On Some Aspects of Mimetic Gravity

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I am among those who think that science has great beauty. A scientist in his laboratory is not only a technician: he is also a child placed before natural phenomena which impress him like a fairy tale.

— Marie Curie

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Zusammenfassung

Wir betrachten die Klasse der Theorien, die „Mimetic Gravity“ (Mimetische Gravitation) genannt werden. Diese sind Weyl-invariante und allgemein kovariante Modifikationen der Allgemeinen Relativitätstheorie. Diese Theorien sind interessant, da manche von ihnen Dunkle Materie (DM) auf großen Skalen nachbilden können und zu mimetischer DM werden, während uns andere eine elegante Formulierung der Unimodularen Gravitation bieten und zu mimetischer Dunkler Energie (DE) werden. Im zweiten Fall tritt die Kosmologische Konstante als eine Integrationskonstante auf und stellt einen globalen Freiheitsgrad dar. Um mimetische DM zu generieren, wird ein Skalarfeld verwendet, wohingegen ein Vektor- respektive Eichvektorfeld zur Konstruktion mimetischer DE benutzt wird. Mimetische Theorien können erzeugt werden, indem man der Raumzeitmetrik ihren Status als dynamische Variable entzieht. In mimetischen Theorien ist die Raumzeitmetrik ein zusammengesetztes Feld, das aus einer Hilfsmetrik und zusätzlichen Feldern besteht. Unter Weyl-Transformationen der Hilfsmetrik ändern sich diese Felder dergestalt, dass die Raumzeitmetrik invariant bleibt, was Weyl-Invarianz in der resultierenden mimetischen Theorie hervorruft. Diese Konstruktion kann äquivalent als Zwangsbedingung formuliert werden, was in der Wirkung mithilfe eines Lagrangemultiplikators erreicht wird. Für die skalaren mimetischen Modelle diskutieren wir auch die möglichen Erweiterungen mittels höherer Ableitungen. In diesem Zusammenhang und unter Verwendung kosmologischer Störungstheorie besprechen wir die Schallgeschwindigkeit und Einschränkungen der Theorie.

Darüber hinaus wenden wir zum ersten Mal in der Literatur Noethers erstes und zweites Theorem auf mimetische Theorien an. Mittels des ersten Noethertheorems zeigen wir für jede der Weyl-invarianten mimetischen Theorien, dass der Noetherstrom identisch verschwindet und daher diese Symmetrie nicht zu einer nicht-trivialen Erhaltungsgröße führt. Das zweite Noethertheorem wird dann verwendet um zu beweisen, dass die vorhandene Weyl-Symmetrie entweder zu trivialen Identitäten führt oder die mimetische Zwangsbedingung wiedergibt.

Im Fall der skalaren Theorien besprechen wir auch die Möglichkeit, diese im UV-Regime zu vervollständigen. Diese Vervollständigung ermöglicht es, das Problem von Kaustiken zu lösen, die auf nicht-linearen Skalen für flüssigkeitsartigen Staub auftreten. Die Vervollständigung im UV wird durch die Einführung eines komplexen Skalarfeldes erreicht, für das der Absolutwert die Energiedichte der mimetischen DM liefert und die Phase das Geschwindigkeitspotential.

Außerdem zeigen wir, wie man eine mimetische Theorie mit einem Eichvektorfeld in

eine Yang-Mills-Theorie einbettet, die an ein Axion gekoppelt ist. Dies ist ein Versuch zu verstehen, warum die Kosmologische Konstante so klein ist, ähnlich zum θ-Parameter der Quantenchromodynamik. Wir betrachten allgemeine $SU(N)$-Gruppen und stellen detaillierte Konstruktionen für die Eichgruppen $SU(2)$ und $SU(3)$ vor, womit wir auch die Existenz von Lösungen für die mimetische Zwangsbedingung in diesen Fällen beweisen.

Abstract

We consider the class of theories called Mimetic Gravity, which are Weyl invariant and generally covariant modifications of General Relativity. These theories are interesting because some of them can emulate dark matter (DM) on large scales and become mimetic dark matter while others can provide us with an elegant formulation of unimodular gravity and become mimetic dark energy (DE). In the latter case, the cosmological constant appears as an integration constant, representing a global degree of freedom. A scalar field is employed to build mimetic DM, whereas a vector or a gauge vector field are used to construct mimetic DE. Mimetic theories can be induced by demoting the spacetime metric from its role as a dynamical variable. In mimetic theories the spacetime metric is a composite field given by an ansatz consisting of an auxiliary metric and additional fields. Under Weyl transformations of the auxiliary metric, these fields change to leave the spacetime metric invariant enforcing Weyl invariance in the resulting mimetic theory. This construction can equivalently be reformulated in terms of a constraint, enforced in the action through a Lagrange multiplier. For the scalar mimetic models we also discuss the possible extensions via higher derivative terms. There, using the framework of cosmological perturbation theory, we discuss the speed of sound and limitations of the theory.

Moreover, for the first time in the literature, we apply Noether's first and second theorems in the context of these mimetic theories. With the help of Noether's first theorem we show that for each of the Weyl invariant mimetic theories the Noether current vanishes identically and therefore, this symmetry does not lead to a non-trivial conserved quantity. The second Noether theorem is then used to prove that the Weyl symmetry either leads to trivial identities or reproduces the mimetic constraint.

In the case of scalar theories we also discuss the possibility of the completion of the theory in the UV regime. This completion allows one to solve the issue of caustics, present on non-linear scales for fluid-like dust. The UV completion is achieved by the introduction of a complex scalar field, for which the absolute value yields the mimetic dark matter energy density, whereas the phase provides the velocity potential.

Furthermore, we show how to embed a mimetic theory with a gauge vector field into a Yang-Mills theory coupled with an axion. This is an attempt at understanding the smallness of the cosmological constant, similarly to the smallness of the θ-parameter in quantum chromodynamics. We consider general $SU(N)$ groups and provide detailed constructions for gauge groups $SU(2)$ and $SU(3)$, also demonstrating the existence of solutions for the mimetic constraint equation in these cases.

Chapter 1

Introduction

In the beginning, we will start with a brief overview over the most important mathematical and physical concepts we will need later on, presenting the main ideas behind gravity and general relativity, cosmology, dark matter and dark energy. This is mainly to introduce conventions and notations, not to give the subject a thorough treatment. Our main sources are a few of the classic textbooks, especially [1–8].

We stress here that we work in the following conventions: The *metric signature* we use will be $(+,\ -,\ -,\ -)$, similar to [4]. Moreover, *Einstein summation convention* over repeated upper and lower indices will be used, where Greek indices appear for spacetime, while Latin indices are only valid for space. Exceptions will be noted.

We will also use *reduced Planck units* $c = 8\pi G_\mathrm{N} = \hbar = k_\mathrm{B} = 1$ with c speed of light, $\hbar$ Planck's constant G_N Newton's constant of gravitation and k_B Boltzmann's constant. Note that the "normal" Planck units are defined by $c = G_\mathrm{N} = \hbar = k_\mathrm{B} = 1$. For convenience and completeness, we will add the base values of the (reduced) Planck units, namely length, time, mass and temperature. Reduced Planck units will be denoted with a tilde $\tilde{}$, Planck units without. Then the Planck units are [5]

$$l_\mathrm{Pl} = \left(\frac{G_\mathrm{N}\hbar}{c^3}\right)^{1/2} \approx 1.161 \times 10^{-33}\,\mathrm{cm} \tag{1.1}$$

$$t_\mathrm{Pl} = \left(\frac{G_\mathrm{N}\hbar}{c^5}\right)^{1/2} \approx 5.391 \times 10^{-44}\,\mathrm{s} \tag{1.2}$$

$$m_\mathrm{Pl} = \left(\frac{\hbar c}{G_\mathrm{N}}\right)^{1/2} \approx 2.177 \times 10^{-5}\,\mathrm{g} \tag{1.3}$$

$$T_\mathrm{Pl} = \left(\frac{\hbar c^5}{G_\mathrm{N} k_\mathrm{B}^2}\right)^{1/2} \approx 1.416 \times 10^{32}\,\mathrm{K} = 1.221 \times 10^{19}\,\mathrm{GeV}\,, \tag{1.4}$$

while the reduced Planck units are

$$\tilde{l}_\mathrm{Pl} = \sqrt{8\pi} l_\mathrm{Pl}\,, \quad \tilde{t}_\mathrm{Pl} = \sqrt{8\pi} t_\mathrm{Pl}\,, \quad \tilde{m}_\mathrm{Pl} = \frac{1}{\sqrt{8\pi}} m_\mathrm{Pl}\,, \quad \tilde{T}_\mathrm{Pl} = \frac{1}{\sqrt{8\pi}} T_\mathrm{Pl}\,. \tag{1.5}$$

1.1 Overview over formulae and conventions in GR and cosmology

The theory of *General Relativity (GR)* was developed by Albert Einstein from 1907 to 1915, resulting in his seminal paper *"Die Feldgleichungen der Gravitation"* [9], where he introduced the field equations describing gravity.

The main idea of his theory is a simple one: While electrodynamics and the other forces of nature are described by fields, gravity is described by the geometry of spacetime, or rather the curvature of spacetime. The mathematical concept behind spacetime is a *differentiable manifold*: It looks locally like Minkowski spacetime, but globally its geometry might be quite different [2, 3].

Let us reiterate that in this section only the most important concepts for the understanding of the thesis will be introduced, mostly based on the classic textbooks [1–8].

1.1.1 Geometry and curvature

A very important tensor in GR is the *metric*, a symmetric and nondegenerate tensor with two lower indices, usually written as $g_{\mu\nu}$ in components. Directly connected to the metric is the *spacetime interval*

$$ds^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{1.6}$$

This gives us the intuitive notion of an infinitesimal squared distance, now generalised to work not only in space, but in spacetime. Notice that it may be positive (for timelike separation), negative (for spacelike separation) and even zero for light. An inverse metric $g^{\mu\nu}$ can also be defined, such that

$$g_{\mu\alpha} g^{\alpha\nu} = \delta^\nu_\mu \,. \tag{1.7}$$

The metric and interval also define the *proper time* τ as

$$d\tau^2 = ds^2 \tag{1.8}$$

in case the separation between them is timelike. It is interesting to consider a *synchronous reference frame* associated with this proper time time [4]

$$ds^2 = \mathrm{d}\tau^2 - \gamma_{ij}(\tau, x^k) \mathrm{d}x^i \mathrm{d}x^j \tag{1.9}$$

with the purely spatial metric γ_{ij}. This is useful for us in the sense that the coordinate time equals the proper time, therefore the time coordinate provides a parametrisation of geodesics for static observers and moreover, the time vectors are hypersurface orthogonal to the spatial slices.

We will need a way to connect two nearby tangent spaces of the same manifold. The crucial object in this context is the *connection*. A connection we can construct from the metric of the manifold is called the *Christoffel symbol*, given by

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \tag{1.10}$$

Note that this is not a tensor, although the notation might suggest so. Then we can define a *covariant derivative* of a vector field v^ν by

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha \,. \tag{1.11}$$

This enables us to go from partial derivatives, which do not transform like tensors, to covariant derivatives, which do. Apart from that, two more properties of the connection are important: The connection is said to be *torsion-free* in case the two lower indices are symmetric, and the covariant derivative is called *metric compatible* if

$$\nabla_\alpha g_{\mu\nu} = 0 \,. \tag{1.12}$$

Now we are equipped to talk about *geodesics*, the shortest path of a particle and the generalization of the concept of a straight line in a curved geometry. The geodesic extremises the length of a curve ℓ parametrised by λ between two points P and Q

$$\ell = \int_P^Q \mathrm{d}\lambda \sqrt{\pm g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \,. \tag{1.13}$$

Under the square root, the positive (negative) signs account for timelike (spacelike) curves. Using the action principle and the Euler-Lagrange equations for the above formulation, we will get the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \,. \tag{1.14}$$

What we have to keep in mind, though, is that the curve has to be *affinely parametrised* in order for the geodesic equation to take this form. That is, the curve parameter λ has to be related to the proper time (or proper distance for spacelike curves) τ by

$$\lambda = a\tau + b \tag{1.15}$$

for constants a and b. Another way to derive the geodesic equation is the following: we have to *parallel transport* a tangent vector $dx^\mu/d\lambda$ to a curve along the same curve and ask that it does not change. This gives us

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \nabla_\alpha \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = 0 \,. \tag{1.16}$$

Expanding this equation with the help of the covariant derivative gives us the geodesic equation (1.14) as above. Another important concept is the *four-velocity* u^μ of a particle along a curve, written as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,. \tag{1.17}$$

We also realise that it is normalised to 1, purely by looking at the definition of the proper time (1.8). Then we can also write the definition of the *four-acceleration*

$$a^\mu = u^\alpha \nabla_\alpha u^\mu \,. \tag{1.18}$$

Expanding this with the appropriate definitions, one realises that the geodesic equation (1.14) is the equivalent expression of $a^\mu = 0$, i.e. vanishing acceleration. The geodesic thus describes a freely falling particle [2, 6].

Another possibility to form a derivative which is a tensor is the so-called *Lie derivative*. The Lie derivative is an even simpler construction than the covariant derivative, as it does not require a connection. It is a generalisation of the directional derivative along a vector field, so the form of the Lie derivative of a function is [2]

$$\mathscr{L}_V f = V^\lambda \partial_\lambda f \,. \tag{1.19}$$

For a vector field U^μ meanwhile, the Lie derivative will be

$$\mathscr{L}_V U^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \equiv [V, U]^\mu \,, \tag{1.20}$$

also defining the *Lie bracket* $[V, U]^\mu$. For a one-form ω_μ the Lie derivative looks like

$$\mathscr{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu \,, \tag{1.21}$$

whereas we can combine both expressions for a rank two tensor T^α_β along a vector field V^μ to give

$$\mathscr{L}_V T^\alpha_\beta = V^\lambda \partial_\lambda T^\alpha_\beta - (\partial_\lambda V^\alpha) T^\lambda_\beta + (\partial_\alpha V^\lambda) T^\alpha_\lambda \,. \tag{1.22}$$

First note that these are indeed tensor expressions, even if it does not look like it at the first glance, but if one substitutes the covariant derivative ∇_μ (1.11) instead of the partial derivative ∂_μ, one can verify that the terms with the connection coefficients indeed drop out. Second, this can easily be generalised to tensors of higher rank. And third, in contrast to the covariant derivative, the Lie derivative of a vector does not produce a tensor. Put another way, the Lie derivative does not increase the rank of the tensor it acts on. Furthermore, if one chooses a coordinate system such that for example the vector V has components $V^\mu = (1, 0, \ldots, 0)$ only along the first direction, or $V = \partial/\partial x^1$, the Lie derivative describes merely the derivative along that direction x^1, or

$$\mathscr{L}_V U^\mu = \frac{\partial U^\mu}{\partial x^1} \,. \tag{1.23}$$

Of course this also generalises to higher rank tensors [2].

The next important geometric object is the *Riemann tensor*, directly describing the intrinsic curvature of the manifold. This comes about by parallel transporting a vector around a closed loop. In flat spacetimes, this vector would be unchanged, as intuition tells us correctly. But in curved spacetimes this is not the case. A good way to visualise this is via the commutator of covariant derivatives of a vector field. The computation will show that

$$[\nabla_\mu, \nabla_\nu] v^\alpha = R^\alpha{}_{\beta\mu\nu} v^\beta \tag{1.24}$$

where $R^\alpha{}_{\beta\mu\nu}$ are the components of the Riemann tensor. Expanded in Christoffel symbols they are

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\beta} \,. \tag{1.25}$$

The Riemann tensor has certain properties, such as antisymmetry under exchange of either the first two or the last two indices. Furthermore, you can swap the first two with the last two indices and the Riemann tensor stays unchanged and the *Bianchi identity* follows from the definition of the Riemann tensor:

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0\,. \tag{1.26}$$

All of these properties restrict the number of independent components. In our usual four dimensions, it has 20 components. By contraction of the first and third index of the Riemann tensor one can form the *Ricci tensor*

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}\,, \tag{1.27}$$

which is symmetric in its indices. One further contraction yields the *Ricci scalar*

$$R = R^\mu_\mu\,. \tag{1.28}$$

Another combination which is very important for GR is the *Einstein tensor*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\,, \tag{1.29}$$

where one can check that contracting the Bianchi identity (1.26) twice is equivalent to

$$\nabla^\mu G_{\mu\nu} = 0\,. \tag{1.30}$$

This will be important later on, if we want to calculate the equations of motion [2, 3].

1.1.2 Energy-momentum tensor and Einstein equations

A simple way to model matter is via a *scalar field*. It is a function that assigns a single number to every point in spacetime, real numbers in case of a real scalar field. It also does not change under coordinate transformations, therefore is the same in every coordinate system. The most general action for a real scalar field ϕ in a curved spacetime with metric $g_{\mu\nu}$ is

$$S_{(\phi)}[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \tag{1.31}$$

with the potential $V(\phi)$ depending only on ϕ. Meanwhile the first term in the action is called the kinetic term. One special and often occurring case of the potential is the mass term, such that the action becomes

$$S_{(\phi)}[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right)\,. \tag{1.32}$$

m describes the mass of the particles that would result from quantising the scalar field. The equations of motion are easily determined using the *Euler-Lagrange equations* in curved spacetime

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0\,, \tag{1.33}$$

resulting in the *Klein-Gordon equation*

$$\Box\phi + \frac{\partial}{\partial\phi}V(\phi) = 0\,. \tag{1.34}$$

for the action with a general potential of the scalar field and the *d'Alembertian* is defined as

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{1.35}$$

A very convenient formula which is used here is provided by rewriting the divergence of a vector field A^μ as [6]

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}A^\mu\right)\,. \tag{1.36}$$

For a complex scalar field Ψ which is invariant under $U(1)$ transformations, meanwhile, we have the action

$$S_{(\Psi)}[g_{\mu\nu},\Psi,\Psi^\dagger] = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi^\dagger - V(\Psi\Psi^\dagger)\right)\,, \tag{1.37}$$

where † denotes the Hermitian conjugate as usual [10,11].

We will also need an *energy-momentum tensor* $T^{\mu\nu}$ describing the matter contents of our theory. If we use the variational principle for the action formulation, the definition of the EMT will be

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}}\,. \tag{1.38}$$

In that formula, S_m describes the action of all matter fields. For a real scalar field ϕ with the form of the matter action (1.31) this results in

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi)\right]\,. \tag{1.39}$$

On the other hand, the EMT for the complex scalar (1.37) will be

$$T^{(\Psi)}_{\mu\nu} = \nabla_\mu\Psi\nabla_\nu\Psi^\dagger - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\nabla_\rho\Psi\nabla_\sigma\Psi^\dagger - V(\Psi\Psi^\dagger)\right]\,. \tag{1.40}$$

In many important cosmological contexts a *perfect fluid* will suffice for homogeneous solutions such as matter in local equilibrium. The perfect fluid form of the EMT is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}\,, \tag{1.41}$$

where ρ and p are the energy density and the pressure of the matter, respectively. u^μ is the four-velocity in which observers travel that measure this energy density and pressure. The general fluid dynamical definitions of these are [12–15]

$$\rho = T_{\mu\nu}u^\mu u^\nu \tag{1.42}$$

$$p = -\frac{1}{3}T^{\mu\nu}\mathcal{P}_{\mu\nu}\,, \tag{1.43}$$

where we used the *projector* $\mathcal{P}_{\mu\nu}$ to the hypersurface orthogonal to u^μ, defined as

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \,. \tag{1.44}$$

But in more general contexts, important for structure formation and non-linear gravitational collapse, one will encounter more than just perfect fluids. The general fluid energy-momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \Pi^{\mu\nu} \,. \tag{1.45}$$

In this case, we additionally need the definitions of the *heat flux* q^μ and the *anisotropic stress* $\Pi^{\mu\nu}$

$$q^\mu = \mathcal{P}^\mu_\alpha T^\alpha_\beta u^\beta \,, \tag{1.46}$$

$$\Pi^{\mu\nu} = \left(\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta - \frac{1}{3}\mathcal{P}^{\mu\nu}\mathcal{P}_{\alpha\beta}\right) T^{\alpha\beta} \,. \tag{1.47}$$

The decomposition of the full energy-momentum tensor (1.45) works for any observers and any matter composition. One simply has to change the velocity to that of the desired observer, let us call it U^μ, and repeat the process while also altering the projector (1.44), of course. In particular what looks like a perfect fluid in one local rest frame may look different from another observer's perspective. Also, it may not even be possible to find a frame where the fluid looks like a perfect one. As an instructive example, let us consider the energy-momentum tensor (1.39) of the classical scalar field and decompose it according to the full fluid-dynamic energy-momentum tensor (1.45). The derivative along the four-velocity is defined as

$$\dot{\phi} = u^\mu \nabla_\mu \phi \,. \tag{1.48}$$

In conjunction with that, we will introduce a purely spatial derivative

$$D_\mu \phi = \mathcal{P}^\nu_\mu \nabla_\nu \phi \,. \tag{1.49}$$

Then we can calculate the energy density (1.42), pressure (1.43), heat flux (1.46) and anisotropic stress (1.47) as

$$\rho = \frac{1}{2}\left(\dot{\phi}^2 - \mathcal{P}^{\alpha\beta} D_\alpha \phi D_\beta \phi\right) + V(\phi) \,, \tag{1.50}$$

$$p = \frac{1}{2}\left(\dot{\phi}^2 + \frac{1}{3}\mathcal{P}^{\alpha\beta} D_\alpha \phi D_\beta \phi\right) - V(\phi) \,, \tag{1.51}$$

$$q^\mu = \dot{\phi} D^\mu \phi \,, \tag{1.52}$$

$$\Pi^{\mu\nu} = D^\mu \phi D^\nu \phi - \frac{1}{3}\mathcal{P}^{\mu\nu}\mathcal{P}^{\alpha\beta} D_\alpha \phi D_\beta \phi \,. \tag{1.53}$$

In the case of $\nabla_\mu \phi$ being timelike $g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi > 0$ and future-directed, i.e. continuously lying in the future half of the light cone [3], one can define a velocity

$$U^\mu = \frac{\nabla^\mu \phi}{\sqrt{2X}} \tag{1.54}$$

with the usual kinetic term

$$X = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \,, \tag{1.55}$$

such that U^μ is properly normalised. Then one may choose the *local rest frame* of this fluid moving with this velocity U^μ and define the projector into the spacelike hypersurface orthogonal to it. As one can easily see, the spatial derivatives $D_\mu \phi$ in this case vanish and so do the heat flux q^μ and the anisotropic stress tensor $\Pi^{\mu\nu}$. Therefore, in the local rest frame moving with ϕ the fluid resulting from the canonical scalar looks like a perfect one.

Returning to the overall properties of the energy-momentum tensor, it is covariantly conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{1.56}$$

This yields the Euler equations of fluid mechanics in the appropriate Newtonian limit [2,16].

Now we are finally equipped to write down the famous Einstein field equations and put them into context. They will replace the Poisson equation for the Newtonian gravitational potential Φ for us, which looks like

$$\Delta \Phi = 4\pi G_{\mathrm{N}} \rho \,, \tag{1.57}$$

with Δ the spatial Laplacian. We also have restored the $8\pi G_{\mathrm{N}}$ for the moment. In our conventions for the units the Poisson equation would rather look like $\Delta\Phi = \frac{1}{2}\rho$. Therefore, the form of the new gravitational equation will be something like $[\Delta g]_{\mu\nu} \propto T_{\mu\nu}$. The first term in this means that we will need second derivatives of the metric, which is encapsulated by the Ricci tensor. To fit the number of indices and to account for energy-momentum conservation (1.56), we will refer to the Bianchi identity (1.26) and finally write down Einstein's field equations of General Relativity

$$G_{\mu\nu} = T_{\mu\nu} \,. \tag{1.58}$$

The proportionality factor between the two tensors is fixed by asking for the recovery of the Newtonian limit [2].

Another way to formulate and derive the Einstein field equations is via the *Einstein-Hilbert action*

$$S_{\mathrm{EH}}[g_{\mu\nu}] = -\frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} R \tag{1.59}$$

and the matter action $S_{\mathrm{m}}[g_{\mu\nu}, \Phi_{\mathrm{m}}]$ with Φ_{m} collectively denoting the matter fields present in the theory. Variation of the total action $S_{\mathrm{EH}} + S_{\mathrm{m}}$ also yields the Einstein equations, using the definition of the energy-momentum tensor from the action (1.38). For completeness, let us also introduce the cosmological constant term Λ here, although we will talk more deeply about it later in 1.2.2. Then the action will look like

$$S[g_{\mu\nu}, \Lambda] = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} R + \Lambda \right) \,. \tag{1.60}$$

This results in the Einstein field equations looking like

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} + \Lambda g_{\mu\nu} \,. \tag{1.61}$$

There are a few subtleties concerning the so-called Gibbons-Hawking-York boundary term [17, 18] of the Einstein-Hilbert action, but we will choose to gloss over them here [2, 4].

1.1.3 Weyl transformations

Next, let us introduce the concept of *Weyl transformations*. A remark is in order for this: Many people and textbooks use different definitions for *conformal transformations*, so we will make an effort to distinguish them from Weyl transformations clearly, primarily based on [19, 20]. Weyl transformations are local rescalings of the metric and the other fields, such that it represents a physical change of the metric, as in

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)\,, \tag{1.62}$$

where it is important to stress that the scalar function $\Omega(x)$ depends on the point of spacetime. Angles are not affected by this transformation, but distances change at each point. Therefore, spacelike and timelike vectors change in length, but lightcones and therefore the causal structure are unaffected. In contrast to Weyl transformations, conformal transformations are a special set of coordinate transformations which leaves the metric unchanged up to a conformal factor. We will talk about Weyl transformations here and the actual change of geometry they cause. What it does not cause, however, are changes in the causal structure. As angles are unchanged, lightcones stay the same, as well as the nature of timelike and spacelike vectors. Geodesics however will change, so what is a geodesic for $g_{\mu\nu}$ will not be a geodesic for $\tilde{g}_{\mu\nu}$. Another difficulty lies in the question which metric to choose for raising and lowering indices, as for example $g_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} \equiv g^{\mu\nu} \neq g_{\alpha\beta} \tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu}$. We will try to be clear about which metric has been used in every case. Furthermore, we will list a few useful formulas for Weyl transformations of much used tensors and scalars, for further convenience. Everything listed here is valid in four dimensions, although [2] does use more general formulas. The new Christoffel symbols will be calculated from the old ones to be

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}\left(\delta^{\rho}_{\mu} \nabla_\nu \Omega + \delta^{\rho}_{\nu} \nabla_\mu \Omega - g_{\mu\nu} g^{\rho\lambda} \nabla_\lambda \Omega\right)\,. \tag{1.63}$$

The Riemann tensor changes under (1.62) as

$$\begin{aligned} \tilde{R}^{\rho}{}_{\sigma\mu\nu} = R^{\rho}{}_{\sigma\mu\nu} &- 2\left(\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - g_{\sigma[\mu}\delta^{\alpha}_{\nu]} g^{\rho\beta}\right)\Omega^{-1}(\nabla_\alpha \nabla_\beta \Omega) \\ &+ 2\left(2\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - 2 g_{\sigma[\mu}\delta^{\alpha}_{\nu]} g^{\rho\beta} + g_{\sigma[\mu}\delta^{\rho}_{\nu]} g^{\alpha\beta}\right)\Omega^{-2}(\nabla_\alpha \Omega)(\nabla_\beta \Omega)\,. \end{aligned} \tag{1.64}$$

Antisymmetrisation of indices was used, e.g.

$$\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]} \equiv \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\alpha}_{\nu} - \delta^{\rho}_{\nu}\delta^{\alpha}_{\mu}\right)\,. \tag{1.65}$$

The Ricci tensor and scalar meanwhile will look like

$$\begin{aligned} \tilde{R}_{\sigma\nu} = &R_{\sigma\nu} - \left(2\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} + g_{\sigma\nu} g^{\alpha\beta}\right)\Omega^{-1}(\nabla_\alpha \nabla_\beta \Omega) \\ &+ \left(4\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} - g_{\sigma\nu} g^{\alpha\beta}\right)\Omega^{-2}(\nabla_\alpha \Omega)(\nabla_\beta \Omega) \end{aligned} \tag{1.66}$$

and

$$\tilde{R} = \Omega^{-2}R - 6g^{\alpha\beta}\Omega^{-3}(\nabla_\alpha\nabla_\beta\Omega)\,. \tag{1.67}$$

Very important and useful will also be the Weyl transformations of the second derivatives of a scalar field, as the first derivative is merely

$$\tilde{\nabla}_\mu\phi = \nabla_\mu\phi = \partial_\mu\phi\,. \tag{1.68}$$

But the second derivative will be

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi = \nabla_\mu\nabla_\nu\phi - \left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu - g_{\mu\nu}g^{\alpha\beta}\right)\Omega^{-1}(\nabla_\alpha\Omega)(\nabla_\beta\Omega) \tag{1.69}$$

while the d'Alembertian (1.35) will have the Weyl transformed form

$$\tilde{\Box}\phi = \Omega^{-2}\Box\phi + 2g^{\alpha\beta}\Omega^{-3}(\nabla_\alpha\Omega)(\nabla_\beta\Omega)\,. \tag{1.70}$$

Moreover, we will need the concept of a *conformal weight* for fields occurring in our theories. We Weyl transform the action of our theory with the help of the Weyl transformation (1.62) of the metric. Then we will look for the transformation of the occurring scalar, vector, etc. fields, such that the action stays unchanged. In the case of a scalar field, for example,

$$\phi \to \tilde{\phi} = \Omega^{-\Delta}(x)\phi\,. \tag{1.71}$$

Then, Δ is called the conformal weight [21]. One interesting example for a scalar-tensor theory can be written as

$$S[g_{\mu\nu},\phi,\Phi_\mathrm{m}] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(-\xi\phi^2 R + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + S_\mathrm{m}[g_{\mu\nu},\Phi_\mathrm{m}] \tag{1.72}$$

with a parameter ξ to determine. S_m denotes the matter action which depends on the metric and the collective matter fields Φ_m. The aim is to find the fitting value such that the action is unchanged under Weyl transformations. To achieve that, we perform the Weyl transformations of the metric (1.62), the Ricci scalar (1.67) and one for the scalar field (1.71), where we want to determine the conformal weight of ϕ. It turns out that the conformal weight should be $\Delta = 1$ in this case for the action to be unchanged. Later in this work, we will encounter a vector field with the unusual conformal weight four. Within the calculation of the Weyl transformation, we need to find the right ξ such that the rest vanishes. This condition will leave us with

$$\xi = \frac{1}{6} \tag{1.73}$$

and therefore the action

$$S[g_{\mu\nu},\phi,\Phi_\mathrm{m}] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(-\frac{\phi^2}{6}R + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + S_\mathrm{m}[g_{\mu\nu},\Phi_\mathrm{m}]\,. \tag{1.74}$$

This scalar-tensor action with non-minimal coupling between gravity and scalar field is said to be *conformally invariant*. This conformal invariance also extends to the field equations of the theory [22–24]. In the above example of the conformally coupled scalar the appropriate Weyl transformation would be

$$g_{\alpha\beta} = \frac{6}{\phi^2}\hat{g}_{\alpha\beta}\,, \tag{1.75}$$

so with the help of the conformal transformation of the Ricci scalar (1.67) the action will become

$$S[\hat{g}_{\mu\nu},\phi,\Phi_\mathrm{m}] = \int \mathrm{d}^4x\sqrt{-\hat{g}}\left(-\frac{1}{2}\hat{R} + \frac{12}{\phi^2}\hat{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + S_\mathrm{m}\left[\frac{6}{\phi^2}\hat{g}_{\mu\nu},\Phi_\mathrm{m}\right]. \tag{1.76}$$

Therefore we learn in the case of a conformally coupled scalar, a Weyl transformation can revert the gravity term to Einstein-Hilbert (1.59), but now with a scalar field term, although not the ususal kinetic term (1.55). And also note the coupling of matter not to $\hat{g}_{\mu\nu}$ we will discuss in the following subsection 1.1.4.

Furthermore, there are certain terms in actions which are not Weyl invariant by construction. This can be the equations of motion derived from the EMT, i. e. $\nabla_\mu T^{\mu\nu} = 0$. They are only Weyl invariant in the case that the trace T^μ_μ vanishes, for example in the case of light. Interesting for us, a cosmological constant spoils Weyl invariance, as it introduces a scale into the theory [25]. Concerning the Weyl invariance of fields of certain spins and spacetime dimensions: (massless) scalar and fermion fields are always Weyl-invariant, but bosons require four spacetime dimensions for Weyl invariance [25, 26].

1.1.4 Jordan and Einstein frames

After discussing Weyl transformations, it is useful to talk more about frames occurring in GR. As we have seen in the action (1.74) above, the scalar field ϕ is directly coupled to the Ricci scalar R. This frame in which the theory is formulated is called the *Jordan frame*. Also a direct coupling of some arbitrary function of the scalar to field to the Ricci scalar is possible. But this theory can be reformulated by a Weyl transformation, such that the gravity part of the action looks like the Einstein-Hilbert action again, and the direct coupling of the scalar field to the Ricci scalar is removed. This frame is then called the *Einstein frame*. In the above example, this is (1.76).

The most important thing to notice about this transformation between the two frames is the following: In the matter action, any kind of matter Φ_m will not be minimally coupled to $\hat{g}_{\mu\nu}$, but to some product of the metric and the scalar field, e. g. $\frac{6}{\phi^2}\hat{g}_{\alpha\beta}$ as in (1.76). This is the price to pay for the Einstein-Hilbert form of the gravitational part of the action [25].

Let us illustrate this mechanism and its consequence with another example, using a widely known scalar-tensor theory called *Brans-Dicke theory* [27], also called *Jordan-Brans-Dicke theory* [28]. As an aside, it was found that a special case of Brans-Dicke theory, namely singular Brans-Dicke theory, is connected to mimetic gravity [29]. The action for Brans-Dicke theory looks like

$$S_\mathrm{BD}[g_{\mu\nu},\varphi,\Phi_\mathrm{m}] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(-\varphi R + \frac{\omega}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right) + \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_\mathrm{m}[g_{\mu\nu},\Phi_\mathrm{m}]\,. \tag{1.77}$$

In this notation, φ is the scalar field, Φ_{m} collectively denotes the matter fields, ω is the so-called fudge factor, a dimensionless constant, and the pure matter Lagrangian is written as $\mathcal{L}_{\mathrm{m}}$. The equation of motion for the scalar field and the modified Einstein field equations follow

$$\frac{2\omega}{\varphi}\Box\varphi - \frac{\omega}{\varphi^2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + R = 0\,, \tag{1.78}$$

$$G_{\mu\nu} = \frac{1}{\varphi}T^{\mathrm{m}}_{\mu\nu} + \frac{\omega}{\varphi^2}\left(\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi\right) + \frac{1}{\varphi}\left(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi\right)\,. \tag{1.79}$$

Covariant derivative, Ricci scalar and Einstein tensor are built with respect to the metric $g_{\mu\nu}$ while $T^{\mathrm{m}}_{\mu\nu}$ denotes the energy-momentum tensor purely coming from the matter Lagrangian. To transform this to the Einstein frame, one uses the following redefinitions [25,30]

$$\varphi \to \tilde{\varphi} = \left(\frac{2\omega+3}{2}\right)^{1/2}\ln\varphi \tag{1.80}$$

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \tag{1.81}$$

to arrive at the action

$$\begin{aligned} S[\tilde{g}_{\mu\nu},\tilde{\varphi},\Phi_{\mathrm{m}}] = &\frac{1}{2}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left(-\tilde{R} + \tilde{g}^{\mu\nu}\nabla_\mu\tilde{\varphi}\nabla_\nu\tilde{\varphi}\right) \\ &+ \int \mathrm{d}^4x\sqrt{-\tilde{g}}\,\exp\left(-\sqrt{\frac{8}{2\omega+3}}\tilde{\varphi}\right)\mathcal{L}_{\mathrm{m}}[\tilde{g}_{\mu\nu},\Phi_{\mathrm{m}}]\,. \end{aligned} \tag{1.82}$$

As we can see, the gravitational part of the action is now no longer coupled directly to the scalar field and has taken on the usual Einstein-Hilbert form. This transformation comes at a price: the matter action is now coupled to φ^2 and therefore non-minimally coupled [25].

Also, the two frames are widely regarded as mathematically equivalent, as also the physical interpretation, if one takes all field redefinitions and the non-minimal coupling of matter into account. This is valid at the classical level, which we will use and discuss here. As an aside, at the quantum level things are not so clear and because of the lack of a full theory of quantum gravity, unsolved as of now [25, 27, 31–34].

1.1.5 Cosmology

We take now a closer look at the *Friedmann equations*, the field equations resulting from the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric* (note that we may often simply refer to it as the *Friedmann metric*) [2]

$$ds^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right) \tag{1.83}$$

with the cosmological *scale factor* $a(t)$ describing the expansion of the universe and the curvature parameter k taking on different values for different homogeneous and isotropic geometries, namely

$$k = \begin{cases} +1, & \text{closed (spherical)} \\ 0, & \text{flat} \\ -1, & \text{open (hyperbolic)}\,. \end{cases} \tag{1.84}$$

Including the cosmological constant Λ, from (1.60), the Friedmann equations will look like

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_{\rm m}}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.85}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_{\rm m} + 3p_{\rm m}) + \frac{\Lambda}{3}\,, \tag{1.86}$$

where the *Hubble parameter* H was defined, written here explicitly again [2,35]

$$H = \frac{\dot{a}}{a}\,. \tag{1.87}$$

The $\dot{}$ signifies a derivative with respect to coordinate time t here, while the energy-momentum tensor has a perfect fluid form as is required by the form of the Einstein tensor, with matter energy density $\rho_{\rm m}$ and matter pressure $p_{\rm m}$.

The EMT conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.88}$$

results in

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}\,. \tag{1.89}$$

with the definition of the *equation of state parameter* w

$$p = w\rho\,. \tag{1.90}$$

Yet the energy conservation equation (1.89) and the Friedmann equations (1.85) and (1.86) are not independent of each other. We only need two of the three, while the third one is a consequence. An often used quantity in cosmology is the *critical density* $\rho_{\rm crit}$, the energy density at which the universe is exactly flat, defined in reduced Planck units as

$$\rho_{\rm crit} = 3H^2\,, \tag{1.91}$$

allowing us to define the *density parameters* Ω_i for each matter component ρ_i, i.e. normal matter, dark energy, etc separately. Then

$$\Omega_i = \frac{\rho_i}{\rho_{\rm crit}} \tag{1.92}$$

and we can write the first Friedmann equation (1.85) as

$$1 = \Omega_\mathrm{m} + \Omega_\Lambda - \frac{k}{a^2 H^2} \,. \tag{1.93}$$

So we see that in the case of a flat universe ($k = 0$), the density parameters of the universe should add up to unity [2].

In measuring cosmological times and distances, the *cosmological redshift* is fairly important. The wavelength of photons is stretched as the universe expands such that

$$\frac{\lambda_\mathrm{obs}}{\lambda_\mathrm{em}} = \frac{a_\mathrm{obs}}{a_\mathrm{em}} \tag{1.94}$$

where a_em refers to the scale factor at the emission of the photon and a_obs to the scale factor at the observation of the photon. The redshift z itself is defined as the fractional difference between the wavelengths, i.e.

$$z = \frac{\lambda_\mathrm{obs} - \lambda_\mathrm{em}}{\lambda_\mathrm{em}} \,. \tag{1.95}$$

So in terms of the scale factor, this results in

$$1 + z = \frac{a_0}{a_\mathrm{em}} \tag{1.96}$$

with a_0 the scale factor at present day [5].

There is another way to define the Friedmann equations and the Hubble constant, namely with the help of *conformal time* η, defined as

$$\eta = \int \frac{\mathrm{d}t}{a(t)} \,. \tag{1.97}$$

To distinguish, we will often call t the *physical time*. It is also natural as a cosmological time, as the that physical time is equivalent to the proper time, if one compares this with a synchronous frame (1.9). While $\dot{(\ )}$ denotes differentiation w.r.t. physical time, $(\ )'$ denotes differentiation w.r.t. conformal time. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\eta} = a\frac{\mathrm{d}}{\mathrm{d}t} \,. \tag{1.98}$$

The Friedmann metric (1.83) will change to

$$ds^2 = a^2(\eta)\left(\mathrm{d}\eta^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \tag{1.99}$$

with spatial metric γ_{ij}. We can see that this metric is Weyl transformed from a flat Minkowski metric, compare (1.62), where $a^2(\eta)$ now plays the role of the conformal factor. The conformal Hubble constant $\mathscr{H}$ can be written as [5]

$$\mathscr{H} = \frac{a'}{a} = aH \,. \tag{1.100}$$

For the background flat Friedmann metric we cite the components of the Einstein tensor as [5]

$$G^0_0 = \frac{3\mathscr{H}^2}{a^2} = 3H^2 \tag{1.101}$$

$$G^0_i = 0 \tag{1.102}$$

$$G^i_j = \frac{1}{a^2}\left(2\mathscr{H}' + \mathscr{H}^2\right)\delta^i_j = \left(2\dot{H} + 3H^2\right)\delta^i_j\,. \tag{1.103}$$

The transformations (1.98) and (1.100) between conformal and physical quantities were used.

In cosmology we sometimes need perturbations of the metric, often as small deviations around flat Friedmann space. The perturbed universe in the *Newtonian gauge*, only containing the scalar perturbations, can be written as [5]

$$ds^2 = (1+2\Phi)\mathrm{d}t^2 - (1-2\Phi)a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1.104}$$

with the Newtonian gravitational potential Φ. Note that this is similar to the conformal metric used in [36], but the scalar potentials $\Phi = \Psi$ are set equal. This can be justified in a case where the energy-momentum tensor does not have purely spatial off-diagonal terms, i.e. the anisotropic stress tensor (1.47) vanishes [36].

So far, Einstein's theory of gravity, the theory of general relativity, has stood the test of time. There are Einstein's three classical solar system tests [37], namely the perihelion precession of Mercury, the deflection of light by the sun, and the gravitational redshift of light in a potential well that support the theory. But also more modern observations like the Hulse-Taylor binary pulsar [38], gravitational waves [39] and the direct observation of the black hole in the center of the galaxy M87 [40] and the one at the center of the Milky Way [41] offer more proof that GR is indeed the theory to describe gravity.

1.2 Introduction to Dark Matter and Dark Energy

But if we indeed consider GR as the valid theory of gravity, then other observations seem to bring up contradictions, provided one assumes that all the matter is luminous. So our next look will be into the *dark universe* and its consequences for observations and theory.

1.2.1 Dark Matter

One of the first examples of those observational contradictions appeared already in 1933 on extragalactic scales, when Fritz Zwicky applied the virial theorem to the motion of galaxies in galaxy clusters. The mass of the cluster inferred by estimating it via the velocity dispersion of single galaxies and using the virial theorem was found to be much larger than that of the luminous matter [42]. Something similar can be observed in the rotation velocities of stars around the centres of spiral galaxies. Also here the luminous matter and the matter inferred by the rotation curves do not add up unless one adds a halo of matter in the outskirts of galaxies [43]. For this invisible mass the name *dark matter (DM)* was coined [42], matter which only interacts gravitationally, but not electromagnetically.

One model to explain the galactic rotation curves are "massive compact halo objects" (MACHOs), astrophysical objects like brown dwarfs or black holes, too dark to be detected by optical telescopes. Here the search has been unsuccessful so far. Microlensing observations have shown that there are not enough MACHOs to account for the amount of dark matter present in our galaxy [44].

On larger scales of galaxy clusters, another effect becomes important, namely *gravitational lensing*, the idea being introduced in [45,46], the first source being Einstein himself. Light of background objects is deflected and curved around large matter concentrations which distorts the image seen on Earth. The effect is especially noticeable on larger scales, where the light of background galaxies is distorted by large dark matter concentrations present in galaxy clusters and the filaments in between. This is the effect of *strong gravitational lensing*, when one can observe the warped image of a galaxy with the naked eye. On the other hand, *weak gravitational lensing* only shows up in statistical samples, when images of the background galaxies are overall distorted in a preferred direction. One can actually map the dark matter distribution by this, and reveal the large scale structure of dark matter in the universe. Another scale of gravitational lensing is that of *microlensing* around smaller dark objects even in our own galaxy, like brown dwarfs or stellar black holes. There, no actual distortion of the image can be observed, as the involved masses are too small, but gravitational lensing also increases the observed brightness of the object. By monitoring the light curve of objects one can see a temporary brightening due to a dark object transversing our line of sight [47].

An extremely important era of cosmological evolution which was affected by dark matter was structure formation in the early universe. Matter — dark as well as baryonic — started to cluster due to gravitational attraction. But at that time, the universe was still hot enough that the baryonic matter was coupled to photons via Thomson scattering. This provided enough pressure that any baryonic overdensities would have been erased quickly. Therefore, before decoupling of baryonic matter from photons only dark matter could form clumps in which the baryonic matter could fall into. Without this added time to form matter structures the universe would not have had developed the large-scale structure of clusters, filaments and voids we observe today [48].

A very important probe for the cosmological parameters and dark matter content of the universe is the *cosmic microwave background (CMB)* and its anisotropies. The CMB was formed at recombination, when electrons and nuclei formed the atoms and photons decoupled from them, now travelling freely through the universe. Although the universe at that time, approximately 380,000 years after the big bang, was extremely homogeneous, tiny temperature fluctuations of $< 0.01\%$ can tell us more about the contents of the universe. The temperature fluctuations and their two-point correlation functions on different angular scales θ give us information about physical processes happening on certain scales. The physical divide between the two scales is the Hubble radius, the causally connected region, at recombination, corresponding to an angular scale of roughly 1° as seen on the CMB today. On scales larger than that, the primordial inhomogeneities from the inflationary period have been conserved, while on the other hand we get information on the gravitational instability and clumping of matter from smaller scales. The results are displayed in the *multipole moments* C_l of the power spectrum. The Hubble radius at recombination sits at a multipole of $l \sim 200$. Some features like a plateau at low multipoles $l < 20$ (or large angular scales), followed by oscillating peaks and valleys, which then decay at very high multipoles, are universal and qualitative features, predicted by inflation. But the exact form of the the power spectrum quantitatively constrains the cosmic parameters such as energy densities and curvature parameter to high accuracy. For example, the locations of the acoustic peaks are sensitive to the matter density, as they are determined by the angular size of the sound horizon at the time of recombination. And the size of the sound horizon as seen today is equally determined by the baryon density, the dark matter density and the spatial curvature. One very important observation based on the height of the first acoustic peak ($l \sim 250$) combined with the existence of the second peak ($l \sim 550$), is that the cold dark matter density is less than the critical density, and it is higher than the baryon density [5, 49, 50].

One more piece of evidence for dark matter comes from the observation of the *bullet cluster*. It is a pair of galaxy clusters that has crossed through each other. The bullet cluster has been observed by means of x-rays and gravitational lensing (among others). The point is that the main mass distribution of the matter as observed by gravitational lensing follows the observed stars and not the hot gas. The hot gas has interacted, as expected, emitting x-rays, whereas the stars and indeed most of the mass has passed through each other without colliding, strongly suggesting that the main component of the matter is indeed dark matter, and not baryonic matter [8, 51, 52]. Although, there also exist other well-founded sources which claim that the bullet cluster poses a challenge to the ΛCDM model. In a cosmological N-body simulation set up according to the accepted cosmological model the cluster infall velocities required to produce the observed x-ray brightness were not found [53].

The technical definition of "matter" comes from the Friedmann equations and more properly the energy conservation equation (1.89). The condition for being considered as matter, be it dark or baryonic, is that its pressure is zero, i.e. the equation of state parameter (1.90) is

$$w = 0\,, \tag{1.105}$$

resulting in the following behaviour for the energy density of matter

$$\rho_{\mathrm{m}} \propto a^{-3}\,. \tag{1.106}$$

This behaviour can be intuitively understood as matter being diluted as the volume of the universe expands, while the mass of the matter itself is conserved. The particles of dark matter moreover have to obey the collisionless Boltzmann equation, as they should not interact too much with each other [8]. Also, the velocity dispersion of cold dark matter particles has to be quite small, as the resulting free-streaming would erase structures being formed from gravitational collapse. This can be seen in the CMB and in the large scale structure of the universe and constrains the temperature to mass ratio of CDM particles to $\frac{T_0}{m} < 1.07 \times 10^{-14}$. This relates the CDM temperature T_0 to the particle mass m and is used as a measure for the velocity dispersion. In case this value is too large, the particles are classified as "hot" dark matter and cannot clump and therefore form structures in the universe [54]. Further requirements for the properties of dark matter can be inferred from other observations. For example, the difference to baryonic matter is that dark matter only interacts gravitationally and not with radiation, such that it cannot be observed by means of any telescopes operating with electromagnetic radiation, at least so far it is known [8].

In general, particle dark matter can be subdivided into thermal relics and nonthermal relics, depending on whether they were ever in a thermal bath before they decoupled or were created through some other nonthermal process. The thermal relics can be further divided into categories of *hot* and *cold dark matter*, or whether they were relativistic or non-relativistic at their decoupling. Relativistic species dampen the clumping of matter as they can free stream before they become non-relativistic. Therefore, structure would first clump on large scales, which is not what is observed in galaxy surveys [5, 52]. The natural conclusion is that hot dark matter can only make up a non-significant part of all dark matter, most of it should be cold, so that the large scale structure can form from smaller structures first. Neutrinos make up some of the hot dark matter as they are light and decoupled from thermal equilibrium at relativistic speeds. One can constrain their masses to $m_\nu < 5\,\mathrm{eV}$ in the case of three neutrino species, if one asks for the universe to be flat and the dark matter not making up more than 30% of the energy density, but also allowing the dark matter to consist entirely of neutrinos, which cannot be due to them being hot dark matter [5]. More realistic assumptions lead to tighter constraints, heavily backed by observational bounds from the Planck satellite [50], where it was found that the sum of the neutrino masses should be $\sum m_\nu < 0.12\,\mathrm{eV}$. Thermal cold relics might be found from supersymmetric models, in their lightest stable electrically neutral particle, the so-called neutralino. Examples for nonthermal relics are *axions*, very light (sub-eV) particles originating from QCD considerations. Their mass bounds may be even calculated from cosmological and astrophysical observations, to be less than $m_a \lesssim 10^{-5} - 10^{-3}\,\mathrm{eV}$ [55]. Their momentum is thought to be very small, so they make good candidates for cold dark matter [5, 52]. Furthermore, they are weakly coupled to ordinary matter, therefore they are thought to be stable on cosmological scales. Another property which makes them an interesting DM candidate for us is the fact that as bosons they can have high occupation numbers. This is crucial for axions to be feasible as DM candidates, as they are extremely light and only through their high occupation numbers they can reach the densities required to explain the observed DM mass. A consequence of this property is that they can interact collectively, almost classically and like a fluid [56]. Our later fluid description of dark matter is therefore very well suited. We will discuss them more in 6.2.

Formerly favoured DM particles were the so-called *weakly interacting massive particles (WIMPs)*. They should have the properties, as their name suggests, of being weakly interacting with the standard model particles, non-relativistic, i.e. cold, and massive enough to account for the observed mass. However, they also have fairly high cross-sections, higher than some other DM candidates like axions, their supersymmetric counterpart axinos or the also supersymmetric gravitinos. Their cross-section are still smaller than neutrino cross-sections, though. In theory this cross-section would make them good candidates to be observed in experiments through recoil energy and scattering, but despite the efforts to discover them in various colliders, x-rays, gamma rays and cosmic rays, nothing conclusive has been found as of now. Moreover, the by now excluded parameter space is getting larger than the allowed parameter space for feasible DM candidates, therefore WIMPs look like [57, 58].

Two general DM production mechanisms are called *freeze-out* and *freeze-in*. The freeze-out mechanism is one possible mechanism that describes how particles drop out of equilibrium with the thermal bath of the surrounding hot plasma in the early universe. At first the DM particles are in thermal equilibrium, until the temperature of the universe has cooled down so much due to its expansion that the annihilation and decay of the DM particles to lighter particles become ineffective. Then the DM number density becomes fixed. In this case, the final abundance decreases with interaction strength, as the annihilation becomes more effective [5, 59].

On the other hand, the freeze-in mechanism is a newer suggestion [59]. This mechanism assumes a thermal bath of particles on the one hand, and a DM particle on the other hand, only feebly interacting with the thermal bath and in negligible abundance, which is a crucial feature. Then the in initial production through collision or decay of standard model particles at temperatures larger than the DM particle mass is very small. The main production occurs around the point where the bath temperature is similar to the DM mass. To sum up, the DM abundance freezes in, and increases with interaction strength, as opposed to the freeze-out mechanism, and produces just enough particles to match the observed dark matter density today [59]. There exist other theories where DM is modelled as a real scalar which interacts with the Standard Model particles via a coupling to a scalar field which is in thermal equilibrium with the standard model particles. This model undergoes first a spontaneous phase transition in which the DM field acquires a non-zero expectation value but the crucial point comes later: Due to cooling down the symmetry gets restored and the system undergoes an inverse phase transition where the DM field starts oscillating shortly before and consequently produces dark matter particles [60].

A good consistency test of the dark matter theory is to ask whether the equation of state parameter for dark matter is actually observed to vanish. This has been tested by using a general dark matter model, to see whether the actual CMB data at various times really coincides with this theoretical prediction [61]. They found that the equation of state parameter w of dark matter is consistent with zero, although it is much better constrained around matter-radiation equality than at present times.

Another property is that it is *non-baryonic*, as has been mentioned before, i.e. it is not comprised of neutrons and protons. Although, there exist speculations that dark matter might be composed of a stable six quark state *uuddss*, hinted at by lattice QCD [62]. However, it has neither been detected in accelerators nor in dark matter searches yet.

Although there are many experiments set up for direct detection of dark matter particles, especially WIMPs, they have been unsuccessful in their search so far [63, 64]. In the end, whatever dark matter may be made out of, its general properties must be fulfilled in any case. As mentioned, dark matter must not interact with electromagnetism, it has to be cold, and be able to reproduce the abundance we observe today [5, 8].

1.2.2 Dark Energy and Cosmological Constant

So far, we have accounted for only about one third of the energy content of the universe. The remaining two thirds are even more elusive than dark matter and are aptly named *dark energy (DE)* in its most general configuration. One of the most important forms of dark energy is called the *cosmological constant (CC)*, characterised by an equation of state parameter (1.90) of $w = -1$. A cosmological constant refers to its energy density not changing in time and space. Moreover, as we will discuss more later in 1.2.3, the CC can be seen as a coupling constant which is added in the action, or as the consequence of the quantum fluctuations of all the fields in the universe in the vacuum state.

The effects of dark energy mostly concern the expansion history of the universe and the relation between redshift and distance. Furthermore, dark energy only started to dominate recently, in cosmological terms, so that high-redshift observations are less important than the phenomena in the relatively late evolution stages of the universe [65]. "Recently" in cosmological terms means that the cosmological constant started to dominate over normal matter at a redshift (1.96) of

$$z_{\Lambda\,\mathrm{dom}} = \left(\frac{1-\Omega_\mathrm{m}}{\Omega_\mathrm{m}}\right)^{1/3} - 1 \approx 0.28\,. \tag{1.107}$$

There the first Friedmann equation (1.85) was used, in terms of the cosmological density parameters (1.92), while using the redshift (1.96) and setting $a_0 = 1$. For the dominating cosmological constant we assume $H = H_0$. As the density parameter for matter we took $\Omega_\mathrm{m} \approx 0.32$ and the assumption that the universe is flat, i.e. $\Omega_\mathrm{m} + \Omega_\Lambda = 1$ [50]. To arrive at the physical time in terms of the redshift, we need to differentiate (1.96) with respect to time and reformulate it to

$$t(z) = \int_z^\infty \frac{\mathrm{d}\tilde{z}}{H(\tilde{z})(1+\tilde{z})}\,, \tag{1.108}$$

inserting the first Friedmann equation as above and integrating results in

$$t(z) = \frac{2}{3H_0} \cdot \frac{1}{\sqrt{1-\Omega_\mathrm{m}}} \operatorname{arcsinh}\left(\frac{1+z_{\Lambda,,\mathrm{dom}}}{1+z}\right)^{3/2} \tag{1.109}$$

or, by inserting $z = 0$ to calculate the age of the universe today $t_0 \approx 13.8\,\mathrm{Gyr}$,

$$t(z) = t_0 \cdot \frac{\operatorname{arcsinh}\left(\frac{1+z_{\Lambda\,\mathrm{dom}}}{1+z}\right)^{3/2}}{\operatorname{arcsinh}\left(\frac{1-\Omega_\mathrm{m}}{\Omega_\mathrm{m}}\right)^{1/2}}\,. \tag{1.110}$$

This translates to matter and CC densities being equal only 3.45 Gyr ago [5]. One can also compare this to the time when the universe started accelerating. In this case, take the second Friedmann equation (1.86) in terms of redshift and density parameters and set it to zero. Therefore,

$$z_{\mathrm{acc}} = \left(\frac{2(1-\Omega_{\mathrm{m}})}{\Omega_{\mathrm{m}}}\right)^{1/3} - 1 \approx 0.62 \tag{1.111}$$

for the above value of Ω_{m}. Inserting z_{acc} and $z_{\Lambda\,\mathrm{dom}}$ in the formula for the time (1.110) gives us the result that the universe started accelerating approximately 6 Gyr ago. Comparing both values to the age of the sun, namely 4.6 Gyr [66], and the age of the universe, 13.8 Gyr [50], we indeed see that dark energy became important only in approximately the last third of the evolution of the universe.

Dark energy was first directly observed in the redshifts of so-called Type Ia supernovae. They are believed to be standard candles with an intrinsic brightness range, such that their distance from us can be inferred. Therefore, any deceleration or acceleration in the expansion history of the universe can be detected. And it was found that the universe not only expands, but that the expansion is accelerating [67, 68]. One important method to probe the geometry of the universe is via the *number density of galaxies* in galaxy clusters. This depends on the comoving volume in which the cluster is observed, i.e. the geometry of the universe at that redshift, and also on the growth rate of the structures, i.e. the history of the density perturbations. Large-scale galaxy surveys can detect this and be compared with predictions from different cosmological models [65, 69]. Another method to observe the effects of dark energy is looking at the *baryon acoustic oscillations (BAOs)*. Those describe sound waves in the early universe, when photons and baryons were still coupled to each other. As the universe had cooled down enough, such that atoms could form, the photons could travel freely and the distribution of the baryons at that time was frozen. This clustering of matter at certain scales can be observed in the cosmic microwave background and also in galaxy clustering surveys, providing us with a standard ruler measuring the geometry of the universe. This preferred distance beween galaxies shows up as a peak in the two-point correlation function of the observed galaxies. The angular distance to the standard ruler can be measured at different redshifts, which in turn gives us information about the expansion history and therefore the energy content of the universe [65, 70]. One more possibility to probe the expansion evolution is via *weak gravitational lensing*. For that, people look at the the statistics of the distortions in galaxy surveys, which are caused by light bending around dark matter distributions. Similar to the number count observation of clusters, the shape in which the background galaxies are distorted due to the dark matter structures depends on the geometry and the growth rate of said structures. From this one can also extract the impact of dark energy on structure formation [65, 71].

We can learn a few things from the Friedmann equations in the case of the cosmological constant: Inserting the equation of state parameter $w = -1$ (1.114) into the second Friedmann equation (1.86), also called *acceleration equation*, we see that the acceleration is indeed positive for a universe filled with a positive cosmological constant. Moreover, also inserting the equation of state into the EMT conservation (1.89), it becomes clear that it is indeed a cosmological *constant*, as its energy density does not change under time evolution [2]. By now, the equation of state parameter w of dark energy has been measured by the Planck satellite, with the current best fit to the cosmological models of

$$w = -1.03 \pm 0.03\,, \tag{1.112}$$

so the observations are consistent with a cosmological constant [50].

The cosmological constant term Λ was first added by hand to the Einstein field equations (1.61) by Einstein himself in 1917 [72]. This was his attempt to find a solution for a static universe, as the velocities of stars were observed to be much smaller than the speed of light, and the expansion of the universe was not yet discovered at that time. This particular form of the field equations does not destroy general covariance and still enforces the conservation of the matter energy-momentum tensor through the Bianchi identity for constant Λ. We can compare this added term

$$T^{\Lambda}_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.113}$$

with the energy-momentum tensor of a perfect fluid (1.41) to realise that the equation of state is, as expected [2, 4]

$$p_\Lambda = -\Lambda \equiv -\rho_\Lambda\,. \tag{1.114}$$

Approximately two months later, also in 1917, de Sitter [73] nevertheless found a solution of the Einstein field equations which was apparently static but did not contain matter. This was in contradiction to what Einstein had expected from setting up a static universe: that matter should set inertial frames in the universe, therefore being directly connected to the geometry of the universe [74]. And at the same time, Slipher [75] observed the cosmological redshift of galaxies (which he called "spiral nebulae"), all but a few moving away from the Milky Way. The explanation was found by Lemaître in 1927 [76, 77]. He concluded from the data that the universe is expanding and from General Relativity he found what was later called *Hubble's law*, stating that the recessional velocities increase with increasing distance. As his report was only published in a Belgian journal, not many physicists ever read it. Even earlier than that, Friedmann already derived from general relativity that the universe might be expanding [78]. Finally, Hubble [79] again found the law which was later named after him, two years after Lemaître. Therefore they showed that the universe is expanding and not static, as Einstein tried to set it up. Nevertheless, the International Astronomical Union voted to rename the law to *Hubble-Lemaître law* in order to honour Lemaître's contribution [80]. In the light of those new observational and theoretical discoveries, Einstein decided in 1931 [81] to abandon his cosmological constant, but as we have seen, it came back [74].

There is a problem with the cosmological constant as it can be observed throughout the cosmic history: Its energy density stays constant over time while the matter energy density gets diluted with the volume increase of the universe. Therefore, there is only a short time during the expansion of the universe where both of them are of the same order of magnitude. As the matter comprises about 32% of today's energy content while dark energy is about 68% [50], this is the case today. So the question is: Why are the two of comparable magnitude exactly during our lifetime? Or put differently: Why did dark energy only fairly recently start to dominate? This is called the *coincidence problem* [82].

When we talk about accelerated expansion, the early universe period of *inflation* comes to mind [5]. Therefore, we will sum up what properties of inflation might be useful to get an idea about the recent dark energy. A stage of accelerated expansion is thought to greatly enlarge a homogeneous, causally connected patch to large sizes, explaining the homogeneity and isotropy of the universe today. Also, inflation flattens the early universe to large precision, such that small deviations from flatness at very early times do not blow up in an un-accelerated Friedmann stage [5]. A crucial property of inflation is the exit from the almost expontential expansion stage. An exact de Sitter stage, i.e. something akin to a cosmological constant, as a consequence seems unreasonable, solely because the universe would have not have dropped out of the exponential expansion stage at any time. Inflation would have continued and hindered all gravitational clustering and structure formation [83]. But a dynamical model is a viable solution to a graceful exit from inflation. Often a scalar field action like (1.31) is taken, with the scalar field called an *inflaton*. The corresponding equation of motion in a Friedmann universe (1.83) can be calculated as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0\,. \tag{1.115}$$

This is of course nothing else than the Klein-Gordon equation (1.34) calculated in a Friedmann metric (1.83). Note that the scalar field should follow the geometry of the universe and be homogeneous, i.e. bear no spatial dependence. The equation of state of the scalar field, using the perfect fluid picture of energy density (1.50) and pressure (1.51) can be written as [84]

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}\,. \tag{1.116}$$

The spatial derivatives of the scalar field are zero, $D_\mu \phi = 0$, as the universe is assumed to be homogeneous on very large scales. One can easily see that for $\dot{\phi}^2 \ll V(\phi)$, the equation of state approaches that of a cosmological constant and produces accelerated expansion as desired. For a general potential we see from the first Friedmann equation (1.85) that

$$H \propto \sqrt{\varepsilon} \sim \sqrt{V}\,, \tag{1.117}$$

so a large potential also might lead to a large Hubble friction term $3H\dot{\phi}$. Together with asking that the acceleration of the scalar field should be small compared to the Hubble friction term $3H\dot{\phi}$ this produces the *slow-roll conditions*

$$\left|\dot{\phi}^2\right| \ll |V|\,, \qquad |\ddot{\phi}| \ll 3H\dot{\phi} \sim \left|\frac{\partial V}{\partial \phi}\right|\,. \tag{1.118}$$

With the help of (1.117) the slow-roll conditions can be rewritten as

$$\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1\,, \qquad \left|\frac{V_{,\phi\phi}}{V}\right| \ll 1\,, \tag{1.119}$$

where $V_{,\phi} \equiv \frac{\partial V}{\partial \phi}$ and $V_{,\phi\phi} \equiv \frac{\partial^2 V}{\partial \phi^2}$, respectively. For a scalar field potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \tag{1.120}$$

the relation between H and the potential suggests, using (1.117), that $H \sim m\phi$. For any power-law potential, the slow-roll conditions (1.119) are satisfied for $|\phi| \gg 1$ [5]. Combining these two relations yields

$$m \ll H \,. \tag{1.121}$$

The mass of the inflaton should therefore be much smaller than the Hubble constant, which is reminiscent of the cosmological constant problem for late stage accelerated expansion: Why is the cosmological constant so small? We will talk about this more in 1.2.3. The length of inflation to solve the horizon problem of causally connected regions is mostly quoted with approximately 60 Hubble times (with the Hubble constant appropriate for the time of inflation). In the context of inflation, one talks about 60 *e-folds*, i.e. the universe expands by a factor of e^{60} [48]. For a relatively long time during inflation, the equation of state is extremely close to -1, before dropping out of that de Sitter stage and exponential expansion [5]. The details of how exactly inflation proceeds depends on the shape of the potential, of course.

But in the end, how the problem of accelerated expansion is approached in the context of inflation may be helpful in the consideration for dark energy problems as well. This is especially true for the case of models with a dark energy equation of state of $w \neq -1$. Some of them are referred to as *quintessence* [85]. It is modelled by a scalar field with our already known action (1.31) and it is counted as new dynamical content of the universe, in contrast to a cosmological constant [84]. In the case of quintessence, the accelerated expansion in the late universe comes about by virtue of the potential of the scalar field. The only requirement here is that the scale factor will evolve such that it provides us with accelerated expansion. One possibility for an exact solution is similar to power-law inflation with a scale factor

$$a(t) \propto t^r \tag{1.122}$$

with the exponent $r > 1$ in order to get accelerated expansion. Then it can be shown that the potential of the scalar field has to have the form

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{r}}\frac{\phi}{m_{\mathrm{Pl}}}\right) \tag{1.123}$$

with the constant V_0 and where the Planck mass m_{Pl} has been restored [48, 84]. Another possibility for the potential would be

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}\,, \tag{1.124}$$

where $\alpha \geq 1$ and M denotes a mass scale which is determined by the observed dark energy density today. The scalar field can be shown to exhibit a tracker behaviour [86, 87]. This means that the quintessence field is drawn towards solutions where many initial conditions lead to similar final solutions. Moreover, the tracking quintessence does not adjust to the background equation of state, i.e. to that of radiation or matter, but stays away from that, tending more to cosmological constant behaviour in the later cosmic history. This would be a desired solution to the coincidence problem [86].

This accelerated expansion of the universe can only be accounted for by an equation of state parameter of $w < -\frac{1}{3}$, as it follows from the second Friedmann equation (1.86). The cosmological constant lies at exactly $w = -1$, as already discussed. A very interesting possibility of the DE equation of state lies in the region of $w < -1$, or in the so-called *phantom energy*. As we have already seen in (1.112), observational data does not exclude this possibility, even seems to slightly favour it. A recent analysis and comparison between various data sources, such as the CMB, supernovae Ia, baryon acoustic oscillations or also the evolution of massive old galaxies show that the dark energy equation of state is indeed

$$w_{\mathrm{DE}} = -1.0131^{+0.038}_{-0.043}\,. \tag{1.125}$$

for the combination of those four data sets. The slight favour of phantom energy seems to mainly come from the CMB, although the authors claim that there is no need to deviate from the standard assumption of a cosmological constant [88]. Furthermore, to fit today's observations, phantom energy should come to dominate later than a possible CC or quintessence. This has consequences especially in the late universe, which should be even observable in the future and aid us to distinguish which form of dark energy permeates our universe. For example, the relation between magnitude and distance changes such that far away supernovae would appear dimmer due to phantom energy [89]. The simplest solution to get phantom energy lies again in a scalar field, but this time the kinetic term should have the "wrong" sign, therefore representing a ghost field [84]. Phantom energy has an interesting property concerning the future of the universe: Its energy density grows with time as opposed to other forms of matter. As it can be shown, phantom energy reaches infinite energy density within finite time, therefore overtaking all other forms of matter and as the accelerated expansion continues, it rips apart galaxies, stars, planets and finally molecules and atoms [84,90]. But note that the idea of the scale factor reaching an infinite value within finite time was already put forward in 1999, see [91]. Phantom energy is also thought to be able to alleviate the Hubble tension problem, that is the statistically significant deviation of the Hubble constant observed from early universe sources such as the CMB, and late universe measurements such as supernovae distances calibrated by variable stars, e.g. Cepheids. Exemplary, we quote [92]

$$H_0^{\mathrm{early}} = 67.27 \pm 0.60\,\mathrm{km/(s\,Mpc)} \tag{1.126}$$
$$H_0^{\mathrm{late}} = 73.2 \pm 1.3\,\mathrm{km/(s\,Mpc)}\,. \tag{1.127}$$

The early universe value is taken directly from the Planck 2018 release [50], whereas the late one comes from a SN Ia sample, with their distances calibrated with Cepheids [93]. This difference seems to be persistent and there is much discussion how to resolve this problem. Most likely this phenomenon is caused by systematics in one of the measurements. If that is indeed the case and there is a physical significance behind this difference, that gap might be bridged by phantom energy. More correctly, it is *phantom crossing* we are searching for, i.e. dark energy crossing the dividing line from non-phantom energy ($w_{\mathrm{DE}} > -1$) to phantom energy ($w_{\mathrm{DE}} < -1$) [94]. The authors assume a general extremum of the dark energy density before the present time and check this model against the available CMB, BAO, lensing and supernova Ia data. As a result, they find that

$$H_0^{\text{phantom}} = 70.25 \pm 0.78\,\mathrm{km/(s\,Mpc)} \tag{1.128}$$

for all data sets. This is still not in full agreement with the Hubble value found from supernova data [95], but the tension is at least alleviated. Of course, phantom energy is just one possibility of many under investigation which could solve the Hubble tension [92, 96].

Another theoretical model for dark energy is *k-essence* [97, 98], a scalar-tensor model of gravity. In this scenario, it is the non-canonical kinetic term of the scalar field which is responsible for the acceleration. It was actually already described in [99] as kinetically driven quintessence. The action for the k-essence model looks like

$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \sqrt{-g}\, p(\phi, X) \tag{1.129}$$

with the usual kinetic term X (1.55). The Lagrange density $p(\phi, X)$ is aptly named, as it is at the same time a pressure density, which is often taken to be factorisable, i.e.

$$p(\phi, X) = K(\phi)\tilde{p}(X)\,. \tag{1.130}$$

Then, the energy density can be calculated to be

$$\varepsilon(\phi, X) = K(\phi)\left(2X\tilde{p}_{,X} - \tilde{p}\right)\,, \tag{1.131}$$

where $_{,X} \equiv \partial_X$ denotes the partial derivative with respect to the kinetic term X, such that the equation of state for the k-essence is

$$w_k = \frac{\tilde{p}}{2X\tilde{p}_{,X} - \tilde{p}}\,. \tag{1.132}$$

This model also exhibits tracker and attractor behaviour. The solutions are called trackers when the k-essence mimics either radiation or matter behaviour. On the other hand there are attractors, when the scalar field runs towards a solution whose equation of state is different from either matter or radiation. Those are important in the extreme cases when the k-essence density is much larger or smaller than the matter or radiation density. For the case that the k-essence density ε_k is much smaller than the matter density ε_{m}, those attractors are called de Sitter attractors and they mimic a cosmological constant where $w_k \to -1$. They are a generic feature of the theory. On the other end of the spectrum, there exist k-attractors for $\varepsilon_k \gg \varepsilon_{\mathrm{m}}$. The whole appeal of this theory is that the scalar field evolves with time, with different possibilities for the scalar field to track and mimic the energy contents of the universe. With this property it is possible for certain choices of parameters to explain the development of the energy density as we observe it. At first, during radiation domination, it is sub-dominant and tracks the radiation energy density. At the onset of matter domination, ε_k drops by several orders of magnitude, whereas w_k quickly approaches that of a cosmological constant, starting the accelerated expansion we see today. And interestingly, ε_k starts to dominate a short time ago, as it is observed. Therefore k-essence can be tweaked to model the behaviour of the observed energy densities just by attractor dynamics, without needing many initial conditions. A feature distinguishing k-essence from other models is the emergence of a speed of sound c_s [100], defined and calculated as

$$c_\mathrm{s}^2 \equiv \frac{p_{,X}}{\varepsilon_{,X}} = \frac{\tilde{p}_{,X}}{\tilde{p}_{,X} + 2X\tilde{p}_{,XX}} \,. \tag{1.133}$$

Therefore, unlike in "normal" quintessence, the speed of sound can be quite different from one. The speed of sound can even be larger than one, though without violating causality [101]. As it can be very small, crucially k-essence can cluster and fall into the gravitational wells formed by dark matter. This might produce results which are actually observable in the CMB, therefore enabling us to distinguish between quintessence and k-essence [102]. But note that k-essence does not allow phantom crossing [103].

One more possible scalar-tensor model is called *kinetic gravity braiding* [14, 104]. This builds up on the k-essence model in the sense that it deals with non-canonical kinetic terms. Its Lagrangian is of the form

$$\mathcal{L} = K(\phi, X) + G(\phi, X)\Box\phi \,, \tag{1.134}$$

where $K(\phi, X)$ and $G(\phi, X)$ are general functions of the scalar field ϕ and its kinetic term X, compare (1.55). $\Box$ denotes the d'Alembertian as usual (1.35). For $G(\phi, X) = 0$ and any function $G(\phi)$ which is independent from X we get back the k-essence from above. But kinetic gravity braiding contains a non-trivial coupling between the scalar kinetic term and the tensor kinetic term, schematically written as $G\partial g\partial\phi$. Non-trivial meaning that it cannot be undone by any field redefinitions. But still, it can be shown that kinetic gravity braiding only leads to second order equations of motion, therefore making it a viable theory containing no ghost fields. The important fact in that context is that the scalar field monitors the external energy density and moves towards attractor solutions that act as a cosmological constant for appropriately chosen functions K and G. One has to notice that in this model the scalar does not behave as a perfect fluid, but an imperfect one [14]. The scalar from kinetic gravity braiding also tends towards phantom energy as an attractor solution, therefore it is also an interesting model to solve the Hubble tension. Moreover, in contrast to k-essence, kinetic gravity braiding can also cross the phantom divide without ghosts and gradient instabilities [104]. Similarly to k-essence, kinetic gravity braiding also does not change the speed of gravitational waves [105]. This fact is very important, as the fractional difference between gravitational wave speed and light speed has been observed to be

$$-3 \times 10^{-15} \leq \frac{c_\mathrm{GW} - c_\mathrm{EM}}{c_\mathrm{EM}} \leq 7 \times 10^{-16} \tag{1.135}$$

from the multimessenger observation of the gravitational wave event GW170817 and the short gamma ray burst GRB 170817A [106]. So observations greatly constrain the deviation of the gravitational wave speed from the speed of light and many models like the most general Horndeski models [107] and beyond Horndeski models, e.g. [108, 109] have been ruled out by this [105].

1.2.3 The Cosmological Constant problem

After we have described the cosmological constant and its properties and the observations made to suggest its existence, we will discuss the so-called *cosmological constant problem.* Because if we ask ourselves where dark energy actually comes from, quantum fluctuations of the vacuum come to mind. If we view each mode present as a quantum mechanical harmonic oscillator, their ground states, i.e. vacuum energies, are certainly not zero. And as there are infinitely many of them, the total vacuum energy should be infinite. This is *not* a problem in the absence of gravity, as only differences in energies play a role quantum field theory in flat spacetime. Radiative corrections also appear, but in the framework of QFT they are correctly dealt with an appropriate renormalisation method. The actual cosmological problem is the following: As soon as we turn on gravity, the absolute value of energy matters, as any kind of energy gravitates. We can try to renormalise the appearing infinities as usual, but we will realise that although the renormalisation works, it predicts a much higher value for the cosmological constant than it is observed [110].

As we have already discussed, the *bare cosmological constant* Λ_B enters the action merely as a free parameter of the theory, which we will repeat here for convenience

$$S[g_{\mu\nu}, \Lambda_\mathrm{B}, \Phi_\mathrm{m}] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R + \Lambda_\mathrm{B}\right) + S_\mathrm{m}[g_{\mu\nu}, \Phi_\mathrm{m}]\,, \tag{1.136}$$

resulting in the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} + \Lambda_\mathrm{B}g_{\mu\nu}\,. \tag{1.137}$$

Then we have to take into account the vacuum state of the quantum fields which results in

$$\langle 0|T_{\mu\nu}|0\rangle = \rho_\mathrm{vac}g_{\mu\nu} \tag{1.138}$$

with the energy density of the vacuum ρ_vac. Due to energy-momentum conservation (1.88) and metric compatibility (1.12) we know that this must be a constant. Under the assumption that this vacuum energy is affected by gravity (as all forms of energy are, according to the equivalence principle), the Einstein field equations now look like

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T^\mathrm{m}_{\mu\nu} + \Lambda_\mathrm{eff}g_{\mu\nu}\,. \tag{1.139}$$

We denote the EMT resulting from ordinary matter from now on as $T^\mathrm{m}_{\mu\nu}$ and have used the expression for an effective cosmological constant

$$\Lambda_\mathrm{eff} = \Lambda_\mathrm{B} + \rho_\mathrm{vac}\,. \tag{1.140}$$

This sum of the bare cosmological constant term and the vacuum energy density is actually the cosmological constant term which is observed in nature. The next problem is that this effective cosmological constant receives corrections from various sources we will at least mention in passing here.

The first possibility is a classical contribution, if we assume the cosmological constant coming from a scalar field with a general potential, as already mentioned in (1.31). At first, one part of the vacuum energy literally comes from the potential at its minimum, such that $\rho_{\text{vac}} = V(\phi_{\text{min}})$. This cannot always be set to zero, as there may be a phase transition in the early universe, causing the potential of the fields to be non-zero. And in the presence of gravity this non-zero potential will contribute to the total energy density and act like a cosmological constant. One such phase transition in the early universe is the electroweak phase transition, another one the QCD phase transition. In the case of the electroweak phase transition the potential our interest lies in is the Higgs potential. The Higgs boson is a complex scalar and therefore can be denoted like we already did in (1.37). Writing the Higgs field as χ, its potential is

$$V(\chi, \chi^\dagger) = \frac{m^2}{2}\chi^\dagger\chi + \frac{\lambda}{4}\left(\chi^\dagger\chi\right)^2 , \tag{1.141}$$

with m as the mass of the Higgs and λ a coupling constant of its self-interaction. This potential has two minima after the spontaneous symmetry breaking, in this case also called *Higgs mechanism*, giving the gauge bosons mass. From $\partial V/\partial(\chi^\dagger\chi) = 0$ one can derive the non-zero vacuum expectation value (VEV) v after the phase transition

$$\langle\chi\rangle = v = \sqrt{-\frac{m^2}{\lambda}} . \tag{1.142}$$

In case we set the minimum of the vacuum to zero before the phase transition, it will be

$$V(\langle\chi\rangle = v) = -\frac{m^4}{4\lambda} \tag{1.143}$$

after. The second possibility is that we chose the value of the vacuum energy before the phase transition to be $\frac{m^4}{4\lambda}$, such that it would vanish afterwards. In both cases, the main point to take away is, that we cannot set the vacuum energy to zero before and after the phase transition, so it will be affected by gravity. If one assumes the vacuum energy to be non-vanishing after the electroweak phase transition, one can calculate it to be

$$\rho_{\text{vac}}^{\text{EW}} \approx -10^{55}\rho_{\text{crit}} \approx -1.2 \times 10^{8}\,\text{GeV}^4 , \tag{1.144}$$

using the critical density ρ_{crit} (1.91) that would yield a flat universe [110]. This prediction contradicts various observations, as the measurement of the vacuum energy density of the Planck satellite [50]. There one can see that the value is

$$\Lambda = (2.846 \pm 0.076) \times 10^{-122}\, m_{\text{pl}}^2 \tag{1.145}$$

with Planck mass $m_{\rm pl}$. Using that

$$\rho_{\rm vac}^{\rm cosm} = \frac{\Lambda}{8\pi G_{\rm N}} = \Lambda \bar{m}_{\rm pl}^2 \tag{1.146}$$

with the reduced Planck mass (1.3) and (1.5) and the value for the Planck mass (rather Planck energy or temperature) (1.4) we can finally calculate the vacuum energy density

$$\rho_{\rm vac}^{\rm cosm} = 2.517 \times 10^{-47}\,{\rm GeV}^4\,. \tag{1.147}$$

This is of course several dozen orders of magnitude smaller than the classical contribution from the electroweak phase transitition (1.144). But even if we manage to fine-tune the classical contribution to effective cosmological constant to the observed value, the problem will come back to us on the level of quantum fluctuations.

Therefore, we will now discuss the *quantum-mechanical contributions to the vacuum energy*, at the level of the zero-point fluctuations of the quantum fields which are present in the universe. We will quickly discuss the main points of this to address a few misconceptions, at first only for a scalar field. Nevertheless, we keep in mind that one would have to consider fermionic and vector fields as well as gravitons. We will skim the topic of those other fields quickly after the scalar fields. Furthermore, for the high energy scales we consider here, we can treat the spacetime as flat, as the high energy modes in the ultraviolet regime are only sensitive to the very local properties of the spacetime. Therefore, curvature does not change those considerations. The whole discussion follows [110] if not stated otherwise. For a scalar field ϕ of mass m the equation of motion is the Klein-Gordon equation (1.34)

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = \ddot{\phi} - \delta^{ij}\partial_i\partial_j\phi + m^2\phi = 0\,. \tag{1.148}$$

The ansatz for a free scalar field will be

$$\hat{\phi}(t,\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{{\rm d}^3\boldsymbol{k}}{\sqrt{2\omega(k)}}\left(\hat{a}_{\boldsymbol{k}}e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^\dagger e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}\right) \tag{1.149}$$

with the 4-momentum vector denoted as

$$k^\mu = (k^0, \boldsymbol{k}) \tag{1.150}$$

and the (positive) frequency ω as defined in the dispersion relation

$$\omega(k) = \sqrt{k^2 + m^2} \tag{1.151}$$

Moreover, the creation and annihilation operators satisfy the usual commutation relations

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k'}}^\dagger\right] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k'})\,. \tag{1.152}$$

Using the formula for the EMT calculated from the scalar field (1.39) and the energy density (1.42) and pressure (1.43) of a perfect fluid, it can be derived that

$$\langle \rho \rangle = \frac{1}{2} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, \omega(k) \tag{1.153}$$

$$\langle p \rangle = \frac{1}{6} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\boldsymbol{k}^2}{\omega(k)}\,. \tag{1.154}$$

We will then transform the measure to be relativistically invariant, using [111]

$$\frac{\mathrm{d}^4 k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k_0) = \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega(k)} \tag{1.155}$$

with the delta functional $\delta(k^2 - m^2)$ setting $k^2 = m^2$ and the Heaviside step function $\theta(k_0)$ selecting positive k_0. k is understood as being the length of the four-vector k^μ, i.e. $k = \sqrt{k^\mu k_\mu}$. Therefore we have

$$\langle \rho \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^3}\, k_0^2 \delta(k^2 - m^2)\theta(k_0) \tag{1.156}$$

$$\langle p \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^3}\, \frac{k_1^2 + k_2^2 + k_3^2}{3} \delta(k^2 - m^2)\theta(k_0)\,. \tag{1.157}$$

explicitly writing out $\boldsymbol{k}^2 = k_1^2 + k_2^2 + k_3^2$. As we can see from the formulae above, if we perform a *Wick rotation* [112] on the spatial coordinates and therefore go to Euclidean coordinates, the $O(4)$-symmetry of the problem becomes apparent. We observe that

$$\langle p \rangle = -\langle \rho \rangle \tag{1.158}$$

for the Euclidean quantities, directly arriving at the equation of state for a cosmological constant.

The usual course of action would have been to realise that the energy density (1.153) goes with $\propto k^4$ for high momenta, impose a sharp cutoff scale μ, often taken at the Planck scale and estimate the vacuum energy like that. It can be shown [110] that this causes a problem: the equation of state parameter after this calculation would be $\langle \rho \rangle / \langle p \rangle = 1/3$ and not -1, as expected from the cosmological constant. The upshot is: The regularisation scheme to sharply cut off the infinities does not reflect the underlying Lorentz symmetry of the theory, as it cuts off the spatial part of momentum alone. This is what would lead to a result for the vacuum energy density of something like

$$\langle \rho \rangle \propto \mu^4 \sim m_{\mathrm{Pl}}^4 \approx 10^{76}\,(\mathrm{GeV})^4\,. \tag{1.159}$$

Compare this to the observed cosmological constant value (1.147) and we arrive at the often quoted difference of 123 orders of magnitude between observation and expectation [110, 113]. As mentioned, this is an artefact of the wrong regularisation scheme. Solutions for this have been proposed, but some of them are to be treated carefully. The often used *dimensional regularisation* [110,114], where one goes from four to d spacetime dimensions to regularise the integrals and find the correct relation between energy density and pressure, does lose terms of the form $\int \mathrm{d}^d k (k^2)^\alpha$ for all powers of α [113]. Therefore, it is to be avoided. More promising is starting from the vacuum expectation value of the trace of the energy-momentum tensor and arriving at

$$\langle\rho\rangle = \frac{1}{4} m^2 \langle 0|\phi^2|0\rangle = \frac{m^2}{4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}\,, \tag{1.160}$$

Wick rotate and use a four-momentum cutoff scale μ [113]. This results in

$$\langle\rho\rangle = \frac{1}{64\pi^2} \left[\mu^2 m^2 - m^4 \ln\left(\frac{\mu^2 + m^2}{m^2}\right)\right]\,. \tag{1.161}$$

As we can see, the vacuum energy density vanishes for massless fields, i.e. for the electromagnetic field as well, as the term quartic in the cutoff scale vanishes identically [113].

One thing to notice is that with this result the massless photons do not add any correction to the cosmological constant. Those "quantum" contributions to the cosmological constant come from the so-called bubble diagrams of quantum field theory, where they are no external legs, in contrast to loop diagrams. Bubble diagrams do not contribute to the energy density in flat spacetimes, but as soon as gravity is taken into account, they will matter. Equivalent to the scalar field, similar considerations lead to similar results in the case of spinor and vector fields. The energy density will always look like

$$\langle\rho\rangle = \frac{1}{(2\pi)^3} \frac{s}{2} \int \mathrm{d}^3\boldsymbol{k}\, \omega(k) \tag{1.162}$$

with different s accounting for the number of the polarisation states and spin, namely $s = 1$ for the scalar field, $s = -4$ for the Dirac spinor field, $s = 3$ for the massive vector field and $s = 2$ for the massless one [110]. Now we can make a simple order of magnitude estimation of the vacuum energy density, based on the first term of (1.161), as the second term is subdominant. We also use the cutoff scale $\mu = m_{\mathrm{Pl}}$ and the mass of the top quark as the heaviest standard model particle with $m_t \approx 176.69\,\mathrm{GeV}$ [115]. We will take the factor of $|s|= 4$ for the spinor field and arrive at

$$|\langle\rho\rangle| \approx 10^{40}\,\mathrm{GeV}^4\,. \tag{1.163}$$

This is still much larger than observed value of $\rho_{\mathrm{vac}}^{\mathrm{cosm}} \approx 10^{-47}\,\mathrm{GeV}^4$, but nevertheless much smaller than the previous naive $\rho_{\mathrm{vac}} \approx 10^{76}\,\mathrm{GeV}^4$. So while the cosmological constant problem is far from being solved, the proper regularization at least alleviates it.

Another observation can be made when looking at the vacuum energy density for fermions in (1.162). It is negative, while it is positive for bosons. If one were now able to find an almost exact copy of the standard model particles, with the only difference that the fermions would have bosonic partners and the bosons fermionic partners, the cosmological constant problem would be solved. This symmetry which relates bosons with fermions is called *supersymmetry*. In its original and simplest model in $3+1$ dimensions standard model particles and their superpartners have equal masses [116, 117]. Unfortunately we know that supersymmetry must be broken, i.e. the masses of the standard model particles and their superpartners cannot be the same, as supersymmetric particles have not been observed in any accelerators yet [118]. Therefore it is not the solution to the cosmological constant problem we have to look for. Finally, the vacuum energy density can be summarised as

$$\rho_{\mathrm{vac}} = \sum_i n_i \frac{m_i^4}{64\pi^2} \ln\left(\frac{m_i^2}{\mu^2}\right) + \rho_{\mathrm{B}} + \rho_{\mathrm{vac}}^{\mathrm{EW}} + \rho_{\mathrm{vac}}^{\mathrm{QCD}} + \dots \,. \tag{1.164}$$

The contributions from the bare cosmological constant ρ_{B} and from the phase transitions are also accounted for, with more possible phase transitions denoted in the dots. The i accounts for the different particles present in the theory, the n_i denotes the polarisation states combined with the different signs for bosons and fermions, as stated above. Then one inserts the particle masses of the standard model in eV, e.g. the Higgs boson with $n_{\mathrm{H}} = 1$ as a scalar, $m_{\mathrm{H}} \simeq 125\,\mathrm{GeV}$. Furthermore, we have the six quarks, each with $n_{\mathrm{quark}} = -4$, the three fermions (without the neutrinos, as their masses are so small), and finally the three massive gauge bosons Z and W$^{\pm}$, each with $n_{\mathrm{gauge}} = 3$. The only question is what to take as the regularisation scale μ, as it should not fundamentally influence the outcome of the calculation. Also the author of [110] is not quite sure what to take, but in the end argues that the result stays stable over many orders of magnitude for μ. So even if we now take the mass of the top quark as a scale, practically only the Higgs boson and the gauge bosons contribute, and the approximate result will be

$$\rho_{\mathrm{vac}} \simeq -10^6\,\mathrm{GeV}^4 + \rho_{\mathrm{B}} + \rho_{\mathrm{vac}}^{\mathrm{EW}} + \rho_{\mathrm{vac}}^{\mathrm{QCD}} + \dots \,. \tag{1.165}$$

[110] takes a different $\mu \sim \sqrt{E_\gamma E_{\mathrm{grav}}} \simeq 3 \times 10^{-25}\,\mathrm{GeV}$, resulting from the scale of the Hubble constant $E_{\mathrm{grav}} \simeq H_0 \simeq 3.7 \times 10^{-41}\,\mathrm{GeV}$ and the photon energy E_γ at a wavelength of $\lambda \simeq 500\,\mathrm{nm}$. A similar calculation will result in

$$\rho_{\mathrm{vac}} \simeq -10^8\,\mathrm{GeV}^4 + \rho_{\mathrm{B}} + \rho_{\mathrm{vac}}^{\mathrm{EW}} + \rho_{\mathrm{vac}}^{\mathrm{QCD}} + \dots \,. \tag{1.166}$$

The important takeaway message is that for both very dissimilar choices of the regularisation scale, the calculated vacuum energy density is far away from the often cited [35] $\rho_{\mathrm{vac}}^{\mathrm{QFT}} \approx 10^{74}\,\mathrm{GeV}^4$, but still much larger than the observed value of $\rho_{\mathrm{vac}}^{\mathrm{cosm}} \approx 10^{-47}\,\mathrm{GeV}^4$ (1.147). Obviously, the observations are still in huge contradiction with the now more carefully theorised value [110]. A very similar argument is made in [119, 120]. The author works with the Pauli sum rules, on exactly the degeneracy factors and polarisations we called n_i here of the bosons and fermions present in the theory, reaching the same vacuum energy density for the zero-point energies as in (1.164). Then the author sets the energy scale for the standard model by asking that

$$\sum_i n_i m_i^4 \ln\left(\frac{m_i^2}{\mu^2}\right) = 0\,. \tag{1.167}$$

With the help of $\hat{m} = m/m_{\mathrm{H}}$ the energy scale for the standard model will be

$$\mu^2 = m_{\mathrm{H}}^2 \exp\left(\frac{\sum_i s_i \hat{m}_i^4 \ln \hat{m}_i^2}{\sum_i s_i \hat{m}_i^4}\right) \simeq m_{\mathrm{H}}^2 (1.442)^2\,, \tag{1.168}$$

resulting in

$$\mu \approx 180.25\,\mathrm{GeV}\,, \tag{1.169}$$

not dissimilar to the value which was taken to obtain (1.165).

The discrepancy between cosmological, and therefore GR, measurements in eq. (1.147) and QFT estimates in eq. (1.166), which are both well tested and accepted theories, is called the *cosmological constant problem.* And even if one assumes that there is an extra effect that would cancel the theorised high value of the vacuum energy, it is difficult to imagine that it would cancel exactly to the observed value, a small but non-vanishing value. This results in a fine-tuning problem, attracting a lot of interest among physicists [74,121].

1.2.4 Trace-free Einstein gravity and the cosmological constant problem

One proposal by Einstein himself was to take the trace-free part of his field equations, although in a slightly different version than we are going to use here [122]. More modern but still pioneering treatments include [123–125]. We will use [126]

$$G_{\mu\nu} - \frac{1}{4} G g_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu}\,. \tag{1.170}$$

The Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ (1.30) still holds, but now the energy-momentum conservation equation (1.56) is a separate assumption and not a consequence of the aforementioned Bianchi identity anymore. An important point to stress is that the trace-free Einstein equations do not contain any information about the cosmological constant. Taking the covariant derivative of those new trace-free equations leaves us with a new integrability condition

$$\nabla_\mu G = \nabla_\mu T\,. \tag{1.171}$$

In other words, the cosmological constant re-enters as a pure integration constant

$$4\Lambda \equiv G - T \tag{1.172}$$

giving us back our well-known Einstein equations (1.61) with the cosmological constant included [121,126].

On the one hand, this theory is no different to normal GR with an unspecified cosmological constant. It also contains the vacuum solutions (e.g. Schwarzschild and Kerr), but with $\Lambda = 0$. Moreover, it does also not affect cosmological solutions such as the Friedmann metric and also the junction conditions used for stellar models remain valid [121]. And lastly, also its effect on inflation was checked and it was found that its outcome is unchanged [127].

On the other hand, equations (1.170) are unchanged under shifts of the vacuum energy density ρ_{vac} of the form

$$T_{\mu\nu} \to T_{\mu\nu} + \rho_{\text{vac}} g_{\mu\nu} \,. \tag{1.173}$$

So in the end, all values of the cosmological constant are mathematically valid at that point, however fixing an effective value for it leaves us with the usual GR Einstein equations (1.61), as the trace-free Einstein equations (1.170) are insensitive to those changes. So it seems we just shifted the cosmological constant problem one step further away without actually solving it. Although, this symmetry in the equations permits us to simply split the energy-momentum tensor into the vacuum contributions and everything else. This might look like

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + T^{\text{vac}}_{\mu\nu} \tag{1.174}$$

with the constant vacuum energy looking like

$$T^{\text{vac}}_{\mu\nu} = \frac{1}{4} T^{\text{vac}} g_{\mu\nu} \,, \tag{1.175}$$

where T^{vac} is a constant throughout spacetime. Then we transfer all of the quantum corrections into the part $T^{\text{vac}}_{\mu\nu}$ of energy-momentum tensor, while $\tilde{T}_{\mu\nu}$ receives none. Inserting this into the traceless Einstein equations (1.170), we see that those quantum corrections exactly drop out. At the very least, the cosmological constant is stable against the quantum vacuum corrections. Summing up this part, the cosmological constant problem is not quite solved, but we have gained more freedom in how to interpret the results by making the cosmological constant not fixed from the beginning [126].

Another reformulation of trace-free gravity is so-called *unimodular gravity*, where the name-giving *unimodular constraint* is [123, 124]

$$\sqrt{-g} = 1 \,. \tag{1.176}$$

This has to be implemented directly into the Einstein-Hilbert action in order to modify the dynamics of the theory. One possibility is to do so with a Lagrange multiplier, such that

$$S[g_{\mu\nu}, \lambda, \Phi_{\text{m}}] = \int \mathrm{d}^4 x \left(-\frac{1}{2}\sqrt{-g}R + \lambda \left(\sqrt{-g} - 1\right) \right) + S_{\text{m}}[g_{\mu\nu}, \Phi_{\text{m}}] \,. \tag{1.177}$$

There are several problems with this formulation. One, this direct implementation of the unimodular constraint explicitly breaks diffeomorphism invariance, which changes the symmetry group of the action to be merely invariant under transverse diffeomorphisms

$$\nabla_\mu \xi^\mu = 0 \tag{1.178}$$

generated by ξ^μ. Those transverse diffeomorphisms are characterised by preserving the metric under variations like

$$\delta_\xi \sqrt{-g} = \mathscr{L}_\xi = \frac{1}{2}\sqrt{-g}\nabla_\mu \xi^\mu = 0 \,, \tag{1.179}$$

with the Lie derivative $\mathscr{L}$ (1.22). Moreover, the resulting Einstein equations are

$$G_{\mu\nu} + \lambda g_{\mu\nu} = T_{\mu\nu} \,. \tag{1.180}$$

If we take the covariant derivative on both sides, this will result in the condition that

$$\nabla_\mu \lambda = 0 \,, \tag{1.181}$$

the reasoning being as follows: The Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ holds, of course, whereas diffeomorphism invariance is still assumed to be valid for the matter sector, with the result of EMT conservation $\nabla_\mu T^{\mu\nu} = 0$. So if we now provide initial conditions for the cosmological constant λ, as it is fit for a dynamical variable of the theory, we end up with the old cosmological constant problem, that we need to fine-tune its value. One could again shift the quantum vacuum density ρ_{vac} to decouple from the theory. But this results in a shift in the Lagrange multiplier and therefore in the value of the initial condition for the cosmological constant. The conclusion is, we are back at fine-tuning the cosmological constant at every loop order [126].

On the other hand, there is also the possibility of carrying out a Weyl transformation of the metric of the form

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu} |g|^{-1/4} \tag{1.182}$$

everywhere but the unimodular constraint term. The goal is to make the quantum vacuum contributions ρ_{vac} decouple automatically. This will give us the resulting action

$$S[g_{\mu\nu}, \lambda, \Phi_{\text{m}}] = \int \mathrm{d}^4 x \left(-\frac{1}{2} \hat{R} + \lambda \left(\sqrt{-g} - 1 \right) \right) + S_{\text{m}}[\hat{g}_{\mu\nu}, \Phi_{\text{m}}] \,. \tag{1.183}$$

Varying this action with respect to the original metric $g_{\mu\nu}$ will lead to the following equations of motion

$$\hat{G}_{\mu\nu} - \frac{1}{4} \hat{G} \hat{g}_{\mu\nu} + \lambda g_{\mu\nu} = \hat{T}_{\mu\nu} - \frac{1}{4} \hat{T} \hat{g}_{\mu\nu} \,. \tag{1.184}$$

But when reinserting the unimodular constraint (1.176), $g_{\mu\nu} = \hat{g}_{\mu\nu}$ so we can drop the hats. From taking the trace of (1.184) we see that $\lambda = 0$ without giving any initial conditions and λ does not play the role of the cosmological constant anymore. But as the matter and gravity Lagrangians depend on the metric $\hat{g}_{\mu\nu}$, the vacuum contributions $\rho_{\text{vac}}\sqrt{-\hat{g}}$ will decouple automatically, as $\sqrt{-\hat{g}} = 1$ by construction, compare to (1.182). Furthermore, this leads to the unimodular action being modified one step further to

$$S[g_{\mu\nu}, \lambda, \Phi_{\text{m}}] = -\frac{1}{2} \int \mathrm{d}^4 x \hat{R} + S_{\text{m}}[\hat{g}_{\mu\nu}, \Phi_{\text{m}}] \,. \tag{1.185}$$

The resulting equations of motion are the traceless Einstein equations. A remark about the difference in the ways implementing the unimodular constraint is in order. When we use the method with the Lagrange multiplier, initial conditions for the constant part of the Lagrange multiplier λ were implied, therefore not varying the zero mode by imposing the constraint under the integral, see

$$\int \mathrm{d}^4x\sqrt{-g} = \int \mathrm{d}^4x\,. \tag{1.186}$$

Therefore, the theory we are left with is still standard GR [126].

As an aside, in the Lagrange multiplier formulation of unimodular gravity one of the problems was that a tensor density, the metric determinant $\sqrt{-g}$, was set equal to a scalar quantity, namely unity, therefore breaking diffeomorphism invariance. This was finally solved by Henneaux and Teitelboim [128], where they introduced a vector density V^μ to write the constraint in the action in the generally covariant form

$$S_{\mathrm{HT}}[g_{\mu\nu}, \lambda, V^\mu, \Phi_{\mathrm{m}}] = \int \mathrm{d}^4x \left[-\frac{1}{2}\sqrt{-g}R - \lambda\left(\partial_\mu V^\mu - \sqrt{-g}\right)\right] + S_{\mathrm{m}}[g_{\mu\nu}, \Phi_{\mathrm{m}}]\,. \tag{1.187}$$

The discussion on this follows the one above about the original unimodular constraint, including the question on how to impose it, whether directly by a Lagrange multiplier or a field redefinition of the metric, but now with the crucial difference about the theory being diffeomorphism invariant [126, 128].

1.2.5 The cosmological concordance model

After the overview of the contents of the universe we will sum up the main results in what is commonly called the *cosmological concordance model*, the accepted standard model of the universe. From observing the cosmic microwave background [50]: Only about 4% of the matter content of the universe make up the usual luminous baryonic matter, about 28% dark matter and the remaining 68% dark energy. This is the dark universe. In summary, the currently accepted cosmological model for our universe is called ΛCDM, a universe containing a *cosmological constant* Λ and cold dark matter (CDM). Furthermore, it is most likely *flat*, i.e. the spatial curvature is zero, expressed by the curvature parameter k (1.84) being zero [8].

But if most of our universe is invisible and not constituted by any particle in the standard model, the question of its composition remains. For dark matter, often particles beyond the standard model are suggested as solutions to that problem, but so far nothing has been detected in experiments searching for new particles [129, 130].

For dark energy, the idea that it is a consequence of the vacuum energy throughout the universe is appealing. But as we have seen above 1.2.3, even with refining the underlying ideas, the estimated orders of magnitude for the energy densities obtained from QFT do not match at all. The cosmological constant problem remains challenging.

The spatial flatness of the universe, meanwhile, is well explained by an early period of *inflation*. Quasi-exponential growth of the scale factor which flattens the universe as well as initial quantum fluctuations which get amplified to provide seeds for later structure formation are the commonly accepted model [5, 36, 131].

1.3 Aim and overview of the thesis

After having detailed the problem of the dark universe, we will come to the core topic of this thesis: *Mimetic gravity* and its attempt at a solution. It is a modified theory of General Relativity, generally covariant and Weyl invariant. In the original theory of mimetic dark matter [132], GR was reformulated by splitting the metric with the help of a scalar field such that the theory acquires an extra longitudinal degree of freedom which can mimic dark matter, therefore providing the name. Note that backreaction of the mimetic dark matter on the surrounding dark matter was never taken into account, which can be viewed as mimetic dark matter not being counted as a full degree of freedom. Later, this original theory was modified and generalised to describe dark energy in the form of a cosmological constant. This can be achieved by using vector fields and gauge vector fields instead of a scalar field.

This thesis aims at a deeper understanding of mimetic theories by investigating its Noether currents and therefore its symmetries and conservation laws. It also provides a placement of mimetic theories within other frameworks, such as embedding it into a complex scalar field theory, or a theory with an axion, in order to address various open questions.

In chapter 2 we will at first focus on the mimetic construction, from the original idea towards more generalised extensions of how a mimetic theory can be built. We will reinforce that it indeed comprises a new theory, not just a reformulation of Einstein's General Relativity.

In chapter 3 we will interrupt our discussion of mimetic theories in order to review Noether's theorems and their importance for theoretical physics in the context of connecting symmetries of the theory with conserved quantities.

Chapter 4 is then dedicated to purely scalar extensions of mimetic gravity, be it via a potential or the introduction of higher derivative terms. Gauge invariant representations of the theory will be discussed, to reveal the different faces of mimetic gravity. Also, the results from 3 will be applied to show that the Weyl symmetry present in the theory does not introduce conserved quantities allowing us to split dynamics into different sectors. This chapter continues in the discussion of an embedding of mimetic gravity into a complex scalar field formulation, which serves to show that caustics can be avoided. Then we continue into describing higher derivative extensions of scalar mimetic gravity. Finally, we discuss a formula for the speed of sound which can be derived via cosmological perturbation theory, with the novel application to models with limiting curvature.

Another modification of mimetic gravity will be introduced in chapter 5. The mimetic construction from 2 can also be used to build a mimetic theory based on a vector field of the unusual conformal weight four. Also for this theory we will discuss different gauge invariant formulations, as well as the Weyl symmetry and the associated Noether currents.

Chapter 6 is another interlude, dedicated to a very short introduction to some concepts of group theory, mostly Lie groups. Moreover, concepts borrowed from quantum chromodynamics, especially the strong CP problem and its solution by the axion are also discussed.

In chapter 7 we continue with the construction of a mimetic theory out of gauge vector fields and how it presents us with a cosmological constant. This formulation contains an axionic coupling, as we will see, and will also be presented in gauge invariant variables. Furthermore, the mimetic construction is also applied for the gauge vector theory, as well as the calculations of the Noether currents associated with the Weyl symmetry. Moreover, we can generalise the abelian formulation of the theory to a non-abelian one and prove the existence of solutions for $SU(N)$, while giving explicit constructions for $SU(2)$ as well as $SU(3)$.

In the final chapter 8 we will summarise the thesis and provide a short outlook for future research and challenges.

This thesis was typeset with LaTeX, while Mathematica and Ti*k*Z were used to make the plots and Mathematica was also used to check some of the longer calculations, whereas the LaTeX bibliography style from the Journal of High Energy Physics (JHEP) was adopted.

Chapter 2

On Mimetic Theory and its Construction

The theory of mimetic matter was introduced [132] via a *Weyl transformation* of the metric, which was already summarised in 1.1.3. In this chapter we will start on discussing the significance and details of it. At first, we will give an introduction to the origins of mimetic dark matter, before continuing to the more general question: What are the conditions to be met to change the content of the theory of General Relativity and when will a Weyl transformation cause merely a reformulation of GR? Explicit calculations can be found in 5.3 for the mimetic theory with a vector field [133] and 7.3 for the case with a gauge vector field [134].

2.1 Original mimetic dark matter

At first we recap the main concepts of mimetic dark matter without any other modifications. The primary source of this section is the original paper [132], unless stated otherwise.

The idea was to covariantly isolate or revive the conformal degree of freedom in Einstein's theory of gravity. We start with the *physical metric* $g_{\mu\nu}$, which is physical in the sense that the geodesics of massive point particles are determined with respect to that metric, as well as causal structure and curvature of that spacetime. To achieve that split of the degrees of freedom, the physical metric $g_{\mu\nu}$ was decomposed into an *auxiliary metric* $h_{\mu\nu}$ and the *mimetic scalar field* ϕ in the following form

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) h_{\mu\nu}\,, \tag{2.1}$$

such that as a result $g_{\mu\nu}(h_{\mu\nu}, \phi)$. In the end, this is nothing more than a Weyl transformation of the metric $h_{\mu\nu}$. However now, under further Weyl transformations of the auxiliary metric $h_{\mu\nu}$, as in (1.62), i.e.

$$h_{\mu\nu} \to \Omega^2(x) h_{\mu\nu}\,, \tag{2.2}$$

the physical metric $g_{\mu\nu}$ stays invariant. We will explore this property and its consequences in the general chapter 3 and the more specialised ones for different mimetic theories in 4.4, 5.4 and 7.4 for the Noether currents and their consequence on symmetries.

As a direct consequence of the decomposition (2.1), the scalar field satisfies a constraint equation with respect to the physical metric $g_{\mu\nu}$, namely

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1\,. \tag{2.3}$$

Thus $\partial_\mu\phi$ is a timelike unit vetor. If it is future-directed it can be used as a four-velocity

$$u_\mu = \partial_\mu\phi\,. \tag{2.4}$$

Moreover, if we compare (2.3) with the relativistic energy-momentum relation

$$g^{\mu\nu}p_\mu p_\nu = m^2 \tag{2.5}$$

with the four-momentum p^μ and mass m of a particle, we can immediately conclude that for mimetic dark matter [14]

$$p_\mu = m\partial_\mu\phi\,. \tag{2.6}$$

Equivalently, this constraint equation (2.3) can be viewed as the Hamilton-Jacobi equation for a relativistic particle of unit mass. Furthermore, comparing with relativistic mechanics we know that [4]

$$p_\mu = -\partial_\mu S\,, \tag{2.7}$$

therefore the mimetic field takes on the role of the action, more precisely

$$\phi = -\frac{S}{m}\,. \tag{2.8}$$

This can be compared with the definition of the action of the free point particle in relativistic mechanics, namely

$$S = -m\int \mathrm{d}\tau \tag{2.9}$$

with the proper time τ. Therefore,

$$\phi = \tau \tag{2.10}$$

without loss of generality, i.e. the mimetic field plays the role of a "clock", slicing the spacetime into hypersurfaces of constant time [132, 135, 136].

The original theory of mimetic dark matter was then constructed by inserting this metric decomposition (2.1) into the usual Einstein-Hilbert "seed" action S_{EH}, such that

$$\begin{aligned} S[h_{\mu\nu},\phi,\Phi_{\mathrm{m}}] &= S_{\mathrm{EH}}\left[g_{\alpha\beta}(h_{\mu\nu},\phi)\right] + S_{\mathrm{m}}\left[g_{\alpha\beta}(h_{\mu\nu},\phi),\Phi_{\mathrm{m}}\right] & (2.11)\\ &= \int \mathrm{d}^4x\sqrt{-g_{\alpha\beta}(h_{\mu\nu},\phi)}\left(-\frac{1}{2}R\left(g_{\alpha\beta}(h_{\mu\nu},\phi)\right) + \mathcal{L}_{\mathrm{m}}[\Phi_{\mathrm{m}},h_{\mu\nu}]\right) & (2.12)\end{aligned}$$

including matter fields Φ_m, their Lagrangian $\mathcal{L}_\mathrm{m}$ and action S_m. The equations of motion resulting from variation with respect to the auxiliary metric $h_{\mu\nu}$, i.e. the modified Einstein equations, were found to be

$$(G_{\mu\nu} - T_{\mu\nu}) - (G - T)\partial_\mu\phi\partial_\nu\phi = 0\,, \tag{2.13}$$

with $G_{\mu\nu}$ as the Einstein tensor (1.29), the Ricci tensor $R_{\mu\nu}$ (1.27) and the Ricci scalar R (1.28). The usual definition of the energy-momentum tensor (EMT) for matter fields is used, cf. (1.38),

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}}\,. \tag{2.14}$$

$G = g^{\mu\nu}G_{\mu\nu}$ and $T = g^{\mu\nu}T_{\mu\nu}$ are their respective traces. Correspondingly, the equation of motion for the scalar field ϕ is

$$\nabla_\mu\left((G - T)g^{\mu\nu}\partial_\nu\phi\right) = 0\,, \tag{2.15}$$

where ∇_μ is the covariant derivative compatible with the physical metric $g_{\mu\nu}$

$$\nabla_\mu g_{\alpha\beta} = 0\,. \tag{2.16}$$

The equation of motion can also be seen as a current conservation equation [15]

$$\nabla_\mu J^\mu = 0 \tag{2.17}$$

with current

$$J_\mu = (G - T)\partial_\mu\phi\,. \tag{2.18}$$

Taking the trace of the new Einstein equations (2.13) yields

$$(G - T)\left(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = 0\,, \tag{2.19}$$

which is identically satisfied due to the mimetic constraint equation (2.3) even for $(G-T) \neq 0$. For standard GR, in the absence of matter, i.e. $T^{\mu\nu} = 0$, the Einstein equations would imply vanishing curvature. However, in the case of mimetic dark matter this is changed.

One can view the modified Einstein equations as

$$G_{\mu\nu} = T_{\mu\nu} + \tilde{T}_{\mu\nu} \tag{2.20}$$

with an additional "matter term"

$$\tilde{T}_{\mu\nu} = (G - T)\partial_\mu\phi\partial_\nu\phi\,. \tag{2.21}$$

This was compared to the standard perfect fluid EMT as in (1.41)

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - p g_{\mu\nu} \tag{2.22}$$

with energy density ε, pressure p and normalised four-velocity u^μ. Then the following identifications were made:

$$p = 0\,, \tag{2.23}$$
$$\varepsilon = G - T\,, \tag{2.24}$$
$$u_\mu = \partial_\mu \phi\,, \tag{2.25}$$

which demonstrates that the system like fluid-like dust, confirming (2.4).

One can also show that in the fluid picture the *vorticity vector* [15, 137]

$$\Omega^\mu(V) = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\mu} V_\gamma \nabla_\alpha V_\beta\,, \tag{2.26}$$

here defined for any timelike vector field V^μ, vanishes. In our case, the timelike vector field is $u_\mu = \nabla_\mu \phi$ and it follows that

$$\frac{1}{2}\varepsilon^{\alpha\beta\gamma\mu}\nabla_\gamma\phi\nabla_\alpha\nabla_\beta\phi \equiv 0\,, \tag{2.27}$$

as covariant derivatives acting on a scalar are interchangeable, i.e. symmetric, while the Levi-Civita tensor $\varepsilon^{\alpha\beta\gamma\mu}$ is totally antisymmetric and defined by

$$\varepsilon^{\alpha\beta\gamma\mu} = \frac{\epsilon^{\alpha\beta\gamma\mu}}{\sqrt{-g}} \tag{2.28}$$

with the Levi-Civita-symbol $\epsilon^{\alpha\beta\gamma\mu}$. The dust-like fluid is then called *irrotational*, as the vorticity is equal to zero. The acceleration (1.18) for our form of the four-velocity vanishes as well [15, 138]

$$a^\mu = u^\nu\nabla_\nu u^\mu = \nabla^\nu\phi\nabla_\nu\nabla^\mu\phi = \frac{1}{2}\nabla^\mu\left(\nabla^\nu\phi\nabla_\nu\phi\right) \equiv 0 \tag{2.29}$$

because of the constraint equation (2.3). Moreover, from the conservation of the energy-momentum tensor (2.21), namely

$$\nabla_\mu \tilde{T}^{\mu\nu} = 0\,, \tag{2.30}$$

one can write

$$0 = u_\nu\nabla_\mu\tilde{T}^{\mu\nu} = u_\nu\nabla_\mu\left(\varepsilon u^\mu u^\nu\right) = \nabla_\mu\left(\varepsilon u^\mu\right) = u^\mu\nabla_\mu\varepsilon + \varepsilon\nabla_\mu u^\mu\,. \tag{2.31}$$

With the definitions

$$\dot{\varepsilon} \equiv u^\mu\nabla_\mu\varepsilon\,, \tag{2.32}$$
$$\theta \equiv \nabla_\mu u^\mu \tag{2.33}$$

of the proper time derivative $\dot{}$ and the *expansion parameter* θ we can write the conservation of energy as

$$\dot{\varepsilon} + \theta\varepsilon = 0\,. \tag{2.34}$$

Shortly after mimetic dark matter was introduced in [132], in [139, 140] it was realised that based on the constraint equation (2.3) there exists a more convenient formulation of the dynamics for the system, namely

$$S[g_{\mu\nu}, \phi, \rho, \Phi_{\mathrm{m}}] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g_{\mu\nu}) + \frac{\rho}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1)\right) + S_{\mathrm{m}}[g_{\mu\nu}, \Phi_{\mathrm{m}}] \,. \quad (2.35)$$

where the constraint is directly implemented via the Lagrange multiplier ρ. This definition of the Lagrange multiplier was chosen suggestively, as becomes apparent when this action is varied with respect to $g_{\mu\nu}$. The resulting Einstein equations are

$$G_{\mu\nu} = T_{\mu\nu} + \rho\partial_\mu\phi\partial_\nu\phi \,. \quad (2.36)$$

By taking the trace we reproduce that $\rho \equiv \varepsilon = G - T$, i.e. the energy density of mimetic dark matter (cf. eqs. (2.21) and (2.24)).

As we mentioned, the mimetic field plays the role of proper time. Therefore, we can insert a general synchronous metric (1.9) into the e.o.m. for the scalar field (2.15). The result is, using ϕ as time (2.10), from

$$\frac{1}{\sqrt{-\gamma}}\partial_t\left(\sqrt{-\gamma}(G-T)g^{tt}\partial_t\phi\right) = 0 \,, \quad (2.37)$$

the equation

$$\varepsilon = G - T = \frac{C(x^i)}{\sqrt{-\gamma}} \,, \quad (2.38)$$

with $C(x^i)$ a constant only depending on space. If we use the definition of the expansion parameter θ (2.33) of a congruence of geodesic curves for such a synchronous frame, we have [6]

$$\theta = \frac{1}{\sqrt{-\gamma}}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{-\gamma} \,. \quad (2.39)$$

Then we see that (2.38) solves the energy conservation equation (2.34).

In a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime we have the special case

$$ds^2 = \mathrm{d}t^2 - a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad (2.40)$$

with the scale factor $a(t)$, from where it follows that the energy density is (2.38)

$$\varepsilon \propto \frac{1}{a^3(t)} \,. \quad (2.41)$$

So the model really behaves as dark matter in an expanding universe, at least on cosmologically large linear scales. As it is well known, on smaller scales the mimetic model develops *caustic singularities*, focal points at which the massive particles moving on geodesics come together under gravitation being attractive [6].

2.2 Disformal transformations and the mimetic construction

After this exposition and discussion on how mimetic matter was originally obtained, we will introduce the more general formalism on how a Weyl transformation between Jordan and Einstein frames 1.1.4 changes the physics proper. If we take General Relativity as a "seed" action and perform a Weyl transformation (1.62) of the physical metric $g_{\mu\nu}$ to the auxiliary metric $h_{\mu\nu}$ with the help of a function Ω which is *only* dependent on the field ϕ but not its derivatives, it will become apparent that the resulting equations of motion will not look like Einstein's equations, but rather a scalar-tensor theory. If the inverse conformal transformation is applied to them however, General Relativity is recovered and and therefore this theory can aptly be named "veiled" GR as in [31]. There is now the question: In which cases is this merely a reformulation of GR in other variables, while in others it is a true transformation to another theory, like in mimetic gravity, as we know.

Therefore, we will at first use the generalization of Weyl transformations, the so-called *disformal transformation*. Bekenstein [141] first introduced those also depending on the kinetic term X (1.55) of the scalar field, described by

$$g_{\mu\nu} = C(\phi, X)h_{\mu\nu} + D(\phi, X)\partial_\mu\phi\partial_\nu\phi \,, \tag{2.42}$$

where the functions $C(\phi, X)$ and $D(\phi, X)$ are general functions of the scalar field ϕ and the kinetic term X. Applying this disformal transformation to the standard GR "seed" theory, one can show that this can in some cases lead to new degrees of freedom on top of the two graviton ones. One step was made in [142], which we will follow for the rest of this section.

Starting from the Einstein-Hilbert action plus matter action

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}R + S_{\mathrm{m}}[\phi_{\mathrm{m}}, g_{\mu\nu}] \,. \tag{2.43}$$

we perform the disformal transformation (2.42). In the original mimetic dark matter model, the occurring functions are $C = 2X$ and $D = 0$ respectively. Varying the action to arrive at the equations of motion results in the well-known expression

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(G^{\mu\nu} - T^{\mu\nu}\right)\delta g_{\mu\nu} \tag{2.44}$$

with Einstein tensor (1.29) and energy momentum tensor (1.38). So the task is now to vary the metric $g_{\mu\nu}$ in terms of its disformal transformations in order to find the equations of motion. This will lead us to the expression

$$\begin{aligned}\delta g_{\mu\nu} = C\delta h_{\mu\nu} &- \frac{1}{2}\left(h_{\mu\nu}\frac{\partial C}{\partial X} + \partial_\mu\phi\partial_\nu\phi\right)\left(h^{\alpha\rho}\partial_\alpha\phi h^{\beta\sigma}\partial_\beta\phi\delta h_{\rho\sigma} - 2h^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi\right)\\ &+ \left(h_{\mu\nu}\frac{\partial C}{\partial\phi} + \partial_\mu\phi\partial_\nu\phi\frac{\partial D}{\partial\phi}\right) + D\left(\partial_\mu\phi\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\phi\right) \,.\end{aligned} \tag{2.45}$$

Multiplying with $(G^{\mu\nu} - T^{\mu\nu})$ we can read off the equations of motion, first the modified Einstein equations resulting from $\delta S/\delta h_{\mu\nu} = 0$

$$C\left(G^{\mu\nu} - T^{\mu\nu}\right) = \frac{1}{2}\left(A\frac{\partial C}{\partial X} + B\frac{\partial D}{\partial X}\right) h^{\alpha\mu}\partial_\alpha\phi h^{\beta\nu}\partial_\beta\phi \tag{2.46}$$

$$\text{with} \quad A = \left(G^{\mu\nu} - T^{\mu\nu}\right) h_{\mu\nu} \quad \text{and} \quad B = \left(G^{\mu\nu} - T^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\,. \tag{2.47}$$

The equation of motion for ϕ coming from $\delta S/\delta\phi = 0$ is then

$$\frac{2}{\sqrt{-g}}\partial_\rho\left\{\sqrt{-g}\partial_\sigma\phi\left[D\left(G^{\rho\sigma} - T^{\rho\sigma}\right) + \frac{1}{2}\left(A\frac{\partial C}{\partial X} + B\frac{\partial D}{\partial X}\right)h^{\rho\sigma}\right]\right\} = A\frac{\partial C}{\partial\phi} + B\frac{\partial D}{\partial\phi}\,. \tag{2.48}$$

As we are going to investigate whether the disformal transformation is invertible or not, the Jacobian determinant will become useful. So we need to take the modified Einstein equations (2.47) and project them along $h_{\mu\nu}$ and $\partial_\mu\phi\partial_\nu\phi$, resulting in the two equations depending on the functions A and B

$$f_1(A,B) = A\left(C - X\frac{\partial C}{\partial X}\right) - BX\frac{\partial D}{\partial X} = 0\,, \tag{2.49}$$

$$f_2(A,B) = 2AX^2\frac{\partial C}{\partial X} - B\left(C - 2X^2\frac{\partial D}{\partial X}\right) = 0\,. \tag{2.50}$$

Now we can easily calculate the determinant of this system as

$$\mathcal{D} = \det\begin{pmatrix} \frac{\partial f_1}{\partial A} & \frac{\partial f_1}{\partial B} \\ \frac{\partial f_2}{\partial A} & \frac{\partial f_2}{\partial B} \end{pmatrix} = 2X^2 C\frac{\partial}{\partial X}\left(D + \frac{C}{2X}\right)\,. \tag{2.51}$$

In order for this determinant to be non-zero, the only generic solution would be $A = B = 0$ in order to fulfill $f_1 = f_2 = 0$. Then, the equations of motion become

$$C\left(G^{\mu\nu} - T^{\mu\nu}\right) = 0 \quad \text{and} \quad \partial_\rho\left[\sqrt{-g}\partial_\sigma\phi D\left(G^{\rho\sigma} - T^{\rho\sigma}\right)\right] = 0\,, \tag{2.52}$$

where it is clear that the e.o.m.s for the metric become the standard Einstein equations in the (obvious) case of $C \neq 0$ and the e.o.m. for the scalar field as a consequence becomes trivially equal to zero. This is exactly the point we wanted to reinforce: in the case of a generic disformation, where the Jacobian of this transformation is invertible, the resulting theory is nothing else than "veiled" General Relativity, a reformulation in variables that nevertheless does not contain new information.

In contrast to that, we will now have a look at the case in which this Jacobian determinant $\mathcal{D}$ is indeed singular. Then, there is from (2.51) the condition that

$$\frac{\partial}{\partial X}\left(D + \frac{C}{2X}\right) = 0\,, \tag{2.53}$$

that is, there exists now a relation between the functions

$$D(X,\phi) = -\frac{C(X,\phi)}{2X} + l(\phi)\,, \tag{2.54}$$

with $l(\phi)$ only depending on ϕ. This relation uniquely solves the system $f_1 = 0,\ f_2 = 0$, yielding

$$B = 2XA\,. \tag{2.55}$$

This in turn can be used to simplify the equations of motion (2.47) and (2.48), giving

$$G^{\mu\nu} - T^{\mu\nu} = \frac{A}{2X} h^{\alpha\mu}\partial_\alpha\phi h^{\beta\nu}\partial_\beta\phi \quad \text{and} \quad \frac{2}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}Alh^{\sigma\rho}\partial_\sigma\phi\right) = 2XA\frac{\mathrm{d}l}{\mathrm{d}\phi}\,. \tag{2.56}$$

One can invert the metric $g_{\mu\nu}$ to arrive at

$$g^{\mu\nu} = \frac{h^{\mu\nu}}{C} + \frac{C - 2Xl}{4X^2Fl} h^{\alpha\mu}\partial_\alpha\phi h^{\beta\nu}\partial_\beta\phi\,, \tag{2.57}$$

but only if $l(\phi) \neq 0$. Using the substitutions

$$A = \frac{G - T}{2Xl} \equiv \frac{g_{\mu\nu}\left(G^{\mu\nu} - T^{\mu\nu}\right)}{2Xl} \tag{2.58}$$

$$h^{\mu\alpha}\partial_\alpha\phi = 2Xl\partial^\mu\phi \equiv 2Xlg^{\mu\alpha}\partial_\alpha\phi \tag{2.59}$$

and a redefinition of the scalar field as φ, such that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varphi} = \sqrt{|h|} \tag{2.60}$$

will transform the equations of motion (2.56) to

$$G_{\mu\nu} - T_{\mu\nu} = \epsilon(G - T)\partial_\mu\varphi\partial_\nu\varphi \quad \text{and} \quad \nabla_\alpha\left[(G - T)\partial^\alpha\varphi\right] = 0\,, \tag{2.61}$$

where $\epsilon = \pm 1$, depending on the sign of $\partial_\mu\varphi$. For timelike $\partial_\mu\varphi$, i.e. $\epsilon = +1$, we can immediately recognize those equations as the equations of motion (2.13) and (2.15) of mimetic matter, as they were originally derived in [132]. Thus it was shown in [142], that if the disformal transformation is of the special case that there exists a certain relation between functions C and D (2.54), the theory reduces to the original mimetic dark matter. Let us stress again that the Jacobian determinant of the disformal transformation needs to be singular for that.

2.3 Extended mimetic construction

But later it was discovered that this argument and result is not the whole story. In [143] they proved that there is a more general condition on the functions C and D, namely on the solutions of the first order nonlinear partial differential equation

$$C = \frac{\partial C}{\partial Y}Y + \frac{\partial D}{\partial Y}Y^2 \tag{2.62}$$

for the scalar field ϕ to introduce novel degrees of freedom. They also show that mimetic dark matter emerging from these constructions is much more common than previously thought.

To prepare that discussion we will actually repeat the argument of the singular Jacobian determinant, now looking more closely what causes the behaviour as described in the previous section. Therefore, we study the Jacobian matrix and determinant of the above disformal transformation (2.42), following [108, 143]. The Jacobian is calculated as

$$\mathcal{J}^{\rho\sigma}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial h_{\rho\sigma}} = \frac{1}{2}C\left(\delta^\sigma_\mu \delta^\rho_\nu + \delta^\sigma_\nu \delta^\rho_\mu\right) - \left(\frac{\partial C}{\partial Y}h_{\mu\nu} + \frac{\partial D}{\partial Y}\partial_\mu\phi\partial_\nu\phi\right) h^{\sigma\alpha}\partial_\alpha\phi h^{\rho\beta}\partial_\beta\phi \tag{2.63}$$

while its eigenvalue equation is

$$\mathcal{J}^{\mu\nu}_{\rho\sigma}\xi^a_{\mu\nu} = \lambda_a \xi^a_{\rho\sigma} \tag{2.64}$$

with eigentensors $\xi^a_{\mu\nu}$ associated with the respective eigenvalue λ_a. No sum is intended over the index a which merely refers to the respective eigenvalues [143]. Similarly to [108], the two eigenvalues and their eigentensors were found to be

$$\lambda_0 = C\,, \qquad\qquad \xi^0_{\mu\nu} = \phi^\perp_{\mu\nu}\,, \tag{2.65}$$

$$\lambda_* = C - \frac{\partial C}{\partial Y}Y - \frac{\partial D}{\partial Y}Y^2\,, \quad \xi^*_{\mu\nu} = \frac{\partial C}{\partial Y}h_{\mu\nu} + \frac{\partial D}{\partial Y}\partial_\mu\phi\partial_\nu\phi \tag{2.66}$$

with $\phi^\perp_{\mu\nu}$ a symmetric tensor perpendicular to $\partial_\mu\phi\partial_\nu\phi$, explicitly

$$\phi^\perp_{\mu\nu}h^{\mu\alpha}\partial_\alpha\phi h^{\nu\beta}\partial_\beta\phi = 0\,. \tag{2.67}$$

Those eigenvalues are called the *conformal eigenvalue* λ_0 and the *kinetic eigenvalue* λ_*. As setting the conformal eigenvalue equal to zero would result in the conformal factor being zero, we will need to find the vanishing kinetic eigenvalue. In other words, $g_{\mu\nu}$ being a valid metric tensor requires it to be invertible, i.e.

$$g^{\mu\nu} = \frac{1}{C}\left(h^{\mu\nu} - \frac{D}{C+DY}h^{\mu\alpha}\partial_\alpha\phi h^{\nu\beta}\partial_\beta\phi\right)\,, \tag{2.68}$$

which would not be the case for $C = 0$, obviously. Therefore, we necessarily need to use the vanishing of the kinetic eigenvalue

$$\lambda_* = C - \frac{\partial C}{\partial Y}Y - \frac{\partial D}{\partial Y}Y^2 = 0 \tag{2.69}$$

in order to find the singular solutions. Those transformations where the Jacobian determinant vanishes are then aptly called *singular disformal transformations* [143].

The value of the singular disformal transformations is that they do not only yield what one might call "veiled General Relativity", i.e. GR merely reformulated in another frame, but an essentially new theory. This theory has equations of motion (2.61) other than the standard GR field equations, as we have seen in the section above. In the coming paragraph we will try to illustrate that, following [108]. At first we note down that there is not only the eigenvalue equation (2.64), but one can also write one for the *dual* or *left eigentensors*, see

$$\zeta_a^{\sigma\rho}\mathcal{J}_{\sigma\rho}^{\mu\nu} = \lambda_a \zeta_a^{\mu\nu} \,. \tag{2.70}$$

Again, there is no summation over a, this again merely denotes conformal and kinetic eigenvalues. The dual eigenvalues and eigenvectors are then

$$\lambda_0 = C \,, \qquad \zeta_0^{\mu\nu} = \phi_{\top}^{\mu\nu} \,, \tag{2.71}$$

$$\lambda_* = C - \frac{\partial C}{\partial Y}Y - \frac{\partial D}{\partial Y}Y^2 \,, \quad \zeta_*^{\mu\nu} = h^{\mu\alpha}\partial_\alpha\phi h^{\nu\beta}\partial_\beta\phi \,, \tag{2.72}$$

the dual eigenvalues being the same as the original eigenvalues, see (2.65) and (2.66). Also the notation $\phi_{\top}^{\mu\nu}$ refers to any symmetric tensor orthogonal to $\xi^*_{\mu\nu}$, i.e.

$$\zeta_0^{\mu\nu}\xi^*_{\mu\nu} = \phi_{\top}^{\mu\nu}\xi^*_{\mu\nu} = 0 \,. \tag{2.73}$$

Now we can pose the actual problem. As mentioned above, we go from a *seed action* $S_{\mathrm{seed}}[g_{\mu\nu}, \Phi_{\mathrm{m}}]$ (in our case the Einstein-Hilbert action) to an action which is reached by a disformal transformation, such that

$$S_{\mathrm{dis}}[h_{\mu\nu}, \phi, \Phi_{\mathrm{m}}] = S_{\mathrm{seed}}[g_{\mu\nu}\left(h_{\mu\nu}, \phi\right), \Phi_{\mathrm{m}}] \,. \tag{2.74}$$

If this is varied, we will arrive at the following variation of the action required to be zero

$$0 = \frac{\delta S_{\mathrm{dis}}}{\delta h_{\mu\nu}} = \frac{\delta S_{\mathrm{seed}}}{\delta g_{\sigma\rho}}\mathcal{J}_{\sigma\rho}^{\mu\nu} \,. \tag{2.75}$$

So the ultimate goal is to find the equations of motion for the disformed action, i.e. $(\delta S_{\mathrm{dis}})/(\delta g_{\sigma\rho}) = 0$. There are two ways to achieve this: Either the Jacobian matrix is invertible, $\mathcal{J}_{\sigma\rho}^{\mu\nu} \neq 0$, then $(\delta S_{\mathrm{seed}})/(\delta g_{\sigma\rho}) = 0$ which restores the Einstein equations of the original seed theory, leading to "veiled GR". The much more interesting possibility is for the Jacobian determinant to be singular, such that the kinetic eigenvalue $\lambda_* = 0$. Therefore, $(\delta S_{\mathrm{seed}})/(\delta g_{\sigma\rho}) \neq 0$ and the equations of motion will not be the ones of GR. By the eigenvalue equation (2.70) we realize that $(\delta S_{\mathrm{seed}})/(\delta g_{\sigma\rho})$ must be proportional to $\zeta_*^{\mu\nu}$ which then results in

$$\frac{\delta S_{\mathrm{seed}}}{\delta g_{\mu\nu}} = \bar{\rho}\zeta_*^{\mu\nu} = \bar{\rho}\partial^\mu\phi\partial^\nu\phi \,, \tag{2.76}$$

using the dual eigentensors and its definition. The right hand side of this variation no longer vanishes, therefore signalling the appearance of a novel degree of freedom, written here as the new proportionality factor $\bar{\rho}$. As a consequence, the usual variation w.r.t. the physical metric will produce the equation of motion

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{seed}}}{\delta g_{\mu\nu}} = G^{\mu\nu} - T^{\mu\nu} \,. \tag{2.77}$$

After inserting (2.76) we arrive at

$$\frac{2}{\sqrt{-g}}\bar{\rho}\partial^\mu\phi\partial^\nu\phi + T^{\mu\nu} = G^{\mu\nu}\,. \tag{2.78}$$

Let us merely notice the form of this equation, details can be found in [143]. This looks quite like the side of the energy momentum tensor in the Einstein equations has gained a term equivalent to pressure-less dust for $u_\mu \propto \partial_\mu\phi$, i.e. the same phenomenon which was found for the original mimetic dark matter.

In the case of the pure Weyl transformation, without the disformal part, i.e. $D = 0$, which is used in mimetic theories, the kinetic eigenvalue and eigentensor reduce to

$$\lambda_* = C - \frac{\partial C}{\partial Y}Y\,, \tag{2.79}$$

$$\xi^*_{\mu\nu} = \frac{\partial C}{\partial Y}h_{\mu\nu}\,. \tag{2.80}$$

One thing to notice here is that the associated eigentensor $\xi^*_{\alpha\beta}$ is proportional to the auxiliary metric $h_{\mu\nu}$, with the prefactor being unimportant (as long as it does not vanish), as we will later set the equation to zero to find the singular solutions. We will use this fact in the calculations following in 5.3 and 7.3. Moreover, not only solutions of eq. (2.79) set to zero and treated as a differential equation will lead to solutions for C, but one can also treat it as an algebraic equations and try to solve it. This approach will be outlined later in 5.3.1.

Chapter 3

Overview over Noether's Theorems

Symmetries in physics are of paramount importance, be it internal or spacetime symmetries. They can greatly simplify calculations and reveal a lot about the underlying structure of the theory. One example we are going to discuss is *Weyl symmetry*, as it leaves the theory unchanged under a local rescaling of the metric and if necessary, an appropriate and similar scaling of the other fields of the theory, as already introduced in 1.1.3.

We have already seen in (2.2) that the original mimetic theory [132] is Weyl invariant, so naturally the question arises whether this has any significance. A few years ago a discussion arose whether this Weyl invariance can help in solving problems in cosmology, i.e. whether it has any significance in further actions arising in cosmology and inflation, e.g. [144, 145]. Yet others claim that this is just a *fake Weyl symmetry* [146–149] with no physical consequences.

While discussing this issue, we consider the overall topic: When talking about symmetries and conservation laws, *Noether's theorems* [150] lie at the heart of the matter. Therefore, in this chapter we will review the Noether theorems in flat and curved spacetimes and discuss their properties and consequences, before we will apply this principles to different mimetic theories in 4.4 for the scalar mimetic theories [132, 135], in 5.4 for the vector case [133] and in 7.4 for the mimetic theory with a field strength term originating from a gauge vector field [134].

3.1 Noether's theorems — Symmetries and conservation laws

We will now investigate the question of the physical relevance of the occurring Weyl invariance. As those invariances are symmetries of the theory, we will have a closer look at *Noether's theorems*. Emmy Noether's seminal paper "*Invariante Variationsprobleme*" [150] was published in 1918. It contains an analysis of a variational problem and presents two new theorems: The first applies to global symmetries, the second to local (or gauge) symmetries [151]. A few introductory remarks seem in order, concerning the general purpose and statement of Noether's theorems.

Both theorems are concerned with the conditions under which an action is invariant, allowing also for a boundary term. In general,

$$S = \int \mathrm{d}^4x \, \mathcal{L}\left(x, \psi(x), \partial_\mu \psi(x)\right) \tag{3.1}$$

with ψ collectively denoting all fields. Notice that the Lagrangian only depends on first derivatives of the fields. This can be generalised, compare [150], but here we refrain from it, preferring simplicity over completeness. Note that we will try to slightly modernise the notation from Noether's paper, but otherwise follow her discussion in the whole section, occasionally also using [151]. Another comment: We have tried to avoid writing spacetime indices, especially on the coordinates, for improved readability. They only appear wherever needed to clarify Einstein summation.

We have to distinguish between two types of variations in the Lagrangian here. On the one hand, there can be a coordinate or spacetime transformation which is also acting on the field itself

$$\tilde{x} = x + \Delta x \tag{3.2}$$

$$\tilde{\psi}(\tilde{x}) = \psi(x) + \Delta\psi \, . \tag{3.3}$$

One the other hand, the field itself changes *at one point*, which is the proper field variation

$$\delta\psi(x) = \tilde{\psi}(x) - \psi(x) \, . \tag{3.4}$$

Taylor-expanding the scalar field as

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(x + \Delta x) \approx \tilde{\psi}(x) + \partial_\mu \tilde{\psi}(x) \Delta x^\mu \tag{3.5}$$

and then employing eqs. (3.3) and (3.4) yields the combined expression

$$\delta\psi(x) = \Delta\psi - \partial_\mu \tilde{\psi}(x) \Delta x^\mu \, . \tag{3.6}$$

So it is worth keeping in mind that there are two different effects at work when varying the action. One generalising remark: So far we have treated ψ as a *scalar* field, but as stated above, it denotes all fields present in the Lagrangian. Fields of spin 1 and 2 would change under the same coordinate transformation as [131]

$$\tilde{A}_\alpha(x + \Delta x) \approx \tilde{A}_\alpha(x) - \tilde{A}_\beta(x) \partial_\alpha \Delta x^\beta - \partial_\beta \tilde{A}_\alpha(x) \Delta x^\beta \, , \tag{3.7}$$

$$\tilde{h}_{\alpha\beta}(x + \Delta x) \approx \tilde{h}_{\alpha\beta}(x) - \tilde{h}_{\alpha\rho}(x) \partial_\beta \Delta x^\rho - \tilde{h}_{\rho\beta}(x) \partial_\alpha \Delta x^\rho - \partial_\rho \tilde{h}_{\alpha\beta}(x) \Delta x^\rho \, . \tag{3.8}$$

Of course, one can express the general rule more concisely as soon as one notices that those are special cases of the Lie derivative $\mathscr{L}$ (1.22) along Δx^μ, so that overall

$$\tilde{\psi}(x + \Delta x) \approx \tilde{\psi}(x) + \mathscr{L}_{\Delta x} \tilde{\psi}(x) \, . \tag{3.9}$$

Let us now retrace the steps to Noether's theorems, before explicitly stating and discussing them. The basis is that the action is invariant under some transformation of the fields and the coordinates, represented by a Lie group G, be it a finite or an infinite group. Here we just take the infinitesimal transformation of the coordinates (3.2) and of the fields (3.3) resulting in

$$0 = \Delta S = \int \mathrm{d}^4\tilde{x}\ \mathcal{L}\left(\tilde{x}, \tilde{\psi}(\tilde{x})\,, \tilde{\partial}_\mu \tilde{\psi}(\tilde{x})\right) - \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \psi(x), \partial_\mu \psi(x)\right)\ . \tag{3.10}$$

Next is a simple coordinate transformation of the first integral term by just using (3.2) and a Taylor expansion. Also note that the integration measure changes as

$$\mathrm{d}^4\tilde{x} = \mathrm{d}^4 x + \mathrm{d}^4 \Delta x \quad \text{and} \quad \mathrm{d}^4 \Delta x = \left(\frac{\partial \Delta x}{\partial x}\right) \mathrm{d}^4 x \tag{3.11}$$

with the Jacobian $\left(\frac{\partial \Delta x}{\partial x}\right)$ of the coordinate transformation. After expanding to first order in Δx we will then get a boundary term

$$\begin{aligned} &\int \mathrm{d}^4\tilde{x}\ \mathcal{L}\left(\tilde{x}, \tilde{\psi}(\tilde{x}), \tilde{\partial}_\mu \tilde{\psi}(\tilde{x})\right) \\ &= \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \tilde{\psi}(x), \partial_\mu \tilde{\psi}(x)\right) + \int \mathrm{d}^4 x\ \partial_\mu \left(\mathcal{L} \Delta x^\mu\right)\ , \end{aligned} \tag{3.12}$$

such that for the moment

$$\begin{aligned} 0 &= \Delta S \\ &= \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \tilde{\psi}(x), \partial_\mu \tilde{\psi}(x)\right) - \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \psi(x), \partial_\mu \psi(x)\right) + \int \mathrm{d}^4 x\ \partial_\mu \left(\mathcal{L} \Delta x^\mu\right)\ . \end{aligned} \tag{3.13}$$

On the other hand, there is the field variation *at one point* x^μ using (3.4) and partial integration yielding

$$\begin{aligned} \delta S &= \int \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\mu \delta \psi\right) \\ &= \int \mathrm{d}^4 x\ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta \psi\right) + \int \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) \delta \psi \\ &\equiv \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \tilde{\psi}(x), \partial_\mu \tilde{\psi}(x)\right) - \int \mathrm{d}^4 x\ \mathcal{L}\left(x, \psi(x), \partial_\mu \psi(x)\right)\ , \end{aligned} \tag{3.14}$$

where the last line denotes just the definition of the field variation. So comparing this with (3.13) and rearranging we can finally write the expression for the Euler-Lagrange equations of motion (leaving out the integral, as the integrand should vanish)

$$\frac{\delta S}{\delta \psi} \delta \psi = -\partial_\mu J^\mu \tag{3.15}$$

with the functional derivative of the action $\frac{\delta S}{\delta\psi}$ (sometimes also called "Lagrangian expression" Ψ, also in Noether's original paper) and the *Noether current* J^μ

$$\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \tag{3.16}$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi - \mathcal{L}\Delta x^\mu \equiv K^\mu - X^\mu \,. \tag{3.17}$$

Eq. (3.15) is the crucial expression on which Noether's theorems are based.

A few comments about the form of the Noether current eq. (3.17): The second term X^μ was derived *without* using the equations of motion. This is to be understood as a boundary term which the Lagrangian picks up under the transformation it undergoes. On the other hand, the first term of the current K^μ we received *using* the equations of motion. This has to be reflected in the calculations we will do after the general introduction. Also, the Noether current J^μ is just conserved on the equations of motion, i.e. in the case $\Psi = \frac{\delta S}{\delta\psi} = 0$, as is obvious from eq. (3.15). One last remark: For multiple fields ψ, the first term of this current should of course be a sum over the expressions for the different fields. Equally clear, for fields occurring in higher derivatives than the first, this has to be included in the pure variation and in the equations of motion in a straightforward way.

3.1.1 Noether's first theorem

Noether's first theorem concerning global symmetries is cited and discussed extensively in every book about mechanics, field theory and quantum field theory, e.g. [16, 152, 153], therefore it is often just called *Noether's theorem*. It deals with *global* symmetries, i.e. transformations under finite dimensional Lie groups G_N with N parameters $\epsilon_1, \dots, \epsilon_N$. As the symmetry transformation is assumed to be linear in those ϵ_a (view them as infinitesimally small parameters, in which coordinates and fields are expanded), all our relevant expressions are linear in these parameters, such that

$$\delta\psi = \delta\psi^{(1)}\epsilon_1 + \cdots + \delta\psi^{(N)}\epsilon_N\,; \quad J^\mu = J^{(1)\mu}\epsilon_1 + \cdots + J^{(N)\mu}\epsilon_N\,. \tag{3.18}$$

Inserting that in our general expression for the Noether current (3.17), it splits in N different expressions, one for each parameter

$$\frac{\delta S}{\delta\psi}\delta\psi^1 = -\partial_\mu J^{(1)\mu}\,;\ \dots;\ \frac{\delta S}{\delta\psi}\delta\psi^N = -\partial_\mu J^{(N)\mu}\,. \tag{3.19}$$

So this is now Noether's first theorem (compare [150, 151]):

> *If the Euler-Lagrange field equations for all fields present are satisfied, i.e.* $\frac{\delta S}{\delta\psi} = 0$*, then there exist* N *conserved currents, one for each parameter of the finite dimensional continuous symmetry group* G_N*, representing a global symmetry.*

3.1.2 Noether's second theorem

Noether's second theorem, on the contrary, is cited much less. It deals with *local* symmetries, i.e. *infinite dimensional* Lie groups $G_{\infty N}$. In this case, $\delta\psi$, its derivatives and J^μ depend linearly on N arbitrary functions $p^{(\alpha)}(x^\mu)$ with $\alpha = 1, \dots, N$. Then, restricting ourselves to first derivatives of $p^{(\alpha)}(x)$ for convenience (as in [151])

$$\delta\psi = \sum_\alpha \left(a^{(\alpha)}(x, \psi, \partial_\mu\psi)\Delta p^{(\alpha)}(x) + b^{(\alpha)\mu}(x, \psi, \partial_\mu\psi)\partial_\mu \left(\Delta p^{(\alpha)}(x)\right)\right) \tag{3.20}$$

with infinitesimal $\Delta p^{(\alpha)}(x)$. Inserting in our general expression (3.17) and partial integration leaves us with

$$\begin{aligned}\Psi\delta\psi &= \sum_\alpha \left(a^{(\alpha)}\Psi\Delta p^{(\alpha)} + b^{(\alpha)\mu}\Psi\partial_\mu \left(\Delta p^{(\alpha)}\right)\right) \\ &= \sum_\alpha \left(a^{(\alpha)}\Psi - \partial_\mu \left(\Psi b^{(\alpha)\mu}\right)\right) \Delta p^{(\alpha)} + \sum_\alpha \partial_\mu \left(\Psi b^{(\alpha)\mu}\Delta p^{(\alpha)}\right) \\ &\equiv -\partial_\mu J^\mu\end{aligned}$$

using the "Lagrangian expression" $\Psi = \frac{\delta S}{\delta\psi}$ for brevity here. Rearranging and writing an integral gives

$$\int \mathrm{d}^4x \sum_\alpha \left(a^{(\alpha)}\Psi - \partial_\mu \left(\Psi b^{(\alpha)\mu}\right)\right) \Delta p^{(\alpha)} = -\int \mathrm{d}^4x\ \partial_\mu \left(J^\mu + \sum_\alpha \Psi b^{(\alpha)\mu}\Delta p^{(\alpha)}\right) . \tag{3.21}$$

As the right hand side is just a boundary term and the functions $p^{(\alpha)}$ arbitrary, we are free to choose them such that this boundary vanishes entirely. It follows that *for each* of the N parameters $p^{(\alpha)}$ the integrand on the left hand side has to vanish separately, so that we eventually have N partial differential equations

$$a^{(\alpha)}\Psi = \partial_\mu \left(\Psi b^{(\alpha)\mu}\right) \tag{3.22}$$

relating Ψ and its partial derivative. Therefore, Noether's second theorem is

> *If an action is invariant under a local gauge symmetry, i.e. a Lie group $G_{\infty N}$, there exist N differential relations between the Lagrangian expressions and their derivatives.*

3.2 Noether's theorems in curved spacetimes

Generalising Noether's theorems to curved spacetimes will lead to a concept of symmetries in GR which we will explore in the following, always drawing from [154], if not stated otherwise. They start with a coordinate transformation

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x) \tag{3.23}$$

resulting in a change of a scalar field

$$\delta\phi(x) = -\xi^\nu(x)\partial_\nu\phi\,. \tag{3.24}$$

The metric does also change, according to the usual tensor transformation law and the above coordinate transformation (3.23)

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}g^{\alpha\beta}(x) = g^{\mu\nu}(x) + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu\,. \tag{3.25}$$

On the other hand, a Taylor expansion to first order yields

$$\tilde{g}^{\mu\nu}(\tilde{x}) = g^{\mu\nu}(x) + \xi^\lambda(x)\partial_\lambda g^{\mu\nu}(x)\,. \tag{3.26}$$

If we combine both, this gives us the variation of the metric under infinitesimal coordinate transformations

$$\begin{aligned}\delta g^{\mu\nu} &= -\xi^\lambda\partial_\lambda g^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu && (3.27)\\ &= \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu\,, && (3.28)\end{aligned}$$

where the definition of the covariant derivative of a tensor was used. As we are looking for symmetries of the metric under coordinate transformations, it means that this variation needs to vanish under them, or explicitly written

$$\mathscr{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0\,, \tag{3.29}$$

using the Lie derivative (1.22) along ξ^μ. This is called the *Killing equation* and vector fields ξ^μ which satisfy this are known as *Killing fields*. They describe the geometric symmetries of a spacetime. So for an action depending on a metric, a scalar field and its first derivative

$$S[g^{\mu\nu},\phi,\partial_\mu\phi] = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}(g^{\mu\nu},\phi,\partial_\mu\phi) \tag{3.30}$$

we have the variation of said action

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\mu\delta\phi\right) + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu} \tag{3.31}$$

using the usual expression for the energy-momentum tensor $T_{\mu\nu}$ (1.38), see also [4]. Inserting (3.24) and (3.26) and simplifying gives

$$\begin{aligned}\delta S = &\int \mathrm{d}^4x\sqrt{-g}\left[-\xi^\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\lambda\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\lambda\partial_\mu\phi\right) - \left(\partial^\mu\xi^\lambda + \partial^\lambda\xi^\mu\right)\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\partial_\lambda\phi\right]\\ &+ \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}\,.\end{aligned} \tag{3.32}$$

With the help of the Killing equation (3.27), i.e. also using that $\delta g^{\mu\nu} = 0$, this will be

$$\delta S = -\int \mathrm{d}^4x\sqrt{-g}\,\xi^\lambda \left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\lambda\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\lambda\partial_\mu\phi + \partial_\lambda g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\partial_\lambda\phi\right) \tag{3.33}$$

$$= -\int \mathrm{d}^4x\sqrt{-g}\,\xi^\lambda\partial_\lambda\mathcal{L}\,. \tag{3.34}$$

Using the Killing equation (3.27) again it is clear that

$$\nabla_\mu\xi^\mu = 0\,, \tag{3.35}$$

so that we can partially integrate (3.34) to give

$$\delta S = \int \mathrm{d}^4x\,\partial_\lambda\left(-\sqrt{-g}\xi^\lambda\mathcal{L}\right)\,. \tag{3.36}$$

This is again the point in the derivation, where we have not used the equations of motion in the variation of the action and it yields a surface term

$$\delta S = \int \mathrm{d}^4x\,\partial_\mu K^\mu\,, \tag{3.37}$$

compare with eq. (3.17).

For the general case including the Einstein-Hilbert action, i.e.

$$S[g^{\mu\nu},\phi,\partial_\mu\phi] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g^{\mu\nu}) + \mathcal{L}(g^{\mu\nu},\phi,\partial_\mu\phi)\right) \tag{3.38}$$

the corresponding variation is

$$\begin{aligned}\delta S = &\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu} \\ &+ \int \mathrm{d}^4x\sqrt{-g}\left(\xi^\lambda\partial_\lambda\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\partial_\nu\phi\delta g^{\mu\nu}\right)\,.\end{aligned} \tag{3.39}$$

Therefore again, for ξ^μ being a Killing field the variation of the action is a surface term

$$\delta S = \int \mathrm{d}^4x\,\partial_\lambda\left(-\sqrt{-g}\xi^\lambda\mathcal{L}\right)\,. \tag{3.40}$$

Following the construction in the flat spacetime example, we can calculate the Noether current, but the second part we need is the variation of the action while using the equations of motion. We first write the variation of the action as

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu} + \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\mu\delta\phi\right) \tag{3.41}$$

The equations of motion for $g^{\mu\nu}$ and ϕ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \tag{3.42}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) \tag{3.43}$$

and therefore the variation of the action reduces to

$$\delta S = \int \mathrm{d}^4 x \, \partial_\mu \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi \right) . \tag{3.44}$$

To assemble the Noether current at last, we subtract this from eq. (3.40) and get

$$\partial_\mu \left[\sqrt{-g} \, \xi^\lambda \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \partial_\lambda \phi - \delta^\mu_\lambda \mathcal{L} \right) \right] = 0 \tag{3.45}$$

with the Noether current in square brackets. The object within the parentheses is obviously the energy-momentum tensor as defined in classical field theory, cf. [4], so finally

$$\partial_\mu \left[\sqrt{-g} \xi^\lambda T^\mu_\lambda \right] = 0 \, . \tag{3.46}$$

This is actually easy to guess, see the following calculation:

$$\nabla_\mu \left(\xi_\lambda T^{\mu\lambda} \right) = \nabla_\mu \xi_\lambda T^{\mu\lambda} + \xi_\lambda \nabla_\mu T^{\mu\lambda} \tag{3.47}$$

$$= \frac{1}{2} \left(\nabla_\mu \xi_\lambda + \nabla_\lambda \xi_\mu \right) T^{\mu\lambda} + \xi_\lambda \nabla_\mu T^{\mu\lambda} \, . \tag{3.48}$$

The first term is zero because of the Killing equation, and the second one is zero due to the equation of motion, so energy-momentum conservation. Therefore the conservation law for the Noether current is very obvious from those lines.

Chapter 4

Scalar Extensions of Mimetic Gravity

Now that we have introduced the original dark matter and the more general mimetic construction in 2, we will discuss further variations of mimetic theories in more detail. We will start by a general discussion on how to write a completion of mimetic theories such that they are more well behaved in the ultraviolet regime 4.1. Then we will go on to discuss the further modifications of mimetic dark matter, but all based on scalar fields 4.2, an expression for the speed of sound for a general higher derivative theory 4.2.4. We also include more in-depth calculations on Noether's theorems for the formulations of mimetic gravity with a scalar field 4.4.

4.1 UV completion through a complex scalar field

As some modified theories of gravity experience problems like caustics [155] and gradient instabilities [156], there have been suggestions to embed those theories of gravity into better behaved theories in the UV regime. This was attempted by promoting the real scalar field ϕ of modified gravity to the phase of a canonical complex scalar field Ψ, such that in general

$$\Psi = |\Psi| e^{i\phi} \,. \tag{4.1}$$

As they also show in [156], the amplitude $|\Psi|$ needs to be frozen out in order to recover the original k-essence model, i.e. its dynamics have to be set to zero up to the point where they should appear in order to alleviate the problems they should.

At first we follow [155] in order to see how the complex scalar could smooth out caustics occurring in the original theory of gravity including a pressureless perfect fluid. The Lagrangian for a complex scalar field Ψ with mass m is defined as

$$\mathcal{L} = \frac{1}{2} |\partial\Psi|^2 - \frac{m^2}{2} |\Psi|^2 \,. \tag{4.2}$$

This is of course equivalent to the action (1.37), just in slightly different notation. To carry on, we can employ a field redefinition similar (but not identical) to the so-called *Madelung* *transformation* [157] of the complex scalar field Ψ, to look like

$$\Psi = \frac{\sqrt{\rho}}{m}\exp(im\phi) \tag{4.3}$$

so that it is rewritten in two real scalar fields ρ and ϕ. It is worth mentioning that the fields have non-standard mass dimensions $[\phi] = 1/M$ and $[\rho] = M^4$, respectively. They are already named appropriately, as we will see. Then, the Lagrangian (4.2) reads

$$\mathcal{L} = \frac{1}{8}\frac{(\partial\rho)^2}{\rho m^2} + \frac{1}{2}\rho\left((\partial\phi)^2 - 1\right) \,. \tag{4.4}$$

In the formal limit $m \to \infty$ we reproduce the mimetic Langrangian (2.35) with constraint

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1\,. \tag{4.5}$$

Physically the complex scalar field Ψ is in this regime $m \to \infty$ when the following conditions hold

$$\frac{(\partial\rho)^2}{\rho m^2} \ll \rho, \qquad \text{i.e.} \qquad \frac{|\partial\rho|}{m} \ll \rho\,, \quad \text{and} \tag{4.6}$$

$$\frac{(\partial\rho)^2}{\rho m^2} \ll \rho(\partial\phi)^2, \qquad \text{i.e.} \qquad \frac{|\partial\rho|}{m} \ll \rho|\partial\phi|\,, \tag{4.7}$$

such that the energy density ρ should not change on length scales $1/m$. So in this regime we have the usual pressureless dust with slowly changing energy density. On the other hand, conditions for the inhomogeneous evolution of the complex scalar while still being a pressureless perfect fluid can be stated, namely

$$\left|\frac{\partial_i^2\sqrt{\rho}}{\sqrt{\rho}}\right| \propto \frac{1}{L^2} \ll m^2v^2\,, \quad \left|\frac{\partial_i\sqrt{\rho}}{\sqrt{\rho}}\right| \propto \frac{1}{L} \ll mv\,, \quad m\left|\partial_i^2\phi\right| \propto \frac{m\frac{1}{m}}{L^2} \ll m^2v^2\,, \tag{4.8}$$

so therefore in total

$$\frac{1}{L^2m^2v^2} \ll 1\,. \tag{4.9}$$

This is satisfied for sufficiently large m, so that the inhomogeneities in the scalar field are small enough on cosmological scales. But when the inhomogeneities of the scalar field ρ grow, the inequality (4.9) becomes less and less accurate and the complex scalar does not resemble a perfect fluid anymore. Looking for an interpretation for the non-relativistic regime, we can insert the Madelung transformation (4.3) in the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Psi}{\partial x^2}\,. \tag{4.10}$$

After separating the real and imaginary parts of this equation and using

$$\boldsymbol{v} = -\frac{\partial}{\partial x}\phi \tag{4.11}$$

we end up with the so-called *Madelung equations*

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\frac{1}{2m^2} \nabla \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \tag{4.12}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{v}) = 0 \tag{4.13}$$

which allow a hydrodynamic interpretation of quantum mechanics. The important part for us is the right hand side of eq. (4.12), which is called the *quantum pressure*. It depends on the spatial derivatives of the energy density and becomes important at small scales and this is what prevents caustics from forming when the following inequality is valid

$$\left| \frac{\nabla \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}}{m^2\, (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}} \right| \propto \frac{1}{L^2 \cdot m^2 \cdot v^2} \ll 1\,. \tag{4.14}$$

This is exactly the same inequality as above (4.9), where the question was when the complex scalar does not resemble a perfect fluid anymore. So mimetic matter or a pressureless perfect fluid can be extended to a complex scalar field which mimics dust on large, homogeneous scales, while on small scales the arising quantum pressure prevents caustic singularities to form.

Furthermore, also in the context of shift-symmetric k-essence [97, 98] it has been shown that one can again use a complex scalar field in order to remove gradient instabilities [156]. As a complex scalar field has two propagating degrees of freedom, there are two branches ω_+ and ω_- in the dispersion relation derived from it. For momenta $k \to \infty$ the standard $\omega^2 = k^2$ is recovered in both branches. On the contrary, for the low-momentum limit $k \to 0$ the dispersion relation should approach $\omega^2 = c_\mathrm{s}^2 k^2$, which is recovered in only one of the branches which is therefore being called the *hydrodynamical* branch of the dispersion relation, whereas the other one is called the *non-hydrodynamical* branch. For the non-hydrodynamical branch to be stable, the speed of sound must be subluminal, i.e. $c_\mathrm{s}^2 < 1$. In the superluminal case, the non-hydrodynamical branch of the dispersion relation develops a tachyonic instability. The branches are associated with parameters M_1 and M_2 with a mass gap in between them. Note that we will only look closer at the cases where $c_\mathrm{s}^2 < 1$, but in [156] also superluminality is discussed in more depth. At first we note down the definition of the speed of sound in terms of the mass parameters:

$$c_\mathrm{s}^2 = \frac{M_2^2}{M_1^2}\,. \tag{4.15}$$

In the low-momentum limit and the case $M_1^2 > 0$ we have the solutions in a cosmological background for the frequencies

$$\omega_-^2 = \frac{M_2^2}{M_1^2} k^2 + \frac{\left(M_1^2 - M_2^2\right)^2}{a^2 M_1^6} k^4 + \mathcal{O}(k^6) \tag{4.16}$$

$$\omega_+^2 = M_1^2 a^2 + \frac{2M_1^2 - M_2^2}{M_1^2} k^2 + \mathcal{O}\left(\frac{k^4}{a^2 M_1^2}\right) \tag{4.17}$$

for the hydrodynamical and non-hydrodynamical branches, respectively [156]. It is important for ω_+ to be real, such that there are different cases to consider for the mass parameters and the sound speed. We need

$$\omega_+ \gg \omega_- \,, \tag{4.18}$$

meaning that both branches are widely separated from each other by a mass gap set by M_1. The non-hydrodynamical mode ω_+ is obviously safe and stable for $M_1^2 > M_2^2$. The hydrodynamical mode however should be treated with more care and the question will be whether M_2^2 is positive or negative, leading to subluminal propagation or imaginary speed of sound, going by the definition of it (4.15). One can use this definition of the sound speed in terms of the mass parameters and rewrite those parameters in a cosmological background, such that

$$M_1^2 = \frac{4\dot{\varphi}^2}{1-c_\mathrm{s}^2} \quad \text{and} \quad M_2^2 = \frac{4c_\mathrm{s}^2\dot{\varphi}^2}{1-c_\mathrm{s}^2} \,. \tag{4.19}$$

For $|c_\mathrm{s}^2| \ll 1$ it is obvious that $M_1^2 > M_2^2$.

4.2 Modifying mimetic “dark matter” as a scalar-tensor theory

The basic model (2.35) was later modified to account for more general and diverse behaviour, first in [135], where a potential $V(\phi)$ for the scalar field was introduced. Second, one can also add a function $f(\Box\phi)$ of higher derivatives of ϕ. The generic form of the action can therefore be written as

$$\begin{aligned} S[g_{\mu\nu},\phi,\rho,\Phi_\mathrm{m}] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g_{\mu\nu}) + \frac{\rho}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + f(\Box\phi)\right) \\ + S_\mathrm{m}[g_{\mu\nu},\Phi_\mathrm{m}] \,. \end{aligned} \tag{4.20}$$

The authors of this paper explain the implications of those further added functions. The potential $V(\phi)$ accounts for non-zero pressure, so that this model can also mimic other cosmological fluids, an inflaton or quintessence. The higher derivative term instead changes the short-wavelength behaviour of the fluid and gives the theory a non-vanishing sound speed. We will have a closer look at the consequences of those terms in the following section.

4.2.1 Mimetic gravity with a potential

At first, we will have a closer look at the potential of the scalar field and how it modifies the behaviour of mimetic dark matter to also imitate other cosmological fluids, following [135]. Therefore, we at first set the higher derivative terms of the scalar field in action (4.20) to zero. The resulting equations of motion are

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V + T_{\mu\nu} \tag{4.21}$$

where $T_{\mu\nu}$ is the EMT resulting directly from an external matter action S_{m}. Therefore, the terms derived from the mimetic scalar field and its potential can be viewed again as a perfect fluid, with energy density $\varepsilon_{\mathrm{mim}}$ and pressure p_{mim}

$$\varepsilon_{\mathrm{mim}} = G - T - 3V \tag{4.22}$$

$$p_{\mathrm{mim}} = -V\,. \tag{4.23}$$

As already described in (2.25), the scalar field acts as a velocity potential. It is obvious that now also other equations of state can be described, with the scalar field potential being the negative pressure.

To discuss the cosmological solutions we are mostly interested in, we will again follow [135] and take a flat ($k = 0$) Friedmann universe (1.83) without ordinary matter, therefore $T_{\mu\nu} = 0$. Once more as already discussed (2.10), the scalar field works as the time coordinate, i.e. $\phi = t$. Then the solution for the energy density reads as

$$\varepsilon_{\mathrm{mim}} = \frac{3}{a^3}\int \mathrm{d}a\, a^2 V \tag{4.24}$$

with the scale factor a as usual. The resulting constant of integration again plays the part of mimetic dark matter, diluting with time as $\propto a^{-3}$, whereas a non-zero potential V can mimic other cosmological fluids on top.

In the for us interesting case of quintessence, i.e. a scalar field producing dynamical and time-varying dark energy, the potential of the scalar field is taken to be

$$V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2}\,. \tag{4.25}$$

The universe itself is taken to be dominated by some other form of energy with a constant equation of state, i.e. $p = w\varepsilon$. The the mimetic component will have

$$\varepsilon_{\mathrm{mim}} = -\frac{\alpha}{wt^2} \tag{4.26}$$

$$p_{\mathrm{mim}} = -\frac{\alpha}{t^2}\,, \tag{4.27}$$

therefore imitating the equation of state of the dominant matter component, while the normal matter energy density is

$$\varepsilon = 3H^2 = \frac{4}{3(1+w)^2t^2}\,, \tag{4.28}$$

such that the mimetic matter is subdominant if $\frac{\alpha}{w} \ll 1$. In the case that we will take the more general solution for the scalar field as the time coordinate, i.e. $\phi = t + t_0$, the mimetic matter will behave as

$$\begin{aligned}&\text{CC for } t < t_0\\ &\text{dominant matter for } t > t_0\,.\end{aligned}$$

This is not the only matter content mimetic matter can imitate, it can also be tweaked to take on the role of an inflaton, or produce a bouncing universe [135].

4.2.2 General higher derivative terms

For the sake of completeness and as a basis for further discussions, we will discuss the action with just a general higher derivative term, following [158], i.e.

$$S[g_{\mu\nu},\phi,\rho,B,\Phi_{\rm m}] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g_{\mu\nu}) + \frac{\rho}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + f(B)\right) + S_{\rm m}[g_{\mu\nu},\Phi_{\rm m}] \tag{4.29}$$

with external matter action $S_{\rm m}[g_{\mu\nu},\Phi_{\rm m}]$. Deriving the equations of motion (e.o.m.) from the purely mimetic part of this action (without the Einstein-Hilbert term and the external matter action) gives

$$\frac{\delta S}{\delta\phi} = -\nabla_\mu\left(\rho\nabla^\mu\phi - \nabla^\mu f_{,B}\right) \equiv -\nabla_\mu J^\mu\,, \tag{4.30}$$

$$\frac{\delta S}{\delta B} = f_{,BB}\left(\Box\phi - B\right)\,, \tag{4.31}$$

where $f_{,B} \equiv \partial_B f$ and $f_{,BB} \equiv \partial_B\partial_B f$ and the d'Alembertian $\Box$ (1.35). From the e.o.m. for B (4.31) it is obvious that we introduced a new field with definition

$$B \equiv \Box\phi \tag{4.32}$$

in the case that $f_{,BB} \neq 0$. Moreover, by the e.o.m. of ϕ, eq. (4.30), we defined a conserved current J^μ.

The EMT of action (4.29) can be derived as usual and reads

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \rho\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(f(B) - \partial^\alpha\phi\partial_\alpha f_{,B} - f_{,B}B\right) - \partial_\mu\phi\partial_\nu f_{,B} - \partial_\mu f_{,B}\partial_\nu\phi\,. \tag{4.33}$$

Again, we will mostly look at synchronous coordinate systems (1.9), such that the mimetic field takes on the role of a clock once more, as discussed in (2.10). Therefore, in general

$$\phi = \pm t + A \tag{4.34}$$

with a constant of integration A. As a consequence, the higher derivative term $B = \Box\phi$ can be calculated as

$$\Box\phi = \frac{\dot\gamma}{2\gamma} \tag{4.35}$$

with the spatial metric γ_{ij} and its determinant γ. Also, $\dot{(\,)} = \partial_0$ denotes a derivative w.r.t. time.

To determine the Lagrange multiplier ρ we can look at the equation of motion for ϕ (4.30) and set it equal to zero. In the synchronous frame this results in

$$\frac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}\rho\right) = \Box f_{,B} = \frac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}f_{,BB}\partial_0 B\right)\,. \tag{4.36}$$

Integrating and solving for ρ yields

$$\rho = \frac{C}{\sqrt{\gamma}} + f_{,BB}\dot{B}\,. \tag{4.37}$$

The first term in this reflects the already discussed mimetic dark matter with a purely spatially dependent constant C (2.38), whereas the second term arises due to the higher derivative terms, but only for $f_{,BB} \neq 0$ [158].

In the original paper where they introduced this generic form of the higher derivative terms in order to find functions f suited to avoiding cosmological [158] and black hole singularities [159], as well as constructing asymptotically free mimetic gravity [160].

4.2.3 Quadratic higher derivatives

As we have seen above, only higher derivative terms with $f_{,BB} \neq 0$ will result in a modification of the mimetic theory. A reasonable and more specific function of the higher derivative term

$$f(\Box\phi) = \frac{1}{2}\gamma\left(\Box\phi\right)^2 \tag{4.38}$$

was introduced in [135], such that the action looks like

$$S_\gamma[g_{\mu\nu}, \phi, \rho] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g_{\mu\nu}) + \frac{\rho}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \frac{1}{2}\gamma\left(\Box\phi\right)^2\right), \tag{4.39}$$

where γ is just assumed to be a numerical constant with $\gamma \ll 1$, which ensures the higher derivative terms to be just a small correction. It was shown in [15] that the action can be rewritten as

$$S_\gamma[g_{\mu\nu}, \phi, \theta, \rho] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R(g_{\mu\nu}) + \frac{\rho}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - \gamma\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\theta + \frac{1}{2}\theta^2\right)\right). \tag{4.40}$$

Then the equations of motion for the two scalar fields are

$$\frac{\delta S_\gamma}{\delta\theta} = \Box\phi - \theta \tag{4.41}$$

$$\frac{\delta S_\gamma}{\delta\phi} = \gamma\Box\theta - \rho\Box\phi - g^{\mu\nu}\partial_\mu\phi\partial_\nu\rho\,. \tag{4.42}$$

The first equation of motion just shows that $\theta = \Box\phi$. Therefore, the second equation of motion can be rewritten as [15, 29]

$$\rho\theta + g^{\mu\nu}\partial_\mu\phi\partial_\nu\rho = \gamma\Box\theta\,. \tag{4.43}$$

As we notice, the theory is shift-symmetric under $\phi \to \phi + c$ with a constant c, there must be a conserved current associated with it, due to Noether's theorems [150]. We will discuss this topic of symmetries and conserved quantities more extensively in 4.4. For the moment, let us write the Noether current associated with the shift symmetry [29]

$$J_\mu = \rho\partial_\mu\phi - \gamma\partial_\mu\rho\,, \tag{4.44}$$

such that the equation of motion (4.43) corresponds to

$$\nabla_\mu J^\mu = 0\,. \tag{4.45}$$

And calculated just as above, the EMT for the “matter” part of this action is

$$T_{\mu\nu} = \rho\partial_\mu\phi\partial_\nu\phi + \gamma\left(g_{\mu\nu}\left(\partial^\alpha\phi\partial_\alpha\theta + \frac{1}{2}\theta^2\right) - \partial_\mu\phi\partial_\nu\theta - \partial_\mu\theta\partial_\nu\phi\right)\,. \tag{4.46}$$

This energy-momentum tensor corresponds to the one for an imperfect fluid [15,29], and the hydrodynamical quantities can be written in the *local rest frame (LRF)* with the velocity

$$u_\mu = \partial_\mu\phi \tag{4.47}$$

and the projector $\mathcal{P}_{\mu\nu}$ (1.44) to the hypersurface orthogonal to this velocity. Then the conserved Noether current (4.44) will look like

$$J_\mu = n u_\mu - \gamma\mathcal{P}_\mu^\lambda\nabla_\lambda\theta\,. \tag{4.48}$$

In this formula we used the shift-charge density

$$n = \rho - \gamma\dot{\theta} \tag{4.49}$$

with the derivative along the velocity $u^\mu\nabla_\mu = \dot{}$. The energy-momentum tensor can also be decomposed in the usual fashion like in (1.45). Then in the LRF the energy density ε (1.42), the pressure p (1.43), the energy flux q_μ (1.46) and the anisotropic stress tensor $\Pi_{\mu\nu}$ (1.47) will be

$$\varepsilon = \rho - \gamma\left(\dot{\theta} - \frac{1}{2}\theta^2\right) \tag{4.50}$$

$$p = -\gamma\left(\dot{\theta} + \frac{1}{2}\theta^2\right) \tag{4.51}$$

$$q_\mu = \mathcal{P}_\mu^\lambda J_\lambda \tag{4.52}$$

$$\Pi_{\mu\nu} = 0\,. \tag{4.53}$$

The anisotropic stress turns out to be vanishing in the local rest frame, whereas the energy flux is equivalent to the spatial components of the Noether current J_μ [15].

4.2.4 Mimetic dark matter and sound speed

In the original mimetic dark matter model [132] the mimetic scalar field provides us with pure dark matter on a cosmological background, with a vanishing speed of sound c_s. This might become a problem in case we want to look at a mimetic scalar field in the context of inflation. A vanishing speed of sound means that perturbations cannot propagate, which is exactly the opposite of what we want from inflation. In that case of $c_\mathrm{s} = 0$, those initial perturbations would not have been able to grow, therefore the dark matter would not have been able to clump and provide us with the large scale structure of the universe we observe today. If we merely want to mimic dark matter in today's universe, a vanishing speed of sound is well within the expectations. But a small c_s might be helpful to prevent caustic instabilities, i.e. geodesics of massive particles converging to a single point [161].

Moreover, one can also look at the speed of the tensor perturbations c_T on that Friedmann background, in order to get the gravitational wave speed in this theory. Due to the already mentioned observations (1.135) of it being extremely close to the speed of light, any large deviations from $c_\mathrm{T} = 1$ are excluded [106]. As it was found, the quadratic higher derivative term $\frac{1}{2}\gamma\left(\Box\phi\right)^2$ is in perfect agreement with this [161].

We will now start by deriving the sound speed for the theory with a general $f(\Box\phi)$, from the action (4.29). To achieve that, we will use perturbation theory to the first order for the energy momentum tensor (4.33), following the calculation as it was done in [135]. The background will be cosmological, a flat Friedmann universe (1.83). The equation of motion for B allows us to calculate in this background

$$B = \Box\phi = 3H \tag{4.54}$$

with the usual Hubble constant H (1.87). We will use that to first order in perturbations

$$\phi \to \phi + \epsilon\delta\phi \tag{4.55}$$

$$B \to B + \epsilon\delta B\,, \tag{4.56}$$

where ϵ is a small number used to keep track of the orders in perturbation theory. Furthermore, as we know, the solution of the mimetic constraint equation (2.3) in a Friedmann universe (1.83) is of the following form

$$\phi(t) = t + A\,, \tag{4.57}$$

compare the discussion around (2.10) with a constant of integration A. Therefore, for the first order perturbations of the scalar field

$$\partial_0\phi = 1, \quad \partial_i\phi = 0 \tag{4.58}$$

$$\partial_0\delta\phi \neq 0, \quad \partial_i\delta\phi \neq 0\,. \tag{4.59}$$

As a convention, ∂_0 denotes the derivative w.r.t. coordinate time, and ∂_i the spatial derivative. $B = \Box\phi$ can be calculated as a background quantity (i.e. in a flat Friedmann universe (1.83)) to be

$$B = \Box\phi = 3H\,. \tag{4.60}$$

Next, we need to determine ρ from the background Einstein equations. Calculating T^0_0 and T^i_j from the EMT (4.33), equating them with G^0_0 (1.101) and G^i_j (1.103) results in

$$\rho = 2\left(\partial_0 f_{,B} - \dot{H}\right) . \tag{4.61}$$

Recall the convention that $f_{,B} \equiv \partial_B f$ and $f_{,BB} \equiv \partial_B \partial_B f$. Also notice that

$$\partial_i f_{,B} = 0 \tag{4.62}$$

as $f_{,B}$ is a background quantity and therefore spatially homogeneous in a Friedmann universe. We will also need the relation for the first order quantities Φ and $\delta\phi$. Calculating the mimetic constraint equation (2.3) in the conformal Newtonian metric (1.104) up to first order yields

$$\Phi = \delta\dot{\phi} . \tag{4.63}$$

Another important first order quantity is

$$\delta B = \delta(\Box\phi) = -3\delta\ddot{\phi} - 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi \tag{4.64}$$

with $\Delta = \delta^{ij}\partial_i\partial_j$. This is again derived from the Newtonian metric (1.104), while also using the relation between Φ and $\delta\phi$ (4.63). The first order perturbation of the Einstein tensor in this metric can be transformed to [36]

$$\delta G^0_i = 2\partial_i\left(H\delta\dot{\phi} + \delta\ddot{\phi}\right) \tag{4.65}$$

with the help of (4.63) and the transformation from conformal to physical quantities (1.98). Now we are well set up to calculate the first order in perturbation theory of the 0-i component of the energy-momentum tensor (4.33). The 0-i component of the energy-momentum tensor will look like

$$\begin{aligned} T^0_i \to &\rho\partial^i(\phi + \epsilon\delta\phi)\partial_0(\phi + \epsilon\delta\phi) - \partial^0(\phi + \epsilon\delta\phi)\partial_i(f_{,B} + \epsilon f_{,BB}\delta B) \\ &- \partial_i(\phi + \epsilon\delta\phi)\partial^0(f_{,B} + \epsilon f_{,BB}\delta B) . \end{aligned} \tag{4.66}$$

The first order in ϵ then follows from the expansions in ϕ (4.55) and B (4.56)

$$\delta T^0_i = \rho\,\partial_i\delta\phi - \partial_0 f_{,B}\,\partial_i\delta\phi - \partial_i\left(f_{,BB}\delta B\right) . \tag{4.67}$$

Expanding and inserting the already calculated quantity δB (4.64) and equating everything to the Einstein tensor component (4.65) then yields up to first order, using also the expression for ρ (4.61)

$$\begin{aligned} \partial_i\left(H\delta\dot{\phi} + \delta\ddot{\phi}\right) = &\frac{1}{2}\partial_0 f_{,B}\partial_i\delta\phi - \dot{H}\partial_i\delta\phi + \frac{3}{2}\partial_i\left(f_{,BB}\delta\ddot{\phi}\right) + \frac{3}{2}H\partial_i\left(f_{,BB}\delta\dot{\phi}\right) \\ &+ \frac{1}{2}\partial_i\left(f_{,BB}\frac{\Delta}{a^2}\delta\phi\right) . \end{aligned} \tag{4.68}$$

The next step is noticing that this equation contains a total spatial derivative. We can drop this and simplify this first order equation for δT^0_i to

$$\delta\ddot{\phi}\left(1-\frac{3}{2}f_{,BB}\right)+H\delta\dot{\phi}\left(1-\frac{3}{2}f_{,BB}\right)-\frac{1}{2}f_{,BB}\frac{\Delta}{a^2}\delta\phi+\left(\frac{1}{2}\partial_0 f_{,B}-\frac{\rho}{2}\right)\delta\phi=0\,. \tag{4.69}$$

Inserting the already calculated ρ (4.61) will give us the equation of motion for $\delta\phi$, i.e.

$$\delta\ddot{\phi}+H\delta\dot{\phi}+\dot{H}\delta\phi-\frac{1}{2}\frac{f_{,BB}}{1-\frac{3}{2}f_{,BB}}\frac{\Delta}{a^2}\delta\phi=0\,. \tag{4.70}$$

The Laplacian appearing in this equation is obviously modified by the scale factor in an expanding universe, such that $\Delta\to\frac{\Delta}{a^2}$. Comparing the equation of motion for $\delta\phi$ with a general wave equation for u (e.g. [162])

$$\ddot{u}-c_{\rm s}^2\Delta u=0 \tag{4.71}$$

we conclude that the sound speed squared for general $f(\Box\phi)$ is

$$c_{\rm s}^2=\frac{f_{,BB}}{2-3f_{,BB}}\,. \tag{4.72}$$

This result was already obtained in [163] and is consistent with the sound speed calculated in [135] for the quadratic higher derivative function (4.38), which is

$$c_{\rm s}^2=\frac{\gamma}{2-3\gamma}\,. \tag{4.73}$$

Now that we have derived the form of the speed of sound for general functions of

$$B=\Box\phi\,, \tag{4.74}$$

we can apply that to function already appearing in the literature. Namely, in [158] they they sought solutions for the mimetic scalar field action which enabled them to avoid cosmological (and black hole [159]) singularities by introducing a limiting curvature B_m. They used the higher derivative function

$$f(B)=B_m^2\left[1+\frac{1}{3}\frac{B^2}{B_m^2}-\sqrt{\frac{2}{3}}\frac{B}{B_m}\arcsin\left(\sqrt{\frac{2}{3}}\frac{B}{B_m}\right)-\sqrt{1-\frac{2}{3}\frac{B^2}{B_m^2}}\right]\,. \tag{4.75}$$

Therefore, the second derivative of this function will be

$$f_{,BB}=\frac{1}{3}\left(1-\frac{1}{\sqrt{1-\frac{2}{3}\frac{B^2}{B_m^2}}}\right) \tag{4.76}$$

and the speed of sound, with the help of (4.72), is then found to be

$$c_{\rm s}^2 = \frac{1}{3}\left(\sqrt{1 - \frac{2}{3}\frac{B^2}{B_m^2}} - 1\right) . \tag{4.77}$$

This only vanishes for for $B = 0$, as it is expected. In all other cases the sound speed squared is strictly negative; for small B the expansion will be

$$c_{\rm s}^2 = -\frac{1}{9}\frac{B^2}{B_m^2} - \frac{1}{54}\frac{B^4}{B_m^4} + \mathcal{O}(B^6) . \tag{4.78}$$

As the curvature invariants are required to be bounded by a maximal value according to the original paper [158], we can also make the following observation: For $B = B_m$ the value of the sound speed squared will be

$$c_{\rm s}^2(B = B_m) = \frac{1}{3}\left(\sqrt{1 - \frac{2}{3}} - 1\right) \approx -0.14 . \tag{4.79}$$

This value can still be considered reasonably small, therefore we can discuss the viability of the theory with an imaginary speed of sound, which corresponds to gradient instabilities [156].

As already described in 4.1, the UV completion by a complex scalar field may alleviate that problem similarly as it was done in the case of k-essence. Both branches (4.16) and (4.17) of the dispersion relation should be stable in UV. The non-hydrodynamical mode ω_+ can easily rendered to be real, as discussed, whereas there is also a possibility for the hydrodynamical one ω_- to become stable for small imaginary speeds of sound. Expressing ω_- in terms of the speed of sound, as in (4.19), this yields

$$\omega_-^2 = c_{\rm s}^2 k^2 + \frac{(1 - c_{\rm s}^2)^3}{4\varphi'^2} k^4 + \mathcal{O}(k^6) . \tag{4.80}$$

As usual, φ' denotes differentiation of φ w.r.t. conformal time (1.98). For our case $c_{\rm s}^2 < 0$ the first of these terms would be negative and lead to gradient instabilities. But for certain values of the momenta the second term is able to overcome the first and render ω_- real and stable. The range of momenta is

$$|M_2| \ll k/a \ll M_1 . \tag{4.81}$$

The first condition comes from asking for the second term in (4.16) to be larger than the first, while the second condition comes from asking that we are in the low-momentum limit. The dispersion relation of that complex scalar field model then deviates from the one in the k-essence model, as the k^4-term dominates over the k^2-term, but that modification of the theory also removes the gradient instabilities. The authors of [156] conclude that a small imaginary speed of sound does not necessarily make the affected theory implausible or even impossible.

4.3 Gauge invariant representations

Based on the Weyl transformation of the original theory of mimetic gravity [132] (2.1)

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) h_{\mu\nu}\,, \tag{4.82}$$

one can set up different representations of the same theory, always using different gauge invariant variables. In the following, we will summarise them for the scalar case.

Using the kinetic term X (1.55) while promoting it to a variable of the Weyl transformed theory results in the action, cf. [29],

$$S_{\text{scalar}}[h_{\mu\nu}, \phi, X, \lambda] = -\int \mathrm{d}^4x\sqrt{-h}\left[XR(h) + \frac{3}{2}\frac{h^{\alpha\beta}\partial_\alpha X\partial_\beta X}{X} + \lambda\left(X - \frac{1}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\right] \tag{4.83}$$

with the Lagrangian multiplier λ enforcing the definition of X as a new dynamical variable. As one can see from that action, it is invariant under the following set of transformations

$$h_{\mu\nu} \to \Omega^2(x)h_{\mu\nu}\,, \tag{4.84}$$
$$\phi \to \phi\,, \tag{4.85}$$
$$X \to \Omega^{-2}(x)X\,, \tag{4.86}$$
$$\lambda \to \Omega^{-2}(x)\lambda \tag{4.87}$$

with function $\Omega(x)$ dependent on the coordinates. From there it is obvious that the conformal weights of λ and X are two. In order to construct gauge invariant variables, one needs to combine these transformations such that they do not change. One such possibility is the following set of transformations

$$\begin{aligned} h_{\mu\nu} &= (2X)^{-1}g_{\mu\nu}\,,\\ \phi &= \phi\,,\\ X &= X\,,\\ \lambda &= 2X\rho\,. \end{aligned} \tag{4.88}$$

Then, action (4.83) becomes

$$S_{\text{scalar}}[g_{\mu\nu}, \phi, \rho] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}R(g) + \frac{\rho}{2}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - 1\right)\right]\,. \tag{4.89}$$

We recognize this as the action which has been described early on in the formulation of mimetic gravity in [139,140], cf. (2.35), which had the mimetic constraint (2.3) enforced via a Lagrange multiplier ρ, later identified with the energy density of reh resulting irrotational fluid [15]. In this case, the matter is minimally coupled to the metric $g_{\mu\nu}$. Moreover, another way to view this set of transformations is the following: One could also simply gauge fix (4.83) with the choice $X = \frac{1}{2}$ [29].

But, as it was found in [29], this set of transformations (4.88) is not the only possibility to construct gauge-invariant variables. Namely, there also exists

$$\begin{aligned} h_{\mu\nu} &= \lambda^{-1}\hat{g}_{\mu\nu}\,, \\ \phi &= \phi\,, \\ X &= \lambda\chi\,, \\ \lambda &= \lambda\,. \end{aligned} \tag{4.90}$$

Then, the resulting action will rather look like

$$S_{\text{scalar}}[\hat{g}_{\mu\nu},\phi,\chi] = \int \mathrm{d}^4x\sqrt{-\hat{g}}\left[\frac{1}{2}\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \left(\chi R(\hat{g}) + \frac{3}{2}\frac{\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi}{\chi}\right) + \chi\right]\,. \tag{4.91}$$

In this case the matter is no longer minimally coupled to $\hat{g}_{\mu\nu}$, clearly visible from the coupling of the Ricci scalar to $2\chi\hat{g}_{\mu\nu}$, compared to the regular Einstein-Hilbert term. And also here one could arrive at the action (4.91) via fixing the gauge in (4.83), this time by setting $\lambda = 1$ [29].

4.4 Noether currents arising in scalar mimetic theory

After we reviewed Noether's theorems in chapter 3, let us come back to our original problem, namely the question: What is the physical significance of the Weyl symmetry present in the theory? And now we can follow the course of [146–149] and argue: We will show that the Noether current vanishes.

So the following part is dedicated to calculating the Noether current for various actions playing a role in mimetic theory. As hinted before, eq. (3.17) will come in handy. Therefore, on the one hand, we will have to calculate the part of the current *without the equations of motion* by just Weyl transforming the action and looking for the boundary. On the other hand, we will have to find the part of the Noether current *involving the equations of motion*.

4.4.1 The mimetic action with the Lagrange multiplier

The first task is to see that the mimetic action including the Lagrange multiplier term (4.89) has a vanishing Noether current. We will take the form

$$S[h_{\mu\nu},\chi,\phi,\lambda] = -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + \frac{\lambda}{12}\chi^2 - \frac{\lambda}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) \tag{4.92}$$

directly from [29], their eq. (2.7), and redefine the kinetic term X (1.55), written w.r.t. the metric $h_{\mu\nu}$, as

$$X = \frac{1}{12}\chi^2\,. \tag{4.93}$$

The first two terms

$$S[h_{\mu\nu},\chi] = -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) . \tag{4.94}$$

are already discussed in [146]. The appropriate Weyl transformation is repeated here, cf. subsection 1.1.3,

$$\begin{aligned}
h^{\alpha\beta} &\to e^{2\theta}h^{\alpha\beta}\,, \quad \delta h^{\alpha\beta} = 2\theta h^{\alpha\beta} \\
\chi &\to e^{\theta}\chi\,, \quad \delta\chi = \theta\chi\,, \qquad \text{so therefore} \\
\sqrt{-h} &\to e^{-4\theta}\sqrt{-h} \\
R(h) &\to e^{2\theta}\left(R(h) - 6h^{\alpha\beta}\nabla_\alpha\theta\nabla_\beta\theta + 6h^{\alpha\beta}\nabla_\alpha\nabla_\beta\theta\right) \\
&= e^{2\theta}\left(R(h) - 6h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta + 6\frac{1}{\sqrt{-h}}\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta\theta\right)\right) \\
\partial_\alpha\chi &\to e^{\theta}\left(\partial_\alpha\chi + \chi\partial_\alpha\theta\right) .
\end{aligned} \tag{4.95}$$

As the authors of this paper approach the calculation of the part of the Noether current without using the equations of motion a bit differently, let us repeat that part. Inserting this Weyl transformation (4.95) into the action (4.94) yields

$$\begin{aligned}
S[h_{\mu\nu},\chi] = &-\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\left(\chi\partial_\alpha\chi\partial_\beta\theta + \chi\partial_\alpha\theta\partial_\beta\chi\right) + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) \\
&-\int \mathrm{d}^4x\left(\frac{1}{2}\chi^2\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta\theta\right)\right) ,
\end{aligned} \tag{4.96}$$

such that we just need to perform the partial integration on the integral in the second line to give us

$$\begin{aligned}
S[h_{\mu\nu},\chi] = &-\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\left(\chi\partial_\alpha\chi\partial_\beta\theta + \chi\partial_\alpha\theta\partial_\beta\chi\right) + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) \\
&-\int \mathrm{d}^4x\;\partial_\alpha\left(\frac{1}{2}\chi^2\sqrt{-h}h^{\alpha\beta}\partial_\beta\theta\right) + \int \mathrm{d}^4x\sqrt{-h}\;\chi\partial_\alpha\chi h^{\alpha\beta}\partial_\beta\theta \\
= &-\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) - \int \mathrm{d}^4x\;\partial_\alpha\left(\frac{1}{2}\chi^2\sqrt{-h}h^{\alpha\beta}\partial_\beta\theta\right) .
\end{aligned}$$

So we see that the theory is indeed Weyl invariant up to a boundary term where $\mathcal{L} \to \mathcal{L} + \partial_\alpha X^\alpha$ and we conclude that

$$\delta\mathcal{L} = \partial_\alpha X^\alpha \tag{4.97}$$

$$X^\alpha = \frac{1}{2}\chi^2\sqrt{-h}h^{\alpha\beta}\partial_\beta\theta\,. \tag{4.98}$$

Comparing this to the current in [146] we just re-derived their result by directly performing the Weyl transformation and looking for the boundary term.

For the calculation of the term using the equations of motion we directly repeat what was done in [146]. We need to consider the fields involved and then write the pure variation of the Lagrangian

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\chi}\delta\chi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\partial_\mu\delta\chi + \frac{\partial\mathcal{L}}{\partial h^{\alpha\beta}}\delta h^{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu h^{\alpha\beta})}\partial_\mu\delta h^{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\partial_\mu\partial_\nu\delta h^{\alpha\beta}\,. \tag{4.99}$$

Notice that this indeed generalises easily to the double derivative terms of the metric originating from the Ricci scalar. But as we need the part of the current *with* the equations of motion, we first write them down:

$$\frac{\partial\mathcal{L}}{\partial\chi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} \tag{4.100}$$

$$\frac{\partial\mathcal{L}}{\partial h^{\alpha\beta}} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu h^{\alpha\beta})} - \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\,. \tag{4.101}$$

Therefore, inserting and rearranging with the help of total derivatives yields

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{4.102}$$

$$K^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\delta\chi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu h^{\alpha\beta})}\delta h^{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\partial_\nu\delta h^{\alpha\beta} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\delta h^{\alpha\beta}\,. \tag{4.103}$$

Following [146], we quote their result

$$X^\mu = K^\mu \tag{4.104}$$

and therefore the total Noether current

$$J^\mu = K^\mu - X^\mu = 0\,. \tag{4.105}$$

In [149] it is also shown that for this part of the action, which is obtained via a Weyl transformation of the metric (see (1.74)), the Noether current for this Weyl symmetry always vanishes. The author argues that this can be applied to any gravitational theory which is locally conformally invariant. So it is further discussed that this Weyl symmetry has no further dynamical consequences and only gives an advantage during the calculation. One might also consider the conformally invariant theory with the scalar field as a more fundamental theory, as GR can be derived by gauge fixing the scalar field. The equivalence between different theories related by Weyl symmetries is established at least at the classical level. Therefore, the question remains whether this equivalence carries over if one considers quantum theories. In case a quantum system breaks a classical symmetry, one talks about an *anomaly* [112]. In [149] they consider whether such a conformal anomaly might exist in the case of unimodular gravity, which we have seen is similar to our model. They cite further work [164–168] that suggests that anomalies do not show up if the local conformal symmetry is broken spontaneously, i.e. the vacuum expectation value of the field should not vanish. But it remains to state that due to the trivial Noether current the dynamics of the system cannot be reduced and the phase space of the system cannot be separated [169].

Then we do the same for the part of the action coming from the mimetic constraint equation

$$S_\phi[h_{\mu\nu},\chi,\phi,\lambda] = -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{\lambda}{12}\chi^2 - \frac{\lambda}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right). \tag{4.106}$$

Keeping the above Weyl transformation (4.95) we immediately see that we have to amend it by adding

$$\begin{aligned} &\lambda \to e^{2\theta}\lambda\,, \quad \delta\lambda = 2\theta\lambda\,,\\ &\phi \to \phi\,, \quad \delta\phi = 0\,. \end{aligned} \tag{4.107}$$

This Weyl transformation is trivial, in the sense that we do not have to perform a partial integration in order to see this action is Weyl invariant. So we do *not* have a boundary term and the current $X^\alpha_{(\phi)} = 0$.

For the total Noether current $J^\alpha_{(\phi)} = K^\alpha_{(\phi)} - X^\alpha_{(\phi)}$ we still have to calculate $K^\alpha_{(\phi)}$ by using the equations of motion

$$\frac{\partial\mathcal{L}_\phi}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\phi)} \tag{4.108}$$

$$\frac{\partial\mathcal{L}_\phi}{\partial\chi} = \partial_\mu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\chi)} \tag{4.109}$$

$$\frac{\partial\mathcal{L}_\phi}{\partial\lambda} = \partial_\mu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\lambda)} \tag{4.110}$$

$$\frac{\partial\mathcal{L}_\phi}{\partial h^{\alpha\beta}} = \partial_\mu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu h^{\alpha\beta})} - \partial_\mu\partial_\nu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})} \tag{4.111}$$

which simplify the variation of $\mathcal{L}_\phi$ to

$$\delta\mathcal{L}_\phi = \partial_\mu K^\mu_{(\phi)} \tag{4.112}$$

with

$$\begin{aligned} K^\mu_{(\phi)} = &\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\chi)}\delta\chi + \frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\lambda)}\delta\lambda + \frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu h^{\alpha\beta})}\delta h^{\alpha\beta}\\ &+ \frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\partial_\nu\delta h^{\alpha\beta} - \partial_\nu\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\delta h^{\alpha\beta}\,. \end{aligned} \tag{4.113}$$

The calculation is absolutely analogous to above. Evaluating this is trivial: All terms vanish immediately, either by looking at the action or the Weyl transformations, such that $K^\mu_{(\phi)} = 0$ identically. Therefore, the Noether current for this part of the action is indeed

$$J^\mu_{(\phi)} = K^\mu_{(\phi)} - X^\mu_{(\phi)} \equiv 0\,. \tag{4.114}$$

4.4.2 The higher derivative term

Next, we want to discuss the term originating from $\frac{1}{2}\gamma(\Box_g\phi)^2$ cf. (4.39). We go the extra step by Weyl transforming from the physical metric $g_{\mu\nu}$ to the auxiliary metric $h_{\mu\nu}$ and the kinetic term X (1.55), defined w.r.t. the metric $h_{\mu\nu}$, such that (with the help of [2] and subsection 1.1.3)

$$\begin{aligned} g_{\mu\nu} &= 2Xh_{\mu\nu} \\ \sqrt{-g} &= (2X)^2\sqrt{-h} \\ \Box_g\phi &= \frac{1}{2X}\Box_h\phi + \frac{1}{2X^2}h^{\alpha\beta}\partial_\alpha X\partial_\beta\phi\,. \end{aligned} \tag{4.115}$$

Using the field redefinition

$$X = \frac{1}{12}\chi^2 \tag{4.116}$$

the $(\Box_g\phi)^2$-term of the action will be

$$S_\gamma[h_{\mu\nu},\phi,\chi] = \frac{1}{2}\gamma\int \mathrm{d}^4x\sqrt{-h}\left(\Box_h\phi + \frac{2}{\chi}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi\right)^2\,. \tag{4.117}$$

Following the same argumentation as above, the variation of the Lagrangian *without* using the equations of motion can be calculated. Use the Weyl transformations (4.95) and (4.107) and keep in mind that

$$\Box_h\phi \to e^{2\theta}\left(\Box_h\phi - 2h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\phi\right)\,. \tag{4.118}$$

Then we see

$$\begin{aligned} S_\gamma[h_{\mu\nu},\phi,\chi] &= \frac{1}{2}\gamma\int \mathrm{d}^4x\sqrt{-h}\left(\Box_h\phi - 2h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\phi + \frac{2}{\chi}h^{\alpha\beta}\partial_\alpha\phi\left(\partial_\beta\chi + \chi\partial_\beta\theta\right)\right)^2 \\ &= \frac{1}{2}\gamma\int \mathrm{d}^4x\sqrt{-h}\left(\Box_h\phi - 2h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\phi + \frac{2}{\chi}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi + 2h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\theta\right)^2 \\ &= \frac{1}{2}\gamma\int \mathrm{d}^4x\sqrt{-h}\left(\Box_h\phi + \frac{2}{\chi}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi\right)^2\,. \end{aligned} \tag{4.119}$$

So also this term is immediately Weyl invariant without a boundary term.

Therefore we just need to explicitly check the current with the equations of motion, such that

$$\delta\mathcal{L}_\gamma = \partial_\mu K^\mu_{(\gamma)} \tag{4.120}$$

and

$$K^\mu_{(\gamma)} = \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\chi)}\delta\chi + \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu h^{\alpha\beta})}\delta h^{\alpha\beta} + \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\partial_\nu\delta h^{\alpha\beta} - \partial_\nu\frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\partial_\nu h^{\alpha\beta})}\delta h^{\alpha\beta} \tag{4.121}$$

as above. Expressing the d'Alembertian $\Box_h\phi$ (1.35) in partial derivatives gives us

$$\Box_h\phi = -\frac{1}{2}h^{\mu\nu}\partial_\nu\phi h_{\alpha\beta}\partial_\mu h^{\alpha\beta} + \delta^\mu_\alpha\partial_\mu h^{\alpha\beta}\partial_\beta\phi + h^{\alpha\beta}\partial_\alpha\partial_\beta\phi\,. \tag{4.122}$$

Also, we have to keep in mind

$$h^{\alpha\beta}\partial_\mu h_{\alpha\beta} = -h_{\alpha\beta}\partial_\mu h^{\alpha\beta}\,. \tag{4.123}$$

Then we see that only the second and third term of $K^\mu_{(\gamma)}$ remain. The first one of them evaluates to

$$\begin{aligned}\frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\chi)} &= \gamma\sqrt{-h}\mathcal{B}\frac{2}{\chi}h^{\mu\nu}\partial_\nu\phi\\ \delta\chi &= \theta\chi\\ \Rightarrow \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu\chi)}\delta\chi &= 2\gamma\sqrt{-h}\theta\mathcal{B}h^{\mu\nu}\partial_\nu\phi\end{aligned} \tag{4.124}$$

whereas the second one is

$$\begin{aligned}\frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu h^{\alpha\beta})} &= \gamma\sqrt{-h}\mathcal{B}\left(-\frac{1}{2}h_{\alpha\beta}h^{\mu\nu}\partial_\nu\phi + \delta^\mu_\alpha\partial_\beta\phi\right)\\ \delta h^{\alpha\beta} &= 2\theta h^{\alpha\beta}\\ \Rightarrow \frac{\partial\mathcal{L}_\gamma}{\partial(\partial_\mu h^{\alpha\beta})}\delta h^{\alpha\beta} &= -2\gamma\sqrt{-h}\theta\mathcal{B}h^{\mu\nu}\partial_\nu\phi\,,\end{aligned} \tag{4.125}$$

always using that

$$\mathcal{B} = \left(\Box_h\phi + \frac{2}{\chi}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi\right)\,. \tag{4.126}$$

So we see that those two remaining terms exactly cancel and $K^\mu_{(\gamma)} = 0$ identically. Also in this case we conclude

$$J^\mu_{(\gamma)} = K^\mu_{(\gamma)} - X^\mu_{(\gamma)} \equiv 0\,. \tag{4.127}$$

4.4.3 Noether's second theorem in scalar mimetic gravity

Up to now, we have not yet applied Noether's second theorem to the varied actions occurring in mimetic theory. To understand the significance of this, we will simply start with the pure Einstein-Hilbert action (1.59) of the *physical* metric, i.e.

$$S_{\mathrm{EH}}[g_{\mu\nu}] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,R(g)\,. \tag{4.128}$$

By varying this action as usual w.r.t. the metric, we get the known Einstein equations of motion (1.58) (without external matter) [4]

$$\frac{\delta S_{\mathrm{EH}}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\left(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\right) \equiv -\frac{1}{2}\sqrt{-g}\,G_{\mu\nu}(g)\,. \tag{4.129}$$

To get to the purely gravitational action

$$S[h_{\mu\nu},\chi] = -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right)\,, \tag{4.130}$$

i.e. (4.94), the appropriate Weyl transformation is

$$g_{\mu\nu} = \frac{\chi^2}{6} h_{\mu\nu} \,. \tag{4.131}$$

This fact can be used to derive the equation of motion for $h_{\mu\nu}$ in a simple way: Just perform this exact Weyl transformation on $G_{\mu\nu}(g)$, yielding (with 1.1.3 and [147])

$$G_{\mu\nu}(g) = G_{\mu\nu}(h) + \frac{4}{\chi^2}\left(\partial_\mu\chi\partial_\nu\chi - \frac{1}{4}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) - \frac{2}{\chi}\left(\nabla_\mu\nabla_\nu\chi - h_{\mu\nu}\Box\chi\right) \tag{4.132}$$

with the Einstein tensor $G_{\mu\nu}(h)$ defined w.r.t. the auxiliary metric $h_{\mu\nu}$, as well as the covariant derivative ∇_μ. This is also valid for the rest of the chapter. Moreover we observe

$$\delta S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\, G_{\mu\nu}(g)\delta g^{\mu\nu} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-h}\left(\frac{\chi^2}{6}\right)^2 \left(G_{\mu\nu}(h) + \dots\right)\cdot\frac{6}{\chi^2}\delta h^{\mu\nu} \tag{4.133}$$

to get the factors of $\frac{\chi^2}{6}$ correct. The ellipsis is just there to simplify notation and signifies the omitted terms from eq. (4.132). Therefore, the equation of motion for the auxiliary metric is

$$\frac{\delta S}{\delta h^{\mu\nu}} = \sqrt{-h}\left(-\frac{1}{12}G_{\mu\nu}(h)\chi^2 - \frac{1}{3}\left(\partial_\mu\chi\partial_\nu\chi - \frac{1}{4}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) + \frac{\chi}{6}\left(\nabla_\mu\nabla_\nu\chi - h_{\mu\nu}\Box\chi\right)\right) \,. \tag{4.134}$$

The equation of motion for the scalar field is found by straightforward variation of the action (4.130), such that

$$\frac{\delta S}{\delta\chi} = \sqrt{-h}\left(-\frac{1}{6}R(h)\chi + \Box\chi\right) \,. \tag{4.135}$$

Now that we have derived the equations of motion, we will apply Noether's second theorem. Considering the necessary Weyl transformations and the definition of small variations of the fields under their influences (3.20), we have

$$\begin{aligned}
&\Delta p^{(\alpha)} \equiv \theta \qquad \text{and therefore} \\
&\quad h^{\mu\nu} \to e^{2\theta}h^{\mu\nu} \quad \Rightarrow \quad \delta h^{\mu\nu} = 2\theta h^{\mu\nu} \quad \Rightarrow \quad a_h^{\mu\nu} = 2h^{\mu\nu},\ \ b_h^{\alpha\,\mu\nu} = 0 \\
&\quad\quad \chi \to e^{\theta}\chi \quad \Rightarrow \quad \delta\chi = \theta\chi \quad \Rightarrow \quad a_\chi = \chi\,,\ \ b_\chi^\alpha = 0\,.
\end{aligned} \tag{4.136}$$

Therefore, Noether's second theorem for this theory looks like

$$\begin{aligned}
\frac{\delta S}{\delta\chi}\delta\chi + \frac{\delta S}{\delta h^{\mu\nu}}\delta h^{\mu\nu} &= \frac{\delta S}{\delta\chi}\cdot\chi\theta + \frac{\delta S}{\delta h^{\mu\nu}}\cdot 2\theta h^{\mu\nu} \\
&= \sqrt{-h}\left(-\frac{1}{6}R(h)\chi^2 + \chi\Box\chi\right)\theta \\
&\quad + \sqrt{-h}\left(-\frac{1}{6}G(h)\chi^2 + \frac{\chi}{3}\left(h^{\mu\nu}\nabla_\mu\nabla_\nu\chi - 4\Box\chi\right)\right)\theta \\
&\equiv 0
\end{aligned} \tag{4.137}$$

using $R(h) = -G(h)$, so here Noether's second theorem does not give us any new information.

After this preparation for pure Einstein-Hilbert gravity we will discuss the remaining terms in scalar mimetic gravity. At first we will look at the action with the Lagrange multiplier λ. As we have already done the calculations for the Weyl transformed Einstein-Hilbert term (4.130) and found that Noether's second theorem amounts to zero, we will concentrate on the remaining terms of the action including the Lagrange multiplier (4.106). Using the Weyl transformations (4.95) and (4.107) the second Noether theorem for these terms has the form

$$\left(\frac{\delta S}{\delta h^{\mu\nu}} + \frac{\delta S_\phi}{\delta h^{\mu\nu}}\right)\delta h^{\mu\nu} + \left(\frac{\delta S}{\delta\chi} + \frac{\delta S_\phi}{\delta\chi}\right)\delta\chi + \frac{\delta S_\phi}{\delta\lambda}\delta\lambda + \frac{\delta S_\phi}{\delta\phi}\delta\phi = 0\,. \tag{4.138}$$

The variations of the action (4.106) for the metric $h^{\mu\nu}$ and the scalar fields χ and λ can be calculated as

$$\frac{\delta S_\phi}{\delta h^{\mu\nu}} = \sqrt{-h}\left(\frac{\lambda}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}h_{\mu\nu}\left(\frac{\lambda}{12}\chi^2 - \frac{\lambda}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\right) \tag{4.139}$$

$$\frac{\delta S_\phi}{\delta\chi} = -\sqrt{-h}\,\frac{\lambda}{6}\chi \tag{4.140}$$

$$\frac{\delta S_\phi}{\delta\lambda} = -\sqrt{-h}\left(\frac{1}{12}\chi^2 - \frac{1}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) \tag{4.141}$$

As $\delta\phi = 0$, only the first three terms of (4.138) remain, which result in

$$-\sqrt{-h}\left(\frac{1}{12}\chi^2 - \frac{1}{2}h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\cdot 2\theta\lambda = 0\,. \tag{4.142}$$

But this only means that

$$\frac{\chi^2}{6} = h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,, \tag{4.143}$$

which reflects the constraint expressed via the Lagrange multiplier λ, see (4.141). This is understandable, as in the original construction of the scalar mimetic theory [132] the mimetic constraint equation is valid by construction and does not need the equations of motion. Therefore, also in the reformulation via a Lagrange multiplier this constraint can be seen as valid by construction. Moreover, as discussed in [151], Noether's second theorem is often found to reproduce already known or somewhat trivial results. The authors of this paper also mention the possibility that the transformations of only the gauge fields depend on $\partial_\mu\left(p^{(\alpha)}(x)\right)$, where the $p^{(\alpha)}(x)$ are the arbitrary functions of the gauge transformations, in our calculations named θ (compare (3.20)). They argue that local gauge symmetry *and* the equations of motion for the gauge fields leads to a conserved current. The matter field equations in this derivation of the current are not a necessary, although a sufficient condition. In the set of Weyl transformations we use ((4.95) and (4.107)) there is no dependence on $\partial_\mu\theta$ to be found and as a consequence, this possibility does not apply to our calculation. Nevertheless, we stress that this result from Noether's second theorem discussed in [151] are worth mentioning.

If we view χ not as a new field constrained with the help of a Lagrange multiplier, but only as a shorthand for

$$\chi = \sqrt{6h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}\,, \tag{4.144}$$

the action S_{HD} expressed in higher derivatives of ϕ looks just like (4.130), but its variables are actually $h^{\mu\nu}$ and ϕ. As ϕ does not transform under the Weyl transformation of the auxiliary metric, Noether's second theorem takes the form of

$$\frac{\delta S_{\mathrm{HD}}}{\delta h^{\mu\nu}} \cdot 2\theta h^{\mu\nu} = 0\,. \tag{4.145}$$

This can be reformulated as

$$\frac{\delta S_{\mathrm{HD}}}{\delta h^{\mu\nu}}\delta h^{\mu\nu} = \left(\left.\frac{\delta S}{\delta h^{\mu\nu}}\right|_{\chi=\mathrm{const}} + \frac{\delta S}{\delta\chi}\frac{\delta\chi}{\delta h^{\mu\nu}} \right) \cdot 2\theta h^{\mu\nu} = 0\,, \tag{4.146}$$

where in the first term χ is taken to be constant. To simplify our calculation, we can take $\frac{\delta S}{\delta h^{\mu\nu}}$ (4.134) and $\frac{\delta S}{\delta\chi}$ (4.135) from above. Additionally, it turns out that

$$\frac{\delta\chi}{\delta h^{\mu\nu}} = \frac{3}{\chi}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \qquad \text{or} \qquad \frac{\delta\chi}{\delta h^{\mu\nu}}h^{\mu\nu} = \frac{1}{2}\chi\,. \tag{4.147}$$

In total,

$$\frac{\delta S_{\mathrm{HD}}}{\delta h^{\mu\nu}} \cdot 2\theta h^{\mu\nu} = 2\theta\sqrt{-h}\left[-\frac{1}{12}G\chi^2 - \frac{1}{2}\chi\Box\chi + \left(-\frac{1}{6}R\chi + \Box\chi \right) \cdot \frac{1}{2}\chi \right] = 0\,, \tag{4.148}$$

once more using that $G = -R$, which can be seen from the definition of the Einstein tensor (1.29). Consequently, Noether's second theorem does not give us any new information in the case of S_{HD}, where no constraint via a Lagrange multiplier is used in the action.

Chapter 5

Vector Mimetic Gravity

Mimetic theory has also been shown to be able to produce dark energy, be it in the form of quintessence or a cosmological constant. For example, one of the earliest constructions was to take the bare scalar field version of mimetic matter and add a suitable potential term for it, as was done in [135, 138] and in this thesis summarised in 4.2.1.

Another possibility to model dark energy with the mimetic formalism was discussed in [135, 161, 170], making use of the higher derivative terms, compare 4.2.2 and 4.2.3. It is compatible with the observational results of the gravitational wave event GW170817 [171].

So in this chapter we focus on one model which was detailed in [133]. (Note that this ansatz has also been introduced in [172].) The authors of [133] devised a vector-tensor theory with higher derivatives. The novel vector field V^μ in the theory has the unusual conformal weight of four and produces one new global degree of freedom, i.e. a constant of integration. In the case of this theory, as we will see, it is not dark matter it mimics, but a cosmological constant, the aforementioned new degree of freedom. The aim of this mimetic construction was to find a mimetic theory with a vector field additional to the usual metric. Moreover, it was found to reproduce unimodular gravity [122] (see also (1.176)) in the formulation by Henneaux and Teitelboim [128] (1.187), which also contains the divergence of a vector field.

This chapter also includes the proof that the Weyl transformation of the theory is the only one possible to be consistent with the described mimetic construction 2, namely in 5.3. Moreover, also the Noether theorems are explicitly used and their currents calculated 5.4.

5.1 Constraint and equations of motion

For the general introduction to this theory we will use [133], so if not mentioned otherwise, this reference will be the source.

The Weyl transformation which is the foundation of this mimetic theory is

$$g_{\mu\nu} = \sqrt{\nabla^{h)}_\rho V^\rho}\, h_{\mu\nu}\,, \tag{5.1}$$

where the covariant derivative $\nabla^{h)}_\rho$ is metric compatible, see (1.12), but with the auxiliary metric $h_{\mu\nu}$. Moreover, under a Weyl transformation of the auxiliary metric

$$h_{\mu\nu} = \Omega^2(x) h'_{\mu\nu} \tag{5.2}$$

with an arbitrary function $\Omega^2(x)$ of the coordinates, while also transforming the vector field as

$$V^\mu = \Omega^{-4}(x) V'^\mu \,, \tag{5.3}$$

the physical metric $g_{\mu\nu}$ will be unchanged. As we can see, comparing with its definition (1.71), the conformal weight of this vector field is four. Then the new theory is implemented by performing the Weyl transformation of the metric in the usual seed action of GR, the Einstein-Hilbert action (1.59), such that

$$S_{\text{vector}}[h_{\mu\nu}, V^\mu, \Phi_{\text{m}}] = S_{\text{seed}}[g_{\mu\nu}(h_{\mu\nu}, V^\mu)\,, \Phi_{\text{m}}] \tag{5.4}$$

with the usual matter fields Φ_{m}. There also exists another gauge invariance

$$V_\mu = V'_\mu + \partial_\mu \theta \qquad \text{with} \qquad \Box\theta = 0\,. \tag{5.5}$$

This is similar to an *unfree gauge symmetry* as presented in [173], where the gauge transformations leave the action unchanged, but only if the gauge parameters are constrained by partial differential equations. The setup in our case is similar to the Lorenz gauge of electrodynamics, where you can also define such a redundant scalar quantity like θ [174].

Performing this Weyl transformation (5.1) explicitly leads to the action (omitting the matter action)

$$S_{\text{vector}}[h_{\mu\nu}, V^\mu] = -\frac{1}{2}\int \mathrm{d}^4x \sqrt{-h}\left[\left(\nabla^{h)}_\alpha V^\alpha\right)^{1/2} R(h) + \frac{3}{8}\cdot\frac{\left(\nabla^{h)}_\mu \nabla^{h)}_\alpha V^\alpha\right)^2}{\left(\nabla^{h)}_\beta V^\beta\right)^{3/2}}\right]\,. \tag{5.6}$$

The equations for motion for this action can be written as

$$\frac{1}{\sqrt{-h}}\frac{\delta S_{\text{vector}}}{\delta V^\mu} = \frac{1}{4}\partial_\mu (T - G) = 0 \tag{5.7}$$

for the vector field and for the Einstein field equations

$$\frac{1}{\sqrt{-h}}\frac{\delta S_{\text{vector}}}{\delta h^{\alpha\beta}} = \frac{\sqrt{\nabla^{h)}_\mu V^\mu}}{2}\left[T_{\alpha\beta} - G_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta}\left(T - G - \frac{V^\nu \partial_\nu (T-G)}{\nabla^{h)}_\mu V^\mu}\right)\right] = 0\,. \tag{5.8}$$

The traces of Einstein tensor (1.29) and energy-momentum tensor (1.38) w.r.t the physical metric $g_{\mu\nu}$ are again denoted by G and T, respectively. Inserting the equation of motion for the vector field into the Einstein equations will result in the tracefree Einstein equations

$$G_{\alpha\beta} - T_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta}(G - T) = 0\,. \tag{5.9}$$

If we now define a cosmological constant Λ as in (1.172) by the e.o.m. for the vector field, we see that Λ indeed emerges as an integration variable, see also the discussion on trace-free Einstein gravity in 1.2.4. As a consequence, the mimetic Weyl transformation provides us with *Mimetic Dark Energy* [133].

One can also define a vector field as

$$W^{\mu} = \frac{V^{\mu}}{\nabla^{h)}_{\lambda} V^{\lambda}}\,, \tag{5.10}$$

such that the new vector field W^{μ} is Weyl invariant. One can then show that the action in the new variables looks like

$$S_{\text{vector}}[g_{\mu\nu}, W^{\mu}, \Lambda, \Phi_{\text{m}}] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}R(g) + \Lambda\left(\nabla^{g)}_{\mu} W^{\mu} - 1\right)\right] + S_{\text{m}}[g_{\mu\nu}, \Phi_{\text{m}}]\,, \tag{5.11}$$

with matter fields Φ_{m} and matter action S_{m}. This is exactly the Henneaux-Teitelboim representation of unimodular gravity [128], also written in a covariant form and now with manifestly gauge invariant variables. But again, let us stress that this vector field V^{μ} which was introduced here is not a U(1) gauge potential, but a vector field of conformal weight four. What one can also easily see from the action is the presence of a constraint term, namely

$$\nabla^{g)}_{\mu} W^{\mu} = 1\,, \tag{5.12}$$

equivalent to the already known mimetic constraint equation (2.3) in the original scalar-tensor theory. This constraint term can also be directly derived using the definition of the divergence of a vector field (1.36). We write

$$\nabla^{g)}_{\mu} W^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}\frac{V^{\mu}}{\nabla^{h)}_{\lambda} V^{\lambda}}\right) \tag{5.13}$$

while also using that

$$\sqrt{-g} = \sqrt{-h}\,(\nabla^{h)}_{\alpha} V^{\alpha}) \tag{5.14}$$

from the Weyl transformation (5.1) by which this theory is obtained. Therefore,

$$\nabla^{g)}_{\mu} W^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-h}\,(\nabla^{h)}_{\alpha} V^{\alpha})\frac{V^{\mu}}{\nabla^{h)}_{\lambda} V^{\lambda}}\right) = \frac{1}{\nabla^{h)}_{\alpha} V^{\alpha}\sqrt{-h}}\partial_{\mu}\left(\sqrt{-h}V^{\mu}\right) \equiv 1\,, \tag{5.15}$$

which proves the point.

5.2 Gauge invariant representation

Also for the mimetic theory with a vector field V^{μ} of weight four [133], we repeat the relevant Weyl transformation, namely (5.1)

$$g_{\mu\nu} = \sqrt{\nabla^{h)}_{\rho} V^{\rho}}\, h_{\mu\nu}\,. \tag{5.16}$$

At first, we need the definition of the scalar field

$$\mathscr{D} = \nabla_\rho^{h)} V^\rho \,, \tag{5.17}$$

such that after its promotion to a dynamic variable, the Weyl transformed action will be

$$S_{\text{vector}}[h_{\mu\nu}, \mathscr{D}, V^\alpha, \lambda] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-h}\left[\sqrt{\mathscr{D}}R(h) + \frac{3}{8}\frac{h^{\alpha\beta}\partial_\alpha\mathscr{D}\partial_\beta\mathscr{D}}{\mathscr{D}^{3/2}} + \lambda\left(\mathscr{D} - \nabla_\rho^{h)}V^\rho\right)\right] \,, \tag{5.18}$$

again with a Lagrange multiplier λ. Even another representation of this action can be written down, if one introduces another scalar field χ, this time of conformal weight one, namely via

$$\mathscr{D} = \left(\frac{\chi^2}{6}\right)^2 \,, \tag{5.19}$$

therefore yielding

$$S_{\text{vector}}[h_{\mu\nu}, \chi, V^\alpha, \lambda] = \int \mathrm{d}^4x\sqrt{-h}\left[-\frac{1}{2}(\partial\chi)^2 - \frac{1}{12}\chi^2 R(h) - \frac{\lambda}{72}\chi^4 + \frac{\lambda}{2}\nabla_\rho^{h)}V^\rho\right] \,. \tag{5.20}$$

The first three terms in this action corresponds to Dirac's theory of Weyl invariant gravity [175]. Furthermore, it is worth stressing that the scalar field χ has a kinetic term of the wrong sign, therefore is a ghost. The variables present in the action (5.18) change under a Weyl transformation as

$$h_{\mu\nu} = \Omega^2(x)h'_{\mu\nu} \,, \tag{5.21}$$

$$\mathscr{D} = \Omega^{-4}(x)\mathscr{D}' \,, \tag{5.22}$$

$$V^\mu = \Omega^{-4}(x)V'^\mu \,, \tag{5.23}$$

$$\lambda = \lambda' \,. \tag{5.24}$$

Then a set of gauge invariant variables can be found, which are

$$\begin{aligned} g_{\mu\nu} &= \mathscr{D}^{1/2}h_{\mu\nu} \,, \\ W^\mu &= \mathscr{D}^{-1}V^\mu \,, \\ \Lambda &= \frac{\lambda}{2} \,. \end{aligned} \tag{5.25}$$

Inserting this into (5.18) will result in

$$S_{\text{vector}}[g_{\mu\nu}, W^\alpha, \Lambda] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}R(g) + \Lambda\left(\nabla_\rho^{g)}W^\rho - 1\right)\right] \,, \tag{5.26}$$

being the Henneaux-Teitelboim form of unimodular gravity [128], as already discussed. Moreover, here the matter is again minimally coupled to the metric $g_{\mu\nu}$. Also here we can clearly see that Λ is a Lagrange multiplier, enforcing a constraint on the divergence of the vector field W^μ we have already encountered in (5.12).

The resulting equations of motion are

$$\frac{1}{\sqrt{-g}} \cdot \frac{\delta S_{\text{vector}}}{\delta W^\mu} = -\partial_\mu \Lambda = 0\,, \tag{5.27}$$

setting up the Lagrange multiplier as a (cosmological) constant, whereas

$$\frac{2}{\sqrt{-g}} \cdot \frac{\delta S_{\text{vector}}}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Lambda g_{\mu\nu} - G_{\mu\nu} = 0 \tag{5.28}$$

are the Einstein field equations in this context. And finally, we have the e.o.m. for the Lagrange multiplier

$$\frac{1}{\sqrt{-g}} \cdot \frac{\delta S_{\text{vector}}}{\delta \Lambda} = \nabla_\mu^{g)} W^\mu - 1 = 0\,, \tag{5.29}$$

giving us back the constraint equation (5.12).

Also note that the equation of motion 5.29 or equivalently the constraint (5.12) can also be viewed as a current non-conservation equation for the current W^μ [133]. This leads to a global charge

$$\mathscr{T}(t) = \int \mathrm{d}^3\boldsymbol{x} \sqrt{-g}\, W^t(t, \boldsymbol{x}) \tag{5.30}$$

after an appropriate foliation of the spacetime. This global mode $\mathscr{T}$ of the charge is often called *cosmic time*. One can calculate

$$\begin{aligned} \dot{\mathscr{T}}(t) &= \int \mathrm{d}^3\boldsymbol{x}\, \partial_t \left(\sqrt{-g} W^t(t, \boldsymbol{x})\right) = \int \mathrm{d}^3\boldsymbol{x} \left(\sqrt{-g} - \partial_i \left(\sqrt{-g} W^i\right)\right) \\ &= \int \mathrm{d}^3\boldsymbol{x} \sqrt{-g} - \oint_{\mathcal{B}} \mathrm{d}s_i \sqrt{-g}\, W^i \end{aligned}$$

with the last integral in the second line taken over the boundary surface $\mathcal{B}$ of three-dimensional space. In case that there is no flux going through that boundary, one can write

$$\mathscr{T}(t_2) - \mathscr{T}(t_1) = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \sqrt{-g}\,, \tag{5.31}$$

representing the invariant spacetime volume between times t_1 and t_2 [128, 133]. Furthermore, this cosmic time is a global degree of freedom present in the theory, additional to the two graviton polarisations of standard GR. It is also canonically conjugated to the cosmological constant. Moreover, $\mathscr{T}$ can be shifted by a constant

$$\mathscr{T}(t) = \mathscr{T}'(t) + c\,, \tag{5.32}$$

resulting from the gauge transformations (5.5) of the vector field V^μ. As a consequence of this shift symmetry of a coordinate, the corresponding canonical momentum, here Λ, is a constant [134].

5.3 Mimetic construction of the vector field theory

After having discussed the mimetic construction in 2, we will now confirm that it is indeed the singular Jacobian determinant of the appropriate Weyl transformation (5.1) which is responsible for introducing a new degree of freedom into the theory. Also, it is only the exact functional form of the appearing derivatives of the vector field leading to a new mimetic theory. To see this, we will calculate the appropriate derivative conformal coupling in the case with the vector divergence term from [133] we will abbreviate as

$$\mathscr{D} = \nabla^{h)}_{\rho} V^{\rho} \tag{5.33}$$

with the vector field V^{ρ} of conformal weight four and the covariant derivative $\nabla^{h)}_{\rho}$ being metric compatible with the auxiliary metric $h_{\mu\nu}$. As a consequence we start with the following Weyl transformation, a generalisation of (5.1),

$$g_{\mu\nu} = C\left(\nabla^{h)}_{\rho} V^{\rho}\right) h_{\mu\nu} = C(\mathscr{D}) h_{\mu\nu}\,, \tag{5.34}$$

$C(\mathscr{D})$ being a general function of $\mathscr{D}$, which we now want to determine by demanding that the Jacobian determinant of this transformation is singular. The Jacobi matrix then looks like

$$\frac{\partial g_{\mu\nu}}{\partial h_{\alpha\beta}} = C\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + h_{\mu\nu}\frac{\partial C}{\partial \mathscr{D}}\frac{\partial \mathscr{D}}{\partial h_{\alpha\beta}} \tag{5.35}$$

where we need to pay attention to the covariant derivative and also the derivatives of the metric contained in it. To find $\frac{\partial \psi}{\partial h_{\alpha\beta}}$, we use formula (3.4) of [133], for reference

$$\delta\mathscr{D} = \nabla^{h)}_{\alpha}\delta V^{\alpha} + \frac{1}{2}h^{\alpha\beta}V^{\lambda}\nabla^{h)}_{\lambda}\delta h_{\alpha\beta}\,. \tag{5.36}$$

As we just need the variation with respect to the metric, we have

$$\delta\mathscr{D} = \frac{1}{2}h^{\alpha\beta}V^{\lambda}\nabla^{h)}_{\lambda}\delta h_{\alpha\beta} \tag{5.37}$$

$$= \nabla^{h)}_{\lambda}\left(\frac{1}{2}h^{\alpha\beta}V^{\lambda}\delta h_{\alpha\beta}\right) - \frac{1}{2}h^{\alpha\beta}\nabla^{h)}_{\lambda}V^{\lambda}\delta h_{\alpha\beta}\,. \tag{5.38}$$

The first term vanishes, as $h^{\alpha\beta}\delta h_{\alpha\beta} = \frac{1}{2}\delta\left(h^{\alpha\beta}h_{\alpha\beta}\right) = 0$ and therefore

$$\frac{\partial \mathscr{D}}{\partial h_{\alpha\beta}} = -\frac{1}{2}h^{\alpha\beta}\mathscr{D}\,. \tag{5.39}$$

Inserting that in the Jacobi matrix (5.35), we have

$$\frac{\partial g_{\mu\nu}}{\partial h_{\alpha\beta}} = C\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \frac{1}{2}h_{\mu\nu}\frac{\partial C}{\partial \mathscr{D}}h^{\alpha\beta}\mathscr{D}\,. \tag{5.40}$$

To evaluate the eigenvalue equation, we need to find the kinetic eigenvalue λ_* (2.79), as discussed there. We use that $\xi^*_{\alpha\beta} \propto h_{\alpha\beta}$ for convenience, as we will set the kinetic eigenvalue equation to zero anyways. Therefore we can simply multiply (5.40) by $h_{\alpha\beta}$ and extract the kinetic eigenvalue

$$\lambda_* = C - 2\mathscr{D}\frac{\partial C}{\partial \mathscr{D}}\,. \tag{5.41}$$

As discussed, the Weyl transformation (5.34) is singular in the case that this kinetic eigenvalue vanishes, which gives us a simple differential equation for the unknown function $C(\mathscr{D})$. The solution is the expected

$$C(\mathscr{D}) = \sqrt{\mathscr{D}} = \sqrt{\nabla^{h)}_\alpha V^\alpha}\,, \tag{5.42}$$

cf. (5.1), as stated in [133], finally proving that this indeed provides us with a new degree of freedom as per the mimetic construction.

5.3.1 Mimetic theory emerging through an algebraic solution

This construction on how to get a new degree of freedom out of Weyl transforming a metric is not the whole story, though. As the authors in [143] have argued, solving (5.41) set to zero as a differential equation is not the only way for the kinetic eigenvalue to vanish. One can also view these equations as algebraic equations and solve for certain values of D. The result will also be a mimetic theory of gravity. In general, we are looking for any solution of (2.79)

$$C = 2\mathscr{D}\frac{\partial C}{\partial \mathscr{D}}\,, \tag{5.43}$$

which we will call $\mathscr{D}^*$. Then, equivalent to the discussion of the case of mimetic matter with a scalar field in [143], the constraint equation for the vector field V^μ will be

$$\nabla^{h)}_\mu V^\mu = \mathscr{D}^*\,. \tag{5.44}$$

This leads to equation (5.43) having more than one solution.

In the following, we show concrete examples how this algebraic equation (5.43) could be solved. We can make an ansatz for the function C, such that

$$C(\mathscr{D}) = e^{a\mathscr{D}} \tag{5.45}$$

with a general constant a. Then, inserting this ansatz into (5.43) will give us

$$\mathscr{D}^* = \frac{1}{2a}\,. \tag{5.46}$$

An ansatz for a general polynomial of degree n looks like

$$C(\mathscr{D}) = a_0 + a_1\mathscr{D} + a_2\mathscr{D}^2 + \cdots + a_n\mathscr{D}^n\,, \tag{5.47}$$

with constants a_i, no Einstein summation convention intended. The equation (5.43) with this ansatz will again give us a polynomial of degree n, namely

$$a_0 - a_1\mathscr{D} - 3a_2\mathscr{D}^2 + \cdots + (1-2n)a_n\mathscr{D}^n = 0\,. \tag{5.48}$$

According to the fundamental theorem of algebra this polynomial has n complex roots, if they are counted with the right multiplicity, but at least one [176]. Moreover, a polynomial of odd degree has at least one real root. Whether there exist more real roots can be checked with Sturm's theorem, for example, but this should suffice to say that in most cases there are solutions of this equation to be found [177].

There can also be found examples which do not admit (real) solutions, e.g.

$$C(\mathscr{D}) = \sqrt{1 - a^2\mathscr{D}^2}\,, \tag{5.49}$$

as the solution of (5.43) results in $1 + a^2\mathscr{D}^2 = 0$, which does not have any solutions for a, $\mathscr{D} \in \mathbb{R}$, not even for a, $\mathscr{D} \in]0,1[$, when $C(\mathscr{D})$ is real. But in the end, it is important that the differential equation (5.43) does admit any solutions at all, be it general ones by directly solving the differential equation, or specific ones by viewing it as an algebraic equation. All of those examples serve to illustrate that there are many more ways to produce mimetic theories than previously thought, like in [142].

5.4 Noether's theorems and the vector field term

Now we will also investigate the question of the occurring Noether currents in the theory with a vector field term from [133], with the basics discussed in 3.

5.4.1 First theorem

We start with the action with scalar field and vector field, cf. (5.20),

$$\begin{aligned} S_V[h_{\mu\nu}, \chi, V^\rho, \Lambda] &= \int \mathrm{d}^4x\sqrt{-h}\left(-\frac{\Lambda}{72}\chi^4 + \frac{\Lambda}{2}\nabla_\alpha V^\alpha\right) \\ &= \int \mathrm{d}^4x\sqrt{-h}\left(-\frac{\Lambda}{72}\chi^4 + \frac{\Lambda}{2}\partial_\alpha V^\alpha - \frac{\Lambda}{4}h_{\mu\nu}V^\alpha\partial_\alpha h^{\mu\nu}\right) \end{aligned} \tag{5.50}$$

with Lagrange multiplier Λ and a vector field V^α with conformal weight four. Additionally to the Weyl transformations (4.95) and (4.107), we can observe that (again following subsection 1.1.3)

$$\begin{aligned} \Lambda &\to \Lambda \\ V^\alpha &\to e^{4\theta}V^\alpha\,, \quad \delta V^\alpha = 4\theta V^\alpha \\ \nabla_\alpha V^\alpha &= \frac{1}{\sqrt{-h}}\partial_\alpha\left(\sqrt{-h}V^\alpha\right) \\ &\to \frac{1}{e^{-4\theta}\sqrt{-h}}\partial_\alpha\left(e^{-4\theta}\sqrt{-h}e^{4\theta}V^\alpha\right) \\ &= e^{4\theta}\nabla_\alpha V^\alpha\,. \end{aligned} \tag{5.51}$$

Inserting all this in (5.50), it is trivial to see that we again do not have a boundary term, so $X^\alpha_{(V)} = 0$.

And as usual, the current involving the equations of motion condenses to

$$K^{\alpha}_{(V)} = \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha V^\mu)} \delta V^\mu + \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha h^{\mu\nu})} \delta h^{\mu\nu} \,. \tag{5.52}$$

This can be easily calculated as

$$\begin{aligned} \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha V^\mu)} &= \sqrt{-h} \cdot \frac{1}{2} \delta^\alpha_\mu \,, \quad \delta V^\alpha = 4\theta V^\alpha \\ \Rightarrow \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha V^\mu)} \delta V^\mu &= 2\sqrt{-h}\theta V^\alpha \end{aligned} \tag{5.53}$$

for the first term and

$$\begin{aligned} \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha h^{\mu\nu})} &= -\sqrt{-h} \cdot \frac{1}{4} h_{\mu\nu} V^\alpha \,, \quad \delta h^{\mu\nu} = 2\theta h^{\mu\nu} \\ \Rightarrow \frac{\partial \mathcal{L}_V}{\partial(\partial_\alpha h^{\mu\nu})} \delta h^{\mu\nu} &= -2\sqrt{-h}\theta V^\alpha \end{aligned} \tag{5.54}$$

for the second term, so that we see that they exactly cancel. And again the Noether current for this Weyl symmetry is zero

$$J^\mu_{(V)} = K^\mu_{(V)} - X^\mu_{(V)} \equiv 0 \,. \tag{5.55}$$

5.4.2 Second theorem

We will have another look at the action (5.20), i.e. the action from [133]

$$\begin{aligned} S_{V,\,\mathrm{total}}[h_{\mu\nu}, \chi, V^\rho, \Lambda] &= S[h_{\mu\nu}, \chi] + S_V[h_{\mu\nu}, \chi, V^\rho, \Lambda] \\ &= -\int \mathrm{d}^4 x \sqrt{-h} \left(\frac{1}{12} R(h) \chi^2 + \frac{1}{2} h^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi \right) \\ &\quad - \int \mathrm{d}^4 x \sqrt{-h} \left(\frac{\Lambda}{72} \chi^4 - \frac{\Lambda}{2} \nabla_\alpha V^\alpha \right) \,, \end{aligned} \tag{5.56}$$

recalling the definition of the action $S[h_{\mu\nu}, \chi]$ (4.130) and the fact that Noether's second theorem is trivial for this part of the action, see (4.137). Noether's second theorem for the whole action $S_{V,\,\mathrm{total}}[h_{\mu\nu}, \chi, V^\rho, \Lambda]$ looks like

$$\left(\frac{\delta S}{\delta \chi} + \frac{\delta S_V}{\delta \chi} \right) \cdot \chi\theta + \frac{\delta S_V}{\delta V^\alpha} \cdot 4 V^\alpha \theta + \left(\frac{\delta S}{\delta h^{\mu\nu}} + \frac{\delta S_V}{\delta h^{\mu\nu}} \right) \cdot 2 h^{\mu\nu} \theta = 0 \,, \tag{5.57}$$

with the Weyl transformations (4.95) and (5.51) as appropriate. Note that the Lagrange multiplier Λ is invariant under the Weyl transformations and does not enter. Again, we use for $\frac{\delta S}{\delta h^{\mu\nu}}$ (4.134) and for $\frac{\delta S}{\delta \chi}$ (4.135). The not yet calculated parts of the equations of motion are

$$\frac{\delta S_V}{\delta h^{\mu\nu}} = \sqrt{-h}\left(\frac{\Lambda}{144}\chi^4 + \frac{1}{4}V^\alpha \partial_\alpha \Lambda\right) h_{\mu\nu}\,, \tag{5.58}$$

$$\frac{\delta S_V}{\delta \chi} = -\sqrt{-h}\frac{\Lambda}{18}\chi^3\,, \tag{5.59}$$

$$\frac{\delta S_V}{\delta V^\alpha} = -\frac{1}{2}\sqrt{-h}\partial_\alpha \Lambda\,. \tag{5.60}$$

Evaluating Noether's second theorem (5.57) results in all terms vanishing. This happens in contrast to the case in the scalar mimetic theory, see (4.142). The reason for this is that for the scalar theory, the Lagrange multiplier λ enters into Noether's second theorem, as it transforms under Weyl transformations of the auxiliary metric. Therefore, Noether's second theorem gives us back the constraint equation. Λ, the Lagrange multiplier in the vector mimetic theory, is invariant under Weyl transformations and does not even enter Noether's second theorem.

Chapter 6

On the Strong CP Problem, QCD and Axions

In the following chapter 7 we will discuss another form of mimetic theory [134], namely with a gauge vector field. As we will see, the cosmological constant will turn up as an integration constant again, but this time with a crucial axion-like coupling. This is similar to the following concept in quantum chromodynamics, as Wilczek proposed [178]: "I would like to briefly mention one idea in this regard, that I am now exploring. It is to do something for the Λ-parameter very similar to what the axion does for the θ-parameter in QCD, another otherwise mysteriously tiny quantity. The basic idea is to promote these parameters to dynamical variables, and then see if perhaps small values will be chosen dynamically." So maybe this is a step in the right direction to resolve the question why the cosmological constant is so small. Therefore, we will give a short exposition on theoretical concepts of group theory, quantum chromodynamics (QCD) and axions we will need later on.

At first, we will need to concentrate on a short introduction to group theory, as this is crucial to understand the structure of non-abelian gauge theories. We will focus on the concepts we need, mainly Lie groups and especially the special unitary group of N dimensions $SU(N)$.

6.1 A short introduction to group theory

The symmetries of laws of physics can be described by *group theory* and *representation theory* of groups, especially those in quantum theories, particle physics and high energy theory. And as we are going to borrow many concepts from quantum field theory, especially quantum chromodynamics, we will give a short introduction to the ones relevant to this work. This part is mostly summarized from [10, 179].

6.1.1 Lie groups and algebras

First and foremost, a *group* G consists of group elements $\{g_\alpha\}$ which can be composed together to form another element, see $g_\gamma = g_\alpha \cdot g_\beta$. This g_γ will also be an element of the group. Note that the order of the group elements matters. In the case that the group elements commute, the group is called *Abelian*. This composition has to satisfy [10]

1. *associativity*, i.e. $(g_\alpha \cdot g_\beta) \cdot g_\gamma = g_\alpha \cdot (g_\beta \cdot g_\gamma)$,
2. existence of an *identity* e, such that $g_\alpha \cdot e = e \cdot g_\alpha = g_\alpha$,
3. existence of an *inverse* g_α^{-1} to every single element g_α, such that $g_\alpha \cdot g_\alpha^{-1} = g_\alpha^{-1} \cdot g_\alpha = e$.

Another important concept is that of a *representation* D of a group. It provides a mapping of the group elements onto linear operators, such that

1. the identity element of the group is mapped to the identity of the operation, $D(e) = \mathrm{Id}$,
2. and the composition of the group is transferred naturally to the multiplication in the linear space of the group representation, i. e. $D(g_1 \cdot g_2) = D(g_1)D(g_2)$.

We will need the concept of a *Lie group*, where the group elements $g(\alpha)$ now depend on one (or more) continuous parameter α, such that for $\alpha = 0$ the group element reverts to the identity, i.e. $g_{\alpha=0} = e$. Now Sophus Lie's idea was to look at infinitesimal group elements Taylor expanded around the identity in a series. To get a finite group element, infinitesimal elements have to be summed up, but only in linear order. This is of course possible because the composition of two group elements comprise another group element. In this Lie group, the representation of such an infinitesimal element $\mathrm{d}\alpha$ looks like

$$D(\mathrm{d}\alpha) = 1 + i\,\mathrm{d}\alpha_a T_a\,, \tag{6.1}$$

such that the representation of the finite group element α can be written as an exponential

$$D(\alpha) = \exp(i\alpha_a T_a)\,. \tag{6.2}$$

In both equations, the T_a are called the *generators* of the group. Those generators form an *algebra* and can be shown to satisfy the following commutation relations

$$[T_a, T_b] = i f_{abc} T_c \tag{6.3}$$

with the so-called *structure constants* f_{abc} [179]. They are unique to the particular group and describe its essential features near its identity. Note that a new group element is formed by composition of two other group elements, while we get a new member of the algebra by commuting two other members. Put more succinctly, a Lie algebra is a vector space with the additional operation called the *Lie bracket* $[\cdot,\cdot]$. The Lie bracket has to fulfil certain properties, such as total antisymmetry and also the *Jacobi identity*

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0\,. \tag{6.4}$$

But note that we do not need anything else than the structure constants. They completely determine the algebra, which in turn determines the Lie group, but only locally [10]. Also note that groups are normally written down with uppercase letters, like $SU(N)$, but the corresponding algebra is often called $\mathfrak{su}(N)$, with lowercase gothic letters. Very often, for reasons discussed above, we will omit the proper distinction and only refer to the group $SU(N)$.

6.1.2 Special unitary groups

Next, we will introduce the *special unitary group* $SU(N)$ in N-dimensional space which can be represented by matrices $\boldsymbol{U}$ by the condition

$$\boldsymbol{U}^\dagger \boldsymbol{U} = \mathrm{Id} \tag{6.5}$$

with † denoting Hermitian conjugation, i.e. the matrices $\boldsymbol{U}$ have to be unitary. Moreover,

$$\det \boldsymbol{U} = 1\,, \tag{6.6}$$

providing us with a subset of unitary matrices. Note that the bigger group $U(N)$, the unitary group without the condition (6.6), contains as subgroups $SU(N)$ and $U(1)$, the Abelian group of phase factors $e^{i\alpha}$, which is for example important for charge conservation in electromagnetism [10, 179].

The group we will mostly need is $SU(2)$. It occurs in quantum mechanics, especially in the electron spin, but also in the electroweak interaction, where the symmetry group can be written as $SU(2) \otimes U(1)$ [10, 112]. Its algebra (6.3), which is sufficient to determine the structure of the group locally, can be written as

$$\left[J^j, J^k\right] = i\epsilon^{jkl} J^l \tag{6.7}$$

with the generators J^i of the algebra. This is the simplest structure we can write, with the totally antisymmetric Levi-Civita symbol ϵ^{jkl} as the structure constants of the algebra [179]. Thus, the group $SU(2)$ is locally isomorphic to the group $SO(3)$ (an example of the special *orthogonal* group), that is the group of rotations in three-dimensional space, meaning that locally an element of $SO(3)$ is related to an element of $SU(2)$. There is one striking difference though: this is only valid locally, as for a group element $\boldsymbol{U} \in SU(2)$ it is valid that $\boldsymbol{U}$ and $-\boldsymbol{U}$ are mapped into the same element of $SO(3)$. It is said that $SU(2)$ is a *double cover* of $SO(3)$. Put differently: In $SU(2)$ we need a rotation by 720° in order to return to the original state. A rotation by only 360° leads to a minus sign. To give an example of a representation of $SU(2)$, we will need the *Pauli matrices*

$$\sigma_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_b = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{6.8}$$

They satisfy the relation

$$\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c\,. \tag{6.9}$$

Therefore we see that evaluating this gives us the commutator

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \tag{6.10}$$

and dividing by 4 on both sides results in

$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\epsilon_{abc}\frac{\sigma_c}{2}\,. \tag{6.11}$$

This shows us that the $\sigma_i/2 \equiv J_i$ are the simplest possibility of representing $SU(2)$. It is the spin 1/2 representation and the defining one of $SU(2)$. There exists another popular formulation of the $SU(2)$ algebra, namely in *raising and lowering operators* $J_\pm$, or alternatively *ladder operators*. The corresponding set of operators are defined as

$$J_\pm = J_a \pm i J_b \quad \text{and} \quad J_3 = J_c\,. \tag{6.12}$$

The corresponding commutation relations will be

$$[J_3, J_\pm] = \pm J_\pm \tag{6.13}$$
$$[J_+, J_-] = 2J_3\,. \tag{6.14}$$

The raising and lowering operators $J_\pm$ are constructed such that one can move between the eigenstates of the J_3 operator in discreet steps, mapping out the full spectrum from the highest eigenstate to the lowest one [10].

Another important Lie group (and algebra) will be $SU(3)$, the group of special unitary matrices in three dimensions. Equivalent to the Pauli matrices of $SU(2)$, there is a set of matrices forming a representation of $SU(3)$, called the *Gell-Mann matrices* [180], often written as [10, 179]

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{6.15}$$

As one can see, the λ_1 to λ_3 are just the Pauli matrices, written in 3×3 matrices. Also note the non-intuitive order in which the matrices were summarised. This is because of their properties and structure. λ_3 and λ_8 both are diagonal. λ_1, λ_2 and λ_3 together form an $SU(2)$ subgroup, as it is intuitive from them being extended Pauli matrices. λ_4 and λ_5 are similar to λ_1 and λ_2, but are found in the (1-3) sector of the 3×3 matrices, so to speak. Similarly, λ_6 and λ_7 are in the (2-3) sector. Moreover, working out the commutators

$$[\lambda_4, \lambda_5] = i(\lambda_3 + \sqrt{3}\lambda_8) \tag{6.16}$$

$$[\lambda_5, i(\lambda_3 + \sqrt{3}\lambda_8)] = -4\lambda_4 \tag{6.17}$$

$$[\lambda_4, i(\lambda_3 + \sqrt{3}\lambda_8)] = -4\lambda_5 \tag{6.18}$$

reveals that this new matrix generated of a linear combination of matrices λ_3 and λ_8 together with λ_4 and λ_5 provides us with a $SU(2)$ subgroup. The same can be done for λ_6 and λ_7 and another linear combination of λ_3 and λ_8. Therefore the conclusion is that $SU(3)$ contains three overlapping $SU(2)$ subgroups. It is important to stress that they are overlapping, though, because $SU(3) \neq SU(2) \otimes SU(2) \otimes SU(2)$ [10].

Similar to the group $SU(2)$, the generators of the algebra will be

$$T_a = \frac{\lambda_a}{2}, \tag{6.19}$$

such that the Lie bracket of the $\mathfrak{su}(3)$ algebra will be (cf. (6.3))

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc}\frac{\lambda_c}{2}. \tag{6.20}$$

The structure constants are not quite as trivial as the ones for $\mathfrak{su}(2)$, but nevertheless they retain their totally antisymmetric property. The non-vanishing structure constants are [10]

$$\begin{aligned} &f_{123} = 1 \\ &f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ &f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \end{aligned} \tag{6.21}$$

To better understand the structure of $SU(3)$, we will redefine the generators T_a, following the idea of the raising and lowering operators of $SU(2)$, see (6.12). This will give us [10]

$$I_\pm = T_1 \pm iT_2 \tag{6.22}$$

$$U_\pm = T_6 \pm iT_7 \tag{6.23}$$

$$V_\pm = T_4 \pm iT_5 \tag{6.24}$$

$$I_3 = T_3 \tag{6.25}$$

$$Y = \frac{2}{\sqrt{3}}T_8. \tag{6.26}$$

Note that in particle physics, Y is known as the hypercharge, and I_3 as the third component of isospin. Commuting the now defined matrices with each other results in the following relations [10]

$$[I_3, I_\pm] = \pm I_\pm\,, \quad [I_3, U_\pm] = \mp\frac{1}{2}U_\pm\,, \quad [I_3, V_\pm] = \pm\frac{1}{2}V_\pm \tag{6.27}$$

$$[Y, I_\pm] = 0\,, \quad [Y, U_\pm] = \pm U_\pm\,, \quad [Y, V_\pm] = \pm V_\pm \tag{6.28}$$

$$[I_+, I_-] = 2I_3 \tag{6.29}$$

$$[U_+, U_-] = \frac{3}{2}Y - I_3 = \sqrt{3}T_8 - T_3 \equiv 2U_3 \tag{6.30}$$

$$[V_+, V_-] = \frac{3}{2}Y + I_3 = \sqrt{3}T_8 + T_3 \equiv 2V_3 \tag{6.31}$$

$$[I_+, V_-] = -U_- \tag{6.32}$$

$$[I_+, U_+] = V_+ \tag{6.33}$$

$$[U_+, V_-] = I_- \tag{6.34}$$

$$[I_+, V_+] = 0 \tag{6.35}$$

$$[I_+, U_-] = 0 \tag{6.36}$$

$$[U_+, V_+] = 0\,. \tag{6.37}$$

The rest of the commutators can be derived from Hermitian conjugation. Furthermore, from the equations (6.29), (6.30) and (6.31) one can also see that $SU(3)$ indeed contains three overlapping $SU(2)$ subalgebras, with the I_+ as the raising operator, I_- as the lowering operator and I_3 as I-spin in the z-direction. This scheme is equivalent for the U and V matrices.

6.2 About QCD, the strong CP problem and the axion

We will now give a short exposition to the strong CP problem and its resolution by axions, as the terminology, method and idea behind it is an integral part of our topic. This is mainly based on [5, 55, 181, 182].

At the heart of this discussion lies the more elaborate vacuum structure of non-abelian gauge theories, alongside with the *chiral anomaly*. This anomaly comes about as follows: *Classically* we would expect an axial symmetry of $U(1)_\mathrm{A}$, corresponding to a conservation of the associated axial current J_5^μ which is expressed by

$$\partial_\mu J_5^\mu = 0\,. \tag{6.38}$$

But experimentally the consequences of that symmetry were not observed, so it was suggested that that symmetry did not truly exist in the first place [55, 183]. This is now where the chiral anomaly comes into play. In the *quantum* picture there are certain loop graphs which modify the classical picture, such as the one-loop triangle diagram consisting of a quark loop [181], shown in 6.1. Carefully calculating the current, analogously to [181], will produce a different result than in the classical case, namely

$$\partial_\mu J_5^\mu = \frac{g^2 N}{32\pi^2} F_a^{\mu\nu} \tilde{F}^a_{\mu\nu} \tag{6.39}$$

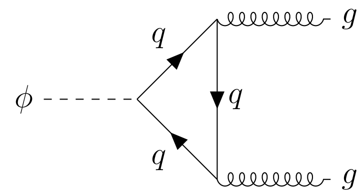

Figure 6.1: Schematic one-loop diagram of gluons responsible for the chiral anomaly, coupling a chiral current ϕ to two gluons g via a quark loop [184]

where N is the number of flavours and g the coupling constant [55]. Also note that $\tilde{F}^a_{\mu\nu}$ describes the dual tensor

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{a\alpha\beta}\,. \tag{6.40}$$

So in the end, the chiral anomaly affects the action even under a $U(1)_{\mathrm{A}}$ transformation

$$q_f \to q_f e^{i\alpha\gamma_5/2}\,, \tag{6.41}$$

namely as

$$\delta S = \alpha\int \mathrm{d}^4x\,\partial_\mu J^\mu_5 = \alpha\frac{g^2N}{32\pi^2}\int \mathrm{d}^4x\,F^{\mu\nu}_a\tilde{F}^a_{\mu\nu}\,. \tag{6.42}$$

But then one notices that this pseudoscalar density $F^{\mu\nu}_a\tilde{F}^a_{\mu\nu}$ can be written as a total divergence

$$F^{\mu\nu}_a\tilde{F}^a_{\mu\nu} \equiv \partial_\mu K^\mu \tag{6.43}$$

with the so-called *Chern-Simons current* [185, 186]

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma}A^a_\alpha\left(F^a_{\beta\gamma} - \frac{g}{3}f^a{}_{bc}A^b_\beta A^c_\gamma\right) \tag{6.44}$$

with the gauge field A^a_α and the structure constants $f^a{}_{bc}$ of the relevant Lie algebra ($SU(3)$ in the case of the colour indices of QCD). But this means that the action can be written as a total surface integral

$$\delta S = \alpha\frac{g^2N}{32\pi^2}\int \mathrm{d}^4x\,\partial_\mu K^\mu = \alpha\frac{g^2N}{32\pi^2}\int \mathrm{d}\sigma_\mu K^\mu. \tag{6.45}$$

Under the boundary condition $A^\mu_a = 0$ at spatial infinity one would assume that this surface integral vanishes. But this turns out to be a bit too short-sighted and one really needs to take into account that the vacuum can also be described by

$$\boldsymbol{A}_0 = 0, \quad \boldsymbol{A}_i = \frac{i}{g}(\partial_i\boldsymbol{U})\boldsymbol{U}^{-1} \tag{6.46}$$

with an arbitrary time-independent and unitary matrix $\boldsymbol{U(x)}$. These configurations are called *pure gauges* [5, 55, 182]. But these $\boldsymbol{U(x)}$ are not all equivalent, instead they can be categorized into *homotopy classes*, depending on whether there exists a nonsingular continuous transformation between them or not. One can prove that the vacua belonging to different homotopy classes are *topologically inequivalent*. Indeed, using the concept of winding numbers, one realises that

$$\boldsymbol{U}(\chi, \boldsymbol{e}) = \cos(n\chi)\mathbf{1} - i(\boldsymbol{e}\cdot\boldsymbol{\sigma})\sin(n\chi) \tag{6.47}$$

with $\boldsymbol{e} = (e_1, e_2, e_3)$ is the unit vector and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the three Pauli matrices organized in a vector. $n \in \mathbb{Z}$ turns out to be the winding number [5].

But those topologically different n-vacua can be combined in the true vacuum, called the θ-vacuum, as follows:

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle\,. \tag{6.48}$$

One can then calculate that the supposed boundary term measures exactly the difference in winding number

$$n|_{t=+\infty} - n|_{t=-\infty} = \frac{g^2}{32\pi^2}\int \mathrm{d}\sigma_\mu K^\mu|_{t=-\infty}^{t=+\infty}\,. \tag{6.49}$$

Therefore, the non-trivial vacuum structure of the QCD vacuum necessitates the inclusion of gauge field configurations with different winding numbers. So one has to incorporate the supposed boundary term proportional to $F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}$ into the QCD action such that [55, 182]

$$S_{\text{eff}}[A] = S_{\text{QCD}} + \theta\frac{g^2}{32\pi^2}\int \mathrm{d}^4x F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}\,. \tag{6.50}$$

This *topologial term* is not only CP-violating, but it also has observational consequences on the electric dipole moment of the neutron (EDM). As so far no EDM has been found, one can measure the upper bound on the parameter θ, which gives current best results as $|d_n| < 0.18 \times 10^{-25} e\,\text{cm}$, such that the constraint on θ itself is $\theta \lesssim 10^{-10}$ [115]. It cannot be exactly zero, because then CP would be conserved, so the question of why this parameter should be so small is known as the *strong CP problem*. This is even made worse by the fact that by including the weak interaction and quark masses with the help of the quark mass matrix M, θ will change to the physically observable

$$\theta = \bar{\theta} + \arg\det M \tag{6.51}$$

where $\bar{\theta}$ is the pure QCD θ-parameter. So the fine tuning problem even amounts to the fact that $\bar{\theta}$ and $\arg\det M$ need to almost precisely cancel against each other, both dimensionless and originating from two different theories, which is highly unlikely [55].

In the search for a "natural" solution to this without any fine tuning involved, a dynamical process has been suggested, resulting in a new particle called the *axion*. This comes about by introducing a new global chiral symmetry $U(1)_{\text{PQ}}$, the so-called *Peccei-Quinn symmetry* [187], which is spontaneously broken and by the *Goldstone theorem* a massless boson is produced, namely the axion $a(x)$, which transforms under $U(1)_{\text{PQ}}$ as

$$a(x) \to a'(x) = a(x) + \alpha f_a \tag{6.52}$$

with the *axion decay constant* or *order parameter* f_a and α a constant to be determined [55, 184]. Therefore, the Lagrangian has to be completed to

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}} + \bar{\theta}\frac{g^2}{32\pi^2}F_a^{\mu\nu}\tilde{F}^a_{\mu\nu} + \frac{1}{2}\partial_\mu a\partial^\mu a + \mathcal{L}_{\mathrm{int}}\left[\frac{\partial^\mu a}{f_a};\Psi\right] + \xi\frac{a}{f_a}\frac{g^2}{32\pi^2}F_a^{\mu\nu}\tilde{F}^a_{\mu\nu} \tag{6.53}$$

with the *anomaly coefficient* ξ depending on the PQ charges of the quarks. The Standard Model Lagrangian $\mathcal{L}_{\mathrm{SM}}$ and the following Chern-Simons term with QCD $\bar{\theta}$ is complemented with the axion kinetic term, an interaction Lagrangian $\mathcal{L}_{\mathrm{int}}$ between axion and matter Ψ and the new term also representing a chiral anomaly

$$\partial_\mu J^\mu_{\mathrm{PQ}} = \xi\frac{g^2}{32\pi^2}F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}\,, \tag{6.54}$$

which is at the same time an effective potential for the axion field. Important to determine is the θ dependence of the QCD vacuum expectation value (VEV) at the value of the shifted axion a'. Therefore we need the expectation value of the effective potential evaluated over the gauge fields A_μ in the state a'

$$V_{\mathrm{eff}}(\langle a'\rangle) = -\frac{\xi}{f_a}\frac{g^2}{32\pi^2}\langle a'|a'F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}|a'\rangle\,. \tag{6.55}$$

Due to *instanton* effects in the QCD vacuum, i.e. the vacuum being a θ-vacuum (6.48), the expectation value of the field strength tensor term can be approximated as a periodic function [55, 182]

$$V_{\mathrm{eff}}(\langle a'\rangle) \sim -\cos\left(\bar{\theta} + \xi\frac{\langle a\rangle}{f_a}\right)\,, \tag{6.56}$$

This vacuum is obviously minimised when the argument of the cosine vanishes, by definition at $\langle a'\rangle = 0$, which is, using (6.52),

$$\langle a\rangle = -\frac{f_a}{\xi}\bar{\theta}\,. \tag{6.57}$$

If we write the Lagrangian $\mathcal{L}$ (6.53) in terms of the physical axion value $a' = a + \langle a\rangle$, we see that the CP violating term proportional to $\bar{\theta}$ vanishes and therefore the strong CP problem is solved dynamically [55, 184].

Chapter 7

Gauge Vector Mimetic Gravity

After the vector field of unusual weight four was introduced to model a cosmological constant in a mimetic theory [133], other possibilities were investigated. Among them is the topic of this chapter: A theory of mimetic gravity obtained by introducing a *gauge vector field*, following [134]. The aim also in this case was to produce a theory with a cosmological constant via the traceless Einstein equations, as discussed in 1.2.4.

Also for this model one can review the mimetic construction, indeed verifying that the proposed Weyl transformation (7.1) is the only one 7.3. Also the Noether currents were calculated 7.4. Moreover, this incarnation of the mimetic theories possesses a crucial axionic coupling, as we will see, opening up new possibilities in constructing a cosmological constant. This will be the topic of 7.2 and 7.5, building on the concepts of chapter 6.

7.1 Constraint and equations of motion

In this introduction to the theory we will follow [134] if not mentioned otherwise. The proposed Weyl transformation is

$$g_{\mu\nu} = h_{\mu\nu}\sqrt{F_{\alpha\beta}\tilde{F}^{\alpha\beta}} \tag{7.1}$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{7.2}$$

is the *field strength tensor* built out of the *U(1) gauge field* A_μ, whereas $\tilde{F}^{\alpha\beta}$ describes the field strength tensor *dual* to $F_{\alpha\beta}$ with definition

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}E^{\alpha\beta\mu\nu}F_{\mu\nu} = \frac{1}{2}\frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-h}}F_{\mu\nu}\,. \tag{7.3}$$

We can see the Levi-Civita symbol $\epsilon^{\alpha\beta\mu\nu}$ and the corresponding *Levi-Civita tensor*

$$E^{\alpha\beta\mu\nu} = \frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-h}}\,, \tag{7.4}$$

defined w.r.t. the auxiliary metric $h_{\mu\nu}$. One can also rewrite (7.1) as

$$g_{\mu\nu} = \frac{h_{\mu\nu}}{(-h)^{1/4}}\sqrt{\mathcal{P}} \tag{7.5}$$

with the *Chern-Pontryagin density* [185]

$$\mathcal{P} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = 2\epsilon^{\alpha\beta\mu\nu}\partial_\alpha A_\beta \partial_\mu A_\nu \tag{7.6}$$

which is obviously insensitive to the metric used. Furthermore, we can rewrite

$$F_{\alpha\beta}\tilde{F}^{\alpha\beta} = E^{\alpha\beta\mu\nu}\left(\nabla^{h)}_\alpha A_\beta\right)F_{\mu\nu} = \nabla^{h)}_\alpha\left(E^{\alpha\beta\mu\nu}A_\beta F_{\mu\nu}\right)\,. \tag{7.7}$$

This second equality holds because $\nabla^{h)}_\alpha\left(E^{\alpha\beta\mu\nu}F_{\mu\nu}\right) = 0$ merely is the Bianchi identity, cf. (1.26). As one can see from the last equality in (7.7), one can actually view this as the divergence of a vector field, the so-called *Chern-Simons current* [185]

$$C^\alpha = E^{\alpha\beta\mu\nu}A_\beta F_{\mu\nu} = 2\tilde{F}^{\alpha\beta}A_\beta = 2E^{\alpha\beta\mu\nu}A_\beta\nabla^{h)}_\mu A_\nu = 2E^{\alpha\beta\mu\nu}A_\beta\partial_\mu A_\nu\,, \tag{7.8}$$

with a few more useful identities included. Comparing this with the original ansatz for this mimetic theory with a gauge vector field, (7.1), this turns into

$$g_{\mu\nu} = h_{\mu\nu}\sqrt{\nabla^{h)}_\alpha C^\alpha}\,, \tag{7.9}$$

now reminiscent of the theory in [133], but now we know that the vector field V^μ introduced there can be viewed as the Chern-Simons current. Nevertheless, we have to stress that in the theory with a gauge vector field the dynamic variables are in truth $A_\mu, h_{\alpha\beta}$ and not $C^\mu, h_{\alpha\beta}$, as it would have been in the theory with the vector field of conformal weight four.

Once more, the Weyl transformation (7.1) is performed on the Einstein-Hilbert action (1.59) of General Relativity as a seed theory, i.e.

$$S_{\text{gauge}}[h_{\mu\nu}, A_\alpha, \Phi_{\text{m}}] = S_{\text{seed}}[g_{\mu\nu}(h_{\alpha\beta}, A_\rho), \Phi_{\text{m}}] \tag{7.10}$$

such that its gravitational part is

$$S_{\text{gauge}}[h_{\mu\nu}, A_\alpha] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-h}\left[\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right)^{1/2}R(h) + \frac{3}{8}\cdot\frac{\left(\nabla^{h)}_\mu\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right)\right)^2}{\left(F_{\sigma\rho}\tilde{F}^{\sigma\rho}\right)^{3/2}}\right]\,, \tag{7.11}$$

comprising a novel theory. The equations of motion for this theory can be derived in the usual way, by varying the action

$$\delta S_{\text{gauge}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(T_{\mu\nu} - G_{\mu\nu}\right)\delta g^{\mu\nu} + \text{boundary terms} \tag{7.12}$$

with the Einstein tensor $G_{\mu\nu}$ (1.29) defined for the physical metric $g_{\mu\nu}$. Calculating the equations of motion for the gauge field A_μ results in

$$\frac{1}{\sqrt{-h}}\frac{\delta S_{\text{gauge}}}{\delta A_\nu} = \tilde{F}^{\mu\nu}\partial_\mu(T-G) = 0\,. \tag{7.13}$$

In case the inverse of the dual field strength tensor exists, this again means that $G - T =$ const, cf. (1.172). The modified Einstein field equations are then

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\text{gauge}}}{\delta h^{\alpha\beta}} = \frac{2}{\sqrt{F_{\alpha\beta}\tilde{F}^{\alpha\beta}}}\left[T_{\alpha\beta} - G_{\alpha\beta} - \frac{1}{4}(T-G)g_{\alpha\beta}\right] = 0\,. \tag{7.14}$$

If one considers the e.o.m. of the gauge vector we get back the trace-less Einstein field equations with the cosmological constant as an integration constant once more, if one combines (7.13) and (7.14)

$$G_{\alpha\beta} - T_{\alpha\beta} - \frac{1}{4}(G-T)g_{\alpha\beta} = 0\,. \tag{7.15}$$

Those are obviously symmetric under the following shifts of the energy-momentum tensor

$$T_{\mu\nu} \to T_{\mu\nu} + \Lambda g_{\mu\nu}\,, \tag{7.16}$$

therefore being equivalent to the equations of motion of unimodular gravity, as discussed in 1.2.4. So also this theory provides us with *Mimetic Dark Energy* [134].

Concerning the built-in constraints inherent in mimetic theory, like (2.3) and (5.12), we first need the definition of a *Hodge-dual tensor* of the field strength

$$F^{\star\alpha\beta} = \frac{1}{2}\frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-g}}F_{\mu\nu}\,, \tag{7.17}$$

this time defined w.r.t. the physical metric $g_{\mu\nu}$. Then, the corresponding mimetic constraint for the theory with a gauge field is

$$F_{\alpha\beta}F^{\star\alpha\beta} = 1\,. \tag{7.18}$$

Also this constraint is an inherent quality of the theory which also holds off-shell, as can be made obvious from the definitions of the quantities involved. Once more, from the Weyl transformation (7.1) the square root of the metric determinant transforms as

$$\sqrt{-g} = \sqrt{-h}\,F_{\alpha\beta}\tilde{F}^{\alpha\beta}\,, \tag{7.19}$$

where $\tilde{F}^{\alpha\beta}$ is defined w.r.t. the metric $h_{\mu\nu}$, for emphasis. Then we can write, with the help of the dual tensor (7.3) and the Hodge star dual (7.17)

$$F_{\alpha\beta}F^{\star\alpha\beta} = F_{\alpha\beta}F_{\mu\nu}\cdot\frac{1}{2}\frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-g}} = F_{\alpha\beta}F_{\mu\nu}\cdot\frac{1}{2}\frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{-h}\cdot\frac{1}{2}F_{\rho\sigma}F_{\lambda\kappa}\frac{\epsilon^{\rho\sigma\lambda\kappa}}{\sqrt{-h}}} \equiv 1\,, \tag{7.20}$$

thereby completing the proof. Further details about this theory and its axionic coupling can be found in 7.2 and 7.5.

Another comment needs to be made on the Chern-Simons current (7.8) being not gauge invariant under the usual U(1) gauge transformations like

$$A_\mu = A'_\mu + \partial_\mu \theta \,. \tag{7.21}$$

If applied to the Chern-Simons current, the current transforms inhomogeneously as

$$C^\mu = C'^\mu + 2\tilde{F}^{\mu\nu}\partial_\nu \theta \,. \tag{7.22}$$

Nevertheless, as

$$\nabla^{h)}_\mu \left(\tilde{F}^{\mu\nu}\partial_\nu \theta\right) = 0 \,, \tag{7.23}$$

the divergence of the Chern-Simons current stays unchanged by U(1) transformations. The importance of this is the following: We recall that the gauge transformations (5.5) of the vector field leads to the shift symmetry of the cosmic time (5.30), which is canonically conjugated to the cosmological constant. So naturally, in our search for a cosmological constant, we would like those gauge transformations work equally well in our theory with the gauge field.

7.2 Gauge invariant representation and axionic cosmological constant

One can reformulate the gauge vector theory in a scalar-vector-tensor form, similar to how it was already done in [133] for the vector field of conformal weight four. Also in this section we will follow [134], if not mentioned otherwise. The appropriate Weyl transformation is, as a reminder, (7.1)

$$g_{\mu\nu} = h_{\mu\nu}\sqrt{F_{\alpha\beta}\tilde{F}^{\alpha\beta}} \,. \tag{7.24}$$

And once more, similar to (5.17) and (5.19), we can redefine the scalar quantity $F_{\alpha\beta}\tilde{F}^{\alpha\beta}$ with the help of

$$F_{\alpha\beta}\tilde{F}^{\alpha\beta} = \left(\frac{\chi^2}{6}\right)^2 \,. \tag{7.25}$$

This will allow us to write the scalar χ as a dynamical variable and at the same time eliminate the higher derivatives. At some point we will need to enforce the definition (7.24) with the help of a Lagrange multiplier λ. Then, the action (7.11) will transform to

$$S_{\text{gauge}}[h_{\mu\nu}, \chi, A_\rho, \lambda] = \int \mathrm{d}^4x\sqrt{-h}\left[-\frac{1}{2}(\partial\chi)^2 - \frac{1}{12}\chi^2 R(h) - \frac{\lambda}{72}\chi^4 + \frac{\lambda}{2}\cdot F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right] \tag{7.26}$$

As the original formulation was Weyl-invariant, this one should be, too. Moreover, as another consequence, this also requires the Lagrange multiplier λ to be Weyl-invariant.

In [133] it was already noticed that the first three terms are equivalent to Dirac's Weyl-invariant gravity [175]. But one should notice that the kinetic term of the scalar field has the wrong sign, i.e. is a ghost. And furthermore, the Lagrange multiplier field λ has the same coupling $\propto \lambda F_{\alpha\beta}\tilde{F}^{\alpha\beta}$ as the one coupling the axion to the chiral anomaly in QCD. Exploring the possibilities of that coupling will be the topic of 7.2 and 7.5.

Under the Weyl-transformation of the $h_{\mu\nu}$ the variables transform like

$$h_{\mu\nu} = \Omega^2(x) h'_{\mu\nu}\,, \tag{7.27}$$

$$\chi = \Omega^{-1}(x)\chi'\,, \tag{7.28}$$

$$A_\mu = A'_\mu\,, \tag{7.29}$$

$$\lambda = \lambda'\,. \tag{7.30}$$

Depending on that information, one can introduce a set of new independent variables $\{g_{\mu\nu}, A_\mu, \Lambda, \chi\}$ with the relations

$$\begin{aligned} g_{\mu\nu} &= \frac{\chi^2}{6}\cdot h_{\mu\nu}\,,\\ \Lambda &= \frac{\lambda}{2}\,. \end{aligned} \tag{7.31}$$

which are now gauge-invariant. Inserting them in the action (7.26) will result in

$$S_{\text{gauge}}[g_{\mu\nu}, A_\rho, \Lambda] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}R(g) + \Lambda\left(F_{\alpha\beta}F^{\star\alpha\beta} - 1\right)\right]\,, \tag{7.32}$$

where the constraint (7.18) with respect to the Hodge dual $F^{\star\alpha\beta}$ is now directly enforced by the Lagrange multiplier, while we recall that this version of the Hodge dual is defined using the physical metric $g_{\mu\nu}$, see (7.17).

In this action (7.32) we also encounter the *axionic coupling* of the Lagrange multiplier Λ to the field strength term $F_{\mu\nu}$ and its Hodge star dual. But note that there is no kinetic term for the gauge field present in the action, unlike for the normal QCD axion. The equations of motion can also be calculated for action (7.32), such that for variation w.r.t. Λ we get

$$F_{\alpha\beta}F^{\star\alpha\beta} = 1\,, \tag{7.33}$$

so exactly the constraint equation (7.18), of course. The equation of motion for the gauge field A_μ yields

$$\frac{1}{\sqrt{-g}}\cdot\frac{\delta S_{\text{gauge}}}{\delta A_\nu} = -4\nabla^{g)}_\mu\left(\Lambda E^{\alpha\beta\mu\nu}\nabla^{g)}_\alpha A_\beta\right) = 4F^{\star\nu\mu}\partial_\mu\Lambda\,, \tag{7.34}$$

with $E^{\alpha\beta\mu\nu}$ being the Levi-Civita tensor as defined in (7.4), but with the metric determinant $\sqrt{-g}$ instead of $\sqrt{-h}$. Meanwhile, the corresponding variant of the Einstein field equations is

$$\frac{2}{\sqrt{-g}}\cdot\frac{\delta S_{\text{gauge}}}{\delta g^{\alpha\beta}} = T_{\alpha\beta} + \Lambda g_{\alpha\beta} - G_{\alpha\beta} = 0\,, \tag{7.35}$$

with the Lagrange multiplier being the cosmological constant. From the e.o.m. for the gauge vector field (7.34) we can conclude that $\Lambda = \text{const}$, if the dual $\tilde{F}^{\alpha\beta}$ is invertible, which we need if the constraint (7.18) holds.

7.3 Mimetic construction of the gauge vector term

Now we can again explicitly use the mimetic construction discussed in 2 to check that this indeed leads to the proposed Weyl transformation (7.1). In this case, we will use the abbreviation

$$\Xi = F_{\alpha\beta}\tilde{F}^{\alpha\beta} \tag{7.36}$$

with the field strength tensor $F_{\alpha\beta}$ (7.2) in terms of the gauge vector A_α and its dual $\tilde{F}^{\alpha\beta}$ (7.3). Then the general Weyl transformation we want to investigate is

$$g_{\mu\nu} = C\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right) h_{\mu\nu} \tag{7.37}$$

with C a general function of Ξ. The calculation of the Jacobian determinant of this Weyl transformation (7.37) is straightforward, starting from

$$\frac{\partial g_{\mu\nu}}{\partial h_{\alpha\beta}} = C\delta^\alpha_\mu \delta^\beta_\nu + h_{\mu\nu}\frac{\partial C}{\partial \Xi}\frac{\partial \Xi}{\partial h_{\alpha\beta}}\,. \tag{7.38}$$

Equivalent to the calculation above, cf. (5.38), we have

$$\frac{\partial \Xi}{\partial h_{\alpha\beta}} = -\frac{1}{2}h^{\alpha\beta}\Xi\,. \tag{7.39}$$

Inserting this in (7.38) results in

$$\frac{\partial g_{\mu\nu}}{\partial h_{\alpha\beta}} = C\delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{2}\frac{\partial C}{\partial \Xi}\,\Xi\, h^{\alpha\beta}h_{\mu\nu}\,. \tag{7.40}$$

To find λ_* we follow the same steps as above, i.e. multiplying by $h_{\alpha\beta}$, yielding the kinetic eigenvalue

$$\lambda_* = C - 2\frac{\partial C}{\partial \Xi}\,\Xi\,. \tag{7.41}$$

Setting this to zero again and solving the differential equation exactly produces what we were looking for, namely

$$C(\Xi) = \sqrt{\Xi} = \sqrt{F_{\alpha\beta}\tilde{F}^{\alpha\beta}}\,, \tag{7.42}$$

just like it was expected from [134]. The discussion of algebraic solutions of (7.39) follows the same path like the one in 5.3.1, just with D replaced by Ξ.

7.4 Noether's theorems and the field strength term

Also for this theory, Noether's theorems can be discussed, while the general introduction to the concepts can be found in 3.

7.4.1 First theorem

We need the action involving the gauge field term, from [134], cf. (7.26),

$$S_F[h_{\mu\nu},\chi,A_\rho,\Lambda] = \int \mathrm{d}^4x\sqrt{-h}\left(-\frac{\Lambda}{72}\chi^4 + \frac{\Lambda}{2}F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right) \tag{7.43}$$

with the field strength tensor $F_{\alpha\beta}$ (7.2) of some gauge vector field A_α and its dual tensor $\tilde{F}^{\alpha\beta}$ (7.3). The definition of th scalar field χ, meanwhile, is enforced via the Lagrange multiplier Λ to be (7.25)

$$\frac{\chi^4}{36} = F_{\alpha\beta}\tilde{F}^{\alpha\beta}\,. \tag{7.44}$$

The action can be rewritten as

$$S_F[h_{\mu\nu},\chi,A_\rho,\Lambda] = \int \mathrm{d}^4x\left(-\frac{\Lambda}{72}\sqrt{-h}\chi^4 + \frac{\Lambda}{4}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right)\,. \tag{7.45}$$

As $\varepsilon^{\alpha\beta\mu\nu}$ stays the same in all frames, we can infer that

$$\begin{aligned} F_{\mu\nu} &\to F_{\mu\nu} \\ \Rightarrow A_\mu &\to A_\mu\,. \end{aligned} \tag{7.46}$$

So the explicit Weyl transformation yields the exact same action, without needing a boundary term, so $X^\alpha_{(F)} = 0$.

Finally, the explicit calculation of the current using the equations of motion is trivial in this case, as there are not even first derivatives of transforming quantities in this action. In the end,

$$J^\mu_{(F)} = K^\mu_{(F)} - X^\mu_{(F)} \equiv 0\,. \tag{7.47}$$

7.4.2 Second theorem

Next, we want to examine whether Noether's second theorem for this action with the gauge vector term gives us any new information. As we have done it for the theory with the vector field, see (5.56), we decompose the action with the field strength tensor $F_{\mu\nu}$ into

$$\begin{aligned} S_{F,\,\mathrm{total}}[h_{\mu\nu},\chi,A_\rho,\Lambda] &= S[h_{\mu\nu},\chi] + S_F[h_{\mu\nu},\chi,A_\rho,\Lambda] \\ &= -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{1}{12}R(h)\chi^2 + \frac{1}{2}h^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi\right) \\ &\quad -\int \mathrm{d}^4x\sqrt{-h}\left(\frac{\Lambda}{72}\chi^4 - \frac{\Lambda}{2}F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right) \end{aligned} \tag{7.48}$$

and repeat this calculation. Again, note that we will take $S[h_{\mu\nu}, \chi]$ as in (4.130) with the known consequence of Noether's second theorem for this part vanishing, see (4.137). The Weyl transformation is essentially the same as above, eq. (4.136), but the transformation of the vector field V^μ gets replaced by

$$A_\mu \to A_\mu \,, \tag{7.49}$$

so Noether's second theorem for the total action $S_{F,\,\mathrm{total}}[h_{\mu\nu}, \chi, A_\rho, \Lambda]$ simplifies in this case to

$$\left(\frac{\delta S}{\delta \chi} + \frac{\delta S_F}{\delta \chi}\right)\theta\chi + \left(\frac{\delta S}{\delta h^{\mu\nu}} + \frac{\delta S_F}{\delta h^{\mu\nu}}\right)\cdot 2\theta h^{\mu\nu} = 0\,. \tag{7.50}$$

We remind ourselves that the action term explicitly including the field strength tensor is $\int \mathrm{d}^4x \ \frac{\Lambda}{4}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$, using the definition of the dual field strength tensor (7.3), and therefore contains neither the metric nor the scalar field χ, so we can neglect this term in our further calculations. The relevant parts of the equations of motion are

$$\frac{\delta S_F}{\delta h^{\mu\nu}} = \sqrt{-h}\frac{\Lambda}{144}\chi^4 h_{\mu\nu}\,, \tag{7.51}$$

$$\frac{\delta S_F}{\delta \chi} = -\sqrt{-h}\frac{\Lambda}{18}\chi^3\,. \tag{7.52}$$

Again, we notice that Noether's second theorem is trivially satisfied, for the same reason that was discussed in 5.4.2 for the theory with the vector field.

7.5 Axionic cosmological constant — non-abelian generalisation

The next task will be to generalize our theory to non-abelian gauge symmetries, especially with $SU(N)$ gauge groups. We will sum up the reasons why this is indeed a natural idea [134].

1. The vacuum structure in non-abelian gauge theories is much more intricate, giving rise to new degrees of freedom.

2. Axionic couplings, like we have in our theory, are much more common in non-abelian gauge theories.

3. Standard couplings of the gauge field to matter are much easier to introduce with the now more usual gauge field A_μ instead of the vector field V^μ.

So at first we will discuss the general structures of a non-abelian theory. As usual, the non-abelian gauge field is expanded in the group generators $\boldsymbol{T}_a$, such that

$$\boldsymbol{A}_\mu = A^a_\mu \boldsymbol{T}_a\,. \tag{7.53}$$

We will also need a covariant derivative

$$\boldsymbol{D}_\mu = \partial_\mu + i\mathrm{g}\boldsymbol{A}_\mu \tag{7.54}$$

with self-coupling constant g. Then we can define the usual field strength tensor

$$\boldsymbol{F}_{\mu\nu} = \boldsymbol{D}_\mu \boldsymbol{A}_\nu - \boldsymbol{D}_\nu \boldsymbol{A}_\mu = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + i\mathrm{g}[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \tag{7.55}$$

alongside its Hodge dual (compare to (7.3))

$$\tilde{\boldsymbol{F}}^{\mu\nu} = \frac{1}{2} E^{\alpha\beta\mu\nu} \boldsymbol{F}_{\alpha\beta} \tag{7.56}$$

with the usual definition of the Levi-Civita tensor (7.4). The Weyl transformation of the physical metric is then generalized to

$$g_{\mu\nu} = \sqrt{\mathrm{Tr}\boldsymbol{F}_{\alpha\beta}\tilde{\boldsymbol{F}}^{\alpha\beta}} \cdot h_{\mu\nu}\,, \tag{7.57}$$

very similar to the abelian theory (7.1), with the trace Tr over the colour indices. The Chern-Simons current

$$C^\mu = \mathrm{Tr}\, E^{\mu\alpha\beta\gamma} \left(\boldsymbol{F}_{\alpha\beta} \boldsymbol{A}_\gamma - \frac{2i\mathrm{g}}{3} \boldsymbol{A}_\alpha \boldsymbol{A}_\beta \boldsymbol{A}_\gamma \right) \tag{7.58}$$

follows, equivalent to the abelian theory (7.8), but note that the second term arises due to the non-abelian nature of the theory. Similar to the cases with the vector and the abelian gauge vector, we can look of gauge transformations of the non-abelian gauge vector, namely [5]

$$\boldsymbol{A}_\mu \to \boldsymbol{U}\boldsymbol{A}_\mu \boldsymbol{U}^{-1} + \frac{i}{\mathrm{g}} \partial_\mu \boldsymbol{U}\, \boldsymbol{U}^{-1} \tag{7.59}$$

with a Hermitian matrix $\boldsymbol{U(x)}$. Also note that the field strength tensor is gauge invariant, i.e. it will transform homogeneously as

$$\boldsymbol{F}_{\alpha\beta} \to \boldsymbol{U}\boldsymbol{F}_{\alpha\beta}\boldsymbol{U}^{-1}\,. \tag{7.60}$$

Then, the Chern-Simons current will change as

$$C^\mu \to C^\mu + \frac{2i}{\mathrm{g}} \mathrm{Tr}\, E^{\mu\alpha\beta\gamma} \partial_\alpha \boldsymbol{A}_\beta \boldsymbol{U}^{-1} \partial_\gamma \boldsymbol{U} \equiv C^\mu + \epsilon^\mu \tag{7.61}$$

using the cyclicity of the trace and the properties of the Levi-Civita symbol. The last equality defines the vector ϵ^μ, which should be divergenceless. We see with the help of the formula (1.36) for the divergence and the definition of the Levi-Civita tensor (7.4) that indeed

$$\nabla_\mu \epsilon^\mu = \frac{2i}{\mathrm{g}} \frac{1}{\sqrt{-g}} \partial_\mu \left(\mathrm{Tr}\epsilon^{\mu\alpha\beta\gamma} \partial_\alpha \boldsymbol{A}_\beta \boldsymbol{U}^{-1} \partial_\gamma \boldsymbol{U} \right) = 0\,, \tag{7.62}$$

as all occurring combinations of double partial derivatives, e.g. $\partial_\mu \partial_\alpha \boldsymbol{A}_\beta$, are symmetric, therefore they are cancelled by contraction with the totally antisymmetric symbol $\epsilon^{\mu\alpha\beta\gamma}$. As already discussed for the abelian case in 7.1, the Chern-Simons current will change under gauge transformations, but the divergence will stay the same. Furthermore, the generalisation of the action in gauge invariant variables is straightforward and results in

$$S[g, \boldsymbol{A}, \Lambda, \Phi_{\mathrm{m}}] = \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} R(g) + \Lambda \left(\mathrm{Tr} \boldsymbol{F}_{\alpha\beta} \boldsymbol{F}^{\star\alpha\beta} - 1 \right) \right] + S_{\mathrm{m}}[g, \Phi_{\mathrm{m}}] \tag{7.63}$$

with matter fields Φ_{m} and the Hodge star dual now defined w.r.t. the metric $g_{\mu\nu}$, just as in (7.17) [134].

At first, we will explain the general ansatz for a field strength tensor of non-abelian gauge fields, defined as

$$\begin{aligned}
\boldsymbol{F}_{\mu\nu} &= \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + i\mathrm{g}[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] && (7.64)\\
&= \partial_\mu \boldsymbol{T}_a A^a_\nu - \partial_\nu \boldsymbol{T}_a A^a_\mu + i\mathrm{g}[\boldsymbol{T}_b, \boldsymbol{T}_c] A^b_\mu A^c_\nu && (7.65)\\
&= \boldsymbol{T}_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathrm{g} f^a{}_{bc} A^b_\mu A^c_\nu \right) && (7.66)\\
&\equiv \boldsymbol{T}_a F^a_{\mu\nu} \,. && (7.67)
\end{aligned}$$

with the group generators $\boldsymbol{T}_a$ and the commutation relation for them, namely

$$[\boldsymbol{T}_a, \boldsymbol{T}_b] = i f^c{}_{ab} \boldsymbol{T}_c \,, \tag{7.68}$$

with the structure constants $f^c{}_{ab}$ of the Lie algebra, see (6.3).

7.5.1 Existence of solutions for $SU(2)$

So far this discussion has been valid for all groups, but let us now specialize to the special unitary group $SU(2)$, see also the discussion in 6.1.2. This part is taken and expanded from [134, 188]. We will set up a general basis e^α_a in our spacetime. The greek indices will denote the usual spacetime indices, while a describes the index reserved for the internal group space, i.e. it refers to the generators of the Lie group. Also note that a $3+1$ split of spacetime into a time function t and several spatial hypersurfaces for each moment in time. In every coordinate system the following is valid [6]

$$\mathscr{L}_t e^\alpha_a = 0 \,. \tag{7.69}$$

In this equation, $\mathscr{L}_t$ describes the Lie derivative (1.22) w.r.t. time t. Note that although the Lie derivative is usually defined along a vector field, one can view this as Lie derivative along a curve parametrised by time. Also notice that from

$$0 = \mathscr{L}_t \delta^b_a = \mathscr{L}_t \left(e^\alpha_a e^b_\alpha \right) = e^\alpha_a \mathscr{L}_t e^b_\alpha + e^b_\alpha \mathscr{L}_t e^\alpha_a \tag{7.70}$$

it follows that this equation is also valid for one-forms, i.e.

$$\mathscr{L}_t e^a_\alpha = 0 \,. \tag{7.71}$$

We can then choose that

$$e^a_0 = 0\,, \tag{7.72}$$
$$\partial_0 e^a_\mu = 0\,, \tag{7.73}$$

such that the time components are set to zero. Therefore we are left with the three spatial components of the basis. So we can use the ansatz for the gauge field

$$\mathrm{g} A^a_\mu = \alpha e^a_\mu\,, \tag{7.74}$$

where g is the coupling constant of the theory while α denotes a scalar field depending on spacetime coordinates. Note that this equation is merely an identification in a certain gauge, as the components of the gauge field A^a_μ transform under $SU(2)$, while the basis vectors e^a_μ do not. Using this decomposition we can calculate the field strength tensor (7.55), using the Levi-Civita symbol as the structure constants of $SU(2)$ (6.7)

$$\begin{aligned} \mathrm{g} F^a_{\mu\nu} &= \partial_\mu \left(\alpha e^a_\nu\right) - \partial_\nu \left(\alpha e^a_\mu\right) - \alpha^2 \varepsilon^a_{\ bc} e^b_\mu e^c_\nu & (7.75)\\ &= 2\partial_{[\mu} \alpha e^a_{\nu]} + 2\alpha \partial_{[\mu} e^a_{\nu]} - \alpha^2 \varepsilon^a_{\ bc} e^b_\mu e^c_\nu & (7.76) \end{aligned}$$

where in the first term of the second line the derivative acts only on α and the antisymmetrisation is defined as

$$\partial_{[\mu} e^a_{\nu]} \equiv \frac{1}{2}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right)\,. \tag{7.77}$$

The Pontryagin class (7.6) can be calculated as follows

$$\begin{aligned} \frac{\mathrm{g}^2}{2} \varepsilon^{\mu\nu\sigma\rho} F^a_{\mu\nu} F^b_{\sigma\rho} \delta_{ab} &= -2\alpha^2 \partial_\mu \alpha e^a_\nu e^b_\sigma e^c_\rho \varepsilon_{abc} \varepsilon^{\mu\nu\sigma\rho} + 4\alpha \partial_\mu \alpha e^a_\nu \partial_\sigma e^b_\rho \varepsilon^{\mu\nu\sigma\rho} \delta_{ab} & (7.78)\\ &= -2\alpha^2 \dot{\alpha} e^a_i e^b_j e^c_k \varepsilon_{abc} \varepsilon^{ijk} + 4\alpha\dot{\alpha} e^a_i \partial_j e^b_k \varepsilon^{ijk} \delta_{ab}\,. & (7.79) \end{aligned}$$

The symmetry of the Levi-Civita symbol was used to simplify calculations. In the last equality the dot $\dot{}$ is another way of writing the time derivative ∂_0 and therefore the four-dimensional Levi-Civita symbol reduces to the three-dimensional one, as those terms are the only ones which remain, as per (7.72) and (7.73). Also notice that in the case of a non-abelian gauge theory, it was used that [10]

$$\mathrm{Tr}\left(\boldsymbol{T}_a \boldsymbol{T}_b\right) = \frac{1}{2}\delta_{ab}\,. \tag{7.80}$$

One can then define a quantity β as

$$2 e^a_i \partial_j e^b_k \varepsilon^{ijk} \delta_{ab} = -\beta e^a_i e^b_j e^c_k \varepsilon_{abc} \varepsilon^{ijk}. \tag{7.81}$$

So switching j and k indices on the left-hand side (but not for the Levi-Civita symbol) gives:

$$\begin{aligned} 2 e^a_i \partial_k e^b_j \varepsilon^{ijk} \delta_{ab} &= -\beta e^a_i e^b_k e^c_j \varepsilon_{abc} \varepsilon^{ijk} & (7.82)\\ &= +\beta e^a_i e^b_j e^c_k \varepsilon_{abc} \varepsilon^{ijk} & (7.83) \end{aligned}$$

after renaming j and k and using the antisymmetric properties of ε^{ijk}. Therefore we can write

$$e_i^a \left(\partial_j e_k^b - \partial_k e_j^b\right) \varepsilon^{ijk} \delta_{ab} = -\beta e_i^a e_j^b e_k^c \varepsilon_{abc} \varepsilon^{ijk} \,. \tag{7.84}$$

At the same time we know by definition [189]

$$\partial_j e_k^b - \partial_k e_j^b \equiv c^b_{\ rs} e_j^r e_k^s \tag{7.85}$$

with the *coefficients of anholonomy* $c^b_{\ rs}$. Despite their appearance, note that they do not form a tensor. Moreover, they are exactly zero when the basis is a coordinate basis. In general, for the dual basis $\boldsymbol{e}_\alpha$, all possible commutators between basis vectors have to vanish

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = 0 \tag{7.86}$$

for a holonomic (coordinate) basis. Combining eqs. (7.85) and (7.84), we arrive at

$$e_i^a e_j^b e_k^c c_{abc} \varepsilon^{ijk} = \beta e_i^a e_j^b e_k^c \varepsilon_{abc} \varepsilon^{ijk} \,. \tag{7.87}$$

With the help of the determinant e of the frame fields

$$6e = e_i^a e_j^b e_k^c \varepsilon_{abc} \varepsilon^{ijk} \tag{7.88}$$

we can write β as

$$\beta = \frac{1}{6e} e_i^a e_j^b e_k^c c_{abc} \varepsilon^{ijk} \,. \tag{7.89}$$

Inserting this together with (7.81) into (7.79) will then result in

$$\frac{\mathrm{g}^2}{2} \varepsilon^{\mu\nu\sigma\rho} F^a_{\mu\nu} F^b_{\sigma\rho} \delta_{ab} = -12\alpha(\alpha+\beta) e\dot{\alpha} \,. \tag{7.90}$$

If we use the mimetic constraint equation (7.18) we can combine the equations into the result

$$-\mathrm{g}^2 \sqrt{-g} = 12\alpha(\alpha+\beta) e\dot{\alpha} \,. \tag{7.91}$$

The general solution will be

$$\alpha^3 + \frac{3}{2}\alpha^2\beta = -\frac{\mathrm{g}^2}{4e} \int \sqrt{-g}\, \mathrm{d}t \,. \tag{7.92}$$

We observe that this is a cubic polynomial, therefore we can try to find real solutions for this. If we define the function

$$f(\alpha, t) = \alpha^3 + \frac{3}{2}\alpha^2\beta + \frac{\mathrm{g}^2}{4e} \int_{t_0}^{t} \sqrt{-g}\, \mathrm{d}\tilde{t} \tag{7.93}$$

we can find solutions for $t = t_0$ and $t < t_0$. In fig. 7.1 we display the solutions for positive β, while in fig. 7.2 the same is done for negative beta. In both cases, the solid line represents

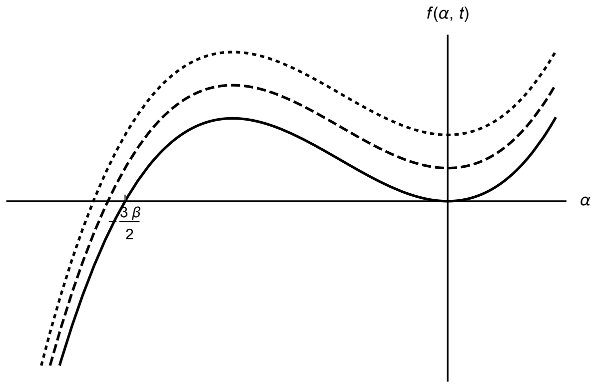

Figure 7.1: Function $f(\alpha, t)$ for $\beta > 0$. Solid line for $t = t_0$, dashed and dotted lines indicate time evolution upwards.

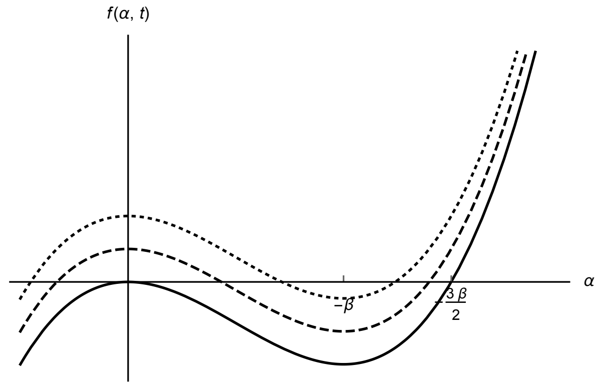

Figure 7.2: Function $f(\alpha, t)$ for $\beta < 0$. Solid line for $t = t_0$, dashed and dotted lines indicate time evolution upwards.

the starting point at $t = t_0$, showing two different solutions for α. If we increase time t, while assuming that

$$\frac{\mathrm{g}^2}{4e} \int_{t_0}^{t} \sqrt{-g}\, \mathrm{d}\tilde{t} > 0 \,, \tag{7.94}$$

we can schematically see in both figures the time evolution of the curve from solid over dashed to dotted line. In other words, time evolution shifts the curve upwards. Notice that for $\beta > 0$ the initial two solutions immediately reduce to only one, leaving us with an unambiguous solution. For $\beta < 0$ this is not so easy however. We see in fig. 7.2 that the initial two solutions will become three for a while, before reducing to two. This second double root will happen for

$$\frac{\mathrm{g}^2}{4e} \int_{t_0}^{t} \sqrt{-g}\, \mathrm{d}\tilde{t} = -\frac{1}{2}\beta \,, \tag{7.95}$$

which can be found by calculating the local minimum of the function at $\alpha = -\beta$. If the whole integral expression finally becomes larger than $-1/2\,\beta$, we will finally have one unambiguous solution once again. On the other hand, note that we do not have a continuous transition from one solution to the next. As soon as the two solutions for positive α have merged into one, they vanish completely and the only solution left is the one for negative α, but the jump between them is discrete. Also, if we consider also going backward in time, in fig. 7.1 we will encounter a similar problem with the continuity between solutions.

7.5.2 Existence of solutions for $SU(3)$

As we have seen, solutions of the mimetic constraint equation (7.18) exist in the case of the non-abelian group $SU(2)$. But now the question arises, whether this is also true for other special unitary groups, so let us consider $SU(3)$. As we have already seen in the introduction to group theory 6.1.2, $SU(3)$ contains three *overlapping* $SU(2)$ subalgebras.

For example, the $SU(2)$ subalgebras consisting of the generators I_3 (6.25), I_+ and I_- (6.22) is formulated exactly as the spin in z direction, as well as its raising and lowering operators, with the corresponding algebra (6.29). Similarly defined are the subalgebras for U (6.30) and for V (6.31). Of course, as $SU(3)$ has eight generators while three $SU(2)$ subalgebras would need nine independent generators, they are overlapping in the sense that in these definitions they share their z direction spins, as it is obvious from (6.29), (6.30) and (6.31), i.e. I_3, U_3 and V_3 are constructed out of only two generators, T_3 and T_8 [10].

Following this idea, we can now choose two different $SU(2)$ solutions from the full $SU(3)$ group. Same as above, we now construct two different solutions

$$\mathrm{g}A^a_\mu = \alpha e^a_\mu \tag{7.96}$$
$$\mathrm{g}\hat{A}^a_\mu = \hat{\alpha}\hat{e}^a_\mu\,, \tag{7.97}$$

where for example A^a_μ corresponds to the solution of the I subalgebra (6.29) and $\hat{A}^a_\mu$ to the U subalgebra (6.30). Note that the manifold we are basing this problem on is still the same, so the bases e^a_μ and $\hat{e}^a_\mu$ are related via a transformation, such that

$$\hat{e}^a_\mu = J^a_j e^j_\mu \tag{7.98}$$

with transformation matrix J^a_j. They will be used to connect the different subalgebras with each other.

At first, we will expand the gauge field $\boldsymbol{A}_\mu$ in the generators as we have already done in the case of $SU(2)$. Therefore, using the definitions of the raising and lowering operators for the I spin subalgebra (6.22) and (6.25)

$$\boldsymbol{A}_\mu = A^a_\mu T_a = A^1_\mu T_1 + A^2_\mu T_2 + A^3_\mu T_3 \tag{7.99}$$
$$= \frac{1}{2}(A^1_\mu - iA^2_\mu)I_+ + \frac{1}{2}(A^1_\mu + iA^2_\mu)I_- + A^3_\mu I_3\,. \tag{7.100}$$

We then can use the identification of the A^a_μ components with the basis vectors e^a_μ from (7.74), such that

$$\boldsymbol{A}_\mu = \frac{\alpha}{\mathrm{g}}\left[\frac{1}{2}(e^1_\mu - ie^2_\mu)I_+ + \frac{1}{2}(e^1_\mu + ie^2_\mu)I_- + e^3_\mu I_3\right]\,. \tag{7.101}$$

But notice that the algebra in the basis of I_+, I_- and I_3 is more noticeably difficult and the structure constants of the resulting $SU(2)$ will not be the usual ε_{abc}, as in the basis of T_1, T_2 and T_3. So we can simply switch back to this basis via ((6.22) and (6.25))

$$\begin{aligned}T_1 &= \frac{1}{2}(I_+ + I_-)\\ T_2 &= \frac{i}{2}(I_- - I_+)\\ T_3 &= I_3\,.\end{aligned} \tag{7.102}$$

This can be analogously done for the U and V subalgebras with the help of ((6.23) and (6.30))

$$\begin{aligned} T_6 &= \frac{1}{2}(U_+ + U_-) \\ T_7 &= \frac{i}{2}(U_- - U_+) \\ \frac{1}{2}\left(\sqrt{3}T_8 - T_3\right) &= U_3 \end{aligned} \tag{7.103}$$

as well as ((6.24) and (6.31))

$$\begin{aligned} T_4 &= \frac{1}{2}(V_+ + V_-) \\ T_5 &= \frac{i}{2}(V_- - V_+) \\ \frac{1}{2}\left(\sqrt{3}T_8 + T_3\right) &= V_3 \,. \end{aligned} \tag{7.104}$$

Therefore, the three possible solutions for the gauge vector field can be written as

$$\boldsymbol{A}_\mu = \frac{\alpha}{\mathrm{g}}\left[e^1_\mu T_1 + e^2_\mu T_2 + e^3_\mu T_3\right] \tag{7.105}$$

$$\hat{\boldsymbol{A}}_\mu = \frac{\hat{\alpha}}{\mathrm{g}}\left[e^1_\mu T_6 + e^2_\mu T_7 + \frac{1}{2}e^3_\mu\left(\sqrt{3}T_8 - T_3\right)\right] \tag{7.106}$$

$$\bar{\boldsymbol{A}}_\mu = \frac{\bar{\alpha}}{\mathrm{g}}\left[e^1_\mu T_4 + e^2_\mu T_5 + \frac{1}{2}e^3_\mu\left(\sqrt{3}T_8 + T_3\right)\right] , \tag{7.107}$$

where the first one corresponds to the I subalgebra, the second to the U subalgebra and the third to the V subalgebra. Furthermore, in each of these cases, the structure constants of the $SU(2)$ subalgebras are given by the Levi-Civita symbol, as can be verified with the help of the structure constants of $SU(3)$ listed in (6.21). To reiterate once more, notice that the third component of $\boldsymbol{A}_\mu$, $\hat{\boldsymbol{A}}_\mu$ and $\bar{\boldsymbol{A}}_\mu$ is made up of only the generators T_3 and T_8. Therefore, only two of the solutions for the gauge field are independent. As a consequence of these discussions, the exact same solution for α, as detailed in 7.5.1, can similarly be found for $\hat{\alpha}$ and $\bar{\alpha}$.

The next question is now, whether the solutions for the three possible subalgebras are equivalent to each other, i.e. can be transformed into each other. Therefore, we try to do a gauge transformation of the gauge field $\boldsymbol{A}_\mu$, as in (7.59), i.e.

$$\boldsymbol{A}_\mu \to \frac{\alpha}{\mathrm{g}}\left[e^1_\mu \boldsymbol{U} T_1 \boldsymbol{U}^{-1} + e^2_\mu \boldsymbol{U} T_2 \boldsymbol{U}^{-1} + e^3_\mu \boldsymbol{U} T_3 \boldsymbol{U}^{-1} + \frac{i}{\mathrm{g}}\partial_\mu \boldsymbol{U}\,\boldsymbol{U}^{-1}\right] \tag{7.108}$$

$$\stackrel{!}{=} \frac{\hat{\alpha}}{\mathrm{g}}\left[\hat{e}^1_\mu T_6 + \hat{e}^2_\mu T_7 + \hat{e}^3_\mu U_3\right] . \tag{7.109}$$

If we simplify this for a rotation matrix $\boldsymbol{U}$ with constant numbers as entries, the term $\partial_\mu \boldsymbol{U}\,\boldsymbol{U}^{-1}$ vanishes and the question is whether we can find a $\boldsymbol{U}$ such that its application to the gauge field results in a linear combination of generators T_6, T_7 and U_3. So let us try to solve the system of equations

$$\boldsymbol{U}T_1\boldsymbol{U}^{-1} = \alpha_1 T_6 + \beta_1 T_7 + \gamma_1 U_3 \tag{7.110}$$
$$\boldsymbol{U}T_2\boldsymbol{U}^{-1} = \alpha_2 T_6 + \beta_2 T_7 + \gamma_2 U_3 \tag{7.111}$$
$$\boldsymbol{U}T_3\boldsymbol{U}^{-1} = \alpha_3 T_6 + \beta_3 T_7 + \gamma_3 U_3 \tag{7.112}$$

which includes neither the scalar fields α and $\hat{\alpha}$ nor the bases e^a_μ or $\hat{e}^a_\mu$. This would offer us only more freedom on how to transform the two sets of generators into each other. In case we write the prefactors of the linear expansion as a matrix, we can define

$$\boldsymbol{R} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} . \tag{7.113}$$

Also setting the rotation matrix as the most general

$$\boldsymbol{U} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \tag{7.114}$$

results in multiple possible solutions, but to show the existence of only one, we will verify that the following ansatz provides us with one solution to our problem. We will take

$$\boldsymbol{U} = \begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix} , \qquad \boldsymbol{R} = \begin{pmatrix} -\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta_2 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{7.115}$$

Of course, the matrix $\boldsymbol{U}$ has to be invertible, therefore in the simplified case

$$\det \boldsymbol{U} = -ceg \neq 0. \tag{7.116}$$

Then we can show that under the condition

$$g = e(i\beta_1 - \beta_2) \tag{7.117}$$

this is indeed a solution of the system of equations (7.110) to (7.112). In case we want the determinant of $\boldsymbol{R}$ to be equal to 1, we have the additional condition of

$$\beta_1^2 + \beta_2^2 = 1 . \tag{7.118}$$

Furthermore, we have to show that $\boldsymbol{U}$ (7.115) is actually a matrix belonging to the group $SU(3)$, therefore we will discuss the conditions under which it is unitary and its determinant is one. For the unitarity condition we have

$$\boldsymbol{U}^\dagger = \boldsymbol{U}^{-1} \tag{7.119}$$

$$\begin{pmatrix} 0 & 0 & -e\left(\beta_1^* - i\beta_2^*\right) \\ 0 & e & 0 \\ c & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{e}\frac{i}{\beta_1 + i\beta_2} \\ 0 & \frac{1}{e} & 0 \\ \frac{1}{c} & 0 & 0 \end{pmatrix} . \tag{7.120}$$

As a consequence,

$$e^2 = 1\,, \quad c^2 = 1 \quad \text{and} \quad (\beta_1 + i\beta_2)\,(\beta_1^* + i\beta_2^*) = 1\,. \tag{7.121}$$

The last of these conditions is solved for

$$\beta_1 = \pm 1 + \Im(\beta_2) - i\Re(\beta_2)\,, \tag{7.122}$$

where $\Re$ and $\Im$ denote real and imaginary parts of β_2, respectively. The matrix $\boldsymbol{U}$ will then be

$$\boldsymbol{U} = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ i & 0 & 0 \end{pmatrix}\,. \tag{7.123}$$

Notice that this matrix is unitary, but its determinant is $-i$, but not 1. Therefore, normalizing this matrix will give us, dropping the $\pm$ option,

$$\boldsymbol{U}_{\text{norm}} = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \tag{7.124}$$

which can be shown to be Hermitian and of determinant 1, i.e. an element of $SU(3)$. Furthermore, the matrix $\boldsymbol{R}$ therefore turns out to be

$$\boldsymbol{R}_{\text{norm}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,, \tag{7.125}$$

such that in other words, the system of matrix equations (7.110) to (7.112) reduces to

$$\boldsymbol{U}_{\text{norm}} T_1\, \boldsymbol{U}_{\text{norm}}^{-1} = -T_7 \tag{7.126}$$
$$\boldsymbol{U}_{\text{norm}} T_2\, \boldsymbol{U}_{\text{norm}}^{-1} = -T_6 \tag{7.127}$$
$$\boldsymbol{U}_{\text{norm}} T_3\, \boldsymbol{U}_{\text{norm}}^{-1} = -U_3\,. \tag{7.128}$$

With this, we have shown that there exists at least one rotation $\boldsymbol{U}_{\text{norm}}$, which is at the same time an element of the group $SU(3)$, which rotates the I subalgebra into the U subalgebra. This can be viewed as a constant gauge transformation, and therefore the solutions from these two $SU(2)$ subalgebras are not distinct from each other.

So in the end let us summarise the main results of these discussions. As we have seen, the gauge group $SU(3)$ contains the gauge group $SU(2)$ as a subgroup, therefore the results of 7.5.1 are directly applicable, if we choose to set five of the eight components of the gauge vector A_μ^a to zero. It may first look like there are three possible solutions because of the three overlapping $SU(2)$ subalgebras, but it turns out that they are in fact equivalent to each other. As far as general $SU(N)$ groups, the exact same discussion applies, as they also all contain $SU(2)$ subgroups, therefore solutions of the mimetic constraint equation can be found.

Another comment is due on the number of spatial dimensions in our problem. In the real four-dimensional spacetime, we are dealing with three spatial dimensions, therefore we need three generators for the relevant subalgebra. This of course means that $SU(2)$ is a natural choice. Had we but two spatial dimensions, a gauge group with only two generators would be harder to find and $SU(2)$ would not be possible. One can, of course, extend the discussion to speculative higher spatial dimensions, where more generators and more complex gauge groups would be feasible. But nevertheless, notice that eight spatial dimensions, in order to justify using all of the generators of $SU(3)$, seems highly improbable and artificial.

Chapter 8

Conclusions

In chapter 1 of this thesis, we introduced major concepts of general relativity, cosmology, dark matter and dark energy. We dedicated chapter 2 to reviewing the mimetic construction, beginning from the Weyl transformation of the physical metric, as it was was introduced in the original paper [132] by Mukhanov and Chamseddine in order to construct mimetic dark matter. This metric redefinition involves new dynamical variables, namely an auxiliary metric and a scalar field. These variables are introduced in such a way as to keep the physical metric invariant under Weyl transformations of the auxiliary metric. After that, we discuss that mimetic theories are more widespread than previously thought and in particular that they can be realised through metric redefinition without keeping the Weyl invariance [143]. Noether's first and second theorems were also reviewed in chapter 3, in flat and curved spacetimes, in order to stress their importance in the context of occurring symmetries and gauge degeneracies. Furthermore, the review in chapter 4 covers various possibilities to extend the scalar mimetic models with higher derivative terms to include a pressure and speed of sound [135]. Other mimetic setups in order to model mimetic dark energy were discussed, such as a construction with a vector field [133], in chapter 5, as well as a $SU(N)$ gauge vector field [134], in chapter 7. Various gauge (Weyl) invariant representations of all of the presented theories were discussed, to highlight the fact that mimetic theories can appear in many different forms. Chapter 6 concerns an overview over Lie groups, in particular special unitary groups, and over quantum chromodynamics (QCD) and axions. This was in order to prepare for chapter 7 and embedding the mimetic dark energy theory with a gauge vector field into a theory with an axion to address the question of the smallness of the cosmological constant, equivalently to the smallness of the θ-parameter of QCD.

In chapter 4, we discussed the UV completion of a scalar mimetic theory by embedding it into a theory with a complex scalar field. There we provide a new and more direct formulation (4.4) of this correspondence, coming from the transformation of the complex scalar field (4.3). Under certain assumptions about its components on the scales of the mass of the scalar field, the mimetic theory with a scalar field is reproduced. This UV complection was introduced in order to avoid caustics which might otherwise appear in theories with fluid-like dust. We also presented the derivation of the speed of sound for general higher derivative extensions of scalar mimetic theories with the focus on models with limiting curvature, constructed to avoid cosmological and black hole singularities [158, 159]. For this we employed the corresponding cosmological perturbation theory. The speed of sound in this special case was found to be imaginary, (4.77), and we speculated under which conditions the theory might still be viable. This also shows the limitations of mimetic theories modelling limiting curvature.

Another topic considered in this thesis was the application of Noether's first and second theorem in the context of mimetic theories. As the latter are Weyl invariant, it is natural to ask whether this symmetry leads to any non-trivial conserved quantities. Noether's first theorem was used to calculate the conserved current in the case of scalar (4.105) (4.114) (4.127), vector (5.55) and gauge vector mimetic theories (7.47). In all cases, the identically vanishing current proves that the Weyl symmetry does not introduce separate sectors in the phase space. Noether's second theorem was used to show that there exist differential relations between the respective equations of motion of the auxiliary metric and either the scalar (4.142) (4.148), vector (5.57) or gauge vector fields (7.50). It was shown that these relations are either trivially fulfilled or reproduce the mimetic constraint equation.

Moreover, in chapter 5, for the case of mimetic gravity with a vector field also algebraic solutions of (5.43) were explicitly written for the first time. These other solutions of the mimetic construction show even more clearly that mimetic dark energy is more widespread and common than previously thought.

Following our paper [134], we proved in chapter 7 that solutions of non-abelian formulations of gauge vector mimetic gravity do exist. We considered $SU(N)$ groups and provided explicit constructions for the gauge vector field in the gauge groups $SU(2)$ (7.92) and $SU(3)$ (7.105) – (7.107). Indeed, as we discussed in chapter 6, the $SU(3)$ algebra has three distinct $SU(2)$ subalgebras for which we can employ the $SU(2)$ ansatz. Then formally employing the $SU(2)$ ansatz for each of the subalgebras we can get three different solutions, however we showed that the three solutions can be transformed into each other by gauge transformations. Furthermore, we also demonstrated that the solutions we found for $SU(2)$ cannot be extended either into the eternal past or the eternal future (7.93). Thus, there is a singularity either in the past or the future. This non-abelian formulation of mimetic gravity is interesting because it points towards a possible solution of the cosmological constant problem, similar to the dynamic resolution of the strong CP problem in quantum chromodynamics. Via proving the existence of mimetic solutions within the Lie group $SU(3)$, we strengthened that speculation.

Remaining open questions concern, amongst others, whether one could quantise mimetic theories. Of course, one can always employ the UV completion for this procedure, but it is less clear for the limiting case of the dust-like fluid. As we have seen in the discussion of caustics and the quantum pressure resolving that issue, this might be a fundamental step closer to a better description, as in reality all fields are quantum.

As we have seen in chapter 4, the higher derivative theory modelling limiting curvature leads to an imaginary speed of sound. Therefore, it is interesting to investigate whether there exists another function of higher derivatives such that one can ameliorate this problem within the models of limiting curvature.

Moreover, as there exist mimetic theories for functions of scalar, vector and gauge vector fields, one could speculate about a mimetic theory constructed with the help of a tensor of rank two, or bimetric mimetic gravity. The mimetic theories discussed in the literature so far yield mimetic dark matter, mimetic dark energy and also inflation. Therefore the question is what bimetric mimetic gravity would produce instead, or indeed, if it is useful to address open questions in cosmology.

Another open question concerns the solutions for mimetic dark energy within general $SU(N)$ Lie groups. One could ask, for example, whether solutions from more than one $SU(2)$ subalgebra could be constructed and whether they are topologically distinct from the found solutions. Also, further properties of those already found solutions could be studied, as for now we merely showed the existence of them. It would also be interesting to study the aforementioned singularity of the solutions and discuss whether one could avoid it by finding other, non-singular solutions. Furthermore, it would also be interesting to understand what happens if one restores the kinetic term that was omitted in the limiting procedure from the complex scalar field.

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