right)\right) \\
&\leq \limsup_{\delta\to 0}\limsup_{n\to\infty}\left((2\pi)^{-d}|B_1(0)|\int_\Omega\left|\frac{V_n}{1-\delta}\right|^{\frac{d}{2}} + C_\Omega\delta^{-\frac{d}{2}}|||V-V_n|||^{\frac{d}{2}}\right) \\
&= (2\pi)^{-d}|B_1(0)|\limsup_{\delta\to 0}\int_\Omega\left|\frac{V}{1-\delta}\right|^{\frac{d}{2}} = (2\pi)^{-d}|B_1(0)|\int_\Omega |V|^{\frac{d}{2}},
\end{aligned} \tag{206}$$

which is the desired upper bound. For the corresponding lower bound, we replace $V$ by $(1-\delta)V_n$, and we replace $V_n$ by $(1-\delta)V$ in (205) to get

$$N\left(-\Delta_\Omega^N + \lambda V\right) \geq N\left(-\Delta_\Omega^N + \lambda(1-\delta)V_n\right) - C_\Omega\left(1+\delta^{-\frac{d}{2}}(1-\delta)^{\frac{d}{2}}\lambda^{\frac{d}{2}}|||V-V_n|||^{\frac{d}{2}}\right), \tag{207}$$

and then proceeding as above.

6.2. **Weyl's law for compactly supported potentials.** Now we prove (202) for $0 \geq W \in C_c(\Omega)$. This result can also be found in [8, Theorem 4.29] for the Laplacian on $\mathbb{R}^d$. For the reader's convenience, we explain the proof below since our setting is slightly different. We follow the proof strategy of Weyl, namely we cover the support of $W$ by small cubes of side-length independent of $\lambda$ such that each cube is completely contained in $\Omega$. We then apply Weyl's law for constant potentials on cubes.

Let $m_0 \in \mathbb{N}$ be such that

$$\sqrt{d}2^{-m_0} < \operatorname{dist}\left(\operatorname{supp} W, \partial\Omega\right). \tag{208}$$

Then every cube of side-length at most $2^{-m_0}$ intersecting $\operatorname{supp} W$ is contained in $\Omega$. For every $m \in \mathbb{N}$, $j \in \mathbb{Z}^d$ let

$$Q_j^m := 2^{-m}\left(j + (0,1)^d\right) \tag{209}$$

be the open cube of side-length $2^{-m}$ whose bottom left corner is at $2^{-m}j \in \mathbb{R}^d$. For every $m \in \mathbb{N}$ let

$$J_m := \left\{j \in \mathbb{Z}^d \mid Q_j^m \cap \operatorname{supp} W \neq \emptyset\right\}. \tag{210}$$

**Upper bound.** We claim that for every $m \geq m_0$, we have

$$N\left(-\Delta_\Omega^N + \lambda W\right) \leq \sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^N + \lambda W\right). \tag{211}$$

This can be seen as follows. By the min-max principle [18, Theorem 12.1, Version 2], we know that for every $j \in J_m$ there exists an $N(-\Delta_{Q_j^m}^N + \lambda W)$-dimensional subspace of $L^2(Q_j^m)$, which we call $M_j$, such that

$$\int_{Q_j^m} |\nabla u|^2 + \int_{Q_j^m} \lambda W|u|^2 \geq 0 \tag{212}$$

for all $u \in H^1(Q_j^m)$ that are in the orthogonal complement of $M_j$ with respect to $L^2(Q_j^m)$. Let $M \subset L^2(\Omega)$ be the span of all $M_j$ for $j \in J_m$, where we extend functions in $L^2(Q_j^m)$

by zero on $\Omega \setminus Q_j^m$. Note that $M$ has dimension at most

$$\sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^N + \lambda W\right). \tag{213}$$

By (212), $\Omega \supset \cup_{j\in J_m}\overline{Q_j} \supset \operatorname{supp} W$ and since the cubes $Q_j$ are disjoint, we get for every $u \in H^1(\Omega)$ in the orthogonal complement of $M$ with respect to $L^2(\Omega)$

$$\int_\Omega |\nabla u|^2 + \int_\Omega \lambda W|u|^2 \geq \sum_{j\in J_m}\left(\int_{Q_j^m} |\nabla u|^2 + \int_{Q_j^m} \lambda W|u|^2\right) \geq 0. \tag{214}$$

By the min-max principle [18, Theorem 12.1, version 2], we obtain (211). Therefore, for every $m \in \mathbb{N}$, $m \geq m_0$

$$N\left(-\Delta_\Omega^N + \lambda W\right) \leq \sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^N + \lambda W\right) \leq \sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^N - \lambda \sup_{x\in Q_j^m} |W(x)|\right). \tag{215}$$

By Weyl's law for constant potentials with Neumann boundary conditions on cubes, see for example [8, Theorem 3.20],

$$\begin{aligned}\limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_\Omega^N + \lambda W\right) &\leq \sum_{j\in J_m} \limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_{Q_j^m}^N - \lambda \sup_{x\in Q_j^m} |W(x)|\right) \\ &= \sum_{j\in J_m}\left((2\pi)^{-d}\, |B_1(0)|\, \left|Q_j^m\right| \left(\sup_{x\in Q_j^m} |W(x)|\right)^{\frac{d}{2}}\right).\end{aligned}$$

Since $W \in C_c(\Omega)$, the right hand side agrees with (202) as $m \to \infty$.

**Lower bound.** For every $m \in \mathbb{N}$, $m \geq m_0$, we have

$$N\left(-\Delta_\Omega^N + \lambda W\right) \geq \sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^D + \lambda W\right). \tag{216}$$

For the proof of (216), note that by the min-max principle [18, Theorem 12.1, Version 3] for every $j \in J_m$ there exists an $N\left(-\Delta_{Q_j^m}^D + \lambda W\right)$-dimensional subspace of $H_0^1(Q_j^m)$, which we call $M_j$, such that

$$\int_{Q_j^m} |\nabla u_j|^2 + \int_{Q_j^m} \lambda W|u_j|^2 < 0 \text{ for all } 0 \not\equiv u_j \in M_j. \tag{217}$$

Since the cubes $Q_j^m$, $j \in J_m$ are disjoint and each $M_j \subset H_0^1(Q_j^m) \subset H^1(\Omega)$, if we denote by $M$ the span of all $M_j$, then $M \subset H^1(\Omega)$ is a subspace of dimension

$$\sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^D + \lambda W\right). \tag{218}$$

By $\operatorname{supp} W \subset \cup_{j\in J_m}\overline{Q_j}$ and (217), we obtain for every $0 \not\equiv u = \sum_{j\in J_m} u_j \in M$ with $u_j \in M_j$ for each $j \in J_m$,

$$\int_\Omega |\nabla u|^2 + \int_\Omega \lambda W|u|^2 = \sum_{j\in J_m}\left(\int_{Q_j^m} |\nabla u_j|^2 + \int_{Q_j^m} \lambda W|u_j|^2\right) < 0. \tag{219}$$

By the min-max principle [18, Theorem 12.1, Version 3], we get (216). Using (216) and Weyl's law for constant potentials on cubes [2, 3], we get

$$\liminf_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_\Omega^N + \lambda W\right) \geq \liminf_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \sum_{j\in J_m} N\left(-\Delta_{Q_j^m}^D + \lambda W\right)$$
$$\geq \sum_{j\in J_m} \liminf_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_{Q_j^m}^D - \lambda \inf_{x\in Q_j^m} |W(x)|\right)$$
$$= \sum_{j\in J_m} \left((2\pi)^{-d} |B_1(0)| \left|Q_j^m\right| \left(\inf_{x\in Q_j^m} |W(x)|\right)^{\frac{d}{2}}\right).$$

Taking $m \to \infty$, we conclude (202). The proof of Theorem 1.3 is complete.

**Remark 6.1.** *In dimension $d \geq 3$, we can obtain the lower bound in Theorem 1.3 for all potentials $V \in L^{\frac{d}{2}}(\Omega)$, $V \leq 0$ by comparing with $-\Delta_{\mathbb{R}^d} + \lambda V$ and using Weyl's law for Schrödinger operators on $\mathbb{R}^d$ [8, Theorem 4.46]. However, this is not true in dimension $d = 2$, see [8, Remark after Theorem 4.46].*

## 7. Example with non-semiclassical behaviour (Theorem 1.1)

In this section we prove Theorem 1.1. We explain the proof strategy in Section 7.1 and the details are given in Section 7.2.

7.1. **General strategy.** In this subsection, we explain for fixed $\gamma \in \left(\frac{d-1}{d}, 1\right)$ how to construct a $\gamma$-Hölder domain $\Omega \subset \mathbb{R}^d$ and a potential $V : \Omega \to (-\infty, 0]$ with $V \in L^{\frac{d}{2}}(\Omega)$ such that (5) holds. We construct the $\gamma$-Hölder domain $\Omega$ in the same way as Netrusov and Safarov [10, Theorem 1.10]. The potential $V$ will be chosen such that it grows near the boundary of $\Omega$. This will allow us for certain values of $\lambda$ going to infinity to find significantly more than $\lambda^{\frac{d}{2}}$ negative eigenvalues of $-\Delta_\Omega^N + \lambda V$.

We start by fixing $M := 2^m$, where $m \in \mathbb{N}$ is chosen large enough depending on $\gamma$. The main part of $\Omega$ will be given by the subgraph

$$\left\{\left(x', x_d\right) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x' \in Q^{(d-1)}, 0 < x_d < f(x')\right\} \tag{220}$$

of a $\gamma$-Hölder continuous function $f$ on the $(d-1)$-dimensional unit cube $Q^{(d-1)}$ that vanishes on the boundary of $Q^{(d-1)}$. For every $j \in \mathbb{N}$ we can decompose $Q^{(d-1)}$ into $M^{(d-1)j}$ small cubes of side-length $M^{-j}$. The function $f$ will be chosen in such a way that for any $j \in \mathbb{N}$ it oscillates on the order of magnitude $M^{-j\gamma}$ on each of the small cubes of side-length $M^{-j}$. Intuitively speaking, $f$ looks no better than a $\gamma$-Hölder continuous function on each of the small cubes for every length scale $M^{-j}$, $j \in \mathbb{N}$.

The potential $V$ will be chosen such that it is large close to the boundary of $\Omega$. We define

$$V\left(x', x_d\right) := -c\left(f(x') - x_d\right)^{\frac{2}{d}(-1+\varepsilon)} \text{ for } \left(x', x_d\right) \in \Omega \subset \mathbb{R}^{d-1} \times \mathbb{R} \tag{221}$$

for a suitably chosen $0 < \varepsilon < (d-1)(1/\gamma - 1)$ and a constant $c = c(d, \gamma, \varepsilon) > 0$. Note that $V \in L^{\frac{d}{2}}(\Omega)$ since $\varepsilon > 0$. In the following, for fixed $j \in \mathbb{N}$, we can for simplicity think of $f$ as a $j$-dependent constant $c(j) > 0$ plus a small spike of height $M^{-j\gamma}$ on each of the $M^{(d-1)j}$ $(d-1)$-dimensional small cubes of side-length $M^{-j}$. We denote these small cubes of side-length $M^{-j}$ by $Q(j,k)$, $k \in \left\{1, \ldots, M^{(d-1)j}\right\}$. Moreover, we define

$$\Omega_{j,k} := \left\{\left(x', x_d\right) \in Q(j,k) \times \mathbb{R} \;\middle|\; c(j) < x_d < f(x')\right\}, \tag{222}$$

and $u_{j,k} \in H^1(\Omega)$ by

$$u_{j,k}\left(x', x_d\right) := \sin\left(M^{j\gamma}\left(x_d - c(j)\right)\right) 1_{\Omega_{j,k}}\left(x', x_d\right). \tag{223}$$

Note that for fixed $j \in \mathbb{N}$, the interior of the support the $\{u_{j,k}\}_{k=1}^{M^{(d-1)j}}$ are disjoint. One can show that for

$$\lambda(j) := M^{2\gamma j\left(1+\frac{1}{d}(-1+\varepsilon)\right)} \tag{224}$$

we have for every $k \in \{1, \ldots, M^{(d-1)j}\}$

$$\int_\Omega |\nabla u_{j,k}|^2 + \int_\Omega \lambda V\, |u_{j,k}|^2 < 0. \tag{225}$$

Hence, for every $j \in \mathbb{N}$

$$N\left(-\Delta_\Omega^N + \lambda(j)V\right) \geq M^{(d-1)j}. \tag{226}$$

A computation shows that since $\varepsilon < (d-1)(1/\gamma - 1)$,

$$\lim_{j\to\infty} \lambda(j)^{-\frac{d}{2}} M^{(d-1)j} = \infty. \tag{227}$$

It follows that

$$\limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} N\left(-\Delta_\Omega^N + \lambda V\right) = \infty. \tag{228}$$

7.2. **Details of the proof of Theorem 1.1.** Let us now come to the details of the proof of Theorem 1.1. For the definition of $\Omega$ and of the orthogonal set of test functions, we closely follow [10, Theorem 1.10].

**Definition 7.1.** *Let $d \geq 2$, $\gamma \in \left(\frac{d-1}{d}, 1\right)$ and let $m \in \mathbb{N}$ be large enough such that $m\gamma \geq 1$ and $m(1-\gamma) \geq 4$. Define $Q^{(d-1)} := (0,1)^{d-1}$ and*

$$\psi : \mathbb{R}^{d-1} \to [0, 1/2]\ ,\ x' \mapsto \frac{1}{2} - \left| x' - \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right|_\infty 1_{Q^{(d-1)}}(x').$$

*For every $j \in \mathbb{N}_0$ let*

$$K_j := \left\{0, 1, 2, 3, \ldots, 2^{jm} - 1\right\}^{d-1} \tag{229}$$

*and for $k \in K_j$ define*

$$Q(j,k) := \left\{ x' \in \mathbb{R}^{d-1} \;\middle|\; 2^{jm}x' - k \in Q^{(d-1)} \right\} \subset Q^{(d-1)}. \tag{230}$$

*Define for $j \in \mathbb{N}_0$ the functions*

$$g_j : Q^{(d-1)} \to [0, 1/2]\ ,\ x' \mapsto \sum_{k\in K_j} \psi\left(2^{jm}x' - k\right), \tag{231}$$

*for $n \in \mathbb{N}_0 \cup \{-1\}$*

$$f_n : Q^{(d-1)} \to [0, \infty)\ ,\ x' \mapsto \sum_{j=0}^{n} 2^{-\gamma jm} g_j(x'). \tag{232}$$

*We also denote $f_{-1} \equiv 0$ and $f = \lim_{n\to\infty} f_n$. Also define for every $n \in \mathbb{N}_0$, $k \in K_n$*

$$a_{n,k} := \sup_{x' \in Q(n,k)} f_{n-1}(x'). \tag{233}$$

**Lemma 7.2.** *(i) $f : \left(Q^{(d-1)}, |\cdot|_\infty\right) \to [0, \infty)$ is $\gamma$-Hölder continuous with constant 3.*
*(ii) For every $n \in \mathbb{N}_0$, $k \in K_n$ and $x', y' \in Q(n,k)$, we have*

$$|f_{n-1}(x') - f_{n-1}(y')| \leq \frac{1}{8} 2^{-\gamma mn}. \tag{234}$$

*(iii) For every* $n \in \mathbb{N}_0$*,* $k \in K_n$ *and* $x' \in Q(n,k)$ *with* $2^{nm}x' - k \in [1/4, 3/4]^{d-1}$*, we have*

$$f(x') - a_{n,k} \geq \frac{1}{8} 2^{-\gamma m n}. \tag{235}$$

*(iv) For every* $n \in \mathbb{N}_0$*,* $k \in K_n$ *and* $x' \in Q(n,k)$*, we have*

$$f(x') - a_{n,k} \leq 2^{-\gamma m n}. \tag{236}$$

*Proof.* **Proof of (i).** Let $x'$, $y' \in Q^{(d-1)}$ with $x' \neq y'$ and denote by $n'$ the largest number in $\mathbb{N}_0$ such that $2^{-n'm} \geq |x' - y'|_\infty$. In particular,

$$2^{-(n'+1)m} < \left|x' - y'\right|_\infty \leq 2^{-n'm}. \tag{237}$$

We have

$$\begin{aligned} |f(x') - f(y')| &= \left| \sum_{j=0}^{\infty} 2^{-\gamma m j} g_j(x') - \sum_{j=0}^{\infty} 2^{-\gamma m j} g_j(y') \right| \\ &\leq \sum_{j=0}^{n'} 2^{-\gamma m j} |g_j(x') - g_j(y')| + \sum_{j=n'+1}^{\infty} 2^{-\gamma m j} |g_j(x') - g_j(y')| \end{aligned}$$

For the first term, we use the Lipschitz continuity of $g_j$, (237) and $m(1-\gamma) \geq 4$ by Definition 7.1 to get

$$\begin{aligned} &\sum_{j=0}^{n'} 2^{-\gamma m j} |g_j(x') - g_j(y')| \leq \sum_{j=0}^{n'} 2^{-\gamma m j} \cdot 2^{jm} \left|x' - y'\right|_\infty \\ &\leq \left|x' - y'\right|_\infty^\gamma 2^{-m(1-\gamma)n'} \sum_{j=0}^{n'} 2^{m(1-\gamma)j} \leq \left|x' - y'\right|_\infty^\gamma \sum_{j=0}^{n'} 2^{-m(1-\gamma)j} \leq 2 \left|x' - y'\right|_\infty^\gamma . \end{aligned}$$

For the second term, we use the Lipschitz continuity of $g_j$, (237) and $m\gamma \geq 1$ to get

$$\sum_{j=n'+1}^{\infty} 2^{-\gamma m j} |g_j(x') - g_j(y')| \leq \frac{1}{2} \sum_{j=n'+1}^{\infty} 2^{-\gamma m j} = \frac{1}{2} 2^{-\gamma m(n'+1)} \sum_{j=0}^{\infty} 2^{-\gamma m j} \leq \left|x' - y'\right|_\infty^\gamma .$$

Combining these two estimates, we obtain the claim.

**Proof of (ii).** Let $n \in \mathbb{N}_0$, $k \in K_n$ and $x', y' \in Q(n,k)$. We have

$$\begin{aligned} |f_{n-1}(x') - f_{n-1}(y')| &= \left| \sum_{j=0}^{n-1} 2^{-\gamma m j} g_j(x') - \sum_{j=0}^{n-1} 2^{-\gamma m j} g_j(y') \right| \leq \sum_{j=0}^{n-1} 2^{-\gamma m j} |g_j(x') - g_j(y')| \\ &\leq \sum_{j=0}^{n-1} 2^{-\gamma m j} \cdot 2^{mj} \left|x' - y'\right|_\infty \leq 2^{-mn} \sum_{j=0}^{n-1} 2^{(1-\gamma)mj} \leq \frac{1}{8} 2^{-\gamma m n}, \end{aligned}$$

where we used Lipschitz continuity of $g_j$ in the third step, and $m(1-\gamma) \geq 4$ in the last step.

**Proof of (iii).** First note that for all $y' \in [1/4, 3/4]^{d-1}$, we have $\psi(y') \geq 1/4$. Let $n \in \mathbb{N}_0$, $k \in K_n$ and $x' \in Q(n,k)$ with $2^{nm}x' - k \in [1/4, 3/4]^{d-1}$. Then,

$$g_n(x') = \sum_{\tilde{k} \in K_n} \psi\left(2^{nm}x' - \tilde{k}\right) = \psi\left(2^{nm}x' - k\right) \geq \frac{1}{4}. \tag{238}$$

Therefore, we have

$$\begin{aligned} f(x') - a_{n,k} &\geq f_n(x') - a_{n,k} = f_n(x') - f_{n-1}(x') + f_{n-1}(x') - a_{n,k} \\ &\geq 2^{-\gamma mn} g_n(x') - |f_{n-1}(x') - a_{n,k}| \geq \frac{1}{4} 2^{-\gamma mn} - \frac{1}{8} 2^{-\gamma mn} = \frac{1}{8} 2^{-\gamma mn}, \end{aligned}$$

where we used (238), (ii) and (233) in the fourth step.

**Proof of (iv).** Let $n \in \mathbb{N}_0$, $k \in K_n$, $x' \in Q(n,k)$. Then by (233), $a_{n,k} \geq f_{n-1}(x')$. By $g_j(x') \leq \frac{1}{2}$ and $m\gamma \geq 1$, it follows that

$$f(x') - a_{n,k} \leq f(x') - f_{n-1}(x') = \sum_{j=n}^{\infty} 2^{-\gamma mj} g_j(x') \leq \frac{1}{2} 2^{-\gamma mn} \sum_{j=0}^{\infty} 2^{-\gamma mj} \leq 2^{-\gamma mn}.$$

□

**Definition 7.3** ($\Omega$ and $\Omega_{n,k}$). *Define the $\gamma$-Hölder domain*

$$\Omega := \left\{ x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \;\middle|\; x' \in Q^{d-1},\ 0 \leq x_d < f(x') \right\} \cup \Big( (-2,2)^{d-1} \times (-2,0) \Big)$$

*and for all $n \in \mathbb{N}_0$, $k \in K_n$ define*

$$\Omega_{n,k} := \left\{ x \in \Omega \;\middle|\; x' \in Q(n,k),\ x_d \in \big(f_{n-1}(x'), f(x')\big) \right\}. \tag{239}$$

**Definition 7.4** ($u_{n,k}, b_2, b_\nabla, b_V$ and $V$). *(i) For $n \in \mathbb{N}_0$, $k \in K_n$, let $u_{n,k} : \Omega \to \mathbb{R}$ with*

$$u_{n,k}(x) := \begin{cases} \sin\left(2^{\gamma mn}\left(x_d - a_{n,k}\right)\right) & \text{for } x' \in Q(n,k),\ x_d \geq a_{n,k} \\ 0 & \text{else.} \end{cases} \tag{240}$$

*(ii) Define*

$$b_2 := 2^{-(d-1)} \int_0^{\frac{1}{8}} \mathrm{d}t\, |\sin(t)|^2, \quad b_\nabla := \int_0^1 \mathrm{d}t\, |\cos(t)|^2 \quad \text{and} \quad b_V := 2\frac{b_\nabla}{b_2}. \tag{241}$$

*(iii) Let $\varepsilon \in (0,1)$. Define $V : \Omega \to \mathbb{R}$ depending on $\varepsilon$ by*

$$V(x) := \begin{cases} -b_V \left(f(x) - x_d\right)^{\frac{2}{d}(-1+\varepsilon)} & \text{for } x' \in Q^{(d-1)},\ 0 \leq x_d < f(x') \\ 0 & \text{else.} \end{cases} \tag{242}$$

**Lemma 7.5** (Estimates for $u_{n,k}$). *Let $n \in \mathbb{N}_0$ and $k \in K_n$. Let $\varepsilon \in (0,1)$.*

*(i) Then $u_{n,k} \in H^1(\Omega)$. Moreover,*

$$\int_\Omega |u_{n,k}|^2 \geq b_2 2^{-(d-1)mn} \cdot 2^{-\gamma mn}, \qquad \int_\Omega |\nabla u_{n,k}|^2 \leq b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn}.$$

*(ii) For all $\lambda > 0$, we have*

$$\int_\Omega |\nabla u_{n,k}|^2 + \int_\Omega \lambda V |u_{n,k}|^2 \leq b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn} \left(1 - 2\lambda 2^{-2\gamma mn} \cdot 2^{-\gamma mn \frac{2}{d}(-1+\varepsilon)}\right). \tag{243}$$

*Proof.* **Proof of (i).** A direct computation shows that $u_{n,k} \in H^1(\Omega)$. Next, we compute

$$\begin{aligned} \int_\Omega |u_{n,k}|^2 &= \int_{Q(n,k)} \mathrm{d}x' \int_0^{f(x') - a_{n,k}} \mathrm{d}s\, \left|\sin\left(2^{\gamma mn} s\right)\right|^2 \\ &\geq \int_{\left\{x' \in Q(n,k) \,\middle|\, 2^{mn}x' - k \in [1/4, 3/4]^{d-1}\right\}} \mathrm{d}x' \int_0^{\frac{1}{8}2^{-\gamma mn}} \mathrm{d}s\, |\sin(2^{\gamma mn} s)|^2 \\ &= 2^{-(d-1)mn} \cdot 2^{-(d-1)} \int_0^{\frac{1}{8}} \mathrm{d}t\, |\sin(t)|^2 2^{-\gamma mn} = b_2 2^{-(d-1)mn} \cdot 2^{-\gamma mn}, \end{aligned}$$

where we used Lemma 7.2(iii) in the third step. Moreover, by Lemma 7.2 (iv),

$$\int_\Omega |\nabla u_{n,k}|^2 = \int_{Q(n,k)} \mathrm{d}x' \int_0^{f(x')-a_{n,k}} \mathrm{d}s\, 2^{2\gamma mn} |\cos(2^{\gamma mn} s)|^2$$
$$\leq \int_{Q(n,k)} \mathrm{d}x' \int_0^{2^{-\gamma mn}} \mathrm{d}s\, 2^{2\gamma mn} |\cos(2^{\gamma mn} s)|^2 = b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn},$$

where we used the change of variables $t = 2^{\gamma mn} s$.

**Proof of (ii).** Let $\lambda > 0$. For all $x \in \Omega(n,k)$, we have $x_d \in (f_{n-1}(x'), f(x'))$, so by $0 \leq g_j \leq \frac{1}{2}$ for all $j \in \mathbb{N}_0$, we get as in the proof of Lemma 7.2 (iv)

$$f(x') - x_d \leq f(x') - f_{n-1}(x') = \sum_{j=n}^{\infty} 2^{-\gamma mj} g_j(x') \leq 2^{-\gamma mn}. \tag{244}$$

For all $x \in \operatorname{supp}(V|u_{n,k}|^2) \subset \Omega_{n,k}$, we have by (244) and $\varepsilon < 1$

$$|V(x)| = b_V \left(f(x') - x_d\right)^{\frac{2}{d}(-1+\varepsilon)} \geq b_V 2^{-\gamma mn \frac{2}{d}(-1+\varepsilon)}. \tag{245}$$

Using (ii) and (iii), we get by (241)

$$\int_\Omega |\nabla u_{n,k}|^2 + \int_\Omega \lambda V |u_{n,k}|^2 \leq b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn} - \lambda b_V 2^{-\gamma mn \frac{2}{d}(-1+\varepsilon)} b_2 2^{-(d-1)mn} 2^{-\gamma mn}$$
$$= b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn} \left(1 - 2\lambda 2^{-\gamma mn \frac{2}{d}(-1+\varepsilon)} \cdot 2^{-2\gamma mn}\right).$$

□

**Remark 7.6.** *Lemma 7.5(ii) will be the starting point for the example that satisfies* (5) *we are looking for in this subsection. If we choose for $n \in \mathbb{N}_0$*

$$\lambda := 2^{2\gamma mn} \cdot 2^{\gamma mn \frac{2}{d}(-1+\varepsilon)} \tag{246}$$

*in Lemma 7.5(ii), then for all $k \in K_n$, we have*

$$\int_\Omega |\nabla u_{n,k}|^2 + \int_\Omega \lambda V |u_{n,k}|^2 \leq b_\nabla 2^{-(d-1)mn} \cdot 2^{\gamma mn} \left(1 - 2\lambda 2^{-\gamma mn \frac{2}{d}(-1+\varepsilon)} \cdot 2^{-2\gamma mn}\right) < 0$$

*Now suppose $-\Delta_\Omega^N + \lambda V$ was a self-adjoint operator with quadratic form domain $H^1(\Omega)$. In fact, this will be shown under suitable assumptions in Lemma 7.7. Then, we deduce by the min-max principle [18, Theorem 12.1, version 3] and since the $\{u_{n,k}\}_{k \in K_n}$ have disjoint interior of their support that*

$$N\left(-\Delta_\Omega^N + \lambda V\right) \geq |K_n| = 2^{(d-1)mn}. \tag{247}$$

*If $\varepsilon = (d-1)(1/\gamma - 1)$, then $|K_n| = \lambda^{\frac{d}{2}}$. But we can apply Lemma 7.5 with $0 < \varepsilon < (d-1)(1/\gamma - 1) < 1$ and $\lambda$ as in* (246)*, then we get*

$$\lambda^{-\frac{d}{2}} N\left(-\Delta_\Omega^N + \lambda V\right) \geq \lambda^{-\frac{d}{2}} |K_n| = \lambda^{-\frac{d}{2}} 2^{(d-1)mn} \to \infty \text{ as } n \to \infty. \tag{248}$$

*Since $\lambda \to \infty$ as $n \to \infty$, we have shown* (5).

**Lemma 7.7** (Self-adjointness of $-\Delta_\Omega^N + \lambda V$ and $\|V\|_{\tilde{p},\beta} = \infty$)**.** *Let $\gamma \in \left(\frac{d-1}{d}, 1\right)$.*

*(i) Then there exists $0 < \varepsilon < (d-1)(1/\gamma - 1)$ such that $V \in L^{p^*}(\Omega) \subset L^{\frac{d}{2}}(\Omega)$ and for every $\lambda > 0$ the operator $-\Delta_\Omega^N + \lambda V$ is bounded from below, has finitely many negative eigenvalues and it is self-adjoint with quadratic form domain $H^1(\Omega)$.*

*(ii) For every $0 < \varepsilon < (d-1)(1/\gamma - 1)$, we have $\|V\|_{\tilde{p},\beta} = \infty$.*

**Remark 7.8.** *Lemma 7.7(ii) shows that this example does not contradict Theorem 1.4.*

*Proof.* **Proof of (i).** Let us first find $0 < \varepsilon < (d-1)(1/\gamma - 1)$ such that $V \in L^{p^*}(\Omega) \subset L^{\frac{d}{2}}(\Omega)$. Note that $L^{p^*}(\Omega) \subset L^{\frac{d}{2}}(\Omega)$ since $\Omega$ is bounded and $p^* > \frac{d}{2}$. By (242), the definition of $V$ and the boundedness of $\Omega$, it suffices to find $0 < \varepsilon < (d-1)(1/\gamma - 1)$ such that

$$p^* \frac{2}{d}(-1+\varepsilon) > -1. \tag{249}$$

By a continuity argument, it suffices to show that (249) holds for $\varepsilon = (d-1)(1/\gamma - 1)$, namely $\mu(\mu - (d+1)) > -d$ with $\mu := \frac{d-1}{\gamma} + 1 \in (d, d+1)$, which is true.

Fix $\varepsilon = \varepsilon(d,\gamma)$ as above and let $\lambda > 0$ be arbitrary. In order to show the self-adjointness of the operator $-\Delta_\Omega^N + \lambda V$, we show that the corresponding quadratic form on $H^1(\Omega)$ is well-defined, bounded from below and that it has the $H^1(\Omega)$-norm as its quadratic form norm, hence it is closed. The claim then follows from Friedrich's theorem.

For every $x \in \Omega$ with $x_d \geq 0$ and $\delta > 0$, let the oscillatory domain $D_x(\delta)$ and $a_x = a$ be defined as in Definition 2.6. Let $M := \delta/a$ as in the proof of Lemma 3.4(ii). Then

$$\begin{aligned} \|V\|_{p^*,D_x(\delta)}^{p^*} &\leq a^{d-1} \int_0^\delta \mathrm{d}t \left| b_V t^{\frac{2}{d}(-1+\varepsilon)} \right|^{p^*} = M^{-(d-1)} \delta^{d-1} |b_V|^{p^*} \int_0^\delta \mathrm{d}t\, t^{p^* \frac{2}{d}(-1+\varepsilon)} \\ &= C(d,\gamma) M^{-(d-1)} \delta^{d+p^*\frac{2}{d}(-1+\varepsilon)} \to 0 \quad \text{ as } \delta \to 0. \end{aligned} \tag{250}$$

In the third step we used that $p^*$, $b_V$, $\varepsilon$ only depend on $d$ and $\gamma$. We also used (249) to ensure that the integral is finite.

Let $K = K(d,\gamma)$ be the constant in the covering theorem for oscillatory domains (Lemma 4.1)). By (250), we can choose $\delta > 0$ small enough such that

$$(4\lambda K)^{p^*} C(d,\gamma) \delta^{d+p^*\frac{2}{d}(-1+\varepsilon)} < \left( \frac{1}{C_{PS}} \right)^{p^*}. \tag{251}$$

Note that $\delta$ only depends on $d$, $\gamma$, $\lambda$ but *not* on $x$. For each $x \in \Omega$ with $x_d \geq 0$ let $D_x := D_x(\delta)$ with $\delta$ defined in (251). By (251) and (250),

$$\|4\lambda K V\|_{p^*,D_x}^{p^*} M^{d-1} < \left( \frac{1}{C_{PS}} \right)^{p^*}, \tag{252}$$

so for every $v \in H^1(\Omega)$ with $\int_{D_x} v = 0$, we have by (102)

$$\int_{D_x} |\nabla v|^2 + \int_{D_x} 4K\lambda V |v|^2 \geq 0. \tag{253}$$

Let $\mathcal{F}_1, \ldots, \mathcal{F}_K$ be the families of oscillatory domains we get from applying the covering theorem for oscillatory domains (Lemma 4.1) to $\{D_x\}_{x\in\Omega \text{ with } x_d \geq 0}$. Note that

$$\bigcup_{k=1}^K \bigcup_{D \in \mathcal{F}_k} D \supset \operatorname{supp}(V). \tag{254}$$

Using (253) and (254), we obtain in exactly the same way as in (192) in the proof of Lemma 1.7

$$-\Delta_\Omega^N + \lambda V \geq \frac{1}{2}(-\Delta_\Omega^N) + \frac{1}{2K} \sum_{k=1}^K \sum_{D\in\mathcal{F}_k} (-\Delta_D^N + 2\lambda K V 1_D)$$

$$\geq \frac{1}{2}(-\Delta_\Omega^N) - 2\lambda \left( \sum_{k=1}^K \sum_{D\in\mathcal{F}_k} \frac{1}{|D|}\int_D |V| \right) 1_\Omega.$$

For each $k \in \{1,\dots,K\}$ and $D \in \mathcal{F}_k$, we have

$$|D| \geq \frac{1}{4}\delta a^{d-1} \geq \frac{1}{4}\delta\left(c_0(c_1\delta)^{\frac{1}{\gamma}}\right)^{d-1} = \frac{1}{4}c_0^{d-1}c_1^{\frac{d-1}{\gamma}}\delta^{1+\frac{d-1}{\gamma}}. \tag{255}$$

Since the domains $D \in \mathcal{F}_k$ for each $k \in \{1,\dots,K\}$ are all disjoint and $\Omega$ is bounded, we deduce that $\sum_{k=1}^K |\mathcal{F}_k| < \infty$. Thus, since $V \in L^{p^*}(\Omega) \subset L^1(\Omega)$, we get

$$\sum_{k=1}^K \sum_{D\in\mathcal{F}_k} \int_D |V| \frac{1}{|D|} < \infty, \tag{256}$$

so $-\Delta_\Omega^N + \lambda V$ is a well-defined quadratic form on $H^1(\Omega)$ that is bounded from below and has the $H^1(\Omega)$-norm as its quadratic form norm. We can deduce as in the proof of Lemma 1.8 that $-\Delta_\Omega^N + \lambda V$ has finitely many negative eigenvalues.

**Proof of (ii).** By the definition of $V$ and of $\|V\|_{\tilde{p},\beta}$, it suffices to consider the case $\varepsilon := (d-1)(1/\gamma - 1)$. Recall that

$$\operatorname{supp}(V) = \left\{ x = (x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \,\middle|\, x_d \geq 0 \right\} \cap \Omega. \tag{257}$$

For all $x \in \Omega$ with $x_d \geq 0$ we use the shorthand notation

$$h_x := f(x') - x_d. \tag{258}$$

By the definition of $\|V\|_{\tilde{p},\beta}$, it suffices to show $|V|_{\tilde{p},\beta}^{\tilde{p}} = \infty$. Recall that

$$|V|_{\tilde{p},\beta}^{\tilde{p}} = \int_\Omega \mathrm{d}x\, |V(x)|^{\tilde{p}} h_x^{-\beta} = \int_\Omega |b_V|^{\tilde{p}} h_x^{\tilde{p}\frac{2}{d}(-1+\varepsilon)} h_x^{-\beta}. \tag{259}$$

Now, recall that by (238), we have

$$f(x') = \sum_{j=0}^\infty 2^{-\gamma j m} g_j(x') \geq g_0(x') \geq \frac{1}{4} \tag{260}$$

for all $x' \in [1/4, 3/4]^{d-1}$, so

$$\begin{aligned} |V|_{\tilde{p},\beta}^{\tilde{p}} &= |b_V|^{\tilde{p}} \int_\Omega h_x^{\tilde{p}\frac{2}{d}(-1+\varepsilon)-\beta} = |b_V|^{\tilde{p}} \int_{Q^{(d-1)}} \mathrm{d}x' \int_0^{f(x')} \mathrm{d}x_d\, \left(f(x') - x_d\right)^{\tilde{p}\frac{2}{d}(-1+\varepsilon)-\beta} \\ &\geq |b_V|^{\tilde{p}} \int_{\left[\frac{1}{4},\frac{3}{4}\right]^{d-1}} \mathrm{d}x' \int_0^{f(x')} \mathrm{d}t\, t^{\tilde{p}\frac{2}{d}(-1+\varepsilon)-\beta} \geq |b_V|^{\tilde{p}} 2^{-(d-1)} \int_0^{\frac{1}{4}} \mathrm{d}t\, t^{\tilde{p}\frac{2}{d}(-1+\varepsilon)-\beta}. \end{aligned}$$

Hence, if we can show that

$$\tilde{p}\frac{2}{d}(-1+\varepsilon) - \beta \leq -1, \tag{261}$$

then we have $|V|_{\tilde{p},\beta}^{\tilde{p}} = \infty$, so $\|V\|_{\tilde{p},\beta} = \infty$. Recalling (22), the bound (261) is equivalent to

$$f(\mu) = \frac{1}{d^2}\mu^2(\mu - (d+1)) - \frac{1}{d+1}\mu\left[\frac{1}{d}\mu^2 - d\right] \leq -1 \tag{262}$$

with $\mu := \frac{d-1}{\gamma} + 1 \in (d, d+1)$. Note that

$$f'(\mu) = \frac{3\mu^2}{d^2(d+1)} - \frac{2(d+1)}{d^2}\mu + \frac{d}{d+1}$$

is a convex function with $f'(d) < 0$ and $f'(d+1) < 0$. Therefore, $f$ is monotone decreasing in $(d, d+1)$, and hence (262) follows from the fact that $f(d) = -1$. The proof of Lemma 7.7 is complete. □

*Proof of Theorem 1.1.* Remark 7.6 combined with Lemma 7.7(i) show Theorem 1.1. □

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# Weyl formulae for some singular metrics with application to acoustic modes in gas giants

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**Abstract**

This paper is motivated by recent works on inverse problems for acoustic wave propagation in the interior of gas giant planets. In such planets, the speed of sound is isotropic and tends to zero at the surface. Geometrically, this corresponds to a Riemannian manifold with boundary whose metric blows up near the boundary. Here, the spectral analysis of the corresponding Laplace-Beltrami operator is presented and the Weyl law is derived. The involved exponents depend on the Hausdorff dimension which, in the supercritical case, is larger than the topological dimension.

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## 1 Introduction

### 1.1 Seismology on gas giant planets

Seismology has played an important role in revealing the (deep) interiors of gas giant planets in our solar system [6, 24]. Indeed, the acoustic spectra and free oscillations have been studied for Saturn and Jupiter over the past few decades [33, 20, 18]. The excitation of acoustic modes in gas giant planets presumably occurs through convection in their interiors. The observation of acoustic eigenfrequencies, that is, the discrete spectrum can be realized, in principle, through visible photometry, thermal infrared photometry, Doppler spectrometry, and ring seismology for nonradial oscillations [25, 26] (in particular, in the case of Saturn). In ring seismology and with the Cassini mission, one measured the "resonances" in the inner C ring of Saturn with visual and infrared mapping spectrometer (VIMS) stellar occultations [22, 14, 17]. The rings are gravitationally coupled to the acoustic modes of the planet (taking self gravitation into account). Detection of Jupiter's acoustic eigenvalues has been attempted with ground-based imaging-spectrometry (seismographic imaging interferometer for monitoring of planetary atmospheres or SYMPA) by measuring line of sight velocity [32, 18]. Recently, Juno spacecraft gravity measurements have provided evidence for normal modes of Jupiter [15].

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### 1.2 Singular Riemannian metrics

On a gas giant planet, unlike a rocky planet, the speed of sound goes to zero at the boundary. In the geometric mathematical model that we employ hereafter, the rate at which this happens follows a power law which determines a specific conformal blow-up rate of a Riemannian metric, thus defining a singular metric. This rate happens to be slower than on asymptotically hyperbolic manifolds and the boundary is at a finite distance from interior points. The rate is implied by an equation of state in the upper part of the planet, in general, in the sense of a fit. (For some models of the speed of sound of Jupiter and Saturn showing this behavior, see [20, Figure 1] and [23, Figure 1].) Only for a polytrope is the rate exact. Polytropes, for which the pressure is proportional to a power of the density of mass, have been viewed as relevant simplifications; models with variable polytrope index have indeed been applied to planet and material models [34]. Typically, an equation of state is computed numerically using density functional molecular dynamics simulations with mixtures of chemical elements: The dominant elements in terms of mass fraction are hydrogen and helium, but also heavy elements are important. The equation of state is different for the upper part and the deep interior as the helium fraction can be higher in the interior due to helium rain (helium becoming immiscible with hydrogen at high pressure). Equations of state play a vital role in the evolution and realization of structure of gas giant planets [28, 29].

More specifically, if $g_e$ is the Euclidean Riemannian metric on a smooth domain $X \subset \mathbb{R}^{n+1}$, then the speed of sound $c$ can be encoded by the conformally Euclidean Riemannian metric $g = c^{-2} g_e$. In local coordinates where the boundary of $X$ is (locally) described by $u = 0$, the polytropic model suggests that $c \sim u^{1/2}$. Indeed, the natural generalization is $c \sim u^{\alpha/2}$, that is, $c^{-2} \sim u^{-\alpha}$; through previous analysis [13] it appears that restricting $\alpha$ according to $\alpha \in (0, 2)$ guarantees the presence of a discrete spectrum as it has been observed. Thus, the Riemannian geometry lies between standard geometry with boundary and asymptotically hyperbolic geometry. Some of the phenomena in this geometry are unlike those seen at either end. The extreme case $\alpha = 0$ corresponds physically to solid bodies and mathematically to manifolds with boundary, and the other extreme $\alpha = 2$ corresponds to asymptotically hyperbolic geometry but is far from all planetary models.

Therefore, following [13, Section 1.1], we model a gas giant planet as a smooth manifold $X$ with a boundary, endowed with a Riemannian metric $g$ on $X \setminus \partial X$ such that, near $\partial X$, we have $g = \bar{g}/u^\alpha$ where $\bar{g}$ is a well-defined Riemannian metric up to the boundary, and $\partial X = \{u = 0\}$ locally. The fact that $\bar{g}$ is neither zero nor infinite at $\partial X$ implies a specific blow-up rate for $g$ near $\partial X$. This conformal power-law blow-up is the key geometric feature of gas giant metrics. The speed of sound might contain jump discontinuities where phase transitions occur (see [27]), that is, the metric can contain conormal singularities while the manifold consists of multiple "layers". A key interior boundary in gas giants corresponds with the transition from molecular to metallic hydrogen. Accounting for discontinuities in an asymptotic formalism for gas giant seismology was developed a few decades ago (see [30]).

The mathematical study of the spectrum associated with gas giants' acoustic modes was initiated in [13]. In this paper, we analyze the relevant Laplace-Beltrami operator and we compute the Weyl law. The study of Weyl asymptotics, which reflects some properties of the singular metric, is a preliminary step towards analyzing some inverse problems, in view of reconstructing some features of the internal structure of gas giant planets.

# 2 Mathematical model and main results

## 2.1 Mathematical model

Let $X$ be a smooth compact manifold of dimension $n+1$ with a boundary $\partial X$. Near $\partial X$, $X$ is diffeomorphic to $[0,1)\times M$, where $M$ is a smooth compact manifold of dimension $n \geqslant 1$ and $\partial X$ is identified with $\{0\}\times M$ and also with $u=0$ where $u$ is a transverse coordinate, locally near $\partial X$, ranging over $[0,1)$. As discussed in Section 1.2, we consider on $X$ a singular Riemannian metric $g$ that is a smooth metric on $X \setminus \partial X$, written near $\partial X$ as

$$g = \bar{g}/u^{\alpha}$$

where $0<\alpha<2$ and $\bar{g}$ is a smooth (non-singular) Riemannian metric on $X$, up to the boundary. Following [13, Proposition 2], which uses a normal form for the metric near the boundary, due to [19, Lemma 5.2], we have

$$g = u^{-\alpha}(du^2 + g_0(u))$$

where $g_0(u)$ is a smooth Riemannian metric on $M$ (pulled back to the level set $u = \mathrm{Cst}$) depending smoothly on $u \in [0,1)$.

We make a change of variable. Setting $x = x(u) = \int_0^u s^{-\alpha/2} ds = (1-\frac{\alpha}{2})^{-1} u^{1-\frac{\alpha}{2}}$, we get

$$g = dx^2 + x^{-\beta} g_1(x) \text{ where } \beta = \frac{2\alpha}{2-\alpha} \tag{1}$$

and $g_1 = g_1(x)$ is a smooth Riemannian metric on $M$ (pulled back to the level set $x = \mathrm{Cst}$) depending smoothly on $x \in [0,1)$. We note that, since $\alpha \in (0,2)$, $\beta$ can take any positive value. We also note that a polytrope (for any index) corresponds to $\beta = \beta_{\mathrm{poly}} = 2$. We have that $g_1(x) = C(\alpha) g_0(u)$ for some constant $C(\alpha) > 0$.

For any $x \in [0,1)$, denoting by $dv_1^x$ the volume measure on $M$ associated to the metric $g_1(x)$, the $g$-volume is $dv_g = x^{-\beta n/2}\,|dx|\,dv_1^x$. The volume is finite if and only if $\beta < \beta_c$, where

$$\beta_c = \frac{2}{n} \tag{2}$$

is a critical value of $\beta$. We will see later that this critical value plays a role in the Weyl asymptotics. At this point, we can note that $\beta_{\mathrm{poly}} > \beta_c$ for $n = 2$.

The following three propositions were proved in [13]. The first proposition concerns the Haussdorff dimenion.

**Proposition 1.** *The Hausdorff dimension of $(X,g)$ is*

$$d_H = \max\left(n+1, n\left(1+\frac{\beta}{2}\right)\right).$$

We define $\delta_H = n(1+\frac{\beta}{2})$, and note that $d_H > n+1$ ($n+1$ is the topological dimension of $X$) if and only if $\beta > \beta_c$. We give in Appendix A.5 a sketch of the proof of Proposition 1, in which we also show that $d_H$ coincides with the Minkowski dimension of $(X,g)$.

**Proposition 2.** *The Laplace-Beltrami operator $\triangle_g$, with core $C_0^\infty(X \setminus \partial X)$, is essentially self-adjoint if and only if $\beta \geqslant \beta_c$.*

For $\beta < \beta_c$, there exist several extensions of $\triangle_g$, with core $C_0^\infty(X\setminus\partial X)$. In the further analysis, we consider its Friedrichs extension (that is, “Dirichlet extension”).

**Proposition 3.** *For every $\beta > 0$, the spectrum of $\triangle_g$ is discrete.*

We denote the eigenvalues of $\triangle_g$ by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$ with associated eigenfunctions $\phi_j$, $j \in \mathbb{N}^*$, normalized in $L^2(X, dv_g)$. We note that, if the volume of $X$ is infinite, i.e., if $\beta \geqslant \beta_c$, we have $\lambda_1 > 0$, in contrast to the usual Riemannian case. The Weyl counting function is defined by

$$N(\lambda) = \#\{j \in \mathbb{N}^* \ \mid \ \lambda_j \leqslant \lambda\}$$

where the eigenvalues are counted with their multiplicity. Our objective is to derive a Weyl law describing the asymptotics of $N(\lambda)$ as $\lambda \to +\infty$.

**Remark 1.** The following fact will be used in Section 3.5. For any $\varepsilon > 0$, there exists $\delta > 0$ such that the metric $g$ is *$\varepsilon$-quasi-isometric* (see Appendix A.1) to a singular Riemannian metric $\tilde{g}$ on $X$, smooth on $X \setminus \partial X$ and given by $\tilde{g} = dx^2 + x^{-\beta} g_1(0)$ on $(0, \delta) \times M$. In order to derive a Weyl law for $(X, g)$ it suffices to derive the corresponding Weyl law for $\tilde{g}$ for any $\varepsilon > 0$ (see, again, Appendix A.1 for details). This remark is important, because it implies that we mainly have to work within the so-called *separable* case.

**Separable case.** We say that we are in the *separable* case if the metric $g_1(x)$ on $M$ (defined by (1)) does not depend on $x$, i.e., $g_1(x) = g_1(0)$ for any $x \in (0, 1)$; we still denote this metric by $g_1$. In the sequel, we consider $[0, 1) \times M$ instead of $[0, \delta) \times M$ for simplicity of notation, while the proofs are similar in both cases.

We denote by $\triangle_M$ the Laplace-Beltrami operator on $(M, g_1)$. We denote the eigenvalues of $\triangle_M$ by $0 \leqslant \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_j \leqslant \cdots$ with an associated orthonormal basis of eigenfunctions $(\psi_j)_{j \in \mathbb{N}^*}$. The Weyl counting function for $\triangle_M$ is defined by

$$N_M(\omega) = \#\{k \in \mathbb{N}^* \ \mid \ \omega_k \leqslant \omega\}.$$

Since $g_1$ is a smooth Riemannian metric on $M$, the classical Weyl law for $(M, g_1)$ yields that $N_M(\omega) = \gamma_n \mathrm{Vol}_{g_1}(M) \omega^{n/2} + \mathrm{O}\left(\omega^{n/2}\right)$ as $\omega \to +\infty$ where

$$\gamma_n = \frac{1}{(4\pi)^{n/2} \Gamma\left(\frac{n}{2} + 1\right)} \tag{3}$$

(see [3, Chapter 3E] for the heat trace and then apply the Karamata tauberian theorem, i.e., Theorem 3 in Appendix A.2).

Denoting by $dv_1$ the volume measure on $M$ associated to the metric $g_1 = g_1(0)$, the $g$-volume is $dv_g = x^{-\beta n/2} \, |dx| \, dv_1$. Making the change of function $f \mapsto x^{-\beta n/4} f$, we get the new volume form $|dx| \, dv_1$; the Laplace-Beltrami operator on $X_1 = (0, 1) \times M$ is now given by

$$\triangle_g = -\partial_x^2 + \frac{C_\beta}{x^2} + x^\beta \triangle_M$$

where $x \in (0, 1)$ and

$$C_\beta = \frac{\beta n}{4} \left( \frac{\beta n}{4} + 1 \right).$$

The proof is straightforward by performing an integration by parts with respect to $x$ in the Dirichlet form defining the Laplace-Beltrami operator, using the Dirichlet boundary condition at $x = 0$. We note that $C_\beta \geqslant 3/4$ if and only if the volume of $X$ is infinite. Using the Weyl criterion (see Appendix A.3), this inequality also implies that $P_\omega$ defined by (5) below is essentially self-adjoint for any $\omega > 0$, but not for $\omega = 0$.

We will need to work first on the non-compact conic manifold $X_\infty = (0, +\infty) \times M$ endowed with the metric $g = dx^2 + x^{-\beta} g_1$. Let $\triangle_\infty$ stand for the Laplace-Beltrami operator on $(X_\infty, g)$. Invoking a separation of variables, we have

$$\triangle_\infty = \bigoplus_{k=1}^{+\infty} (\mathrm{id} \otimes \pi_k)\,(P_{\omega_k} \otimes \mathrm{id})\,(\mathrm{id} \otimes \pi_k) \tag{4}$$

where

$$P_\omega = -\partial_x^2 + \frac{C_\beta}{x^2} + \omega x^\beta \tag{5}$$

is a Schrödinger operator on $L^2((0, +\infty), dx)$ for any $\omega \geqslant 0$, and where $\pi_k$ is the orthogonal projection of $L^2(M, dv_1)$ onto the subspace generated by $\psi_k$ and id denotes the identity operator on $L^2(X_\infty, |dx|\, dv_1)$ (resp., on $L^2(M, dv_1)$). Hence, $\triangle_\infty$ is unitarily equivalent to $\bigoplus_{k=1}^{+\infty} P_{\omega_k}$.

## 2.2 Main results

Recalling that $g_1$ is defined by (1), we set $G = g_1(0)$ and denote by $v_G$ the corresponding volume form on $M$. We also recall that $\beta_c$ is defined in (2) and that $\gamma_n$ is defined in (3).

**Theorem 1. (Weyl asymptotics)**

- *If* $\beta > \beta_c$ *then*
$$N(\lambda) \sim A(\beta, n) v_G(M)\, \lambda^{d_H/2}$$
*as* $\lambda \to +\infty$*, with*
$$A(\beta, n) = \frac{n\gamma_n(\beta + 2)}{4\Gamma(1 + d_H/2)} \int_0^{+\infty} \mathcal{Z}_1(\tau) \tau^{\frac{d_H}{2} - 1} d\tau$$
*where* $\mathcal{Z}_1(\tau) = \mathrm{Tr}(\exp(-\tau P_1))$ *and* $P_1$ *is the Schrödinger operator on* $L^2((0, +\infty), dx)$ *defined by* (5).

- *If* $\beta = \beta_c = 2/n$ *then*
$$N(\lambda) \sim C_n v_G(M)\, \lambda^{(n+1)/2} \ln \lambda$$
*as* $\lambda \to +\infty$*, with*
$$C_n = \frac{1}{(n + 1)(4\pi)^{(n+1)/2}\Gamma\left((n + 1)/2\right)}.$$
*In particular,* $C_1 = 1/8\pi$.

- *If* $\beta < \beta_c$ *then*
$$N(\lambda) \sim \gamma_{n+1} v_g(X)\, \lambda^{(n+1)/2}$$
*as* $\lambda \to +\infty$.

**Remark 2.**

- When $M = \mathbb{R}/2\pi\mathbb{Z}$ and $X$ is diffeomorphic to the hemisphere, endowed with the so-called Grushin metric, the authors of [4] derived the Weyl law using an explicit computation of the spectrum. We recover their result as a particular case with $n = 1$ and $\beta = \beta_c = 2$.

- To prove Theorem 1, we make use of heat kernels. Alternatively, it is possible to use Dirichlet-Neumann bracketing. Both methods allow to treat conormal jump singularities of the metric $\bar{g}$ inside $X$ that model layering in the gas planet.

**Remark 3.** A natural question is whether $\beta$ can be determined from the Weyl asymptotics. Indeed, when $n$ is known, $\beta$ can be determined in the case where $\beta \geqslant \beta_c$. When $\beta < \beta_c$ the question remains open. To shed light on this, it would be useful and interesting to get the next term in the small-time heat trace expansion (see Section 3.2.2) when $\beta \leqslant \beta_c$.

We next compute the *Weyl measures*, which are the probability measures $w_g$ on $X$, defined, if the limit exists, by

$$\int_X f\,dw_g = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j\leqslant\lambda} \int_X f|\phi_j|^2\,dv_g$$

for any function $f : X \to \mathbb{R}$ that is continuous up to the boundary of $X$. Such measures have been introduced in [9, 11] in the framework of sub-Riemannian geometry in order to provide an account of how the high-frequency eigenfunctions concentrate.

**Theorem 2. (Weyl measures)**

- *If $\beta \geqslant \beta_c$ then the Weyl measure is $\delta_{x=0} \otimes dv_G/v_G(M)$.*
- *If $\beta < \beta_c$ then the Weyl measure is the uniform probability distribution given by the normalized volume of $(X, g)$, that is $dv_g/v_g(X)$.*

Using [11, Corollary 7.1], we obtain the following consequence.

**Corollary 1.** *If $\beta \geqslant \beta_c$ then there exists a density-one subsequence $(\phi_{j_k})_{k\in\mathbf{N}^*}$ of the sequence of eigenfunctions that concentrates on $\partial X$, meaning that for any compact subset $K \subset X \setminus \partial X$, we have*

$$\lim_{k\to+\infty} \int_K |\phi_{j_k}|^2\,dv_g = 0.$$

# 3 Proofs of Theorems 1 and 2

Our strategy of proof is the following. We first treat the separable case (Sections 3.1 to 3.4). As a preliminary, we perform in Section 3.1 a spectral study of the 1D Schrödinger operator $P_\omega$ defined by (5), deriving exponential estimates for truncated heat traces. Then, in Section 3.2, we estimate the small-time asymptotics of the truncated heat trace of $\triangle_g$, near the boundary (actually, on a cone); the three cases $\beta > \beta_c$, $\beta = \beta_c$, $\beta \leqslant \beta_c$, must be treated in different ways. In Section 3.3, using a heat parametrix, we glue together the heat kernel near the boundary and the Riemannian heat kernel far from the boundary. Finally in Section 3.4 we prove Theorem 1 in the separable case.

In Section 3.5, we show how to pass from the separable to the general case by using the fact that the metric $g$ is quasi-isometric to a separable metric. In Section 3.6, we prove Theorem 2. Our approach uses again heat traces.

## 3.1 Spectral study of the 1D Schrödinger operators

We consider the family of Schrödinger operators,

$$P_1 = -\partial_x^2 + q_{C,\beta}(x)$$

where $q_{C,\beta}(x) = Cx^{-2} + x^\beta$, $C > 0$ and $\beta > 0$, acting on $L^2((0, +\infty), dx)$. The operators $P_1$ are essentially self-adjoint if and only if $C \geqslant 3/4$; when $C < 3/4$ we consider the Friedrichs extension of $P_1$ with core $C_0^\infty((0, +\infty))$ (see Appendix A.3). The spectrum of $P_1$ is discrete; we denote it by

$0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \cdots$. We derive precise semi-classical asymptotics for the associated truncated heat trace.

Let $\chi : [0,+\infty) \to [0,1]$ be a smooth decreasing function with $\chi \equiv 1$ on $[0,a]$ with $a > 0$ and $\chi' \leqslant 0$ everywhere. We note that $\chi \equiv 1$ is included. We define the corresponding truncated heat trace by

$$\mathcal{Z}_\chi(\tau) = \operatorname{Tr}\left(e^{-\tau P_1}\chi\right) \qquad \forall \tau > 0.$$

Let $\gamma = \max(1/\beta, 1/2)$.

**Proposition 4.** *Given any $0 < \tau \leqslant 1$, we have*

$$\mathcal{Z}_\chi(\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x)dx + \mathrm{O}\left(\tau^{-\gamma}\right)$$

*and for $\tau \geqslant 1$,*

$$\mathcal{Z}_\chi(\tau) = \mathrm{O}\left(e^{-\mu_1 \tau}\right)$$

*uniformly with respect to $\chi$ in both cases.*

*The counting function $N_1(\mu) = \#\{j \in \mathbb{N}^* \mid \mu_j \leqslant \mu\}$ satisfies $N_1(\mu) \sim A\mu^{\frac{1}{2}+\frac{1}{\beta}}$ as $\mu \to +\infty$ with $A = \sqrt{\frac{2}{\pi}}\frac{1}{\beta}B(3/2, 1+1/\beta)$ where $B$ is the Beta function.*

*Proof of Proposition 4.* We first establish an elementary lemma. We denote by $Q^N$ (resp., $Q^D$) the self-adjoint operator $-\partial_x^2$ on an interval of length 1 with Neumann (resp., Dirichlet) boundary condition.

**Lemma 1.** *For $0 < \tau \leqslant 1$ and $\star \in \{D, N\}$, we have* $\operatorname{Tr}(\exp(-\tau Q^\star)) = (4\pi\tau)^{-1/2} + \mathrm{O}(1)$.

*Proof of Lemma 1.* The estimate does not depend on the chosen interval. The spectrum of $Q^N$ is $\{n^2\pi^2 \mid n \in \mathbb{N}\}$ and the spectrum of $Q^D$ is $\{n^2\pi^2 \mid n \in \mathbb{N}^*\}$. Hence both traces differ by 1, and it suffices to prove the estimate for $Q^N$. Writing

$$\operatorname{Tr}(\exp(-\tau Q^N)) = \frac{1}{2}\left(1 + \sum_{n\in\mathbb{Z}} e^{-\tau n^2\pi^2}\right)$$

and applying the Poisson summation formula gives the result. ∎

We now prove the proposition. We first consider the case where $\tau \leqslant 1$. We are going to apply Dirichlet-Neumann bracketing with the decomposition $(0,+\infty) = \cup_{j=0}^{+\infty} J_k$ where the intervals $J_k$ are defined below.

Let $x_0$ be defined by $q_{C,\beta}(x_0) = \min q_{C,\beta}(x)$. Then $q_{C,\beta}(x) = Cx^{-2} + x^\beta$ is increasing on $[x_0, +\infty)$. Let $J_k = [x_0 + k, x_0 + k + 1]$ with $k \geqslant 1$ and $J_0 = ]0, x_0 + 1]$. We have the following estimates for the Dirichlet and Neumann heat traces $\mathcal{Z}^\star_{k,\chi}$ on $J_k$: for $k \geqslant 1$,

$$\mathcal{Z}^N_{k,1}(\tau) \leqslant \left(\frac{1}{\sqrt{4\pi\tau}} + \mathrm{O}(1)\right) e^{-\tau q(x_0+k)} \leqslant \left(\frac{1}{\sqrt{4\pi\tau}} + \mathrm{O}(1)\right) \int_{x_0+k-1}^{x_0+k} e^{-\tau q_{C,\beta}(x)}dx$$

and

$$\begin{aligned}\mathcal{Z}^D_{k,\chi}(\tau) &\geqslant \left(\frac{1}{\sqrt{4\pi\tau}} + \mathrm{O}(1)\right) e^{-\tau q(x_0+k+1)}\chi(x_0+k+1) \\ &\geqslant \left(\frac{1}{\sqrt{4\pi\tau}} + \mathrm{O}(1)\right) \int_{x_0+k+1}^{x_0+k+2} e^{-\tau q_{C,\beta}(x)}\chi(x)dx,\end{aligned} \tag{6}$$

while

$$\mathcal{Z}_0^N(\tau) = \mathrm{O}(\tau^{-1/2}).$$

The minimax principle implies that each eigenvalue $\mu_j$ is larger than the $j^{\text{th}}$-eigenvalue of the union for all $k$ of the Neumann problem on the intervals $J_k$. In this way, we obtain the following upper bound for the trace with $\chi = 1$:

$$\mathcal{Z}_1(\tau) \leqslant \mathrm{O}\left(\frac{1}{\sqrt{\tau}}\right) + \left(\frac{1}{\sqrt{4\pi\tau}} + \mathrm{O}(1)\right) \int_{x_0}^{+\infty} e^{-\tau q_{C,\beta}(x)} dx. \tag{7}$$

Noting that

$$\left| \int_0^{+\infty} \left(e^{-\tau q_{C,\beta}(x)} - e^{-\tau x^\beta}\right) dx \right| = \mathrm{O}(1),$$

we infer that

$$\mathcal{Z}_1(\tau) \leqslant \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} dx + \mathrm{O}\left(\tau^{-\gamma}\right).$$

Similarly, using the fact that the Dirichlet heat kernel is smaller than the global kernel (see [12, Theorem 2.1.6]), we get from (6) the following lower bound:

$$\mathcal{Z}_\chi(\tau) \geqslant \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx + \mathrm{O}\left(\tau^{-\gamma}\right). \tag{8}$$

Note that the same lower bound is valid when replacing $\chi$ by $1 - \chi$.

Now, we use a variant of the fact that

$$(A + B = A' + B',\ A \geqslant A',\ B \geqslant B') \quad \Rightarrow \quad (A = A',\ B = B').$$

We take $A = \mathcal{Z}_\chi$, $B = \mathcal{Z}_{1-\chi}$, $A' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx$ and $B' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} (1 - \chi(x)) dx$. By (7), we have

$$A + B = A' + B' + \mathrm{O}\left(\tau^{-\gamma}\right)$$

and, from (8),

$$A \geqslant A' + \mathrm{O}\left(\tau^{-\gamma}\right),\ B \geqslant B' + \mathrm{O}\left(\tau^{-\gamma}\right).$$

It follows that $A = A' + \mathrm{O}\left(\tau^{-\gamma}\right)$ and $B = B' + \mathrm{O}\left(\tau^{-\gamma}\right)$. In particular,

$$\mathcal{Z}_1(\tau) \sim \frac{1}{(4\pi\tau)^{\frac{1}{2}}} \int_0^{+\infty} e^{-\tau x^\beta} dx.$$

Using the Karamata tauberian Theorem (recalled in Appendix A.2), we get

$$N_1(\mu) \sim A\mu^{\frac{1}{2} + \frac{1}{\beta}}$$

as $\mu \to +\infty$, with the constant $A$ defined in Proposition 4.

We now prove the exponential upper bound for $\tau \geqslant 1$. We note that $\mathcal{Z}_\chi(\tau) \leqslant \mathcal{Z}_1(\tau)$. Moreover all eigenvalues $\mu_j$ of $P_1$ are larger than the minimum $q(x_0)$ of $q$. Then, for $\tau \geqslant 1$, we have

$$\mathcal{Z}_1(\tau) = e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-\tau(\mu_j - \mu_1)} \leqslant e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-(\mu_j - \mu_1)}.$$

The Weyl law applied to $P_1$ implies that the sum at the right-hand side converges and thus $\mathcal{Z}_\chi(\tau) \leqslant ce^{-\tau\mu_1}$ for some $c > 0$. □

**Corollary 2.** *There exists $C_1 > 0$ only depending on $C$ and $\beta$ but not on $\chi$, such that, setting $J = \int_0^{+\infty} \chi(x)dx$, we have, for every $\tau > 0$,*

$$\mathcal{Z}_\chi(\tau) \leqslant \frac{C_1}{\sqrt{\tau}} \min\left(J, \tau^{-1/\beta}\right).$$

*Proof.* The bound involving $J$ follows from the estimate $e(\tau, x, x) \leqslant (4\pi\tau)^{-\frac{1}{2}}$ which is valid for any positive potential $q$ (see [12, Theorem 2.1.6]). The other bound follows from Proposition 4 and from the estimate $\int_0^{+\infty} e^{-\tau x^\beta} dx = \mathrm{O}\left(\tau^{-1/\beta}\right)$. □

Given any $\omega > 0$, we set

$$P_\omega = -\partial_x^2 + Cx^{-2} + \omega x^\beta. \tag{9}$$

**Proposition 5.** *For any $\omega > 0$, the operator $P_\omega$ is unitarily equivalent to $\omega^{2/(2+\beta)} P_1$. In particular, the spectrum of $P_\omega$ is $\omega^{2/(2+\beta)}$ times the spectrum of $P_1$.*

*Proof.* Considering the unitary map $U : L^2(\mathbb{R}^+, dx) \to L^2(\mathbb{R}^+, dx)$ defined by

$$Uf(x) = \omega^{\frac{1}{2(2+\beta)}} f(\omega^{\frac{1}{2+\beta}} x),$$

we find that $U^\star P_\omega U = \omega^{2/(2+\beta)} P_1$. □

## 3.2 Truncated heat asymptotics for the cone $X_\infty$

In this subsection, we compute the small-time asymptotics of the truncated heat trace,

$$Z_{\infty,\chi}(t) = \mathrm{Tr}\left(e^{-t\triangle_g}\chi\right)$$

where $\chi$ is as in Section 3.1 and moreover is compactly supported in $[0, +\infty)$, and $g$ is the metric $g = dx^2 + x^{-\beta} g_1$ on the cone $X_\infty = (0, +\infty) \times M$. The manifold $M$ is equipped with the metric $g_1$ that is independent of $x$. Here, we do not assume that $\partial M$ is empty: this will be useful in the proof of Theorem 2. We will only use the Weyl asymptotics on $M$.

Using the direct sum decomposition given in (4), we have

$$Z_{\infty,\chi}(t) = \sum_{k=1}^{+\infty} \mathrm{Tr}\left(e^{-tP_{\omega_k}}\chi\right) \tag{10}$$

where we recall that the $\omega_k$ are the eigenvalues of $\triangle_M$ and $P_\omega$ is defined by (9).

We make the following two preliminary observations:

- For $k$ fixed and $t \to 0^+$, we have $\mathrm{Tr}(e^{-tP_{\omega_k}}\chi) = \mathrm{O}(t^{-1/2})$. This term will be negligible in the sequel because the global trace is not less than $C/t$ for some $C > 0$ (since $\dim(X) \geqslant 2$).
- For $t > 0$ fixed, the smooth function $f : \omega \mapsto \mathrm{Tr}(e^{-tP_\omega}\chi)$ has a fast decay at infinity: by Proposition 5,
$$f(\omega) \leqslant \mathrm{Tr}(e^{-tP_\omega}) = \mathrm{Tr}(e^{-t\omega^{2/(2+\beta)}P_1}).$$
The claim then follows from the second assertion given in Proposition 4.

We split the sum (10) into two parts,

$$Z_{\infty,\chi}(t) = \sum_{\omega_k < 1} + \sum_{\omega_k \geqslant 1} = Z^0_{\infty,\chi}(t) + Z^1_{\infty,\chi}(t).$$

The first part, $Z^0_{\infty,\chi}(t)$, is $\mathrm{O}(t^{-1/2})$ by the first preliminary observation and we thus only have to estimate the second part, $Z^1_{\infty,\chi}(t)$. Using Proposition 5 and its proof, we have

$$Z^1_{\infty,\chi}(t) = \sum_{\omega_k \geqslant 1} \mathrm{Tr}\left(e^{-t\omega_k^{2/(2+\beta)}P_1}\chi(\cdot/\omega_k^{1/(2+\beta)})\right) = \sum_{\omega_k \geqslant 1} \mathcal{Z}_{\chi(\cdot/\omega_k^{1/(2+\beta)})}(t\omega_k^{2/(2+\beta)})$$

and we remark that, for $\omega \geqslant 1$, the function $\chi(\cdot/\omega^{1/(2+\beta)})$ is identically equal to 1 on $[0, a]$ (where $a$ was introduced Section 3.1), so that we can use the estimate of Section 3.1.

Converting this sum into an integral (see Appendix A.2), using the Weyl law on $M$, we obtain

$$\#\{\omega_k \leqslant \omega\} \sim \gamma_n \mathrm{Vol}(M)\omega^{n/2} \quad \text{as} \quad \omega \to +\infty.$$

Using Proposition 7 and the definition of $f$, we get

$$Z^1_{\infty,\chi}(t) \sim \frac{n\gamma_n \mathrm{Vol}(M)}{2} \int_1^{+\infty} \mathcal{Z}_{\chi(./\omega^{1/(2+\beta)})}(t\omega^{2/(2+\beta)})\omega^{\frac{n}{2}-1}\, d\omega.$$

Making the change of variable $\tau = t\omega^{2/(2+\beta)}$, we arrive at the following lemma, recalling the $\delta_H$ was introduced below Proposition 1.

**Lemma 2.** *The following holds,*

$$Z^1_{\infty,\chi}(t) \sim \frac{n\gamma_n(\beta+2)\mathrm{Vol}(M)}{4t^{\delta_H/2}} \int_t^{+\infty} \mathcal{Z}_{\chi(.\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau \ \text{as}\ t \to 0^+ \ .$$

The integral,

$$I(t) = \int_t^{+\infty} \mathcal{Z}_{\chi(.\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1}\, d\tau \tag{11}$$

is convergent at $\tau = \infty$ for all $\beta > 0$ but, in general, not at $\tau = 0$ because if $\beta \leqslant \beta_c$ then $\frac{\delta_H}{2} - 1 \leqslant \frac{1}{2} + \frac{1}{\beta} - 1$. We can compare this with the estimate in Corollary 2.

#### 3.2.1 Case $\beta > \beta_c$

We estimate the small-time behavior of $I(t)$ defined by (11). By the monotone convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} \int_0^{+\infty} e(\tau, x, x)\chi(\varepsilon x)\, dx = \int_0^{+\infty} e(\tau, x, x)\, dx$$

where $e$ is the heat kernel of $P_1$. Hence, for any $\tau > 0$,

$$\lim_{t\to 0^+} \mathcal{Z}_{\chi(.\sqrt{t/\tau})}(\tau) = \mathcal{Z}_1(\tau).$$

Using, again, the monotone convergence theorem, we conclude that

$$\lim_{t\to 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau)\tau^{\frac{\delta_H}{2}-1}\, d\tau.$$

From Corollary 2, we get

$$\mathcal{Z}_1(\tau)\tau^{\delta_H/2-1} \leqslant C\tau^{\frac{n}{2}(1+\frac{\beta}{2})-1-\frac{1}{2}-\frac{1}{\beta}}e^{-\mu_1\tau}$$

and $\beta > \beta_c = 2/n$ implies $\frac{n}{2}(1+\frac{\beta}{2}) - 1 - \frac{1}{2} - \frac{1}{\beta} > -1$. Thus, the corresponding limit is finite:

$$\lim_{t\to 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau)\tau^{\frac{\delta_H}{2}-1}\, d\tau < +\infty.$$

### 3.2.2 Case $\beta \leqslant \beta_c$

By the second estimate in Proposition 4, the contribution to the integral from 1 to $+\infty$ in the expression for $I(t)$ (which was introduced in Lemma 2) is $\mathrm{O}(1)$ uniformly with respect to $t$ and, hence, the corresponding part of $Z^1_{\infty,\chi}(t)$ is $\mathrm{O}(t^{-\delta_H/2})$, which will be negligible. We only need to estimate the asymptotics of

$$J(t) = \int_t^1 \mathcal{Z}_{\chi(\cdot\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1}\, d\tau.$$

**Sub-case $\beta < \beta_c$.** We prove that there exists a $\delta > 0$ such that

$$\mathrm{Tr}\left(e^{-t\triangle_g}\chi\right) = \mathrm{O}\left(J^\delta t^{-(n+1)/2}\right) \tag{12}$$

as $t \to 0^+$. We split the integral,

$$J(t) = \int_t^1 \mathrm{Tr}\left(e^{-\tau P_1}\chi\left(\sqrt{t/\tau}.\right)\right)\tau^{\frac{\delta_H}{2}-1} d\tau = J_1(t) + J_2(t) = \int_t^{\tau_0} + \int_{\tau_0}^1$$

where $\tau_0$ satisfies $\tau_0^{-1/\beta} = J\sqrt{\tau_0/t}$, i.e., $\tau_0 = \left(t/J^2\right)^{\beta/(2+\beta)}$ with $J = \int_0^{+\infty}\chi(x)\,dx$ as in Corollary 2. We get upper bounds for $J_1$ and $J_2$ using the upper bounds given in Corollary 2 as follows. Using the first argument in the minimum, we have

$$J_1(t) \leqslant C\frac{J}{\sqrt{t}}\tau_0^{\delta_H/2} = CJ^{1-(n\beta/2)}t^{(n\beta/4)-(1/2)}$$

Similarly, using the second argument in the minimum, we find that

$$J_2(t) \leqslant C\int_{\tau_0}^1 \tau^{-1/\beta}\tau^{(\delta_H/2)-(3/2)}\, d\tau \leqslant CJ^{1-(n\beta/2)}t^{(n\beta/4)-(1/2)}.$$

Finally,

$$t^{-\delta_H/2}I(t) \leqslant CJ^{1-(n\beta/2)}t^{-(n+1)/2}$$

so that we can take $\delta = 1-(n\beta/2)$. We will use this further in Section 3.4 by choosing $J$ small.

**Sub-case $\beta = \beta_c$.** When $\beta = 2/n$, we have to estimate the asymptotics of

$$J(t) = \int_t^1 \mathrm{Tr}\left(e^{-\tau P_1}\chi(\cdot\sqrt{t/\tau})\right)\tau^{(n-1)/2}\, d\tau.$$

Using the estimate of Proposition 4, we get

$$J(t) \sim \frac{1}{\sqrt{4\pi}}\int_t^1 \tau^{\frac{n}{2}-1}\, d\tau \int_0^\infty e^{-\tau x^{2/n}}\chi(x\sqrt{t/\tau})\, dx$$

modulo terms of smaller order in $\tau$. Using the change of variable $y = \tau x^{2/n}$, we get

$$J(t) \sim \frac{n}{4\sqrt{\pi}}\int_t^1 \frac{d\tau}{\tau}\int_0^\infty e^{-y}y^{\frac{n}{2}-1}\chi\left(\sqrt{t}\tau^{-(n+1)/2}y^{n/2}\right)\, dy = \frac{n}{4\sqrt{\pi}}\int_t^1 \frac{d\tau}{\tau}F\left(\tau^{(n+1)/2}/t^{1/2}\right)$$

where the function $F$, defined by

$$F(X) = \int_0^{+\infty} e^{-y}y^{\frac{n}{2}-1}\chi\left(y^{n/2}/X\right)\, dy,$$

is smooth and satisfies $F(X) = \mathrm{O}_{X\to 0}(X)$ and $\lim_{X\to+\infty} F(X) = \Gamma(n/2)$. Using the new variable $u = \tau^{(n+1)/2}/t^{1/2}$, we get

$$J(t) \sim \frac{n}{2(n+1)\sqrt{\pi}} \int_{t^{n/2}}^{t^{-1/2}} F(u)\, \frac{du}{u},$$

and finally

$$J(t) \sim \frac{n\Gamma(n/2)}{4(n+1)\sqrt{\pi}} |\ln t|$$

as $t \to 0^+$.

### 3.3 The heat parametrix in the separable metric case

We adapt the method of [5] and we use Appendix A.4. We denote by $z, z'$ some generic points of $X$ and by $z = (x, m)$, $z' = (x', m')$ generic points of $[0, 1) \times M \subset X$. Let $\chi$ be as in the previous sections, vanishing near $x = 1$ and extended by 0 inside $X$. Let $\eta \in C_0^\infty([0, 1))$ so that $\eta = 1$ near the support of $\chi$, and $\eta_0 \in C_0^\infty(X)$, vanishing near $\partial X$ and equal to 1 near the support of $1 - \chi$. We choose $a > 0$ so that $\eta_0$ vanishes for $x \leqslant 2a$. We claim that

$$p(t; z, z') = \eta(x) e_\infty(t; z, z'))\chi(x') + \eta_0(x) e_0(t; z; z')(1 - \chi(x'))$$

where $e_\infty$ is the heat kernel on the cone $X_\infty$ and $e_0$ the Riemannian heat kernel generated by the Laplacian $\triangle_g$ on $X \setminus \{x \leqslant a\}$ with Dirichlet boundary conditions, is a good approximation of the heat kernel on $X$ as $t \to 0$.

**Proposition 6.** *Let $P(t)$ be the operator of Schwartz kernel $p(t, \cdot, \cdot)$. We have*

$$\mathrm{Tr}\left(P(t) - e^{-t\triangle_g}\right) = \mathrm{O}\left(t^\infty\right)$$

*as $t \to 0^+$.*

*Proof.* We set $r(t, z, z') = (\partial_t + (\triangle_g)_z)\, p(t, z, z')$. The kernel $r$ vanishes if $x$ is small enough. By the local nature of the small-time asymptotics of Riemannian heat kernels (see Appendix A.4), $e_\infty(t, \cdot, \cdot)$ and $e_0(t, \cdot, \cdot)$ are $\mathrm{O}(t^\infty)$ close in $C^\infty$ topology on $[0, 1) \times M$. Moreover, if $x \in \mathrm{supp}(\eta')$ or $x \in \mathrm{supp}(\eta_0')$, $p(t, \cdot, \cdot)$ and $e_0(t, \cdot, \cdot)$ are $\mathrm{O}\left(t^\infty\right)$ in the $C^\infty$ topology, because $z \neq z'$. It follows that $r(t, \cdot, \cdot) = \mathrm{O}\left(t^\infty\right)$ in $C^\infty$ topology. Therefore, denoting by $R(t)$ the operator of Schwartz kernel $r(t, \cdot, \cdot)$, the trace norm of $R(t)$ is a $\mathrm{O}\left(t^\infty\right)$. By the Duhamel formula, using that $P(t) \to \mathrm{id}$ as $t \to 0^+$, we have

$$P(t) - e^{-t\triangle_g} = \int_0^t e^{-(t-s)\triangle_g} r(s)\, ds.$$

The result follows because the operator norm of $e^{-t\triangle_g}$ is not greater than 1. □

### 3.4 Completion of the proof of Theorem 1 in the separable metric case

Thanks to the previous section, we only have to estimate the trace of $P(t)$. We use the local nature of the heat asymptotics to show that the contribution of the term $\eta_0 e_0(t)(1 - \chi)$ is equivalent to $(4\pi t)^{-(n+1)/2} \int_X (1 - \chi)\, dv_g$. We are left to estimate the term that corresponds to the truncated cone as in Section 3.2. This gives the conclusion when $\beta \geqslant \beta_c$.

When $\beta < \beta_c$, the first term can be made smaller than $\varepsilon t^{-(n+1)/2}$ for any $\varepsilon > 0$ by choosing $J = \int_0^{+\infty} \chi(x)\, dx$ small enough as mentioned in Section 3.2.2.

### 3.5 From the separable to the general case

We prove that, for any given $\varepsilon > 0$, the metric $g$ on $X$ is $\varepsilon$-quasi-isometric to a separable metric $g_s$. We choose $\delta > 0$ so that

$$|g_0(u) - g_0(0)| \leqslant \varepsilon(du^2 + g_0(u))$$

for any $u \in [0, \delta]$. Then, we choose $\eta \in C_0^\infty([0, \delta))$, identically equal to 1 near $u = 0$. We consider the separable metric $g_s$ which coincides with $g$ outside $u \leqslant \delta$ and is given near $\partial X$ by

$$g_s = \eta u^{-\alpha}\left(du^2 + g_0(0)\right) + (1 - \eta)g.$$

Then

$$\left|\frac{g_s}{g} - 1\right| = \left|\eta\frac{g_0(u) - g_0(0)}{du^2 + g_0(u)}\right| \leqslant \varepsilon.$$

Using Appendix A.1, this concludes the proof of Theorem 1 in the general (non-separable) case.

### 3.6 Proof of Theorem 2

#### 3.6.1 Case $\beta < \beta_c$

We consider the heat traces $Z_f(t) = \mathrm{Tr}\left(e^{-t\triangle_g} f\right)$ where $f : X \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$. We choose a smooth function $\chi : X \to [0, 1]$ that is identically equal to 1 near $\partial X$ and such that $\int_X \chi\, dv_g \leqslant \varepsilon$. Writing $g = \chi f + (1 - \chi)f$, we get

$$\mathrm{Tr}\left(e^{-t\triangle_g} f\right) = \mathrm{Tr}\left(e^{-t\triangle_g}(1 - \chi)f\right) + \mathrm{Tr}\left(e^{-t\triangle_g}\chi f\right) = J(t) + K(t).$$

By the local nature of the heat trace asymptotics (see Appendix A.4), we have

$$J(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X (1 - \chi)f\, dv_g$$

as $t \to 0^+$. Besides,

$$K(t) \leqslant \|f\|_\infty \mathrm{Tr}\left(e^{-t\triangle_g}\chi\right).$$

By (12), and since we can choose $\epsilon > 0$ arbitrarily small, it follows that

$$Z_f(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X f\, dv_g$$

as $t \to 0^+$, which gives the expected result.

#### 3.6.2 Case $\beta \geqslant \beta_c$

We give the proof in the case $\beta > \beta_c$. The case $\beta = \beta_c$ is treated similarly. Let us first prove that the support of the Weyl measure is contained in $\partial X$. If $\mathrm{supp}(f) \cap \partial X = \emptyset$, we get again by the local nature of the heat asymptotics that

$$\mathrm{Tr}\left(e^{-t\triangle_g} f\right) = \mathrm{O}\left(t^{-(n+1)/2}\right)$$

while, by the Weyl law given in Theorem 1, we have

$$t^{-(n+1)/2} = \mathrm{o}\left(\mathrm{Tr}\left(e^{-t\triangle_g}\right)\right)$$

as $t \to 0^+$. Hence, it suffices to consider functions $f$ of the form $f = \mathbb{1}_D$ where $D = [0, a] \times D_1$ with $D_1$ a piecewise smooth domain in $M$.

We recall that, according to [12, Chapter 5, Theorem 2.1.6], we have

$$0 \leqslant e_D(t,m,m) \leqslant e_X(t,m,m) \qquad \forall m \in D \tag{13}$$

where $e_D$ is the Dirichlet heat kernel on $D$.

We set $D' = [0,a] \times (M \setminus D)$ and $D'' = X \setminus [0,a] \times M$. For any domain $K$, we denote by $Z_K$ the Dirichlet heat trace and by $Z'_K(t) = \int_K Z_X(t,m,m)\, dv_g(m)$. We have

$$Z_D(t) \sim C\,\mathrm{Vol}(D_1) t^{-d_H/2}, \qquad Z_{D'}(t) \sim C\,\mathrm{Vol}(M \setminus D_1) t^{-d_H/2} \tag{14}$$

and $Z_{D''}(t) = \mathrm{o}(t^{-d_H/2})$ as $t \to 0^+$. Note that for (14), we used the proofs in Section 3.2 for the case where the $n$-dimensional manifold $M$ (here: $D_1$ and $M \setminus D_1$) can have a boundary. The sum $(Z'_D + Z'_{D'} + Z'_{D''})(t) = Z_X(t)$ is equivalent to $C\,\mathrm{Vol}(M)t^{-d_H/2}$ by Theorem 1. Hence, $(Z'_D + Z'_{D'})(t) \sim C\mathrm{Vol}(M)t^{-d_H/2}$. On the other hand, thanks to (13) we have

$$Z_D \leqslant Z'_D, \qquad Z_{D'} \leqslant Z'_{D'}, \qquad Z_{D''} \leqslant Z'_{D''}.$$

It follows that

$$Z'_D(t) \sim C\mathrm{Vol}(D_1) t^{-d_H/2}$$

as $t \to 0^+$, which yields the desired result.

# 4 Discussion and open problems

In this article, motivated by the propagation of acoustic waves in gas giant planets, we derived the Weyl law for the Laplace-Beltrami operator on a smooth compact Riemannian $(n+1)$-dimensional manifold $X$ with boundary whose metric blows up near the boundary. Many new questions emerge. We present some of them.

**Quantum Ergodicity and Quantum Limits.** We have seen in Corollary 1 that, if $\beta \geqslant \beta_c$, then a density-one subsequence of eigenfunctions concentrates on $\partial X$. This is a preliminary result towards Quantum Ergodicity (QE).

Recall that, on a locally compact space $U$ endowed with a probability Radon measure $\mu$, given a self-adjoint nonnegative operator $T$ on $L^2(U,\mu)$, of discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty$ associated with an orthonormal eigenbasis $\Phi = (\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(U,\mu)$, a Quantum Limit (QL) of $\Phi$ is a probability Radon measure $\nu$ on $U$ that is a weak limit of a subsequence of the probability measures $|\phi_j|^2\,\mu$, i.e., there exists a subsequence $(j_k)_{k \in \mathbb{N}^*}$ such that

$$\int_U f |\phi_{j_k}|^2 \, d\mu \underset{k \to +\infty}{\longrightarrow} \int_U f\, d\nu \qquad \forall f \in C^0_c(U). \tag{15}$$

We say that QE holds for $(T,\Phi)$ if there exists a QL $\nu$ on $U$ and a subsequence $(j_k)_{k \in \mathbb{N}^*}$ of density one such that (15) holds.

One may wonder whether, when $\beta \geqslant \beta_c$, a QE property on $M$ would imply a QE property on $X$. Proving this fact certainly requires fine spectral properties of Schrödinger operators (see [1]). Besides, inspired by [9, Theorem B], we wonder what can be said on QLs supported on $\partial X = \{0\} \times M$: are they invariant under the geodesic flow of $(M,G)$ (where $G = g_1(0)$)? Defining QLs on $T^\star M$ will already be a challenge.

**Inverse problems on spectra.** A natural question is: does the spectrum of $X$ determine the spectrum of $M$? Attacking this problem certainly requires developing appropriate trace formulas, as in [8].

**Closed geodesics.** Recalling that $G = g_1(0)$ where $g_1$ is defined by (1), it is natural to view geodesics on $(M, G)$ as limits, in an appropriate sense, of geodesics on $(X, g)$. A natural question is then: do there exist some closed geodesics of $X$ accumulating on (converging to) closed geodesics of $\partial X = M$? We refer to [7] for a similar question investigated in the framework of contact sub-Riemannian 3D manifolds. Here again, having appropriate trace formulas might be useful.

**Observability properties.** The study of the Weyl asymptotics is a first step towards solving some inverse problems. As explained in Section 1, the knowledge of spectrum properties can already be used to check the validity of some models, but the main objective in the physical context would be the ability to reconstruct some features of the internal structure of the planets, based on the observation of acoustic waves. The feasibility of such an inverse problem is mathematically modeled by an observability inequality, which can be settled as follows for half-waves. Given any $T > 0$ and any subset $\omega$ of $X$, we say that the observability property holds true for $(\omega, T)$ if there exists a positive constant $C_T(\omega)$ such that

$$\int_0^T \left\| \mathbb{1}_\omega \, e^{it\sqrt{\triangle_g}} \phi \right\|^2_{L^2(X,dv_g)} dt \geqslant C_T(\omega) \|\phi\|^2_{L^2(X,dv_g)} \qquad \forall \phi \in L^2(X, dv_g). \tag{16}$$

– When $\beta < \beta_c$, we expect that (16) holds as soon as $\omega$ is open and $(\omega, T)$ satisfies the Geometric Control Condition (GCC, see [2]), like in the classical case of a non-singular Riemannian metric.

– When $\beta \geqslant \beta_c$, an obvious necessary condition for (16) to hold is that $\omega$ contain an open neighborhood of a subset of $\partial X$. Indeed, take $\phi$ in (16) to be a highfrequency eigenfunction and apply Corollary 1. We think that this condition is sufficient if moreover $(\omega, T)$ satisfies GCC.

We note that, when $X$ is a closed ball in $\mathbb{R}^{n+1}$ (an idealized situation for an exactly round planet), GCC is never satisfied unless $\omega$ contains an open neighborhood of the *whole* boundary of $X$, which is certainly not relevant for applications from the physical point of view. In this case where $X$ is a round ball, it is more interesting to take a small observation subset $\omega$, containing a small open subset of $\partial X$. But, as soon as $\omega$ is a proper subset of a half-ball, GCC (and thus (16)) obviously fails due to trapped rays, propagating along a diameter never meeting $\omega$. In this deteriorated context, we wonder, however, whether (16) is anyway satisfied if we restrict the inequality to radial waves or to surface waves, which are the most physically meaningful waves to be observed.

**Metrics that are singular on larger codimension submanifolds.** In this paper, we have considered a class of singular metrics blowing up at the boundary of $X$, where the boundary can be seen as a codimension-one submanifold of $X$.

In more general, let $X$ be a smooth compact manifold and let $Z$ be a submanifold of $X$ of codimension $m \in \mathbb{N}^*$, and consider the class of singular metrics $g$ on $X$ that are smooth on $X \setminus Z$ and that, near $Z$, are written as

$$g = h + g_Z(x) r^{-\beta}$$

in a neighborhood of $Z$ assumed to be diffeomorphic to $Z \times B_m$ where $B_m$ is the unit ball of $\mathbb{R}^m_x$ (this holds if the normal bundle of $Z$ is trivial) equipped with the Euclidean metric $h$ and the polar coordinates $(r, \sigma)$, and $g_Z(x)$ is a metric on $Z$, parametrized by $x \in B_m$ and depending smoothly on $x$. The techniques developed in our paper can certainly be extended to compute the Weyl asymptotics in such cases.

# A Appendix

## A.1 Quasi-isometries

Let $X$ be a smooth manifold of dimension $n+1$, with boundary. Two metrics $g_1$ and $g_2$, smooth on $X \setminus \partial X$, are said to be $\varepsilon$-quasi-isometric if

$$\left| \frac{g_1}{g_2} - 1 \right| \leqslant \varepsilon$$

uniformly on $X \setminus \partial X$. For $i \in \{1,2\}$, let $\triangle_{g_i}$ be the Friedrichs extension of the Laplace-Beltrami operator on $(X, g_i)$ with core $C_0^\infty(X \setminus \partial X)$.

If $\triangle_{g_1}$ has a discrete spectrum $(\lambda_j^1)_{j\in\mathbb{N}^*}$ then $\triangle_{g_2}$ has also a discrete spectrum $(\lambda_j^2)_{j\in\mathbb{N}^*}$ and, for $\varepsilon \leqslant \frac{1}{2}$ (this condition is to get bounds on the inverse of $g_i$), there exists $C(n) > 0$ such that, for every $j \in \mathbb{N}^*$,

$$\left| \frac{\lambda_j^1}{\lambda_j^2} - 1 \right| \leqslant C(n)\varepsilon.$$

Indeed, this estimate follows from the minimax characterization of the eigenvalues and from the comparison of the Rayleigh quotients, i.e., of the volumes and co-metrics.

## A.2 Karamata tauberian theorem and converse

We recall the Karamata tauberian theorem (see [16, Chapter XIII, Theorem 2]).

**Theorem 3.** *Let $\mu$ be a positive Radon measure on $\mathbb{R}^+$. If there exists $\alpha > 0$ such that*

$$\int_0^{+\infty} e^{-t\lambda} d\mu(\lambda) \sim At^{-\alpha} \quad (\textit{resp.},\ A|\ln t|t^{-\alpha})$$

*as $t \to 0^+$, then*

$$\mu([0,\lambda]) \sim \frac{A}{\Gamma(\alpha+1)}\lambda^\alpha \quad \left(\textit{resp.},\ \frac{A}{\Gamma(\alpha+1)}\lambda^\alpha \ln\lambda\right)$$

*as $\lambda \to +\infty$.*

We need a converse of Theorem 3. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function of class $C^1$, such that $f$ and $f'$ have a fast decay at infinity. Let $(\lambda_j)_{j\in\mathbb{N}^*}$ be a nondecreasing sequence of positive real numbers. We define the counting function $N(\lambda) = \#\{j \in \mathbb{N}^* \mid \lambda_j \leqslant \lambda\}$, for any $\lambda \in \mathbb{R}$. The objective is to estimate the sum

$$S = \sum_{j=1}^{+\infty} f(\lambda_j).$$

**Proposition 7.** *Assume that there exist $C > 0$ and $\alpha > 0$ such that $N(\lambda) \sim C\lambda^\alpha$ as $\lambda \to +\infty$. For any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$, depending on the counting function $N$ but not on $f$, such that*

$$\left| S - C\alpha \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1}\, d\lambda \right| \leqslant K(\varepsilon) f(\lambda_1) + \varepsilon \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1}\, d\lambda.$$

*Proof.* Given any $\varepsilon > 0$, let $\Lambda_0 > 0$ such that, for every $\lambda \geqslant \Lambda_0$,

$$(1-\varepsilon)C\lambda^\alpha \leqslant N(\lambda) \leqslant (1+\varepsilon)C\lambda^\alpha. \tag{17}$$

Noting that $dN(\lambda) = \sum_{j=1}^{+\infty} \delta_{\lambda_j}$, using the Stieltjes integral, we have

$$S = \sum_{\lambda_j < \Lambda_0} f(\lambda_j) + \int_{\Lambda_0}^{+\infty} f(\lambda)\, dN(\lambda).$$

Now, since $\sum_{\lambda_j < \Lambda_0} f(\lambda_j) \leqslant N(\Lambda_0) f(\lambda_1)$, we get by integration by parts, using the fast decay of $f$ at infinity, that

$$S \leqslant N(\Lambda_0)\left(f(\lambda_1) - f(\Lambda_0)\right) - \int_{\Lambda_0}^{+\infty} f'(\lambda) N(\lambda) d\lambda.$$

We derive an upper bound for $S$. A lower bound is obtained similarly. Using (17), integrating by parts and using that $f(\Lambda_0) \leqslant f(\lambda_1)$, we obtain

$$\begin{aligned} -\int_{\Lambda_0}^{+\infty} f'(\lambda) N(\lambda)\, d\lambda &\leqslant -(1+\varepsilon) C \int_{\Lambda_0}^{+\infty} f'(\lambda) \lambda^{\alpha}\, d\lambda \\ &\leqslant (1+\varepsilon) C \left( f(\lambda_1) \Lambda_0^{\alpha} + \alpha \int_{\Lambda_0}^{+\infty} f(\lambda) \lambda^{\alpha-1}\, d\lambda \right). \end{aligned}$$

Therefore,

$$S \leqslant f(\lambda_1)\left(N(\Lambda_0) + C(1+\varepsilon)\Lambda_0^{\alpha}\right) + (1+\varepsilon) C \alpha \int_{\lambda_1}^{+\infty} f(\lambda) \lambda^{\alpha-1}\, d\lambda$$

and the result follows with $K(\varepsilon) = N(\Lambda_0) + C(1+\varepsilon)\Lambda_0^{\alpha}$. □

## A.3 Weyl circle-point limit criterion

We consider the Schrödinger operator $P = -\partial_x^2 + q(x)$ on $C_0^\infty((0,+\infty))$, where $q$ is a smooth function on $(0,+\infty)$. According to the Weyl circle-point limit criterion (see [31, Theorem X.7]), $P$ is essentially self-adjoint if and only if there exists at least one solution of $Pu = 0$ that is not square integrable at 0 and at least one solution of $Pu = 0$ that is not square integrable at $+\infty$.

When $q(x) = Cx^{-2} + \omega x^{\beta}$ for some $C \geqslant 0$, $\beta > 0$ and $\omega > 0$, there is only one solution of $Pu = 0$ that is square integrable at $+\infty$. Near 0, the solutions of $Pu = 0$ are equivalent to linear combinations of $x^{\gamma_+}$ and $x^{\gamma_-}$ where $\gamma_+$ and $\gamma_-$ are the two solutions of $-\gamma(\gamma-1) + C = 0$. It follows that $P$ is essentially self-adjoint if and only if $\gamma_- \leqslant -\frac{1}{2}$, that is, if and only if $C \geqslant 3/4$.

## A.4 Local nature of the small-time asymptotics of heat kernels

Let $(U, g)$ be a smooth Riemannian manifold and let $\triangle$ be the Laplace-Beltrami operator. For our needs (see Section 3.3), $U = X \setminus \partial X$ with the metric $g$ .

Let $e_1$ and $e_2$ be two solutions of $(\partial_t + \triangle_x) e_i(t,x,y) = 0$ for $t > 0$, satisfying $e_i(t,x,y) = e_i(t,y,x)$ for all $t > 0$ and $(x,y) \in U \times U$ and

$$\lim_{t \to 0^+} \int_U e_i(t,x,y) f(y)\, dv_g(y) = f(x) \qquad \forall x \in U \qquad \forall f \in C_0^\infty(U),$$

for $i \in \{1,2\}$.

**Lemma 3.** *We have $e_1(t,\cdot,\cdot) - e_2(t,\cdot,\cdot) = \mathrm{O}(t^\infty)$ as $t \to 0^+$ in $C^\infty$ topology on $U \times U$. Moreover, denoting by $D$ the diagonal of $U \times U$, for $i \in \{1,2\}$, we have $e_i(t,\cdot,\cdot) = \mathrm{O}(t^\infty)$ as $t \to 0^+$ in $C^\infty$ topology on $U \times U \setminus D$.*

This result reflects Kac's principle of "not feeling the boundary", showing that the small-time asymptotic behavior of heat kernels is purely local. A detailed proof can be found in [10, Section 3.2.1]. The idea comes from the paper [21]. The proof uses the fact that the Hörmander operator $P = 2\partial_t + \triangle_x + \triangle_y$ is hypoelliptic. Extending the kernels $e_i$ by 0 for $t < 0$, we have $Pe_i = 0$ on $\mathbb{R} \times U \times U \setminus D$ and $P(e_1 - e_2) = 0$ on $\mathbb{R} \times U \times U$, in the distributional sense. The result then follows by hypoellipticity.

### A.5 $\triangle_g$ as a nonsmooth Hörmander operator

Based on the mathematical model provided in Section 2.1, near any point of the boundary of $X$ we have $X \simeq [0,1) \times \mathbb{R}^n$ with a local system of coordinates $(x, y)$, with $x \in [0,1)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and we can write (locally)

$$\triangle_g = -\sum_{i=0}^{n} X_i^* X_i + V \tag{18}$$

where $V(x,y) = \frac{C(x,y)}{x^2}$ is a potential and the $X_i$'s are vector fields given by

$$X_0 = a_0(x,y)\,\partial_x, \qquad X_i = x^{\beta/2} a_i(x,y)\,\partial_{y_i}, \quad i \in \{1, \ldots, n\}.$$

The functions $C$ and $a_i$, $i \in \{0, \ldots, n\}$ are smooth on $\mathbb{R} \times \mathbb{R}^n$ and $C(0,\cdot) = C_\beta$ and $a_i(0,\cdot) = 1$ (they can be expressed in terms of the coefficients of the smooth Riemannian metric $g_1(x)$ on $M$ defined by (1)). The separable case corresponds to $a_0 = 1$ and $a_i$ not depending on $x$.

Expressed as (18), the operator $\triangle_g$ is then a Hörmander operator, however nonsmooth unless $\beta \in 2\mathbb{N}^*$. Because of this lack of smoothness, many classical results cannot be applied here.

When $\beta \in 2\mathbb{N}^*$, the above vector fields are smooth and define an almost-Riemannian geometry, in which the Weyl asymptotics of the almost-Riemannian Laplacian $\triangle_{aR} = -\sum_{i=0}^{n} X_i^* X_i$ (i.e., (18) with $V = 0$), of Grushin type, has been established in [11].

With these preliminary remarks in mind, we then mention a few interesting facts hereafter.

**Homogeneity.** In the above local coordinates, given any $\varepsilon > 0$, we define the *dilation*

$$\delta_\varepsilon(x,y) = (\varepsilon x, \varepsilon^{1+\beta/2} y) \qquad \forall (x,y) \in [0,1) \times \mathbb{R}^n.$$

In the separable case where $a_0 = 1$ and $a_i$ does not depend on $x$, for any $i \in \{1, \ldots, n\}$, we define $\widehat{X}_i = \lim_{\varepsilon \to 0} \varepsilon \delta_\varepsilon^* X_i = x^{\beta/2} a_i(0)\,\partial_{y_i}$, and we have

$$\varepsilon \delta_\varepsilon^* X_0 = X_0 \qquad \text{and} \qquad \varepsilon \delta_\varepsilon^* \widehat{X}_i = \widehat{X}_i \quad \forall i \in \{1, \ldots, n\}.$$

In sR geometry, $\widehat{X}_i$ is the *nilpotentization* of the vector field $X_i$ at the point identified with $(0,0)$. Extrapolating results of sub-Riemannian geometry that one can find in [10] to the case of $\beta > 0$, denoting by $d_g$ the $g$-distance on $X$, one can show that $d_g((0,0),(x,y))$ divided by $|x| + \sum_{i=1}^{n} |y_i|^{1/(1+\beta/2)}$ is bounded above and below by some positive constants in a neighborhood of $(0,0)$. Noting that $1/(1+\beta/2) = 1 - \alpha/2$, we thus recover [13, Proposition 13] and thus the result of Proposition 1 and the fact that Hausdorff and Minkowski dimensions coincide. In the non-separable case, we obtain the result by using quasi-isometries.

**Weyl law when $\beta \in 2\mathbb{N}^*$.** When $\beta \in 2\mathbb{N}^*$, we always have $\beta \geqslant \beta_c$, and $\beta = \beta_c$ if and only if $n = 1$. Since the potential $1/x^2$ is homogeneous, combining results of [10, 11], we recover the Weyl law established in Theorem 1.

**Weyl law when $\beta \notin 2\mathbb{N}^*$.** To establish the Weyl law in general sub-Riemannian cases, the approach developed in [11] consists of estimating singular integrals involving the heat kernel, by performing the so-called $(J + K)$-decomposition. Applying this approach to the nonsmooth operator in (18) cannot be done directly because we miss a general hypoellipticity theory, valid for nonsmooth vector fields as above, and a generalization of Lemma 3 (see Appendix A.4) to that context.

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# Isoperimetric inequalities for inner parallel curves

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**Abstract.** We prove weighted isoperimetric inequalities for smooth, bounded, and simply-connected domains. More precisely, we show that the moment of inertia of inner parallel curves for domains with fixed perimeter attains its maximum for a disk. This inequality, which was previously only known for convex domains, allows us to extend an isoperimetric inequality for the magnetic Robin Laplacian to non-convex centrally symmetric domains. Furthermore, we extend our isoperimetric inequality for moments of inertia, which are second moments, to $p$-th moments for all $p$ smaller than or equal to two. We also show that the disk is a strict local maximiser in the nearly circular, centrally symmetric case for all $p$ strictly less than three, and that the inequality fails for all $p$ strictly bigger than three.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. For any $t \geq 0$, we define the corresponding inner parallel curve $S_t$ by

$$S_t := S_t(\Omega) = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) = t\}. \tag{1.1}$$

The systematic study of the geometric structure and regularity of inner parallel curves was initiated in [4, 11, 16], see also [27, 28] and references therein.

By [28, Theorem 4.4.1] and [27, Proposition A.1], the inner parallel curve $S_t$ is a finite union of piecewise smooth simple curves for almost every $t \geq 0$. Hartman [16, Corollary 6.1] showed that

$$|S_t| \leq |\partial\Omega| - 2\pi t \quad \text{for almost every } t \geq 0 \text{ with } S_t \neq \varnothing, \tag{1.2}$$

where $|S_t|$ and $|\partial\Omega|$ denote the length of $S_t$ and $\partial\Omega$, respectively.

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*Keywords:* inner parallel curves, isoperimetric inequalities, moments of inertia, spectral inequalities.

The moment of inertia of the inner parallel curve $S_t$ computed with respect to its centroid

$$c(t) := \frac{1}{|S_t|} \int_{S_t} x \, \mathrm{d}\mathcal{H}^1(x) \tag{1.3}$$

is given by

$$\int_{S_t} |x - c(t)|^2 \, \mathrm{d}\mathcal{H}^1(x), \tag{1.4}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. In this paper, we address the following question.

**Question.** *Fixing the perimeter of $\Omega$, what shape of $\Omega$ maximises the moment of inertia of the inner parallel curve $S_t$ as defined in (1.4) for given $t \geq 0$?*

Our main result states that the optimal shape is attained for a disk.

**Theorem 1.1** (An isoperimetric inequality for moments of inertia). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Then, for almost every $t \geq 0$,*

$$\int_{S_t(\Omega)} |x - c(t)|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq \int_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\mathcal{H}^1(x), \tag{1.5}$$

*where $\mathcal{B}$ is the disk centred at the origin and with the same perimeter as $\Omega$. Here $S_t(\cdot)$ and $c(t)$ are defined in (1.1) and (1.3), respectively. When $t \in \left[0, \frac{|\partial\Omega|}{2\pi}\right)$, the equality in (1.5) is attained if and only if $\Omega$ is a disk.*

Note that $c(t) = 0$ if $\Omega$ is a disk centred at the origin. In the setting where $S_t$ is a closed curve, the statement of Theorem 1.1 can be deduced from a result due to Hurwitz [18, pp. 396–397] combined with (1.2). For instance, this argument applies for convex domains $\Omega$, see [19, p. 12] for further details. In general, $S_t$ can consist of several connected components, see for example Figure 1.1, and Theorem 1.1 is novel in this case. The classical result by Hurwitz itself provides an isoperimetric inequality for the moment of inertia of the boundary of a planar domain under fixed perimeter constraint and essentially coincides with the statement of Theorem 1.1 in the special case $t = 0$. The recent contribution [23] proves a quantitative version of the inequality by Hurwitz and addresses the higher-dimensional setting.

The smoothness assumption for the domain is not optimal. It is used in the proof for Proposition 2.1 below, see [16], see also [28, Theorem 4.4.1] and [27, Proposition A.1]. For piecewise $C^2$-domains satisfying a similar result to Proposition 2.1 below, we can also obtain (1.5) using the same rest of the proof. In general, Theorem 1.1 fails for non-simply-connected domains; see Remark 4.1 below.

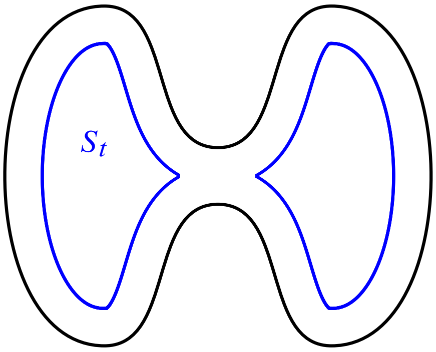


**Figure 1.1.** A schematic representation of the inner parallel curve $S_t$ in a dumbbell-like domain. Note that $S_t$ can be disconnected.

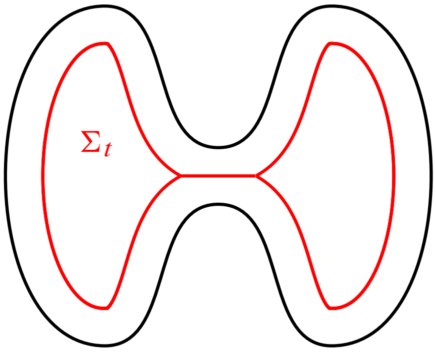


**Figure 1.2.** A schematic illustration of the connected curve $\Sigma_t$ in the case of a dumbbell-like domain. The proof of Theorem 1.2 will show that the segment connecting the two connected components of $S_t$ is doubly covered.

Our proof of Theorem 1.1 relies on an explicit construction of a *closed* curve $\Sigma_t$. An illustration for $\Sigma_t$ is shown in Figure 1.2 (where $\Sigma_t$ inherits the symmetry of $\Omega$).

**Theorem 1.2** (Covering inner parallel curves with a closed curve). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Then, for almost every $t \geq 0$ with $S_t \neq \varnothing$, there exists a closed and piecewise smooth curve $\Sigma_t$ with*

$$S_t \subset \Sigma_t \quad \textit{and} \quad |\Sigma_t| \leq |\partial\Omega| - 2\pi t,$$

*where $S_t$ was defined in* (1.1).

The technical result Theorem 1.2 is of independent interest as we obtain an improved version of (1.2) taking the distance between different connected components of $S_t$ into account, see Corollary 7.1 below.

More generally, we can consider $p$-th moments and ask for which $p \in (0, \infty)$ we have

$$\int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\mathcal{H}^1(x) \leq \int_{S_t(\mathcal{B})} |x|^p \, \mathrm{d}\mathcal{H}^1(x) \quad \text{for almost every } t \geq 0, \tag{1.6}$$

where $c(t)$ is the centroid of $S_t(\Omega)$ and $\mathcal{B}$ is a disk centred at the origin with $|\partial\Omega| = |\partial\mathcal{B}|$. For $t \geq 0$ small enough, we have $|S_t(\Omega)| = |S_t(\mathcal{B})|$ and $S_t(\Omega)$ is a closed curve, see Lemma 2.3 (ii) below, so (1.6) reduces to

$$\frac{(2\pi)^p}{|S_t(\Omega)|^{p+1}} \int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\mathcal{H}^1(x) \leq 1.$$

So we may ask if we have

$$\frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_\Gamma |x|^p \, \mathrm{d}\mathcal{H}^1(x) \le 1 \tag{1.7}$$

for all closed Lipschitz curves $\Gamma$ with the origin as its centroid.

Note that the centroid $c(t)$ is independent of $t$ for all centrally symmetric domains $\Omega$, or for example for domains with two not necessarily orthogonal axes of symmetry. To keep things simple, we focus on the centrally symmetric case.

**Theorem 1.3** (An isoperimetric inequality for $p$-th moments). *The followings statements hold.*

(i) *The statement of Theorem 1.1 extends to (1.6) in the case $p \in (0, 2]$.*

(ii) *For $p < 3$, the boundary of a disk is a strict local maximiser among nearly circular, centrally symmetric closed Lipschitz curves $\Gamma$ of the left-hand side in the inequality (1.7).*

(iii) *For $p > 3$, (1.7) does not hold, not even locally near boundary of the disk. More precisely, there exists a sequence of nearly circular, centrally symmetric closed Lipschitz curves $(\Gamma_n)_{n\in\mathbb{N}}$ converging uniformly to the boundary of the disk for which*

$$\frac{(2\pi)^p}{|\Gamma_n|^{p+1}} \int_{\Gamma_n} |x|^p \, \mathrm{d}\mathcal{H}^1(x) > 1.$$

This naturally leads to the following conjecture.

**Conjecture 1.4.** (1.7) *holds for all $p \le 3$ and all closed Lipschitz curves $\Gamma$ with the origin as its centroid.*

In the case $p \in (0, 2]$, Theorem 1.3 follows from Theorem 1.1 and Theorem 1.2 using Jensen's inequality. For the local optimality for $p < 3$ in Theorem 1.3 (ii), we follow a Fuglede-type argument [12]. From these computations, we also obtain Theorem 1.3 (iii), where symmetry breaking occurs for $p > 3$.

Theorem 1.1 and Theorem 1.3 are of general interest as (weighted) isoperimetric inequalities have recently received great attention [1, 3, 7, 10], see also [5, 9, 13, 24, 25] on quantitative isoperimetric inequalities. In the present paper, we consider the moment of inertia of the inner parallel curves $S_t$ and compare it with the corresponding quantity for a disk of the same perimeter. This is a relatively unusual setting as our constraints do not involve the area of the domain $\Omega$, but only its perimeter. Note that under fixed area constraint the $p$-th moment of the boundary is minimised by the disk for all $p \ge 1$ (cf. [3, Theorem 2.1]), which is in contrast to our result under fixed perimeter.

In the case of centrally symmetric domains, or more generally for domains $\Omega$ for which the centroid $c(t)$ defined in (1.3) is independent of $t$, we can deduce from Theorem 1.1 a result going back to Hadwiger [14], see (7.1) by integrating over $t$.

As an application of Theorem 1.1, we obtain an isoperimetric inequality for the magnetic Robin Laplacian. More precisely, considering the magnetic Robin Laplacian with a negative boundary parameter $\beta$ and a sufficiently small constant magnetic field $b$, the ground state energy is expressed as follows:

$$\lambda_1^{\beta,b}(\Omega) = \inf_{\substack{u\in H^1(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \Bigg( \int\limits_{\Omega} |(-\mathrm{i}\nabla - b\mathbf{A})u|^2 + \beta \int\limits_{\partial\Omega} |u|^2 \,\mathrm{d}\mathcal{H}^1(x)\Bigg),$$

where $\mathbf{A}$ is a vector field in $\Omega$ with $\operatorname{curl}\mathbf{A} = 1$. It was shown in [19, Theorem 4.8] that the corresponding ground state energies for convex and centrally symmetric domains $\Omega$ and a disk $\mathcal{B}$ of the same perimeter satisfy $\lambda_1^{\beta,b}(\Omega) \le \lambda_1^{\beta,b}(\mathcal{B})$. Using Theorem 1.1, we can remove the convexity assumption on $\Omega$.

**Theorem 1.5** (An isoperimetric inequality for the magnetic Robin Laplacian). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Assume that $\Omega$ is centrally symmetric or, more generally, that the centroid of $S_t(\Omega)$ is independent of $t$ for all $t \ge 0$ with $S_t(\Omega) \ne \varnothing$. Let $\beta < 0$ be the negative Robin parameter, and let $0 < b < b_0(|\partial\Omega|, \beta)$, where $b_0(|\partial\Omega|, \beta)$ depends on $|\partial\Omega|$ and $\beta$. Then the lowest eigenvalue of the magnetic Robin Laplacian on $\Omega$ with constant magnetic field of strength $b$ and Robin boundary conditions with parameter $\beta$ satisfies*

$$\lambda_1^{\beta,b}(\Omega) \le \lambda_1^{\beta,b}(\mathcal{B}), \tag{1.8}$$

*where $\mathcal{B} \subset \mathbb{R}^2$ is the disk having the same perimeter as $\Omega$. Equality in (1.8) occurs if and only if $\Omega$ is a disk.*

If $|\partial\Omega| = 2\pi$, then we have the explicit expression $b_0(|\partial\Omega|, \beta) = \min(1, 4\sqrt{-\beta})$.

**Structure of the paper.** The rest of the paper is organised as follows. In Section 2, we introduce some notation and auxiliary results on inner parallel curves. In Section 3, we prove Theorem 1.2. We use this in Section 4 to prove Theorem 1.1. In Section 5, we show Theorem 1.3. More precisely, the proof and the precise statement of Theorem 1.3 (i) can be found in Corollary 5.1 and for Theorem 1.3 (ii), (iii) we refer to Proposition 5.6. Some background material on the magnetic Robin Laplacian and the proof of Theorem 1.5 are given in Section 6. In Section 7, we present two simple applications of Theorem 1.1 and Theorem 1.2, namely Corollary 7.1 on an improved version of (1.2), and Section 7.2 on moments of inertia of domains.

## 2. Preliminaries

### 2.1. Notation

We introduce for a piecewise-$C^1$ mapping $\gamma\colon [0, L] \to \mathbb{R}^2$ the length of the closed and not necessarily simple curve parametrised by $\gamma$,

$$\ell(\gamma) := \int_0^L |\gamma'(s)| \,\mathrm{d}s.$$

We also use the notation $\gamma([a, b]) = \{\gamma(s)\colon s \in [a, b]\}$ for $a, b \in [0, L]$, $a < b$. We say that $\gamma_1, \gamma_2\colon [0, L] \to \mathbb{R}^2$ *parametrise the same curve* if there exists a continuous bijection $\psi\colon [0, L] \to [0, L]$ such that $\gamma_1 = \gamma_2 \circ \psi$. A subset $U \subset \mathbb{R}^2$ is said to be *centrally symmetric* if it coincides with its reflection $\{-x\colon x \in U\}$ with respect to the origin.

### 2.2. Inner parallel curves

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected smooth domain. In this subsection we recall some properties of the inner parallel curves of $\Omega$.

**Parametrisation of the boundary.** Let us denote by $L = |\partial\Omega|$ the perimeter of $\Omega$. Consider the arc-length parametrisation of $\partial\Omega$ oriented in the counter-clockwise direction,

$$s \in \mathbb{R}/(L\mathbb{Z}) \mapsto \gamma(s) = (\gamma_1(s), \gamma_2(s))^\top \in \mathbb{R}^2,$$

which identifies $\partial\Omega$ with $\mathbb{R}/(L\mathbb{Z}) \simeq [0, L)$; the function $\gamma$ is smooth which matches with the smoothness hypothesis we imposed on $\partial\Omega$.

The vector $\gamma'(s) = (\gamma_1'(s), \gamma_2'(s))^\top$ is the unit tangent vector to $\partial\Omega$ at $\gamma(s)$ and points in the counter-clockwise direction. The unit normal vector at $\gamma(s)$ pointing inwards the domain $\Omega$ is given by

$$\mathbf{n}(s) = (-\gamma_2'(s), \gamma_1'(s))^\top.$$

We introduce the curvature

$$\kappa(s) := \gamma_2''(s)\gamma_1'(s) - \gamma_1''(s)\gamma_2'(s) \tag{2.1}$$

of $\partial\Omega$ at the point $\gamma(s)$. In particular, the Frenet formula

$$\gamma''(s) = \kappa(s)\mathbf{n}(s), \tag{2.2}$$

holds. Recall that, since $\partial\Omega$ is a smooth closed simple curve, the total curvature identity [20, Corollary 2.2.2] yields

$$\int_0^L \kappa(s)\,\mathrm{d}s = 2\pi. \tag{2.3}$$

We remark that within the chosen sign convention the curvature of a convex domain is non-negative.

**Properties of inner parallel curves.** We define the *in-radius* of $\Omega$ by

$$r_{\mathrm{i}}(\Omega) := \max_{x\in\Omega} \rho(x),$$

where $\rho$ is the distance function given by

$$\rho\colon \Omega \to \mathbb{R}_+, \quad \rho(x) := \inf_{y\in\partial\Omega} |x-y|. \tag{2.4}$$

Recall that the inner parallel curve for $\Omega$ is the level set of the distance function

$$S_t = \{x \in \overline{\Omega}\colon \rho(x) = t\}, \quad t \in [0, r_{\mathrm{i}}(\Omega)).$$

For almost every $t \in (0, r_i(\Omega))$, the inner parallel curve $S_t$ is a finite union of disjoint piecewise smooth simple closed curves, and the curve $S_t$ admits a parametrisation as in Proposition 2.1 below, which was proved in [16], see also [28, Theorem 4.4.1] and [27, Proposition A.1] for more modern presentations and further refinements.

**Proposition 2.1.** *There exists a subset $\mathcal{L} \subset [0, r_{\mathrm{i}}(\Omega))$, whose complement is of Lebesgue measure zero, such that for any $t \in \mathcal{L}$, there exist $m \in \mathbb{N}$ and*

$$0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \le L,$$

*such that the inner parallel curve $S_t$ consists of the union of finitely many smooth curves parametrised by*

$$[a_k, b_k] \ni s \mapsto \gamma(s) + t\mathbf{n}(s), \quad k \in \{1, 2, \ldots, m\},$$

*which forms a union of finitely many piecewise-smooth simple closed curves.*

Consider the mapping

$$(s,t) \in \mathbb{R}/(L\mathbb{Z}) \times \big(0, r_{\mathrm{i}}(\Omega)\big) \mapsto \Phi(s,t) := \gamma(s) + t\mathbf{n}(s) \in \mathbb{R}^2 \tag{2.5}$$

According to [21, Theorem 5.25], there exists $t_\star \in (0, r_{\mathrm{i}}(\Omega))$ such that the restriction of the mapping $\Phi$ in (2.5) to the set $\mathbb{R}/(L\mathbb{Z}) \times (0, t_\star)$ is a smooth diffeomorphism onto its range. The range of this restriction is then given by a tubular neighbourhood of $\partial\Omega$. It is not difficult to verify that $S_t = \Phi(\mathbb{R}/(L\mathbb{Z}), t)$ for all $t \in (0, t_\star)$. However, for $t \geq t_\star$ the same property, in general, does not hold. It can also be easily checked that for all $t \in (0, t_\star)$, the inner parallel curve $S_t$ is connected and $|S_t| = L - 2\pi t$.

**Lemma 2.2.** *Let $t \in \mathcal{L}$ and the associated numbers $m \in \mathbb{N}$, $\{a_k\}_{k=1}^m$, $\{b_k\}_{k=1}^m$, be as in Proposition 2.1. Then, for any $k \in \{1, 2, \ldots, m\}$ and any $s_0 \in [a_k, b_k]$, it holds that $\kappa(s_0) \leq \frac{1}{t}$.*

*Proof.* Let us introduce the notation

$$\mathsf{m}(t) := \gamma(s_0) + t\mathbf{n}(s_0).$$

By the Frenet formula (2.2), one has $\gamma''(s_0) = \kappa(s_0)\mathbf{n}(s_0)$ and by Taylor's formula near $s_0$ we get

$$\gamma(s) = \gamma(s_0) + (s - s_0)\gamma'(s_0) + \frac{1}{2}(s - s_0)^2\kappa(s_0)\mathbf{n}(s_0) + \mathcal{O}(|s - s_0|^3), \quad s \to s_0.$$

Consequently, using orthogonality of $\gamma'(s_0)$ and $\mathbf{n}(s_0)$, we get

$$\operatorname{dist}(\gamma(s), \mathsf{m}(t))^2 = t^2 + (1 - t\kappa(s_0))(s - s_0)^2 + \mathcal{O}(|s - s_0|^3), \quad s \to s_0.$$

Since $\mathsf{m}(t) \in S_t$, then $\operatorname{dist}(\gamma(s), \mathsf{m}(t)) \geq t$ for $s$ in a neighbourhood of $s_0$, which is possible only when $1 - t\kappa(s_0) \geq 0$. ∎

In the next lemma we provide a simple construction of a closed but not necessarily simple curve which contains $S_t$. The geometric bound as in Theorem 1.2 on its length will only hold for $t$ not larger than the inverse of the maximum of the curvature for the curve $\gamma$:

$$\kappa_{\max}(\Omega) := \max_{s \in \mathbb{R}/(L\mathbb{Z})} \kappa(s)$$

**Lemma 2.3.** *For $t \in (0, r_{\mathrm{i}}(\Omega))$, the mapping*

$$s \in \mathbb{R}/(L\mathbb{Z}) \mapsto \alpha_t(s) := \gamma(s) + t\mathbf{n}(s) \in \mathbb{R}^2$$

*parametrises a smooth closed, not necessarily simple curve such that*

(i) $S_t \subset \alpha_t([0, L])$ *for all* $t \in \mathcal{L}$;

(ii) $\ell(\alpha_t) = L - 2\pi t$ *for all* $t \leq \frac{1}{\kappa_{\max}(\Omega)}$;

(iii) $\ell(\alpha_t) > L - 2\pi t$ *for all* $\frac{1}{\kappa_{\max}(\Omega)} < t < r_{\mathrm{i}}(\Omega)$.

**Remark 2.4.** According to [26], the domain $\Omega$ contains a disk of radius $\frac{1}{\kappa_{\max}(\Omega)}$. In other words, it holds that

$$r_{\mathrm{i}}(\Omega) \geq \frac{1}{\kappa_{\max}(\Omega)}. \tag{2.6}$$

The equality occurs for some special types of domains such as a disk or (going beyond smooth domains) for a convex hull of two disjoint disks of equal radius. The original work [26] is hardly available and the complete proof can be found in [17, Proposition 2.1]. Inequality (2.6) shows that, in general, a more sophisticated method than in Lemma 2.3 is needed to construct for any $t \in (0, r_{\mathrm{i}}(\Omega))$ a closed curve of length not larger than $L - 2\pi t$, which contains the inner parallel curve $S_t$.

*Proof of Lemma* 2.3. Notice that smoothness of $\gamma$ on $\mathbb{R}/(L\mathbb{Z})$ ensures that $\alpha_t$ is smooth on $\mathbb{R}/(L\mathbb{Z})$ as well. It is also clear from Proposition 2.1 that $S_t \subset \alpha_t([0, L])$ for all $t \in \mathcal{L}$.

Using the identity (2.3), we get for any $t \leq \frac{1}{\kappa_{\max}(\Omega)}$

$$\ell(\alpha_t) = \int_0^L |\dot{\alpha}(s)| \,\mathrm{d}s = \int_0^L |1 - t\kappa(s)| \,\mathrm{d}s = \int_0^L (1 - t\kappa(s)) \,\mathrm{d}s = L - 2\pi t.$$

Analogously, we get for any $t \in \left(\frac{1}{\kappa_{\max}(\Omega)}, r_{\mathrm{i}}(\Omega)\right)$

$$\ell(\alpha_t) = \int_0^L |\dot{\alpha}(s)| \,\mathrm{d}s = \int_0^L |1 - t\kappa(s)| \,\mathrm{d}s > \int_0^L (1 - t\kappa(s)) \,\mathrm{d}s = L - 2\pi t. \qquad \blacksquare$$

In the remainder of this subsection we will discuss the properties of $S_t$ for a centrally symmetric domain $\Omega$. The central symmetry of $\Omega$ is inherited by $\partial\Omega$. Consequently, if $y = \gamma(s) \in \partial\Omega$, we know that $-y \in \partial\Omega$ too; moreover the centroid of $\partial\Omega$ is the origin, so

$$\int_0^L \gamma(s) \,\mathrm{d}s = 0.$$

**Lemma 2.5.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected, centrally symmetric smooth domain. Then, for all $t \in (0, r_{\mathrm{i}}(\Omega))$, the inner parallel curve $S_t \subset \Omega$ is centrally symmetric.*

*Proof.* Let $x \in S_t$ be fixed. Then $\rho(x) = t$ and there exists a point $y \in \partial\Omega$ such that $|x - y| = t$. Observe now that $\rho(-x) \leq t$, because $-y \in \partial\Omega$. In the case that $\rho(-x) < t$, there would exist a point $z \in \partial\Omega$ such that $|z + x| < t$. Since $-z \in \partial\Omega$, we would get that $\rho(x) < t$, leading to a contradiction. Thus, we infer that $\rho(-x) = t$ and hence $-x \in S_t$. $\blacksquare$

### 2.3. An auxiliary geometric inequality

The aim of this subsection is to provide a geometric inequality, which will be used in the proof of Theorem 1.2.

**Hypothesis 2.6.** Let $c_1, c_2 \in \mathbb{R}^2$ and $t > 0$ be fixed. Let a smooth simple non-closed curve $\Gamma \subset \mathbb{R}^2$ be parametrised by the arc-length via the mapping $\gamma\colon [s_1, s_2] \to \mathbb{R}^2$, $s_1 < s_2$. Assume that the following properties hold:

(i) $p_j := \gamma(s_j) \in \partial\mathcal{B}_t(c_j)$ for $j = 1, 2$;

(ii) $\gamma'(s_j)$ is tangent to $\partial\mathcal{B}_t(c_j)$ in the counterclockwise direction for $j = 1, 2$.y;

(iii) $\Gamma$ can be extended up to a closed simple curve so that $\mathcal{B}_t(c_1) \cup \mathcal{B}_t(c_2)$ is surrounded by this extension.

**Proposition 2.7.** *Under Hypothesis 2.6 the following geometric inequality holds:*

$$|\Gamma| \geq |c_1 - c_2| + t \int_{s_1}^{s_2} \kappa(s)\,\mathrm{d}s, \tag{2.7}$$

*where $\kappa$ is the curvature of $\Gamma$ defined as in (2.1).*

*Proof.* The proof relies on an abstract result due to Chillingworth [8, Theorem 3.3], which states that two closed homotopic curves are regularly homotopic if the curves are direct, that is, the corresponding curves in the covering space are simple.

The curve $\Gamma$ is homotopic to its projection $\Sigma$ on the convex hull of the two disks. After modifying $\Sigma$ suitably so we avoid nullhomotopic loops, we can extend $\Gamma$ and $\Sigma$ to closed direct curves $\widetilde{\Gamma}$, $\widetilde{\Sigma}$ using the same extension. By [8, Theorem 3.3], $\widetilde{\Gamma}$ and $\widetilde{\Sigma}$ are regularly homotopic, so the integral over their curvatures agree: $\int_{\widetilde{\Gamma}} \kappa = \int_{\widetilde{\Sigma}} \kappa$, see for example [30]. Since $\Gamma$ and $\Sigma$ were extended using the same extension, we also get $\int_{\Gamma} \kappa = \int_{\Sigma} \kappa$. Finally, one can check that $\Sigma$ satisfies (2.7) and by $|\Gamma| \geq |\Sigma|$, we obtain (2.7) for $\Gamma$. ■

## 3. Proof of Theorem 1.2 – Covering inner parallel curves

The aim of this section is to prove the following theorem, which yields Theorem 1.2.

**Theorem 3.1.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain and let $S_t$ for $t \geq 0$ be the corresponding inner parallel curves defined in (1.1). Then there exists a subset $\mathcal{L} \subset [0, r_{\mathrm{i}}(\Omega))$ such that $(0, r_{\mathrm{i}}(\Omega)) \setminus \mathcal{L}$ is of Lebesgue measure zero, and for any $t \in \mathcal{L}$, there exists a piecewise smooth continuous mapping $\sigma_t\colon \mathbb{R}/(L\mathbb{Z}) \to \mathbb{R}^2$ such that*

(i) $S_t \subset \sigma_t([0, L])$;

(ii) $\ell(\sigma_t) \leq L - 2\pi t$;

(iii) *for centrally symmetric domains* $\Omega$, *the curve* $\sigma_t([0, L])$ *is centrally symmetric too.*

In the following, let the set $\mathcal{L}$ be as in Proposition 2.1. Let $t \in \mathcal{L}$ be fixed. Then, there exist $m \in \mathbb{N}$ and

$$0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \leq L$$

such that the inner parallel curve $S_t$ is given by

$$S_t = \bigcup_{k=1}^{m} \{\gamma(s) + t\mathbf{n}(s) \colon s \in [a_k, b_k]\}.$$

Without loss of generality, we can always reparametrise the boundary of $\Omega$ so that $a_1 = 0$. In the following we assume that such a re-parametrisation is performed and that $a_1 = 0$. Moreover, for the sake of convenience we also set $a_{m+1} := L$ and $b_{m+1} := b_1$.

The inner parallel curve $S_t$, $t \in \mathcal{L}$, is not necessarily connected and, in general, it consists of finitely many piecewise-smooth, simple, closed curves. Note that, while $S_t$ consists of finitely many piecewise smooth simple closed curves, the pieces $C_k := \{\gamma(s) + t\mathbf{n}(s) \colon s \in [a_k, b_k]\}$ are not necessarily closed curves. Our aim is to construct a piecewise smooth, closed, not necessarily simple curve, which contains $S_t$ and whose length is not larger than $L - 2\pi t$. The idea is to connect the terminal point of $\{\gamma(s) + t\mathbf{n}(s) \colon s \in [a_k, b_k]\}$ with the starting point of $\{\gamma(s) + t\mathbf{n}(s) \colon s \in [a_{k+1}, b_{k+1}]\}$ for all $k \in \{1, 2, \ldots, m\}$.

Using the computation in the proof of Lemma 2.3, and that $\kappa(s) \leq \frac{1}{t}$ for all $s \in [a_k, b_k]$ by Lemma 2.2, we get the following expression for the length of $S_t$:

$$|S_t| = \sum_{k=1}^{m} \int_{a_k}^{b_k} |1 - t\kappa(s)| \,\mathrm{d}s = \sum_{k=1}^{m} \int_{a_k}^{b_k} (1 - t\kappa(s)) \,\mathrm{d}s. \tag{3.1}$$

Let us define the points

$$\mathfrak{p}_k = \gamma(a_k) + t\mathbf{n}(a_k), \quad \mathfrak{q}_k := \gamma(b_k) + t\mathbf{n}(b_k), \quad \text{for } k \in \{1, 2, \ldots, m+1\},$$

and the line segments connecting them

$$\mathcal{I}_k := \{(1 - s)\mathfrak{q}_k + s\mathfrak{p}_{k+1} \colon s \in [0, 1]\}, \quad k \in \{1, 2, \ldots, m\}.$$

In the case that $b_m = L$, the line segment $\mathcal{I}_m$ reduces to a single point. Note that, even when $b_k < a_{k+1}$, the line segment $\mathcal{I}_k$ could reduce to a single point. The piecewise smooth continuous mapping $\sigma_t : \mathbb{R}/(L\mathbb{Z}) \to \mathbb{R}^2$ defined by

$$\sigma_t(s) = \begin{cases} \gamma(s) + t\mathbf{n}(s), & s \in [a_k, b_k] \quad \text{for } k \in \{1, \dots, m\}, \\ \dfrac{a_{k+1} - s}{a_{k+1} - b_k}\mathfrak{q}_k + \dfrac{s - b_k}{a_{k+1} - b_k}\mathfrak{p}_{k+1}, & s \in [b_k, a_{k+1}] \quad \text{for } k \in \{1, \dots, m\}, \end{cases}$$

parametrises a closed, not necessarily simple curve in $\mathbb{R}^2$. The property (i) in the formulation of Theorem 3.1 follows from Proposition 2.1 and the construction of the mapping $\sigma_t$.

Note that the curve $\gamma([b_k, a_{k+1}])$, $k \in \{1, \dots, m\}$, satisfies Hypothesis 2.6 with $c_1 = \mathfrak{q}_k$ and $c_2 = \mathfrak{p}_{k+1}$. Thus, it follows from Proposition 2.7 and since $\gamma$ is parametrised by arc-length that for any $k \in \{1, 2, \dots, m\}$

$$|\mathcal{I}_k| \le |\gamma([b_k, a_{k+1}])| - t\int_{b_k}^{a_{k+1}} \kappa(s)\,\mathrm{d}s = a_{k+1} - b_k - t\int_{b_k}^{a_{k+1}} \kappa(s)\,\mathrm{d}s.$$

Combining formula (3.1) for the length of $S_t$ with the upper bounds on the lengths of line segments $\{\mathcal{I}_k\}_{k=1}^m$ we get

$$\begin{aligned} \ell(\sigma_t) = |S_t| + \sum_{k=1}^m |\mathcal{I}_k| &\le \sum_{k=1}^m \int_{a_k}^{b_k} (1 - t\kappa(s))\,\mathrm{d}s + \sum_{k=1}^m \int_{b_k}^{a_{k+1}} (1 - t\kappa(s))\,\mathrm{d}s \\ &\le L - t\int_0^L \kappa(s)\,\mathrm{d}s = L - 2\pi t, \end{aligned}$$

where we used the total curvature identity (2.3) in the last step. Hence, we get the property (ii) in the formulation of Theorem 3.1.

Finally, if $\Omega$ is centrally symmetric, then by Lemma 2.5 so is $S_t$. Consequently, to every piece $C_k$ of $S_t$ joining $\mathfrak{p}_k$ to $\mathfrak{q}_k$, there corresponds a curve $C_{k^*}$ which is the symmetric of $C_k$ about the origin. This forces the number $m$ of the curves $C_k$ to be even, unless it is equal to one, and therefore we get that the corresponding joining segments $(\mathcal{I}_k)_{1 \le k \le m}$ constitute a centrally symmetric set. This proves that the image of $\sigma_t$ is centrally symmetric, thereby establishing (iii) in the formulation of Theorem 3.1. This proves Theorem 1.2.

## 4. Proof of Theorem 1.1 – An isoperimetric inequality for moments of inertia

Let $t \in \mathcal{L}$ and the mapping $\sigma_t$ be as constructed in the proof of Theorem 3.1, which defines a closed curve $\Sigma_t$. Since the moment of inertia of a curve about a point $p$ is minimal when $p$ is the centroid of the curve, it suffices to prove (1.5) with $c(t)$ the centroid of $\Sigma_t$. Let us introduce the notation $L_t := \ell(\sigma_t)$ and re-parametrise the curve $\Sigma_t$ by the arc-length via the mapping $\widetilde{\sigma}_t : \mathbb{R}/(L_t\mathbb{Z}) \to \mathbb{R}^2$. Clearly, we have $\widetilde{\sigma}_t \in H^1(\mathbb{R}/(L_t\mathbb{Z}))$ thanks to the regularity of $\sigma_t$. Furthermore, by centring the coordinates at the centroid $c(t)$ of $\Sigma_t$, we can assume that $c(t) = 0$, and consequently

$$\int_0^{L_t} \widetilde{\sigma}_t(s)\,\mathrm{d}s = 0. \tag{4.1}$$

Using the inclusion $S_t \subset \widetilde{\Sigma}_t$ and applying the Wirtinger inequality [15, Section 7.7],

$$\int_0^{L_t} |\widetilde{\sigma}_t(s)|^2\,\mathrm{d}s \le \frac{|L_t|^2}{4\pi^2}\int_0^{L_t} |\widetilde{\sigma}_t'(s)|^2\,\mathrm{d}s, \tag{4.2}$$

we get

$$\int_{S_t} |x|^2\,\mathrm{d}\mathcal{H}^1(x) \le \int_0^{L_t} |\widetilde{\sigma}_t(s)|^2\,\mathrm{d}s \le \frac{|L_t|^3}{4\pi^2} \le \frac{(L-2\pi t)^3}{4\pi^2}, \tag{4.3}$$

where we employed that $L_t \le L - 2\pi t$ in the last step. Therefore, (1.5) is proved.

Assuming that there is equality in (1.5), then we get from (4.3) that $L_t = L - 2\pi t$ and there is equality is (4.2). Under the conditions (4.1) and $|\tilde{\sigma}_t'(s)| = 1$, equality happens in (4.2) if and only if $\tilde{\sigma}_t(s) = \frac{L_t}{2\pi}\mathrm{e}^{\pm \mathrm{i}2\pi(s-s_0)/L_t}$ for some $s_0 \in \mathbb{R}$ and $\Sigma_t$ is a circle (here we identify $\mathbb{C}$ and $\mathbb{R}^2$). Moreover, knowing that $L_t = L - 2\pi t$ and $S_t = \Sigma_t$, we get that $S_t$ is the circle of centre 0 and radius $\frac{L}{2\pi} - t$, and consequently, the domain $\Omega$ with perimeter $L$ contains the disk $\mathcal{B}$ of radius $\frac{L}{2\pi}$, hence $\Omega = \mathcal{B}$ is a disk, thanks to the geometric isoperimetric inequality.

Finally, if $\Omega$ is a disk, then $S_t$ is a circle of radius $\frac{L}{2\pi} - t$ and equality in (1.5) occurs.

**Remark 4.1.** The hypothesis of simple connectivity is necessary in Theorem 1.1 as we demonstrate in the following example. Let $a \in \left(0, \frac{1}{4}\right)$ be a parameter, which we will send to zero later. Consider the annulus $\Omega_a := B_{1-a}(0) \setminus \overline{B_a(0)}$, where we denote by $B_r(x)$ the open disk centred at $x \in \mathbb{R}^2$ of radius $r > 0$. Then the corresponding $\mathcal{B}$ with the same perimeter as $\Omega_a$ is $\mathcal{B} := B_1(0)$. Furthermore, for all $t < \frac{1}{4}$, we have $S_t(\Omega_a) = \partial B_{a+t}(0) \cup \partial B_{1-(a+t)}(0)$, and $S_t(\mathcal{B}) = \partial B_{1-t}(0)$. Also note that the centroid of each $S_t(\Omega_a)$ is the origin. Then

$$\frac{1}{2\pi} \int\limits_{S_t(\Omega_a)} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) = (a+t)^3 + (1-(a+t))^3$$

and

$$\frac{1}{2\pi} \int\limits_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) = (1-t)^3.$$

Letting $a \to 0$, we find that

$$\frac{1}{2\pi} \int\limits_{S_t(\Omega_a)} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) \to t^3 + (1-t)^3 > \frac{1}{2\pi} \int\limits_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x),$$

where the convergence is uniform for $t \in [t_1, t_2]$ for any $0 < t_1 < t_2 < \frac{1}{4}$. This shows that Theorem 1.1 cannot hold in this case.

## 5. Proof of Theorem 1.3 – An isoperimetric inequality for $p$-th moments

In this section, we study $p$-th moments of inner parallel curves and prove Theorem 1.3. We show Theorem 1.3 (i) (extension of Theorem 1.1 to $p$-th moments, for $0 \le p \le 2$) in Corollary 5.1 with the help of Jensen's inequality. Proposition 5.6 yields Theorem 1.3 (ii) and (iii).

**Corollary 5.1.** *Let $p \in [0,2]$ and suppose that $\Omega \subset \mathbb{R}^2$ is a smooth, bounded and simply-connected domain. Then, for almost every $t \ge 0$,*

$$\int\limits_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\,\mathcal{H}^1(x) \le \int\limits_{S_t(\mathcal{B})} |x|^p \, \mathrm{d}\,\mathcal{H}^1(x), \tag{5.1}$$

*where $c(t) \in \mathbb{R}^2$ is the centroid of $S_t(\Omega)$ and where $\mathcal{B}$ is the disk centred at the origin and with the same perimeter as $\Omega$. Here $S_t(\cdot)$ is defined in (1.1), and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. For $p \neq 0$ and $t \in \left[0, \frac{|\partial\Omega|}{2\pi}\right)$, the equality in (5.1) is attained if and only if $\Omega$ is a disk*

*Proof.* For $p = 0$, (5.1) reduces to the well-known bound $|S_t| \le L - 2\pi t$, see (1.2).

Let us take $p \in (0,2]$. Since $\frac{2}{p} \ge 1$, we write by Jensen's inequality,

$$\left( \int\limits_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\,\mathcal{H}^1(x) \right)^{2/p} \le |S_t(\Omega)|^{\frac{2}{p}-1} \int\limits_{S_t(\Omega)} |x - c(t)|^2 \, \mathrm{d}\,\mathcal{H}^1(x).$$

To finish the proof, we use that $|S_t| \leq L - 2\pi t$, apply Theorem 1.1, and note that

$$\int_{S_t(\mathcal{B})} |x|^p \, \mathrm{d}\mathcal{H}^1(x) = 2\pi\Big(\frac{L}{2\pi} - t\Big)^{1+p}.$$

Since the inequality in Theorem 1.1 is strict for all $t \in \big[0, \frac{|\partial\Omega|}{2\pi}\big)$ when $\Omega$ is not a disk, this also holds for (5.1). ∎

Secondly, we formulate the corresponding variational problem to Theorem 1.3.

**Definition 5.2.** Given $p > 0$, we define

$$C_p := \sup_{\Gamma} \frac{\int_{\Gamma} |x|^p \, \mathrm{d}\mathcal{H}^1(x)}{\int_{\partial\mathcal{B}} |x|^p \, \mathrm{d}\mathcal{H}^1(x)} = \sup_{\Gamma} \frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_{\Gamma} |x|^p \, \mathrm{d}\mathcal{H}^1(x), \tag{5.2}$$

where the supremum is taken over all centrally symmetric, closed Lipschitz curves $\Gamma$ and $\mathcal{B}$ is a disk centred at the origin with $|\partial\mathcal{B}| = |\Gamma|$.

**Remark 5.3.** By scaling, we find that $C_p \in (0, \infty)$. By testing with $\Gamma = \partial\mathcal{B}$, we find $C_p \geq 1$ for all $p$. We have already shown that $C_p = 1$ for all $p \in (0, 2]$, see the end of the proof of Theorem 1.1 (or alternatively the result by Hurwitz [18]) combined with Jensen's inequality as in the proof of Corollary 5.1. In fact, using Jensen's inequality as in the proof of Corollary 5.1, one can show that $C_p$ is non-decreasing in $p$.

**Remark 5.4** (Existence of an optimising curve in (5.2))**.** We only sketch a way of proving the existence of an optimising curve in (5.2) here. First note that by scaling, we can restrict ourselves to smooth centrally symmetric curves satisfying $|\Gamma| = 1$. We can approximate these curves by piecewise linear centrally symmetric curves with $|\Gamma| = 1$, so we can take the supremum over such curves instead. Using a finite-step mirroring argument, the supremum stays the same if we only consider convex piecewise linear centrally symmetric curves with $|\Gamma| = 1$. More precisely, in this context we say that a piecewise linear curve is convex if it is the boundary of a convex polygon.

Now, consider a sequence of such curves $(\Gamma_n)_{n\in\mathbb{N}}$ such that the corresponding expression in the supremum in (5.2) converges to $C_p$. Assume that the $\Gamma_n$ are parametrised by arc-length by $\gamma_n\colon [0, 1] \to \mathbb{R}^2$. By $|\Gamma_n| = 1$ and the central symmetry, we have $\Gamma_n \subset B_1(0)$. Due to the arc-length parametrisation of the $\gamma_n$, we obtain that up to a subsequence, they converge uniformly to a Lipschitz continuous function $\gamma\colon [0, 1] \to \mathbb{R}^2$ with Lipschitz constant at most one. To see this, one can for instance use the Arzelà–Ascoli theorem. Since the $\Gamma_n$ are convex, the corresponding curve $\Gamma$ is convex, too.

Furthermore, we have $|\Gamma| = 1$. Proving this is the key step and it uses the convexity. It can be seen by approximating $\Gamma$ by piecewise linear convex centrally symmetric curves with the same length as $\Gamma$, and then showing that the lengths of these piecewise linear convex curves have to be close to $|\Gamma_n| = 1$. Finally, using that $\gamma$ is Lipschitz continuous with Lipschitz constant at most one and $|\Gamma| = 1$, we get $|\gamma'(s)| = 1$ almost everywhere, so

$$\lim_{n\to\infty} \int_{\Gamma_n} |x|^p \,\mathrm{d}\mathcal{H}^1(x) = \lim_{n\to\infty} \int_0^1 |\gamma_n(s)|^p \,\mathrm{d}s = \int_0^1 |\gamma(s)|^p \,\mathrm{d}s = \int_{\Gamma} |x|^p \,\mathrm{d}\mathcal{H}^1(x).$$

Together with $|\Gamma| = 1$, this proves the optimality of the convex curve $\Gamma$. Also note that any optimal curve needs to be convex.

The next proposition shows that $C_p$ is not constant.

**Proposition 5.5.** *We have $C_p > 1$ for all $p$ large enough.*

*Proof.* Consider the curve $\Gamma$ parametrised by $\gamma\colon [0,1] \to \mathbb{R}^2$, $\gamma(s) = (\gamma_1(s), \gamma_2(s))^\top$ with $\gamma_2(s) = 0$ for all $s \in [0,1]$ and

$$\gamma_1(s) = \begin{cases} -\dfrac{1}{4} + s & \text{for } s \in \left[0, \dfrac{1}{2}\right], \\[2mm] \dfrac{3}{4} - s & \text{for } s \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

$\Gamma$ is a closed curve that is parametrised by arc-length, centrally symmetric, its shape is the doubly covered interval $[-\frac{1}{4}, \frac{1}{4}] \times \{0\}$, and its length is $|\Gamma| = 1$. We have

$$\lim_{p\to\infty} \left( \int_{\Gamma} |x|^p \,\mathrm{d}\mathcal{H}^1(x) \right)^{1/p} = \lim_{p\to\infty} \left( \int_0^1 |\gamma_1(s)|^p \,\mathrm{d}s \right)^{1/p} = \sup_{s\in[0,1]} |\gamma_1(s)| = \frac{1}{4},$$

and therefore, by $|\Gamma| = 1$,

$$\lim_{p\to\infty} \left( \frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_{\Gamma} |x|^p \,\mathrm{d}\mathcal{H}^1(x) \right)^{1/p} = \frac{2\pi}{4} > 1.$$

This proves $C_p > 1$ for all $p$ large enough. ∎

This leads to the question of determining the critical value

$$p_* := \sup\{p : p > 0 \text{ and } C_p = 1\}.$$

The following proposition shows that $p_* \leq 3$, see (ii), which we conjecture to be optimal (see Conjecture 1.4) since the disk is a local optimiser for $p < 3$, see (i) below.

**Proposition 5.6.** *The following statements hold.*

(i) *Let $p < 3$. Then the disk is a local optimiser among centrally symmetric curves in the following sense. If $r : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is continuous with $r(\theta) = r(\theta + \pi)$ for all $\theta$, and the curve $\Gamma_\varepsilon$ is parametrised by $\gamma_\varepsilon : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ with*

$$\gamma_\varepsilon(\theta) = \begin{pmatrix} R_\varepsilon(\theta)\cos(\theta) \\ R_\varepsilon(\theta)\sin(\theta) \end{pmatrix}, \quad \text{where } R_\varepsilon(\theta) = 1 + \varepsilon r(\theta), \tag{5.3}$$

*then*

$$\frac{(2\pi)^p}{|\Gamma_\varepsilon|^{p+1}} \int_{\Gamma_\varepsilon} |x|^p \, \mathrm{d}\mathcal{H}^1(x) \leq 1$$

*for all $\varepsilon > 0$ small enough; furthermore, the inequality is strict if $r$ is non-constant.*

(ii) *Let $p > 3$. Then the disk is not optimal, not even locally: There exists a sequence of nearly circular, centrally symmetric closed Lipschitz curves $(\Gamma_n)_{n\in\mathbb{N}}$ converging uniformly to the boundary of the disk for which*

$$\frac{(2\pi)^p}{|\Gamma_n|^{p+1}} \int_{\Gamma_n} |x|^p \, \mathrm{d}\mathcal{H}^1(x) > 1.$$

*Proof.* Consider the curve $\Gamma_\varepsilon$ defined by the parametrisation of $\Gamma_\varepsilon$ in (5.3). Since we assume that $r(\theta) = r(\theta + \pi)$, the curve $\Gamma_\varepsilon$ is centrally symmetric.

It is straightforward to check that with $\gamma_\varepsilon$ defined as in (5.3),

$$\begin{aligned} |\gamma_\varepsilon(\theta)|^p &= 1 + \varepsilon p r(\theta) + \frac{\varepsilon^2}{2} p(p-1)|r(\theta)|^2 + \mathcal{O}(\varepsilon^3), \\ |\gamma_\varepsilon'(\theta)| &= 1 + \varepsilon r(\theta) + \frac{\varepsilon^2}{2}|r'(\theta)|^2 + \mathcal{O}(\varepsilon^3), \\ |\Gamma_\varepsilon| &= 2\pi + \varepsilon \int_0^{2\pi} r(\theta)\,\mathrm{d}\theta + \frac{\varepsilon^2}{2} \int_0^{2\pi} |r'(\theta)|^2 \,\mathrm{d}\theta + \mathcal{O}(\varepsilon^3). \end{aligned}$$

With the above formulas in hand, we get

$$\frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} = 2\pi + \varepsilon(p+1)\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta + \varepsilon^2(p+1)\Big(\frac{1}{2}\int_0^{2\pi}|r'(\theta)|^2\,\mathrm{d}\theta + \frac{p}{4\pi}\Big(\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\Big) + \mathcal{O}(\varepsilon^3),$$

and

$$\int_{\Gamma_\varepsilon} |x|^p\,\mathrm{d}\mathcal{H}^1(x) - \frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} = \frac{\varepsilon^2 p}{2}(\mathcal{F}(r) + \mathcal{O}(\varepsilon)), \tag{5.4}$$

where

$$\begin{aligned}\mathcal{F}(r) &= (p+1)\int_0^{2\pi}|r(\theta)|^2\,\mathrm{d}\theta - \frac{p+1}{2\pi}\Big(\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2 - \int_0^{2\pi}|r'(\theta)|^2\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\Big[(p+1)\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2 - |r'(\theta)|^2\Big]\,\mathrm{d}\theta.\end{aligned}$$

We expand $r$ as a Fourier series and notice that the coefficients of the odd indices will vanish, thanks to the symmetry condition on $r$. More precisely, we have

$$\begin{aligned} r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta &= \sum_{n\geq 2} a_n\cos(n\theta) + \sum_{n\geq 2} b_n\sin(n\theta),\\ r'(\theta) &= -\sum_{n\geq 2} n a_n\sin(n\theta) + \sum_{n\geq 2} n b_n\cos(n\theta).\end{aligned}$$

By Parseval's identity, we write

$$\begin{aligned}\int_0^{2\pi}\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\,\mathrm{d}\theta &= \pi\sum_{n\geq 2}\big(|a_n|^2 + |b_n|^2\big),\\ \int_0^{2\pi}|r'(\theta)|^2\,\mathrm{d}\theta &= \pi\sum_{n\geq 2} n^2\big(|a_n|^2 + |b_n|^2\big).\end{aligned}$$

Hence,

$$\int_0^{2\pi}|r'(\theta)|^2\,\mathrm{d}\theta \geq 4\int_0^{2\pi}\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\,\mathrm{d}\theta, \tag{5.5}$$

and consequently, for $p < 3$

$$\mathcal{F}(r) \leq (p-3) \int_0^{2\pi} \left( r(\theta) - \frac{1}{2\pi} \int_0^{2\pi} r(\theta)\,\mathrm{d}\theta \right)^2 \mathrm{d}\theta \leq 0.$$

For $p < 3$, $\mathcal{F}(r)$ vanishes if and only if the function $r$ is constant. To conclude the proof of (i), we take $\varepsilon \to 0$ and note that the disk is a strict local maximiser if and only if we have for all non-constant $r$ and $\varepsilon$ small enough

$$\int_{\Gamma_\varepsilon} |x|^p \,\mathrm{d}\mathcal{H}^1(x) - \frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} < 0.$$

Taking $\varepsilon \to 0$ and using (5.4), we obtain the desired result.

For (ii), note that choosing $r(\theta) := \sin(2\theta)$, we have equality in (5.5), which leads to

$$\mathcal{F}(r) = (p-3) \int_0^{2\pi} \left( r(\theta) - \frac{1}{2\pi} \int_0^{2\pi} r(\theta)\,\mathrm{d}\theta \right)^2 \mathrm{d}\theta.$$

And for $p > 3$, we get $\mathcal{F}(r) > 0$, which yields the claim by (5.4) for $\varepsilon$ small enough. ∎

**Remark 5.7.** Proposition 5.6 does not address the case $p = 3$ since it is a degenerate case: the quantity $\mathcal{F}(r)$ is zero for the optimal choice $r(\theta) := \sin(2\theta)$.

Note that, by Remark 5.3, $C_p$ is non-decreasing in $p$. Furthermore, we deduce that $C_p$ is left-continuous by the continuity in $p$ of the expression inside the supremum in (5.2) for every fixed $\Gamma$. Hence,

$$\{p \in [0, \infty) : C_p = 1\} = [0, p_*].$$

So in order to prove Conjecture 1.4, namely that $p_* = 3$, it suffices to show $C_p = 1$ for all $p < 3$.

**Remark 5.8** (Optimal curve for large $p$). The curve $\Gamma$ considered in the proof of Proposition 5.5 is the optimal curve in the case $p = \infty$ when replacing the $L^p$ norms by the corresponding supremum norms. For values of $p \in (3, \infty)$, we conjecture the optimal curve to be a deformed circle that degenerates into the curve from the proof of Proposition 5.5 as $p$ increases, compare also with the proof of Proposition 5.6 (ii) above.

## 6. Proof of Theorem 1.5 – Applications to the magnetic Robin Laplacian

In this section, we show that Theorem 1.1 can be used to relax the assumptions on the domain in the isoperimetric inequality for the lowest eigenvalue of the magnetic Robin Laplacian on a bounded domain with a negative boundary parameter, recently obtained in [19] by the second and the third authors of the present paper.

The operator we study involves the vector potential (magnetic potential)

$$\mathbf{A}(x) := \frac{1}{2}(-x_2, x_1)^\top \quad (x = (x_1, x_2)).$$

and two parameters, $b \geq 0$ standing for the intensity of the magnetic field and $\beta \leq 0$, the Robin parameter, appearing in the boundary condition. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected smooth domain. Our magnetic Robin Laplacian, $\mathsf{H}_\Omega^{\beta,b}$, is the self-adjoint operator defined by the closed, symmetric, densely defined and lower semi-bounded quadratic form

$$\mathfrak{q}_\Omega^{\beta,b}[u] := \|(\nabla - \mathrm{i}b\mathbf{A})u\|^2_{L^2(\Omega;\mathbb{C}^2)} + \beta\|u\|^2_{L^2(\partial\Omega)}, \quad \operatorname{dom}\mathfrak{q}_\Omega^{\beta,b} := H^1(\Omega),$$

and it is characterised by

$$\begin{aligned}\operatorname{dom}\mathsf{H}_\Omega^{\beta,b} =& \{u \in H^1(\Omega) : \text{there exists } w \in L^2(\Omega) \text{ such that} \\ & \qquad \mathfrak{q}_\Omega^{\beta,b}[u,v] = (w,v)_{L^2(\Omega)} \text{ for all } v \in \operatorname{dom}\mathfrak{q}_\Omega^{\beta,b}\}, \\ \mathsf{H}_\Omega^{\beta,b}u :=& -(\nabla - \mathrm{i}b\mathbf{A})^2 u = w.\end{aligned}$$

Denoting by $\nu$ the unit inward normal vector on $\partial\Omega$, we observe that functions in $\operatorname{dom}\mathsf{H}_\Omega^{\beta,b}$ satisfy the (magnetic) Robin boundary condition

$$\nu \cdot (\nabla - \mathrm{i}b\mathbf{A})u = \beta u \quad \text{on } \partial\Omega.$$

The isoperimetric inequality obtained in [19] concerns the lowest eigenvalue of $\mathsf{H}_\Omega^{\beta,b}$, which we express in the variational form as follows:

$$\lambda_1^{\beta,b}(\Omega) := \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\mathfrak{q}_\Omega^{\beta,b}[u]}{\|u\|^2_{L^2(\Omega)}}.$$

Denoting by $\mathcal{B}$ the disk in $\mathbb{R}^2$ centred at the origin, with radius $R$ and having the same perimeter $2\pi R = |\partial\Omega|$ as the domain $\Omega$, it is known that the following inequality holds (see [19, Theorem 4.8, Corollary 4.9]):

$$\lambda_1^{\beta,b}(\Omega) \leq \lambda_1^{\beta,b}(\mathcal{B}),$$

provided that

(i) $\beta < 0$ and $0 < b < \min(R^{-2}, 4\sqrt{-\beta}\, R^{-3/2})$ (i.e., the magnetic field's intensity $b$ is of moderate strength); and

(ii) the *inner parallel curves* of $\Omega$ obey the condition

$$\int_{S_t} |x - x_0|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq \frac{(L - 2\pi t)^3}{4\pi^2} \tag{6.1}$$

for some fixed point $x_0 \in \mathbb{R}^2$ and almost all $t \in (0, r_{\mathrm{i}}(\Omega))$. This condition holds for instance, when $\Omega \subset \mathcal{B}$ or when $\Omega$ is *convex* and *centrally symmetric* (see [19, Proposition 4.4])

*Proof of Theorem* 1.5. In view of Theorem 1.1, the condition in (6.1) holds with $x_0 = 0$ for all bounded centrally symmetric simply-connected smooth domains or, more generally, with $x_0$ being the centroid of all $S_t(\Omega)$ for all simply-connected smooth domains $\Omega$ such that the centroid of the inner parallel curve $S_t(\Omega)$ is independent of $t$. Thus, we relaxed the convexity assumption on the domain $\Omega$. We obtain Theorem 1.5 with the choice $b_0(|\partial\Omega|, \beta) = \min\{R^{-2}, 4\sqrt{-\beta} R^{-3/2}\}$, where $R := \frac{|\partial\Omega|}{2\pi}$. ■

## 7. Some direct consequences of Theorems 1.1 and 1.2

### 7.1. A refined bound on the length of the disconnected inner parallel curve

In this subsection we use Theorem 3.1 to get a refined upper bound on the length of the inner parallel curve $S_t$ in the situation when $S_t$ consists of several connected components.

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected smooth domain with perimeter $L > 0$. Let the inner parallel curve $S_t \subset \Omega$ be as in (1.1). For any $t \in \mathcal{L}$, we have by Proposition 2.1 for some $N \in \mathbb{N}$

$$S_t = \bigcup_{n=1}^{N} \Gamma_n, \quad |S_t| \leq L - 2\pi t,$$

where $\{\Gamma_n\}_{n=1}^N$ are piecewise-smooth closed simple curves that are pairwise disjoint. In the case that $S_t$ is connected, one has $N = 1$. However, in general, $N$ can be an arbitrarily large integer number. In the case that $N = 2$, we immediately get as a consequence of Theorem 3.1

$$|S_t| + 2\,\mathrm{dist}(\Gamma_1, \Gamma_2) \leq L - 2\pi t.$$

This observation can be generalised to the case of arbitrary $N \in \mathbb{N}$ to improve Hartman's bound (1.2), see [16], on the length of the inner parallel curve $S_t$.

**Corollary 7.1.** *For all $t \in \mathcal{L}$, it holds that*

$$|S_t| + \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n) \leq L - 2\pi t.$$

*Proof.* The length of the closed piecewise-smooth curve parametrised by the mapping $\sigma_t$ constructed in the proof of Theorem 3.1 is given by

$$\ell(\sigma_t) = |S_t| + \sum_{k=1}^{m} |\mathcal{I}_k|.$$

Every $\mathcal{I}_k$ connects some $\Gamma_{n(k)}$ with some $\Gamma_{n(k+1)}$, and for each $n \in \{1, \ldots, N\}$ there is at least one $k \in \{1, \ldots, m\}$ such that $n = n(k)$. Hence, we get that

$$\sum_{k=1}^{m} |\mathcal{I}_k| \geq \sum_{k=1}^{m} \operatorname{dist}(\Gamma_{n(k)}, \Gamma_{n(k+1)}) \geq \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n).$$

Thus, we conclude that

$$|S_t| + \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n) \leq \ell(\sigma_t) \leq L - 2\pi t. \qquad \blacksquare$$

### 7.2. Moments of inertia of domains

In this subsection, we apply Theorem 1.1 to recover an isoperimetric upper bound on the moment of inertia for the domain $\Omega$ itself leading to an alternative proof of a result due to Hadwiger [14].

Assume that $\Omega \subset \mathbb{R}^2$ is a bounded, simply-connected, centrally symmetric domain. Then

$$\int_{\Omega} |x|^2 \,\mathrm{d}x \leq \int_{\mathcal{B}} |x|^2 \,\mathrm{d}x, \tag{7.1}$$

where $\mathcal{B} \subset \mathbb{R}^2$ is a disk centred at the origin with the same perimeter as $\Omega$.

**Remark 7.2.** The isoperimetric inequality (7.1) can be derived from the inequality by Hadwiger [14], where only convex domains were considered. Let $\mathcal{K} \subset \mathbb{R}^2$ be a bounded convex domain with a Lipschitz boundary. We can translate the domain $\mathcal{K}$ so that the origin becomes the centroid of $\mathcal{K}$ in the sense that

$$\int_{\mathcal{K}} x \,\mathrm{d}x = 0.$$

Let $\mathcal{B}' \subset \mathbb{R}^2$ be the disk centred at the origin of the same perimeter as $\mathcal{K}$. It is proved in [14] that

$$\int_{\mathcal{K}} |x|^2 \, \mathrm{d}x \leq \int_{\mathcal{B}'} |x|^2 \, \mathrm{d}x. \tag{7.2}$$

Let us define the domain $\mathcal{K}$ as the convex hull of the bounded simply-connected centrally symmetric smooth $\Omega \subset \mathbb{R}^2$. Then, the perimeter of $\mathcal{K}$ does not exceed the perimeter of $\Omega$. This is a well-known fact, whose proof can be found, e.g., in [29]. Moreover, the convex domain $\mathcal{K}$ is centrally symmetric as well. Therefore, the origin is the centroid of $\mathcal{K}$. Hence, we get from (7.2)

$$\int_{\Omega} |x|^2 \, \mathrm{d}x \leq \int_{\mathcal{K}} |x|^2 \, \mathrm{d}x \leq \int_{\mathcal{B}'} |x|^2 \, \mathrm{d}x \leq \int_{\mathcal{B}} |x|^2 \, \mathrm{d}x,$$

where we used that the perimeter of $\Omega$ is larger than or equal to the perimeter of $\mathcal{K}$, so the radius of $\mathcal{B}'$ does not exceed the radius of $\mathcal{B}$.

**Remark 7.3.** If we furthermore assume that $\Omega$ is smooth, we have the following proof of (7.1) using Theorem 1.1. Recall that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected smooth domain with the perimeter $L > 0$ and the origin being the centroid of $S_t$ for almost every $t \in (0, r_{\mathrm{i}}(\Omega))$, and that $\mathcal{B} \subset \mathbb{R}^2$ is the disk of radius $R = \frac{L}{2\pi}$, having thus the same perimeter as $\Omega$. By the geometric isoperimetric inequality, we have $|\Omega| \leq |\mathcal{B}|$ and therefore it holds that $R \geq r_{\mathrm{i}}(\Omega)$.

Recall the co-area formula in two dimensions (see [2, Theorem 4.20] and [22]). If $\mathcal{A} \subset \mathbb{R}^2$ is an open set, $f\colon \mathcal{A} \to \mathbb{R}$ is a Lipschitz continuous real-valued function, and $g\colon \mathcal{A} \to \mathbb{R}$ is an integrable function, then we have

$$\int_{\mathcal{A}} g(x) |\nabla f(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(t)} g(x) \, \mathrm{d}\mathcal{H}^1(x) \, \mathrm{d}t. \tag{7.3}$$

Applying the co-area formula (7.3) with $\mathcal{A} = \Omega$, $g(x) = |x|^2$ and $f(x) = \rho(x)$ (the distance function to the boundary of $\Omega$ defined in (2.4)) we get using the inequality in Theorem 1.1,

$$\begin{aligned} \int_{\Omega} |x|^2 \, \mathrm{d}x &= \int_0^{r_{\mathrm{i}}(\Omega)} \int_{S_t} |x|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq \int_0^{r_{\mathrm{i}}(\Omega)} \frac{(L - 2\pi t)^3}{4\pi^2} \, \mathrm{d}t \\ &\leq \int_0^R \frac{(L - 2\pi t)^3}{4\pi^2} \, \mathrm{d}t = 2\pi \int_0^R (R - t)^3 \, \mathrm{d}t = \frac{\pi R^4}{2} = \int_{\mathcal{B}} |x|^2 \, \mathrm{d}x. \end{aligned}$$

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**Calculus of Variations**

# Hardy–Sobolev interpolation inequalities

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**Abstract**
We derive a family of interpolation estimates which improve Hardy's inequality and cover the Sobolev critical exponent. We also determine all optimizers among radial functions in the endpoint case and discuss open questions on nonrestricted optimizers.

**Mathematics Subject Classification** 35A23 · 35Q55

## 1 Introduction

The classical Hardy inequality states that for every dimension $d \geq 3$,

$$h[u] := \int_{\mathbb{R}^d} |\nabla u(x)|^2 \,\mathrm{d}x - \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x \geq 0, \quad \forall u \in \dot{H}^1(\mathbb{R}^d). \tag{1}$$

In this short article we are interested in interpolation inequalities involving the quadratic form $h[u]$ and $L^p$-norms of $u$. A classical result in this direction is the Gagliardo–Nirenberg type inequality

$$h[u]^\theta \|u\|_{L^2}^{2(1-\theta)} \geq C\|u\|_{L^q}^2, \quad \forall u \in H^1(\mathbb{R}^d) \tag{2}$$

for a constant $C > 0$ independent of $u$, which holds for every

$$d \geq 3, \quad 2 < q < 2^* = \frac{2d}{d-2}, \quad \theta = d\left(\frac{1}{2} - \frac{1}{q}\right).$$

The inequality (2) can be deduced from the results of Brezis and Vázquez [3, Theorem 4.1 and Extension 4.3]; see [23] for related results. The bound (2) can be also derived from Sobolev's embedding theorem and the kinetic estimate

$$h[u]^\theta \|u\|_2^{2(1-\theta)} \geq C\|(-\Delta)^{s/2} u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^d), \tag{3}$$

for $s \in (0, 1)$ and $\theta = \theta(s)$, which was proved by Frank [6, Theorem 1.2]. Both of (2) and (3) have been extended to the fractional Laplacian in [6], motivated by applications in the

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asymptotic behavior of large Coulomb systems [20] and the stability of relativistic matter [6, 8].

Note that the restriction $q < 2^*$ in (2) is necessary, namely the quadratic form $h[u]$ is really weaker than $\|\nabla u\|_{L^2}^2$. Here we are interested in a replacement of (2) which covers the critical power $q = 2^*$, with the expense that the $L^2$-norm is replaced by the energy associated with the inverse square potential. We have

**Theorem 1** *(Hardy-Sobolev interpolation inequality) If $d = 3$ and $\theta = 1/3$, then the inequality*

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}\,\mathrm{d}x\right)^{\theta} \left(\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}\,\mathrm{d}x\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2 \tag{4}$$

*holds with a constant $C = C(d,\theta) > 0$ independent of $u \in \dot{H}^1(\mathbb{R}^d)$. Moreover, (4) does not hold if $d \geq 4$ or if $\theta \neq 1/3$.*

**Remark 1** The bound (4) is invariant under translations and dilations. Note that for the first term on the left-hand side, Hardy's inequality (1) is equivalent to

$$\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}\,\mathrm{d}x \geq 0, \quad \forall u \in \dot{H}^1(\mathbb{R}^d).$$

For the second term, it is important to include $\sup_{y\in\mathbb{R}^d}$ since otherwise this term can be made arbitrarily small by translation $u \mapsto u(\cdot - z)$ with $|z| \to \infty$.

**Remark 2** For all $d \geq 3$ and $1 - 2/d \leq \theta \leq 1$ we have

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\theta} \left(\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}\,\mathrm{d}x\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2, \quad \forall u \in H^1(\mathbb{R}^d). \tag{5}$$

This is a consequence of the improved Sobolev inequality involving Morrey norms

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\theta} \left(\sup_{R>0, x\in\mathbb{R}^d} R^{-2} \int_{B(x,R)} |u|^2\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2, \quad \forall u \in \dot{H}^1(\mathbb{R}^d), \tag{6}$$

which was proved by Palatucci–Pisante [17, Theorem 1], using subtle weighted $L^p$-estimates for Riesz potentials in [19] and Calderón-Zygmund type techniques in the spirit of the Fefferman–Phong argument [5]. The bound (6) is helpful to obtain the compactness of minimizing sequences of the critical Sobolev inequality; see [17, Theorem 3] for details. In contrast, our inequality (4) is stronger than (5) and it only holds for the special case $d = 1/\theta = 3$.

In the next result, we extend (4) by replacing the gradient term $\|\nabla u\|_{L^2}$ by $\|\nabla u\|_{L^p}$, as well as replacing the $L^{2^*}$-norm by the $L^{p^*,r}$-Lorentz norm. Recall that (see [9, Definition 1.4.6 and Proposition 1.4.9])

$$\|u\|_{L^{p,r}} = \|u\|_{p,r} = \begin{cases} \left(p\int_0^\infty s^{r-1}|\{|u|>s\}|^{r/p}\,\mathrm{d}s\right)^{1/r}, & 0 < r < \infty, \\ \sup_{s>0} s|\{|u|>s\}|^{1/p}, & r = \infty. \end{cases}$$

**Theorem 2** *[Hardy-Sobolev inequalities with Lorentz norms] Let* $d \geq 2$, $p \in [2, d)$, $p^* = pd/(d-p)$, $r \in [p, \infty]$ *and*

$$\theta \in \left[\frac{p}{\min(r, p^*)}, \frac{1}{p} - \frac{1}{r}\right]. \tag{7}$$

*Then*

$$\left(\int_{\mathbb{R}^d} |\nabla u|^p - \left(\frac{d-p}{p}\right)^p \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \,\mathrm{d}x\right)^\theta \left(\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \,\mathrm{d}x\right)^{1-\theta} \geq C \|u\|^p_{L^{p^*, r}} \tag{8}$$

*with a constant* $C = C(d, p, r, \theta) > 0$ *independent of* $u \in \dot{W}^{1,p}(\mathbb{R}^d)$. *The bound* (8) *does not hold if* $\theta < p/\min(r, p^*)$ *(with arbitrary* $p \geq 2$*), or if* $\theta > 1/p - 1/r$ *and* $p = 2$. *In particular, when* $p = 2$, *the range of* $\theta$ *in* (7) *is optimal.*

Theorems 1 and 2 naturally lead to the question of determining optimizers of the relevant inequalities. We expect that in the non-endpoint cases

$$\frac{p}{\min(r, p^*)} < \theta < \frac{1}{p} - \frac{1}{r},$$

the existence of optimizers of (8) follows from the standard concentration compactness method. Below we focus on the endpoint cases. While the existence of optimizers in this case is open in general, we are able to give a partial answer under the restriction to radial functions. We will limit ourselves to the choice $p = 2$ and $r \in \{2^*, \infty\}$, for which the right-hand side of (8) becomes either the usual $L^{2^*}$-norm or the $L^{2^*,\infty}$-weak norm. The relevant functional space is

$$\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d) = \{u \in \dot{H}^1(\mathbb{R}^d) : u \text{ is radially symmetric}\}.$$

In the radial case, we can work directly with the quadratic form $h[u]$ in (1). We have

**Theorem 3** *(Radial optimizers) Let* $d \geq 3$ *and* $p = 2$.

*(i) Let* $r = 2^* = 2d/(d-2)$ *and* $\theta = 1/p - 1/r = 1/d$. *Then all optimizers of the inequality*

$$h[u]^\theta \left(\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x\right)^{1-\theta} \geq C_{\mathrm{rad},2^*} \|u\|^2_{L^{2^*}}, \quad \forall u \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d) \tag{9}$$

*are given by the family*

$$u_\eta(x) = \frac{1}{(|x|^{1-\eta}(1 + |x|^{2\eta}))^{(d-2)/2}}, \quad \eta \in (0, \infty), \tag{10}$$

*up to dilation* $u_\eta(x) \mapsto a u_\eta(bx)$ *with* $a \in \mathbb{C}$, $b > 0$. *Furthermore,* (9) *does not hold if* $\theta \neq 1/d$.

*(ii) Let* $r = \infty$ *and* $\theta = 1/p - 1/r = 1/2$. *Then all optimizers of the inequality*

$$h[u]^\theta \left(\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x\right)^{1-\theta} \geq C_{\mathrm{rad},\infty} \|u\|^2_{L^{2^*,\infty}}, \quad \forall u \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d) \tag{11}$$

*are given by the family*

$$u_c(x) = \begin{cases} |x|^{c-d/2+1} & , |x| \leq 1, \\ |x|^{-c-d/2+1} & , |x| > 1, \end{cases} \quad c \in (0, d/2 - 1], \tag{12}$$

*up to dilation.*

**Remark 3** The classification of all optimizers in Theorem 3 (i) is consistent with Terracini's study in [21] where all radial positive solution of the Euler-Lagrange equation

$$-\Delta u(x)-\frac{(d-2)^2}{4}(1-\eta^2)\frac{u(x)}{|x|^2}=u^{2^*-1}(x),\quad x\in\mathbb{R}^d, \tag{13}$$

with a given constant $\eta>0$, were derived. In particular, according to [21, Eq. (4.6)], the only regular solutions (i.e., belonging to $L^{2^*}$), up to rescaling, are of the form $(d(d-2)\eta^2)^{(d-2)/4}u_\eta$ with $u_\eta$ given in (10).

**Remark 4** We leave the following **open questions**: Do optimizers of (4) exist? And if exist, are they radial? The same questions for the simpler inequality (5) in the endpoint case $\theta=1-2/d$ remain unsolved. Note that both in (4) and (5), all quantities scale in the same way. Moreover, by taking several bubbles travelling far from each other, one can construct optimizing sequences that do not converge weakly to a nonzero limit after any choice of dilations and translations. It seems that a novel concentration-compactness argument will be needed to resolve the existence problem of optimizers for these inequalities.

We prove Theorem 3 in Sect. 2, and then prove Theorems 1 and 2 in Sect. 3.

## 2 Radial case

In this section we prove Theorem 3. Let $u\in H^1_{\mathrm{rad}}(\mathbb{R}^d)$ with $d\geq 3$.

**Proof of (i):** Using the ground state representation for Hardy's inequality (see e.g. [7, Eq. (2.14)]), we denote

$$u(x)=\frac{f(|x|)}{|x|^{(d-2)/2}} \tag{14}$$

and rewrite

$$\int_{\mathbb{R}^d}|u|^{2^*}=|\mathbb{S}^{d-1}|\int_0^\infty\frac{|f(r)|^{2^*}}{r}\,\mathrm{d}r,\quad \int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x=|\mathbb{S}^{d-1}|\int_0^\infty\frac{|f(r)|^2}{r}\,\mathrm{d}r,$$
$$\int_{\mathbb{R}^d}|\nabla u|^2-\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x=\int_{\mathbb{R}^d}\frac{|\nabla f(x)|^2}{|x|^{d-2}}\,\mathrm{d}x=|\mathbb{S}^{d-1}|\int_0^\infty|f'(r)|^2r\,\mathrm{d}r.$$

Here $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere in $\mathbb{R}^d$. Thus (9) is equivalent to

$$\left(\int_0^\infty r|f'(r)|^2\mathrm{d}r\right)^\theta\left(\int_0^\infty\frac{|f(r)|^2}{r}\mathrm{d}r\right)^{1-\theta}\geq C_{\mathrm{rad},2^*}|\mathbb{S}^{d-1}|^{2/2^*-1}\left(\int_0^\infty\frac{|f(r)|^{2^*}}{r}\,\mathrm{d}r\right)^{2/2^*}. \tag{15}$$

The bound (15) can be interpreted as a Caffarelli–Kohn–Nirenberg type inequality [4], namely

$$\||x|^{1/2}\nabla f\|^\theta_{L^2(\mathbb{R}_+)}\||x|^{-1/2}f\|^{1-\theta}_{L^2(\mathbb{R}_+)}\geq\sqrt{C_{\mathrm{rad},2^*}}|\mathbb{S}^{d-1}|^{1/2^*-1/2}\||x|^\gamma f\|_{L^r(\mathbb{R}_+)} \tag{16}$$

with $r=2^*=2d/(d-2)$, $\gamma=-1/r$. Actually it is a limiting case as $1/r+\gamma/n=0$ in dimension $n=1$, which does not seem available from the literature (see [13, Theorem 3.1], [14, Theorem 1.2] and [15, Theorem 2.2] for recent results in the limiting case).

Inspired by [21], we make the following changes of variables

$$f(r) = \psi(\log r), \quad s = \log r \in \mathbb{R}, \quad \mathrm{d}s = \frac{\mathrm{d}r}{r},$$

which give

$$\int_0^\infty \frac{|f(r)|^2}{r}\,\mathrm{d}r = \int_{\mathbb{R}} |\psi(s)|^2\,\mathrm{d}s, \quad \int_0^\infty \frac{|f(r)|^{2^*}}{r}\,\mathrm{d}r = \int_{\mathbb{R}} |\psi(s)|^{2^*}\,\mathrm{d}s,$$
$$\int_0^\infty |f'(r)|^2 r\,\mathrm{d}r = \int_{\mathbb{R}} |\psi'(s)|^2\,\mathrm{d}s.$$

Therefore, (15) is equivalent to the Gagliardo–Nirenberg interpolation inequality

$$\left(\int_{\mathbb{R}} |\psi'(s)|^2\,\mathrm{d}s\right)^\theta \left(\int_{\mathbb{R}} |\psi(s)|^2\,\mathrm{d}s\right)^{1-\theta} \geq C_{\mathrm{rad},2^*}|\mathbb{S}^{d-1}|^{2/2^*-1}\left(\int_{\mathbb{R}} |\psi(s)|^{2^*}\,\mathrm{d}s\right)^{2/2^*}. \tag{17}$$

The optimal constant of the one-dimensional inequality (17) was already obtained by Nagy in 1941 [16], with $2^* = 2d/(d-2)$ replaced by a general positive power. The existence and uniqueness of optmizers of the analogue of (17) in higher dimensions are also well-known; we refer to the classical works of Weinstein [24] and Kwong [11] for instance. The uniqueness of optimizers of (17) can be translated straightforwardly to the classification of optmizers of (9) as stated in Theorem 3 (i); we refer to [21, Eq. (4.6)] for a similar analysis.

**Remark 5** In the special case $d = 3$ (which is relevant to Theorem 1), the interpolation inequality (17) with $2^* = 6$ goes back to the (1D, one-body) Lieb–Thirring inequality [12] as well as Keller's lower bound on the lowest eigenvalue of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ on $L^2(\mathbb{R})$ [10]; see also [1, Section 2] for a simple derivation of the optimal constant in this special case.

**Unique choice of $\theta$ for** (9): Consider (15) with the trial function

$$f(r) = \begin{cases} r^\varepsilon, & r \in (0,1], \\ r^{-\varepsilon}, & r \in [1,\infty), \end{cases} \tag{18}$$

where $\varepsilon > 0$ is a parameter. Then we have

$$\int_0^\infty r|f'(r)|^2\,\mathrm{d}r = \varepsilon, \quad \int_0^\infty \frac{|f(r)|^2}{r}\,\mathrm{d}r = \frac{1}{\varepsilon}, \quad \int_0^\infty \frac{|f(r)|^{2^*}}{r}\,\mathrm{d}r = \frac{2}{2^*\varepsilon}. \tag{19}$$

Therefore, (15) requires to have

$$\varepsilon^\theta \varepsilon^{-(1-\theta)} \geq C\varepsilon^{-2/2^*} \tag{20}$$

for all $\varepsilon > 0$. By letting $\varepsilon \to 0$ and $\varepsilon \to \infty$, we find that

$$\theta = \frac{1 - 2/2^*}{2} = \frac{1}{d}.$$

Thus, (9) holds only if $\theta = 1/d$.

**Proof of (ii):** Let us consider (11). Using again the ground state representation (14), we have

$$\|u\|_{L^{2^*,\infty}} = \sup_{t>0} t|\{x : |u(x)| > t\}|^{1/2^*} = \sup_{t>0} t|\{x : |f(x)| > t|x|^{d/2^*}\}|^{1/2^*}$$

$$\leq \sup_{t>0} t|\{x : \|f\|_{L^\infty} > t|x|^{d/2^*}\}|^{1/2^*} = |B(0,1)|^{1/2^*}\|f\|_{L^\infty}. \tag{21}$$

Here $|B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^d$. Therefore, (11) holds if we can show that

$$\left(\int_0^\infty r|f'(r)|^2\,\mathrm{d}r\right)\left(\int_0^\infty \frac{|f(r)|^2}{r}\,\mathrm{d}r\right) \geq C_{\mathrm{rad},\infty}^2 |\mathbb{S}^{d-1}|^{-2}|B(0,1)|^{4/2^*}\|f\|_{L^\infty}^4. \tag{22}$$

From (14) and $u \in \dot{H}^1(\mathbb{R}^d)$, we deduce that the function $f : (0,\infty) \to [0,\infty)$ is continuous and satisfies

$$\lim_{r\to 0} f(r) = 0, \quad \lim_{r\to\infty} f(r) = 0.$$

Therefore, up to dilation, we may without loss of generality assume that

$$f(1) = \|f\|_{L^\infty}.$$

By the Cauchy-Schwarz inequality and the fundamental theorem of calculus, we have

$$\begin{aligned}\left(\int_0^\infty r|f'(r)|^2\,\mathrm{d}r\right)\left(\int_0^\infty \frac{|f(r)|^2}{r}\,\mathrm{d}r\right) &\geq \left(\int_0^\infty |f'(r)f(r)|\,\mathrm{d}r\right)^2 \\ = \left(\frac{1}{2}\left(\int_0^1 \left|\frac{\partial}{\partial r}(|f(r)|^2)\right|\,\mathrm{d}r + \int_1^\infty \left|\frac{\partial}{\partial r}(|f(r)|^2)\right|\,\mathrm{d}r\right)\right)^2 &\geq |f(1)|^4 = \|f\|_{L^\infty}^4.\end{aligned} \tag{23}$$

Thus (22) holds, and consequently (11) holds, with

$$C_{\mathrm{rad},\infty}^2 |\mathbb{S}^{d-1}|^{-2}|B(0,1)|^{4/2^*} = 1. \tag{24}$$

To have the equality in (11), we need to ensure all equalities in (21) and (23). The bound (23) contains two inequalities where the first equality occurs if there exists a constant $c > 0$ such that

$$r|f'(r)|^2 = c\frac{|f(r)|^2}{r}, \qquad \text{a.e. } r \in (0,\infty),$$

while the second equality occurs if $|f|^2$ is monotone increasing on $(0,1)$ and monotone decreasing on $(1,\infty)$. Thus, we have all equalities in (23) if and only if

$$f(r) = \begin{cases} r^c & , r \in (0,1], \\ r^{-c} & , r \in [1,\infty). \end{cases} \tag{25}$$

It remains to determine the range of $c$ in (25) to get the equality in (21). If $0 < c \leq (d-2)/2$, then the function

$$u(x) = f(|x|)|x|^{-\frac{d-2}{2}} = \begin{cases} |x|^{c-\frac{d-2}{2}} & , |x| \leq 1, \\ |x|^{-c-\frac{d-2}{2}} & , |x| \geq 1 \end{cases} \tag{26}$$

is radially symmetric decreasing, and the equality in (21) occurs since

$$\|u\|_{L^{2^*,\infty}} = \sup_{t>0} t|\{x : |u(x)| > t\}|^{1/2^*} \geq |\{x : |u(x)| > 1\}|^{1/2^*} = |B(0,1)|^{1/2^*}. \tag{27}$$

On the other hand, if $c > (d-2)/2$, then the inequality in (21) is strict: since $u$ defined in (26) is bounded by 1, we have

$$\|u\|_{L^{2^*,\infty}}^{2^*} = \sup_{0<t<1} t^{2^*}|\{x : |u(x)| > t\}|$$

$$
\begin{aligned}
&= \sup_{0<t<1} t^{2^*} \left( |\{|x| \le 1 : |u(x)| > t\}| + |\{|x| > 1 : |u(x)| > t\}| \right) \\
&= \sup_{0<t<1} t^{2^*} \left( |\{x : 1 \ge |x| > t^{\frac{2}{2c-(d-2)}}\}| + |\{x : t^{-\frac{2}{2c+d-2}} > |x| > 1\}| \right) \\
&= |B(0,1)| \sup_{0<t<1} t^{2^*} \left( t^{-\frac{2d}{2c+d-2}} - t^{\frac{2d}{2c-(d-2)}} \right) < |B(0,1)|,
\end{aligned}
$$

where the latter estimate can be easily seen using the fact that

$$
2^* = \frac{2d}{d-2} > \frac{2d}{2c+d-2}.
$$

Thus, in summary, (11) holds with the optimal constant $C_{\mathrm{rad},\infty}$ given in (24), and all optimizers are uniquely characterized up to dilation by (26) with $c \in (0, (d-2)/2]$.

The proof of Theorem 3 is complete.

## 3 General case

In this section we prove Theorem 1 and Theorem 2.

### 3.1 Proof of Theorem 1

We divide the proof into two parts. First, we prove (4) for $d = 3 = 1/\theta$. Then we show that the condition $d = 3 = 1/\theta$ is necessary.

***Proof*** (Proof of (4) for $d = 3 = 1/\theta$) Let $u \in \dot{H}^1(\mathbb{R}^d)$ and denote

$$
A = \int_{\mathbb{R}^d} |\nabla u|^2, \quad B = \frac{(d-2)^2}{4} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2} \, \mathrm{d}x.
$$

We consider two cases.

**Case 1:** $(1-\theta)A \ge B$. Then using

$$
B \ge \frac{(d-2)^2}{4} \sup_{r>0, y \in \mathbb{R}^d} \int_{B(y,r)} \frac{|u(x)|^2}{|x-y|^2} \, \mathrm{d}x \gtrsim \sup_{r>0, y \in \mathbb{R}^d} \frac{1}{r^2} \int_{B(y,r)} |u|^2,
$$

and $A - B \ge \theta A$, we conclude from (6) (see [17, Theorem 1]) that

$$
(A-B)^\theta B^{1-\theta} \gtrsim \|\nabla u\|_{L^2}^\theta \left( \sup_{r>0, y \in \mathbb{R}^d} \frac{1}{r^2} \int_{B(y,r)} |u|^2 \right)^{1-\theta} \gtrsim \|u\|_{L^{2^*}}^2.
$$

**Case 2:** $(1-\theta)A < B$. Then $B \mapsto (A-B)^\theta B^{1-\theta}$ is monotone decreasing since

$$
\frac{\mathrm{d}}{\mathrm{d}B} ((A-B)^\theta B^{1-\theta}) = ((1-\theta)A - B)(A-B)^{\theta-1} B^{-\theta} < 0.
$$

Moreover, by the Hardy-Littlewood and Pólya-Szegö rearrangement inequalities (see e.g. [2, Lemma 1.6 and Theorem 4.7]) we have

$$
B^* = \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u^*(x)|^2}{|x|^2} \, \mathrm{d}x \ge B, \quad A^* = \|\nabla u^*\|_{L^2}^2 \le A,
$$

where $u^*$ denotes the radially symmetric decreasing rearrangement of $u$. Therefore,

$$(A-B)^\theta B^\theta \geq (A-B^*)^\theta (B^*)^{1-\theta} \geq (A^*-B^*)^\theta (B^*)^{1-\theta}.$$

Thus it remains to consider (4) in the case when $u$ is radially symmetric decreasing. In this case,

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}. \tag{28}$$

by the Hardy-Littlewood rearrangement inequality. Now the desired bound has been already proved in Theorem 3 (i).

Thus in all cases, (4) holds for $d = 3 = 1/\theta$. □

***Proof*** (Proof of the necessity of $d = 3 = 1/\theta$) Let us show that (4) fails if $d \geq 4$ or if $\theta \neq 1/3$. First, the necessity of $\theta \leq 1/d$ can be seen from the radial case as explained in Theorem 3 (i). To be precise, we consider the example in (18) with $\varepsilon > 0$ small. In this case, $u$ is radially symmetric decreasing, and hence (28) holds. Therefore, (4) requires (20) for $\varepsilon > 0$ small, which implies that $\theta \leq 1/d$.

In order to complete the proof, we consider another (non-radial) example. Fix $\varphi \in C_c^\infty\backslash\{0\}$, $z \in \mathbb{R}^d\backslash\{0\}$ and choose

$$u_N(x) = \sum_{n=1}^N \varphi(x + nNz)$$

with $N \to \infty$. Then by replacing $u$ by $u_N$, we find that

$$A \sim N, \quad B \sim 1, \quad \|u\|_{L^{2^*}}^{2^*} \sim N.$$

for large $N$. Therefore, (4) requires

$$N^\theta \gtrsim N^{2/2^*},$$

which implies that $\theta \geq 2/2^* = 1 - 2/d$. Combining with the upper bound $\theta \leq 1/d$ and the constraint $d \geq 3$, we find that the only possibility is $d = 3$ and $\theta = 1/3$. □

The proof of Theorem 1 is complete.

### 3.2 Proof of Theorem 2

We first prove (8) and then explain the necessity of the constraint of $\theta$.

***Proof*** (Proof of (8)) Let $d \geq 2$ and $p \in [2, d)$. Assume that $r \in [p, \infty]$ and

$$\frac{p}{\min(r, p^*)} \leq \theta \leq \frac{1}{p} - \frac{1}{r}. \tag{29}$$

Fix a small constant $\varepsilon = \varepsilon(d, p, r, \theta) \in (0, 1)$. We consider two cases.

**Case 1:** Assume

$$(1-\varepsilon)\int_{\mathbb{R}^d} |\nabla u|^p \geq \left(\frac{d-p}{p}\right)^p \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p}\,\mathrm{d}x. \tag{30}$$

From [17, Theorem 1], see also [18, Eq. (1.4)], we have

$$\left(\int_{\mathbb{R}^d} |\nabla u|^p\right)^{p/p^*} \left(\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u(x)|^p\,\mathrm{d}x\right)^{1-p/p^*} \gtrsim \|u\|^p_{L^{p^*}}. \tag{31}$$

A simplified proof of (31) based on sharp maximal functions can be obtained by following the analysis in [22]. We can extend this bound to the Lorentz norm on $L^{p^*,r}$ with $r \in [p, p^*]$, namely

$$\left(\int_{\mathbb{R}^d} |\nabla u|^p\right)^{p/r} \left(\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u(x)|^p\,\mathrm{d}x\right)^{1-p/r} \gtrsim \|u\|^p_{L^{p^*,r}}. \tag{32}$$

To prove (32), let us use the standard dyadic decomposition: recalling that $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported on the annulus $1/2 \le |t| \le 2$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1, \quad \forall t \in \mathbb{R}\backslash\{0\},$$

we write

$$u = \sum_{j\in\mathbb{Z}} u_j, \quad u_j = u\varphi(2^{-j}|u|).$$

Then

$$\begin{aligned}\|u\|^{p^*}_{L^{p^*,r}} &\sim \left(\sum_{j\in\mathbb{Z}} \|u_j\|^r_{L^{p^*}}\right)^{p^*/r}\\ &\lesssim \left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^d} |\nabla u_j|^p\right)^{r/p^*} \left(\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u_j(x)|^p\,\mathrm{d}x\right)^{r/d}\right)^{p^*/r},\end{aligned}$$

where we used (31) for $u_j$ and the fact that $(1 - p/p^*)r/p = r/d$ (as $1/p - 1/d = 1/p^*$). On the other hand, using

$$\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u_j(x)|^p\,\mathrm{d}x \lesssim \min\left\{\int_{\mathbb{R}^d} |\nabla u_j|^p,\ \sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u(x)|^p\,\mathrm{d}x\right\}$$

and splitting the power

$$r/d = r(1/p - 1/p^*) = (1 - r/p^*) + (r/p - 1),$$

we find that

$$\begin{aligned}&\left(\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u_j(x)|^p\,\mathrm{d}x\right)^{r/d}\\ &\lesssim \left(\int_{\mathbb{R}^d} |\nabla u_j|^p\right)^{1-r/p^*} \left(\sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R^p}\int_{B(y,R)} |u(x)|^p\,\mathrm{d}x\right)^{r/p-1}.\end{aligned}$$

Here we used the constraint $r \in [p, p^*]$ to ensure that both $(1 - r/p^*)$ and $(r/p - 1)$ are nonnegative. Thus, we obtain

$$\|u\|_{L^{p^*,r}}^{p^*} \lesssim \left( \sum_{j \in \mathbb{Z}} \left( \int_{\mathbb{R}^d} |\nabla u_j|^p \right) \left( \sup_{y \in \mathbb{R}^d, R>0} \frac{1}{R^p} \int_{B(y,R)} |u(x)|^p \, dx \right)^{r/p-1} \right)^{p^*/r}$$
$$\lesssim \left( \left( \int_{\mathbb{R}^d} |\nabla u|^p \right) \left( \sup_{y \in \mathbb{R}^d, R>0} \frac{1}{R^p} \int_{B(y,R)} |u(x)|^p \, dx \right)^{r/p-1} \right)^{p^*/r},$$

which is equivalent to (32).

From (31), (32) and the obvious bound $\|u\|_{L^{p^*}}^p \gtrsim \|u\|_{p^*,r}^p$ for $r \geq p^*$, we get

$$\left( \int_{\mathbb{R}^d} |\nabla u|^p \right)^{\frac{p}{\min(r,p^*)}} \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx \right)^{1 - \frac{p}{\min(r,p^*)}} \gtrsim \|u\|_{p^*,r}^p. \tag{33}$$

By Hardy's inequality and the condition $\theta \geq p/\min(r, p^*)$, we also get

$$\left( \int_{\mathbb{R}^d} |\nabla u|^p \right)^{\theta} \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx \right)^{1-\theta} \gtrsim \|u\|_{p^*,r}^p. \tag{34}$$

Combining (30) and (34), we obtain

$$\begin{aligned} &\left( \int_{\mathbb{R}^d} |\nabla u|^p - \left( \frac{d-p}{p} \right)^p \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx \right)^{\theta} \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx \right)^{1-\theta} \\ &\geq \left( \varepsilon \int_{\mathbb{R}^d} |\nabla u|^p \right)^{\theta} \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx \right)^{1-\theta} \gtrsim \|u\|_{p^*,r}^p. \end{aligned} \tag{35}$$

**Case 2:** Assume

$$(1-\varepsilon) \int_{\mathbb{R}^d} |\nabla u|^p \leq \left( \frac{d-p}{p} \right)^p \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \, dx.$$

Then by a monotonicity argument as in the proof of Theorem 1, we can reduce to the case when $u$ is radially symmetric decreasing. We have for radially symmetric decreasing functions $u$,

$$\|u\|_{q,s}^s \sim \int_{\mathbb{R}^d} \frac{|u(x)|^s}{|x|^{\alpha}} \, dx \tag{36}$$

if $s/q = 1 - \alpha/d$ (see e.g. [7, Lemma 4.3] for the case $q = p^*$, $s = p$). In particular, for $q = p^* = dp/(d-p)$ and $s = \alpha = d$, we obtain

$$\|u\|_{p^*,s}^s \sim \int_{\mathbb{R}^d} \frac{|f(x)|^s}{|x|^d} \, dx \sim \int_0^{\infty} \frac{|f(r)|^s}{r} \, dr \tag{37}$$

with $u(x) = |x|^{1-d/p} f(x)$. Moreover,

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} \, dx = \int_{\mathbb{R}^d} \frac{|f(x)|^p}{|x|^d} \, dx \sim \int_0^{\infty} \frac{|f(r)|^p}{r} \, dr,$$

and by the ground state representation [7, Eq. (2.14)] (here we use that $p \geq 2$)

$$\int_{\mathbb{R}^d} |\nabla u|^p - \left( \frac{d-p}{p} \right)^p \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} \, dx \gtrsim \int_{\mathbb{R}^d} \frac{|\nabla f(x)|^p}{|x|^{d-p}} \, dx \sim \int_0^{\infty} |f'(s)|^p s^{p-1} \, ds.$$

By Hölder's inequality, we have, with $1/p + 1/p' = 1$,

$$\left(\int_0^\infty |f'(s)|^p s^{p-1}\,\mathrm{d}s\right)^{1/p}\left(\int_0^\infty \frac{|f(s)|^p}{s}\,\mathrm{d}s\right)^{1/p'} \geq \int_0^\infty |f'(s)||f(s)|^{p-1} = p^{-1}\int_0^\infty |(f^p(s))'|\,\mathrm{d}s \gtrsim_p \|f\|_\infty^p. \tag{38}$$

Under the constraint (29), there exists $\tilde r \in [p, r]$ such that $\theta = 1/p - 1/\tilde r$. Let $\beta \geq 0$ such that $\tilde r = p(1+\beta)$. Then

$$\left(\int_0^\infty |f'(s)|^p s^{p-1}\,\mathrm{d}s\right)^{\beta/p}\left(\int_0^\infty \frac{|f(s)|^p}{s}\,\mathrm{d}s\right)^{1+\beta/p'} \geq p^{-1}\|f\|_\infty^{\beta p}\int_0^\infty \frac{|f(s)|^p}{s}\,\mathrm{d}s = p^{-1}\int_0^\infty \frac{|f(s)|^{\tilde r}}{s}\,\mathrm{d}s \sim \|u\|_{p^*,\tilde r}^{\tilde r} \gtrsim \|u\|_{p^*,r}^{\tilde r}, \tag{39}$$

where we used (37) with $s = \tilde r$ and $\tilde r \leq r$. This implies (8) by the choice of $\tilde r$. □

***Proof*** (Proof of the necessity of the range of $\theta$) Let us explain why (8) fails for certain values of $\theta$ as indicated in Theorem 2. First we consider $\theta < p/\min(r, p^*)$ for general $p \geq 2$ where we split into two cases $r \geq p^*$ and $r < p^*$, and then we focus on the case $p = 2$.

**Counterexample for the case** $r \geq p^*$ **and** $\theta < p/\min(r, p^*) = p/p^*$. The idea is to consider $N \in \mathbb{N}$ identical bubbles that travel away from each other and to let $N \to \infty$. Let $0 \neq \varphi \in C_c^\infty(B(0,1))$, $0 \neq z \in \mathbb{R}^d$ and choose

$$u_N(x) = \sum_{n=1}^N \varphi(x + nNz) \tag{40}$$

for every $N \in \mathbb{N}$. The translation by $Nz$ ensures that the functions $\{\varphi(\cdot + nNz)\}_{n=1}^N$ have disjoint support for $N$ large. We have

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u_N(x)|^p}{|x-y|^p}\,\mathrm{d}x = \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\varphi(x)|^p}{|x-y|^p}\,\mathrm{d}x + o(1) \quad \text{as} \quad N \to \infty \tag{41}$$

and

$$\int_{\mathbb{R}^d} |\nabla u_N|^p - \left(\frac{d-p}{p}\right)^p \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u_N(x)|^p}{|x-y|^p}\,\mathrm{d}x = N\int_{\mathbb{R}^d} |\nabla\varphi|^p + O(1) \quad \text{as} \quad N \to \infty. \tag{42}$$

Moreover, it is straightforward to see that for $N$ large

$$\|u_N\|_{p^*,r} = p^*\int_0^\infty s^{r-1}|\{|u_N| > s\}|^{r/p^*}\,\mathrm{d}s = N^{r/p^*}p^*\int_0^\infty s^{r-1}|\{|\varphi| > s\}|^{r/p^*}\,\mathrm{d}s, \tag{43}$$

if $r < \infty$, and

$$\|u_N\|_{p^*,\infty} = \sup_{s>0} s|\{|u_N| > s\}|^{1/p^*} = N^{1/p^*}\sup_{s>0} s|\{|\varphi| > s\}|^{1/p^*}. \tag{44}$$

if $r = \infty$. Hence, if (8) holds, then by taking $u = u_N$, we get

$$N^\theta \gtrsim N^{\frac{p}{p^*}} \tag{45}$$

for $N$ large, which implies that $\theta \geq p/p^*$.

**Counterexample for the case $r < p^*$ and $\theta < p/\min(r, p^*) = p/r$.** Let $0 \neq \varphi \in C_c^\infty(B(0,1))$, $0 \neq z \in \mathbb{R}^d$ and define

$$v_N(x) = \sum_{j=1}^{N} 2^j \varphi(2^{p^* j/d}(x + jNz)) \tag{46}$$

for every $N \in \mathbb{N}$. The scaling is chosen such that $\|2^j \varphi(2^{p^* j/d}\cdot)\|_{L^{p^*}(\mathbb{R}^d)} = \|\varphi\|_{L^{p^*}(\mathbb{R}^d)}$ for all $j$, and similarly all relevant terms in (8) are invariant when changing $\varphi \mapsto 2^j \varphi(2^{p^* j/d}\cdot)$. Again, the translation by $Nz$ ensures that the supports of the functions $\{2^j \varphi(2^{p^* j/d}(\cdot + jNz))\}_{j=1}^N$ are far away from each other for $N$ large. Then

$$\int_{\mathbb{R}^d} |\nabla v_N|^p = N \int_{\mathbb{R}^d} |\nabla \varphi|^p, \quad \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v_N(x)|^p}{|x-y|^p} \mathrm{d}x \sim \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\varphi(x)|^p}{|x-y|^p} \mathrm{d}x \sim 1$$

and

$$\|v_N\|_{L^{p^*,r}}^p \sim \left( \sum_j \|2^j \varphi(2^{p^* j/d}\cdot)\|_{L^{p^*}(\mathbb{R}^d)}^r \right)^{p/r} \sim N^{p/r}.$$

Thus inserting $v_N$ in (8), we find that for $N$ large,

$$N^\theta \gtrsim N^{p/r},$$

which requires $\theta \geq p/r$.

**Counterexample for the case $p = 2$ and $\theta > 1/p - 1/r$.** Define $u : \mathbb{R}^d \to \mathbb{R}$ by

$$u(x) = \frac{f(|x|)}{|x|^{(d-p)/p}}. \tag{47}$$

where $f(r)$ is chosen as in (18) with $\varepsilon > 0$ small. Then $u$ is radially symmetric decreasing and hence (28) holds. Therefore,

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \mathrm{d}x = \int_{\mathbb{R}^d} \frac{|f(|x|)|^p}{|x|^d} \mathrm{d}x = |\mathbb{S}^{d-1}| \int_0^\infty \frac{|f(r)|^p}{r} \mathrm{d}r = |\mathbb{S}^{d-1}| \frac{2}{\varepsilon p}. \tag{48}$$

We also have by the ground state representation for $p = 2$,

$$\int_{\mathbb{R}^d} |\nabla u|^p - \left(\frac{d-p}{p}\right)^p \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x-y|^p} \mathrm{d}x = \int_{\mathbb{R}^d} \frac{|f(|x|)|^p}{|x|^{d-p}} \mathrm{d}x = \varepsilon^{p-1} |\mathbb{S}^{d-1}| \frac{2}{p}. \tag{49}$$

Note that the analogue of (49) is more complicated for $p \neq 2$ (see [7]), which is why our counterexample only works for $p = 2$.

By (36), we have for $r < \infty$

$$\|u\|_{L^{p^*,r}}^r = \int_{\mathbb{R}^d} \frac{|f(x)|^r}{|x|^d} \mathrm{d}x = |\mathbb{S}^{d-1}| \int_0^\infty \frac{|f(s)|^r}{s} \mathrm{d}s = |\mathbb{S}^{d-1}| \frac{2}{\varepsilon r}.$$

Moreover, for $r = \infty$, by (21) and (27),

$$\|u\|_{p^*,\infty} = |B(0,1)|^{1/p^*}.$$

Therefore, if (8) holds, then

$$\varepsilon^{(p-1)\theta}\varepsilon^{-(1-\theta)} \gtrsim \varepsilon^{-p/r}, \tag{50}$$

for $\varepsilon > 0$ small, which requires that $p\theta - 1 \geq -p/r$ namely

$$\theta \geq \frac{1}{p} - \frac{1}{r}.$$

The proof of Theorem 2 is complete. □

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