Cobordism and String Theory

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Zusammenfassung

Kobordismus-Theorie hat sich schnell zu einem wichtigen Instrument entwickelt, um nicht-perturbative Physik in der Stringtheorie aufzudecken. Ursprünglich als konzeptioneller Rahmen in der algebraischen Topologie entwickelt, um bestimmte Äquivalenzklassen von Mannigfaltigkeiten zu untersuchen, hat sie in den letzten Jahren Eingang in die Untersuchung von konsistenten Quantengravitationstheorien gefunden. In dieser Arbeit werden wir uns auf eine bestimmte Anwendung konzentrieren – die Kobordismus-Vermutung.

Konkret bindet die Kobordismus-Vermutung nicht-triviale Kobordismus Äquivalenzklassen an eine nicht verschwindende globale Symmetrie, die geeicht oder gebrochen werden muss, um Konsistenz mit Quantengravitation zu gewährleisten. Diese Vermutung stammt aus dem Swampland-Programm und ist Teil eines ganzen Netzwerks miteinander verbundener Vermutungen. Ziel dieses Netzes von Vermutungen ist es, diejenigen effektiven Theorien, die konsistent zu einer Theorie der Quantengravitation im UV-Limit vervollständigt werden können, von denjenigen abzugrenzen, für welche dies nicht möglich ist. Als Paradebeispiel für eine konsistente Theorie der Quantengravitation bietet die Stringtheorie einen Rahmen zum Sammeln von Beweismaterial für oder gegen eine bestimmte Vermutung. Insbesondere im Hinblick auf die Kobordismus-Vermutung kann sie sogar unbekannte Eigenschaften der Stringtheorie aufdecken, die in traditionellen Ansätzen der Stringtheorie viel undurchsichtiger sind.

Im Hauptteil untersuchen wir zunächst das Verhalten der Kobordismus-Vermutung bei Dimensionsreduktion auf Hintergrundmannigfaltigkeiten. Daraufhin untersuchen wir zwei eng verwandte Stringtheorien – Typ I und ihr stark gekoppeltes, heterotisches Dual – mittels der Kobordismus-Vermutung. Zugleich erweitern wir die traditionelle Beschreibung von Dp-Branen in Typ I Stringtheorie, um die Beschreibung derjenigen Objekte zu erleichtern, die erforderlich sind, um die Kobordismus-Vermutung zu erfüllen. Schließlich verwenden wir unsere Erkenntnisse aus der vorangegangenen Rechnung, um eine Konstruktion vorzustellen, die Shenker-Effekte in heterotischen Stringtheorien erklärt, ein nicht-perturbativer Effekt, der zwar für jede geschlossene Stringtheorie vermutet wurde, für den es aber in beiden heterotischen Stringtheorien jahrzehntelang keine Konstruktion gab.

Abstract

Cobordism theory has quickly developed into a substantial tool to uncover non-perturbative physics in string theory. Initially developed as a framework in algebraic topology to study certain equivalence classes of manifolds, it has entered the study of consistent quantum gravity theories in recent years. In this thesis, we will focus on one application – the Cobordism Conjecture.

Concretely, the Cobordism Conjecture ties non-trivial cobordism equivalence classes to a non-vanishing global symmetry, which has to be gauged or broken to ensure quantum gravitational consistency. This conjecture originates from the Swampland Program as part of a whole network of interconnected conjectures. The aim of this web of conjectures is to universally delineate the effective theories coupled to gravity that can be consistently completed to a theory of quantum gravity in the UV from those that cannot. As the prime example of a consistent theory of quantum gravity string theory provides a framework for gathering evidence in favor or against a particular conjecture. Especially in regards to the Cobordism Conjecture it can even uncover unknown features of string theory, which are much more obscure in traditional approaches to string theory.

In the main part, we first explore the behavior of the Cobordism Conjecture under dimensional reduction on background manifolds. We then investigate two closely related string theories – type I and its strong coupling heterotic dual – through the lens of the Cobordism Conjecture. Coincidentally, we extend the traditional description of Dp-branes in type I to facilitate our description of the objects required to satisfy the Cobordism Conjecture. Finally, we use our insights from the previous calculation to present a construction that explains Shenker effects in heterotic string theories, a peculiar non-perturbative effect that has been conjectured to exist for any closed string theory, but has lacked a construction in both heterotic string theories for decades.

So ist's mit aller Bildung auch beschaffen:
Vergebens werden ungebundne Geister
Nach der Vollendung reiner Höhe streben.
Wer Großes will, muß sich zusammenraffen;
In der Beschränkung zeigt sich erst der Meister,
Und das Gesetz nur kann uns Freiheit geben.

Johann Wolfgang Goethe

1
Introduction

1.1 Quantum (and) Gravity

Our current theoretical understanding of our universe relies on two main pillars: Quantum Field Theory (QFT) – further developed into the Standard Model [1–3] of particle physics and General Relativity [4] providing a framework describing gravity. Both have been incredibly useful in making remarkable predictions matched to an astonishing degree of precision by numerous experimental tests. However, maybe the most glaring shortcoming here is that General Relativity is not quantized, i.e. a classical theory. This makes it impossible to treat the four fundamental forces we have experimentally verified so far – electromagnetism, weak and strong force subsumed by the Standard Model and gravitation on the other side – on the same footing. Therefore, formulating a consistent theory of quantum gravity has been a longstanding goal in theoretical physics dating back to the 1930s. However, in the early 70s it was revealed that conventional quantization of General Relativity fails due to the non-renormalizability of the theory already at one-loop, when coupled to matter fields[1] [6–8]. Unavoidably, this meant that new approaches were necessary to make progress. An intermediate step is so called semi-classical gravity, where gravity

[1] Pure GR actually suffers from the same issue, albeit at the two-loop level [5].

is treated classically, while interacting with a quantum matter system. Even though such a treatment falls short, it allowed for significant advances in our understanding of black holes and cosmology. On one side it became clear that black holes are not the stable accumulations of mass they were thought to be, but thermodynamical objects with a temperature just dependant on their mass, the so called Hawking temperature $T_H(M) = \frac{1}{8\pi M}$. An immediate consequence of this is that a black hole emits black body radiation and thereby decays with a mass-dependent lifetime [9]. Its conjugate thermodynamical variable the Bekenstein-Hawking entropy of a black hole might be even more insightful as it has a counter-intuitive area law scaling, $S_{bh} = \frac{A}{4G}$ [10, 11]. On the other side important parts of the cosmological Standard Model, the ΛCMB [12–14], rely on the semi-classical approximation, as well. Perhaps the most important example would be the production mechanism of the large scale structure of our universe within the ΛCMB model(see [15, 16] for reviews), namely quantum fluctuations in the inflation spectrum. Just like the description of black holes, a better understanding of these cosmological results like further corrections, a full description of gravity at the smallest scales etc. necessitates an actual theory of quantum gravity.

1.2 String Theory – a theory of Quantum Gravity

Since General Relativity is not renormalizable, what are our ways out? One approach would be to ignore the divergences for now and look for a UV fixed point of the renormalization group flow such that non-perturbatively the number of relevant couplings necessary to renormalize the theory becomes finite (instead of the infinite number necessary in the perturbative picture). However, this is not the direction we would like to pursue. Our path starts with a different approach to the ultraviolet divergences. Namely, we take the position that they are an indicator of a fundamentally different theory at short distances. String Theory provides an exceptionally elegant resolution by smearing out the singular interaction of gravity in the UV. In particular, string theory accomplishes this feat by supplanting 0+1 dimensional point particles with 1+1 dimensional strings as the fundamental objects of quantum gravity. To illustrate how string theory avoids these treacherous short distance divergences we will follow the discussion in [17] of the illustrative example of the one-loop cosmological constant.

In quantum field theory the diagram calculating the one-loop cosmological constant is a simple circle parametrized by the proper time τ:

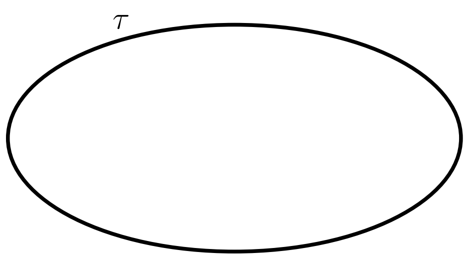

Figure 1.1: Feynman diagram for one-loop cosmological constant in QFT

This translates into this expression for the one-loop cosmological constant:

$$\Gamma_1 = \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr}\, e^{-\tau \mathcal{H}}\,, \tag{1.1}$$

where $\mathcal{H}$ denotes the Hamiltonian of the specific quantum field theory at hand. The ultraviolet divergence is precisely in the $\tau \to 0$ region and only gets worse, when unfolding the momentum integral hidden inside the trace. Now, let us turn to string theory. Instead of just integrating over the proper time we have to integrate both over the proper space and time parametrizing the worldsheet the string is sweeping out in space-time. The corresponding Feynman diagram now looks like a torus instead of a circle.

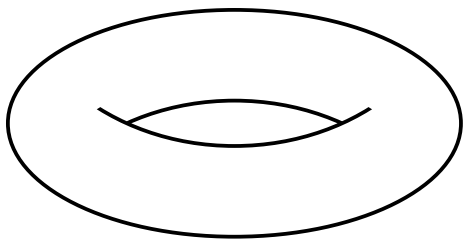

Figure 1.2: Feynman diagram for one-loop cosmological constant in string theory

This brings us to one of the reasons that makes string theory so powerful – conformal invariance. While we are going to elaborate on this issue in our background section on string theory, here is the short version. The action describing the fundamental string is invariant under any transformations that preserve the angle between any two lines locally, i.e. string theory is conformally invariant. For the torus diagram this turns out to become even more constraining. Our theory on the torus becomes modular invariant and to avoid overcounting we have to limit the region of integration. Usually one picks the so-called fundamental domain:

$$-\frac{1}{2} \leq \sigma_1 \leq \frac{1}{2}\,, \quad \sigma_2 > 0 \quad \text{and} \quad |\sigma| > 1\,, \tag{1.2}$$

where σ_1 and σ_2 are the proper space and time coordinates parametrizing the string worldsheet. Therefore, looking at the one-loop cosmological constant calculation in string theory given by the torus diagram

$$\Gamma_1^{string} = \int \frac{d^2\sigma}{\sigma_2} \mathcal{Z}(\sigma, \overline{\sigma}) \tag{1.3}$$

we see that precisely the UV divergent region $\sigma_2 \to 0$ is excluded, because whenever σ_2 would dip into this region it turns out that it is already accounted for due to the modular symmetry! While this is just a particular example for the built-in UV-finiteness of string theory, it actually holds generally.

1.3 The Landscape – Solution space of quantum gravity

In the preceding section we have encountered the consequence of one of string theories main staples – conformal invariance, so let's introduce the next one – the absence of tunable dimensionless parameters. The only adjustable parameter string theory has is the Regge slope α'. However, it is dimensionful and therefore should rather be viewed as defining the fundamental length scale of string theory – the string length $l_s = 2\pi\sqrt{\alpha'}$. This sounds just like what one would expect for a unifying theory of all fundamental interactions. Now, curiously, string theory does not have a unique solution to its equations of motion. Moreover, string theory in its most accessible sector is 10 dimensional. This means that the geometry of these solutions is in general extremely intricate, in particular for space-time solutions with four large non-compact dimensions as in our universe and six small, compact dimensions. The precise geometry of the compact space is specified by parameters, the so-called moduli. The number of moduli can easily reach $\mathcal{O}(100)$ to $\mathcal{O}(1000)$ (see e.g. [18]). Intriguingly, these moduli in turn define the parameters of the effective field theory at low energies, i.e. coupling constants, masses of fields etc. Among the plethora of solutions the most interesting for string phenomenology are, of course, those solutions that closely resemble our own universe, i.e. four large, non-compact dimensions combined with a Minimally Supersymmetric Standard Model-like effective field theory description. Getting all details about our observed universe just right is a very challenging and ongoing task as it necessitates a deep understanding of the underlying mathematics to keep control over all kinds of subtleties. For recent reviews on this vast topic see [19–21]. Concerning the techniques for obtaining these results [22] and the textbook [23] provide excellent insight that goes way beyond the scope of this introduction. Now, does it even matter to precisely construct a low energy effective theory with all the known observed features of our universe? Or asked differently, can string theory provide predictions guiding experimental searches for beyond the Standard Model physics? After all, the solution space appears to be so large, every kind of consistent effective field theory could be realized, our Standard Model among them. However, recent work strongly disagrees, and the effective field theories are far more constrained than previously thought.

1.4 The Swampland – Constraints on effective field theories

The program to formalize these constraints in a general framework is the Swampland Program initiated almost 20 years ago in [24]. By now, the framework has developed into a network of interconnected conjectures on conditions a low energy effective theory coupled to gravity has to fulfill to embed into quantum gravity in the UV. These conjectures cover a wide range of subjects from algebraic topology, geometry, or logic to very specific applied topics in cosmology like the compatibility of eternal de Sitter space with quantum gravity. A highly important principle underlying the Swampland Program is that of UV/IR mixing [25–27], namely that high energy and low energy physics do not decouple and can not be discussed separately. This implies that the imprints of quantum gravity can already be observed at energies way below the quantum gravity scale. Consequently, the Swampland Program can be utilized to explain naturalness issues of our Standard Models including hierarchy problems coming from quantum gravity constraints and make predictions on what kind of generic experimental signatures we should expect of UV-completable beyond the Standard Model physics. A concrete example is the "Dark Dimension Scenario" [28], which is centered around providing an explanation of the unnaturally small dark energy from a QFT point of view we observe in our universe by proposing a single mesoscopic dimension among the compactified extra dimensions in string theory. Interestingly, this scenario seems to implicate rather constrained phenomenological predictions concerning, for example, dark matter (realized as higher-dimensional gravitons [29] or primordial black holes [30]) and axions [31]. Throughout this thesis, we will turn to the more mathematical sector of the Swampland Program, in particular the Cobordism Conjecture.

1.5 Outline of the thesis

In the main part we seek to explore the linkage of cobordism and string theory. Calculating specific cobordism groups will help us unravel new aspects of the non-perturbative sectors of string theory required by the Cobordism Conjecture for quantum gravitational consistency. Before we get to the main part of this thesis however, we want to introduce the core background material. We start with a chapter 2 on superstring theory basics accompanied with a more advanced background on its non-perturbative sector with a particular emphasis on dualities and Dp-branes (and NSp-branes). In the succeeding chapter 3 we are then going to discuss some core Swampland Conjectures, in particular the Cobordism Conjecture, and highlight the importance of cobordism theory for describing quantum gravity. We conclude our background chapters with a primer 4 on hand-picked topics in algebraic topology essential to the following chapters, namely generalized cohomology theories, spectral sequences and topological invariants. Chapters 5, 6 and 7 are based on papers developed by the author and collaborators. The papers presented here were specifically selected due to their close topical proximity illustrating how Swampland principles can guide us to reveal new corners of string theory or how they allow us to tackle unresolved issues of string theory. We start out by exploring the Cobordism Conjecture in the context of dimensional reduction by calculating specific cobordism groups of manifolds commonly occurring in dimensional reductions utilizing the Atiyah-Hirzebruch Spectral Sequence in chapter 5. Having completed these computations we observe that these cobordism groups indeed reproduce in simple cases pattern of global symmetries we expect from dimensionally reducing global symmetries in the unreduced theory. We also present some more general cases, which refine this pattern of global symmetries. Furthermore, we examine the gauging mechanism for the cobordism groups proposed in [32]. We find that the correspondence between certain cobordism groups and K-theory leading to tadpole cancellation of Dp-brane charges within K-theory still hold in this more generalized setting and can account for gauging a substantial amount of non-trivial cobordism groups.

In the subsequent chapter 6 we first present a rather formal computation of the spin cobordism groups of the classifying space of the precise gauge group of type I and its strong-coupling dual heterotic dual string theory. We proceed by providing a string theoretic interpretation on how to trivialize these cobordism groups. This involves both gauging and breaking. As the cobordism groups arise as K-theory building blocks we propose that large parts of this subsector can be gauged by Dp-brane tadpole charge cancellation. Moreover, we suggest that the K-theory groups including the classifying space actually give the full Dp-brane classification for type I string theory extending the traditional one.

In the final chapter 7 we take advantage of the preceding computation and present a construction of specific stringy heterotic instantons. We propose that these instantons resolve a longstanding mystery regarding non-perturbative $\exp(1/g_s)$ effects in heterotic string theory put forward by Shenker in [33]. Ultimately, we conclude with a summary and an outlook on future directions 8.

Relevant Publications

This thesis, especially chapters 5, 6 and 7 is based on the work published in the following papers:

- **Dimensional Reduction of Cobordism and K-theory** [34]
  R. Blumenhagen, N. Cribiori, C. Kneißl and A. Makridou.
  arXiv: 2208.01656 [hep-th]; published in: JHEP 03 (2023) 181.

- **Spin cobordism and the gauge group of type I/heterotic string theory** [35]
  C. Kneißl.
  arXiv: 2407.20333 [hep-th]; published in: JHEP 01 (2025) 181.

- **Spin cobordism and the gauge group of type I/heterotic string theory** [36]
  R. Álvarez-García, C. Kneißl, J. M. Leedom and N. Righi.
  arXiv: 2407.20319 [hep-th];

The following two papers were published as well during the author's doctoral studies. Their topics are not covered in this thesis.

- **Dynamical cobordism of a domain wall and its companion defect 7-brane** [37]
  R. Blumenhagen, N. Cribiori, C. Kneißl and A. Makridou.
  arXiv: 2205.09782 [hep-th]; published in: JHEP 08 (2022) 204.

- **Dynamical Cobordism Conjecture: solutions for end-of-the-world branes** [38]
  R. Blumenhagen, C. Kneißl, C. Wang.
  arXiv: 2303.03423 [hep-th]; published in: JHEP 05 (2023) 123.

2

Superstring Essentials

We have already seen some core features of string theory. Nevertheless, in order to be self-contained and provide a proper foundation for the subsequent chapters we are going to start with the construction of the five superstring theories. In the process, we are going to discuss the differences between them, highlighting their distinct perturbative and non-perturbative features. Finally, we are going to discuss their unlikely unification through dualities that transform these seemingly disparate theories into one another. In the following, we loosely follow the classic textbooks [39], [40, 41] and [42, 43], supplemented by more specialized literature.

2.1 The Superstring Action

Generally speaking, the construction of the superstring action is fairly straightforward. The central idea is that the simple bosonic string, which is described by a collection of scalar fields coupled to 2d gravity and provides us with space-time gravity as a particular excitation in its spectrum. While the bosonic string achieves this central feature of quantum gravity effortlessly, it has a very notable shortcoming: There are no fermionic excitations in its spectrum. This means that we should think about extending the simple bosonic string action by other degrees of freedom. An notably natural extension is adding fermions to the worldsheet description. It turns out that the most interesting choice is a d-plet of Majorana fermions transforming in the vector representation of the Lorentz group $SO(d-1,1)$ joining the d-plet of scalar fields of the bosonic part. This results into the following action

$$S = -\frac{1}{8\pi}\int d^2\sigma \left[\frac{2}{\alpha'}\partial_\alpha X^\mu\,\partial^\alpha X_\mu - 2i\,\bar{\psi}^{\,\mu}\rho^\alpha\,\partial_\alpha\psi_{\,\mu}\right], \tag{2.1}$$

where we pick the following basis for the 2d Dirac matrices ρ^α and the components of ψ in this basis:

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \psi^{\,\mu} = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix}. \tag{2.2}$$

One particularly nice feature about this action is that it retains the local conformal symmetry of just the bosonic part, moreover it is also supersymmetric.[1] The supersymmetry transformations leaving the action invariant are

$$\sqrt{\frac{2}{\alpha'}}\,\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^{\,\mu}\,, \tag{2.3}$$

$$\delta_\epsilon\psi^{\,\mu} = \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\,\rho^\alpha\partial_\alpha X^\mu\epsilon\,, \tag{2.4}$$

where ϵ is another Majorana spinor and the infinitesimal parameter for the supersymmetry transformations. It turns out that switching to so called light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ is a lot more convenient. The Minkowski metric and partial derivatives become

$$\eta_{ij} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad \eta^{ij} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)\,. \tag{2.5}$$

This results into this very convenient form of the superstring action

$$S = \frac{1}{2\pi}\int d^2\sigma \left[\frac{2}{\alpha'}\,\partial_+ X\cdot\partial_- X - i\big(\psi_+\cdot\partial_-\psi_+ + \psi_-\cdot\partial_+\psi_-\big)\right] \tag{2.6}$$

[1] The action can also be constructed as supersymmetric matter coupling to 2d supergravity. At first glance one attains a more extensive and more complicated action. However, by using its equations of motion and removing gauge redundancy (picking the superconformal gauge) it reduces precisely to the action we have introduced.

from which these simple equation of motions arise:

$$\partial_+ \partial_- X^\mu = 0\,, \tag{2.7}$$

$$\partial_- \psi_+^\mu = \partial_+ \psi_-^\mu = 0\,. \tag{2.8}$$

We already broached the local superconformal symmetry of our superstring action. To properly impose these symmetries on the spectrum we require the generators of them, the energy-momentum tensor $\mathcal{T}_{ij}$ and the supercurrents $\mathcal{J}_\pm$

$$\mathcal{T}_{++} = -\frac{1}{\alpha'} \partial_+ X \cdot \partial_+ X - \frac{i}{2} \psi_+^\mu \, \partial_+ \psi_{+\,\mu}\,, \tag{2.9}$$

$$\mathcal{T}_{--} = -\frac{1}{\alpha'} \partial_- X \cdot \partial_- X - \frac{i}{2} \psi_-^\mu \, \partial_- \psi_{-\,\mu}\,, \tag{2.10}$$

$$\mathcal{J}_\pm = -\frac{1}{2} \sqrt{\frac{2}{\alpha'}} \psi_\pm^\mu \, \partial_\pm X_\mu\,. \tag{2.11}$$

Conformal symmetry means that in light-cone coordinates the energy-momentum tensor's off-diagonal components are trivial. Conservation of the energy-momentum tensor implies $\partial_+ \mathcal{T}_{--} = 0 = \partial_- \mathcal{T}_{++}$. Conversely, supersymmetry means that $\partial_- \mathcal{J}_+ = 0 = \partial_+ \mathcal{J}_-$. Additionally, we have to impose the superconformal Virasoro constraints

$$\mathcal{T}_{\pm\pm} \stackrel{!}{=} 0\,, \quad \mathcal{J}_\pm \stackrel{!}{=} 0\,. \tag{2.12}$$

These constraints can be carefully derived from the general superstring action (before going to the superconformal gauge), which we are going to omit here. They are going to be central to properly study the spectrum of the superstring as they are part of general equation of motions and crucial to remove clearly unphysical solutions. Now, we can fully discuss the equations of motion. In particular, we have to specify boundary conditions. When varying the action (2.1) the following boundary term arises from the fermionic part and has to vanish on its own

$$\delta S = \frac{1}{2\pi} \int_{\tau_0}^{\tau_1} d\tau \left[\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \right] \Bigg|_{\sigma=0}^{\sigma=l}\,, \tag{2.13}$$

where the length of our string is specified by l. Now, we can split the boundary conditions into two further categories, namely whether we want the variation to vanish at each endpoint separately or not. Requiring triviality at each endpoint just means that our fundamental strings are open, whereas joint triviality means our string is closed. Let us first go over the simpler case of the closed string.

2.2 Solving the equations of motion

2.2.1 The closed string

Here, to respect the periodicity of a closed string we require that the two endpoints cancel each other

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_-\big|_{\sigma=0} \stackrel{!}{=} \psi_+\delta\psi_+ - \psi_-\delta\psi_-\big|_{\sigma=l} \tag{2.14}$$

The Poincaré invariant solution then takes the following form

$$\psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma + l), \tag{2.15}$$

$$\psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma + l), \tag{2.16}$$

where both signs are admissible because we are dealing with a spinor allowing for both periodic or anti-periodic fermionic boundary conditions once we complete a full closed string loop $\sigma \to \sigma + l$. Therefore, we get two different sectors depending on the sign we choose. These are conventionally referred to as Ramond (R) sector for the periodic and Neveu-Schwarz (NS) sector for the anti-periodic boundary condition. Solving the equations of motion leads us to integer mode expansion for the R sector and a half-integer mode expansion for NS sector

$$\begin{aligned} \psi_+^\mu(\tau,\sigma) &= \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} \bar{b}_r^\mu e^{-2\pi i\, r\, \sigma_+/l} \\ \psi_-^\mu(\tau,\sigma) &= \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} b_r^\mu e^{-2\pi i\, r\, \sigma_-/l} \end{aligned} \qquad \text{where } \phi = \begin{cases} 0 \text{ (R)} \\ \frac{1}{2} \text{ (NS)}. \end{cases} \tag{2.17}$$

Since we can choose Ramond or Neveu-Schwarz boundary conditions separately for ψ_+^μ and ψ_-^μ, we get four combinations in total: R-R, R-NS, NS-R and NS-NS. The bosonic part on the other hand is rather straightforward for the closed string as there is no sign choice, but the equation of motion for $X^\mu(\tau,\sigma)$ implies that it describes waves moving left or right respectively, i.e. we can split it these two parts $X^\mu(\tau,\sigma) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-)$ and we obtain the following oscillator expansion:

$$\begin{aligned} X_L^\mu(\sigma_+) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{l} p^\mu\, \sigma_+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \bar{\alpha}_n^\mu\, e^{-\frac{2\pi}{l} i n \sigma_+}\ , \\ X_R^\mu(\sigma_-) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu\, \sigma_- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \alpha_n^\mu\, e^{-\frac{2\pi}{l} i n \sigma_-}\ . \end{aligned} \tag{2.18}$$

2.2.2 The open string

Requiring the boundary terms to vanish individually entails of course that ψ^μ_+ and ψ^μ_- have to be related at the endpoints now

$$\psi^\mu_+(\sigma = 0, l) = \pm\psi^\mu_-(\sigma = 0, l)\,. \tag{2.19}$$

While we could treat the bosonic part separately for the closed string the sign choices and the supersymmetry relations between bosonic and fermionic fields matter now, since the open string has its own boundary term

$$\delta S = \int_{\tau_0}^{\tau_1} d\tau \partial_\sigma X \cdot \delta X \bigg|_{\sigma=0}^{\sigma=l} \tag{2.20}$$

leading to these two possibilities for both endpoints, called Neumann (N) and Dirichlet (D) boundary conditions respectively:

$$\partial_\sigma X^\mu|_{\sigma=0,l} = 0 \quad \text{(Neumann)}\,, \tag{2.21}$$

$$\delta X^\mu|_{\sigma=0,l} = 0 \quad \text{(Dirichlet)}\,. \tag{2.22}$$

All in all we get three different oscillator expansion depending on whether we have (NN), (DD) or mixed (ND)/(DN) boundary conditions for the open string endpoints

$$\begin{aligned}
X^\mu(\tau,\sigma) &= x^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau\, \sigma \\
&\quad + i\sqrt{2\alpha'} \sum_{n\,\in\,\mathbb{Z}\backslash\{0\}} \frac{1}{n}\,\alpha^\mu_n\, e^{-\frac{\pi}{l} in\tau} \cos\left(\frac{n\pi\sigma}{l}\right) \qquad \text{(NN)}\,, \\
X^\mu(\tau,\sigma) &= x^\mu_0 + \frac{1}{l}(x^\mu_1 - x^\mu_0)\,\sigma \\
&\quad + \sqrt{2\alpha'} \sum_{n\,\in\,\mathbb{Z}\backslash\{0\}} \frac{1}{n}\,\alpha^\mu_n\, e^{-\frac{\pi}{l} in\tau} \sin\left(\frac{n\pi\sigma}{l}\right) \qquad \text{(DD)}\,, \\
X^\mu(\tau,\sigma) &= x^\mu + i\sqrt{2\alpha'} \sum_{n\,\in\,\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\,\alpha^\mu_n\, e^{-\frac{\pi}{l} in\tau} \cos\left(\frac{n\pi\sigma}{l}\right) \qquad \text{(ND/DN)}\,.
\end{aligned} \tag{2.23}$$

With the bosonic oscillator solutions in place, we can shift our attention back to the fermionic oscillator modes. Whenever we choose (NN) boundary conditions for a coordinate μ, we fix the overall sign of the spinors such that

$$\psi^\mu_+(\sigma = 0) = \psi^\mu_-(\sigma = 0)\,. \tag{2.24}$$

Then we get the two possibilities

$$\psi^\mu_+(\sigma = l) = \pm\psi^\mu_-(\sigma = l)\,, \tag{2.25}$$

where the (anti-)periodic option gives us the open string (Neveu-Schwarz) Ramond sector with (half-)integer mode expansion

$$\psi^\mu_\pm(\tau,\sigma) = \sqrt{\frac{\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} b^\mu_r e^{-2\pi i\, r\, \sigma_\pm/l} \quad \text{where } \phi = \begin{cases} 0 \text{ (R)} \\ \frac{1}{2} \text{ (NS)} \end{cases} . \tag{2.26}$$

Now, if we have (DD) boundary conditions for a coordinate μ we get a sign flip for our bosonic fields: $X^\mu_+ \to X^\mu_+$, $X^\mu_- \to -X^\mu_-$. Due to supersymmetry this also flips the sign of ψ^μ_-. Since this is a sign at both endpoints, it does not affect the oscillator expansion of $\psi^\mu_\pm$ for a (DD) coordinate and we get the same (half-)integer expansion for the (Neveu-Schwarz) Ramond sector. For mixed (ND)/(DN) boundary conditions this no longer true. Here, we have a (half-)integer expansion for the (Ramond) Neveu-Schwarz sector.

2.3 Quantizing the superstring – Canonical quantization

With the classical solution in place our next task is to quantize the superstring. There are a couple of different avenues to achieve this feat. In the following we are going to choose the so-called old canonical quantization. Just as any other canonical quantization we start with promoting the classical Poisson brackets in the bosonic and the Dirac brackets in the fermionic case to quantum commutators and anti-commutators respectively. First, we get for our bosonic scalars:

$$\begin{aligned} [\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] &= 2\pi i\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma')\,, \\ [\hat{X}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] &= [\hat{P}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = 0 \end{aligned} \tag{2.27}$$

with $P^\mu = \dot{X}^\mu$. Consequently, our expansion coefficients in (2.18) and (2.23) also obey corresponding commutator relations

$$\begin{aligned} [\hat{x}^\mu, \hat{p}^\nu] &= i\eta^{\mu\nu}\,, \\ [\hat{\alpha}^\mu_m, \hat{\alpha}^\nu_n] = [\hat{\tilde{\alpha}}^\mu_m, \hat{\tilde{\alpha}}^\nu_n] &= m\delta_{m+n,0}\,\eta^{\mu\nu}\,, \\ [\hat{\alpha}^\mu_m, \hat{\tilde{\alpha}}^\nu_n] &= 0\,, \end{aligned} \tag{2.28}$$

where the oscillator operators follow hermiticity conditions: $(\hat{\alpha}^\mu_m)^\dagger = \hat{\alpha}^\mu_{-m}$ and $(\hat{\tilde{\alpha}}^\mu_m)^\dagger = \hat{\tilde{\alpha}}^\mu_{-m}$. Therefore, we can proceed by treating $\hat{\alpha}^\mu_m$, $m > 0$ as annihilation and $\hat{\alpha}^\mu_{-m}$, $m > 0$ as creation operators with respect to the vacuum state $|0; p^\mu\rangle$. On top, a state can also have momentum eigenvalues p^μ, such that

$$\begin{aligned} \hat{\alpha}^\mu_m|0; p^\mu\rangle &= 0 \quad \text{for } m > 0\,, \\ \hat{p}^\mu|0; p^\mu\rangle &= p^\mu|0; p^\mu\rangle\,. \end{aligned} \tag{2.29}$$

There is an interesting complication however, namely $\eta^{\mu\nu}$ is of Lorentzian signature, i.e. $[\hat{\alpha}^0_m, \hat{\alpha}^0_{-m}] = -m$.

This entails that we have negative norm states $\langle 0|\hat{\alpha}^0_m\hat{\alpha}^0_{-m}|0\rangle = -m\langle 0|0\rangle < 0$, which poses a serious conflict with unitarity. However, this is where we should remember that we did not solve the full equations of motion and still have to impose the (super-)conformal Virasoro constraints. Before we do so let us complete the introduction of the commutator relations by including the fermionic oscillators. Since we are dealing with fermions, we are of course dealing with anti-commutators:

$$\begin{aligned} \{\hat{\psi}^\mu_\pm(\tau,\sigma), \hat{\psi}^\nu_\pm(\tau,\sigma')\} &= 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma')\,, \\ \{\hat{\psi}^\mu_\pm(\tau,\sigma), \hat{\psi}^\nu_\mp(\tau,\sigma')\} &= 0\,. \end{aligned} \tag{2.30}$$

The corresponding mode operators then follow these anti-commutator relations:

$$\begin{aligned} \{\hat{b}^\mu_r, \hat{b}^\nu_s\} = \{\hat{\tilde{b}}^\mu_r, \hat{\tilde{b}}^\nu_s\} &= \delta_{r+s,0}\,\eta^{\mu\nu}\,, \\ \{\hat{b}^\mu_r, \hat{\tilde{b}}^\nu_s\} &= 0\,. \end{aligned} \tag{2.31}$$

Their action on the vacuum is almost alike the bosonic mode operators

$$\hat{b}^\mu_r|0\rangle = 0 \quad \forall\, r > 0 \quad \text{with } r \in \begin{cases} \mathbb{Z} & (R) \\ \mathbb{Z} + \frac{1}{2} & (NS) \end{cases} \tag{2.32}$$

This means that the Ramond sector zero modes $\hat{b}^\mu_0$ do not annihilate the vacuum, but in turn are annihilated by the annihilation operators. In fact, the Ramond ground state is therefore degenerate. Moreover, the (anti-)commutator relations of $\hat{b}^\mu_0$ indicate what we are actually working with. Namely, they satisfy (upon proper normalization) a Clifford algebra with $\hat{b}^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu$, i.e. the Ramond ground state $|0\rangle_R$ is a d-dimensional spinor. This closes the question on how space-time fermions arise in superstring theories.

To impose the superconformal Virasoro constraints (2.12) we need to write down the (anti-)commutator relations of $\mathcal{T}_{ij}$ and $\mathcal{J}_\pm$. We would also like to define the number operator $\mathcal{N}$:

$$\mathcal{N} = \mathcal{N}_{(\alpha)} + \mathcal{N}_{(b)} = \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n + \sum_{r \in \mathbb{Z}_+ + \phi}^{\infty} r\, b_{-r} \cdot b_r \,. \tag{2.33}$$

Then we obtain the energy-momentum modes from

$$L_m = L_m^{(\alpha)} + L_m^{(b)} = -\frac{l}{2\pi^2} \int_0^l d\sigma \left(e^{i\frac{\pi}{l} m\sigma} \mathcal{T}_{++} + e^{-i\frac{\pi}{l} m\sigma} \mathcal{T}_{--} \right) \,, \tag{2.34}$$

which evaluates to[2]

$$\begin{aligned} L_m^{(\alpha)} &= \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{-n} \cdot \alpha_{m+n} : \,, \\ L_m^{(b)} &= \frac{1}{2} \sum_{r \in \mathbb{Z} + \phi} \left(r + \frac{m}{2} \right) : b_{-r} \cdot b_{m+r} : \,. \end{aligned} \qquad \text{with} \quad m \in \mathbb{Z} \tag{2.35}$$

The zero-mode just reads $L_0 = \frac{1}{2}\alpha_0^2 + \mathcal{N}$. The supercurrent modes arise alike

$$G_r = -\frac{1}{\pi} \sqrt{\frac{l}{\pi}} \int_0^l d\sigma \left(e^{i\frac{\pi}{l} m\sigma} \mathcal{J}_+ + e^{-i\frac{\pi}{l} m\sigma} \mathcal{J}_- \right) \,, \tag{2.36}$$

$$G_r = \sum_{m \in \mathbb{Z}} \alpha_{-m} b_{r+m} \quad \text{with} \quad r \in \mathbb{Z} + \phi \,. \tag{2.37}$$

All these modes together follow an intricate scheme of (anti-)commutator relations, they form the so-called super-Virasoro algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n) L_{m+n} + \frac{c}{12} m(m^2 - 2\phi) \delta_{m+n,0} \,, \\ [L_m, G_r] &= \left(\frac{m}{2} - r \right) G_{m+r} \,, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12} (4r^2 - 2\phi) \delta_{r+s,0} \end{aligned} \tag{2.38}$$

with $c = \frac{3}{2}\, d$. The (super-)Virasoro algebra arises as the central extension of the classical Witt-algebra, the corresponding Lie algebra of the conformal symmetry group

[2] As with any other canonical quantization we have to introduce normal ordering, which we denote as usual with : :.

$\mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$. Central extensions arise in the quantization of classical system, whenever its symmetry group allows for such. We start by looking directly at the short exact sequence on the algebra level:

$$0 \to \mathbb{R} \to \mathfrak{g} \to \mathfrak{h} \to 0 \tag{2.39}$$

corresponding to the central extension on the group level

$$1 \to U(1) \to G \to H \to 1\,. \tag{2.40}$$

The point is that as we are quantizing our system we want to map to the corresponding projective representations in the projective unitary group of our Hilbert space $PU(\mathbb{H})$. In the case where our Lie algebra has a non-trivial central extension the distinction between the group of unitary operators on the Hilbert space $\mathbb{H}$ and $PU(\mathbb{H})$ becomes actually relevant

$$1 \to U(1) \to U(\mathbb{H}) \to PU(\mathbb{H}) \to 1 \tag{2.41}$$

as there are phases we have to take care of. We refer to chapter 3 of [44] for much more detailed account of the mathematics involved. In particular that the second cohomology $H^2(\mathfrak{h}, \mathbb{R})$ is in one-to-one correspondence with the equivalence classes of central extensions of $\mathfrak{h}$ by $\mathbb{R}$. As it so happens the authors of [45] have shown that the Lie algebra of the conformal symmetry on our string worldsheet, the Witt algebra, has a non-trivial second cohomology with coefficients in $\mathbb{R}$: $H^2(\mathfrak{h}, \mathbb{R}) = \mathbb{R}$.[3] Back to the energy-momentum and supercurrent modes, we can translate the super-Virosoro constraints into the action L_m and G_r on our quantum states. For the energy-momentum modes L_n we have to be careful about the normal ordering, namely for $n = 0$ we have an ambiguity. We introduce a normal ordering constant $a_{NS/R}$ shifting L_0 to $L_0 - a_{NS/R}$, which have yet to be determined.[4] In the NS-sector we get:

$$\begin{aligned}(L_0 - a_{NS})|\phi\rangle &= 0\,,\\ L_m|\phi\rangle &= 0 \quad \text{with } m > 0,\ m \in \mathbb{Z} \\ G_r|\phi\rangle &= 0 \quad \text{with } r > 0,\ r \in \mathbb{Z} + \tfrac{1}{2}\,.\end{aligned} \tag{2.42}$$

[3] An even more nuanced discussion is necessary when actually addressing the complex central extension of the Witt algebra by $\mathbb{C}$ instead of $\mathbb{R}$, which we are actually working with. Again we refer to [44] for more details.

[4] Since the Hamiltonian depends linearly on L_0, the correct physical interpretation of a is that it represents the Casimir energy of the string.

a_{NS} denotes a normal ordering constant of the NS-sector. Below we will determine a_{NS} from requiring a unitary theory. Alternatively, it can also be determined directly. Let us demonstrate how we can obtain it in the case of a periodic boson:

$$
\begin{aligned}
L_0^{(\alpha)} &= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=-\infty}^{-1}\alpha_{-n}\alpha_n \\
&= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} \\
&= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty}n
\end{aligned}
\tag{2.43}
$$

Therefore, $\frac{1}{2}\sum_{n=1}^{\infty} n$ is exactly $-a$. While this sum appears to be divergent, one gets a finite result deploying zeta-function regularization. In fact, the periodic boson sum evaluates to $a = \frac{1}{24}$. For anti-periodic bosons one gets $a = -\frac{1}{48}$ and for periodic/anti-periodic fermions $-\frac{1}{24}$ and $\frac{1}{48}$ respectively.[5] The Ramond sector turns out to be simpler, we get:

$$
\begin{aligned}
(L_0 - a_R)|\phi\rangle &= 0\,, \\
L_m|\phi\rangle &= 0 \quad \text{with } m > 0,\ m \in \mathbb{Z} \\
G_r|\phi\rangle &= 0 \quad \text{with } r \geq 0,\ r \in \mathbb{Z}\,.
\end{aligned}
\tag{2.44}
$$

Because of $G_0|\phi\rangle = 0$ and the super-Virasoro algebra (2.38) telling us that $G_0^2 = L_0$, the normal ordering constant has to be trivial $a_R = 0$. Quantization of the superstring would not be complete without showing off some core features of superstring theory, namely the critical dimension and the graviton arising natural from its spectrum. First, we will take a look at the critical dimension. For canonical quantization, it arises from banishing the unitarity violating ghosts, i.e. negative norm states. The first possible ghosts arise from the level $|\phi'\rangle = G_{-\frac{1}{2}}|\phi\rangle$, where $|\phi\rangle$ is annihilated by $G_{\frac{1}{2}}$ and $G_{\frac{3}{2}}$. Then, $G_{\frac{1}{2}}|\phi'\rangle = 2L_0|\phi\rangle = 2(a_{NS} - \frac{1}{2})|\phi\rangle = 0$, fixing $a_{NS} = \frac{1}{2}$ renders it massless. The critical dimension arises at the next level, we begin by constructing the following family of states $|\phi''\rangle = G_{-\frac{3}{2}} + \lambda G_{-\frac{1}{2}}L_{-1}|\phi\rangle$, where $|\phi\rangle$ again denotes

[5] The final result for the normal ordering constant can in fact be written as a function of the space-time dimensions and we get $a_{NS} = (D-2)(\frac{1}{24} + \frac{1}{48}) = \frac{3(D-2)}{48}$.

a state annihilated by $G_{\frac{1}{2}}$ and $G_{\frac{3}{2}}$. With the super-Virasoro algebra (2.38) we get

$$\begin{aligned} G_{\frac{1}{2}}|\phi''\rangle &= (2-\lambda)L_{-1}|\phi\rangle\,, \\ G_{\frac{3}{2}}|\phi''\rangle &= (D-2-4\lambda)|\phi\rangle\,. \end{aligned} \tag{2.45}$$

For exact annihilation we require $\lambda = 2$ and the famous critical dimension $D_{crit} = 10$.[6]

2.4 The spectrum of the RNS superstring

Our next step is to analyze the actual spectrum of superstring theory. We have briefly mentioned that the Ramond ground state turns out to be fermionic and degenerate. Moreover, these spinors respect the Lorentz symmetry of our target space-time. We define the generators of the Lorentz symmetry built out of the Gamma matrices $\Gamma^\mu = \sqrt{2}\,\hat{b}_0^\mu$ as

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]\,, \tag{2.46}$$

whose creation/annihilation operators are

$$S^a = i^{\delta_{a,0}}\Sigma^{2a,\,2a+1} \quad \text{with } a \in \{0,\dots,4\}\,. \tag{2.47}$$

Due to this Lorentz symmetry we can take a basis of eigenstates of the S^a-operators, i.e.

$$|\mathbf{s}\rangle_R = |s_0, \dots s_4\rangle_R\,. \tag{2.48}$$

To help us keep track of the anti-commutation properties with respect to the Dirac-matrices we define the mod 2 operator based on the worldsheet fermion number $\mathcal{F}$:

$$(-1)^{\mathcal{F}} \quad \text{with} \quad \mathcal{F} = \sum_{a=0}^{4} S^a\,. \tag{2.49}$$

Applied to our mode operators, we see that $\hat{b}_r^\mu$ raises $\mathcal{F}$ by 1 and thereby anti-commutes with our $(-1)^{\mathcal{F}}$-operator. We also define a chirality operator Γ as the

[6] Technically, absence of ghosts can be achieved by having $D < 10$ [42]. Going beyond tree-level however poses serious unitarity issues for $D < 10$ string theories. One observes that below the critical dimension the one-loop amplitudes show unitarity-violating branch-cuts instead of simple poles in case of the critical dimension. This is reviewed for example in [46] back when superstring theory was not yet recognized as a theory of quantum gravity.

product of the Dirac matrices:

$$\Gamma = i^{-k}\,\Gamma^0 \dots \Gamma^9 \,. \tag{2.50}$$

Now, let's look at the spectrum of the NS and R sectors. We start with the open NS-sector:

- The ground state $|0, p^\mu\rangle$ is tachyonic with $\alpha' M^2 = -a_{NS} = -\frac{1}{2}$ transforming as the $\mathbb{1}$ of the little group $SO(9)$
- The first excited state by acting with $\hat{b}^\mu_r$ on the ground state is massless and transforms as a vector $\mathbf{8_v}$ in $SO(8)$
- The higher massive states transform in the tensor of $SO(9)$

Conversely, the open Ramond sector shakes out as follows

- We know that the ground state is a spinor of the 10d Lorentz group $SO(1,9)$ and in 10 dimensions in particular we have 32 components satisfying the Dirac equation, which arises as the supercurrent zero-mode constraint $G_0|0\rangle_R$. Now, we can impose that the spinors are Majorana due to the reality condition for ψ^μ and their oscillator modes and that they are Weyl to assign a definite chirality, i.e. if we act by $\gamma = \gamma_0\gamma_1$ the $\psi_\pm$s are unaffected: $\gamma\psi_\pm = \psi_\pm$. These conditions can only be simultaneously imposed in dimension d = 2 mod 8 for Lorentzian signature.[7] Imposing both conditions, while simultaneously satisfying the Dirac equation the Ramond ground state decomposes as

$$|0\rangle_R = \left(\frac{1}{2}, \mathbf{8}\right) \oplus \left(\frac{1}{2}, \mathbf{8}'\right) \,, \tag{2.51}$$

  where $\mathbf{8}$ and $\mathbf{8}'$ are the positive and negative chirality Weyl spinors transforming in the $SO(8)$ stemming from the $SO(1,9) \to SO(1,1) \oplus SO(8)$ decomposition. The $\frac{1}{2}$s come from the Dirac equation. The massive states can be shown to transform as irreducible representations in the little group $SO(9)$.

The closed string spectrum is computed as a product of two open string theories – one for the left-movers and one for the right-movers – and imposing the level matching

[7] The lowest cases of course being the dimension of the worldsheet and the critical (target spacetime) dimension, which highlights their outstanding nature once more.

condition $(L_0 - \bar{L}_0)|\phi\rangle = 0$.[8] The level matching condition necessitates that we can only combine certain sectors. Concretely, a left-moving NS_- sector can only be combined with another NS_- sector. The minus sign is with respect to the mod 2 operator $(-1)^{\mathcal{F}+1}$ and denotes the sector containing the tachyonic ground state. For the Ramond sector it is also convenient to introduce a form of parity, here with respect to the operator $\Gamma\,(-1)^{\mathcal{F}}$. After the level matching condition [9] we are left with 10 combinations of $\mathrm{NS}_\pm$ and $\mathrm{R}_\pm$ instead of 16. Therefore, we would get $\sum_{k=0}^{10}\binom{10}{k} = 2^{10}$ potential string theories. However, it turns out only a tiny subset is actually consistent. The reason behind this is the GSO-projection named after its authors [47], who worked out the correct consistency condition for the theory to be modular invariant. We omit the details here and just state the results:

- Clearly, the two supersymmetric and tachyon-free string theories, named type IIA and type IIB, should be the headliner arising from the following combination of sectors:

  type IIA: (NS_+, NS_+), (R_+, R_-), (NS_+, R_-) and (R_+, NS_+)

  type IIB: (NS_+, NS_+), (R_+, R_+), (NS_+, R_+) and (R_+, NS_+)

  From this structure it is clear that type IIB is a chiral theory, while type IIA is not.

- Furthermore, there are two tachyonic string theories due to the presence of a (NS_-, NS_-)-sector, but nevertheless they are modular invariant:

  type 0A: (NS_+, NS_+), (NS_-, NS_-), (R_+, R_-) and (R_-, R_+)

  type 0B: (NS_+, NS_+), (NS_-, NS_-), (R_+, R_+) and (R_-, R_-)

Now, let us look at the state decomposition of our closed string sectors into irreducible representations of $SO(8)$.

[8] The physical meaning of this condition is simply that there should not exist a distinguished point on the closed string.

[9] Interchanging NS-R to R-NS does not change the spectrum.

Sector	SO(8)	tensors	dimension
$(\text{NS}_+, \text{NS}_+)$	$\mathbf{8_v} \otimes \mathbf{8_v}$	[0] + [2] + (2)	1 + 28 + 35
(R_+, R_+)	$\mathbf{8} \otimes \mathbf{8}$	$[0] + [2] + [4]_+$	$1 + 28 + 35_+$
(R_+, R_-)	$\mathbf{8} \otimes \mathbf{8'}$	[1] + [3]	$8_v + 56_t$
(R_-, R_-)	$\mathbf{8'} \otimes \mathbf{8'}$	$[0] + [2] + [4]_-$	$1 + 28 + 35_-$
$(\text{NS}_+, \text{R}_+)$	$\mathbf{8_v} \otimes \mathbf{8}$		8' + 56
$(\text{NS}_+, \text{R}_-)$	$\mathbf{8_v} \otimes \mathbf{8'}$		8 + 56'

Table 2.1: State decomposition of closed string sectors at massless level.

The clear highlight is the symmetric, traceless 2-tensor (2), i.e. the massless spin-2 graviton $G_{\mu\nu}$ in the $(\text{NS}_+, \text{NS}_+)$-sector. The other massless objects in this sector are called dilaton Φ, a scalar field [0], and Kalb-Ramond field $B_{\mu\nu}$, an anti-symmetric 2-form [2]. This sector arises in all of the 4 consistent sector combinations leading to superstring theories we discussed above. In the following we will focus on the non-tachyonic theories as they are physically much more interesting. As a concluding thought we would like to mention that the tachyon in the the type 0 string theories has been better understood recently and there exists a tachyon condensation mechanism to their non-tachyonic 2d type 0 counterparts [48].[10] Back to our stable 10d string theories we can summarize their massless spectrum as follows, which will prepare us for their low-energy effective descriptions.

type IIA	type IIB
$(\text{NS}_+, \text{NS}_+) : \Phi, B_{[\mu\nu]}, G_{(\mu\nu)}$	$(\text{NS}_+, \text{NS}_+) : \Phi, B_{[\mu\nu]}, G_{(\mu\nu)}$
$(\text{R}_+, \text{R}_-) : C^{(1)}_{\mu_1}, C^{(3)}_{[\mu_1\mu_2\mu_3]}$	$(\text{R}_+, \text{R}_+) : C^{(0)}, C^{(2)}_{[\mu_1\mu_2]}, C^{(4)+}_{[\mu_1\mu_2\mu_3\mu_4]}$
$(\text{NS}_+, \text{R}_-) : \tilde{\lambda}_a, \tilde{\psi}^\mu_a$	$(\text{NS}_+, \text{R}_+) : \lambda_a, \psi^\mu_a$
$(\text{R}_+, \text{NS}_+) : \lambda_a, \psi^\mu_a$	$(\text{R}_+, \text{NS}_+) : \lambda_a, \psi^\mu_a$

Table 2.2: Massless states of type IIA/IIB string theory

[10] Similar observations can be made about other tachyonic 10d string theories, as well [48, 49]. However, not all of them decay to two dimensions.

The Ramond-Ramond sector comes with a couple of anti-symmetric $C^{(p)}_{\mu_1\ldots\mu_p}$-fields, which will be of particular importance, when we introduce the concept of Dp-branes. In the other sectors we encounter the supersymmetric partners of the dilaton Φ the spin-$\frac{1}{2}$ dilatino λ^a and the spin-$\frac{3}{2}$ partner of the graviton $G_{(\mu\nu)}$ the gravitino ψ^μ_a. While it is not obvious in this presentation of our superstring theories, they are in fact space-time supersymmetric, as well. In particular, they give rise to two independent supersymmetry algebras, i.e. their massless sectors are $\mathcal{N}=2$ supergravity theories explaining our two sets of dilatino and gravitinos respectively in a non-chiral and chiral arrangement.

Before presenting the effective actions of the type II string theories let us briefly discuss an important principle of string theory – its two-fold perturbative expansion in terms of worldsheet diagrams weighed by the string coupling and in terms of the Regge slope α'.

2.5 String theory's two-fold expansion

To get some intuition it suffices to look at just the bosonic part here. We have omitted so far that there is an additional term allowed by coordinate and Poincaré invariance that we can add to the string worldsheet action (2.1):

$$S' = S + \lambda\chi\,. \tag{2.52}$$

χ is the Euler number – a topological invariant, which according to the Gauss-Bonnet theorem for a two-dimensional Riemannian manifold can be expressed as

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}\,R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k\,, \tag{2.53}$$

where γ_{ab} is the 2d worldsheet metric, R the Ricci scalar and k the extrinsic curvature or equivalently[11]

$$\chi(\Sigma) = 2 - 2g - b \tag{2.54}$$

with g handles and b boundaries. Now, consider the string path-integral running over the space-time coordinates X and the Euclidean worldsheet metrics γ:

$$Z[\lambda] \sim \int [dX\,d\gamma] e^{-S'}\,. \tag{2.55}$$

[11] For non-orientable surfaces we get $2-2g-b-c$ with g handles, b holes and c crosscaps.

This entails that we have a natural summation over worldsheet topologies weighed by $e^{-\lambda\chi}$

$$Z[\lambda] \sim \sum_{topologies} e^{-\lambda\chi} \int [dX\, d\gamma] e^{-S} . \tag{2.56}$$

The constant e^{λ} is defined to be the string coupling constant

$$g_s := e^{\lambda} \tag{2.57}$$

Below we illustrate what the expansion of worldsheet diagrams would look like for a closed (oriented) string.

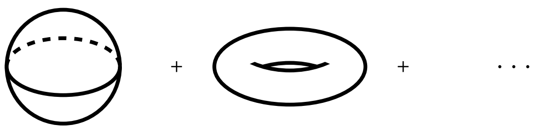

Figure 2.1: Closed (oriented) string worldsheet expansion – first two topologies

Actually, what we stated above would contradict our previous claim that string theory has no dimensionless parameters, if the string coupling is defined as (2.57). For the full story we need to consider non-trivial backgrounds, for the bosonic part we can consider the backreaction of the space-time metric $G_{\mu\nu}(X)$, the Kalb-Ramond field $B_{\mu\nu}(X)$ and the dilaton $\Phi(X)$. The most general worldsheet action for the bosonic part then becomes

$$\begin{aligned} -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} [\gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \\ + \alpha' R\, \Phi(X)] . \end{aligned} \tag{2.58}$$

These background fields unfortunately spoil the local conformal symmetry of our worldsheet theory as can observed from the non-vanishing energy-momentum tensor trace:

$$2\alpha' T^a_a = \alpha' \beta^\Phi R(X) + \beta^G_{\mu\nu} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu + \beta^B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu \tag{2.59}$$

To restore our local conformal invariance requires to kill all three β-functions. These have to be determined order by order in α'. These are truly stringy corrections and give rise to the second perturbative expansion of string theory, the α'-expansion. To first order the beta function are:

$$\begin{aligned}\beta^G_{\mu\nu} &= \alpha'\left(R_{\mu\nu} - H_\mu^{\rho\sigma}H_{\nu\rho\sigma} + 2\nabla_\mu\nabla_\nu\Phi\right) + \mathcal{O}(\alpha'^2)\,,\\ \beta^B_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\nabla_\lambda H^\lambda_{\mu\nu} + 2H^\lambda_{\mu\nu}\nabla_\lambda\Phi\right) + \mathcal{O}(\alpha'^2)\,,\\ \beta^\Phi &= \frac{1}{4}(d - d_{\text{crit}}) + \alpha'\left((\nabla\Phi)^2 - \frac{1}{2}\nabla^2\Phi - \frac{1}{24}H^2\right) + \mathcal{O}(\alpha'^2)\,,\end{aligned} \tag{2.60}$$

where $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is the totally anti-symmetric field strength of the Kalb-Ramond field. Notice that $\beta^G_{\mu\nu} = 0$ gives us the Einstein's equation paired with Kalb-Ramond and dilaton sources. In the β-function of the dilaton we also get the critical dimension condition, if we do not want to consider a background, such that it compensates $d - d_{\text{crit}} \neq 0$.

Now, circling back to the issue with defining the string coupling to be the constant e^λ we see the general picture here. The flat space limit, in which we encountered this potential issue, is obtained by considering the background $G_{\mu\nu}(X) = \eta_{\mu\nu}$, $B_{\mu\nu}(X) = 0$ and $\Phi(X) = \Phi_0 = \text{const}$, where $\lambda = \Phi_0$. This entails that different values of λ do not correspond to different theories, but different backgrounds within the *same* string theory. Therefore, λ (or g_s) is not actually a free parameter. Generally, we define the string coupling as $g_s := e^{\Phi_0}$, where Φ_0 is the vacuum expectation value $\langle\Phi\rangle$ of Φ.

In the next section we will now change gears and switch to the space-time point of view and look at the effective action of type II string theory.

2.6 The effective actions of type II string theories

With the help of string perturbation theory one can determine the space-time effective theory based on the massless spectrum by calculating the relevant amplitudes. Alternatively, one can look at the β-functions of the generalized worldsheet theory we discussed above and construct an effective action to reproduce $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ as its equations of motion. We consider here the effective action to the lowest order in both the string coupling g_s and α'. In general, the action of the EFT splits into an NS-part and a R-part and a topological Chern-Simons contribution $S_{\text{typeIIA/B}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}$. The bosonic NS-action is the same for both and looks as follows

$$S_{\rm NS} = \frac{1}{2\,\kappa_{10}} \int d^{10}x \sqrt{-G}\, e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \,, \tag{2.61}$$

where again $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$ is the totally anti-symmetric field strength of the Kalb-Ramond field. in the following we are going to use the much more elegant notation through differential forms and thereby drop the indices. For $H_{\mu\nu\rho}$ we can simply denote $H_3 = dB_2$ and for the Ramond-Ramond sector we are denoting the field strengths associated to the C_p-fields as $F_{p+1} = dC_p$. In type IIA theory we then get these kinetic terms for the p-form fields in the Ramond-Ramond sector

$$S_{\rm R} = -\frac{1}{4\,\kappa_{10}} \int d^{10}x\, F_2 \wedge *F_2 + \tilde{F}_4 \wedge *\tilde{F}_4 \tag{2.62}$$

with $\tilde{F}_4$ being the H-twisted four form $\tilde{F}_4 = dC_3 - C_1 \wedge H_3$. The Chern-Simons action for type IIA consists of just one term

$$S_{\rm CS} = -\frac{1}{4\,\kappa_{10}} \int d^{10}x\, B_2 \wedge F_4 \wedge F_4 \,. \tag{2.63}$$

In type IIB we arrive at

$$S_{\rm R} = -\frac{1}{4\,\kappa_{10}} \int d^{10}x\, F_1 \wedge *F_1 + \tilde{F}_3 \wedge *\tilde{F}_3 + \tilde{F}_5 \wedge *\tilde{F}_5 \,, \tag{2.64}$$

where both $\tilde{F}_3$ and $\tilde{F}_5$ are deformed versions and are defined as $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 - \frac{1}{2} B_2 \wedge F_3$ respectively and the Chern-Simons term is given by

$$S_{\rm CS} = -\frac{1}{4\,\kappa_{10}} \int d^{10}x\, C_4 \wedge H_3 \wedge F_3 \,. \tag{2.65}$$

Furthermore, we have a self-duality constraint for $\tilde{F}_5$, namely $\tilde{F}_5 = *\tilde{F}_5$. This concludes our presentation of type IIA and type IIB string theory for now as we turn to the construction of the remaining three stable and supersymmetric 10d string theories.

2.7 Type I string theory – spectrum and effective action

One of the three missing string theories, so-called type I string theory, can be attained by taking the quotient of type IIB by one of its fundamental symmetries, namely the invariance under exchange of left- and right-movers. From the worldsheet point of view we denote:

$$\Omega : \sigma \to l - \sigma \,, \tag{2.66}$$

which translates to the worldsheet fields as

$$\begin{aligned} \Omega^\dagger \, X^\mu(\tau,\sigma)\, \Omega &= X^\mu(\tau, l-\sigma)\,, \\ \Omega^\dagger \, \psi^\mu(\tau,\sigma)\, \Omega &= \psi^\mu(\tau, l-\sigma)\,. \end{aligned} \tag{2.67}$$

This operation, a so-called orientifold, means that we are removing the orientation of our strings, i.e. we obtain our first unoriented string theory. From (2.67) we can determine the outcome of the operation on the worldsheet oscillators:

$$\begin{aligned} &\Omega^\dagger \, \alpha_n^\mu \, \Omega = \tilde{\alpha}_n^\mu \,, \\ &\Omega^\dagger \, b_r^\mu \, \Omega = e^{2\pi i \phi} \, b_r^\mu \quad \text{with } \phi = \begin{cases} 0 & \text{R-sector}\,, \\ \frac{1}{2} & \text{NS-sector}\,. \end{cases} \quad . \end{aligned} \tag{2.68}$$

The ramifications for the spectrum are as follows

- In the (NS, NS)-sector both the dilaton and the graviton are Ω-even and remain in the spectrum, whereas the Kalb-Ramond field $B_{\mu\nu}$ is Ω-odd and therefore projected out.

- In the (R, R)-sector we have an interesting twist as the ground state of this sector is Ω-odd, therefore only the Ω-odd $C_{(p)}$-fields survive. Concretely, this is just $C_{(2)}$, which transforms in the antisymmetric representation of $SO(8)$, while both $C_{(0)}$ and $C^+_{(4)}$ transform in symmetric representations of $SO(8)$.

- In the (NS, R)- and (R, NS)-sectors the following happens

$$\Omega \; \left(|0\rangle_{L,\,NS} \otimes |\mathbf{s}\rangle_{R,\,R}\right) = |\mathbf{s}\rangle_{L,\,R} \otimes |0\rangle_{R,\,NS} \,. \tag{2.69}$$

  This means that we are projecting down to just the diagonal combination of λ_a and ψ^μ_a. Consequently, type I preserves just half of the supersymmetries, namely $\mathcal{N} = 1$ supersymmetry, and its low-energy limit turns out to be a particular case of $\mathcal{N} = 1$ supergravities.

Whilst nothing in the orientifold construction points towards an inconsistency once we check the one-loop interactions a tadpole becomes apparent rendering the theory in its current state inconsistent. This is due to the fact that summing over the two unoriented closed string contributions to the one-loop-level amplitude, whose worldsheet geometry are the cylinder and the Klein-bottle respectively, lead to a UV-divergence. The cylinder amplitude gives us the following crucial contribution:

$$Z_{\mathrm{C}_2} \sim N^2\, 2^{-6} \int_0^\infty ds \tag{2.70}$$

To understand the cylinder contribution going with N^2 better we need to recognize a fundamental principle of closed string amplitudes, namely the equivalence of one-loop open and tree-level closed string amplitudes. We already used this in our construction of the closed string sector, namely that closed strings are a product of two open strings. Let's look at the massless open string spectrum. We have already encountered that there are two types of boundary conditions for the open string we can choose from: Neumann or Dirichlet.

Now, consider an open string with $(p+1)$ Neumann direction and $9-p$ Dirichlet boundary conditions. With this set of boundary conditions the open strings ends on a $p+1$-dimensional hyperplane embedded in our 10d space-time, breaking our $SO(8)$-symmetry to $SO(p-1) \times SO(8-p)$. These $p+1$-dimensional hyperplanes were named Dp-branes and constitute a very important ingredient to our understanding of the non-perturbative sector of string theory. They get their own dedicated chapter in this thesis. Now, including these hyperplanes necessarily breaks our Lorentz symmetry. Concretely, the first excited massless states of our open string transform in the $SO(p-1)$ Lorentz group of the hyperplane, $\alpha^i_{-1}|0,p^i\rangle$ transform as vectors and $\alpha^a_{-1}|0,p^i\rangle$ as $(9-p)$ scalars, where we denote the directions parallel to the hyperplane with index i and the directions normal to it with a. Now, we don't have to require that an open string ends on the same hyperplane, it can equally as well be extended between two of those. To keep track of the general case, where we have N such hyperplanes we label our open string states $\alpha^\mu_{-1}|k,l,p^i\rangle$ to indicate that the open string ends on the k-th and l-th hyperplane. These are called Chan-Paton labels [50]. Moreover, these N^2 degrees of freedom can be conveniently encoded in $N \times N$-dimensional matrices:

$$|k,l;p^i\rangle = \lambda^a_{k,l}\,|a,p^i\rangle\,. \tag{2.71}$$

It turns out that the massless vector fields are in fact gauge bosons living on the worldvolume. In both type II theories they transform in a $U(N)$ Lie group and the $N \times N$-dimensional matrices are just the Hermitian generators of $U(N)$ (with $\lambda^\dagger = \lambda$). In type I on the other hand the orientifold changes the picture, there are two choices

$$\Omega|k, l; p^i\rangle = \pm|k, l; p^i\rangle \,, \tag{2.72}$$

that translates to $\lambda^a_{k,l} = \pm(\lambda^a_{k,l})^T$ describing the generators of symplectic $\mathfrak{sp}(N)$ or orthogonal $\mathfrak{so}(N)$ Lie algebras respectively.

Finally, this brings us back to the cylinder amplitude, which should be viewed as a one-loop open string diagram. Therefore, we are dealing with a trace over our Chan-Paton-factors giving us the crucial factor N^2 in front of the divergent integral (2.70). In type II this leads to the consistency condition that there mustn't be any space-time filling D9-brane present, i.e. $N = 0$.

In type I we do not have this option, since the orientifold forces us to include more worldsheet topologies. The first one is the Klein bottle, which can be viewed as a cylinder with two crosscaps at the endpoints. However, the Klein bottle does not have a boundary, i.e. we can not view it as an open string geometry. Nevertheless, one can understand it in a similar fashion as the cylinder diagram, namely as our unoriented closed string coupling to some 10d hyperplane. It is not a D9-brane, but a so called orientifold Op-plane. The contribution we get looks as follows

$$Z_{\text{KB}} \sim 2^4 \int_0^\infty ds \,. \tag{2.73}$$

Lastly, there is one worldsheet geometry missing at this level, the Möbius strip, which can be viewed as a cylinder with only one boundary replaced by a crosscap. With this open string geometry we get our final contribution scaling as

$$Z_{\text{Mb}} \sim \mp N \int_0^\infty ds \,, \tag{2.74}$$

where the sign choice is coupled to our choice of gauge group, $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$ respectively. Putting everything together we see that

$$Z_{\text{full}} \sim 2^{-6}(N^2 \mp 2^6 N + 2^{10}) \int_0^\infty ds = 2^{-6}(N \mp 2^5)^2 \int_0^\infty ds \,. \tag{2.75}$$

Consequently, to cancel the divergence type I string theory has to feature 32 space-time-filling D9-branes with a worldvolume Lie algebra of $\mathfrak{so}(32)$. So far we have been careful about the exact Lie group associated to the algebra as there are multiple options. To settle this we actually need more input, namely heterotic string theory and dualities.

2.8 Heterotic string theories

The two heterotic string theories are the youngest members of the five consistent, tachyon-free superstring theories in 10 dimensions. From the construction it should become obvious, why these two happen to be the last ones to be discovered. Namely, the heterotic string is a curious amalgamation of a left-moving bosonic string and a right-moving superstring. The heterotic string is of course textbook material by now [39, 41, 42], but also the original references [51, 52] are rather introductory in its approach. Like for the other superstring theories we start with the action of the heterotic string. Interestingly, there are two equivalent formulations, which arise from encoding the mismatch in degrees of freedom between the left-moving bosonic string and the right-moving superstring. The critical dimension of the bosonic string is 26, while we have seen that the superstring becomes critical in only 10 dimensions. As [51] have shown one can either use 16 bosonic coordinates to account for the additional, internal, degrees of freedom of the left-moving sector or 32 fermionic coordinates.[12] We will now show both formulations as it is more convenient for us to use the bosonic formulation to show some of the quintessential features, but in the main part of this thesis we are going to require the fermionic version. Starting with the latter one, we have:

$$S = -\frac{1}{8\pi}\int d^2\sigma \left[\frac{2}{\alpha'}\partial_\alpha X^\mu\,\partial^\alpha X_\mu - 2i\,\psi^{\,\mu}\rho^+\,\partial_+\psi_{\,\mu} - 2i\,\lambda^{\,a}\rho^-\,\partial_-\lambda_{\,a}\right]\,, \quad (2.76)$$

where $\mu \in \{0,\ldots,9\}$ goes over the same range as in the RNS case, $\psi_{\,\mu}$ purely right-moving fermionic fields and another set of fermionic fields $\lambda_{\,a}$, which are left-moving with a 32 dimensional index $a \in \{1,\ldots,32\}$. On the other hand, we have as the bosonic formulation:

$$S = -\frac{1}{8\pi}\int d^2\sigma \left[\frac{2}{\alpha'}\partial_\alpha X^\mu\,\partial^\alpha X_\mu - 2i\,\psi^{\,\mu}\rho^+\,\partial_+\psi_{\,\mu} + \frac{2}{\alpha'}\,\partial_\alpha X^I\,\partial^\alpha X_I\right]\,, \quad (2.77)$$

[12] This equivalence can be understood through the concept of bosonization of fermions in 2d conformal field theory, leading to half as many bosonic degrees of freedom. For a modern, generalized treatment of bosonization involving cobordism theory we refer to [53].

where the only change with respect to the fermionic action above is the bosonic left-moving fields with a 16 dimensional index $I \in \{1, \ldots, 16\}$. To enforce the left-moving constraint on X^I we have to require that

$$\partial_- X^I \stackrel{!}{=} 0\,. \tag{2.78}$$

The worldsheet theory is still supersymmetric although it only acts on the right-moving fields. We should highlight that only closed string boundary conditions appear to be achievable as these do not mix left-moving and right-moving degrees of freedom. In the last chapter 7 we will come back to this point.

Canonical quantization of the closed string proceeds along very similar lines as the RNS-superstring. In fact the external part is unchanged with the bosonic oscillator expansions

$$\begin{aligned} X^\mu_L(\sigma_+) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{l} p^\mu\,\sigma_+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\,\bar\alpha^\mu_n\, e^{-\frac{2\pi}{l} in\sigma_+}\,, \\ X^\mu_R(\sigma_-) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu\,\sigma_- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\,\alpha^\mu_n\, e^{-\frac{2\pi}{l} in\sigma_-}\,. \end{aligned} \tag{2.79}$$

and the fermionic oscillators

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} b^\mu_r e^{-2\pi i\, r\, \sigma_-/l} \quad \text{where } \phi = \begin{cases} 0\ (\mathrm{R}) \\ \frac{1}{2}\ (\mathrm{NS}) \end{cases} \tag{2.80}$$

The internal part however deviates. Properly taking the constraint on the internal coordinates into account (2.78) we get the following oscillator expansion

$$X^I(\sigma_+) = x^I + \alpha' p^I\,\sigma_+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\,\bar\alpha^I_n\, e^{-\frac{2\pi}{l} in\sigma_+}\,, \tag{2.81}$$

with adjusted commutator relations accounting for the vanishing of all right-movers and the absence of time-like coordinates

$$\begin{aligned} [\hat x^\mu, \hat p^\nu] &= \frac{i}{2}\delta^{\mu\nu}\,, \\ [\hat{\bar\alpha}^\mu_m, \hat{\bar\alpha}^\nu_n] &= m\,\delta_{m+n,0}\,\delta^{\mu\nu}\,. \end{aligned} \tag{2.82}$$

As demonstrated in the original heterotic string construction [51, 52] the internal coordinates can be interpreted as the bosonic string compactified on a 16-dimensional torus T^{16}. The requirement of matching the above expansion turns out to be extremely constraining. We start by considering a general d-dimensional torus compactification of the bosonic string. The torus can be treated by quotienting the Euclidean space $\mathbb{R}^d$ by a lattice Γ^d generated by d independent basis vectors e_i^I, $i \in \{0, \dots, d\}$, which we choose to be of length $\sqrt{2}$. Taking the circumference of each torus coordinate to be $2\pi R_i$ with an individual radius for each of the d directions, the compactness condition on X^I equates to identifying

$$X^I \sim X^I + 2\pi L^I \,, \tag{2.83}$$

where

$$L^I = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} n_i R_i e_i^I \tag{2.84}$$

are topological winding numbers (valued by the radius) classifying the homotopy class of embedding a circle parametrized by $0 \leq \sigma \leq 2\pi$ into $0 \leq X^I \leq 2\pi R^I$. The oscillator expansion for X^I then takes the form:

$$X^I(\sigma, \tau) = x^I + \alpha' p^I \tau + L^I \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^I \, e^{-\frac{2\pi}{l} in\sigma_-} + \bar{\alpha}_n^I \, e^{-\frac{2\pi}{l} in\sigma_+} \right) \tag{2.85}$$

with the center of mass $x^I = \sqrt{2}\,\pi \, \sum_{i=1}^{d} n_i R_i e_i^I$.

Since the momenta p^I generate the translations of x^I and we require single-valuedness of $e^{i\,x^I p_I}$, we get that

$$p^I = \sqrt{2} \sum_{i=1}^{d} \frac{m_i}{R_i} e_i^{*I} \,, \tag{2.86}$$

where e_i^{*I} are the basis vectors of the dual lattice Γ^* such that

$$\sum_{I=1}^{d} e_i^I e_j^{*I} = \delta_{ij} \,. \tag{2.87}$$

Now, decomposing the bosonic oscillators into left- and right-movers again $X^I(\tau, \sigma) = X_L^I(\sigma_+) + X_R^I(\sigma_-)$, we define

$$\begin{aligned} X_L^I(\sigma_+) &= \frac{1}{2} x^I + (\frac{\alpha'}{2} p^I + L^I)\,\sigma_+ + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \, \bar{\alpha}_n^I \, e^{-\frac{2\pi}{l} in\sigma_+} \,, \\ X_R^I(\sigma_-) &= \frac{1}{2} x^I + (\frac{\alpha'}{2} p^I - L^I)\,\sigma_- + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha_n^I \, e^{-\frac{2\pi}{l} in\sigma_-} \,. \end{aligned} \tag{2.88}$$

To obtain a purely left-moving sector the momentum modes $\frac{1}{2}p^I$ have to equal the winding modes L^I and we obtain the following condition

$$\frac{\alpha'}{2}p^I - L^I = \frac{1}{\sqrt{2}}\sum_{i=1}^{d}\left(\alpha'\frac{m_i}{R_i}e_i^{*I} - n_i R_i e_i^I\right) \stackrel{!}{=} 0\,. \tag{2.89}$$

From this we get our first conditions, namely $R_i = \sqrt{\alpha'}$, $m_i = n_i$ and a self-dual lattice $\Gamma = \Gamma^*$. The second condition comes from the fact that we are dealing with closed strings without a distinguished point on the worldsheet. Differently put, the unitary operator responsible for shifts $U(\Delta)$ on the worldsheet has to be the identity. With $U(\Delta)$ given by

$$U(\Delta) = e^{i\,\Delta(N-\bar{N}+1-\frac{1}{2}\sum_{I=1}^{16}(p^I)^2)}\,, \tag{2.90}$$

where N and $\tilde{N}$ are the respective number operators, we get our second condition on the physical states, namely

$$N - \bar{N} + 1 - \frac{1}{2}\sum_{I=1}^{16}(p^I)^2 = 0\,. \tag{2.91}$$

This entails that our self-dual lattice has to be even, as well, for $\frac{1}{2}\sum_{I=1}^{16}(p^I)^2 \in \mathbb{Z}$. Even and self-dual lattices are extremely rare in (relatively) low dimensions. For one, they only occur in dimensions 0 mod 8. In dimension 16 there exist just two of them, which we label $\Gamma_8 \times \Gamma_8$ and Γ_{16}^+. The first one is the lattice of $E_8 \times E_8$ and the latter the lattice corresponding to $Spin(32)/\mathbb{Z}_2$, which we will refer to as the Semispin group, abbreviated as $Ss(32)$.[13] Of course, there is clear intent behind our introduction of these Lie groups. Namely, the massless spectrum of our heterotic strings contains gauge bosons transforming in the adjoint of these Lie groups. This happens as follows, first of all a generic toroidal compactification of the bosonic worldsheet fields X^I gives rise to massless fields

$$\begin{aligned} &\alpha^\mu_{-1}\bar{\alpha}^I_{-1}|0\rangle\,, \\ &\bar{\alpha}^\mu_{-1}\alpha^I_{-1}|0\rangle\,, \end{aligned} \tag{2.92}$$

[13] We do this to explicitly avoid any misconceptions about the Lie group as the quotient $Spin(32)/\mathbb{Z}_2$ could also mean $SO(32)$, which is not the Lie group corresponding to Γ_{16}^+. The first crucial difference is that instead of the vector representations as for $SO(32)$, one of the two spinor representations of $Spin(32)$ survives the quotient explaining the name of the Lie group. Both of them share the same Lie algebra $\mathfrak{so}(32)$, though.

the gauge bosons of $U(1)^d_L \times U(1)^d_R$. However, at the torus radii $R = \sqrt{\alpha'}$ we get a gauge group enhancement due to additional massless gauge bosons. This can be seen from the mass formula:

$$M^2 \sim N + \bar{N} - 2 + \frac{1}{2}\sum_I (p^I)^2 \,. \tag{2.93}$$

We get massless modes for $N - \bar{N} = \pm 1$ and $(p^I)^2 = 2$

$$|(p^I)^2 = 2\rangle \,, \tag{2.94}$$

providing the non-Abelian gauge bosons for the gauge group enhancement. For our 16-dimensional self-dual lattices we get 480 vectors with length squared 2, yielding 480 massless vector bosons combining with the 16 Abelian massless vectors to account for the 496 representations in the adjoint of either $E_8 \times E_8$ or $Ss(32)$.[14] The full spectrum of the heterotic string arises as it is a closed string spectrum from combining the right-moving fermionic and the left-moving bosonic string. Applying the GSO-projection once more projects out the tachyon in the NS-sector and ensures one-loop consistency, while the tachyonic vacuum in the bosonic sector is removed by invoking the left-right level matching condition:

$$N_L + \frac{1}{2}p_L^2 - 1 = \begin{cases} N_R & \text{R sector} \\ N_R + \frac{1}{2} & \text{NS sector}\,. \end{cases} \tag{2.95}$$

The full massless spectrum can be summarized as follows:

- 10d graviton, dilaton and Kalb-Ramond field

$$\bar{\alpha}^\mu_{-1}|0\rangle \otimes b^\mu_{-\frac{1}{2}}|0\rangle_{NS} \,. \tag{2.96}$$

- The supersymmetric partners of the above states, the gravitino and dilatino,

$$\bar{\alpha}^\mu_{-1}|0\rangle \otimes |\mathbf{s}\rangle_R \,. \tag{2.97}$$

[14] For brevity we will usually call the two heterotic string theories by their common abbreviations HE and HO theory. The first refers to the one with gauge group $E_8 \times E_8$ and the second alludes to the underlying $\mathfrak{so}(32)$ Lie algebra for the $Ss(32)$ gauge group of the other heterotic string.

- The gauge bosons of $E_8 \times E_8$ or $Ss(32)$

$$\begin{aligned} \bar{\alpha}^I_{-1}|0\rangle \otimes b^\mu_{-\frac{1}{2}}|0\rangle_{NS} \,, \\ |(p^I)^2 = 2\rangle \otimes b^\mu_{-\frac{1}{2}}|0\rangle_{NS} \,. \end{aligned} \tag{2.98}$$

- And also their supersymmetric partners, the gaugini:

$$\begin{aligned} \bar{\alpha}^I_{-1}|0\rangle \otimes |\mathrm{s}\rangle_R \,, \\ |(p^I)^2 = 2\rangle \otimes |\mathrm{s}\rangle_R \,. \end{aligned} \tag{2.99}$$

From this spectrum we can already anticipate a major insight into non-perturbative aspects. Namely, the HO and type I string theory spectra look very similar. Eventually, they turn out to be two sides of the same coin merely expressing the same degrees of freedom at weak or strong coupling respectively. Before, we can dive into these non-perturbative aspects of the five supersymmetric and stable string theories however we need to elucidate the dynamical nature of Dp-branes.

2.9 Dp-branes as dynamical objects

In the preceding chapters we made contact with Dp-branes as the boundaries of open strings, but they turn out to be far more eclectic than that. For one, as Polchinski pointed out in [54] they have a tension and are charged under the Ramond-Ramond C_{p+1}-fields. Actually, we have already encountered that. The divergence of the one-loop open string cylinder diagram in type I can also be viewed through the lens of an closed string exchange, since it would lead to exactly the same worldsheet topology. In fact the closed string exchange can be decomposed into two disk tadpoles and a closed string propagator.

However, there are only very few massless fields, that could be responsible for such a tadpole. In fact, we know that it is generated by a Ramond-Ramond field, because the Kalb-Ramond field would not cause a Lorentz-invariant tadpole. The only possible explanation is that the D9-brane emitting the closed string couples to the R-R 10-form C_{10}. More generally, it can be shown that every Ramond-Ramond $p+1$-form field pairs up with a Dp-brane that couples to this field as

$$\mu_p \int_{\mathcal{W}} C_{p+1} \tag{2.100}$$

with a charge $\mu_p = (2\pi)^{-p}(\alpha')^{-\frac{p+1}{2}}$. Moreover, the fields strengths of the $p+1$-form fields turn out to be Hodge dual to one another:

$$F_{10-p} = *F_p \,. \tag{2.101}$$

Therefore, a p-brane sources the same field as a $(6-p)$-brane – one as an electric, one as a magnetic source. Finally, these charges have to satisfy Dirac quantization. For the unphysical Dirac string to vanish the Dp-brane charges have to fulfill:

$$\frac{\mu_{6-p}\mu_p}{2\kappa_0^2} = 2\pi n \,, \tag{2.102}$$

which is realized in string theory with the minimal quantum number $n = 1$.

The electric/magnetic coupling of the Dp-branes are not the only D-brane dynamics one can detect from disk amplitudes, i.e. open-string tree level of order g_s^{-1}. We learn that a Dp-brane couples to the NS-NS sector fields as well, which takes the form of the Dirac-Born-Infeld action, which was developed initially much like string theory itself to solve a completely different physical problem [55, 56]:

$$S_{DBI} = -T_p \int_{\mathcal{W}} d^{p+1}\xi \, e^{-\Phi} \left(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})\right)^{\frac{1}{2}} \,, \tag{2.103}$$

where $T_p = |\mu_p|$ is the tension of the brane and Φ, G_{ab} and B_{ab} the familiar massless fields in the closed string spectrum, whereas F_{ab} is the field strength of the $U(N)$ gauge group for a stack of N Dp-branes. To record the universal g_s^{-1} scaling of the D-brane directly in the tension one can define $\tau_p = g_s^{-1}T_p$. For type I we have a field strength of an either orthogonal or symplectic gauge group due to the orientifold as we have already seen for the D9-brane. Moreover, the orientifold lowers the tension by a factor of $\frac{1}{\sqrt{2}}$. This reveals a pretty drastic virtue of string theory, strings are not the only dynamical objects of the theory, but branes, as well.

So far, we have approached Dp-branes through string perturbation theory. As it turns out they can also be described as BPS solutions to the low-energy effective action. The solutions are surprisingly simple and straightforward:

$$\begin{aligned} ds^2 &= H_p^{-\frac{1}{2}} \, \eta_{\mu\nu} dx^\mu dx^\nu + H_p^{\frac{1}{2}} \, \delta_{ij} dy^i dy^j \,, \\ e^{2\Phi} &= g_s^2 \, H_p^{\frac{3-p}{2}} \,, \\ C_{p+1} &= (H_p^{-1} - 1) \, g_s^{-1} \, dx^0 \wedge \cdots \wedge dx^p \,, \end{aligned} \tag{2.104}$$

where x^μ and x^ν are the D-brane worldvolume coordinates and y^i, y^j the coordinates transverse to the brane. Furthermore, H_p are harmonic functions given by the following expression:

$$
\begin{aligned}
H_p &= 1 + \frac{(2\pi)^{p-2}\,(\alpha')^{\frac{7-p}{2}}\,g_s\,N}{r^{7-p}}\,c(p) \quad \text{with} \\
c(p) &= 2^{7-2p}\,\pi^{\frac{9-3p}{2}}\,\Gamma\left(\frac{7-p}{2}\right)
\end{aligned}
\tag{2.105}
$$

with N the number of Dp-branes sitting at the origin $r = 0$. Immediately, the non-trivial curvature of this set of solutions should make us suspicious. So far, we have described Dp-branes as flat hyperplanes in a Minkowski background, so how do the two descriptions even fit together? It is all about regimes of validity. The perturbative string description is valid for $g_s\,N < 1$, where the curvature is high in the above solution pushing the supergravity approximation outside of its regime of validity. Conversely, for $g_s\,N > 1$ we can work perfectly fine with just the low-energy effective action.

Now, let us turn the logic around: Do there exist solutions to the low-energy effective action that we have not yet identified from a string perturbation perspective? Indeed, there exists another large class of extended objects, the NSp-branes. Their name refers to them carrying charges with respect to NS-NS fields. In fact, we have already encountered one of them – the NS1-brane, of course better known as our fundamental string that couples (electrically) to the B_2-field. We omit the solution here and refer for a discussion of this solution to e.g. [39, 57].

Using the Dp-branes as a kind of guiding principle, does an extended object exist that couples to B_2 magnetically and electrically to the Hodge dual of B_2? Again, the answer is affirmative. Based on Hodge duality it is clear that it has to be a 5-brane coupling to the B_6-field. For both the magnetic and the electric coupling we can write down a solution for the bosonic part of the low-energy effective action shared by both type II string theories. Type I does not feature such a object, since the B_2-field is projected out. Both heterotic string theories do also contain a NS5-brane, but we have to seriously take the background gauge field into account. We therefore begin with the type II solution and subsequently augment the solution for the heterotic string theories. For brevity, we just show the magnetic solution here:

$$
\begin{aligned}
ds^2 &= e^{-\frac{(\Phi-\Phi_0)}{2}}\,\eta_{\mu\nu}dx^\mu dx^\nu + e^{\frac{3(\Phi-\Phi_0)}{2}}\,\delta_{ij}dy^i dy^j\,,\\
e^{2\Phi} &= e^{2\Phi_0}\left(1+\frac{k_6}{y^2}\right)\,,\\
H_3 &= 2\,k_6\,e^{\frac{\Phi_0}{2}}\,\epsilon_3\,.
\end{aligned}
\tag{2.106}
$$

The coordinates are split between those on the worldvolume of the NS5-brane x and those transverse to it y. For a more convenient notation of the metric we have not converted the vacuum expectation value of the dilaton Φ_0 into the string coupling constant g_s. The H_3-flux is supported on a three-sphere S^3 with volume element ϵ_3. The parameter k_6 is just a composition of a bunch of constants:

$$
k_6 = \frac{\kappa_0 g_6}{\sqrt{2}\,\Omega_6}\,e^{-\frac{\Phi_0}{2}}
\tag{2.107}
$$

with g_6 the topological magnetic charge. Interestingly, the NS5-brane – just like its Dp-brane counterparts – satisfies the same Dirac quantization condition (2.102), which can be translated into a version involving the tensions τ_p. We then have this Dirac quantization condition between the NS5-brane and the fundamental F1 string

$$
\frac{\tau_{NS5}\,\tau_{F1}}{2\kappa_0^2} = \frac{2\pi n}{g_s^2}\,.
\tag{2.108}
$$

Since the fundamental string tension does not scale with the string coupling, the NS5-brane has a novel, characteristic g_s^{-2} scaling. As we mentioned before the NS5-brane solution for both heterotic string theories is more complicated, because the bosonic part of the low-energy effective action is enhanced by the topological term:

$$
S_{het,\,top} = -\frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\,\frac{\alpha'}{4}\Big(tr(F\wedge F) - tr(R\wedge R)\Big)\,.
\tag{2.109}
$$

The topological piece modifies the Bianchi identity for H_3, as well:

$$
dH_3 = \frac{\alpha'}{4}\Big(tr(R\wedge R) - tr(F\wedge F)\Big)\,.
\tag{2.110}
$$

There are two solutions going beyond the "elementary" NS5-brane from before, which have been coined "gauge" and "symmetric" 5-brane in the literature. For the first one we set $tr(R\wedge R) = 0$ and construct a $SU(2)$ instanton, embedded in $E8\times E8$ or

$Ss(32)$, in the 4-dimensional transverse space to the 5-brane. The solution looks as follows:

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2\Phi} \delta_{ij} dy^i dy^j \,, \\
e^{2\Phi} &= e^{2\Phi_0} + 8\alpha' \frac{(y^2 + 2\rho^2)}{(y^2+\rho^2)^2} \,, \\
H_{\mu\nu\lambda} &= \pm \epsilon_{\mu\nu\lambda\rho} \nabla^\rho \Phi \,, \\
F^{\pm}_{\mu\nu} &= \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F^{\pm}_{\rho\sigma} \,.
\end{aligned}
\tag{2.111}
$$

The NS5-brane now carries an additional quantized topological charge:

$$
\nu = \frac{1}{480\,\pi^2} \int_{M_4} tr(F \wedge F) \in \pi_4(BG) \,, \tag{2.112}
$$

where M_4 is the transverse manifold, which is either compact or at least has compact support, and $\pi_*(BG)$ denotes the homotopy groups of the classifying space of the gauge group G with G either $E8 \times E8$ or $Ss(32)$. For the symmetric solution we cancel the gravitational contribution with the gauge contribution such that $tr(F \wedge F) - tr(R \wedge R) = 0$. Both are taken to be self-dual. The solution is almost the same but the dilaton solution, which can be written as follows:

$$
e^{2\Phi} = e^{2\Phi_0} + \frac{n\,\alpha'}{y^2} \,. \tag{2.113}
$$

It should be highlighted that we have another charge, which is rarely mentioned in the literature, the gravitational charge:

$$
\nu_{\text{grav}} = \frac{1}{480\,\pi^2} \int_{M_4} tr(R \wedge R) \,. \tag{2.114}
$$

This is not a homotopy charge and actually reveals something deeper. We will come back to this point later.

Other NSp-branes are discussed to a much lesser extend in the literature, one particular example we would like to highlight is the NS-NS sector counterpart of the D9-brane, the NS9-brane. The effective action was first determined in [58] to be:

$$
\begin{aligned}
S_{NS9} = S_{DBI} + S_{CS} = &- T_9 \int_{\mathcal{W}_{10}} d^{10}\xi \, e^{-4\Phi} \left(-\det(G_{ab} + 2\pi\alpha' \, e^{\Phi} \, F_{ab}) \right)^{\frac{1}{2}} \\
&- T_9 \int_{\mathcal{W}_{10}} (B_{10} + \ldots) \,.
\end{aligned}
\tag{2.115}
$$

Now, it is time to expand on a particular aspect of Dp-branes that will allow us to classify Dp-branes more generally. What we are referencing is the presence of so called anomalous couplings of Dp-branes in the Chern-Simons type action. Initially, they were inferred from anomaly inflow on intersections of Dp-branes necessitating the presence of these couplings to cancel the anomaly [59]. Two years later they were calculated from string amplitudes directly and the results could be extended to the Chern-Simons term for Op-planes, as well [60, 61]. They read as follows:

For type II

$$S_{CS}^{Dp} = \mu_p \int_{\mathcal{W}_{p+1}} \left[\bigoplus_q C_q \wedge \mathrm{ch}(\mathcal{F}) \wedge \sqrt{\frac{Td(T\mathcal{W})}{Td(N\mathcal{W})}} \right] \Bigg|_{p+1} , \tag{2.116}$$

for type I

$$S_{CS}^{Dp} = \mu_p \int_{\mathcal{W}_{p+1}} \left[\bigoplus_q C_q \wedge \mathrm{ch}(F) \wedge \sqrt{\frac{\hat{A}(T\mathcal{W})}{\hat{A}(N\mathcal{W})}} \right] \Bigg|_{p+1} \tag{2.117}$$

and the Op-plane action

$$S_{CS}^{Op} = 2^{p-4} \mu_p \int_{\mathcal{W}_{p+1}} \left[\bigoplus_q C_q \wedge \sqrt{\frac{L(T\mathcal{W})}{L(N\mathcal{W})}} \right] \Bigg|_{p+1} . \tag{2.118}$$

$ch(F)$ is the Chern character D of the appropriate Chan-Paton gauge bundle F on the respective Dp-brane. For type IIB the Chan-Paton bundle is modified to include the B-field $\mathcal{F} = 2\pi\alpha' F + B$.

The gravitational couplings detect the curvature of the tangential space TM and the normal space NM with respect to the worldvolume manifold. The difference in the gravitational couplings between type I and type II Dp-branes is rarely acknowledged. Td is the Todd genus of a manifold, whereas type I Dp-branes feature the $\hat{A}$ genus. On manifolds with almost complex structure the two are in fact closely related $Td(M_{ac}) = e^{c_1/2}\hat{A}(M_{ac})$ [62]. Moreover, complete equivalence is for example realized on Calabi-Yau manifolds, which by definition are almost complex and have a vanishing first Chern class $c_1 = 0$. We expand on the properties of these topological invariants in the appendix D.

These gauge and gravitational couplings are a quite remarkable result realizing "branes within branes" in string theory [63]. For example a D9-brane couples to C_6 – like a D5-brane with a 4 dimensional gauge or gravitational instanton supported on the worldvolume of the D9-brane.

K-theory Dp-brane charge classification

The first account of the proper Dp-brane charge classification by K-theory groups is [64]. Due to the Atiyah-Singer index of the (twisted) Dirac operator $(E_1, E_2)_K = \mathrm{ind}(D_{E_1,E_2})$ living in K-theory, the authors of [64] were led to the natural identification of the Dp-brane charge as the image of the modified Chern isomorphism $E \to ch(E)\sqrt{\hat{A}(T\mathcal{W})}$ from K-theory to deRham cohomology.

Witten [65] then expanded on this natural mathematical suggestion by linking the charge classification of Dp-branes with K-theory to Sen's construction of Dp-branes from space-time-filling D9 and anti-D9 branes [66]. This can be understood from the basic definition of K-theory as follows. As we are going to review in the mathematical background chapter K-theory is defined as the equivalence class of pairs of vector bundles (E, F), which are essentially the difference of the two vector bundles $E - F$. This models precisely the simultaneous occurrence of D9-branes carrying Chan-Paton bundle E and $\overline{\text{D9}}$-branes carrying the bundle F. Further assuming that a collection of n D9- and n $\overline{\text{D9}}$-branes sharing the same bundle H can be created or annihilated can be directly identified with the equivalence relation of K-theory:

$$(E, F) \sim (E \oplus H, F \oplus H) . \tag{2.119}$$

This means we can model space-time-filling Dp-branes, what about lower dimensional Dp-branes? Sen's construction provides just the means to understand what is going on. Let's start with type II string theory. Consider a coincident pair of a D$p+2$- and $\overline{\text{D}p+2}$-brane on whose worldvolumes support a $U(1) \times U(1)$ gauge field and a tachyon carrying a charge $(1, -1)$. Rather then just inducing the annihilation of brane and anti-brane, the tachyon forms a vortex configuration vanishing on a codimension 2 subspace. The tachyon field is complex and therefore can attain a topologically non-trivial winding. The associated winding number is nothing else than the magnetic charge the codimension 2 subspace carries with respect to the $U(1)$ that our initial $U(1) \times U(1)$ configuration is broken to by the tachyon. Since the D$p+2$ charges vanish the configuration is indistinguishable from a Dp-brane.

This tachyon construction is in one-to-one correspondence to the notion of "branes within branes", where we annihilate the initial D$(p+2k)$-brane on which the lower dimensional Dp lives. This mechanism can be repeated iteratively, i.e. we can create a Dp-brane from a D$(p+2)$-$\overline{\text{D}(p+2)}$-pair, which itself can be created by two D$(p+4)$-$\overline{\text{D}(p+4)}$-pairs. Generally, a Dp-brane is therefore created from 2^{k-1} D$(p+2k)$-$\overline{\text{D}(p+2k)}$-pairs.

This spacing by 2 for stable Dp-branes matching the pattern of C_p-fields in type II string theory is reflected on the complex K-theory side classifying type II Dp-branes by Bott-periodicity [67, 68]

$$K^{-n}(X) \cong K^{-n\pm 2}(X) \tag{2.120}$$

What we have covered so far directly applies for type IIB strings as this theory feature space-time-filling D9-branes, which we identified as being modeled by $K^0(X)$. For type IIA the Dp-branes of lowest codimension are D8 branes. [65] therefore proposed to identify $K^0(S^1 \times X)$ with the K-theory group classifying D8-branes. Bott periodicity then leads to the correct full spectrum of Dp-branes in type IIA. In flat space we take the trivial limit for $X = pt$, such that we have the following classification of Dp-branes in type II string theory:

n	0	1	2	3	4	5	6	7	8	9	10
$K^{-n}(\text{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	D9		D7		D5		D3		D1		D(-1)

Table 2.3: Flat space type IIB classification through complex K-theory.

n	0	1	2	3	4	5	6	7	8	9	10
$K^{-n+1}(\text{pt})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
		D8		D6		D4		D2		D0	

Table 2.4: Flat space type IIA classification through shifted complex K-theory.

Now we are left with a final superstring theory featuring Dp-branes, namely type I. We have seen in the previous discussion on type I string theory that the orientifold projection changes the Chan-Paton bundle to transform under the orthogonal group. This means that we can apply the same reasoning as for type IIB string theory once we change from complex vector bundles to real vector bundles. We should stress here that this real vector bundle is not a principal bundle, which will be necessary to encode the background gauge group arising from the 32 tadpole-canceling D9-branes. This subtle difference will be important for a more nuanced discussion in the main part of this thesis.

By constructing a K-theory based on pairs of real vector bundles over a background space X we obtain real K-theory $KO^{-n}(X)$ with its own version of Bott periodicity [67, 68]

$$KO^{-n}(X) \cong KO^{-n\pm 8}(X). \tag{2.121}$$

Here, something very intriguing happens. Considering again type I on flat space we get the following classification:

n	0	1	2	3	4	5	6	7	8	9	10
$KO^{-n}(\text{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	D9	$\widehat{\text{D}}8$	$\widehat{\text{D}}7$		D5				D1	$\widehat{\text{D}}0$	$\widehat{\text{D}}(-1)$

Table 2.5: Flat space type I classification through real K-theory.

We immediately see that we recover the expected D9-, D5- and D1-branes coupling to the RR-fields surviving the orientifold projection. However, we also get torsional pieces corresponding to previously unknown Dp-branes. These actually classify charges of non-BPS-branes. The non-BPS $\widehat{D}0$ actually has been (partially) discovered slightly before [69]. This proposal can in fact be extended to construct general non-BPS $\widehat{\text{D}}p$-branes matching the KO-theory classification exactly [70]. The construction works as follows: Consider again a pair of type II Dp+1-brane and anti-brane, this time with a Z_2 Wilson line on top or alternatively modding out $(-1)^{\mathcal{F}_L}$ in the theory. This configuration condenses to a non-BPS Dp-brane. For type IIB we get non-BPS Dp-branes with $p \in \{0, 2, 4, 6, 8\}$. As the complex K-theory classification indicates these objects are unstable. In fact they contain an open string tachyon for open strings starting and ending on the non-BPS Dp-brane. However, it turns out that for $p \in \{0,4,8\}$ the orientifold projection can eliminate the tachyon and stabilize the object. Additionally, non-BPS objects in type I can be obtained from Dp-$\overline{\text{D}p}$ pairs without a Wilson line, merely projecting out the tachyon between the two through orientifold projection. This is possible for $p \in \{-1,3,7\}$. Therefore, there are potentially stable type I non-BPS Dp-branes for $p \in \{-1,0,3,4,7,8\}$. Still, there can exist tachyonic modes. To check their absence we need to look at the spectrum of open strings in the DD, starting and ending on the non-BPS brane and the DN sector, connecting the non-BPS brane with the 32 background D9-branes. The leading term of the total amplitude of the open strings in the DD sector reads:

$$\mathcal{A}_{tot} \sim \int_0^\infty \frac{ds}{2s}\, s^{-\frac{p+1}{2}}\, q^{-1} \left[\mu_p^2 - 2\mu_p \sin\left(\frac{\pi}{4}(9-p)\right)\right] , \tag{2.122}$$

where the q^{-1}-dependence indicates a tachyonic instability. Therefore, we need to require:

$$\mu_p = 2\sin\left(\frac{\pi}{4}(9-p)\right) . \tag{2.123}$$

Since μ_p is a positive coefficient entering the tension of the non-BPS Dp-brane, we have to discard $p = 3, 4$. This accounts precisely for the previously unaccounted Dp-branes in our KO-theory classification. For the DN sector the presence of tachyons can be directly deduced from the normal ordering constant in the NS sector, which for ν mixed boundary conditions reads:

$$a_{NS} = \frac{1}{2} - \frac{\nu}{4} \tag{2.124}$$

Here, only $\nu > 2$ leads to massive modes. Correspondingly, the $\widehat{\text{D}}7$ and $\widehat{\text{D}}8$ are unstable in the DN sector and decay. This is exactly what was observed in [71]. Essentially, the $\widehat{\text{D}}7$ and $\widehat{\text{D}}8$ decay to just a non-trivial gauge field configuration on the background D9-branes. On the opposite end non-BPS branes $\widehat{\text{D}}p$ with $p \leq 6$ do not contain a tachyon in their DN sector.

2.10 Dualities

Next, we come to a topic intimately tied to the non-perturbative sector of string theory, populated by branes, namely dualities. The notion of duality loosely describes some form of equivalence between seemingly distinct theories. Especially in string theory they are quite ubiquitous and appear in all kinds of flavor. Here, we will limit ourselves to introducing T- and S-duality, both of which are instrumental to a unified understanding of all five superstring theories. Also for this section we will rely on the textbook accounts in [39–43]. An excellent introductory review on just duality can for example be found in [72].[15]

2.10.1 T-duality

To explain T-duality let us go back to the heterotic string. We observed that the gauge theoretic degrees of freedom giving rise to either an $E_8 \times E_8$ or $Ss(32)$ gauge group can be understood from compactifying internal directions on a torus. Let us now consider the simplest example, namely a compactification on just a circle. The closed string oscillator now periodic under shifts on the circle. Moreover, we seen that it receives contributions from winding around the circle (2.85). Now, evaluating the mass operator we obtain:

$$\alpha' m^2 = \alpha' \frac{M^2}{R^2} + \frac{L^2 R^2}{\alpha'} + 2(N_L + N_R - 2)\,, \tag{2.125}$$

where M are the integer momentum values we named m_i before and L is the 1d version of the winding numbers n_i for the torus. The sector with $M = 0$ and $L = 0$ we obtain the uncompactified mass formula. M is just Kaluza-Klein (KK) modes going back to the earliest examples of dimensional reduction on compact spaces. Now this spectrum has a curious symmetry we call T-duality:

$$M \leftrightarrow L \quad \text{and} \quad R \leftrightarrow R' = \frac{\alpha'}{R}\,. \tag{2.126}$$

This means that closed strings perceive space-time topology and geometry very different from particles. For one they have a minimal length resolution, since at $R = \sqrt{\alpha'}$ we get a strict equality. All radii below this minimal radius are mapped back to much larger radii, therefore revealing no new information. With regards to the left and right-moving bosonic oscillators exchanging KK modes with winding modes $M \leftrightarrow L$ leads to a parity transformation

$$X_L^{S^1}(\sigma_+) \leftrightarrow X_L^{S^1}(\sigma_+) \quad \text{and} \quad X_R^{S^1}(\sigma_-) \leftrightarrow -X_R^{S^1}(\sigma_-)\,. \tag{2.127}$$

[15] For the supergravity perspective on dualities we would like to refer the reader to the recent review [73].

For the superstring this means that superconformal transformations on the world-sheet force us to have a parity flip for the right-moving fermionic coordinates, as well:

$$\psi_-^{S^1}(\sigma_-) \leftrightarrow -\psi_-^{S^1}(\sigma_-) \tag{2.128}$$

However, this means that in particular the zero-modes $\hat{b}_0^\mu$ are affected. If we take for example the 9th spacial direction to be the one about to be T-dualized we get a sign flip

$$\hat{b}_0^9 \leftrightarrow -\hat{b}_0^9 \tag{2.129}$$

This however entails that this sign flip proceeds right to our modified fermion number operator in the right-moving Ramond sector $\Gamma\,(-1)^{\mathcal{F}}$. More specifically, Γ flips its sign. Therefore the closed string sectors transform as

$$\begin{aligned} (NS_+, R_\pm) &\xleftrightarrow{\text{T-duality}} (NS_+, R_\mp) \\ (R_+, R_\pm) &\xleftrightarrow{\text{T-duality}} (R_+, R_\mp) \end{aligned} \tag{2.130}$$

This has a fascinating consequence: T-duality transforms type IIB into type IIA and vice versa! For the open string we have already recognized that changing (DD) boundary conditions to (NN) boundary conditions amounts to a sign flip for the right-moving sector. Consider the situation, where we have all (NN) boundary conditions in 10d amounting to a D9-brane setup. Now, we compactify on a circle and T-dualize along this dimension. We obtain a D8-brane. Analogous scenarios lead us to the full spectrum of type IIA Dp-branes, which is a shift for p to $p-1$ from type IIB to type IIA. T-duality can be generalized straightforwardly from a circle S^1 to a torus T^d. Then, T-dualizing along an even number of dimensions d gives us back the type II theory we started with, while an odd number interchanges the theories.

T-duality not only connects the two type II string theories, but also the heterotic ones. Here however, we need to add a Wilson line on the S^1 to break the respective gauge group to a matching subgroup. Contrary to claims in the literature this is not possible and the actual construction is more subtle. While we can break $E8 \times E8$ to its $Ss(16) \times Ss(16)$ subgroup, $Ss(32)$ does not feature such a subgroup as was discussed in depth in [74]. Instead, we can only look for a common double cover lift for both subgroups. This leads to a non-trivial consistency condition, the vanishing of a particular topological obstruction [75].

Suppose the construction delivers a common subgroup, we can have a look at the momenta in the presence of a Wilson line on just an S^1. We get the following shifted momenta:

$$\begin{aligned} k_L &= \frac{M}{R} + \frac{L\,R}{\alpha'} - q^I A^I - \frac{L\,R}{2}(A^I)^2\,, \\ k_L^I &= (q^I + L\,R\,A^I)(2\alpha')^{\frac{1}{2}}\,, \\ k_R &= \frac{M}{R} - \frac{L\,R}{\alpha'} - q^I A^I - \frac{L\,R}{2}(A^I)^2\,. \end{aligned} \tag{2.131}$$

Here, q^I denote the internal momentum modes on the internal lattice corresponding to the common subgroup. For respective choices of

$$R\,A^I = \text{diag}\left(\frac{1}{2}^8,\, 0^8\right) \tag{2.132}$$

for the HO string and

$$R'\,A^I = \text{diag}\left(1,\, 0^7,\, 1,\, 0^7\right) \tag{2.133}$$

for the HE string we focus on the states with $k_L^I = 0$. This can only happen for even integer winding modes $L \in 2\mathbb{Z}$. Consequently, the corresponding momenta read:

$$k_{L,R} = \frac{\tilde{M}}{R} \pm \frac{L\,R}{\alpha'}\,, \quad k'_{L,R} = \frac{\tilde{M}'}{R'} \pm \frac{L'\,R'}{\alpha'}\,, \tag{2.134}$$

where $\tilde{M}$ denotes the shifted $M + L$. And again we obtain a dual spectrum under interchanging winding modes L and shifted Kaluza-Klein Modes $\tilde{M}$ and inverting the radius:

$$\begin{aligned} (\tilde{M}, L) &\leftrightarrow (L', \tilde{M}')\,, \\ (k_L, k_R) &\leftrightarrow (k_L, -k_R)\,, \\ R &\leftrightarrow \frac{\alpha'}{R'}\,. \end{aligned} \tag{2.135}$$

The notion of T-duality can also be extended beyond just toroidal spaces. On Calabi-Yau manifolds T-duality was linked to a curious geometric symmetry called mirror symmetry [76]. Moreover, T-duality goes beyond just geometry. In [77] so-called T-folds were constructed, where a string theory on a geometric T^n fibration gets T-dualized to another T^n-fibration, which only has a local geometric formulation, but not a global one.

Drawing back to our K-theory classification, T-duality is simply encoded in one of the Eilenberg-Steenrod axioms for a generalized (co-)homology theory such as K-theory, namely the suspension axiom. It tells us that under the reduced suspension

Σ, which is homeomorphic to the smash product with a circle $\Sigma X \cong S^1 \wedge X$, our generalized (co-)homology theory is shifted by a dimension:

$$\tilde{K}^{-n}(\Sigma X) \cong \tilde{K}^{n-1}(X)\,. \tag{2.136}$$

We can then write the following transformation from type IIB to type IIA:

$$\underbrace{K^{-n}(pt) \cong \tilde{K}(S^n) \cong \tilde{K}(\Sigma S^{n-1})}_{\text{type IIB}} \xleftrightarrow{\text{T-duality}} \underbrace{\tilde{K}(S^{n-1}) \cong K^{-n+1}(pt)}_{\text{type IIA}}\,. \tag{2.137}$$

2.10.2 S-duality

The second fundamental duality of string theory we want to introduce here is the so-called S-duality. Simply put, it relates a theory at weak coupling and a dual theory at strong coupling:

$$g \xleftrightarrow{\text{S-duality}} \frac{1}{g}\,. \tag{2.138}$$

Such a duality was first conjectured for non-Abelian QFTs with magnetic monopole solutions by Montonen and Olive [78]. At its core the idea of Montonen-Olive duality is simple: electric and magnetic states appear on the same footing in Dirac's quantization condition and exchange roles under the duality, while simultaneously inverting the coupling strength. It became clear later that this proposal can be put on much firmer ground in the context of supersymmetry [79] and especially $\mathcal{N}=4$ [80]. The first concrete example of a precise realization was then worked out in [81] for $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions. Sen explained how the spectrum and the generalized coupling constant $\lambda = \theta + ig^{-2}$, a convenient combination of the topological theta angle θ and the gauge coupling g, is invariant under $SL(2,\mathbb{Z})$-transformations:

$$\begin{pmatrix} k \\ k \end{pmatrix} \to \begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \quad \text{and} \quad \lambda \to \frac{p\lambda - q}{-r\lambda + s}\,, \tag{2.139}$$

where $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$ given that $\det(M) = 1$. This transformation was shown to relate electric states to bound states of magnetic and/or dyonic states, all satisfying the BPS condition. The general mass formula takes the following S-duality invariant mass-formula:

$$m^2 \sim \left(k^2 g^2 + \frac{(4\pi\hbar)^2 l^2}{g^2} \right)\,. \tag{2.140}$$

Moreover, full $SL(2,\mathbb{Z})$-invariance is generated by the transformations $\theta \to \theta + 1$ and $\lambda \to -\frac{1}{\lambda}$, where the latter precisely realizes the strong-weak coupling duality we were looking for. That this $SL(2,\mathbb{Z})$-invariance and thereby S-duality should be embedded into string theory was first pointed out in [82][16] due to $\mathcal{N} = 4$ super-Yang-Mills theory arising as a particular limit of heterotic strings compactified to four dimensions on a six-torus, where the $SL(2,\mathbb{Z})$-invariance arises from the internal sector.

By the mid 90s S-duality became part of the understanding of ten-dimensional, uncompactified string theory [83–85]. Let us start of with type IIB string theory. Its low-energy effective supergravity description 2.6 has a $SL(2,\mathbb{R})$ symmetry, which uplifts to its quantized version for full type IIB string theory.[17]. Similar to Montonen-Olive duality between electric and magnetic objects supersymmetry gives rise to mass (tension) formulae for the NSp- and Dp-branes sorting them into $SL(2,\mathbb{Z})$-invariant orbits [87]. For example the fundamental type IIB string F1 coupling to the NSNS two-form B_2 transforms into the D1-brane coupling to the RR two-form C_2. They can form bound states, so-called (p,q)-strings with a generalized tension (in string frame) of the form:

$$T_{(p,q)} = T_{F1} \sqrt{(p + C_0\, q)^2 + \left(\frac{q}{g_s}\right)^2}\,. \tag{2.141}$$

The contribution of C_0 originates from the non-perturbative contribution of D-instantons to the path integral as $\exp(2\pi\, i\, \tau)$ in the Einstein frame [88, 89]. For $C_0 = 0$ the $(0,1)$-string is just the D1-brane. Similar orbits arise for the D5- and NS5-brane transforming into one another. The D3-brane however is a notable exception as it is self-dual under the $SL(2,\mathbb{Z})$-duality. As the massless worldvolume theory is given by the $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group $U(N)$ this closes the circle as the self-duality of the D3-brane is nothing else than the string theoretic completion of the Montonen-Olive duality in the low-energy limit on its worldvolume.

For type IIA the story takes a different turn. Here, we follow the discussion in [41]. To get an intuition of what we are dealing with at strong string coupling let

[16] The terms S- and T-duality actually originate from this paper.

[17] In fact the full type IIB symmetry group of type IIB is an extension of $SL(2,\mathbb{Z})$, namely the Pin^+ double-cover of $GL(2,\mathbb{Z})$ [86]. In this chapter we will refrain from discussing the full symmetry group and stick to just $SL(2,\mathbb{Z})$.

us look at the object with the lowest mass scale at strong coupling. This is not the string, but the D0-brane because of the characteristic mass scale of Dp-branes:

$$m \sim g_s^{-\frac{1}{p+1}} \alpha'^{-\frac{1}{2}} \,. \tag{2.142}$$

Thus, we expect a collection of n D0 branes to contribute as

$$n\,\tau_{D0} = \frac{n}{g_s\,\alpha'^{\frac{1}{2}}} \,. \tag{2.143}$$

We have seen such a scaling before. It looks just like a Kaluza-Klein tower of an eleven dimensional theory compactified on a radius $R_{10} = g_s\,\alpha'^{\frac{1}{2}}$! In eleven dimensions there is a unique supergravity theory that can describe type IIA at strong coupling but low energies. Just like its 10d analogs 11d supergravity features extended objects. Since there is only a three-form field $C_{\mu\nu\rho}$ there are two associated dynamical objects, one electric brane with respect to the three-form, the M2-brane, and one magnetic brane, the M5-brane. Additionally, we also have to consider the magnetic dual to the Kaluza-Klein-particles, a Kaluza-Klein magnetic monopole of codimension 3 in the non-compact directions.

Then all the non-perturbative objects we have seen so far in type IIA the Dp-branes with $p \in \{0, 2, 4, 6\}$ and the NS5-brane are described by wrapped or unwrapped configurations of the aforementioned extended objects in 11d. We summarize the configurations below:

type IIA	11d realization
D0-brane	KK-modes on S^1
F1-string	M2-brane wrapped on S^1
D2-brane	M2-brane transversal to the S^1
D4-brane	M5-brane wrapped on S^1
NS5-brane	M5-brane transversal to the S^1
D6-brane	magnetic KK monopole dual to KK-modes on S^1

Table 2.6: type IIA NSp- and Dp-branes and their M-theory dual

We specifically left out the D8-brane as its 11d counterpart will show up naturally, when looking for a strong coupling description of the HE string theory.

Before we do that let us explore the type I and HO string theory strong coupling descriptions. It turns out that they are the strong coupling description of one another. At first glance this looks very natural, for example they both have a gauge group with Lie algebra $\mathfrak{so}(32)$. But is this indeed the same Lie group?

As we have seen from the construction of the $Ss(32)$ heterotic string its gauge group arises just from mathematical consistency applied to the internal degrees of freedom. One particular aspect we want to highlight in regards to the SemiSpin-group are the representation theoretic properties of the $Spin(4n)$ quotients. Since the center of $Spin(4n)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ we get three variants $SO(4n)$ by quotienting the diagonal element and two equivalent Lie groups the $Ss(32)$-groups, which inherit either the positive or negative chirality spinor representations, while these do not survive the quotient to $SO(4n)$ [90, 91].

This means that there are non-BPS spinor states in the spectrum of the heterotic string. However, until the advent of explicit constructions of non-BPS Dp-branes the respective S-dual state in type I was missing. The non-BPS $\widehat{\text{D0}}$ and its eventual interpretation within real K-theory closed precisely this gap [65,69]. The other major gap that had to be figured out was which string object on the type I side would carry the intricate worldsheet degrees of freedom of the fundamental heterotic string. This was answered in [54]: The type I D1-brane has just the right worldsheet structure. This should be reminiscent of type IIB, where in the absence of the C_0-field, which is conveniently projected out in type I, the fundamental string is S-dual to the type IIB D1-brane and vice versa.

To analyze the degrees of freedom on the D1-brane, we need to look at the spectrum of both the open fundamental strings with both ends on the D1-brane (DD sector) and those with one end on the D1-brane and the other on the background D9-brane (DN sector). Summarizing the discussion of [54], the massless bosonic modes in the DD sector survive the orientifold projection and we get 10 massless bosonic scalar degrees of freedom both left- and right-moving. The fermionic massless degrees of freedom in the DD sector on the other hand transform in the $\mathbf{16}$ of $\mathfrak{so}(1, 9)$. More specifically, under the decomposition $\mathbf{16} = \mathbf{8}'_+ \oplus \mathbf{8}''_-$ they transform into the negative part, where the sign is with respect to $\mathfrak{so}(1, 1)$ indicating $\Gamma^0\,\Gamma^1\,\chi = -\chi$, i.e. the massless fermions are right-moving. This matches precisely with the heterotic string except for the internal fermionic degrees of freedom giving us the gauge group. These actually come in from the DN sector. Here, only fermions are massless, which are in fact left-moving $\Gamma^0 \Gamma^1 \lambda = \lambda$. Since they carry due to the Neumann boundary Chan-Paton factors from the D9-branes, we get 32 massless left-moving fermions transforming in the $\mathbf{32}$ of $\mathfrak{so}(32)$ matching precisely the heterotic worldsheet description of the fundamental string in the fermionic presentation. Similarly, the D5-brane in type I becomes the NS5-brane of HO theory under S-duality.

Moreover, Hull [92, 93] provides a (non-perturbative) orientifold construction of HO theory from type IIB string theory. The orientifold is a combination of the S-generator of the type IIB $SL(2, \mathbb{Z})$-invariance and the orientifold projection, which yields type I string theory:

$$Ss(32) - \text{heterotic string theory} \cong \text{type IIB}/\tilde{\Omega}\,, \tag{2.144}$$

where $\tilde{\Omega} = S\,\Omega S^{-1}$. The key idea here is the following: the S-duality of type I and HO theory can be understood as a remainder of the full $SL(2, \mathbb{Z})$-invariance of type IIB under orientifold. While HO at face value seems like a completely different from type I, Hull poses that we can understand a lot more about HO theory by treating it as type IIB/$\tilde{\Omega}$. For one just like for type I we would need to include background p-branes for tadpole cancellation of this heterotic O9-plane, in this case the S-dual counterpart of D9-branes, 32 NS9-branes. Just like for type I these provide a background gauge field. However, from the heterotic worldsheet construction we know that these gauge degrees of freedom are part of the worldsheet – how is this arising from the orientifold construction?

The answer is we have to consider the S-dual of the type I setup that gave us the right worldsheet degrees of freedom for the D1-brane to properly match the fundamental heterotic string. On the type I side it is the DD and DN sectors of the fundamental open string that is responsible, which means we have to consider an S-dual open D1-string on the heterotic side either with both endpoints on the fundamental string or with one endpoint on the fundamental string and one on the NS9-brane stack. Since the D1-string has a tension of g_s^{-1} in the perturbative heterotic limit $g_s \to 0$ the string becomes infinitely heavy and retracts into the fundamental heterotic string, which matches our perturbative description of heterotic string theory never featuring a D1-string. Furthermore, in this limit only the massless sector of the heterotic D-string spectrum can survive and it has to match the S-dual massless sector of the excitations of type I strings ending on a type I D1-string. Curiously, this entails that at finite heterotic string coupling the worldsheet degrees of freedom are free to move off from the worldsheet along an open heterotic D1-string attached to the fundamental worldsheet. We will explore the consequences of this further in the main part of the thesis and see how the underlying K-theory structure remedies this.

At this point we have now found strong coupling descriptions for four of the five superstring theories: type IIB is self-dual, type I and $Ss(32)$-heterotic string theory are dual to another and type IIA has an eleven-dimensional strong coupling description. Based on the fact that T-duality carries us from type IIB to type IIA and $Ss(32)$ to $E_8 \times E_8$ heterotic string theory, one might expect a symmetric behavior and conjecture an eleven-dimensional strong coupling description for $E_8 \times E_8$ heterotic string theory, as well. This expectation turns out to be just right! However, due to the background gauge group this is even more complicated than for type IIA. In fact, the compact eleventh dimension is given by an interval, the $\mathbb{Z}_2$-orbifold of a circle. This has two important consequences. For one, it breaks half of the supersymmetries, which means we should get an $\mathcal{N} = 1$ theory in 10d as compared to an $\mathcal{N} = 2$ theory (type IIA), when compactified on the circle. On the other hand the orbifold has two fixed points, i.e. two 10d boundaries at the endpoints of the interval, which have to carry an E_8 gauge group each [94, 95]. The argument is mainly based on an extension of anomaly cancellation in 10d $\mathcal{N} = 1$ string theories to eleven dimensions. Since we created two boundaries through the $S^1/\mathbb{Z}_2$ orbifold, only with $E_8 \times E_8$ we have the necessary factorization to make the anomaly cancellation on two distinct boundaries work.

Finally, we want to put both T- and S-dualities together. On the type II side, we have argued that type IIA arises as the compactification of an eleven dimensional theory, but we also know that compactifying type IIA on another S^1 and T-dualizing along this direction gives us type IIB string theory. Combining both of these dualities we can realize type IIB string theory compactifying the 11d theory on a 2-torus and then decompactifying the T-duality circle, taking its radius to infinity. Curiously, it turns out that the torus-structure survives this operation as the radius of the T-duality circle is a combination of the torus radii. Moreover, we have already seen the imprint of this torus structure in type IIB, the $SL(2, \mathbb{Z})$-duality group, which is nothing else than the modular group of the 2-torus.

Analogously, we can construct type I/HO theory from eleven dimensions, if we compactify on $S^1/\mathbb{Z}_2 \times S^1$. Just we discussed a Wilson line has to be turned on for T-duality between the two heterotic string theories, such that we have an intermediate gauge group with Lie algebra $\mathfrak{so}(16) \times \mathfrak{so}(16)$.

Thus, we have connected all five superstring theories to an eleven dimensional theory featuring 2-branes, 5-branes and some form of 9-brane (the 10d boundary for the strong coupling description of the $\mathcal{N} = 1$ superstring theories) with eleven dimensional supergravity as its low-energy limit. This underlying theory unifying all superstring theories was dubbed M-theory by Witten [87]. Visualizing this leads to the famous duality star in figure 2.2.

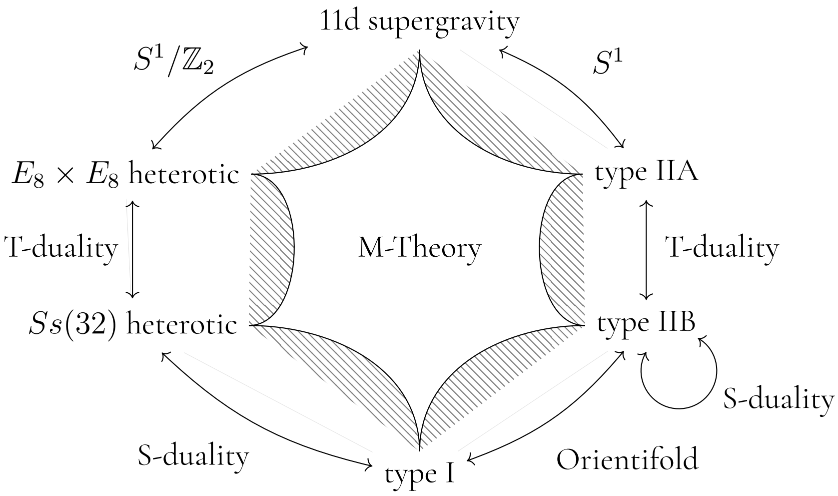

Figure 2.2: The M-theory duality star

3
The Swampland Program

As we already addressed in the introduction, the fact that the solution space of string theory appears to be incredibly vast, leads to the natural question, if every consistent-looking low-energy effective theory can arise top-down from string theory. The answer turns out to be negative. Delineating these two classes of effective field theories is the core objective of the Swampland Program [24]. Our knowledge about the boundary between the two classes is organized in conjectures, which are usually based on either generic behavior of string theory or gedankenexperiments involving black holes as laboratories of quantum gravity at lower energies. While it provides a useful basis to explore the interplay between string theory restrictions and beyond the (cosmology and particle physics) Standard Model, it can be also used to refine our fundamental understanding of string theory.

In this chapter we will focus on this aspect of the Swampland Program as it will be a central theme of the remaining thesis. We therefore refer to these introductions [96–99] for a much more generic primer on the Swampland Program. The first conjecture we want to look at is in fact probably the oldest, the No Global Symmetries Conjecture going back at least to the 1950s [100, 101].

3.1 The No Global Symmetries Conjecture

Let's start by properly defining the conjecture. To this end we will follow the discussion in [97].

Definition (Global Symmetry)**.** *A global symmetry with symmetry group G is conventionally defined based on unitary local operators $U(g)$ depending on a group element g of the symmetry group G. Then the unitary local operator has to satisfy these criteria:*

- *Doesn't violate the group law: $U(g)\,U(g') = U(g \circ g')$.*
- *Transforms charged operators $\mathcal{O}(x)$ with x the position in space-time living in an Hilbert space in a non-trivial way: $U^\dagger(g)\,\mathcal{O}(x)\,U(g) \neq \mathcal{O}(x)$.*
- *Satisfies the local energy conservation condition: $U^\dagger\,T_{\mu\nu}(x)\,U = T_{\mu\nu}(x)$.*
- *Maps a local operator to another local operator.*

We are going to use this as a basis from which we are going to generalize the notion of a global symmetry. An immediate generalization still within the definition above is the notion of a p-form symmetry, which due to the ubiquity of p-forms in string theory is very natural for us to consider.

The unitary local operator of a p-form A_p is formulated through its conserved current $J = F_{p+1} = dA_p$ as

$$U(M^{d-p-1}) = \exp\left(i\,a \oint_{M^{d-p-1}} *J_{p+1}\right) \tag{3.1}$$

and is supported on a $d-p-1$ submanifold M^{d-p-1} within a d-dimensional space-time. The charge operator this symmetry operator is supposed to act on is a Wilson line supported on a complementary p-dimensional submanifold W^p:

$$\mathcal{O}(W^p) = \exp\left(i\,n \oint_{W^p} A_p\right). \tag{3.2}$$

To get rid of such a global symmetry, one can either break the conservation law by including a defect, such that

$$d * J_{p+1} = 0 \rightarrow d * J_{p+1} \neq 0 \tag{3.3}$$

or alternatively couple the current to another $p+1$-form field to gauge the symmetry:

$$\int C_p \wedge *J_{p+1}\,. \tag{3.4}$$

Then we state the No Global Symmetries Conjecture as

Definition (No Global Symmetries Conjecture)**.** *There are no global symmetries in quantum gravity. They are either broken or gauged.*

Let's look at an example of a potentially non-trivial global symmetry in type IIB string theory [99]: We have already seen that type IIB has a complex scalar (the axio-dilaton) $\tau = C_0 + ie^{-\phi}$. The associated combined kinetic term becomes

$$\frac{\partial_\mu \tau \, \partial_\nu \bar{\tau}}{e^{-2\,\phi}}\,. \tag{3.5}$$

Naively, we could think that there is a continuous global shift symmetry $C_0 \to C_0+\epsilon$. However, we have already seen that this is not compatible with the BPS-mass of for example the (p,q)-strings (2.141). The only allowed shift is the one compatible with the $SL(2,\mathbb{Z})$ symmetry, i.e. the T-transformation $C_0 \to C_0 + 1$. Now, is this an admissible global symmetry?

The answer is still no, as this symmetry is actually gauged. The fact that we have D7-branes, which are magnetically charged under C_0 means that $C_0 \to C_0 + 1$ is a gauge symmetry. More generally, Polchinski shows in [41] that in perturbative string theory every global symmetry is associated to a global charge on the worldsheet. However, on the worldsheet vertex operators associated to the charge create gauge degrees of freedom in space-time, such that the putative global symmetry becomes a gauge symmetry. For discrete symmetries like the example we showed above it becomes more subtle, but there is no known counterexample, where the global symmetry persists [97]. They are either broken or gauged, which are the only two possibilities of getting rid of a global symmetry.

Apart from string theory, there exists also a bottom-up reasoning without specifying the UV-completion of the effective theory. It goes as follows: Consider throwing a particle charged under a global symmetry into a black hole. Once it is beyond the horizon we lose the information as the Hawking black body radiation doesn't carry any information with regards to the content of the black hole. Now, either the black hole completely vanishes through radiation and the global charge is violated or alternatively the radiation process stops at some point and the left over remnant still carries the global charge we have thrown in. However, for continuous symmetries this doesn't work either as this would give rise to an arbitrary number of distinct remnants, which poses a serious threat to the effective gravitational theory [102].

The final piece of evidence in favor of this conjecture we want to mention is the perspective of holography on this issue. Relying on the AdS/CFT correspondence the authors of [103, 104] could prove that a global symmetry in the bulk is inconsistent if the representative symmetry operator on the CFT boundary can be split into a product of operators, each sitting in different subregions of the boundary. Based on the Ryu-Takayanagi formula (see [105] for a review) each of the boundary subregions in turn has access to a certain entanglement wedge in the bulk. However, all these entanglement wedges together do not cover the whole bulk anymore, such that we can have a global symmetry charge operator sitting in the undetected region of the bulk, which would clearly be inconsistent.

3.2 The Completeness Hypothesis

Connected to this is another folk theorem, the Completeness Hypothesis, which is tough to accredit to a first source. It can be found however for example in [106] and [107]. The statement is as follows

Definition (Completeness Conjecture)**.** *The spectrum of physical states of a gauge theory coupled to gravity is complete with respect to the charge lattice compatible with Dirac quantization.*

In the references above it was mostly argued by virtue of no known counterexamples especially within string theory. However, this is not the end of the story as completeness was recently connected to the No Global Symmetries Conjecture [108]. In particular, the authors could extend known results about the connection between the completeness of the spectrum of gauge theories with a compact and connected symmetry group G and 1-form global symmetries with the center $Z(G)$ as its symmetry group (see for example [104, 106]) to a notion for all compact gauge groups. For finite gauge groups there might not be a center symmetry to distinguish the complete and incomplete cases. However, the topological Gukov-Witten operators of codimension 2 are in 1-1 correspondence with an incomplete spectrum [109]. Since these operators do not satisfy an invertible fusion algebra anymore, but only a non-invertible one,

$$T_a^{\mathrm{GW}} \times T_b^{\mathrm{GW}} = \sum_c N_{ab}^c T_c^{\mathrm{GW}}, \tag{3.6}$$

this notion of incompleteness actually extend nicely to compact, but disconnected gauge groups, where the 1-form global symmetry detecting the incompleteness of the spectrum is non-invertible, as well [108]. Consequently, the Completeness Hypothesis follows from requiring absence of non-invertible symmetries. This is not covered by our definition of the No Global Symmetry Conjecture as it violates our first assumption. This issue was clarified soon afterwards in [110] explaining that gravitational solitons break the non-invertible symmetry to its maximal group-like sub-symmetry. To achieve such a breaking the new gravitational solitons have to account for exactly those electric charges on which the center of G acts trivially, such that the non-invertible operators become endable on precisely those charges. This is the precise match that was established in [110].

Non-invertible symmetries have presented us with a first example of generalizing the notion of a global symmetry by revoking one of the criteria from our definition. Now, we are going to look at another generalization for which we remove our last requirement to transform local operators to local operators.

3.3 The Cobordism Conjecture

In this section we would like to attend to a quintessential feature of quantum gravity – non-trivial topology. String theory as our prime example of a consistent theory of quantum gravity is no different. Among the myriad of solutions to the string theoretic equations of motion most of them involve a compact manifold with complicated topology. Core examples are Calabi-Yau manifolds like $2n$-tori, the four dimensional complex surface K3, or Calabi-Yau threefolds. Now, imagine the following situation: We are considering a quantum gravitational system on a flat space $\mathbb{R}^d$. Next, we are cutting out a small ball within $\mathbb{R}^d$ and glue in a topologically non-trivial compact manifold M^n on which our quantum gravity theory is still consistent. Far away from the glued part quantum gravity will perceive the space-time as flat, we therefore have introduced a $(d-k-1)$-dimensional defect into the theory. Due to the non-trivial topology of the defect we can associate topological numbers invariant under homeomorphic deformations to such manifolds. Hence, we have associated a global charge to the defect. Repeating the black hole argument, we can throw this global defect into a black hole, respectively a black brane of appropriate dimension, and we would run into the same problems as with a global charge carried by a pointlike defect.

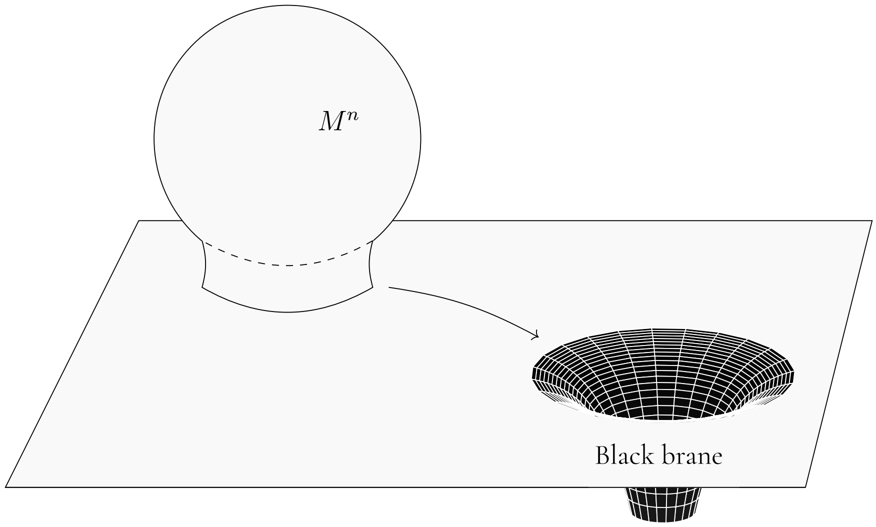

Figure 3.1: A topological defect thrown into a black brane.

In order to avoid such inconsistencies, our quantum gravity theory has to be able to perceive the defect as equivalent to the initial configuration without M^n glued in.

Such a classification is provided by cobordism theory, a mathematical framework sorting closed manifolds into equivalence classes given some input on the topological characteristics of these manifolds. We explain this in more detail in the following chapters on the mathematical background, where we go into more detail on cobordism theory.

What we need for now is that these equivalence classes form Abelian groups and are specified by the dimension of the manifolds within the equivalence classes and some structure we need to choose such that our theory is well-defined on these manifolds. This is formally denoted as Ω_n^{QG}, where n is the number of dimensions and QG is the choice of structure compatible with quantum gravity. Associated to these groups we have cobordism invariants detecting non-trivial classes taking values in these groups. Thereby we have a refinement of which topological invariants are carried by these problematic defects. Namely, if their cobordism invariant is trivial they are nullbordant, i.e. they are cobordant to nothing[1].

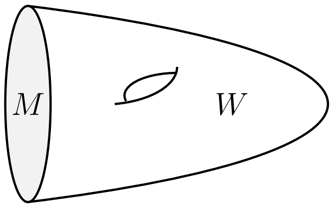

Figure 3.2: Vanishing cobordism group – nullbordant manifold

Then the non-trivial cobordism groups tell us about a non-trivial global symmetry. Since they are inconsistent with quantum gravity, the Cobordism Conjecture can be reduced to the following simple statement:

Definition (Cobordism Conjecture)**.** *The cobordism groups of a quantum gravity theory consistently defined on a background space containing a closed n-dimensional manifold M with tangential structure QG, such that $[M] \in \Omega_n^{QG}$ have to be trivial:*

$$\Omega_n^{QG} = 0\,. \tag{3.7}$$

Just like for conventional global symmetries there are two options quantum gravity can take to achieve this: Either gauging or breaking the global symmetry. This can be formally written as follows: Let's consider a proto-quantum gravity structure $\widetilde{QG}$, which we take to be some approximation to the full quantum gravity structure. Then we describe the two options as follows [113]:

- *Gauging* the global symmetry amounts to including a gauge field, such that integration over the compact manifold within the cobordism equivalence class leads to the condition that the only consistent configurations are:

$$[0] \in \Omega^{\widetilde{QG}}\,. \tag{3.8}$$

  Differently put, there exists a forgetful map from the quantum gravity structure including the gauge field back to the proto-quantum gravity structure itself

$$\Omega^{\widetilde{QG}+\text{ gauge field}} \to \Omega^{\widetilde{QG}}\,, \tag{3.9}$$

[1] Another motivation leading us to the Cobordism Conjecture is the Holographic Principle [111, 112]. If every consistent quantum gravity theory is holographic, i.e. all of its degrees of freedom are captured by a theory in one dimension higher, then all consistent backgrounds of the theory have to be nullbordant.

Then the classes in the co-kernel of this map are gauged or co-killed.

- *Breaking* the global symmetry means we include a defect into the theory, such that there exists a nullbordant configuration pictured below. In this case we

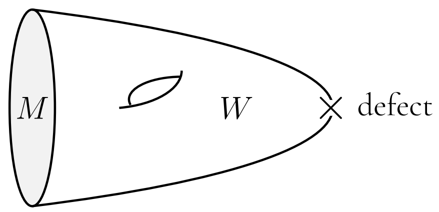

Figure 3.3: Breaking the global symmetry

kill the classes in the kernel of the inclusion map

$$\Omega^{\widetilde{QG}} \to \Omega^{\widetilde{QG} + \text{ defect}} . \tag{3.10}$$

This concludes our introduction to the Swampland Conjectures most pertinent to this thesis. We continue with an exposition to the mathematical tools necessary for the main part of the thesis.

4
Mathematical Tools

In this chapter we want to look at the mathematical tools we have already encountered, as well as those we will work with in the remainder of the thesis. We start by giving a bit more background on two generalized (co-)homology theories, K- and cobordism theory. Afterwards, we will introduce the homological variant of K-theory, which we will call K-homology. Furthermore, we are going to introduce the tools required to compute these groups, spectral sequences.

4.1 K-theory

K-theory is a generalized cohomology theory classifying vector bundles possibly over some background space X [114, 115]. For introductions to K-theory from a physicist's point of view we refer the reader to [116–118]. On the mathematical side see for example [119, 120] and chapter 24 of [121]. Depending on the type of vector bundle there exist several closely related variants. The two we are going to work with are complex and real K-theory, denoted as $K(X)$ and $KO(X)$, arising from complex and real vector bundles respectively.

Now consider two complex vector bundles E and F over a background manifold X. We organize them into a pair (E, F) and require an equivalence relation under adding a third complex vector bundle H to both

$$(E,F) \sim (E \oplus H, F \oplus H)\,. \tag{4.1}$$

Since this directly implies

$$(0,0) \sim (E,E)\,, \tag{4.2}$$

it is clear that this definition resembles a subtraction. Furthermore, we can define addition and subtraction of pairs

$$\begin{aligned}(E,F)\,+\,(E',F') &= (E \oplus E', F \oplus F')\,,\\ (E,F)\,-\,(E',F') &= (E \oplus F', F \oplus E')\,,\end{aligned} \tag{4.3}$$

which means that the inverse of (E,F) is (F,E) with the identity $(0,0)$. This defines the Abelian group $K(X)$. As mentioned above K-theory is the prime example of a generalized cohomology theory, satisfying all Eilenberg-Steenrod axioms for (co-)homology theories [122] except for the dimension axiom (see appendix A). This entails that if we choose X to be just the point, $K(pt)$ does not vanish.

In fact, there is a very important relationship between $K(X)$ and $K(pt)$. This is known as the splitting principle.

Lemma 4.1.1 (Splitting Principle)**.** *Let's consider the map*

$$i^* : K(X) \to K(pt) \tag{4.4}$$

induced by the inclusion map

$$i : pt \hookrightarrow X\,. \tag{4.5}$$

We then obtain the following short exact sequence:

$$0 \to \tilde{K}(X) \to K(X) \to K(pt) \to 0\,, \tag{4.6}$$

where the reduced K-theory group of X, $\tilde{K}(X)$, is defined as the kernel of i^*. In fact, this short exact sequence splits, such that we can always split off $K(pt)$ from $K(X)$:

$$K(X) = K(pt) \oplus \tilde{K}(X)\,. \tag{4.7}$$

Further we can define higher K-theory groups $K^{-n}(X)$ based on the behavior of a generalized cohomology theory under suspension. First, we use that

$$K(pt) = \tilde{K}(S^0) \tag{4.8}$$

Then, we define the higher reduced K-theory groups as

$$\tilde{K}^{-n}(X) := \tilde{K}(\Sigma^n X) \tag{4.9}$$

with the reduced suspension $\Sigma^n X = S^n \wedge X$. Using the splitting principle the construction of the higher (unreduced) K-theory groups follows, too:

$$K^{-n}(X) := \tilde{K}(\Sigma^n S^0) \oplus \tilde{K}(\Sigma^n X)\,. \tag{4.10}$$

As we already mentioned in the previous chapter complex K-theory is periodic. This is known as so-called Bott periodicity [67, 68]:

$$K^{-n}(X) \cong K^{-n\pm 2}(X)\,. \tag{4.11}$$

Constructing a K-theory out of real instead of complex vector bundles yields us so-called KO-theory. Importantly its higher groups are mod 8 periodic [67, 68]:

$$KO^{-n}(X) \cong KO^{-n\pm 8}(X)\,. \tag{4.12}$$

4.2 Cobordism

For cobordism[1] we will mostly follow the notes [125, 126], the textbook [127] and the background material in [128], which also has an eye on the application to the Cobordism Conjecture. Very similar to K-theory we begin with an equivalence relation:

Definition 4.2.1 (Cobordism equivalence classes)**.** *Two n-dimensional closed manifolds M and N are said to be cobordant, if there exists a third manifold W of one dimension higher such that the boundary of W is the disjoint union of M and N:*

$$\partial W = M \cup \bar{N}\,, \tag{4.13}$$

where $\bar{N}$ denotes the inverse of N.

[1] To clarify the notation here: We are going to refer with the term cobordism to the generalized homology theory as originally coined by Thom [123], a neologism based on the French word for boundary "bord" and the prefix "co-" to signify the joining of manifolds under the disjoint union. Atiyah later discovered [124] that there exists both a generalized homology and cohomology cobordism theory and therefore renamed the homological version to bordism. Since the cohomological version has remained largely irrelevant for physics, we are going to stick to the initial naming convention, which was also largely adopted in the physics literature.

If two manifolds are cobordant crucially depends on if we want to preserve some mathematical structure on these manifolds.

Furthermore, we can add equivalence classes of manifolds by considering their disjoint union:

$$[M_1] + [M_2] = [M_1 \cup M_2] \,. \tag{4.14}$$

We already mentioned that there is a notion of an inverse of an equivalence class, which leads to

$$[M_1] + [\bar{M}_1] = [0] \,. \tag{4.15}$$

Thus, these equivalence classes organize into Abelian groups and more over just like K-theory cobordism satisfies all the Eilenberg-Steenrod axioms A except for the dimensionality axiom, thereby providing us with another example of a generalized (co-)homology theory. We denote them these groups as $\Omega_n^O(pt)$, where n is the dimension.

So far we have been rather lenient with the kinds of manifolds we want to organize in these groups. The key concept here are tangential structures. This is essentially the modern version of so-called (B, f)-structures one encounters in older accounts like [125]. Here, we follow [129]. In general a tangential structure ξ is based on a Serre fibration:

$$\xi :\, B \to BGL(d, \mathbb{R}) \simeq BO(d) \,, \tag{4.16}$$

where $BGL(d, \mathbb{R})$ and $BO(d)$ are the classifying spaces of $GL(d, \mathbb{R})$ and $O(d)$ respectively. Classifying spaces for some topological group G denoted as BG are used to classify G-principal bundles and are constructed as quotient spaces EG/G. As they will continue to play an important role throughout this thesis we will collect some basic information about them in the appendix B. Take the vector bundle classification on a manifold X

$$f :\, X \to BGL(d, \mathbb{R}) \tag{4.17}$$

corresponding to viewing the tangent bundle as a $GL(d, \mathbb{R})$ (or $O(d)$) vector-bundle thereby explaining the name. Now, a ξ-structure on the tangent bundle is a map $i :\, X \to B$ with $\xi \circ i = f$. The neat thing is we can now use representations of a group G

$$\rho :\, G \to GL(d, \mathbb{R}) \tag{4.18}$$

to define general G-structures by utilizing the induced Serre fibration:

$$B\rho:\ BG \to BGL(d,\mathbb{R})\,. \tag{4.19}$$

In the following we will be working with the stable version defined by replacing $O(d)$ with the stable orthogonal group O:

$$O := \mathrm{colim}(O(1) \hookrightarrow O(2) \hookrightarrow \dots) \tag{4.20}$$

A particularly important stable G-structure is $Spin$-structure, i.e. the stable limit of the Lie group $Spin(N)$, whose classification map $BSpin \to BO$ is achieved by subsequent maps from $Spin \to SO$ and embedding $SO \hookrightarrow O$.

This actually points towards a convenient principle to organize these structures, so called Postnikov systems. Here, we will be working with the closely related concept of a Whitehead tower. As is nicely elucidated in [128] the key idea is to decompose some X in our case BO into a series of spaces $X\langle n\rangle$, whose higher homotopy groups agree with X, but whose lower homotopy groups vanish:

$$\pi_i(X\langle n\rangle) = \begin{cases} 0 & \forall\, i \leq n \\ \pi_n(X) & \forall\, i > n\,. \end{cases} \tag{4.21}$$

In this context it is very convenient to introduce Eilenberg-MacLane spaces $K(A,n)$, they are defined by having just a single non-trivial homotopy group

$$\pi_i(K(A,n)) = \begin{cases} 0 & \forall\, i \neq n \\ A & \forall\, i = n\,. \end{cases} \tag{4.22}$$

What we also need is that the homotopy classes $[M, K(A,n)]$ are isomorphic to $H^n(M,A)$, if M is a pointed CW complex. With the homotopy fibration

$$X\langle n\rangle \to X\langle n-1\rangle \to K(\pi_n(X),n) \tag{4.23}$$

we can start organizing our tangential structures into these nice triangle maps

$$\begin{array}{ccccc} & & X\langle n\rangle & & \\ & \nearrow & \downarrow & & \\ M & \xrightarrow[f]{} & X\langle n-1\rangle & \longrightarrow & K(\pi_n(X),n)\,. \end{array} \tag{4.24}$$

Now, let's specify to our tangential structure, for which we want to classify maps from manifolds M, or cobordism equivalence classes, to $X = BO = BO\langle 0\rangle$. The first non-trivial step is captured by $\pi_1(BO) = \mathbb{Z}_2$. We get a diagram

$$\begin{array}{ccccc} & \nearrow & BO\langle 1\rangle \simeq BSO & & \\ & & \downarrow & & \\ M & \xrightarrow[c]{} & BO\langle 0\rangle \simeq BO & \xrightarrow[w_1]{} & K(\mathbb{Z}_2, 1)\,. \end{array} \tag{4.25}$$

A natural question now becomes, when does our classifying map lift to BSO? The answer of course is that $w_1 \circ c$ should be homotopically trivial. However, since $[M, K(\mathbb{Z}_2, 1)] \simeq H^1(M, \mathbb{Z}_2)$, this is detected by its cohomology invariant, the first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}_2)$. We then call a manifold orientable, if $w_1(M) = 0$. The next structure, associated to the obstruction in $\pi_2(BO) \simeq \mathbb{Z}_2$ is the so called spin structure, which will become very important throughout this thesis. Analogously, a manifold is called spinnable if its second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. This condition is tantamount for fermions to be anomaly free on such a manifold, see e.g. [130]. These subsequent refinements form a tower-like structure, the aforementioned Whitehead tower. The one based on BO looks as follows:

$$\begin{array}{rc} & \vdots \\ & \downarrow \\ & BO\langle 9\rangle = BFivebrane \\ \tfrac{1}{6}p_2 = 0 \; \big(& \downarrow \\ & BO\langle 8\rangle = BString \\ \tfrac{1}{2}p_1 = 0 \; \big(& \downarrow \\ & BO\langle 4\rangle = BSpin \\ w_2 = 0 \; \big(& \downarrow \\ & BO\langle 2\rangle = BSO \\ w_1 = 0 \; \big(& \downarrow \\ & BO\langle 1\rangle = BO \end{array}$$

Figure 4.1: Whitehead tower of the orthogonal group O.

A slight spin-off of the $spin$-structure is $spin^c$-structure, which is a fiber product $spin \times_{\mathbb{Z}_2} U(1)$. A similar construction as for $spin$-structure factoring in the $U(1)$ leads to the relevant obstruction invariant being the integral lift of the second Stiefel-Whitney class w_2 to $W_3 = \beta(w_2)$, where β is the Bockstein homomorphism part of the long exact sequence in cohomology induced from the coefficient short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ [91]:

$$\cdots \to H^{n+1}(X,\mathbb{Z}) \xrightarrow{\times 2} H^{n+1}(X,\mathbb{Z}) \to H^n(X,\mathbb{Z}_2) \xrightarrow{\beta} H^{n+1}(X,\mathbb{Z}) \to \ldots \tag{4.26}$$

Coming back to our cobordism groups we now have two organizing inputs, the number of dimensions n and the tangential structure ξ. We write our cobordism groups as Ω_n^ξ. However, there is a third input we want to introduce, namely background spaces X, to which there is a map preserved throughout the cobordism equivalence.

Concretely, we are considering a manifold M with a map to this background space $f : M \to X$ to be cobordant to another manifold N also endowed with a map to this same background space $g : N \to X$, if there exists a manifold of one dimension higher W equipped with a map $h : W \to X$, such that the W has M and N as its boundaries and h reduces to f and g on the respective boundaries. We can visualize it in the following way:

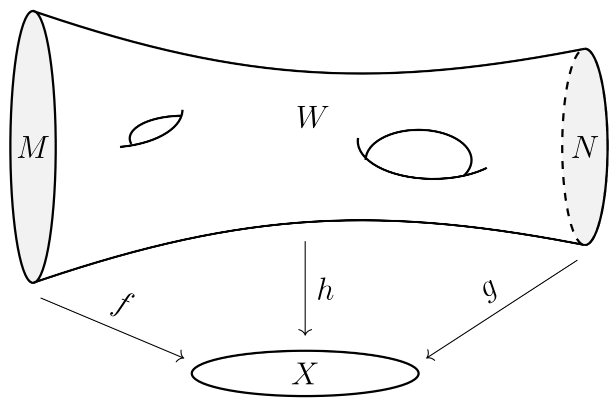

Figure 4.2: Cobordism (W, h) between (M, f) and (N, g).

The cobordism groups satisfying these conditions are then written: $\Omega_n^\xi(X)$. We can also repeat the construction of the splitting principle we introduced for K-

theory (4.7) for our cobordism groups and get:

$$\Omega_n^\xi(X) = \Omega_n^\xi(pt) \oplus \tilde{\Omega}_n^\xi(X)\,. \tag{4.27}$$

4.3 K-homology

To the reader it might seem curious that we didn't introduce K-homology directly together with its cohomological twin K-theory. The reason is that the theory turns out to be quite closely related to cobordism, too. K-homology was introduced by Baum and Douglas [131,132]. The connection of K-homology to Dp-branes was made in [133], which also provides a very useful introduction to the topic that we follow in our introduction here. The basis of K-homology $K_n(X)$ are so called K-cycles on a space X.

Definition 4.3.1. *K-cycles on a space X are triples (M, E, f) fulfilling the following properties:*

- *M is a compact, closed n-dimensional $spin^c$ manifold.*
- *E is a complex vector bundle over M.*
- *f is a continuous map $M \to X$.*

Then we can define a disjoint union for K-cycles:

$$(M, E, f) \cup (N, F, g) := (M \cup N, E \cup F, f \cup g)\,. \tag{4.28}$$

Our final ingredient is so-called vector bundle modification: First, one constructs a $spin^c$-manifold $\hat{M}$ from a real $spin^c$ vector bundle F with even-dimensional fibers:

$$\widehat{M} = \mathbb{B}^+(F) \cup_{\mathbb{S}_F} \mathbb{B}^-(F)\,, \tag{4.29}$$

where $\mathbb{B}^\pm(F)$ are unit ball bundles with $spin^c$-structures inverse to another. Then there is the projection back to M:

$$\pi:\ \widehat{M} \to M \tag{4.30}$$

In a similar fashion one designs a vector bundle $H(F)$ through the half-spinor bundles projected on the unit ball bundles $\Delta^\pm(F) = S_\pm(F)|_{\mathbb{B}^\pm(F)}$ (the sign referring to their chirality):

$$H(F) = \Delta^+(F) \cup_\sigma \Delta^-(F) \tag{4.31}$$

with σ the vector bundle map $S_+(F) \to S_-(F)$ covering the identity of F.

By this procedure, we can obtain another K-cycle $(\widehat{M}, H(F) \otimes \pi^*(E), f \circ \pi)$ from (M, E, f) with π^* the pullback of π. This operation is called vector bundle modification[2].

With all necessary ingredients lined up, we are in position to define K-homology. First, we define equivalence classes $\Gamma_n(X)$ of K-cycles.

Definition 4.3.2 (Equivalence classes of K-cycles)**.** *Two K-cycles (M, E, f) and (N, F, g) are said to be equivalent, if there exists a diffeomorphism ϕ from M to N, such that $spin^c$-structure is preserved, the pullback of F is E and the triangle diagram*

$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & N \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & X \end{array} \tag{4.32, 4.33}$$

commutes.

Then we can define K-homology groups $K_n(X)$ as follows.

Definition 4.3.3 (K-homology groups)**.** *Consider $\Gamma_n(X)$ equivalence classes and quotient by the following three operations:*

- *$spin^c$-Cobordism with the intermediate $(n+1)$-dimensional manifold carrying a complex vector bundle reducing to E and F on the respective boundaries.*
- *Direct sum of vector bundles, i.e. if $E = E_1 \oplus E_2$, then we identify (M, E, f) with $(M, E_1, f) \cup (M, E_2, f)$.*
- *The previously introduced vector bundle modification.*

This defines the generalized homology groups $K_n(X) := \Gamma_n(X)/\sim$.

[2] If we consider the simplest K-cycle $(pt, \mathbb{1}_{\mathbb{C}}, i)$ vector bundle modification gives us the K-cycle $(\mathbb{S}^{2n}, H(F), i \circ f)$. However, this modification is nothing else than a basic example of Myers' dielectric effect [134] applied to a D(-1)-brane. It gets puffed up into a spherical configuration of D$(2n-1)$-branes [133]. From a string theoretic point of view this effect makes it completely natural to identify the two configurations, which is actually part of the mathematical definition of K-homology, too, as we will see next.

What we should immediately notice is the close relation between $spin^c$-cobordism and (complex) K-homology. Analogously, $spin$-cobordism and KO-homology are very similar as the definition of KO-homology is the same as for K-homology once we exchange $spin^c$- for $spin$-structure and complex vector bundles for real vector bundles.

4.4 Connection between cobordism, K-theory and K-homology

The fundamental results by Anderson, Brown and Peterson [135] allow us to understand the connection between *spin* (*spin*[c]) and *ko*-homology[3] (*k*-homology) at a more precise level. While the results become even more potent in context with the Adams Spectral Sequence as we will see at the end of this chapter, we can already point out the decomposition of $spin$-cobordism groups into ko-homology groups:

Theorem 4.4.1 (Anderson-Brown-Peterson [135]). *$Spin$-cobordism at the prime 2 is isomorphic to the following shifted ko-homology pieces*

$$\Omega_n^{Spin}(X)_{\widehat{2}} \simeq ko_n(X)_{\widehat{2}} \oplus ko_{n-8}(X)_{\widehat{2}} \oplus ko_{n-10}\langle 2\rangle(X)_{\widehat{2}} \oplus \ldots \, . \tag{4.34}$$

The subscript $\widehat{2}$ denotes localization at the prime $p = 2$, which means for an Abelian group G:

$$G_{\widehat{2}} := \varprojlim_{s} G/2^s \, . \tag{4.35}$$

For $G = \mathbb{Z}$ this gives us the 2-adic integers for example.

Often times though this is enough to determine parts or even all $spin$-cobordism groups as higher p-torsion is usually very sparse. For example, Milnor showed that $\Omega_n^{Spin}(pt)$ lacks odd torsion [136] and therefore the decomposition (4.34) gives us the full result for $X = pt$.

A very similar result holds for $spin^c$-cobordism and complex k-homology

Theorem 4.4.2 (Anderson-Brown-Peterson [135]). *At prime 2 spin*[c] *cobordism is isomorphic to k-homology terms shifted in degree:*

$$\Omega_n^{Spin^c}(X)_{\widehat{2}} = \bigoplus_{4n(I)} k_n(X)_{\widehat{2}} \oplus T(X) \, , \tag{4.36}$$

[3] We use the lowercase notation to denote the connective version of K-homology meaning that groups with negative dimension are vanishing.

where $n(I) := \sum_k i_k$ *is the sum over an ascending collection of integers* i_k $I = (i_1, \ldots, i_k)$ *and we have abbreviated the part based on the Eilenberg-Maclane spectrum* $H\mathbb{Z}_2$ *as* $T(X)$, *which can be understood additively.*

Based on this [137] calculated $\Omega_n^{Spin^c}(pt)$ for $n \leq 59$. The first non-trivial contribution of $T(pt)$ comes in dimension 10. Alternatively, we can determine $\Omega_n^{Spin^c}(pt)$ through the isomorphism $\Omega_n^{Spin^c}(pt) \simeq \tilde{\Omega}_{n-2}^{Spin}(\mathbb{CP}^\infty) = \tilde{\Omega}_{n-2}^{Spin}(BU(1))$ [125].

Away from the localization at $p = 2$ Hopkins and Hovey [138] constructed a map between $\Omega_*^{Spin}(X)$ and $KO_*(X)$:

Theorem 4.4.3 (Hopkins-Hovey [138])**.** *There exists an isomorphism of rings*

$$\Omega_*^{Spin}(X) \otimes_{\Omega_*^{Spin}(pt)} KO_*(pt) \simeq KO_*(X)\,. \tag{4.37}$$

And analogously, they showed an isomorphism between $\Omega_*^{Spin^c}(X)$ and $K_*(X)$ as well:

Theorem 4.4.4 (Hopkins-Hovey [138])**.** *There exists an isomorphism of rings*

$$\Omega_*^{Spin^c}(X) \otimes_{\Omega_*^{Spin^c}(pt)} K_*(pt) \to K_*(X)\,. \tag{4.38}$$

Both of these are based on the natural homomorphisms, the Atiyah-Bott-Shapiro orientation [139] provided by the maps we denote as α and Todd genus α_c as a placeholder for now:

$$\begin{aligned} \alpha &: \Omega_*^{Spin}(pt) \to KO_*(pt)\,, \\ \alpha_c &: \Omega_*^{Spin^c}(pt) \to K_*(pt)\,. \end{aligned} \tag{4.39}$$

Explicitly, the two ABS orientations at fixed degree n are given by the Todd genus, i.e.t̃he index of the Spin^c Dirac operator

$$\alpha_n^c([M]) = \mathrm{Td}(M) \equiv \int_M \mathrm{td}_n(M)\,, \tag{4.40}$$

and by the index of the Dirac operator on M, respectively [140]

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8m, \\ \hat{A}(M)/2 & n = 8m+4, \\ \dim H \quad \bmod 2 & n = 8m+1, \\ \dim H^+ \quad \bmod 2 & n = 8m+2, \\ 0 & \text{otherwise}, \end{cases} \tag{4.41}$$

where $\hat{A}(M)$ is the $\hat{A}$ genus and H (H^+) the space of (positive) harmonic spinors. We discuss the construction of both the $\hat{A}$ genus and the Todd genus in appendix D. Since these invariants detect the cobordism subgroups equivalent to the K-theory groups, this offers a unique window into quite a lot of interesting cobordism groups through a K-theoretic lens.

We have now related cobordism to K-homology, but what about K-theory and K-homology?

Based on the the well-known Poincaré duality for the integral (co)homology of a connected, compact and oriented manifold X (see e.g. chapter 20. of [121])

$$H^n(X) \simeq H_{k-n}(X) \tag{4.42}$$

we would expect something similar to hold for K-theory and K-homology. And indeed such a generalized Poincaré duality can be defined for both complex and real K-theory. The two isomorphisms are

$$K^{-n}(X) \simeq K_{k+n}(X)\,, \tag{4.43}$$

if the k-dimensional manifold X is K-oriented, and

$$KO^{-n}(X) \simeq KO_{k+n}(X)\,, \tag{4.44}$$

if the k-dimensional manifold X is KO-oriented [127]. Moreover, K-orientation and KO-orientation are in one-to-one correspondence with $spin^c$- and $spin$-structure [127].

4.5 Spectra and the Pontryagin-Thom construction

To extend and deepen the preceding part of the chapter we want to introduce spectra, a much more encompassing framework to work with generalized (co)homology theories. To this end we will follow the excellent introduction in [127] closely.

First off, a spectrum E is simply defined as a sequence $\{E_n, s_n\}_{n\in\mathbb{Z}}$, where E_n are CW-complexes and embeddings $s_n : \Sigma E_n \to E_{n+1}$ (with ΣE_n the reduced suspension of E_n, i.e. $\Sigma E_n := S^1 \wedge E_n$), such that ΣE_n is a subcomplex of E_{n+1}.

Then, we are able to define a subspectrum $\{F_n, t_n\}_{n\in\mathbb{Z}}$, such that F_n are subcomplexes of E_n and $t_n : \Sigma F_n \to F_{n+1}$ is the appropriate restriction of s_n.

This has a remarkably powerful consequence that is actually the basis of the next section, namely filtrations. For this purpose we take a family of subspectra $\{E(i)\}$ of E and create another subspectrum of E as $\bigcup_i E(i)$ by means of $(\bigcup_i E(i))_n := \bigcup_i E(i)_n$. Then a filtration of a spectrum E is defined as the family

$$\{\cdots \subset E(i) \subset E(i+1) \subset \cdots \subset E\} \tag{4.45}$$

with each $E(i)$ being a subspectrum of E and finally $\bigcup E(i) = E$.

Furthermore, one can define for a spectrum X a spectrum $\Sigma^k X$ by taking $(\Sigma^k X)_n$ to be X_{n+k} and $s_{n+k} : \Sigma(\Sigma^k X)_n \to (\Sigma^k X)_{n+1}$. From this we can construct the so-called suspension spectrum $\Sigma^\infty X$, such that:

$$(\Sigma^\infty X)_n = \begin{cases} pt & \text{for } n < 0, \\ \Sigma^n X & \text{for } n \geq 0. \end{cases} \tag{4.46}$$

The smash product with which we constructed the spectrum itself is also meaningful between a spectrum E and a CW-complex X, we denote by $E \wedge X$ the spectrum based on $\{E_n \wedge X\}$ and analogously $X \wedge E := \{X \wedge E_n\}$[4].

Now, we get to the most important point of this section. Based on the Eilenberg-Steenrod axioms one can show that every spectrum yields a (possibly generalized) (co)homology theory. A (generalized) homology theory of X $G_n(X)$ based on a spectrum E is defined as

$$G_n(X) := \pi_n(E \wedge X) \tag{4.47}$$

or equivalently as homotopy classes $G_n(X) := [\Sigma^n E, X]$. Consequently, a (generalized) cohomology theory is defined based on the cohomotopy groups

$$G^n(X) := \pi^n(E \wedge X) = [X, \Sigma^n E]\,. \tag{4.48}$$

Then for any spectrum E G_* and G^*. As an example, we can demonstrate the power of Eilenberg-MacLane spaces in yet another context. Let's consider the spectrum $H\mathbb{Z}$[5], such that

$$\pi_n(H\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \tag{4.49}$$

[4] Correspondingly, one can also define smash products of spectra, on which we do not expedite here. We directly refer to chapter II, section 2 in [127].

[5] This can be worked out analogously for any other Abelian group.

Then this spectrum defines a homology theory:

$$H\mathbb{Z}_n(pt) = \widetilde{H\mathbb{Z}}_n(S^0) = \pi_n(H\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n \neq 0, \end{cases} \tag{4.50}$$

where $\widetilde{H\mathbb{Z}}_n(S^0)$ is the reduced homology theory. By applying the Eilenberg-Steenrod axioms one can show that this is just ordinary integral homology $H\mathbb{Z}_n(X) = H_n(X,\mathbb{Z})$. Moreover, the Eilenberg-MacLane spectrum $H\mathbb{Z}$ can then be expressed through Eilenberg-MacLane spaces as follows:

$$H_n(X,\mathbb{Z}) = \pi_n(H\mathbb{Z} \wedge X^+) = \lim_{k\to\infty} \pi_{n+k}(K(\mathbb{Z},k) \wedge X^+)\,. \tag{4.51}$$

Analogously, we can define ordinary (integral) cohomology through the Eilenberg-MacLane spectrum $H\mathbb{Z}$.

Finally, we would like to express our three generalized (co)homology theories by using spectra. For complex and real K-theory, the spectra are denoted K and KO respectively. The spectrum of complex K-theory then gives us [127] for the homology theory

$$K_*(pt) = \pi_*(K) = \mathbb{Z}[t, t^{-1}]\,, \tag{4.52}$$

where $dim(t) = 2$ and t generates Bott periodicity, such that $\Sigma^2 K \simeq K$. Equivalently we get complex K-theory from the same spectrum by deploying the cohomology construction. The connective covering of K $p : k \to K$ works so that $\pi_*(k) = \mathbb{Z}[t]$ and $p_* : \pi_n(k) \to \pi_n(K)$ is an isomorphism for $n \geq 0$. Similarly, one gets the spectrum KO with $\Sigma^8 KO \simeq KO$ and its connective version ko. The story for cobordism equipped with a tangential structure requires us to construct so called Thom spectra. Thom spectra were introduced by Thom [123] to construct the spectrum MO corresponding to unoriented cobordism, i.e.

$$\Omega_n^O = \pi_n(MO)\,. \tag{4.53}$$

MO is constructed as a series of Thom spaces $MO(n)$:

$$MO := \{MO(1), MO(2), \dots\}. \tag{4.54}$$

The Thom space $MO(n)$ is constructed by taking the universal vector bundle of dimension n $\gamma_n \to BO(n)$ and construct the Thom space

$$MO(n) = Th(\gamma_n) := D_{\leq 1}(\gamma_n)/S(\gamma_n), \tag{4.55}$$

where $D_{\leq 1}(\gamma_n)$ and $S(\gamma_n)$ denote the sub-bundle of γ_n, such that their fibers are the unit disk and the unit sphere respectively. The inclusion $O(n) \overset{O}{\hookrightarrow} (n+1)$ can be understood as the addition of a trivial bundle, such that

$$\Sigma MO(n) = Th(\gamma_n + \mathbb{R}) \to MO(n+1), \tag{4.56}$$

which defines our spectrum. Now, this definition can be nicely extended precisely through obstruction theory, which we introduced in section 4.2. This is known as the Pontryagin-Thom theorem stating:

Theorem 4.5.1 (Pontryagin-Thom [123, 141])**.** *There exists an isomorphism*

$$\Omega_n^{\xi}(X) \simeq \pi_n(M\xi \wedge X)\,, \tag{4.57}$$

where $M\xi$ denotes the Thom spectrum corresponding to a tangential structure ξ.

Given a universal vector bundle of dimension n $\tilde{\gamma}_n \to BG(n)$. These are precisely the maps we studied before when defining tangential structures. Let us for example take $G(n) = Spin(n)$ or $Spin^c(n)$, we can then define corresponding Thom spaces $MG(n) = Th(\tilde{\gamma}_n)$ with which we obtain the Thom spectra and get our cobordism groups, e.g.

$$\begin{aligned} \Omega_n^{Spin} &= \pi_n(MSpin)\,, \\ \Omega_n^{Spin^c} &= \pi_n(MSpin^c)\,. \end{aligned} \tag{4.58}$$

As our final step in this chapter on the mathematical background we have to introduce the proper tools to actually compute these, to a physicist abstract looking, objects, if we are not working with $X = pt$, but more general background spaces. The tool we are talking about are spectral sequences.

4.6 Spectral sequences

Spectral sequences come in a wide variety of variants and applications. So let's start by recollecting the fundamental, shared properties loosely following the nice presentation in [142]. Since spectral sequences take such an important place in algebraic topology, there are many more excellent textbook accounts, see for example [143, 144]. For the Adams spectral sequence specifically [145] provides a brilliant introduction.

Especially in the last few years spectral sequences have also spread in physics, in particular high energy physics, with a wide variety of different applications, see for example these references [34, 35, 146–164].

The aim of deploying a spectral sequence is usually to calculate some object G_*, which in our case will be a (generalized) (co-)homology theory, like a K-theory variant or some cobordism theory, which satisfy the Eilenberg-Steenrod axioms (see appendix A) except for the dimensionality axiom.

First, we are restricting ourselves to computing a homological object, which we denote by a lower index. The central idea to a spectral sequence is then to approximate G_* through a series of steps getting closer each step. Ideally, this requires just a finite amount of steps. To this end, we require our object G_* to be filtered (bounded below), i.e. we can define a series of sub-objects organized in the following way:

$$\{0\} = F_{-1}G_* \supset F_0G_* \supset \cdots \supset F_nG_* \supset \cdots \supset G_* \,. \tag{4.59}$$

This filtration can be used to define the so called associated graded vector space as an approximation to G_*:

$$E_{p,q} = F_pG_{p+q}/\, F_{p-1}G_{p+q} \,. \tag{4.60}$$

Then we can get G_n by summing over p, the filtration degree, and q, a complementary degree:

$$G_n = \bigoplus_{p+q=n} E_{p,\,q} \,. \tag{4.61}$$

Now, a spectral sequence consists of a sequence of differential bigraded vector spaces, which means that we have $r \in \mathbb{N}$ bigraded vector spaces, called pages, $E^r_{p,q}$. For all the connective versions of our (generalized) (co-)homology theories first quadrant spectral sequences, i.e. $p, q \geq 0$, would be sufficient for our computations, since in our cases n is just the dimension. For full K-theory we have to loosen this requirement and can not restrict the sign of n. Due to the critical dimension of string theory we don't need to calculate to arbitrarily high n for our applications, though.

As we outlined we also want these bigraded vector spaces to be differential, i.e. we equip them with a linear mapping within a spectral sequence page $d_r : E^r_{p,\,q} \to E^r_{p-r,\,q+r-1}$, where we call the linear map d_r differential because of its property $d_r \circ d_r = 0$. Then we can calculate the next page in the following way:

$$E^{r+1}_{p,q} = \frac{\ker d_r : E^r_{p,\,q} \to E^r_{p-r,\,q+r-1}}{\operatorname{im} d_r : E^r_{p+r,\,q-r+1} \to E^r_{p,\,q}} \,. \tag{4.62}$$

In certain instances, e.g. the Adams spectral sequence for unoriented cobordism [160] or the Atiyah-Hirzebruch spectral sequence for complex K-theory [148], the authors were able to assign a clear physical interpretation to the differentials. Generically, the physical interpretation is tied to tachyonic behavior, i.e. the differential tells us about the physical instabilities that our theory has in order to arrive at the true stable description G_n from some approximation $G_n^r = \bigoplus_{p+q=n} E_{p,q}^r$. We have visualized the second page of such a homological spectral sequence and its differentials in the figure below 4.3. We assume some non-trivial differentials, which are killing the entries they are acting on completely (green), reducing them (orange) or acting trivially (black).

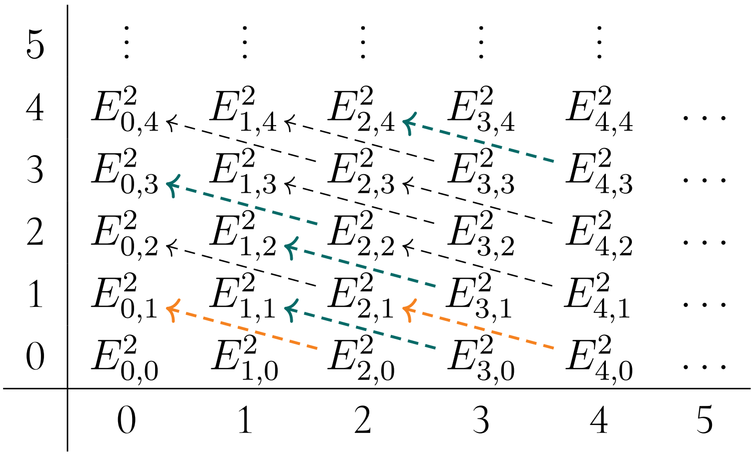

Figure 4.3: Example of a second page E^2 of a first quadrant homological spectral sequence and all possible d_2 differentials. The non-vanishing d_2 are shown by orange and green arrows.

After having evaluated the action of the differentials we can then look at the third page, where we use orange, green and black colors to indicate which kind of differential led to the respective entry on the third page 4.4. We notice that there are way fewer non-trivial differentials possible, since a bunch of entries are zero and therefore cannot facilitate any non-trivial differentials.

Coming back to our introduction to spectral sequences, suppose G_* has a filtration and we converge to an “infinity page", the endpoint of our spectral sequence,

$$E_{p,q}^{\infty} = F_p G_{p+q} / F_{p-1} G_{p+q} \,, \tag{4.63}$$

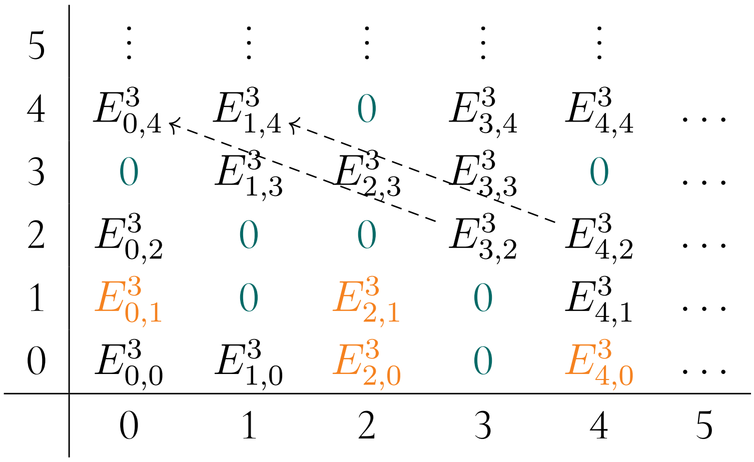

Figure 4.4: Third page E^3 of the same spectral sequence and all possible d_3 differentials. The green differentials have (co-)killed the page elements they were acting on, while the orange ones let them partially survive. The black elements, on which no differential acted, carried over intact to the next page, i.e. $E^3_{p,q} \cong E^2_{p,q}$.

which is our approximation to G_*. Finally, since going through infinitely many pages isn't tractable, we want to focus on spectral sequences collapsing at some page $r = N$, such that $d_r = 0$ for $r \geq N$. In physics applications this is usually achieved by taking a finite upper dimension, such as the critical dimension in string theory. Then

$$E^N_{*,*} \cong E^{N+1}_{*,*} \cong \ldots \cong E^\infty_{*,*} \,. \tag{4.64}$$

Given, of course, the fixed input by choosing a certain G_* we have now reached a stable configuration for the objects classified by G_*. As mentioned before, from a physics point of view we have taken all possible decay channels (given the input) into account.

(Dual) Cohomological Spectral Sequence

In many respects a cohomological spectral sequence is analogous to its homological counterpart. The goal here is to approximate an object G^* with a reversed filtration compared to the homological filtration:

$$G^* = F^0 G^* \supset \cdots \supset F^n G^* \supset \cdots \supset \{0\} \,. \tag{4.65}$$

The cohomological spectral sequence then is again composed of a sequence of differential bigraded vector spaces $E_r^{p,q}$ with a differential $d_r : E_r^{p,\,q} \to E_r^{p+r,\,q-r+1}$ going in the opposite direction of the homological differentials. A corresponding pictorial depiction for a cohomological spectral sequence would look like this:

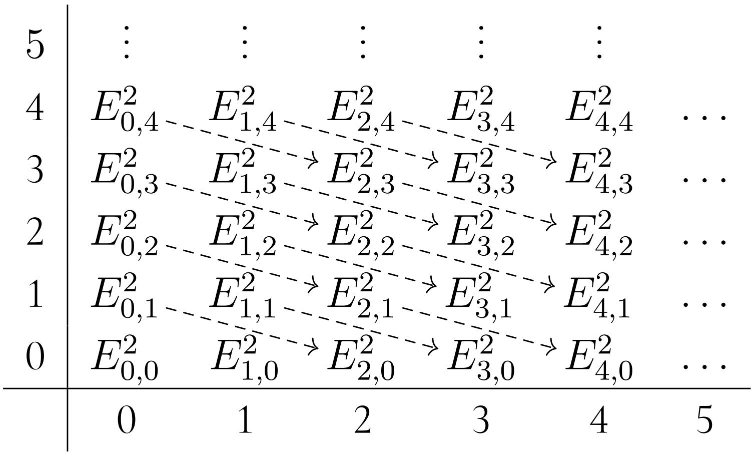

Figure 4.5: Example of a second page E^2 of a first quadrant cohomological spectral sequence and all possible d_2 differentials.

Extension problems

Besides determining the action of the differentials, there is a second major obstruction to determining the final result G_* based on the necessary sum over the grading

$$G_n = \bigoplus_{p+q=n} E_{p,q} \,. \tag{4.66}$$

called extension problem stemming from our usage of filtrations. The point is that for each filtration degree p there exists a short exact sequence

$$0 \to F_{p-1}G_{p+q} \to F_pG_{p+q} \to F_pG_{p+q}/\,F_{p-1}G_{p+q} \to 0 \tag{4.67}$$

and simultaneously we have

$$E^{\infty}_{p,q} = F_pG_{p+q}/\,F_{p-1}G_{p+q} \,. \tag{4.68}$$

Since we consider a filtration, which is degree-wise bounded, at least we only have to worry about finitely many such problems. Consider the simple example in degree $n = p + q = 2$, where we have three entries $E^{\infty}_{2,0}$, $E^{\infty}_{1,1}$ and $E^{\infty}_{0,2}$. Due to the boundedness the short exact sequences can be reduced to just two

$$\begin{aligned} 0 \to E^{\infty}_{0,2} \to E \to E^{\infty}_{1,1} \to 0\,, \\ 0 \to E \to F \to E^{\infty}_{2,0} \to 0\,, \end{aligned} \tag{4.69}$$

where E is the extension of $E^{\infty}_{1,1}$ by $E^{\infty}_{0,2}$, which we denote $e(E^{\infty}_{1,1}, E^{\infty}_{0,2})$. Correspondingly, F is a nested extension of $E^{\infty}_{2,0}$ by $e(E^{\infty}_{1,1}, E^{\infty}_{0,2})$ so $e(E^{\infty}_{2,0}, e(E^{\infty}_{1,1}, E^{\infty}_{0,2}))$, whose notation we will shorten to just $e(E^{\infty}_{2,0}, E^{\infty}_{1,1}, E^{\infty}_{0,2})$. A cohomological spectral sequence, which is degree-wise bounded analogously runs into the same type of extension problems. We are going to deepen the discussion of extension problems in appendix E a bit.

With this short introduction to the notion of a spectral sequence we will now move on to a concrete example, the Leray–Serre–Atiyah–Hirzebruch spectral sequence (LSAHSS), which is one of the main tools we will be working with.

4.6.1 The Leray–Serre–Atiyah–Hirzebruch spectral sequence

As the Leray–Serre–Atiyah–Hirzebruch spectral sequence (LSAHSS) we understand a tool for calculating (generalized) (co)homology groups $G_*(X)$ ($G^*(X)$) based on a Serre fibration, i.e. a fibration

$$F \hookrightarrow X \xrightarrow{p} B\,, \tag{4.70}$$

where F and B are fiber and base respectively and B is a path-connected $\pi_0(B) = 0$ CW-complex, such that F_b and $F_{b'}$ are homotopy equivalent for $b,\ b' \in B$. For this section we follow the presentation in [165] from which we also adopt the slightly cumbersome naming convention[6]. We also need to distinguish the case, where B is not simply connected ($\pi_1(B) \neq 0$), since we would get G_n as some $\mathbb{Z}[\pi_1(B)]$-module in such a case. In this thesis we will not encounter a case of this type, then we get the following spectral sequence in the homological case

$$E^2_{p,q} \cong H_p(B; G_q(F)) \Rightarrow G_{p+q}(E)\,. \tag{4.71}$$

and

$$E^{p,q}_2 = H^p(B; G^q(F)) \Rightarrow G^{p+q}(E)\,. \tag{4.72}$$

[6] This convention is used to accredit both the work by Leray and Serre on the spectral sequence computing (co)homology groups $H_*(X)$ ($H^*(X)$) and Atiyah's and Hirzebruch's extension to generalized (co)homology groups, but with a trivial fibration.

in the cohomological case.

In the homological case the spectral sequence is built upon a filtration of the form:

$$F_pG_n = \mathrm{Im}(G_n(f^{-1}(B^p)) \to G_n(E))\,, \tag{4.73}$$

where $f : E \to B$ and B^p is the p-skeleton of the CW-complex of B. The cohomological version on the other hand is built from the filtration [115]

$$F^pG^n = \ker(G^n(E) \to G^n(E^{p-1}))\,. \tag{4.74}$$

In the next chapter we will use the homological AHSS to determine the cobordism groups $\Omega_n^\xi(X)$ and the cohomological AHSS for real and complex K-theory groups $KO^{-n}(X)$ and $K^{-n}(X)$, with X a compact manifold of dimension up to ten. We will specialize to particular choices of X, which very common string theory backgrounds, namely $X = \{S^k, T^k, K3, CY_3\}$, with a cobordism structure groups $\xi = \mathrm{Spin}, \mathrm{Spin}^c$.

Edge homomorphisms, vanishing differentials and trivial fibration

Here, we would like to further exploit the boundedness of our filtration underlying the LSAHSS. We first consider differentials starting or ending on edges beyond which the page entries vanish and thereby the differential does, as well. For a general first quadrant homological spectral sequence differentials starting from $E^r_{0,n}$ and differentials ending on $E^r_{n,0}$ necessarily vanish. Again because of the short exact sequences from the filtration we get two homomorphisms (see for example section XV.5. in [166]):

$$\begin{aligned} E^2_{0,n} &\to G_n\,, \\ G_n &\to E^2_{n,0}\,. \end{aligned} \tag{4.75}$$

For the cohomological version everything reverses again:

$$\begin{aligned} E_2^{n,0} &\to G^n\,, \\ G^n &\to E_2^{0,n}\,. \end{aligned} \tag{4.76}$$

Now, applying this to the LSAHSS based on the general Serre fibration $F \hookrightarrow X \to B$ the first edge homomorphism becomes [165]:

$$G_n(F) \to H_0(B; G_n(F)) = E^2_{0,n} \to E^\infty_{0,n} \to G_n(X), \tag{4.77}$$

which equals the map between $G_n(F)$ and $G_n(X)$ based on the inclusion map $F \hookrightarrow X$. The other edge homomorphism consequently takes the form:

$$G_n(X) \to E^2_{n,0} = H_n(B; G_0(F)). \tag{4.78}$$

The first homological homomorphisms becomes especially powerful, if we consider trivial fibrations, and we can use it to rule out a bunch of differentials.

To see this consider the trivial fibration

$$\text{pt} \hookrightarrow X \overset{\text{id}}{\to} X. \tag{4.79}$$

The inclusion pt $\hookrightarrow X$ is split by the constant map $X \to$ pt, implying that

$$G_n(\text{pt}) \to G_n(X) \tag{4.80}$$

is a split injection (G_* again being a generalized homology theory). Choosing now $F =$ pt and $B = X$ in (4.77), one should recover the split injection (4.80) and thus

$$E^2_{0,n} \cong E^\infty_{0,n}. \tag{4.81}$$

This is nothing else than the splitting principle we saw for example already for K-theory (4.7) appearing within the spectral sequence. Here this has a strong imprint, since in this case the entries on this edge have to survive to the final page and any differential acting on them,

$$d^r : E^r_{r,q} \to E^r_{0,q+r-1}, \tag{4.82}$$

has to be trivial. This observation greatly simplifies the calculation of the related spectral sequences and will have a direct application in the upcoming computation of cobordism groups. Because of the reverse direction of the differentials in the cohomological case the second homomorphism leads under a trivial fibration to trivial differentials starting from $E^r_{0,n}$.

4.6.2 The Adams spectral sequence

While the LSAHSS is an extremely versatile tool to compute K-theory and cobordism groups, for some of the groups we are interested in this is not quite enough. For example, in chapter 6 we are going to determine $\Omega^{Spin}_n(BSs(32))$. The main tool of choice for us to achieve that will be the Adams spectral sequence[7]. While this spectral sequence was invented to compute the stable homotopy groups of the spheres, i.e. for the spectrum $\Sigma^\infty S^0$, it is also particularly useful for the calculation of generalized homology theories by calculating the homotopy groups of the associated spectrum. However, the input necessary to fill the second page of the Adams spectral sequence might be tougher to attain than for the LSAHSS.

In general terms Adams [167] devised the following spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X,\mathbb{Z}_p),\mathbb{Z}_p) \Rightarrow \pi_{t-s}(X)_{\widehat{p}}\,, \tag{4.83}$$

where $\mathcal{A}$ denotes the Steenrod algebra and the index $\widehat{p}$ again denotes localization at the prime p, i.e. for an Abelian group $G_{\widehat{p}} := \lim_{\overleftarrow{s}} G/p^s$. While we will refrain from giving an extensive introduction to the Steenrod algebra (and its subalgebras), we will collect some useful information about the Steenrod squares and the algebra they generate in the appendix C. The differentials in the Adams spectral sequence are slightly different from the ones we introduced in the above sections, namely the differential d_r has grading $(r, r-1)$. Now, for the computations in this thesis it turns out we can just focus just on the 2-torsion part as we will proof later on. So for concreteness, we will focus on the $p = 2$ case, while we should keep in mind that in a different case all odd primes p might become important. Here, by utilizing the Anderson-Brown-Peterson decomposition [135] we can express spin cobordism in terms of connective ko-homology at the prime $p = 2$ (4.34). This means that as a first step we need to calculate the connective ko-homology of $BSs(32)$. In particular, the calculation of real connective k-homology $ko_*(pt) = \pi_*(ko)$ or more generally $ko_*(X) = \pi_*(ko \wedge X)$ simplifies due to work of Stong [168]:

$$H^*(ko,\mathbb{Z}_2) \cong \mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{Z}_2\,. \tag{4.84}$$

We get by change of rings, see for example [145]:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_1}^{s,t}(H^*(X,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow ko_{t-s}(X)_{\widehat{2}}\,. \tag{4.85}$$

Unfortunately, the mod 2 cohomology of the classifying space of the $SemiSpin$ group $H^*(BSs(4n),\mathbb{Z}_2)$ (specifically for $BSs(32)$) is not fully known. However,

[7] We are going to use the Adams spectral sequence for a simple example, namely $\Omega^{Spin}(\mathbb{CP}_2)$, already in chapter 5.

since we are only interested in the cobordism groups up to dimension $n = 12$, it will suffice to calculate $H^*(BSs(4n), \mathbb{Z}_2)$ up to degree $* \leq 13$. In particular, to set up the second page of the Adams spectral sequence we will need to determine the Steenrod operations within $H^*(BSs(4n), \mathbb{Z}_2)$ in order to get the structure of $H^*(BSs(4n), \mathbb{Z}_2)$ in terms of $\mathcal{A}_1$-modules. Before we tackle this question let us introduce a couple more useful concepts related to the Adams spectral sequence, which will be of importance in the following.

From $H^*(X, \mathbb{Z}_2)$ to $\text{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2)$ – The second page of the Adams spectral sequence

Suppose that we have determined the Steenrod algebra for $H^*(X, \mathbb{Z}_2)$. As we will be interested in $spin$ cobordism the subalgebra $\mathcal{A}_1$ is sufficient due to the power of Stong's theorem and the ABP decomposition (with the absence of odd primes). To tailor this introduction towards our subsequent applications of the methods we are mostly focusing to $\mathcal{A}_1$. In order to fill the second page of the Adams spectral sequence, which means that we want to determine $\text{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2))$. Hence, we need to recollect some facts about $\text{Ext}^{s,t}_{\mathcal{R}}(M, N)$ and especially how to get $\text{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2)$ from the $\mathcal{A}_1$-module structure of $H^*(X, \mathbb{Z}_2)$. Again, we refer to [145] for a more general discussion of this topic as we will keep this part concise. $\text{Ext}^{s,t}_{\mathcal{R}}(M, N)$ can be understood as equivalence classes of extensions with $s \geq 1$. An element of $\text{Ext}^{s,t}_{\mathcal{R}}(M, N)$ then represents the following extension:

$$0 \to \Sigma^t N \to P_1 \to \cdots \to P_s \to M \to 0\,, \tag{4.86}$$

where $\Sigma^t N$ denotes the t-th reduced suspension of N, both M and N are $\mathcal{R}$-modules and P_s is a filtration of M. We further discuss Ext-groups (specifically for $\mathcal{R} = \mathbb{Z}$ and no filtration) in E applied to the extension problems arising in the LSAHSS. Let us first introduce the two most important classes in $\text{Ext}_{\mathcal{A}_1}(\mathbb{Z}_2, \mathbb{Z}_2)$ that are conventionally used to depict the second page $\text{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2))$. These are $h_0 = \text{Ext}^{1,1}_{\mathcal{A}_1}(\mathbb{Z}_2, \mathbb{Z}_2)$

$$0 \to \Sigma\mathbb{Z}_2 \to \Sigma^{-1} H^*(\mathbb{RP}^2, \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0 \tag{4.87}$$

and $h_1 = \text{Ext}^{1,2}_{\mathcal{A}_1}(\mathbb{Z}_2, \mathbb{Z}_2)$

$$0 \to \Sigma^2\mathbb{Z}_2 \to \Sigma^{-2} H^*(\mathbb{CP}^2, \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0\,. \tag{4.88}$$

The usual convention is to use coordinates $(t - s, s)$ for all of the pages of the Adams spectral sequence, such that each homotopy group $\pi_{t-s}(X)$ can be read off as a column on the infinity page. Each $\mathbb{Z}_2$ summand within $\text{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2))$ amounts to a dot on the second page, while h_0's are vertical lines raising s and t by 1 and h_1's are diagonal lines raising s by 1, while t gets raised by 2. There are two more classes of $\text{Ext}_{\mathcal{A}_1}(\mathbb{Z}_2, \mathbb{Z}_2)$, which are customarily not depicted to avoid cluttering the Adams pages, namely v of degree $(7, 3)$, whose action raises s by 3 and $t - s$ by 4, and w of degree $(12, 4)$ raising s by 4 and $t - s$ by 8. All of these actions have (geometric) interpretations, e.g. as multiplication by 2 in the case of h_0 or multiplication by certain manifolds of appropriate degree in $t - s$. We will go into more detail once we actually utilize these properties.

There are two major pathways to determine $\text{Ext}^{s,t}_{\mathcal{R}}(M, N)$: Minimal resolutions and long exact sequences. Since for most of the $\mathcal{A}_1$-modules $\mathcal{M}$ we are going to encounter in $H^n(BSs(32), \mathbb{Z}_2)$ their $\text{Ext}^{s,t}_{\mathcal{A}_1}(\mathcal{M}, \mathbb{Z}_2)$ can be found explicitly in the literature, we will try to kill two birds with one stone. We are going to explain the long exact sequence method for the concrete example of $\tilde{R}_2$, an $\mathcal{A}_1$-module that can not be easily found in the literature. A lot more information on minimal resolutions can be found for example in Beaudry-Campbell [145], which we will follow in the presentation of the long exact sequence method.

Below we include a depiction of $\tilde{R}_2$, figure 4.6. Since $\mathcal{A}_1$ is only generated by Sq^1 and Sq^2 the different nodes are connected by just two types of lines: straight ones raising degree by 1 corresponding to Sq^1 and curved lines representing Sq^2s raising the degree by 2.

Consider the exact sequence of (in our specific case) $\mathcal{A}_1$-modules:

$$0 \to \Sigma^3 C\eta \to \tilde{R}_2 \to J \to 0\,, \tag{4.89}$$

which we depict as 4.7.

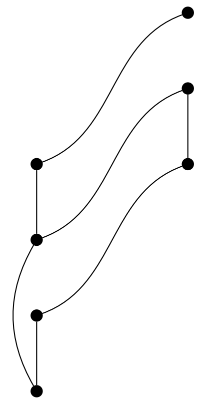

Figure 4.6: The $\mathcal{A}_1$-module – $\tilde{R}_2$

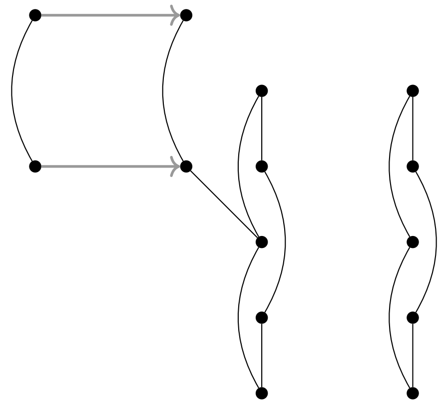

Figure 4.7: Exact sequence for $\tilde{R}_2$

Now, this exact sequence leads to this dual long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}_1}(J, \mathbb{Z}_2)) \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\tilde{R}_2, \mathbb{Z}_2)) \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\Sigma^3 C\eta, \mathbb{Z}_2)) \xrightarrow{\delta} \mathrm{Ext}^{s+1,t}_{\mathcal{A}_1}(J, \mathbb{Z}_2)) \longrightarrow \mathrm{Ext}^{s+1,t}_{\mathcal{A}_1}(\tilde{R}_2, \mathbb{Z}_2)) \longrightarrow \ldots \tag{4.90}$$

By splicing our exact sequence (4.89) into the extension corresponding to

$\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\mathcal{A}_1/\mathcal{E}_1, \mathbb{Z}_2)$ we extend it to

$$\begin{array}{ccccc} 0 \to \Sigma^t\mathbb{Z}_2 \to P_1 \to \cdots \to P_s & & \longrightarrow & & \tilde{R}_2 = P_{s+1} \to J \to 0\,. \\ & \searrow & & \nearrow & \\ & & \Sigma^3 C\eta & & \end{array} \tag{4.91}$$

Therefore, we can understand δ as mapping elements in $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\Sigma^3 C\eta, \mathbb{Z}_2)$ to their boundary representative in $\mathrm{Ext}^{s+1,t}_{\mathcal{A}_1}(J, \mathbb{Z}_2)$. Since we know $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\Sigma^3 C\eta, \mathbb{Z}_2)$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(J, \mathbb{Z}_2)$, see for example [145], we can get $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\tilde{R}_2, \mathbb{Z}_2)$ as well. To represent the Ext-functors we transition to the so called Adams charts, which use the same conventions as the second page of the Adams spectral sequence. A key difference is that we want to track the effect of our boundary map δ, which can be understood as a differential d_1. This is, of course, by definition never part of a second page of a spectral sequence. The differentials d_r in an Adams chart (on a page E_r in the spectral sequence) then go from a node at $(t-s, s)$ to $(t-s-1, s+1)$. So in our example possible differentials go from nodes coming from $\Sigma^3 C\eta$ to nodes coming from J. They eliminate all of the nodes connected via a differential. Keeping this in mind we get the following Adams chart in figure 4.8, where we encircled all of the nodes, which are not affected by a differential.

For the final result below in figure 4.9 we added an extension (dashed) not seen through this specific exact sequence. There are two different methods to see this extension. Either by working with minimal resolutions, which can be a bit tedious, or alternatively, we can look at the following exact sequence involving $\tilde{R}_2$:

$$0 \to \Sigma^6\mathbb{Z}_2 \to \mathcal{A}_1 \to \tilde{R}_2 \to 0\,. \tag{4.92}$$

Since we know the Adams charts for $\mathbb{Z}_2$ and $\mathcal{A}_1$, one can get $\tilde{R}_2$ as well, albeit without the extension problem.

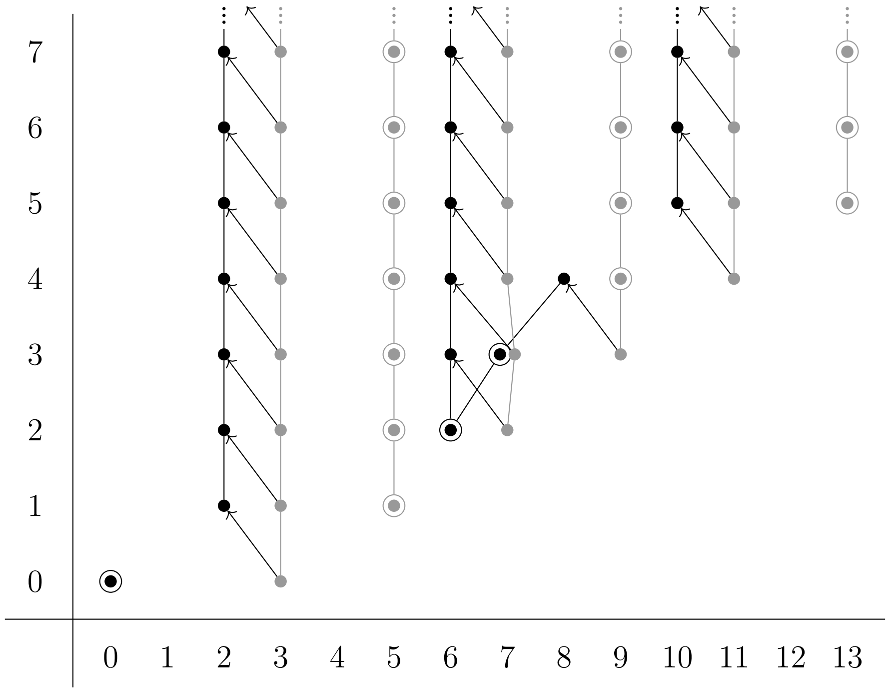

Figure 4.8: Adams chart for $\tilde{R}_2$ extension including δ

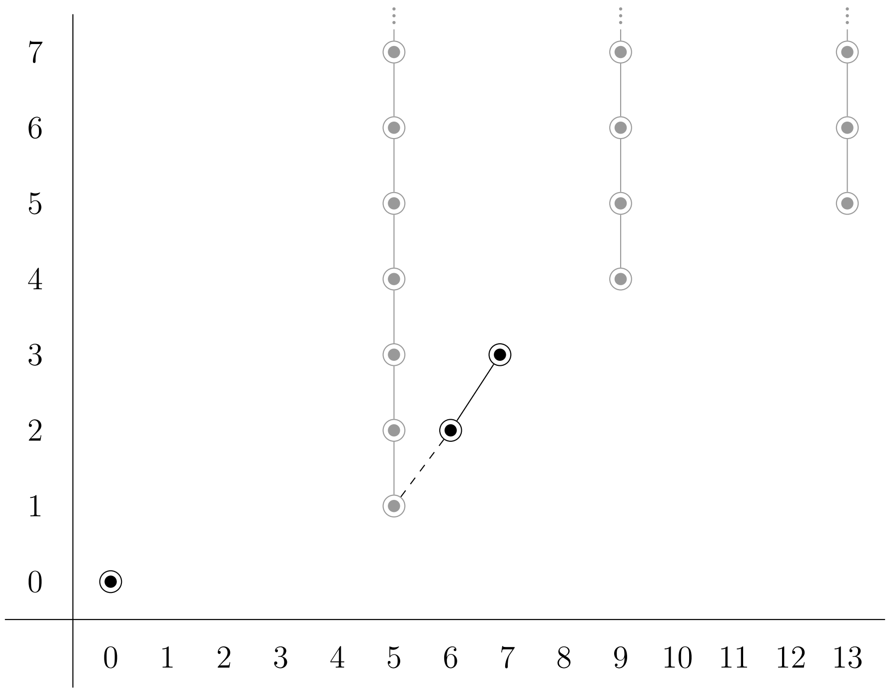

Figure 4.9: Final Adams chart for $\tilde{R}_2$ extension

4.6.3 The Eilenberg-Moore spectral sequence

Since we need the mod 2 cohomology of the classifying space of the $SemiSpin(32)$ group $BSs(32)$ to fill out the second page of the Adams spectral sequence, which has not been fully determined in the literature yet, we need a tool to compute this mod 2 cohomology from something we already know. Luckily, the so called Eilenberg-Moore spectral sequence [169–171] provides a way to compute the mod 2 cohomology of the classifying space of a compact Lie group BG from the mod 2 cohomology of the compact Lie group:

$$E_2 = \text{Cotor}_{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \Longrightarrow H^*(BSs(n), \mathbb{Z}_2)\,. \tag{4.93}$$

First, let's quickly introduce the Cotor functor $\text{Cotor}_R(M, N)$ following [121]. The construction of the Cotor functor is similar to the Ext functor, which we have just encountered as a primary ingredient of the Adams spectral sequence. We start with the cotensor product for a right comodule A and a left comodule B over C. With a coalgebra C we define the cotensor product between A and B as:

$$A \,\square_C\, B := ker(\phi_A \otimes id_B - id_A \otimes \phi_B)\,, \tag{4.94}$$

where $\phi_A : A \to A \otimes C$ and $\phi_B : B \to C \otimes B$ are the respective comultiplications. Then considering the following injective resolution for A with comodules A_* over C

$$A \to A_0 \to A_1 \to \dots A_n \to \dots \tag{4.95}$$

we can introduce the following series of cotensor products:

$$A_0 \,\square_C\, B \to A_1 \,\square_C\, B \to \dots \to A_n \,\square_C\, B \to \dots \tag{4.96}$$

Finally, the Cotor functor is now the cohomology of this resolution

$$\text{Cotor}^n_C(A, B) = H^n(A_0 \,\square_C\, B \to A_1 \,\square_C\, B \to \dots \to A_n \,\square_C\, B \to \dots)\,. \tag{4.97}$$

For the spectral sequence itself there arises another index q from the fact that A, B and C can be graded, such that $\bigoplus_q \text{Cotor}^{p,q}_C(A, B) = \text{Cotor}^p_C(A, B)$. In our specific spectral sequence (4.93) $C = H^*(Ss(32), \mathbb{Z}_2)$ introduces the grading. While the Eilenberg-Moore spectral sequence is a fair bit more general, we will focus on a particular application, which was expounded in [170]. Rothenberg and Steenrod proved that there is a convergent spectral sequence:

$$E_2 = \mathrm{Cotor}_{H^*(G,\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \Longrightarrow H^*(BG, \mathbb{Z}_2)\,. \tag{4.98}$$

The underlying idea here is that we can build an injective resolution for a classifying space BG through the group G itself, by constructing G-invariant closed subspaces of BG. We recapitulate that $BG = EG/G$, where EG is constructed by Milnor's construction as the limit of joined G-spaces, i.e. $EG = G \star G \star \dots$. Now, by first considering

$$EG_n = \underbrace{G \star G \star \dots}_{n+1 \text{ times}} \tag{4.99}$$

we can then construct a filtration $\{BG_n\}$, where BG_n is the projection $p(EG_n)$ associated to the universal bundle $p : EG \to BG$. This filtration is underlying the Rothenberg-Steenrod variant of the more general Eilenberg-Moore spectral sequence. We will close our short introduction to the Eilenberg-Moore spectral sequence here and point to chapter 7 and 8 in [142] for a much deeper discussion of the topic.

5

Dimensional reduction of Cobordism and K-theory

In this chapter we want to explore the meaning of including finite dimensional background manifolds X into cobordism groups, i.e. $\Omega^\xi(X)$, and their interplay with the Cobordism Conjecture. In particular we are going to focus on gauging the non-trivial cobordism groups. To this end we are going to build upon the proposal [32] that the gauging of *spin* and *spin*^{c} cobordism groups crucially involves the Atiyah-Bott-Shapiro map to KO- or K-theory respectively. Moreover, they argued that the familiar tadpole cancellation conditions for Dp-branes amount to gauging the cobordism groups in the co-image of this map.

We are going to notice that $\Omega^\xi(X)$ seems to account for the proper cobordism groups under dimensional reduction. The concept of dimensional reduction is ubiquitous in String Phenomenology and the Swampland Program as at low energies we want to have a four dimensional EFT to match with our observed universe at such an energy scale. Dimensional reduction should be clearly distinguished from just compactification as in the latter case we still working with a ten dimensional description albeit with a partially compact space-time.

To set everything up, we are first going to discuss the dimensional reduction of symmetries. Next, we will turn on the spectral sequence machinery for the first time to compute the necessary cobordism and K-theory groups.

5.1 Dimensional reduction of symmetries

Before we delve into the mathematically more involved evaluation of cobordism and K-theory groups under dimensional reduction, it is instructive to review how dimensional reduction is usually performed in (co)homology. This will help us appreciating what we really gain from using the description in terms of cobordism and K-theory.

Let us consider an effective field theory in general d dimensions, where of course we have the critical dimension $d = 10$ in mind. Suppose we have a continuous global p-form symmetry with a field A_p, then there exists an corresponding current $J_n = dA_p$ with $n = d - p - 1$. We have already seen both options, gauging and breaking, to rid the theory of such a global symmetry, which is incompatible with quantum gravity.

When performing a dimensional reduction over a compact space X, we need to expand the currents and gauge fields, in a cohomological basis of X. This works as follows, see for example Appendix B. in [23] for a concise introduction: Let's take a p-form field A_p and expand its components in coordinates:

$$A_p = \frac{1}{p!} A_{m_1 \dots m_p}(x)\, dx^{m_1} \wedge \cdots \wedge dx^{m_p} \tag{5.1}$$

with the antisymmetrized exterior product $\wedge$. Of course, this entails that the exterior product of a p- and a q-form behaves as $A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p$. The familiar exterior derivative d, satisfying the usual requirement for a differential $d \circ d = 0$, maps of course a p-form to a $p+1$-form:

$$dA_p = \frac{1}{p!} \partial_{m_0} A_{m_1 \dots m_p}(x)\, dx^{m_0} \wedge dx^{m_1} \wedge \cdots \wedge dx^{m_p} \tag{5.2}$$

Now, we want to look at two classes of p-forms:

- closed p-forms A_p, such that $dA_p = 0$,
- exact p-forms $A_p = dB_{p-1}$.

By definition, exact p-forms are always closed, but this doesn't hold generally in the other direction. Since we would like to treat currents J_n arising as an exact form from a gauge field A_p, we construct the equivalence classes:

$$A_p \sim A_p + dB_{p-1}\,. \tag{5.3}$$

It should not surprise us that we can build a cohomology theory out of that. The quotient of the set of exact p-forms on a space X $\Gamma^p(X)$ by the set of closed p-forms on X $\Pi^p(X)$ gives us so called de Rham cohomology:

$$H^p_{\mathrm{dR}}(X) = \Gamma^p(X)/\Pi^p(X)\,. \tag{5.4}$$

In fact, de Rham cohomology can be shown to be isomorphic to conventional singular cohomology with real coefficients. This isomorphism is known as de Rham's theorem, see for example chapter 0 in [172]. The meaning of this isomorphism sometimes appears to be obscured in the physics literature. Often times, it is stated that the p-form fields or currents take values in singular cohomology $H^p(X,\mathbb{R})$. By looking at the actual map from de Rham to singular cohomology

$$\begin{aligned} H^p_{\mathrm{dR}}(X) &\to H^p(X,\mathbb{R}) : \\ [\omega][M] &\mapsto \int_M \omega \end{aligned} \tag{5.5}$$

we see that we have mapped our current class to its charge Q by integrating over a representative M of the homology class $[M] \in H_p(X,\mathbb{R})$.

Once we quantize however, the Dirac quantization condition, we have already seen in previous chapters, will impose charges to be constrained to some lattice, and thus one has to consider integral cohomology. To be more specific, we are considering the torsion-free integral cohomology to keep things simple, $H^p(X,\mathbb{Z})/\mathrm{Torsion}$. However, it is not only the charges and the cohomology theory, they take values in, that gets modified by quantization. The currents and p-forms also adapt and then take values in differential cohomology $\check{H}^p(X)$ as constructed in [173], also known as the groups of Cheeger-Simons differential characters [174] or Deligne cohomology [175]. For a lot more background and the eventual uplift to generalized cohomology theories please have a look at [176].

In the following we are sidestepping these issues a bit, as we are ultimately interested in charges valued in cobordism and K-theory groups, which are quantum gravitational refinements of integral (co)homology. So we are taking the current classes

$[*J_n]$ eventually mapped to $H^p(X,\mathbb{Z})/\text{Torsion}$, such that the charges $Q_n = \int_X *J_n$ are quantized in a free Abelian group (Torsional charges valued in some $\mathbb{Z}_p$-group would not arise from the conventional continuous p-forms, which why we are focusing on just the free part).

Let us now dimensionally reduce the theory on a compact k-dimensional space X. We obtain a lower $D = d - k$ dimensional effective theory inheriting global symmetries from the upstairs, higher dimensional theory. In general, a given p-form symmetry in D dimensions can receive contributions from different $(p+q)$-form symmetries of the d-dimensional theory. To these contributions we associate currents J_{n+m}, now with $p = D - n - 1$ and $q = k - m$, wrapping $m = 0, 1, \dots, k$ cycles in X and extending along n directions in the non-compact space.

To expand our objects properly, we choose a basis in (torsion-free) integral cohomology, $\omega_{(m)a} \in H^m(X;\mathbb{Z})$, where $a = 1, \dots, b_m$, with $b_m = \dim(H^m(X;\mathbb{Z}))$ – the Betti numbers. Then, we can decompose the current classes as

$$[*J_{n+m}] = \sum_{a=1}^{b_m} [*j_n^{(m)a}] \wedge \omega_{(m)a}\,. \tag{5.6}$$

Thus, p-form symmetries in the dimensionally reduced theory of D dimensions are tied to the set of currents $[*j_n^{(m)a}]$, for $a = 1, \dots b_m$ and $m = 0, \dots, k$. Since we are performing an expansion in cohomology, we see that if the currents $*J_{n+m}$ are closed, $d*J_{n+m} = 0$, they produce a whole global p-form symmetry (charge) lattice in D dimensions, corresponding to the lower dimensional current classes

$$d * j_n^{(m)a} = 0, \qquad \forall\, a = 1, \dots b_m, \quad \forall\, m = 0, \dots k\,. \tag{5.7}$$

Breaking or gauging global symmetries in the d dimensional parent theory leads to a cascade of lower dimensional broken or gauged symmetries organized in a lattice drawing back from the original theory. Similar to the currents classes themselves the defects, signified by delta functions, or gauge fields are to be expanded in differential cohomology, but after integration we also have to map to integral cohomology, in which we are formally expanding.

To break the currents $*J_{n+m}$ we need forms $\delta^{(n+m+1)}$ in the d-dimensional theory such that $d * J_{n+m} = \delta^{(n+m+1)}(\Delta_{p+q}) \neq 0$, with $p + q = d - n - m - 1$. These forms represent defects living on a product manifold $\Delta_{p+q} = \Pi_p \times \Sigma_q$ in the d-dimensional space, where Π_p is a p-dimensional slice of the non-compact space, while we take Σ_q to be a q-dimensional homological cycle of X. To break the global symmetry in the downstairs D-dimensional theory, we take $p = D - n - 1$ and $q = k - m$, such that the defect in the dimensionally reduced theory has codimension $n + 1$.

We can then formally expand in cohomology

$$\delta^{(n+m+1)}(\Delta_{p+q}) = \sum_{a=1}^{b_m} \delta^{(n+1)}(\Pi_p)^{(m)a} \wedge \omega_{(m)a}. \tag{5.8}$$

Thus, from any defect $\delta^{(n+m+1)}$ in d dimensions we generate a lattice of codimension $n+1$ defects in D dimensions, $\delta^{(n+1)}(\Pi_p)^{(m)a}$. They can be used to break the lattice of global currents (5.7), i.e.

$$d * j_n^{(m)a} = \delta^{(n+1)}(\Pi_{D-n-1})^{(m)a} \neq 0, \tag{5.9}$$

where again this is really a set of equations for $a = 1, \dots b_m$ and $m = 0, \dots, k$.

To gauge the currents $*J_{n+m}$ we need gauge field strengths B_{n+m-1} in the d-dimensional theory such that $J_{n+m} = dB_{n+m-1}$. The dimensional reduction of these Bianchi identities can be performed in analogy to what was previously done. One thus finds a lattice of $(n-1)$-form field strengths $b_{n-1}^{(m)a}$ in D dimensions which are gauging the n-form currents $*j_n^{(m)a}$, thus giving the Bianchi identities

$$*j_n^{(m)a} = db_{n-1}^{(m)a}. \tag{5.10}$$

The integrated version leads to a tadpole cancellation condition as the integration leads to a vanishing total charge $\int_M db_{n-1}^{(m)a} = 0$.

Next, we are computing the cobordism and K-theory groups for a set of exemplary background spaces and will demonstrate, that they show this same pattern for dimensionally reduced broken and gauged symmetries on X. We will see that the description in terms of cobordism and K-theory provides by itself an organizing principle for the various symmetries in the dimensionally reduced theory, something which is not transparent from the above analysis. Indeed, contributions to a given (broken or gauged) p-form symmetry in D-dimensions and its corresponding charged objects will be encoded into $K^{-n}(X)$ and $\Omega_{k+n}^{\mathrm{Spin^c}}(X)$, for $p = D - 1 - n$ and $n \geq 0$. We will see that for $-k \leq n \leq 0$ the corresponding D-brane, respectively gravitational soliton, does not consistently fit into the D-dimensional space so that there does not exist any obvious physical interpretation of the cobordism and K-theory groups. We will provide some ideas on a possible, rather offbeat interpretation of these groups in string theory. For type I, we have a similar story for $\Omega^{\mathrm{Spin}}_{k+n}(X)$ and $KO^{-n}(X)$. This behavior under compactification gives further support to the interpretation of K-theory and cobordism groups as higher-form charges in an effective field theory.

As said, the above analysis was the classical dimensional reduction using (the differential uplift of) singular (co)homology without torsion. Therefore, all objects in D-dimensions are the result of a naive dimensional reduction along homological cycles in X, nothing is lost and nothing new arises in D-dimensions. However, the appearance of torsion through the refinement to generalized (co)homology theories can open up new decay channels of non-BPS branes, and it is known that new stable torsion branes can appear on X, even if they were not present in d dimensions. Additionally, non-trivial extensions can lead to fusion of branes, where instead of two objects valued in two separate groups they form a single object, whose charge lives in a single group. Moreover, for wrapped D-branes there can be quantum effects that spoil these simple (classical) expectations. For instance, some wrapped branes can develop a Freed-Witten anomaly so that they should actually not be present in the D-dimensional theory. All these effects are taken into account by the description in terms of cobordism and K-theory rather than (co)homology.

5.2 Application of the LSAHSS to cobordism

In this section, we employ our first spectral sequence, the homological variant of the LSAHSS to compute cobordism groups $\Omega^{\xi}_n(X)$ for non-trivial k-dimensional spaces X. We are considering the trivial fibration $\mathrm{pt} \to X \to X$, such that the LSAHSS to determine $\Omega^{\xi}_n(X)$ needs just the homology of our set of background spaces and the known cobordism groups of the point given in table 5.1 as input. The second page of

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega^{\mathrm{Spin}}_n(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$
$\Omega^{\mathrm{Spin}^c}_n(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	0	$4\mathbb{Z} \oplus \mathbb{Z}_2$

Table 5.1: Spin and Spin^c cobordism groups of the point up to $n = 10$.

the the LSAHSS is given by

$$E^2_{p,q} = H_p(X; \Omega^\xi_q) \,. \tag{5.11}$$

To avoid cluttering the expressions, in the remainder of this section we use the shorthand notation $\Omega^\xi_n(\mathrm{pt}) \equiv \Omega^\xi_n$. Note that we will only show the parts of the pages with $p, q \leq 10$, as this is sufficient to study the manifolds of interest for physical applications.

5.2.1 Computing $\Omega^\xi_n(S^k)$

Before passing to higher-dimensional spheres, we start with the straightforward, yet illustrative, computation of $\Omega^\xi_n(S^2)$. We present here the case where $\xi = \mathrm{Spin}$, while the similarly computed results for $\xi = \mathrm{Spin}^c$ follow from the general formula (G.1) proved in the the appendix G.

While a direct computation of $H_p(S^2, \Omega^{\mathrm{Spin}}_q)$ is straightforward for low q, in general one turns to the universal coefficient theorem (see for instance [121]), according to which there is a short exact sequence

$$0 \to H_n(S^2;\mathbb{Z}) \otimes \Omega^{\mathrm{Spin}}_q \to H_n(S^2;\Omega^{\mathrm{Spin}}_q) \to \mathrm{Tor}_1(H_{n-1}(S^2;\mathbb{Z}), \Omega^{\mathrm{Spin}}_q) \to 0. \tag{5.12}$$

Recalling the well known homology groups

$$H_n(S^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

and the fact that $H_n(S^2;\mathbb{Z})$ is torsion-free, (5.11) directly evaluates as

$$E^2_{p,q} = H_p(S^2;\Omega^{\mathrm{Spin}}_q) \cong H_p(S^2;\mathbb{Z}) \otimes \Omega^{\mathrm{Spin}}_q = \begin{cases} \Omega^{\mathrm{Spin}}_q & \text{for } p = 0, 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

Hence, the second page of the LSAHSS takes the following form pictured in 5.1.

We see that in the degree range we are considering there exist four differentials that could kill some of the page entries. However, they all end on the first column of the page and thus they vanish according to the edge homomorphism reviewed in section 25 in accordance with the Splitting Principle (4.27). Thus, one can immediately conclude that $E^2_{p,q} \cong E^3_{p,q}$. From the third page, no differentials can act on the

10	$\Omega_{10}^{\text{Spin}}$	0	$\Omega_{10}^{\text{Spin}}$	0	0	0
9	Ω_{9}^{Spin}	0	Ω_{9}^{Spin}	0	0	0
8	Ω_{8}^{Spin}	0	Ω_{8}^{Spin}	0	0	0
7	Ω_{7}^{Spin}	0	Ω_{7}^{Spin}	0	0	0
6	Ω_{6}^{Spin}	0	Ω_{6}^{Spin}	0	0	0
5	Ω_{5}^{Spin}	0	Ω_{5}^{Spin}	0	0	0
4	Ω_{4}^{Spin}	0	Ω_{4}^{Spin}	0	0	0
3	Ω_{3}^{Spin}	0	Ω_{3}^{Spin}	0	0	0
2	Ω_{2}^{Spin}	0	Ω_{2}^{Spin}	0	0	0
1	Ω_{1}^{Spin}	0	Ω_{1}^{Spin}	0	0	0
0	Ω_{0}^{Spin}	0	Ω_{0}^{Spin}	0	0	0
	0	1	2	3	4	5

$\Rightarrow$

10	$3\mathbb{Z}_2$	0	$3\mathbb{Z}_2$	0	0	0
9	$2\mathbb{Z}_2$	0	$2\mathbb{Z}_2$	0	0	0
8	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	0	0
7	0	0	0	0	0	0
6	0	0	0	0	0	0
5	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0
1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
	0	1	2	3	4	5

Figure 5.1: Second (and final) page of LSAHSS for $\Omega_n^{\text{Spin}}(S^2)$.

page elements, as its degree would be larger than any possible difference of degree between non-zero elements of the page. Therefore, $E^2_{p,q} \cong E^\infty_{p,q}$ and we arrive at the results in table 5.2.[1]

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z},\mathbb{Z}_2)$	$\mathbb{Z}_2$	$e(\mathbb{Z}_2,\mathbb{Z})$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$e(2\mathbb{Z},3\mathbb{Z}_2)$

Table 5.2: Cobordism groups $\Omega_n^{\text{Spin}}(S^2)$, $n = 0, \dots, 10$, up to extensions.

Let us now tackle the extension problems one by one. Our main tools are briefly reviewed in appendix E.

- $e(\mathbb{Z},\mathbb{Z}_2)$: We have $\text{Ext}^1(\mathbb{Z},\mathbb{Z}_2) = 0$ and thus there is only the trivial extension, $e(\mathbb{Z},\mathbb{Z}_2) = \mathbb{Z} \oplus \mathbb{Z}_2$.
- $e(\mathbb{Z}_2,\mathbb{Z})$: We have from (E.8) that $\text{Ext}^1(\mathbb{Z}_2,\mathbb{Z}) = \mathbb{Z}_2$. The two possible extensions are $\mathbb{Z}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}$, so we need some additional input to select the ap-

[1] We denote by $e(A,B)$ the extension of A by B as already established in our mathematical tools chapter. It should be stressed that there isn't a uniform convention, sometimes the opposite convention is used e.g. in [151].

propriate one. One simple strategy would be to use the splitting lemma (4.27), which tells us that $\Omega_4^{\text{Spin}}(S^2)$ should contain a factor $\Omega_4^{\text{Spin}} = \mathbb{Z}$. Unfortunately such factor is present in both extension options, so we cannot draw any conclusion. In appendix G, we show (indirectly) that for $\Omega_n^{\xi}(S^k)$ the extension is always trivial, therefore even in this case $e(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.

- $e(2\mathbb{Z}, 3\mathbb{Z}_2)$: We have $\text{Ext}^1(2\mathbb{Z}, 3\mathbb{Z}_2) = 2\text{Ext}^1(\mathbb{Z}, 3\mathbb{Z}_2) = 5\text{Ext}^1(\mathbb{Z}, \mathbb{Z}_2) = 0$, so the trivial extension must be chosen, in accordance with the general proof of appendix G.

We summarize our findings in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$2\mathbb{Z} \oplus 3\mathbb{Z}_2$

Table 5.3: Cobordism groups $\Omega_n^{\text{Spin}}(S^2)$.

The calculation of $\Omega_n^{\text{Spin}}(S^k)$ for higher k proceeds similarly. Since the only non-vanishing homology classes are $H_0(S^k; \mathbb{Z}) = H_k(S^k; \mathbb{Z}) = \mathbb{Z}$ and the universal coefficient theorem applies, the second page for the trivial fibration pt $\to S^k \to S^k$ looks very similar to the one for S^2, with the non-vanishing entries along the $p = 0, k$ columns. The only possibly non-vanishing differentials are d_k, but since they end on the first column they vanish due to the edge homomorphism. Hence, the computation proceeds exactly as before. For S^1 the computation is even simpler, since for degree reasons no differential can act. As explained at the beginning of the present section, the fact that $\pi_1(S^1) \neq 0$ does not concern us since we are using a trivial fibration.

For the computation of the Spin^c cobordism groups $\Omega_n^{\text{Spin}^c}(S^k)$ one follows similar steps. Now the second page is

$$E^2_{p,q} = H_p(S^k; \Omega_q^{\text{Spin}^c}) \cong H_p(S^k; \mathbb{Z}) \otimes \Omega_q^{\text{Spin}^c} = \begin{cases} \Omega_q^{\text{Spin}^c} & \text{for } p = 0, k, \\ 0 & \text{otherwise,} \end{cases} \tag{5.15}$$

and the same arguments as for the $\Omega_n^{\text{Spin}}(S^k)$ computation still go through. As proven in appendix G, for both structures $\xi = \text{Spin}, \text{Spin}^c$ the final result can be compactly written as

$$\Omega_n^{\xi}(S^k) = \Omega_n^{\xi}(\text{pt}) \oplus \Omega_{n-k}^{\xi}(\text{pt})\,. \tag{5.16}$$

5.2.2 Computing $\Omega_n^\xi(T^2)$

For the two-torus, $T^2 = S^1 \times S^1$, we present the computation for both $\xi = \text{Spin}$ and $\xi = \text{Spin}^c$ in parallel. Starting from the known homology groups (recall the Betti numbers of the torus $b_0 = b_2 = 1, b_1 = 2$)

$$H_n(T^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ 2\mathbb{Z} & \text{for } n = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{5.17}$$

and using the universal coefficient theorem again (with vanishing Tor_1 group), one can compute the second page

$$E^2_{p,q} = H_p(T^2;\Omega_q^\xi) \cong H_p(T^2;\mathbb{Z}) \otimes \Omega_q^\xi = \begin{cases} \Omega_q^\xi & \text{for } p = 0, 2, \\ 2\Omega_q^\xi & \text{for } p = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{5.18}$$

The second pages for the two structures $\xi = \text{Spin}, \text{Spin}^c$ are shown in figure 5.2. For the Spin case we have four differentials which could be non-trivial, but they vanish due to the edge homomorphism for the trivial fibration. For the Spin^c case, no differential can act for degree reasons. Hence, the second pages are in fact the final pages and we have the results displayed in table 5.4, where we stick to the notation $e(A, B, C) = e(A, e(B, C))$.

Two facts are crucial to solve the extension problem for these cobordism groups. First, the extensions of all free Abelian groups are trivial. Second, $e(m\mathbb{Z}, n\mathbb{Z}_k) = m\mathbb{Z} \oplus n\mathbb{Z}_k$ since $\text{Ext}^1(m\mathbb{Z}, n\mathbb{Z}_k) = 0$. However, since $\text{Ext}^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, we cannot conclude anything about $e(\mathbb{Z}_2, \mathbb{Z}_2)$, which is either $2\mathbb{Z}_2$ or $\mathbb{Z}_4$. A similar story applies for $e(\mathbb{Z}_2, \mathbb{Z})$. Up to this point, our results are shown in table 5.5. According to the general proof given in appendix G, the remaining extension problems should be trivial. Indeed, there we generically show that the cobordism groups of k-dimensional tori have a simple decomposition,

$$\Omega_n^\xi(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} \Omega_{n-m}^\xi(\text{pt}), \tag{5.19}$$

for a generic structure ξ, which can be taken to be Spin or Spin^c. The binomial coefficient can be interpreted as the number of m-cycles on T^k.

10	$3\mathbb{Z}_2$	$6\mathbb{Z}_2$	$3\mathbb{Z}_2$	0	0	0
9	$2\mathbb{Z}_2$	$4\mathbb{Z}_2$	$2\mathbb{Z}_2$	0	0	0
8	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0	0	0
7	0	0	0	0	0	0
6	0	0	0	0	0	0
5	0	0	0	0	0	0
4	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
0	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	0	0
	0	1	2	3	4	5

10	$4\mathbb{Z} \oplus \mathbb{Z}_2$	$8\mathbb{Z} \oplus 2\mathbb{Z}_2$	$4\mathbb{Z} \oplus \mathbb{Z}_2$	0	0
9	0	0	0	0	0
8	$4\mathbb{Z}$	$8\mathbb{Z}$	$4\mathbb{Z}$	0	0
7	0	0	0	0	0
6	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0	0
5	0	0	0	0	0
4	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0	0
3	0	0	0	0	0
2	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	0
	0	1	2	3	4

Figure 5.2: Second (and final) pages of LSAHSS for $\Omega_n^{\mathrm{Spin}}(T^2)$ (above) and $\Omega_n^{\mathrm{Spin}^c}(T^2)$ (below).

As we have just seen things become more difficult the more structure the background manifolds we employ have. For this reason we have to deploy some more mathematical tools for the cobordism groups of $K3$ and CY_3. Due to the absence of low-dimensional torsional groups in the $spin^c$-case and the distribution of non-

n	0	1	2	3	4
$\Omega_n^{\mathrm{Spin}}(T^2)$	$\mathbb{Z}$	$e(2\mathbb{Z}, \mathbb{Z}_2)$	$e(\mathbb{Z}, 2\mathbb{Z}_2, \mathbb{Z}_2)$	$e(\mathbb{Z}_2, 2\mathbb{Z}_2)$	$e(\mathbb{Z}_2, \mathbb{Z})$
$\Omega_n^{\mathrm{Spin^c}}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z}$	$e(\mathbb{Z}, \mathbb{Z})$	$2\mathbb{Z}$	$e(\mathbb{Z}, 2\mathbb{Z})$

n	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(T^2)$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$e(4\mathbb{Z}, 2\mathbb{Z}_2)$	$e(2\mathbb{Z}, 4\mathbb{Z}_2, 3\mathbb{Z}_2)$
$\Omega_n^{\mathrm{Spin^c}}(T^2)$	$4\mathbb{Z}$	$e(2\mathbb{Z}, 2\mathbb{Z})$	$4\mathbb{Z}$	$e(2\mathbb{Z}, 4\mathbb{Z})$	$8\mathbb{Z}$	$e(4\mathbb{Z}, 4\mathbb{Z} \oplus \mathbb{Z}_2)$

Table 5.4: Cobordism groups $\Omega_n^{\mathrm{Spin}}(T^2)$ and $\Omega_n^{\mathrm{Spin^c}}(T^2)$, $n = 0, \ldots, 10$, up to extensions.

n	0	1	2	3	4
$\Omega_n^{\mathrm{Spin}}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z} \oplus \mathbb{Z}_2$	$e(\mathbb{Z}, 2\mathbb{Z}_2, \mathbb{Z}_2)$	$e(\mathbb{Z}_2, 2\mathbb{Z}_2)$	$e(\mathbb{Z}_2, \mathbb{Z})$
$\Omega_n^{\mathrm{Spin^c}}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$3\mathbb{Z}$

n	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(T^2)$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$4\mathbb{Z} \oplus 2\mathbb{Z}_2$	$e(2\mathbb{Z}, 4\mathbb{Z}_2, 3\mathbb{Z}_2)$
$\Omega_n^{\mathrm{Spin^c}}(T^2)$	$4\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}$	$6\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z} \oplus \mathbb{Z}_2$

Table 5.5: Cobordism groups $\Omega_n^{\mathrm{Spin}}(T^2)$ and $\Omega_n^{\mathrm{Spin^c}}(T^2)$, $n = 0, \ldots, 10$.

trivial entries in even degrees leads to easier computations, where we can show that the differentials vanish. Therefore, we start for both cases with $Spin^c$ to familiarize ourselves with the problem and then go to the more complicated *spin* case.

5.2.3 Computing $\Omega_n^{\xi}(K3)$

For the determination of the $spin^c$ cobordism groups of $K3$ we again start with the known result for $H_n(K3; \mathbb{Z})$.

$$H_n(K3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 4, \\ 22\mathbb{Z} & \text{for } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.20}$$

h!

10	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0	$88\mathbb{Z}\oplus 22\mathbb{Z}_2$	0	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0
9	0	0	0	0	0	0
8	$4\mathbb{Z}$	0	$88\mathbb{Z}$	0	$4\mathbb{Z}$	0
7	0	0	0	0	0	0
6	$2\mathbb{Z}$	0	$44\mathbb{Z}$	0	$2\mathbb{Z}$	0
5	0	0	0	0	0	0
4	$2\mathbb{Z}$	0	$44\mathbb{Z}$	0	$2\mathbb{Z}$	0
3	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0
1	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0
	0	1	2	3	4	5

Figure 5.3: Second (and final) page of the LSAHSS for the computation of $\Omega_n^{\mathrm{Spin}^c}(K3)$.

where the non-vanishing Betti numbers of $K3$ are $b_0 = b_4 = 1$, $b_2 = 22$. Once again using the trivial fibration and the universal coefficient theorem we compute the second page entries shown in figure 5.3. For Spin^c all differentials are trivial for degree reason, so that we can conclude $E^2_{p,q} = E^\infty_{p,q}$ with

$$E^2_{p,q} = H_p(K3;\Omega_q^{\mathrm{Spin}^c}) \cong H_p(K3;\mathbb{Z})\otimes\Omega_q^{\mathrm{Spin}^c} = \begin{cases} \Omega_q^{\mathrm{Spin}^c} & \text{for } p=0,4, \\ 22\,\Omega_q^{\mathrm{Spin}^c} & \text{for } p=2, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.21}$$

We depict the second page explicitly in 5.3. Up to $n = 10$ all extension problems are trivial, so that we can express the final result as

$$\begin{aligned} \Omega_n^{\mathrm{Spin}^c}(K3) &= \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \,\oplus\, \tilde{\Omega}_n^{\mathrm{Spin}^c}(K3) \\ &= \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \,\oplus\, 22\,\Omega_{n-2}^{\mathrm{Spin}^c}(\mathrm{pt}) \,\oplus\, \Omega_{n-4}^{\mathrm{Spin}^c}(\mathrm{pt})\,. \end{aligned} \tag{5.22}$$

In this formula, it is understood that cobordism groups with negative index are set to zero. The explicit groups resulting from the formula above are reported in table 5.6.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}^c}(K3)$	$\mathbb{Z}$	0	$23\mathbb{Z}$	0	$25\mathbb{Z}$	0	$47\mathbb{Z}$	0	$50\mathbb{Z}$	0	$94\mathbb{Z} \oplus \mathbb{Z}_2$

Table 5.6: Cobordism groups $\Omega_n^{\mathrm{Spin}^c}(K3)$, $n = 0, \ldots, 10$.

On the flip side, we can have a look at the second page for the LSAHSS to compute $\Omega^{Spin}(K3)$:

10	$3\mathbb{Z}_2$	0	$66\mathbb{Z}_2$	0	$3\mathbb{Z}_2$	0
9	$2\mathbb{Z}_2$	0	$44\mathbb{Z}_2$	0	$2\mathbb{Z}_2$	0
8	$2\mathbb{Z}$	0	$44\mathbb{Z}$	0	$2\mathbb{Z}$	0
7	0	0	0	0	0	0
6	0	0	0	0	0	0
5	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0
	0	1	2	3	4	5

Figure 5.4: Second page of the LSAHSS for the computation of $\Omega_n^{\mathrm{Spin}}(K3)$ with a potentially non-trivial differential.

As we can understand from the second page of this spectral sequence 5.4 there are potentially a few non-trivial differentials and extension problems. So instead of working through these one by one we are going to utilize a homotopy decomposition theorem for the suspension of closed, smooth, connected, orientable four dimensional manifolds, which have only torsion pieces $T = \bigoplus_{j=1}^{n} \mathbb{Z}_{k_j}$ with k_j an odd prime in their homology groups [177]. There is also a complementary, related result for manifolds with only 2-torsion in their integral homology groups [178]. For our purposes we are going to focus on just the case, where M is spinnable. This might appear overly specific, but we know that a $K3$ fulfills exactly these criteria. In fact it is one of the simplest examples of such a manifold. Then the theorem states the following:

Theorem 5.2.1 (So-Theriault [177]). *Given a closed, smooth, connected, spinnable 4-manifold M with an integral homology*

$$H_n(M,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } n=0,4, \\ b_1\mathbb{Z}\oplus T & \textit{for } n=1, \\ b_2\mathbb{Z}\oplus T & \textit{for } n=2, \\ b_1\mathbb{Z} & \textit{for } n=3, \\ 0 & \textit{otherwise}, \end{cases} \tag{5.23}$$

b_p the p-th Betti number, the (reduced) suspension of M has the following homotopy equivalence:

$$\Sigma M \simeq \left(\bigvee_{i=1}^{b_1}(S^2\vee S^4)\right)\vee\left(\bigvee_{j=1}^{n}(P^3(\mathbb{Z}_{k_j})\vee P^5(\mathbb{Z}_{k_j}))\right)\vee\left(\bigvee_{l=1}^{b_2}(S^3)\right)\vee S^5\,, \tag{5.24}$$

where $P^n(\mathbb{Z}_{k_j})$ is the n-dimensional Moore space of $\mathbb{Z}_{k_j}$.[2]

We can now nicely apply this theorem to the reduced $spin$ cobordism $\tilde{\Omega}^{Spin}_n(K3) = \tilde{\Omega}^{Spin}_{n+1}(\Sigma K3)$. Since we have no torsion and $b_1 = 0$ in the integral homology groups of $K3$ (5.20), we get the simple decomposition that combined with our previous results for the $spin$ cobordism of spheres yields:

$$\begin{aligned} \tilde{\Omega}^{Spin}_{n+1}(\Sigma K3) &\simeq \tilde{\Omega}^{Spin}_{n+1}(\bigvee_{l=1}^{22}(S^3)\vee S^5) \\ &= 22\,\tilde{\Omega}^{Spin}_{n+1}(S^3)\oplus\tilde{\Omega}^{Spin}_{n+1}(S^5) = 22\,\Omega^{Spin}_{n-2}\oplus\Omega^{Spin}_{n-4} \end{aligned} \tag{5.26}$$

Using the splitting lemma we get the same result as for $spin^c$-structure, namely:

$$\Omega^{Spin}_n(K3) = \Omega^{Spin}_n\oplus\tilde{\Omega}^{Spin}_n(K3) = \Omega^{Spin}_n\oplus 22\,\Omega^{Spin}_{n-2}\oplus\Omega^{Spin}_{n-4}\,. \tag{5.27}$$

[2] The Moore space $P^n(G)$ for an Abelian group G is defined through its homology groups, such that:

$$\begin{aligned} H_i(P^n(G),\mathbb{Z}) &\simeq G \quad \text{for } i=n, \\ \tilde{H}_i(P^n(G),\mathbb{Z}) &= 0 \quad \forall\, i\neq n. \end{aligned} \tag{5.25}$$

Since our result for the spheres hold for any structure ξ, this result generalizes to $\Omega^{\xi}_{n}(K3)$ and thereby shows our result for $spin^c$ to hold for higher dimensions, as well.

It also means that there are no non-trivial differentials or extensions for the LSAHSS 5.4 we have considered before.

Next, we are going to work through the computation of $\Omega^{\xi}_{n}(CY_3)$ in a similar manner. First, we look at the $spin^c$-case for which the LSAHSS significantly easier and then utilize a homotopy decomposition theorem to obtain the final result in the $spin$ case.

5.2.4 Computing $\Omega^{\xi}_{n}(CY_3)$

The computation for the cobordism groups of a Calabi-Yau threefold are obtained similarly to those of $K3$. We start from the known result[3]

$$H_n(CY_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,6, \\ b_2\,\mathbb{Z} & \text{for } n = 2,4, \\ b_3\,\mathbb{Z} & \text{for } n = 3, \\ 0 & \text{otherwise}, \end{cases} \tag{5.28}$$

where again b_p are the CY_3 Betti numbers (with $b_p = b_{6-p}$). The second page is then given by

$$\begin{aligned} E^2_{p,q} &= H_p(CY_3;\Omega^{\mathrm{Spin^c}}_q) \\ &\cong H_p(CY_3;\mathbb{Z}) \otimes \Omega^{\mathrm{Spin^c}}_q = \begin{cases} \Omega^{\mathrm{Spin^c}}_q & \text{for } p = 0,6, \\ b_2\,\Omega^{\mathrm{Spin^c}}_q & \text{for } p = 2,4, \\ b_3\,\Omega^{\mathrm{Spin^c}}_q & \text{for } p = 3, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5.29}$$

and shown explicitly in figure 5.5. One realises that this time five non-vanishing columns $E^2_{p,q}$ exist in the second page, the elements of which are given by $b_p\,\Omega^{\mathrm{Spin^c}}_q$.

[3] By assumption, the Calabi-Yau threefolds we consider in this work are such that $\pi_1(CY_3) = 0$. In general, there exist Calabi-Yau manifolds with $\pi_1(CY_3) = \mathbb{Z}_n$, for some integer n, i.e. with torsion in $H^1(CY_3;\mathbb{Z})$. Typical examples are free quotient of Calabi-Yau n-folds without torsion, such as the free quotient of the quintic $\mathbb{P}_4[5]/\mathbb{Z}_5$. They have been investigated, especially in a K-theory context, for instance in [149, 179]. For Calabi-Yau twofolds, one has $\pi_1(K3) = 0$. Taking a free quotient by $\mathbb{Z}_n$ reduces the Euler number to $\chi/24n$, so that the quotient manifold is not $K3$ anymore.

10	$4\mathbb{Z} \oplus \mathbb{Z}_2$	0	$b_2(4\mathbb{Z} \oplus \mathbb{Z}_2)$	$b_3(4\mathbb{Z} \oplus \mathbb{Z}_2)$	$b_2(4\mathbb{Z} \oplus \mathbb{Z}_2)$	0	$4\mathbb{Z} \oplus \mathbb{Z}_2$	0
9	0	0	0	0	0	0	0	0
8	$4\mathbb{Z}$	0	$4b_2\mathbb{Z}$	$4b_3\mathbb{Z}$	$4b_2\mathbb{Z}$	0	$4\mathbb{Z}$	0
7	0	0	0	0	0	0	0	0
6	$2\mathbb{Z}$	0	$2b_2\mathbb{Z}$	$2b_3\mathbb{Z}$	$2b_2\mathbb{Z}$	0	$2\mathbb{Z}$	0
5	0	0	0	0	0	0	0	0
4	$2\mathbb{Z}$	0	$2b_2\mathbb{Z}$	$2b_3\mathbb{Z}$	$2b_2\mathbb{Z}$	0	$2\mathbb{Z}$	0
3	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
	0	1	2	3	4	5	6	7

Figure 5.5: Second (and final) page of the LSAHSS for the computation of $\Omega_n^{\mathrm{Spin}^c}(CY_3)$. One of the possibly non-vanishing differentials $d^3 : E^3_{6,q} \to E^3_{3,q+2}$ is displayed (for $q = 0$). They eventually vanish for $q \leq 6$.

None of the differentials d_r with even r can act for degree reasons. However, there are two kinds of third differentials that can be non-trivial. The first class is

$$d^3 : E^3_{3,q} \to E^3_{0,q+2} \,, \tag{5.30}$$

which vanish due to the edge homomorphism (see section 25). The second class acts as

$$d^3 : E^3_{6,q} \to E^3_{3,q+2} \,, \tag{5.31}$$

which is in principle non-vanishing.[4]. That this differential is trivial up to $q = 6$, too, follows from Lemma 3.1 of [181]. We thus get the results in table 5.7.

For *spin*-cobordism we first look at the second page of the LSAHSS $E^2_{p,q} = H_p(CY_3; \Omega_q^{\mathrm{Spin}})$ to get an overview of the situation. Straightforwardly, we obtain the second page shown in figure 5.6.

Based on [182] the first differentials can be classified as follows:

[4] This differential is given by the homological dual of the cohomology operation $Sq^3_{\mathbb{Z}}$, the (integral) third Steenrod square (the operations Sq^i are introduced briefly later on; see also the appendix C). Interestingly, its triviality is the homological dual statement of the Freed-Witten anomaly cancellation [148, 180], which we are going to discuss later on in the K-theory calculations.

n	0	1	2	3	4	5
$\Omega_n^{\mathrm{Spin^c}}(CY_3)$	$\mathbb{Z}$	0	$(b_2+1)\mathbb{Z}$	$b_3\mathbb{Z}$	$(2+2b_2)\mathbb{Z}$	$b_3\mathbb{Z}$

n	6	7	8	9	10
$\Omega_n^{\mathrm{Spin^c}}(CY_3)$	$(3+3b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$	$(5+4b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$	$(6+6b_2)\mathbb{Z}\oplus\mathbb{Z}_2$

Table 5.7: Cobordism groups $\Omega_n^{\mathrm{Spin^c}}(CY_3)$, $n = 0, \ldots, 10$.

10	$3\mathbb{Z}_2$	0	$3b_2\mathbb{Z}_2$	$3b_3\mathbb{Z}_2$	$3b_2\mathbb{Z}_2$	0	$3\mathbb{Z}_2$	0
9	$2\mathbb{Z}_2$	0	$2b_2\mathbb{Z}_2$	$2b_3\mathbb{Z}_2$	$2b_2\mathbb{Z}_2$	0	$2\mathbb{Z}_2$	0
8	$2\mathbb{Z}$	0	$2b_2\mathbb{Z}$	$2b_3\mathbb{Z}$	$2b_2\mathbb{Z}$	0	$2\mathbb{Z}$	0
7	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
3	0	0	0	0	0	0	0	0
2	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
	0	1	2	3	4	5	6	7

Figure 5.6: Second page of the LSAHSS for the computation of $\Omega_n^{Spin}(CY_3)$. We display two kinds of possibly non-vanishing differentials: $d^2 : E^2_{4,0} \to E^2_{2,1}$ and $d^2 : E^2_{4,1} \to E^2_{2,2}$.

- The differential $d_2 : H_p(X, \Omega_1^{Spin}) \to H_{p-2}(X, \Omega_2^{Spin})$ is the dual of the second Steenrod square $Sq^2 : H^{p-2}(X, \mathbb{Z}_2) \to H^p(X, \mathbb{Z}_2)$.

- The differential $d_2 : H_p(X, \Omega_0^{Spin}) \to H_{p-2}(X, \Omega_1^{Spin})$ is the dual of the second Steenrod square composed with reduction mod 2 $Sq^2\rho : H^{p-2}(X, \mathbb{Z}) \to H^p(X, \mathbb{Z}_2)$.

In general these differentials are tedious to evaluate. Moreover, there are likely extension problems. Therefore, we bypass this problem again by exploiting a theorem determining the homotopy type of twice suspended simply connected, closed and orientable 6-manifolds [183]. Like for the theorem for 4-manifolds, we will be content with the more stringent requirement that the 6-manifold is spinnable. Our torsion-free Calabi-Yau threefold will be a simple example of such a manifold. The statement goes as follows:

Theorem 5.2.2 (Huang [183]). *Consider a closed, smooth, connected, spinnable 6-manifold M with an integral homology*

$$H_n(M,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } n = 0, 6, \\ b_2\mathbb{Z} \oplus T & \textit{for } n = 2, \\ b_3\mathbb{Z} \oplus T & \textit{for } n = 3, \\ b_2\mathbb{Z} & \textit{for } n = 4, \\ 0 & \textit{otherwise}, \end{cases} \tag{5.32}$$

where b_p is the p-th Betti number and T is a finitely generated Abelian torsion group. Then depending on an integer $0 \le c \le b_2$, determined by the specific second Steenrod square $Sq^2 : H^2(X,\mathbb{Z}_2) \to H^4(X,\mathbb{Z}_2)$, the double (reduced) suspension of M has one of the following homotopy equivalences:

- *For $c = 0$*

$$\Sigma^2 M \simeq \Sigma W_0 \vee \bigvee_{i=1}^{b_2-1} (S^4 \vee S^6) \vee \bigvee_{j=1}^{b_2} (S^5) \vee (P^5(T) \vee P^6(T)) \,, \tag{5.33}$$

  where $W_0 \simeq (S^3 \vee S^5) \cup e^7$.

- *For $c = b_2$*

$$\Sigma^2 M \simeq \Sigma W_{b_2} \vee \bigvee_{i=1}^{b_2-1} \Sigma^2\mathbb{CP}_2 \vee \bigvee_{j=1}^{b_2} (S^5) \vee (P^5(T) \vee P^6(T)) \,, \tag{5.34}$$

  where $W_{b_2} \simeq \Sigma\mathbb{CP}_2 \cup e^7$.

- *For $1 \le c \le b_2$*

$$\Sigma^2 M \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2\mathbb{CP}_2 \vee \bigvee_{j=1}^{b_2-c-1} (S^4 \vee S^6) \vee \bigvee_{l=1}^{b_2} (S^5) \vee (P^5(T) \vee P^6(T)) \,, \tag{5.35}$$

  where $W_c \simeq (S^3 \vee S^5 \vee \Sigma\mathbb{CP}_2) \cup e^7$.

Applying this to *spin*-cobordism is fairly straightforward. We already know *spin*-cobordism for the spheres. The 7-cell e^7 can be understood from our results for the spheres as well since an n-sphere can be understood as attaching an n-cell to a 0-cell. From decomposing our sphere result, i.e. $\Omega_n^{Spin}(S^k) = \Omega_n^{Spin}(e^0 \cup e^k) = \Omega_n^{Spin} \oplus \Omega_{n-k}^{Spin}$, we can read of the outcome of attaching an n-cell. Finally, we need to determine $\mathbb{CP}_2$. Since $\mathbb{CP}_2$ is not *spin*, we can not simply use the homotopy decomposition 5.2.1. We would need to use its generalization, which is not as straightforward. However, we can simply compute $\tilde{\Omega}_n^{Spin}(\mathbb{CP}_2)$ with the Adams Spectral Sequence. As highlighted before (4.34) the Anderson-Brown-Peterson decomposition theorem gives us a way to obtain *spin* cobordism groups through connective *ko*-homology[5]. To compute the latter we utilize the Adams Spectral Sequence:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_1}^{s,t}(H^*(\mathbb{CP}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow ko_{t-s}(\mathbb{CP}_2)_{\hat{2}}\,. \tag{5.36}$$

Luckily, for us $\mathrm{Ext}_{\mathcal{A}_1}^{s,t}(H^*(\mathbb{CP}_2, \mathbb{Z}_2), \mathbb{Z}_2)$ is absolutely standard and we can just read it off from for example [145]. We will just picture the reduced part in figure 5.7 as we can easily add in the rest with the splitting lemma.

Since differentials can only end on nodes in the column right next to theirs in the Adams Spectral Sequence, we are done and can read off the results.

$$\widetilde{ko}_n(\mathbb{CP}_2) \simeq \mathbb{Z} \quad \forall n \geq 2 \text{ and even.} \tag{5.37}$$

Because $ko_{0,1}\langle 2\rangle(X)_{\hat{2}} \simeq H_{0,1}(X, \mathbb{Z}_2)$ [161] and the first new entry compared to $X = pt$ of $H_n(\mathbb{CP}_2, \mathbb{Z}_2)$ at $n = 2$, we only get the standard contribution
$ko_0\langle 2\rangle(pt)_{\hat{2}} \simeq \mathbb{Z}_2$ in $n = 10$. We get:

[5] This of course is only true for *spin* cobordism localized at 2. This is enough however, as one can observe from the LSAHSS. Since the integral homology groups of $\mathbb{CP}_2$ only contain free Abelian pieces at $n = 0, 2$ and 4, there is no way to generate odd torsion. This is because differentials are group homomorphisms and therefore cannot generate odd torsion mapping from or to $\mathbb{Z}_2$. The only remaining possibility would be through differentials between free Abelian pieces, but there are none due to the even spacing of $\mathbb{Z}$s.

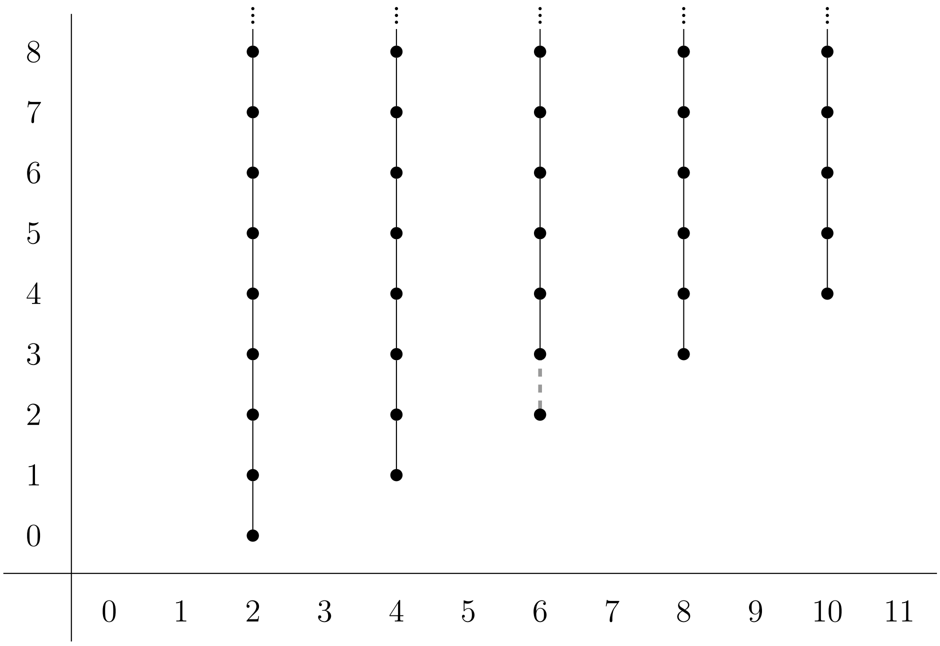

Figure 5.7: Second (and final) page of Adams Spectral Sequence for $\widetilde{ko}_*(\mathbb{CP}_2)_{\hat{2}}$, where we highlighted the non-trivial extension in degree 6.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{Spin}(\mathbb{CP}_2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	$\mathbb{Z}$	0	$3\mathbb{Z}$	$2\mathbb{Z}_2$	$2\mathbb{Z}\oplus 3\mathbb{Z}_2$
$\tilde{\Omega}_n^{Spin}(\mathbb{CP}_2)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$

Table 5.8: Reduced and unreduced Cobordism groups $\tilde{\Omega}_n^{Spin}(\mathbb{CP}_2)$ and $\Omega_n^{Spin}(\mathbb{CP}_2)$, $n = 0, \dots, 10$.

Returning to the decomposition theorem 5.2.2, let's insert the decomposition (we take $1 \le c \le b_2 - 1$) into the reduced *spin* cobordism groups:

$$\begin{aligned}\tilde{\Omega}_n^{Spin}(CY_3) &\simeq \tilde{\Omega}_{n+2}^{Spin}(\Sigma^2 CY_3) \simeq \tilde{\Omega}_{n+1}^{Spin}(W_c) \,\oplus\, (c-1)\,\tilde{\Omega}_n^{Spin}(\mathbb{CP}_2) \\ &\quad \oplus (b_2 - c - 1)\,(\tilde{\Omega}_{n+2}^{Spin}(S^4) \,\oplus\, \tilde{\Omega}_{n+2}^{Spin}(S^6)) \,\oplus\, b_3\,\tilde{\Omega}_{n+2}^{Spin}(S^5) \\ &\simeq \tilde{\Omega}_n^{Spin}(\mathbb{CP}_2) \,\oplus\, \Omega_{n-2}^{Spin} \,\oplus\, \Omega_{n-4}^{Spin} \,\oplus\, \Omega_{n-6}^{Spin} \,\oplus\, (c-1)\,\tilde{\Omega}_n^{Spin}(\mathbb{CP}_2) \\ &\quad \oplus (b_2 - c - 1)\,(\Omega_{n-2}^{Spin} \,\oplus\, \Omega_{n-4}^{Spin}) \,\oplus\, b_3\,\Omega_{n-3}^{Spin}\,.\end{aligned} \tag{5.38}$$

To keep it concise here we omit the other decompositions, the final result with c running over its full value range $0 \le c \le b_2$ reads:

n	0	1	2	3	4
$\Omega_n^{Spin}(CY_3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$b_2\mathbb{Z} \oplus \mathbb{Z}_2$	$(b_2 - c)\mathbb{Z}_2 \oplus b_3\mathbb{Z}$	$(b_2+1)\mathbb{Z}$ $\oplus(b_2 + b_3 - c)\mathbb{Z}_2$

n	5	6	7
$\Omega_n^{Spin}(CY_3)$	$(b_2 - c + b_3)\mathbb{Z}_2$	$(b_2+1)\mathbb{Z} \oplus (b_2 - c)\mathbb{Z}_2$	$b_3\mathbb{Z} \oplus \mathbb{Z}_2$

n	8	9	10
$\Omega_n^{Spin}(CY_3)$	$(b_2+2)\mathbb{Z} \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2$	$(2b_2+1)\mathbb{Z} \oplus 3\mathbb{Z}_2$

Table 5.9: Cobordism groups $\Omega_n^{Spin}(CY_3)$, $n = 0, \ldots, 10$.

From the result we see nicely the imprint of the second differential equivalent to the second Steenrod square Sq^2. To our knowledge the number c is not known to be constrained for general Calabi-Yau threefolds. As a final remark we want to highlight the strange behavior of the torsional piece in $(b_2 - c)\mathbb{Z}_2 \subset \Omega_6^{Spin}(CY_3)$. From the second page of the LSASS we can observe that the reduction by $c\,\mathbb{Z}_2$ groups is not caused by a differential as there is none starting or ending on $E^r_{4,2}$. Instead, the reason is a non-trivial extension $\mathbb{Z}_2 \to e(\mathbb{Z}, \mathbb{Z}_2) \to \mathbb{Z}$ solved by $e(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}$. By using the homotopy decomposition we mapped this extension problem into the calculation $\tilde{\Omega}_6^{Spin}(\mathbb{CP}_2)$. Indeed, it does have a non-trivial extension in degree 6! So instead of having $\tilde{\Omega}_6^{Spin}(\mathbb{CP}_2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$, which would lead to the full $b_2\mathbb{Z}_2$ piece in $\tilde{\Omega}_6^{Spin}(CY_3)$, we get $\tilde{\Omega}_6^{Spin}(\mathbb{CP}_2) \simeq \mathbb{Z}$. We will observe a related non-trivial extension in $KO^0(CY_3)$.

5.3 Application of the LSAHSS to K-theory

Next, we perform similar computations for the K- and KO-theory groups of spheres, tori and Calabi-Yau manifolds. For this purpose we employ the cohomological version of the LSAHSS. Real K-theory turns out to be more involved similar to the related spin cobordism groups as consequence of its torsion groups. Nevertheless, we manage to fully compute these groups. The starting point for the LSAHSS are the K-theory groups of the point, which we list again below in table 5.10. We point out the characteristic Bott periodicity.

n	0	1	2	3	4	5	6	7	8	9	10
$KO^{-n}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$K^{-n}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 5.10: K- and KO-groups of the point up to $n = 10$.

5.3.1 Computing $K^{-n}(S^k)$

The K-theory groups of spheres S^k are known to be [184]

$$K^{-n}(S^k) = \begin{cases} \mathbb{Z} & \text{for } k \text{ odd,} \\ 2\mathbb{Z} & \text{for } n, k \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.39}$$

but it is instructive to reproduce these results using the cohomological LSAHSS (4.72). As usual, we use the trivial fibration pt $\to S^k \to S^k$ and we do not have to worry about local coefficients. Recalling that

$$K^{-n}(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.40}$$

we have the second page

$$E_2^{p,q} = H^p(S^k; K^q(\mathrm{pt})) = \begin{cases} \mathbb{Z}, & \text{for } q \text{ even}, p = 0, k, \\ 0, & \text{otherwise}\,. \end{cases} \tag{5.41}$$

Note that it is essential to include the bottom quadrant (with $q < 0$) to arrive at reasonable results. Limiting our spectral sequence to the first quadrant only, as in the homological case, is not consistent as it would violate Bott periodicity.

For concreteness, let us consider $X = S^3$. We are interested in the groups $K^{-n}(X)$, with $n > 0$, so the relevant page elements lie on the $p + q = -n$ bands of the final page, which now intersect the axes only once. The d_2 differential vanish so that $E_3^{p,q} = E_2^{p,q}$, but d_3 may act non-trivially

$$d_3 : E_3^{0,q} \to E_3^{3,q-2}, \qquad q \text{ even}\,. \tag{5.42}$$

This differential was found by Atiyah and Hirzebruch [185] to be an instance of a cohomological operation known as the integral Steenrod square ($Sq^i_{\mathbb{Z}}$)

$$Sq^3_{\mathbb{Z}} : H^n(X;\mathbb{Z}) \to H^{n+3}(X;\mathbb{Z}). \tag{5.43}$$

6	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
5	0	0	0	0	0
4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
3	0	0	0	0	0
2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-1	0	0	0	0	0
-2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-5	0	0	0	0	0
-6	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0

Figure 5.8: Second (and final) page of the LSAHSS for the computation of $K^{-n}(S^3)$. One of the d_3 differentials is shown explicitly. They all eventually vanish.

Explicitly, it is given by the composition

$$d_3 = Sq^3_{\mathbb{Z}} = \beta \circ Sq^2 \circ \rho, \tag{5.44}$$

where ρ is the reduction modulo 2 and β the Bockstein homomorphism, namely

$$Sq^3_{\mathbb{Z}} : H^n(X;\mathbb{Z}) \xrightarrow{\rho} H^n(X;\mathbb{Z}_2) \xrightarrow{Sq^2} H^{n+2}(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+3}(X;\mathbb{Z}). \tag{5.45}$$

We refer the reader to the appendix C for a more precise definition of Steenrod squares and of the Bockstein homomorphism, together with a short summary of their main properties.

Fortunately, since no torsion is involved, according to Theorem 4.8 of [186] all differentials (including d_3) vanish. This fact will be used systematically in the other computations of $K^{-n}(X)$ groups below.[6] Moreover, the extension problem is always

[6] This is a consequence of the Chern isomorphism

$$K^0(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigoplus_n H^{2n}(X;\mathbb{R}), \qquad K^{-1}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigoplus_n H^{2n+1}(X;\mathbb{R}), \tag{5.46}$$

which implies that if there is no torsion in cohomology, the LSAHSS for K-theory terminates already at the second page.

trivial, since only free Abelian groups are present. Thus, for every odd value of k we recover $K^{-n}(S^{2k+1}) = \mathbb{Z}$. The situation for even k is simpler as for degree reasons no differentials can act, so that $E_2^{p,q} = E_\infty^{p,q}$. We recover then $K^{-2n-1}(S^{2k}) = 0$ and $K^{-2n}(S^{2k}) = 2\mathbb{Z}$. Notice that the final result can be expressed as

$$K^{-n}(S^k) = K^{-n}(\mathrm{pt}) \oplus K^{-k-n}(\mathrm{pt})\,. \tag{5.47}$$

5.3.2 Comment on Freed-Witten anomalies

Let us comment more on the role of $d_3 = Sq^3_{\mathbb{Z}}$ and on its physical consequences, beyond the computation of $K^{-n}(S^k)$. From [187], it is known that type II D-branes (in absence of the B field) must wrap a Spin^c manifold Y, otherwise there is a global Freed–Witten anomaly. Given an element $y \in H^n(X;\mathbb{Z})$, one has (see appendix C)

$$Sq^3_{\mathbb{Z}}(y) = W_3(N) \cup y, \tag{5.48}$$

where N is the normal bundle of the codimension n submanifold Poincaré dual to y, which we call Y below, while $\cup$ is the cup product. Since $W_3(N) = 0$, iff Y is Spin^c,[7] one can relate a trivial action of d_3 in the LSAHSS to the absence of Freed–Witten anomalies for a D-brane wrapping Y [148,180]. Indeed, if $E^4 = \ker d_3/\mathrm{Im}\, d_3$ is given in terms of the groups $H^n(X;\mathbb{Z})$ without further restrictions, all cohomology classes (and their dual cycles) survive. Otherwise, either some are removed when passing from cohomology to K-theory or they change to a torsion group [148]. Physically, they would correspond to D-branes which are anomalous or unstable.

5.3.3 Computing $K^{-n}(T^k)$

Next we consider the k-dimensional torus $T^k = (S^1)^k$. To proceed, one can either compute the groups by using the LSAHSS in a similar manner as done for the sphere (extending also the page to include the fourth quadrant) or use the known results for the reduced K-theory groups $\widetilde{K}^{-n}(T^k)$ and the decomposition (4.7).

Starting with the second approach, we observe that according to [184] we have

$$\widetilde{K}^{-n}(T^k) = \begin{cases} 2^{k-1}\mathbb{Z} & \text{for } n \text{ odd}, \\ (2^{k-1}-1)\mathbb{Z} & \text{for } n \text{ even}. \end{cases} \tag{5.49}$$

[7] The obstruction to Spin^c structure on Y is really $W_3(Y) = \beta(w_2(Y))$. However, since in our case X is Spin and Y is oriented by assumption (in type II), one can show that $w_2(N) = w_2(Y)$, implying $W_3(N) = W_3(Y)$ [65, 187].

Since $K^{-2n}(\mathrm{pt}) = \mathbb{Z}$ and $K^{-2n-1}(\mathrm{pt}) = 0$, it follows that

$$K^{-n}(T^k) = 2^{k-1}\mathbb{Z}, \tag{5.50}$$

for n any integer. For the trivial case $k = 1$, where the torus is just a circle, the above result coincides with the expected one from the sphere computation, i.e. $K^{-n}(T^1) = \mathbb{Z}$.

Let us also comment on the calculation of $K^{-n}(T^k)$ using the spectral sequence. One has the second page

$$E_2^{p,q} = H^p(T^k; K^q(\mathrm{pt}))\,. \tag{5.51}$$

The computation using the LSAHSS for the trivial fibration gives the same result (5.50), upon realising that once again all differentials vanish since there is no torsion, so $E_2^{p,q} = E_\infty^{p,q}$, and the extension problem is trivial. We note that the final result can be elegantly written as

$$K^{-n}(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} K^{-m-n}(\mathrm{pt})\,, \tag{5.52}$$

where the binomial coefficient can be interpreted as the number of m-cycles on T^k.

5.3.4 Computing $K^{-n}(K3)$

The LSAHSS also allows to straightforwardly compute the K-theory groups on $K3$. The second page of the sequence is given by

$$E_2^{p,q} = H^p(K3; K^q(\mathrm{pt})) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 4,\, q \text{ even,} \\ 22\,\mathbb{Z} & \text{for } p = 2,\, q \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.53}$$

This is explicitly shown in figure 5.9. It is evident that no differentials can act non-trivially on the second page for degree reasons so that the sequence promptly terminates. Thus, the final result reads

$$K^{-n}(K3) = \begin{cases} 0 & \text{for } n \text{ odd,} \\ 24\mathbb{Z} & \text{for } n \text{ even.} \end{cases} \tag{5.54}$$

Note that the factor 24 arises as $b_0 + b_2 + b_4 = 1 + 22 + 1$ with b_m being the Betti numbers of $K3$. Therefore, we can also express the K-theory groups on K3 as

$$K^{-n}(K3) = \bigoplus_{m=0}^{4} b_{4-m}(K3)\, K^{-m-n}(\mathrm{pt})\,. \tag{5.55}$$

6	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
5	0	0	0	0	0
4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0
2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-1	0	0	0	0	0
-2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0
-6	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$0\mathbb{Z}$

Figure 5.9: Second (and final) page of the LSAHSS for the computation of $K^{-n}(K3)$.

5.3.5 Computing $K^{-n}(CY_3)$

The computation of $K^{-n}(CY_3)$ is similar to that of $K3$. Omitting unnecessary details, we present directly the second page in figure 5.10. The only possibly non-vanishing differential is $d_3 : E_3^{1,q} \to E_3^{4,q-2}$. However, due to lack of torsion it is in fact vanishing and, given also that the extension problem is trivial, we conclude that

$$K^{-n}(CY_3) = \begin{cases} b_3\,\mathbb{Z} & \text{if n odd,} \\ (2+2b_2)\,\mathbb{Z} & \text{if n even.} \end{cases} \tag{5.56}$$

Notice the factor $(2+2b_2)$ arises as $b_0+b_2+b_4+b_6$, with $b_0 = b_6 = 1$ and $b_2 = b_4$, b_p being the Betti numbers of the CY_3. The result can also be found in Corollary 1.9 of [188]. Again, we can elegantly express the K-theory groups on (simply connected) Calabi-Yau threefolds as

$$K^{-n}(CY_3) = \bigoplus_{m=0}^{6} b_{6-m}(CY_3)\, K^{-m-n}(\mathrm{pt})\,. \tag{5.57}$$

6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
5	0	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-1	0	0	0	0	0	0	0
-2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-3	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0	0	0
-6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$

Figure 5.10: Second (and final) page of LSAHSS for computation of $K^{-n}(CY_3)$.

5.3.6 KO-groups of spheres and tori

The KO groups can similarly be computed using the LSAHSS. However, in this case there is torsion, so the differentials can be non-vanishing. For spheres S^k, one can use the splitting lemma and determine the relevant groups as

$$KO^{-n}(S^k) = \widetilde{KO}(S^{n+k}) \oplus \widetilde{KO}(S^n) = KO^{-n-k}(\text{pt}) \oplus KO^{-n}(\text{pt})\,. \tag{5.58}$$

For tori, it was shown in [189] that

$$KO^{-n}(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} KO^{-m-n}(\text{pt})\,. \tag{5.59}$$

5.3.7 Computing $KO^{-n}(K3)$

For real K-theory, computations involving higher dimensional manifolds with a more complex topology than the torus or the sphere become more complicated, due to more involved differentials and extension problems. Indeed, for Calabi-Yau threefolds the computation turned out to be fairly subtle. However, in the case of $K3$, as we will show now, all differentials are vanishing and the computations can be performed, up to extensions. The second page of the spectral sequence is the following:

$$E_2^{p,q} = H^p(K3; KO^q(\text{pt})) = \begin{cases} KO^q(\text{pt}) & \text{for } p = 0, 4, \\ 22\, KO^q(\text{pt}) & \text{for } p = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.60}$$

We realise that the differentials d_2 and d_4 (depending on how d_2 acts) can possibly

7	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
6	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
5	0	0	0	0	0
4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-1	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-2	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0
-6	0	0	0	0	0
-7	0	0	0	0	0

Figure 5.11: Second (and final) page of the LSAHSS for the computation of $KO^{-n}(K3)$.

be non-vanishing. At second degree, we have

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}, \tag{5.61}$$

for $p = 0, 2$ and $q = 0, -1$, together with all of its periodic copies. The explicit form of this differential is known to be [190, 191]

$$d_2 = \begin{cases} Sq^2\rho : & H^p(K3; KO^0(\text{pt})) \to H^{p+2}(K3; KO^{-1}(\text{pt}))\,, \\ Sq^2 : & H^p(K3; KO^{-1}(\text{pt})) \to H^{p+2}(K3; KO^{-2}(\text{pt})) \end{cases} \tag{5.62}$$

corresponding to $q = 0, -1$ respectively. Here, $Sq^2 : H^p(X;\mathbb{Z}_2) \to H^{p+2}(X;\mathbb{Z}_2)$ is the second Steenrod square and ρ is the reduction modulo 2. We argue now that d_2 is vanishing for $X = K3$. We discuss the case $q = -1$, but the analysis can be extended to $q = 0$ in a similar way. For any element $y \in H^p(X;\mathbb{Z}_2)$, we can represent $Sq^2(y) = \iota_*(w_2(N)) \cup y$ [192]. Here, N is the normal bundle of the submanifold $Y \subset X$ Poincaré dual to y and $\iota_* : H^p(Y) \to H^p(X)$ the cohomological push-forward. For $p = 0$, the differential d_2 vanishes since y is dual to the whole four dimensional manifold $X = K3$ which is Spin, thus $w_1(N) = w_2(N) = 0$. Alternatively, it vanishes since $Sq^2(y) = 0$ for $y \in H^0(X;\mathbb{Z}_2)$ (see the properties of Sq^i listed in appendix C). For $p = 2$, the differential vanishes as well, since from the condition $w_2(X) = w_1(X) = 0$ (i.e. X is Spin), one can then prove $w_2(N) = 0$ for a two dimensional manifold Y not necessarily orientable [180]. Alternatively, for $p = 2$ we can also write $Sq^2(y) = \nu_2 \cup y$ (see equation (C.9)) and then the second Wu class, $\nu_2 = w_2(X) + w_1(X)^2$, vanishes since $X = K3$ is Spin. Thus, d_2 is trivial.

At degree four, we have the differential

$$d_4 : E_4^{0,-1} \to E_4^{4,-4}. \tag{5.63}$$

However, since there cannot be non-trivial homomorphisms[8] $\mathbb{Z}_k \to \mathbb{Z}$ for $k \geq 2$ also this differential must vanish and $E_2^{p,q} \cong E_\infty^{p,q}$.

Thus, one can read off the $KO^{-n}(K3)$ groups, which we present in table 5.11. Note that we have already made use of the splitting lemma (4.7) to simplify the results. There is a remaining extension problem for which we deploy the decomposition theorem 5.2.1 we already used for the *spin* cobordism groups of $K3$. This yields the complete splitting of $KO^0(K3)$ as well, since we get:

$$\widetilde{KO}^1(\Sigma K3) \simeq 22\widetilde{KO}^1(S^3) \oplus \widetilde{KO}^1(S^5) \simeq 22\mathbb{Z}_2 \oplus \mathbb{Z}. \tag{5.64}$$

[8] This can be seen directly as follows. Consider the case $\phi : \mathbb{Z}_2 \to \mathbb{Z}$, the generalization to $k > 2$ being straightforward. ϕ cannot be a non-trivial homomorphism since, choosing $\phi(0) = 0$ and $\phi(1) = 1$, one is lead to the contradiction $0 = \phi(0) = \phi(2) = \phi(1) + \phi(1) = 2$. Thus, the only option is to set also $\phi(1) = 0$ and ϕ is trivial.

n	0	1	2	3	4	5	6	7
$KO^{-n}(K3)$	$2\mathbb{Z} \oplus 22\mathbb{Z}_2$	$\mathbb{Z}_2$	$22\mathbb{Z} \oplus \mathbb{Z}_2$	0	$2\mathbb{Z}$	$\mathbb{Z}_2$	$22\mathbb{Z} \oplus \mathbb{Z}_2$	$22\mathbb{Z}_2$

Table 5.11: KO-groups $KO^{-n}(K3)$, $n = 0, \ldots, 7$. The result can be extrapolated to $n \geq 8$ by Bott periodicity.

5.3.8 Computing $KO^{-n}(CY_3)$

Finally, we come to the most complicated KO-theory computation we attempt in this chapter, which is fairly similar to the one for $\Omega_n^{Spin}(CY_3)$. To get some intuition for what is happening let us look at the second page of the corresponding LSASS:

0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
-1	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
-2	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
-3	0	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
-5	0	0	0	0	0	0	0	0
-6	0	0	0	0	0	0	0	0
-7	0	0	0	0	0	0	0	0
-8	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
-9	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
-10	$\mathbb{Z}_2$	0	$b_2\mathbb{Z}_2$	$b_3\mathbb{Z}_2$	$b_2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
-11	0	0	0	0	0	0	0	0
-12	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
-13	0	0	0	0	0	0	0	0

Figure 5.12: Second page of the LSAHSS for the computation of $KO^{-n}(CY_3)$.

The first non-trivial differentials d_2 are again given by (5.62).

To circumvent the explicit evaluation of all the differentials and extension problems we utilize the decomposition theorem 5.2.2. We may write:

$$\begin{aligned}\widetilde{KO}^{-n}(CY_3) &\simeq \widetilde{KO}^{-n+2}(\Sigma^2 CY_3) \simeq \widetilde{KO}^{-n+1}(W_c) \oplus (c-1)\,\widetilde{KO}^{-n}(\mathbb{CP}_2) \\ &\quad \oplus (b_2-c-1)\,(\widetilde{KO}^{-n+2}(S^4) \oplus \widetilde{KO}^{-n+2}(S^6)) \oplus b_3\,\widetilde{KO}^{-n+2}(S^5) \\ &\simeq KO^{-n-6} \oplus c\,\widetilde{KO}^{-n}(\mathbb{CP}_2) \oplus (b_2-c)\,(KO^{-n-2} \oplus KO^{-n-4}) \\ &\quad \oplus b_3\,KO^{-n-3}\,.\end{aligned} \tag{5.65}$$

By applying the results of [193] for the reduced KO-theory groups of $\mathbb{CP}_2$

$$\widetilde{KO}^{-n}(\mathbb{CP}_2) \simeq \begin{cases} \mathbb{Z} & \text{for } n \text{ even}\,, \\ 0 & \text{for } n \text{ odd} \end{cases} \tag{5.66}$$

we are lead to the following result for $KO^{-n}(CY_3)$ 5.12:

n	0	1	2	3	4
$KO^{-n}(CY_3)$	$(b_2+1)\mathbb{Z}$ $\oplus(b_2-c)\mathbb{Z}_2$	$b_3\mathbb{Z}\oplus\mathbb{Z}_2$	$(b_2+1)\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$(b_2+1)\mathbb{Z}\oplus\mathbb{Z}_2$

n	5	6	7
$KO^{-n}(CY_3)$	$b_3\mathbb{Z}\oplus(b_2-c)\mathbb{Z}_2$	$b_2\mathbb{Z}\oplus(b_2-c+b_3)\mathbb{Z}_2$	$(b_2-c+b_3)\mathbb{Z}_2$

Table 5.12: KO-theory groups $KO^{-n}(CY_3)$, $n = 0,\ldots,7$, all other groups follow from Bott periodicity.

Here, we want to highlight a few interesting observations on the above results. First, we see the imprint of the second Steenrod square $Sq^2 : H^2(CY_3,\mathbb{Z}_2) \to H^4(CY_3,\mathbb{Z}_2)$ directly determining the second differential of (5.62) captured by the integer c. However, the presence of c is not just explained by d_2. For instance, $KO^0(CY_3)$ contains a $(b_2-c)\mathbb{Z}_2$ subgroup[9], but there is no differential acting on $E_r^{2,-2}$. Instead, this is due to a non-trivial extension. Under the decomposition we see that this extension problem is answered through the outcome for $\widetilde{KO}^0(\mathbb{CP}_2)$. From writing down the second page of the LSAHSS for $KO^{-n}(\mathbb{CP}_2)$ one gets a very analogous extension problem. But we know the eventual answer from [193]: $\widetilde{KO}^{-n}(\mathbb{CP}_2)$

[9] We should point out that our result for $KO^0(CY_3)$ deviates from the one reported in [183]. This deviation stems from different $\widetilde{KO}^0(\mathbb{CP}_2)$. [183] assumes $\widetilde{KO}^0(\mathbb{CP}_2) \simeq \mathbb{Z}\oplus\mathbb{Z}_2$, which contradicts the result of [193].

has only free Abelian entries, so the extension of $e(\mathbb{Z}_2, \mathbb{Z})$ is the non-trivial one: $e(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}$.

5.4 Physical interpretation

In this section, we show that the cobordism and K-theory groups of X previously calculated with the LSAHSS can be interpreted in terms of the dimensional reduction of global symmetries, thus making contact with section 5.1. For $X \in \{S^k, T^k, K3\}$, the analysis turns out to be particularly simple, since all the differentials in the LSAHSS vanish and extensions are trivial, as we explicitly showed. For the more involved CY_3 background, these simplifications do not occur, but the LSAHSS (paired with some tricks) gives the correct answer.

We expect the story to become substantially more involved if one turns on fluxes. For instance, allowing for non-trivial NS-NS three-form flux H leads to the computation of H-twisted K-theory groups $K_H^{-n}(X)$ and the corresponding cobordism groups $\Omega^{\mathrm{Spin}^c, H}(X)$. In this case where $W_3 = 0$, i.e. we have a Spin^c-structure, the absence of Freed–Witten anomalies $W_3 + [H] = 0$ implies that the H-flux class $[H]$ through a D-brane must vanish. This will result in non-trivial maps $d_r : E_r^{p,q} \to E^{p+r, q-r+1}$ in the evaluation of the LSAHSS.

5.4.1 General aspects

All of the analyzed examples have in common that the final results of the LSAHSS can be expressed in a convenient, compact manner. For the K-theory groups $K^{-n}(X)$ of a k-dimensional manifold $X \in \{S^k, T^k, K3, CY_3\}$, we have in fact

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X)\, K^{-n-m}(\mathrm{pt})\,, \tag{5.67}$$

with $n \geq 0$. The interpretation of this result in terms of D-branes is as follows. Say we are in $d = 10$ dimensions and compactify the theory on the k-dimensional manifold X, so that the total space is $\mathbb{R}^{1,d-k-1} \times X$. Then, $K^{-n}(X)$ classifies all type II D-branes that are of codimension n in the flat space $\mathbb{R}^{1,d-k-1}$. From the d-dimensional point of view, these are given by the set of all codimension $n + m$ branes wrapping $(k - m)$-cycles on the compact space X. Hence, the result (5.67) just reflects that the dimensional reduction performed following this perhaps naive geometrical reasoning is already the correct answer on these manifolds. Nevertheless, thanks to the LSAHSS we also learn that in the complex K-theory case none of the wrapped Dp-branes experiences a Freed–Witten anomaly nor that there is an instantonic decay-channel or a non-trivial Dp-brane fusion.

The relation (5.67) has a nice connection to the Completeness Hypothesis 3.2. The right hand side of (5.67) is indeed a lattice of charges $(\mathbf{q_1}, \ldots, \mathbf{q_k})$, where each entry $\mathbf{q}_m$ is a charge vector with b_{k-m} components.[10] The fact that they can and indeed they are populated independently of one another means that in general the full spectrum of charges (or rather stable states with that given charge) is complete. To understand the point, consider the simple two-dimensional situation in which the lattice is just $\mathbb{Z} \oplus \mathbb{Z}$. In this case, one not only has stable bound states of branes associated to say $(1,0)$ and $(0,1)$, but also to $(1,1)$. Thus, what the relation (5.67) is telling us is that to any non-vanishing element $(\mathbf{q_1}, \ldots, \mathbf{q_k})$ must be associated a stable object and, in this sense, the spectrum is complete. In general, especially in the presence of multicharged or non-BPS branes, the situation might become highly involved, but K-theory should give the correct answer.

For cobordism groups we found an analogous result, namely that for $n \geq 0$ they can also be expressed as

$$\Omega^{\mathrm{Spin}^c}_{n+k}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X)\, \Omega^{\mathrm{Spin}^c}_{n+m}(\mathrm{pt})\,. \tag{5.68}$$

The case $-k \leq n < 0$ will be discussed later. We propose the following intuitive interpretation of this result. First, recall that in the definition of $\Omega_n(X)$ one introduces continuous maps $f : M \to X$, for every n-dimensional compact manifold M, such that $[M, f] \in \Omega_n(X)$. A non-vanishing term labeled by m in the sum on the right hand side indicates that the map $f : M \to X$ from the $(n+k)$-dimensional manifold M into the k-dimensional manifold X is such that it wraps M around a non-trivial $(k-m)$-cycle of X, while no other obstruction is introduced by the map in the remaining $(n+m)$ directions of M. Since there are b_{k-m} different $(k-m)$-cycles on X, we get b_{k-m} factors of $\Omega^{\mathrm{Spin}^c}_{n+m}(\mathrm{pt})$ in the total cobordism group $\Omega^{\mathrm{Spin}^c}_{n+k}(X)$.

[10] For other generalized (co)homology theories this would be slightly inaccurate, since the groups of the point might be actually direct sums, so one of them could correspond to more sites in the lattice. Here, we neglect this complication for the sake of simplicity. The analysis can be directly adapted.

Taking into account that the objects charged under the cobordism groups $\Omega_n(\mathrm{pt})$ are the $(d-n)$-dimensional gravitational solitons mentioned in section 3.3, one can provide a similar interpretation as for the K-theory groups. Accordingly, $\Omega^{\mathrm{Spin}^c}_{n+k}(X)$ classifies all gravitational solitons that are of codimension n in the flat space $\mathbb{R}^{1,d-k-1}$. From the d-dimensional point of view, they are given by the set of all codimension $(n+m)$ objects wrapping $(k-m)$-cycles on the compact space X.

Concretely, defining a basis $\{\Sigma^a_m\}$ of m-cycles on X, with $a = 1, \dots, b_m(X)$, and taking into account that $\Omega^{\mathrm{Spin}^c}_{\mathrm{even}}(\mathrm{pt}) = \mathbb{Z}$, for a given m-charge vector

$$\mathbf{q_m} = (q^1_m, \dots, q^{b_m}_m) \in b_m\mathbb{Z}, \tag{5.69}$$

the map f is such the $(n+k)$-dimensional manifold M_{n+k} is wrapped q^a_m times around the m-cycle Σ^a_m of X. Hence, one can think of such an m-cycle to be shared between M and X.

For all values of the index $n+k$, our goal is to explain how to organize the information contained in K-theory and cobordism groups of X and then reconstruct tadpole cancellation conditions known from string theory. As we will see, for $n \geq 0$ this is quite straightforward, whereas in the regime $-k \leq n < 0$ we will encounter some new issues. We thus assume $n \geq 0$ for the time being. Given the previous results, we can understand how the ABS-map applies to cobordism and K-theory groups of manifolds X which are not just a point. Via the relation (4.43)

$$K^{-n}(X) = K_{n+k}(X), \tag{5.70}$$

valid for X a k-dimensional Spin^c manifold, the K-theory result (5.67) can be formally brought into the same form as (5.68), namely we can pass from generalized cohomology to homology. Therefore (for $n \geq 0$) the ABS orientation can be extended to a map

$$\alpha^c_X : \Omega^{\mathrm{Spin}^c}_{n+k}(X) \to K_{n+k}(X), \tag{5.71}$$

acting as Td in (4.39) on each term $\Omega^{\mathrm{Spin}^c}_{n+k-m}(\mathrm{pt})$. Dividing by the kernel of this map provides an isomorphism between cobordism and K-theory classes on X. Hence, at least for these simple cases, the latter isomorphism is directly inherited from the isomorphism between $\Omega^{\mathrm{Spin}^c}_n(\mathrm{pt})$ and $K_n(\mathrm{pt})$. As we have shown, the LSAHSS gives analogous simple results for the Spin cobordism groups $\Omega^{\mathrm{Spin}}_{n+k}(X)$ and the real K-theory classes $KO_{n+k}(X)$ for $X \in \{S^k, T^k, K3\}$. This implies that the above structure carries over to such cases, as well.

For $X = CY_3$, we have seen that the result crucially depends on c. To still denote the result in this compact form, we can introduce some "generalized" Betti numbers $\tilde{b}_{k-m}(CY_3)$, such that whenever $KO^{-1,-2 \bmod 8}(pt)$ would naively be paired with a b_2/b_4, we replace it by $\tilde{b}_2 = (b_2 - c)$. For low-dimensional *spin*-cobordism groups $n \leq 10$ we can essentially do the analogous replacement. However at higher degrees, when $ko\langle 2\rangle_{n-10}(X)$ and higher correction terms in the ABP decomposition (4.34) become impactful, this replacement likely doesn't hold up in this simple form. Overall, the interpretation we have put forward for the simpler K-theory and $spin^c$ cobordism groups still applies. The main difference is that c cycles become unstable for non-BPS $\widehat{\mathrm{D}p}$-branes, not being able to support the *spin* condition for these branes. Recall, that the KO-homology construction of type I Dp-branes requires them to live on/wrap a *spin* submanifold. As outlined above the $KO^0(CY_3)$ subgroup $(b_2 - c)\mathbb{Z}_2$ is special, since it arises from a non-trivial extension. So seemingly c of the b_2 non-BPS $\widehat{\mathrm{D}7}$ fuse with the b_2 D5-branes. It would be interesting to investigate even more general setups of such fusions. Probably the KO-theory of Moore spaces of some torsional group containing 2-torsion provides such an example where a non-trivial extension of for example $e(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_4$ fuse two separate $\widehat{\mathrm{D}p}$-branes into a single brane with charge valued in $\mathbb{Z}_4$.

We can also give an interpretation in terms of global symmetries. In our examples, the groups $K_{n+k}(X)$ and $\Omega^{\mathrm{Spin}^c}_{n+k}(X)$ classify all global $(D-1-n)$-form charges in the non-compact $D = d - k$ dimensions. These can be thought of as arising from the dimensional reduction of global $d - 1 - n, d - 2 - n, \ldots, d - 1 - k - n$ form charges along the $k, k - 1, \ldots, 0$ cycles of X. Due to the simple underlying structure, it is now clear that the fate of these global symmetries will follow in the simple cases the standard rules of the dimensional reduction, while in more complicated setups global symmetries are lost or modified due to non-trivial topology on certain background manifolds. As already laid out in section 5.1, if a global symmetry in D dimensions descends from a global symmetry in d dimensions, then its gauging involves the dimensionally reduced gauge field in d dimensions and also the corresponding dimensionally reduced Dp-branes (defects). In fact, the whole tadpole cancellation condition in D dimensions arises from the dimensional reduction of the tadpole cancellation condition in d dimensions. We will provide more concrete examples in section 5.4.2.

5.4.2 Example of a Calabi-Yau threefold

Let us now focus on the case of the ten-dimensional type IIB superstring compactified on a Calabi-Yau threefold X. In the previous section, we computed

$$K^0(X) = b_6 \underbrace{K^0(\text{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{K^{-2}(\text{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{K^{-4}(\text{pt})}_{\mathbb{Z}} \oplus b_0 \underbrace{K^{-6}(\text{pt})}_{\mathbb{Z}}, \tag{5.72}$$

with $b_0 = b_6 = 1$. The corresponding Dp-branes are all of codimension zero in the flat $\mathbb{R}^{1,3}$ space. In particular, in subsequent order, the four types of (single charged) Dp-branes corresponding to the $K^{-2n}(\text{pt})$ groups are: D9-branes wrapping the entire CY_3, D7-branes wrapping the b_4 4-cycles of the CY_3, D5-branes wrapping the b_2 2-cycles of the CY_3 and finally D3-branes being point-like on the CY_3. At the next level, we found

$$K^{-1}(X) = b_3 \underbrace{K^{-4}(\text{pt})}_{\mathbb{Z}}, \tag{5.73}$$

corresponding to a codimension one brane in $\mathbb{R}^{1,3}$ and given by D5-branes wrapping any of the b_3 three-cycles on the CY_3. As already explained, for *all* multi-charges there should exist corresponding bound states of the single charged states. This is consistent with the Completeness Hypothesis (see section 3.2).

As we have inferred from the LSAHSS, the corresponding cobordism groups split in a very similar manner

$$\Omega_6^{\text{Spin}^c}(X) = b_6 \underbrace{\Omega_0^{\text{Spin}^c}(\text{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{\Omega_2^{\text{Spin}^c}(\text{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{\Omega_4^{\text{Spin}^c}(\text{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} \oplus b_0 \underbrace{\Omega_6^{\text{Spin}^c}(\text{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} \tag{5.74}$$

and

$$\Omega_7^{\text{Spin}^c}(X) = b_3 \underbrace{\Omega_4^{\text{Spin}^c}(\text{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} . \tag{5.75}$$

As mentioned, this pattern is related to dimensional reduction of global symmetries. Indeed, from

$$\Omega_6^{\text{Spin}^c}(X) = \mathbb{Z} \oplus b_4\mathbb{Z} \oplus b_2(\mathbb{Z}\oplus\mathbb{Z}) \oplus (\mathbb{Z}\oplus\mathbb{Z}) \tag{5.76}$$

we infer that there is a $3b_2 + 3$ dimensional lattice of $\mathbb{Z}$-valued global 3-form charges in $\mathbb{R}^{1,3}$ (recall $b_4 = b_2$). These are the dimensional reduction of the ten-dimensional

9-form, 7-form, 5-form and 3-form global symmetries along the 6, 4, 2, 0-cycles of the CY_3.

We now explain how to organize the information in the groups above and reconstruct tadpole cancellation conditions. Consider a six-dimensional Spin^c-manifold M_6 that lies in the contribution $b_m \Omega^{\text{Spin}^c}_{6-m}(\text{pt})$ to $\Omega^{\text{Spin}^c}_6(X)$ but, contrary to the Calabi-Yau X, it is not necessarily a solution to the string theory equations of motion. Hence, in this sense, M_6 can be off-shell. Since there must exist a continuous map $f : M_6 \to X$, the manifold M_6 shares some of the m-cycles with the background space X. Which m-cycles are shared depends on the non-zero entries in the charge vector (5.69). Then, the magnetic $(6-m)$-form currents are obtained from the $spin^c$ cobordism invariants

$$\begin{aligned}
\tilde{J}_0(M_6) &= \text{td}_0(M_6)\,, \\
\tilde{J}_2(M_6) &= \text{td}_2(M_6) = \frac{1}{2} c_1(M_6)\,, \\
\tilde{J}_{4,1}(M_6) &= \text{td}_4(M_6) = \frac{1}{12}\left(c_2(M_6) + c_1^2(M_6)\right)\,, \quad \tilde{J}_{4,2}(M_6) = c_1^2(M_6)\,, \\
\tilde{J}_{6,1}(M_6) &= \text{td}_6(M_6) = \frac{1}{24} c_2(M_6)\, c_1(M_6)\,, \qquad \tilde{J}_{6,2}(M_6) = \frac{1}{2} c_1^3(M_6)\,.
\end{aligned} \tag{5.77}$$

Concretely, we propose that the magnetic $(6-m)$-form currents are defined by expanding the right hand sides into a basis of those $(6-m)$-forms in $H^{6-m}(M_6;\mathbb{Z})$ that also lie in $H^{6-m}(X;\mathbb{Z})$ (again depending on the entries in the charge vector). For the Poincaré dual of the currents, denoted with hat, this means that we expand

$$\hat{\tilde{J}}_{m,i}(M_6) = \sum_{a=1}^{b_m} \alpha^a_{m,i}\, q^a_m\, \Sigma^a_m + \ldots \tag{5.78}$$

where the dots indicate more contributions along m-cycles of M_6 that do not lie in X. Note that the (co)homology of M_6 can in principle be bigger than that of X. Since this expansion is also valid for $M_6 \neq X$, it allows us to go slightly off-shell. More general, topological K-theory and cobordism groups classify all global charges that can be present in principle, irrespective of properties like supersymmetry or being on-shell.

Recall that the Todd classes also define the ABS orientation at fixed degree, i.e. $\alpha^c_n(M_6) = \text{Td}_n(M_6)$. Due to this map and the fact that all K-theory global symmetries are gauged as we will explain below, we can infer that at fixed $n = 0, 2, 4, 6$

the coimage of the ABS map is gauged. We can understand this as K-theory Dp-brane tadpole cancellation conditions. This might appear surprising, since Dp-brane K-theory charge that has to be canceled is just the square root of the Todd genus (together with the Chern character of the Chan-Paton bundle, which we shelf for this discussion). The resolution turns out to be quite subtle as the index of the Dirac operator, i.e. the Todd genus in the $spin^c$ case and the $\hat{A}$ genus in the spin case, induces a bilinear pairing between K-homology and K-theory [133, 194]. This is analogous to the bilinear pairing in homology and cohomology between electric and magnetic currents that gives us the anomaly theory

$$I_{d+2} = \int j_e \wedge j_m \,. \tag{5.79}$$

There are many intricacies piling up here, due to the asymmetry with which the pairing and our traditional understanding of electric and magnetic charges attach. In a certain sense we are gauging the global symmetries arising from the cobordism subclasses mapped to K-theory with a combination of electric and magnetic charges. This distinction between electric and magnetic objects arises once we arrive in K-theory and the index becomes a bilinear pairing. In the following, we are taking a simplistic point of view and identify the magnetic Dp-brane as the one responsible for the gauging of the global symmetry. One final general comment we should make before we go into the details is the observation [63] the contribution of Dp-branes and topological invariants to a given tadpole cancellation is indistinguishable and should be identified. So while we tend to treat the Dp-branes as different from the cobordism invariants we attempt to cancel with them, such a distinction is actually not quite correct and should be understood as a merely semantic trick. With all these necessary warnings about the subtleties in place we now discuss below all four classes of global symmetries in turn.

- First, we have $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt}) \xrightarrow{\simeq} K_0(\mathrm{pt}) \xrightarrow{\simeq} K^0(\mathrm{pt})$. The factor $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z}$ gives rise to a single global 3-form symmetry in four dimensions, with the trivial magnetic current $\tilde{J}_0(M_6) = \mathrm{td}_0(M_6)$. In ten dimensions, the corresponding 9-form symmetry is gauged with the charged objects being $D9$-branes, classified by $K^0(\mathrm{pt}) = \mathbb{Z}$. Since the cobordism invariant $\int_{pt} \mathrm{td}_0(M_6) = \sum_p 1 = N_{pt} \in \mathbb{Z}$ the number of points, detects also the K-theory groups $K_0(\mathrm{pt}) \simeq K^0(\mathrm{pt})$, in which the D9-brane charges take value in, we can understand the gauging of $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt})$ as the tadpole constraint:

$$\int_{pts} \mathrm{td}_0(M_6) = N_{D9/\overline{D9}} \in \mathbb{Z} \to N_{D9/\overline{D9}} + N_{\overline{D9}/D9} \int_{pt} \delta^{(0)}(M_6) = 0\,, \tag{5.80}$$

where $\delta^{(0)}(M_6)$ denotes the 0-form Poincaré dual to the 6-cycle M_6 wrapped by the stack of additional N $D9$-branes.[11] This is the familiar $D9$-tadpole cancellation condition in type IIB string theory, where we need a net zero of D9-branes for consistency[12]. Thus, the $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt}) \xrightarrow{\simeq} K^0(\mathrm{pt}) = \mathbb{Z}$ global symmetries is gauged. Breaking this global symmetry would amount to introducing a domain wall in type II string theory as was explored in [113]. While this is understood in type IIA as the Hořava-Witten ETW wall, in the chiral type IIB theory such a codimension one domain wall is currently unknown. (see [195] for a recent discussion of lower dimensional chiral ETW-branes).

- Second, we have $\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^{-2}(\mathrm{pt})$. The factor $b_4\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt}) = b_4\mathbb{Z}$ gives rise to b_4 global 3-form symmetries in four dimensions, whose preserved magnetic 0-form currents $\tilde{j}_0^{(2)a}$ (again in $D = 4$ and with the notation of section 5.1) are given by the expansion of the ten-dimensional 2-form current $\tilde{J}_2(M_6) = \mathrm{td}_2(M_6)$ in a cohomological basis $\omega_{(2)a} \in H^2(X;\mathbb{Z})$, namely

$$\tilde{J}_2(M_6) = \sum_{a=1}^{b_4} \tilde{j}_0^{(2)a} \wedge \omega_{(2)a}\,. \tag{5.81}$$

Note that $b_4 = b^2$, so that this is the Poincaré dual to the expansion (5.78), where we included the charges q_4^a into the coefficients. Similarly, for a $D7$-brane classified by $K^{-2}(\mathrm{pt})$ and wrapping 4-cycles $\Sigma_4 \in H_4(M_6;\mathbb{Z})$ that are contained in X (times the flat space $\mathbb{R}^{1,3}$), we can expand its Poincaré dual 2-form as

$$\delta^{(2)}(\mathbb{R}^{1,3}\times\Sigma_4) = \sum_{a=1}^{b_4} \delta^{(0)}(\mathbb{R}^{1,3})^{(2)a} \wedge \omega_{(2)a}\,. \tag{5.82}$$

In ten dimensions, the gauging of the corresponding 7-form global symmetry is associated to a tadpole constraint, such that we add D7-branes transverse to

[11] Formally, this 0-form arises from the ten-dimensional delta $\delta^{(0)}(\mathbb{R}^{1,3}\times M_6) = \delta^{(0)}(\mathbb{R}^{1,3}) \wedge \delta^{(0)}(M_6)$.

[12] This should be contrasted with type I for which the cobordism group including the $Ss(32)$ background gauge field gives us rank($Ss(32)$)

the 2-dimensional manifold to cancel the initial cobordism/K-theory invariant

$$\sum_{j\in\text{def}} N_j \int_{X_2} \delta^{(2)}(\mathbb{R}^{1,3}\times\Sigma_{4,j}) + \int_{X_2} \frac{c_1(M_6)}{2} = 0 \tag{5.83}$$

which upon expansion in a cohomological basis of $H^2(X;\mathbb{Z}) = b_4\mathbb{Z}$ leads to $b_4 = b^2$ tadpole cancellation conditions. Hence, a subgroup $b_4\mathbb{Z}$ of the initially present global symmetry $b_4\left(\Omega_2^{\text{Spin}^c}(\text{pt})\oplus K^{-2}(\text{pt})\right) = b_4\left(\mathbb{Z}\oplus\mathbb{Z}\right)$ is gauged while the orthogonal $b_4\mathbb{Z}$ group is broken. Of course, for $M_6 = CY_3$ we have $c_1(CY_3) = 0$ and the tadpole cancellation condition simplifies, but the power of K-theory and cobordism is that they allow us to go off-shell and see terms that could appear in principle, even if they are absent for the on-shell configurations.

- Third, we have $\Omega_4^{\text{Spin}^c}(\text{pt})$ and $K^{-4}(\text{pt})$. The summand $b_2\Omega_4^{\text{Spin}^c}(\text{pt}) =$ $b_2\left(\mathbb{Z}\oplus\mathbb{Z}\right)$ gives rise to $2\,b_2$ global 3-form symmetries in four dimensions. Notice that this time the ABS orientation between K-theory and cobordism is not an isomorphism. The preserved magnetic 0-form currents $\tilde{j}^{(4)a}_{0,i}$, $i = 1,2$, in $D = 4$ are given by the expansion of the ten-dimensional 4-form currents $\tilde{J}_{4,i}(M_6)$ in a cohomological basis $\hat{\omega}_{(4)a}$ of $H^4(X;\mathbb{Z})$

$$\tilde{J}_{4,i}(M_6) = \sum_{a=1}^{b_2} \tilde{j}^{(4)a}_{0,i} \wedge \hat{\omega}_{(4)a}\,. \tag{5.84}$$

The defects classified by $K^{-4}(\text{pt})$ are $D5$-branes wrapping 2-cycles $\hat{\Sigma}_2$ on M_6 that are shared with X (times the flat space $\mathbb{R}^{1,3}$). Again their Poincaré duals can be expanded similarly to (5.84). Apart from the natural direct cancellation of the Todd genus contribution by D5-branes, we can also consider the gauging of the diagonal component (sitting in $\mathbb{Z}\oplus\mathbb{Z}$) of the ten-dimensional 5-form symmetry, which would imply a tadpole condition of the form

$$\sum_{j\in\text{def}} N_j \int_{X_4} \delta^{(4)}(\mathbb{R}^{1,3}\times\hat{\Sigma}_{2,j}) + \int_{X_4}\left(\frac{c_2(M_6)+c_1^2(M_6)}{12}\right) + \int_{X_4} c_1^2(M_6) = 0. \tag{5.85}$$

Upon expansion in a cohomological basis of $H^4(X;\mathbb{Z}) = b_2\mathbb{Z}$, one obtains $b_2 = b^4$ tadpole cancellation conditions. Hence, a subgroup $b_2\mathbb{Z}$ of the initially present global symmetry is gauged while the orthogonal group is broken.

So far we have argued that inclusion Dp-branes is responsible for the coimage of the ABS-map. But what about the second invariant? Curiously, this invariant becomes relevant for backgrounds outside of the immediate type IIB realm. A particular example is F-theory on an elliptically-fibered Calabi-Yau threefold with a Hirzebruch surface $\mathbb{F}_n$ [196] as its base. As discussed in [32] this manifold the 4d $spin^c$ cobordism invariants take the values $\int_{\mathbb{F}_n} td_4 = 1$ and $\int_{\mathbb{F}_n} c_1^2 = 8$. These F-theory compactifications involving these curious manifolds are dual to the type I/$Ss(32)$-heterotic string on a $K3$-manifold [197,198]. Since the $K3$ itself also has $c_1 = 0$, the rise of these extra cobordism invariants is only detected by F-theory itself, but not from the string theory perspective. The ability of F-theory to detect cobordism invariants that the dual string theory isn't able to detect appears to be a general phenomenon and would be very interesting to explore further.

- Fourth, we have $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^{-6}(\mathrm{pt})$. The factor $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z} \oplus \mathbb{Z}$ gives rise to 2 global 3-form symmetries in four dimensions, whose preserved magnetic 0-form currents $\tilde{j}^{(6)}_{0,i}$, $i = 1, 2$ (again in $D = 4$) are given by the reduction of the ten-dimensional 6-form currents $\tilde{J}_{6,i}(M_6)$ along the volume 6-form of M_6,
$$\tilde{J}_{6,i}(M_6) = \tilde{j}^{(6)}_{0,i} \ \mathrm{vol}(M_6)\,. \tag{5.86}$$
The defects classified by $K^{-6}(\mathrm{pt})$ are $D3$-branes being point-like on M_6. Again, we skip the gauging of just the coimage of the ABS map and look at the gauging of the diagonal ten-dimensional 3-form symmetry, which would imply a tadpole condition of the general form
$$\sum_{j\in \mathrm{def}} N_j \int_{X_6} \delta^{(6)}(\mathbb{R}^{1,3} \times \mathrm{pt}_j) + \int_{X_6} \frac{c_2(M_6)\, c_1(M_6)}{24} + \int_{X_6} \frac{c_1^3(M_6)}{2} = 0\,. \tag{5.87}$$
Again, for a Calabi-Yau manifold, such as $M_6 = X$, the two contributions from cobordism are vanishing but the off-shell nature of cobordism itself makes them visible in the general case. Interestingly, in F-theory such a configuration seems to be realized in the following manner [199]: The D7-brane in this setup has a worldvolume $\mathbb{R} \times K$, where K is the discriminant of the fibration $N \to B$ with K a complex surface in the base B such that it is valued as $12c_1$.

Then the induced D3-brane charge becomes

$$\int_B (12c_1)\,(c_2(K)/24), \tag{5.88}$$

such that $c_2(K)$ expressed as an invariant of the base instead of the discriminant becomes a linear expression of $c_1^2(B)$ and $c_2(B)$, namely

$$\int_B (12c_1)\,(c_2(K)/24) = \int_B (12c_1)\,(c_2(B) - 30c_1^2(B))\,. \tag{5.89}$$

This is only possible due to the extra structure in F-theory, where we able to differentiate between K and B.

Finally, let us discuss the four-dimensional global 2-form symmetries related to $K^{-1}(X) = b_3K^{-4}(\mathrm{pt}) = b_3\mathbb{Z}$ and $\Omega_7^{\mathrm{Spin}^c}(X) = b_3\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt}) = b_3(\mathbb{Z}\oplus\mathbb{Z})$. From the ten-dimensional perspective, these arise from the reduction of the global 5-form symmetries along the b_3 3-cycles of X. Concerning $\Omega_7^{\mathrm{Spin}^c}(X)$, the $2b_3$ preserved magnetic 1-form currents $\tilde{j}_{1,i}^{(3)a}$, with $i = 1, 2$, in $D = 4$ are given by the dimensional reduction of the ten-dimensional 4-form currents $\tilde{J}_{4,i}(M_6)$ along the basis 3-forms $\omega_{(3)a} \in H^3(X;\mathbb{Z})$,

$$\tilde{J}_{4,i}(M_6) = \sum_{a=1}^{b_3} \tilde{j}_{1,i}^{(3)a} \wedge \omega_{(3)a}\,. \tag{5.90}$$

Note that this is meant in principle and that the currents $\tilde{j}_{1,i}^{(3)a}$ can also be vanishing. The $D5$-brane defects wrapping 3-cycles Σ_3 on M_6 shared with X times a three-dimensional submanifold Π_3 of the flat space $\mathbb{R}^{1,3}$ can be expanded in a similar fashion

$$\delta^{(4)}(\Pi_3\times\Sigma_3) = \sum_{a=1}^{b_3} \delta^{(1)}(\Pi_3)^{(3)a} \wedge \omega_{(3)a}. \tag{5.91}$$

In ten dimensions, the global symmetry of $K^{-4}(\mathrm{pt})$ is gauged leading to a magnetic Bianchi identity

$$d\tilde{F}_3 = \sum_{j\in\mathrm{def}} N_j\,\delta^{(4)}(\Pi_{3,j}\times\Sigma_{3,j}) + a_1^{(4)}\,\tilde{J}_{4,1}(M_6) + a_2^{(4)}\,\tilde{J}_{4,2}(M_6)\,. \tag{5.92}$$

Expanding now also the magnetic field strength as

$$\tilde{F}_3 = \sum_{a=1}^{b_3} \tilde{f}_0^{(3)a} \,\wedge\, \omega_{(3)a}, \tag{5.93}$$

we arrive at b_3 Bianchi identities for the four-dimensional 0-forms

$$d\tilde{f}_0^{(3)a} = \sum_{j\in\mathrm{def}} N_j\,\delta^{(1)}(\Pi_{3,j})^{(3)a} + a_1^{(4)}\,\tilde{j}_{1,1}^{(3)a} + a_2^{(4)}\,\tilde{j}_{1,2}^{(3)a}\,. \tag{5.94}$$

Thus, everything fits nicely together once more. The discussion for higher groups, such as $\Omega_8^{\mathrm{Spin}^c}(X)$ and $\Omega_9^{\mathrm{Spin}^c}(X)$, together with their K-theory counterparts, follows a similar logic, however the kernel of the ABS map becomes a lot bigger the higher we go. It is unclear, if F-theory is enough to account for this.

Summary of results

We demonstrated that, for the example of a Calabi-Yau space X, the K-theory and cobordism classes on X for $n \geq 0$ are to be interpreted from the point of view of global symmetries and their subsequent gauging. In the $spin^c$ situation, the LSAHSS is simple in the sense that no non-trivial maps, i.e. differentials, appear and the outcomes reproduce the naive expectation from dimensional reduction. Of course, a more involved task is to compute K-theory and cobordism classes where maps can be non-trivial and D-branes become inconsistent or unstable. We have demonstrated this in the case of spin cobordism and KO-theory of X, where non-trivial differentials and extensions appear and we had to rely on additional, supplemental mathematics to fully solve the LSAHSS. However, even if in these cases the Dp-brane spectrum for a background space X changes, the map between K-theory and cobordism is proven to be intact, so that the related global symmetries are guaranteed to disappear simultaneously. Therefore, the interpretation in terms of gauging is very similar to what we discussed above.

Our results carry over to the correspondence of KO-groups and Spin-cobordism. A new aspect is the appearance of $\mathbb{Z}_2$ torsion groups related to non-BPS branes on the KO-theory side. Their appearance highlights what we explained before: The cobordism and K-theory groups classify charges, not p-forms, and one has to be careful with statements about the latter. As these torsional charges feature prominently in the next chapter, we do not expedite on them any further here.

Let us also mention that the homotopy decomposition theorems we have used above [177, 178, 183] (see also [200] for 6-manifolds) offer substantial generalization, when the homology of the 4- or 6-manifold under consideration contains torsion. Since this chapter is more about the more common background spaces for string compactifications $K3$ and CY_3, we have left out the homology torsion pieces. However, as demonstrated beautifully by the aforementioned theorems the torsion group T is fully captured by Moore spaces $P^n(T)$. Therefore including torsion of background manifolds compatible with the aforementioned theorems becomes a straightforward objective, namely we have to calculate the cobordism and K-theory groups of Moore spaces. We performed these calculations in appendix H, so one just needs to add these in for a choice of torsion compatible with the theorems (5.2.1) or (5.2.2).

5.4.3 Fate of low-dimensional $\Omega_n^\xi(X)$

It remains to discuss what happens in the regime $-k \leq n < 0$, for which the cobordism groups $\Omega^\xi_{n+k}(X)$ are still non-vanishing. Let us consider $\xi = Spin^c$ for now. The discussion for $\xi = Spin$ (paired with KO-theory) would be very similar. To get a better idea on what is different with respect to the regime $n \geq 0$, we start by asking what the corresponding K-theory groups are, namely $K_{n+k}(X) = K^{-n}(X)$ with $-k \leq n < 0$, and what they physically mean. For concreteness, consider e.g. the class $K^2(CY_3)$. Extrapolating the relation (5.67) to $n = -2$, we would get

$$K^2(X) = \bigoplus_{m=0}^{6} b_{6-m}(X)\, K^{2-m}(\mathrm{pt}) = \boldsymbol{K^2(\mathrm{pt})} \oplus b_4(X)\, K^0(\mathrm{pt}) \oplus \dots\,, \quad (5.95)$$

where we highlight the odd term $K^2(\mathrm{pt})$, associated to $m = 0$. The latter can be defined via Bott periodicity to be equal to $\mathbb{Z}$, but it is not clear what it should represent physically. This is because effectively we are looking at a negative dimensional vector bundle. This fits in with the story on the cobordism side as we are looking at $\Omega_4^{\mathrm{Spin}^c}(X)$ meaning we are considering cobordism equivalence classes with a continuous map from a four dimensional manifold to a six dimensional one. However, looking at the corresponding expansion for $\Omega_4^{\mathrm{Spin}^c}(X)$ there is no matching term with coefficient b_6 as $\Omega_{-2}^{\mathrm{Spin}^c}(pt)$ is vanishing in our definition. Therefore, we could take the point of view that we should only consider the connective version of K-theory with $k^{-n}(pt) = 0$ for $n < 0$ and have cobordism and K-theory match again.

If we do not want to take this route, we have to consider so called derived manifolds [201] to make sense of this mathematically. While we can not give the topic proper justice in this limited subsection, we can at least give some intuition about the map $M_n \to X_k$, which is called an *imbedding*. The rough idea is to define local models $\mathcal{U}$, such that there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}^k$, where $\mathcal{U} \simeq \mathbb{R}^n_{f=0}$. Then we can associate a virtual dimension to $\mathcal{U}$: $n - k$. Then for an object $\mathcal{X}$ in the category of derived manifolds, the underlying space $X = U(\mathcal{X})$ can be covered by open subsets in such a way that the corresponding open subobjects of $\mathcal{X}$ are all local models. More generally, any open cover of $U(\mathcal{X})$ by open sets can be refined to an open cover whose corresponding open subobjects are local models.

Now let X be a k-dimensional and Y be an n-dimensional smooth manifolds and consider a smooth map $g : X \to Y$. Then g is an *imbedding* if and only if there is a cover of Y by open submanifolds U^i such that, if we set $X_i = g^{-1}(U^i)$, each of the induced maps $g|_{X_i} : X_i \to U^i$ is such that there exists a smooth function $h : U^i \to \mathbb{R}^n$, transverse to $0 : \mathbb{R}^0 \to \mathbb{R}^n$ with $X_i \simeq U^i_{f=0}$ over U. Example 2.7 in [201] provides a nice intuitive picture of what we have constructed. Consider the structureless cobordism ring of the projective space $\mathbb{P}^k$ $\Omega_*(\mathbb{P}^k)$. Moreover, let us consider p- and l-dimensional planes M and N within $\mathbb{P}^k$, such that they meet transversely in a $(p+l-k)$-plane A. As a derived manifold we can understand A as a fiber product:

$$A \simeq M \times_{\mathbb{P}^k} N \,. \tag{5.96}$$

The example most informative for the interpretation of our low-dimensional manifolds is actually as follows: We take a set $f_1, \ldots, f_k : M \to \mathbb{R}$ of smooth functions on a manifold M, then their zero set is a derived manifold $\mathcal{X}$. To see this, let $f = (f_1, ..., f_k) : M \to R^k$ and realize $\mathcal{X}$ as the fiber product in the diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{R}^0 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle 0} \\ M & \xrightarrow{\;f\;} & \mathbb{R}^k \,. \end{array} \tag{5.97}$$

The (virtual) dimension of $\mathcal{X}$ is $m-k$, where m is the dimension of M. This means we can construct in such a way the "missing" equivalence classes of manifolds within $\Omega^\xi_{n+k}(X)$, where $-k \leq n < 0$ and k the dimension of X, which have negative virtual dimension.

Let us conclude with a few remarks. Looking at the K-theory side the groups $K^n(pt)$ with $n > 0$ apparently classifies charges of Dp-branes transversal to a $(-n)$-dimensional space. So for example the Dp-brane captured by $K^2(pt)$ we mentioned above would be associated to an imbedding $M_4 \to X_6$ and the brane would live on some type of fibration with a fiber of dimension -2.

We do not claim that these configurations are actually physical. To make further progress it would be necessary to construct them from the traditional worldsheet approach. Since the dimension is intimately tied to the central charge, it would be prudent to start from there. If such imbeddings can be physically realized, it would enlarge our tool box significantly especially for phenomenological applications, since the dimension of our universe is a key observable for any quantum gravity theory.

After this conceptually rather odd subject related to cobordism groups of finite background spaces we are going to look into infinite-dimensional background spaces next. More specifically classifying spaces capable of modelling the precise topological imprint of background gauge fields. In particular, we want to look at type I and heterotic string theory and their shared Semispin background gauge group $Ss(32)$.

6

Spin cobordism and the gauge group of type I/heterotic string theory

By now it is a well-established assertion in the string literature that both the type I and one of the two supersymmetric heterotic string theories – the HO theory – in ten dimensions actually share the same gauge group, denoted as $Spin(32)/\mathbb{Z}_2$ in the string theory literature, out of the various Lie groups with a $\mathfrak{so}(32)$ Lie algebra. As we have already reviewed this in our string theory introduction 2 we will just recall the most salient facts. On the heterotic side consistency of the worldsheet theory immediately singles out this Lie group [202] from the even, self dual lattice on which we have to compactify the bosonic degrees of freedom. for type I string theory the story is a bit more intricate as perturbatively the symmetry group is $O(32)/\mathbb{Z}_2$ [65]. The story changes when accounting for the D9-branes canceling the Ramond-Ramond tadpole (as well as the NS-NS tadpole) leading to a SO(32) gauge symmetry group. However, this is still not the full picture.

The discovery of S-duality between the $Spin(32)/\mathbb{Z}_2$ heterotic string (HO string) and type I string theory [54, 85], i.e. they provide the weak/strong coupling description of the same complete theory, suggests that both theories have to be built upon the same gauge group [65][1]. As we have mentioned in section 2.10.2 subsequent analysis of nonperturbative objects in type I showed that certain non-BPS Dp-branes are responsible for amending the perturbative gauge group $O(32)/\mathbb{Z}_2$ to $Spin(32)/\mathbb{Z}_2$ on the type I side [65], such that S-duality between type I and HO can be seriously considered. In particular, the non-BPS D0-brane is the one carrying spinor charge [69].

As we also reviewed earlier the correct quotient of $Spin(32)/\mathbb{Z}_2$ is the one killing just half of the spinors (the two choices killing either the positive or negative chirality spinors are isomorphic), which is usually called SemiSpin because of that. Importantly, this group $Ss(32)$ has fundamentally different properties than $SO(32)$. For example, the two groups are of different homotopy type [204]. A lot of other subtle differences, especially in the context of string dualities, were highlighted in [74, 75]. In the following we want to actually calculate the cobordism groups relevant to nonperturbative objects in the type I/HO string theory considering the correct gauge group.

Incorporating gauge groups into cobordism considerations

As already mentioned above the concept we want to introduce is classifying spaces BG as "background gauge" manifolds for the cobordism groups, i.e. we would like to study $\Omega_n^\xi(X = BG)$. This is quite distinct from the case of a finite dimensional background manifold X, which models dimensional reduction [34], that we have seen earlier.

More specifically, the map we are fixing to be preserved throughout the cobordism equivalence is a classifying map $M \to BG$. As reviewed in appendix B this assigns a G-principal bundle to the manifolds in our equivalence classes within each cobordism group due to the bijection between the homotopy class of the classifying map and the isomorphism class of numerable G-principal bundles [205]. From a physics point of view the G-principal bundle is the topological backbone of the gauge theory we want to fix. In our case this is the unique gauge group arising in type I/HO string theory.

[1] It's quite insightful to contrast S-duality between the two $Spin(32)/\mathbb{Z}_2$ string theories with Montonen-Olive duality [78, 203], since the strong coupling dual gauge group of the latter duality is the Langlands-dual of the gauge group one started with. Therefore, our physical expectation that this also holds true in string theory matches up nicely with the mathematical fact that $Spin(32)/\mathbb{Z}_2$ is in fact Langlands-self-dual, see for example appendix D of [203].

It should be emphasized that the cobordism groups of some classifying space should be considered a specific, intermediate step in the approximation of the "final" quantum gravity structure the Cobordism Conjecture 3.3 is referring to. Each step reveals a different aspect of quantum gravity. In the case at hand we gain insight into the D-brane bound states in correspondence with the presence of a Yang-Mills theory, introduced from the type I perspective by the tadpole canceling stack of 32 D9-branes and one O9-plane.

In what follows we will utilize rather heavy mathematical machinery to calculate the spin cobordism groups $\Omega_{n\leq 12}^{Spin}(BSs(32))$ below dimension 13 of the classifying space of $Ss(32)$ in 6.1. The spectral sequences have been introduced in the mathematical background chapter 4.6. Afterwards, we interpret the result from a string theoretic point of view, especially in the context of the Cobordism Conjecture, in section 6.2.

6.1 The calculation of $\Omega_n^{Spin}(BSs(32))$

After our short introduction to the main mathematical tools we are going to use we will now work through the computation of $\Omega_n^{Spin}(BSs(32))$. This section is divided into the two main steps of the calculation: Determining $H^n(BSs(4n), \mathbb{Z}_2)$ via the Eilenberg-Moore spectral sequence and then followed up by computing $\Omega_n^{Spin}(BSs(32))$.

6.1.1 Determining $H^n(BSs(4n), \mathbb{Z}_2)$

To access the second page of the Adams spectral sequence we will now partially follow and extend the calculation of $H^{*\leq 11}(BSs(n), \mathbb{Z}_2)$ demonstrated by [206]. In particular the computation exploited the fact that while we do not know $H^n(BSs(n), \mathbb{Z}_2)$, we actually do know $H^n(Ss(n), \mathbb{Z}_2)$ [207]. The aforementioned calculation consists of two subsequent spectral sequences. First, the May spectral sequence [208] sets up the next spectral sequence

$$\mathrm{Cotor}_{A'}(\mathbb{Z}_2, \mathbb{Z}_2) \Longrightarrow \mathrm{Cotor}_{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2)\,, \tag{6.1}$$

where A' is is a Hopf algebra such that it is isomorphic as an algebra with A' such that every generator is primitive. Then, the result is fed as the second page into the

Eilenberg-Moore spectral sequence (4.93) [169–171]

$$E_2 = \mathrm{Cotor}_{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \Longrightarrow H^*(BSs(n), \mathbb{Z}_2)\,. \tag{6.2}$$

However, we will follow a different route to determine $\mathrm{Cotor}_{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2)$, namely the "twisted tensor product" method of [209].

We start with $H^*(Ss(32), \mathbb{Z}_2)$ as a Hopf algebra as determined in [207]. Up to degree $n = 15$ $H^n(Ss(n), \mathbb{Z}_2)$ the authors showed that as an algebra it is isomorphic to

$$\Delta(w_3, w_5, w_6, w_7, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}) \otimes \mathbb{Z}_2[\bar{v}]\,. \tag{6.3}$$

Importantly, there are a couple nontrivial coproducts in the degree range we are interested in:

$$\bar{\psi}(w_7) = \bar{v} \otimes w_6 + \bar{v}^2 \otimes w_5 + \bar{v}^4 \otimes w_3\,, \tag{6.4}$$

$$\bar{\psi}(w_{11}) = \bar{v} \otimes w_{10} + \bar{v}^2 \otimes w_9 + \bar{v}^8 \otimes w_3\,, \tag{6.5}$$

$$\bar{\psi}(w_{13}) = \bar{v} \otimes w_{12} + \bar{v}^4 \otimes w_9 + \bar{v}^8 \otimes w_5\,, \tag{6.6}$$

$$\bar{\psi}(w_{14}) = \bar{v}^2 \otimes w_{12} + \bar{v}^4 \otimes w_{10} + \bar{v}^8 \otimes w_6\,. \tag{6.7}$$

As our first step of determining $\mathrm{Cotor}_{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2)$, we define the $\mathbb{Z}_2$-submodule L of $H^*(Ss(n), \mathbb{Z}_2)$ generated by

$$\{\bar{v}, \bar{v}^2, \bar{v}^4, \bar{v}^8, w_3, w_5, w_6, w_7, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}\}\,. \tag{6.8}$$

Then we have the projection $\theta : H^*(Ss(32), \mathbb{Z}_2) \to L$, the inclusion $\iota : L \to H^*(Ss(32), \mathbb{Z}_2)$ and the suspension s uplifting the elements of L to:

$$sL = \{a_2, a_3, a_5, a_9, b_4, b_5, b_7, c_8, b_{10}, b_{11}, c_{12}, b_{13}, c_{14}, c_{15}\}\,. \tag{6.9}$$

Now we extend the maps θ and ι to $\bar{\theta} = s \circ \theta : H^*(Ss(32), \mathbb{Z}_2) \to sL$ and $\bar{\iota} = \iota \circ s^{-1}$. Next, we construct $\bar{X}$ as $\bar{X} := T(sL)/I$, where $T(sL)$ is the tensor algebra with the natural product ψ and I is the two-sided ideal of $T(sL)$ generated by $Im(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi) \circ Ker(\bar{\theta})$. Consequently, $\bar{X}$ is given by

$$\bar{X} = \mathbb{Z}_2[a_i, b_j, c_k]/I\,, \tag{6.10}$$

where I is generated by

$$
\begin{aligned}
&[a_i, a_j],\ [b_i, b_j],\ [a_i, b_j],\ [b_i, c_j],\ [a_i, c_j] \text{ for } (i, j) \in \{(5, 12), (3, 14), (2, 15)\}\\
&[a_2, c_8] + a_3 b_7,\ [a_3, c_8] + a_5 b_6,\ [a_5, c_8] + a_9 b_4,\\
&[a_2, c_{12}] + a_3 b_{11},\ [a_3, c_{12}] + a_5 b_{10},\\
&[a_2, c_{14}] + a_3 b_{13},\ [a_5, c_{14}] + a_9 b_{10},\\
&[a_3, c_{15}] + a_5 b_{13},\ [a_5, c_{15}] + a_9 b_{11},
\end{aligned}
\tag{6.11}
$$

[,] denotes the commutator. From here we construct our twisted tensor product. Now, we define a differential $\bar{d}$ as a map $\bar{d} = \psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi\, \bar{\iota} : sL \to T(sL)$ that is uniquely extended to $\bar{d} : T(sL) \to T(sL)$ with $\bar{d}(I) \subset I$, such that $\bar{X}$ becomes a differential algebra.

Consequently, following [210] we construct through the triviality of $\bar{d} \circ \bar{\theta} + \psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi = 0$ a twisted tensor product $W = H^*(Ss(32), \mathbb{Z}_2) \otimes \bar{X}$ with respect to $\bar{\theta}$. W is a differential $H^*(Ss(32), \mathbb{Z}_2)$-comodule with the differential:

$$
d = 1 \otimes \bar{d} + (1 \otimes \psi) \circ (1 \otimes \bar{\theta} \otimes 1) \circ (\phi \otimes 1) \tag{6.12}
$$

with the product ψ over $T(sL)$ and ϕ the product over $H^*(Ss(32), \mathbb{Z}_2)$. The differential acts in the following way on the elements:

$$
\begin{aligned}
dw_i &= a_{i+1} \text{ for all } i \text{ except for } i = \{7, 11, 13, 14\}\,,\\
d\bar{v}^j &= b_{j+1} \text{ for } j = \{1, 2, 4, 8\}\,,\\
dw_7 &= c_8 + \bar{v} \otimes a_7 + \bar{v}^2 \otimes a_6 + \bar{v}^4 \otimes a_4\,,\\
dw_{11} &= c_{12} + \bar{v} \otimes a_{11} + \bar{v}^2 \otimes a_{10} + \bar{v}^8 \otimes a_4\,,\\
dw_{13} &= c_{14} + \bar{v} \otimes a_{13} + \bar{v}^4 \otimes a_{10} + \bar{v}^8 \otimes a_6\,,\\
dw_{14} &= c_{15} + \bar{v}^2 \otimes a_{13} + \bar{v}^4 \otimes a_{11} + \bar{v}^8 \otimes a_7\,.
\end{aligned}
\tag{6.13}
$$

As a result, we get the following action on $\{a_i, b_j, c_k\}$:

$$
\begin{aligned}
da_i &= \bar{d}a_i = 0\,,\\
db_j &= \bar{d}b_j = 0\,,\\
dc_8 &= \bar{d}c_8 = b_2 a_7 + b_3 a_6 + b_5 a_4\,,\\
dc_{12} &= \bar{d}c_{12} = b_2 a_{11} + b_3 a_{10} + b_9 a_4\,,\\
dc_{14} &= \bar{d}c_{14} = b_2 a_{13} + b_5 a_{10} + b_9 a_6\,,\\
dc_{15} &= \bar{d}c_{15} = b_3 a_{13} + b_5 a_{11} + b_9 a_7\,.
\end{aligned}
\tag{6.14}
$$

Consequently, in accordance to the procedure outlined by [209,210] we define weights in W, which in our case is 1 for the "pairs" with respect to the suspension

$$(w_7,\, c_8),\, (w_{11},\, c_{12}),\, (w_{13},\, c_{14}),\, (w_{14},\, c_{15}) \tag{6.15}$$

and zero for all the other pairs.

This allows us to define a filtration with respect to these weights:

$$F_r = \{x \in W \text{ with weight } \leq r\}\,. \tag{6.16}$$

Then we have essentially achieved the same as a spectral sequence and we define $E_\infty(W) = \sum_r F_r/F_{r-1}$. The point is that since $\bar{d}(F_r) \subset F_r$ the homology groups of $E_\infty(W)$ vanish, i.e. $E_\infty(W)$ is acyclic. Thus, W itself is acyclic, as well. Then $W = H^*(Ss(32), \mathbb{Z}_2) \otimes \bar{X}$ is an acyclic injective comodule resolution of $\mathbb{Z}_2$ over $H^*(Ss(32), \mathbb{Z}_2)$, which is reminiscent of the definition we gave for the Cotor functor. The point is that the cohomology of $\bar{X}$ together with the map $\bar{d}$ gives us the Cotor functor:

$$\text{Cotor}^{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong H(\bar{X} \,:\, \bar{d}) = ker(\bar{d})/im(\bar{d}) \tag{6.17}$$

in the notation of [210]. Therefore, we now get:

$$\begin{aligned}\text{Cotor}^{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \,&[\bar{x}_2, \bar{x}_3, \bar{x}_5, \bar{x}_9, \bar{y}_4, \bar{y}_6, \bar{y}_7, \bar{y}_{10}, \bar{y}_{11}, \bar{y}_{13}] \,/ \\ &\big(\bar{x}_2\bar{y}_7 + \bar{x}_3\bar{y}_6 + \bar{x}_5\bar{y}_4,\, \bar{x}_2\bar{y}_{11} + \bar{x}_3\bar{y}_{10} + \bar{x}_9\bar{y}_4, \\ &\bar{x}_2\bar{y}_{13} + \bar{x}_5\bar{y}_{10} + \bar{x}_9\bar{y}_6,\, \bar{x}_3\bar{y}_{13} + \bar{x}_5\bar{y}_{11} + \bar{x}_9\bar{y}_7\big)\,.\end{aligned} \tag{6.18}$$

With $\text{Cotor}^{H^*(Ss(32),\mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2)$ determined we can proceed with the main goal of this section resolving the Eilenberg-Moore spectral sequence (4.93). Actually, we can make the task a lot easier as all the generators can be written as Steenrod squares acting on either $\bar{x}_2$ or $\bar{y}_4$:

$$\begin{aligned}&\bar{x}_3 = Sq^1\bar{x}_2,\, \bar{x}_5 = Sq^2Sq^1\bar{x}_2,\, \bar{x}_9 = Sq^4Sq^2Sq^1\bar{x}_2, \\ &\bar{y}_6 = Sq^2\bar{y}_4,\, \bar{y}_7 = Sq^3\bar{y}_4,\, \bar{y}_{10} = Sq^4Sq^2\bar{y}_4, \\ &\bar{y}_{11} = Sq^5Sq^2\bar{y}_4,\, \bar{y}_{13} = Sq^6Sq^3\bar{y}_4.\end{aligned} \tag{6.19}$$

Since $\bar{x}_i \in E_2^{1,i-1}$ and $y_j \in E_2^{1,j-1}$ we can see that there are no nontrivial differentials acting on $\bar{x}_2$ and $\bar{y}_4$:

$$d_r(\bar{x}_2) : E_r^{1,1} \to E_r^{1+r,1-(r-1)} = 0\,, \tag{6.20}$$

$$d_r(\bar{y}_4) : E_r^{1,3} \to E_r^{1+r,3-(r-1)} = 0\,. \tag{6.21}$$

As a consequence all of the elements up to the degree we are studying are actually permanent cycles as both $\bar{x}_2$ and $\bar{y}_4$ are and subsequently every element related by a cohomology operation (in this case Steenrod squares) to a permanent cycle. Therefore, our spectral sequence collapses. Now, proceeding to $H^*(BSs(n), \mathbb{Z}_2)$, let's define the elements. Analogous to [206] we choose y_4 instead of $y_4 + x_2^2$ as the representative of $\bar{y}_4$ besides x_2 as the representative of $\bar{x}_2$, the other representatives in the same order as in (6.18):

$$\begin{aligned} &x_3 = Sq^1 x_2,\ x_5 = Sq^2 x_3,\ x_9 = Sq^4 x_5, \\ &y_6 = Sq^2 y_4,\ y_7 = Sq^1 y_6,\ y_{10} = Sq^4 y_6, \\ &y_{11} = Sq^1 y_{10},\ y_{13} = Sq^2 y_{11}. \end{aligned} \tag{6.22}$$

At this point let us denote the action of the Steenrod squares, which is going to be very important for the Adams spectral sequence:

	Sq^1	Sq^2	Sq^3	Sq^4
x_2	x_3	x_2^2		
x_3	/	x_5	x_3^2	
x_5	x_3^2	/	/	x_9
x_9	x_5^2	/	/	/
y_4	/	y_6	y_7	y_4^2
y_6	y_7	/	/	y_{10}
y_7	/	/	/	y_{11}
y_{10}	y_{11}	/	/	/
y_{11}	/	y_{13}	/	/
y_{13}	/	/		

Table 6.1: Elements of $H^n(BSs(4n), \mathbb{Z}_2)$ and their transformation under Steenrod squares.

As alluded to before obtaining the higher relations r_i is a lot simpler as they are related to r_1 by Steenrod operations. The first relation is given by:

$$r_1 = x_2 y_7 + x_3 y_6 + \tilde{x}_5 y_4\,, \tag{6.23}$$

where we defined $\tilde{x}_5 = x_5 + x_2x_3$. Now, we can bootstrap the next relations, while crosschecking that Steenrod squares acting on the relations actually vanish. The first few Steenrod operations acting on r_1 are pretty simple:

$$\begin{aligned} Sq^1(r_1) &= x_3y_7 + x_3y_7 = 0\,, \\ Sq^2(r_1) &= x_2^2y_7 + x_5y_6 + x_2\tilde{x}_5y_4 + \tilde{x}_5y_6 = x_2r_1 = 0\,, \\ Sq^3(r_1) &= x_5y_7 + x_3^2y_6 + \tilde{x}_5y_7 + (x_3x_5 + x_2x_3^2)y_4 = x_3r_1 = 0\,. \end{aligned} \tag{6.24}$$

Finally with Sq^4 we reach the next relation r_2:

$$r_2 = Sq^4(r_1) = x_2y_{11} + x_3y_{10} + \tilde{x}_5y_4^2 + x_3^2y_7 + x_2\tilde{x}_5y_6 + \tilde{x}_9y_4\,, \tag{6.25}$$

where we defined $\tilde{x}_9 = x_9 + x_2^2x_5 + x_3^3$. This definition is very convenient, since we have $Sq^4(\tilde{x}_5) = \tilde{x}_9$ and $Sq^1(\tilde{x}_9) = \tilde{x}_5^2$. So let's look at the next couple of Steenrod squares building onto r_2, as well:

$$\begin{aligned} Sq^1(r_2) &= x_3y_{11} + x_3y_{11} + \tilde{x}_5r_1 = 0\,, \\ r_3 = Sq^2(r_2) &= x_2y_{13} + \tilde{x}_5y_{10} + x_3x_5y_7 + x_2^2\tilde{x}_5y_6 + \tilde{x}_9y_6 + x_2\tilde{x}_9y_4\,. \end{aligned} \tag{6.26}$$

Subsequently, we get r_4 from r_3 (or as $Sq^3(r_2)$ from r_2):

$$\begin{aligned} r_4 = Sq^1(r_3) = x_3y_{13} &+ \tilde{x}_5y_{11} + x_2^2\tilde{x}_5y_7 + x_3^3y_7 + \tilde{x}_9y_7 \\ &+ \tilde{x}_5^2y_6 + x_3\tilde{x}_9y_4 + x_2\tilde{x}_5^2y_4\,. \end{aligned} \tag{6.27}$$

As a final consistency check we calculate Sq^1 and Sq^2 of r_4 to check that both actually vanish[2].

$$\begin{aligned} Sq^1(r_4) &= \tilde{x}_5^2y_7 + \tilde{x}_5^2y_7 + x_3\tilde{x}_5^2y_4 + x_3\tilde{x}_5^2y_4 = 0\,, \\ Sq^2(r_4) &= \tilde{x}_9r_1 + \tilde{x}_5^2(x_2y_6 + x_2^2y_4 + x_2y_6 + x_2^2y_4) = 0\,. \end{aligned} \tag{6.28}$$

Now, with the action of the Steenrod squares set up and the relations determined we can go ahead and identify the $\mathcal{A}_1$-module structure of $H^*(BSs(4n), \mathbb{Z}_2)$. First, from (6.1.1) we recognize that the elements of $H^*(BSs(4n), \mathbb{Z}_2)$ split into two different parts a "x-part" consisting of $x_2, \ldots, x_9$ and a "y-part" composed of $y_4, \ldots, y_{13}$ as the Steenrod squares never transforms them into each other. The third part, a mixed part, of course is comprised of the combination of x- and y-elements. This is where the relations come into play.

[2] There is an additional relation r_5 in degree 17, but we don't hit it with $Sq^1(r_4)$.

Notice that the "x-part" is up to the degrees we are working with isomorphic to the algebra of $H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the "y-part" can be identified with $H^*(BE_8, \mathbb{Z}_2)$. The "x-part" can be understood as coming from the fibration

$$BSpin(4n) \to BSs(4n) \to B^2\mathbb{Z}_2 \,. \tag{6.29}$$

From a string theory perspective the "y-part" has a very natural interpretation as the lead actor of the T-Duality between the two supersymmetric heterotic string theories with gauge groups $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ and $Ss(32)$.

Interestingly, the coproducts in $H^*(Ss(4n), \mathbb{Z}_2)$ precisely cause the "y-part" in the range relevant to both string theories to change from being identical to $H^*(BSpin(4n), \mathbb{Z}_2)$ to being identical to $H^*(BE_8, \mathbb{Z}_2)$.

Let us also point out the close relation to $H^*(BSO(4n), \mathbb{Z}_2)$ as sometimes the gauge group of type I/HO string theory is wrongfully identified as SO(32). Here, additionally to removing the coproducts we would also need to couple the "x-" and the "y-part", such that after renaming the "x-elements": $x_i \to y_i$ ($i \in 2, 3, 5, 9$) we would have the following action of Sq^i, $i \in 1, 2$:

$$Sq^1(y_i) = (i-1)\, y_{i+1} \,, \tag{6.30}$$

$$Sq^2(y_i) = \binom{i-1}{2} y_{i+2} + y_2 y_i \,. \tag{6.31}$$

We leave a string theoretic interpretation of the coproducts and the consequent relations in $H^*(BSs(4n), \mathbb{Z}_2)$ to future work.

Let's start with looking at the $\mathcal{A}_1$-module structure of the "x-part". Since $\mathbb{Z}_2$ is a discrete group, $B^2\mathbb{Z}_2$ is nothing else than the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$. Up to degree $n = 40$ the ko-homology $ko_n(K(\mathbb{Z}_2, 2))$, which is more than sufficient for the string theoretic applications we have in mind, has been calculated in [211]. As already hinted upon before we will use some particular combinations of x_2, x_3, x_5 and x_9. Namely, we will use

$$\begin{aligned}
\tilde{x}_5 &= x_5 + x_2 x_3 \,, \\
\tilde{x}_9 &= x_9 + x_2^2 x_5 + x_3^3 \,, \\
\tilde{x}_9' &= x_2^3 x_3 + x_2^2 x_5 + x_3^3 \,, \\
\tilde{x}_{11} &= x_2 x_9 + x_3^2 x_5 + x_2 x_3^3
\end{aligned}$$

matching the definitions of [211]. With the action of Sq^1 and Sq^2 on the elements of the "x-part" we get the following $\mathcal{A}_1$-module structure:

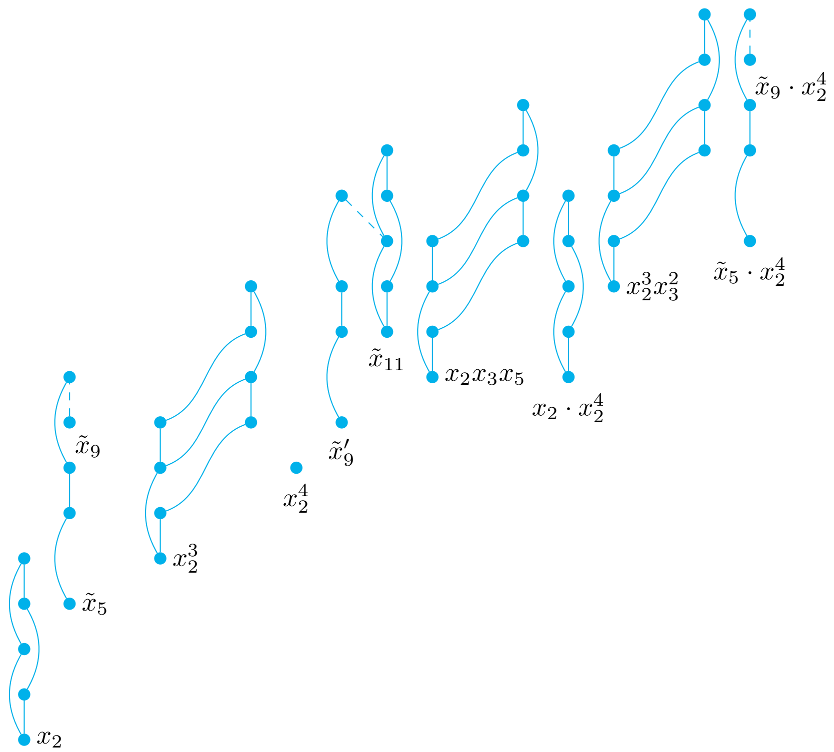

Figure 6.1: $\mathcal{A}_1$-module structure – x-part

To match the notation of [211] we included the Sq^1 connecting $\tilde{x}_9$ with $Sq^2\, Sq^1\, Sq^2(\tilde{x}_5)$ as a dashed line in the figure above. For the subsequent Adams charts this Sq^1 does not makes a difference and we can effectively treat $\mathbb{Z}_2[\tilde{x}_9]$ and $M_0[\tilde{x}_5]$ as separate modules following [211]. In our degree range there is another non-trivial extension, namely a $Sq^3 = Sq^1Sq^2$ between $\tilde{x}_{11}$ and $Sq^2\, Sq^1\, Sq^2(\tilde{x}_9')$ [211], which we indicate with a dashed line. Once more, we can treat $M_0[\tilde{x}_9']$ and $J[\tilde{x}_{11}]$ as different modules in the Adams charts.

Before we discuss setting up the Adams spectral sequence by filling the second page, we'll look at the $\mathcal{A}_1$-module structure of the "y-part" and the mixed "x-y-part". Whereas the structure of the "y-part" is simple compared to the other parts as can be seen below, the mixed "x-y-part" has a highly involved $\mathcal{A}_1$-module structure.

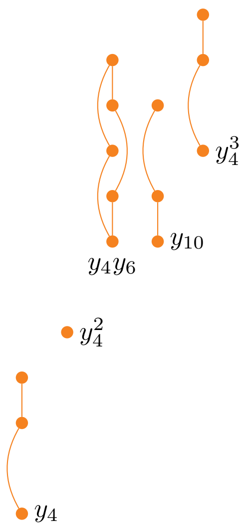

Figure 6.2: $\mathcal{A}_1$-module structure – y-part

After carefully incorporating the relations r_i, $i \in 1, \ldots, 4$ (6.23)-(6.27) we finally arrive at the following structure:

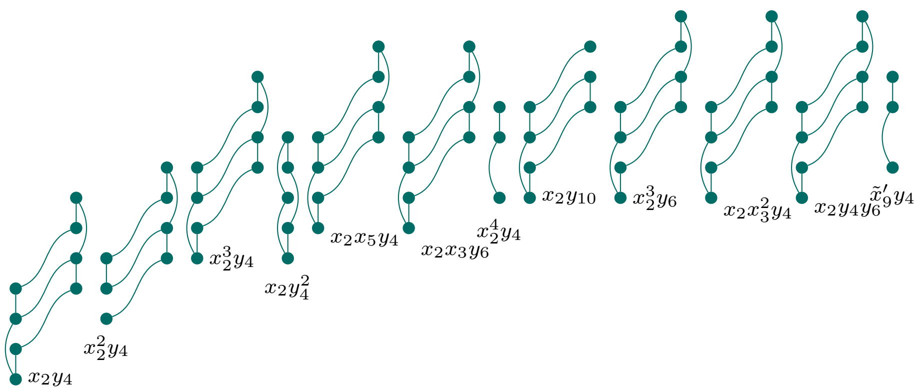

Figure 6.3: $\mathcal{A}_1$-module structure – mixed x-y-part

6.1.2 Determining the spin cobordism groups of $BSs(32)$ up to degree 12

Theorem 6.1.1. *The 2-completed ko-homology of $BSs(32)$ up to degree 12 takes the following form:*

n	0	1	2	3	4	5	6	7	8
$ko_n(BSs(32))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	0	$2\,\mathbb{Z}\oplus\mathbb{Z}_2$	0	$2\,\mathbb{Z}_2$	0	$4\,\mathbb{Z}\oplus\mathbb{Z}_8$

n	9	10	11	12
$ko_n(BSs(32))$	$4\,\mathbb{Z}_2$	$8\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$6\,\mathbb{Z}\oplus 7\,\mathbb{Z}_2\oplus\mathbb{Z}_8$

Table 6.2: ko-homology groups $ko_n(BSs(32))_{\hat{2}}$.

Proof. As outlined earlier the tool of our choice to calculate $ko_n(BSs(32))_{\hat{2}}$ is the Adams spectral sequence. With the $\mathcal{A}_1$-module structure of $H^*(BSs(32),\mathbb{Z}_2)$ we can easily write down the second page E_2 of the spectral sequence. Essential to this proof are the two maps we already alluded to in the discussion of the $\mathcal{A}_1$-module structure, namely $ko_*(BSs(32)) \to ko_*(B^2\mathbb{Z}_2)$ from the fibration $BSpin(n) \to BSs(n) \to B^2\mathbb{Z}_2$ and $ko_{n<16}(BSs(32)) \to ko_{n<16}(BE_8)$ from the inclusion $Ss(16) \hookrightarrow E_8$. Again, the first one is highlighted by blue color and the second one by orange color in the following. The Adams spectral sequences for both ko-homologies are well studied, i.e. differentials and extensions are solved (or solvable) up to at least dimension 12, and due to the naturality of the Adams spectral sequence we can connect differentials that arise in our spectral sequence to known ones via the aforementioned maps. To make the spectral sequence easier to follow we discuss the spectral sequences for $ko_*(B^2\mathbb{Z}_2)_{\hat{2}}$ and $ko_*(BE_8)_{\hat{2}}$ separately and we just show the reduced ko-homologies, since we can use the splitting principle to complete them.

The Adams spectral sequence for $ko_*(B^2\mathbb{Z}_2)_{\hat{2}}$

The spectral sequence for $ko_*(B^2\mathbb{Z}_2)_{\hat{2}} = ko_*(K(\mathbb{Z}_2,2))_{\hat{2}}$, where $K(\mathbb{Z}_2,2)$ is the Eilenberg-MacLane space of the pair $(\mathbb{Z}_2,2)$[3], was studied in [211] and we will briefly

[3] The equivalence between classifying spaces B^nG of a discrete group G and Eilenberg-MacLane spaces $K(G,n)$ is for example discussed in chapter 16.5. of [121].

recount their results for this spectral sequence. We start by transferring the $\mathcal{A}_1$-modules, which are well known in the literature, see for example [145], into an Adams chart. We get the following second page, where we have encircled nodes stemming from a full $\mathcal{A}_1$-module, which will not partake in neither a non-trivial differential nor an extension due to Margolis' theorem [212]:

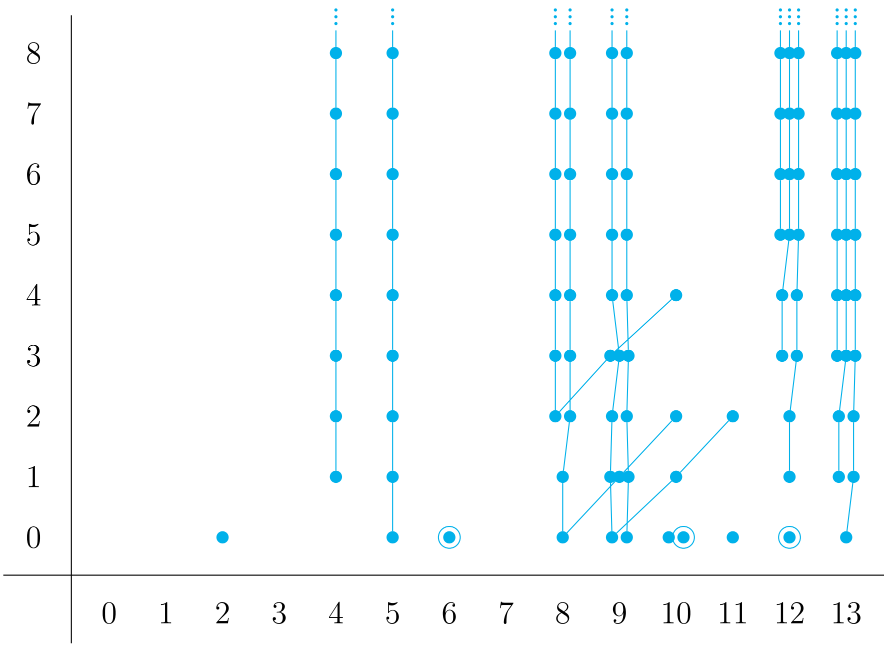

Figure 6.4: E_2 for $\widetilde{ko}_*(B^2\mathbb{Z}_2)_{\hat{2}}$ without differentials

Now we can start looking at the differentials. As mentioned before a differential d_r on the r-th page goes from a node $(s, t-s)$ to another one at $(s+r, t-s-1)$. Interestingly, differentials in the Adams spectral sequence have a very peculiar property, namely they are equivariant under elements of $\text{Ext}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ meaning that in our case acting with the h_i's with $i = 0, 1$ from before results in:

$$d_r(h_i x) = h_i d_r(x)\,. \tag{6.32}$$

With the properties of differentials in the Adams spectral sequence taken into account we see that only "tower killing" differentials can be present going from one tower of nodes to another one.

Lemma 6.1.2. *(Wilson [211]) The tower killing differentials up to degree 40 can be identified with Bocksteins. Up to degree 13 we have the following differentials:*

- *There is a d_2 starting from towers in degree $4k+5$ coming from ΣM_0 and $\Sigma^9\mathbb{Z}_2[\tilde{x}_9]$ killing the towers in degree $4k+4$ from $\Sigma^2 J[x_2]$.*
- *There is another d_2 between towers generated by the same $\mathcal{A}_1$-modules as above multiplied by x_2^4.*
- *A d_3 kills the towers in degree $4k+8$ associated to $\Sigma^8\mathbb{Z}_2[x_2^4]$ and starts from the towers coming from $\Sigma^9 M_0[\tilde{x}_9']$ and $\Sigma^{11}J[\tilde{x}_{11}]$.*

With these results on the differentials we are able to reach the infinity page. Since there are no higher differentials than d_3 up to degree 13, page 4 is already equivalent to the infinity page. The second, third and final fourth page look as follows:

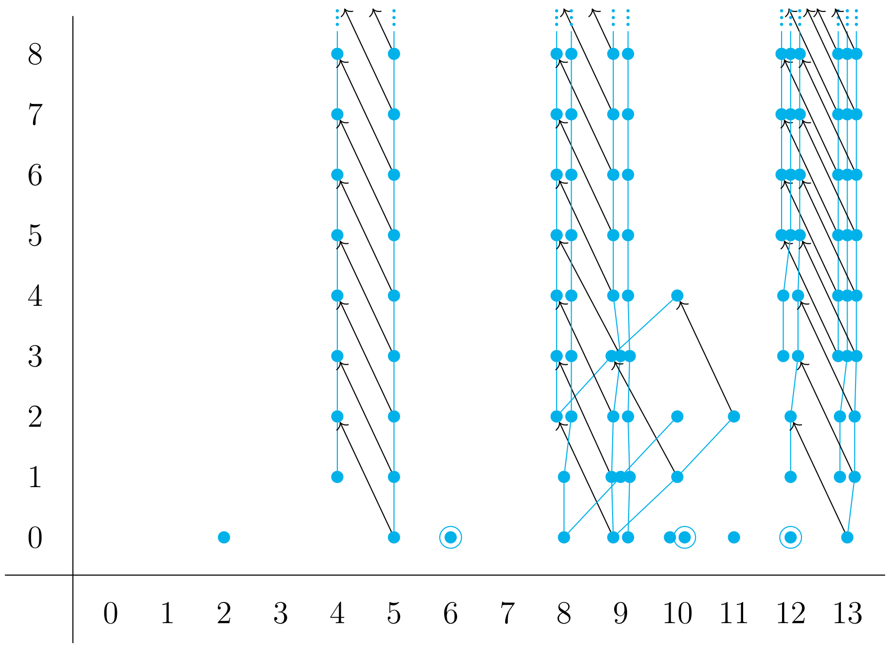

Figure 6.5: Second page E_2 for $\widetilde{ko}_*(B^2\mathbb{Z}_2)_{\hat{2}}$ including d_2 differentials

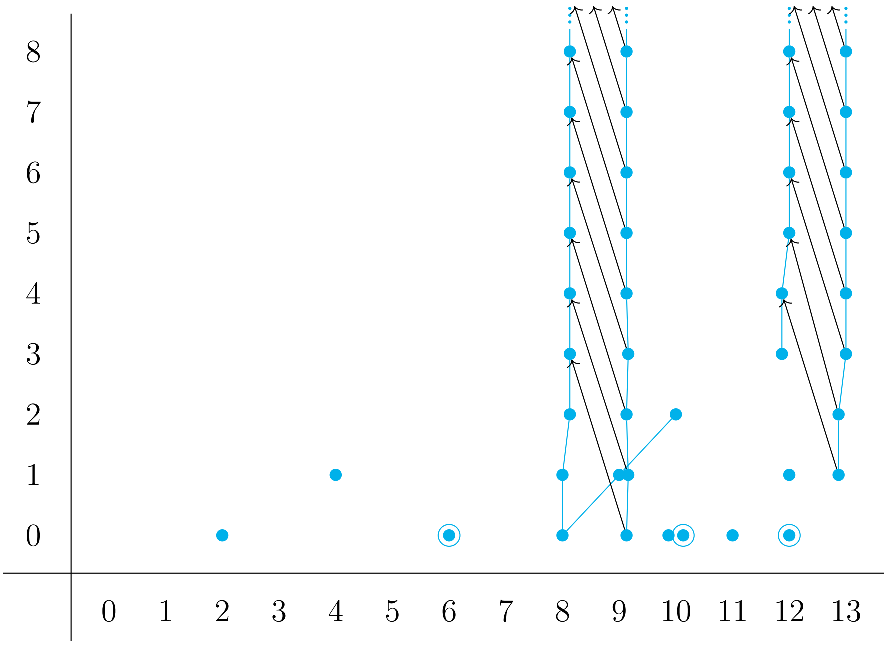

Figure 6.6: Third page E_3 for $\widetilde{ko}_*(B^2\mathbb{Z}_2)_{\widehat{2}}$

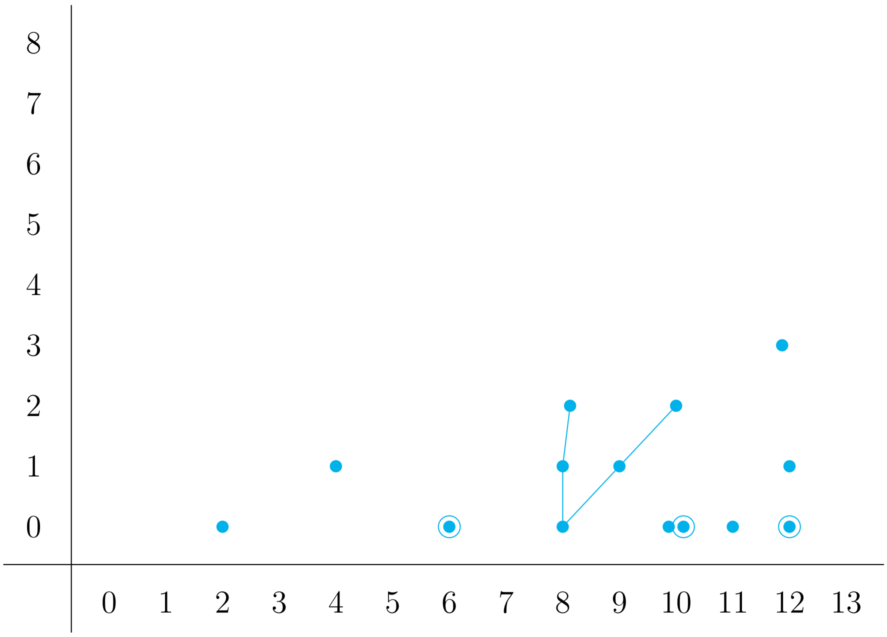

Figure 6.7: Final page E_∞ for $\widetilde{ko}_*(B^2\mathbb{Z}_2)_{\widehat{2}}$

At this point we will not discuss any extension problems, we will do so once we assembled the final page of the full Adams spectral sequence for $\widetilde{ko}_*(BSs(32))_{\widehat{2}}$.

The Adams spectral sequence for $ko_*(BE_8)_{\widehat{2}}$

As we stated before, the $\mathcal{A}_1$-structure of the "y-part" is completely equivalent to the one for BE_8 in the degrees we are interested in. Since we can also map the differentials by naturality of the Adams spectral sequence, we again first want to study the isolated case of $ko_*(BE_8)_{\widehat{2}}$ to infer a lot about the actual spectral sequence we care about. To compute the low dimensional spin cobordism or ko-homology groups usually the isomorphism between BE_8 and the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$ in degrees ≤ 15 is exploited. The associated Adams spectral sequence for $ko_*(K(\mathbb{Z}, 4))$ was studied in detail in [213]. Firstly, there are only a few $\mathcal{A}_1$-modules, which are quickly translated into an Adams chart, which looks as follows:

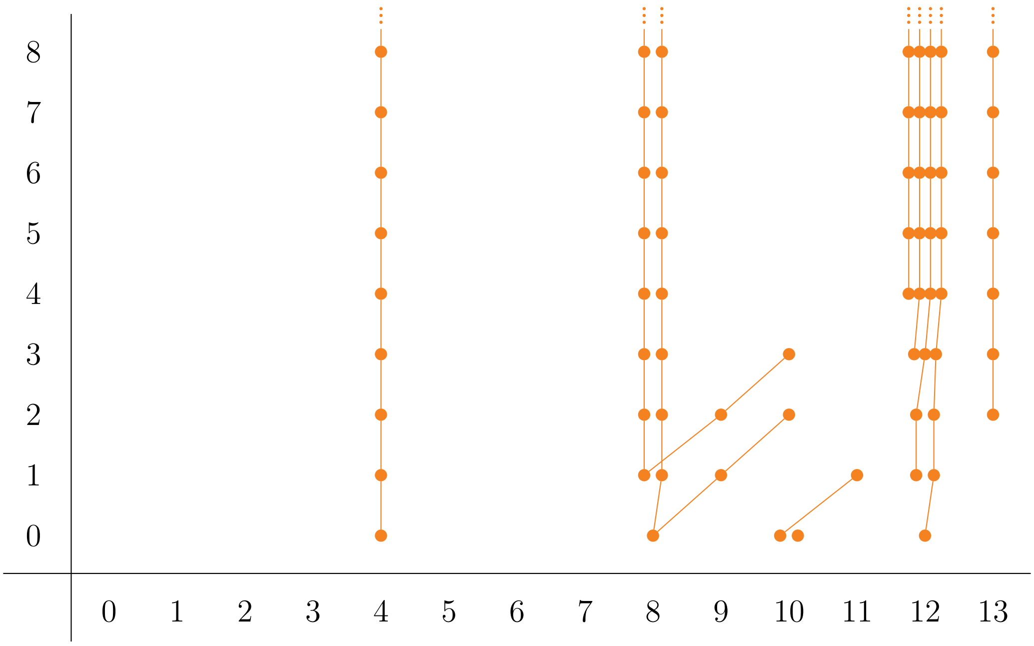

Figure 6.8: Second page E_2 for $\widetilde{ko}_*(BE_8)_{\widehat{2}}$ without differentials

As we can see from the second page (without differentials) the $\mathcal{A}_1$-module structure of the "y-part" doesn't leave much room for differentials. Still, there are two distinct differentials possible, which are in fact realized.

Lemma 6.1.3. *(Francis [213]) There is a d_2 in degree 10 (and 11 linked by h_1 action) and a* *tower killing differential in degree 13 such that there is no torsion in degree 12. Both differentials are linked by a higher cohomology operation, namely a Massey product.*

Since there are no more possible differentials in this range, we are done.

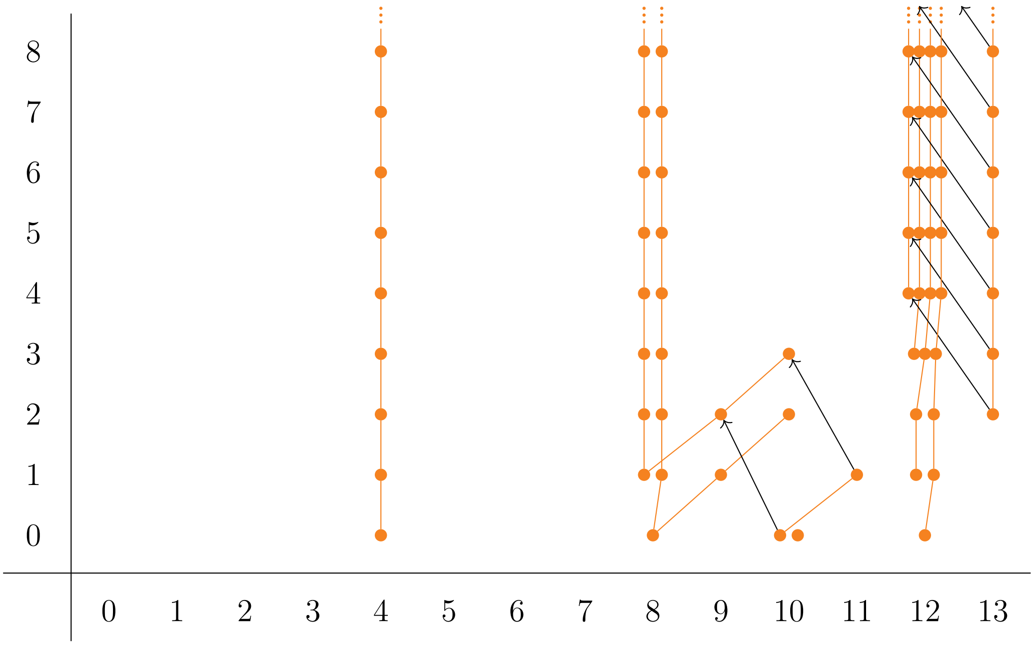

Figure 6.9: Second page E_2 for $\widetilde{ko}_*(BE_8)_{\hat{2}}$ with differentials

Completing the Adams spectral sequence for $ko_*(BSs(32))_{\hat{2}}$

For the final "x-y-part" the $\mathcal{A}_1$-modules are again well known, see for example [145]. Additionally, we demonstrated the computation of the Adams chart for the $\tilde{R}_2$-module in the introduction. So, we get the following second page without differentials depicted in 6.11.

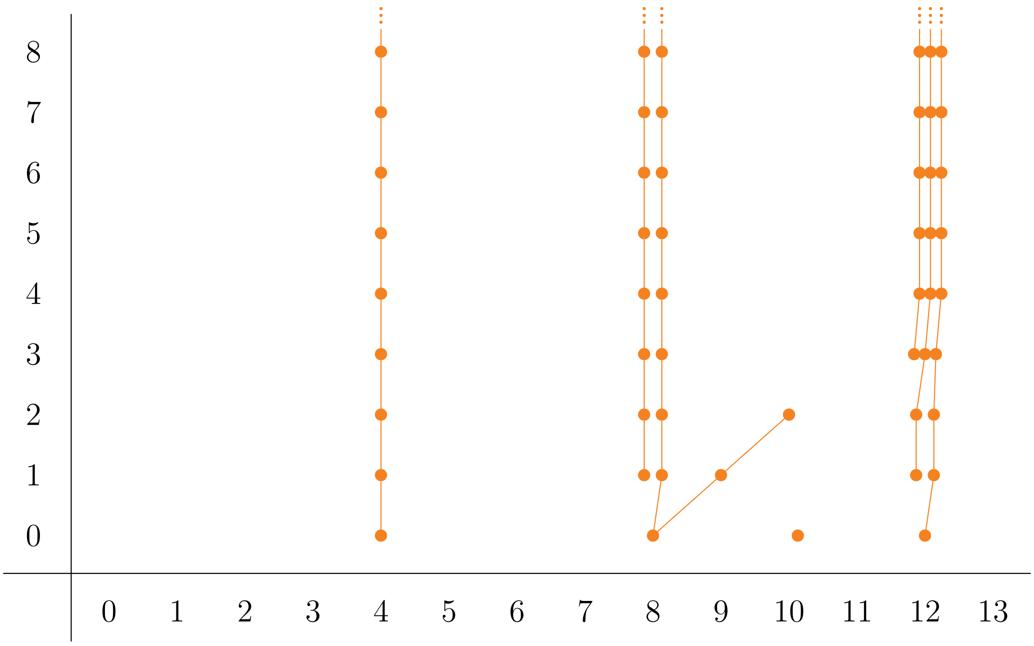

Figure 6.10: Final page E_∞ for $\widetilde{ko}_*(BE_8)_{\hat{2}}$

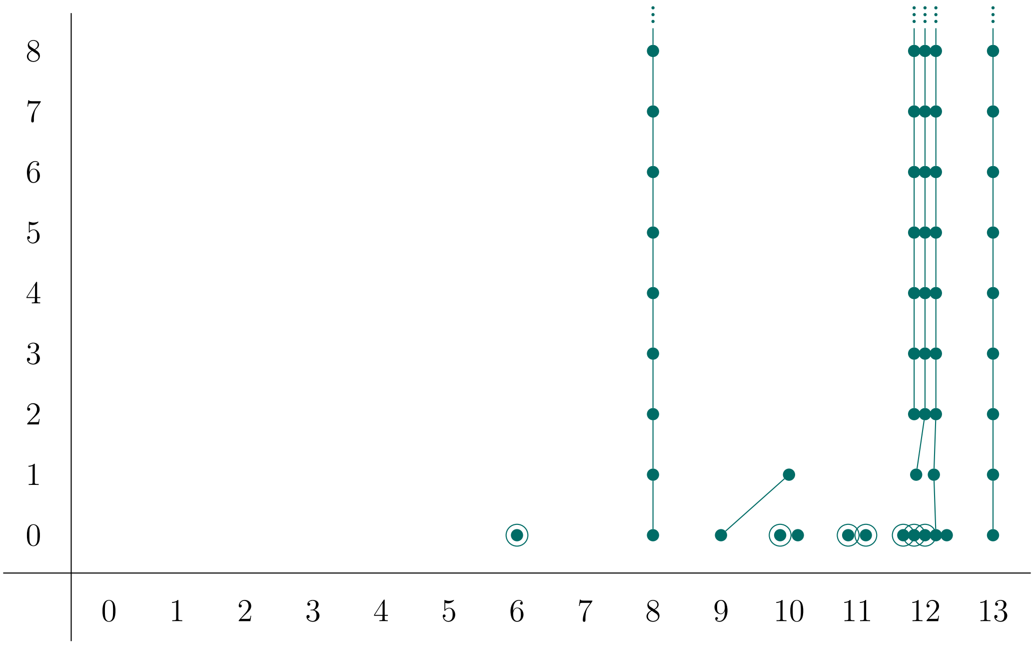

Figure 6.11: Second page E_2 for the (reduced) "x-y-part" without differentials

Again, the encircled nodes denote $\mathbb{Z}_2$'s stemming from full $\mathcal{A}_1$, which do not participate in any differentials (or non-trivial extensions).

Lemma 6.1.4. *Up to degree 13 there is only one differential d_3 coming from the $\widetilde{Q}[\tilde{x}_9' y_4]$*

$$d_3(\tilde{x}_9'\, y_4) = d_3(\tilde{x}_9')\, y_4\,. \tag{6.33}$$

reducing the $\widetilde{Q}[x_2^4 y_4]$ tower to a $\mathbb{Z}_8$-torsion piece.

Proof. From the Adams chart and equivariance of differentials under h_0- and h_1-actions in particular it is an immediate consequence that the only possible differential would come from the $\widetilde{Q}[\tilde{x}_9' y_4]$ tower in degree 13. Then we can use the fact that the cup product of the cohomology ring of $H^*(BSs(32), \mathbb{Z}_2)$ induces a multiplicative structure on the Adams Spectral sequence [214]. This entails that we can determine the differential from the $\widetilde{Q}[\tilde{x}_9' y_4]$ tower via the Leibniz rule induced by the multiplicative structure. Of course we have to be careful with the relations (6.23)-(6.27). With this taken into account we get that the differential in question is in fact a d_3 as it is the only non-trivial possibility respecting the Leibniz rule

$$d_3(\tilde{x}_9'\, y_4) = d_3(\tilde{x}_9')\, y_4\,. \tag{6.34}$$

Consequently this cuts the $\widetilde{Q}[x_2^4\, y_4]$ tower in degree 12 to just a $\mathbb{Z}_8$ torsion piece. □

We depict the resulting pages of the spectral sequence in figures 6.12 and 6.13.

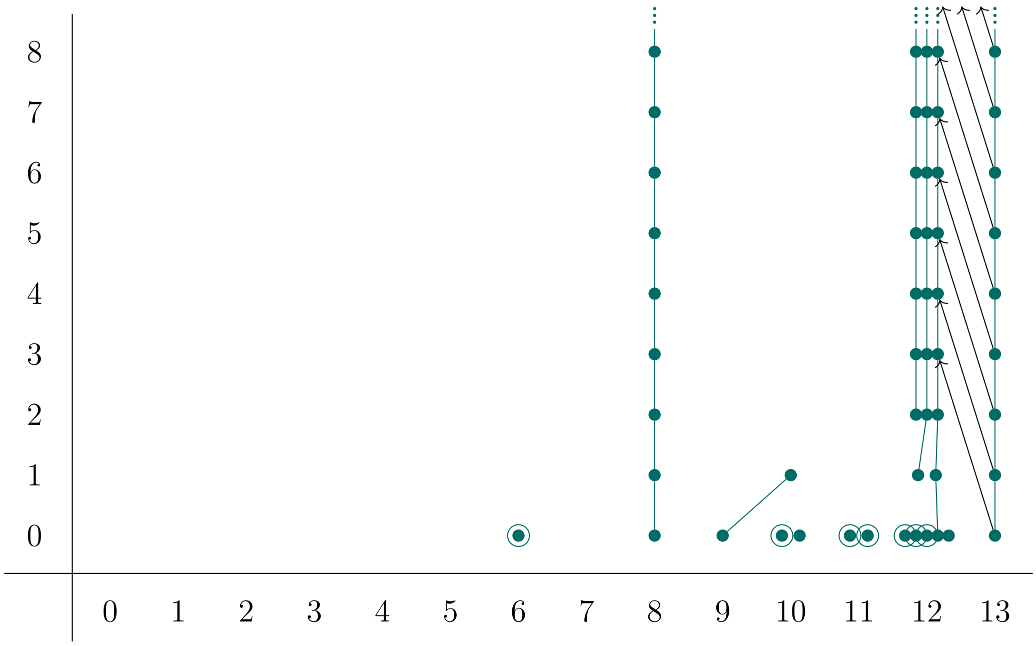

Figure 6.12: Third page E_3 for the (reduced) "x-y-part" with differentials

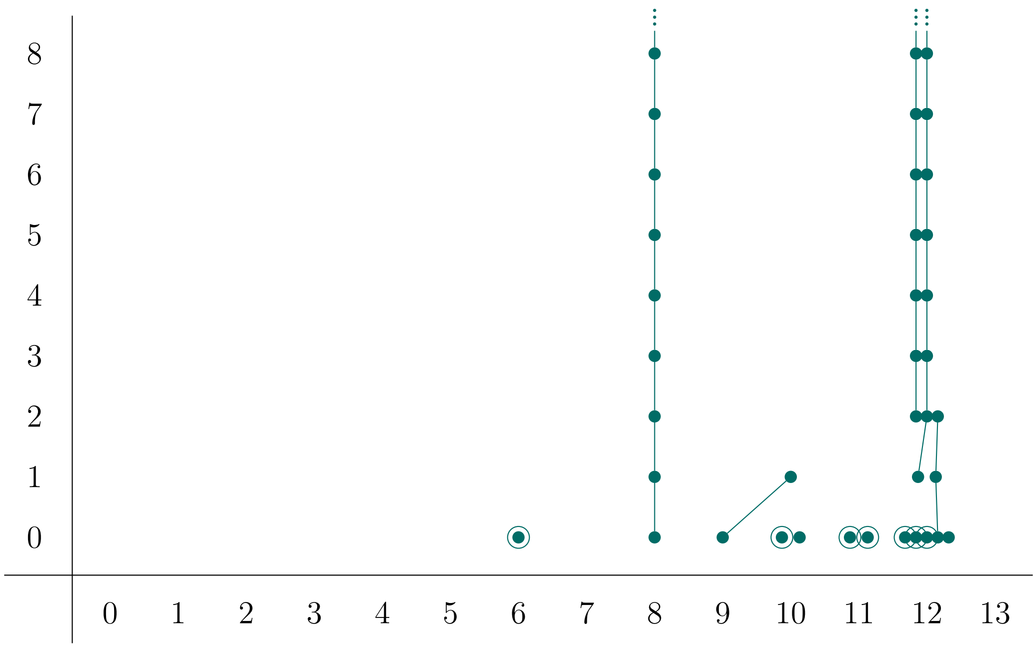

Figure 6.13: Final page E_∞ for the (reduced) "x-y-part"

Eventually, putting all parts together again we obtain the final page for the Adams spectral sequence for $\widetilde{ko}_*(BSs(32))_{\hat{2}}$ in figure 6.14.

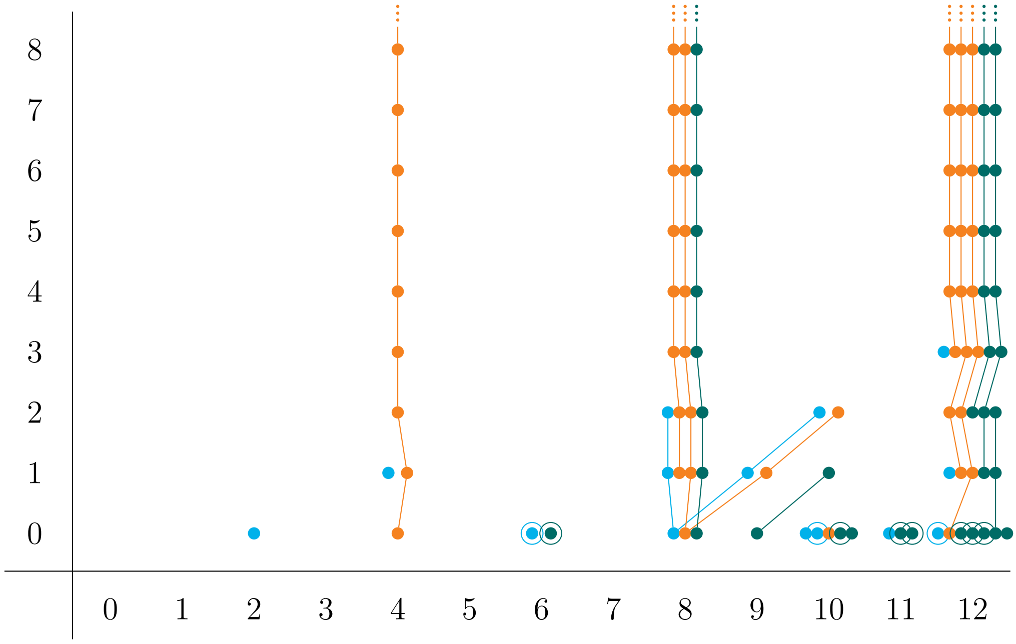

Figure 6.14: Final page E_∞ for $\widetilde{ko}_*(BSs(32))_{\hat{2}}$

Reading off the ko-homology of $BSs(32)$ is straightforward at this point: each node corresponds to a $\mathbb{Z}_2$ and each vertical line, i.e. h_0, to multiplication by 2. Consequently, the towers are depicting a $\mathbb{Z}$-summand. Unfortunately, we are not quite done yet with the determination of the ko-homology groups as there are potentially non-trivial so called hidden extensions. This means that $\mathbb{Z}_2$-nodes in appropriate degree could either split into a direct sum of each other or be connected by a so far undetected h_0. In the case at hand the extensions all split and we do not uncover any hidden extensions. Non-trivial extensions are possible in degrees 9, 10 and 12. So let's tackle them case by case.

Lemma 6.1.5. $\widetilde{ko}_9(BSs(32)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. All of the nodes in degree 9 are connected via a non-trivial h_1-action to nodes in degree 10. Now suppose there is a hidden extension between the green node and either the blue or orange node. Then these two nodes combine to a $\mathbb{Z}_4$. Let's call the generator of this $\mathbb{Z}_4$ x and the image of the h_1 action on $2x$ in degree 10 y, such that $h_1\, 2x \neq 0$. Moreover, the h_1 action lifts to an η-action, corresponding to multiplication by the "anti-periodic" circle in ko-homology. However, since $2\,\eta = 0$, this is contradicting $h_1\,2x = \eta\,2x \neq 0$. Therefore, all $\mathbb{Z}_2$'s must split in $ko_9(BSs(32))$. □

Lemma 6.1.6. $\widetilde{ko}ko_{10}(BSs(32)) \cong 3\,\mathbb{Z}_2 \oplus 2\,\mathbb{Z}_2 \oplus 3\,\mathbb{Z}_2$.

Proof. First of all, the Margolis theorem tells us that the two circled nodes do not participate in any non-trivial extensions and just split off. Furthermore, since all extensions in degree 9 are split, η carries this splitting into degree 10: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset ko_{10}(BSs(32))$. Since the remaining nodes can not form non-trivial extension amongst each other, we're done. □

Lemma 6.1.7. *There are no hidden extensions in* $ko_{12}(BSs(32))$.

Proof. We start by looking at the three blue nodes coming from $H^*(BSs(32), \mathbb{Z}_2)$. Of course, the circled node splits off completely due to Margolis theorem. To see the splitting between the other two nodes we need to put a bit more work in. In our proof before we have exploited a non-trivial η-action.

Here, we will use that the top blue node in degree 12 coming from the $\Sigma^8\,\mathbb{Z}_2[x_2^4]$-module is connected to a node stemming from the same module in degree 20 by a ω-action, corresponding to multiplication by a so-called Bott-manifold B_8. Just like the node in degree 12 the one in degree 20 is the only surviving after taking the tower-killing differential d_3 into account. The same can not be said about the other blue node, which leads to the following situation:

Let's denote the generator of the bottom $\mathbb{Z}_2$ x and the generator of the top blue node y. Assuming the extension doesn't split we have $2x = y$. Now, since $\omega \cdot y \neq 0$, we get $\omega \cdot 2x \neq 0$. However, this is a contradiction, because the ω-action on x vanishes. Therefore, the extension must split.

Since we can map the blue nodes isomorphically into the Adams spectral sequence for $ko_*(B^2\mathbb{Z}_2)$, the blue nodes have to split in the case of $BSs(32)$ as well. The remaining possible hidden extension is between the bottom single green $\mathbb{Z}_2$ node and one of the two $\mathbb{Z}$-towers.

Again, we will make use out of non-trivial ω-actions. Namely, both towers are linked by ω-actions to corresponding towers in degree 20. Even though we have not determined the differentials in that degree we know that there cannot be any non-trivial differential acting on these towers as $d_r(\omega\,x) = \omega\,d_r(x)$ and there is no differential acting on our towers in degree 12. Moreover, the ω-action doesn't just imply the existence of towers in degree 20, but since there is no non-trivial ω-action on the green bottom node (not circled), which can be determined from the short exact sequence used to calculate the Adams chart for the $\tilde{R}_2$, we know that both possible extensions do in fact split. □

Now that we have covered the 2-torsion part for $ko_*(BSs(32))$ the natural next step is to calculate the odd-torsion Adams spectral sequence. However, as we show next this not necessary, in fact $ko_*(BSs(32))_{\hat{2}} = ko_*(BSs(32))$ for $* \leq 12$.

Lemma 6.1.8. *Neither $ko_*(BSs(32))$ nor $\Omega_*^{Spin}(BSs(32))$ contains any odd torsion up to at least degree 12 and therefore $ko_*(BSs(32))_{\hat{2}} = ko_*(BSs(32))$ as well as $\Omega_*^{Spin}(BSs(32))_{\hat{2}} = \Omega_*^{Spin}(BSs(32))$ below at least degree 13.*

Proof. One way to see this is to deploy another type of spectral sequence capable of computing groups of generalized homology theories like connective ko-homology, namely the Atiyah-Hirzebruch spectral sequence (AHSS) [115]

$$E^2_{p,q} = H_p(B, G_q(F)) \Rightarrow G_{p+q}(X)\,. \tag{6.35}$$

for a Serre fibration $X \to B$ with fiber F and a generalized homology theory like cobordism or K-homology, i.e. satisfying all the Eilenberg-Steenrod axioms except for the dimension axiom.

In particular we will use the following Serre fibration

$$BSpin(4n) \to BSs(4n) \to B^2\mathbb{Z}_2\,, \tag{6.36}$$

which can be seen as a consequence of the Puppe sequence for the fibration $B\mathbb{Z}_2 \to BSpin(4n) \to BSs(4n)$. Now, since the differentials are group homomorphisms and since $E^{r+1} = \frac{Ker(d^r)}{Im(d^r)}$, they can not generate odd torsion groups from 2-torsion groups on subsequent pages except for differentials between different $\mathbb{Z}$ entries.

Therefore, if there is no odd torsion on the second page of the aforementioned spectral sequence and we can exclude differentials between free Abelian entries, the infinity page isn't going to contain odd torsion either.

Both $ko_*(BSpin(n))$ and $\Omega_*^{Spin}(BSpin(n))$ do not contain any odd torsion, see e.g. [158] for this statement. Therefore, as long as the integral homology of $B^2\mathbb{Z}_2$ does not contain odd torsion groups the second page of the AHSS is free of odd torsion. Indeed, the integral homology groups for $B^2\mathbb{Z}_2 = K(\mathbb{Z}_2, 2)$ have been computed up to degree 200 in [215], showing absence of odd torsion groups below at least degree 13.

Additionally, the integral homology groups of $K(\mathbb{Z}_2, 2)$ do not contain any free Abelian groups in degree $n \geq 1$, thereby excluding any differentials between different $\mathbb{Z}$ on any page of the spectral sequence. □

□

From $ko_*(BSs(32))$ to $\Omega_*^{Spin}(BSs(32))$

As the final step we can now complete our calculation of $\Omega_*^{Spin}(BSs(32))$

Theorem 6.1.9. *As the result of the ABP-splitting* (4.34) *we get the following cobordism groups*

n	0	1	2	3	4	5	6	7	8
$\Omega_n^{Spin}(BSs(32))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	0	$2\mathbb{Z} \oplus \mathbb{Z}_2$	0	$2\mathbb{Z}_2$	0	$5\mathbb{Z} \oplus \mathbb{Z}_8$

n	9	10	11	12
$\Omega_n^{Spin}(BSs(32))$	$5\mathbb{Z}_2$	$10\mathbb{Z}_2$	$3\mathbb{Z}_2$	$8\mathbb{Z} \oplus 9\mathbb{Z}_2 \oplus \mathbb{Z}_8$

Table 6.3: Spin cobordism groups $\Omega_n^{Spin}(BSs(32))$.

Proof. Let's write the ABP-splitting (4.34) in a more convenient form for us to see what we mean by $ko_{n-10}\langle 2\rangle(X)_{(2)}$:

$$H^*(MSpin, \mathbb{Z}_2) \cong \mathcal{A} \otimes_{\mathcal{A}_1} (\mathbb{Z}_2 \oplus \Sigma^8\mathbb{Z}_2 \oplus \Sigma^{10}J \oplus \dots) \tag{6.37}$$

Here, J denotes the so-called "Joker"-module, which we encountered already, for example directly as the first module within the "x-part". For a detailed account of the next terms we refer to appendix D.1 in [150].

From the expression above we can see that we get:

$$\begin{aligned}\Omega_n^{Spin}(BSs(32))_{\widehat{2}} = &\pi_{t-s}(ko \wedge BSs(32))_{\widehat{2}} \oplus \pi_{8+t-s}(ko \wedge BSs(32))_{\widehat{2}} \\ &\oplus \pi_{10+t-s}(ko \wedge J \wedge BSs(32))_{\widehat{2}} \oplus \dots ,\end{aligned} \tag{6.38}$$

where $\pi_{t-s}(ko \wedge J \wedge BSs(32))_{\widehat{2}}$ is calculated by the Adams spectral sequence $Ext^{s,t}_{\mathcal{A}_1}(J \otimes H^*(BSs(32), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(ko \wedge J \wedge BSs(32))_{\widehat{2}}$.

We see that apart from the two ko-homology building blocks we just get $\Sigma^{10} J \otimes H^*(BSs(32)) = \Sigma^{10} J \otimes (\Sigma^2 J \oplus \dots)$ on top in the degree range we are interested in. This amounts to a full $\mathcal{A}_1$-module in degree 12 [216]. The rest is just utilizing our result on the connective ko-homology of $BSs(32)$. Finally, we can just apply our knowledge from the AHSS that $\Omega^{Spin}_{n\leq 12}(BSs(32))$ doesn't contain odd torsion, so we are done. □

6.2 The non-vanishing cobordism groups and string theory

Finally, we want to sort out the interpretation of the non-trivial cobordism groups in regards to the Cobordism Conjecture requiring some physical mechanism trivializing them. Before we dive into the trivialization mechanism let us provide some explanation on the physics of the ko-homology building blocks under the ABP decomposition (4.34).

6.2.1 The ko-homology building blocks $ko_n(BSs(32))$

We begin by noting that we can split $ko_n(BSs(32)) = ko_n(pt) \oplus \widetilde{ko}_n(BSs(32))$, where $\widetilde{ko}_n(X)$ are the so called reduced ko-homology groups of X. With $n \geq 0$ the first part under the splitting $ko_n(pt)$ can be nicely mapped to $KO^{-n}(pt)$ under Poincaré duality. These groups specifically were famously proposed to classify type I D-branes [65]. This raises the question of how to properly understand the latter groups under the splitting. The refinement we propose is that the ko_n-homology groups at hand detect the magnetic charges of the type I Dp-branes measured in the n-dimensional space transverse to the D-brane worldvolume.

Specifically, the $ko_n(pt)$ detects the set of gravitational magnetic charges, whereas $\widetilde{ko}_n(BSs(32))$ classifies the gauge-theoretic magnetic charges, which arise from open fundamental strings connecting the D-brane to the background D9-branes.

To exemplify this let us look at the magnetic charge of the D5-brane in type I string theory. Let us consider its contribution to the Bianchi identity of C_2, which we integrate over the compact transverse space:

$$N_{grav} = N_{gauge} + N_{D5}\,, \tag{6.39}$$

where $N_{grav} = -\frac{p_1}{2} = \frac{1}{16\pi^2}\int_{M_4} \mathrm{Tr}\, R \wedge R$ denotes the curvature contribution and $N_{gauge} = -p_1(E) = \int_{M_4} \mathrm{Tr}\, F \wedge F$ stands for the gauge instanton contribution. This puts the gravitational and the gauge instanton on the same footing as the number of D5-branes making them really indistinguishable from one another [63]. This behavior should not surprise us, if our refinement to the K-theory – D-brane relationship holds true. And precisely, $\frac{1}{16\pi^2}\int_{M_4} \mathrm{Tr}\, R \wedge R$ is just a multiple of $\hat{A}_4$, i.e. the index of the Dirac operator, and detects non-triviality of $\mathbb{Z} \cong ko_4(pt)$. Fittingly, $\int_{M_4} \mathrm{Tr}\, F \wedge F$ uplifts to the cobordism invariant detecting the other free Abelian piece in codimension 4: $\mathbb{Z} \subseteq \widetilde{ko}_4(BSs(32))$. After carefully implementing electric-magnetic duality (Poincaré duality) on a fixed background space there is a further refinement for the electric charges. This involves a few more steps, which will be worked through in [217].

Of course, we should ask for an interpretation of the same ko-homology building blocks for the $SemiSpin(32)$ heterotic string as well. Actually, we can see this as the K-theoretic realization of Hull's proposal for the non-perturbative sector of the HO string [93], which is based on the description of the $SemiSpin(32)$ heterotic string as a composite orientifold of type IIB [92]

$$\widetilde{\Omega} = S\,\Omega\,S^{-1}\,, \tag{6.40}$$

consisting of the S, the S-duality $SL(2,\mathbb{Z})$-transformation of type IIB, and Ω the standard type IIB orientifold giving rise to type I string theory. Consistency of the theory requires now 32 NS9-branes giving rise to the background gauge field just like their S-dual twins. Still, why does string perturbation theory look so fundamentally different for the $SemiSpin(32)$ heterotic string as compared to the type I string?

The resolution lies in the fact that the fundamental open string on the type I side gets turned into a D-string on the heterotic side, whose tension scales like $\frac{1}{g_s}$. This entails that in the perturbative limit $g_s \to 0$ the D-string, becoming infinitely heavy, retracts into the object it is ending on. The massless sector of the D-string survives this limit and for example for D-strings attached to the fundamental closed HO string these massless modes provide the worldsheet structure we are familiar with.

Our ko-homology groups are, of course, not sensitive to the string coupling, and a heterotic interpretation entails that the groups classify the charges of the heterotic NSp-branes associated with its open D-string sector. In particular $\widetilde{ko}_n(BSs(32))$ assorts the charges resultant from the background NS9-branes connected to the lower dimensional NSp-branes through the D-strings. So, while we would naively expect K-theory to play no role in the interpretation of NSp-branes, it actually does, since it is only susceptible to the open string endpoints. Whether it is Dp-brane–NS1 or NSp–D1, does not matter. We will see the importance of K-theory for the description of the D-string imprint in HO-theory in the next chapter 7. Form this perspective it is not surprising, that we get for example a tadpole condition for NS5-branes on a compact transverse space mirroring (6.39). So S-duality between type I and HO string theory manifests itself as self-duality of the ko-homology groups:

$$ko_n(BSs(32)) \xleftrightarrow{S-\text{duality}} ko_n(B^L Ss(32)) \cong ko_n(BSs(32))\,. \tag{6.41}$$

In the following we will usually take the type I perspective on the cobordism groups. Of course, there exists a S-dual heterotic perspective along the lines outlined above, which we sometimes highlight as well.

$$\begin{array}{ccccccc}
\text{type I} & & \widetilde{ko}_n(BSs(32)) & & \widetilde{ko}_n(BSs(32)) & & \text{HO} \\
 & & \uparrow{\scriptstyle \text{DN}} & & \uparrow & & \\
\Omega_n^{Spin}(BSs(32)) & \xrightarrow{\text{ABS}} & ko_n(BSs(32)) & \xleftrightarrow{\text{S-duality}} & ko_n(BSs(32)) & \xleftarrow{\text{ABS}} & \Omega_n^{Spin}(BSs(32)) \\
 & & \downarrow{\scriptstyle \text{DD/NN}} & & \downarrow & & \\
 & & ko_n(pt) & & ko_n(pt) & &
\end{array} \tag{6.42}$$

Next we want to take a look at the torsional pieces of our spin cobordism groups up until $n = 8$.

6.2.2 The torsional spin cobordism subgroups $\tilde{\Omega}_{n\leq 8}^{Spin}(BSs(32))$

As we have discussed before certain $\mathbb{Z}_{2^p}$ summands of $\Omega_2^{Spin}(BSs(32))$ are in the image of the map to $\Omega_n^{Spin}(B^2\mathbb{Z}_2)$. In particular, we want to look at the image under the map between the reduced cobordism groups. Their associated cobordism invariants are $\int_{M_{2k}} x_2^k$ with $k = 1,3,4$, where x_2^k are the "generalized" Stiefel-Whitney classes referenced in [218, 219] and $\frac{1}{2}\mathcal{P}_2(x_2)$ in dimension $n = 4$ [152], where $\mathcal{P}_2(x)$ is the Pontryagin square of x. Furthermore, we find a $\mathbb{Z}_2$ in $\Omega_6^{Spin}(BSs(32))$ detected by $x_2 y_4$.

n	0	1	2	3	4	5	6	7	8
$\widetilde{\Omega}_n^{Spin}(BSs(32))_{tors}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_8$
invariants	-	-	$\int_{M_2} x_2$	-	$\frac{1}{2}\mathcal{P}_2(x_2)$	-	$\int_{M_6} x_2^3$, $\int_{M_6} x_2 y_4$	-	$\int_{M_8} x_2^4$

Table 6.4: Reduced torsional cobordism groups and their invariants

- n = 2: Following [150, 161] we know that the nontrivial $\mathbb{Z}_2$ is detected by the corresponding cohomology class, since the $\mathbb{Z}_2$ stems from filtration $s = 0$ in the Adams Spectral sequence. The convention in the physics literature on this topic is to call the cohomology class $\widetilde{w}_2$ in analogy to the Stiefel-Whitney-classes w_i. However, we will stick to our convention of calling it x_2. First, we note that $\widetilde{\Omega}_2^{Spin}(BSs(32)) \cong \widetilde{ko}_2(BSs(32))$ and therefore expect a D-brane interpretation. The simplest example we can look at is a type I compactification on a T^2 with full $Ss(32)$-gauge group, i.e. without vector structure in the language of [219]. As laid out in [220] the non-trivial $\mathbb{Z}_2$ ko-theory charge arising here should be understood as the charge of a non-BPS $\widehat{\text{D7}}$-brane. Now, why is that? The charge of the "conventional" $\widehat{\text{D7}}$-brane in type I was identified with $KO^{-2}(pt)$ in [65], which is of course Poincaré dual to $ko_2(pt)$. We can draw an analogy to the D5-brane case we saw before. While the $\widehat{\text{D7}}$-brane does not couple to a dynamical gauge field, the K-theoretic tadpole cancellation (6.39) still arises [220]. The ingredients are again invariants of the ko-homology groups in question:

$$\int_{M_2} x_2 + \hat{A}_2 + N_{\widehat{D7}} = 0 \mod 2\,, \tag{6.43}$$

  where we refer to the mod 2 index of the Dirac operator on the transverse compact space M_2 as $\hat{A}_2$ [221]. Similarly to the D5-brane case the charge of the $\widehat{\text{D7}}$-brane is really indistinguishable from the gauge and gravitational contribution. This should not surprise us, since both invariants have been identified with the non-BPS $\widehat{\text{D7}}$-brane. The gravitational invariant showed up as a "Berry's phase" in the system of a probe non-BPS $\widehat{D0}$-brane and the $\widehat{\mathrm{D7}}$-brane itself [222]. The complementary statement can be found in [220]: The $\widehat{D7}$ shows up as a toron gauge-field configuration in the background presence of D9-brane(s), matching precisely our expectation that this charge would not be realized without the background gauge field provided by the D9-branes. The torsional tadpole constraint (6.43) can be fulfilled in a couple of different ways. The simplest case is, if every term vanishes separately (mod 2). In the language of [113], the higher form symmetries are gauged, i.e. the physical system is in the trivial cobordism configuration.

However, this makes the opposite configuration, i.e. both the gauge and gravitational charge contributions are odd and $N_{\widehat{D7}} = 0 \mod 2$, more subtle. While cancellation against each other ensures string theoretic tadpole cancellation, none of the two nontrivial global symmetries are resolved by gauging.

So the remaining pathway to quantum gravitational consistency is breaking both of those by codimension 3 defects. Remarkably, there seem to exist supergravity solutions describing these codimension 3 defects. The defect 6-brane breaking $\widetilde{\Omega}_2^{Spin}(BSs(32))$ was identified in [223] (we also refer to [224, 225] for a more extensive discussion) as the extremal limit of the non-supersymmetric heterotic black 6-brane solution [226], which looks like a
4-dimensional magnetic black hole with 6 flat dimensions added:

$$ds^2 = dx^\mu dx_\mu + dy^2 + r_0 d\Omega_2^2, \quad e^{-2\phi} = g_s^{-2} e^{y/r_0}\,, \tag{6.44}$$

additionally the S^2 horizon comes equipped with non trivial $\int_{S_2} x_2$ charge.

Moreover, the authors of [223] propose a detailed worldsheet description of this 6-brane through a $(SU(16)/\mathbb{Z}_4)_1$ spin-CFT with $c_L = 15$ on the S^2-part, such that in total one gets:

$$\mathbb{R}^{(1,6)} \times \mathbb{R}_{\text{linear dilaton}} \times \text{CFT}((SU(16)/\mathbb{Z}_4)_1)\,. \tag{6.45}$$

The 6-brane, which has to accompany the non-BPS $\widehat{\mathrm{D7}}$-brane and breaks
$\Omega_2^{Spin}(pt)$, is expected to have a supergravity solution as well. It should look somewhat similar to its twin, the ETW-7-brane, arising when studying the consistency of the backreacted geometry of a single non-BPS $\widehat{\mathrm{D8}}$-brane [37]. There, one gets a spontaneously compactified dimension, in form of a S^1, in the space transversal to the $\widehat{\mathrm{D8}}$-brane, which matches that the generator of $\widetilde{\Omega}_1^{Spin}(pt)$ is precisely

the circle with periodic boundary condition for fermions. As expected one finds that the topology of the space transverse to the ETW-7-brane solution is that of a disk. The ETW-6-brane subsequently would feature a transverse topology $S^1 \times D^2$ bounding the 2-torus generating $\Omega_2^{Spin}(pt)$[4].

While the geometry of the respective solutions reflects nicely the expected properties from the cobordism viewpoint, the purely bosonic Lagrangian utilized to construct these solutions does not detect the periodic fermionic boundary conditions. One option would be to use the aforementioned $\widehat{D0}$-brane probe [222]. It should be emphasized however that this $\mathbb{Z}_2$ arises from the necessary spin-structure and not from the background D9-branes introducing the background gauge theory and as already discussed above has to be combined with $\widetilde{ko}_2(BSs(32)) \cong \mathbb{Z}_2$ to describe the full K-theoretic charges associated to the non-BPS $\widehat{\mathrm{D7}}$-brane.

Finally, we can ask, if we can understand the transformation of these defects under duality. In particular the $Ss(32)$-heterotic string is T-dual to the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string. In particular it is known that the $Ss(32)$-string on a T^2-compactification with nontrivial x_2 is T-dual to the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string, where the two E_8 s are exchanged, when going around one circle in the 2-torus [219]. Precisely this exchange symmetry is detected by $\Omega_1^{Spin}(B\mathbb{Z}_2)$ stemming from the fibration $E_8 \times E_8 \to (E_8 \times E_8) \rtimes \mathbb{Z}_2 \to \mathbb{Z}_2$ as explained in [162]. While we work at a different level of structural refinement, i.e. not yet incorporating the twisted string structure of the $Ss(32)$ heterotic string, these classes would also be present, if one would work with $\Omega_n^{Spin}(B((E_8 \times E_8) \rtimes \mathbb{Z}_2))$ instead, matching our level of refinement.

The T-duality between two NS5-branes and the non-supersymmetric heterotic 6-brane proposed in [227] is not in reach at this level of refinement as the T-duality appears to map defects breaking nontrivial cobordism groups. The NS5-brane first shows up as a breaking defect at String-structure breaking $\Omega_3^{String}(pt) \cong \mathbb{Z}_{24}$ generated by a S^3 with H-flux [113]. Therefore, we should expect to find the S^3 as the generator of the cobordism with twisted string structure for the HO-string,

[4] We have not commented on the interval in the geometry created by the dynamical tadpole associated to the non-BPS $\widehat{D8}/\widehat{\mathrm{D7}}$-brane. However, it turns out, when properly accounting for the precise topology, that only the boundary of the interval has a nontrivial cobordism group. Therefore, we can think of the ETW-brane as the defect necessary to cap off the S^1 boundaries of the cylinder with disks.

probably even
$\Omega_3^{String-Ss(32)}(pt) \cong \Omega_3^{String}(pt)$.

- n = 4: The natural setup to look at here is type I/HO string compactifications without vector structure on a four dimensional manifold. In particular the orientifold of type IIB on the $T^4/\mathbb{Z}_2$ orbifold limit of K3 studied in [228] comes to mind. Tadpole cancellation requires us to introduce 8 dynamical D5-branes with its corresponding collective Chan-Paton index value being 32 due the orientifold and the orbifold. There are a multitude of different solutions based on the position of the D5-branes, i.e. whether they reside at one of the 16 fixed points or not. Now, does this orbifold construction impact our analysis as we are not working with $\mathbb{Z}_2$ equivariant ko-homology/spin cobordism necessary to properly take the orbifold into account?

  The answer turns out to be no. As demonstrated in [218] the fixed points of this precise orbifold[5] can actually all be blown up and the spectra fully agree with the smooth K3. More importantly for our discussion here, the fixed points were shown to each carry a hidden instanton and when blown up the associated gauge bundle supported on the S^2 replacing the singularity indeed stems from a Ss(32)-bundle. This can be seen as follows: The construction in [228] necessitates a non-trivial twist acting on the Chan-Paton label by a matrix M in 16×16 block form:

$$M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} . \tag{6.46}$$

  Since this does not square to 1, it would be causing an inconsistency, if the gauge group were $SO(32)$. But we can take $M^2 = w$, where $w \in Z(Spin(32))$ and $w = -1$ in the vector and one spinor representation of $Spin(32)$, but $w = 1$ in the second spinor representation. So for the actual $Ss(32)$ gauge group we have exactly $M^2 = 1$ and topologically trivial paths around the fixed point are well defined.

  Now, upon blowing up the singularity the authors of [218] showed that the resulting two-sphere S with self-intersection -2 supports a gauge field obeying Dirac quantization for the adjoint or spinor, but not for the vector. Subsequently the first Chern number of the gauge bundle on S has to be normalized as $\int_S \frac{F}{2\pi} = \frac{1}{2}$.

[5] This does not hold true for other orbifold limits of K3 [229].

After blowing up the singularities the final instanton number on the Eguchi-Hanson space X[6] turns out to be precisely the one matching the requirement that the 16 fixed points provide the missing 16 to the K3 tadpole cancellation condition $24 = n_{D5} + 16$. Therefore, the GP model after singularities have been blown up can be understood from a non-equivariant cobordism perspective.

However, while this example is very instructive to construct an instanton exhibiting the key physical feature of absence of vector structure as expected for a $Ss(32)$-instanton, it is not yet what we should aim for. It turns out that the "background" gauge group provided by the D9-branes is not $Ss(32)$ anymore, but broken to $U(16)/\mathbb{Z}_2$. Therefore, we will pivot to a setup, where the full $Ss(32)$-group is preserved. A detailed account of this setup as the general F-theory construction of type I/HO K3 compactifications with fully preserved 9-brane gauge group is provided by [230]. It turns out that the instanton construction of [218] has to be adapted just slightly. In general we can write the integral over the curvature of the gauge bundle as

$$\int_{C_i} \frac{F}{2\pi} = \frac{1}{2}(\widetilde{w}_2 \cdot C_i) + k \,, \tag{6.47}$$

where we adapted our notation to the one in [230], denoting with C_i one of the 16 exceptional divisors associated to the 16 fixed points of the $\mathbb{Z}_2$ orbifold limit of K3 and $\widetilde{w}_2 \in H^2(C_i, \mathbb{Z}_2)$ arising from the classifying map $f : X \to BSs(32)$.

[230] now argues that since the curvature of the instanton should arise from the local geometry the "generalized" Stiefel-Whitney class $\widetilde{w}_2$ has to be proportional to C_i. While the instanton of [218] is defined by $\widetilde{w}_2 = \frac{1}{2}C_i$, the instanton in the case of unbroken $Ss(32)$ is constructed through $\widetilde{w}_2 = C_i$ [230]. Now, the obstruction can not be detected by C_i itself, as $C_i \cdot C_i = 0 \mod 2$. Nevertheless, we can consider a dual exceptional divisor C_i', which can detect the obstruction, i.e. $C_i' \cdot \widetilde{w}_2 = C_i' \cdot C_i = 1$. Therefore, this results in a contribution of four per instanton to the instanton number on K3. Based on the dual description of the type I/HO string on K3 in terms of F-theory compactified on a Calabi-Yau threefold X with elliptic fibration $f : X \to \mathbb{F}_n$, where $\mathbb{F}_n$ denotes a Hirzebruch surface, [230] further derived a precise correspondence between the number of

[6] Here X is used to approximate the space close to S and can be treated as the total space of the line bundle $\mathcal{O}(-2)$, when we regard the two-sphere S as the complex space $\mathbb{P}_1$.

these unconventional instantons present and the integer n defining the Hirzebruch surfaces as $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(n))$, which are $\mathbb{P}_1$-fibrations over $\mathbb{P}_1$ [196]. The tadpole cancellation on the K3 for the $Ss(32)$-heterotic string then becomes

$$\sum_i^{\mu} k_i + 4(4-n) = 24\,, \tag{6.48}$$

where the first contribution comes from μ groups of conventional heterotic instantons and the second contributions from the special instantons of instanton number 4 associated to the $4-n$ collisions between the discriminant and the zero section of $\mathbb{F}_n$ in the F-theory description. Equally, we can phrase this as a contribution from

$$\widetilde{w}_2 \cdot \widetilde{w}_2 = 2(n-4) \tag{6.49}$$

through the $\widetilde{w}_2$ correction to the integral over the curvature of the gauge bundle (6.47). This entails that there are three succinct equivalence classes[7] of the type I/HO string on K3 enumerated by elements in integral homology $H_4(B^2\mathbb{Z}_2, \mathbb{Z})$, namely

$$\widetilde{w}_2 = 0\,, \tag{6.50}$$

$$\widetilde{w}_2 \neq 0 \text{ and } \widetilde{w}_2^2 = 0 \mod 4\,, \tag{6.51}$$

$$\widetilde{w}_2 \neq 0 \text{ and } \widetilde{w}_2^2 = 2 \mod 4\,. \tag{6.52}$$

Now we will argue that the charge of the instanton can be understood from gauging $\mathbb{Z}_2 \cong \widetilde{\Omega}_4^{Spin}(B^2\mathbb{Z}_2) \subset \widetilde{\Omega}_4^{Spin}(BSs(32))$, i.e. the only physically acceptable configurations are the ones with $\frac{1}{2}\mathcal{P}(x_2) = 0 \mod 2$, where $\frac{1}{2}\mathcal{P}(x_2)$ is the cobordism invariant associated to the nontrivial $\mathbb{Z}_2$.

By employing the associated Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_0}^{s,t}(H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow H_{t-s}(B^2\mathbb{Z}_2, \mathbb{Z})_{\widehat{2}}\,. \tag{6.53}$$

we can examine the close connection between $H_4(B^2\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_4$ and $\widetilde{ko}_4(B^2\mathbb{Z}_2) = \mathbb{Z}_2$, that both arise from the same d_2 cutting down the initial

[7] To emphasize the difference between models "with" and "without vector structure", the case $\widetilde{w}_2 = 0$ is conventionally split off from the $\widetilde{w}_2^2 = 0 \mod 4$ equivalence class. Although, just by looking at $\widetilde{w}_2^2$ we cannot detect this difference.

h_0 tower going from the 2nd to the 3rd page in their respective Adams spectral sequence.

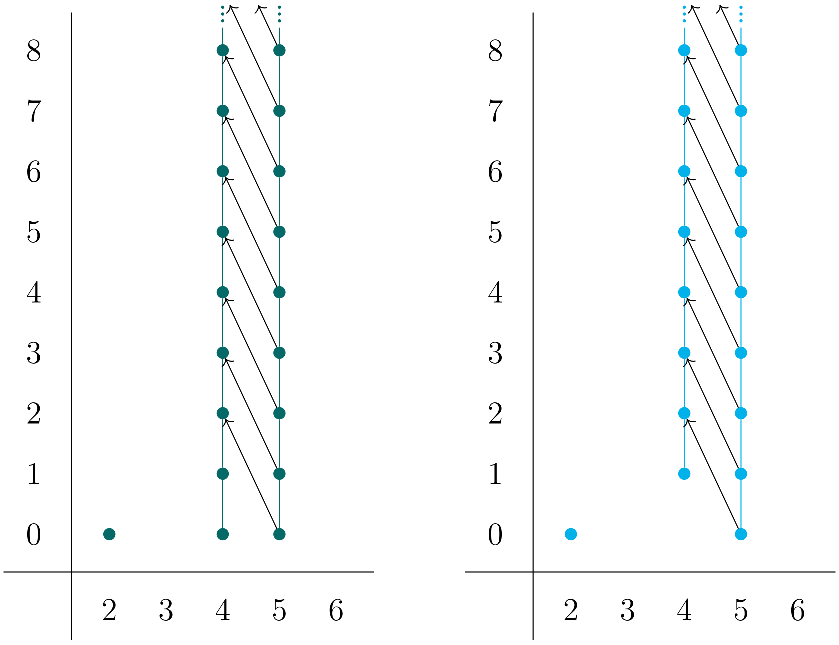

Figure 6.15: Second pages E_2 for $H_4(B^2\mathbb{Z}_2, \mathbb{Z})_{\widehat{2}}$ and $ko_4(B^2\mathbb{Z}_2)_{\widehat{2}}$

We lose the class in filtration zero, when going from $H_4(B^2\mathbb{Z}_2, \mathbb{Z})_{\widehat{2}}$ to $\widetilde{ko}_4(B^2\mathbb{Z}_2)_{\widehat{2}}$, because of a non-trivial second Steenrod Square Sq^2 enforcing the spin condition[8]. At this point we can exploit the fact that the two $\mathbb{Z}_2$ in degree 4 and filtration 0 and 1, let's call them a and b, are connected by a h_0 and therefore detect corresponding classes $\alpha, \beta \in H_4(B^2\mathbb{Z}_2, \mathbb{Z})$ with $\beta = 2\,\alpha$. This means that b exactly detects the equivalence classes $\widetilde{w}_2^2 = 0, 2 \mod 4$ we are interested in. Now, the map $\mathcal{A}_1 \to \mathcal{A}_0$ induces a map

$$\phi : \mathrm{Ext}^{1,3}_{\mathcal{A}_1}(H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \to \mathrm{Ext}^{1,3}_{\mathcal{A}_0}(H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2)\,, \tag{6.54}$$

which is an isomorphism due to mapping a $\mathbb{Z}_2$ to another $\mathbb{Z}_2$(see appendix D. of [161] for a very similar map). Therefore, our cobordism invariant detects the same equivalence classes as $\widetilde{w}_2^2 = 0, 2 \mod 4$. Invoking gauging the nontrivial

[8] This Sq^2 is the first non-trivial differential d_2 for the corresponding AHSS $H_p(B^2\mathbb{Z}_2, ko_q(pt)) \Rightarrow ko_{p+q}(B^2\mathbb{Z}_2)$ and can be understood as an instability due to D-instantons as discussed in [148].

$\mathbb{Z}_2 \subset \Omega_4^{Spin}(BSs(32))$ we need even numbered magnetic charges $\frac{1}{2}\mathcal{P}(x_2) = 0 \mod 2$ carried by the NS5-branes/D5-branes in the respective descriptions of the theory. Going backwards through the chain we can map this to $\widetilde{w}_2^2 = 0 \mod 4$ or expressed differently the contribution of the NS5-branes/D5-branes, i.e. the $SemiSpin(32)$-instantons in the full F-theory framework, to the instanton number measured within $H_4(BSs(32), \mathbb{Z})$ has to be a multiple of 4, such that $\widetilde{w}_2^2 = 0 \mod 4$. As we reviewed above this is exactly the charge such an instanton contributes [230].

- n = 6: Additionally to $\int_{M_6} x_2^3 \in \mathbb{Z}_2$ we also encounter $\int_{M_6} x_2 y_4 \in \mathbb{Z}_2$, which originates from the "x-y-part" in the Adams spectral sequence. Both stem from a $\mathbb{Z}_2$ in filtration zero and are therefore detected by their cohomological counterparts. Curiously, we encounter a somewhat unexpected asymmetry between $\Omega_6^{Spin}(pt) \cong ko_6(pt) = 0$ and $\widetilde{\Omega}_6^{Spin}(BSs(32)) \cong \widetilde{ko}_6(BSs(32)) = 2\,\mathbb{Z}_2$. We have already seen generally and for $ko_2(BSs(32))$ in particular that the splitting principle $ko_n(BSs(32)) \cong ko_n(pt) \oplus \widetilde{ko}_n(BSs(32))$ assorts the Abelian groups into different open string sectors. Consequently, this suggests that our nontrivial $\widetilde{ko}_6(BSs(32))$ groups capture the charges of a non-BPS $\widehat{\text{D3}}$-brane arising because of open fundamental strings connecting the $\widehat{\text{D3}}$-brane to the background stack of 32 D9-branes and an O9-plane.
  Thus, our expectation is that this non-BPS D3-brane would only exist as a boundary condition for DN strings supplying it with the necessary gauge degrees of freedom to stabilize it, while for the DD sector there is no topological obstruction for a decay. [70] provides a general construction of the type I non-BPS $\widehat{Dp}$-branes as type IIB Dp-$\overline{\text{D}p}$-brane bound states, where the generic tachyon can be projected out by the orientifold, which we introduced in chapter 2. By analyzing the DD sector of the open string amplitude precisely this anticipated tachyonic instability was found. One gets the following condition for the absence of tachyons
  $$\mu_p = 2\sin(\frac{\pi}{4}(9-p)) \, > \, 0\,, \tag{6.55}$$
  which is only true for all the non-BPS $\widehat{Dp}$-branes classified by $ko_n(pt)$, i.e. $p = -1,\ 0,\ 7,\ 8$. However, things change once we look at the DN sector. [70] also give a criterion for tachyonic instability in the DN sector, namely:
  $$a_{NS} = \frac{1}{2} - \frac{\nu}{8} \, < \, 0\,. \tag{6.56}$$

Here, ν is the number of Dirichlet-Neumann directions available to the open string. Concretely, we see that our non-BPS $\widehat{\text{D3}}$-brane would be free of tachyons in this sector. Interestingly, this is completely orthogonal to the non-BPS $\widehat{D7}$ and $\widehat{\text{D8}}$-branes, which are unstable in this sector while they are stable in the DD sector. The consequences of this DN sector instability was explored in [71]. It would be very interesting to further explore the DD instability in the presence of the D9-branes/O9-plane background with $\int_{M_6} x_2 y_4$ or $\int_{M_6} x_2^3$ non-trivial, saving the configuration from completely decaying to the vacuum. In [219] the case of $\int_{T^6} x_2^3 \neq 0$ realized as a type IIB orientifold on $T^6/\mathbb{Z}_2$ was briefly mentioned. Of course this raises the question if this charge is cancelled. This is a bit different from the $n = 4$ case, where the cohomology class x_2^2 does not detect the spin cobordism class. In the aforementioned orientifold model there are 2 D3-brane pairs present on the type IIB side. It seems reasonable to expect that the charge $\int_{T^6} x_2^3$ can be associated to a non-BPS $\widehat{\text{D3}}$-brane on the type I side and triviality is achieved by a tadpole cancelling configuration of them.

From the perspective of the cobordism conjecture one might reasonably expect that the non-triviality is resolved by uplifting from spin-cobordism to a twisted string structure by properly taking the Bianchi identity into account. It might happen that both $\int_{M_6} x_2^3$ and $\int_{M_6} x_2 y_4$ do not detect any cobordism classes after the uplift. While this scenario provides a satisfying answer to the non-triviality of the spin cobordism classes, we would naively violate K-theoretic charge cancellation in the DN-sector. In particular in accordance to the non-BPS $\widehat{\text{D7}}$-brane case [220] we would expect some form of non-perturbative inconsistency for the type I/HO dual of the aforementioned type IIB model. Besides the gauging of the charge there is also the pathway of breaking the corresponding global symmetries by suitable codimension 7 defects left to be explored.

So far we have not commented on the close connection between discrete flux choices and the cohomology classes we have encountered up until this point. In particular $\int_{M_2} x_2 \neq 0$ and $\int_{M_2} x_2^2 \neq 0$ are dual to type IIB orientifold configurations with discrete values for B_2 flux (or B_2^2 respectively) [231–233]. Consequently, one might be able to explore compactifications of type I with nontrivial $\int_{M_6} x_2^3$ or $\int_{M_6} x_2 y_4$ from a dual type IIB perspective corresponding to specific discrete fluxes turned on. In [234] the authors explored a possible discrete 6-form flux and remarked the absence of a degree 6 “generalized" Stiefel-Whitney class in the cohomology of $BSs(32)$, i.e. there is no x_6, complicating the identification of the discrete flux with a cohomology class.

- n = 8: Alike the $n = 6$ case we attain a torsional cobordism group detected by a cohomological invariant $\int_{M_8} x_2^4$, which can also be tracked from the ko-homological viewpoint. It would be very interesting to investigate whether such compactifications with nontrivial torsional K-theory charge arise from F-theory fivefold compactifications [235], presumably alike the other type I/HO compactifications “without vector structure" in its frozen phase [65, 236, 237].

6.2.3 The remaining torsional spin cobordism subgroups $\Omega^{Spin}_{n>8}(BSs(32))$

As we have seen in table 6.3, the groups $\Omega^{Spin}_{n>8}(BSs(32))$ are crowded with torsional subgroups, which makes it tough to decipher the physical meaning of each one of them. Therefore, we will just make some general remarks. First let us mention $\Omega^{Spin}_{11}(BSs(32))$. Based on its non-triviality one would expect global anomalies [146]. While the same calculation for the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ heterotic string approximated to $\Omega^{Spin}_{11}(BE_8)$ lead to a vanishing group, this is not the end of the story as the full computation has to take the Bianchi identity of the heterotic string into account and results in a non-trivial group [162]. Still, by relying on the Segal-Stolz-Teichner conjecture [238] the authors of [239] provide a general proof for absence of global symmetries in both supersymmetric heterotic string theories. Also, the expected absence of global anomalies in type I string theory has been confirmed in [240] by utilizing KO-theory.

The ko-homology subsector of the cobordism groups of dimension 9 and 10 once again can be linked to Dp-branes. In dimension 9 we expect a correspondence to particle-like defects, whereas dimension 10 should classify instanton effects. Specifically, we meet the gravitational and gauge-theoretic instanton of [241] classified by $ko_{10}(pt) \subset \Omega^{Spin}_{10}(BSs(32))$ and $\pi_{10}(BSs(32)) \subset \Omega^{Spin}_{10}(BSs(32))$ respectively. Those groups are detected by the mod 2 index of the Dirac operator and $\int_{S^{10}} trF^5 \in \mathbb{Z}_2$. The first one is argued to be gauged in [241] as this particular mod 2 index is always even in string theory. The latter one would lead to a similar anomaly (in 9d) as the prime example $\pi_4(SU(2))$ [242]. The corresponding heterotic instantons are instrumental to resolving the origin of Shenker's $1/g_s$-effects in heterotic string theory [36], which we detail in the next chapter.

6.3 Conclusions

The conjectured incompatibility of quantum gravity with global symmetries leaves distinct imprints in its topological sector. The Cobordism Conjecture is a recent formalization of this general principle. In particular it relates non-trivial cobordism groups to higher-form global symmetries in an effective field theory coupled to gravity, whose physics gets encoded in both the tangential structure and the background space of the relevant cobordism groups. In this work we specifically took a look at the consequences of the Cobordism Conjecture for type I and its S-dual formulation as the $Spin(32)/\mathbb{Z}_2$, i.e. $Ss(32)$ in an unambiguous language, heterotic string theory.

To this end we computed the mod 2 cohomology of the classifying space for $Ss(32)$ $H^n(BSs(32), \mathbb{Z}_2)$ via the Eilenberg-Moore spectral sequence in order to feed the Adams Spectral sequence to reach our final goal of calculating $\Omega^{Spin}(BSs(32))$. The physics behind the nontrivial spin cobordism groups can be nicely tracked through its ko-homology building blocks. Here, we observe that the splitting principle $ko_n(BSs(32)) = ko_n(pt) \oplus \widetilde{ko}_n(BSs(32))$ divides the charges classified by the K-theory groups into the ones arising from the Dirichlet-Dirichlet (Neumann-Neumann for $ko_0(pt)$) sector and Dirichlet-Neumann sector respectively. It should be stressed that at this level of refinement we specifically observe the open string sector of both type I/HO string as we account for the objects the endpoint(s) of the type I fundamental open string and the HO D-string live on. This matches nicely with [93] and will be explored further in the next chapter 7. The close relation to ko-homology also carries over to the observation that the interpretation connects nicely to type I/HO string compactifications known as compactifications without vector structure. Furthermore, we have also seen that for string theory setups with multiple simultaneously non-trivial cobordism groups experience stronger constraints than just from (K-theoretic) tadpole cancellation as it can only account for cancellation of a diagonal component of the groups. In the future, clearly the twisted string cobordism groups encoding the Bianchi identity of type I/HO string theory directly in the tangential structure have to be calculated. Also, extending the computations of [243] to

$\Omega_n^{String-((Spin(16)*Spin(16))\rtimes\mathbb{Z}_2)}$[9] would be very interesting as it could shed some light on non-perturbative objects of the unique non-supersymmetric heterotic string theory.

[9] $(Spin(16)*Spin(16))\rtimes\mathbb{Z}_2$ was brought forward in [75] as the refinement to $O(16)\times O(16)$ to account for one of the many subtleties involving the non-supersymmetric heterotic string. Concretely, it is a cover for both $(Ss(16)\times Ss(16))\rtimes\mathbb{Z}_2$, which embeds into $(E_8\times E_8)\rtimes\mathbb{Z}_2$, and the other quotient $(\frac{Spin(16)\times Spin(16)}{\mathbb{Z}_2\times\mathbb{Z}_2})\rtimes\mathbb{Z}_2$, which embeds into $Ss(32)$. Hopefully, the calculation we have detailed in this work can facilitate the computations for all three groups.

7

Open Strings and Heterotic instantons

This chapter is close in spirit the previous one, in which we determined the $spin$ cobordism and ko-homology groups relevant to both type I and the HO string. Here, we want to explore the domain left out in the last chapter, namely that of instantons particularly in heterotic string theory. Endowed with the tools from algebraic topology we present a resolution for a long-standing problem in string theory, Shenker's effects in heterotic string theory [33]. In a classic paper [33], Shenker argued that, to make sense of the asymptotic series for amplitudes produced in closed string perturbation theory, there should be universal leading non-perturbative corrections of order $\mathcal{O}\left(e^{-1/g_s}\right)$.

This amplitudes argument applies to all closed string theories, and hence in particular to the heterotic ones. However, since these lack D-branes, the possible origin of their Shenker effects remains unclear. Early on, Silverstein argued for heterotic $\mathcal{O}\left(e^{-1/g_s}\right)$ effects as the S-duals of type I worldsheet instantons [244], but a fundamental understanding of the effects on the HO side is still absent. Moreover, Shenker effects should be present already in the 10d heterotic theories, in which they cannot be S-dual to type I worldsheet instantons.

One may be tempted to dismiss Shenker's argument as perhaps only relevant for a subset of theories, but the existence of the effects he predicted in the 10d HO theory has been demonstrated by Green and Rudra [245]. Via compactifications of 11D supergravity, they computed the non-perturbative corrections to the HO R^4-term, concluding that its string coupling-dependent coefficient contains precisely a e^{-1/g_s} factor. Interestingly, they also found the Shenker effects in 10d type I, as required by S-duality, but concluded that they are absent from the 10d HE R^4-term; only after compactifying on S^1 with an appropriate Wilson line do they find the relevant terms in the HE theory. While the homotopy groups of the heterotic gauge groups hint at the existence of some objects that could explain the results of Green and Rudra, these remain mysterious, and it is not clear how the theory realizes them. Nevertheless, the Shenker effects in the R^4-term were recently found to be essential for the heterotic theories to align with our expectations on the behavior of the quantum gravity cutoff [246].

In this chapter, we seek to shed some light on the nature of heterotic Shenker effects by describing the existence of heterotic "D-instantons," i.e. heterotic open disk diagrams, by extending Polchinski's idea for open heterotic strings [247]. These are possible thanks to an inflow mechanism arising from fermion zero modes in spacetime, which is naturally captured by K-theory. We start in 7.1 by reviewing Shenker's argument and Polchinski's explanation for open string theories in terms of disk amplitudes of D-instantons in 7.2. As an exemplary model of Shenker effects we will look at the coefficient of the R^4 term in 10d string theory and thereby sharpen the puzzle of Shenker effects in heterotic string theory.

In 7.3, we will then argue that the necessary heterotic objects are inherited from type IIB by regarding the HO theory as its non-perturbative quotient, following [92, 93]. We continue in 7.4 by expounding their topological properties and integrating them into the previous, pertinent cobordism discussion, as well as connecting them to concepts in the Swampland Program. Referring back to the disk amplitude explanation we argue that such diagrams are also possible in the heterotic theories thanks to the aforementioned inflow mechanism for space-time fermion zero modes. While the preceding sections are mostly concerned with the HO theory, we apply these ideas to the HE theory in 7.6, and briefly comment on other heterotic string theories. Finally, we summarize and conclude in 7.7.

7.1 Shenker's effects

The reasoning begins at the general perturbative expansion of closed string amplitudes, where each amplitude contribution is weighed by g_s^2:

$$g_s^2 \mathcal{A}_{\text{closed}} = \sum_{n=0}^{\infty} \mathcal{A}_n g_s^{2n} . \tag{7.1}$$

Like for any perturbative expansion resurgence theory can provide crucial insights into the non-perturbative objects we would expect (see for example [248] or [249] for valuable introductions). The above perturbative series has a convergence radius zero. So to obtain a physically sensible convergent amplitude non-perturbative effects have to contribute non-trivially such that the corrected result is finite. In line with resurgence theory we first take the Borel transform

$$B(t) = \sum_n \mathcal{A}_n g_s^{2n} \frac{t^{2n}}{(2n)!} \tag{7.2}$$

and then we recover $g_s^2 \mathcal{A}_{\text{closed}}$ by integrating the Borel transform over the auxiliary variable t:

$$g_s^2 \mathcal{A}_{\text{closed}} = \int_0^{\infty} dt \, e^{-t} \, B(t) . \tag{7.3}$$

Crucially, non-perturbative effects are in one-to-one correspondence with singularities of the Borel transform. Then, a singularity at some t_0 gives rise to non-perturbative effects weighed by e^{-t_0}. Based on a volume estimate of the moduli space of once punctured Riemannian surfaces Shenker proposed a general scaling of the first singularities

$$\mathcal{A}_n \sim C^{-2n} (2n)! , \tag{7.4}$$

where C is a constant. This entails that the leading non-perturbative effects are weighed by e^{-C/g_s}. In superstring theories including an open string sector we have an obvious suspect with such a scaling – Dp-branes – as their tensions behave as $\tau_p \sim 1/g_s$. On a flat, uncompactified background we would only expect D-instantons to contribute. This is precisely the case studied by Polchinski in [250], where he worked out the disk amplitudes of the 26-dimensional bosonic string corresponding to an open string with both ends on a D-instanton. Moreover, this argument carries over to the open superstring case.

7.2 Disk amplitudes and D-instantons

D-instantons are a somewhat extreme case as all boundaries have to be Dirichlet in a Euclidean space-time. The corresponding contribution to the path integral now involves a sum of the following type [250]:

$$\sum_{N=0}^{\infty}\prod_{a=1}^{N}\left[\int d^d X_a \sum_{n_a}^{\infty}\right] . \tag{7.5}$$

Here, N denotes the number of D-instantons, a the Chan-Paton degrees of freedom, X_a the position of the D-instanton, which has a Chan-Paton dependence, and n_a the number of worldsheets attaching to each D-instanton. We specify to the superstring case $d = 10$. Now, we want to check the amplitude contribution of D-instantons. To this end we further follow Polchinski by considering the simplest case, the one-instanton amplitude $\mathcal{A}_1$. The key insight here is that we get a series of disconnected worldsheet topologies with only Dirichlet boundaries. The leading such contribution is a simple disk $\langle 1\rangle_{D_2}$, whose boundary is the D-instanton, with no strings attached. For M such disks a $1/M!$ symmetry factor has to be included, which leads after summing over it to an exponential [250]:

$$\mathcal{A}_1 = \exp\left(\langle 1\rangle_{D_2} + \dots\right)\mathcal{A}_1^{\text{connected}} . \tag{7.6}$$

Since a string worldsheet is weighed by a string coupling factor $g_s^{m-\chi}$ with m the number of strings attached and χ the Euler number of the worldsheet topology, $\langle 1\rangle_{D_2}$ gets a $1/g_s$ factor. This is precisely the effect predicted by Shenker. However, only in type II string theories such an argument appears to apply as we have D(-1)-branes in type IIB and we could obtain D-instantons in lower dimensions by fully wrapping (Euclidean) branes on some cycle. In 10d type I string theory such an effect only appeared feasible with the discovery of the non-BPS D-instantons, whose charges are classified by $KO^{-10}(pt) \simeq \mathbb{Z}_2$. We will come back to the type I case later on.

One may be tempted to dismiss Shenker's argument as perhaps only relevant for a subset of theories, but the existence of the effects he predicted in the 10d $Ss(32)$ heterotic string theory has been demonstrated by Green and Rudra [245]. In this paper they extended earlier work on calculating non-perturbative corrections to the R^4-term in type II by using the duality to 11d supergravity [251]. By deploying the Hořava-Witten construction they computed the non-perturbative corrections to the

R^4-term of the $\mathcal{N}=1$ superstring theories, as well. Contrasting the story in type II with the one in heterotic/type I documents nicely the dualities and non-perturbative objects involved. So let us first elaborate on the type II side. In that case we start with the one-loop four graviton scattering amplitude in 11d supergravity compactified on an n-torus T^n, which takes the form:

$$\mathcal{A}_4^{(n)} = \pi^{\frac{3}{2}} \tilde{K} \int_0^\infty d\hat{t}\, \hat{t}^{\frac{1}{2}} \sum_{\{\hat{l}_I\}} e^{-\pi \hat{t} G_{IJ} \hat{l}_I \hat{l}_J} , \tag{7.7}$$

where $\hat{t} = 1/t$ is the inverted worldline coordinate, $\tilde{K}$ a kinematical factor, G_{IJ} the metric on the T^n and $\hat{l}_I$ the winding modes on the torus. Compactifying just the 11th dimension on an S^1 (with radius R_{11}), one can read off the general type IIA correction. Therefore we get for $n=1$:

$$\mathcal{A}_4^{(1)} = C\tilde{K} + \tilde{K}\zeta(3)\frac{1}{\pi R_{11}^3} . \tag{7.8}$$

The first term is technically divergent corresponding to the zero winding contribution, where $C = \int_0^\infty d\hat{t}\hat{t}^{\frac{1}{2}}$. From duality considerations one can infer that $C = \frac{\pi}{3}$. The second contribution can be converted to the type IIA frame by identifying the string coupling $g_{IIA} = R_{11}^{\frac{3}{2}}$ matching the precise result that had been determined earlier from type IIA tree-level computations [252, 253]. As anticipated by the K-theory classification there are no D-instanton contributions. Although, as we argue later this result is expected to change on a non-trivial background topology. For type IIB we have to compactify on a T^2 and then take an appropriate 10d decompactification limit. For $n=2$ we get the beautiful result:

$$\mathcal{A}_4^{(2)} = V_2 C\tilde{K} + \frac{1}{2\pi} V_2^{-\frac{1}{2}} \tilde{K} \sum_{(\hat{l}_1,\hat{l}_2)\neq(0,0)} \frac{\tau_2^{\frac{3}{2}}}{|\hat{l}_1 + \hat{l}_2\tau|^3} , \tag{7.9}$$

where V_2 is the torus volume and the sum is nothing else than the non-holomorphic Eisenstein series associated to $SL(2,\mathbb{Z})$:

$$E_s^{SL(2,\mathbb{Z})}(\tau) = \sum_{(\hat{l}_1,\hat{l}_2)\neq(0,0)} \frac{\tau_2^s}{|\hat{l}_1 + \hat{l}_2\tau|^{2s}} , \tag{7.10}$$

where we have $s = \frac{3}{2}$. In the type IIB frame the torus variable are identified with:

$$\begin{aligned} V_2 &= R_{10}R_{11} = g_{IIB}^{\frac{1}{3}}\, r_B^{-\frac{4}{3}} , \\ \tau &= \tau_1 + i\tau_2 = C_0 + i\, g_{IIB}^{-1} . \end{aligned} \tag{7.11}$$

Taking the 10d limit, we get that only the second term proportional to $E_{\frac{3}{2}}^{SL(2,\mathbb{Z})}(\tau)$. This is a truly astounding result, since Eisenstein series appear as fundamental tools in number theory and more specifically the Langlands Program, see for example [254, 255]. As we have mentioned earlier the fact that we run into the Langlands Program is not a coincidence as it appears to be deeply linked with the notion of S-duality (Montonen-Olive duality). Unsurprisingly, $E_{\frac{3}{2}}^{SL(2,\mathbb{Z})}(\tau)$ is fully $SL(2,\mathbb{Z})$-invariant, i.e. it reflects nicely the self-S-duality of type IIB. By taking the Fourier-Bessel expansion of the Eisenstein series in the string coupling we can nicely dissect it into perturbative and non-perturbative contributions:

$$\begin{aligned} E_{\frac{3}{2}}\left(C_0 + i g_{IIB}^{-1}\right) = & 2\zeta(3) g_{IIB}^{-\frac{3}{2}} + 2\zeta(2)\sqrt{g_{IIB}} + \sum_{n\in\mathbb{Z}^+} 8\pi\sigma_{-1}(|n|) \\ & \times \exp(2\pi i n C_0)\exp(-2\pi|n|/g_{IIB})(1+\mathcal{O}(g_{IIB}))\,. \end{aligned} \tag{7.12}$$

Here, we get Shenker's predicted effects in form of a whole tower of $|n|$ D-instantons. Since type IIB D-instantons carry a charge $K^{-10}(pt) \simeq \mathbb{Z}$, this was anticipated.

Now, a very similar calculation can be performed in the Hořava-Witten setup, where the compact dimension is an interval $S^1/\mathbb{Z}_2$ between two 10d boundaries carrying E_8 gauge degrees of freedom. To obtain the type I/$Ss(32)$ heterotic string correction we have to compactify on $S^1 \times S^1/\mathbb{Z}_2$. This choice of geometry leads to a crucial difference, namely the two compact directions with radii R_{10} and R_{11} are distinct:

$$\mathcal{A}_4^{(2)} = \pi^{\frac{3}{2}}\tilde{K}\int_0^\infty d\hat{t}\,\hat{t}^{\frac{1}{2}} \sum_{m_1,m_2} e^{-\pi^2\hat{t}\,l_{11}\left(\frac{m_1^2}{R_{10}^2}+\frac{m_2^2}{R_{11}^2}\right)}\,. \tag{7.13}$$

Then taking the proper limits [245] conclude that the string coupling-dependent coefficient is given both in the type I and in the $Ss(32)$ heterotic case by the real-analytic Eisenstein series only depending on g_s, whose Fourier-Bessel expansion is

$$\begin{aligned} E_{\frac{3}{2}}\left(i g_{I/HO}^{-1}\right) = & 2\zeta(3) g_{I/HO}^{-\frac{3}{2}} + 2\zeta(2)\sqrt{g_{I/HO}} + \sum_{n\in\mathbb{Z}^+} 8\pi\sigma_{-1}(|n|) \\ & \times \exp(-2\pi|n|/g_{I/HO})(1+\mathcal{O}(g_{I/HO}))\,. \end{aligned} \tag{7.14}$$

There are a couple of comments necessary here. First, the fact that we obtain the same result for the coefficient encodes precisely the remaining S-duality after taking the orientifold of type I as the Eisenstein series is completely invariant under $g_I \leftrightarrow g_{HO}^{-1}$. We will detail the orientifold aspect of this further in the next section. While we would expect a D-instanton contribution on the type I side, a whole tower seems surprising as their charge takes value in $\mathbb{Z}_2$. So how does it square with our understanding of D-instanton amplitudes? As reviewed in the previous section Polchinski demonstrated that the contribution leading to precise $e^{-g_s^{-1}}$ effects are disconnected disk diagrams. However, the tachyon leading to the eventual decay of a pair of type I D-instantons appears in the spectrum of an open string connecting two different D-instantons [256], i.e. the associated diagram is that of an annulus. The Euler characteristic χ of the annulus is zero. Therefore, we would associate a $e^{-g_s^0}$ contribution to such a worldsheet topology. Consistently, that contribution as opposed to type IIB (7.12), where it is proportional to C_0, vanishes for the type I R^4 coefficient (7.14). That is in line with understanding type I as a type IIB orientifold, which projects out C_0. This means that the amplitude contribution of D-instantons works counterintuitively, in that a tower of n disk amplitudes contributes for any string theory featuring a D-instanton that is stable on its own.

As a final comment for this section let us mention that the R^4 coefficient in 10d $E_8 \times E_8$ heterotic string theory does not exhibit a D-instanton contribution in the Green and Rudra computation analogous to type IIA.

In the following section we would like to recapitulate Hull's proposal for $Ss(32)$-heterotic string theory as a non-perturbative orientifold of type IIB, which provides some crucial intuition on the K-theoretic inflow mechanism we describe in the subsequent chapter.

7.3 Branes in type IIB quotients

As we reviewed earlier HO string theory can be regarded as a non-perturbative orientifold of type IIB string theory [92, 93]. Extended objects of a quotient theory can be understood by studying how the ones of its cover theory, type IIB, are affected by the relevant group action; in this section, we analyze the HO theory from this perspective. Let us commence by recalling the most salient aspects of the two relevant 10d quotients of type IIB string theory for our discussion, type I and HO string theory.

Considering the quotient of type IIB by Ω, the worldsheet parity symmetry, leads to type I string theory. The F-string becomes unorientable, signaling the presence of a space-time-filling O9-plane; an accompanying stack of 32 D9-branes ensures tadpole cancellation in the consistent background. The Ω-even type IIB Dp-branes, i.e. those with $p \in \{1,5,9\}$, comprise the BPS spectrum of type I. The Dp-branes with $p \in \{-1,0\}$ are non-BPS stable configurations, as can be seen from a K-theoretic [65] or a BCFT [70, 257, 258] analysis. The type I D-instanton is of particular interest, corresponding to a D(-1)-$\overline{\text{D}(-1)}$ superposition in type IIB, for which not only the (-1)–(-1) tachyons are projected out under Ω, but also those in the (-1)–$9 \oplus 9$–(-1) sector, see [65, 70, 258]. The stability of the type I D-instanton is associated with the K-theory charge $\text{KO}(S^{10}) = \mathbb{Z}_2$, meaning that even numbers of them can annihilate [65].

Having exhausted the perturbative quotient constructions descending from type IIB, we turn our attention to the non-perturbative part of its duality group. In particular, following the works by Hull [92, 93], we consider taking the quotient by the operator $\widetilde{\Omega} := S\Omega S^{-1}$. Since $\widetilde{\Omega}$ is obtained by the conjugation action of S-duality on Ω, quotienting the perturbative limit of type IIB by it leads, in view of heterotic/type I duality [54, 85], to the HO theory. The type IIB D-string becomes unorientable, indicating the presence of the S-dual pair of the O9-plane alongside 32 NS9-branes canceling the tadpole. While the parallels with the type I construction are clear, the role played by S-duality in this quotient construction means that we must abandon the perturbative worldsheet paradigm and regard it as an orientifold of the complete, space-time theory.

However, this picture can be connected to the perturbative HO frame as follows. Recall that in the type I frame, the F-string can extend between D-branes, in particular between a D-string and the background stack of D9-branes. At finite values of the string coupling g_s^{I}, said non-BPS string becomes unstable, with a lifetime inversely proportional to $\sqrt{g_s^{\text{I}}}$. Considering the HO theory at perturbative, but finite values of the coupling g_s^{HO}, an analogous picture arises through Hull's orientifold [92, 93]: An unstable non-BPS HO "D-string" can extend between the F-string and the background NS9-branes, the latter providing it with Chan-Paton factors. This D-string tethers the gauge charges to the fundamental string and, in the strict perturbative limit, completely retracts onto it; its massless spectrum provides 32 left-handed Majorana-Weyl worldsheet fermions transforming under the gauge group, i.e. the asymmetry in degrees of freedom of the heterotic worldsheet construction [93]. From the perspective of Hull's orientifold, this inflow of degrees of freedom of the D-string is what distinguishes the heterotic F-string.

In those compactifications of the HO theory for which the internal space has non-trivial 2-cycles, wrapped Euclidean HO D-strings will lead to instanton corrections that can be identified with Shenker effects. The argument for HO Shenker effects as the S-duals of type I worldsheet instantons [244] finds a natural explanation within Hull's orientifold picture, resolving one of the puzzles raised in the beginning of this chapter. Note that the instability of these configurations does not prevent them from contributing to the path integral; indeed, all saddle points must be summed over in the quantum theory, and D-instantons with tachyonic modes can provide sensible contributions [259–261].

Earlier, we reviewed how a tower of D-instantons descends from type IIB to type I, as can be understood from the orientifold construction of the latter. Similarly, Hull's orientifold picture leads us to believe that the HO theory inherits a tower of "D-instantons" from type IIB as well. These were not discussed in [92, 93], and are, in fact, more subtle to track in heterotic/type I duality than the strings we just examined. Since heterotic/type I duality stems from type IIB S-duality, we can appreciate why in the cover theory. To regard 10d type IIB as an appropriate limit of M-theory on T^2 we need to make a choice of F- and D-string, i.e. a marking of the 1-cycles of T^2. This determines a concrete type IIA dual frame and allows us to understand the tower of D-instantons as wrapped D0-particles. S-duality then corresponds to a different marking of the 1-cycles, and hence a different pair of type IIA/IIB frames with its own tower of wrapped D0-particles/D-instantons. The resummed tower of D-instantons of one type IIB frame reorganizes collectively into the one of its S-dual, but the instantons cannot be individually tracked along the process. The appearance of the Eisenstein series in the R^4-term of type IIB showcases how this works: The resummed tower of instantons leads to the Eisenstein series, but individual instantons are only identified after we perform its Fourier-Bessel expansion, i.e. after we make a concrete choice of frame.

In spite of this, Hull's orientifold allows us to gain a heuristic intuition about the HO instantons. On the type I side, we have a tower of $\mathbb{Z}_2$-charged D-instantons probed by the F-string. Slightly deviating from the strict perturbative limit, the F-string starts to retract into the instantons. At strong coupling, we can take the perturbative HO point of view. While the instantons cannot be individually tracked, the instanton tower of the HO frame can be thought of as a collective reorganization of the type I instantons, onto which the HO "D-string" has completely retracted. Said

D-string is also the object tethering the instantons to the background NS9-branes, meaning that we should have $\mathbb{Z}_2$-charged HO instantons with a gauge profile. While heuristic in nature, this argument points towards the properties of the HO instantons that we will independently motivate in the upcoming sections. These are the sources of Shenker effects that we propose are at play in the 10d HO theory and that, in particular, explain the results of [245] for the R^4-term.

7.4 From homotopy to K-theory, cobordism and the Swampland

In the HO theory we have a perturbative $\mathrm{SemiSpin}(32)$-bundle and associated to it the non-trivial homotopy group

$$\pi_{10}(B\mathrm{SemiSpin}(32)) = \pi_9(\mathrm{SemiSpin}(32)) = \mathbb{Z}_2 \,. \tag{7.15}$$

Based on the fact that such a topologically non-trivial gauge configuration is viable in HO-theory, and simultaneously $\pi_9(E_8 \times E_8) \simeq 0$ [245] proposed that this gauge instanton might be the relevant one responsible for the Shenker effect.

It is tempting to associate a purely gauge instanton with the non-trivial class above, but this would be incorrect. An extension of Derrick's theorem [262] implies that a purely gauge configuration for the instanton, which would be characterized by an action scaling like $g_{\mathrm{YM}}^{-2} \sim \left(g_s^{\mathrm{HO}}\right)^{-2}$, is untenable: the HO instanton must be stringy in nature. As explained in the previous chapter we should really refine the discussion to real K-theory because of the presence of Hull's D1-string in non-perturbative HO theory.

We will return to the delicate interplay between the non-trivial gauge configuration and the heterotic strings below, but for the moment let us further expound the topological properties of the HO instanton. We will approach this topic from the same angle as in the previous chapter by looking at the spin cobordism groups of the classifying space of the HO gauge group that we just calculated. There are technically two possibilities for the Cobordism Conjecture to require us to include instantons, i.e. (-1)-branes:

- First, there might be a need to include them to break a non-trivial group corresponding to a 0-form global symmetry in degree 9, which requires a codimension 10 defect:

$$\Omega_9^{\text{Spin}}(B\text{SemiSpin}(32)) \simeq 5\mathbb{Z}_2 \,. \tag{7.16}$$

Ultimately, this is not exactly what we want. To match with Green and Rudra's result in 10d flat space we don't want to break the 10d isometry, which would be necessary to take the 10th dimension to be the "nullbordant" direction.

- Therefore, we look at gauging the (-1)-form global symmetry

$$\Omega_{10}^{\text{Spin}}(B\text{SemiSpin}(32)) \simeq 10\mathbb{Z}_2 \,, \tag{7.17}$$

which, of course, parallels the type I side.

In addition to the information coming from applying the Cobordism Conjecture to $\Omega_{10}^{\text{Spin}}(B\text{SemiSpin}(32))$ the group also informs us about a potentially non-trivial Dai-Freed anomaly. A 9d theory of fermions charged under the group $G = Ss(32)$ will have a global anomaly characterized by the respective cobordism invariant, for example an η-invariant [263]. When the theory is defined over a sphere, the anomaly reduces to being characterized by $\pi_9(Ss(32)) = \mathbb{Z}_2$ and is measured by the mod 2 Atiyah-Singer index theorem [264–266]. This is a variation of the quintessential example of such a global anomaly, namely Witten's $\text{SU}(2)$ anomaly [242]. For us, it is important because such an anomaly heralds, as explained by Witten for the $\text{SU}(2)$ case, the existence of fermion zero modes in one dimension higher, i.e. for the gauge instanton background in the HO theory. These will play a crucial role in 7.5. As the Atiyah-Singer index crucially is a K-theory invariant [267], the more general statement involving the fermion zero modes will require us to rely on K-theory.

Secondly, some of the instantons required for gauging $\Omega_{10}^{\text{Spin}}(BSs(32))$ will also carry K-theory charge. This due to the decomposition of Anderson-Brown-Peterson [268], which we expounded on in chapter 6. We will argue that it is indeed not a coincidence that some of the instantons we have to include because of the Cobordism Conjecture in fact "carry" fermion zero modes as indicated by the AS index.

As we are familiar $\text{ko}_{10}(BSs(32))$ isomorphic to the direct sum $\text{ko}_{10}(\text{pt}) \oplus \widetilde{\text{ko}}_{10}(BSs(32))$. The HO instanton taking values in this group has indeed non-trivial $\mathbb{Z}_2$-valued K-theory charge: A purely gravitational piece corresponding to $\text{ko}_{10}(\text{pt}) = \widetilde{\text{ko}}(S^{10}) = \mathbb{Z}_2$, that we study in 7.6, and a gauge piece associated with $\mathbb{Z}_2 \subseteq \widetilde{\text{ko}}_{10}(BSs(32))$, to which we turn our attention in 7.5. Both groups actually signal a nontrivial mod 2 index counting the aforementioned fermion zero modes. We should also recognize that this not disconnected from $\pi_{10}(BSs(32))$ as the (generalized) Hurewicz homomorphism (see for example chapter II. in [127]) maps this charge into $\mathrm{ko}_{10}(BSs(32))$. Therefore K-theory allows us to have the more nuanced discussion.

7.5 Heterotic disks and D-instantons

We now argue that the instanton, captured by $ko_{10}(BSs(32))$, gives rise to $\mathcal{O}\left(e^{-1/g_s}\right)$ effects in the HO theory. In the previous sections we have established that Shenker's prediction amounts to the statement that the type I and type II closed string perturbation theories must be supplemented by an open string sector, which includes strings with endpoints on a D(-1)-brane responsible for the $\mathcal{O}\left(e^{-1/g_s}\right)$ effects in 10d. This conclusion relies on arguments agnostic to the amount of supersymmetry, and therefore holds also for the 26d bosonic string.

We propose that this logic extends to the HO theory and its instanton — the HO theory must contain a (-1)-brane with open HO strings ending on it. However, at first glance, an open string sector appears antithetic to the very notion of heterotic CFTs. The variation of the worldsheet action 2.76 for an heterotic string with open boundary conditions is

$$\delta S_\Sigma = \frac{1}{2\pi} \int d\tau \, \left(\lambda^a \delta\lambda_a - \psi^\mu \delta\psi_\mu\right)\Big|_{\sigma=0}^{\sigma=\ell} . \tag{7.18}$$

Here $\{\lambda^a\}_{a\in\{1,\ldots,32\}}$ constitute the left-moving current algebra and $\{\psi^\mu\}_{\mu\in\{0,\ldots,9\}}$ are the right-moving superpartners of the worldsheet bosons. The cancellation of (7.18) requires that $\lambda^i = \pm\psi^i$ for all i at the endpoints of the string. However, this condition *cannot* be satisfied in plain heterotic theories due to the asymmetric CFT field content.

Despite this difficulty, open heterotic strings in the HO theory were shown to exist in Lorentzian space-time [247]. The key to their consistency lies in the 0-branes present at the endpoints of the HO string. Critically, the S^8 enclosing a 0-brane supports a vector bundle with $Ss(32)$-structure in the adjoint representation and associated with a non-trivial homotopy class of $\pi_7(Ss(32)) \cong \mathbb{Z}$. We can choose this vector bundle such that its structure group reduces to a subgroup of $Ss(32)$

with $\mathfrak{so}(8)$ Lie algebra. In this background, the gauginos give rise to space-time zero modes $\{\Lambda^b\}_{b\in\{1,\dots,24\}}$ that transform under the $\mathbf{24}$ dimensional representation of the gauge subgroup with trivial bundle.[1] These zero modes can also be enumerated via the Atiyah-Singer index theorem [267,270] applied to the twisted Dirac operator defined on the enclosing sphere S^8. The proposal of [247] is that these zero modes latch onto the endpoints of the HO string and satisfy $\Lambda^b = \pm\lambda^b$ for $b \in \{1,\dots,24\}$, with the remaining 8 current algebra fermions matched to the 8 (physical gauge) fermions $\{\psi^\mu\}_{\mu\in\{2,\dots,9\}}$.

We now extend this logic to Euclidean space-time and propose that a similar mechanism exists to ensure the consistency of HO endpoints on the (-1)-brane. As described above, the HO instanton is characterized by a gauge profile associated with the non-trivial class of $\pi_9(Ss(32)) \cong \mathbb{Z}_2$. As in the case of the type I instanton discussed in [65], and in analogy with the preceding discussion, we can choose the vector bundle such that its structure group reduces to a subgroup with $\mathfrak{so}(10)$ Lie algebra. The mod 2 Atiyah-Singer index theorem [264–266] ensures that the number of gaugino zero modes in such a background is $1 \mod 2$. Furthermore, each of these zero modes $\{\Lambda^{b'}\}_{b'\in\{1,\dots,22\}}$ transform as a $\mathbf{22}$ dimensional representation of the gauge subgroup with trivial bundle.[2] We then patch up the inconsistency of the heterotic CFT by demanding that $\Lambda^{b'} = \pm\lambda^{b'}$, for $b' \in \{1,\dots,22\}$, at the string endpoints. The remaining 10 current algebra fermions are handled by all $\{\psi^\mu\}_{\mu\in\{0,\dots,9\}}$, since here we do not employ physical gauge.

The above suggests indeed that open HO strings can end on the instanton due to a inflow of fermionic zero modes measured by K-theory from space-time onto the worldsheet. Below we would like to illustrate the inflow and how it fits in with Hull's proposal. As we have stated before the reason why real K-theory shows up for the classification of branes in HO theory is Hull's D1-string. Since it's a D-object, its endpoints can take values in real K-theory and can serve as a conductor of fermionic zero modes. At finite g_s we see an extended D1-string tethering the heterotic instanton to the (open) fundamental string

[1] The $\mathbf{492}$ faithful representation of PSO(32) lifts to $Ss(32)$. The embedding $\mathfrak{so}(32) \supset \mathfrak{so}(8) \oplus \mathfrak{so}(24)$ leads to the branching rule $\mathbf{496} = (\mathbf{28}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{276}) \oplus (\mathbf{8_v}, \mathbf{24})$ [269], the r.h.s. corresponding to a representation of $(\mathrm{Spin}(8) \times \mathrm{Spin}(24))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, see [74,75] for details on the global structure of the subgroups of $Ss(32)$.

[2] The same considerations as in 1 hold *mutatis mutandis* for this case.

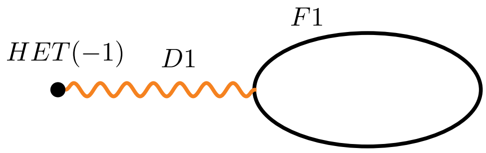

Figure 7.1: Heterotic instanton configuration at finite g_s

In the $g_s \to 0$ limit however, the D1-string becomes infinitely heavy and pulls the instanton into the fundamental string, such that the D1-string gets reduced to a K-theoretic gluing condition between the zero mode carrying instanton and the fundamental (open) heterotic string.

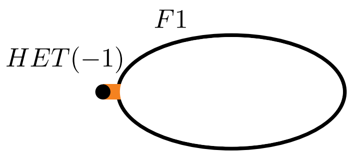

Figure 7.2: Heterotic instanton configuration at $g_s \to 0$

Furthermore, the path integral argument by Polchinski ensures that disconnected disks appear. As expected from S-duality we recognize a lot of similarities with type I. Disk diagrams corresponding to an open string starting and ending on an instanton are possible and moreover we can consider a whole tower of them. However, trying to include two instantons will annihilate the zero modes as they can only exist mod 2 and open heterotic boundary conditions become impossible again. This parallels the tachyon instability for the annulus diagram in type I.

It is instructive to contrast the role of space-time zero modes and the (-1)-brane in the type I and HO frames. In type I, the F-string can end on the (-1)-brane without issue due to the symmetric worldsheet CFT field content, but the space-time fermion zero modes were argued to be necessary in order to remove the disconnected piece of the perturbative $\mathrm{O}(32)$ gauge group [65]. Instead, the HO frame knows the correct gauge group at the perturbative level, but the fermion zero modes are required for consistency of the F-string endpoints on the (-1)-brane.

The key question remaining is: What should the action of this instanton be? If we consider an HO disk diagram with both endpoints on the instanton, the Euler number of the disk suggests that the action is proportional to g_s^{-1}. Such a scaling, paired with the path integral argument, would provide a precise realization of Shenker's prediction in the HO theory and an explanation to the instanton correction to the R^4 term (7.14) demonstrated by Green and Rudra [245]. This inverse g_s scaling is of course the universal feature of D-branes in the type I and type II theories. It is tempting to conclude the same holds in heterotic theories, but our situation is far more subtle — in matching the space-time fermion zero modes with the worldsheet fields, we have gone beyond the usual paradigm of worldsheet (B)CFTs.

Nonetheless, we argue that this naive answer appears correct. First, it seems that the g_s^{-1} scaling is universal in the HO theory as well. The heterotic "D-string" from 7.3 has a tension proportional to g_s^{-1}, which is necessary to match the S-duality arguments of [244]. Furthermore, the tentative (gauge) 0-brane in the 9d HE theory, discussed below, also follows this scaling relation. This suggests that HO disk diagrams, just like their type I and type II counterparts, should also be associated with a g_s^{-1} scaling. Secondly, we can motivate the scaling by appealing to the known results of [245] — if the HO instanton does indeed give rise to the Eisenstein series in (7.14), then its action must necessarily scale as g_s^{-1}.

While these arguments are quite suggestive, they are not a proof. To settle the g_s scaling, one must develop the tools necessary to *calculate* the effect of heterotic disks. We leave this as a task for the future. For the present, we state that, provided one accepts the above arguments, heterotic disks account for Shenker effects in the HO theory.

7.6 Other heterotic theories

We have explained the origin of Shenker effects in the HO theory, but one 10d superstring theory remains: the HE theory. Immediately we should recognize that there is no plain gauge configuration, i.e. in flat space with a compact support, giving the space-time fermion zero modes required for the consistency of HE disk diagrams as $\pi_9(\mathrm{E}_8) = 0$. This appears consistent with the absence of a tower of instanton corrections to the HE R^4-term in 10d flat space [245]: The non-trivial gauge profile was crucial for the instantons we just identified as contributing to this term in the HO theory, but these must disappear in the T-dual decompactification frame. A similar statement was made in [247] forbidding the existence of open HE cosmic strings in Minkowski space-time due to $\pi_7(\mathrm{E}_8) = 0$.

Nonetheless, Shenker's argument applies to the HO and HE theories equally well;

the lack of the effects it predicts for the latter appears problematic. A resolution can be found by drawing an analogy with the type IIA theory, which has a non-BPS, uncharged, and unstable D(-1)-brane. This object does not contribute to the type IIA R^4-term in 10d, but should contribute as an unstable saddle to some set of processes [261].

This motivates us to consider *gravitational* configurations, that are potentially unstable, to justify HE string endpoints. The inflow then occurs due to the combined zero modes of the 10d gauginos, dilatino, and gravitino in the non-trivial space-time.

In Lorentzian space-time, we can consider an open HE cosmic string with endpoints on a 0-brane associated with a non-trivial gravitational charge. The 0-brane exists in a space-time that supports fermionic zero modes for the gauginos, dilatino, and gravitino. Applying the index theorems for the differential operators acting on Weyl spinors and a Rarita-Schwinger field [271] supported on an 8-dimensional manifold surrounding one of the HE string endpoints, consistency requires

$$\begin{aligned}\pm 24n = \frac{1}{2}\int_{M_8}\Bigg[&495\left[\hat{A}(M_8)\right]_8\\ &+\left[\hat{A}(M_8)\left(\mathrm{tr}e^{iR/2\pi}-1\right)\right]_8\Bigg],\end{aligned} \tag{7.19}$$

where $n\in\mathbb{Z}$ and R is the Riemann tensor suitably contracted with $\mathrm{SO}(8)$ generators in its fundamental representation. The required manifold M_8 should lie in the coimage of the ABS map $\hat{A}:\Omega_8^{Spin}(pt)\to KO_8(pt)$ for $\int_{M_8}\hat{A}(M_8)$ to be non-trivial. The generator of this cobordism group is the Bott manifold B with $\int_B\hat{A}(B)=1$. Since cobordism is defined to work additively under the disjoint union, we can either use an appropriate number of Bott manifolds joined together or another manifold in the same equivalence class. Due to Lichnerowicz's and Hitchin's works [272, 273] we know that the scalar curvature of whichever closed manifold ends up supplying the zero modes can not be positive. This is a general statement that $\hat{A}(M)=0$, if M has positive scalar curvature. This also applies to the mod 2 case in dimensions $8k+1$ and $8k+2$.

We expect that a similar situation holds for gravitational instantons in the HE theory arising from (-1)-branes. Endpoints of the HE string on this object are made consistent by a Euclidean 10d analogue of (7.19), but with the zero modes adding up to a multiple of 22. A potential source for such effects are the 10d Hitchin spheres [273], the higher-dimensional cousins of Milnor's exotic spheres [274]. Heterotic supergravity has an even number of fermionic zero modes on such manifolds [241], making them candidates that could realize purely gravitational Shenker effects in the 10d HE theory.

This completes our discussion of Shenker effects for the 10d superstring theories, but does not exhaust the landscape of heterotic theories. First, upon compactification the situation appears significantly enriched thanks to the possibility of breaking the heterotic gauge group or having Euclidean objects wrap the cycles of the compactification variety. Indeed, the results of [245] indicate that, in the HE theory on S^1 with an appropriate Wilson line, a 0-brane contributes Shenker effects to the 9d R^4-term via Euclidean worldlines wrapping the circle. This gauge profile is possible due to the fact that the Wilson line breaks the HE theory gauge group from $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ to $(Ss(16) \times Ss(16)) \rtimes \mathbb{Z}_2$ through the embedding $Ss(16) \subset E_8$, which has

$$\pi_9((Ss(16) \times Ss(16)) \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \,. \tag{7.20}$$

We leave the study of open heterotic strings in compactifications to future work.

Furthermore, Shenker's argument should apply to other heterotic theories beyond the HO and HE theories and their supersymmetry. For example, in 10d we have the non-supersymmetric tachyon-free $\mathrm{SO}(16) \times \mathrm{SO}(16)$ heterotic string [275, 276], whose global gauge group is $[(\mathrm{Spin}(16) \times \mathrm{Spin}(16))/\mathbb{Z}_2] \rtimes \mathbb{Z}_2$ [74, 277]. Heterotic instantons similar to the ones discussed for the HO theory should be possible. Studying the lower-dimensional analogues of this theory [278] would also be of interest. Another sector worth discussing is that populated by the non-critical string theories. The analogue of Polchinski's long string was considered in [279] for the 2d non-critical heterotic strings, finding that it plays a role in the cancellation of the gauge and gravitational anomalies of the twisted orbifold version of the HO theory. It would be appealing to also discuss open heterotic strings and Shenker effects in the context of the supercritical HO^{+} and $\mathrm{HO}^{+/}$ theories [280]. These are closely related to the conventional HO theory, enabling, e.g., a K-theoretic description of its NS5-brane via tachyon condensation [281].

7.7 Conclusions

Heterotic non-perturbative effects of order $\mathcal{O}\left(e^{-1/g_s}\right)$ were predicted by Shenker [33] and confirmed to exist by Green and Rudra [245], but an explanation of the objects that give rise to them remained elusive. Here we have motivated that their origin is found in heterotic "D-instantons," i.e. heterotic disk diagrams. These are highly non-perturbative configurations, forcing us to go well beyond the usual worldsheet (B)CFT paradigm by mixing worldsheet and space-time degrees of freedom, expanding on the ideas of [247].

The above discussion reveals several key lessons. The first is that those heterotic branes on which the F-string can end must feature an inflow mechanism for consistency, as exemplified by the D-string in [92, 93], the 0-brane in [247] and the (-1)-brane discussed in this work.

Second, while heterotic disks lie outside the purview of typical worldsheet (B)CFTs, it appears one should nonetheless associate them with a scaling proportional to g_s^{-1}. This is required to match the contributions calculated in [245] arising from the HO(-1)-brane and 9d HE (gauge) 0-brane. The same scaling for the heterotic disks is required for the HO D-string to match the duality arguments of [244]. This scaling agrees nicely with our intuition from the Euler number of a disk.

Finally, the reciprocal picture of an old string theory adage arises: It is common lore that a theory of open strings must contain closed strings due to the possibility of endpoint reconnection. Our discussion above indicates that Shenker's argument, drawn to its natural conclusion, implies that a theory of closed strings must also incorporate open strings.

The path to this conclusion involved a non-trivial confluence of several distinct topics, including dualities between string theories and quotients thereof, K-theory, fermion zero mode inflow arising from index theorems, and the theory of resurgence applied to string amplitudes.

It would be desirable to establish a consistent framework to calculate the effects of open heterotic strings from first principles, a question that merits future investigation. A more fundamental treatment of such effects may rely on non-perturbative approaches to quantum gravity, among which dualities, string field theory and matrix models have proven to be very useful elsewhere. Indeed, there exist hints of a connection between matrix models and heterotic Shenker effects [282].

Compactifications to lower dimensions and broken gauge groups may alter the way in which the heterotic Shenker effects are concretely realized. In 4D compactifications they may play a role in moduli stabilization [283–288], cosmology [289] and string phenomenology more broadly, making their study a crucial target.

Les vrais paradis sont les paradis qu'on a perdus.

Marcel Proust

8
Closing Words

Finally, after a lot of very formal string theory and mathematics we would like to take the opportunity to give some concluding thoughts on the results presented in this thesis and also would like give some outlook on steps ahead. The bulk of this thesis was dedicated to the very rich subfield of cobordism theory applied to string theory. By now it has developed into the quintessential tool to understand topological, non-perturbative phenomena within the theory, specifically the detection of possible global anomalies and the classification of defects and solitons. The latter is a consequence of the Cobordism Conjecture, which poses that non-trivial cobordism groups are in one-to-one correspondence with global symmetries and therefore all cobordism groups with a tangential structure compatible with quantum gravity have to vanish as a consequence of the No Global Symmetries Conjecture.

Often times the cobordism groups we encounter do not vanish, which means we are missing crucial physics that we have to include. There is two branches of trivializing cobordism groups (within quantum gravity), gauging, which amounts to a *tadpole cancellation condition* or breaking, which is accomplished by including *defects*. We started out by exploring the implications of introducing exemplary background spaces commonly considered for string compactifications into the cobordism groups. We observe that these background spaces indeed give rise to a multitude of lower dimensional global symmetries, which matches with the expectation from dimensional reduction, but the fact that we working with a generalized homology theory gives a more refined result, including more complicated torsional groups.

Our primary computational tool for this part of the thesis was the Leray-Serre-Atiyah-Hirzebruch spectral sequence, which has the excellent property that it takes integral (co)homology as its input and eventually converges to a generalized (co)homology. The LSAHSS offers a detailed answer on how good of an approximation for cobordism (or K-theory) ordinary (co)homology is. In the case of spin cobordism of Calabi-Yau threefolds we have shown a prime example for when the homology approximation fails. This means global symmetries in the uncompactified theory can be significantly altered by the dimensional reduction. Thus it is essential to calculate $\Omega_n^{\widetilde{QG}}(X)$, when discussing dimensional reductions as it may contain a lot of information that is not visible through just $\Omega_n^{\widetilde{QG}}(pt)$. One particular effect we would like to study is the apparent fusion of global symmetries through non-trivial extensions. For example we encountered $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ in appendix H relevant to 4- and 6-manifolds containing 2-torsion in their homology.

By utilizing the deep mathematical connection between cobordism and K-theory we could further establish that a significant subset of cobordism groups are trivialized by invoking tadpole cancellations of Dp-brane charges, which take values in K-theory groups.

For low-dimensional cobordism groups, whose dimension is lower than the dimension of the background space, we saw that after the same map to K-theory gauging these groups would correspond to tadpole cancellations for Dp-branes with negative transversal dimensions understood by means of so-called imbeddings. Studying whether these Dp-branes are physically realized would be very interesting.

In the subsequent chapter we were primarily focused on closing a particular gap in the string theory and math literature, concerning the classifying space of the SemiSpin group. While it plays a main role in 10d superstring theory, from a mathematical point of view it is maybe the most obscure of the simple, connected Lie groups. For practically all other simple and connected Lie groups the mod p cohomology groups of their classifying spaces have been calculated in classic papers. However, for the classifying space of the Semispin groups these results are missing. By means of the Eilenberg-Moore spectral sequence we were able to compute the intricate structure of the mod 2 cohomology of the classifying space of $Ss(32)$ in the dimensions we require for our string theory applications (and also above to check some more relations between cohomology invariants).

By solving the Adams spectral sequence at the prime 2 completely, which starts from the data of the mod 2 cohomology of the classifying space we could fully determine the spin cobordism groups of $BSs(32)$ in the dimensions relevant to string theory.

Extending the ideas from the previous chapter, we specifically scrutinize the ko-homology subsector of the spin cobordism groups. We find that the full Dp-brane classification actually requires us to take the classifying space into account, as this enables us to describe the charges arising from open string connecting the Dp-brane to the background D9-brane stack. For , these additional ko-homology groups reveal a non-BPS D3-brane in type I that only carries charges in the DN sector, which was missed before. It turns out to be consistent with the boundary state classification of stable configurations as an open string with DN boundary conditions ending on a 3-brane does not have a tachyon in its spectrum. Another important lesson is that there are instances where satisfying the tadpole cancellation condition only gauges the diagonal component of the direct sum of two (or more) Abelian groups and needs to be supplemented by breaking the remaining global symmetry. This should have some major phenomenological implications, such as circumventing No-Go-Theorems for de Sitter constructions. Hopefully, we can explore this phenomenon in the future.

Furthermore, our calculation of the mod 2 cohomology is a necessary step for the calculation of the twisted string cobordism groups, which will reveal even more about the intricacies of type I/HO theory. An aspect that has to be resolved first is how the 2-torsion enters the Bianchi identity of type I/HO theory, i.e. the precise twist of the string structure.

Furthermore, we have seen that our proposal for extending the K-theory description of type I/HO theory seems fully compatible with S-duality. So a natural question would be, if we can find an S-duality compatible extension for the type II string theories, as well. Just complex K-theory of the point does not achieve that (see e.g. [116]).

Our final chapter 6 can be seen as a smooth extension of the previous one, where we take a particular interest in the 10th dimensional cobordism group, which can be gauged by instantons. More specifically, we looked at the simplest topological configuration that gives rise to a non-trivial charge in $\Omega^{Spin}(BSs(32))$, namely a ten-sphere furnished with a topologically non-trivial $Ss(32)$-principal-bundle. We show that this instanton configuration provides zero modes that by inflow onto the worldsheet allow for open string boundary conditions in HO-theory. Furthermore, we argue that the inflow is the residue of Hull's D1-string (the S-dual of the open fundamental string in type I) in the HO weak coupling limit. Since it is a D-object it is allowed to carry K-theory charges, i.e. the zero modes, and serve as a conductor between the heterotic instanton and the worldsheet. With this construction we can answer a classic question in string theory, namely what the objects behind Shenker's $\exp(g_s^{-1})$-effects in heterotic string theory are. As we discussed in the traditional string theories containing an open string sector (type II, type I, bosonic string theory) these effects come from disk amplitudes, i.e. an open string with both ends on a D-instanton (or a fully wrapped Euclidean D-brane). With this unconventional inflow configuration disk amplitudes become possible in HO-theory, as well. However, this goes beyond this one specific instanton configuration, which we picked as our prime example, because it could be matched to actual amplitude calculations on flat, topologically trivial space-time backgrounds. The HE string in 10d provides a nice contrast. Purely gauge zero modes do not work, but gravitational instantons can supply the necessary zero modes and therefore account for Shenker effects in 10d HE theory. Uncovering them in string amplitudes would require us to calculate the amplitudes on gravitationally highly non-trivial backgrounds.

As we have demonstrated cobordism theory provides a holistic framework to study topological solitons in string theory/quantum gravity non-perturbatively. Obstruction theory and the Cobordism Conjecture step by step reveal necessary non-perturbative physics to achieve quantum gravitational consistency. This makes cobordism theory a sharp tool to reexamine old problems in string theory and to push for a deeper understanding of the non-perturbative sector of string theory.

Finally, we would like to mention a few avenues to proceed that we had no opportunity to present yet.

To actually match with real world physics we technically have to change to Lorentzian signature, which is not nearly as well understood as the Euclidean case. For plain Lorentzian spin cobordism it seems that the results should carry over, but the story is already more subtle for $X = pt$. Especially, for more complicated background spaces or higher structure this is very interesting to explore further. We might also want to change another assumption of cobordism theory, namely that of considering just closed manifolds within the equivalence classes. Indeed, it is feasible to define cobordism theory based on compact manifolds with boundaries. For M-theory compactifications on open manifolds like $S^1/\mathbb{Z}_2$ or even non-oriented open manifolds are probably the most well-understood cases of them appearing in string theory. Presumably, a more general understanding of these backgrounds should give us deeper insight on string theories with broken supersymmetry, since such open background are quite universal in non-supersymmetric string theory and in the M-theory case get us from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry.

The most ambitious step is to proceed to the realm of algebraic geometry and study the algebraic siblings of cobordism and K-theory. Amazingly, much of the material covered in this thesis like spectra, the LSAHSS, the Adams Spectral Sequence etc. has a motivic analog in so-called $\mathbb{A}_1$-homotopy theory. We expect to obtain a direct access to periods valued within the algebraic analogs instead of topological charges (see for example for periods valued in algebraic K-theory [290]). Moreover string amplitudes in general seem to possess a motivic structure [291]. Matching this from the $\mathbb{A}_1$-homotopy side motivated by an extension of the (topological) Cobordism Conjecture would be a huge conceptual step.

I hope that this thesis has provided some insight on why cobordism theory is such an exciting tool for studying quantum gravity, whose capabilities we have not yet exhausted. Hopefully I can return to some of the issues raised here.

A

Generalized (co)homology theories – The Eilenberg-Steenrod axioms

In this appendix we want to state the Eilenberg-Steenrod axioms defining a (generalized) (co)homology, introduced in the foundational work [122]. First, we pick some Abelian group π and a pair of topological spaces (X, A). Thereby we define so-called relative (co)homology groups, which are a more general version of what we are using in the main text. It will reduce to the latter by picking the pair $(X, \emptyset)$. For the precise statement of the axioms, which we will propound in a categorical language, we will rely on the concise discussion in [121]. First, we give the homology case:

Theorem A.0.1. *For a given integer p there exist a functor $H_p(X, A; \pi)$, which we call (relative) homology from the homotopy category of pairs of topological spaces to the category of Abelian groups together with natural transformations $\partial : H_p(X, A; \pi) \to H_{p-1}(A; \pi)$. Further we denote by $H_p(X; \pi)$ the homology of the pair $(X, \varnothing)$: $H_p(X, \varnothing; \pi)$. Then the functor, including the natural transformations, satisfy and characterized by the following axioms:*

- ***Dimension**: For $X = pt$:*

$$H_p(X;\pi) = \begin{cases} \pi & \text{for } p = 0, \\ 0 & \text{for } p \neq 0\,. \end{cases} \tag{A.1}$$

If this is axiom is violated, but all subsequent are obeyed we call this homology theory a generalized homology theory.

- ***Exactness**: The following sequence is exact, where the arrows are induced by the inclusions $A \hookrightarrow X$ and $(X, \varnothing) \hookrightarrow (X, A)$, as well as the natural transformation ∂ introduced above:*

$$\cdots \to H_p(A;\pi) \to H_p(X;\pi) \to H_p(X, A;\pi) \to H_{p-1}(X, A;\pi) \to \dots \tag{A.2}$$

- ***Excision**: Suppose X is the union of the interior of spaces A and B, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism:*

$$H_*(A, A \cap B;\pi) \simeq H_*(X, B;\pi)\,. \tag{A.3}$$

- ***Additivity**: Let the pair (X, A) be a disjoint union of a set of pairs (X_i, A_i) then the inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism:*

$$\bigoplus_i H_*(X_i, A_i;\pi) \simeq H_*(X, A;\pi)\,. \tag{A.4}$$

- ***Weak Equivalence**: If there exist weak equivalences $X \to Y$ and $A \to B$ (or on the level of pairs $(X, A) \to (Y, B)$), then there exists an isomorphism:*

$$H_*(X, A;\pi) \simeq H_*(Y, B;\pi)\,. \tag{A.5}$$

We should remark that for pairs of CW-complexes the last axiom is turned into a definition of the homology theory of a CW-complex by choosing a CW-approximation functor Γ, such that $(X, A) \to (\Gamma X, \Gamma A)$ is a weak equivalence and $H_*(\Gamma X, \Gamma A;\pi)$ becomes the (relative) homology of a CW-pair.

The cohomology case is fairly similar, but contains a few subtle differences:

Theorem A.0.2. *For a given integer* p *there exists a functor* $H^p(X, A; \pi)$*, which we call (relative) cohomology from the homotopy category of pairs of topological spaces to the category of Abelian groups together with natural transformations* $\delta : H^p(A; \pi) \to H^{p+1}(X, A; \pi)$*. Further we denote by* $H^p(X; \pi)$ *the cohomology of the pair* $(X, \emptyset)$*:* $H^p(X, \emptyset; \pi)$*. Then the functor, including the natural transformations, satisfy and characterized by the following axioms:*

- ***Dimension****: For* $X = pt$*:*

$$H^p(X; \pi) = \begin{cases} \pi & \text{for } p = 0, \\ 0 & \text{for } p \neq 0\,. \end{cases} \tag{A.6}$$

  If this is axiom is violated, but all subsequent are obeyed we call this homology theory a generalized homology theory.

- ***Exactness*** *The following sequence is exact, where the arrows are induced by the inclusions* $A \hookrightarrow X$ *and* $(X, \emptyset) \hookrightarrow (X, A)$*, as well as the natural transformation* ∂ *introduced above:*

$$\cdots \to H^p(X, A; \pi) \to H^p(X; \pi) \to H^p(A; \pi) \to H^{p+1}(X, A; \pi) \to \dots \tag{A.7}$$

- ***Excision****: Suppose* X *is the union of the interior of spaces* A *and* B*, then the inclusion* $(A, A \cap B) \hookrightarrow (X, B)$ *induces an isomorphism:*

$$H^*(X, B; \pi) \simeq H^*(A, A \cap B; \pi)\,. \tag{A.8}$$

- ***Additivity****: Let the pair* (X, A) *be a disjoint union of a set of pairs* (X_i, A_i) *then the inclusions* $(X_i, A_i) \hookrightarrow (X, A)$ *induce an isomorphism:*

$$H^*(X, A; \pi) \simeq \bigoplus_i H^*(X_i, A_i; \pi)\,. \tag{A.9}$$

- ***Weak Equivalence****: If there exist weak equivalences* $X \to Y$ *and* $A \to B$ *(or on the level of pairs* $(X, A) \to (Y, B)$*), then there exists an isomorphism:*

$$H_*(Y, B; \pi) \simeq H_*(X, A; \pi)\,. \tag{A.10}$$

  Again the last axiom can be turned into a definition of the cohomology theory of a CW-complex by choosing a CW-approximation functor Γ*, such that* $(X, A) \to (\Gamma X, \Gamma A)$ *is a weak equivalence and* $H^*(\Gamma X, \Gamma A; \pi)$ *becomes the (relative) homology of a CW-pair.*

B

Classifying Spaces

As mentioned in the main part of the thesis classifying spaces BG occupy an important place in algebraic topology to classify G-principal bundles and are also central to define characteristic classes. For this exposition we will rely heavily on the excellent [205]. The construction of classifying spaces for a topological group G goes as follows: First, we construct the universal space EG through Milnor's construction [292] as the join of a countably infinite number of copies of G

$$EG := G \star G \star \dots \tag{B.1}$$

The join of topological spaces $A_1 \star \dots \star A_n$ is defined as follows: A point in the join is specified by

- Real numbers $t_1, \dots, t_n$ with $t_i \geq 0$, such that $t_1 + \dots + t_n = 1$ and
- points $a_i \in A_i$ with non-vanishing t_i then assemble to a point in the join as $t_1 a_1 \oplus \dots \oplus t_n a_n$, where a_i can be chosen arbitrarily, if $t_i = 0$.

Then we construct coordinate maps:

$$t_i : A_1 \star \dots \star A_n \to [0,1] \quad \text{and} \quad a_i : t_i^{-1}\,]0,1] \to A_i\,. \tag{B.2}$$

Furthermore, the Milnor topology on $A_1\star\cdots\star A_n$ is defined as the strongest topology for which both coordinate maps are continuous. Back to EG the join of infinitely many topological spaces is defined the same, but we equire that all but a finite number of t_is are vanishing. Finally, for A_1 are right G-spaces we can still define a continuous action of G on the join with the map:

$$((t_i a_i), g) \mapsto (t_i a_i g)\,. \tag{B.3}$$

EG turns out to be contractible, i.e. all homotopy groups vanish. Then we can define the classifying space BG as the quotient EG/G and construct the universal bundle p_U:

$$p_U\,:\; EG \to BG\,. \tag{B.4}$$

Where this construction becomes particularly powerful for physics is that we can classify G-principal bundles over some space X, since a G-principal bundle models the topological structure of the gauge field of G, see for example [293]. Assigning to the isomorphism classes of numerable G-principal bundles over X $\mathcal{B}(X, BG)$ the homotopy classes of the classifying map $X \to BG$ $[X, BG]$ we get a bijection between the two: $\mathcal{B}(X, BG) \leftrightarrow [X, BG]$. Then the inverse assigns to the classifying map $b : X \to BG$ the bundle $E \to X$, where E is a numerable free G-space, induced by the classifying map b from the universal bundle p_U.

Classifying spaces have a few nice further properties. By constructing the path fibration $p : P \to BG$ with a contractible total space and fiber ΩBG, the (based) loop space of BG we can turn the homotopy equivalence $EG \to P$ into a fiberwise homotopy equivalence. Since $\Omega BG \simeq G$, we get the isomorphism $\pi_n(BG) \simeq \pi_{n-1}(G)$.

Furthermore, one can show that the classifying space of a discrete group H is nothing else than the Eilenberg-MacLane space $K(H, 1)$. The two above statements can be combined. An example important for the main part is $B^2\mathbb{Z}_2 = B(B\mathbb{Z}_2)$, which is isomorphic to $K(\mathbb{Z}_2, 2)$.

C
The Steenrod algebra

In this appendix, we collect some useful facts about Steenrod squares and the Steenrod algebra. For a nice, pedagogical review with a particular emphasis on the Adams spectral sequence, we refer the reader to [145]. A very useful, general discussion on the topic of cohomology operations and specifically the Steenrod algebra can be found in [144]. Let's start with an axiomatic definition of the Steenrod squares. A cohomology operation of degree i is a map

$$H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2). \tag{C.1}$$

It is said to be stable if it commutes with the suspension isomorphism. Steenrod squares, Sq^i, are stable cohomology operations of degree i satisfying the following properties, for any $i \geq 0$:

0. Sq^i is a natural homomorphism $H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$;

1. For $i < j$ $Sq^i(x) = 0$, for all $x \in H^j(X;\mathbb{Z}_2)$;

2. $Sq^i(x) = x \cup x$, for all $x \in H^i(X;\mathbb{Z}_2)$;

3. $Sq^0 = \mathrm{Id}$;

4. $Sq^1 = \beta$ is the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$;

5. Cartan formula: $Sq^i(x \cup y) = \sum_{m+n=i} Sq^m(x) \cup Sq^n(y)$.

6. Adem relation: $Sq^i \circ Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \begin{pmatrix} j-k-1 \\ i-2k \end{pmatrix}_{\text{mod } 2} Sq^{i+j-k} \circ Sq^k$,
   for $0 < i < 2j$.

Then the Steenrod algebra $\mathcal{A}$ is a $\mathbb{Z}_2$ tensor algebra, whose elements are polynomials in Sq^i satisfying both $Sq^0 = \text{Id}$ and the Adem relations. Importantly, $\mathcal{A}$ is generated by just Sq^{2^n} with $n \geq 0$. Furthermore, even though the Steenrod algebra is infinitely generated, in each degree it is finitely generated. These subalgebras, denoted $\mathcal{A}_n$, are then generated by Sq^1, ..., Sq^{2^n}.

C.1 Some additional background on Steenrod squares

The map $Sq^1 \equiv \tilde{\beta}$ is an example of a Bockstein homomorphism. It is associated to the short exact sequence

$$0 \to \mathbb{Z}_2 \stackrel{\times 2}{\to} \mathbb{Z}_4 \stackrel{\rho}{\to} \mathbb{Z}_2 \to 0, \tag{C.2}$$

where the first map is multiplication by 2 and the second (ρ) is the reduction modulo 2, which induces the long exact sequence

$$\ldots \stackrel{\tilde{\beta}}{\to} H^n(X;\mathbb{Z}_2) \stackrel{\times 2}{\to} H^n(X;\mathbb{Z}_4) \stackrel{\rho}{\to} H^n(X;\mathbb{Z}_2) \stackrel{\tilde{\beta}}{\to} H^{n+1}(X;\mathbb{Z}_2) \to \ldots . \tag{C.3}$$

Here, $\tilde{\beta}$ is the connecting homomorphism between cohomology groups of different degree. Another Bockstein homomorphism, called β in the main text, can be constructed in association with the short exact sequence

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \stackrel{\rho}{\to} \mathbb{Z}_2 \to 0, \tag{C.4}$$

inducing in turn the long exact sequence

$$\ldots \stackrel{\beta}{\to} H^n(X;\mathbb{Z}) \stackrel{\times 2}{\to} H^n(X;\mathbb{Z}) \stackrel{\rho}{\to} H^n(X,\mathbb{Z}_2) \stackrel{\beta}{\to} H^{n+1}(X,\mathbb{Z}) \to \ldots . \tag{C.5}$$

The two Bocksteins are related by

$$\tilde{\beta} = \rho \circ \beta. \tag{C.6}$$

At odd degree $i = 2k+1$, one can define an integral lift of the Steenrod squares,

$$Sq_{\mathbb{Z}}^{2m+1} = \beta \circ Sq^{2m}, \tag{C.7}$$

which is such that $\rho \circ Sq_{\mathbb{Z}}^{2m+1} = Sq^{2m+1}$ and maps $H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z})$. One further gets a map between integral cohomology by first reducing modulo 2 and then acting with $Sq_{\mathbb{Z}}^i$,

$$Sq_{\mathbb{Z}}^i \circ \rho : H^n(X;\mathbb{Z}) \to H^{n+i}(X;\mathbb{Z}). \tag{C.8}$$

An integral lift of Sq^i for even $i = 2m$ does not exist.[1]

Given an element $x \in H^{k-i}(X;\mathbb{Z}_2)$, with $k = \dim(X)$, the action of the Steenrod squares can be defined as

$$Sq^i(x) = \nu_i \cup x, \tag{C.9}$$

where $\nu_i \in H^i(X;\mathbb{Z}_2)$ is the i-th Wu class of X (more precisely, of a real vector bundle over X of rank k, which we generically take to be the tangent bundle), such that $\nu_i = 0$ if $i > k - i$. Since the total Wu class is the Steenrod square of the total Stiefel-Whitney class, one can express each of the single Wu classes in terms of Stiefel-Whitney classes. At lower degree, one has

$$\begin{aligned} \nu_1 &= w_1, \\ \nu_2 &= w_2 + w_1 \cup w_1, \\ \nu_3 &= w_1 \cup w_2. \end{aligned} \tag{C.10}$$

In certain cases, one can give an alternative action of Sq^i, namely (see e.g. [148, 180])

$$Sq^i(y) = \iota_*(w_i(N)) \cup y, \tag{C.11}$$

[1] This can be proven as follows. Suppose it exists an integral lift for the even case, $Sq^{2m} = \rho \circ Sq_{\mathbb{Z}}^{2m}$. Exactness of the sequence (C.5) means that $\ker \beta = \operatorname{Im} \rho$, implying in turn $\beta \circ Sq^{2m} = \beta(\rho(Sq_{\mathbb{Z}}^{2m})) = 0$. However, this is contradiction with the Adem relation $Sq^1 \circ Sq^{2m} = Sq^{2m+1} \neq 0$ (recall $Sq^1 = \rho \circ \beta$). Thus, such an integral lift $Sq_{\mathbb{Z}}^{2m}$ cannot exist.

where $y \in H^n(X;\mathbb{Z}_2)$, N is the normal bundle of the submanifold $Y \subset X$ Poincaré dual to y and $\iota : Y \to X$ is the inclusion.[2] This is most convenient for physical purposes, such as checking the absence of Freed–Witten anomalies for branes wrapping Y, on which we comment in section 5.3.2 (there, following [148, 180], we directly employ the integral lift $W_3(N)$ of $w_3(N)$ and omit the pushforward ι_*).

[2] In general, one can define an action $Sq^i(u) = \pi^*(w_i(\xi)) \cup u$, with $u \in H^k(E;\mathbb{Z}_2)$ and $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$, for any k-plane bundle $\xi : F \to E \xrightarrow{\pi} B$ of which the normal bundle $N(B)$ is a particular case [192].

D

Characteristic Numbers and Genera of Polynomial Sequences

In this appendix we want to detail the construction of some very important topological invariants that also appear as invariants of some cobordism groups.

We will initially follow [121] for defining Stiefel-Whitney, Chern and Pontryagin classes and numbers. We define a characteristic number of a manifold M as a characteristic class of the tangent bundle of M $s(TM)$[1] evaluated with the fundamental class $\mu \in H_n(M;R)$

$$s[M] := \langle s(TM), \mu \rangle \,, \tag{D.1}$$

which we formally denote as the integration over M $\langle s(TM), \mu \rangle = \int_M s(TM)$.

Stiefel-Whitney classes

We start off by defining Stiefel-Whitney classes axiomatically. Since they appear in the main part of the thesis via obstruction theory we choose to define them as

[1] One can also use the normal bundle of M to define it. Whenever we do use the characteristic number of the normal bundle, we emphasize it explicitly.

elements w_i of $H^iBO(n); \mathbb{Z}_2$ with $n \geq 1$. With slightly different axioms they can be defined for real vector bundles as well.

1. $w_0 = 1$ and $w_i = 0$ for all $i > n$.
2. $w_1 \neq 0$ when $n = 1$
3. For the classifying map $i_n : BO(n) \to BO(n+1)$ we have under the pullback: $i_n^*(w_i) = w_i$
4. Under the pullback of the classifying map $p_{m,n} : BO(m) \times BO(n) \to BO(m+n)$ we have: $p_{m,n}^*(w_i) = \sum_{j=0}^{i} w_j \otimes w_{i-j}$.

Then $H^iBO(n); \mathbb{Z}_2$ is generated as an polynomial algebra $\mathbb{Z}_2[w_1, \ldots, w_n]$.

Chern classes

For Chern classes we take a complex vector bundle $E \to M$ then the Chern classes C_i are elements in $H^i(M; \mathbb{Z})$ satisfying the properties [120]:

1. $c_i = 0$ for all $i > \dim(E)$.
2. $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback $f^*(E)$
3. $c(E_1 \otimes E_2) = c(E_1) \cup c(E_2)$, where $c(E)$ is the total Chern class $c(E) = 1 + c_1 + c_2 + \cdots \in H^*M; \mathbb{Z}$
4. c_1 is the canonical generator of $H^2(BU(1); \mathbb{Z})$ (with $BU(1) = \mathbb{CP}^\infty$).

Pontryagin classes

The simplest definition of Pontryagin classes is through Chern classes. As opposed to Chern classes they are based on real vector bundles. For a real vector bundle $E \to M$ we can take the complexification $E^{\mathbb{C}}$, which is defined by applying complex structure on the fiber $\mathbb{R} \oplus \mathbb{R}$ via the rule $i(x, y) = (-y, x)$. Then the Pontryagin classes p_i living in $H^{4i}(M; \mathbb{Z})$ are defined as:

$$p_i = (-1)^i c_{2i}(E^{\mathbb{C}}) \,. \tag{D.2}$$

The genera of multiplicative sequences

The three genera we are about to introduce play an important role in the main part of the thesis as both cobordism invariants of cobordism with oriented, $spin$ or $spin^c$-structure and charges of Dp-branes and Op-planes. We rely on [120, 121, 294] for this brief introduction and refer to them for a thorough discussion. Generally, a genus $G(M)$ of some manifold M is defined by three properties [294]:

1. $G(M \times N) = G(M) \cdot G(N)$.
2. $G(M \cup N) = G(M) + G(N)$.
3. $G(M) = 0$, if M bounds.

These properties already indicate that they should play a major role in cobordism theory.

Chern character

Before we give definitions for the aforementioned genera we look at a similar object defined as a polynomial of Chern classes, the Chern character of a vector bundle F:

$$ch(F) := \mathrm{rk}(F) + \sum_{k>0} \frac{1}{k!}\, s_k(c_1(F), \ldots, c_k(F)) \,, \tag{D.3}$$

where $s_k(\tau_1, \ldots, \tau_k)$ can be written recursively as:

$$s_k = \tau_1\, s_{k-1} - \tau_2\, s_{k-2} + \cdots + (-1)^{k-2}\tau_{k-1}\, s_1 + (-1)^{k-1} k\, \tau_k \tag{D.4}$$

The Chern character also satisfies some similar properties to that a genus, namely

1. $ch(E \times F) = ch(E) \cdot ch(F)$.
2. $G(E \oplus F) = ch(E) + ch(F)$.

The $\hat{A}$ genus

The genera we want to look at next arise from characteristic classes defined as multiplicative sequences. The first one is the cobordism invariant of $\Omega^{Spin}_{4k}(pt)$, the $\hat{A}$-genus. Taking the following formal power series, we define a characteristic class

$$\hat{A}(x) = \frac{\sqrt{x}/2}{\sinh\left(\sqrt{x}/2\right)} = 1 - \frac{1}{24}x + \frac{7}{5760}x^2 + \ldots \tag{D.5}$$

Then, one can translate x into Pontryagin classes [91], which evaluates for the first three terms to:

$$\begin{aligned}\hat{A}_1 &= -\frac{1}{24}p_1\,,\\ \hat{A}_2 &= \frac{1}{5760}(-4p_2 + 7p_1^2)\\ \hat{A}_3 &= \frac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3)\,.\end{aligned} \tag{D.6}$$

The $\hat{A}$-genus of a manifold M is subsequently given by evaluating the characteristic class on the manifold M: $\hat{A}(M)$. Interestingly, while $\hat{A}$-genus does not give an integer on every manifold, on spin manifolds it does.

The Todd genus

Very closely related to the $\hat{A}$ genus we have the Todd genus, which is based on Chern instead of Pontryagin classes and whose class is given the following formal power series:

$$Td(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \ldots \tag{D.7}$$

Converting x into Chern classes we get for the first three terms:

$$\begin{aligned}Td_1 &= -\frac{1}{2}c_1\\ Td_2 &= \frac{1}{12}(c_2 + c_1^2)\\ Td_3 &= \frac{1}{24}(c_2c_1)\,.\end{aligned} \tag{D.8}$$

Again, the Todd genus is the Todd class evaluated on a manifold M. As we alluded to the Todd genus is intimately related to the $\hat{A}$-genus. For instance, on an almost complex manifold M the Todd genus can be written as [62]

$$Td(M) = e^{c_1/2}\hat{A}\,. \tag{D.9}$$

As a cobordism invariant it turns up for $spin^c$ cobordism.

The Hirzebruch genus

The final genus we would like to introduce in this appendix, an oriented cobordism invariant, the Hirzebruch genus $L(M)$, which is associated to the formal power series:

$$L(x) = \frac{\sqrt{x}}{\tanh\left(\sqrt{x}\right)} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \dots \tag{D.10}$$

Expressed in terms of Pontryagin classes we get for the first three terms:

$$\begin{aligned} L_1 &= -\frac{1}{3}p_1\,, \\ L_2 &= \frac{1}{45}(7p_2 - p_1^2) \\ L_3 &= \frac{1}{315}(62p_3 - 13p_2p_1 + 2p_1^3)\,. \end{aligned} \tag{D.11}$$

E

Short exact sequences, extensions and Ext

Consider the Abelian groups A, B and C. A short sequence

$$0 \longrightarrow B \xrightarrow{\beta} C \xrightarrow{\alpha} A \longrightarrow 0 \tag{E.1}$$

is exact if the map β is injective and the map α surjective, i.e. if $\ker(\alpha) = \mathrm{Im}(\beta)$. In this case, we say that C is an extension of A by B and we denote it as

$$C = e(A, B). \tag{E.2}$$

The Splitting Lemma for abelian groups tells us that the extension is trivial,

$$C = A \oplus B, \tag{E.3}$$

iff there is a left inverse to β iff there is a right inverse to α. In this case, one says that the short exact sequence is split. In general, the extension might not be unique and there can be more extensions besides the trivial one. Equivalence classes of extensions of A by B are in one-to-one correspondence with elements of the group $\mathrm{Ext}^1(A, B)$, with the trivial extension corresponding to 0 (see e.g. Theorem 3.4.3 of [295]).

The definition and main properties of the groups $\mathrm{Ext}^n(A,B)$ can be found e.g. in [295], chapter 3. We recall some of them below. As stated in Lemma 3.3.1, if A and B are Abelian (as we assume) we have that $\mathrm{Ext}^n(A,B)=0$ for $n\geq 2$. Therefore, only the groups associated to $n=0,1$ are relevant for us. We have that $\mathrm{Ext}^0(A,B)=\mathrm{Hom}(A,B)$, while $\mathrm{Ext}^1(A,B)$ classifies extensions of A by B, as anticipated above. Two useful properties of these groups are

$$\mathrm{Ext}^n(\oplus_i A_i, B) = \Pi_i \mathrm{Ext}^n(A_i, B), \tag{E.4}$$

$$\mathrm{Ext}^n(A, \Pi_i B_i) = \Pi_i \mathrm{Ext}^n(A, B_i), \tag{E.5}$$

and we recall that for Abelian groups direct product and direct sum coincide. For cyclic groups, we recall the results

$$\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}) = 0, \tag{E.6}$$

$$\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}_n) = 0, \tag{E.7}$$

$$\mathrm{Ext}^1(\mathbb{Z}_n,\mathbb{Z}) = \mathbb{Z}_n, \tag{E.8}$$

$$\mathrm{Ext}^1(\mathbb{Z}_m,\mathbb{Z}_n) = \mathbb{Z}_k, \tag{E.9}$$

where $k=\mathrm{GCD}(m,n)$. All of this is used in the calculations of chapter 5, when we have to resolve extension problems of the LSAHSS.

Let us give two simple examples to illustrate how everything works in a combined way. Let us consider the short exact sequence

$$0\to\mathbb{Z}_2\to e(\mathbb{Z}_2,\mathbb{Z}_2)\to\mathbb{Z}_2\to 0. \tag{E.10}$$

Since $\mathrm{Ext}^1(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$, $e(\mathbb{Z}_2,\mathbb{Z}_2)$ is not split, instead we have two possible extensions. Indeed, it is well-known that there are two short exact sequences

$$0\to\mathbb{Z}_2\to\mathbb{Z}_4\to\mathbb{Z}_2\to 0, \tag{E.11}$$

$$0\to\mathbb{Z}_2\to\mathbb{Z}_2\oplus\mathbb{Z}_2\to\mathbb{Z}_2\to 0. \tag{E.12}$$

Instead, the short exact sequence

$$0\to\mathbb{Z}_3\to\mathbb{Z}_6\to\mathbb{Z}_2\to 0, \tag{E.13}$$

is split, since $\mathrm{Ext}^1(\mathbb{Z}_2,\mathbb{Z}_3)=0$.

F

Wedge sum, smash product and reduced suspension

Consider two pointed topological spaces (X, x_0) and (Y, y_0). The wedge sum, $X \vee Y$, is defined as

$$X \vee Y = X \sqcup Y / \sim, \tag{F.1}$$

where the equivalence relation identifies the two base points x_0 and y_0. The smash product, $X \wedge Y$, is defined as the quotient of the Cartesian product by the wedge sum

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \tag{F.2}$$

It satisfies the properties

$$X \wedge Y \cong Y \wedge X, \tag{F.3}$$

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge X), \tag{F.4}$$

where the symbol $\cong$ means homeomorphic as topological spaces.

Consider then the n-sphere S^n. The reduced suspension of X is defined as

$$\Sigma X \cong S^1 \wedge X. \tag{F.5}$$

The construction can be iterated

$$\Sigma^n X \cong S^n \wedge X. \tag{F.6}$$

An important case is when $X = S^k$, thus giving

$$\Sigma^n S^k \cong S^{n+k}. \tag{F.7}$$

We also recall that

$$\Sigma^0 \wedge X \cong S^0 \wedge X \cong X, \tag{F.8}$$

where $S^0 \cong \mathrm{pt} \sqcup \mathrm{pt}$. Another useful formula is

$$\Sigma(X \times Y) \cong \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \tag{F.9}$$

G
Cobordism groups of spheres and tori

We can prove that for a generic structure ξ the cobordism groups of spheres (S^k) and tori (T^k) have a simple decomposition in terms of the respective cobordism groups of the point, namely

$$\Omega_n^\xi(S^k) = \Omega_n^\xi(\mathrm{pt}) \oplus \Omega_{n-k}^\xi(\mathrm{pt}), \tag{G.1}$$

$$\Omega_n^\xi(T^k) = \bigoplus_{i=0}^{k} \binom{k}{i} \Omega_{n-i}^\xi(\mathrm{pt}), \tag{G.2}$$

where we implicitly assume the groups with negative index to be vanishing. This explains for example why, when computing Spin and Spin^c cobordism groups of spheres in section 5.2, even if we found that in general

$$\Omega_n^{\mathrm{Spin}}(S^k) = \begin{cases} \Omega_n^{\mathrm{Spin}}(\mathrm{pt}) & n < k, \\ e(\Omega_{n-k}^{\mathrm{Spin}}(\mathrm{pt}), \Omega_n^{\mathrm{Spin}}(\mathrm{pt})) & n \geq k, \end{cases} \tag{G.3}$$

(and similarly for Spin^c cobordism), every time the information at our disposal was enough to solve the extension problem, it turned out to be trivial. We now prove (G.1) and (G.2) by induction. The proof only necessitate a generalized homology theory $G_n(pt)$ like cobordism groups with tangential structure ξ $\Omega_n^\xi(pt)$ satisfying the

Eilenberg-Steenrod axioms, but can also be straightforwardly extended to a generalized cohomology theory $G^{-n}(pt)$. If we choose the latter one, we can handily prove the analogous results

$$G^{-n}(S^k) = G^{-n}(\mathrm{pt}) \oplus G^{-n-k}(\mathrm{pt}), \tag{G.4}$$

$$G^{-n}(T^k) = \bigoplus_{i=0}^{k} \binom{k}{i} G^{-n-i}(\mathrm{pt}), \tag{G.5}$$

which match our results in the main part.

We start from the cobordism groups of spheres, S^k. For S^1, we have

$$\begin{aligned} G_n(S^1) &= G_n(\mathrm{pt}) \oplus \tilde{G}_n(S^1) \\ &= G_n(\mathrm{pt}) \oplus \tilde{G}_n(\Sigma(S^0)) \\ &= G_n(\mathrm{pt}) \oplus \tilde{G}_{n-1}(S^0) \\ &= G_n(\mathrm{pt}) \oplus G_{n-1}(\mathrm{pt}). \end{aligned} \tag{G.6}$$

In passing from the second to the third line we used the suspension axiom $\tilde{G}_n^(\Sigma X) = \tilde{G}_{n-1}(X)$ A, while in the last step we employed that $\tilde{G}_n(S^0) = G_n(\mathrm{pt})$, which follows from

$$G_n(S^0) = G_n(\mathrm{pt} \sqcup \mathrm{pt}) = G_n(\mathrm{pt}) \oplus G_n^*(pt) = G_n(\mathrm{pt}) \oplus \tilde{G}_n(S^0). \tag{G.7}$$

Then, we assume the formula to hold for S^k and we prove it for S^{k+1}. Using again the Splitting Lemma (4.27) and the suspension axiom, we have

$$\begin{aligned} G_n(S^{k+1}) &= G_n(\mathrm{pt}) \oplus \tilde{G}_n(S^{k+1}) \\ &= G_n(\mathrm{pt}) \oplus \tilde{G}_n(\Sigma(S^k)) \\ &= G_n(\mathrm{pt}) \oplus \tilde{G}_{n-1}(S^k) \\ &= G_n(\mathrm{pt}) \oplus G_{n-k-1}(\mathrm{pt}). \end{aligned} \tag{G.8}$$

This proves (G.1) by induction.

Then, we look at the cobordism groups of tori, T^k. The result for $T^1 = S^1$ is already proven in (G.6). We thus assume the formula to hold for T^k and we prove it for T^{k+1}. To this purpose, using (F.9) we can write

$$\Sigma(T^k \times S^1) = \Sigma(T^k) \vee \Sigma(S^1) \vee \Sigma(T^k \wedge S^1) \tag{G.9}$$

and therefore

$$
\begin{aligned}
G_n(T^{k+1}) &= G_n(\mathrm{pt}) \,\oplus\, \tilde{G}_{n+1}(\Sigma(T^{k+1})) \\
&= G_n(\mathrm{pt}) \,\oplus\, \tilde{G}_{n+1}(\Sigma(T^{k} \times S^1)) \\
&= G_n(\mathrm{pt}) \,\oplus\, \tilde{G}_{n+1}(\Sigma(T^{k})) \,\oplus\, \tilde{G}_{n+1}(\Sigma(S^1)) \,\oplus\, \tilde{G}_{n+1}(\Sigma^2(T^{k})) \\
&= G_n(\mathrm{pt}) \,\oplus\, \tilde{G}_{n}(T^{k}) \,\oplus\, \tilde{G}_{n}(S^1) \,\oplus\, \tilde{G}_{n-1}(T^{k}) \\
&= G_n(T^k) \,\oplus\, G_n(T^k),
\end{aligned}
\tag{G.10}
$$

where we used $\tilde{G}_n(X \vee Y) = \tilde{G}_n(X) \oplus \tilde{G}_n(Y)$, valid for reduced generalized homology theories [165]. We can finally demonstrate that

$$
\begin{aligned}
G_n(T^{k+1}) &= G_{n-1}(T^k) \,\oplus\, G_n(T^k) \\
&= \bigoplus_{i=0}^{k} \binom{k}{i} G_{n-1-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k} \binom{k}{i} G_{n-i}(\mathrm{pt}) \\
&= \bigoplus_{i=1}^{k+1} \binom{k}{i-1} G_{n-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k} \binom{k}{i} G_{n-i}(\mathrm{pt}) \\
&= \bigoplus_{i=0}^{k+1} \binom{k}{i-1} G_{n-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k+1} \binom{k}{i} G_{n-i}(\mathrm{pt}) \\
&= \bigoplus_{i=0}^{k+1} \binom{k+1}{i} G_{n-i}(\mathrm{pt}).
\end{aligned}
\tag{G.11}
$$

In passing from the third to the fourth line we just added zero to both terms, while in the last step we used Pascal's formula. This concludes our proof of (G.2) by induction.

An alternative proof can be given by exploiting some more advanced mathematical constructions. In particular, one can use that Spin and Spin^c cobordism are generalized homology theories classified by Thom spectra $MSpin$ and $MSpin^c$ respectively. One can thus write

$$
\tilde{\Omega}_n^{\mathrm{Spin}}(X) := [\mathbb{S}, MSpin \wedge X]_n, \tag{G.12}
$$

where X is a generic topological space, and similarly for Spin^c. Considering for example $X = S^k$, by exploiting the properties of the smash product and the suspension given in appendix F, we have

$$\begin{aligned}\tilde{\Omega}_n^{\mathrm{Spin}}(S^k) &:= [\mathbb{S}, MSpin \wedge S^k]_n \\ &= [\mathbb{S}, MSpin]_{n-k} \\ &= \pi_{n-k}(MSpin) \\ &= \Omega_{n-k}^{\mathrm{Spin}}(\mathrm{pt}).\end{aligned} \tag{G.13}$$

In passing from the first to the second line we used that $[\Sigma X, Y] = [X, \Omega Y]$ and then that $\Omega\Sigma X = X$, with ΩX the loop space, while in the last step we used the Pontryagin-Thom isomorphism. Combining this with the Splitting Lemma (4.27), one gets (G.1).

H

$Spin/Spin^c$ cobordism and K/KO-theory of Moore spaces

As demonstrated by the theorems (5.2.1) and (5.2.2) certain torsion pieces in the (co)homology of 4- or 6-manifolds are captured by Moore spaces $P^k(T)$. As an addendum to the calculations demonstrated in the main part we are going to compute the necessary cobordism and K-theory groups to get the most general result that the aforementioned theorems allow for. We will use the LSAHSS with support from the Adams Spectral Sequence for these calculations. Since $Spin$-cobordism and KO-theory contains 2-torsion we need to compute the mod 2 (co)homology of Moore spaces. This can be easily done by using the Universal Coefficient Theorem:

$$0 \to H_n(P^k(T);\mathbb{Z})\otimes A \to H_n(X;A) \to \mathrm{Tor}_1(H_{n-1}(P^k(T);\mathbb{Z}),A) \to 0. \quad \text{(H.1)}$$

By definition the only non-trivial entries of $H_n(P^k(T);\mathbb{Z})$ are

$$H_n(P^k(T);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0 \\ T & \text{for } n=k\,. \end{cases} \quad \text{(H.2)}$$

Therefore, for $A = \mathbb{Z}_2$ we get only non-trivial entries for $n > 0$, if T contains 2-torsion, because of $\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_{\gcd(m,n)}$ and $\mathrm{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n) \simeq \mathbb{Z}_{\gcd(m,n)}$. Then one obtains:

$$H_n(P^k(T);\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } n=0 \\ \mathbb{Z}_2 & \text{for } n=k\,, \\ \mathbb{Z}_2 & \text{for } n=k+1\,. \end{cases} \tag{H.3}$$

The $Spin^c$-cobordism case is pretty simple. It is very similar to the k-sphere case in dimensions below 10 we can write[1]

$$\Omega_n^{Spin^c}(P^k(T)) = \Omega_n^{Spin^c}(\text{pt}) \oplus (\Omega_{n-k}^{Spin^c}(\text{pt}) \otimes T)\,. \tag{H.4}$$

The Spin case is a lot more intricate. For $n \leq 10$ we obtain

$$\Omega_n^{Spin}(P^k(T)) = \Omega_n^{Spin}(\text{pt}) \oplus A_{n-k}(T)\,, \tag{H.5}$$

where $A_{n-k}(T)$, if T contains 2-torsion, is either given by

$$A_{n-k}(T) = \begin{cases} T & \text{for } (n-k) = 0,\ 4,\ 8\,, \\ \mathbb{Z}_2 & \text{for } (n-k) = 1,\ 9\,, \\ \mathbb{Z}_4 & \text{for } (n-k) = 2,\ 10\,, \\ \mathbb{Z}_2 & \text{for } (n-k) = 3\,. \end{cases} \tag{H.6}$$

or if T contains no 2-torsion

$$A_{n-k}(T) = T \quad \text{for } (n-k) = 0,\ 4,\ 8\,. \tag{H.7}$$

The $\mathbb{Z}_4$ piece in the 2-torsion case is the result of a nontrivial extension $e(\mathbb{Z},\mathbb{Z}_2)$, which can be determined from the Adams spectral sequence computing $ko_*(\Sigma^k\mathbb{RP}^2)_{\hat{2}}$ (and by using the ABP decomposition (4.34) also spin cobordism), since the result of the mod 2 contributions doesn't depend on the precise torsion group T and we obtain the same result for these three groups by looking at just the mod 2 Moore space $P^k(\mathbb{Z}_2)$, which is equivalent $\Sigma^k\mathbb{RP}^2$. The $\mathcal{A}_1$-module for $\mathbb{RP}^2$ is simply two $\mathbb{Z}_2$s one degree apart connected by an Sq^1. The Adams chart looks just as follows:

[1] For higher dimensions one has to be careful with torsion.

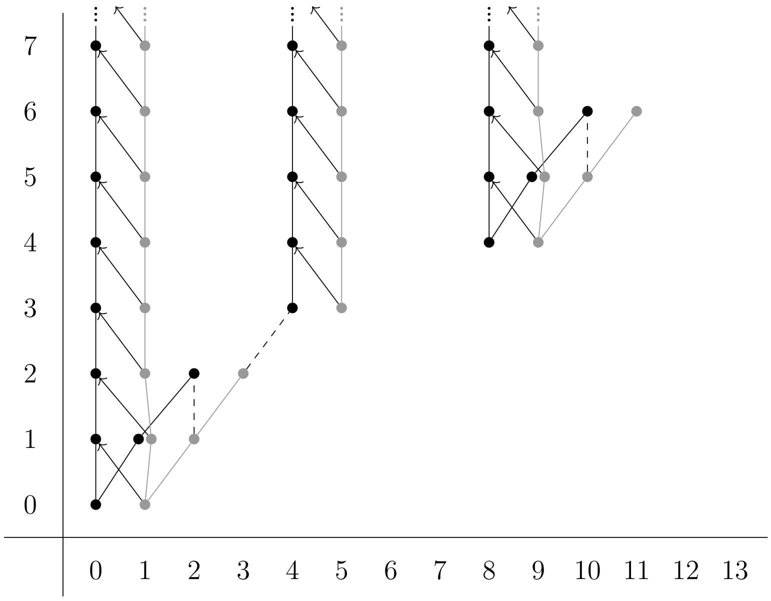

Figure H.1: Adams chart for $\mathbb{RP}_2$-module extension including δ

And the final product looks like, where we get the two non-trivial extensions

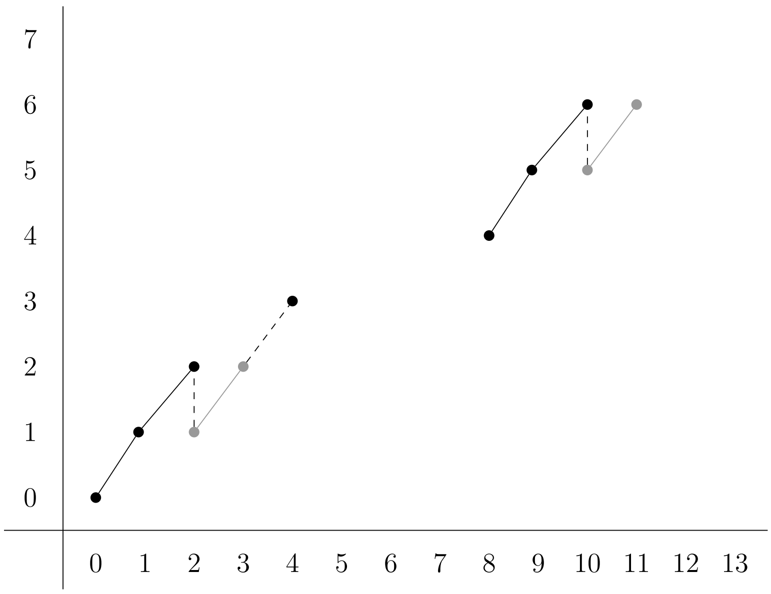

Figure H.2: Final Adams chart for $\mathbb{RP}_2$-module extension

from running the extension with well-known $\mathcal{A}_1$-modules

$$0 \to \Sigma^2\mathbb{RP}_2 \to \tilde{Q} \to \mathbb{Z}_2 \to 0\,. \tag{H.8}$$

shown below:

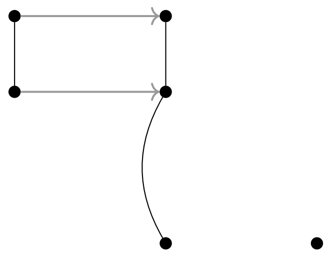

Figure H.3: Alternative exact sequence for $\mathbb{RP}^2$

The Adams chart can be simply plugged into the ko-homology Adams spectral sequence and there are no nontrivial differentials or extension. With the equivalence we explained above

$$\widetilde{ko}_{(n-k)=1,2,3 \bmod 8}(\Sigma^k \mathbb{RP}^2) \simeq \widetilde{ko}_{(n-k)=1,2,3 \bmod 8}(P^k(T)) \tag{H.9}$$

we get the aforementioned result (H.6). We see that the $\mathbb{Z}_4$ piece is a result of the nontrivial extension visible already on the $\mathcal{A}_1$-module level (as a result of the Sq^1). The complex K-theory is due to its lack of torsion completely straightforward, we get:

$$K^{-n}(P^k(T)) = K^{-n}(\text{pt}) \oplus (K^{-n+k}(\text{pt}) \otimes T)\,. \tag{H.10}$$

For the LSAHSS computing KO-theory we need the mod 2 cohomology groups of $P^k(T)$, which we get from the (cohomological) universal coefficient theorem:

$$\begin{aligned} 0 &\to \text{Ext}^1_{\mathbb{Z}}(H_{n-1}(P^k(T);\mathbb{Z}),\mathbb{Z}_2) \to H^n(P^k(T);\mathbb{Z}_2) \\ &\to \text{Hom}^1_{\mathbb{Z}}(H_n(P^k(T);\mathbb{Z}),\mathbb{Z}_2) \to 0. \end{aligned} \tag{H.11}$$

Since $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n) = \mathbb{Z}_{\gcd(m,n)}$ and $\text{Hom}^1_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n) = \mathbb{Z}_{\gcd(m,n)}$, we get analogous to the homological case provided T contains 2-torsion:

$$H^n(P^k(T);\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } n = 0 \\ \mathbb{Z}_2 & \text{for } n = k\,, \\ \mathbb{Z}_2 & \text{for } n = k+1\,. \end{cases} \tag{H.12}$$

For the subsequent LSAHSS we list in the result below. There is another non-trivial extension $e(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_4$, which we can determine by utilizing the fact that $\Sigma^k \mathbb{RP}_2$ detects the 2-torsion information responsible for the bottom three groups in the list. The results for KO-theory of the real projective spaces $\mathbb{RP}^n$ are well-known [296], including $KO^{-2}(\mathbb{RP}^2) \simeq \mathbb{Z}_4$.

$$KO^{-n}(P^k(T)) = KO^{-n}(\mathrm{pt}) \oplus B^{-n-k}(T)\,, \tag{H.13}$$

where $A_{n-k}(T)$, if T contains 2-torsion, is either given by

$$B^{-n-k}(T) = \begin{cases} T & \text{for } (n+k) = 0,\ 4,\ \bmod 8\,, \\ \mathbb{Z}_2 & \text{for } (n+k) = 1,\ \bmod 8\,, \\ \mathbb{Z}_4 & \text{for } (n+k) = 2,\ \bmod 8\,, \\ \mathbb{Z}_2 & \text{for } (n+k) = 3 \bmod 8\,. \end{cases}\,, \tag{H.14}$$

or if T contains no 2-torsion

$$B^{-n+k}(T) = T \quad \text{for } (n-k) = 0,\ -4,\ -8\,. \tag{H.15}$$

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