Interaction of Closed Strings with Non-Perturbative Objects

Scattering Closed Strings off Dp-Branes and Op-Planes in the Pure Spinor Formalism

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Zusammenfassung

Streuamplituden stellen eine Verbindung zwischen experimenteller und theoretischer Physik her, weil der resultierende Wirkungsquerschnitt zur Überprüfung theoretischer Vorhersagen in Experimenten verwendet werden kann. Darüber hinaus können Streuamplituden genutzt werden, um Erkenntnisse über die Struktur der zugrunde liegenden Theorie zu erhalten. Da die Stringtheorie eine konsistente Theorie der Quantengravitation darstellt, sind insbesondere Gravitationsamplituden phänomenologisch interessant. Die ersten Quantenkorrekturen zur Einstein-Hilbert-Wirkung der Stringtheorie können durch Streuamplituden mit der Kreisscheibe oder der reellen projektiven Ebene als Weltflächen beschrieben werden und beinhalten nicht-perturbative Objekte wie Dp-Branen bzw. Op-Ebenen.

In dieser Dissertation untersuchen wir diese Amplituden. Für beide Konstellationen geben wir eine Vorschrift zur Berechnung der allgemeinen n-Punkt-Amplitude an und diskutieren die Auswirkungen dieser nicht-perturbativen Objekte auf die Welftflächen-Felder. Explizit berechnen wir die Streuamplituden von drei geschlossenen Strings auf der Kreisscheibe und der reellen projektiven Ebene im Pure Spinor Formalismus. Für beide Amplituden wird die analytische Fortsetzung berechnet, um die holomorphen und antiholomorphen Koordinaten der geschlossenen Strings auf den Weltflächen unabhängig voneinander betrachten zu können. Aufgrund der Verzweigungsstruktur des Koba-Nielsen-Faktors müssen Monodromien berücksichtigt werden, so dass wir kompakte Ergebnisse für die Streuung von drei geschlossenen Strings für beide Weltflächen erhalten. Letztendlich können beide Streuprozesse durch Amplituden von sechs offenen Strings auf der Kreisscheibe beschrieben werden. Mit Hilfe dieser Ergebnisse versuchen wir, unsere Erkenntnisse zu verallgemeinern und einen Ansatz für die Streuung von n geschlossenen Strings an einer Dp-Brane herzuleiten, der die allgemeine Struktur der Amplitude auf der Kreisscheibe beschreibt. Diese Verallgemeinerung für eine beliebige Anzahl geschlossener Strings kann durch Amplituden mit $2n$ offenen Strings ausgedrückt werden.

Diese Amplituden beschreiben gravitative Wechselwirkungen auf Baumniveau in Anwesenheit einer Dp-Brane oder Op-Ebene. Deshalb entwickeln wir die Amplitude von drei geschlossenen Strings auf der Kreisscheibe in der inversen Stringspannung und können so relevante gravitative Dp-Branen-Kopplungen analysieren, die mit Korrekturen zur Einstein-Hilbert-Wirkung verbunden sind. Außerdem vergleichen wir die String-Amplitude mit den feldtheoretischen Ergebnissen, die wir von der Dirac-Born-Infeld-Wirkung erhalten haben. Gleichzeitig können wir damit die Konsistenz unserer Berechnungen überprüfen.

Abstract

Scattering amplitudes provide a connection between experimental and theoretical physics, as the corresponding cross section can be used to check theoretical predictions in experiments. In addition, scattering amplitudes can be used to gain insight on the structure of the underlying theory. Since string theory provides a consistent theory of quantum gravity, especially gravitational amplitudes are phenomenologically interesting in this context. The first quantum corrections to the Einstein-Hilbert action of string theory can be captured by scattering amplitudes with the disk or the real projective plane as worldsheets and involve non-perturbative objects namely Dp-branes and Op-planes, respectively.

In this thesis we investigate these kinds of amplitudes: For both setups we provide an amplitude prescription for the general n-point amplitude and discuss the implications of these non-perturbative objects on the worldsheet fields. Explicitly, we calculate the scattering amplitudes of three closed strings on the disk and real projective plane in the pure spinor formalism. We analytically continue both amplitudes to disentangle the holomorphic and antiholomorphic closed string worldsheet coordinates on the disk and real projective plane. By introducing monodromy phases arising form the branch cut structure of the Koba-Nielsen factor we arrive at compact expressions for the scattering of three closed strings on the disk and real projective plane. In the end, both scattering processes can be described in terms color ordered amplitudes of six open strings on the disk. Using these results we try to generalize our findings and provide an ansatz for the scattering of n closed strings from a Dp-brane, which encompasses the general structure of the disk amplitude. This generalization for any number of closed strings can be written in terms of color ordered open string amplitudes involving $2n$ open strings.

Since these amplitudes probe tree-level gravitational interactions in the presence of a Dp-brane or Op-plane, we carry out the low energy expansion in the inverse string tension of the three-point disk amplitude and analyse some relevant gravitational Dp-brane couplings associated to corrections of the Einstein-Hilbert action. In particular, we compare the string amplitude to the analogue field theory calculation obtained from the Dirac-Born-Infeld action and thereby provide a consistency check for our calculations.

Chapter 1

Introduction

At the beginning of the last century, our understanding of the universe underwent a fundamental change with the development of at that time groundbreaking theories: Quantum mechanics [1], special relativity [2] and general relativity (GR) [3, 4]. Originally, they were introduced to solve specific problems: Quantum mechanics provides a description for the energy spectrum of a black body in terms of harmonic oscillators. Moreover, it was observed that also energy carried by light appears in discrete quanta and is not following a continuos spectrum [5], which previously could not be explained by Maxwell's theory. On the other hand, general relativity as a refinement of Newton's law can explain the precession of Mercury [6].

Combining the principles of special relativity and quantum mechanics led to the framework of quantum field theory (QFT), which allowed for a consistent description of fundamental non-gravitational interactions in nature. This progress led to the Standard Model (SM) of particle physics, which successfully explains the unification of electromagnetism, the weak and strong fundamental interactions, which are all described by the exchange of gauge bosons. However, any approach to incorporate gravity in a perturbative quantum field theory has to break down at the Planck scale $M_p \simeq 10^{19}$ GeV [7], because quantum gravitational corrections describe an irrelevant interaction. Meaning that the gravitational coupling grows weaker at low energies and becomes negligible at the energy scale of current particle physics. On the other hand, the interaction becomes stronger at high energies, which leads ultimately to a break down for energies above the Planck scale. Hence, QFT when accommodating gravity into the formalism is a non-renormalizable theory.

Although, the success of the SM and general relativity is undeniable, i.e. they can explain a large number of phenomena, they do not suffice to explain all observations in nature. The observed positive and small value of the cosmological constant deviated from the prediction of the SM by 120 orders of magnitude [8]. Recent measurements [9] might even suggest that the cosmological constant is not-constant at all and the story is more involved. But this is still under investigation and further measurements are needed. Furthermore, a purely phenomenological approach is not satisfying. In theoretical physics understanding the bigger, more complete and perhaps even fundamental picture has always been pursued.

A first step to gain insights in cases, where the quantum nature of gravity becomes non-negligible, is provided by semiclassical gravity as an approximation that treats matter as quantum fields, while simultaneously spacetime is considered to be classical. Moreover, the perturbative expansion for the gravitational interaction has to be truncated at the one-loop level to avoid the issue of non-renormalizability. This ansatz led to important insights in physical processes, where the quantum nature of gravity becomes relevant. Examples of these progress are cosmological perturbation theory [10] and black hole (BH) physics. Especially, the latter one has become of great interest due to recent experimental observations via gravitational waves [11, 12] and direct imaging [13, 14]. The semiclassical approximation of BHs resulted in a thermodynamical description for BHs, which allows to express their entropy such that it depends only on the surface area of the BH [15, 16, 17, 18, 19]. Despite all of this progress the question about the correct and consistent theory of quantum gravity remains unanswered in any of these frameworks.

1.1 String theory as a theory of quantum gravity

String theory is based on a rather simple geometric idea with dramatic consequences: The fundamental degrees of freedom in perturbative string theory are described by one dimensional objects moving through a D dimensional spacetime. The time evolution sweeps out a two dimensional surface called the worldsheet of the string. Similar as the action of a point particle corresponds to the worldline, the string action originates from the according worldsheet. In principle, there are open and closed strings, which can be distinguished depending on whether their endpoints are identified or not. This distinction has huge impact on the spectrum describing the oscillation modes on each string. The lowest massless excitations in the open string spectrum have spin-one and show the same properties as gauge bosons of super Yang-Mills theory. On the other hand, after quantization the closed string spectrum naturally contains a massless spin-two mode, which is identified as the graviton, i.e. the gauge boson mediating gravity. Thus, string theory is a theory of quantum gravity. Because of the extended nature of the string the UV-divergences of point particles in QFT are absent. The interaction between strings is non-local and of purely topological nature, i.e. it is described by the joining and splitting of strings. Schematically, Feynman diagrams in QFT become the propagation of strings as depicted in figure 1.1.

The quantization of strings gives rise to a QFT on the worldsheet which is invariant under conformal transformations. This symmetry makes string theory very powerful, as the underlying conformal field theory (CFT) of string theory is exactly solvable in simple backgrounds. However, this CFT is only anomaly free in ten spacetime dimensions for the superstring. Note that from a worldsheet point of view spacetime and the fields of general relativity and quantum field theories are emergent phenomena, because they arise from string excitations. The number of spacetime dimensions $D = 10$ in string theory is in contradiction to the four spacetime dimensions we observe. While there are attempts to investigate non-critical string theory [20, 21], i.e. string theory with $D \neq 10$, the common approach is to compactify the extra dimensions on a small internal manifold in order to make contact with phenomenology. Essentially, the six extra directions predicted by string theory are curled up on a volume small enough such that they become invisible for current experiments, because the energy necessary to resolve the curled up dimensions is out of reach. This process is a generalization of the idea by Kaluza [22] and Klein [23], where in their proposal the fifth extra dimension is compactified on a circle. To curl up six dimensions in string theory one has to choose a suitable compactification manifold. The physics of the lower dimensional theory, i.e. the couplings, masses and quantum numbers of the particles in the low energy effective theory, is determined by the topology and geometry of the according internal space. Following from the string equations of motion the geometry for this compactification manifold is required to be Ricci-flat or equivalently it has to be a Calabi-Yau manifold. The number of known Calabi-Yau manifolds is large, see for example [24]. Moreover, there were attempts to estimate the total number of Calabi-Yau manifolds [25], which led to astronomically large numbers and it still remains an open question whether this number is finite or not. Hence, finding the correct Calabi-Yau that gives rise to the four dimensional theory we observes, seems like a hopeless task due to the enormous amount of possibilities. However, by imposing finiteness criteria following from universal properties of quantum gravity the swampland program [26] tries to find the landscape of consistent (string theory) vacua and to rule out the inconsistent effective theories in the swampland.

In addition to strings as the fundamental degrees of freedom, in a perturbative regime at weak coupling the consistency of string theory requires $(p+1)$ dimensional objects [27], which are not quantized as fundamental objects, but have to be treated as non-perturbative excitations with their own dynamics. These Dp-branes resemble the endpoints of open strings, which have to satisfy either Dirichlet or Neumann boundary conditions. If for $9-p$ directions the open string has endpoints with Dirichlet boundary conditions, the other endpoints in the remaining $p+1$ dimensions have to lie on the worldvolume of the Dp-brane. Moreover, these object provide an explanation why we observe only four spacetime dimension. This is, because we are confined to live on a D3-brane, which extends along the four visible dimensions and is embedded in some internal, higher dimensional manifold.

1.2 Scattering amplitudes in string theory

String theory is an S-matrix theory and therefore scattering amplitudes, which are the elements of the S-matrix, are the fundamental quantities that originate from a perturbative expansion in the small coupling regime. They describe the transition probability from one state of the asymptotic Hilbert space to another. These final and initial states are asymptotically free states such that they have to be located infinitely far away from the interaction.

Concretely, the elements of the S-matrix in string theory can be obtained from the Polyakov path integral over gauge inequivalent worldsheet metrics. In the exponential of this path integral only the free worldsheet action appears. Adding cubic or higher order interaction terms in the matter fields is not compatible with the residual superconformal gauge transformations, which are necessary for unitarity and anomaly cancellation. However, as already stated above considering locally only freely propagating superstring is sufficient, as interaction originate from the global properties, i.e. the topology, of the worldsheet, which cannot be captured by any higher order interactions in the action. Therefore, the closed string interaction shown in figure 1.1 encompasses the joining and splitting of a worldsheet that locally looks like a free string.

The first scattering amplitude was computed before string theory even existed [28, 29] in order to describe hadron scattering in terms of a dual four point scattering process with special crossing symmetries. In the high energy regime dual models behave much softer than any quantum field theory, which provides a suitable description of hadrons exhibiting a soft high energy behaviour. In the end, it turned our that this amplitudes arises from (bosonic) string theory.

In perturbative QFT computing scattering amplitudes corresponds to summing Feynman diagrams, whereas the perturbative expansion in string theory requires a sum over different worldsheet topologies. Thereby, one worldsheet contributing to each order in perturbation theory covers a wide range of Feynman diagrams of the low energy effective QFT, which is depicted in figure 1.1. In a field theory a single Feynman diagram is only

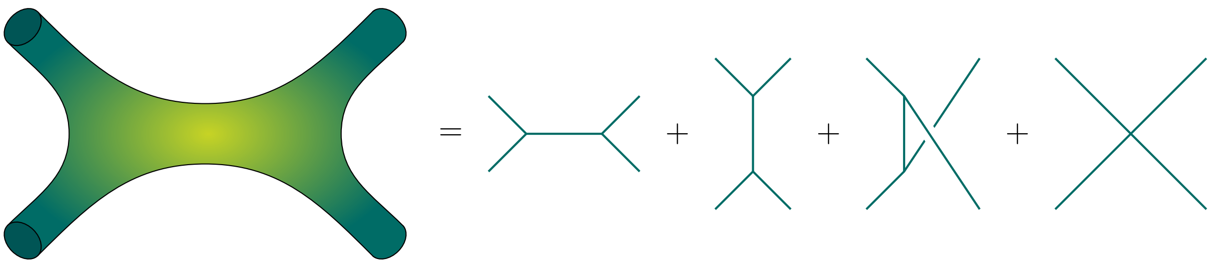

Figure 1.1: A single worldsheet encompasses several Feynman diagrams.

part of the bigger picture so that various kinds of symmetries can be obscured in individual diagrams that would otherwise be present in the complete amplitude. This is absent in string theory, because the interaction of string states is uniquely determined by the free worldsheet theory.

However, string theory is connected to QFT: As physics allows to decouple different energy regimes, we can model two different energy scales by different theories, which provides us with an effective description. Hence, string theory can be viewed as a high energy completion of quantum field theory such that in the limit $\alpha' \to 0$, where the string length

$\ell_s = \sqrt{\alpha'}$ goes to zero, we should recover QFT amplitudes from string scattering processes. Performing an expansion of a string scattering amplitude in α' yields a complex pole structure in Mandelstam variables, where each pole can be associated to different field theoretic scattering channels, i.e. Feynman diagrams. In figure 1.1 the scattering of four closed strings decomposes in the point particle limit into s, t, u channels and a four-point interaction. By taking this limit the Lagrangians of many different theories such as general relativity and super Yang-Mills (SYM) theory can be constructed from superstring scattering amplitudes.

In super Yang-Mills theory the duality between color and kinematics in gauge theories [30], which is hidden in a Lagrangian description, allows to rearrange kinematical factors in amplitudes and to even interchange the role of color and kinematics in the full color decomposition of an amplitude [31]. This duality between color and kinematics originates from the fact that kinematic variables in scattering amplitudes of gauge theories satisfy the same Jacobi identity as the gauge algebra. Replacing the gauge factors by the according kinematic variables in the amplitude establishes a relation between gauge and gravity amplitudes, which can schematically be written as

$$\text{gravity} \sim \text{gauge} \otimes \text{gauge}\ . \tag{1.1}$$

such that some of these properties also carry over to gravitational scattering processes. Since SYM amplitudes can also be derived from string theory in the low energy limit, we expect that they have a stringy origin. This suggests that we might get a deeper understanding of these symmetries from string theory. Indeed, string theory provides the color decomposition of gauge amplitudes almost as a definition, as in any string amplitude the Chan-Paton factors are isolated and the amplitude can be split into gauge invariant pieces. In the end, identities like the Bern-Carrasco-Johansson relations [30, 32, 33] or Kawai-Lewellen-Tye (KLT) relations [34] can be derived and understood from string theory, as they originate form the monodromy properties of the string worldsheet [35, 36].

Eventually, computing scattering amplitudes boils down to presenting the final result in a compact and short form that encodes the symmetries of the underlying theory. For string theory tree-level amplitudes this is achieved by grouping the worldsheet dependence and the kinematics into individual building blocks and expressing both in a minimal basis. Therefore, the calculation of string scattering amplitudes is a intriguing area of research: The result obtained by any means necessary are analysed to find the underlying structure leading to a compact expression. In the end, we try to understand why the simplified results takes this form, which usually does not happen by accident.

1.3 Pure spinor formalism

The underlying degrees of freedom in string theory admit a variety of different formulations: For the calculation of scattering amplitudes the most commonly known formulations of the superstring are the Ramond-Neveu-Schwarz [37, 38, 39, 40] and the Green-Schwarz [41, 42] formalisms. At the beginning of this millennium, a new, consistent formulation of the superstring came to life. Berkovits developed the pure spinor formalism resulting in the first covariantly quantised super string theory that is manifestly super-Poincaré invariant [43, 44, 45]. All of these formulations vary with respect to their implementation of the worldsheet and supersymmetry, but are widely expected to be equivalent [46, 47, 48], which is explicitly verified for the leading orders in string perturbation theory, i.e. all amplitudes computed in RNS or GS formalism are in agreement with the corresponding expressions obtained in PSF.

Moreover, the pure spinor formalism exhibits a framework to organise the kinematic and worldsheet dependencies of a scattering process in a way that allows for a more efficient calculation of string scattering amplitudes compared to the other two formalisms. An outstanding example is the computation of the scattering of four closed strings at two-loops, which required hundred pages in the RNS formalism [49, 50, 51, 52] and [53, 54], see also [55, 56, 57, 58, 59], and was done in the PSF in only ten pages [60, 61]. This efficiency also allowed to compute amplitudes that were out of reach in RNS and GS formalism: Two examples are the scattering of n open strings on the disk [62, 63] and the three-loop scattering process of four closed strings [64].

Except for computing scattering amplitudes the pure spinor formalism can be used to study the propagation of strings in curved backgrounds [65, 66, 67, 68, 69, 70] and more recently [71], which incorporates for example the derivation of equations of motion for non-linear Born-Infeld theory [72]. In addition, strings in $AdS_5 \times S^5$ backgrounds can be studied [73, 74, 75, 76, 77], which led to the construction of vertex operators [78, 79, 80] and the computation of string scattering amplitudes [81, 82] in $AdS_5 \times S^5$. Furthermore, it is possible to investigate Chern-Simons corrections, which are necessary for anomaly cancellation [83].

1.4 Overview and organization of the thesis

There was remarkable progress (partly due to the pure spinor formalism) in the understanding and capabilities to compute scattering amplitudes over the last decades and it is the goal of this thesis to contribute here. We focus on scattering amplitudes at tree-level, as they give rise to the first quantum corrections in the limit $\alpha' \to 0$ to the effective action in string theory. In this context, our main interest are amplitudes with the disk or real projective plane as worldsheet, which describe the interaction of closed strings with non-perturbative objects like Dp-branes and Op-planes. For the scattering of closed strings off a Dp-brane they provide the first gravitational corrections to the Dirac-Born-Infeld (DBI) action and are therefore phenomenologically very interesting. For this reason, there exists already considerable body of literature on disk amplitudes: For example, in [84] higher derivative gravitational corrections to the DBI action are derived from disk amplitudes.

Furthermore, the dilation one-point function in bosonic string theory was calculated in [85, 86] and a generalization for the superstring was found in [87, 88]. The scattering of two superstrings from a Dp-brane was first computed in the RNS formalism and performed in [89, 90, 91, 92]. A detailed review of these computation and the calculation in the PSF can be found in [93, 94, 95, 96] and [88], respectively. For the special case where the external states are described by one RR field and two NSNS fields there are even computations of scattering amplitudes involving three closed strings on the disk [97, 98, 99], but all of the are formulated in the RNS framework. However, none of these compute the full superstring amplitude of three closed strings on the disk. Finally, there is a variety of calculations of disk amplitudes involving open and closed strings in both pure spinor [100, 101] and RNS formalism [35, 102, 103, 104], see also [105, 106, 107, 108].

In this thesis we want to push the analysis of scattering processes of closed strings from non-perturbative objects like Dp-branes and Op-planes. Therefore, we generalize the previous work on amplitudes involving D-branes to arbitrary external states, i.e. states from either NSNS, RNS, NSR or RR sector, for three closed strings and outline the procedure that can be used to compute an n-point closed string amplitude on the disk. Furthermore, we are preforming these calculations on the upper half plane contrary to [35], where the scattering of three closed strings is computed on the double cover, i.e. the sphere. So far, on the disk only the interaction of one closed string with an arbitrary number of open strings was investigated in [104]. But working on the upper half plane has consequences: In the scattering amplitude on the double cover certain poles in the kinematic invariants are missing. The result obtained on the upper half plane in [109] shows all the expected poles.

The scattering of closed strings on the real projective plane was only considered for two external NSNS states in [110] and reviewed in [96]. Here, we provide a prescription for the general n-point amplitude and compute the scattering of three closed strings of an Op-plane utilizing the results from the disk calculation.

This thesis is organized as follows: We start with an overview of $\mathcal{N} = 1$ super Yang-Mills theory in ten spacetime dimensions in chapter 2. There we derive the equation of motions first of the non-linear and afterwards of the linear superfields, which will be used in various pure spinor computations. Moreover, from the equations of motion we compute the power series of the linearized superfields in the fermionic coordinates of pure spinor superspace.

In chapter 3 we provide an introduction in the pure spinor formalism, which includes a derivation of the fundamental degrees of freedom and their CFT originating from Siegel's formulation of the GS superstring. By introducing the pure spinor ghost sector Siegel's ansatz for the superstring becomes anomaly free and consistent with the RNS formalism. We analyse these ghost fields and their CFT in detail and use them to formulate integrated and unintegrated vertex operators, which are used in scattering amplitudes to describe the massless string excitation of the external states.

Chapter 4 discusses the general procedure for the evaluation of scattering amplitudes starting with a prescription for open and closed string scattering amplitudes at tree-level. The non-zero modes of the conformal primaries with weight one can be integrated out using Wick's theorem. Afterwards, the zero modes of the primaries with conformal weight zero give rise to a zero mode correlator, which is evaluated by a zero mode prescription. We conclude this section with a sample calculation that demonstrates this procedure.

Using Wick's theorem in the procedure described in chapter 4 is not very efficient, but Wick contractions can be reorganised in terms of composite superfields. These are defined recursively in chapter 5 and can be used to simplify the organisation of kinematic terms. Then, using the cohomology properties of the pure spinor formalism and the simple form of the BRST charge composite superfields give rise to BRST building blocks, which have an interpretation in terms of tree graphs.

In chapter 6 a generalisation of Berends-Giele currents is constructed, which include also the superpartner of the gluon. Similar as the BRST building blocks they have an interpretation in terms of tree graphs, which allows to express the Berends-Giele supercurrents in terms of expressions obtained from the cohomology of the pure spinor formalism. Furthermore, we derive SYM amplitudes from Berends-Giele supercurrents.

Closed string amplitudes on the sphere are discussed in chapter 7, where we explain how the closed string can be decomposed into two open string amplitudes by using analytic continuation. Thereby, the complex worldsheet integration over the sphere can be split into two integrations over parts of the real line. Furthermore, the open string n-point amplitudes are not independent, which allows to express them in a minimal basis with $(n-3)!$ elements. Therefore, one has to use symmetry properties of the amplitude and monodromy relations, which can be derived from worldsheet properties of the open string amplitudes.

In chapter 8 we analyse the scattering of three closed string off a Dp-brane. First, we discuss the boundary conditions imposed by the D-brane and give an amplitude prescription for the n-point amplitude. Afterwards, we use monodromy relations of the worldsheet, analytic continuation and a $PSL(2,\mathbb{R})$-transformation to express the closed string amplitude in terms of open string six-point partial amplitudes. By performing these contour deformations we generalized the sphere calculation of chapter 7 to the disk. In the end, this leads to a compact formula in which the scattering of three closed strings on the disk is written in terms of only two independent open string subamplitudes instead of six, which is the minimal basis for six open strings.

Using the final result of the scattering of three closed strings on the disk we want to generalise this calculation to an arbitrary number of external states and present an ansatz for the n-point function in chapter 9. Therefore, we comment on the n-point scattering amplitude of open strings on the disk, which are the building blocks of the closed string generalisation.

Similar as for the computation on the disk, we start the discussion in chapter 10 on the scattering of three closed strings on the real projective plan by introducing the conditions imposed by the Op-plane on the worldsheet fields at the T-dual point. Then, we continue again by relating the closed string amplitudes to open string subamplitude using analytic continuation. It turns out that the scattering of three closed string from an Op-plane can be described by four open string partial amplitudes.

In chapter 11 we perform an expansion in α' to obtain the low energy effective description of the scattering of three closed strings on the disk. In addition, we compare our findings to some of the leading terms in the Dirac-Born-Infeld action and comment on the absence of disk-corrections to the Einstein-Hilbert action.

Finally, in chapter 12 we present some concluding remarks and an outlook of possible future work, which includes the generalization of the results in this thesis to higher genera or massive external states.

The appendices contain some technical details: In appendix A we explain some details about the $U(5)$ decomposition of the Wick rotated Lorentz group in ten spacetime dimension, which are relevant for the analysis of the pure spinor constraint and the derivation of the CFT of the ghost sector in chapter 3. Appendix B discusses the invariance of a correlator of holomorphic and antiholomorphic fields on the disk under global conformal transformations. These results are then used in appendix C to explicitly perform a $PSL(2,\mathbb{R})$-transformation of the correlator, which is obtained by performing the vertex operator contractions in the closed string three-point function. Moreover, we express the amplitude in terms of hypergeometric functions and Berends-Giele supercurrents, i.e. SYM amplitudes, which carry the α'-dependence and the kinematic terms, respectively. Finally, in appendix D we give some details on the derivation of the monodromy phase, which ensures that a disk amplitude is well defined after analytic continuation.

This thesis, especially the chapters 8, 9, 10 and 11, are based on the author's work partly published in:

Scattering three closed strings off a Dp-brane in pure spinor formalism [109]
Andreas Bischof, Michael Haack and Stephan Stieberger
JHEP 10 (2023) 184; arXiv: 2308.04175 [hep-th]

Superstring scattering on the real projective plane
Andreas Bischof and Stephan Stieberger
Work in progress

Chapter 2

Super Yang-Mills theory in ten spacetime dimensions

In the low energy limit $\alpha' \to 0$ open superstring excitations describe the interaction of only gluons and gluinos. Hence, it might not be surprising or a coincidence that the degrees of freedom of open strings in the pure spinor formalism are essentially described by $\mathcal{N} = 1$ super Yang-Mills (SYM) theory in ten spacetime dimensions. Its spectrum contains only a gluon and gluino, which are related by sixteen supercharges [111, 112]. These degrees of freedom can be packaged into superfields that are defined on superspace which is spanned by ten spacetime coordinates X^m and their associated sixteen superpartners θ^α. The Grassmann odd spinor variables form a right handed Majorana-Weyl spinor of $SO(1,9)$. In addition, the super-Poincaré covariant formulation [113] of this SYM theory allows the pure spinor formalism to be a very efficient tool in computing scattering amplitudes. The presentation of the superfields here follows the corresponding chapters in [114, 115, 116, 117].

2.1 Non-linear superfields

To define the gauge theory we start by introducing covariant derivatives. Because this theory is defined on superspace, there exist derivatives in the spacetime directions X^m and the spinor space coordinates θ^α

$$\nabla_m = \partial_m - \mathbb{A}_m \ , \qquad \nabla_\alpha = D_\alpha - \mathbb{A}_\alpha \ , \tag{2.1}$$

which are Lie-algebra valued connections for the superfields $\mathbb{A}_m = \mathbb{A}_m(X,\theta)$ and $\mathbb{A}_\alpha = \mathbb{A}_\alpha(X,\theta)$ and obey the Lie-bracket [112, 113]

$$\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m \ . \tag{2.2}$$

Above we have defined the spacetime derivative $\partial_m = \frac{\partial}{\partial X^m}$ and the superspace derivative

$$D_\alpha = \frac{\partial}{\partial \theta_\alpha} + \frac{1}{2}(\gamma^m \theta)_\alpha \partial_m \ , \tag{2.3}$$

which satisfies a Lie-bracket on its own $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$. Moreover, we have introduced the 16×16 Pauli matrices $\gamma^m_{\alpha\beta}$.

The Pauli matrices originate from the 32×32 Dirac matrices Γ^m in ten dimensional Minkowski space $\mathbb{R}^{1,9}$, which satisfy the Clifford algebra

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{1}_{32\times 32} \qquad \text{for } m = 0, 1, \ldots, 9 \ , \tag{2.4}$$

where the signature of the metric η^{mn} is mostly plus $(-, +, +, \ldots, +)$. The chiral matrices γ^m are given by the off-diagonal components of Γ^m in the Weyl representation

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & \end{pmatrix} \tag{2.5}$$

and satisfy a Clifford algebra on their own

$$\gamma^m_{\alpha\beta}(\gamma^n)^{\beta\gamma} + \gamma^n_{\alpha\beta}(\gamma^m)^{\beta\gamma} = 2\eta^{mn}\delta^\delta_\alpha \ . \tag{2.6}$$

More details and the explicit form of the γ^m matrices can be found in [114].

We can define field strength tensors $\mathbb{W}^\alpha$ and $\mathbb{F}_{mn}$ for the superfields $(\mathbb{A}_\alpha, \mathbb{A}_m)$, i.e. for the gluino and gluon, respectively. These non-linear superfields have to obey the equations of motion[1] [112, 113]

$$\begin{aligned} &\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m \ , && [\nabla_\alpha, \nabla_m] = -(\gamma_m \mathbb{W})_\alpha \ , \\ &\{\nabla_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} \ , && [\nabla_\alpha, \mathbb{F}_{mn}] = -(\mathbb{W}^{[m}\gamma^{n]})_\alpha \ , \end{aligned} \tag{2.8}$$

where we have introduced

$$\mathbb{F}_{mn} = -[\nabla_m, \nabla_n] \ , \qquad \mathbb{W}^\alpha_m = [\nabla_m, \mathbb{W}^\alpha] \ . \tag{2.9}$$

These on-shell constraints can be derived from (2.2) and the associated Bianchi identities.

The equations of motion (2.8) are invariant under infinitesimal gauge transformations of the gluon and gluino superfields $(\mathbb{A}_m, \mathbb{A}_\alpha)$. These gauge transformations can be described by a Lie algebra-valued gauge parameter $\Omega = \Omega(X, \theta)$ such that

$$\delta_\Omega \mathbb{A}_\alpha = [\nabla_\alpha, \Omega] \ , \qquad \delta_\Omega \mathbb{A}_m = [\nabla_m, \Omega] \ . \tag{2.10}$$

[1] Except otherwise stated, in expression, where we symmetrize or antisymmetrize indices, we use the convention

$$\begin{aligned} A^1_{[a_1}A^2_{a_2}\cdots A^k_{a_k]} &= A^1_{a_1}A^2_{a_2}\cdots A^k_{a_k} \pm \text{permutations} \ , \\ A^1_{(a_1}A^2_{a_2}\cdots A^k_{a_k)} &= A^1_{a_1}A^2_{a_2}\cdots A^k_{a_k} + \text{permutations} \ , \end{aligned} \tag{2.7}$$

where we do not include a factor of $\frac{1}{k!}$.

Similarly, the associated field strengths transform as

$$\delta_\Omega \mathbb{W}^\alpha = [\Omega, \mathbb{W}^\alpha] \ , \qquad \delta_\Omega \mathbb{F}^{mn} = [\Omega, \mathbb{F}^{mn}] \ , \qquad \delta_\Omega \mathbb{W}^\alpha_m = [\Omega, \mathbb{W}^\alpha_m] \ . \tag{2.11}$$

Moreover, we can derive the massless Dirac and Yang-Mills equations

$$\gamma^m_{\alpha\beta}[\nabla_m, \mathbb{W}^\beta] = 0 \ , \qquad [\nabla_m, \mathbb{F}^{mn}] = \gamma^n_{\alpha_\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \tag{2.12}$$

using the constraint in (2.2) and the equations of motion

$$\begin{aligned}\gamma^m_{\alpha\beta}[\nabla_m, \mathbb{W}^\beta] &= [\{\nabla_\alpha, \nabla_\beta\}, \mathbb{W}^\beta] \\ &= -[\{\mathbb{W}^\beta, \nabla_\alpha\}, \nabla_\beta] - [\{\nabla_\beta, \mathbb{W}^\beta\}, \nabla_\alpha] \\ &= -\frac{1}{4}(\gamma^{mn})_\alpha{}^\beta[\mathbb{F}^{mn}, \nabla_\beta] \\ &= \frac{1}{4}(\gamma^{mn}\gamma_n\mathbb{W}_m)_\alpha - \frac{1}{4}(\gamma^{mn}\gamma_m\mathbb{W}_n)_\alpha \\ &= \frac{9}{2}\gamma^m_{\alpha\beta}[\nabla_m, \mathbb{W}^\beta] \ ,\end{aligned} \tag{2.13}$$

where we furthermore used that γ^{mn} is a traceless matrix, i.e. $(\gamma^{mn})_\alpha{}^\alpha = 0$ and the identity $\gamma^{mn}\gamma^n = 9\gamma^m$. Hence, equation (2.13) implies that $\gamma^m_{\alpha\beta}[\nabla_m, \mathbb{W}^\beta] = 0$. The Yang-Mills equation in (2.12) can be obtained by taking the anticommutator of the Dirac equation with $\gamma^{\alpha\delta}_n\nabla_\delta$. By utilizing the Bianchi identity this anticommutator evaluates to

$$\begin{aligned}0 &= \frac{1}{8}\gamma^{\alpha\delta}_n\gamma^m_{\alpha\beta}\{\nabla_\delta, [\nabla_m, \mathbb{W}^\beta]\} \\ &= \frac{1}{8}\gamma^{\alpha\delta}_n\gamma^m_{\alpha\beta}\Big(\{\mathbb{W}^\beta, [\nabla_\delta, \nabla_m]\} + \{\nabla_m, [\mathbb{W}^\beta, \nabla_\delta]\}\Big) \\ &= -\frac{1}{8}\gamma^{\alpha\delta}_n\gamma^m_{\alpha\beta}\Big((\gamma_m)_{\delta\sigma}\{\mathbb{W}^\beta, \mathbb{W}^\sigma\} - \frac{1}{4}(\gamma^{rs})_\delta{}^\beta[\nabla_m\mathbb{F}_{rs}]\Big) \\ &= \gamma^n_{\beta\sigma}\{\mathbb{W}^\beta, \mathbb{W}^\sigma\} - [\nabla_m, \mathbb{F}^{mn}] \ .\end{aligned} \tag{2.14}$$

To derive this result we have used that $-(\gamma^m\gamma^n\gamma_m)_{\beta\sigma} = 8\gamma^n_{\beta\sigma}$, which follows from the Clifford algebra of the γ-matrices (2.6), $\gamma^m\gamma_m = 10$ and that $\frac{1}{4}\operatorname{Tr}(\gamma_m\gamma_n\gamma^{rs}) = 4(\delta^r_n\delta^s_m - \delta^r_m\delta^s_n)$.

The equations of motion (2.8) of the superfields can be written in an alternative form

$$\begin{aligned}\{\nabla_\alpha, \mathbb{A}_\alpha\} + \{\nabla_\beta, \mathbb{A}_\alpha\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m - \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} \ , \qquad & [\nabla_\alpha, \mathbb{A}_m] &= [\partial_m, \mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha \ , \\ \{\nabla_\alpha, \mathbb{W}^\beta\} &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} \ , & [\nabla_\alpha, \mathbb{F}^{mn}] &= (\mathbb{W}^{[m}\gamma^{n]})_\alpha \ .\end{aligned} \tag{2.15}$$

When substituting (2.1) in (2.15) the above equations become

$$\begin{aligned}\{D_\alpha, \mathbb{A}_\alpha\} + \{D_\beta, \mathbb{A}_\alpha\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} \ , \\ [D_\alpha, \mathbb{A}_m] &= [\partial_m, \mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m] \ ,\end{aligned}$$

$$
\begin{aligned}
\{D_\alpha, \mathbb{W}^\beta\} &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\} \ , \\
[D_\alpha, \mathbb{F}^{mn}] &= (\mathbb{W}^{[m}\gamma^{n]})_\alpha + [\mathbb{A}_\alpha, \mathbb{F}^{mn}] \ .
\end{aligned}
\tag{2.16}
$$

More details on the non-linear superfields and their higher mass dimension generalisation can be found in [114].

2.2 Linearized superfields

The asymptotic states in string scattering processes are described by the linearized description of the ten dimensional superfields, because interactions will be introduced later by a perturbative approach based on string theory. These linearized superfields are obtained by discarding the non-linear terms in (2.16) that are quadratic in the superfields

$$
\begin{aligned}
D_\alpha A_\beta + D_\beta A_\alpha &= \gamma^m_{\alpha\beta} A_m \ , & D_\alpha A_m &= (\gamma_m W)_\alpha + \partial_m A_\alpha \ , \\
D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn} \ , & D_\alpha F_{mn} &= \partial_{[m}(\gamma_{n]} W)_\alpha \ .
\end{aligned}
\tag{2.17}
$$

The linearized superfields are invariant under the linearized version of the gauge transformations in (2.10), which are given by

$$
\delta_\Omega A_\alpha = D_\alpha \Omega, \qquad \delta_\Omega A_m = \partial_m \Omega \ . \tag{2.18}
$$

The linearized gauge transformations will be important for the definition of the massless vertex operators in the pure spinor formalism in section 3.6.

The superfield A_m is an auxiliary field and not independent of A_α. Because a bispinor $D_{(\alpha} A_{\beta)}$ contains a one-form and a five-form, we can rephrase the first equation in (2.17) as $\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0$, which is a constraint that puts the fields on-shell. With the on-shell constraint A_m can be expressed in terms of the spinorial superfield A_α by

$$
A_m = \frac{1}{8}\gamma_m^{\alpha\beta} D_\alpha A_\beta \ . \tag{2.19}
$$

Note that the vanishing five form does not put the fields on-shell for $D < 10$, i.e. if this computation was carried out in lower dimensions the constraint would not eliminate any p form component from $D_{(\alpha} A_{\beta)}$ [115, 116].

Moreover, the fields (W_α, F_{mn}), which are the associated field strengths to (A_α, A_m), can be expressed in terms of (A_α, A_m): By antisymmetrizing either two spinorial or one spinorial and one spacetime component of the (A_α, A_m)-Jacobi matrix we can define the the gauge invariant field strength tensors

$$
W^\alpha = \frac{1}{10}\gamma_m^{\alpha\beta}(D_\beta A_m - \partial^m A_\beta) \ , \tag{2.20}
$$

$$
F^{mn} = \partial_m A_n - \partial_n A_m \ . \tag{2.21}
$$

These linearized field strength tensors satisfy the linearized version of (2.12), which can be obtained from (2.17). Therefore, we have to act on $D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}$ with D_γ and symmetrize the expression with respect to the indices (α, γ). If we then take the δ^γ_β trace, this yields a massless Dirac equation for the linear superfield W^α

$$\gamma^m_{\alpha\beta}\partial_m W^\beta = 0 \ . \tag{2.22}$$

Acting with the derivative $\gamma^{\alpha\gamma}_n D_\gamma$ on the Dirac equation (2.22) and also using the equation of motions of the linear superfields (2.17) gives

$$\partial_m F^{mn} = 0 \ . \tag{2.23}$$

Since (2.22) and (2.23) describe equations of motion for the gluino and gluon components of the superfields, we can already conclude that the lowest order θ-components in the θ-expansion of W^α and A_m correspond to the gaugino and gauge boson, respectively [116].

2.3 The θ-expansion of linearized superfields

The equations of motion (2.17) of the $\mathcal{N} = 1$ SYM superfields $(A_\alpha, A_m, W^\alpha, F^{mn})$ can be solved separately for X^m and θ^α. We can expand the superfields in θ^α, where at each order $\mathcal{O}(\theta^k)$ the expansion coefficient $\Phi^{(k)}(X)$ is a spacetime function. Because of the Grassmann odd nature of the spinorial coordinates any power series expansion in θ will terminate at order $\mathcal{O}(\theta^{16})$. Moreover, we find recursive relations between the expansion coefficients $\Phi^{(k)}(X)$ at neighbouring orders of θ^k. When making an appropriate gauge choice (Harnad–Shnider gauge) for Ω in (2.18) we can enforce

$$\theta^\alpha A_\alpha(X, \theta) = 0 \ , \tag{2.24}$$

which is the supersymmetric analogue of choosing normal coordinates and simplifies the θ-expansion of the superfields. In addition, by imposing this gauge we can convert the canonical differential operator D_α of the spinorial coordinates into an ordinary derivative $\theta^\alpha D_\alpha = \theta^\alpha \frac{\partial}{\partial\theta^\alpha}$. We can organize the X^m-dependence of the superfields in terms of plane waves with momentum k^m and parametrize the solution to the equations of motion by the polarization vector e^m and spinor χ^α for the gluon and gluino, i.e. we can set

$$e_m = A_m\Big|_{\theta=0} \ , \qquad \chi^\alpha = W^\alpha\Big|_{\theta=0} \ . \tag{2.25}$$

Putting everything together this leads to the following recursion relations for the coefficients $\Phi^{(k)}$ in the θ-expansion

$$\begin{aligned} A^{(k)}_\alpha &= \frac{1}{n+1}(\gamma^m\theta)_\alpha A^{(k-1)}_m \ , \\ A^{(k)}_m &= \frac{1}{n}(\theta\gamma_m W^{(k-1)}) \ , \end{aligned}$$

$$(W^{(k)})^\alpha = -\frac{1}{2n}(\gamma^{mn}\theta)^\alpha \partial_m A_n^{(k-1)} \ . \tag{2.26}$$

These recursive relations above can be solved by the following ansatz

$$\begin{aligned} A_m^{(2k)} &= \frac{1}{(2k)!}[\mathcal{O}^k]_m{}^q e_q \ , \\ A_m^{(2k+1)} &= \frac{1}{(2k+1)!}[\mathcal{O}^k]_m{}^q (\theta\gamma_q u) \ , \end{aligned} \tag{2.27}$$

where we have introduced the expression

$$[\mathcal{O}]_m{}^q = \frac{1}{2}(\theta\gamma_m{}^{qp})\partial_p \ . \tag{2.28}$$

The equations (2.26) together with (2.27) fully determine the θ-expansion of all superfields [118]. Then, the first terms in the θ^α power series of the superfields up to order $\mathcal{O}(\theta^6)$ are given by [119, 120]

$$\begin{aligned} A_\alpha(X,\theta) = e^{ik\cdot X}\Big\{ &\frac{e_m}{2}(\gamma^m\theta)_\alpha - \frac{1}{3}(\chi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta)ik_{[m}e_{n]} \\ &+\frac{1}{60}(\gamma_m\theta)_\alpha ik_n(\chi\gamma_p\theta)(\theta\gamma^{mnp}\theta) \\ &+\frac{1}{1152}(\gamma^m\theta)_\alpha ik^r(\theta\gamma_{mrs}\theta)(\theta\gamma^{spq}\theta)ik_{[p}e_{q]} + \mathcal{O}(\theta^6)\Big\} \ , \end{aligned} \tag{2.29}$$

$$\begin{aligned} A_m(X,\theta) = e^{ik\cdot X}\Big\{ &e_m - (\chi\gamma_m\theta) - \frac{1}{4}ik_p(\theta\gamma_m{}^{pq}\theta)e_q + \frac{1}{12}ik_p(\theta\gamma_m{}^{pq}\theta)(\chi\gamma_q\theta) \\ &+\frac{1}{96}ik_n(\theta\gamma_m{}^{np}\theta)ik_q(\theta\gamma_p{}^{qr}\theta)e_r \\ &-\frac{1}{480}ik_n(\theta\gamma_m{}^{np}\theta)ik_q(\theta\gamma_p{}\ \theta)(\theta\gamma_r\chi) + \mathcal{O}(\theta^6)\Big\} \ , \end{aligned} \tag{2.30}$$

$$\begin{aligned} W^\alpha(X,\theta) = e^{ik\cdot X}\Big\{ &\chi^\alpha - \frac{1}{4}ik_{[m}e_{n]}(\gamma^{mn}\theta)^\alpha + \frac{1}{4}ik_m(\gamma^{mn}\theta)^\alpha(\chi\gamma_n\theta) \\ &+\frac{1}{48}ik_m(\gamma^{mn}\theta)^\alpha(\theta\gamma_n{}^{pq}\theta)ik_{[p}e_{q]} - \frac{1}{96}(\gamma^{mn}\theta)^\alpha ik_m(\theta\gamma_n{}^{pq}\theta)ik_p(\chi\gamma_q\theta) \\ &-\frac{1}{768}(\gamma^{mn}\theta)^\alpha ik_m(\theta\gamma_n{}^{pq}\theta)ik_p(\theta\gamma_q{}^{rs}\theta)ik_r e_s + \mathcal{O}(\theta^6)\Big\} \ , \end{aligned} \tag{2.31}$$

$$\begin{aligned} F_{mn}(X,\theta) = e^{ik\cdot X}\Big\{ &ik_{[m}e_{n]} - ik_{[m}(\chi\gamma_{n]}\theta) - \frac{1}{4}ik_{[p}e_{q]}ik_{[m}(\theta\gamma_{n]}{}^{pq}\theta) \\ &+\frac{1}{12}ik_{[m}(\theta\gamma_{n]}{}^{pq}\theta)ik_p(\chi\gamma_q\theta) + \frac{1}{96}ik_{[m}(\theta\gamma_{n]}{}^{tp}\theta)ik_t ik_q(\theta\gamma_p{}^{qr}\theta)e_r \\ &-\frac{1}{480}ik_{[m}(\theta\gamma_{n]}{}^{tp}\theta)ik_t(\theta\gamma_p{}^{qr}\theta)ik_q(\theta\gamma_r\chi) + \mathcal{O}(\theta^6)\Big\} \ . \end{aligned} \tag{2.32}$$

For the purpose of calculating tree-level scattering amplitudes an expansion up to $\mathcal{O}(\theta^5)$ is sufficient. Higher order terms don't contribute to scattering amplitudes due to the zero mode prescription (4.15).

Although, the series expansion in (2.32) terminates at $\mathcal{O}(\theta^{16})$, it can be formally resumed to all orders in θ^α as [118]

$$A_m = e^{ik\cdot X}\Big\{[\cosh\sqrt{\mathcal{O}}]_m{}^q e_q + [\sqrt{O}^{-1}\sinh\sqrt{\mathcal{O}}]_m{}^q(\theta\gamma_q\chi)\Big\}\ . \tag{2.33}$$

The expansion in (2.32) follows an alternating pattern with respect to the gauge boson and gaugino. For the spinorial superfields (A_α, W^α) the gluon polarization vector appears along an odd power of θ and the gluino wave function with an even power of θ. For the bosonic superfields (A_m, F_{mn}) this behaviour is exactly opposite [116].

Chapter 3

Pure spinor formalism

The pure spinor formalism is an alternative description of the superstring to the RNS and GS formalism. The discovery by Berkovits in [43] led to an efficient method for computing superstring scattering amplitudes: It combined various convenient properties of the RNS [37, 38, 39, 40] and GS [41, 42] formulations such that calculations of scattering amplitudes are possible, which were previously out of reach. The pure spinor formalism is inspired by the GS superstring, but is quantized in a $U(5)$ covariant way such that it is not necessary to go to light cone gauge. Due to this Lorentz covariant quantization one can circumvent drawbacks like a non-covariant gauge and restricted momenta. Moreover, the pure spinor superstring is manifestly supersymmetric and does not contain worldsheet spinors, which makes an explicit summation over spin structures obsolete. Compared to the RNS string this shortens the calculation of higher loop amplitudes dramatically [116]. Nevertheless, the two formalism can be related via field redefinitions [121].

In this chapter we will review the basic aspects of the CFT of the pure spinor formalism. We will set the stage for computing superstring scattering amplitudes on genus zero Riemann surfaces. The presentation will follows [114, 117], which are based on the PhD theses [115, 116].

3.1 Origins of the pure spinor formalism

Heterotic strings [122, 123, 124], type I superstrings [125, 126, 127] and type II superstrings [128] are supersymmetric in ten space time dimensions. Therefore, one would also like to find a manifestly supersymmetric description of their worldsheet action. In principal, this is achieved by the GS formalism, but unfortunately the classical action cannot be quantized in a Lorentz covariant way.

In 1986 Warren Siegel [129] made another approach for covariant quantization of the GS superstring. This ansatz uses the spacetime coordinates X^m of the bosonic string, which are complemented by the spinorial fields θ^α. The fermionic coordinates are Majorana-Weyl spinors that transform under $SO(1,9)$ and have 16 real components. These are basically the same degrees of freedom as in the GS formalism, but here the conjugated momenta p_α for θ^α are treated as independent variables in contrast to the GS formalism. We can propose the following action for this ansatz[1]

$$S_{\text{Siegel}} = \frac{1}{\pi}\int \mathrm{d}^2 z\left[\frac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial}\theta^\alpha + \overline{p}_{\hat{\alpha}}\partial\overline{\theta}^{\hat{\alpha}}\right] , \tag{3.1}$$

where the spinor indices $\hat{\alpha}, \hat{\beta}, \ldots$ of the antiholomorphic fields have the same (opposite) chirality as $\alpha, \beta, \ldots$ for type IIB (type IIA). Moreover, for open strings we would only have the left-moving sector of (3.1).[2] Since the holomorphic and antiholomorphic sectors are independent, we will only focus on the holomorphic sector in the remaining chapter. Moreover, for closed strings the antiholomorphic expressions are analogous to the holomorphic ones.

Similar as for p_α the fermionic constraint d_α in the GS formalism becomes a unconstrained variable

$$d_\alpha = p_\alpha - \frac{1}{2}\Big(\partial X^m + \frac{1}{4}(\theta\gamma^m\partial\theta)\Big)(\gamma_m\theta)_\alpha , \tag{3.2}$$

which makes the problem of mixed first and second class constraints of the GS superstring absent in Siegel's approach. Note that the constraint of the GS formalism differs slightly from the field (3.2) by an expression that is proportional to $\overline{\partial}\theta^\alpha$ and vanishes by the equation of motion for p_α.

The spacetime supersymmetry transformations that leave the action (3.1) invariant are given by

$$\begin{aligned}
\delta X^m &= \frac{1}{2}(\eta\gamma^m\theta) , \\
\delta\theta^\alpha &= \eta^\alpha , \\
\delta p_\alpha &= -\frac{1}{2}\partial X_m(\eta\gamma^m)_\alpha + \frac{1}{8}(\eta\gamma^m\theta)(\partial\theta\gamma_m)_\alpha ,
\end{aligned} \tag{3.3}$$

where η^α is a Grassmann odd infinitesimal parameter. These supersymmetry transformations are generated by the charge

$$\mathcal{Q}_\alpha = \oint \frac{\mathrm{d}y}{2\pi i}\left[p_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\Big(\partial X_m + \frac{1}{12}(\theta\gamma_m\partial\theta)\Big)\right] , \tag{3.4}$$

[1] We have chosen different values for α' for open and closed strings, i.e. $\alpha'_{\text{closed}} = 2$ and $\alpha'_{\text{open}} = \frac{1}{2}$, because we are always considering open and closed strings seperately. Moreover, this allows for a unified treatment of open and closed strings in the following, e.g. we can use the same OPEs for open and closed strings.

[2] In the spirit of [27] we use the following words synonymous

holomorphic = left-moving
antiholomorphic = right-moving

for the degrees of freedom on the string.

which satisfies a supersymmetry algebra

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \oint \frac{\mathrm{d}z}{2\pi i}\, \gamma^m_{\alpha\beta}\partial X_m \ . \tag{3.5}$$

We can check that the action (3.1) is invariant under the transformations in (3.3): The infinitesimal variation of the action is given by

$$\begin{aligned}\delta S_{\text{Siegel}} = \frac{1}{\pi}\int \mathrm{d}z \Big[&\frac{1}{4}(\eta\gamma^m\partial\theta)\overline{\partial}X_m + \frac{1}{4}(\eta\gamma^m\overline{\partial}\theta)\partial X_m - \frac{1}{2}(\eta\gamma^m\overline{\partial}\theta)\partial X_m \\ &-\frac{1}{8}(\overline{\partial}\theta\gamma_m\partial\theta)(\eta\gamma^m\theta)\Big]\end{aligned} \tag{3.6}$$

and we can show that δS_{Siegel} vanishes. The three terms in the first line cancel against each other, which follows after integration by parts of the first term in (3.6). Hence, the last term in (3.6) has to be identical zero, which can be seen by also integrating this term by parts twice [115]

$$\begin{aligned}\int \mathrm{d}^2z\,(\overline{\partial}\theta\gamma^m\partial\theta)(\eta\gamma_m\theta) &= -\int \mathrm{d}^2z\,(\theta\gamma^m\partial\theta)(\eta\gamma_m\overline{\partial}\theta) - \int \mathrm{d}^2z\,(\theta\gamma^m\overline{\partial}\partial\theta)(\eta\gamma_m\theta) \\ &= -\int \mathrm{d}^2z\,(\theta\gamma^m\partial\theta)(\eta\gamma_m\overline{\partial}\theta) + \int \mathrm{d}^2z\,(\partial\theta\gamma^m\overline{\partial}\theta)(\eta\gamma_m\theta) \\ &\quad + \int \mathrm{d}^2z\,(\theta\gamma^m\overline{\partial}\theta)(\eta\gamma_m\partial\theta) \\ &= -\int \mathrm{d}^2z\,\theta^\alpha\partial\theta^\beta\eta^\rho\overline{\theta}^\sigma\left(\gamma^m_{\alpha\beta}(\gamma_m)_{\rho}\sigma - \gamma^m_{\beta\rho}(\gamma_m)_{\rho\alpha} + \gamma^m_{\alpha\sigma}(\gamma_m)_{\rho\beta}\right) \\ &= 2\int \mathrm{d}^2z\,\theta^\alpha\partial\theta^\beta\eta^\rho\overline{\theta}^\sigma\left(\gamma^m_{\beta\sigma}(\gamma_m)_{\rho\alpha}\right) \\ &= -2\int \mathrm{d}^2z\,(\overline{\partial}\gamma^m\partial\theta)(\eta\gamma_m\theta)\ ,\end{aligned} \tag{3.7}$$

where we used that $\gamma^m_{\alpha(\beta}(\gamma_m)_{\rho\sigma)} = 0$. After bringing the right hand side of the above equation to the left hand side we have shown that $\int \mathrm{d}^2z\,(\overline{\partial}\theta\gamma^m\partial\theta)(\eta\gamma_m\theta) = 0$. Hence, the variation (3.6) of the action (3.1) vanishes under supersymmetry transformations (3.3) and we can conclude that Siegel's approach is indeed spacetime supersymmetric.

For the action (3.1) the holomorphic component $T = T(z)$ of the energy momentum tensor is given by

$$T = -\frac{1}{2}\partial X^m\overline{\partial}X_m - p_\alpha\partial\theta^\alpha = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha\ , \tag{3.8}$$

where the supersymmetric momentum is defined as

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta)\ . \tag{3.9}$$

Moreover, the action yields a Lorentz current for the spinor variables

$$\Sigma^{mn} = -\frac{1}{2}(p\gamma^{mn}\theta)\ , \tag{3.10}$$

which can be found using Noether's method for the infinitesimal Lorentz transformations of p_α and θ^α

$$\delta p_\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta, \qquad \delta\theta^\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\theta^\beta \tag{3.11}$$

and defining the variation of the action (3.1) to be

$$\delta S_{\text{Siegel}} = -\frac{1}{\pi}\int \mathrm{d}^2 z\, \frac{1}{2}\overline{\partial}\varepsilon_{mn}\Sigma^{mn} \,. \tag{3.12}$$

The calculation is straightforward and using the antisymmetry $(\gamma^{mn})_\alpha{}^\beta = -(\gamma^{mn})^\beta{}_\alpha$ gives

$$\begin{aligned}\delta S_{\text{Siegel}} &= \frac{1}{\pi}\int \mathrm{d}^2 z\, \delta(p_\alpha\overline{\partial}\theta^\alpha)\\ &= \frac{1}{\pi}\int \mathrm{d}^2 z \left[\frac{1}{4}\varepsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta\overline{\partial}\theta^\alpha + \frac{1}{4}p_\alpha\overline{\partial}(\varepsilon_{mn}(\gamma^{mn}\theta)^\alpha)\right]\\ &= \frac{1}{\pi}\int \mathrm{d}^2 z \left[\frac{1}{4}\overline{\partial}\varepsilon_{mn}p_\alpha(\gamma^{mn})^\alpha{}_\beta\theta^\beta\right]\end{aligned} \tag{3.13}$$

such that when we compare (3.12) with the actual variation under Lorentz transformations (3.13), we recover the Lorentz current (3.10).

The action (3.1) is invariant under conformal transformations and therefore defines a conformal field theory. With standard path integral methods we can derive the following operator product expansions (OPEs) among the fields in (3.1) [129]

$$\begin{aligned} X^m(z,\overline{z})X^n(w,\overline{w}) &\sim -\eta^{mn}\ln|z-w|^2\,, & p_\alpha(z)\theta^\beta(w) &\sim \frac{\delta_\alpha{}^\beta}{z-w}\,,\\ d_\alpha(z)d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\Pi_m(w)}{z-w} & d_\alpha(z)\Pi^m(w) &\sim \frac{(\gamma^m\partial\theta(w))_\alpha}{z-w}\,,\\ \Pi^m(z)\Pi^n(m) &\sim -\frac{\eta^{mn}}{(z-w)^2}\,, & d_\alpha(z)\theta^\beta(w) &\sim \frac{\delta^{\alpha\beta}}{z-w}\,, \end{aligned} \tag{3.14}$$

where $\sim$ indicates that we dropped the regular terms on the right hand side of the OPEs as $z \to w$. In principle, it would be sufficient to only have the OPEs between X^m and itself and between p_α and θ^α, since these are the fundamental fields and Π^m and d_α are composite fields. But for calculating scattering amplitudes it is helpful to have the OPEs involving Π^m and d_α, because in the pure spinor formalism these are conformal primary fields that appear in the vertex operators. Using the energy momentum tensor (3.8) and the OPEs (3.14) we find that the holomorphic conformal weight of the primaries $\partial X^m, \Pi^m, p_\alpha$ and d_α is $h = 1$ and θ^α is $h = 0$.

Even though, this approach has some advantages one also has to deal with difficulties, which will later be addressed by the pure spinor formalism introduced in section 3.2. When quantized the theory becomes anomalous, because it has a non-vanishing central charge, which raises a first major concern. Each component of the bosonic spacetime coordinates X^m contributes $c_X = 1$ [27] and each pair of $(p_\alpha, \theta^\alpha)$ contributes $c_{p,\theta} = -2$ to the total central charge, where we have used that the two fermionic fields correspond to a bc ghost system with $\lambda = 1$ [27, 37]. Therefore, in ten spacetime dimensions the total central charge of the energy momentum tensor (3.8) is

$$c_{\text{total}} = 10 \cdot c_X + 16 \cdot c_{p,\theta} = -22 \ . \tag{3.15}$$

In [129] a supersymmetric integrated vertex operator for a massless open string was proposed

$$U^{\text{Siegel}} = \int \mathrm{d}z(\partial\theta^\alpha A_\alpha(X,\theta) + \Pi^m A_m(X,\theta) + d_\alpha W^\alpha(X,\theta)) \ , \tag{3.16}$$

where $\{A_\alpha, A_m, W^\alpha\}$ are the linearized SYM fields from section 2.2. But U^{Siegel} cannot lead to the same results for amplitudes computed in the RNS or GS formalism, because it does not satisfy the same OPEs [43]. When using the θ-expansions (2.32) of the superfields the vertex operator (3.16) becomes

$$U^{\text{Siegel}}_{\text{gluon}} = \int \mathrm{d}z \left(e^m \partial X_m - \frac{1}{4}(p\gamma^{mn}\theta) f_{mn} + \mathcal{O}(\theta^2) \right) e^{ik\cdot X} \tag{3.17}$$

up to order θ^2, where we only kept the terms of the θ-expansion describing a gluon with polarization e^m and $f_{mn} = ik_m e_n - ik_n e_m$ is the field strength of the gluon. In the RNS formalism a gluon vertex operator is given by [130]

$$U^{\text{RNS}}_{\text{gluon}} = \int \mathrm{d}z \left(e^m \partial X_m - \frac{1}{2}\psi^m \psi^n f_{mn} \right) e^{ik\cdot X} \ . \tag{3.18}$$

The fields ψ^m are the worldsheet fermions of the RNS formalism, which have conformal weight $h_\psi = \frac{1}{2}$. Comparing the vertex operators in (3.17) and (3.18) one can read of contributions to the Lorentz currents coming from the fermionic fields, which are the operators multiplied by $\frac{1}{2}f_{mn}$, in each formalism

$$\Sigma^{mn}_{\text{RNS}} = -\psi^m\psi^n, \qquad \Sigma^{mn}_{\text{Siegel}} = -\frac{1}{2}(p\gamma^{mn}\theta). \tag{3.19}$$

Computing the OPE between the currents shows a significant difference. In the RNS formalism we find that the level of the Kač-Moody current algebra is $+1$, which can be read of from the coefficient of the double pole term of the OPE

$$\Sigma^{mn}_{\text{RNS}}\Sigma^{pq}_{\text{RNS}} \sim \frac{\eta^{p[m}\Sigma^{n]q}_{\text{RNS}} - \eta^{q[m}\Sigma^{n]p}_{\text{RNS}}}{z-w} + \frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2} \ . \tag{3.20}$$

Using the OPE between θ^α and p_α in (3.14) the OPE between Lorentz currents in the PSF takes the form

$$\Sigma^{mn}_{\text{Siegel}}\Sigma^{pq}_{\text{Siegel}} \sim \frac{1}{4}\frac{p(\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn})\theta}{z-w} + \frac{1}{4}\frac{\text{Tr}(\gamma^{mn}\gamma^{pq})}{(z-w)^2}$$

$$= \frac{\eta^{p[m}\Sigma^{n]q}_{\text{Siegel}} - \eta^{q[m}\Sigma^{n]p}_{\text{Siegel}}}{z-w} + 4\frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2} \ . \tag{3.21}$$

Above we used that $\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn} = 2\eta^{np}\gamma^{mq} - 2\eta^{nq}\gamma^{mp} + 2\eta^{mq}\eta^{np} - 2\eta^{mp}\gamma^{nq}$ and $\text{Tr}(\gamma^{mn}\gamma^{pq}) = 16\delta^m_{[q}\delta^n_{p]}$ to obtain (3.21). The double pole has a coefficient of +4 such that the level of the Kač-Moody current algebra in the PSF differs from the RNS formalism. When computing gluon scattering amplitudes with the two vertex operators in (3.16) and (3.18) the different current algebra levels will lead to discrepancies.

Finally, the spectrum of Siegel's superstring (3.1) is not in agreement with the RNS formalism. One would need to include an appropriate set of first class constraints to reproduce the RNS spectrum. The set of constraints should contain the Virasoro constraint and κ-symmetry generator

$$T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha \ , \qquad G^\alpha = \Pi^m(\gamma_m d)^\alpha \tag{3.22}$$

of the GS formalism expressed in terms of the supersymmetric momentum Π^m and GS constraint d_α. So far, the whole set of constraints was never found for the superstring. Nevertheless, there was a successful description of the superpartical using Siegel's ansatz [131, 132]. In the end, all this effort was not lost. Berkovits used this ansatz in his proposal [43] to construct the pure spinor formalism.

3.2 Fundamentals of the pure spinor formalism

We can see that it is possible to circumvent the problem associated to the mixed first and second class constraints in the GS formalism by the approach in section 3.1. Nevertheless, one does not arrive at a consistent theory: The non-vanishing central charge $c_X + c_{p,\theta} = -22$ and the level +4 of the Lorentz current algebra makes it impossible for this theory to describe the superstring known from RNS and GS formalism.

In [43] Berkovits modified Siegel's approach by adding a ghost sector. These pure spinor ghosts contribute +22 to the central charge of the energy momentum tensor and −3 to the double pole of the Lorentz current OPE. This ansatz leads us to a consistent theory – the *pure spinor formalism* – by fixing the issues of Siegel's idea.[3]

We start to construct this theory by proposing a BRST operator[4]

$$Q = \oint \mathrm{d}z\, \lambda^\alpha(z) d_\alpha(z) \ , \tag{3.23}$$

where λ are commuting $SO(1,9)$ Weyl spinors and therefore ghosts of the theory and d_α corresponds to the former GS constraint (3.2). The BRST operator (3.23) must be

[3] In this thesis we always refer to the minimal formulation of the pure spinor formalism. The non-minimal pure spinor formalism including additional worldsheet fields can be found for example in [45].

[4] The BRST charge can be derived from first principles [133] and see also [134, 135, 136, 137] for previous attempts.

nilpotent $Q^2 = 0$ such that the BRST charge is invariant under gauge constraints [27]. Using the OPEs (3.14) and Cauchy's formulas we can show that

$$Q^2 = \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} \lambda^\alpha(z) d_\alpha(z) \lambda^\beta(w) d_\beta(w) = -\oint \frac{\mathrm{d}z}{2\pi i} (\lambda\gamma^m\lambda)\Pi_m \ . \tag{3.24}$$

Imposing the consistency condition $Q^2 = 0$ for the BRST operator the above expression vanishes if the bosonic ghost fields λ^α obey the *pure spinor* constraints

$$(\lambda\gamma^m\lambda) = 0 \quad \text{for } m = 1, 2, \ldots, D \ . \tag{3.25}$$

Hence, a pure spinor λ^α in D spacetime dimensions is defined to be a Weyl spinor that satisfies the pure spinor constraint (3.25). These were first studied by Cartan from a geometrical perspective [138].

3.2.1 The pure spinor constraint

It is important to understand the pure spinor constraint (3.25) in more detail: Naively, one would assume that the constraints in (3.25) associated to $m = 0, 1, \ldots, 9$ would eliminate ten degrees of freedom of a pure spinor λ^α of $SO(1,9)$. But for a pure spinor λ^α with $16 - 10 = 6$ degrees of freedom the ghost sector would not cancel the anomaly generated by the non-vanishing central charge (3.15) of the matter sector, see below for more details.

However, a pure spinor has 11 independent degrees of freedom, i.e. the pure spinor constraint reduces the degrees of freedom of λ^α by five. In order to proof this statement we Wick rotate the Lorentz group from $SO(1,9)$ to $SO(10)$ and in addition break the manifest Euclidean Lorentz symmetry down to its $U(5)$ subgroup as described in appendix A. Using the 32×32 dimensional representation of the Γ^m-matrices the pure spinor constraint can be written as [138]

$$\Lambda^{\mathrm{T}} C \Gamma^m \Lambda = 0 \ , \tag{3.26}$$

where C is the charge conjugation matrix (A.8) and Λ is a bosonic Weyl spinor that satisfies $\Gamma^{11}\Lambda = -\Lambda$. Because the chirality matrix (A.9) is block diagonal we find $\Lambda = (\lambda \ \ 0)^{\mathrm{T}}$, where λ^α is a 16 dimensional bosonic spinor. Therefore, we recover the constraint $\lambda^\alpha\gamma_{\alpha\beta}\lambda^\beta = 0$. Including the conjugate momentum w_α of λ^α we obtain the $16 \oplus 16'$ states which are the ten dimensional Weyl spinors $|\lambda\rangle$ and anti-Weyl spinors $|\omega\rangle$. Because these states are anti-chiral $\Gamma_{11}|\lambda\rangle = -|\lambda\rangle$ and chiral $\Gamma_{11}|\omega\rangle = |\omega\rangle$ the $U(5)$ decomposition of them is given by

$$\begin{aligned} |\lambda\rangle &= \lambda^+ |0\rangle + \frac{1}{2}\lambda_{ab} b^b b^a |0\rangle + \frac{1}{4!}\lambda^a \epsilon_{abcde} b^e b^d b^c b^c |0\rangle \ , \\ |\omega\rangle &= \frac{1}{5!}\omega_+ \epsilon_{abcde} b^a b^b b^c b^d b^e |0\rangle + \frac{1}{2!3!}\omega^{ab}\epsilon_{abcde} b^c b^s b^e |0\rangle + w_a b^a |0\rangle \end{aligned} \tag{3.27}$$

in terms of the creation operators (A.21). Note that $\lambda_{ab} = -\lambda_{ba}$ is an complex anti-symmetric tensor that parametrizes a $SO(10)/U(5)$ coset [46]. The expansions in b^a follows from the properties of Γ_{11}: When expressing the chirality matrix (A.24) in terms of the creation operators it follows that $\Gamma_{11}\left|0\right\rangle = -\left|0\right\rangle$ and $\{\Gamma_{11}, b^a\} = 0$. Hence, states with an even or odd number of creation operators acting on the vacuum have eigenvalue -1 or $+1$ under Γ_{11}, respectively.

The $U(5)$ components of $|\lambda\rangle$ and $|\omega\rangle$ in (3.27) can be obtained by the following projections

$$\lambda^+ = \langle 0|\lambda\rangle \ , \qquad \lambda_{ab} = \langle 0|b_a b_b|\lambda\rangle \ , \qquad \lambda^a = \epsilon^{abcde}\frac{1}{4!}\langle 0|b_b b_c b_d b_e|\lambda\rangle \tag{3.28}$$

and

$$\omega_+ = \frac{1}{5!}\epsilon^{abcde}\langle 0|b_a b_b b_c b_d b_e|\omega\rangle \ , \quad \omega^{ab} = -\frac{1}{3!}\epsilon^{abcde}\langle 0|b_c b_d b_e|\omega\rangle \ , \quad \omega_a = \langle 0|b_a|\omega\rangle \ . \tag{3.29}$$

Due to the fermionic nature of the creation operators b^a the number of independent degrees of freedom of the components in (3.29) is given by $\#(b^{a_1}\cdots b^{a_n}) = \binom{5}{n}$. Hence, the $SO(10)$ Weyl λ^α and anti-Weyl ω_α spinors have been decomposed into irreducible $U(5)$ representations

$$\begin{aligned} &\lambda^\alpha \to (\lambda^+, \lambda_{ab}, \lambda^a) \ , && \omega_\alpha \to (\omega_+, \omega^{ab}, \omega_a) \ , \\ &\mathbf{16} \to (\mathbf{1}_{\frac{5}{2}}, \overline{\mathbf{10}}_{\frac{1}{2}}, \mathbf{5}_{-\frac{3}{2}}) \ , && \mathbf{16'} \to (\mathbf{1}_{-\frac{5}{2}}, \mathbf{10}_{-\frac{1}{2}}, \overline{\mathbf{5}}_{\frac{3}{2}}) \ . \end{aligned} \tag{3.30}$$

Before we start to analyse (3.25) we will take a look at some useful results. Let's consider two states ϕ and ψ which are generated by acting with an arbitrary number of creation operators b^i on the vacuum

$$|\phi\rangle = \phi_{i_1 i_2\dots i_m} b^{i_1} b^{i_2}\cdots b^{i_m}\left|0\right\rangle \ , \qquad |\psi\rangle = \psi_{j_1 j_2\dots j_n} b^{j_1} b^{j_2}\cdots b^{j_n}\left|0\right\rangle \ . \tag{3.31}$$

Then, the product of these two states with the charge conjugation matrix $\langle\phi|\, C\,|\psi\rangle$ is only non-vanishing and proportional to ϵ^{abcde} if and only if $\phi^\dagger\psi$ is proportional to $b^a b^b b^c b^d b^e$. We can bring all b_a in $\langle\phi|$ to the right of C by using that

$$C b_a = b^a C \ , \qquad C b^a = b_a C \ , \tag{3.32}$$

which follows from (A.7) together with (A.22) and thereby obtain

$$\begin{aligned} \langle\phi|\, C\,|\psi\rangle &= \phi^*_{i_1 i_2\dots i_m}\psi_{j_1 j_2\cdots j_n}\left\langle 0\right| (b_{i_m}\cdots b_{i_2} b_{i_1}) C (b^{j_1} b^{j_2}\cdots b^{j_n})\left|0\right\rangle \\ &= \phi^*_{i_1 i_2\dots i_m}\psi_{j_1 j_2\cdots j_n}\left\langle 0\right| C (b^{i_m}\cdots b^{i_2} b^{i_1})(b^{j_1} b^{j_2}\cdots b^{j_n})\left|0\right\rangle \\ &= \phi^*_{i_1 i_2\dots i_m}\psi_{j_1 j_2\cdots j_n}\left\langle 0\right| (b_5 b_4 b_3 b_2 b_1)(b^{i_m}\cdots b^{i_2} b^{i_1})(b^{j_1} b^{j_2}\cdots b^{j_n})\left|0\right\rangle \ , \end{aligned} \tag{3.33}$$

where we used (A.22) and that $\left\langle 0\right| C = \left\langle 0\right| b_5 b_4 b_3 b_2 b_1$. The above expression is only non-vanishing, if all the annihilation operators $(b_5 b_4 b_3 b_2 b_1)$ match the creation operators $(b^{i_m}\cdots b^{i_2} b^{i_1})(b^{j_1} b^{j_2}\cdots b^{j_n})$ exactly. So far, we can conclude that if $\langle\phi|\, C\,|\psi\rangle$ is different from zero then

$$\langle\phi|\, C\,|\psi\rangle = (\phi^*\otimes\psi)_{abcde}\left\langle 0\right| C b^a b^b b^c b^d b^e\left|0\right\rangle \tag{3.34}$$

for $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$, which are all mutually different. Moreover, by using (A.23) and the normalization $\langle 0|0\rangle = 1$ we can recognize that $\langle 0| Cb^1b^2b^3b^4b^5 |0\rangle = 1$ and that $\langle 0| Cb^ab^bb^cb^db^e |0\rangle$ is totally antisymmetric in all indices. Hence, we find

$$\langle 0| Cb^ab^bb^cb^db^e |0\rangle = \epsilon^{abcde} , \tag{3.35}$$

which concludes the proof, i.e.

$$\langle \phi| C |\psi\rangle = (\phi^* \otimes \psi)_{abcde}\epsilon^{abcde} . \tag{3.36}$$

To give an example we choose $\langle\phi| = \{\langle 0| b_a, \langle 0| b_ab_bb_c, \langle 0| b_ab_bb_cb_db_e\}$ and $|\psi\rangle = |\lambda\rangle$ and find from the spinor decomposition (3.27) that

$$\begin{aligned} \langle 0| Cb^a |\lambda\rangle &= \lambda^a , \\ \langle 0| Cb^ab^bb^c |\lambda\rangle &= -\frac{1}{2}\epsilon^{abcde}\lambda_{de} , \\ \langle 0| Cb^ab^bb^cb^db^e |\lambda\rangle &= \epsilon^{abcde}\lambda^+ . \end{aligned} \tag{3.37}$$

Under the decomposition $SO(10) \to U(5)$ the pure spinor constraint (3.26) splits into two independent equations: Plugging Γ^i in terms of b^i and b_i

$$\Gamma^i = b^i + b_i , \qquad \Gamma^{i+5} = -i(b^i - b_i) \tag{3.38}$$

into (3.26) and taking suitable linear combination of the constraints gives

$$\langle\lambda| Cb^i |\lambda\rangle = 0 , \qquad \langle\lambda| Cb_i |\lambda\rangle = 0 \tag{3.39}$$

for $i = 1, 2, \ldots, 5$. If we use the component expansion $\langle\lambda| = \langle 0| \lambda_+ + \frac{1}{2}\langle 0| b_ab_b\lambda_{ab} + \frac{1}{24}\langle 0| b_bb_cb_db_e\lambda^a\epsilon^{abcde}$ the first equation in (3.39) becomes

$$\begin{aligned} 0 = \langle\lambda| Cb^a |\lambda\rangle &= \lambda^+ \langle 0| Cb^a |\lambda\rangle + \frac{1}{2}\lambda_{bc}\langle 0| b_bb_cCb^a |\lambda\rangle + \frac{1}{24}\lambda^b\epsilon_{bcdef}\langle 0| b_cb_db_eb_fCb^a |\lambda\rangle \\ &= 2\lambda^+\lambda^a - \frac{1}{4}\epsilon^{abcde}\lambda_{bc}\lambda_{de} , \end{aligned} \tag{3.40}$$

where we have used (3.32) and (3.37). Therefore, the constraint $\langle\lambda| Cb^i |\lambda\rangle = 0$ allows us to express the five vector components λ^a in terms of λ^+ and λ^{ab}:

$$\lambda^a = \frac{1}{8\lambda^+}\epsilon^{abcde}\lambda_{bc}\lambda_{de} \tag{3.41}$$

for $\lambda^+ \neq 0$. In addition, this solution automatically solves the second set of constraints in (3.39)

$$\langle\lambda| Cb_a |\lambda\rangle = -2\lambda_{ab}\lambda^b . \tag{3.42}$$

When substituting the solution into the above equation we get

$$-2\lambda_{ab}\lambda^b = -\frac{1}{4\lambda^+}\lambda_{ab}\epsilon^{bcdef}\lambda_{cd}\lambda_{ef} = 0 \ , \tag{3.43}$$

which vanishes due to the antisymmetric nature of the Levi-Civita symbol and λ_{ab} and the fact that the indices only take values between one and five: For any value of a one of the other indices b, c, d, e or f must take the same value. For $b = a$ the statement is trivially true because $\lambda_{aa} = 0$. If $c = a$ we find that

$$\lambda_{ab}\epsilon^{badef}\lambda_{ad}\lambda_{ef} = -\lambda_{ad}\epsilon^{dcbef}\lambda_{ab}\lambda_{ef} \ , \tag{3.44}$$

which vanishes after renaming $b \leftrightarrow d$. Proceeding similar with the remaining indices we can conclude that also the second set of constraints in (3.39) is automatically satisfied by (3.41).

Therefore, a pure spinor in ten spacetime dimensions is given by (3.27) together with the solution of the pure spinor constraint (3.41). The relation in (3.41) eliminates five degrees of freedom of $\lambda^a \in \mathbf{5}$ and hence out of the 16 degrees of freedom of λ^α a total of 11 independent components are left in a pure spinor of $SO(10)$. These remaining components split into an antisymmetric 2-from $\lambda_{ab} \in \overline{\mathbf{10}}$ with ten and a scalar $\lambda^+ \in \mathbf{1}$ with one degree of freedom.

3.2.2 Lorentz current for the ghost sector

The pure spinor ghost λ^α and the dual field w_α are $SO(1,9)$ Weyl and anti Weyl spinors. Therefore, they contribute to the to the $SO(1,9)$ Lorentz current

$$M^{mn} = \Sigma^{mn} + N^{mn} \ , \tag{3.45}$$

where N^{mn} is the contribution coming from the pure spinor ghost sector. The coefficient of the double pole in the OPE of N^{mn} with itself is -3, which implies that N^{mn} satisfies the following OPEs[5]

$$N^{mn}(z)N^{pq}(w) \sim \frac{\eta^{p[m}N^{n]q} - \eta^{q[m}N^{n]p}}{z-w} - 3\frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2} \ , \tag{3.46}$$

$$\Sigma^{mn}(z)N^{pq}(w) \sim \text{regular} \ . \tag{3.47}$$

Therefore, the level of the Kač-Moody current algebra of M^{mn} matches the one in the RNS formalism, i.e. M^{mn} satisfies the same OPE as in (3.20).

As for the pure spinor constraint to understand the OPE (3.46) we have to break the manifest $SO(10)$ symmetry down to its $U(5)$ subgroup. In terms of $U(5) = SU(5) \otimes U(1)$ variables the pure spinor Lorentz current decomposes into irreducible representation as

$$N^{mn} \to (n, n^a_b, n_{ab}, n^{ab}) \ , \tag{3.48}$$

[5] In [139] the OPE for the ghost Lorentz current is derived form the decomposition of λ^α and w_α.

which have the $U(1)$ charges $(0,0,+2,-2)$. After identifying (3.48) with the Lorentz generators (A.13) and (A.14) with a traceless n^a_b, c.f. (A.18), as

$$(n, n^a_b, n^{ab}, n_{ab}) \to -\left(\frac{m}{\sqrt{5}}, m^a_b - \frac{1}{5}m, m^{ab}, m_{ab}\right) \tag{3.49}$$

the OPE in (3.46) decomposes under $SO(10) \to SU(5) \otimes U(1)$ into the following OPEs for (3.48)

$$\begin{aligned}
n_{ab}(z)n_{cd}(w) &\sim \text{regular} , & n^{ab}(z)n^{cd}(w) &\sim \text{regular} ,\\
n_{ab}(z)n^{cd}(w) &\sim \frac{-\delta^{[c}_{[a}n^{d]}_{b]}(w) - \frac{2}{\sqrt{5}}\delta^c_{[a}\delta^d_{b]}n(w)}{z-w} - 3\frac{\delta^c_b\delta^d_a - \delta^c_a\delta^d_b}{(z-w)^2} , & n(z) &\sim \text{regular} ,\\
n^a_b(z)n^c_d(w) &\sim \frac{-\delta^c_b n^a_d(w) + \delta^a_d n^c_b(w)}{z-w} - 3\frac{\delta^a_d\delta^c_b - \frac{1}{5}\delta^a_b\delta^c_d}{(z-w)^2} , & n(z)n_{ab}(w) &\sim \frac{2}{\sqrt{5}}\frac{n_{ab}(w)}{z-w} ,\\
n^{ab}(z)n^c_d(w) &\sim \frac{-\delta^a_d n^{bc}(w) + \delta^b_d n^{ac}(w) - \frac{2}{5}\delta^c_d n^{ab}(w)}{z-w} , & n(z)n^{ab}(w) &\sim \frac{2}{\sqrt{5}}\frac{n^{ab}(w)}{z-w} ,\\
n^{ab}(z)n^c_d(w) &\sim \frac{-\delta^c_b n_{ad}(w) + \delta^c_a n^{bd}(w) + \frac{2}{5}\delta^c_d n_{ab}(w)}{z-w} , & n(z)n(w) &\sim -\frac{3}{(z-w)^2} ,
\end{aligned} \tag{3.50}$$

where more details can be found in appendix A and [140, 141]. With the redefined $U(1)$ generator n the corresponding charge is then defined by $[n, R] = \frac{q_R}{\sqrt{5}}R$. According to (3.50) the fields $(n, n^a_b, n^{ab}, n_{ab})$ transform in the $(\mathbf{1}_0, \mathbf{24}_0, \mathbf{10}_{-2}, \overline{\mathbf{10}}_2)$ representation of $SU(5) \otimes U(1)$.

There is a further constraint on the total Lorentz current (3.45): The pure spinor λ^α has to transform as a spinor under Lorentz transformations. These are generated by the action of M^{mn} on λ^α, c.f. (A.10):

$$\delta\lambda^\alpha = \frac{1}{2}\left[\oint \mathrm{d}z\, \varepsilon_{mn}M^{mn}, \lambda^\alpha\right] = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn}\lambda)^\alpha . \tag{3.51}$$

Since a pure spinor λ^α can only have a non-regular OPE with the Lorentz current N^{mn}, the transformation (3.51) implies

$$N^{mn}(z)\lambda^\alpha(w) \sim \frac{1}{2}\frac{(\gamma^{mn})^\alpha{}_\beta\lambda^\beta(w)}{z-w} . \tag{3.52}$$

Following from appendix A a $SO(10)$ pure spinor transforms under Lorentz transformations in terms of its $U(5)$ representations as [140]

$$\begin{aligned}
n(z)\lambda^+(w) &\sim -\frac{\sqrt{5}}{2}\frac{\lambda^+(w)}{z-w} , & n(z)\lambda_{cd}(w) &\sim -\frac{1}{2\sqrt{5}}\frac{\lambda_{cd}(w)}{z-w} ,\\
n(z)\lambda^a(w) &\sim \frac{3}{2\sqrt{5}}\frac{\lambda^a(w)}{z-w} , & n^a_b(z)\lambda_{cd}(w) &\sim \frac{\delta^a_d\lambda_{cb}(w) - \delta^a_c\lambda_{db}(w)}{z-w} - \frac{2}{5}\frac{\delta^a_b\lambda_{cd}(w)}{z-w} ,
\end{aligned}$$

$$
\begin{aligned}
n^a_b(z)\lambda^+(w) &\sim \text{regular}\ , & n^a_b(z)\lambda^c(w) &\sim \frac{1}{5}\frac{\delta^a_b\lambda^c(w)}{z-w} - \frac{\delta^c_b\lambda^a(w)}{z-w}\ ,\\
n_{ab}(z)\lambda^+(w) &\sim \frac{\lambda_{ab}(w)}{z-w}\ , & n_{ab}(z)\lambda_{cd}(w) &\sim \frac{\epsilon_{abcde}\lambda^e(w)}{z-w}\ ,\\
n_{ab}(z)\lambda^c(w) &\sim \text{regular}\ , & n^{ab}(z)\lambda^+(w) &\sim \text{regular}\ ,\\
n^{ab}(z)\lambda_{cd}(w) &\sim -\frac{\delta^a_{[c}\delta^b_{d]}\lambda^+(w)}{z-w}\ , & n^{ab}(z)\lambda^c(w) &\sim -\frac{1}{2}\frac{\epsilon^{abcde}\lambda_{de}(w)}{z-w}\ .
\end{aligned} \tag{3.53}
$$

3.3 A parametrization of the pure spinor ghosts

The solution for the pure spinor constraint and the Kač-Moody current algebra allows us to write the $U(5)$ components of the pure spinors $(\lambda^+, \lambda_{ab}, \lambda^a)$ and the Lorentz current $(n, n^a_b, n_{ab}, n^{ab})$ in terms of the ghost variables $s(z), u^{ab}(z)$ and their conjugate momenta $t(z), v_{ab}(z)$. The action of this parametrization of the pure spinor ghosts is given by [43, 46, 142]

$$
S_\lambda = \frac{1}{2\pi}\int \mathrm{d}^2z \left(\frac{1}{2}v^{ab}\overline{\partial}u_{ab} - \partial t\overline{\partial}s\right)\ ,\quad a,b = 1,\ldots 5\ . \tag{3.54}
$$

Because $s(z)$ and $t(z)$ are chiral bosons, we have to impose their equations of motion by hand: $\overline{\partial}s = \overline{\partial}t = 0$. Moreover, the OPEs between the ghost fields and their conjugate momenta are given by

$$
\begin{aligned}
t(z)s(w) &\sim \ln(z-w)\ ,\\
v^{ab}(z)u_{cd}(w) &\sim \frac{\delta^a_{[c}\delta^b_{d]}}{z-w}\ .
\end{aligned} \tag{3.55}
$$

This parametrization of the pure spinor formalism has to respect the group-theoretic relations of λ^α and N^{mn}. Hence, the OPEs of λ^α and N^{mn} in terms of the ghost fields have to satisfy (3.50) and (3.53). With this restriction the solution for the $U(5)$ components is given by [43, 46, 142]

$$
\begin{aligned}
n &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{ab}v^{ab} + \frac{5}{2}\partial t - \frac{5}{2}\partial s\right)\ , & \lambda^+ &= e^s\ ,\\
n^b_a &= u_{ac}v^{bc} - \frac{1}{5}\delta^b_a u_{cd}v^{cd}\ , & \lambda_{ab} &= u_{ab}\ ,\\
n^{ab} &= -e^s v^{ab}\ , & \lambda^a &= \frac{1}{8}e^{-s}\epsilon^{abcde}u_{bc}u_{de}\ ,\\
n_{ab} &= e^s\left(2\partial u_{ab} - u_{ab}\partial t - 2u_{ab}\partial s + u_{ac}u_{bd}v^{cd} - \frac{1}{2}u_{ab}u_{cd}v^{cd}\right)\ .
\end{aligned} \tag{3.56}
$$

It is straightforward to check that these definitions reproduce (3.50) and (3.53), if the ghost fields $s(z), t(z), u_{ab}(z)$ and $v^{ab}(z)$ satisfy (3.55). For example, we can compute the OPE

$$
n(z)n^{ab}(w) = \frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{cd}v^{cd} + \frac{5}{2}\partial t - \frac{5}{2}\partial s\right)(z)e^{s(w)}v^{ab}(w)
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\frac{1}{4}e^{s(w)}v^{cd}(z)\overbrace{u_{cd}(z)v^{ab}}(w) - \frac{\sqrt{5}}{2}\overbrace{\partial t(z)e^{s(w)}}v^{ab}(w) \\
&\sim \frac{1}{\sqrt{5}}\frac{1}{4}e^{s(w)}v^{cd}(z)\frac{-\delta^a_{[c}\delta^b_{d]}}{z-w} - \frac{\sqrt{5}}{2}\frac{1}{z-w}e^{s(w)}v^{ab}(w) \\
&\sim -\frac{2}{\sqrt{5}}\frac{n^{ab}(w)}{z-w}
\end{aligned} \tag{3.57}
$$

and in addition

$$
\begin{aligned}
n^{ab}(z)\lambda^c(w) &= -\frac{1}{8}e^{s(z)}e^{-s(w)}v^{ab}(z)\epsilon^{cdefg}u_{de}(w)u_{fg}(w) \\
&= -\frac{1}{8}\epsilon^{cdefg}e^{s(z)}e^{-s(w)}\left(\overbrace{v^{ab}(z)u_{de}}(w)u_{fg}(w) + u_{de}(w)\overbrace{v^{ab}(z)u_{fg}}(w)\right) \\
&\sim -\frac{1}{8}\epsilon^{cdefg}e^{s(z)}e^{-s(w)}\left(\frac{-\delta^a_{[d}\delta^b_{e]}}{z-w}u_{fg}(w) + u_{de}(w)\frac{-\delta^a_{[f}\delta^b_{g]}}{z-w}\right) \\
&= -\frac{1}{8}e^{s(z)}e^{-s(w)}\frac{2\epsilon^{cabfg}u_{fg}(w) + 2\epsilon^{cdeab}u_{de}(w)}{z-w} \\
&\sim -\frac{1}{2}\epsilon^{abcde}\frac{\lambda_{de}(w)}{z-w} ,
\end{aligned} \tag{3.58}
$$

where we have used the OPE for $u_{ab}(z)v^{cd}(w) \sim \frac{-\delta^a_{[c}\delta^b_{d]}}{z-w}$ and $\partial t(z)e^{s(w)} \sim \frac{1}{z-w}e^{s(w)}$, which follow from the OPEs in (3.55) of the ghost fields. Moreover, we have discarded the non-singular terms that arise from the Taylor expansions of the fields around z and w. The computations above reproduce the OPEs in (3.50) and (3.53), which were derived from the group-theoretic decomposition of the $SO(10)$ covariant OPEs. All other OPEs can be obtained in a similar way. Although, the action S_λ of the ghosts is not manifestly Lorentz invariant, we have constructed this parametrization such that all OPEs involving the $U(5)$ Lorentz current and pure spinors originate from manifestly $SO(10)$ covariant expressions. Therefore, the pure spinor formalism is manifestly Lorentz covariant.

The energy-momentum tensor $T_\lambda(z)$ for the ghost action (3.54) can be obtained by Noether's theorem for the continuos symmetry corresponding to a shift in the worldsheet coordinates, i.e. the variation of the action under $\delta z = -\varepsilon$ and $\delta\overline{z} = -\overline{\varepsilon}$, which is given by

$$
\delta S_\lambda = \frac{1}{2\pi}\int \mathrm{d}^2z \left(\overline{\partial}\varepsilon T_\lambda(z) + \partial\overline{\varepsilon}\overline{T}_\lambda(\overline{z})\right) . \tag{3.59}
$$

The conformal transformations for the ghost fields $(\partial s, \overline{\partial}t)$ are obtained by assuming that they depend on both $(z,\overline{z})$ and performing a Taylor expansion for $s(z,\overline{z})$ and $t(z,\overline{z})$ in the infinitesimal parameters ε and $\overline{\varepsilon}$ [140][6]

$$
\delta\partial s = \partial\varepsilon\partial s + \varepsilon\partial^2 s + \partial\overline{\varepsilon}\overline{\partial}s + \overline{\varepsilon}\partial\overline{\partial}s , \qquad \delta\overline{\partial}t = \overline{\partial}\overline{\varepsilon}\overline{\partial}t + \overline{\varepsilon}\overline{\partial}^2 t + \overline{\partial}\varepsilon\partial t + \varepsilon\overline{\partial}\partial t . \tag{3.60}
$$

[6]Note that the transformation for ∂s in (3.60) differs from equation (3.28) in [140], but $\delta\partial s$ in (3.60) is obtained by following the steps in [140] and gives the correct contribution to the energy momentum tensor, see (3.62).

The other ghosts fields (v_{ab}, u^{ab}) have conformal weights $(h,\overline{h}) = ((1,0),(0,0))$ such that their conformal transformations are given by [143]

$$\delta v^{ab} = \partial\varepsilon v^{ab} + \varepsilon\partial v^{ab} + \overline{\varepsilon}\overline{\partial} v^{ab} \ , \qquad\qquad \delta u_{ab} = \varepsilon\partial u_{ab} + \overline{\varepsilon}\overline{\partial} u_{ab} \ . \tag{3.61}$$

For the variation of the action S_λ for s and t under the above conformal transformations we find

$$\begin{aligned}\int \mathrm{d}^2 z\, \delta\Big(\overline{\partial} t \partial s\Big) &= \int \mathrm{d}^2 z \, \Big[\Big(\overline{\partial}\overline{\epsilon}\overline{\partial} t + \overline{\epsilon}\overline{\partial}^2 t + \overline{\partial}\epsilon\partial t + \epsilon\overline{\partial}\partial t\Big)\partial s \\ &\quad + \overline{\partial} t\Big(\partial\epsilon\partial s + \epsilon\partial^2 s + \partial\overline{\epsilon}\overline{\partial} s + \overline{\epsilon}\partial\overline{\partial} s\Big)\Big] \\ &= \int \mathrm{d}^2 z \, \Big[\partial\Big(\epsilon\overline{\partial} t\partial s\Big) + \overline{\partial}\Big(\overline{\epsilon}\overline{\partial} t \partial s\Big) + \overline{\partial}\epsilon\partial t\partial s + \partial\overline{\epsilon}\overline{\partial} s\overline{\partial} t\Big] \\ &= \int \mathrm{d}^2 z \, \Big[\overline{\partial}\epsilon\partial t\partial s + \partial\overline{\epsilon}\overline{\partial} s\overline{\partial} t\Big] \ . \end{aligned} \tag{3.62}$$

At the boundary of integration the surface terms in (3.62) vanish such that the contribution of s and t to the holomorphic part of the energy momentum tensor is $T_{st}(z) = \partial s(z)\partial t(z)$, which follows after comparing (3.59) with (3.62). For u_{ab} and v^{ab} we find the contribution to the energy-momentum tensor by identifying them with a bc ghost system with $\lambda = 1$. Hence, for $b = -\frac{1}{2} v^{ab}$ and $c = u_{ab}$ the holomorphic part of the energy-momentum tensor is $T_{uv}(z) = \frac{1}{2} v^{ab}(z)\partial u_{ab}(z)$. Adding both pieces together we end up with

$$T_\lambda(z) = \frac{1}{2} v^{ab}(z)\partial u_{ab}(z) + \partial s(z)\partial t(z) \ . \tag{3.63}$$

When computing the OPE of $T_\lambda(z)$ with $n(w)$

$$\begin{aligned} T_\lambda(z)n(w) &= -\frac{1}{\sqrt{5}}\Big(\frac{1}{2} v^{ab}\partial u_{ab} + \partial s\partial t\Big)(z)\Big(\frac{1}{4} u_{cd} v^{cd} + \frac{5}{2}\partial t - \frac{5}{2}\partial s\Big)(w) \\ &= \frac{1}{8\sqrt{5}}\Big(v^{ab}(z)\partial u_{ab}(z) u_{cd}(w) v^{cd}(w) + v^{ab}(z)\partial u_{ab}(z) u_{cd}(w) v^{cd}(w) \\ &\quad + v^{ab}(z)\partial u_{ab}(z) u_{cd}(w) v^{cd}(w)\Big) - \frac{\sqrt{5}}{2}\partial s(z)\partial t(z)\partial t(w) + \frac{\sqrt{5}}{2}\partial s(z)\partial t(z)\partial s(w) \\ &\sim -\frac{\sqrt{5}}{(z-w)^3} + \frac{n(w)}{(z-w)^2} + \frac{\partial n(w)}{z-w} \ , \end{aligned} \tag{3.64}$$

where the triple pole $\frac{1}{(z-w)^3}$ comes from the contraction

$$\begin{aligned} \frac{1}{8\sqrt{5}} v^{ab}(z)\partial u_{ab}(z) u_{cd}(w) v^{cd}(w) &\sim -\frac{1}{8\sqrt{5}}\frac{\delta^a_{[c}\delta^b_{d]}}{z-w}\frac{\delta^c_{[a}\delta^d_{b]}}{(z-w)^2} \\ &= -\frac{\sqrt{5}}{(z-w)^3} \ . \end{aligned} \tag{3.65}$$

The triple pole $\frac{1}{(z-w)^3}$ implies that $n(w)$ is not a primary field, which is not the case. However, by adding $\partial^2 s$ to the energy momentum tensor the OPE (3.64) is corrected by

$$\begin{aligned}\partial^2 s(z)n(w) &= -\frac{1}{sqrt5}\partial^2 s(z)\Big(\frac{1}{2}v^{ab}(z)\partial u_{ab}(z) + \partial s(z)\Big)\\ &= -\frac{\sqrt{5}}{2}\partial^2 s(z)\partial t(w)\\ &\sim \frac{\sqrt{5}}{(z-w)^3}\end{aligned} \tag{3.66}$$

such that $n(w)$ is a primary field of conformal dimension $h = 0$. Note that it is possible to add $\partial^2 s$, because it drops out of (3.59).

Finally, we can conclude that the holomorphic energy-momentum tensor of the ghost sector is given by

$$T_\lambda = -\frac{1}{2}v^{ab}\partial u_{ab}\partial t\partial s + \partial^2 s \ . \tag{3.67}$$

The central charge of the ghost sector is determined by the coefficient of the fourth order pole $\frac{1}{(z-w)^4}$ in the OPE $T_\lambda(z)T_\lambda(w) \sim \frac{c_\lambda}{2}\frac{1}{(z-w)^4}$. The terms that contribute to this pole are

$$\begin{aligned}\frac{1}{4}v^{ab}(z)\partial u_{ab}(z)v^{cd}(w)\partial u_{cd}(w) &\sim \frac{1}{4}\frac{\delta^a_{[c}\delta^b_{d]}\delta^c_{[a}\delta^d_{b]}}{(z-w)^4} = \frac{10}{(z-w)^4}\ ,\\ \partial t(z)\partial s(z)\partial t(w)\partial s(w) &\sim \frac{1}{(z-w)^4}\ .\end{aligned} \tag{3.68}$$

Adding both contributions implies that $c_\lambda = 22$. Therefore, the pure spinor formalism will not exhibit a conformal anomaly: The total central charge of the energy-momentum tensor in the pure spinor formalism

$$T_{\text{PS}} = -\frac{1}{2}\partial X^m\partial X_m - p_\alpha\partial\theta^\alpha + \frac{1}{2}v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s \tag{3.69}$$

vanishes, i.e. $c_{\text{total}} = c_x + c_{p\theta} + c_\lambda = 10 - 32 + 22 = 0$, because all OPEs between the pure spinor ghosts and the matter fields are regular.

3.4 The action of the pure spinor formalism

When adding the pure spinor ghost action (3.54) to the action of the matter fields (3.1) we obtain the $U(5)$ covariant action [43]

$$S_{\text{PS}} = \frac{1}{\pi}\int \mathrm{d}^2 z\left(\frac{1}{2}\partial X^m\overline{\partial}X_m + p_\alpha\overline{\theta}^\alpha - \partial t\overline{\partial}s + \frac{1}{2}v^{ab}\overline{\partial}u_{ab}\right)\ . \tag{3.70}$$

The action S_{PS} of the pure spinor formalism can also be written in terms of the $SO(10)$ covariant fields as

$$S_{\text{PS}} = \frac{1}{\pi}\int \mathrm{d}^2 z\left(\frac{1}{2}\partial X^m\overline{\partial}X_m + p_\alpha\overline{\theta}^\alpha - w_\alpha\overline{\partial}\lambda^\alpha\right)\ . \tag{3.71}$$

Although, we have added the pure spinor ghosts, the action (3.71) is still supersymmetric, because the pure spinor ghosts transform under supersymmetry as[7]

$$\delta\omega_\alpha = 0, \qquad \delta\lambda^\alpha = 0 \tag{3.72}$$

and the action of the matter fields is invariant under supersymmetry transformations, as was shown in section 3.1.

From dimensional analysis we can reinstate the α'-dependence, where variables in (3.71) have the following length dimensions [144, 145]

$$[\alpha'] = 2\ , \qquad [X^m] = 1\ , \qquad [\theta^\alpha] = [\lambda^\alpha] = \frac{1}{2}\ , \qquad [p_\alpha] = [w_\alpha] = -\frac{1}{2}\ . \tag{3.73}$$

Moreover, the $SO(10)$ covariant version of the energy-momentum tensor and the Lorentz current of the fermionic fields are given by

$$T_{\rm PS} = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial^\alpha + w_\alpha\partial\lambda^\alpha\ , \qquad M^{mn} = -\frac{1}{2}(p\gamma^{mn}\theta) + \frac{1}{2}(w\gamma^{mn}\lambda) \tag{3.74}$$

and can be derived from the action (3.71). Above we have written $T_{\rm PS}$ in terms of the supersymmetric momentum Π^m and the GS constraint d_α, which are defined as

$$\begin{aligned} \Pi^m &= \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta)\ , \\ d_\alpha &= p_\alpha - \frac{1}{2}\Big(\partial X^m + \frac{1}{4}(\theta\gamma^m\partial\theta)\Big)(\gamma_m\theta)_\alpha\ , \end{aligned} \tag{3.75}$$

see also section 3.1 for more details.

3.5 Operator product expansions in the pure spinor formalism

In this section we summarize the OPEs that underlie the CFT of the pure spinor formalism. The matter fields, the supersymmetric momentum and the GS constraint satisfy

$$\begin{aligned} X^m(z,\overline{z})X^n(w,\overline{w}) &\sim -\eta^{mn}\ln|z-w|^2\ , & p_\alpha(z)\theta^\beta(w) &\sim \frac{\delta_\alpha{}^\beta}{z-w}\ , \\ d_\alpha(z)d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\Pi_m(w)}{z-w}\ , & d_\alpha(z)\Pi^m(w) &\sim \frac{(\gamma^m\partial\theta(w))_\alpha}{z-w}\ , \\ \Pi^m(z)\Pi^n(m) &\sim -\frac{\eta^{mn}}{(z-w)^2}\ , & d_\alpha(z)\theta^\beta(w) &\sim \frac{\delta^{\alpha\beta}}{z-w}\ . \end{aligned} \tag{3.76}$$

[7]Similarly, the $U(5)$ covariant fields transform as $\delta s = \delta t = \delta u_{ab} = \delta v^{ab} = 0$ under supersymmetry transformations.

The pure spinor constraint implies that w_α and λ^α are not free fields such that it is not straightforward to give a $SO(10)$ covariant OPE, i.e. the OPE $w_\alpha(z)\lambda^\beta \sim \frac{\delta_\alpha^\beta}{z-w}$ obtained from the pure spinor action (3.71) is not correct. Nevertheless, by decomposing the ghost fields into their $U(5)$ covariant variables it is possible to find the OPE

$$w_\alpha(z)\lambda^\beta \sim \frac{\delta_\alpha^\beta}{z-w} + \dots \tag{3.77}$$

where $\dots$ are non-covariant $U(5)$ corrections, which are needed to make the OPE of the conjugated momentum w_α with the pure spinor constraint $(\lambda\gamma^m\lambda)$ non-singular. However, the corrections do not contribute to the OPE of a pure spinor λ^α with the ghost Lorentz current N^{mn}. [43, 114].

From the θ-expansions (2.32) it follows that the superfields $K(X,\theta)$ only depend on X^m via an overall plane wave factor $e^{ik\cdot X}$

$$K(X,\theta) = e^{ik\cdot X}K(\theta)\ . \tag{3.78}$$

The other factor $K(\theta)$ of the superfields depends only on θ^α such that it has a non-vanishing OPE only with p_α. Then, by using the OPEs (3.76) we find that the non-vanishing OPEs of Π^m and d_α with θ^α and $e^{ik\cdot X}$ are

$$\begin{aligned} \Pi^m(z)e^{ik\cdot X(w)} &\sim \partial X^m(z)e^{ik\cdot X(w)} \sim \frac{-ik^m}{z-w}e^{ik\cdot X(w)}\ , \\ d_\alpha(z)e^{ik\cdot X(w)} &\sim -\frac{1}{2}(\gamma_m\theta)_\alpha\partial X^m(z)e^{ik\cdot X(w)} \sim \frac{1}{2(z-w)}(\gamma_m\theta)_\alpha\partial^m e^{ik\cdot X}(w)\ , \\ d_\alpha(z)\theta^\beta(w) &\sim p_\alpha(z)\theta^\beta(w) \sim \frac{1}{z-w}\frac{\partial}{\partial\theta^\alpha}\theta^\beta(w)\ . \end{aligned} \tag{3.79}$$

We used in the second line that $ik^m e^{ik\cdot X} = \partial^m e^{ik\cdot X}$ and in the third line that $\delta_\alpha^\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta$ to rewrite the results of the OPEs in a form that suits our purpose. Now we can calculate the OPE of Π^m and d_α with an arbitrary superfield

$$\begin{aligned} \Pi^m(z)K(X(w,\overline{w}),\theta(w)) &\sim \frac{-\partial K(X(w,\overline{w}),\theta(w))}{z-w}\ , \\ d_\alpha(z)K(X(w,\overline{w}),\theta(w)) &\sim \frac{D_\alpha K(X(w,\overline{w}),\theta(w))}{z-w}\ , \end{aligned} \tag{3.80}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ is the canonical derivative, we have defined in (2.3). Moreover, the OPEs involving the fermionic Lorentz current are given by

$$\begin{aligned} M^{mn}(z)M^{pq}(w) &\sim \frac{\eta^{p[m}M^{n]q} - \eta^{q[m}M^{n]p}}{z-w} + \frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\ , \\ N^{mn}(z)N^{pq}(w) &\sim \frac{\eta^{p[m}N^{n]q} - \eta^{q[m}N^{n]p}}{z-w} - 3\frac{\eta^{m[q}\eta^{p]n}}{(z-w)^2}\ , \end{aligned}$$

$$N^{mn}(z)\lambda^\alpha(w) \sim \frac{1}{2}\frac{(\gamma^{mn})^\alpha{}_\beta\lambda^\beta(w)}{z-w} \ . \tag{3.81}$$

Using these OPEs one can show that the fields $\{\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}\}$ are conformal primaries of weight $h = +1$: The OPEs of these fields with the energy momentum tensor (3.74) are given by

$$T_{\text{PS}}(z)\{\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}\}(w) \sim \frac{\{\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}\}(w)}{(z-w)^2} + \frac{\partial\{\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}\}(w)}{z-w} \ . \tag{3.82}$$

3.6 Massless vertex operators for the $\mathcal{N} = 1$ SYM multiplet

The information about asymptotic states in string scattering amplitudes is contained in vertex operators. The integrated massless vertex operators (3.16) in Siegel's formulation of the superstring lead to discrepancies with the RNS formalism, because of the double pole coefficients of the Lorentz current of the fermionic variables, see section 3.1. By adding a correction proportional to the pure spinor Lorentz current N^{mn} to $U_{\text{Siegel}}(z)$ we obtain [43]

$$U(z) = \Big[\partial\theta^\alpha A_\alpha(X,\theta) + \Pi^m A_m(X,\theta) + d_\alpha W^\alpha(X,\theta) + \frac{1}{2}N^{mn}F_{mn}(X,\theta)\Big](z) \tag{3.83}$$

to resolve these discrepancies. The integrated vertex operator is parametrized by the linearized superfields $A_\alpha(X,\theta), A_m(X,\theta), W^\alpha(X,\theta)$ and $F^{mn}(X,\theta)$ of section 2.2, which are worldsheet functions through the superspace coordinates $X^m = X^m(z)$ and $\theta^\alpha = \theta^\alpha(z)$ such that we can introduce the shorthand notation $K(z) = K(X(z),\theta(z))$ for any linearized superfield K. Hence, the integrated massless vertex operator describes the degrees of freedom of a massless gauge multiplet. The superfields of this multiplet have the following length dimensions [144, 145]

$$[A_\alpha] = \frac{1}{2} \ , \qquad [A_m] = 0 \ , \qquad [W^\alpha] = -\frac{1}{2} \ , \qquad [F_{mn}] = -1 \ , \tag{3.84}$$

which implies for the vertex operators

$$[V] = [U] = 1 \ , \tag{3.85}$$

where V is the unintegrated vertex operator introduced below in (3.86). The gluon vertex operator obtained form (3.83) contains the complete fermionic Lorentz current $M^{mn} = \Sigma^{mn} + N^{mn}$ as a coefficient of the component field strength f_{mn} when using the θ-expansions (2.32) of the superfields. This implies that the double pole of the vertex operator in (3.83) is in agreement with the RNS vertex operator.

Note that the vertex operator (3.83) has conformal weight +1, which means that it has to appear in any superstring amplitude in the conformally invariant combination $\int \mathrm{d}z\, U(z)$, where it is integrated over (parts of) the worldsheet.

For computing scattering amplitudes we also need a massless vertex operator with conformal dimension zero. To remove the redundancies of the conformal Killing group (Möbius transformations) we require vertex operators at fixed positions on the worldsheet. This unintegrated vertex operator is given by

$$V(z) = [\lambda^\alpha A_\alpha(X,\theta)](z) \ . \tag{3.86}$$

The physical states in the pure spinor formalism, described by the unintegrated and integrated vertex operators, have to be in the cohomology of the BRST operator Q of (3.23). A state Ψ is in the cohomology of the BRST operator Q, if it is BRST closed $Q\Psi = 0$, but not BRST exact $\Psi \neq Q\Lambda$ for some state Λ:

$$\mathcal{H}_{\mathrm{BRST}} = \frac{\mathcal{H}_{\mathrm{closed}}}{\mathcal{H}_{\mathrm{exact}}} \ . \tag{3.87}$$

For massless states with $k^2 = 0$ the unintegrated vertex operator V is BRST closed when the superfield A_α is on-shell[8]

$$\begin{aligned} QV(w) &= \oint \frac{\mathrm{d}z}{2\pi i}\, \lambda^\alpha(z) d_\alpha(z) \lambda^\beta A_\beta(w) \\ &= \oint \frac{\mathrm{d}z}{2\pi i}\, \frac{1}{z-w} \Big(\lambda^\alpha \lambda^\beta D_\alpha A_\beta\Big)(w) \\ &= \lambda^\alpha \lambda^\beta D_{(\alpha} A_{\beta)} = \frac{1}{2} \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta A_m = 0 \ , \end{aligned} \tag{3.88}$$

which follows when using the pure spinor constraint (3.25) and the OPE (3.80) between d_α and a superfield K. The conformal dimension of the unintegrated vertex operator is determined by the OPE with the energy-momentum tensor: In a conformal field theory the OPE of the energy-momentum T with a conformal primary ϕ_h of conformal weight h is given by [143]

$$T(z)\phi_h(w) \sim \frac{h\phi_h}{(z-w)^2} + \frac{\partial\phi_h}{z-w} \ . \tag{3.89}$$

Using (3.76) we find for the OPE of the unintegrated vertex operator (3.86) with the energy-momentum tensor T_{PS} in (3.74)

$$T_{\mathrm{PS}}(z)V(w) \sim \frac{1}{2}\frac{\partial^m \partial_m V(w)}{(z-w)^2} + \frac{(\Pi^m \partial_m + \partial\theta^\alpha D_\alpha)V(w) + \partial\lambda^\alpha A_\alpha(w)}{z-w}$$

[8] The relationship between pure spinors and the equations of motion in super Yang-Mills theory was already pointed out in [146, 147]. For a more recent overview for the use of pure spinors in off-shell supersymmetric theories see [148, 149]. Moreover, an early implementation of pure spinors in classical superstring theory in ten spacetime dimensions can be found in [150].

$$= \frac{\partial V(w)}{z-w} \ . \tag{3.90}$$

The first term above vanishes due to $k^2 = 0$ and we used the chain rule for the worldsheet derivative

$$(\Pi^m \partial_m + \partial\theta^\alpha D_\alpha)V + \partial\lambda^\alpha A_\alpha = \lambda^\alpha \partial A_\alpha + \partial\lambda^\alpha A_\alpha = \partial V \ , \tag{3.91}$$

which follows from the definitions of Π^m and d_α in (3.75) and

$$(\Pi^m \partial_m + \partial\theta^\alpha D_\alpha)K(X,\theta) = \left(\partial X^m \partial_m + \partial\theta^\alpha \frac{\partial}{\partial\theta_\alpha}\right)K(X,\theta) = \partial K(X,\theta) \tag{3.92}$$

for a superfield $K(X,\theta)$, which does not depend on the pure spinor λ^α or derivates of X^m or θ^α.

Because the unintegrated vertex operator is in the cohomology of Q, we have to exclude pure gauge superfields: Any gauge variation (2.18) of the linearized superfield A_α corresponds to adding a BRST exact piece $\lambda^\alpha \delta_\Omega A_\alpha = \lambda^\alpha D_\alpha \Omega = Q\Omega$ to the vertex operator. Hence, the gauge variation of V is not in the cohomology of the BRST operator.

In the PSF the unintegrated and integrated vertex operator for massless states are related by

$$QU = \partial V \ . \tag{3.93}$$

Acting with the BRST charge Q on the unintegrated vertex operator gives

$$\begin{aligned} Q(\partial\theta^\alpha A_\alpha) &= \partial\lambda^\alpha A_\alpha - \partial\theta^\alpha \lambda^\beta D_\beta A_\alpha \\ &= \partial\lambda^\alpha A_\alpha - \partial\theta^\alpha \lambda^\beta(-D_\alpha A_\beta + \gamma^m_{\alpha\beta} A_m) \ , \end{aligned} \tag{3.94}$$

$$\begin{aligned} Q(\Pi^m A_m) &= (\lambda\gamma^m \partial\theta)A_m + \Pi^m \lambda^\alpha D_\alpha A_m \\ &= (\lambda\gamma^m \partial\theta)A_m + \Pi^m \lambda^\alpha((\gamma_m W)_\alpha + \partial_m A_\alpha) \ , \end{aligned} \tag{3.95}$$

$$\begin{aligned} Q(d_\alpha W^\alpha) &= -(\lambda\gamma^m W)\Pi_m - d_\alpha \lambda^\beta D_\beta W^\alpha \\ &= -(\lambda\gamma^m W)\Pi_m - \frac{1}{4} d_\alpha \lambda^\beta (\gamma^{mn})_\beta{}^\alpha \mathcal{F}_{mn} \ , \end{aligned} \tag{3.96}$$

$$\begin{aligned} Q\Big(\frac{1}{2} N^{mn} \mathcal{F}_{mn}\Big) &= \frac{1}{4} d_\alpha (\gamma^{mn}\lambda)^\alpha \mathcal{F}_{mn} + \frac{1}{2} N^{mn} \lambda^\alpha D_\alpha \mathcal{F}_{mn} \\ &= \frac{1}{4} d_\alpha (\gamma^{mn}\lambda)^\alpha \mathcal{F}_{mn} + N^{mn} \lambda^\alpha \partial_m (\gamma_n W)_\alpha \ , \end{aligned} \tag{3.97}$$

where we used the OPEs in (3.76) and (3.80) and in addition the equations of motion of the super Yang-Mills fields (2.17). Adding the individual contributions yields

$$QU = \partial\lambda^\alpha A_\alpha + \partial\theta^\alpha \lambda^\beta D_\alpha A_\beta + \Pi^m \lambda^\alpha \partial_m A_\alpha + N^{mn}(\lambda\gamma_n \partial_m W) \ . \tag{3.98}$$

The last term $N^{mn}(\lambda\gamma_n \partial_m W)$ vanishes due to the pure spinor constraint

$$(\lambda\gamma^n)_\alpha (\lambda\gamma_n)_\beta = -\frac{1}{2} \gamma^m_{\alpha\beta} (\lambda\gamma_m \lambda) = 0 \tag{3.99}$$

and the Dirac equation $\gamma^m_{\alpha\beta}\partial_m W^\beta = 0$ such that

$$N^{mn}\lambda^\alpha\partial_m(\gamma_n W)_\alpha = \frac{1}{2}(w\gamma^m\gamma^n\lambda)(\lambda\gamma_n\partial_m W) - \frac{1}{2}(w\lambda)(\lambda\gamma^m\partial_m W) = 0 \ , \tag{3.100}$$

where we used the definition of the ghost Lorentz current $N^{mn} = \frac{1}{2}(w\gamma^{mn}\lambda)$ following from (3.74). Using the chain rule (3.91) we obtain for the action of the BRST operator on the integrated vertex operator

$$QU = \partial\lambda^\alpha A_\alpha + \lambda^\alpha(\partial\theta^\beta D_\beta A_\alpha + \Pi^m\partial_m A_\alpha) = \partial\lambda^\alpha A_\alpha + \lambda^\alpha\partial A_\alpha = \partial(\lambda A) = \partial V \ . \tag{3.101}$$

Therefore, we conclude that the integrated vertex operator $\int \mathrm{d}z\, U(z)$ is BRST closed up to surface terms. Surface terms do not contribute to string scattering amplitudes: They vanish because of the cancelled propagator argument. In addition, the cancellation of surface terms implies the invariance of scattering amplitudes under linearized gauge transformations. For the transformations $\delta_\Omega A_\alpha = D_\alpha\Omega$ and $\delta_\Omega A_m = \partial_m\Omega$ in (2.18) with some gauge scalar superfield Ω the variation of the unintegrated vertex operator $\delta_\Omega V = \lambda^\alpha D_\alpha\Omega = Q\Omega$ vanishes in the cohomology of the BRST charge. After using the chain rule (3.91) the variation of the integrated vertex operator reduces to the surfaces term

$$\delta_\Omega U = \Pi^m\partial_m\Omega + \partial\theta^\alpha D_\alpha\Omega = \partial\Omega \ , \tag{3.102}$$

which vanishes upon integration.

Chapter 4

Tree level amplitudes in the pure spinor formalism

At tree-level the interaction of massless superstring states, described by the vertex operators in section 3.6, corresponds to scattering amplitudes on the sphere and disk for closed and open strings, respectively. Both worldsheets have no moduli and therefore we only have to take care of the residual symmetry of the conformal Killing group (CKG) of the worldsheet topology. Because we have three (six) conformal Killing vectors (CKV) on the disk (sphere), we have to fix the position of three (six) real worldsheet positions [130]. Fixing the reparametrization invariance of the Möbius group of the worldsheet leads to the insertion of unintegrated vertex operators at these positions, while the other vertex operator are integrated over. It is a convenient choice to fix the vertex operators $i = 1, n-1$ and n, where n is the number of external states, to some arbitrary positions z_1, z_{n-1} and z_n. In principle, the amplitude is independent of the assignment of the integrated and unintegrated vertex operators. For n massless open strings we find that the scattering amplitude prescription is given by the following correlation function of vertex operators[1] [43]

$$\mathcal{A}(\sigma) = \int_{D_2(\sigma)} \mathrm{d}z_2\, \mathrm{d}z_3\, \cdots \mathrm{d}z_{n-2}\, \langle\!\langle V_1(z_1) U_2(z_2) U_3(z_3) \cdots U_{n-2}(z_{n-2}) V_{n-1}(z_{n-1}) V_n(z_n) \rangle\!\rangle\ , \tag{4.1}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ denotes the path integral over the variables in the pure spinor action (3.71). The integration domain, which is the boundary of the disk, can be parametrized by (parts

[1] As discussed in [114] the computation of gauge theory scattering amplitudes simplifies tremendously when considering ordered gauge invariants that depend only on kinematics [151, 152], i.e. color stripped or color ordered amplitudes. The color dressed S-matrix elements can be recovered by summing over color ordered open string amplitudes with appropriate color weights. For partial open string amplitudes the number of local diagrams grows factorial instead of exponentially [114]. Moreover, we are interested in closed string amplitudes, where the Chan-Paton factors of open string amplitudes are irrelevant. Therefore, we are only considering color stripped or ordered open string amplitudes.

of) the compactified real line

$$D_2(\sigma) = \{(z_1, z_2, \dots, z_n) \in \mathbb{R}^n | -\infty < z_{p_1} < z_{p_2} < \dots < z_{p_n} < \infty\} \ , \tag{4.2}$$

where $\sigma \equiv \sigma(1, 2, \dots, n)$ is the permutation of the labels that corresponds to the color ordering of the n external string states. Note that three vertex operator positions (z_1, z_{n-2}, z_n) in $D_2(\sigma)$ are position fixed due to the $PSL(2,\mathbb{R})$ invariance of the worldsheet. A convenient choice is $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$.

Similarly, the scattering amplitudes for n massless closed string states is given by [114]

$$\begin{aligned}\mathcal{A}_n = \int_{\mathcal{M}_{0,n}} \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3 \cdots \mathrm{d}^2 z_{n-2} \, \langle\!\langle V_1(z_1, \overline{z}_1) U_2(z_2, \overline{z}_2) \\ \times U_3(z_3, \overline{z}_3) \cdots U_{n-2}(z_{n-2}, \overline{z}_{n-2}) V_{n-1}(z_{n-1}, \overline{z}_{n-1}) V_n(z_n, \overline{z}_n) \rangle\!\rangle \ . \end{aligned} \tag{4.3}$$

Due to the action of the automorphism group $PSL(2,\mathbb{C})$ of the sphere we have fixed three punctures (z_1, z_{n-1}, z_n) and the integration over the remaining worldsheet coordinates $z_2, z_3, \dots, z_{n-2}$ is realized by the integral over the moduli space of the punctured Riemann sphere [153]

$$\mathcal{M}_{0,n} = \left\{(z_2, z_2, \dots, z_{n-2}) \in (\mathbb{CP}^1)^{n-3} | z_i \neq z_j \text{ for all } i \neq j\right\} \ . \tag{4.4}$$

Using KLT relations [34] it is possible to express the n closed string amplitude (4.3) in terms of two n open string amplitudes (4.1), which results in [34, 154, 155]

$$\begin{aligned}\mathcal{A}_n = \sum_{\sigma,\rho \in S_{n-3}} \mathcal{S}[\sigma(2, 3, \dots, n-2) | \rho(2, 3, \dots, n-2)]_1 \\ \times \mathcal{A}(1, \sigma(2, 3, \dots, n-2), n, n-1) \tilde{\mathcal{A}}(1, \rho(2, 3, \dots, n-2), n-1, n) \ , \end{aligned} \tag{4.5}$$

where σ and ρ are permutations of $\{2, 3, \dots, n-2\}$ and $\mathcal{S}[\cdot|\cdot]_p$ is the KLT momentum kernel, see chapter 7 for more details. Hence, we will focus only on open string amplitudes in the following sections, which are based on [114, 116, 117].

4.1 Wick's theorem in the pure spinor formalism

The correlation function $\langle\!\langle \dots \rangle\!\rangle$ in (4.1) is evaluated by integrating out the non-zero modes of the $h = 1$ conformal primaries $\partial\theta^\alpha(z_i), \Pi^m(z_i), d_\alpha(z_i)$ and $N^{mn}(z_i)$, which is done by applying Wick's theorem and using the OPEs of section 3.5. Similarly, one has to compute the contractions of non-zero modes of plane wave factors $e^{ik_i \cdot X(z_i, \overline{z}_i)}$, which gives the Koba-Nielsen factor of the corresponding amplitude. Thereby, one replaces the conformal primaries by their singularities with the other fields in the correlator and obtains [156]

$$\mathcal{A}(\sigma) = \int_{D_2(\sigma)} \mathrm{d}z_2\, \mathrm{d}z_3 \cdots \mathrm{d}z_{n-2} \, \langle\!\langle V_1(z_1) U_2(z_2) U_3(z_3) \cdots U_{n-2}(z_{n-2}) V_{n-1}(z_{n-1}) V_n(z_n) \rangle\!\rangle$$

$$= \int_{D_2(\sigma)} \mathrm{d}z_2 \, \mathrm{d}z_3 \cdots \mathrm{d}z_{n-2} \; \mathrm{KN}(\{z_i\}) \langle \mathcal{K}_n(\{z_i\}) \rangle \;, \tag{4.6}$$

where we have used (3.78) to strip off the open string Koba-Nielsen factor from the vertex operators and introduced the zero mode correlator $\langle \ldots \rangle$. The contraction of the plane wave factors is given by

$$\mathrm{KN}(\{z_i\}) = \prod_{i<j}^{n} |z_{ij}|^{s_{ij}} \;, \tag{4.7}$$

where $s_{ij} = \frac{1}{2}(k_i + k_j)^2 = k_i \cdot k_j$ for massless states, i.e. $k_i^2 = 0$. Moreover, the exact dependence of $\mathcal{K}_n(\{z_i\})$ on z_i is determined by the OPEs of the superfields. Nevertheless, this determines the correlator as a unique function of the worldsheet coordinates z_i on the disk [27]. More explicitly, the correlator can be expressed as

$$\langle \mathcal{K}_n(\{z_i\}) \rangle = \left\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta; \{z_i\}) \right\rangle \;. \tag{4.8}$$

Note that $f_{\alpha\beta\gamma}(\theta; \{z_i\})$ contains all the information of the external states like momenta and polarization vectors/spinors. These enter $f_{\alpha\beta\gamma}(\theta; \{z_i\})$ via the θ-expansions in (2.32).

According to Wick's theorem to compute $f_{\alpha\beta\gamma}(\theta; \{z_i\})$ in (4.8) explicitly we have to sum over all possible contractions of the integrated and unintegrated vertex operators. For example, the contraction between an unintegrated and integrated vertex operator follows from the OPEs [116]

$$\begin{aligned}
(\Pi^m A^j_m)(z_j) V_i(z_i) &\sim -V_i(z_i) i k_i^m A^j_m \lambda^\alpha A^i_\alpha = -\frac{1}{z_{ji}} (i k_i \cdot A^j) V_i \;, \\
(d_\beta W^\beta_j)(z_j) V_i(z_i) &\sim -\frac{1}{z_{ji}} \lambda^\alpha D_\beta A^i_\alpha W^\beta_j = -\frac{1}{z_{ji}} \lambda^\alpha (-D_\alpha A^i_\beta + \gamma^m_{\alpha\beta} A^i_m) W^\beta_j \\
&= \frac{1}{z_{ji}} \Big((Q A^i_\alpha) W^\alpha_j - A^i_m (\lambda \gamma^m W_j) \Big) \;, \\
\frac{1}{2} N^{mn} F^j_{mn} V_i(z_i) &\sim -\frac{1}{4} \frac{1}{z_{ji}} \lambda^\alpha \gamma^{mn}{}_\alpha{}^\beta A^i_\beta F^j_{mn} = -\frac{1}{z_{ji}} A^i_\alpha (Q W^\alpha_j) \;,
\end{aligned} \tag{4.9}$$

which were compute using (3.76). Because $(\partial\theta A_j)$ has no OPE with V_i the OPE residue between U_j and V_i is given by

$$U_j(z_j) V_i(z_i) \sim \frac{1}{z_{ji}} \Big(-(i k_i \cdot A^j) V_i - A^i_m (\lambda \gamma^m W_j) + Q(A_i W_j) \Big) \;. \tag{4.10}$$

From this analysis the single contraction between the j^{th} integrated vertex operator with the i^{th} unintegrated vertex operator follows [109]

$$\begin{aligned}
K_{ji} &= z_{ji} \overrightarrow{U_j(z_j) V_i}(z_i) \\
&\sim -(i k_i \cdot A_j) V_i - A^i_m (\lambda \gamma^m W_j) + Q(A_i W_j) \;.
\end{aligned} \tag{4.11}$$

Moreover, the contraction between two integrated vertex operators can be denoted by

$$
\begin{aligned}
K_{ji} &= z_{ji}\overrightarrow{U_j(z_j)U}_i(z_i) \\
&\sim -(ik_i \cdot A_j)U_i + \partial\theta^\alpha D_\alpha A^i_\beta W^\beta_j + \Pi^m ik^i_m(A_i W_j) + (\partial\theta\gamma^m W_i)A^j_m \\
&\quad + \frac{1}{4}(d\gamma^{mn}W_j)F^i_{mn} + N^{mn}\Big(k^i_m(W_i\gamma_n W_j + \eta^{ab}F^j_{ma}F^i_{nb})\Big) ,
\end{aligned}
\tag{4.12}
$$

where the arrow indicates that we are contracting the conformal primaries with $h = 1$ in U_j with U_i or V_i but not the $h = 1$ primaries of U_i with U_j. Note that even after the contraction the superfields $K_i(z_i)$ and conformal primaries still depend on the corresponding vertex operator position z_i. This will be important when we are computing Ks involving more than two vertex operators, i.e. more than one contraction. Although, Wick's theorem is the most fundamental way to perform this task, it is not very efficient. During the evaluation of correlation functions we will encounter the same contractions over and over again but with different labels for the external states in the amplitude, which can be exploited in these calculations by introducing composite superfields. Hence, we will connect Wick contractions to the composite superfields of [62] in chapter 5, which provide a more suitable framework for computing scattering amplitudes in the pure spinor formalism.

4.2 The zero mode prescription

At tree-level the fields with conformal weight one appearing in the integrated and unintegrated vertex operators have no zero modes. Therefore, after integrating them out the correlator will only depend on the zero modes of the conformal weight $h = 0$ fields λ^α and θ^α,[2] which have a single zero mode at genus zero [38]. This zero mode correlator is denoted by $\langle \ldots \rangle$ in (4.8) and will be analysed in this section.

The left side of equation (4.8) is BRST closed, because it is made out of BRST closed objects $V(z_i)$ and $\int \mathrm{d}z_i\, U(z_i)$. This implies that the function $f_{\alpha\beta\gamma}(\theta; \{z_i\})$ has to satisfy the constraint

$$
\int_{D_2(\sigma)} \mathrm{d}z_2\, \mathrm{d}z_3 \cdots \mathrm{d}z_{n-2}\, \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta D_\delta f_{\alpha\beta\gamma}(\theta; \{z_i\}) = 0 . \tag{4.13}
$$

In general, $f_{\alpha\beta\gamma}(\theta; \{z_i\})$ is a power series in θ^k for $k = 0, 1, \ldots, 16$ according to the θ-expansions (2.32) of the super Yang-Mills fields. But only terms proportional to θ^5 give a non-vanishing contribution to the zero mode correlator, i.e.

$$
\Big\langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta; \{z_i\}) \Big\rangle = \Big\langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta; \{z_i\}) \Big|_{\theta^5} \Big\rangle . \tag{4.14}
$$

[2] There is actually a further contribution of conformal primaries with conformal weight $h = 0$, which is the plane wave factor $e^{ik_i \cdot X(z_i, \bar{z}_i)}$. The zero modes of the plane wave factor give a momentum preserving δ-function, which is left implicit, but we always assume momentum conservation.

Because at ghost number three there is only one element in the BRST cohomology that is proportional to θ^5, in ten dimensions all non-vanishing contributions in (4.14) to a zero mode correlator are proportional to

$$\Big\langle(\lambda^3\theta^5)\Big\rangle = 2880 \ , \tag{4.15}$$

where we have introduced the notation

$$(\lambda^3\theta^5) = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \ . \tag{4.16}$$

The normalization 2880 was chosen such that tree-level amplitude results match in PSF and RNS formalism [157].

General properties of the zero mode prescription

The zero mode prescription (4.15) is BRST closed

$$\begin{aligned} Q(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) &= 3(\lambda\gamma^m\lambda)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \\ &\quad -2(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\lambda\gamma_{mnp}\theta) = 0 \ , \end{aligned} \tag{4.17}$$

where the first term vanishes due to the pure spinor constraint $\lambda\gamma^m\lambda = 0$ and after decomposing $\gamma_{mnp} = \gamma_m\delta_{np} - \delta_{mn}\gamma_p + \delta_{mp}\gamma_n$ and using $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$ also the second term is zero. Furthermore, the zero mode prescription $\langle(\lambda^3\theta^5)\rangle$ is not BRST exact

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \neq Q\Omega \ , \tag{4.18}$$

because there is no Lorentz scalar build from two λs and six θs: If there was a scalar $\Omega(\lambda,\theta)$ with $Q\Omega(\lambda,\theta) = (\lambda^3\theta^5)$ it would be constructed out of two λs and six θs, because $Q\theta^\alpha = \lambda^\alpha$ and ∂_m in D_α vanishes for functions depending only on λ and θ. From the Fierz identity

$$\psi^\alpha\phi^\beta = \frac{1}{16}\gamma^{\alpha\beta}_{m_1}(\psi\gamma^{m_1}\phi) + \frac{1}{96}(\gamma_{m_1\ldots m_3})^{\alpha\beta}(\psi\gamma^{m_1\ldots m_3}\phi) + \frac{1}{3840}(\gamma_{m_1\ldots m_5})^{\alpha\beta}(\psi\gamma^{m_1\ldots m_5}\phi) \ , \tag{4.19}$$

where the gamma-matrices are defined as

$$\gamma^{m_1 m_2 \ldots m_k} = \frac{1}{k!}\gamma^{[m_1}\gamma^{m_2}\cdots\gamma^{m_k]} \ , \tag{4.20}$$

it follows that the combination $\lambda^\alpha\lambda^\beta = \frac{1}{3840}(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$ only contains a five form component.[3] After the tensor product with an antisymmetric spinor $\theta^{\alpha_1}\cdots\theta^{\alpha_6}$ the combination $\lambda^2 \otimes \theta^6$ does not incorporate a Lorentz scalar. Explicitly, this can be seen from the $SO(10)$ representation of the tensor product $\lambda^2 \otimes \theta^6$ in terms of their Dynkin labels

$$(00002) \otimes ((01020) \oplus (201000)) = (00011) \oplus (00022) \oplus 2(00120) \oplus \ldots \ , \tag{4.21}$$

[3] In $\lambda^\alpha\lambda^\beta$ the only $SO(10)$ irreducible component is the five form: The vector $(\lambda\gamma^m\lambda)$ of $\lambda^\alpha\lambda^\beta$ vanishes due to the pure spinor constraint (3.25) and the 3-form is not present because of the antisymmetry of gamma matrices $\gamma^{mnp}_{\alpha\beta} = -\gamma^{mnp}_{\beta\alpha}$.

where two λs are represented by the Dynkin labels (00002) and six θs are characterized by (01020) $\oplus$ (201000). Hence, the tensor product in (4.21) does not exhibit a scalar representation (00000).

The OPEs used to compute contractions between vertex operators transform appropriately under the generator $\mathcal{Q}_\alpha$ in (3.5) and will not break supersymmetry, because the action of the pure spinor formalism is spacetime supersymmetric as was shown in section 3.1 and 3.2. It remains to show that the open string scattering amplitude prescription (4.1) is supersymmetric. The zero mode prescription (4.15) should preserve the supersymmetric nature of the formalism: Due to the zero mode prescription (4.15) we only get non-vanishing contributions after the supersymmetry transformation $\delta\theta^\alpha = \eta^\alpha$ and $\delta\lambda^\alpha = 0$ from terms of the form

$$\mathcal{A}(\sigma) = \int_{D_2(\sigma)} \mathrm{d}z_2\, \mathrm{d}z_3\, \cdots \mathrm{d}z_{n-2}\, \langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\theta^\alpha \Phi_\alpha(\{z_i\})\rangle \ , \tag{4.22}$$

where Φ_α is a θ^α independent spinor that contains the momenta and polarization vectors/spinors of the external states. The transformation replaces one θ^α by η^α and therefore leaves us again with three λs and five θs such that after the zero mode integration the supersymmetry variation of the amplitude becomes

$$\delta\mathcal{A}(\sigma) = 2880 \int_{D_2(\sigma)} \mathrm{d}z_2 \mathrm{d}z_3 \cdots \mathrm{d}z_{n-2} \eta^\alpha \Phi_\alpha(\{z_i\}) \ . \tag{4.23}$$

In (4.22) we have constructed one example of a function

$$f_{\alpha\beta\gamma}(\theta, z_i) = (\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta)\theta^\delta\Phi_\delta(\{z_i\}) \ . \tag{4.24}$$

By demanding BRST closure for (4.22), i.e. plugging the function $f_{\alpha\beta\gamma}(\theta;\{z_i\})$ in (4.24) into (4.13), we get

$$\int_{D_2(\sigma)} \mathrm{d}z_2\, \mathrm{d}z_3\, \cdots \mathrm{d}z_{n-2}\, \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\Phi_\delta(\{z_i\}) = 0 \ , \tag{4.25}$$

which only vanishes if Φ_α is a total derivative of the worldsheet coordinates: $\Phi_\alpha = \partial(\ldots)$. Thus, the supersymmetry variation (4.23) of the amplitude vanishes after integration and the zero mode prescription preserves supersymmetry.

Pure spinor superspace

The last step in the computation of scattering amplitudes is to extract the contractions between polarizations and momenta from pure spinor superspace expressions (4.8) by integrating out the zero modes of three λs and five θs utilizing the zero mode prescription (4.15). The most general form of a pure spinor superspace expression [156] in (4.8) containing only five θs can be written as

$$f_{\alpha\beta\gamma}(\theta;\{z_i\}) = \lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\delta_1}\theta^{\delta_1}\theta^{\delta_2}\theta^{\delta_3}\theta^{\delta_4}\theta^{\delta_5} f_{\alpha\beta\gamma|\delta_1\delta_2\delta_3\delta_4\delta_5}(\{z_i\}) \ , \tag{4.26}$$

where the momentum and polarization dependence is stored in $f_{\alpha\beta\gamma|\delta_1\delta_2\delta_3\delta_4\delta_5}$. To perform the zero mode integration and extract the explicit dependence on momenta and polarizations we use that there is only one scalar representation in $SO(10)$ in the decomposition of three pure spinors λ^α and five Weyl spinors θ^α [114]

$$(00003) \otimes ((00030) \oplus (11010)) = 1 \times (00000) \oplus 2 \times (00011) \otimes \ldots \ , \tag{4.27}$$

where in the tensor product the $SO(10)$ representation (00003) and (00030) $\oplus$ (11010) correspond to $\lambda^\alpha\lambda^\beta\lambda^\gamma$ and $\theta^{\delta_1}\theta^{\delta_1}\theta^{\delta_2}\theta^{\delta_3}\theta^{\delta_4}\theta^{\delta_5}$, respectively. Since the scalar (00000) in (4.27) appears with multiplicity one, any expression (4.8) containing only three λs and five θs is proportional to the scalar component $(\lambda^3\theta^5)$. Moreover, the proportionality constants can be fully expressed in terms of Kronecker deltas, gamma functions and Levi-Civita tensors.

In section 4.4 we will find a superspace expression $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{abc}\theta)\rangle$ with free vector indices m, n, p and a, b, c. We have to extract the unique scalar component $(\lambda^3\theta^5)$ before we can use the zero mode prescription (4.15). Using symmetry arguments we arrive at

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{abc}\theta)\rangle = \frac{24}{3!}\delta_a^{[m}\delta_b^n\delta_c^{p]} \ . \tag{4.28}$$

The combination of Kronecker deltas has the same symmetry properties as the superspace expression $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{abc}\theta)\rangle$: Both are antisymmetric in the indices $[mnp]$ and $[abc]$. Moreover, we can determine the proportionality constant by fully contracting the vectorial indices using $\delta_m^a\delta_n^b\delta_p^c$. Thereby, for the left hand side of (4.28) we get $\langle(\lambda^3\theta^5)\rangle = 2880$ and the contracted Kronecker delta $\frac{1}{3!}\delta_m^{[m}\delta_n^n\delta_p^{p]} = \binom{10}{3} = 120$ gives for the right hand side $24 \times 120 = 2880$. According to the discussion above there is only one scalar representation for three λs and five θs in an arbitrary superspace expression such that (4.28) is the unique choice.

The other zero mode prescription needed in section 4.4 is given by the expression

$$\langle(\lambda^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\chi\gamma_n\theta)(\phi\gamma_p\theta)\rangle = -\frac{1}{96}(\chi\gamma_n\gamma^{rst}\gamma_p\psi)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{rst}\theta) \ , \tag{4.29}$$

where χ and ψ are arbitrary Weyl spinors and we used the Fierz identity (4.19), i.e. $\theta^\alpha\theta^\beta = \frac{1}{96}\gamma_{rst}^{\alpha\beta}(\theta\gamma^{rst}\theta)$. Thus, after applying (4.28) and $\gamma_n\gamma^{mnp}\gamma_p = -72\gamma^m$ we obtain

$$\langle(\lambda^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\chi\gamma_n\theta)(\phi\gamma_p\theta)\rangle = -\frac{1}{4}(\chi\gamma_n\gamma^{mnp}\gamma_p\psi) = 18(\chi\gamma^m\psi) \ . \tag{4.30}$$

More details and an in-depth discussion of pure spinor zero mode correlators can be found in [114].

4.3 Independence of scattering amplitudes on the assignment of unintegrated and integrated vertex operators

In the formulation of the scattering amplitude prescription for open and closed strings in (4.1) and (4.3) we have chosen the external states $1, n-1$ and n to be represented by an unintegrated vertex operator V at a fixed locations z_1, z_{n-1} and z_n on the worldsheet. Moreover, we stated that the amplitude is independent of this assignment, i.e. that the prescriptions in (4.1) and (4.3) do not depend on which external states $\{i, j, k\}$ appear as unintegrated vertex operators V_i, V_j and V_k at fixed punctures z_i, z_j and z_k on the worldsheet.

Open superstring amplitudes

In an open string scattering amplitude it is possible to swap the representation of neighbouring states i and $i+1$ from $V_i \int \mathrm{d}z_{i+1}\, U_{i+1}$ to $\int \mathrm{d}z_i\, U_i V_{i+1}$ [157]. For the amplitude prescription (4.1) this implies that

$$\begin{aligned}&\left\langle\!\!\left\langle V_1(z_1) \int_{z_1}^{z_{n-1}} \mathrm{d}z_2\, U_2(z_2) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(z_{n-1}) V_n(z_n) \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \int_{z_n}^{z_1} \mathrm{d}y\, U_1(y) V_2(z_1) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(z_{n-1}) V_n(z_n) \right\rangle\!\!\right\rangle . \end{aligned} \tag{4.31}$$

As before, we integrate the vertex operator positions over the compactified real line including the point at $\pm\infty$. The integrated vertex operator itself is not BRST closed $QU_i(w) = \oint \lambda^\alpha(z) d_\alpha(z) U_i(w) = \partial V_i(w)$ such that

$$V_1(z_1) V_n(z_n) = \int_{z_n}^{z_1} \mathrm{d}y\, \partial V_1(y) V_n(z_n) = \int_{z_n}^{z_1} \mathrm{d}y\, Q(U_1(y)) V_n(z_n) \ , \tag{4.32}$$

where the contribution $V_1(z_n)V_n(z_n)$ coming from the lower integration boundary vanishes by the cancelled propagator argument. Terms containing two vertex operators at the same position can be discarded, because for two states i and j with sufficiently large and positive $k_i \cdot k_j$ the contraction of their vertex operators is given by [157]

$$V_i(z) V_j(z+\epsilon) \to \epsilon^{k_i \cdot k_j} \to 0 \quad \text{for} \quad \epsilon \to 0 \ . \tag{4.33}$$

Since a scattering amplitude is analytic in the momenta (except for the poles), the above statement (4.33) holds for all values of k_i and k_j after analytic continuation, if it does for some region of k_i and k_j [27]. Next, we have to deform the integration contour of the BRST operator Q in order to encircle all other vertex operators instead of U_1:

$$\left\langle\!\!\left\langle V_1(z_1) \int_{z_1}^{z_{n-1}} \mathrm{d}z_2\, U_2(z_2) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(z_{n-1}) V_n(z_n) \right\rangle\!\!\right\rangle$$

$$
\begin{aligned}
&= -\left\langle\!\!\left\langle \int_{z_n}^{z_1} \mathrm{d}y\, U_1(y) \int_{z_1}^{z_{n-1}} \mathrm{d}z_2\, Q\left[U_2(z_2) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(z_{n-1}) V_n(z_n) \right] \right\rangle\!\!\right\rangle \\
&= -\left\langle\!\!\left\langle \int_{z_n}^{z_1} \mathrm{d}y\, U_1(y) \int_{z_1}^{z_{n-1}} \mathrm{d}z_2\, \partial V_2(z_2) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(1) V_n(z_n) \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \int_{z_n}^{z_1} \mathrm{d}y\, U_1(y) V_2(z_1) \prod_{i=3}^{n-2} \int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, U_i(z_i) V_{n-1}(z_{n-1}) V_n(z_n) \right\rangle\!\!\right\rangle . \qquad (4.34)
\end{aligned}
$$

Above terms, where the BRST charge Q acts on vertex operators $U(z_i)$ for $3 \leq i \leq n-2$, were discarded, since the boundary terms of the integrals $\int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i\, \partial V_i(z_i)$ vanish according to the cancelled propagator argument:

$$
\ldots U_{i-1}(z_{i-1})(V_i(z_{n-1}) - V_i(z_{i-1})) \ldots V_{n-1}(z_{n-1}) \ldots = 0 \; . \qquad (4.35)
$$

Moreover, the unintegrated vertex operators are BRST closed, $QV_i = 0$ for $i = n-1, n$. The remaining integral over $QU_2 = \partial V_2$ vanishes at the upper integration boundary $z_2 = z_n$

$$
\ldots V_2(z_{n-1}) \ldots V_{n-1}(z_{n-1}) \ldots = 0 \; . \qquad (4.36)
$$

However, the lower one at z_1 leads to a non-trivial contribution in (4.34), because it does not coincide with any other vertex operator position, which proofs the claim.

Closed superstring amplitudes

From the decomposition (4.5) it immediately follows that closed superstring amplitudes at sphere level are independent of the choice which vertex operators are integrated and which are unintegrated, because the two open string amplitudes in (4.5) do not depend on that choice. Though, it is possible to show this without using KLT relations such that the closed string amplitude is manifestly independent of the assignment. Therefore, we follow the lines of [157] to show that in the pure spinor formalism the integrated and unintegrated vertex operators for closed strings are not BRST closed

$$
Q\overline{Q} U_i(z, \overline{z}) = \partial \overline{\partial} V_i(z, \overline{z}) \; , \qquad (4.37)
$$

where Q and $\overline{Q}$ are the holomorphic and antiholomorphic BRST operators. Therefore, we write a closed string amplitude at sphere level following from the prescription (4.3) as

$$
\begin{aligned}
&\left\langle\!\!\left\langle V_1(z_1, \overline{z}_1) \prod_{i=2}^{n-2} \int \mathrm{d}^2 z_i\, U_i(z_i, \overline{z}_i) V_{n-1}(z_{n-1}, \overline{z}_{n-1}) V_n(z_n, \overline{z}_n) \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \int \mathrm{d}^2 y \int \mathrm{d}^2 z_2\, (\delta^2(y - z_1) - \delta^2(y - z_2)) V_1(y, \overline{y}) U_2(z_2, \overline{z}_2) \right.\right. \\
&\qquad \left.\left. \times \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_i\, U_i(z_i, \overline{z}_i) V_{n-1}(z_{n-1}, \overline{z}_{n-1}) V_n(z_n, \overline{z}_n) \right\rangle\!\!\right\rangle . \qquad (4.38)
\end{aligned}
$$

We have shifted the vertex operator $V_1(z_1,\overline{z}_1) \to V_1(z_1,\overline{z}_1) - V_1(z_2,\overline{z}_2)$, which is possible due to the cancelled propagator argument, i.e. the term containing $V_1(z_2,\overline{z}_2)$ gives a vanishing contribution. Moreover, two δ-functions were introduced to rewrite the shifted vertex operator as

$$V_1(z_1,\overline{z}_1) - V_1(z_2,\overline{z}_2) = \int \mathrm{d}y^2\,(\delta^2(y-z_1) - \delta^2(y-z_2))V_1(y,\overline{y}) \ . \tag{4.39}$$

These δ functions can be written in terms of a worldsheet derivative

$$\begin{aligned}\delta^2(y-z_1) - \delta^2(y-z_2) &= \frac{1}{2\pi}(\partial_y\partial_{\overline{y}}\ln|y-z_1| - \partial_y\partial_{\overline{y}}\ln|y-z_2|)\\ &= \frac{1}{2\pi}\Big(\partial_y\partial_{\overline{y}}\ln|y-z_1| - \partial_y\partial_{\overline{y}}\ln|y-z_2| + \partial_y\partial_{\overline{y}}\ln|z_2-z_1|\Big)\\ &= \frac{1}{2\pi}\partial_y\partial_{\overline{y}}\ln\left|\frac{(y-z_1)(z_2-z_1)}{(y-z_2)}\right| \ ,\end{aligned} \tag{4.40}$$

where we added the expression $\frac{1}{2\pi}\partial_y\partial_{\overline{y}}\ln|z_2-z_1| = 0$ for later convenience and obtain

$$\begin{aligned}&\left\langle\!\!\left\langle V_1(z_1,\overline{z}_1)\prod_{i=2}^{n-2}\int \mathrm{d}^2 z_i\, U_i(z_i,\overline{z}_i)V_{n-1}(z_{n-1},\overline{z}_{n-1})V_n(z_n,\overline{z}_n)\right\rangle\!\!\right\rangle\\ &= \left\langle\!\!\left\langle\int \mathrm{d}^2 y \int \mathrm{d}^2 z_2\,\frac{1}{2\pi}\partial_y\partial_{\overline{y}}\ln\left|\frac{(y-z_1)(z_2-z_1)}{(y-z_2)}\right| V_1(y,\overline{y})U_2(z_2,\overline{z}_2)\right.\right.\\ &\qquad\left.\left.\times\prod_{i=3}^{n-2}\int \mathrm{d}^2 z_i\, U_i(z_i,\overline{z}_i)V_{n-1}(z_{n-1},\overline{z}_{n-1})V_n(z_n,\overline{z}_n)\right\rangle\!\!\right\rangle \ .\end{aligned} \tag{4.41}$$

After integration by parts such that the derivatives with respect to y and $\overline{y}$ act on $V_1(y,\overline{y})$ and using (4.37) we arrive at

$$\begin{aligned}&\left\langle\!\!\left\langle V_1(z_1,\overline{z}_1)\prod_{i=2}^{n-2}\int \mathrm{d}^2 z_i\, U_i(z_i,\overline{z}_i)V_{n-1}(z_{n-1},\overline{z}_{n-1})V_n(z_n,\overline{z}_n)\right\rangle\!\!\right\rangle\\ &= \left\langle\!\!\left\langle\int \mathrm{d}^2 y \int \mathrm{d}^2 z_2\,\frac{1}{2\pi}\ln\left|\frac{(y-z_1)(z_2-z_1)}{(y-z_2)}\right| Q\overline{Q}U_1(y,\overline{y})U_2(z_2,\overline{z}_2)\right.\right.\\ &\qquad\left.\left.\times\prod_{i=3}^{n-2}\int \mathrm{d}^2 z_i\, U_i(z_i,\overline{z}_i)V_{n-1}(z_{n-1},\overline{z}_{n-1})V_n(z_n,\overline{z}_n)\right\rangle\!\!\right\rangle \ .\end{aligned} \tag{4.42}$$

Similar as for the open string amplitude, we deform the integration contour of the BRST charges Q and $\overline{Q}$ such that they encircle the other vertex operators. If Q or $\overline{Q}$ acts on an integrated vertex operator $U_i(z_i,\overline{z}_i)$ we obtain $\int \mathrm{d}^2 z_i\,\partial V_i(z_i,\overline{z}_i)$ or $\int \mathrm{d}^2 z_i\,\overline{\partial} V_i(z_i,\overline{z}_i)$, respectively, which vanish for $i \neq 2$, because they are integrated over the sphere, which has no boundary. The only non-vanishing contribution comes from $U_2(z_2,\overline{z}_2)$ when both BRST charges Q and $\overline{Q}$ encircle the punctures $(z_2,\overline{z}_2)$:

$$\left\langle\!\!\left\langle\int \mathrm{d}^2 y\, U_1(y,\overline{y})\int \mathrm{d}^2 z_2\,\frac{1}{2\pi}\ln\left|\frac{(y-z_1)(z_2-z_1)}{(y-z_2)}\right| Q\overline{Q}\Big[U_2(z_2,\overline{z}_2)\right.\right.$$

$$\times \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_j \, U_i(z_i,\overline{z}_i) V_{n-1}(z_{n-1},\overline{z}_{n-1}) V_n(z_n,\overline{z}_n) \Big] \Big\rangle\!\!\Big\rangle$$
$$= \Big\langle\!\!\Big\langle \int \mathrm{d}^2 y \, U_1(y,\overline{y}) \int \mathrm{d}^2 z_2 \, \frac{1}{2\pi} \ln \left| \frac{(y-z_1)(z_2-z_1)}{(y-z_2)} \right| \partial_{z_2} \partial_{\overline{z}_2} V_2(z_2,\overline{z}_2)$$
$$\times \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_j \, U_i(z_i,\overline{z}_i) V_{n-1}(z_{n-1},\overline{z}_{n-1}) V_n(z_n,\overline{z}_n) \Big\rangle\!\!\Big\rangle$$
$$= \Big\langle\!\!\Big\langle \int \mathrm{d}^2 y \, U_1(y,\overline{y}) \int \mathrm{d}^2 z_2 \, \frac{1}{2\pi} \partial_{z_2} \partial_{\overline{z}_2} \ln \left| \frac{(y-z_1)(z_2-z_1)}{(y-z_2)} \right| V_2(z_2,\overline{z}_2)$$
$$\times \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_j \, U_i(z_i,\overline{z}_i) V_{n-1}(z_{n-1},\overline{z}_{n-1}) V_n(z_n,\overline{z}_n) \Big\rangle\!\!\Big\rangle$$
$$= \Big\langle\!\!\Big\langle \int \mathrm{d}^2 y \, U_1(y,\overline{y}) \int \mathrm{d}^2 z_2 \, (\delta(z_2-z_1) - \delta(y-z_2)) V_2(z_2,\overline{z}_2)$$
$$\times \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_j \, U_i(z_i,\overline{z}_i) V_{n-1}(z_{n-1},\overline{z}_{n-1}) V_n(z_n,\overline{z}_n) \Big\rangle\!\!\Big\rangle$$
$$= \Big\langle\!\!\Big\langle \int \mathrm{d}^2 y \, U_1(y,\overline{y}) V_2(z_1,\overline{z}_1) \prod_{i=3}^{n-2} \int \mathrm{d}^2 z_j \, U_i(z_i,\overline{z}_i) V_{n-1}(z_{n-1},\overline{z}_{n-1}) V_n(z_n,\overline{z}_n) \Big\rangle\!\!\Big\rangle \; , \qquad (4.43)$$

where we used the same steps as before but in a reversed order and the cancelled propagator argument

$$\ldots V_1(y,\overline{y}) V_2(y,\overline{y}) \ldots = 0 \; . \qquad (4.44)$$

Hence, we obtained the original amplitude with a different assignment of vertex operators.

4.4 Computing scattering amplitudes in the pure spinor formalism

We want to demonstrate the computation of scattering amplitudes (4.1) by applying the steps discussed in this chapter. Therefore, we choose the simplest example, i.e. the color ordered three-point amplitude, which is given by

$$\mathcal{A}(1,2,3) = \langle\!\langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle\!\rangle = \langle\!\langle (\lambda A_1)(z_1) (\lambda A_2)(z_2) (\lambda A_3)(z_3) \rangle\!\rangle \; . \qquad (4.45)$$

The vertex operator positions z_1, z_2 and z_3 are fixed to points on the boundary of the disk and we do not integrate over them. The tree-level prescription does not contain any conformal fields of weight $h = 1$ for less than four massless external states, because there are only unintegrated vertex operators of conformal dimension $h = 0$ in (4.45). Hence, the amplitude gets no contribution from OPE contractions except for the Koba-Nielsen factor

$$\mathrm{KN}_3 = \Big\langle\!\!\Big\langle e^{ik_1 \cdot X(z_1)} e^{ik_1 \cdot X(z_2)} e^{ik_1 \cdot X(z_3)} \Big\rangle\!\!\Big\rangle = \prod_{1 \le i < j}^{3} |z_i - z_j|^{k_i \cdot k_j} = 1 \; , \qquad (4.46)$$

because for massless external states $k_i^2 = 0$ momentum conservation $k_1^m + k_2^m + k_m^3 = 0$ implies that

$$k_i \cdot k_j = \frac{1}{2}(k_i + k_j)^2 = \frac{1}{2}k_k^2 = 0 \quad \text{for } i, j, k = 1, 2, 3\ . \tag{4.47}$$

Therefore, the only contribution to the amplitude (4.45) comes from the zero modes of λ^α and θ^α. The evaluation of the component expansions of the three-point amplitude boils down to plugging in the θ-expansion (2.32) and selecting the terms containing precisely five θs. From the spinorial superpotential A_α^i only the following bosonic terms at order θ^1 and θ^3

$$A_\alpha^i(X,\theta) \to \left\{\frac{1}{2}e_m^i(\gamma^m\theta)_\alpha - \frac{1}{32}f_{mn}^i(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta)\right\}e^{ik_i\cdot X} \tag{4.48}$$

or the fermionic term at order θ^2

$$A_\alpha^i(X,\theta) \to -\frac{1}{3}(\gamma^m\theta)_\alpha(\theta\gamma_m\chi_i)e^{ik_i\cdot X} \tag{4.49}$$

can contribute to (4.45) and lead to the possibilities listed in table 4.1 to saturate θ^5. All other terms in the θ-expansion of order at least θ^4 drop out of the amplitude, since

Superfield:	$A_{1\alpha}(\theta)$	$A_{2\alpha}(\theta)$	$A_{3\alpha}(\theta)$
Number of θs from each superfield:	θ^3	θ^1	θ^1
	θ^1	θ^3	θ^1
	θ^1	θ^1	θ^3
	θ^1	θ^2	θ^2
	θ^2	θ^1	θ^2
	θ^2	θ^2	θ^1

Table 4.1: Terms containing five θs in (4.45).

these result in superspace expressions of the form $\lambda^3\theta^{\geq 6}$ when taking the other vertex operators (each contributing at least one θ^α) into account, which vanish due to the zero mode prescription (4.15). Taking only the terms from table 4.1, which have three λs and five θs leads to

$$\begin{aligned}\mathcal{A}(1,2,3) = \Big\langle \lambda^\alpha\lambda^\beta\lambda^\gamma\Big\{&-\frac{e_m^1}{2}(\gamma^m\theta)_\alpha\frac{e_n^2}{2}(\gamma^n\theta)_\beta\frac{f_3^{pq}}{32}(\gamma^r\theta)_\gamma(\theta\gamma_{pqr}\theta)\\ &+\frac{e_1^m}{2}(\gamma_m\theta)_\alpha\frac{(\chi_2\gamma^n\theta)}{3}(\gamma_n\theta)_\beta\frac{(\chi_3\gamma^p\theta)}{3}(\gamma_p\theta)_\gamma\Big\rangle + \text{cyclic}(123)\\ = &-\frac{1}{128}e_1^m e_2^n f_{pq}^3\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^r\theta)(\theta\gamma_{pqr}\theta)\rangle\\ &+\frac{1}{18}e_1^m\langle(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(e_2\gamma^n\theta)(\chi_3\gamma^p\theta)\rangle + \text{cyclic}(123)\ .\end{aligned} \tag{4.50}$$

To evaluate the zero mode correlators we can use (4.28) with $\delta^r_r = 10$

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^r\theta)(\theta\gamma_{pqr}\theta)\rangle = 24\delta^{mrn}_{pqr} = -64\delta^{mn}_{pq} \tag{4.51}$$

and (4.30). Hence, we find for the supersymmetric three-point amplitude

$$\mathcal{A}(1,2,3) = \frac{1}{2}e^m_1 e^n_2 f^3_{mn} + e^m_1(\chi_2\gamma_m\chi_3) + \text{cyclic}(123) \ , \tag{4.52}$$

where we have applied momentum conservation $k^m_1 + k^m_2 + k^m_3 = 0$ and transversality $e_i \cdot k_i = 0$. Note that $\mathcal{A}(1,2,3)$ is independent of α' and therefore equal to the three-point SYM amplitude.

Chapter 5

BRST bulding blocks for pure spinor superspace

In string theory only physical states contribute to scattering amplitudes. These states are elements of the cohomology of the BRST operator Q, which has a simple form $Q = \lambda^\alpha D_\alpha$ in the pure spinor formalism. Inspired by that we want to exploiting the BRST properties of objects, which naturally arise in the computation of super string scattering amplitudes. Thereby, the pure spinor formalism provides an efficient method to organize the computation of scattering amplitudes.

We will define composite super fields $\tilde{L}_{jiki...pi}$ obtained form OPEs between vertex operators in (4.1), which can also be fundamentally defined by Wick contractions and derive their BRST properties. Because of their recursive definition these composite superfields will contain terms that originate from the contraction of BRST closed terms with integrated vertex operators. These contractions will not contribute to scattering amplitudes, because BRST closed/exact terms are not in the BRST cohomology, see below for more details. This was explicitly shown for the scattering of up to six open strings on the disk [158] and also for two and three closed strings on the disk in [88] and [109], respectively. Moreover, it was conjectured that this pattern persists also for an n-point open string amplitude [62, 114]. Note that terms originating from the contraction of a BRST closed expression with an integrated vertex operator are BRST exact. Even though, they drop out of the scattering amplitude, they contribute to the CFT correlation function, which is crucial for conformal invariance of the correlator.

Using integration by parts relations one can eliminate (some but not all of) the BRST exact terms in $\tilde{L}_{jiki...pi}$ and obtain the composite superfields $L_{jiki...pi}$, which transform covariantly under the action of the BRST charge [159]. Furthermore, they still contain BRST exact pieces, which can be remove due to corrections originating from double pole integrals arising from contractions between integrated vertex operators.[1] In the end, we

[1]Evaluating the CFT correlator of an amplitude with $n \geq 5$ open strings gives rise to single and double poles in the worldsheet coordinates of the vertex operators. These double poles can be used to correct the

obtain a prescription for defining the BRST building blocks $T_{ijk\dots p}$. The derivation of these building blocks in this chapter is based on [62].

5.1 Composite superfields $\tilde{L}_{2131\dots p1}$ and $L_{2131\dots p1}$

As mentioned in section 4.1 using Wick's theorem and expressing all contractions in terms of $K_{2131\dots p1}$ is not very efficient. Therefore, the composite superfields $\tilde{L}_{2131\dots p1}$ were introduced, which can be defined recursively as[2] [62, 158, 159]

$$\begin{aligned}
\lim_{z_2\to z_1} V_1(z_1)U_2(z_2) &\to \frac{\tilde{L}_{21}(z_1)}{z_{21}} \; , \\
\lim_{z_3\to z_1} \tilde{L}_{21}(z_1)U_3(z_3) &\to \frac{\tilde{L}_{2131}(z_1)}{z_{31}} \; , \\
\lim_{z_4\to z_1} \tilde{L}_{2131}(z_1)U_4(z_4) &\to \frac{\tilde{L}_{213141}(z_1)}{z_{41}} \; , \\
&\vdots \\
\lim_{z_p\to z_1} \tilde{L}_{2131\dots p1}(z_1)U_p(z_p) &\to \frac{\tilde{L}_{2131\dots p1}(z_1)}{z_{p1}}
\end{aligned} \tag{5.1}$$

for an unintegrated vertex operator 1 and integrated vertex operators $2, 3, 4, \dots, p$. Composite superfields $\tilde{L}_{2131\dots p1}$ containing contractions among integrated vertex operators are defined using the same pattern. Note that by definition all $\tilde{L}$s in (5.1) depend on only one worldsheet position z_1. At first glance, this could mean that the superfields $\tilde{L}_{2131\dots p1}$ cannot be related to Wick contractions $K_{2131\dots p1}$. But after all conformal weight one fields are integrated out the correlator depends only on the zero modes of λ and θ, which do not depend on the worldsheet positions. Therefore, using partial fractioning it is possible to combine Wick contractions K to form the composite superfields $\tilde{L}$ inside a correlator, see appendix C. Nevertheless, the approach in (5.1) is a priori different from Wick's theorem.

Using the OPEs (3.76) the composite superfields (5.1) can be expressed in terms of the SYM fields of section 2.2. Starting with one integrated and one unintegrated vertex

numerators of single poles. In the end, this allows to transform the composite superfields $L_{jiki\dots pi}$ into BRST building blocks $T_{ijk\dots p}$. Explicitly, this was done in [158, 160] and the general picture is presented in [62]. For three closed strings on the disk we provide the calculation in appendix C.

[2] In a correlator as in (4.1) there are also composite superfields $\tilde{L}$ that involve contractions between integrated vertex operators. These can usually be expressed through superfields in (5.1) by using equations like (C.9) or are corrections to (5.1) in order to obtain the BRST building blocks $T_{ijk\dots p}$ [114, 62, 158]. Therefore, we will only need $\tilde{L}_{jiki\dots pi}$ for the computation of scattering amplitudes.

Moreover, without loss of generality we have chosen specific labels for the vertex operators. All other combinations can simply be obtained by relabelling.

operator we find[3] [161]

$$\begin{aligned}\tilde{L}_{21} &= \lim_{z_2\to z_1} z_{21}V_1(z_1)U_2(z_2)\\ &\sim -A^1_m(\lambda\gamma^m W_2) - V_1(ik_1\cdot A_2) + Q(A_1W_2)\\ &= L_{21} + Q(A_1W_2)\ .\end{aligned} \tag{5.2}$$

The composite superfield $\tilde{L}_{21}$ (or the contraction (4.11)) contains the BRST exact piece $Q(A_1W_2)$, which decouples from the amplitude (4.1). For four external states the contribution $\langle Q(A_1W_2)V_3V_4\rangle = -\langle(A_1W_2)Q(V_3V_4)\rangle = 0$, because $QV_i = 0$. Further, for higher point amplitudes the BRST exact term is multiplied by more integrated vertex operators such that we get $Q(A_1W_2)U_3\cdots U_{n-2}V_{n-1}V_n$. The U_i are not BRST closed at the level of the correlator, but rather satisfy $QU_i = \partial V_i$, which implies

$$\langle Q(A_1W_2)U_3\cdots U_{n-2}V_{n-1}V_n\rangle = -\sum_{i=1}^{n-2}\langle(A_iW_j)U_{m_1}\cdots\partial V_i\cdots U_{n-2}V_{n-1}V_n\rangle \neq 0\ . \tag{5.3}$$

Once we integrate over the vertex operator positions z_i of $U_i(z_i)$ in the amplitude the expression becomes BRST closed

$$\begin{aligned}&\int_{D_2(P)} \mathrm{d}z_2\mathrm{d}z_3\cdots\mathrm{d}z_{n-2}\Big\langle Q(A_1W_2)U_3(z_3)\cdots U_{n-2}(z_{n-2})V_{n-1}(z_{n-1})V_n(z_n)\Big\rangle\\ &= -\int_{D_2(P)} \mathrm{d}z_2\mathrm{d}z_3\cdots\mathrm{d}z_{n-2}\Big\langle (A_1W_2)Q[U_3(z_3)\cdots U_{n-2}(z_{n-2})V_{n-1}(z_{n-1})V_n(z_n)]\Big\rangle\\ &= 0\ ,\end{aligned} \tag{5.4}$$

because of the cancelled propagator argument and the BRST properties of the vertex operators (3.88) and (3.93). Following the same steps as in (5.2) we find expressions in terms of the superfields for $\tilde{L}_{2131}$ and $\tilde{L}_{213141}$ in (5.1) [109]

$$\begin{aligned}\tilde{L}_{2131} \sim\ & L_{2131} - s_{12}[(A_1W_3)V_2 - (A_2W_3)V_1] - (s_{13}+s_{23})(A_1W_2)V_3\\ &-Q[(ik_1\cdot A_2)(A_1W_3)] - Q[A^1_m(W_2\gamma^m W_3)] - Q\{U_3,(A_1W_2)\}\ ,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\tilde{L}_{213141} \sim\ & L_{213141} + (A_1W_4)[s_{12}V_2(ik_1\cdot A_3) - s_{12}L_{32} + (s_{13}+s_{23})(ik_1\cdot A_2)V_3]\\ &+(A_2W_4)[-s_{12}(ik_2\cdot A_3)V_1 + s_{12}L_{31}] + (s_{13}+s_{23})(A_3W_4)L_{21}\\ &-(s_{13}+s_{23})(W_2\gamma^m W_4)A^1_m V_3 + (W_3\gamma^m W_4)[-s_{12}A^2_m V_1 + s_{12}A^1_m V_2]\\ &-s_{12}[U_4(A_1W_3)V_2 - \{U_4,(A_2W_3)\}V_1 + (A_1W_3)\tilde{L}_{42} - (A_2W_3)\tilde{L}_{41}]\\ &-(s_{13}+s_{23})[U_4(A_1W_2)V_3 + (A_1W_2)\tilde{L}_{43}]\\ &+(s_{14}+s_{24}+s_{34})[(ik_1\cdot A_2)(A_1W_3) + (W_2\gamma^m W_3)A^1_m - \{U_3,(A_1W_2)\}]V_4\ ,\end{aligned} \tag{5.6}$$

[3] Note that we treat the contractions of the plane wave factors in the superfields separate by using (3.78), since they result in an overall factor in the correlator of any scattering amplitude. Therefore, they are not included in the composite superfields.

where L_{2131} L_{213141} and are defined below. Moreover, we have introduced the shorthand notation

$$\{U_i, (A_j W_k)\} = -(ik_{jk} \cdot A_i)(A_j W_k) + D_\alpha A^j_\beta W^\beta_k W^\alpha_i + \frac{1}{4}(A_j \gamma^{mn} W_i) F^k_{mn} \ , \tag{5.7}$$

for the contraction of U_i with $(A_j W_k)$ and we used $k^m_{jk} = (k_j)^m + (k_k)^m$. To arrive at (5.5) and (5.6) we had to integrate the BRST operator by parts. Consequently, we assumed that these computation take place inside of a correlator, which allows us to drop terms like $Q(\ldots)$ for a generic superfield expression $(\ldots)$, where the BRST charge acts on all fields in the correlator, because $\langle Q(\ldots)\rangle = 0$. Moreover, we have already discarded the BRST exact terms in $\tilde{L}_{213141}$, because in the amplitudes, which we consider in chapter 8 and 10, the superfields $\tilde{L}_{213141}$ and relabelling thereof are accompanied by two unintegrated vertex operators, which are BRST closed.

The superfields L are obtained by discarding all BRST exact terms that arise from the contraction with BRST exact expressions in $\tilde{L}$, i.e. all terms in (5.5) and (5.6) that are not contained in L. Using the OPEs (3.76) the composite superfields (5.1) evaluate to [62]

$$L_{2131} = -L_{21}(ik_{12} \cdot A_3) + (\lambda\gamma^m W_3)\Big[F^2_{mn} A^n_1 + (ik_1 \cdot A_2) A^1_m - (W_1 \gamma_m W_2)\Big], \tag{5.8}$$

$$\begin{aligned} L_{213141} = &-L_{2131}(ik_{123} \cdot A_4) + (\lambda\gamma^m W_4)\Big[F^2_{nm} A^n_1 (ik_{12} \cdot A_3) - (A_1 \cdot ik_2)(W_2 \gamma_m W_3) \\ &+ F^2_{pq} A^p_1 A^q_3 k^3_m - F^2_{pq} A^q_1 k^p_3 A^3_m + F^3_{mn}(W_1 \gamma^n W_2) - F^3_{mn} A^n_1 (ik_1 \cdot A_2) \\ &+ (W_1 \gamma_m W_3)(ik_1 \cdot A_2) + \Big[(W_1 \gamma_m W_2) - A^1_m (ik_1 \cdot A_2)\Big](ik_{12} \cdot A_3) \\ &+ \frac{1}{4}(W_2 \gamma_{pq} \gamma_m W_3) F^{pq}_1 - \frac{1}{4}(W_1 \gamma_{pq} \gamma_m W_3) F^{pq}_2\Big] \ . \end{aligned} \tag{5.9}$$

The composite superfields $L_{2131\ldots p1}$ do not have any symmetries under the exchange of labels $1, 2, \ldots p$. However, the lack of symmetry properties results in BRST exact terms [114]. Removing them gives rise to redefinitions of $L_{1,2,\ldots,p}$ in the next section 5.2, which leave the amplitude invariant [62, 162]. Under the action of the pure spinor BRST charge the composite superfields transform covariantly [62, 159] such that their BRST variation is given in terms of lower order composite superfields. From the recursive definition of the superfields, $QV_i = 0$ and $QV_i = \partial V_i$ it follows that

$$QL_{2131\ldots p1} = \lim_{z_p \to z_1} z_{p1}\Big[\Big(QL_{2131\ldots(p-1)1}(z_1) U_p(z_p) - L_{2131\ldots(p-1)1}(z_1)\partial V_p(z_p)\Big)\Big] \ , \tag{5.10}$$

where the first term can be computed using again the recursive definition (5.1) (after discarding the BRST exact terms). By applying the OPEs (3.76) the contraction of $\partial V_p = \partial\lambda^\alpha A^p_\alpha + \Pi_m i(k_p)^m V_p + \partial\theta^\alpha D_\alpha V_p$ with the composite superfield $L_{2131\ldots(p-1)1}$ yields $\sum_{i=1}^{p-1} s_{ip} L_{2131\ldots(p-1)1} V_p$ for the second term. Explicitly, we find for the BRST variation of the superfields defined in (5.1) [62]

$$QL_{21} = -s_{12} V_1 V_2 \ , \tag{5.11}$$

$$QL_{2131} = -(s_{13}+s_{23})L_{21}V_3 - s_{12}(L_{31}V_2 + V_1L_{32}) \ , \tag{5.12}$$

$$\begin{aligned} QL_{213141} &= -(s_{14}+s_{24}+s_{34})L_{2131}V_4 - (s_{13}+s_{23})(L_{21}L_{43} + L_{2141}V_3) \\ &\quad -s_{12}(L_{3141}V_2 + L_{31}L_{42} + L_{41}L_{32} + V_1L_{3242}) \ . \end{aligned} \tag{5.13}$$

Hence, after dropping $Q(A_1W_2)$ in order to obtain L_{21} from (5.2) the action of the BRST charge on L_{2131} suggest that we have to drop all BRST exact terms in (5.5) that arise from the contraction with BRST exact terms as well, since the BRST variation (5.12) only holds for L_{2131} but not for $\tilde{L}_{2131}$. Consequently, we proceed in the same way with the higher order composite super fields.

5.2 BRST building blocks $T_{123\ldots p}$

As described in [62] the composite superfields can be used to define the BRST building blocks $T_{123\ldots p}$ in essentially two steps (i) and (ii)

$$L_{2131\ldots p1} \xrightarrow{(i)} \tilde{T}_{123\ldots p} \xrightarrow{(ii)} T_{123\ldots p} \ , \tag{5.14}$$

which remove all remaining BRST exact terms and make the amplitude manifestly invariant under BRST transformations. Moreover, these steps still preserve the BRST variation identities like (5.12) and (5.13) but for the BRST building blocks $T_{123\ldots p}$, i.e. these equations hold after the substitution $L_{123\ldots p} \to T_{123\ldots p}$ on both sides of (5.12) and (5.13).

In the first step (i) the substitution of $L_{2131\ldots p1}$ by $\tilde{T}_{123\ldots p}$ depends on the previous redefinitions $L_{2131\ldots q1} \to T_{123\ldots q}$ for $q < p$. This step ensures that extra terms on the right hand side of the BRST variation $QL_{2131\ldots p1}$ arising from the substitution $L_{2131\ldots q1} \to T_{123\ldots q}$ are absorbed on the left hand side of the BRST variation. Therefore, the composite superfields $L_{2131\ldots p1} \to \tilde{T}_{123\ldots p}$ are redefined such that their BRST variation $Q\tilde{T}_{1234\ldots p}$ is written in term of the BRST building blocks $T_{123\ldots q}$ rather than $L_{2131\ldots q1}$. Explicitly, for (5.12) and (5.13) we find

$$Q\tilde{T}_{123} = -s_{12}(T_{13}V_2 + V_1T_{23}) - (s_{13}+s_{23})T_{12}V_3 \ , \tag{5.15}$$

$$\begin{aligned} Q\tilde{T}_{1234} &= -(s_{14}+s_{24}+s_{34})T_{123}V_4 - (s_{13}+s_{23})(T_{12}T_{34} + T_{124}V_3) \\ &\quad -s_{12}(T_{134}V_2 + T_{13}T_{24} + T_{14}T_{23} + V_1T_{234}) \ . \end{aligned} \tag{5.16}$$

Using the BRST variations $Q\tilde{T}_{123\ldots p}$ one can find BRST closed expressions, which are linear combinations of $\tilde{T}_{123\ldots p}$. For examples from (5.15) it follows that $Q(\tilde{T}_{123} + \tilde{T}_{231} + \tilde{T}_{312}) = 0$. In addition, these combinations of $\tilde{T}_{123\ldots p}$ are BRST exact at the same time, because the cohomology of Q at ghost number $+1$ is non-trivial (non-empty) only at conformal weight $h = 0$ and $h(\tilde{T}_{123\ldots p}) \neq 0$ [62].

Finally, we are ready to remove the remaining BRST exact terms: In the second step (ii) we have to find all BRST exact linear combinations of $\tilde{T}_{123\ldots p}$, which are given by

$$\sum_{\text{perm.}} \tilde{T}_{123\ldots p} = QR^{(I)}_{123\ldots p} \ , \quad I = 1,2,3,\ldots,p-1 \ , \tag{5.17}$$

in order to subtract the corresponding $p-1$ BRST exact pieces $R^{(I)}_{123\dots p}$ from $\tilde{T}_{123\dots p}$, which gives rise to the definitions of $T_{123\dots p}$. Note that the sums include different permutations of the labels of $\tilde{T}_{123\dots p}$ with positive and negative signs. After completing step (ii) of (5.14), which will be explicitly done in section 5.3, the BRST closed sums (5.17) become symmetries

$$\sum_{\text{perm.}} T_{123\dots p} = 0 \tag{5.18}$$

of the BRST building blocks. In order to identify the exact structure of the sum over permutations for $T_{123\dots p}$ in (5.18) it is useful to consider the diagrammatic interpretation of the building blocks. Simultaneously, this procedure enables us to retrieve the BRST exact parts $R^{(I)}_{123\dots p}$ of $\tilde{T}_{123\dots p}$ in (5.17), because by definition both sums (5.17) and (5.18) run over the same permutations. To derive the BRST symmetries of T_{123} we have to consider the diagrams associated to the building blocks in figure 5.1, where we interpret them as a tail-end graph and as a branch. Because both interpretations have to agree, this implies

Figure 5.1: The same diagram interpreted as a tail-end graph and as a branch.

that the BRST symmetries (5.18) of T_{123} can be written as

$$T_{123} - T_{321} + T_{312} = T_{123} + T_{231} + T_{312} = 0 \ . \tag{5.19}$$

The relative sign between the two interpretations of the diagram follows from the fact that the diagram corresponding to $T_{123\dots p}$ picks up a sign $(-1)^p$ under the inversion $(1,2,3,\dots,p-1,p) \leftrightarrow (p,p-1,\dots,3,2,1)$. Hence, in the BRST symmetry identity (5.18) there has to be a relative sign $(-1)^p$ between $T_{123\dots p}$ and $T_{p(p-1)\dots 321}$, c.f. (5.19). The generalization for higher rank composite superfields $T_{123\dots p}$ of (5.19) can be obtained using the same idea, see [62] for more details and is given by

$$\begin{aligned} p = 2n+1: \qquad & 0 = T_{12\dots n+1[n+2[\dots[2n-1[(2n)(2n+1)]]\dots]]} - 2T_{2n+1\dots n+2[n+1[\dots[3[21]]\dots]]} \ , \\ p = 2n: \qquad & 0 = T_{12\dots n[n+1[\dots[2n-2[(2n-1)(2n)]]\dots]]} + T_{2n\dots n+1[n[\dots[3[21]]\dots]]} \ , \end{aligned} \tag{5.20}$$

where the notation $T_{\dots[i[jk]]\dots}$ means that we consecutively antisymmetrize pairs of labels starting from the outer bracket

$$T_{\dots[i[jk]]\dots} = \frac{1}{2}\Big(T_{\dots i[jk]\dots} - T_{\dots[jk]i\dots}\Big) = \frac{1}{4}(T_{\dots ijk\dots} - T_{\dots ikj\dots} - T_{\dots jki\dots} + T_{\dots kji\dots}) \ . \tag{5.21}$$

Moreover, the BRST symmetries of lower rank building blocks $T_{123\dots q}$ transfer to higher rank building blocks $T_{123\dots p}$ with $p > q$ for the first q label and the last labels $q+1, q+2, \dots p$

remain unaffected by the lower rank identities, which follows from the recursive definition of (5.1). The $p-1$ relations between the building blocks at rank p allow us to express any $T_{i_1 i_2 \ldots i_p}$ as a combination of $T_{1 j_1 j_2 \ldots j_{p-1}}$ such that there are $(p-1)!$ independent building blocks $T_{i_1 i_2 \ldots i_p}$ at rank p.

To sum up the building blocks $T_{123\ldots p}$ can be defined in two steps [62]

(i) Rewrite $L_{2131\ldots p1} \to \tilde{T}_{123\ldots p}$ in such a way that $Q\tilde{T}_{123\ldots p}$ is expressed in terms of the lower rank building blocks $T_{123\ldots q}$ for $q < p$.

(ii) Remove the BRST exact pieces (5.17) from $\tilde{T}_{123\ldots p}$ so that the thereby obtained $T_{123\ldots p}$ satisfy the identities (5.18).

The building blocks derived via this procedure transform under BRST variations as

$$QT_{123\ldots p} = -\sum_{j=2}^{p} \sum_{\alpha \in P(\beta_j)} (s_{1j} + s_{2j} + \ldots + s_{j-1,j}) T_{12\ldots j-1} T_{j,\{\beta \backslash \alpha\}} \ , \tag{5.22}$$

where $\beta_j = \{j+1, j+2, \ldots, p\}$ and $P(\beta_j)$ is the power set of β_j. Moreover, a BRST building block T_i with a single index is given by an unintegrated vertex operator V_i. For $p \leq 4$ the building blocks obey the following BRST identities

$$\begin{aligned} QT_{12} &= -s_{12} V_1 V_2 \ , \\ QT_{123} &= -s_{12}(T_{13} V_2 + V_1 T_{23}) - (s_{13} + s_{23}) T_{12} V_3 \ , \\ QT_{1234} &= -(s_{14} + s_{24} + s_{34}) T_{123} V_4 - (s_{13} + s_{23})(T_{12} T_{34} + T_{124} V_3) \\ &\quad - s_{12}(T_{134} V_2 + T_{13} T_{24} + T_{14} T_{23} + V_1 T_{234}) \ . \end{aligned} \tag{5.23}$$

5.3 Explicit construction of T_{12}, T_{123} and T_{1234}

The construction of T_{12} only requires the second step (ii), because the redefinition of L_{12} to $\tilde{T}_{12}$ is trivial, i.e. $\tilde{T}_{12} = L_{12}$, since there are no lower rank redefinitions to consider in the first step (i). Using the action of the BRST charge (5.11) on L_{12} and the equations of motion (2.17) we find that

$$Q(\tilde{T}_{12} + \tilde{T}_{21}) = -s_{12}(V_1 V_2 + V_2 V_1) = 0 \ , \tag{5.24}$$

because the unintegrated vertex operator anticommutes with itself. Hence, the expression $\tilde{T}_{12} + \tilde{T}_{21}$ is BRST closed and moreover also BRST exact

$$\tilde{T}_{12} + \tilde{T}_{21} = -Q(A_1 \cdot A_2) = -QD_{12} \ . \tag{5.25}$$

The BRST building block T_{12} is then defined by satisfying $T_{12} + T_{21} = 0$ according to (5.18), which is achieved by [159]

$$T_{12} = \tilde{T}_{21} = \tilde{T}_{21} + \frac{1}{2} QD_{12} \tag{5.26}$$

and concludes the derivation of the lowest order building block.

For the BRST building block T_{123} we have to carry out both steps of (5.14). The substitution $L_{ij} = T_{ij} - \frac{1}{2}QD_{12}$ in (5.12) executes the first step $L_{2131} \xrightarrow{(i)} \tilde{T}_{123}$, which yields

$$Q\Big(L_{2131} - \frac{1}{2}s_{12}(D_{13}V_2 - D_{23}V_1) - \frac{1}{2}(s_{13}+s_{23})D_{12}V_3\Big)$$
$$= -s_{12}(T_{13}V_2 + V_1T_{23}) - (s_{13}+s_{23})T_{12}V_3 \ . \quad (5.27)$$

Therefore, we find that $\tilde{T}_{123}$ takes the following form

$$\tilde{T}_{123} = L_{2131} - \frac{1}{2}s_{12}(D_{13}V_2 - D_{23}V_1) - \frac{1}{2}(s_{13}+s_{23})D_{12}V_3 \ . \quad (5.28)$$

We consider two BRST closed sums for $\tilde{T}_{123}$ to identify the remaining BRST exact terms

$$Q(\tilde{T}_{123} + \tilde{T}_{213}) = 0 \ , \qquad Q(\tilde{T}_{123} + \tilde{T}_{312} + \tilde{T}_{231}) = 0 \ , \quad (5.29)$$

where the first sum is inherited from the antisymmetry of T_{12} and the second one can be found according to section 5.2. The combinations in (5.29) can be derived by acting with the BRST charge on the ghost number zero superfields $R^{(1)}_{123} = D_{12}(ik_{12}\cdot A_3)$ and $R^{(2)}_{123} = D_{12}(ik_2 \cdot A_3) + \text{cyclic}(123)$ so that [158, 162]

$$\tilde{T}_{123} + \tilde{T}_{213} = QR^{(1)}_{123} \ , \qquad \tilde{T}_{123} + \tilde{T}_{312} + \tilde{T}_{231} = QR^{(2)}_{123} \ , \quad (5.30)$$

which can be shown using the SYM equations of motion (2.17). Furthermore, $R^{(1)}$ and $R^{(2)}$ can be motivated by the residues of the double pole contractions of integrated vertex operators. Finally, after subtracting the BRST exact part in (5.30) from $\tilde{T}_{123}$ the corresponding building block is given by

$$T_{123} = \tilde{T}_{123} - QS^{(1)}_{123}$$
$$= \frac{1}{3}\Big(\tilde{T}_{123} - \tilde{T}_{213}\Big) + \frac{1}{6}\Big(\tilde{T}_{321} - \tilde{T}_{312} + \tilde{T}_{132} - \tilde{T}_{231}\Big) \quad (5.31)$$

with $S^{(1)}_{123} = \frac{1}{2}R^{(1)}_{123} + \frac{1}{6}\Big(R^{(2)}_{123} - R^{(2)}_{213}\Big)$. For the BRST symmetries of (5.31) we find

$$T_{123} + T_{213} = T_{123} + T_{312} + T_{231} = 0 \ , \quad (5.32)$$

which are in agreement with (5.18).

The definition of T_{1234} requires the lower rank redefinitions of L_{21} and L_{2131}. We proceed similarly as before: Executing the first step demands the substitutions $L_{ji} \to T_{ij}$ and $L_{jiki} \to T_{ijk}$ in the right hand side of the BRST variation (5.13), which gives

$$\tilde{T}_{1234} = L_{213141} + \frac{1}{4}[(s_{13}+s_{23})D_{12}QD_{34} + s_{12}(D_{13}QD_{24} + D_{14}QD_{23})]$$
$$-\frac{1}{2}[(s_{13}+s_{23})(D_{12}T_{34} - D_{34}T_{12}) + s_{12}(D_{13}T_{24} + D_{14}T_{23} - D_{23}T_{14} - D_{24}T_{13})]$$

$$+(s_{14}+s_{24}+s_{34})S^{(1)}_{123}V_4 + (s_{13}+s_{23})S^{(1)}_{124}V_3 - s_{12}(S^{(1)}_{234}V_1 - S^{(1)}_{134}V_2) \ , \tag{5.33}$$

whose transformation under the BRST charge is given in (5.16). Using (5.33) one can show that the first labels of $\tilde{T}_{1234}$ inherit the lower order identities of $\tilde{T}_{12}$ and $\tilde{T}_{123}$ in (5.24) and (5.29), respectively. In addition, there is a further BRST closed sum involving also the fourth label

$$\begin{aligned} Q(\tilde{T}_{1234}+\tilde{T}_{2134}) = 0 \ , \qquad Q(\tilde{T}_{1234}+\tilde{T}_{3124}+\tilde{T}_{2314}) = 0 \ , \\ Q(\tilde{T}_{1234}-\tilde{T}_{1243}+\tilde{T}_{3412}-\tilde{T}_{3421}) = 0 \ . \end{aligned} \tag{5.34}$$

From the equations of motion of the linearized SYM fields it follows that the BRST closed combinations are actually also BRST exact

$$\begin{aligned} \tilde{T}_{1234}+\tilde{T}_{2134} &= QR^{(1)}_{1234} \ , \\ \tilde{T}_{1234}+\tilde{T}_{3124}+\tilde{T}_{2314} &= QR^{(2)}_{1234} \ , \\ \tilde{T}_{1234}-\tilde{T}_{1243}+\tilde{T}_{3412}-\tilde{T}_{3421} &= QR^{(3)}_{1234} \ , \end{aligned} \tag{5.35}$$

where we have defined the ghost number zero fields $R^{(I)}_{1234}$ with $I=1,2,3$ as

$$R^{(1)}_{1234} = -R^{(1)}_{123}(ik_{123}\cdot A_4) + \frac{1}{4}s_{12}[D_{13}D_{24}+D_{14}D_{23}] \ , \tag{5.36}$$

$$R^{(2)}_{1234} = -R^{(2)}_{123}(ik_{123}\cdot A_4) + \frac{1}{4}[s_{12}D_{23}D_{14}+s_{23}D_{24}D_{13}+s_{13}D_{34}D_{12}] \ , \tag{5.37}$$

$$\begin{aligned} R^{(3)}_{1234} = {} & (ik_1\cdot A_2)[D_{14}(ik_4\cdot A_3)-D_{13}(ik_3\cdot A_4)] - (ik_2\cdot A_1)[D_{24}(ik_4\cdot A_3)-D_{23}(ik_3\cdot A_4)] \\ & -\frac{1}{4}D_{12}D_{34}(s_{14}+s_{23}-s_{13}-s_{24}) + D_{12}[(ik_4\cdot A_3)(ik_2\cdot A_4)-(ik_3\cdot A_4)(ik_2\cdot A_3)] \\ & +D_{34}[(ik_2\cdot A_1)(ik_4\cdot A_2)-(ik_1\cdot A_2)(ik_4\cdot A_1)] + (W_1\gamma^m W_2)(W_3\gamma_m W_4) \ , \end{aligned} \tag{5.38}$$

and $k^m_{ijk}=(k_i)^m+(k_j)^m+(k_k)^m$. Removing the BRST exact parts above the redefinitions $\tilde{T}_{1234} \xrightarrow{(ii)} T_{1234}$ in the second step lead to the rank four BRST building block

$$T_{1234} = \tilde{T}_{1234} - QS^{(2)}_{1234} \ , \tag{5.39}$$

where we have introduce the field $S^{(2)}_{1234}$, which is defined recursively by

$$\begin{aligned} S^{(2)}_{1234} &= \frac{3}{4}S^{(1)}_{1234} + \frac{1}{4}\Big(S^{(1)}_{1243} - S^{(1)}_{3412} + S^{(1)}_{3421}\Big) + \frac{1}{4}R^{(3)}_{1234} \ , \\ S^{(1)}_{1234} &= \frac{1}{2}R^{(1)}_{1234} + \frac{1}{6}\Big(R^{(2)}_{1234} - R^{(2)}_{2134}\Big) \ . \end{aligned} \tag{5.40}$$

The BRST exact sums in (5.35) become the BRST symmetries of T_{1234}

$$T_{1234}+T_{2134} = T_{1234}+T_{3124}+T_{2314} = T_{1234}-T_{1243}+T_{3412}-T_{3421} = 0 \ . \tag{5.41}$$

To check that (5.41) holds, we have to express $R^{(I)}_{1234}$ in terms of $S^{(2)}_{ijkp}$ and use (5.39) together with (5.40).

For the an n-point amplitudes we do not need to perform step (ii) for $L_{j_1 i j_2 i \ldots j_{n-2} i}$, because after substituting $L_{i j_1 j_2 \cdots j_{n-2}} \xrightarrow{(i)} \tilde{T}_{i j_1 j_2 \cdots j_{n-2}}$ this building block is already BRST exact in the correlator: The building block $\tilde{T}_{i j_1 j_2 \cdots j_{n-2}}$ is always accompanied by two unintegrated vertex operators in the CFT correlator of the n-point amplitude. Therefore, we can drop the BRST exact terms in $\tilde{T}_{i j_1 j_2 \cdots j_{n-2}}$ in the correlator without executing step (ii). After integrating the BRST charge by part we can use $QV = 0$ such that the BRST exact parts of $\tilde{T}_{i j_1 j_2 \cdots j_{n-2}}$ decouple from the amplitude. Hence, in this setup the BRST exact sums (5.30) lead to vanishing correlators, i.e. for $n = 6$ we find that

$$
\begin{aligned}
\Big\langle \Big(\tilde{T}_{1234} + \tilde{T}_{2134}\Big) V_5 V_6 \Big\rangle &= \Big\langle Q R^{(1)}_{1234} V_5 V_6 \Big\rangle = 0 \ , \\
\Big\langle \Big(\tilde{T}_{1234} + \tilde{T}_{3124} + \tilde{T}_{2314}\Big) V_5 V_6 \Big\rangle &= \Big\langle Q R^{(2)}_{1234} V_5 V_6 \Big\rangle = 0 \ , \\
\Big\langle \Big(\tilde{T}_{1234} - \tilde{T}_{1243} + \tilde{T}_{3412} - \tilde{T}_{3421}\Big) V_5 V_6 \Big\rangle &= \Big\langle Q R^{(3)}_{1234} V_5 V_6 \Big\rangle = 0 \ . \qquad (5.42)
\end{aligned}
$$

Therefore, performing the second step (ii) for $\tilde{T}_{ijkp}$ for the scattering of three closed strings on the disk in appendix C is not strictly necessary and in some sense obsolete, because $\langle \tilde{T}_{ijkp} V_m V_n \rangle = \langle T_{ijkp} V_m V_n \rangle$.

Chapter 6

SYM amplitudes from pure spinor superspace expressions

All SYM tree-level amplitudes in ten dimensions can be recovered in the field theory limit for $\alpha' \to 0$ from open superstring amplitudes at tree-level [163, 164, 165, 166]. Therefore, their correlators are in the cohomology of the BRST charge so that it is possible to use the BRST properties of superstring amplitudes to find pure spinor superspace expressions for SYM amplitudes. By doing so we obtain a recursive definition of SYM amplitudes based on supersymmetric Berends-Giele currents [162], which can be constructed from the BRST building blocks of section 5.2. These are the generalizations of the gluonic Berends-Giele currents [167] for YM tree amplitudes, which can be reproduced from their supersymmetric generalizations by truncating them to their bosonic components [168].

Following the lines of [62] we want to construct supersymmetric Berends-Giele currents, which is inspired by the fact that the BRST building blocks $T_{12\ldots p}$ correspond to color ordered diagrams of cubic vertices with p on-shell and one off-shell leg, see figure 6.1. The additional leg has to be off-shell, because at rank p each of the $\frac{1}{p}\binom{2p-2}{p-1}$ diagrams

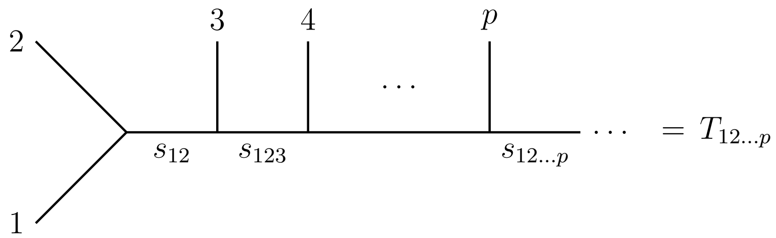

Figure 6.1: A cubic graph corresponding to the BRST building block $T_{12\ldots p}$.

has $p-2$ internal poles in the Mandelstam variables $s_{12\ldots q}$ for $q < p$ and one external propagator $\frac{1}{s_{12\ldots p}}$, which would diverge if all legs are put on-shell [116]. The pure spinor supersymmetric generalization $M_{12\ldots p}$ of the p-point Berends-Giele currents $J_{12\ldots p}$ will be obtained by combining the diagrams of $T_{12\ldots p}$ to form $p+1$-point field theory amplitudes including one off-shell leg. Furthermore, the (super)currents $M_{12\ldots p}$ share the symmetries of the gluonic currents and provide a compact representations of n-point SYM amplitudes in ten spacetime dimensions. To derive the n-point prescription for SYM amplitudes we will use the recursive nature of the Berends-Giele supercurrents, which enables a recursive method for the computation of n-point SYM amplitudes.

6.1 Gluonic Berends-Giele currents

In [167] a recursive relation for gluon scattering amplitudes in QCD at tree-level was proposed[1]

$$A_{\mathrm{YM}}(1,2,\ldots,n,n+1) = s_{12\ldots n} J^m_{12\ldots n} J^m_{n+1} \ , \tag{6.1}$$

where $J_{12\ldots n}$ are Berends-Giele currents. They are defined via the recursive relation in the number of external gluons and start with the polarization vector $J^m_i = e^m_i$ of a single gluon i. For higher rank currents $J_{12\ldots p}$ with one off-shell leg $p+1$ we find

$$\begin{aligned} J^m_i &= e_i \ , \\ s_{12\ldots p} J^m_{12\ldots p} &= \sum_{i=1}^{p-1} [J_{1\ldots i}, J_{i+1\ldots p}]^m + \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1} \{J_{1\ldots i}, J_{i+1\ldots j} J_{j+1\ldots p}\}^m \ , \end{aligned} \tag{6.2}$$

where the brackets $[\cdot,\cdot]^m$ and $\{\cdot,\cdot\}^m$ are defined such that they strip off one gluon with vector index m from the cubic and quartic vertex of the Yang-Mills Lagrangian [114]. Explicitly, they are given by

$$\begin{aligned} [J_{1\ldots i}, J_{i+1\ldots p}]^m &= (k_{i+1\ldots p}\cdot J_{1\ldots i}) J^m_{i+1\ldots p} + \frac{1}{2} k^m_{i+1\ldots p} (J_{1\ldots i}\cdot J^m_{i+1\ldots p}) \\ &\quad -(\{1\ldots i\} \leftrightarrow \{i+1\ldots p\}) \ , \\ \{J_{1\ldots i}, J_{i+1\ldots j}, J_{j+1\ldots p}\}^m &= (J_{1\ldots i}\cdot J_{j+1\ldots p}) J^m_{i+1\ldots j} - \frac{1}{2}(J_{1\ldots i}\cdot J_{i+1\ldots j}) J^m_{j+1\ldots p} \\ &\quad -\frac{1}{2}(J_{i+1\ldots j}\cdot J_{j+1\ldots p}) J^m_{1\ldots i} \ . \end{aligned} \tag{6.3}$$

From the diagrammatic representation of $J^m_{12\ldots p}$ in figure 6.2 one can immediately see, that the brackets $[\cdot,\cdot]^m$ and $\{\cdot,\cdot\}^m$ correspond to the three and four gluon interaction.

From the recursive definition it follows that the Berends-Giele currents at rank p are conserved with respect to the total momentum [167]

$$k^m_{1\ldots p} J^m_{1\ldots p} = 0 \ , \tag{6.4}$$

where the total momentum is defined as $k^m_{1\ldots p} = k^m_1 + k^m_2 + \ldots + k^m_p$. Moreover, the currents are invariant under reflections

$$J^m_{12\ldots p} + (-1)^p J_{p(p-1)\ldots 1} = 0 \tag{6.5}$$

[1]For simplicity, we are considering color stripped amplitudes here.

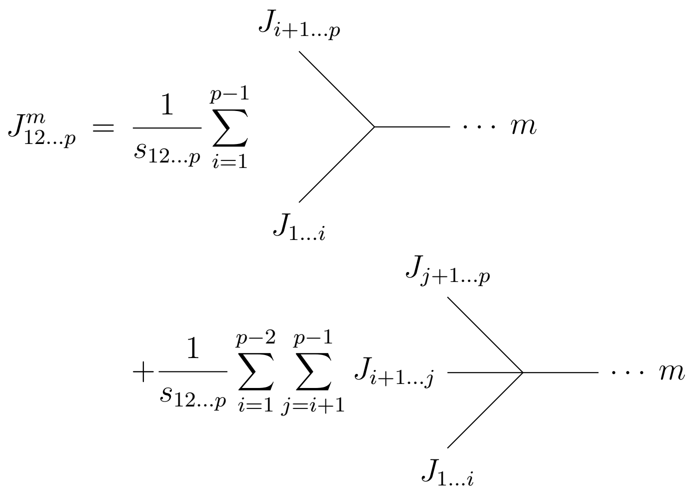

Figure 6.2: Diagrammatic representation of the gluonic Berends-Giele currents.

and the cyclic sum of $J_{12\ldots p}$ is vanishing

$$\sum_{\sigma\in\text{cyclic}} J_{\sigma(1,2,\ldots,p)} = 0 \ . \tag{6.6}$$

This description of gluon tree-level scattering amplitudes is very efficient: The recursive definition utilizes the lower order result and the ansatz (6.1) captures all Feynman diagrams. In addition, it can be used to proof cyclicity, gauge invariance, photon decoupling and covers the factorization in soft and collinear limits.

6.2 Supersymmetric Berends-Giele currents $M_{123\ldots p}$

The supersymmetric Berends-Giele currents $M_{12\ldots p}$ can be expressed in terms of $T_{12\ldots p}$ of section 5.2 and the Mandelstam variables $\{s_{12}, s_{123}, \ldots, s_{12\ldots p}\}$ of the kinematic poles.[2] With the starting point $M_1 = V_1$ their recursive definition is given by

$$\begin{aligned} E_{12\ldots p} &= \sum_{j=1}^{p-1} M_{1\ldots j} M_{j+1\ldots p} \ , \\ QM_{12\ldots p} &= E_{12\ldots p} \ . \end{aligned} \tag{6.7}$$

Even though, the definition of the Berends-Giele $M_{12\ldots p}$ currents is of purely algebraic nature, similar as for the BRST building blocks they have a diagrammatic interpretation and can be related to the sum of $\frac{(2p-2)!}{p!(p-1)!}$ cubic graphs of SYM amplitudes, which are associated to a $p+1$ point amplitude with one off-shell leg. Each of these cubic graphs can

[2] The kinematic invariant are given by $s_{12\ldots p} = \sum_{i<j}^{p} s_{ij}$ with $s_{ij} = k_i \cdot k_j$.

be connected to sums of BRST building blocks $T_{12\ldots p}$, where the relative signs between the building blocks are determined by (6.7).

The first supersymmetric Berends-Giele supercurrents up to rank 4 correspond to the cubic graphs of the color ordered amplitudes for $n \leq 5$ in figure 6.3. The individual cubic

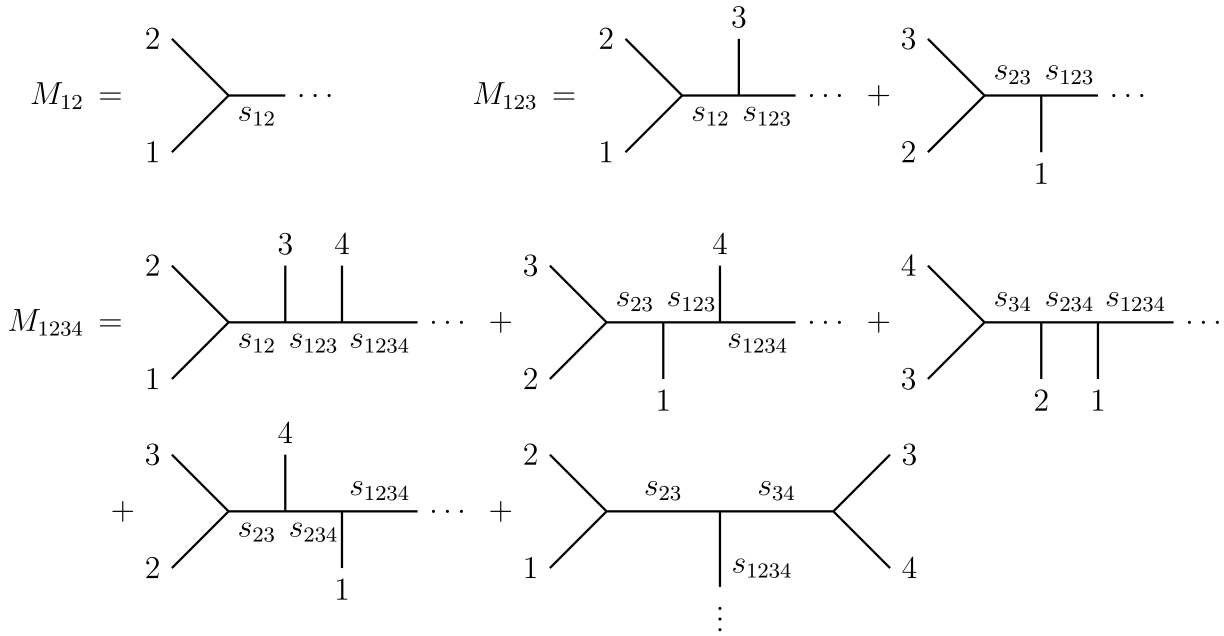

Figure 6.3: Supersymmetric Berends-Giele currents $M_{12\ldots p}$ for $p \leq 4$ expressed in terms of the corresponding cubic graphs.

graphs are associated to BRST building blocks. Therefore, the supersymmetric Berends-Giele currents in figure 6.3 can be expressed in terms of the building blocks of section 5.3 as

$$M_{12} = -\frac{T_{12}}{s_{12}} , \qquad M_{123} = \frac{1}{s_{123}}\left(\frac{T_{123}}{s_{12}} + \frac{T_{321}}{s_{23}}\right) ,$$
$$M_{1234} = -\frac{1}{s_{1234}}\left(\frac{T_{1234}}{s_{12}s_{123}} + \frac{T_{3214}}{s_{23}s_{123}} + \frac{T_{3421}}{s_{34}s_{234}} + \frac{T_{3241}}{s_{23}s_{234}} + \frac{T_{12[34]}}{s_{12}s_{34}}\right) . \tag{6.8}$$

The relative signs are fixed by demanding that they satisfy (6.7). Using (5.22) the BRST variations of (6.8) are given by

$$\begin{aligned} QM_{12} &= V_1V_2 = M_1M_2 , \\ QM_{123} &= -\frac{V_1T_{23}}{s_{23}} + \frac{T_{12}V_3}{s_{12}} = M_1M_{23} + M_{12}M_3 , \\ QM_{1234} &= \frac{V_1}{s_{234}}\left(\frac{T_{234}}{s_{23}} + \frac{T_{432}}{s_{34}}\right) + \left(\frac{T_{12}T_{34}}{s_{12}s_{34}}\right) + \left(\frac{T_{123}}{s_{12}} + \frac{T_{321}}{s_{23}}\right)\frac{V_4}{s_{123}} \\ &= M_1M_{234} + M_{12}M_{34} + M_{123}M_4 , \end{aligned} \tag{6.9}$$

which imply that the ansatz (6.8) for M_{12}, M_{123} and M_{1234} forms a solution of (6.7) up to $p = 4$. Continuing in this manner we can construct higher point Berends-Giele currents for arbitrary p in terms of BRST building blocks. The generalization of the above BRST variations follows immediately from the definition of $M_{12\ldots p}$ [162]

$$QM_{12\ldots p} = \sum_{j=1}^{p-1} M_{1\ldots j} M_{j+1\ldots p} \tag{6.10}$$

and was proven in [169]. This formula is the supersymmetric pure spinor analogue of the recursive definition for gluonic Berends-Giele currents (6.2). The action of Q on $M_{12\ldots p}$ can be interpreted as cutting the Berends-Giele current in every possible way that is consistent with the color ordering, which is depicted in figure (6.4).

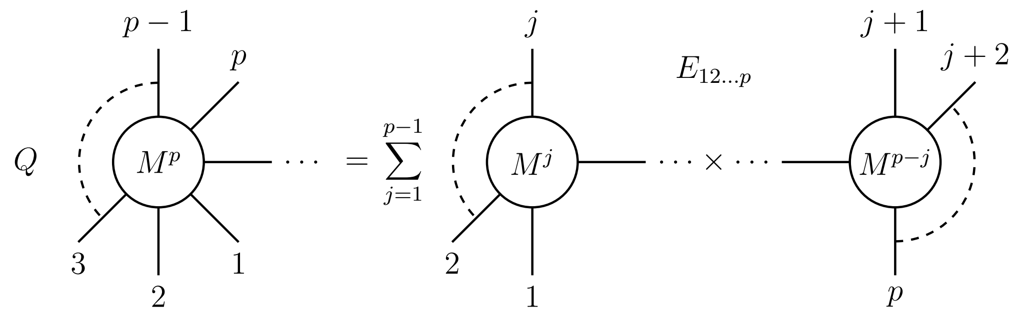

Figure 6.4: Factorisation of $M_{12\ldots p}$ into two channels under the action of the BRST operator.

6.3 Symmetries of Berends-Giele currents

The supersymmetric Berends-Giele currents $M_{12\ldots p}$ share symmetry properties with their gluonic partners $J^m_{12\ldots p}$. These will be useful for the computation of superstring scattering amplitudes in the pure spinor formalism. For the rank $p = 2$ supercurrent we find that M_{12} satisfies $M_{12} + M_{21} = 0$, because the building block T_{ij} is antisymmetric. We obtain an analogue symmetry for M_{123} as for T_{123}:

$$M_{123} + M_{231} + M_{312} = 0 \ , \qquad M_{123} - M_{321} = 0 \ , \tag{6.11}$$

which can be checked by plugging in the explicit expression for M_{ijk} in (6.8) and using (5.19). These identities generalize for higher rank Berends-Giele currents $M_{12\ldots p}$ to

$$M_{12\ldots p} = (-1)^p M_{p(p-1)\ldots 21} \ , \qquad \sum_{\sigma \in \text{cyclic}} M_{\sigma(1,2,\ldots,p)} = 0 \ , \tag{6.12}$$

where the sum runs over all cyclic permutations σ of the labels $(1, 2, \ldots, p)$. By construction all BRST closed and exact terms have been removed from $T_{12\ldots p}$ and therefore also from

$M_{12\ldots p}$, because the Berends-Giele current can be constructed using the BRST building blocks. This implies that $E_{12\ldots p}$ inherits all the symmetry properties of its ancestor $M_{12\ldots p}$ and we can use $E_{12\ldots p}$ to proof the above identities. The reflection symmetry follows by induction and that the cyclic sum vanishes can be shown as follows

$$\begin{aligned}\sum_{\sigma\in\text{cyclic}} E_{\sigma(1,2,\ldots p)} &= \sum_{\sigma\in\text{cyclic}}\sum_{j=1}^{p} M_{\sigma(1,\ldots,j)}M_{\sigma(j+1,\ldots,p)}\\ &= \sum_{\sigma\in\text{cyclic}}\sum_{j=1}^{p}\frac{1}{2}\Big(M_{\sigma(1,\ldots,j)}M_{\sigma(j+1,\ldots,p)} + M_{\sigma(j+1,\ldots,p)}M_{\sigma(1,\ldots,j)}\Big) = 0\ , \qquad (6.13)\end{aligned}$$

where we have shifted all labels in the second term by j, which is possible due to the overall cyclic sum and used that Berends-Giele currents anticommute.

The symmetries (6.12) can naturally be explained by the fact that the Berends-Giele currents $M_{12\ldots p}$ correspond to $p+1$-point amplitudes with one off-shell leg. Inspired by this interpretation we can find further relations between the Berends-Giele currents by removing the $(p+1)^{\text{th}}$ leg from the $p+1$-point Kleiss-Kuijf identity [170]

$$M_{\{\beta\},1,\alpha} = (-1)^{n_\beta} \sum_{\sigma\in\text{OP}(\{\alpha\},\{\beta^{\text{T}}\})} M_{1,\{\sigma\}}\ , \qquad (6.14)$$

which were explicitly checked up to $p=7$ in [62]. The sum over $\text{OP}(\{\alpha\},\{\beta^{\text{T}}\})$ runs over all permutations of the set $\{\alpha\}\bigcup\{\beta^{\text{T}}\}$ that preserve the order of the individual elements in both $\{\alpha\}$ and $\{\beta^{\text{T}}\}$, which are subsets of $\{2,3,\ldots,p\}$. Moreover, the expression $\{\beta^{\text{T}}\}$ denotes the set $\{\beta\}$ with reversed ordering of the n_β elements.

The cyclic and reflection symmetries reduces the number of independent color ordered $p+1$-point amplitudes down to $\frac{p!}{2}$. This number can be further decreased to $(p-1)!$ independent amplitudes by the Kleiss-Kuijf relations. Since these identities do not involve any kinematic factors, they also hold for amplitudes with one off-shell leg. This statement is not surprising: The the Berends-Giele currents $M_{12\ldots p}$ satisfy similar relations and can be identified with $p{+}1$-point amplitudes with one off-shell leg. Therefore, we expect that there are only $(p-1)!$ independent $M_{12\ldots p}$. For $n_\beta = 1$ the special case $\{\beta\} = \{p\}$ reproduces the vanishing cyclic sum in (6.12) from the Kleiss-Kuijf relation (6.14). However, for $p \geq 6$ to reduce the number of independent $M_{i_1 i_2\ldots i_p}$ down to $(p-1)!$ the dual Ward identity or photon decoupling identity (6.12) alone is not sufficient [62, 170]. Nevertheless there are only $(p-1)!$ independent BRST building blocks $T_{i_1 i_2\ldots i_p}$, which suggests that $M_{i_1 i_2\ldots i_p}$ also have a basis of precisely $(p-1)!$ elements. Hence, the Kleiss-Kuijf relations (6.14) should hold also for higher rank $p \geq 7$ Berends-Giele currents. Note that it is not possible to reduce the number of independent $M_{12\ldots p}$ to $(p-2)!$ using the field theory analogues of monodromy relations [35, 36]. These are only valid if all momenta are on-shell and therefore the Berends-Giele currents do not obey similar relations.

6.4 From Berends-Giele currents to SYM amplitudes

The expressions for Berends-Giele currents look like lower order field theory amplitudes. For three external states the SYM amplitude is given by [43]

$$A_{\text{SYM}}(1,2,3) = \langle V_1 V_2 V_3 \rangle = \langle M_1 M_2 M_3 \rangle = \langle E_{12} V_3 \rangle \ . \tag{6.15}$$

At first, one might think that this amplitudes is BRST exact, because $E_{12} = QM_{12} = Q\frac{T_{12}}{s_{12}}$, and thus vanishes. However, for three massless particles all Mandelstam variables vanish, see section 4.4, which implies that E_{12} is not BRST exact. Hence, $E_{12} \neq Q\frac{T_{12}}{s_{12}}$ such that the amplitude is not BRST trivial. The four- and five-point SYM amplitudes can be written in terms of Berends-Giele currents as [161, 160]

$$\begin{aligned} A_{\text{SYM}}(1,2,3,4) &= -\frac{\langle T_{12} V_3 V_4 \rangle}{s_{12}} - \frac{\langle V_1 T_2 3 V_4 \rangle}{s_{23}} = \langle M_{12} M_3 M_4 \rangle + \langle M_1 M_{23} M_4 \rangle = \langle E_{123} V_4 \rangle \ , \\ A_{\text{SYM}}(1,2,3,4,5) &= \frac{\langle T_{123} V_4 V_5 \rangle}{s_{12} s_{45}} + \frac{\langle T_{321} V_4 V_5 \rangle}{s_{12} s_{45}} + \frac{\langle T_{12} T_{34} V_5 \rangle}{s_{12} s_{34}} + \frac{\langle V_1 T_{234} V_5 \rangle}{s_{23} s_{15}} + \frac{\langle V_1 T_{432} V_5 \rangle}{s_{23} s_{15}} \\ &= \langle M_{123} M_4 M_5 \rangle + \langle M_{12} M_{34} M_5 \rangle + \langle M_1 M_{234} M_5 \rangle = \langle E_{1234} V_5 \rangle \ . \end{aligned} \tag{6.16}$$

Also for these higher point examples E_{123} and E_{1234} are not BRST exact, which follows from the more general statement: If the momentum $k_{p+1} = -\sum_{i=1}^{p} k_i$ of an external state $p+1$ goes on-shell $k_{p+1}^2 = 0$, the prefactor in $M_{12\ldots p} \sim \frac{1}{s_{12\ldots p}}$ diverges. Hence, we conclude that the field $E_{12\ldots p}$ in (6.7) is conditionally BRST exact

$$\begin{aligned} QE_{12\ldots p} &= 0 \ , \quad \text{if } s_{12\ldots p} \neq 0 \ , \\ QE_{12\ldots p} &\neq 0 \ , \quad \text{if } s_{12\ldots p} = 0 \ , \end{aligned} \tag{6.17}$$

because $E_{12\ldots p} = QM_{12\ldots p}$ in (6.7) containing the propagator $\frac{1}{s_{12\ldots p}}$ on the right hand side only holds for $s_{12\ldots p} \neq 0$ and becomes ill defined for $s_{12\ldots p} = 0$. For an n-point SYM amplitude involving only massless states with $k_i^2 = 0$ momentum conservation implies $\sum_{i=1}^{n} k_i = 0$ such that $s_{12\ldots n-1} = 0$. In this case $E_{12\ldots n-1}$ is not BRST exact and $E_{12\ldots n-1} V_n$ is in the cohomology of the BRST operator Q. Therefore, we propose the n-point generalization of the SYM amplitudes above [162]:

$$A_{\text{SYM}}(1,2,\ldots n) = \langle E_{12\ldots n-1} V_n \rangle = \sum_{j=1}^{n-2} \langle M_{1\ldots j} M_{j+1\ldots n-1} V_n \rangle \ . \tag{6.18}$$

The n-point amplitude is in the cohomology of the BRST charge, because for n massless states with $s_{12\ldots n-1} = 0$ the superspace expression $E_{12\ldots} V_n$ is BRST closed. The action of the BRST charge on the composite superfield $E_{12\ldots p}$ is given by

$$QE_{12\ldots p} = \sum_{i=1}^{p-1} Q(M_{1\ldots i} M_{i+1\ldots p}) = \sum_{i=1}^{p-1} (M_{1\ldots i} E_{i+1\ldots n} + E_{1\ldots i} M_{i+1\ldots n})$$

$$
\begin{aligned}
&= \sum_{i=1}^{p-1}\sum_{j=1}^{i-1} M_{1\ldots j}M_{j+1\ldots i}M_{i+1\ldots p} - \sum_{i=1}^{p-1}\sum_{j=i+1}^{p-1} M_{1\ldots i}M_{i+1\ldots j}M_{j+1\ldots p} \\
&= \sum_{1\leq j<i}^{p-1} M_{1\ldots j}M_{j+1\ldots i}M_{i+1\ldots p} - \sum_{1\leq i<j}^{p-1} M_{1\ldots i}M_{i+1\ldots j}M_{j+1\ldots p} \\
&= \sum_{i=1}^{p-1}\sum_{j=i+1}^{p-1} (M_{1\ldots i}M_{i+1\ldots j}M_{j+1\ldots p} - M_{1\ldots i}M_{i+1\ldots j}M_{j+1\ldots p}) \\
&= 0
\end{aligned}
\tag{6.19}
$$

such that together with $QV_n = 0$ we obtain $Q(E_{12\ldots n-1}V_n) = 0$ [114].

The diagrammatic representation of $A_{\text{SYM}}(1, 2, \ldots n)$ in figure 6.5 corresponds to the two currents $M_{1\ldots j}$ and $M_{j+1\ldots n-1}$ of rank j and $n-j-1$, respectively, which are connected via a cubic interaction to the n^{th} on-shell leg. The formula (6.18) is the supersymmetric

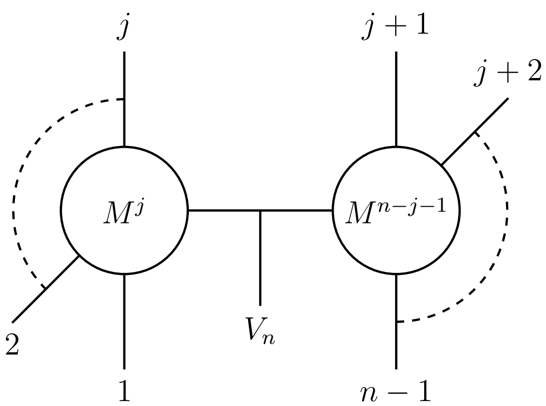

Figure 6.5: Decomposition of SYM amplitudes using pure spinor cohomology methods.

generalization of the amplitude prescription (6.1) for color ordered n gluon amplitudes of [167]. Compared to (6.1) the $n-1$ rank current in (6.18), which is multiplied by the Mandelstam invariant $s_{12\ldots n-1}$, is substituted by $E_{12\ldots n-1}$ and the current of rank one J_n^m can be identified with V_n.

6.5 BRST integration by parts and cyclic symmetries

Because one external state was singled out in the prescription (6.18) for the n-point SYM amplitude, the manifest cyclic symmetry of color stripped field theory amplitudes were hidden. Nevertheless, the amplitude (6.18) is invariant under cyclic permutations

$$A_{\text{SYM}}(1, 2, \ldots n) = A_{\text{SYM}}(2, 3, \ldots n, 1) \ , \tag{6.20}$$

which follows after regrouping terms in $QM_{12\ldots n} = \sum_{i=1}^{n-1} M_{1\ldots i} M_{i+1\ldots n}$. Therefore, we can consider the BRST variation

$$\begin{aligned} Q\sum_{i=2}^{n-2} M_{1\ldots i}M_{i+1\ldots n} &= M_1 \sum_{i=2}^{n-2} M_{2\ldots i}M_{i+1\ldots n} + \sum_{2\le j<i}^{n-2} M_{1\ldots j}M_{j+1\ldots i}M_{i+1\ldots n} \\ &\quad - \sum_{2\le i<j}^{n-2} M_{1\ldots i}M_{i+1\ldots j}M_{j+1\ldots n} - \sum_{i=2}^{n-2} M_{1\ldots i}M_{i+1\ldots n-1}M_n \\ &= M_1(E_{23\ldots n} - M_{23\ldots n-1}M_n) - (E_{12\ldots n-1} - M_1 M_{23\ldots n-1})M_n \\ &= V_1 E_{12\ldots n} - E_{12\ldots n-1}V_n \ , \end{aligned} \tag{6.21}$$

where the sums in the first line cancel each other and also $M_1 M_{23\ldots n-1} M_n$ in the third line drops out. All combinations $M_{1\ldots i}M_{i+1\ldots n}$ are well defined for n massless external states, because the highest rank supercurrents have non-singular poles in $s_{12\ldots n-2}$ and $s_{12\ldots n-1}$ such that

$$\langle E_{12\ldots n-1}V_n\rangle - \langle E_{2\ldots n}V_1\rangle = -\langle Q(M_{12}M_{3\ldots n} + M_{123}M_{4\ldots n} + \ldots + M_{1\ldots n-2}M_{(n-1)n})\rangle = 0 \ , \tag{6.22}$$

because a BRST exact superspace expression vanishes under the pure spinor bracket. Therefore, the amplitude (6.18) is invariant under cyclic permutations $i \to i+1 \mod n$ [114] and we obtain

$$\langle E_{12\ldots n-1}V_n\rangle = \langle E_{2\ldots n}V_1\rangle \ . \tag{6.23}$$

To make the cyclic symmetry of (6.18) manifest we can take advantage of cohomological properties of the pure spinor superspace expression of $A_{\rm SYM}(1,2,\ldots n)$. In this process we will derive an alternative expression for the n-point SYM amplitudes with manifest cyclic symmetry and reduce the rank of the Berends-Giele currents needed for the amplitude, which makes the evaluation of amplitudes more efficient.

Since for three external states the cyclic symmetry is already manifest

$$A_{\rm SYM}(1,2,3) = \frac{1}{3}\langle M_1 M_2 M_3\rangle + \mathrm{cyclic}(123) \ , \tag{6.24}$$

we start by considering the four-point amplitude: Because the combination $M_i M_j$ is BRST exact $M_i M_j = E_{ij} = QM_{ij}$ and BRST exact terms of the form $\langle Q(\ldots)\rangle = 0$ decouple from the cohomology of Q, we find

$$\begin{aligned} A_{\rm SYM}(1,2,3,4) &= \langle M_{12}M_3M_4 + M_1M_{23}M_4\rangle = \langle M_{12}E_{34} + E_{41}M_{23}\rangle \\ &= \langle M_{12}QM_{34} + QM_{14}M_{23}\rangle = \langle E_{12}M_{34} + M_{41}E_{23}\rangle \\ &= \frac{1}{2}\langle M_{12}E_{34}\rangle + \mathrm{cyclic}(1234) \end{aligned} \tag{6.25}$$

after integrating the BRST charge by parts. This integration by parts identity of the BRST charge can be generalized to

$$\langle M_{i_1 i_2\ldots i_p}E_{j_1 j_2\ldots j_q}\rangle = \langle M_{i_1 i_2\ldots i_p}QM_{j_1 j_2\ldots j_q}\rangle = \langle E_{i_1 i_2\ldots i_p}M_{j_1 j_2\ldots j_q}\rangle \ . \tag{6.26}$$

Next, we want to continue with the five-point amplitude to demonstrate that the pattern persists also for higher point $n \geq 5$ amplitudes. Therefore, we utilize (6.26), which for instance implies that

$$\langle M_{123}E_{45}\rangle = \langle E_{123}M_{45}\rangle = \langle (M_1M_{23} + M_{12}M_3)M_{45}\rangle \ , \tag{6.27}$$

where we have used $QM_{123} = M_1M_{23} + M_{12}M_3$ in order to write the amplitude in a form that is manifest cyclic symmetric

$$\begin{aligned} A_{\text{SYM}}(1,2,3,4,5) &= \langle M_{123}M_4M_5 + M_{12}M_{34}M_5 + M_1M_{234}M_5\rangle \\ &= \langle M_{123}E_{45} + M_{12}M_{34}M_5 + E_{51}M_{234}\rangle \\ &= \langle E_{123}M_{45} + M_{12}M_{34}M_5 + M_{51}E_{234}\rangle \\ &= \langle (M_1M_{23} + M_{12}M_3)M_{45} + M_{12}M_{34}M_5 + M_{51}(M_2M_{34} + M_{23}M_4)\rangle \\ &= \langle M_{12}M_3M_{45}\rangle + \text{cyclic}(12345) \ . \end{aligned} \tag{6.28}$$

The highest rank Berends-Giele currents M_{123} and M_{234} of rank three in (6.28) have been traded for rank two M_{ij}. For an n-point SYM amplitude the cohomology formula (6.18) involves Berends-Giele currents $M_{12\ldots n-2}$ with rank up to $n-2$. Using BRST integration by parts (6.26) one can lower the rank of the highest rank building block down to at most $\lfloor \frac{n}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the Gauß bracket $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$, which selects the nearest integer that is smaller or equal to its argument [162].

By applying BRST integration by parts and/or exploiting the fermionic nature of the Berends-Giele currents the $n = 6$ SYM amplitude takes the following cyclic symmetric form

$$\begin{aligned} A_{\text{SYM}}(1,2,3,4,5,6) &= \langle M_{12}M_{34}M_{56}\rangle + \langle M_{23}M_{45}M_{61}\rangle + \langle M_{123}(M_{45}M_6 + M_4M_{56})\rangle \\ &\quad + \langle M_{234}(M_{56}M_1 + M_5M_{61})\rangle + \langle M_{345}(M_{61}M_2 + M_6M_{12})\rangle \\ &= \frac{1}{2}\langle M_{123}E_{456}\rangle + \frac{1}{3}\langle M_{12}M_{34}M_{56}\rangle + \text{cyclic}(123456) \ . \end{aligned} \tag{6.29}$$

The fractional coefficients in (6.29) and also (6.24) and (6.25) are important to avoid overcounting in the explicit sum over cyclic permutation. For particular superfield kinematics the cyclic orbits are shorter than the number of legs n [62, 114]: Due to (6.26) the cyclic sum over $\langle M_{i_1 i_2 \ldots i_p} E_{j_1 j_2 \ldots j_q}\rangle$ results in an overcounting by a factor two for $p = q$. To compensate this we have to introduce a factor of $\frac{1}{2}$ for terms like $\langle M_{1\ldots k}E_{k+1\ldots 2k}\rangle$ for an even number of external states $n = 2k$. For $\langle M_{i_1 i_2 \ldots i_p} M_{j_1 j_2 \ldots j_q} M_{k_1 k_2 \ldots k_r}\rangle$ the cyclic sum leads to an overcounting by a factor three for $p = q = r$. Hence, we have to multiply this expression by a factor of $\frac{1}{3}$ for $n \in 3\mathbb{N}$ external states. If n is not dividable by 2 or 3 we do not have to introduce any fractional coefficients in the manifestly cyclic form of the n-point amplitude.

Chapter 7

Superstring KLT and monodromy relations

In chapter 4 we have already exploited that n closed string scattering amplitudes on the sphere factorise into two n open string amplitudes on the disk. In this chapter we want to construct this double copy for closed string amplitudes and moreover find relations between open string amplitudes. Here, we follow the presentation in [114, 171, 155] and the review [172] for the derivation of the KLT relations and [35, 109] for the discussion on monodromy relations.

7.1 KLT relations for closed string scattering amplitudes

The closed string Hilbert space is a tensor product of two open string Hilbert spaces

$$H_{\text{closed}} = H_{\text{open}} \otimes H_{\text{open}} \ . \tag{7.1}$$

Immediately one can imagine that this property carries over to the vertex operators describing the states in the Hilbert space H_{closed}. Indeed, using the operator state correspondence and the factorisation of the Hilbert space (7.1) the closed string vertex operators $V_i(z_i, \overline{z}_i) = V_i(z_i) \otimes \overline{V}_i(\overline{z}_i)$ and $U_i(z_i, \overline{z}_i) = U_i(z_i) \otimes \overline{U}_i(\overline{z}_i)$ are double copies of open string vertex operators. Hence, this amounts to

$$\begin{aligned} V_i(z_i, \overline{z}_i) &= |\lambda^\alpha A^i_\alpha|^2(z_i, \overline{z}_i) \ , \\ U_i(z_i, \overline{z}_i) &= \left|\partial\theta^\alpha A^i_\alpha + \Pi^m A_m + d_\alpha W^\alpha_i + \tfrac{1}{2} N^{mn} F^i_{mn}(z_i, \overline{z}_i)\right|^2 (z_i, \overline{z}_i) \ , \end{aligned} \tag{7.2}$$

where we used the notation $|\lambda^\alpha A^i_\alpha|^2(z_i, \overline{z}_i) = \Big(\lambda^\alpha A^i_\alpha(\theta)\Big)(z_i)\Big(\overline{\lambda}^{\hat\alpha}\overline{A}^i_{\hat\alpha}(\overline{\theta})\Big)(\overline{z}_i) e^{ik_i\cdot X(z_i,\overline{z}_i)}$.[1] Overlined SYM or worldsheet fields are the antiholomorphic counterparts of the corresponding

[1] Here, we have used (3.78) to separate the plane wave factor from the θ-dependent part of the superfields. Alternatively, we could separate $X^m(z, \overline{z}) = X^m(z) + \overline{X}^m(\overline{z})$ into left- and right movers. This would imply

holomorphic fields, whose spinorial indices with hats $\hat{\alpha}, \hat{\beta}, \ldots$ have the opposite (same) chirality as $\alpha, \beta, \ldots$ for type IIA (type IIB) superstring theory. Moreover, we have used (3.78) to emphasise that the closed string vertex operator depends on a plane wave factor $e^{ik_i\cdot X(z_i,\overline{z}_i)}$. The $\overline{\theta}$-expansions for the antiholomorphic fields are as in (2.32) but with independent polarizations $\overline{e}_i^m, \overline{\chi}_i^\alpha$ instead of e_i^m, χ_i^α, whose tensor product forms the closed string polarization. For example, the purely bosonic (NSNS) closed string polarization tensor is given by $\epsilon_i^{mn} = e_i^m \otimes e_i^n$.

For antiholomorphic fields the OPEs on the sphere are the same as on the disk, but they depend on $\overline{z}$ instead of z. Moreover, there are no OPEs involving holomorphic and antiholomorphic fields, i.e. no cross contractions between holomorphic and antiholomorphic fields on the sphere. This implies that the calculation (OPE contractions) of the correlator $\langle\!\langle \ldots \rangle\!\rangle$ in (4.3) on the sphere separates for the holomorphic and antiholomorphic parts of (7.2) and the plane wave factors lead to the Koba-Nielsen factor $\langle\!\langle \prod_{i=1}^n e^{ik_i\cdot X(z_i,\overline{z}_i)} \rangle\!\rangle \sim \mathrm{KN}(\{z_p, \overline{z}_p\}) = \prod_{i<j}^n |z_{ij}|^{2s_{ij}}$ and a momentum preserving delta function $\delta(\sum_{i=1}^n k_i)$. Similar, the zero mode integration via (4.15) is performed independently for $\lambda^\alpha, \theta^\alpha$ and $\overline{\lambda}^{\hat{\alpha}}, \overline{\theta}^{\hat{\alpha}}$. Due to this factorization the scattering amplitude (4.3) can be written as

$$\begin{aligned}\mathcal{A}_n &= \int_{\mathcal{M}_{0,n}} \mathrm{d}^2 z_2 \mathrm{d}^2 z_3 \cdots \mathrm{d}^2 z_{n-2} \, \langle\!\langle V_1(z_1) U_2(z_2) \cdots U_{n-2}(z_{n-2}) V_{n-1}(z_{n-1}) V_n(z_n) \rangle\!\rangle \\ &\quad \times \langle\!\langle \overline{V}_1(\overline{z}_1) \overline{U}_2(\overline{z}_2) \cdots \overline{U}_{n-2}(\overline{z}_{n-2}) \overline{V}_{n-1}(\overline{z}_{n-1}) \overline{V}_n(\overline{z}_n) \rangle\!\rangle \\ &= \int_{\mathcal{M}_{0,n}} \mathrm{d}^2 z_2 \mathrm{d}^2 z_3 \cdots \mathrm{d}^2 z_{n-2} \prod_{i<j}^n |z_{ij}|^{2s_{ij}} \langle \mathcal{K}_n(\{z_p\}) \rangle \langle \overline{\mathcal{K}}_n(\{\overline{z}_p\}) \rangle \, ,\end{aligned} \tag{7.3}$$

where $\mathcal{K}_n(\{z_p\})$ and $\overline{\mathcal{K}}_n(\{\overline{z}_p\})$ contain the zero modes of $\lambda^\alpha, \theta^\alpha$ and their antiholomorphic counterparts, respectively. Moreover, the singularities $\overline{z}_{ij}^{-1}$ in $\overline{\mathcal{K}}_n(\{\overline{z}_p\})$ obtained by OPE contractions are the complex conjugates of z_{ij}^{-1} in $\mathcal{K}_n(\{z_p\})$.

Next, we want to discuss the decomposition of the sphere integrals in (4.3) into open string integrals. This double copy has a geometrical interpretation: The open string integrals run over the punctured disk, which can be deformed into a punctured hemisphere. Taking two punctured hemispheres and gluing them together along their boundary and matching punctures results in a punctured sphere, which is depicted in figure 7.1. This gluing process corresponds to a map from two integrals over punctured real lines to an integral over the punctured complex plane: $\int_{\mathbb{R}^n} \otimes \int_{\mathbb{R}^n} \to \int_{\mathbb{C}^n}$, where we have to ensure that this map is single valued. More formally, we have to split the Koba-Nielson factor $\mathrm{KN}(z_i, \overline{z}_i)$ into products of meromorphic and antimereomorphic functions, i.e. $|z_{ij}|^{2s_{ij}} = (z_{ij})^{s_{ij}} (\overline{z}_{ij})^{s_{ij}}$ such that (7.3) is an integral over a holomorphic square of a

that the plane wave factor for the left-moving part of a vertex operator only depends holomorphically on z via $X^m = X^m(z)$ in (2.32). Nevertheless, the full closed string vertex operator (7.2) contains the plane wave factor $e^{ik_i\cdot X(z_i,\overline{z}_i)}$, because the plane wave factors of the holomorphic and antiholomorphic sector in (7.2) can be combined into $e^{ik_i\cdot X(z_i,\overline{z}_i)}$ [109].

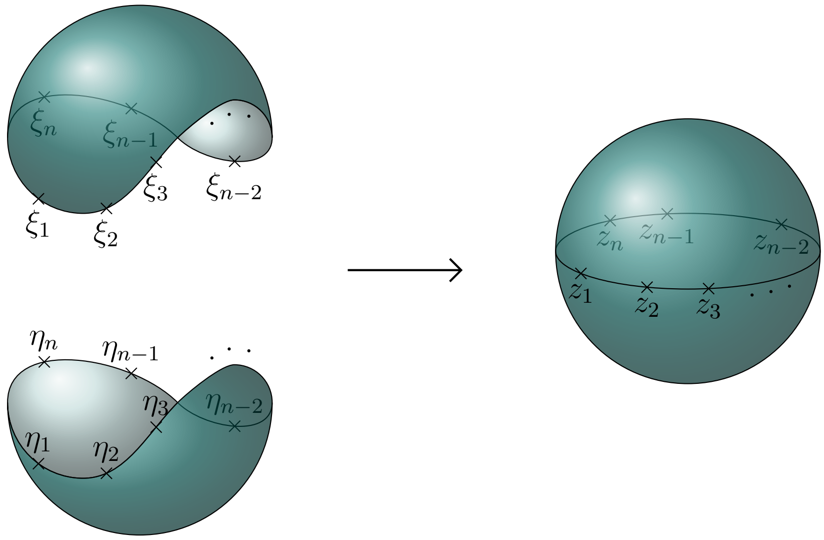

Figure 7.1: Gluing two open string amplitudes together along the boundary of the disk to form one closed string amplitude on the sphere.

multivalued function $\langle \mathcal{K}_n(\{z_p\})\rangle \prod_{i<j}^n (z_{ij})^{s_{ij}}$ with branch points at vertex operator positions $z_i = z_j$. This double copy structure requires monodromy phases $e^{\pm i\pi s_{ij}}$ to become single valued and well defined. These monodromy phases will give rise to the KLT kernel $\mathcal{S}[\cdot|\cdot]_i$ in (4.3) and were first discovered by Kawai, Lewellen and Tye in 1986 [34].

To realize this double copy relation we have to manipulate the integrals over the complex plane in (7.3). Therefore, we start by rewriting the $n-3$ integrations over complex worldsheet positions into integrals over the real line by using $z_i = x_i + iy_i$ and $\overline{z}_i = x_i - iy_i$

$$\prod_{i=2}^{n-2}\int_{\mathbb{C}} \mathrm{d}^2 z_i = \left(\frac{i}{2}\right)^{n-3}\prod_{i=2}^{n-2}\int_{\mathbb{C}} \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i = \prod_{i=2}^{n-2}\int_{\mathbb{R}} \mathrm{d}x_i \int_{\mathbb{R}} \mathrm{d}y_i \ . \tag{7.4}$$

Although, the integration was split into two real integrals, the amplitude (7.3) does not factorize into two open string amplitudes, because the vertex operators still depend on the complex worldsheet positions $x_i \pm iy_i$. To regroup the integrals into open string integrals we have to substitute $x_i \pm iy_i$ with two real variables [34]. We can take the imaginary part of z_i and $\overline{z}_i$ and recognize that the amplitude is an analytic function in y_i except for the branch points at $z_i - z_j = 0$ and $\overline{z}_i - \overline{z}_j = 0$. Thus, without changing the amplitude we can analytically continue the real variable y_i to the complex plane by rotating the integration contour simultaneously for all y_i from the real line to the purely imaginary axis

$$y_i \longrightarrow i e^{-2i\varepsilon} y_i \simeq i(1-2i\varepsilon) y_i \ , \tag{7.5}$$

where $\varepsilon > 0$ is a small constant. The small shift in ε away from the purely imaginary axis is introduced to avoid the branch points of the integrand. Note that this is possible, because there are no poles, genus or other obstructions along the rotation. The only contribution comes from the monodromy of the integrand during the rotation of the integration contour. To determine these let us consider the Koba-Nielsen factor in (7.3) up to linear order in ε after the rotation[2]

$$\begin{aligned} |z_i|^{2s_{1i}} &= \left(x_i^2+y_i^2\right)^{s_{1i}} \longrightarrow \left(x_i^2-y_i^2+4i\varepsilon y_i^2\right)^{s_{1i}} , \\ |z_i-1|^{2s_{i(n-1)}} &= \left((x_i-1)^2+y_i^2\right)^{s_{i(n-1)}} \longrightarrow \left((x_i-1)^2-y_i^2+4i\varepsilon y_i^2\right)^{s_{i(n-1)}} , \\ |z_i-z_j|^{2s_{ij}} &= \left((x_i-x_j)^2+(y_i-y_j)^2\right)^{s_{ij}} \longrightarrow \left((x_i-x_j)^2-(1-4i\varepsilon)(y_i-y_j)^2\right)^{s_{ij}} , \end{aligned} \tag{7.6}$$

where without loss of generality we have made a particular choice for $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ to simplify the discussion and $i, j = 2, 3, \ldots, n-2$. After the transformation

$$\xi_i = x_i - y_i \ , \qquad \eta_i = x_i + y_i \tag{7.7}$$

the above expressions in (7.6) of the KN-factor become

$$\begin{aligned} |z_i|^{2s_{1i}} &= (\xi_i+i\varepsilon\delta_i)^{s_{1i}}(\eta_i-i\varepsilon\delta_i)^{s_{1i}} , \\ |z_i-1|^{2s_{i(n-1)}} &= (\xi_i-1+i\varepsilon\delta_i)^{s_{1i}}(\eta_i-1-i\varepsilon\delta_i)^{s_{i(n-1)}} , \\ |z_i-z_j|^{2s_{ij}} &= (\xi_i-\xi_j+i\varepsilon(\delta_i-\delta_j))^{s_{1i}}(\eta_i-\eta_j-i\varepsilon(\delta_i-\delta_j))^{s_{ij}} , \end{aligned} \tag{7.8}$$

where we have defined $\delta_i = \eta_i - \xi_i$. Thereby, the amplitude (7.3) takes the following form

$$\begin{aligned} \mathcal{A}_n = \left(\frac{1}{2}\right)^{n-3} \int_{-\infty}^{\infty} \prod_{p=2}^{n-2} \mathrm{d}\xi_i \mathrm{d}\eta_i \, \langle \mathcal{K}_n(\{\xi_p\})\rangle \langle \overline{\mathcal{K}}_n(\{\eta_p\})\rangle \prod_{i=2}^{n-2} (\xi_i+i\varepsilon\delta_i)^{s_{1i}}(\eta_i-i\varepsilon\delta_i)^{s_{1i}} \\ \times(\xi_i-1+i\varepsilon\delta_i)^{s_{1i}}(\eta_i-1-i\varepsilon\delta_i)^{s_{i(n-1)}} \\ \times \prod_{2\leq j<i}^{n-2} (\xi_i-\xi_j+i\varepsilon(\delta_i-\delta_j))^{s_{1i}}(\eta_i-\eta_j-i\varepsilon(\delta_i-\delta_j))^{s_{ij}} . \end{aligned} \tag{7.9}$$

The factor $\left(\frac{1}{2}\right)^{n-3}$ appears due to the Jacobian of the coordinate transformation $(x_i, y_i) \to (\xi_i, \eta_i)$ in the integral.

For the following discussion we want to assume that at least one $\eta_i \in]-\infty, 0[$. For the terms in (7.9) containing a specific ξ_i, which are given by

$$\int_{-\infty}^{\infty} \mathrm{d}\xi_i \, \langle \mathcal{K}_n(\{\xi_p\})\rangle (\xi_i+i\varepsilon\delta_i)^{s_{1i}}(\xi_i-1+i\varepsilon\delta_i)^{s_{1i}} \prod_{2\leq j<i}^{n-2} (\xi_i-\xi_j+i\varepsilon(\delta_i-\delta_j))^{s_{1i}} , \tag{7.10}$$

the terms linear in ε show the following behaviour near the branch points

$$\xi_i \approx 0 : \qquad \delta_i = \eta_i - \xi_i \approx \eta_i < 0 ,$$

[2]The branch points at $z_{ij} = 0$ and $\overline{z}_{ij} = 0$ of the integrand in (7.3) are contained in $\mathrm{KN}(\{z_p, \overline{z}_p\})$, because the poles in $\mathcal{K}_n$ and $\overline{\mathcal{K}}_n$ have integer powers.

$$\begin{aligned} \xi_i \approx 1: &\qquad \delta_i \approx \eta_i - 1 < 0 \ , \\ \xi_i \approx \xi_j: &\qquad \delta_i - \delta_j \approx \eta_i - \eta_j < 0 \quad \text{for } \eta_i < \eta_j \ . \end{aligned} \tag{7.11}$$

If $\eta_i > \eta_j$ we have to look at a different ξ_k integral, which would correspond to the smallest η_k. This η_k is again in the interval $]-\infty, 0[$ due to the assumption that at least one η_i is in this interval. Hence, we conclude that the integral of ξ_i can be closed by analytic continuation in the lower half plain and the integral vanishes, because the closed integration contour does not encircle any poles of the integrand. More generally, we evade branch points $\xi_i = \xi_j$ below or above the real axis for $\eta_i < \eta_j$ or $\eta_i > \eta_j$, respectively. Hence, at least one integration contour of ξ_i can be entirely below or above the real axis, if the corresponding η_i is either in the range $]-\infty, 0[$ or $]1, \infty[$ such that all $\eta_i \in]0, 1[$ in order to get a non-vanishing contribution from (7.10), i.e. all η_i-integrations can be written as

$$\int_{0<\eta_2<\eta_3\ldots<\eta_{n-2}<1} \prod_{i=2}^{n-2} \mathrm{d}\eta_i \, \langle \overline{\mathcal{K}}_n(\{\eta_p\}) \rangle (\eta_i)^{s_{1i}} (\eta_i - 1)^{s_{1i}} \prod_{2\le j<i}^{n-2} (\eta_i - \eta_j)^{s_{1i}} \ . \tag{7.12}$$

Note that we made a choice for the ordering $0 < \eta_2 < \eta_3 \ldots < \eta_{n-2} < 1$. To obtain the entire closed string amplitude we have to sum over all permutations $\sigma(2, 3, \ldots, n-2)$ of this ordering for η_i. Moreover, the integrals in (7.12) can be identified with the n-point color ordered open string amplitude (4.1) with fixed vertex operator positions at $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$.[3] In the end, the η_i-integration will result in summing the open string amplitudes $\mathcal{A}(1, \sigma(2, 3, \ldots, n-2), n-1, n)$ over all permutations σ.

In a similar way, we want to examine which integration contour for ξ_i is dictated by the imaginary parts of (7.8) proportional to ε. For $\eta_i \in]0, 1[$ we find

$$\begin{aligned} \eta_i \approx 0: &\qquad \delta_i \approx \xi_i > 0 \ , \\ \eta_i \approx 1: &\qquad \delta_i \approx \xi_i - 1 < 0 \ , \\ \eta_i \approx \eta_j: &\qquad \delta_i - \delta_j \approx \eta_i - \eta_j < 0 \quad \text{for } \xi_i < \xi_j \end{aligned} \tag{7.13}$$

such that the integration contour of $\xi_i + i\varepsilon\delta_i$ lies above and below the real axis for $\eta_i < 0$ and $\eta_i > 1$, respectively. In the interval between 0 and 1 the contour of $\xi_i + i\varepsilon\delta$ is above the one for $\xi_j + i\varepsilon\delta_j$ for $i > j$, which is depicted in figure 7.2. Because the individual ξ_i-contours cannot intersect each other, beginning with the rightmost (leftmost) we need to deform the integration contours for $\xi_i + i\varepsilon\delta_i$ to the left around the branch cut at $\xi_i = 1$ (the right around the branch cut at $\xi_i = 0$), which closes the contour below (above) the real axis. Thereby, we obtain integration regions corresponding to open string scattering

[3] Usually, for this identification we have to rescale $\alpha' \to \frac{\alpha'}{4}$ in the open string amplitude in order to get the correct closed string Koba-Nielsen factor from open strings, i.e. such that the kinematic invariants in the KN-factor agree $s_{ij}^{\text{open}} = 2\alpha' k_i \cdot k_j \to s_{ij}^{\text{closed}} = \frac{\alpha'}{2} k_i \cdot k_j$. Alternatively, we could have defined the open string momenta as $k^{\text{open}} = \frac{1}{2} k^{\text{closed}}$, which is known as the doubling trick.

With our choice for $\alpha' = 2$ for closed strings and $\alpha' = \frac{1}{2}$ for open strings, we don't have to introduce any additional factor, since we have thereby accounted for the relative factor.

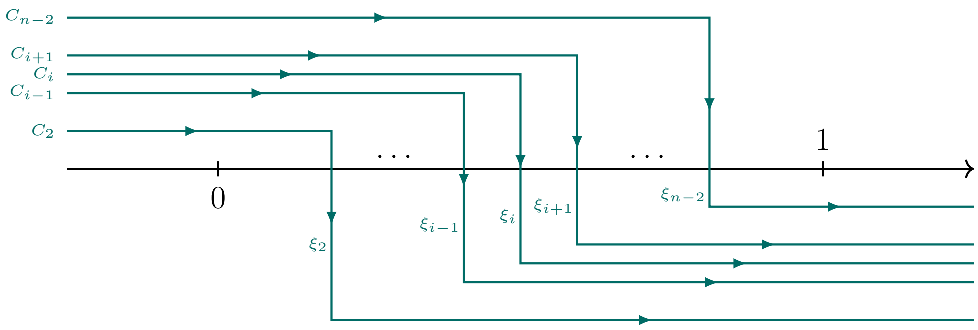

Figure 7.2: Nested structure for the integration contours C_i for $\xi_i + i\varepsilon\delta_i$ corresponding to the ordering $0 < \eta_2 < \eta_3 < \ldots < \eta_{n-2} < 1$.

amplitudes. In this process we have the freedom to chose the number of contours we want to close to the left or right. For example, for $2 \leq j \leq n-1$ all contours from 2 to $j-1$ can be pulled to the left and the reaming contours from j to $n-2$ can be closed to the right. Moreover, it is also possible to close all of the contours either to the left ($j = 2$) or the right ($j = n-1$).

In addition, we have to ensure that the integrand takes the correct form such that we can also identify the integrand with an open string correlator. This can be done by pulling out phase factors $e^{\pm i\pi s_{ij}}$, but we have to be careful not to cross a branch cut. Therefore, we chose the branch cut to lie on the negative real axis and restrict the power function to $z^c = |z|^c e^{ic\theta}$ with $-\pi < \theta < \pi$. According to [172] this implies that

$$z^c = \begin{cases} e^{i\pi c}(-z)^c & \text{for } \Im(z) \geq 0 \,, \\ e^{-i\pi c}(-z)^c & \text{for } \Im(z) < 0 \,. \end{cases} \tag{7.14}$$

Note that (7.14) is valid for both signs of $\Re(z)$ and not only $\Re(z) < 0$ as stated in [172]. For more details see appendix A of [172].

We want to demonstrate the contour deformation for an arbitrary j and start by closing the integration contour C_2 (see figure 7.2) of ξ_2 to the right, which gives

$$\begin{aligned} &\int_{C_2} \mathrm{d}\xi_2\, (\xi_2)^{s_{12}} (1-\xi_2)^{s_{2(n-1)}} \prod_{j=3}^{n-2} (\xi_j - \xi_2)^{s_{2j}} \\ &= \left(e^{i\pi s_{12}} - e^{-i\pi s_{12}}\right) \int_{-\infty}^{0} \mathrm{d}\xi_2 (-\xi_2)^{s_{12}} (1-\xi_2)^{s_{2(n-1)}} \prod_{j=3}^{n-2} (\xi_j - \xi_2)^{s_{2j}} \\ &= 2i \sin(\pi s_{12}) \int_{-\infty}^{0} \mathrm{d}\xi_2 (-\xi_2)^{s_{12}} (1-\xi_2)^{s_{2(n-1)}} \prod_{j=3}^{n-2} (\xi_j - \xi_2)^{s_{2j}} \end{aligned} \tag{7.15}$$

where we have only displayed the Koba-Nielsen factor and shown terms with branch cuts in ξ_2 for simplicity. We can continue by closing the contour for ξ_3 to the left and get

$$\int_{C_3} \mathrm{d}\xi_3\, (\xi_3)^{s_{13}} (1-\xi_3)^{s_{3(n-1)}} (\xi_3 - \xi_3)^{s_{23}} \prod_{j=4}^{n-2} (\xi_j - \xi_2)^{s_{2j}}$$

$$
\begin{aligned}
&= 2i\sin(\pi s_{13}) \int_{\xi_2}^{0} \mathrm{d}\xi_3 (-\xi_3)^{s_{13}} (1-\xi_3)^{s_{3(n-1)}} (\xi_3-\xi_2)^{s_{23}} \prod_{j=4}^{n-2} (\xi_j-\xi_3)^{s_{2j}} \\
&\quad +2i\sin(\pi(s_{12}+s_{13})) \int_{-\infty}^{\xi_3} \mathrm{d}\xi_3 (-\xi_3)^{s_{13}} (1-\xi_3)^{s_{3(n-1)}} (\xi_2-\xi_3)^{s_{23}} \prod_{j=4}^{n-2} (\xi_j-\xi_2)^{s_{2j}} \ . \qquad (7.16)
\end{aligned}
$$

We proceed in this way until we arrive at the last contour ξ_{j-1}, which we have pulled to the left.

On the other side, we also have the possibility to deform the integration contours around the branch point $\xi_i = 1$ to the right. Therefore, we start with closing the contour for ξ_{n-2}, which gives

$$
\begin{aligned}
&\int_{C_{n-2}} \mathrm{d}\xi_{n-2}\, (\xi_{n-2})^{s_{1(n-2)}} (1-\xi_{n-2})^{s_{(n-2)(n-1)}} \prod_{j=3}^{n-3} (\xi_{n-2}-\xi_j)^{s_{jn-2}} \\
&= 2i\sin\Big(\pi s_{(n-2)(n-1)}\Big) \int_{1}^{\infty} \mathrm{d}\xi_{n-2} (\xi_{n-2})^{s_{1(n-2)}} (\xi_{n-2}-1)^{s_{(n-2)(n-1)}} \prod_{j=3}^{n-3} (\xi_{n-2}-\xi_j)^{s_{jn-2}} \qquad (7.17)
\end{aligned}
$$

Next, we take the ξ_{n-3} contour and deform it such that

$$
\begin{aligned}
&\int_{C_{n-3}} \mathrm{d}\xi_{n-3}\, (\xi_{n-3})^{s_{1(n-3)}} (1-\xi_{n-3})^{s_{(n-2)(n-1)}} (\xi_{n-2}-\xi_{n-3})^{s_{(n-3)(n-2)}} \prod_{j=3}^{n-4} (\xi_{n-3}-\xi_j)^{s_{jn-3}} \\
&= 2i\sin\Big(\pi s_{(n-3)(n-1)}\Big) \int_{1}^{\xi_{n-2}} \mathrm{d}\xi_{n-3} (\xi_{n-3})^{s_{1(n-3)}} (\xi_{n-3}-1)^{s_{(n-3)(n-1)}} \\
&\quad \times (\xi_{n-2}-\xi_{n-3})^{s_{(n-3)(n-2)}} \prod_{j=3}^{n-4} (\xi_j-\xi_{n-3})^{s_{j(n-4)}} \\
&\quad +2i\sin\Big(\pi\big(s_{(n-3)(n-1)}+s_{(n-3)(n-2)}\big)\Big) \int_{\xi_{n-2}}^{\infty} \mathrm{d}\xi_{n-3} (\xi_{n-3})^{s_{1(n-3)}} (\xi_{n-3}-1)^{s_{(n-3)(n-1)}} \\
&\quad \times (\xi_{n-3}-\xi_{n-2})^{s_{(n-3)(n-2)}} \prod_{j=3}^{n-4} (\xi_{n-3}-\xi_j)^{s_{j(n-4)}} \qquad (7.18)
\end{aligned}
$$

Again, we can continue with this pattern until we reach the contour for ξ_j, which is the last one, we want to close to the right. Hence, the integrations over the worldsheet variables ξ_i result in a sum over color ordered open string amplitudes $\tilde{\mathcal{A}}(\gamma(\sigma(2,\ldots,j-1)),1,n-1,\beta(\sigma(j,\ldots,n-2)),n)$, which are multiplied by monodromy phases.

In the end, the different ways of closing the contours lead to the following expression for the n-point closed string amplitude on the sphere

$$
\begin{aligned}
\mathcal{A}_n = &\sum_{\sigma,\rho,\gamma\in S_{n-3}} \mathcal{A}(1,\sigma(2,\ldots,n-2),n-1,n)\mathcal{S}[\rho(\sigma(2,\ldots,j-1))|\sigma(2,\ldots j-1)]_1 \\
&\times\mathcal{S}[\sigma(j+1,\ldots,n-2))|\beta(\sigma(j+1,\ldots n-2))]_{n-1} \\
&\times\tilde{\mathcal{A}}(\gamma(\sigma(2,\ldots,j-1)),1,n-1,\beta(\sigma(j,\ldots,n-2)),n) \qquad (7.19)
\end{aligned}
$$

with $2 \leq j \leq n-1$ as before. Note that the momentum kernels in (7.19) depend on the momenta k_1 and k_{n-1} of the external states at the branch points at 0 and 1. They can be defined recursively as [114, 155]

$$\mathcal{S}[p_1,\ldots,p_s,j|q_1,\ldots q_r,j,q_{r+1},\ldots,q_s]_i = \frac{2}{\pi}\sin(\pi k_j \cdot k_{iq_1,\ldots q_r})\mathcal{S}[p_1,\ldots,p_s|q_1,\ldots q_s]_i \ , \tag{7.20}$$

where the starting point of this recursion is given by $\mathcal{S}[\emptyset|\emptyset] = 1$. Moreover, the normalization of the momentum kernel is such that after reinstating α' we recover the field theory momentum kernel $S[\cdot|\cdot]_i$ of [154, 173, 174] in the field theory limit $\alpha' \to 0$.

Equation (7.19) demonstrates how an n-point closed string amplitude factorises into two open string color ordered amplitudes on the disk. It can be interpreted as gluing two amplitudes $\mathcal{A}(\ldots)$ and $\tilde{\mathcal{A}}(\ldots)$ together by the kinematic factors in the KLT kernel $\mathcal{S}$ such that they form one closed string amplitude $\mathcal{A}_n$ similar as in figure 7.1. The amplitude (7.19) is independent of j, which reflects the arbitrariness in the number of contours that are either closed to the left or right. The j-independence originates from the fact that the open string amplitudes satisfy monodromy relations, see section 7.2. In total there are $(n-3)! \times (j-2)! \times (n-1-j)!$ terms in the sum (7.19). Therefore, the maximum number of terms is achieved for $j = 2$ or $j = n-1$ and given by $(n-3)! \times (n-3)!$. For this choice we recover (4.5):

$$\begin{aligned}\mathcal{A}_n = -\sum_{\sigma,\rho\in S_{n-3}} &\mathcal{A}(1,\sigma(2,3,\ldots,n-2),n,n-1)\\ &\times\mathcal{S}[\sigma(2,3,\ldots,n-2)|\rho(2,3,\ldots,n-2)]_1\tilde{\mathcal{A}}(1,\rho(2,3,\ldots,n-2),n-1,n) \ .\end{aligned} \tag{7.21}$$

On the other hand, in [34] they have chosen to close one half of the contours to the left and the other half to the right, which corresponds to the minimum value $(n-3)! \times (\lceil \frac{n}{2} \rceil - 2)! \times (\lfloor \frac{n}{2} \rfloor - 1)!$ of terms in the sum over permutations and is obtained for $j = \lceil \frac{n}{2} \rceil$.[4]

7.2 Monodromy relations for open string scattering amplitudes

In the previous section we have seen that any n-point closed string amplitude on the sphere decomposes in two n-point color ordered open string amplitudes on the disk, see (7.21). Since these are the components of the closed string amplitudes, in which we are interested, it is important to study them in more detail. In general, there are $(n-1)!$ inequivalent color stripped subamplitudes, but they are not independent and can be expressed via a basis with $(n-3)!$ elements [35, 36].

[4] The ceiling function is defined as $\lceil x \rceil = \min\{n \in \mathbb{N}|n \geq x\}$ similar to the floor function.

7.2.1 Relating massless open string subamplitudes

In string theory the worldsheet properties of open string amplitudes imply that under reflection the vertex operators have eigenvalues ± 1 such that the same holds for the amplitude itself [116]. Hence, by applying reflection and parity symmetries we find that the partial amplitudes satisfy

$$\mathcal{A}(1,2,\ldots,n) = (-1)^n \mathcal{A}(n,n-1\ldots,1) \ , \tag{7.22}$$

which reduces the number of independent amplitudes from $(n-1)!$ to $\frac{1}{2}(n-1)!$. Note that this is a symmetry that is also well known from field theory amplitudes and follows form studying the sum of Feynman diagrams [35].

Further, algebraic relations between subamplitudes can be derived by applying worldsheet methods. Therefore, let us consider the canonically ordered amplitude in (4.6) and change the vertex operator fixing to some arbitrary positions $(z_{i_1}, z_{i_2}, z_{i_3})$ such that we obtain

$$\mathbb{A}(1,2,\ldots,n) = \int_{-\infty<z_1<z_2<\ldots<z_n<\infty} \prod_{\substack{i=1\\ i\neq i_1,i_2,i_3}}^{N} \mathrm{d}z_i \prod_{k<l}^{n} |z_k - z_l|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\})\rangle \ , \tag{7.23}$$

where we introduced the notation $\mathbb{A}$ to distinguish the amplitude above from (4.1) with fixed positions (z_1, z_{n-1}, z_n). In the remainder of this section we want to find relations between $\mathbb{A}(1,2,\ldots,n)$ and permutations thereof following the presentation in [109, 35]. Note that (7.23) also satisfies (7.22).

We are only concerned with the Koba-Nielsen factor, since it contains the branch points of the amplitude and therefore prevents the amplitude from being an analytic function of the worldsheet variables z_i. For a specific integration variable z_1, where $i_1, i_2, i_3 \neq 1$, the corresponding terms in the KN-factor $\prod_{j=2}^{n} |z_{1j}|^{s_{ij}}$ can be related to a holomorphic function by using (7.14), i.e. [116]

$$\prod_{j=2}^{n} (z_{1j})^{s_{ij}} = \prod_{j=2}^{n} |z_{1j}|^{s_{ij}} \times \begin{cases} 1 & : \ -\infty < z_1 < z_2 \ , \\ e^{i\pi s_{12}} & : \ z_2 < z_1 < z_3 \ , \\ e^{i\pi s_{12}} e^{i\pi s_{13}} & : \ z_3 < z_1 < z_4 \ , \\ \vdots & \qquad \vdots \\ \prod_{j=1}^{n-1} e^{i\pi s_{1j}} & : \ z_{n-1} < z_1 < z_n \ , \\ 1 & : \ z_n < z_1 < \infty \ . \end{cases} \tag{7.24}$$

Then, we can analytically continue the z_1 integral to the entire complex plane: We integrate z_1 along the real axis followed by a semicircle of infinite radius in the upper half plane rather than over $]-\infty, z_2[$, which is depicted in figure 7.3. The semicircle in the upper half plane vanishes at infinity, since the integrand in (7.23) scales as $z_1^{-2h_1} \to 0$ for $|z_1| \to \infty$, where $h_1 = 1$ is the conformal weight of the integrated vertex operator $U_1(z_1)$ in the pure

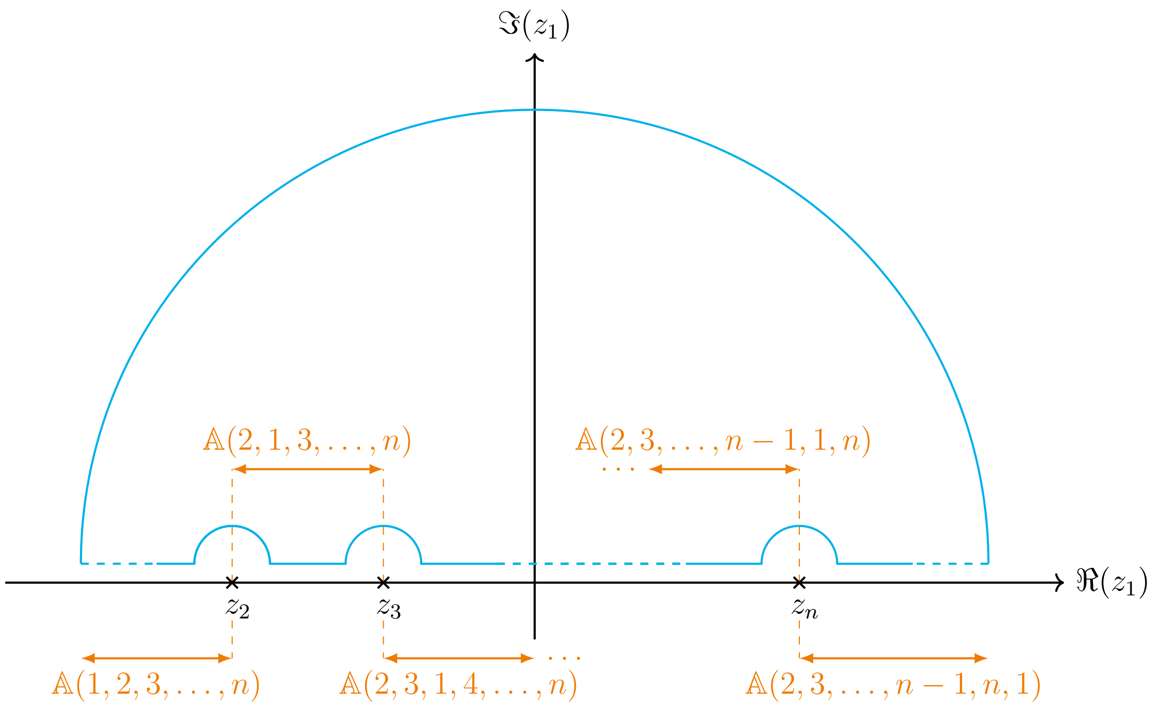

Figure 7.3: Contour integral in the complex z_1-plane.

spinor formalism. From Cauchy's theorem it follows that the integral over the holomorphic integrand in z_1 vanishes along the closed contour of z_1 in figure 7.3. Hence, we obtain [116]

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \mathrm{d}z_1 \int_{z_2<z_3<\ldots<z_n} \prod_{\substack{i=2\\ i\neq i_1,i_2,i_3}}^n \mathrm{d}z_i \prod_{j=2}^n (z_{1j})^{s_{1j}} \prod_{2\leq k<l}^n |z_{kl}|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\})\rangle \\
&= \int_{-\infty}^{z_2} \mathrm{d}z_1 \int_{z_2<z_3<\ldots<z_n} \prod_{\substack{i=2\\ i\neq i_1,i_2,i_3}}^n \mathrm{d}z_i \prod_{j=2}^n |z_{1j}|^{s_{1j}} \prod_{2\leq k<l}^n |z_{kl}|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\})\rangle \\
&\quad + \sum_{q=3}^n e^{i\pi(s_{12}+\ldots s_{1(q-1)})} \int_{z_{q-1}}^{z_q} \mathrm{d}z_1 \int_{z_2<z_3<\ldots<z_n} \prod_{\substack{i=2\\ i\neq i_1,i_2,i_3}}^n \mathrm{d}z_i \prod_{j=2}^n |z_{1j}|^{s_{1j}} \prod_{2\leq k<l}^n |z_{kl}|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\})\rangle \\
&\quad + \int_{z_n}^{\infty} \mathrm{d}z_1 \int_{z_2<z_3<\ldots<z_n} \prod_{\substack{i=2\\ i\neq i_1,i_2,i_3}}^n \mathrm{d}z_i \prod_{j=2}^n |z_{1j}|^{s_{1j}} \prod_{2\leq k<l}^n |z_{kl}|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\})\rangle \\
&= \mathbb{A}(1,2,\ldots,n) + \sum_{q=3}^n e^{i\pi(s_{12}+\ldots s_{1(q-1)})} \mathbb{A}(2,\ldots,q-1,1,q,\ldots n) + \mathbb{A}(2,\ldots,n,1)\ .
\end{aligned} \tag{7.25}$$

Here, we have divided the z_1-integration along $\mathbb{R}$ into smaller intervals $]-\infty,z_2[$, $]z_{q-1},z_q[$ and $]z_n,\infty[$ for $q=3,4,\ldots,n$. Moreover, we used (7.24) to relate the holomorphic factors $(z_{1j})^{s_{ij}}$ to $|z_{1j}|^{s_{1j}}$ in the Koba-Nielsen factor of the open string amplitude. This process required to introduce the phase factors $e^{i\pi(s_{12}+\ldots s_{1(p-1)})}$ for the individual subsets of $\mathbb{R}$.

Further, this can be interpreted as follows: Each time when we encircle another vertex operator position z_i for $i = 2, 3, \dots n$ while integrating z_1 along the real axis, we pick up a phase. The phase arises when we express the integrand of $\mathbb{A}(1, 2, \dots, n)$ in terms of the integrand of $\mathbb{A}(2, \dots, j, 1, j+1, \dots, n)$ by applying (7.14). For example the subamplitude $\mathbb{A}(1, 2, \dots, n)$ contains a factor $(z_1 - z_2)^{s_{12}}$ whereas $\mathbb{A}(2, 1, 3, \dots, n)$ has a factor $(z_2 - z_1)^{s_{12}}$, which can be related via (7.14) [175].

This discussion leads us to relations among open string subamplitudes [109]

$$\begin{aligned} 0 = &\mathbb{A}(1, 2, \dots, n-1, n) + e^{i\pi s_{12}} \mathbb{A}(2, 1, 3, \dots, n-1, n) \\ &+ e^{i\pi(s_{12}+s_{13})} \mathbb{A}(2, 3, 1, \dots, n-1, n) + \dots \\ &+ e^{i\pi(s_{12}+s_{13}+\dots+s_{1(n-1)})} \mathbb{A}(2, 3, \dots, n-1, 1, n) + \mathbb{A}(2, \dots, n-1, n, 1) \ , \end{aligned} \tag{7.26}$$

which are an analogue to the dual Ward identity in field theory. Compared to [35] we obtained a slightly different monodromy relation in (7.26), where we have $\mathbb{A}(2, \dots, n-1, n, 1)$ corresponding to $z_n < z_1 < \infty$, which is not equivalent to $\mathbb{A}(1, 2, \dots, n-1, n)$ corresponding to $-\infty < z_1 < z_2$, because there is no vertex operator with position $z_i \to \infty$. Moreover, the open string subamplitudes $\mathbb{A}(1, 2, \dots, n-1, n)$ and $\mathbb{A}(2, \dots, n-1, n, 1)$ appear with the same phase in the monodromy relation (7.26). Hence, we do not pick up a phase when jumping from $+\infty$ to $-\infty$, since there is no vertex operator localized at infinity. This suggests that they can be combined into one amplitude and in fact they are only parts of the same open string subamplitude,[5] which becomes clear in the next subsection if one unintegrated vertex operator i_p is fixed to $z_{i_p} \to \infty$ for $p \in \{1, 2, 3\}$ [109].

7.2.2 The minimal basis of subamplitudes

Before we use the relations obtained from the monodromy of the worldsheet to reduce the number of inequivalent partial amplitudes, we want to consider the case where one vertex operators is fixed to infinity, which can be obtained form (7.23) by performing a $PSL(2, \mathbb{R})$-transformation,[6] see section 8.3.2 for an example. For simplicity, we choose $z_n \to \infty$ such that the partial amplitudes can be written as

$$\mathcal{A}(1, 2, \dots, n) = \int_{-\infty<z_1<z_2<\dots<z_{n-1}<\infty} \prod_{\substack{i=1 \\ i \neq i_1, i_2}}^{n} \mathrm{d}z_i \prod_{k<l}^{n-1} |z_k - z_l|^{s_{kl}} \langle \mathcal{K}_n(\{z_p\}) \rangle \ , \tag{7.27}$$

where two other vertex operator positions (z_{i_1}, z_{i_2}) are fixed to $(0, 1)$. By taking $(z_{i_1}, z_{i_2}) = (z_1, z_{n-1})$ we would recover (4.6) in the $PSL(2, \mathbb{R})$-frame $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$.

[5]Hence, some $\mathbb{A}(\rho(1, 2, \dots, n))$ do not immediately correspond to open string partial amplitudes with some color ordering ρ of the n open strings, but they can be combined and rewritten such that they will be promoted to open string subamplitudes.

[6]This statement is true up to the subtlety that some of the subamplitudes (7.23) have to be combined to yield one open string partial amplitude (7.27).

By following the same steps as before we obtain the monodromy relations [35, 36]

$$
\begin{aligned}
0 = &\mathcal{A}(1,2,\dots,n-1,n) + e^{i\pi s_{12}}\mathcal{A}(2,1,3,\dots,n-1,n) \\
&+ e^{i\pi(s_{12}+s_{13})}\mathcal{A}(2,3,1,\dots,n-1,n) \\
&+ \dots + e^{i\pi(s_{12}+\dots+s_{1(n-1)})}\mathcal{A}(2,3,\dots,n-1,1,n) \ .
\end{aligned} \tag{7.28}
$$

If we now consider the amplitude $\mathcal{A}(2,3,\dots,n-1,n,1)$ this would correspond to $z_n = -\infty < z_1 < z_2$, because the boundary of the disk corresponds to the compactified real line, i.e. $-\infty$ and $+\infty$ describe the same point on the boundary of the worldsheet. Therefore, the subamplitudes $\mathcal{A}(1,2,3,\dots,n-1,n)$ and $\mathcal{A}(2,3,\dots,n-1,n,1)$ are the same, which was not the case for (7.23). Furthermore, the open string subamplitude $\mathcal{A}(1,2,3,\dots,n-1,n)$ is given by the combined partial amplitudes $\mathbb{A}(1,2,\dots,n)$ and $\mathbb{A}(2,\dots,n,1)$ after performing a $PSL(2,\mathbb{R})$-transformation, where the fixed vertex operators are (z_{i_1}, z_{i_2}, z_n) in both cases. From $\mathbb{A}(1,2,\dots,n)$ one could have guessed that there are $n!$ inequivalent amplitudes, but this identification shows that there are really only $(n-1)!$ independent amplitudes to begin with.

Due to the monodromy relations the dimension of the basis of independent subamplitudes is smaller than $\frac{1}{2}(n-1)!$ suggested by (7.22). To derive the minimal basis following the discussion in [116] we write the set of monodromy relations (7.28) in a more general way

$$
\mathcal{A}(1,\alpha_1,\dots,\alpha_r,n,\beta_1,\dots\beta_s) = (-1)^s \prod_{i<j}^{s} e^{i\pi s_{\beta_i\beta_j}} \sum_{\sigma\in \mathrm{OP}(\{\alpha\},\{\beta\})} \prod_{k=0}^{r}\prod_{l=1}^{s} e^{i\pi s_{\alpha_i\beta_j}}\mathcal{A}(1,\sigma,n) \tag{7.29}
$$

where $\alpha_0 = 1$ and a definition of $\mathrm{OP}(\{\alpha\},\{\beta\})$ is given below (6.14). After using (7.28) and (7.29) the only independent amplitudes left have the external states 1 and n at positions next to each other. The total number of these amplitudes is given by $(n-2)!$ due to the S_{n-2} permutations of the remaining states $2,3,\dots,n-1$. So far, we have neglected that the amplitudes are real $\mathcal{A}(1,2,\dots,n) \in \mathbb{R}$, which makes it possible to take only the real part of (7.29) to carry out the reduction to $(n-2)!$ amplitudes $\mathcal{A}(1,\sigma,n)$. Further, the imaginary parts of these relations can be used to find a simpler set of identities. They include one term less than (7.26) and are given by

$$
\begin{aligned}
0 = &\sin(\pi s_{12})\mathcal{A}(2,1,3,\dots,n-1,n) + \sin(\pi(s_{12}+s_{13}))\mathcal{A}(2,3,1,\dots,n-1,n) \\
&+ \dots + \sin(\pi(s_{12}+\dots+s_{1(n-1)}))\mathcal{A}(2,3,\dots,n-1,1,n)
\end{aligned} \tag{7.30}
$$

and relabellings thereof. With (7.30) we are able to write any subamplitude as a linear combination of $(n-3)!$ basis elements. Note that this number is identical to the dimension of the basis of generalized Gaussian hypergeometric functions, which can be used to characterize the open string n-point amplitude [176, 177, 178].

The CFT correlator $\langle \mathcal{K}_n(\{z_p\})\rangle$ is independent on any permutation of the external states in $\mathcal{A}(1,2,\dots,n)$ and can be evaluated before specifying the partial amplitude. Only the integration region is different for each subamplitude. Moreover, the branch cuts originate from the Koba-Nielsen factor and not from $\langle \mathcal{K}_n(\{z_p\}) \rangle$, which has only poles with integer powers in the worldsheet coordinates and therefore does not influence the analytic properties of the amplitude. Hence, the results in this section are universal for all amplitudes consisting of a correlator with these properties and a Koba-Nielsen factor similar (with the same branch cut structure) to the open string amplitudes considered in this section. Furthermore, the discussion in this section did not depend on the number of spacetime dimension or the amount of spacetime supersymmetry.

Chapter 8

Scattering three closed strings off a Dp-brane

The first quantum corrections to the effective action obtained in the limit $\alpha' \to 0$ from string theory can be captured among others by disk amplitudes. For example, they give rise to higher derivative gravitational corrections to the Dirac-Born-Infeld action [84] and it should be possible to infer disk level corrections to the Einstein-Hilbert term in four dimensions from graviton scattering with only external polarizations. These could originate from a disk level term like $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ when compactifying on a Calabi-Yau with non-vanishing Euler number and were conjectured in [179] to exist in the worldvolume theory of a D9-brane. Hence, it would be interesting to explicitly check for the existence of these terms in the scattering of closed strings off a Dp-brane. In this chapter, which is based on [109], we present the procedure for the computation of these amplitudes: As the scattering of two closed strings did not exhibit the desired gravitational corrections we continue by investigating the scattering of three closed strings on the disk. However, in chapter 11 we will show that also our results do not show any hints of such a disk level corrections to the Einstein-Hilbert term. Nevertheless, we build the foundation for future computations to investigate this further.

8.1 Boundary conditions on the disk

In chapter 7 we considered only closed string states that interact on the sphere. As was argued in section 7.1 on the sphere left- and right-movers can be treated independently, i.e. there is no interaction between the holomorphic and antiholomorphic sector. But the boundary of the disk imposes non-vanishing correlators between the left- and right-moving parts of the worldsheet fields: The holomorphic and antiholomorphic part of closed string vertex operators are not independent any more. In general, an operator $\mathcal{O}_{h,\bar{h}}(z,\bar{z}) =$

$O_h(z) \otimes \overline{O}_{\overline{h}}(\overline{z})$ with conformal weight $(h, \overline{h})$ at the boundary has to satisfy [180]

$$0 = \Big\langle B \Big| \Big[(\overline{z}')^h \overline{O}_h(\overline{z}') - z^h e^{i\pi h} O_h(z) \Big] \Big|_{z=\frac{1}{\overline{z}'}} , \tag{8.1}$$

where $\langle B|$ is the boundary state of the disk. For the computation of the three-point amplitude we can use the doubling trick, which according to (8.1) becomes [27]

$$\overline{O}_{\overline{h}}(\overline{z}) = \left(\frac{\partial z'}{\partial \overline{z}}\right)^{\overline{h}} O_{\overline{h}}(z') \quad \text{for } z' = \frac{1}{\overline{z}} . \tag{8.2}$$

to rewrite the right-moving part of the vertex operators in (7.2) and allow for a unified treatment of the holomorphic and antiholomorphic sector. Note that the $\mathbb{Z}_2$ identification $z' = \frac{1}{\overline{z}}$ can be used to obtain the disk as the quotient $S_2/\mathbb{Z}_2$ form the sphere. Moreover, this identification leaves the disk invariant. Furthermore, we can map the disk to the upper half plane $\mathbb{H}_+$ by using

$$w = i\frac{1-z}{1+z} , \tag{8.3}$$

where w are the worldsheet coordinates on the upper half of the complex plane. This conformal transformation maps the boundary of the disk onto the boundary of $\mathbb{H}_+$, which is the real line $\mathbb{R}$. Consequently, we map $z' = \frac{1}{\overline{z}} \mapsto w' = \overline{w}$ such that (8.2) on $\mathbb{H}_+$ becomes

$$\overline{O}_{\overline{h}}(\overline{w}) = \left(\frac{\partial w'}{\partial \overline{w}}\right)^{\overline{h}} O_{\overline{h}}(w') \quad \text{for } w' = \overline{w} . \tag{8.4}$$

Since we can express right-moving fields via the left-moving counterpart, the action of the boundary state $\langle B|$ on an operator $\mathcal{O}_{(h,\overline{h})}$ imposes an interaction between O_h and $\overline{O}_{\overline{h}}$. Moreover, applying the doubling trick extends the field O from the upper half plane to the entire complex plane.

The scattering amplitude of three closed strings is described by type IIB string theory in a flat ten dimensional spacetime, which contains a Dp-brane that is spanned in the $X_1 \times X_2 \times \ldots X_p$ dimensions. Therefore, at the boundary of $\mathbb{H}_+$ the first $p+1$ components of the worldsheet fields have to satisfy Neumann boundary conditions and the remaining $9-p$ components Dirichlet boundary conditions. By using the doubling trick (8.4) we can replace the antiholomorphic spacetime vectors and spinors by

$$\begin{aligned} &\text{vectors: } \overline{X}^m(\overline{z}) = D^m{}_n X^n(\overline{z}) , \\ &\text{spinors: } \overline{\Psi}^\alpha(\overline{z}) = M^\alpha{}_\beta \Psi^\beta(\overline{z}) \quad \text{ or } \quad \overline{\Psi}_\alpha(\overline{z}) = N_\alpha{}^\beta \Psi_\beta(\overline{z}) , \end{aligned} \tag{8.5}$$

where $\Psi^\alpha \in \{\theta^\alpha, \lambda^\alpha\}$ and $\Psi_\alpha \in \{p_\alpha, w_\alpha\}$ and D, M and N are constant matrices, which account for the Neumann or Dirichlet boundary conditions. Moreover, as stated above these fields are now defined on the entire complex plane. This was first derived in [88] in the pure spinor formalism and the corresponding discussion for the RNS formalism can be found in [92]. In the remaining section we will derive the consequences of (8.5) following the lines of [88].

The matrix D^{mn} describing the boundary condition for spacetime vectors is the same as in the RNS formalism [91, 92]. In a flat background D^{mn} is a diagonal matrix with diagonal components

$$D^{mn} = \begin{cases} \eta^{mn} & \text{for } m,n \in \{0,1,\ldots,p\} \ , \\ -\eta^{mn} & \text{for } m,n \in \{p+1,\ldots,9\} \ , \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{8.6}$$

In the small coupling regime the D-brane is infinitely heavy and is capable of absorbing an arbitrarily large amount of momentum in the directions $X_{p+1}, \ldots X_9$ transverse to the D-brane. Hence, in this regime momentum is only conserved along the D-brane. We can introduce a parallel and transverse momentum

$$k_{i\|} = \frac{1}{2}(k_i + D{\cdot}k_i) \ , \qquad k_{i\perp} = \frac{1}{2}(k_i - D{\cdot}k_i) \ , \tag{8.7}$$

for $i = 1, 2, \ldots, n$ such that only the momenta $k_{i\|}$ are conserved

$$\sum_{i=1}^{n} k_{i\|} = 0 \ , \tag{8.8}$$

where n is the number of external closed string states. Moreover, we are considering massless states so that the momenta satisfy $k_i^2 = 0$ and all k_i are orthogonal to the corresponding polarization tensors $k_i^m \epsilon^i_{mn} = \epsilon^i_{mn} k_i^n = 0$.

For only holomorphic fields we use the correlators on the upper half plane

$$\begin{aligned} \langle X^m(z) X^n(w) \rangle &= -\eta^{mn} \ln(z-w) \ , \\ \langle p_\alpha(z) \theta^\beta(w) \rangle &= \frac{\delta_\alpha^\beta}{z-w} \ , \\ \langle w_\alpha(z) \lambda^\beta(w) \rangle &= \frac{\delta_\alpha^\beta}{z-w} \end{aligned} \tag{8.9}$$

derived from the OPEs in (3.76) and the antiholomorphic part is analogous. Following from the boundary conditions of the worldsheet fields for the interaction between the two sectors we need the correlators

$$\begin{aligned} &\langle X^m(z) \overline{X}^n(\overline{w}) \rangle = -D^{mn} \ln(z-\overline{w}) \ , \\ &\langle p_\alpha(z) \overline{\theta}^\beta(\overline{w}) \rangle = \frac{M^\beta{}_\alpha}{z-\overline{w}} \ , \qquad \langle \overline{p}_\alpha(\overline{z}) \theta^\beta(w) \rangle = \frac{N_\alpha{}^\beta}{\overline{z}-w} \ , \\ &\langle w_\alpha(z) \overline{\lambda}^\beta(\overline{w}) \rangle = \frac{M^\beta{}_\alpha}{z-\overline{w}} \ , \qquad \langle \overline{w}_\alpha(\overline{z}) \lambda^\beta(w) \rangle = \frac{N_\alpha{}^\beta}{\overline{z}-w} \ . \end{aligned} \tag{8.10}$$

The matrices M and N are different from the RNS formalism, because we use a different spinor representation in the pure spinor formalism. According to [88] M and N are not independent. From the two versions of the OPE

$$\begin{aligned}\overline{p}_\alpha(\overline{z})\overline{\theta}^\beta(\overline{w}) &= N_\alpha{}^\gamma p_\gamma(\overline{z})M^\beta{}_\delta\theta^\delta(\overline{w}) = \frac{N_\alpha{}^\gamma M^\beta{}_\gamma}{\overline{z}-\overline{w}} \ , \\ \overline{p}_\alpha(\overline{z})\overline{\theta}^\beta(\overline{w}) &= \frac{\delta_\alpha^\beta}{\overline{z}-\overline{w}}\end{aligned} \tag{8.11}$$

it follows that $N_\alpha{}^\gamma M^\beta{}_\gamma = \delta_\alpha^\beta$ or $N = (M^{\mathrm{T}})^{-1}$. Moreover we would expect that also the supersymmetric momentum $\overline{\Pi}^m$ and GS constraint $\overline{d}_\alpha$ satisfy (8.5). Explicitly, we find[1]

$$\overline{\Pi}^m(\overline{z}) = \Big(D^m{}_n\overline{\partial}X + \frac{1}{2}(\theta M^{\mathrm{T}}\gamma^m M\overline{\partial}\theta)\Big)(\overline{z}) = D^m{}_n\Pi^n(\overline{z}) \ , \tag{8.12}$$

which holds if

$$M^\gamma{}_\alpha\gamma^m_{\gamma\delta}M^\delta{}_\beta = D^m{}_n\gamma^m_{\alpha\beta} \ , \quad \text{i.e.} \quad (M^{\mathrm{T}}\gamma^m M)_{\alpha\beta} = D^m{}_n\gamma^n_{\alpha\beta} \tag{8.13}$$

and we get a similar relation for $N_\alpha{}^\gamma\gamma^m_{\gamma\delta}N_\beta{}^\delta = D^m{}_n\gamma^m_{\alpha\beta}$. Furthermore, the fermionic part of the Lorentz current should respect these boundary conditions as well. Hence, by demanding for the ghost contribution $\overline{N}^{mn}(\overline{z}) = D^m{}_kD^m{}_lN^{kl}(\overline{z})$ we find

$$N_\gamma{}^\alpha(\gamma^m)^{\gamma\delta}N_\delta{}^\beta = D^m{}_n(\gamma^n)^{\alpha\beta} \ , \quad \text{i.e.} \quad (N^{\mathrm{T}}\gamma^m N)^{\alpha\beta} = D^m{}_n(\gamma^n)^{\alpha\beta} \ , \tag{8.14}$$

because otherwise

$$\begin{aligned}\overline{N}^{mn}(\overline{z}) &= \frac{1}{2}(\overline{w}\gamma^{mn}\overline{\lambda}) = \frac{1}{2}(wN^{\mathrm{T}}\gamma^{mn}M\lambda) \\ &= \frac{1}{4}(wN^{\mathrm{T}}\gamma^{[m}NM^{\mathrm{T}}\gamma^{n]}M\lambda) = \frac{1}{4}D^m{}_kD^m{}_lN^{kl}(w\gamma^{[k}\gamma^{l]}\lambda) = \\ &= D^m{}_kD^m{}_lN^{kl}(\overline{z})\end{aligned} \tag{8.15}$$

would not be satisfied. Again, we also can find the corresponding equation for M, which is given by

$$M^\alpha{}_\gamma(\gamma^m)^{\gamma\delta}M^{\beta^\delta} = D^m{}_n(\gamma^n)^{\alpha\beta} \ , \quad \text{i.e.} \quad (M\gamma^m M^{\mathrm{T}})^{\alpha\beta} = D^m{}_n(\gamma^n)^{\alpha\beta} \ . \tag{8.16}$$

Note that these relations between M and N are sufficient such that the OPEs (3.76) can be used with the doubling trick and we will not find any further conditions.

Instead of working with the correlators (8.10) we can use the doubling trick and replace the antiholomorphic fields in the right-moving superfields $\overline{K}[\overline{e},\overline{\chi},k](\overline{X},\theta)$ by X and θ.

[1] In the contraction of fermions like $(\overline{\theta}\gamma^m\overline{\partial\theta})$ the left spinor has to be transposed, which is left implicit in the notation of the pure spinor formalism.

Then, we only need the correlators in (8.9), where the prior antiholomorphic fields are still at positions $\overline{z}$. Thus, the $\overline{\theta}$-expansion in (2.32) for the gluonic part of $\overline{A}_\alpha(\overline{X},\overline{\theta})$ is given by

$$\begin{aligned}
&\overline{A}_\alpha[\overline{e},k](\overline{X},\overline{\theta}) = \overline{A}_\alpha[\overline{e},k](D{\cdot}X, M\theta) = \\
&= e^{ik\cdot D\cdot X}\Big\{\overline{e}_m(\gamma^m M\theta)_\alpha - \frac{1}{16}(\gamma_p M\theta)_\alpha(\theta M^{\mathrm{T}}\gamma^{mnp}M\theta)ik_{[m}\overline{e}_{n]}\Big\} \\
&= e^{ik\cdot D\cdot X}\Big\{\overline{e}_m((M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma^m M\theta)_\alpha - \frac{1}{16}((M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma_p M\theta)_\alpha(\theta M^{\mathrm{T}}\gamma^{mnp}M\theta)ik_{[m}\overline{e}_{n]}\Big\} \\
&= e^{ik\cdot D\cdot X}((M^{\mathrm{T}})^{-1})_\alpha{}^\beta\Big\{(D{\cdot}\overline{e})_m(\gamma^m\theta)_\beta - \frac{1}{16}(\gamma_p\theta)_\beta(\theta\gamma^{mnp}\theta)i(D{\cdot}k)_{[m}(D{\cdot}\overline{e})_{n]}\Big\} \\
&= ((M^{\mathrm{T}})^{-1})_\alpha{}^\beta A_\beta[D{\cdot}\overline{e}, D{\cdot}k](X,\theta)\ ,
\end{aligned} \tag{8.17}$$

and the gluino components of $\overline{A}_\alpha(\overline{X},\overline{\theta})$ can be written as

$$\begin{aligned}
&\overline{A}_\alpha[\overline{\chi},k](\overline{X},\overline{\theta}) = \overline{A}_\alpha[\overline{\chi},k](D{\cdot}X, M\theta) = \\
&= e^{ik\cdot D\cdot X}\Big\{-\frac{1}{3}(\overline{\chi}\gamma_m M\theta)(\gamma^m M\theta)_\alpha + \frac{1}{60}(\gamma_m M\theta)_\alpha ik_n(\overline{\chi}\gamma_p M\theta)(\theta M^{\mathrm{T}}\gamma^{mnp}M\theta)\Big\} \\
&= e^{ik\cdot D\cdot X}\Big\{-\frac{1}{3}(\overline{\chi}(M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma_m M\theta)((M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma^m M\theta)_\alpha \\
&\quad + \frac{1}{60}((M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma_m M\theta)_\alpha ik_n(\overline{\chi}(M^{\mathrm{T}})^{-1}M^{\mathrm{T}}\gamma_p M\theta)(\theta M^{\mathrm{T}}\gamma^{mnp}M\theta)\Big\} \\
&= e^{ik\cdot D\cdot X}((M^{\mathrm{T}})^{-1})_\alpha{}^\beta\Big\{-\frac{1}{3}(\overline{\chi}(M^{\mathrm{T}})^{-1}\gamma_m\theta)(\gamma^m\theta)_\beta \\
&\quad + \frac{1}{60}(\gamma_m\theta)_\beta i(D{\cdot}k)_n(\overline{\chi}(M^{\mathrm{T}})^{-1}\gamma_p\theta)(\theta\gamma^{mnp}\theta)\Big\} \\
&= ((M^{\mathrm{T}})^{-1})_\alpha{}^\beta A_\beta[M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)
\end{aligned} \tag{8.18}$$

where we have used $(M^{\mathrm{T}})^{-1}M^{\mathrm{T}} = 1$ and displayed only the first terms in the θ-expansion, but this holds also for the higher order terms. For the other superfields we can perform a similar calculation and find analogously

$$\begin{aligned}
\overline{A}_m[\overline{e},\overline{\chi},k](\overline{X},\overline{\theta}) &= D_m{}^n A_n[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)\ , \\
\overline{W}_\alpha[\overline{e},\overline{\chi},k](\overline{X},\overline{\theta}) &= M_\alpha{}^\beta W_\beta[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)\ , \\
\overline{F}_{mn}[\overline{e},\overline{\chi},k](\overline{X},\overline{\theta}) &= D_m{}^a D_n{}^b F_{ab}[D{\cdot}\overline{e},\overline{\chi}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)\ .
\end{aligned} \tag{8.19}$$

Replacing also the remaining antiholomorphic worldsheet fields according to (8.5)

$$\overline{\Pi}^m = D^m{}_n\Pi^n\ , \qquad \overline{d}_\alpha = (M^{-1})_\alpha{}^\beta d_\beta\ , \qquad \overline{\lambda}^\alpha = M^\alpha{}_\beta\lambda^\beta\ , \qquad \overline{N}^{mn} = D^m{}_a D^n{}_b N^{ab} \tag{8.20}$$

in the right-moving part of the vertex operators in (7.2) we get after applying the doubling trick

$$\overline{V}(\overline{z}) \to V(\overline{z}) = \Big(\lambda^\alpha A_\alpha[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)\Big)(\overline{z})\ ,$$

$$\overline{U}(\overline{z}) \to U(\overline{z}) = \Big(\overline{\partial}\theta^{\alpha} A_{\alpha}[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta) + \Pi^{m} A_{m}[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta) \\ + d_{\alpha} W^{\alpha}[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta) + \frac{1}{2} N^{mn}\mathcal{F}_{mn}[D{\cdot}\overline{e}, M^{-1}\overline{\chi}, D{\cdot}k](X,\theta)\Big)(\overline{z}) \ . \tag{8.21}$$

To sum up the doubling trick boils down to replacing each antiholomorphic superfield by its holomorphic counterpart and at the same time multiplying gluonic polarisation vector and momentum by D and the polarization spinors with M^{-1} to account for the boundary conditions.

To simplify the notation we omit the dependence of the superfields on polarisation and momentum, but introduce the following notation instead

$$K_{\overline{\imath}}(\overline{z}) \equiv \overline{K}_i[\overline{e}_i, \overline{\chi}_i, k_i](\overline{X}(\overline{z}_i), \overline{\theta}(\overline{z}_i)) = K_i[D{\cdot}\overline{e}_i, M^{-1}\overline{\chi}_i, D{\cdot}k_i](X(\overline{z}_i), \theta(\overline{z}_i)) \tag{8.22}$$

for any superfield K_i of an external string state i. We also use this notation for vertex operators, composite superfields, Berends-Giele currents and to label string states in open string subamplitudes, see for example section 8.3 and appendix C. In general, an overlined label indicates that we employed the doubling trick (8.22) for the antiholomorphic part of the corresponding state.

8.2 The disk correlator of closed strings

The prescription to compute open superstring amplitudes on the disk was already presented in chapter 4 and the results derived using (4.1), see for example [62, 158, 160], are well tested. Therefore, this is also the case for closed strings, as the tree-level scattering amplitude on the sphere can be computed via KLT relations [34], which were presented in section 7.1. The prescription for both cases is very straight forward, since the worldsheets do not have moduli and their CKVs of $PSL(2,\mathbb{C})$ and $PSL(2,\mathbb{R})$ can be used to fix three closed or open string vertex operator positions, respectively. But for closed strings on the disk the conformal Killing group $PSL(2,\mathbb{R})$ of the disk does not allow for this possibility. Instead, the three real CKVs of the disk allow to gauge fix one real position each. Hence, we can only gauge fix the position of one and a half vertex operators corresponding to three real coordinates of the vertex operator positions $z_i = x_i + iy_i$ on the disk. The vertex operator with half fixed position has to be a product of an unintegrated and integrated vertex operator, e.g. $V_i \otimes \overline{U}_i$ or $U_i \otimes \overline{V}_i$, which was derived in [88, 117] and also discussed in [101, 181]. Compared to the sphere this is different, i.e. the vertex operators in (7.2) are of the form $V_i \otimes \overline{V}_i$ or $U_i \otimes \overline{U}_i$.

For the scattering of n-closed strings on the disk we can take the first two states and place their vertex operator insertions on the disk at positions $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, we fix the real parameters of these insertions to the positions $x_1 = 0, y_1 = 1$ and

$x_2 = 0$ while keeping the integration over $y_2 = y$. As was argued in [88, 117] this leads to the prescriptions for n-closed strings on the disk

$$\mathcal{A}_n^{D_2} = 2ig_c^n T_p \int_0^1 \mathrm{d}y \left\langle\!\!\left\langle V_1(i)\overline{V}_1(-i)V_2(iy)\overline{U}_2(-iy) \prod_{j=3}^{n} \int_{\mathbb{H}_+} \mathrm{d}^2 z_j \, U_j(z_j)\overline{U}_j(\overline{z}_j) \right\rangle\!\!\right\rangle , \qquad (8.23)$$

where g_c is the closed string coupling constant and T_p the tension of the Dp-brane. The $PSL(2,\mathbb{R})$ frame above restricts the integration over z_2 and $\overline{z}_2$ to the purely imaginary axis. Here, we integrate y form 0 to 1 and not over the entire real line, because we have to restrict the y-integration to the moduli space of a punctured disk. To determine the moduli space we consider the disk with punctures at the vertex operator positions at $z_1 = x_1 + iy_i$ and $z_2 = x_2 + iy_2$, where $x \in]-\infty, \infty[$ and $y_i \in [0, \infty[$ are the real and imaginary parts of z_i respectively. If the two points z_1 and z_2 in the upper half plane are different, we can define two $PSL(2,\mathbb{R})$-transformations

$$\begin{aligned} &f_\pm(z; z_1, z_2) = \\ &= \frac{(x_2 - x_1)y_1 z + ((x_1 - x_2)x_2 + (y_1 y_\pm - y_2)y_2)y_1}{((x_1 - x_2)^2 + (y_2 - y_1 y_\pm)y_2)z - x_1^3 + 2x_1^2 x_2 + x_2 y_1^2 - x_1(x_2^2 + y_1^2 + y_2^2 - y_1 y_2 y_\pm)} \end{aligned} \qquad (8.24)$$

with

$$y_\pm = \frac{((x_1 - x_2)^2 + y_1^2 + y_2^2) \pm \sqrt{4(x_1 - x_2)^2 y_1^2 + ((x_1 - x_2)^2 - y_1^2 + y_2^2)^2}}{2y_1 y_2} , \qquad (8.25)$$

which map z_1 to i and z_2 to $iy_\pm$. Moreover, we notice that $y_- \in [0, 1]$ and $y_+ \in [1, \infty[$ and for the spacial case where $x_1 = x_2$ and for some particular values for y_1 and y_2 the entire range of the intervals $[0, 1[$ and $]1, \infty[$ are covered by y_- and y_+, respectively. We excluded the limiting value 1 in the intervals, because it would require $x_1 = x_2$ and $y_1 = y_2$ simultaneously, which is forbidden by $z_1 \neq z_2$. Hence, we find that the moduli space of the disk with two closed string punctures is given by $[0, 1[$ or equivalently $]1, \infty[$. However, we still need to check whether a disk with punctures at i and iy can be mapped to another disk with punctures at i and iy', where $y \neq y'$. For concreteness, we focus on the case $y, y' \in [0, 1[$. If there exists such a $PSL(2,\mathbb{R})$-transformation, the two variables y and y' would describe the same punctured disk and therefore the moduli space would be smaller than $[0, 1[$, which is not the case. By performing a $PSL(2,\mathbb{R})$-transformation two points i and iy with $y \in [0, 1[$ can only be mapped to two points i and iy' with $y' \in [0, 1[$ if they are already the same point $y = y'$ on the disk.

8.3 Three closed strings as six opens strings

According to the prescription in (8.23) the scattering of three closed strings off a Dp-brane is given by

$$\begin{aligned}\mathcal{A}_3^{D_2} &= 2ig_c^3 T_p \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z \,\langle\!\langle V_1(i)\overline{V}_1(-i)V_2(iy)\overline{U}_2(-iy)U_3(z)\overline{U}_3(\overline{z})\rangle\!\rangle \\ &= 2ig_c^3 T_p \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z \,\langle\!\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}(-iy)U_3(z)U_{\overline{3}}(\overline{z})\rangle\!\rangle \ . \end{aligned} \quad (8.26)$$

where we have employed the doubling trick (8.22). The computation of the correlator can be done following the steps in [62, 158] and is explicitly carried out in appendix C. When performing the contractions of the $h = 1$ fields and the plane wave factors the amplitude becomes schematically

$$\begin{aligned}\mathcal{A}_3^{D_2} = \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z \, 2^{s_{1\overline{1}}} |2y|^{s_{2\overline{2}}} |1-y|^{2s_{12}} |1+y|^{2s_{1\overline{2}}} |i-z|^{2s_{13}} |i+z|^{2s_{1\overline{3}}} \\ \times |iy-z|^{2s_{23}} |iy+z|^{2s_{2\overline{3}}} |z-\overline{z}|^{s_{3\overline{3}}} \langle \mathcal{K}(y,z,\overline{z})\rangle \ . \end{aligned} \quad (8.27)$$

The correlator of the amplitude (8.26) looks similar to (4.1) for $n = 6$ and also the computation in appendix C suggests that (8.26) can be connected to the scattering of six open strings on the disk when using the identification

$$\overline{1} \leftrightarrow 1 \ , \qquad \overline{2} \leftrightarrow 2 \ , \qquad 3 \leftrightarrow 3 \ , \qquad \overline{3} \leftrightarrow 4 \ , \qquad 2 \leftrightarrow 5 \ , \qquad 1 \leftrightarrow 6 \quad (8.28)$$

between closed and open strings. But the complex integral over the upper half plane does not correspond to an open string integral, which are defined as integrals over parts of the real line. Thus, in this section we want to use the method that was proposed in [34] and explicitly applied in [104] to write the closed integral over $\mathbb{H}_+$ as open string integrals that arise from the color ordered scattering of six open strings on the disk.

For the amplitude in (8.26) we already start with one completely position fixed vertex operator and another vertex operator whose worldsheet position is integrated over parts of the real line. Therefore, we have to split only the integration over z and $\overline{z}$ of the third vertex operator into two real integrals by applying the same method as for the derivation of the KLT relations in section 7.1. Afterwards we will use monodromy relations of section 7.2 to simplify the result similar as in [35]. Together with the calculation in appendix C for the correlator this allows us to identify the scattering of three closed strings on the disk as open string partial amplitudes with a certain colour ordering of the open string vertex operators.

We want to remark that the discussion in [35] was performed on the double cover, which simplifies the computation and has some physical implications as discussed below such that both computations actually described different scattering processes.

8.3.1 Analytic continuation and monodromy relations

The analytic continuation of (8.26) can be performed analogously to the derivation of the KLT relations in section 7.1. We start by writing the complex integral over the upper half plane as two integrals over (parts of) the real line. Therefore, we split the integration variable z in real and imaginary part $z = z_1 + iz_2$ such that the integrand becomes an analytic function in z_1 except for the branch points at $\pm i(1-z_2), \pm i(1+z_2), \pm i(y-z_2)$ and $\pm i(y+z_2)$. Next, we analytically continue the z_1-integration to the complex plane by deforming the integration contour for z_1 from the real axis at $\Im(z_1) = 0$ to the purely imaginary axis $\Re(z_1) = 0$, which is depicted in figure 8.1. As discussed in section 7.2 and

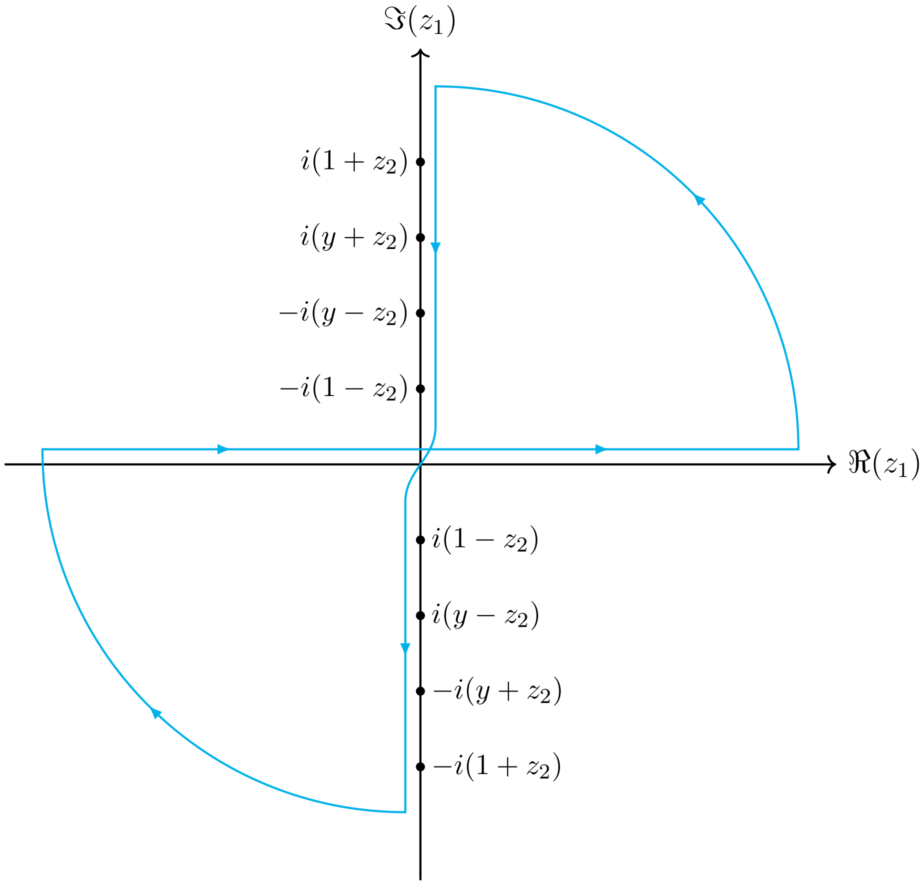

Figure 8.1: Branch point structure and contour deformation in the complex z_1-plane for $z_2 > 1$.

similar as in figure 7.3 both arcs vanish for $|z| \to \infty$. After the contour deformation the amplitude becomes

$$\mathcal{A}_3^{D_2} =$$

$$= -2ig_c^3 T_p \int_0^1 \mathrm{d}y \int_{-i\infty}^{i\infty} \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \, \langle\!\langle V_1(i)V_{\bar{1}}(-i)V_2(iy)U_{\bar{2}}(-iy)U_3(z_1+iz_2)U_{\bar{3}}(z_1-iz_2)\rangle\!\rangle$$
$$= 2g_c^3 T_p \int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \, \langle\!\langle V_1(i)V_{\bar{1}}(-i)V_2(iy)U_{\bar{2}}(-iy)U_3(i(z_1+z_2))U_{\bar{3}}(i(z_1-z_2))\rangle\!\rangle \,, \tag{8.29}$$

where we used (7.4), but the imaginary part $z_2 \in [0, \infty[$, because $z \in \mathbb{H}_+$. Then, we can define the real variables

$$\xi = z_1 + z_2 \,, \qquad \eta = z_1 - z_2 \,, \tag{8.30}$$

which are constrained by $\xi - \eta \geq 0$ to make sure that we are still integrating over the upper half plane, i.e. ξ and η preserve $z_2 \geq 0$. After performing the change of variables $(z_1, z_2) \to (\xi, \eta)$ in the integral we obtain

$$\mathcal{A}_3^{D_2} = g_c^3 T_p \int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \Pi(y,\xi,\eta) \langle\!\langle V_1(i)V_{\bar{1}}(-i)V_2(iy)U_{\bar{2}}(-iy)U_3(i\xi)U_{\bar{3}}(i\eta)\rangle\!\rangle \,, \tag{8.31}$$

where we included the Jacobian $\left|\frac{\partial(z_1,z_2)}{\partial\xi,\partial\eta}\right| = \frac{1}{2}$ of the transformation.

We can pull out the factor of i in all of the vertex operators in (8.31), because the correlator is invariant under rotations and dilatations, i.e. the transformations that are generated by L_0, see (B.2) and [130]. From (B.2) it follows that each conformal primary of dimension h satisfies $\phi(az) = a^h \phi(z)$. Since the unintegrated and integrated vertex operators have conformal dimension $h = 0$ and $h = 1$ we find

$$V(az) = V(z) \,, \qquad U(az) = aU(z) \tag{8.32}$$

and a similar scaling relation for the antiholomorphic parts such that using conformal invariance for $a = i$ the amplitude becomes

$$\mathcal{A}_3^{D_2} = ig_c^3 T_p \int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \Pi(y,\xi,\eta) \langle\!\langle V_1(1)V_{\bar{1}}(-1)V_2(y)U_{\bar{2}}(-y)U_3(\xi)U_{\bar{3}}(\eta)\rangle\!\rangle \,. \tag{8.33}$$

Already in (8.31) we have introduce the monodromy phase $\Pi(y, \xi, \eta)$, which is necessary to split the integration over the upper half plane into the integration of two real variables ξ and η while avoiding branch cuts. More details on how to derive this phase can be found in appendix D or [35, 104]. Including this phase the amplitude becomes a holomorphic and well defined function in ξ and η, because it accounts for the correct branch of the integrand. In addition, the phase factor is independent on the kinematical structure of the correlator and depends only on the worldsheet coordinates. Explicitly, the monodromy phase is given by

$$\Pi(y,\xi,\eta) = e^{i\pi s_{13}\Theta(-(1-\xi)(1+\eta))} e^{i\pi s_{1\bar{3}}\Theta(-(1+\xi)(1-\eta))} e^{i\pi s_{23}\Theta(-(y-\xi)(y+\eta))}$$
$$\times e^{i\pi s_{2\bar{3}}\Theta(-(y+\xi)(y-\eta))} e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))} \,, \tag{8.34}$$

where Θ is the Heaviside step function. Because the phase factor depends on the worldsheet fields via Heaviside step functions, we can split the (ξ,η)-integration into smaller integration regions. Then, for each integration region the phase becomes independent of the particular value of the worldsheet coordinates, but depends on the ordering of ξ and η relative to the other vertex operator insertions. The kinematic invariants in (8.34) are defined as

$$s_{ij}=\frac{1}{2}(k_i+k_j)^2=k_i{\cdot}k_j\ ,\qquad s_{i\bar{j}}=\frac{1}{2}(k_i+D{\cdot}k_j)^2=k_i{\cdot}D{\cdot}k_j\ ,\tag{8.35}$$

which are not independent and can be related via momentum conservation:

$$\begin{aligned}s_{1\bar{1}}&=-s_{12}-s_{1\bar{2}}-s_{13}-s_{1\bar{3}}\ ,\\ s_{2\bar{2}}&=-s_{12}-s_{1\bar{2}}-s_{23}-s_{2\bar{3}}\ ,\\ s_{3\bar{3}}&=-s_{13}-s_{1\bar{3}}-s_{23}-s_{2\bar{3}}\ .\end{aligned}\tag{8.36}$$

Hence, there are six independent Mandelstam variables for the scattering of three closed strings off a Dp-brane [35].

Compared to appendix D we have added $e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))}$ for completeness in (8.34). Even though, we are integrating over $\eta<\xi$ such that the contributions from $e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))}=1$, we will use monodromy relations (7.26) later and thereby encounter integration regions with $\eta>\xi$ such that $e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))}$ is necessary to get the correct monodromy phase.

We evaluate the phase factor by performing the (ξ,η)-integration over the Θ-functions and thereby split the integral $\eta<\xi$ into smaller integration regions. As argued above the monodromy phase becomes constant and therefore independent of the worldsheet coordinates in each integration patch. In the end, the integral in (8.33) is divided into 15 smaller regions in the (ξ,η)-plane, which are listed in table 8.1.

The amplitude in (8.33) together with the integration regions in table 8.1 are in correspondence with the open string subamplitudes (7.23) of section 7.2.[2] In (8.33) we have singled out the integration over y from 0 to 1 and chosen the $PSL(2,\mathbb{R})$-frame, where $z_{i_1}\equiv\overline{z}_1=-1, z_{i_2}\equiv z_1=1$ and $z_{i_3}\equiv z_2=y$. Taking all the phases and the according integration regions in table 8.1 into account and writing them in terms of the partial amplitudes $\mathbb{A}$ the amplitude (8.33) becomes[3]

$$\begin{aligned}\mathcal{A}_3^{D_2}=&\,e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}\mathbb{A}(\overline{3},3,\overline{1},\overline{2},2,1)+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}\mathbb{A}(\overline{3},\overline{1},3,\overline{2},2,1)\\ &+e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}\mathbb{A}(\overline{1},\overline{3},3,\overline{2},2,1)+e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{3},\overline{1},\overline{2},3,2,1)+e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{3},\overline{2},3,2,1)\end{aligned}$$

[2] One might be concerned that the closed string amplitude does not match the open string subamplitudes (7.23) perfectly, e.g. the closed string amplitude is purely imaginary, whereas the open string amplitudes are real, but the discussion in section 7.2 holds for any amplitude with the same branch cut structure as (7.23). Since the amplitude (8.33) fulfils precisely this requirement, we can utilize the results of section 7.2.

[3] As stated at the end of section 7.2 some $\mathbb{A}$ are only part of open string subamplitudes, which is the case if $\mathbb{A}$ starts or ends with 3 or $\overline{3}$, e.g. $\mathbb{A}(\overline{3},\dots)$ or $\mathbb{A}(\dots,3)$.

	$\eta < \xi$	
$\xi < -1$	$e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$	

	$\eta < -1$	$-1 < \eta < \xi$	
$-1 < \xi < -y$	$e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$	$e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$	

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < \xi$	
$-y < \xi < y$	$e^{i\pi s_{13}}e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	1	

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < y$	$y < \eta < \xi$	
$y < \xi < 1$	$e^{i\pi s_{13}}$	1	$e^{i\pi s_{23}}$	$e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$	

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < y$	$y < \eta < 1$	$1 < \eta < \xi$
$1 < \xi$	1	$e^{i\pi s_{13}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$	$e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}$

Table 8.1: $\Pi(y, \xi, \eta)$ for each integration region in the (ξ, η)-plane.

$$\begin{aligned}
&+\mathbb{A}(\overline{1},\overline{2},\overline{3},3,2,1) + e^{i\pi s_{13}}\mathbb{A}(\overline{3},\overline{1},\overline{2},2,3,1) + \mathbb{A}(\overline{1},\overline{3},\overline{2},2,3,1)\\
&+e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{2},\overline{3},2,3,1) + e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{2},2,\overline{3},3,1) + \mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3)\\
&+e^{i\pi s_{13}}\mathbb{A}(\overline{1},\overline{3},\overline{2},2,1,3) + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{2},\overline{3},2,1,3)\\
&+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{2},2,\overline{3},1,3) + e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{2},2,1,\overline{3},3) \ , \qquad (8.37)
\end{aligned}$$

where we made use of (8.22) and also the comment below (8.22). We know from the discussion in section 7.2 that the subamplitudes are not independent, but related via the monodromy relations (7.26). Using the identification (8.28) we obtain permutations of (7.26)

$$\begin{aligned}
0 =\ &\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3) + e^{i\pi s_{13}}\mathbb{A}(\overline{1},\overline{3},\overline{2},2,1,3) + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{2},\overline{3},2,1,3)\\
&+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{2},2,\overline{3},1,3) + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{13}}\mathbb{A}(\overline{1},\overline{2},2,1,\overline{3},3)\\
&+\mathbb{A}(\overline{1},\overline{2},2,1,3,\overline{3}) \ ,\\
0 =\ &\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3) + e^{i\pi s_{13}}\mathbb{A}(\overline{3},\overline{1},\overline{2},2,3,1) + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{3},\overline{1},\overline{2},3,2,1)\\
&+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{3},\overline{1},3,\overline{2},2,1) + e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{3},3,\overline{1},\overline{2},2,1)\\
&+\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1) \qquad (8.38)
\end{aligned}$$

to reduce the number of partial amplitudes in (8.37). The second monodromy relation in (8.38) was derived by complex conjugation of (7.26) and multiplication by a factor of (-1) to account for the reversal of the contour

$$\begin{aligned}
0 = &-\overline{\mathbb{A}}(3,\overline{3},\overline{1},\overline{2},2,1) - e^{-i\pi s_{3\overline{3}}}\overline{\mathbb{A}}(\overline{3},3,\overline{1},\overline{2},2,1) - e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}\overline{\mathbb{A}}(\overline{3},\overline{1},3,\overline{2},2,1)\\
&-e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}e^{-i\pi s_{2\overline{3}}}\overline{\mathbb{A}}(\overline{3},\overline{1},\overline{2},3,2,1) - e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}e^{-i\pi s_{2\overline{3}}}e^{-i\pi s_{23}}\overline{\mathbb{A}}(\overline{3},\overline{1},\overline{2},2,3,1)\\
&-\overline{\mathbb{A}}(\overline{3},\overline{1},\overline{2},2,1,3) \ . \qquad (8.39)
\end{aligned}$$

The partial amplitudes in (7.23) are real, whereas the subamplitudes in (8.37) are purely imaginary due to the overall factor of i in (8.33). Hence, the complex conjugate $\overline{\mathbb{A}} = -\mathbb{A}$ such that we find

$$\begin{aligned}0 = &\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1) + e^{-i\pi s_{3\overline{3}}}\mathbb{A}(\overline{3},3,\overline{1},\overline{2},2,1) + e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}\mathbb{A}(\overline{3},\overline{1},3,\overline{2},2,1)\\ &+e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}e^{-i\pi s_{2\overline{3}}}\mathbb{A}(\overline{3},\overline{1},\overline{2},3,2,1) + e^{-i\pi s_{3\overline{3}}}e^{-i\pi s_{1\overline{3}}}e^{-i\pi s_{2\overline{3}}}e^{-i\pi s_{23}}\mathbb{A}(\overline{3},\overline{1},\overline{2},2,3,1)\\ &+\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3)\ . \end{aligned}\tag{8.40}$$

Using momentum conservation results in the second monodromy relation in (8.38)

$$\begin{aligned}0 = &\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1) + e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{3},3,\overline{1},\overline{2},2,1)\\ &+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{3},\overline{1},3,\overline{2},2,1) + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{3},\overline{1},\overline{2},3,2,1) + e^{i\pi s_{13}}\mathbb{A}(\overline{3},\overline{1},\overline{2},2,3,1)\\ &+\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3)\ . \end{aligned}\tag{8.41}$$

By applying the first monodromy relation in (8.38) the subamplitudes in (8.37) corresponding to the bottom row with $1 < \xi$ in table 8.1 combine into $\mathbb{A}(\overline{1},\overline{2},2,1,3,\overline{3})$: The integration regions for $1 < \xi$ form a closed contour in the complex η-plane with one missing piece along the real line. This missing piece is the subamplitude $\mathbb{A}(\overline{1},\overline{2},2,1,3,\overline{3})$, which closes the integration contour. Also the integration region in the left column with $\eta < -1$ in table 8.1 can be reduced by using the second monodromy relation in (8.38) to the subamplitudes $\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3)$ and $\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1)$. In the end, the scattering amplitude of three closed strings can be written as

$$\begin{aligned}\mathcal{A}_3^{D_2} = &-\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1) + e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{3},3,\overline{2},2,1) + e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{3},\overline{2},3,2,1)\\ &+\mathbb{A}(\overline{1},\overline{2},\overline{3},3,2,1) + \mathbb{A}(\overline{1},\overline{3},\overline{2},2,3,1) + e^{i\pi s_{23}}\mathbb{A}(\overline{1},\overline{2},\overline{3},2,3,1)\\ &+e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{1},\overline{2},2,\overline{3},3,1) - \mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3) - \mathbb{A}(\overline{1},\overline{2},2,1,3,\overline{3})\ , \end{aligned}\tag{8.42}$$

and we have also listed the integration regions of the subamplitudes in table 8.2. After performing a suitable $PSL(2,\mathbb{R})$-transformation the amplitude (8.42) can be written in terms of the open string subamplitudes (7.27).

8.3.2 $PSL(2,\mathbb{R})$-transformation and monodromy relations

Comparing the fixed vertex operator positions in (8.33) with (7.27) we want to perform a $PSL(2,\mathbb{R})$-transformation that maps $(-1,y,1)$ to $(0,1,\infty)$, which was already done for the scattering of two closed strings off a Dp-brane in [88, 91, 92], see for example equation (3.9) in [88]. To find this transformation we consider a general fractional linear transformation $z' := f(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$ and determine the parameters a, b, c and d by solving $f(z) = z'$ for $z \in \{-1, y, 1\}$ and $z' \in \{0, 1, \infty\}$. The computation results in the transformation

$$\frac{1}{\sqrt{2(1-y^2)}}\begin{pmatrix} 1-y & 1-y \\ -(1+y) & 1+y \end{pmatrix} \in PSL(2,\mathbb{R})\ . \tag{8.43}$$

	$\xi < \eta < -1$	
$\xi < -1$	-1	

		$-1 < \eta < \xi$	
$-1 < \xi < -y$		$e^{i\pi s_{23}} e^{i\pi s_{2\bar{3}}}$	

		$-1 < \eta < -y$	$-y < \eta < \xi$	
$-y < \xi < y$		$e^{i\pi s_{23}}$	1	

		$-1 < \eta < -y$	$-y < \eta < y$	$y < \eta < \xi$	
$y < \xi < 1$		1	$e^{i\pi s_{23}}$	$e^{i\pi s_{23}} e^{i\pi s_{2\bar{3}}}$	

	$\eta < -1$		$\xi < \eta$
$1 < \xi$	-1		-1

Table 8.2: $\Pi(y, \xi, \eta)$ for each integration region in the (ξ, η)-plane after applying monodromy relations.

The above $PSL(2, \mathbb{R})$ matrix gives rise to the fractional linear transformation for the coordinates of the vertex operators

$$z' = f(z) = \frac{(1-y)(1+z)}{(1+y)(1-z)} \ , \tag{8.44}$$

which is a special case of (C.11). To change the vertex operator positions from $(-1, y, 1)$ to $(0, 1, \infty)$ in the amplitude we define the new variables

$$\begin{aligned} x = f(-y) &= \frac{(1-y)^2}{(1+y)^2} \ , \\ \tilde{\xi} = f(\xi) &= \frac{(1-y)(1+\xi)}{(1+y)(1-\xi)} \ , \\ \tilde{\eta} = f(\eta) &= \frac{(1-y)(1+\eta)}{(1+y)(1-\eta)} \end{aligned} \tag{8.45}$$

and it is also helpful to have the inverse transformations, which are given by[4]

$$y = \frac{1-\sqrt{x}}{1+\sqrt{x}} \ ,$$

[4] The transformation of y here is the same as in (3.9) in [88].

$$\xi = \frac{\tilde{\xi} - \sqrt{x}}{\tilde{\xi} + \sqrt{x}} \,,$$
$$\eta = \frac{\tilde{\eta} - \sqrt{x}}{\tilde{\eta} + \sqrt{x}} \,. \tag{8.46}$$

In principal, from (8.45) we can find a second solution $y = \frac{1+\sqrt{x}}{1-\sqrt{x}}$ for $x = f(-y)$. We ignore this second solution here, because for $x \in [0,1]$ we would require $y \in [1,\infty[$, but it will be important for the computation of the scattering of three closed strings on the real projective plane in chapter 10.

Because the unintegrated vertex operators have conformal weight zero and the integrated vertex operators have weight one, they transform under global conformal transformations as

$$V(z) \to V'(z') \,, \qquad U(z) \to \left(\frac{\partial z}{\partial z'}\right)^{-1} U'(z') \,. \tag{8.47}$$

The explicit $PSL(2,\mathbb{R})$-transformation of the integrand of each of the different integration regions in table 8.2 was performed in appendix C and can be schematically presented as

$$\begin{aligned}
&\int \mathrm{d}y\, \mathrm{d}\xi\, \mathrm{d}\eta \, \langle\!\langle V_1(1) V_{\bar{1}}(-1) V_2(y) U_{\bar{2}}(-y) U_3(\xi) U_{\bar{3}}(\eta) \rangle\!\rangle \\
&\to \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}\tilde{\xi}\, \mathrm{d}\tilde{\eta}\, \frac{\partial(y,\xi,\eta)}{\partial(x,\tilde{\xi},\tilde{\eta})} \left\langle\!\!\left\langle V_1'(\infty) V_{\bar{1}}'(0) V_2'(1) \left(\frac{\partial y}{\partial x}\right)^{-1} U_{\bar{2}}'(x) \left(\frac{\partial \xi}{\partial \tilde{\xi}}\right)^{-1} U_3'(\tilde{\xi}) \left(\frac{\partial \eta}{\partial \tilde{\eta}}\right)^{-1} U_{\bar{3}}'(\tilde{\eta}) \right\rangle\!\!\right\rangle \\
&= \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}\tilde{\xi}\, \mathrm{d}\tilde{\eta} \, \langle\!\langle V_1(\infty) V_{\bar{1}}(0) V_2(1) U_{\bar{2}}(x) U_3(\tilde{\xi}) U_{\bar{3}}(\tilde{\eta}) \rangle\!\rangle \,.
\end{aligned} \tag{8.48}$$

To get from the second to the last line we have used (B.1), i.e. that the correlator is invariant under $PSL(2,\mathbb{R})$-transformations. Moreover, we have the choice to consider either vertex operator 2 at y or $\overline{2}$ at $-y$ as position fixed and integrate over the other one [88]. Hence this leads to an additional factor two, which we have taken into account by introducing a factor $\frac{1}{2}$ in (8.48). In addition, the worldsheet derivatives coming from the transformation of the integrated vertex operators cancel against the Jacobian of the measure. Moreover, in appendix C we explicitly check that the integrand of each of the integration regions in table 8.2 is mapped correctly.

From (8.45) it follows immediately that for $y \in [0,1]$ the variable x is integrated from 0 to 1. Since we have singled out the y-integration due to the gauge fixing of the amplitude, also the x-integration is the same for all subamplitudes. In contrast to the y-integration the integral over x is not singled out any more, as there is a vertex operator at position $z_{\bar{1}} = 0$ and $z_2 = 1$, which promotes it to the integration region of an open string subamplitude. Next, we want to transform the integration regions of the worldsheet coordinates of the third vertex operator in table 8.2. We would expect that after the transformation the integration boundaries would be determined by the position of other vertex operators as in (7.27).[5] However, for the interval $[1,\infty[$ (or $]-\infty,-1]$) in table 8.2 the upper (lower) integration boundary $\pm\infty$ is mapped under (8.45) onto $-\sqrt{x}$. Hence, the upper (lower) boundary of the ξ- and/or η-integration is not a worldsheet position of another vertex operator and the transformed integration region does not resemble an open string partial amplitude.[6] But as discussed in section 7.2 in the $PSL(2,\mathbb{R})$-frame, where no vertex operator is at infinity, the amplitudes $\mathbb{A}$ can be combined to form one open string subamplitude (7.27), because they are only parts of partial amplitudes. Indeed, after the transformation (8.44) we can combine for example[7]

$$\mathbb{A}(3,\overline{3},\overline{1},\overline{2},2,1)+\mathbb{A}(\overline{1},\overline{2},2,1,3,\overline{3})+\mathbb{A}(\overline{3},\overline{1},\overline{2},2,1,3)\xrightarrow{PSL(2,\mathbb{R})}\mathcal{A}(3,4,1,2,5,6) \qquad (8.50)$$

into one open string integration region corresponding to a subamplitude (7.27). Performing the $PSL(2,\mathbb{R})$-transformation for all integration regions in table 8.2 and combining them if necessary allows us to write (8.42) in terms of (7.27)

$$\begin{aligned}\mathcal{A}_3^{D_2} = &-\mathcal{A}(3,\overline{3},\overline{1},\overline{2},2,1)+e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathcal{A}(\overline{1},\overline{3},3,\overline{2},2,1)+e^{i\pi s_{23}}\mathcal{A}(\overline{1},\overline{3},\overline{2},3,2,1)\\ &+\mathcal{A}(\overline{1},\overline{2},\overline{3},3,2,1)+\mathcal{A}(\overline{1},\overline{3},\overline{2},2,3,1)+e^{i\pi s_{23}}\mathcal{A}(\overline{1},\overline{2},\overline{3},2,3,1)\\ &+e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathcal{A}(\overline{1},\overline{2},2,\overline{3},3,1)\ ,\end{aligned} \qquad (8.51)$$

whose integration regions are given in table 8.3. Under permutations $3\leftrightarrow 2$ and $\overline{2}\leftrightarrow\overline{3}$ the representation of the amplitude in (8.51) is invariant, whereas the symmetries under $\overline{1}\leftrightarrow\overline{2}, 2\leftrightarrow 1$ and $\overline{1}\leftrightarrow\overline{3}, 3\leftrightarrow 1$ can only be seen after applying monodromy relations.

As argued in section 7.2 there are $(n-1)!=120$ open string subamplitudes (7.27) for $n=6$, which can be reduced down to $(n-3)!=6$ independent amplitudes by applying cyclic symmetries, reflection (7.22) and monodromy relations (7.28) [35, 36]. This suggests that the seven open string subamplitudes in (8.51) are not written in terms of a minimal basis and we can rewrite one of them in terms of the others via (7.28). Therefore, we consider the three monodromy relations

$$W_1 := \mathcal{A}(\overline{1},\overline{2},2,1,3,\overline{3})+e^{i\pi s_{13}}\mathcal{A}(\overline{1},\overline{2},2,3,1,\overline{3})+e^{i\pi(s_{13}+s_{1\overline{3}})}\mathcal{A}(\overline{1},\overline{2},2,3,\overline{3},1)$$

[5] This is also crucial for the computation of the correlator in section C.2, as the integration by parts relations rely heavily on the integration boundaries being vertex operator positions.

[6] This reflects the fact that some $\mathbb{A}$ are not open string subamplitudes to begin with.

[7] Explicitly, this can be seen by transforming the integrals

$$\begin{aligned}&\int_{-\infty}^{-1}\mathrm{d}\xi\int_{\xi}^{-1}\mathrm{d}\eta+\int_{1}^{\infty}\mathrm{d}\xi\int_{\xi}^{\infty}\mathrm{d}\eta+\int_{1}^{\infty}\mathrm{d}\xi\int_{-\infty}^{-1}\mathrm{d}\eta\\ &\overset{(8.44)}{\sim}\int_{-\sqrt{x}}^{0}\mathrm{d}\tilde{\xi}\int_{\tilde{\xi}}^{0}\mathrm{d}\tilde{\eta}+\int_{-\infty}^{-\sqrt{x}}\mathrm{d}\tilde{\xi}\int_{\tilde{\xi}}^{-\sqrt{x}}\mathrm{d}\tilde{\eta}+\int_{-\infty}^{-\sqrt{x}}\mathrm{d}\tilde{\xi}\int_{-\sqrt{x}}^{0}\mathrm{d}\tilde{\eta}\\ &=\int_{-\infty}^{0}\mathrm{d}\tilde{\xi}\int_{\tilde{\xi}}^{0}\mathrm{d}\tilde{\eta}\ .\end{aligned} \qquad (8.49)$$

	$\tilde{\xi} < \tilde{\eta} < 0$	
$\tilde{\xi} < 0$	-1	

		$0 < \tilde{\eta} < \tilde{\xi}$	
$0 < \tilde{\xi} < x$		$e^{i\pi\hat{s}_{35}}e^{i\pi\hat{s}_{45}}$	

		$0 < \tilde{\eta} < x$	$x < \tilde{\eta} < \tilde{\xi}$	
$x < \tilde{\xi} < 1$		$e^{i\pi\hat{s}_{35}}$	1	

		$0 < \tilde{\eta} < x$	$x < \tilde{\eta} < 1$	$1 < \tilde{\eta} < \tilde{\xi}$
$1 < \tilde{\xi}$		1	$e^{i\pi\hat{s}_{35}}$	$e^{i\pi\hat{s}_{35}}e^{i\pi\hat{s}_{45}}$

Table 8.3: $\Pi(x, \tilde{\xi}, \tilde{\eta})$ for each integration region in the $(\tilde{\xi}, \tilde{\eta})$-plane.

$$
\begin{aligned}
&+e^{i\pi(s_{1\bar{1}}+s_{13}+s_{1\bar{3}})}\mathcal{A}(\bar{1},1,\bar{2},2,3,\bar{3}) + e^{-i\pi s_{12}}\mathcal{A}(\bar{1},\bar{2},1,2,3,\bar{3}) = 0 \ , \qquad (8.52)\\
W_2 &:= \mathcal{A}(\bar{1},\bar{2},2,3,1,\bar{3}) + e^{i\pi s_{13}}\mathcal{A}(\bar{1},\bar{3},\bar{2},2,3,1) + e^{i\pi(s_{13}+s_{23})}\mathcal{A}(\bar{1},\bar{2},\bar{3},2,3,1)\\
&+e^{i\pi(s_{13}+s_{23}+s_{2\bar{3}})}\mathcal{A}(\bar{1},\bar{2},2,\bar{3},3,1) + e^{-i\pi s_{1\bar{3}}}\mathcal{A}(\bar{1},\bar{2},2,3,\bar{3},1) = 0 \ , \qquad (8.53)\\
W_3 &:= \mathcal{A}(\bar{1},\bar{3},3,2,\bar{2},1) + e^{i\pi s_{1\bar{2}}}\mathcal{A}(\bar{1},\bar{3},3,2,1,\bar{2}) + e^{i\pi(s_{12}+s_{1\bar{2}})}\mathcal{A}(\bar{1},\bar{2},\bar{3},3,2,1)\\
&+e^{i\pi(s_{12}+s_{1\bar{2}}+s_{23})}\mathcal{A}(\bar{1},\bar{3},\bar{2},3,2,1) + e^{-i\pi s_{2\bar{2}}}\mathcal{A}(\bar{1},\bar{3},3,\bar{2},2,1) = 0 \ , \qquad (8.54)
\end{aligned}
$$

which combined as $W_1 - e^{i\pi s_{13}}\overline{W}_2 - e^{i\pi(s_{13}+s_{1\bar{3}}+s_{1\bar{1}})}W_3 = 0$ and subject to cyclic reflection symmetries of the open string subamplitudes yield

$$
\begin{aligned}
\mathcal{A}(\bar{1},\bar{2},2,1,3,\bar{3}) &= \mathcal{A}(\bar{1},\bar{3},\bar{2},2,3,1) + e^{-i\pi s_{23}}\mathcal{A}(\bar{1},\bar{2},\bar{3},2,3,1)\\
&+e^{i\pi(s_{13}+s_{1\bar{3}}+s_{3\bar{3}})}\mathcal{A}(\bar{1},\bar{2},2,\bar{3},3,1) + \mathcal{A}(\bar{1},\bar{2},\bar{3},3,2,1)\\
&+e^{i\pi s_{23}}\mathcal{A}(\bar{1},\bar{3},\bar{2},3,2,1) + e^{-i\pi(s_{12}+s_{1\bar{2}}+s_{2\bar{2}})}\mathcal{A}(\bar{1},\bar{3},3,\bar{2},2,1) \ . \qquad (8.55)
\end{aligned}
$$

Plugging (8.55) into (8.51) and using momentum conservation as well as $k_i^2 = 0$ leads to the compact result

$$
\mathcal{A}_3^{D_2} = 2i\sin(\pi s_{23})\mathcal{A}(\bar{1},\bar{2},\bar{3},2,3,1) + 2i\sin(\pi(s_{23}+s_{2\bar{3}}))\mathcal{A}(\bar{1},\bar{2},2,\bar{3},3,1) \ . \qquad (8.56)
$$

Note that the expansion in α' of (8.56) only starts at subleading order in α', because the lowest order in α' vanishes, c.f. chapter 11. After applying monodromy relations for open strings and performing a global conformal transformation we arrive at the following integration regions in the $(\tilde{\xi}, \tilde{\eta})$-plane

$$
\begin{aligned}
\mathcal{A}(\bar{1},\bar{2},\bar{3},2,3,1): &\qquad \mathcal{I}_1 = \left\{(\tilde{\xi},\tilde{\eta}) \in \mathbb{R}^2 \,|\, 1 < \tilde{\xi} < \infty, x < \tilde{\eta} < 1\right\} \ ,\\
\mathcal{A}(\bar{1},\bar{2},2,\bar{3},3,1): &\qquad \mathcal{I}_2 = \left\{(\tilde{\xi},\tilde{\eta}) \in \mathbb{R}^2 \,|\, 1 < \tilde{\xi} < \infty, 1 < \tilde{\eta} < \tilde{\xi}\right\} \qquad (8.57)
\end{aligned}
$$

for each of the subamplitudes $x \in \mathbb{R}$ and always $0 < x < 1$. Hence, we could express the scattering of closed strings off a Dp-brane in terms of scattering amplitudes of six open strings on the disk. More importantly, we were able to write the closed string scattering process in terms of *only two* independent open string amplitudes instead of six as was originally predicted by [35].

From the computation in appendix C it follows that we can express the correlator of the partial amplitudes in terms of SYM amplitudes of section 6.4 as[8]

$$\begin{aligned} \mathcal{A}(\overline{1},\overline{2},\overline{3},2,3,1) &= -\frac{i}{2} g_c^3 T_p \sum_{\sigma\in S_3} A_{\rm SYM}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1) F_{\mathcal{I}_1}^{\sigma(\overline{2},3,\overline{3})} \ , \\ \mathcal{A}(\overline{1},\overline{2},2,\overline{3},3,1) &= -\frac{i}{2} g_c^3 T_p \sum_{\sigma\in S_3} A_{\rm SYM}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1) F_{\mathcal{I}_2}^{\sigma(\overline{2},3,\overline{3})} \ , \end{aligned} \tag{8.58}$$

where the sum $\sigma \in S_3$ runs over all permutations of the labels $(\overline{2},3,\overline{3})$. Eventually, inserting (8.58) into (8.56) gives

$$\mathcal{A}_3^{D_2} = g_c^3 T_p \sum_{\sigma\in S_3} \Big\{ \sin(\pi s_{23}) F_{\mathcal{I}_1}^{\sigma(\overline{2},3,\overline{3})} + \sin[\pi(s_{23}+s_{2\overline{3}})] F_{\mathcal{I}_2}^{\sigma(\overline{2},3,\overline{3})} \Big\} A_{\rm SYM}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1) \ , \tag{8.59}$$

where the hypergeometric integrals $F_{\mathcal{I}_p}^{\sigma(\overline{2},3,\overline{3})}$ are given by

$$F_{\mathcal{I}_p}^{\sigma(\overline{2},3,\overline{3})} = -\int_{\mathcal{I}_p} dz_{\overline{2}}\, dz_3\, dz_{\overline{3}} \left(\prod_{i<j} |z_{ij}|^{s_{ij}} \right) \frac{s_{\overline{1}\sigma(\overline{2})}}{z_{\overline{1}\sigma(\overline{2})}} \frac{s_{\sigma(\overline{3})2}}{z_{\sigma(\overline{3})2}} \left(\frac{s_{\overline{1}\sigma(3)}}{z_{\overline{1}\sigma(3)}} + \frac{s_{\sigma(\overline{2})\sigma(3)}}{z_{\sigma(\overline{2})\sigma(3)}} \right) \ , \qquad p = 1,2 \ . \tag{8.60}$$

The product above over $i < j$ also involves overlined indices. In (8.60) the two integration domains (8.57) can be written as

$$\begin{aligned} \mathcal{I}_1 &: z_{\overline{1}} < z_{\overline{2}} < z_{\overline{3}} < z_2 < z_3 < z_1 \ , \\ \mathcal{I}_2 &: z_{\overline{1}} < z_{\overline{2}} < z_2 < z_{\overline{3}} < z_3 < z_1 \end{aligned} \tag{8.61}$$

and are subject to the vertex operator position fixing

$$z_{\overline{1}} = 0 \ , \qquad z_2 = 1 \ , \qquad z_1 = \infty \ , \tag{8.62}$$

which we have also applied in the amplitude (8.59) itself.

The final result (8.59) is independent of whether the external states in the scattering process are fermions or bosons. In the language of the RNS formalism the external states could be from either of the four sectors, i.e. we consider NSNS, RR, RNS, NSR states in the amplitude. For the derivation of (8.59) we have only assumed that the external states are massless $k_i^2 = 0$. Therefore, the open string subamplitudes contain both gluons and gluinos, which will form closed string states.

[8] Alternatively, one could have used equation (2.5) in [63] together with the identification (8.28) and included overall factors like the closed string coupling or the tension of the Dp-brane.

In [35] the scattering of three closed strings was computed on the double cover of the disk, i.e. $\mathbb{H}_+ \cup \mathbb{H}_- = \mathbb{C}$, which introduces a manifest symmetry between left- and right-movers. The result for the string amplitude on the double cover can be found in (3.65) in [35] and is given by

$$\begin{aligned}\mathcal{A}_3^{\mathrm{DC}} &= \sin(\pi s_{2\overline{3}})\mathcal{A}(\overline{1},\overline{2},3,2,\overline{3},1) + \sin(\pi s_{23})\mathcal{A}(\overline{1},2,3,\overline{2},\overline{3},1)\\ &\quad+\sin(\pi s_{13})[\mathcal{A}(\overline{1},\overline{2},2,3,1,\overline{3}) + A(\overline{1},2,\overline{2},3,1,\overline{3})]\\ &\quad+[\sin(\pi(s_{23}+s_{2\overline{3}})) + \sin(\pi(s_{13}+s_{1\overline{3}}))][\mathcal{A}(\overline{1},\overline{2},2,3,\overline{3},1) + A(\overline{1},2,\overline{2},3,\overline{3},1)],\end{aligned} \tag{8.63}$$

where we used the identification (8.28) to write (3.65) in [35] in terms of closed string labels. Although, this result is different from (8.56) they can be connected for a specific kinematical configuration. If we impose the symmetries $3 \leftrightarrow \overline{3}$ and $2 \leftrightarrow \overline{2}$ on (8.56) by hand, we get the amplitude

$$\begin{aligned}\mathcal{A}_3^{S^2} := &\,2i\sin(\pi s_{23})\mathcal{A}(1,\overline{1},\overline{2},\overline{3},2,3) + 2i\sin(\pi(s_{23}+s_{2\overline{3}}))\mathcal{A}(1,\overline{1},\overline{2},2,\overline{3},3)\\ &+2i\sin(\pi s_{2\overline{3}})\mathcal{A}(1,\overline{1},\overline{2},3,2,\overline{3}) + 2i\sin(\pi(s_{23}+s_{2\overline{3}}))\mathcal{A}(1,\overline{1},\overline{2},2,3,\overline{3})\\ &+2i\sin(\pi s_{2\overline{3}})\mathcal{A}(1,\overline{1},2,\overline{3},\overline{2},3) + 2i\sin(\pi(s_{23}+s_{2\overline{3}}))\mathcal{A}(1,\overline{1},2,\overline{2},\overline{3},3)\\ &+2i\sin(\pi s_{23})\mathcal{A}(1,\overline{1},2,3,\overline{2},\overline{3}) + 2i\sin(\pi(s_{23}+s_{2\overline{3}}))\mathcal{A}(1,\overline{1},2,\overline{2},3,\overline{3})\ .\end{aligned} \tag{8.64}$$

Then, using monodromy relations (7.29) for open string subamplitudes subject to (8.36) the amplitudes in (8.63) and (8.64) agree up to an overall factor:

$$\mathcal{A}_3^{S_2} = 2i\mathcal{A}_3^{\mathrm{DC}}\ . \tag{8.65}$$

Hence, we have found a connection between the worldsheet integrals on the disk and those on its double cover for each specific kinematical factor. Furthermore, the symmetrization, which is equivalent to extending the disk amplitude to the double cover, corresponds to extending the integration over y and η in (8.31) to $[-1,1]$ and $]-\infty,\infty[$, respectively. Compared to the disk amplitude the resulting integrals miss some poles in the α'-expansion. More concretely, by going to the double cover we lose the s-channel poles. In the disk amplitude the pole $\frac{1}{s_{2\overline{2}}}$ originates from the y-integration over terms proportional to $y^{s_{2\overline{2}}-1}$. On the double cover this pole is absent and in the limit $\alpha' \to 0$ the corresponding singularity becomes finite, because we integrate a term with odd power and pole at 0 over both positive and negative values of y.[9] Explicitly, we can see that by going to the double cover we discard the s-channel poles by symmetrizing the disk amplitude: If we impose the symmetry $3 \leftrightarrow \overline{3}$, the first term

$$\langle M_{\overline{12}}M_{3\overline{3}}M_{21}\rangle = \left\langle \frac{T_{\overline{12}}}{s_{\overline{12}}}\frac{T_{3\overline{3}}}{s_{3\overline{3}}}\frac{T_{21}}{s_{21}}\right\rangle \tag{8.66}$$

[9]This was explicitly demonstrated for the scattering of two closed strings off a Dp-brane in appendix E of [109].

in the SYM amplitude, c.f. (6.29) with the identification (8.28),

$$\begin{aligned} A_{\mathrm{SYM}}(\overline{1},\overline{2},3,\overline{3},2,1) &= \langle M_{\overline{12}} M_{3\overline{3}} M_{21}\rangle + \langle M_{\overline{2}3} M_{\overline{3}2} M_{1\overline{1}}\rangle + \langle M_{\overline{12}3}(M_{\overline{3}2} M_1 + M_{\overline{3}} M_{21})\rangle \\ &\quad + \langle M_{\overline{2}3\overline{3}}(M_{21} M_{\overline{1}} + M_2 M_{1\overline{1}})\rangle + \langle M_{3\overline{3}2}(M_{1\overline{1}} M_{\overline{2}} + M_1 M_{\overline{12}})\rangle \\ &= \frac{1}{2}\langle M_{\overline{12}3} E_{\overline{3}21}\rangle + \frac{1}{3}\langle M_{\overline{12}} M_{3\overline{3}} M_{21}\rangle + \mathrm{cyclic}(\overline{12}3\overline{3}21) \end{aligned} \tag{8.67}$$

vanishes due to the antisymmetry of the BRST building block $T_{3\overline{3}}$. Simultaneously, we drop the pole in $s_{3\overline{3}}$.

We conclude that going to the double cover has dramatic consequences for the low energy theory. Therefore, the scattering of three closed strings on the disk and the double cover of the disk are two different physical processes. This is not surprising: Gluing together $\mathbb{H}_+$ and $\mathbb{H}_-$ is a non-trivial process, since the amplitude (8.26) has poles along the real line $z-\overline{z}=0$. Hence, promoting the real line from the boundary to the bulk of the integration region by going from $\mathbb{H}_+$ and $\mathbb{H}_-$ to $\mathbb{C}$ changes the pole structure of the amplitude. More details on the pole structure in the field theory limit $\alpha' \to 0$ of the disk amplitude (8.59) can be found in chapter 11.

Chapter 9

Higher multiplicity of closed string amplitudes on the disk

To conclude the scattering of closed strings off a Dp-brane we want to conjecture an ansatz for the n_c-point function of closed strings on the disk by following the lines of [109]. Since open string partial amplitudes are the building blocks of closed string scattering amplitudes at tree-level, we start by generalising the discussion for open string amplitudes and give the prescription for the n-point open string amplitude on the disk following the idea presented in [62].

9.1 Higher multiplicity open string scattering amplitudes

As we can see in appendix C for the correlator of closed strings on the disk and as was discussed for various open string amplitudes [62, 158, 160], integrating out the non-zero modes of the $h = 1$ fields corresponds to summing over their OPE singularities. For n external states this results in a sum over $(n-2)!$ single pole terms and a number of double pole integrands that will be used as corrections to the single pole integrands to form the BRST building blocks $T_{ijk\ldots p}$ from the associated OPE residue $L_{jiki\ldots pi}$. The composite superfields $L_{2131\ldots p1}$ of the single pole residues are derived from the contraction of vertex operators, when the integrated vertex operators $U_2 U_3 \cdots U_p$ approach an unintegrated vertex operator V_1, where we have chosen the external states $1, n-1$ and n to be position fixed. The final result is independent on the order of integrating out the $h = 1$ primaries and we can choose for example the order $z_1 \to z_3 \to \ldots \to z_p \to z_1$, which is reflected in the z_{ij} in the denominator:

$$V_1(z_1) U_2(z_2) U_3(z_3) \ldots U_p(z_p) \sim \frac{2^{p-2} L_{[p1,[(p-1)1,[\ldots,[41,[31,21]]\ldots]]]}}{z_{23} z_{34} \cdots z_{(p-1)p} z_{p1}} \, . \tag{9.1}$$

Performing these steps to express all single pole composite superfields in terms of $L_{jiki\dots pi}$ and after the double pole correction are absorbed to transform the composite superfields in their according building blocks $T_{ijk\dots p}$ the scattering amplitude of n open strings on the disk takes the form

$$\begin{aligned}
&\mathcal{A}(\rho(1,2,\dots,n)) = \\
&= \prod_{q=2}^{n-2}\int_{D_2(\rho)} \mathrm{d}z_q \,\langle V_1(0)U_2(z_2)U_3(z_3)\cdots U_{n-2}(z_{n-2})V_{n-1}(1)V_n(\infty)\rangle \\
&= \prod_{q=2}^{n-2}\int_{D_2(\rho)} \mathrm{d}z_q \prod_{i<j}|z_{ij}|^{s_{ij}} \sum_{p=1}^{n-2}\Bigg\langle \frac{T_{12\dots p}T_{(n-1)(n-2)\dots p+1}V_n}{(z_{12}z_{23}\cdots z_{(p-1)p})(z_{(n-1)(n-2)}z_{(n-2)(n-3)}\cdots z_{(p+2)(p+1)})} \\
&\quad +\mathcal{P}(2,3,\dots,n-2)\Bigg\rangle \,, \qquad (9.2)
\end{aligned}$$

which is manifestly symmetric in the labels $2,3,\dots,n-2$ of the integrated vertex operators U_i. This is denote above by the sum over all $(n-3)!$ permutations of these labels. Remarkably, the poles in z_{ij} associated to the BRST building block in the numerator follow a pattern. From the above correlator we find that

$$T_{12\dots p} \longleftrightarrow \frac{1}{z_{12}z_{23}\cdots z_{(p-1)p}} \,. \qquad (9.3)$$

Because of the $(n-3)!$ permutations of the labels of the integrated vertex operators and the sum over p that collects $n-2$ distinct permutation orbits, this special structure (9.3) of the open string amplitude allows to express (9.2) in terms of $(n-2)!$ kinematic numerators and hypergeometric integrals.

To simplify the scattering of n open strings further we exchange the BRST building blocks $T_{12\dots p}$ for the Berends-Giele supercurrents $M_{12\dots p}$, which is possible due to the pattern (9.3), i.e. the z_{ij}-dependence in the denominator of the associated $T_{12\dots p}$. This connection relies on the synergy of different terms in the sum over permutations and the BRST symmetries (5.20) of the building blocks. At level p this interplay reduced the number of independent building blocks $T_{ijk\dots p}$ down to $(p-1)!$. Therefore, we find that the building blocks and currents are related inside the amplitude as

$$\begin{aligned}
\frac{T_{12\dots p}}{z_{12}z_{23}\cdots z_{(p-1)p}} + \mathcal{P}(2,3,\dots,p) &= (-1)^{p-1}\prod_{k=2}^{p}\sum_{i=1}^{k-1}\frac{s_{ik}}{z_{ik}}M_{12\dots p} + \mathcal{P}(2,3,\dots,p) \,, \\
\frac{T_{(n-1)(n-2)\dots p+1}}{z_{(n-1)(n-2)}\cdots z_{(p+2)(p+1)}} + \mathcal{P}(2,3,\dots,p) &= (-1)^{n-p-1}\prod_{k=p+1}^{n-2}\sum_{j=k+1}^{n-1}\frac{s_{ik}}{z_{ik}}M_{(n-1)(n-2)\dots p+1} \\
&\quad +\mathcal{P}(2,3,\dots,p) \,, \\
&= (-1)^{n-p-1}\prod_{k=p+1}^{n-2}\sum_{j=k+1}^{n-1}\frac{s_{kj}}{z_{kj}}M_{(p+1)(p+2)\dots n-1}
\end{aligned}$$

$$+\mathcal{P}(2,3,\ldots,p) \ , \tag{9.4}$$

where we have used the reflection symmetry (6.12) of the rank $n-1-p$ Berends-Giele currents in the last line to rewrite $M_{(n-1)(n-2)\ldots p+1} = (-1)^{n-p-2} M_{(p+1)(p+2)\ldots n-1}$.

The structure of the substitution (9.4) is precisely of the form that we can apply integration by parts relations to the chain of $\frac{s_{ik}}{z_{ik}}$ sums, which arise after exchanging $T_{12\ldots p}$ for $M_{12\ldots p}$ using (9.4). The basic idea is that the integration boundaries correspond to zeros in the Koba-Nielson factor, i.e. vertex operator positions. Hence, the boundary terms in the worldsheet integrals vanish:

$$\int_{D_2(\rho)} \mathrm{d}z_2 \cdots \mathrm{d}z_{n-2} \frac{\partial}{\partial z_k} \frac{\prod_{i<j} |z_{ij}|^{s_{ij}}}{z_{i_1 j_1} z_{i_2 j_2} \cdots z_{i_{n-4} j_{n-4}}} = 0 \ , \tag{9.5}$$

which relates different integrals in the n open string amplitude with $n-3$ powers of $z_{i_m j_m}$ in the denominator. In the case where the differentiation variable z_k does not appear in any $z_{i_m j_m}$, i.e. $k \notin \{i_m, j_m\}$ for $m = 2, 3, \ldots n-4$, the derivative $\frac{\partial}{\partial z_k}$ acts only on the Koba-Nielsen factor such that

$$\int_{D_2(\rho)} \mathrm{d}z_2 \cdots \mathrm{d}z_{n-2} \frac{\prod_{i<j} |z_{ij}|^{s_{ij}}}{z_{i_1 j_1} z_{i_2 j_2} \cdots z_{i_{n-4} j_{n-4}}} \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{s_{ik}}{z_{ik}} = 0 \ . \tag{9.6}$$

Leaving the first $\frac{n}{2} - 1$ factors $\sum_{i=1}^{k-1} \frac{s_{ik}}{z_{ik}}$ untouched and using these relations to integrate the other $\frac{n-3}{2}$ factors by parts yields

$$\begin{aligned}
&\prod_{q=2}^{n-2} \int \mathrm{d}z_q \prod_{i<j} |z_{ij}|^{s_{ij}} \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \cdots \left(\frac{s_{1(n-2)}}{z_{1(n-2)}} + \ldots + \frac{s_{(n-1)(n-2)}}{z_{(n-1)(n-2)}} \right) \\
&= \prod_{q=2}^{n-2} \int \mathrm{d}z_q \prod_{i<j} |z_{ij}|^{s_{ij}} \left(\prod_{k=2}^{\frac{n}{2}} \sum_{i=1}^{k-1} \frac{s_{ik}}{z_{ik}} \right) \left(\prod_{k=\frac{n}{2}+1}^{n-2} \sum_{j=k+1}^{n-1} \frac{s_{kj}}{z_{kj}} \right) \ .
\end{aligned} \tag{9.7}$$

After introducing Berends-Giele currents in the n-point function the above relation can be used to write (9.2) as

$$\begin{aligned}
\mathcal{A}(1,2,\ldots,n) &= \prod_{q=2}^{n-2} \int_{D_2(\rho)} \mathrm{d}z_q \prod_{i<j} |z_{ij}|^{s_{ij}} \Bigg\langle \sum_{p=1}^{n-2} \left(\prod_{k=2}^{p} \sum_{i=1}^{k-1} \frac{s_{ik}}{z_{ik}} M_{12\ldots p} \right) \\
&\quad \times \left(\prod_{k=p+1}^{n-2} \sum_{j=k+1}^{n-1} \frac{s_{kj}}{z_{kj}} M_{p+1,\ldots,n-2,n-1} \right) V_n + \mathcal{P}(2,3,\ldots,n-2) \Bigg\rangle \\
&= \prod_{q=2}^{n-2} \int_{D_2(\rho)} \mathrm{d}z_q \prod_{i<j} |z_{ij}|^{s_{ij}} \Bigg\{ \left(\prod_{k=2}^{\frac{n}{2}} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=\frac{n}{2}+1}^{n-2} \sum_{n=k+1}^{n-1} \frac{s_{kn}}{z_{kn}} \right) \\
&\quad \times \sum_{p=1}^{n-2} \langle M_{12\ldots p} M_{p+1\ldots n-2,n-1} V_n \rangle + \mathcal{P}(2,3,\ldots,n-2) \Bigg\}
\end{aligned}$$

$$= \prod_{q=2}^{n-2} \int_{D_2(\rho)} \mathrm{d}z_q \prod_{i<j} |z_{ij}|^{s_{ij}} \Bigg\{ \Bigg(\prod_{k=2}^{\frac{n}{2}} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \Bigg) \Bigg(\prod_{k=\frac{n}{2}+1}^{n-2} \sum_{n=k+1}^{n-1} \frac{s_{kn}}{z_{kn}} \Bigg)$$
$$\times A_{\mathrm{SYM}}(1,2,3,\ldots,n-1,n) + \mathcal{P}(2,3,\ldots,n-2) \Bigg\} \ . \tag{9.8}$$

In the final result of these manipulations the kinematic building blocks can be expressed as a linear combination of $(n-3)!$ field theory amplitudes, which are each multiplied by a hypergeometric integral $F^{\sigma(2,3,\ldots,n-2)}_{\mathcal{I}_\rho}$ with $\mathcal{I}_\rho = D_2(\rho)$, which is given by the worldsheet dependent part of (9.8) and $\sigma \in S_{n-3}$ refers to the sum over permutations. The string integrands are integrated over parts of the real line given by $D_2(\rho)$ depending on the color ordering ρ.

9.2 Scattering n closed strings from a Dp-brane

Similar, the pure closed string n_c-point amplitude on the disk can be expanded in terms of $(2n_c-3)!$ SYM amplitudes

$$A_{\mathrm{SYM}}(\overline{1},\sigma(\overline{2},3,\overline{3},\ldots,n_c,\overline{n}_c),2,1) \ , \qquad \sigma \in S_{2n_c-3} \ , \tag{9.9}$$

which are inherited from the open superstring amplitude (9.8). Here, we again employ the notation defined in (8.22). But different as for the open string amplitude, where each kinematical building block A_{SYM} is accompanied by a single form factor $F^{\sigma(2,3,\ldots,n-2)}_{\mathcal{I}_\rho}$ referring to the color ordering ρ under consideration, in the closed string amplitude there are L_{n_c} form factors $F^{\sigma(\overline{2},3,\overline{3}\ldots,n_c,\overline{n}_c)}_{\mathcal{I}_{\rho_l}}$ for each (9.9), where each is integrated over a different open string color ordering ρ_l with $l=1,2,\ldots,L_{n_c}$. Moreover, each of these combinations is multiplied by a chain of n_c-2 phase factors, which appear hear in form of sin-functions. These arise as a consequence of disentangling the left- and right-moving part of the vertex operators on the disk.

To get a better feeling for the underlying systematic we consider the simplest scattering amplitude containing only $n_c=2$ closed strings and find for $L_2=1$ [88]

$$\mathcal{A}^{D_2}_2 \sim g_c^2 T_p F^{(\overline{2})}_{\mathcal{I}_1} \ A_{\mathrm{SYM}}(\overline{1},\overline{2},2,1) \ , \tag{9.10}$$

which contains the form factor:

$$F^{(\overline{2})}_{\mathcal{I}_1} = \int \mathrm{d}z_{\overline{2}} \prod_{i<j} |z_{ij}|^{s_{ij}} \frac{s_{12}}{z_{\overline{12}}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{2\overline{2}})}{\Gamma(1+s_{12}+s_{2\overline{2}})} \ . \tag{9.11}$$

For an additional external state, i.e. $n_c=3$, we can find that $L_3=2$ in chapter 8 and the final result for the scattering of three closed strings is presented in (8.59), c.f. also (8.58), such that

$$\mathcal{A}^{D_2}_3 \sim g_c^3 T_p \sum_{\sigma\in S_3} \Bigg\{ \sum_{l=1}^{2} \sin(\pi s_{\rho_l}) \ F^{\sigma(\overline{2}3\overline{3})}_{\mathcal{I}_{\rho_l}} \Bigg\} A_{\mathrm{SYM}}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1) \ , \tag{9.12}$$

where the form factors $F_{\mathcal{I}_{\rho_l}}^{\sigma(\overline{2}3\overline{3})}$ are given in (8.60). For the permutations $\rho_1 = (\overline{1}, \overline{2}, \overline{3}, 2, 3, 1)$ and $\rho_2 = (\overline{1}, \overline{2}, 2, \overline{3}, 3, 1)$ the kinematic invariants take the form $s_{\rho_1} = s_{23}$ and $s_{\rho_2} = s_{23} + s_{2\overline{3}}$, respectively.

Generalizing this for $n_c \geq 4$ the result may be summarized in the following way:

$$\mathcal{A}_{n_c}^{D_2} \sim g_c^n T_p \sum_{\sigma \in S_{2n_c-3}} \left\{ \sum_{l=1}^{L_{n_c}} \left[\prod_{k=1}^{n_c-2} \sin(\pi s_{\rho_{l,k}}) \right] F_{\mathcal{I}_{\rho_l}}^{\sigma(\overline{2},3,\overline{3},\ldots,n_c,\overline{n}_c)} \right\}$$
$$\times A_{\mathrm{SYM}}(\overline{1}, \sigma(\overline{2}, 3, \overline{3}, \ldots, n_c, \overline{n}_c), 2, 1) \ . \tag{9.13}$$

The angles s_{ρ_l} inside the sin-functions are linear combination of kinematic invariants. The sum, which is running over the L_{n_c} different integration regions ρ_l, can be obtained by applying $2n_c$-point open string monodromy relations. Therefore, the number $L_{n_c} < (2n_c - 3)!$, which can be seen in (8.56), where we find for $n_c = 3$ that $L_{n_c} = 2$, but it still remains an open question what the exact n_c-dependence of L_{n_c} is.

Chapter 10

Scattering three closed strings off an Op-plane

The scattering of three closed strings off an Op-plane is described by a scattering amplitude on the real projective plane. As the disk this worldsheet topology contributes to the first quantum corrections obtained from string theory. Hence, terms like $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ could arise (partially) from the scattering of closed strings on the real projective plane such that these amplitudes are phenomenologically very interesting.

The real projective plane can be defined as $S^2/\mathbb{Z}_2$ where the action of $\mathbb{Z}_2$ identifies antipodal points. Here, $\mathbb{Z}_2 : z \mapsto -\frac{1}{\overline{z}}$ is the identification used to obtain the real projective plane as the quotient $S^2/\mathbb{Z}_2$ form the sphere. Moreover, this identification preserves the real projective plane and acts without fixed points such that the real projective plane has no boundary. Topologically $\mathbb{RP}^2$ is equivalent to a sphere with a crosscap. The conformal Killing group of this configuration is $SU(2)$ with three real parameters and corresponds to the subgroup of $PSL(2,\mathbb{C})$, which commutes with the $\mathbb{Z}_2$ action [130]. Therefore, the real projective plane has three conformal Killing vectors, which allow us to fix one and a half, i.e. three real, vertex operator positions as on the disk. With similar arguments as in section 8.2 the n-point closed string amplitude becomes

$$\mathcal{A}_n^{\mathbb{RP}^2} = 2ig_c^n T_p' \int_0^1 \mathrm{d}y \left\langle\!\!\left\langle V_1^{\mathbb{RP}^2}(i,-i)(V\otimes\overline{U})_2^{\mathbb{RP}^2}(iy,-iy)\prod_{j=3}^{n}\int_{\mathbb{H}_+}\mathrm{d}^2 z_j\, U_j^{\mathbb{RP}^2}(z_j,\overline{z}_j)\right\rangle\!\!\right\rangle\,, \quad (10.1)$$

where T_p' is the tension of the Op-plane representing the crosscap state and $(V\otimes\overline{U})^{\mathbb{RP}^2}$ is the half position fixed vertex operator. In addition, we integrate the integrated vertex operators over the upper half plane, because the fundamental domain of the real projective plane is the disk, which can be mapped to the upper half plane. Therefore, the discussion in section 8.2 is also relevant here to derive (10.1).

10.1 Scattering amplitude prescription for the real projective plane

Type II string theory can be obtained from unoriented type I string theory via T-duality. At the fixed points of the T-dual space this procedure gives rise to Op-planes. In the original type I theory states are invariant under the world sheet parity Ω, which requires $\Omega = +1$ to form unoriented closed strings [110]. According to [27] this can be interpreted as gauging Ω such that we include orientation reversal to the transition functions that form the worldsheet. Hence, this produces unoriented worldsheets like the real projective plane. For the spacetime coordinates X^m the T-dual theory can be found by using $X'^m(z,\overline{z}) = X^m(z) - X^m(\overline{z})$ instead of $X^m(z,\overline{z}) = X^m(z) + X^m(\overline{z})$, where $X^m(z)$ is the left-moving and $X^m(\overline{z})$ is the right-moving part of $X^m(z,\overline{z})$. In the original type I theory the worldsheet parity Ω, which acts on X^m as [27]

$$\Omega: \quad X^m(z) \longleftrightarrow X^m(\overline{z}) \ , \tag{10.2}$$

becomes a gauge symmetry. In addition, the T-dual spacetime coordinates transform as [27]

$$\begin{aligned} \Omega: \quad & X^\mu(z,\overline{z}) \longleftrightarrow X^\mu(\overline{z},z) \quad \text{for } \mu = 0,1,\dots p \ , \\ & X'^{\tilde{\mu}}(z,\overline{z}) \longleftrightarrow -X'^{\tilde{\mu}}(\overline{z},z) \quad \text{for } \tilde{\mu} = p+1, p+2, \dots 9 \ , \end{aligned} \tag{10.3}$$

which combines a worldsheet parity transformation with a spacetime reflection. Moreover, we use lower case Latin letters to describe the entire ten dimensional space time, lower case Greek letters represent directions in the world volume of the O-plane and lower case Greek letter with a tilde correspond to the directions transverse to that space, i.e. the coordinates on which the T-dual has been taken.

After splitting the string wave function into its internal part and the center of mass dependent piece $x^{\tilde{\mu}}$, where the former one is an eigenstate of Ω, the projection onto $\Omega = +1$ determines the wave function at the points $-x^{\tilde{\mu}}$ and $x^{\tilde{\mu}}$ to be the same up to a sign [27]. The components of massless states corresponding to the NSNS sector in the RNS formalism, i.e. the spacetime metric and the antisymmetric Kalb-Ramond B-field, obey [110]

$$\begin{aligned} G_{\mu\nu}(x^\alpha, -x^{\tilde{\alpha}}) &= G_{\mu\nu}(x^\alpha, x^{\tilde{\alpha}}) \ , & B_{\mu\nu}(x^\alpha, -x^{\tilde{\alpha}}) &= -B_{\mu\nu}(x^\alpha, x^{\tilde{\alpha}}) \ , \\ G_{\mu\tilde{\nu}}(x^\alpha, -x^{\tilde{\alpha}}) &= -G_{\mu\tilde{\nu}}(x^\alpha, x^{\tilde{\alpha}}) \ , & B_{\mu\tilde{\nu}}(x^\alpha, -x^{\tilde{\alpha}}) &= B_{\mu\tilde{\nu}}(x^\alpha, x^{\tilde{\alpha}}) \ , \\ G_{\tilde{\mu}\tilde{\nu}}(x^\alpha, -x^{\tilde{\alpha}}) &= G_{\tilde{\mu}\tilde{\nu}}(x^\alpha, x^{\tilde{\alpha}}) \ , & B_{\tilde{\mu}\tilde{\nu}}(x^\alpha, -x^{\tilde{\alpha}}) &= -B_{\tilde{\mu}\tilde{\nu}}(x^\alpha, x^{\tilde{\alpha}}) \ , \end{aligned} \tag{10.4}$$

where the orientifold fixed plane is at $x^{\tilde{\alpha}} = 0$. These relations can be written in a compact way [130]

$$G_{mn}(x^\alpha, -x^{\tilde{\alpha}}) = D_m{}^r D_n{}^s G_{rs}(x^\alpha, x^{\tilde{\alpha}}) \ , \qquad B_{mn}(x^\alpha, -x^{\tilde{\alpha}}) = -D_m{}^r D_n{}^s B_{rs}(x^\alpha, x^{\tilde{\alpha}}) \ , \tag{10.5}$$

where the matrix D describing the conditions imposed by an Op-plane is the same as for a Dp-brane, c.f. equation (8.6) and section 8.1 in general. The graviton and Kalb-Ramond wave functions that solve the above conditions are given by

$$\begin{aligned}
G_{\mu\nu}(x^\alpha,-x^{\tilde\alpha}) &= \epsilon_{\mu\nu}e^{ik_\alpha x^\alpha}\cos(k_{\tilde\alpha}x^{\tilde\alpha}) \ , \qquad & B_{\mu\nu}(x^\alpha,-x^{\tilde\alpha}) &= i\epsilon_{\mu\nu}e^{ik_\alpha x^\alpha}\sin(k_{\tilde\alpha}x^{\tilde\alpha}) \ , \\
G_{\mu\tilde\nu}(x^\alpha,-x^{\tilde\alpha}) &= i\epsilon_{\mu\tilde\nu}e^{ik_\alpha x^\alpha}\sin(k_{\tilde\alpha}x^{\tilde\alpha}) \ , \qquad & B_{\mu\tilde\nu}(x^\alpha,-x^{\tilde\alpha}) &= \epsilon_{\mu\tilde\nu}e^{ik_\alpha x^\alpha}\cos(k_{\tilde\alpha}x^{\tilde\alpha}) \ , \\
G_{\tilde\mu\tilde\nu}(x^\alpha,-x^{\tilde\alpha}) &= \epsilon_{\tilde\mu\tilde\nu}e^{ik_\alpha x^\alpha}\cos(k_{\tilde\alpha}x^{\tilde\alpha}) \ , \qquad & B_{\tilde\mu\tilde\nu}(x^\alpha,-x^{\tilde\alpha}) &= i\epsilon_{\tilde\mu\tilde\nu}e^{ik_\alpha x^\alpha}\sin(k_{\tilde\alpha}x^{\tilde\alpha}) \ .
\end{aligned} \tag{10.6}$$

As for the disk the polarization tensors are orthogonal to their momenta $k_i^m\epsilon^i_{mn}=\epsilon^i_{mn}k_i^n=0$ and the momenta satisfy $k_i^2=0$, since we are considering massless states. Combining the above wave functions these can be rewritten in all ten spacetime dimensions as

$$\begin{aligned}
G_{mn} &= \frac{1}{2}\Big(\epsilon_{mn}e^{ik\cdot x}+(D\cdot\epsilon^{\mathrm{T}}\cdot D)_{mn}e^{ik\cdot D\cdot x}\Big) \ , \\
B_{mn} &= \frac{1}{2}\Big(\epsilon_{mn}e^{ik\cdot x}+(D\cdot\epsilon^{\mathrm{T}}\cdot D)_{mn}e^{ik\cdot D\cdot x}\Big) \ .
\end{aligned} \tag{10.7}$$

Therefore, a vertex operator representing the insertion of a closed string state on the real projective plane takes the form

$$\begin{aligned}
V_i^{\mathbb{RP}^2}(z_i,\overline{z}_i) &= \frac{1}{2}\Big(V_i[e_i,k_i](z_i)\overline{V}_i[\overline{e}_i,k_i](\overline{z}_i)+V_i[D\cdot\overline{e}_i,D\cdot k_i](z_i)\overline{V}_i[D\cdot e_i,D\cdot k_i](\overline{z}_i)\Big) \ , \\
U_i^{\mathbb{RP}^2}(z_i,\overline{z}_i) &= \frac{1}{2}\Big(U_i[e_i,k_i](z_i)\overline{U}_i[\overline{e}_i,k_i](\overline{z}_i)+U_i[D\cdot\overline{e}_i,D\cdot k_i](z_i)\overline{U}_i[D\cdot e_i,D\cdot k_i](\overline{z}_i)\Big) \ , \\
(V\otimes\overline{U})_i^{\mathbb{RP}^2}(z_i,\overline{z}_i) &= \frac{1}{2}\Big(V_i[e_i,k_i](z_i)\overline{U}_i[\overline{e}_i,k_i](\overline{z}_i)+V_i[D\cdot\overline{e}_i,D\cdot k_i](z_i)\overline{U}_i[D\cdot e_i,D\cdot k_i](\overline{z}_i)\Big) \ ,
\end{aligned} \tag{10.8}$$

where V and U are the usual unintegrated and integrated vertex operators in the pure spinor formalism, respectively, and we used $\epsilon_i=e_i\otimes\overline{e}_i$.

Similar as for the disk in (8.1) an operator $\mathcal{O}_{(h,\overline{h})}=O_h(z)\otimes\overline{O}_{\overline{h}}(\overline{z})$ approaching a crosscap state is a non-trivial process and puts constraints on the operator [180]:

$$0=\Big\langle C\Big|\Big[(\overline{z}')^h\overline{O}_h(\overline{z}')-z^hO_h(z)\Big]\Big|_{z=-\frac{1}{\overline{z}'}} \ , \tag{10.9}$$

where $\langle C|$ represents a crosscap state, which again imposes an interaction between the holomorphic and antiholomorphic sector. Moreover, in (10.9) the real projective plane is parametrized by the disk. Therefore, the worldsheet coordinates $z,\overline{z}\in D_2$. The above relation implies that when using the doubling operators the different sectors can be related as [27]

$$\overline{O}_{\overline{h}}(\overline{z})=\left(\frac{\partial z'}{\partial\overline{z}}\right)^{\overline{h}}O_{\overline{h}}(z') \quad \text{for } z'=-\frac{1}{\overline{z}} \ . \tag{10.10}$$

When mapping the fundamental domain of $\mathbb{RP}^2$ from the disk to the upper half plane via (8.3) the $\mathbb{Z}_2$ identification is invariant: $z' = -\frac{1}{\overline{z}} \mapsto w' = -\frac{1}{\overline{w}}$, where w are the coordinates on the upper half plane. Hence, the doubling trick becomes

$$\overline{O}_{\overline{h}}(\overline{w}) = \left(\frac{\partial w'}{\partial \overline{w}}\right)^{\overline{h}} O_{\overline{h}}(w') \quad \text{for } w' = -\frac{1}{\overline{w}} \ . \tag{10.11}$$

When using the doubling trick we have to account for the spacetime boundary conditions enforced by the Op-plane: As we have seen above for NSNS states we had to introduce the matrix D in (10.8). More generally, we will again substitute the right-moving fields according to (8.5) such that they also satisfy the constraints imposed by the O-plane. In addition, applying the doubling trick not only relates the holomorphic and antiholomorphic sector, but also converts the correlator on $\mathbb{RP}^2$ to a correlator on the disk, which makes it possible to use the two point functions in (8.9) to integrate out the $h = 1$ primary fields.

As in section 8.1 in (8.22) substituting the antiholomorphic fields by the holomorphic counterparts in (10.8) using the doubling trick (10.11) results in the following vertex operators for massless NSNS states

$$\begin{aligned}
V_i^{\mathbb{RP}^2}(z_i, \overline{z}_i) &= \frac{1}{2}\Big(V_i[e_i, k_i](z_i) V[D{\cdot}\overline{e}_i, D{\cdot}k_i]\Big(-\frac{1}{\overline{z}_i}\Big) + V_i[D{\cdot}\overline{e}_i, D{\cdot}k_i](z_i) V_i[e_i, k_i]\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \\
&= \frac{1}{2}\Big(V_i(z_i) V_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) + V_{\overline{\imath}}(z_i) V_i\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \ , \\
U_i^{\mathbb{RP}^2}(z_i, \overline{z}_i) &= \frac{1}{2}\frac{1}{\overline{z}_i^2}\Big(U_i[e_i, k_i](z_i) U_i[D{\cdot}\overline{e}_i, D{\cdot}k_i]\Big(-\frac{1}{\overline{z}_i}\Big) + U_i[D{\cdot}\overline{e}_i, D{\cdot}k_i](z_i) U_i[e_i, k_i]\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \\
&= \frac{1}{2}\frac{1}{\overline{z}_i^2}\Big(U_i(z_i) U_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) + U_{\overline{\imath}}(z_i) U_i\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \ .
\end{aligned} \tag{10.12}$$

Moreover, for the half unintegrated and half integrated vertex operator we get

$$\begin{aligned}
&(V \otimes \overline{U})_i^{\mathbb{RP}^2}(z_i, \overline{z}_i) = \\
&= \frac{1}{2}\frac{1}{\overline{z}_i^2}\Big(V_i[e_i, k_i](z_i) U_i[D{\cdot}\overline{e}_i, D{\cdot}k_i]\Big(-\frac{1}{\overline{z}_i}\Big) + V_i[D{\cdot}\overline{e}_i, D_i{\cdot}k_i](z_i) U_i[e_i, k_i]\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \\
&= \frac{1}{2}\frac{1}{\overline{z}_i^2}\Big(V_i(z_i) U_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) + V_{\overline{\imath}}(z_i) U_i\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \ ,
\end{aligned} \tag{10.13}$$

where we made use of the notation in (8.22). The derivative $\left(\frac{\partial z'}{\partial \overline{z}}\right)^{\overline{h}} = \left(\frac{1}{\overline{z}^2}\right)^{\overline{h}}$ for $z' = -\frac{1}{\overline{z}}$ coming from (10.11) explains the prefactor for the integrated vertex operator with $\overline{h} = 1$. Moreover, the vertex operator $U_i(z_i, \overline{z}_i)$ is integrated over the upper half plane such that

$$\begin{aligned}
\int_{\mathbb{H}_+} \mathrm{d}^2 z_i\, U_i^{\mathbb{RP}^2}(z_i, \overline{z}_i) &= \frac{1}{2}\int_{\mathbb{H}_+} \mathrm{d}^2 z_i\, \frac{1}{\overline{z}_i^2}\Big(U_i(z_i) U_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) + U_{\overline{\imath}}(z_i) U_i\Big(-\frac{1}{\overline{z}_i}\Big)\Big) \\
&= \frac{1}{2}\int_{\mathbb{H}_+} \mathrm{d}^2 z_i\, \frac{1}{\overline{z}_i^2} U_i(z_i) U_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) + \frac{1}{2}\int_{\mathbb{H}_-} \mathrm{d}^2 z_i\, \frac{1}{\overline{z}_i^2} U_{\overline{\imath}}\Big(-\frac{1}{\overline{z}_i}\Big) U_i(z_i)
\end{aligned}$$

$$= \frac{1}{2} \int_{\mathbb{C}} \mathrm{d}^2 z_i \, \frac{1}{\overline{z}_i^2} U_i(z_i) U_{\overline{i}}\Big(-\frac{1}{\overline{z}_i}\Big) \tag{10.14}$$

where we have performed the coordinate transformation $(z_i, \overline{z}_i) \to \big(-\frac{1}{\overline{z}_i}, -\frac{1}{z_i}\big)$ in the second term. On the disk it is not possible to combine $\mathbb{H}_+ \cup \mathbb{H}_- = \mathbb{C}$, because the disk amplitude (8.23) has poles along the real axis at $z_i - \overline{z}_i = 0$, c.f. the discussion at the end of section 8.3.2. For the amplitude on the real projective plane we can take $\mathbb{H}_+ \cup \mathbb{H}_- = \mathbb{C}$, because this amplitude does not exhibit poles on the real line. As we will see below, the corresponding term obeys $1 + z_i \overline{z}_i \neq 0$, because $|z_i|^2 \geq 0$.

Applying the results derived in this section the amplitude (10.1) becomes

$$\begin{aligned} \mathcal{A}_n^{\mathbb{RP}^2} = &-\Big(\frac{1}{2}\Big)^n i g_c^n T_p' \int_0^1 \mathrm{d}y \, \Big\langle\!\!\Big\langle (V_1(i) V_{\overline{1}}(-i) + V_{\overline{1}}(i) V_1(-i)) \\ &\times \frac{1}{y^2} \Big(V_2(iy) U_{\overline{2}}\Big(-\frac{i}{y}\Big) + V_{\overline{2}}(iy) U_2\Big(-\frac{i}{y}\Big)\Big) \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \, . \end{aligned} \tag{10.15}$$

After the change of variables in the amplitude $(y, z, \overline{z}) \to (-y, -z, -\overline{z})$ in the terms proportional to $V_{\overline{1}}(i) V_1(-i)$ in (10.15) we can utilize that the correlator is invariant under conformal transformations, i.e. we can rescale all vertex operators via (8.32) with $a = -1$ and get

$$\begin{aligned} \mathcal{A}_n^{\mathbb{RP}^2} = &-\Big(\frac{1}{2}\Big)^n i g_c^n T_p' \Big\langle\!\!\Big\langle \Big(\int_0^1 \mathrm{d}y \, V_1(i) V_{\overline{1}}(-i) - \int_{-1}^0 \mathrm{d}y \, V_{\overline{1}}(-i) V_1(i) \Big) \\ &\times \frac{1}{y^2} \Big(V_2(iy) U_{\overline{2}}\Big(-\frac{i}{y}\Big) + V_{\overline{2}}(iy) U_2\Big(-\frac{i}{y}\Big)\Big) \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \\ = &-\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-1}^1 \mathrm{d}y \, \frac{1}{y^2} \Big\langle\!\!\Big\langle V_1(i) V_{\overline{1}}(-i) \Big(V_2(iy) U_{\overline{2}}\Big(-\frac{i}{y}\Big) + V_{\overline{2}}(iy) U_2\Big(-\frac{i}{y}\Big)\Big) \\ &\times \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \, , \end{aligned} \tag{10.16}$$

where we used that the unintegrated vertex operators anticommute to proceed from the first to the second line. Afterwards, we perform the substitution $y \to -\frac{1}{y}$ in the terms proportional to $V_{\overline{2}}(iy) U_2\big(-\frac{i}{y}\big)$, which become after the transformation

$$\begin{aligned} &\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-1}^1 \mathrm{d}y \, \frac{1}{y^2} \Big\langle\!\!\Big\langle V_1(i) V_{\overline{1}}(-i) V_{\overline{2}}(iy) U_2\Big(-\frac{i}{y}\Big) \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \\ &= -\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-\infty}^{-1} \mathrm{d}y \, \Big\langle\!\!\Big\langle V_1(i) V_{\overline{1}}(-i) V_{\overline{2}}\Big(-\frac{i}{y}\Big) U_2(iy) \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \\ &\quad -\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_1^{\infty} \mathrm{d}y \, \Big\langle\!\!\Big\langle V_1(i) V_{\overline{1}}(-i) V_{\overline{2}}\Big(-\frac{i}{y}\Big) U_2(iy) \prod_{j=3}^n \int_{\mathbb{C}} \mathrm{d}^2 z_j \, \frac{1}{\overline{z}_j^2} U_j(z_j) U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) \Big\rangle\!\!\Big\rangle \, , \end{aligned} \tag{10.17}$$

in order to use the independence of the assignment of integrated and unintegrated vertex operators inside an amplitude, which was explicitly shown in [88] for closed string amplitudes on the disk, to exchange $V_{\overline{2}}\big(-\frac{i}{y}\big)U_2(iy) \to -\frac{1}{y^2}U_{\overline{2}}\big(-\frac{i}{y}\big)V_2(iy)$. Here, we provide the most important steps as the calculation is analogue to [88]. Without loss of generality we omit vertex operators with $n > 2$, as their BRST variation vanishes when integrating over them, i.e. when carefully applying Cauchy's theorem such that $QU_{\overline{j}}\big(-\frac{1}{\overline{z}_j}\big) = \overline{z}_j^2\overline{\partial}_j U_{\overline{j}}\big(-\frac{1}{\overline{z}_j}\big)$,

$$\begin{aligned}
Q\int_{\mathbb{C}} \mathrm{d}^2 z_j\, \frac{1}{\overline{z}_j^2} U_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) &= \int_{\mathbb{C}} \mathrm{d}^2 z_j\, \Big(\frac{1}{\overline{z}_j^2}\partial_j V_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big) + U_j(z_j)\overline{\partial}_j V_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\Big) \\
&= \int_{\mathbb{C}} \mathrm{d}^2 z_j\, \Big[\partial_j\Big(\frac{1}{\overline{z}_j^2}V_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\Big) + \overline{\partial}_j\Big(U_j(z_j)V_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\Big)\Big] \\
&= 0 \ ,
\end{aligned} \tag{10.18}$$

because the complex plane has no boundary. Hence, along the lines of the computation in appendix C of [88] we find

$$\begin{aligned}
\int_1^\infty \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_{\overline{2}}\Big(-\frac{i}{y}\Big)U_2(iy)\right\rangle\!\!\right\rangle &= \int_1^\infty \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{-\frac{i}{y}} \mathrm{d}z\, QU_{\overline{2}}(z)U_2(iy)\right\rangle\!\!\right\rangle \\
&= -i\int_1^\infty \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{-\frac{i}{y}} \mathrm{d}z\, U_{\overline{2}}(z)\partial_y V_2(iy)\right\rangle\!\!\right\rangle \\
&= -i\int_1^\infty \mathrm{d}y\, \partial_y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{-\frac{i}{y}} \mathrm{d}z\, U_{\overline{2}}(z)V_2(iy)\right\rangle\!\!\right\rangle \\
&\quad +i\int_1^\infty \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\partial_y\int_{-i}^{-\frac{i}{y}} \mathrm{d}z\, U_{\overline{2}}(z)V_2(iy)\right\rangle\!\!\right\rangle \\
&= -i\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{0} \mathrm{d}z\, U_{\overline{2}}(z)U_2(\infty)\right\rangle\!\!\right\rangle \\
&\quad - \int_1^\infty \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)U_{\overline{2}}\Big(-\frac{i}{y}\Big)V_2(iy)\right\rangle\!\!\right\rangle \ .
\end{aligned} \tag{10.19}$$

With a similar calculation we get

$$\begin{aligned}
\int_{-\infty}^{-1} \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_{\overline{2}}\Big(-\frac{i}{y}\Big)U_2(iy)\right\rangle\!\!\right\rangle &= i\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{0} \mathrm{d}z\, U_{\overline{2}}(z)V_2(-\infty)\right\rangle\!\!\right\rangle \\
&\quad - \int_{-\infty}^{-1} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)U_{\overline{2}}\Big(-\frac{i}{y}\Big)V_2(iy)\right\rangle\!\!\right\rangle
\end{aligned} \tag{10.20}$$

such that the two terms cancel:

$$-i\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{0} \mathrm{d}z\, U_{\overline{2}}(z)U_2(\infty)\right\rangle\!\!\right\rangle + i\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)\int_{-i}^{0} \mathrm{d}z\, U_{\overline{2}}(z)U_2(-\infty)\right\rangle\!\!\right\rangle = 0 \ , \tag{10.21}$$

because the amplitude does not distinguish whether a vertex operator is at ∞ or $-\infty$. The remaining terms after the reassignment of vertex operators are given by

$$\begin{aligned}
&\int_{-\infty}^{-1} \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_{\overline{2}}\Big(-\frac{i}{y}\Big)U_2(iy)\right\rangle\!\!\right\rangle + \int_1^{\infty} \mathrm{d}y \left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_{\overline{2}}\Big(-\frac{i}{y}\Big)U_2(iy)\right\rangle\!\!\right\rangle \\
&= -\int_{-\infty}^{-1} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}\Big(-\frac{i}{y}\Big)\right\rangle\!\!\right\rangle - \int_1^{\infty} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}\Big(-\frac{i}{y}\Big)\right\rangle\!\!\right\rangle
\end{aligned} \tag{10.22}$$

and imply that (10.17) becomes

$$\begin{aligned}
&\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-1}^{1} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_{\overline{2}}(iy)U_2\Big(-\frac{i}{y}\Big)\prod_{j=3}^{n}\int_{\mathbb{C}} \mathrm{d}^2 z_j\, \frac{1}{\overline{z}_j^2}U_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\right\rangle\!\!\right\rangle \\
&= \Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-\infty}^{-1} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}\Big(-\frac{i}{y}\Big)\prod_{j=3}^{n}\int_{\mathbb{C}} \mathrm{d}^2 z_j\, \frac{1}{\overline{z}_j^2}U_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\right\rangle\!\!\right\rangle \\
&\quad + \Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{1}^{\infty} \mathrm{d}y\, \frac{1}{y^2}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}\Big(-\frac{i}{y}\Big)\prod_{j=3}^{n}\int_{\mathbb{C}} \mathrm{d}^2 z_j\, \frac{1}{\overline{z}_j^2}U_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\right\rangle\!\!\right\rangle .
\end{aligned} \tag{10.23}$$

Adding the individual contributions, which are integrated over $y \in]-\infty,-1[$ and $y \in]1,\infty[$ together with $y \in]-1,1[$, provides the scattering amplitude prescription for closed strings on the real projective plane

$$\begin{aligned}
\mathcal{A}_n^{\mathbb{RP}^2} = &-\Big(\frac{1}{2}\Big)^{n-1} i g_c^n T_p' \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{y^2} e^{i\pi(\Theta(y-1)+\Theta(-y-1))}\left\langle\!\!\left\langle V_1(i)V_{\overline{1}}(-i)V_2(iy)U_{\overline{2}}\Big(-\frac{i}{y}\Big)\right.\right. \\
&\times \left.\left.\prod_{j=3}^{n}\int_{\mathbb{C}} \mathrm{d}^2 z_j\, \frac{1}{\overline{z}_j^2}U_j(z_j)U_{\overline{j}}\Big(-\frac{1}{\overline{z}_j}\Big)\right\rangle\!\!\right\rangle ,
\end{aligned} \tag{10.24}$$

where $e^{i\pi(\Theta(y-1)+\Theta(-y-1))}$ accounts for the different sign of the integration regions of $y \in]-1,1[$ and $y \in]-\infty,-1[\cup]1,\infty[$, which arises from the reassignment of vertex operators.

10.2 Analytic continuation and monodromy relations on unoriented surfaces

The scattering amplitude of three closed strings on the real projective plane follows from the prescription (10.24) for $n = 3$:

$$\begin{aligned}
\mathcal{A}_3^{\mathbb{RP}^2} &= 2 i g_c^3 T_p' \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z\, \langle\!\langle V_1^{\mathbb{RP}^2}(i,-i)(V\otimes\overline{U})_2^{\mathbb{RP}^2}(iy,-iy)U_3^{\mathbb{RP}^2}(z,\overline{z})\rangle\!\rangle \\
&= -\frac{1}{2} i g_c^3 T_p' \int_{-\infty}^{\infty} \mathrm{d}y \int_{\mathbb{C}} \mathrm{d}^2 z\, \frac{1}{y^2}\frac{1}{\overline{z}^2} e^{i\pi(\Theta(y-1)+\Theta(-y-1))}
\end{aligned}$$

$$\times \left\langle\!\!\left\langle V_1(i) V_{\overline{1}}(-i) V_2(iy) U_{\overline{2}}\Big(-\frac{i}{y}\Big) U_3(z) U_{\overline{3}}\Big(-\frac{1}{\overline{z}}\Big) \right\rangle\!\!\right\rangle , \tag{10.25}$$

whose Koba-Nielsen factor is given by [27]

$$\begin{aligned} \mathrm{KN}(y,z,\overline{z}) = {} & 2^{s_{\overline{1}1}} |1-y|^{2s_{21}} |1-y|^{2s_{\overline{2}1}} |1+y^2|^{s_{\overline{2}2}} |i-z|^{2s_{13}} |-i-z|^{2s_{\overline{3}1}} \\ & \times |iy-z|^{2s_{23}} |1-iyz|^{2s_{\overline{2}3}} |1+z\overline{z}|^{s_{3\overline{3}}} \ . \end{aligned} \tag{10.26}$$

The correlator of the amplitude is very similar as for three closed strings on the disk. Hence, the computation of the correlator in (10.25) is analogous as in appendix C subject to the additional factor $\frac{1}{y^2}\frac{1}{\overline{z}^2}$ and the modified positions $\overline{z}_2 = -\frac{i}{y}$ and $\overline{z}_3 = -\frac{1}{\overline{z}}$. The analytic continuation of (10.25) follows the same steps as in section 8.3.1. Therefore, we write the integral over the complex plane as two integrals over the real line. By splitting $z = z_1 + iz_2$ into real and imaginary part the integrand in (10.25) becomes an analytic function in z_1 with branch points at $z_1 = \pm i(1-z_2), \pm i(1+z_2), \pm i(y-z_2)$ and $\pm\frac{i}{y}(1+yz_2)$. Because all branch points are purely imaginary, we can rotate the z_1 contour from the real to the purely imaginary axis similar as in figure 8.1. Introducing the real variables

$$\xi = z_1 + z_2 \ , \qquad \eta = z_1 - z_2 \ , \tag{10.27}$$

which are not constrained here, i.e. $\xi, \eta \in \mathbb{R}$ as $z, \overline{z} \in \mathbb{C}$, leaves us with[1]

$$\begin{aligned} \mathcal{A}_3^{\mathbb{RP}^2} = {} & -\frac{1}{4} g_c^3 T_p' \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{1}{y^2}\frac{1}{\eta^2} \Pi(y,\xi,\eta) \\ & \times \left\langle\!\!\left\langle V_1(i) V_{\overline{1}}(-i) V_2(iy) U_{\overline{2}}\Big(-\frac{i}{y}\Big) U_3(i\xi) U_{\overline{3}}\Big(\frac{i}{\eta}\Big) \right\rangle\!\!\right\rangle \ . \end{aligned} \tag{10.28}$$

Moreover, including the monodromy phase $\Pi(y,\xi,\eta)$ ensures again that the integrand is holomorphic in ξ and η and accounts for the correct branch of the integrand. After absorbing $e^{i\pi(\Theta(y-1)+\Theta(-y-1))}$ into the phase factor we find with a similar derivation as in appendix D that the total phase takes the form

$$\begin{aligned} \Pi(y,\xi,\eta) = {} & e^{i\pi(\Theta(y-1)+\Theta(-y-1))} e^{i\pi s_{13}\Theta(-(1-\xi)(1+\eta))} e^{i\pi s_{1\overline{3}}\Theta(-(1+\xi)(1-\eta))} \\ & \times e^{i\pi s_{23}\Theta(-(y-\xi)(y+\eta))} e^{i\pi s_{2\overline{3}}\Theta(-(1+y\xi)(1-y\eta))} e^{i\pi s_{3\overline{3}}\Theta(-(1-\xi\eta))} \ . \end{aligned} \tag{10.29}$$

Furthermore, we pull out the factor of i in each of the vertex operators using (8.32) and conformal invariance of the correlator:

$$\begin{aligned} \mathcal{A}_3^{\mathbb{RP}^2} = {} & -\frac{1}{4} i g_c^3 T_p' \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{1}{y^2}\frac{1}{\eta^2} \Pi(y,\xi,\eta) \\ & \times \left\langle\!\!\left\langle V_1(1) V_{\overline{1}}(-1) V_2(y) U_{\overline{2}}\Big(-\frac{1}{y}\Big) U_3(\xi) U_{\overline{3}}\Big(\frac{1}{\eta}\Big) \right\rangle\!\!\right\rangle \ . \end{aligned} \tag{10.30}$$

[1] Even though, the vertex operator position $-\frac{i}{y}$ and $\frac{1}{\eta}$ diverge for $y, \eta \to 0$ the integrand in (10.28) is well defined, as the prefactor $\frac{1}{y^2}\frac{1}{\eta^2}$ cancels these divergences.

Due to the Θ-functions in the phase factor the integration over ξ and η splits into smaller intervals, in which the monodromy phase becomes constant and independent of the world-sheet coordinates. The individual integration patches and the according phases are listed in table 10.1 for $0 < y < 1$. The integration regions for the amplitude with $-\infty < y < -1$

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta < 0$	$0 < \eta < 1$	$1 < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta$
$\xi < -\frac{1}{y}$	1	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{2\bar{3}}}$	1

	$\eta < -1$	$-1 < \eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta < -y$	$-y < \eta < 0$	$0 < \eta < 1$	$1 < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta$
$-\frac{1}{y} < \xi < -1$	$e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}$	1	$e^{i\pi s_{2\bar{3}}}$

	$\eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta < -1$	$-1 < \eta < -y$	$-y < \eta < 0$	$0 < \eta < 1$	$1 < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta$
$-1 < \xi < 0$	$e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	1	1	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}$

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < 0$	$0 < \eta < 1$	$1 < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta$
$0 < \xi < y$	$e^{i\pi s_{13}}e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	1	1	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}$

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < 0$	$0 < \eta < \frac{1}{\xi}$	$1 < \eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta$
$y < \xi < 1$	$e^{i\pi s_{13}}$	1	$e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}$

	$\eta < -1$	$-1 < \eta < -y$	$-y < \eta < 0$	$0 < \eta < \frac{1}{\xi}$	$\frac{1}{\xi} < \eta < 1$	$1 < \eta < \frac{1}{y}$	$\frac{1}{y} < \eta$
$1 < \xi$	1	$e^{i\pi s_{13}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}$	1

Table 10.1: $\Pi(y, \xi, \eta)$ for each integration region in the (ξ, η)-plane for $0 < y < 1$.

have almost the same monodromy phases, but one has to exchange $k_2 \leftrightarrow D{\cdot}k_2$ and simultaneously $y \leftrightarrow -\frac{1}{y}$, because

$$\begin{aligned} & y \in]0,1[&& \text{and} && -\tfrac{1}{y} \in]-\infty,-1[&& \text{for } 0 < y < 1 \,, \\ & y \in]-\infty,-1[&& \text{and} && -\tfrac{1}{y} \in]0,1[&& \text{for } -\infty < y < -1 \,, \end{aligned} \tag{10.31}$$

such that the vertex operators 2 and $\overline{2}$ switch places. Therefore, the analysis of these two intervals of the y-integration is very similar and we will only discuss $0 < y < 1$ explicitly. Note that the other two y-integration regions, which are given by $y \in]-1, 0[$ and $y \in]1, \infty[$, behave similar and we only discuss $-1 < y < 0$ below.

Not all of these integration patches in table 10.1 are in correspondence with open string subamplitudes, as the integration boundary of η is not always given by another vertex operator position, i.e. they are only parts of subamplitudes. Similar as for the scattering of three closed strings on the disk in section 8.3.1 we encounter integration regions, which begin or end at $\pm\infty$. Since also here there is no vertex operator positioned at infinity, the corresponding integration regions are only parts of partial amplitudes, c.f. the discussion at the end of section 7.1 and footnote 3. Strictly speaking, splitting the y-integration into four integration regions, which is necessary to perform the analytic continuation, means that all integration regions are no longer complete partial amplitudes. But this gets easily resolved when combining two y-integrations. We explicitly discuss this below and show that taking all four integration regions for y leads to well defined open string subamplitudes. To streamline the following analysis we ignore this fact, as it does not influence any analytic properties or the branch cut structure of the (ξ, η)-integration except for the position of the y-dependent poles on the real line. The explicitly y-integration in the subamplitudes (7.23) follows from the context and is always denoted when considering the total amplitude, c.f. for example (10.32).

Nevertheless, the amplitude has the correct branch cut structure such that we can use Cauchy's theorem and build closed integration contours in the complex plane. This becomes clear when performing a coordinate transformation of the integration regions in table 10.2: We change $\eta \to \frac{1}{\eta}$ which puts η and ξ on equal grounds in the sense that after the change of variables the ξ- and η-integration have poles at the same vertex operator positions $(-\frac{1}{y}, -1, y, 1)$ and we obtain[2]

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1} = \frac{1}{4} i g_c^3 T_p' \Pi \int_0^1 \mathrm{d}y \int \mathrm{d}\xi \int \mathrm{d}\eta \, \frac{1}{y^2} \left\langle\!\!\left\langle V_1(1) V_{\bar{1}}(-1) V_2(y) U_{\bar{2}}\Big(-\frac{1}{y}\Big) U_3(\xi) U_{\bar{3}}(\eta) \right\rangle\!\!\right\rangle , \tag{10.32}$$

where the integration regions are given in table 10.2. As mentioned above these integration

	$-1<\eta<0$	$-\frac{1}{y}<\eta<-1$	$\xi<\eta<-\frac{1}{y}$	$\eta<\xi$	$1<\eta$	$y<\eta<1$	$0<\eta<y$
$\xi<-\frac{1}{y}$	1	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{2\bar{3}}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{2\bar{3}}}$	1

	$-1<\eta<0$	$\xi<\eta<-1$	$\frac{1}{y}<\eta<\xi$	$\eta<-\frac{1}{y}$	$1<\eta$	$y<\eta<1$	$0<\eta<y$
$-\frac{1}{y}<\xi<-1$	$e^{i\pi s_{13}} e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}}$	1	$e^{i\pi s_{2\bar{3}}}$

	$\xi<\eta<0$	$-1<\eta<\xi$	$-\frac{1}{y}<\eta<-1$	$\eta<-\frac{1}{y}$	$1<\eta$	$y<\eta<1$	$0<\eta<y$
$-1<\xi<0$	$e^{i\pi s_{13}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{13}} e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	1	1	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}}$

	$-1<\eta<0$	$-\frac{1}{y}<\eta<-1$	$\eta<-\frac{1}{y}$	$1<\eta$	$y<\eta<1$	$\xi<\eta<y$	$0<\eta<\xi$
$0<\xi<y$	$e^{i\pi s_{13}} e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	1	1	$e^{i\pi s_{1\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{2\bar{3}}} e^{i\pi s_{3\bar{3}}}$

	$-1<\eta<0$	$-\frac{1}{y}<\eta<-1$	$\eta<-\frac{1}{y}$	$1<\eta$	$\xi<\eta<1$	$y<\eta<\xi$	$0<\eta<y$
$y<\xi<1$	$e^{i\pi s_{13}}$	1	$e^{i\pi s_{23}}$	$e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{2\bar{3}}} e^{i\pi s_{3\bar{3}}}$

	$-1<\eta<0$	$-\frac{1}{y}<\eta<-1$	$\eta<-\frac{1}{y}$	$\xi<\eta$	$1<\eta<\xi$	$y<\eta<1$	$0<\eta<y$
$1<\xi$	1	$e^{i\pi s_{13}}$	$e^{i\pi s_{13}} e^{i\pi s_{23}}$	$e^{i\pi s_{13}} e^{i\pi s_{23}}$	$e^{i\pi s_{13}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	$e^{i\pi s_{13}} e^{i\pi s_{1\bar{3}}} e^{i\pi s_{23}} e^{i\pi s_{3\bar{3}}}$	1

Table 10.2: $\Pi(y, \xi, \eta)$ for each integration region in the (ξ, η)-plane for $0 < y < 1$ after the transformation $\eta \to \frac{1}{\eta}$.

region can be written as open string subamplitudes (7.23) or parts of partial amplitudes after the transformation, where again the integration over y from 0 to 1 is singled out.[3] The $PSL(2, \mathbb{R})$ position fixing of (10.32) corresponds to $z_{i_1} \equiv \bar{z}_1 = -1, z_{i_2} \equiv z_1 = 1$

[2]When performing a coordinate transformation only the integration patches changes but not their respective phases, since the phase factor becomes independent of the worldsheet coordinates. This can be seen explicitly by comparing table 10.1 and table 10.2. Hence, for the phase Π we dropped the dependence on the worldsheet coordinates but kept Π to indicate that each integration patch enters with a monodromy phase.

[3]For concreteness, we are only considering $0 < y < 1$ here, but this also holds for the other y-integration regions.

and $z_{i_3} \equiv z_2 = y$ in the open string subamplitude. The integration where either the ξ or η-integration begins or ends at 0 are not partial amplitudes, because 0 is not a vertex operator position.[4] Therefore, we introduce partial subamplitudes like

$$\mathbb{A}(\overline{2},\overline{1},\overline{3}|2,3,1) = \frac{1}{4} i g_c^3 T_p' \int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}\xi \int_{-1}^0 \mathrm{d}\eta \, \frac{1}{y^2} \left\langle\!\!\left\langle V_1(1) V_{\overline{1}}(-1) V_2(iy) U_{\overline{2}}\left(-\frac{1}{y}\right) U_3(\xi) U_{\overline{3}}(\eta) \right\rangle\!\!\right\rangle \tag{10.33}$$

such that the notation $\mathbb{A}(\ldots,\overline{3}|\ldots)$ implies that the integration region for η ends at 0. Similar, for $\mathbb{A}(\ldots|\overline{3},\ldots)$ the corresponding integration region for η begins at 0. The same holds for ξ, but there it would be in principle possible to combine some of these integration regions into proper open string subamplitudes. Hence, if we have two subamplitudes with the same monodromy phase, they can be joined as $\mathbb{A}(\ldots,3|\ldots) + \mathbb{A}(\ldots|3,\ldots) = \mathbb{A}(\ldots,3,\ldots)$. With this notation and momentum conservation (10.32) becomes

$$\begin{aligned}
\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1} = {} & \mathbb{A}(3,\overline{2},\overline{1},\overline{3}|2,1) + e^{-i\pi s_{13}} \mathbb{A}(3,\overline{2},\overline{3},\overline{1},2,1) + e^{-i\pi(s_{13}+s_{23})} \mathbb{A}(3,\overline{3},\overline{2},\overline{1},2,1) \\
& + e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \mathbb{A}(\overline{3},3,\overline{2},\overline{1},2,1) + e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \mathbb{A}(3,\overline{2},\overline{1},2,1,\overline{3}) \\
& + e^{i\pi s_{2\overline{3}}} \mathbb{A}(3,\overline{2},\overline{1},2,\overline{3},1) + \mathbb{A}(3,\overline{2},\overline{1}|\overline{3},2,1) + e^{-i\pi s_{2\overline{3}}} \mathbb{A}(\overline{2},3,\overline{1},\overline{3}|2,1) \\
& + e^{-i\pi(s_{13}+s_{2\overline{3}})} \mathbb{A}(\overline{2},3,\overline{3},\overline{1},2,1) + e^{i\pi(s_{1\overline{3}}+s_{23})} \mathbb{A}(\overline{2},\overline{3},3,\overline{1},2,1) \\
& + e^{i\pi s_{1\overline{3}}} \mathbb{A}(\overline{3},\overline{2},3,\overline{1},2,1) + e^{i\pi s_{1\overline{3}}} \mathbb{A}(\overline{2},3,\overline{1},2,1,\overline{3}) + \mathbb{A}(\overline{2},3,\overline{1},2,\overline{3},1) \\
& + e^{i\pi s_{2\overline{3}}} \mathbb{A}(\overline{2},3,\overline{1}|\overline{3},2,1) + e^{i\pi(s_{13}+s_{23}+s_{3\overline{3}})} \mathbb{A}(\overline{2},\overline{1},3,\overline{3}|2,1)+ \\
& + e^{i\pi(s_{13}+s_{23})} \mathbb{A}(\overline{2},\overline{1},\overline{3},3|2,1) + e^{i\pi s_{23}} \mathbb{A}(\overline{2},\overline{3},\overline{1},3|2,1) + \mathbb{A}(\overline{3},\overline{2},\overline{1},3|2,1) \\
& + \mathbb{A}(\overline{2},\overline{1},3|2,1,\overline{3}) + e^{i\pi s_{1\overline{3}}} \mathbb{A}(\overline{2},\overline{1},3|2,\overline{3},1) + e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \mathbb{A}(\overline{2},\overline{1},3|\overline{3},2,1) \\
& + e^{i\pi(s_{13}+s_{23})} \mathbb{A}(\overline{2},\overline{1},\overline{3}|3,2,1) + e^{i\pi s_{23}} \mathbb{A}(\overline{2},\overline{3},\overline{1}|3,2,1) + \mathbb{A}(\overline{3},\overline{2},\overline{1}|3,2,1) \\
& + \mathbb{A}(\overline{2},\overline{1}|3,2,1,\overline{3}) + e^{i\pi s_{1\overline{3}}} \mathbb{A}(\overline{2},\overline{1}|3,2,\overline{3},1) + e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \mathbb{A}(\overline{2},\overline{1}|3,\overline{3},2,1) \\
& + e^{-i\pi(s_{13}+s_{23})} \mathbb{A}(\overline{2},\overline{1}|\overline{3},3,2,1) + e^{i\pi s_{13}} \mathbb{A}(\overline{2},\overline{1},\overline{3}|2,3,1) + \mathbb{A}(\overline{2},\overline{3},\overline{1},2,3,1) \\
& + e^{i\pi s_{23}} \mathbb{A}(\overline{3},\overline{2},\overline{1},2,3,1) + e^{i\pi s_{23}} \mathbb{A}(\overline{2},\overline{1},2,3,1,\overline{3}) + e^{i\pi(s_{1\overline{3}}+s_{23})} \mathbb{A}(\overline{2},\overline{1},2,3,\overline{3},1) \\
& + e^{-i\pi(s_{13}+s_{2\overline{3}})} \mathbb{A}(\overline{2},\overline{1},2,\overline{3},3,1) + e^{-i\pi s_{13}} \mathbb{A}(\overline{2},\overline{1}|\overline{3},2,3,1) + \mathbb{A}(\overline{2},\overline{1},\overline{3}|2,1,3) \\
& + e^{i\pi s_{13}} \mathbb{A}(\overline{2},\overline{3},\overline{1},2,1,3) + e^{i\pi(s_{13}+s_{23})} \mathbb{A}(\overline{3},\overline{2},\overline{1},2,1,3) \\
& + e^{i\pi(s_{13}+s_{23})} \mathbb{A}(\overline{2},\overline{1},2,1,3,\overline{3}) + e^{i\pi(s_{13}+s_{23}+s_{3\overline{3}})} \mathbb{A}(\overline{2},\overline{1},2,1,\overline{3},3) \\
& + e^{-i\pi s_{2\overline{3}}} \mathbb{A}(\overline{2},\overline{1},2,\overline{3},1,3) + \mathbb{A}(\overline{2},\overline{1}|\overline{3},2,1,3) \; .
\end{aligned} \tag{10.34}$$

Even though, there are integration regions like (10.33) we can find closed contours appearing in (10.34), which give rise to monodromy relations. For the integration patches for which $-1 < \eta < 0$ and $0 < \eta < y$, i.e. the first and last column in table 10.2, we find the

[4] For the y-integration we do not use this notation, because for $y \in]0,1[$ the vertex operator position $\overline{z}_2 = -\frac{1}{y} \in]-\infty,1[$ and furthermore we have already mentioned that this integration is singled out.

relations

$$\begin{aligned}0 &= \mathbb{A}(3,\overline{2},\overline{1},\overline{3}|2,1) + e^{i\pi s_{13}}e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},3,\overline{1},\overline{3}|2,1)\\ &\quad + e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},\overline{1},3,\overline{3}|2,1) + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{2},\overline{1},\overline{3},3|2,1)\\ &\quad + e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\overline{2},\overline{1},\overline{3}|3,2,1) + e^{i\pi s_{13}}\mathbb{A}(\overline{2},\overline{1},\overline{3}|2,3,1) + \mathbb{A}(\overline{2},\overline{1},\overline{3}|2,1,3)\ ,\\ 0 &= \mathbb{A}(3,\overline{2},1|\overline{3},2,1) + e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},3,1|\overline{3},2,1) + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},1,3|\overline{3},2,1)\\ &\quad + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},1|3,\overline{3},2,1) + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},1|\overline{3},3,2,1)\\ &\quad + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},1|\overline{3},2,3,1) + \mathbb{A}(\overline{2},1|\overline{3},2,1,3)\end{aligned} \tag{10.35}$$

and for the upper most and lowest row in table 10.2, which correspond to $\xi < -\frac{1}{y}$ and $1 < \xi$ the following equations hold

$$\begin{aligned}0 &= \mathbb{A}(\overline{3},3,\overline{2},\overline{1},2,1) + e^{i\pi s_{3\overline{3}}}\mathbb{A}(3,\overline{3},\overline{2},\overline{1},2,1) + e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(3,\overline{2},\overline{3},\overline{1},2,1)\\ &\quad + e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(3,\overline{2},\overline{1},\overline{3},2,1) + e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(3,\overline{2},\overline{1},2,\overline{3},1)\\ &\quad + \mathbb{A}(3,\overline{2},\overline{1},2,1,\overline{3})\ ,\\ 0 &= \mathbb{A}(\overline{3},\overline{2},\overline{1},2,1,3) + e^{-i\pi s_{23}}\mathbb{A}(\overline{2},\overline{3},\overline{1},2,1,3) + e^{-i\pi s_{13}}e^{-i\pi s_{23}}\mathbb{A}(\overline{2},\overline{1},\overline{3},2,1,3)\\ &\quad + e^{-i\pi s_{13}}e^{-i\pi s_{23}}e^{-i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},\overline{1},2,\overline{3},1,3) + e^{-i\pi s_{13}}e^{-i\pi s_{1\overline{3}}}e^{-i\pi s_{23}}e^{-i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},\overline{1},2,1,\overline{3},3)\\ &\quad + \mathbb{A}(3,\overline{2},\overline{1},2,1,\overline{3})\ .\end{aligned} \tag{10.36}$$

Finally, we want to rewrite the remaining η-integration for $-\frac{1}{y} < \xi < -1$ and the left over ξ-integration for $y < \eta < 1$ by applying the monodromy relations

$$\begin{aligned}0 &= \mathbb{A}(3,\overline{2},\overline{1},2,\overline{3},1) + e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},3,\overline{1},2,\overline{3},1) + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},\overline{1},3,2,\overline{3},1)\\ &\quad + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},\overline{1},2,3,\overline{3},1) + e^{i\pi s_{1\overline{3}}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},\overline{1},2,\overline{3},3,1)\\ &\quad + \mathbb{A}(\overline{2},3,\overline{1},2,1,3)\ ,\\ 0 &= \mathbb{A}(\overline{3},\overline{2},3,\overline{1},2,1) + e^{i\pi s_{23}}\mathbb{A}(\overline{2},\overline{3},3,\overline{1},2,1) + e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},3,\overline{3},\overline{1},2,1)\\ &\quad + e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},3,\overline{1},\overline{3},2,1) + e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\overline{3}}}e^{i\pi s_{3\overline{3}}}\mathbb{A}(\overline{2},3,\overline{1},2,\overline{3},1)\\ &\quad + \mathbb{A}(\overline{2},3,\overline{1},2,1,\overline{3})\ .\end{aligned} \tag{10.37}$$

Contrary to the other monodromy relations above we have to multiply the first equation in (10.37) by $e^{-i\pi s_{2\overline{3}}}$ and the second relation by $e^{i\pi s_{1\overline{3}}}$ to reduce the number of partial subamplitudes in (10.34). In the end, we obtain

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1} &= -\mathbb{A}(3,\overline{2},\overline{1},\overline{3},2,1) - e^{i\pi s_{2\overline{3}}}\mathbb{A}(3,\overline{2},\overline{1},2,\overline{3},1) - e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},3,\overline{1},\overline{3},2,1)\\ &\quad - \mathbb{A}(\overline{2},\overline{1},2,\overline{3},1) + \mathbb{A}(\overline{3},\overline{2},\overline{1},3,2,1) + e^{i\pi s_{23}}\mathbb{A}(\overline{2},\overline{3},\overline{1},3,2,1) + \mathbb{A}(\overline{2},\overline{1},3,2,1,\overline{3})\\ &\quad + e^{i\pi s_{23}}\mathbb{A}(\overline{3},\overline{2},\overline{1},2,3,1) + \mathbb{A}(\overline{2},\overline{3},\overline{1},2,3,1) + e^{i\pi s_{23}}\mathbb{A}(\overline{2},\overline{1},2,3,1,\overline{3})\\ &\quad - \mathbb{A}(\overline{2},\overline{1},\overline{3},2,1,3) - e^{i\pi s_{2\overline{3}}}\mathbb{A}(\overline{2},\overline{1},2,\overline{3},1,3)\ ,\end{aligned} \tag{10.38}$$

which is written in terms of only open subamplitudes (7.23). The integration regions of the partial amplitudes are listed in table 10.3.

	$\eta<\xi$	$\xi<\eta<-\frac{1}{y}$	$-\frac{1}{y}<\eta<-1$	$-1<\eta<y$	$y<\eta<1$	$1<\eta$
$\xi<-\frac{1}{y}$				-1	$-e^{-i\pi s_{2\bar{3}}}$	

	$\eta<-\frac{1}{y}$	$-\frac{1}{y}<\eta<\xi$	$\xi<\eta<-1$	$-1<\eta<y$	$y<\eta<1$	$1<\eta$
$-\frac{1}{y}<\xi<-1$				$-e^{-i\pi s_{2\bar{3}}}$	-1	

	$\eta<-\frac{1}{y}$	$-\frac{1}{y}<\eta<-1$	$-1<\eta<\xi$	$\xi<\eta<y$	$y<\eta<1$	$1<\eta$
$-1<\xi<y$	1	$e^{i\pi s_{23}}$				1

	$\eta<-\frac{1}{y}$	$-\frac{1}{y}<\eta<-1$	$-1<\eta<y$	$y<\eta<\xi$	$\xi<\eta<1$	$1<\eta$
$y<\xi<1$	$e^{i\pi s_{23}}$	1				$e^{i\pi s_{23}}$

	$\eta<-\frac{1}{y}$	$-\frac{1}{y}<\eta<-1$	$-1<\eta<y$	$y<\eta<1$	$1<\eta<\xi$	$\xi<\eta$
$1<\xi$			-1	$-e^{i\pi s_{2\bar{3}}}$		

Table 10.3: $\Pi(y,\xi,\eta)$ for each integration region in the (ξ,η)-plane for $0<y<1$ after applying monodromy relations.

Similar as for the scattering of three closed strings off a Dp brane, we want to change the $PSL(2,\mathbb{R})$-frame to perform the transition $\mathbb{A}\to\mathcal{A}$ of the subamplitudes. Therefore, we apply the $PSL(2,\mathbb{R})$-transformation (8.44) and thereby introduce the new variables (8.45) such that the amplitude can be written as

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1}=\frac{1}{8}ig_c^3T_p'\Pi\int_0^1\mathrm{d}x\int\mathrm{d}\tilde{\xi}\int\mathrm{d}\tilde{\eta}\,\langle\!\langle V_{\bar{1}}(0)U_{\bar{2}}(-x)U_3(\tilde{\xi})U_{\bar{3}}(\tilde{\eta})V_2(1)V_1(\infty)\rangle\!\rangle \tag{10.39}$$

where the boundaries of the $\tilde{\xi}$ and $\tilde{\eta}$-integration can be found in table 10.4. Moreover, in (10.39) we made use of (8.48) and that $-\frac{1}{y}$ gets mapped to $-x$ by (8.44). The integration over x is still singled out, as the integration range $-x\in]-1,0[$ does not correspond to an open string integral.[5] Nevertheless, this is not a problem, because the integration over y from $-\infty$ to -1 will be mapped to $-x\in]-\infty,-1[$ such that they can be combined to form proper subamplitudes.

The explicit calculation to get (10.39) is similar as in appendix C. After combining partial amplitudes $\mathbb{A}$ along (8.50), c.f. also section 7.2, the individual integration patches

[5] As for the scattering of three closed strings on the disk the $PSL(2,\mathbb{R})$-transformation (8.48) maps the integration over $0<y<1$ onto $0<x<1$.

of the amplitude (10.39) are given in table 10.4 and can be written in terms of (7.27) as[6]

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1} = {} & e^{i\pi s_{23}}\mathcal{A}(\overline{3},\overline{2},\overline{1},2,3,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},2,3,1) + \mathcal{A}(\overline{3},\overline{2},\overline{1},3,2,1)) \\ & + e^{i\pi s_{23}}\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1) - e^{-i\pi s_{2\overline{3}}}\mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1) - \mathcal{A}(\overline{2},3,\overline{1},2,\overline{3},1) \\ & - \mathcal{A}(3,\overline{2},\overline{1},\overline{3},2,1) - e^{-i\pi s_{2\overline{3}}}\mathcal{A}(3,\overline{2},\overline{1},2,\overline{3},1)\ . \end{aligned} \tag{10.40}$$

As discussed in section 8.3.2 these partial amplitudes can be written in a basis with di-

	$\tilde\eta < \tilde\xi$	$\tilde\xi < \tilde\eta < -x$	$-x < \tilde\eta < 0$	$0 < \tilde\eta < 1$	$1 < \tilde\eta$
$\tilde\xi < -x$				-1	$-e^{-i\pi s_{2\overline{3}}}$

	$\tilde\eta < -x$	$-x < \tilde\eta < \tilde\xi$	$\tilde\xi < \tilde\eta < 0$	$0 < \tilde\eta < 1$	$1 < \tilde\eta$
$-x < \tilde\xi < 0$				$-e^{-i\pi s_{2\overline{3}}}$	-1

	$\tilde\eta < -x$	$-x < \tilde\eta < 0$	$0 < \tilde\eta < \tilde\xi$	$\tilde\xi < \tilde\eta < 1$	$1 < \tilde\eta$
$0 < \tilde\xi < 1$	1	$e^{i\pi s_{23}}$			

	$\tilde\eta < -x$	$-x < \tilde\eta < 0$	$0 < \tilde\eta < 1$	$1 < \tilde\eta < \tilde\xi$	$\tilde\xi < \tilde\eta$
$1 < \tilde\xi$	$e^{i\pi s_{23}}$	1			

Table 10.4: $\Pi(x,\tilde\xi,\tilde\eta)$ for each integration region in the $(\tilde\xi,\tilde\eta)$-plane for $0 < x < 1$.

mension six. Therefore, we can use monodromy relations between the subamplitudes in (10.40) to simplify the expression further. One can recognize that all of the subamplitudes have a neighbouring subamplitude, i.e. a partial amplitude that can be obtained by one permutation of the labels, which in addition has the correct monodromy phase so that we can consider the relations

$$\begin{aligned} 0 = {} & \mathcal{A}(3,\overline{2},\overline{1},2,\overline{3},1) + \mathcal{A}(\overline{2},3,\overline{1},2,\overline{3},1)e^{i\pi s_{2\overline{3}}} + \mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \\ & + \mathcal{A}(\overline{2},\overline{1},2,3,\overline{3},1)e^{i\pi(s_{1\overline{3}}+s_{23}+s_{2\overline{3}})} + \mathcal{A}(\overline{2},\overline{1},2,\overline{3},3,1)e^{i\pi(s_{1\overline{3}}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \\ 0 = {} & \mathcal{A}(3,\overline{2},\overline{1},\overline{3},2,1) + \mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)e^{-i\pi s_{2\overline{3}}} + \mathcal{A}(\overline{2},\overline{1},3,\overline{3},2,1)e^{-i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \\ & + \mathcal{A}(\overline{2},\overline{1},\overline{3},3,2,1)e^{-i\pi(s_{1\overline{3}}+s_{2\overline{3}}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)e^{-i\pi(s_{1\overline{3}}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \\ 0 = {} & \mathcal{A}(\overline{3},\overline{2},\overline{1},3,2,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1)e^{i\pi s_{23}} + \mathcal{A}(\overline{2},\overline{1},\overline{3},3,2,1)e^{i\pi(s_{13}+s_{23})} \\ & + \mathcal{A}(\overline{2},\overline{1},3,\overline{3},2,1)e^{i\pi(s_{13}+s_{23}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)e^{i\pi(s_{13}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \\ 0 = {} & \mathcal{A}(\overline{3},\overline{2},\overline{1},2,3,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},2,3,1)e^{-i\pi s_{23}} + \mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)e^{-i\pi(s_{13}+s_{23})} \end{aligned}$$

[6]Also here the amplitudes are only parts of subamplitudes, as the x-integration runs from 0 to 1, c.f. also the comment above. As before we ignore this subtlety here, but make it explicit in the end result (10.42) of this computation.

$$+\mathcal{A}(\overline{2},\overline{1},2,\overline{3},3,1)e^{-i\pi(s_{13}+s_{23}+s_{2\overline{3}})}+\mathcal{A}(\overline{2},\overline{1},2,3,\overline{3},1)e^{-i\pi(s_{13}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \quad (10.41)$$

and express the amplitude (10.40) in terms of only two open string partial amplitudes

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1}=2i\sin(\pi s_{1\overline{3}})\mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)\Big|_{0<x<1}+2i\sin(\pi s_{13})\mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)\Big|_{0<x<1}\ . \quad (10.42)$$

With an analogous calculation we get for $-\infty<y<-1$

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-\infty<y<-1}=2i\sin(\pi s_{1\overline{3}})\mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)\Big|_{1<x<\infty}+2i\sin(\pi s_{13})\mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)\Big|_{1<x<\infty}\ , \quad (10.43)$$

which is not the same as (10.42), because the x-integration in (10.43) goes from 1 to ∞. After performing the coordinate change $x\to -x$ we can combine the two results and obtain[7]

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-\infty<y<-1}+\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{0<y<1}&=2i\sin(\pi s_{1\overline{3}})\mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)+2i\sin(\pi s_{13})\mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)\\&=\frac{1}{4}g_c^3T_p'\sum_{\sigma\in S_3}\Big\{\sin(\pi s_{1\overline{3}})F_{\mathcal{I}_1}^{\sigma(\overline{2},3,\overline{3})}+\sin(\pi s_{13})F_{\mathcal{I}_2}^{\sigma(\overline{2},3,\overline{3})}\Big\}\\&\quad\times A_{\rm YM}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1)\ ,\end{aligned} \quad (10.44)$$

where we have used the results from appendix C to express the correlator of (10.42) and (10.43) in terms of YM amplitudes and hypergeometric integrals. The hypergeometric integrals are defined as in (8.60), but their integration regions are given by

$$\begin{aligned}\mathcal{I}_1&=\Big\{x\in\mathbb{R}\cup(\xi,\eta)\in\mathbb{R}^2|-\infty<x<0,0<\xi<1,1<\eta<\infty\Big\}\ ,\\\mathcal{I}_2&=\Big\{x\in\mathbb{R}\cup(\xi,\eta)\in\mathbb{R}^2|-\infty<x<0,1<\xi<\infty,0<\eta<1\Big\}\ .\end{aligned} \quad (10.45)$$

The computation for $-1<y<0$ is very similar as for $0<y<1$, which also holds for the integration over $1<y<\infty$ and $-\infty<y<-1$ so that we will only display the most important steps as before only for $-1<y<0$. After the transformation $\eta\to\frac{1}{\eta}$ the amplitude

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0}&=\frac{1}{4}ig_c^3T_p'\int_{-1}^0\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\mathrm{d}\eta\,\frac{1}{y^2}\Pi(y,\xi,\eta)\\&\quad\times\left\langle\!\!\left\langle V_1(1)V_{\overline{1}}(-1)V_2(iy)U_{\overline{2}}\Big(-\frac{1}{y}\Big)U_3(\xi)U_{\overline{3}}(\eta)\right\rangle\!\!\right\rangle\end{aligned} \quad (10.46)$$

can be written in terms of open string subamplitudes (7.23) as

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0}=\mathbb{A}(3,\overline{1},2,\overline{3}|1,\overline{2})+e^{-i\pi s_{2\overline{3}}}\mathbb{A}(3,\overline{1},\overline{3},2,1,\overline{2})+e^{-i\pi(s_{13}+s_{2\overline{3}})}\mathbb{A}(3,\overline{3},\overline{1},2,1,\overline{2})$$

[7] This is possible, because the amplitude has no pole at $x=-1$ after the transformation. Moreover, we are considering complete subamplitudes (7.27) here.

$$
\begin{aligned}
&+e^{i\pi(s_{1\bar{3}}+s_{23})}\mathbb{A}(\bar{3},3,\bar{1},2,1,\bar{2})+e^{i\pi(s_{1\bar{3}}+s_{23})}\mathbb{A}(3,\bar{1},2,1,\bar{2},\bar{3})\\
&+e^{i\pi s_{1\bar{3}}}\mathbb{A}(3,\bar{1},2,1,\bar{3},\bar{2})+\mathbb{A}(3,\bar{1},2|\bar{3},1,\bar{2})+e^{-i\pi s_{1\bar{3}}}\mathbb{A}(\bar{1},3,2,\bar{3}|1,\bar{2})\\
&+e^{i\pi(s_{13}+s_{23}+s_{3\bar{3}})}\mathbb{A}(\bar{1},3,\bar{3},2,1,\bar{2})+e^{i\pi(s_{13}+s_{23})}\mathbb{A}(\bar{1},\bar{3},3,2,1,\bar{2})\\
&+e^{i\pi s_{23}}\mathbb{A}(\bar{3},\bar{1},3,2,1,\bar{2})+e^{i\pi s_{23}}\mathbb{A}(\bar{1},3,2,1,\bar{2},\bar{3})+\mathbb{A}(\bar{1},3,2,1,\bar{3},\bar{2})\\
&+e^{i\pi s_{1\bar{3}}}\mathbb{A}(\bar{1},3,2|\bar{3},1,\bar{2})+e^{i\pi(s_{13}+s_{2\bar{3}}+s_{3\bar{3}})}\mathbb{A}(\bar{1},2,3,\bar{3}|1,\bar{2})\\
&+e^{i\pi(s_{13}+s_{2\bar{3}})}\mathbb{A}(\bar{1},2,\bar{3},3|1,\bar{2})+e^{i\pi s_{13}}\mathbb{A}(\bar{1},\bar{3},2,3|1,\bar{2})+\mathbb{A}(\bar{3},\bar{1},2,3|1,\bar{2})\\
&+\mathbb{A}(\bar{1},2,3|1,\bar{2},\bar{3})+e^{i\pi s_{23}}\mathbb{A}(\bar{1},2,3|1,\bar{3},\bar{2})+e^{i\pi(s_{1\bar{3}}+s_{23})}\mathbb{A}(\bar{1},2,3|\bar{3},1,\bar{2})\\
&+e^{i\pi(s_{13}+s_{2\bar{3}})}\mathbb{A}(\bar{1},2,\bar{3}|3,1,\bar{2})+e^{i\pi s_{13}}\mathbb{A}(\bar{1},\bar{3},2|3,1,\bar{2})+\mathbb{A}(\bar{3},\bar{1},2|3,1,\bar{2})\\
&+\mathbb{A}(\bar{1},2|3,1,\bar{2},\bar{3})+e^{i\pi s_{23}}\mathbb{A}(\bar{1},2|3,1,\bar{3},\bar{2})+e^{i\pi(s_{1\bar{3}}+s_{23})}\mathbb{A}(\bar{1},2|3,\bar{3},1,\bar{2})\\
&+e^{-i\pi(s_{13}+s_{2\bar{3}})}\mathbb{A}(\bar{1},2|\bar{3},3,1,\bar{2})+e^{i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,\bar{3}|1,3,\bar{2})+\mathbb{A}(\bar{1},\bar{3},2,1,3,\bar{2})\\
&+e^{i\pi s_{13}}\mathbb{A}(\bar{3},\bar{1},2,1,3,\bar{2})+e^{i\pi s_{13}}\mathbb{A}(\bar{1},2,1,3,\bar{2},\bar{3})+e^{i\pi(s_{13}+s_{23})}\mathbb{A}(\bar{1},2,1,3,\bar{3},\bar{2})\\
&+e^{i\pi(s_{13}+s_{23}+s_{3\bar{3}})}\mathbb{A}(\bar{1},2,1,\bar{3},3,\bar{2})+e^{-i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2|\bar{3},1,3,\bar{2})+\mathbb{A}(\bar{1},2,\bar{3}|1,\bar{2},3)\\
&+e^{i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},\bar{3},2,1,\bar{2},3)+e^{i\pi(s_{13}+s_{2\bar{3}})}\mathbb{A}(\bar{3},\bar{1},2,1,\bar{2},3)\\
&+e^{i\pi(s_{13}+s_{2\bar{3}})}\mathbb{A}(\bar{1},2,1,\bar{2},3,\bar{3})+e^{i\pi(s_{13}+s_{2\bar{3}}+s_{3\bar{3}})}\mathbb{A}(\bar{1},2,1,\bar{2},\bar{3},3)\\
&+e^{-i\pi s_{1\bar{3}}}\mathbb{A}(\bar{1},2,1,\bar{3},\bar{2},3)+\mathbb{A}(\bar{1},2|\bar{3},1,\bar{2},3)\ .
\end{aligned}
\tag{10.47}
$$

The six variations of the monodromy relations in (7.26) for $n=6$

$$
\begin{aligned}
0=\;&\mathbb{A}(3,\bar{1},2,\bar{3}|1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{23}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},3,2,\bar{3}|1,\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},2,3,\bar{3}|1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,\bar{3},3|1,\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{23}}\mathbb{A}(\bar{1},2,\bar{3}|3,1,\bar{2})+e^{i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,\bar{3}|1,3,\bar{2})+\mathbb{A}(\bar{1},2,\bar{3}|1,\bar{2},3)\ ,\\
0=\;&\mathbb{A}(3,\bar{1},2|\bar{3},1,\bar{2})+e^{i\pi s_{1\bar{3}}}\mathbb{A}(\bar{1},3,2|\bar{3},1,\bar{2})+e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}\mathbb{A}(\bar{1},2,3|\bar{3},1,\bar{2})\\
&+e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}\mathbb{A}(\bar{1},2|3,\bar{3},1,\bar{2})+e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},2|\bar{3},3,1,\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},2|\bar{3},1,3,\bar{2})+\mathbb{A}(\bar{1},2|\bar{3},1,\bar{2},3)\ ,\\
0=\;&\mathbb{A}(\bar{3},3,\bar{1},2,1,\bar{2})+e^{i\pi s_{3\bar{3}}}\mathbb{A}(3,\bar{3},\bar{1},2,1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(3,\bar{1},\bar{3},2,1,\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(3,\bar{1},2,\bar{3},1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(3,\bar{1},2,1,\bar{3},\bar{2})\\
&+\mathbb{A}(3,\bar{1},2,1,\bar{2},\bar{3})\ ,\\
0=\;&\mathbb{A}(\bar{3},\bar{1},2,1,\bar{2},3)+e^{-i\pi s_{13}}\mathbb{A}(\bar{1},\bar{3},2,1,\bar{2},3)+e^{-i\pi s_{13}}e^{-i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,\bar{3},1,\bar{2},3)\\
&+e^{-i\pi s_{13}}e^{-i\pi s_{1\bar{3}}}e^{-i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,1,\bar{3},\bar{2},3)+e^{-i\pi s_{13}}e^{-i\pi s_{1\bar{3}}}e^{-i\pi s_{23}}e^{-i\pi s_{2\bar{3}}}\mathbb{A}(\bar{1},2,1,\bar{2},\bar{3},3)\\
&+\mathbb{A}(\bar{1},2,1,\bar{2},3,\bar{3})\ ,\\
0=\;&\mathbb{A}(3,\bar{1},2,1,\bar{3},\bar{2})+e^{i\pi s_{1\bar{3}}}\mathbb{A}(\bar{1},3,2,1,\bar{3},\bar{2})+e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}\mathbb{A}(\bar{1},2,3,1,\bar{3},\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}\mathbb{A}(\bar{1},2,1,3,\bar{3},\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{23}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},2,1,\bar{3},3,\bar{2})\\
&+\mathbb{A}(\bar{1},2,1,\bar{3},\bar{2},3)\ ,\\
0=\;&\mathbb{A}(\bar{3},\bar{1},3,2,1,\bar{2})+e^{i\pi s_{13}}\mathbb{A}(\bar{1},\bar{3},3,2,1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},3,\bar{3},2,1,\bar{2})\\
&+e^{i\pi s_{13}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},3,2,\bar{3},1,\bar{2})+e^{i\pi s_{13}}e^{i\pi s_{1\bar{3}}}e^{i\pi s_{2\bar{3}}}e^{i\pi s_{3\bar{3}}}\mathbb{A}(\bar{1},3,2,1,\bar{3},\bar{2})
\end{aligned}
$$

$$+\mathbb{A}(\overline{1},3,2,1,\overline{2},\overline{3}) \tag{10.48}$$

are used to rewrite the corresponding partial amplitudes in (10.47). Thus, the amplitude is reduced to the integration patches

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0} = &-\mathbb{A}(3,\overline{1},2,\overline{3},1,\overline{2}) - e^{i\pi s_{1\overline{3}}}\mathbb{A}(3,\overline{1},2,1,\overline{3},\overline{2}) - e^{i\pi s_{1\overline{3}}}\mathbb{A}(\overline{1},3,2,\overline{3},1,\overline{2})\\ &-\mathbb{A}(\overline{1},3,2,1,\overline{3},\overline{2}) + \mathbb{A}(\overline{3},\overline{1},2,3,1,\overline{2}) + e^{i\pi s_{13}}\mathbb{A}(\overline{1},\overline{3},2,3,1,\overline{2})\\ &+\mathbb{A}(\overline{1},2,3,1,\overline{2},\overline{3}) + e^{i\pi s_{13}}\mathbb{A}(\overline{3},\overline{1},2,1,3,\overline{2}) + \mathbb{A}(\overline{1},\overline{3},2,1,3,\overline{2})\\ &+e^{i\pi s_{13}}\mathbb{A}(\overline{1},2,1,3,\overline{2},\overline{3}) - \mathbb{A}(\overline{1},2,\overline{3},1,\overline{2},3) - e^{i\pi s_{1\overline{3}}}\mathbb{A}(\overline{1},2,1,\overline{3},\overline{2},3)\ .\end{aligned} \tag{10.49}$$

Carrying out the $PSL(2,\mathbb{R})$-transformation (8.44) maps the interval $-1 < y < 0$ to $1 < x < \infty$. Therefore, the inverse transformation requires for y that

$$y = \frac{1+\sqrt{x}}{1-\sqrt{x}} \tag{10.50}$$

in order to map $1 < x < \infty$ back to $-1 < y < 0$. Consequently, the inverse transformations for η and ξ are given by

$$\begin{aligned}\xi &= \frac{\tilde{\xi}+\sqrt{x}}{\tilde{\xi}-\sqrt{x}}\ ,\\ \eta &= \frac{\tilde{\eta}+\sqrt{x}}{\tilde{\eta}-\sqrt{x}}\ .\end{aligned} \tag{10.51}$$

With these the amplitude yields in the new $PSL(2,\mathbb{R})$-frame

$$\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0} = \frac{1}{8} i g_c^3 T_p' \Pi \int_1^\infty \mathrm{d}x \int \mathrm{d}\tilde{\xi} \int \mathrm{d}\tilde{\eta}\ \langle\!\langle V_{\overline{1}}(0) U_{\overline{2}}(-x) U_3(\tilde{\xi}) U_{\overline{3}}(\tilde{\eta}) V_2(1) V_1(\infty) \rangle\!\rangle\ , \tag{10.52}$$

where the integration regions of $\tilde{\xi}$ and $\tilde{\eta}$ correspond to the partial amplitudes in (10.53) below. After the transition $\mathbb{A} \to \mathcal{A}$ the above result in (10.49) becomes

$$\begin{aligned}\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0} = &\, e^{i\pi s_{13}}\mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},2,3,1) + \mathcal{A}(3,\overline{2},\overline{1},\overline{3},2,1))\\ &+e^{i\pi s_{13}}\mathcal{A}(3,\overline{2},\overline{3},\overline{1},2,1) - e^{-i\pi s_{1\overline{3}}}\mathcal{A}(\overline{3},\overline{2},3,\overline{1},2,1) - \mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)\\ &-\mathcal{A}(\overline{3},\overline{2},\overline{1},3,2,1) - e^{-i\pi s_{2\overline{3}}}\mathcal{A}(\overline{3},\overline{2},\overline{1},2,1,\overline{3})\ ,\end{aligned} \tag{10.53}$$

which can be further simplified using the following monodromy relations originating form (7.28) for the subamplitudes $\mathcal{A}$

$$\begin{aligned}0 = &\,\mathcal{A}(3,\overline{2},\overline{3},\overline{1},2,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},2,1)e^{i\pi s_{2\overline{3}}} + \mathcal{A}(\overline{2},\overline{3},3,\overline{1},2,1)e^{i\pi(s_{2\overline{3}}+s_{3\overline{3}})}\\ &+\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1)e^{i\pi(s_{1\overline{3}}+s_{2\overline{3}}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},\overline{3},\overline{1},2,3,1)e^{i\pi(s_{1\overline{3}}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})}\end{aligned}$$

$$
\begin{aligned}
0 &= \mathcal{A}(3,\overline{2},\overline{1},\overline{3},2,1) + \mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)e^{-i\pi s_{2\overline{3}}} + \mathcal{A}(\overline{2},\overline{1},3,\overline{3},2,1)e^{-i\pi(s_{1\overline{3}}+s_{2\overline{3}})} \\
&\quad + \mathcal{A}(\overline{2},\overline{1},\overline{3},3,2,1)e^{-i\pi(s_{1\overline{3}}+s_{2\overline{3}}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1)e^{-i\pi(s_{1\overline{3}}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \\
0 &= \mathcal{A}(\overline{3},\overline{2},\overline{1},3,2,1) + \mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1)e^{i\pi s_{23}} + \mathcal{A}(\overline{2},\overline{1},\overline{3},3,2,1)e^{i\pi(s_{13}+s_{23})} \\
&\quad + \mathcal{A}(\overline{2},\overline{1},3,\overline{3},2,1)e^{i\pi(s_{13}+s_{23}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1)e^{i\pi(s_{13}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})} \\
0 &= \mathcal{A}(\overline{3},\overline{2},3,\overline{1},2,1) + \mathcal{A}(\overline{2},\overline{3},3,\overline{1},2,1)e^{-i\pi s_{23}} + \mathcal{A}(\overline{2},3,\overline{3},\overline{1},2,1)e^{-i\pi(s_{23}+s_{3\overline{3}})} \\
&\quad + \mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)e^{-i\pi(s_{13}+s_{23}+s_{3\overline{3}})} + \mathcal{A}(\overline{2},3,\overline{1},2,\overline{3},1)e^{-i\pi(s_{13}+s_{23}+s_{2\overline{3}}+s_{3\overline{3}})}\ . \qquad (10.54)
\end{aligned}
$$

These reduce the number of partial amplitudes in (10.53) down to two such that

$$
\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0} = 2i\sin(\pi s_{2\overline{3}})\mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)\Big|_{1<x<\infty} + 2i\sin(\pi s_{23})\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1)\Big|_{1<x<\infty}\ , \qquad (10.55)
$$

where $1 < x < \infty$. With a similar calculation the contribution coming from $1 < y < \infty$ is described by the subamplitudes

$$
\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{1<y<\infty} = 2i\sin(\pi s_{2\overline{3}})\mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1)\Big|_{0<x<1} + 2i\sin(\pi s_{23})\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1)\Big|_{0<x<1} \qquad (10.56)
$$

and we integrate x from 0 to 1. With the coordinate transformation $x \to -x$ we end up with

$$
\begin{aligned}
\mathcal{A}_3^{\mathbb{RP}^2}\Big|_{-1<y<0} + \mathcal{A}_3^{\mathbb{RP}^2}\Big|_{1<y<\infty} &= 2i\sin(\pi s_{2\overline{3}})\mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1) + 2i\sin(\pi s_{23})\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1) \\
&= \frac{1}{4}g_c^3T_p'\sum_{\sigma\in S_3}\Big\{\sin(\pi s_{2\overline{3}})F_{\mathcal{I}_3}^{\sigma(\overline{2},3,\overline{3})} + \sin(\pi s_{23})F_{\mathcal{I}_4}^{\sigma(\overline{2},3,\overline{3})}\Big\} \\
&\quad \times A_{\mathrm{YM}}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1)\ , \qquad (10.57)
\end{aligned}
$$

where the two integration regions of the hypergeometric functions are given by

$$
\begin{aligned}
\mathcal{I}_3 &= \Big\{x\in\mathbb{R}\cup(\xi,\eta)\in\mathbb{R}^2 \,|\, -\infty<x<0, -x<\xi<0, 0<\eta<1\Big\}\ , \\
\mathcal{I}_4 &= \Big\{x\in\mathbb{R}\cup(\xi,\eta)\in\mathbb{R}^2 \,|\, -\infty<x<0, 0<\xi<1, -x<\eta<0\Big\}\ . \qquad (10.58)
\end{aligned}
$$

In the end, combining the two results for the different y-integrations we find that the scattering of three closed strings off an Op-plane is given by

$$
\begin{aligned}
\mathcal{A}_3^{\mathbb{RP}^2} &= 2i\sin(\pi s_{1\overline{3}})\mathcal{A}(\overline{2},\overline{1},3,2,\overline{3},1) + 2i\sin(\pi s_{13})\mathcal{A}(\overline{2},\overline{1},\overline{3},2,3,1) \\
&\quad + 2i\sin(\pi s_{2\overline{3}})\mathcal{A}(\overline{2},3,\overline{1},\overline{3},2,1) + 2i\sin(\pi s_{23})\mathcal{A}(\overline{2},\overline{3},\overline{1},3,2,1) \\
&= \frac{1}{4}g_c^3T_p'\sum_{\sigma\in S_3}\Big\{\sin(\pi s_{1\overline{3}})F_{\mathcal{I}_1}^{\sigma(\overline{2},3,\overline{3})} + \sin(\pi s_{13})F_{\mathcal{I}_2}^{\sigma(\overline{2},3,\overline{3})} \\
&\quad + \sin(\pi s_{2\overline{3}})F_{\mathcal{I}_3}^{\sigma(\overline{2},3,\overline{3})} + \sin(\pi s_{23})F_{\mathcal{I}_4}^{\sigma(\overline{2},3,\overline{3})}\Big\}A_{\mathrm{YM}}(\overline{1},\sigma(\overline{2},3,\overline{3}),2,1)\ . \qquad (10.59)
\end{aligned}
$$

Hence, it was possible to write the scattering of three closed strings off an Op-plane in terms of four open string subamplitudes. Moreover, this result is manifestly symmetric under the exchange $3 \leftrightarrow \overline{3}$.

Chapter 11

Low energy expansion and effective action expansion of three closed strings on the disk

In this chapter, which follows the presentation in [109], we take the field theory limit of the amplitude in (8.59). The α'-dependence in (8.59) is completely captured by $F_{\mathcal{I}_p}^{\sigma(\overline{2},3,\overline{3})}$, because the SYM amplitudes are independent of the inverse string tension such that the field theory limit results in performing an expansion of the hypergeometric integrals for $\alpha' \to 0$.

Furthermore, we want to compare the scattering of three gravitons in the presence of a Dp-brane with the low energy limit of (8.59) in field theory, which can be obtained form the DBI-action. For simplicity, we consider only terms where the polarisations are fully contracted among themselves and not with any momenta, otherwise we would have to deal with an enormous amount of terms, which makes the discussion unnecessary complicated.

11.1 Expansion of hypergeometric integrals in the inverse string tension α'

To evaluate the α'-expansion of the amplitude (8.59) we have to express the integrals $F_{\mathcal{I}_p}^{\sigma(\overline{2},3,\overline{3})}$ subject to the integration regions (8.61) for $p = 1, 2$ in terms of a power series with respect to small α'. Therefore, we undo the vertex operator fixing and introduce the volume of the CKG by

$$\int_{\mathcal{I}_p} \frac{\prod_{i=1}^{3} \mathrm{d}z_i \, \mathrm{d}\overline{z}_i}{V_{\text{CKG}}} = |z_{\overline{1}1} z_{\overline{1}2} z_{12}| \int_{\mathcal{I}_p} \mathrm{d}z_{\overline{2}} \, \mathrm{d}z_3 \, \mathrm{d}z_{\overline{3}} \tag{11.1}$$

such that the hypergeometric integrals become

$$F^{\sigma(\overline{2},3,\overline{3})}_{\mathcal{I}_p} = -V^{-1}_{\mathrm{CKG}} \int_{\mathcal{I}_p} \prod_{k=1}^{3} \mathrm{d}z_k \, \mathrm{d}\overline{z}_k \left(\prod_{i<j} |z_{ij}|^{s_{ij}} \right) \frac{1}{z_{\overline{1}2} z_{\overline{1}1} z_{21}} \frac{s_{\overline{1}\sigma(\overline{2})}}{z_{\overline{1}\sigma(\overline{2})}} \frac{s_{\sigma(\overline{3})2}}{z_{\sigma(\overline{3})2}} \left(\frac{s_{\overline{1}\sigma(3)}}{z_{\overline{1}\sigma(3)}} + \frac{s_{\sigma(\overline{2})\sigma(3)}}{z_{\sigma(\overline{2})\sigma(3)}} \right) , \tag{11.2}$$

where $\sigma \in S_3$ and $p = 1, 2$. The order of the worldsheet coordinates in (8.61) suggest a more suitable gauge choice

$$z_{\overline{1}} = 0 \, , \quad z_3 = 1 \, , \quad z_1 = \infty \, . \tag{11.3}$$

Then, we can drop the volume of the CKG again by implementing the new gauge choice

$$F^{\sigma(\overline{2},3,\overline{3})}_{\mathcal{I}_p} = - \int_{\mathcal{I}_n} \mathrm{d}z_2 \, \mathrm{d}z_{\overline{2}} \, \mathrm{d}z_{\overline{3}} \frac{z_{\overline{1}\sigma(3)}}{z_{\overline{1}2}} \left(\prod_{i<j} |z_{ij}|^{s_{ij}} \right) \frac{s_{\overline{1}\sigma(\overline{2})}}{z_{\overline{1}\sigma(\overline{2})}} \frac{s_{\sigma(\overline{3})2}}{z_{\sigma(\overline{3})2}} \left(\frac{s_{\overline{1}\sigma(3)}}{z_{\overline{1}\sigma(3)}} + \frac{s_{\sigma(\overline{2})\sigma(3)}}{z_{\sigma(\overline{2})\sigma(3)}} \right) \tag{11.4}$$

and integrate over $z_2, z_{\overline{2}}$ and $z_{\overline{3}}$. After introducing the maps for the two integration regions

$$\begin{aligned} \varphi_1 : \quad (\overline{1}, \overline{2}, \overline{3}, 2, 3, 1) \quad &\mapsto \quad (1, 2, 3, 4, 5, 6) \qquad \text{for } \mathcal{I}_1 \, , \\ \varphi_2 : \quad (\overline{1}, \overline{2}, 2, \overline{3}, 3, 1) \quad &\mapsto \quad (1, 2, 3, 4, 5, 6) \qquad \text{for } \mathcal{I}_2 \end{aligned} \tag{11.5}$$

the hypergeometric integrals yield

$$F^{\sigma(\overline{2},3,\overline{3})}_{\mathcal{I}_p} = \begin{cases} F^{(\sigma(\varphi_1(\overline{2})\varphi_1(3),\varphi_1(\overline{3})),4)}\Big|_{\hat{s}_{ij} \to \varphi_1^{-1}(\hat{s}_{ij})} & \text{for } p = 1 \, , \\ F^{(\sigma(\varphi_2(\overline{2})\varphi_2(3),\varphi_2(\overline{3})),3)}\Big|_{\hat{s}_{ij} \to \varphi_2^{-1}(\hat{s}_{ij})} & \text{for } p = 2 \, . \end{cases} \tag{11.6}$$

The kinematic invariant with a hat are defined as

$$\hat{s}_{ij} = \frac{1}{2}(p_i + p_j)^2 = p_i \cdot p_j \tag{11.7}$$

and the momenta p_i can be obtained from left-and right-moving momenta k_i and $D \cdot k_i$ via the maps φ_p:

$$\begin{aligned} \varphi_1 : \quad (D{\cdot}k_{\overline{1}}, D{\cdot}k_{\overline{2}}, D{\cdot}k_{\overline{3}}, k_2, k_3, k_1) \quad &\mapsto \quad (p_1, p_2, p_3, p_4, p_5, p_6) \qquad \text{for } \mathcal{I}_1 \, , \\ \varphi_2 : \quad (D{\cdot}k_{\overline{1}}, D{\cdot}k_{\overline{2}}, k_2, D{\cdot}k_{\overline{3}}, k_3, k_1) \quad &\mapsto \quad (p_1, p_2, p_3, p_4, p_5, p_6) \qquad \text{for } \mathcal{I}_2 \, . \end{aligned} \tag{11.8}$$

Hence, the kinematic invariants can be related using the inverse map φ_p^{-1} as

$$\varphi_1^{-1} : \begin{cases} \hat{s}_{12} \mapsto s_{12} \, , \\ \hat{s}_{23} \mapsto s_{23} \, , \quad \hat{s}_{123} \mapsto s_{12} + s_{23} + s_{13} \, , \\ \hat{s}_{34} \mapsto s_{2\overline{3}} \, , \quad \hat{s}_{234} \mapsto s_{23} + s_{2\overline{3}} + s_{2\overline{2}} \, , \\ \hat{s}_{45} \mapsto s_{23} \, , \quad \hat{s}_{345} \mapsto s_{23} + s_{2\overline{3}} + s_{3\overline{3}} \\ \hat{s}_{56} \mapsto s_{13} \, , \\ \hat{s}_{61} \mapsto s_{1\overline{1}} \, , \end{cases} \tag{11.9}$$

and

$$\varphi_2^{-1}: \begin{cases} \hat{s}_{12} \mapsto s_{12} \ , \\ \hat{s}_{23} \mapsto s_{2\bar{2}} \ , \quad \hat{s}_{123} \mapsto s_{12} + s_{1\bar{2}} + s_{2\bar{2}} \ , \\ \hat{s}_{34} \mapsto s_{2\bar{3}} \ , \quad \hat{s}_{234} \mapsto s_{23} + s_{2\bar{3}} + s_{2\bar{2}} \ , \\ \hat{s}_{45} \mapsto s_{3\bar{3}} \ , \quad \hat{s}_{345} \mapsto s_{23} + s_{2\bar{3}} + s_{3\bar{3}} \ . \\ \hat{s}_{56} \mapsto s_{13} \ , \\ \hat{s}_{61} \mapsto s_{1\bar{1}} \ , \end{cases} \tag{11.10}$$

Explicitly, using the map φ_p the integrals in (11.6) are given by

$$\begin{aligned} F^{(\sigma(2),\sigma(3),\sigma(5),4)} &= -\int\limits_{0<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4 \left(\prod_{i<j} |z_{ij}|^{\hat{s}_{ij}}\right) \frac{1}{z_{41}} \frac{\hat{s}_{1\sigma(2)}}{z_{1\sigma(2)}} \frac{\hat{s}_{4\sigma(5)}}{z_{\sigma(5)4}} \left(\frac{\hat{s}_{1\sigma(3)}}{z_{1\sigma(3)}} + \frac{\hat{s}_{\sigma(2)\sigma(3)}}{z_{\sigma(2)\sigma(3)}} \right) , \\ F^{(\sigma(2),\sigma(4),\sigma(5),3)} &= -\int\limits_{0<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4 \left(\prod_{i<j} |z_{ij}|^{\hat{s}_{ij}}\right) \frac{1}{z_{31}} \frac{\hat{s}_{1\sigma(2)}}{z_{1\sigma(2)}} \frac{\hat{s}_{3\sigma(5)}}{z_{\sigma(5)3}} \left(\frac{\hat{s}_{1\sigma(4)}}{z_{1\sigma(4)}} + \frac{\hat{s}_{\sigma(2)\sigma(4)}}{z_{\sigma(2)\sigma(4)}} \right) . \end{aligned} \tag{11.11}$$

Finally, we can build on the results in [63], as the pair of six functions $F^{(p_1,p_2,p_3,4)}$ with $p_j = 2,3,4$ and $F^{(q_1,q_2,q_3,3)}$ with $q_j = 2,4,5$ given in (11.11) are part of an extended set of 24 hypergeometric functions introduced in [63], which originate from the six point open superstring amplitude with canonical color ordering $(1,2,3,4,5,6)$. Due to (dual) monodromy relations all 24 triple hypergeometric functions can be written in terms of the six dimensional basis $F^{(i_1,i_2,i_3)} \equiv F^{(i_1,i_2,i_3,5)}$ with $i_j = 2,3,4$ as [62]

$$\begin{pmatrix} F^{(2354)} \\ F^{(3254)} \\ F^{(5324)} \\ F^{(3524)} \\ F^{(5234)} \\ F^{(2534)} \end{pmatrix} = K_1^* \begin{pmatrix} F^{(234)} \\ F^{(324)} \\ F^{(432)} \\ F^{(342)} \\ F^{(423)} \\ F^{(243)} \end{pmatrix} , \qquad \begin{pmatrix} F^{(2453)} \\ F^{(4253)} \\ F^{(5423)} \\ F^{(4523)} \\ F^{(5243)} \\ F^{(2543)} \end{pmatrix} = K_2^* \begin{pmatrix} F^{(234)} \\ F^{(324)} \\ F^{(432)} \\ F^{(342)} \\ F^{(423)} \\ F^{(243)} \end{pmatrix} , \tag{11.12}$$

respectively and $K_p^* = (K_p^{\mathrm{T}})^{-1}$. The 6×6 matrices can be determined via the (dual) monodromy relations and take the following form [63]

$$\begin{aligned} &(K_1)^i_j = \\ &= \hat{s}_{46}^{-1} \begin{pmatrix} \hat{s}_5 - \hat{s}_{123} & 0 & 0 & 0 & \hat{s}_{14} & -\hat{d}_1 \\ 0 & \hat{s}_5 - \hat{s}_{123} & \hat{s}_{14} & \hat{s}_3 + \hat{s}_{14} & 0 & 0 \\ \frac{\hat{s}_1\hat{s}_4\hat{d}_0}{\hat{s}_{15}\hat{s}_{246}} & \frac{\hat{s}_4\hat{s}_{13}(\hat{s}_{25}-\hat{s}_{46})}{\hat{s}_{15}\hat{s}_{246}} & \frac{-\hat{s}_{13}\hat{s}_{14}\hat{s}_{25}}{\hat{s}_{15}\hat{s}_{246}} & \frac{-\hat{s}_{13}\hat{s}_{25}(\hat{s}_3+\hat{s}_{14})}{\hat{s}_{15}\hat{s}_{246}} & \frac{\hat{s}_{14}(\hat{s}_{46}-\hat{s}_1)\hat{d}_0}{\hat{s}_{15}\hat{s}_{246}} & \frac{\hat{s}_1(\hat{s}_3+\hat{s}_4)\hat{d}_0}{\hat{s}_{15}\hat{s}_{246}} \\ \frac{-\hat{s}_1\hat{s}_4}{\hat{s}_{246}} & \frac{-\hat{s}_4(\hat{s}_1+\hat{s}_2)}{\hat{s}_{246}} & \frac{\hat{s}_{14}\hat{d}_4}{\hat{s}_{246}} & \frac{(\hat{s}_{14}+\hat{s}_3)\hat{d}_4}{\hat{s}_{246}} & \frac{\hat{s}_{14}(\hat{s}_1-\hat{s}_{46})}{\hat{s}_{246}} & \frac{-\hat{s}_1(\hat{s}_3+\hat{s}_4)}{\hat{s}_{246}} \\ \frac{\hat{s}_1\hat{s}_4(\hat{s}_{35}-\hat{s}_{46})}{\hat{s}_{15}\hat{s}_{125}} & \frac{\hat{s}_4\hat{s}_{13}\hat{d}_3}{\hat{s}_{15}\hat{s}_{125}} & \frac{(\hat{s}_{46}-\hat{s}_{13})\hat{d}_3\hat{s}_{14}}{\hat{s}_{15}\hat{s}_{125}} & \frac{(\hat{s}_4+\hat{s}_{24})\hat{s}_{13}\hat{d}_3}{\hat{s}_{15}\hat{s}_{125}} & \frac{-\hat{s}_1\hat{s}_{14}\hat{s}_{35}}{\hat{s}_{15}\hat{s}_{125}} & \frac{\hat{s}_1\hat{s}_{35}\hat{d}_1}{\hat{s}_{15}\hat{s}_{125}} \\ \frac{\hat{s}_4(\hat{s}_1-\hat{s}_{123})}{\hat{s}_{125}} & \frac{-\hat{s}_4\hat{s}_{13}}{\hat{s}_{125}} & \frac{\hat{s}_{14}(\hat{s}_{13}-\hat{s}_{46})}{\hat{s}_{125}} & \frac{-\hat{s}_{13}(\hat{s}_4+\hat{s}_{24})}{\hat{s}_{125}} & \frac{-\hat{s}_{14}\hat{d}_2}{\hat{s}_{125}} & \frac{\hat{d}_1\hat{d}_2}{\hat{s}_{125}} \end{pmatrix} \end{aligned} \tag{11.13}$$

and

$$(K_2)^i_j = \\ = \hat{s}_{36}^{-1} \begin{pmatrix} \hat{s}_{123} - \hat{s}_1 & \hat{s}_{13} & 0 & 0 & 0 & \hat{d}_{14} \\ 0 & 0 & \hat{s}_3 + \hat{s}_{13} & \hat{s}_{13} & \hat{d}_{14} & 0 \\ \frac{\hat{s}_1(\hat{s}_{345}-\hat{s}_4)\hat{d}_{13}}{\hat{s}_{145}\hat{s}_{15}} & \frac{(\hat{s}_{36}-\hat{s}_1)\hat{s}_{13}\hat{d}_{13}}{\hat{s}_{145}\hat{s}_{15}} & \frac{-(\hat{s}_3+\hat{s}_{13})\hat{s}_{14}\hat{s}_{25}}{\hat{s}_{145}\hat{s}_{15}} & \frac{-\hat{s}_{13}\hat{s}_{14}\hat{s}_{25}}{\hat{s}_{145}\hat{s}_{15}} & \frac{\hat{d}_8\hat{s}_{14}\hat{s}_{35}}{\hat{s}_{145}\hat{s}_{15}} & \frac{\hat{s}_1\hat{s}_{35}\hat{d}_{13}}{\hat{s}_{145}\hat{s}_{15}} \\ \frac{\hat{s}_1(\hat{s}_4-\hat{s}_{345})}{\hat{s}_{145}} & \frac{(\hat{s}_1-\hat{s}_{36})\hat{s}_{13}}{\hat{s}_{145}} & \frac{(\hat{s}_3+\hat{s}_{13})\hat{d}_5}{\hat{s}_{145}} & \frac{\hat{s}_{13}\hat{d}_5}{\hat{s}_{145}} & \frac{-(\hat{s}_1+\hat{s}_{24})\hat{s}_{35}}{\hat{s}_{145}} & \frac{-\hat{s}_1\hat{s}_{35}}{\hat{s}_{145}} \\ \frac{\hat{s}_1\hat{s}_4(\hat{s}_1-\hat{s}_{123})}{\hat{s}_{125}\hat{s}_{15}} & \frac{-\hat{s}_1\hat{s}_4\hat{s}_{13}}{\hat{s}_{125}\hat{s}_{15}} & \frac{\hat{s}_{14}(\hat{s}_2+\hat{s}_{35})\hat{d}_3}{\hat{s}_{125}\hat{s}_{15}} & \frac{\hat{s}_{13}\hat{d}_3\hat{d}_7}{\hat{s}_{125}\hat{s}_{15}} & \frac{\hat{s}_{14}\hat{s}_{35}\hat{d}_3}{\hat{s}_{125}\hat{s}_{15}} & \frac{\hat{s}_1(\hat{s}_4-\hat{s}_{36})\hat{s}_{35}}{\hat{s}_{125}\hat{s}_{15}} \\ \frac{(\hat{s}_{123}-\hat{s}_1)\hat{d}_6}{\hat{s}_{125}} & \frac{\hat{s}_{13}\hat{d}_6}{\hat{s}_{125}} & \frac{-\hat{s}_{14}(\hat{s}_2+\hat{s}_{35})}{\hat{s}_{125}} & \frac{-\hat{d}_7\hat{s}_{13}}{\hat{s}_{125}} & \frac{-\hat{s}_{14}\hat{s}_{35}}{\hat{s}_{125}} & \frac{\hat{d}_1\hat{s}_{35}}{\hat{s}_{125}} \end{pmatrix} , \tag{11.14}$$

where in the above matrices we have defined combinations of kinematic invariants as

$$\begin{aligned} &\hat{d}_0 = \hat{s}_{15} + \hat{s}_{35} \ , && \hat{d}_1 = \hat{s}_3 - \hat{s}_5 + \hat{s}_{123} \ , && \hat{d}_2 = \hat{s}_1 - \hat{s}_4 - \hat{s}_5 \ , \\ &\hat{d}_3 = \hat{s}_3 - \hat{s}_5 - \hat{s}_{345} \ , && \hat{d}_4 = \hat{s}_4 + \hat{s}_5 - \hat{s}_{13} \ , && \hat{d}_5 = \hat{s}_1 + \hat{s}_{24} - \hat{s}_{36} \ , \\ &\hat{d}_6 = -\hat{s}_1 + \hat{s}_5 + \hat{s}_{35} \ , && \hat{d}_7 = \hat{s}_1 - \hat{s}_5 + \hat{s}_{24} - \hat{s}_{35} \ , && \hat{d}_8 = \hat{s}_6 - \hat{s}_4 + \hat{s}_{13} - \hat{s}_{24} \ , \\ &\hat{d}_{13} = \hat{s}_{15} + \hat{s}_{45} \ , && \hat{d}_{14} = \hat{s}_{123} - \hat{s}_1 + \hat{s}_3 \end{aligned} \tag{11.15}$$

and $\hat{s}_{ijk} = \hat{s}_{ij} + \hat{s}_{ik} + \hat{s}_{jk}$.

Finally, we can move on to the α'-expansion of the twelve integrals in (8.60). According to the discussion above the latter can be expressed in terms of the six hypergeometric basis functions

$$\begin{pmatrix} F^{(\overline{23}3)}_{\mathcal{I}_p} \\ F^{(\overline{32}3)}_{\mathcal{I}_p} \\ F^{(3\overline{32})}_{\mathcal{I}_p} \\ F^{(\bar{3}3\bar{2})}_{\mathcal{I}_p} \\ F^{(3\overline{23})}_{\mathcal{I}_p} \\ F^{(\bar{2}3\bar{3})}_{\mathcal{I}_p} \end{pmatrix} = K_p^* \left. \begin{pmatrix} F^{(234)} \\ F^{(324)} \\ F^{(432)} \\ F^{(342)} \\ F^{(423)} \\ F^{(243)} \end{pmatrix} \right|_{\hat{s}_{ij} \to \varphi_p^{-1}(\hat{s}_{ij})} \qquad \text{for } p = 1, 2 \ . \tag{11.16}$$

Applying the method proposed in [176] and thereafter refined and systematized in [177, 182] the low energy expansion of (11.16) is realized by

$$\begin{aligned} F^{(234)} &= 1 - \zeta_2(s_{45}s_{56} + s_{12}s_{61} - s_{45}s_{123} - s_{12}s_{345} + s_{123}s_{345}) + \zeta_3(\ldots) + \mathcal{O}(\alpha'^4) \ , \\ F^{(324)} &= -\zeta_2 s_{13}(s_{23} - s_{61} + s_{345}) + \zeta_3(\ldots) + \mathcal{O}(\alpha'^4) \ , \\ F^{(432)} &= -\zeta_2 s_{14}s_{25} + \zeta_3 s_{14}s_{25}(-s_{23} - s_{34} + s_{56} + s_{61} + s_{123} + s_{234} + s_{345}) + \mathcal{O}(\alpha'^4) \ , \\ F^{(342)} &= \zeta_2 s_{13}s_{25} + \zeta_3 s_{13}s_{25}(-s_{12} + s_{23} + 2s_{34} - s_{16} - s_{123} - 2s_{234} - s_{345}) + \mathcal{O}(\alpha'^4) \ , \\ F^{(423)} &= \zeta_2 s_{14}s_{35} + \zeta_3 s_{14}s_{35}(2s_{23} + s_{34} - s_{45} - s_{56} - s_{123} - 2s_{234} - s_{345}) + \mathcal{O}(\alpha'^4) \ , \\ F^{(243)} &= -\zeta_2 s_{35}(s_{34} - s_{56} + s_{123}) + \zeta_3 s_{35}[-2s_{12}s_{23} - 2s_{12}s_{34} + s_{34}^2 + s_{34}s_{45} - s_{45}s_{56} \end{aligned}$$

$$+s_{56}^2 + s_{123}(2s_{23} + s_{45} + s_{123}) + 2s_{12}s_{234} + s_{345}(s_{34} - s_{56} + s_{123})] + \mathcal{O}(\alpha'^4) \; . \tag{11.17}$$

Eventually, inserting (11.16) and (11.17) into (8.59) yields the field theory limit of the α'-dependent part of the scattering amplitude of three closed strings on the disk

$$\sin(\pi s_{23}) \begin{pmatrix} F_{\mathcal{I}_1}^{(\bar{2}\bar{3}3)} \\ F_{\mathcal{I}_1}^{(\bar{3}\bar{2}3)} \\ F_{\mathcal{I}_1}^{(3\bar{3}\bar{2})} \\ F_{\mathcal{I}_1}^{(\bar{3}3\bar{2})} \\ F_{\mathcal{I}_1}^{(3\bar{2}\bar{3})} \\ F_{\mathcal{I}_1}^{(\bar{2}3\bar{3})} \end{pmatrix} + \sin(\pi(s_{23} + s_{2\bar{3}})) \begin{pmatrix} F_{\mathcal{I}_2}^{(\bar{2}\bar{3}3)} \\ F_{\mathcal{I}_2}^{(\bar{3}\bar{2}3)} \\ F_{\mathcal{I}_2}^{(3\bar{3}\bar{2})} \\ F_{\mathcal{I}_2}^{(\bar{3}3\bar{2})} \\ F_{\mathcal{I}_2}^{(3\bar{2}\bar{3})} \\ F_{\mathcal{I}_2}^{(\bar{2}3\bar{3})} \end{pmatrix} = \pi \begin{pmatrix} 0 \\ s_{23} \\ 0 \\ s_{23} + s_{2\bar{3}} \\ 0 \\ 0 \end{pmatrix}$$

$$+\pi\zeta_2 \begin{pmatrix} -s_{12}s_{23}\,(s_{13} + s_{1\bar{3}} - s_{2\bar{3}}) \\ s_{23}[s_{12}^2 + s_{12}s_{1\bar{2}} + s_{12}s_{13} + s_{1\bar{2}}s_{13} + s_{13}^2 + s_{13}s_{1\bar{3}} + 2s_{12}s_{23} + s_{1\bar{2}}s_{23} + s_{13}s_{23} + s_{23}^2 + s_{12}s_{2\bar{3}} + s_{23}s_{2\bar{3}}] \\ -s_{1\bar{3}}(s_{12}s_{23} + s_{1\bar{2}}s_{23} - s_{23}s_{2\bar{3}} - s_{2\bar{3}}^2) \\ (s_{23} + s_{2\bar{3}})[s_{12}^2 + s_{12}s_{1\bar{2}} + s_{13}^2 + s_{13}s_{1\bar{3}} + 2s_{12}s_{23} + s_{1\bar{2}}s_{23} + s_{13}s_{23} + s_{23}^2 + (s_{12} + s_{23})s_{2\bar{3}}] + s_{13}s_{23}(s_{12} + s_{1\bar{2}}) \\ s_{1\bar{3}}s_{2\bar{3}}^2 \\ -s_{12}s_{2\bar{3}}^2 \end{pmatrix}$$

$$+\mathcal{O}(\alpha'^4) \; . \tag{11.18}$$

11.2 Interpretation of the low energy expansion

The scattering of three closed strings off a Dp brane features an intricate structure of poles, which is displayed in (C.33) together with (C.30). These arise at the boundaries of moduli space where two vertex operators approach each other or the boundary of the disk, respectively. Even though, we do not analyse the pole structure in detail, the relevant regions in moduli space can be visualized as in figures 11.1–11.3. In the expansion for small momenta all field theory Feynman digrams contribute at order $\mathcal{O}(k^0)$, where k stand schematically for an arbitrary combination of external momenta.

In all Feynman diagrams in this section the dotted lines represent the propagation of position scalar X^i or vector fields A_a living on the D-brane worldvolume. In both cases the fields originate from massless open string excitations on the D-brane. The dashed lines stand for the propagation of massless closed string excitations, which could be a dilaton, a Kalb-Ramond B-field, a graviton or an RR field depending on the external states of the closed string amplitude. However, if the external states are all dilaton and graviton

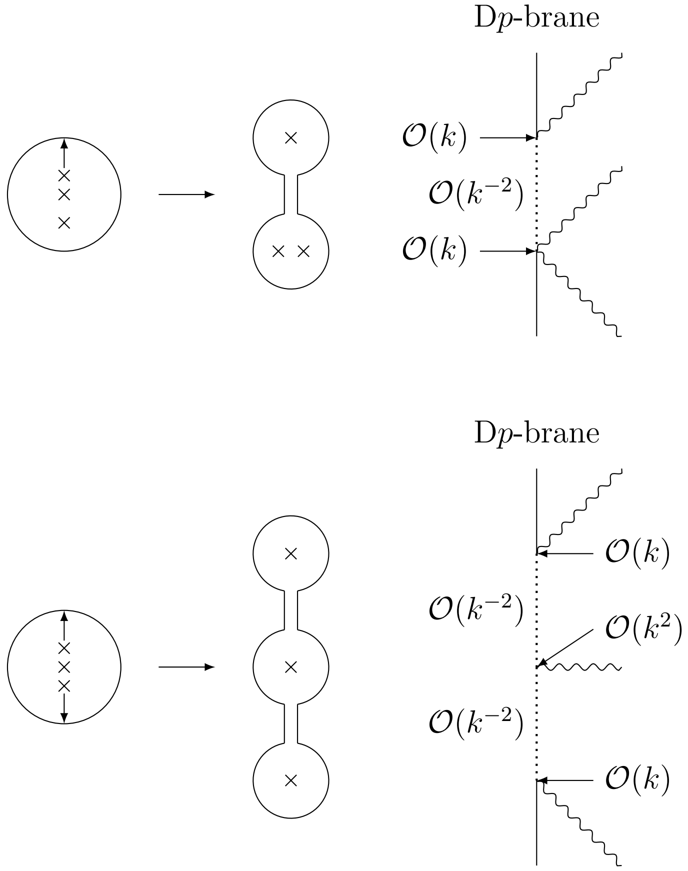

Figure 11.1: Degeneration limits with no vertex collisions.

excitations, also all internal (dashed) lines describe only the propagation of dilatons and gravitons. Since, the propagating fields are all massless their propagator scales as $\mathcal{O}\left(\frac{1}{k^2}\right)$, where again k represents some combination of external momenta. The scaling with respect to the external momenta of the vertex operators in the figures 11.1–11.3 can be derived following the steps in [92, 102]. The vertices off the D-brane, which are in the bulk, can be read off form the bulk action

$$S_{\mathrm{NS-NS}} = \int \mathrm{d}^{10}x \sqrt{-g}\Big[\frac{1}{2\kappa^2}R - \frac{1}{2}(\nabla\phi)^2 - \frac{3}{2}e^{-\sqrt{2}\kappa\phi}H^2\Big] + \dots \tag{11.19}$$

in Einstein-frame, where we have focused on the NSNS sector (in the RNS formalism) and omitted higher derivative α'-corrections. The string-frame metric $G_{\mu\nu}$ is related to the Einstein-frame metric $g_{\mu\nu}$ by the rescaling $G_{\mu\nu} = e^{\Phi/2}g_{\mu\nu}$. Because all terms involve at least one derivate, the corresponding bulk vertices scale as $\mathcal{O}(k^2)$, which is also true when adding RR fields. On the other hand, the vertices on the D-brane can be found from the DBI-action

$$S_{\mathrm{DBI}} = -T_p \int d^{p+1}\sigma \,\mathrm{Tr}\Big(e^{\frac{p-3}{4}\Phi}\sqrt{-\det(\tilde{g}_{ab} + e^{-\Phi/2}\tilde{B}_{ab} + 2\pi\ell_s^2\, e^{-\Phi/2}F_{ab})}\Big) , \tag{11.20}$$

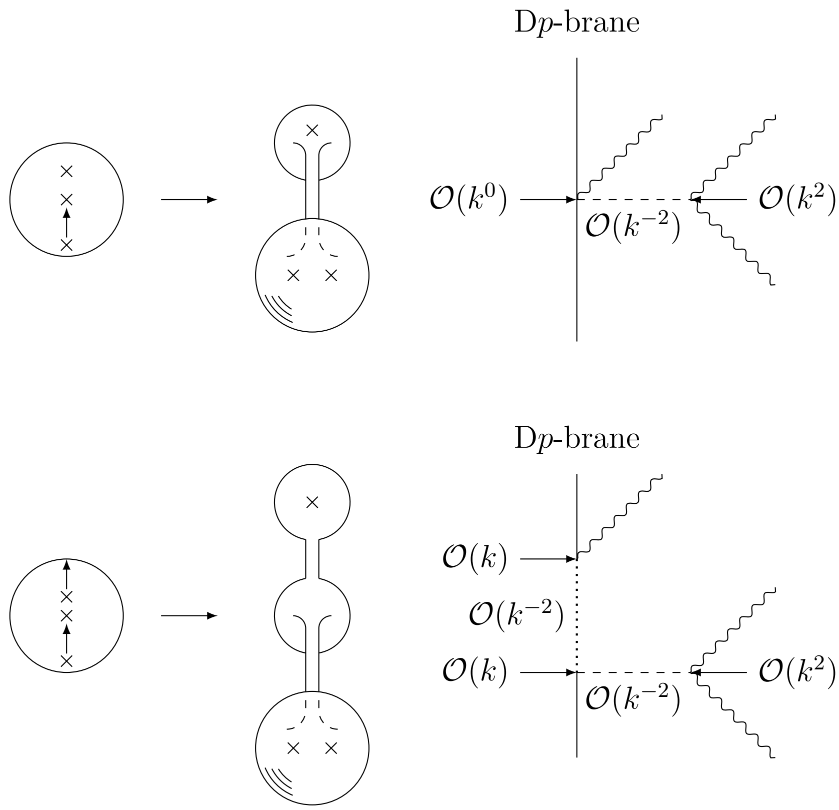

Figure 11.2: Degeneration limits with one vertex collisions.

where the pull-back of the bulk metric in the Einstein-frame is given by

$$\tilde{g}_{ab} = g_{ab} + 2g_{i(a}\partial_{b)}X^i + g_{ij}\partial_a X^i \partial_b X^j \tag{11.21}$$

and we can find a similar formula for the the pull-back of the Kalb-Ramond B-field. At leading order the vertex operators scale as $\mathcal{O}(k^n)$ in the momenta, where n is the number of open string legs attached to a vertex. This is due to the fact that the number of derivatives in an interaction vertex is directly connected to the amount of open string legs going in or out of the vertex. This relation follows directly from the terms in the DBI action (11.20), including the pull-back (11.21), or from performing a series expansion of the closed string fields in the coordinates transverse to the D-brane.

So far, we have only discussed the scenario involving one D-brane. In the case of a stack of D-branes, which corresponds to a non-Abelian gauge group, we have to add terms containing commutators of non-Abelian D-brane scalars to the DBI action (11.20) [183]. In addition, the pull-back has to be computed with the covariant derivative with respect to the D-brane scalars in the non-Abelian case [184]. In figures 11.1 and 11.2 all open string excitations, which propagate along the D-brane, have either $U(1)$ gauge group or are center-of-mass fluctuations, because they couple linearly to the external closed string states [102]. Since there is no non-vanishing vertex involving three open string excitations

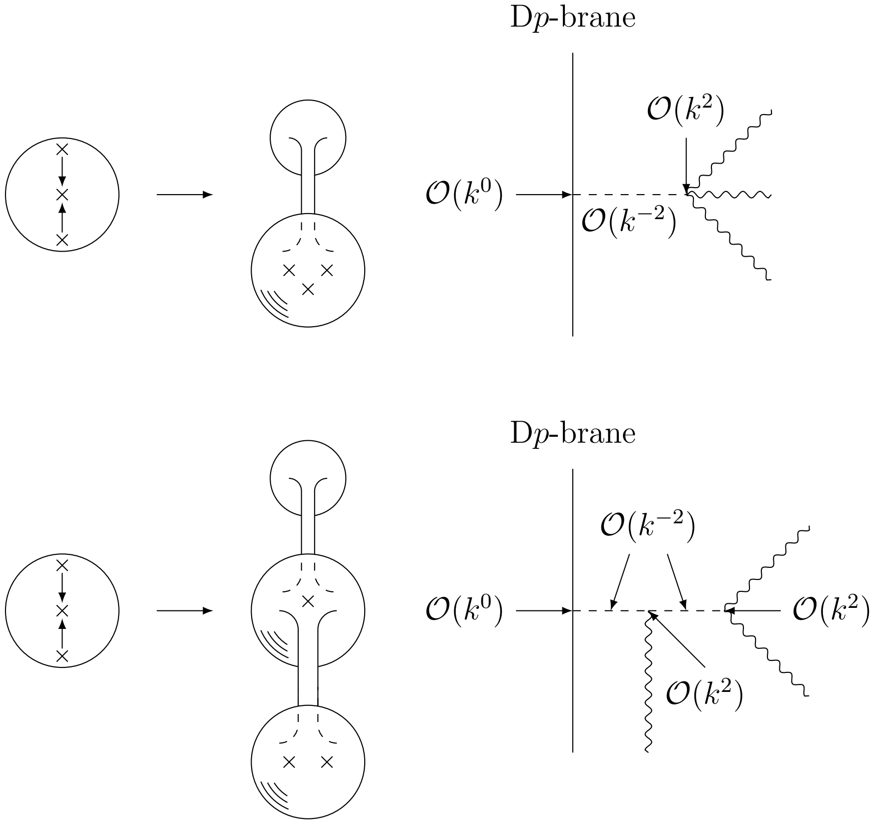

Figure 11.3: Degeneration limits with two vertex collisions. In the upper diagram the three vertex operators approach each other at a uniform pace whereas in the lower diagram the lower two vertex operators collide first and are then approached by the third one.

with gauge group $U(1)$ or center-of-mass fluctuations on the D-branes [102], we do not get any contribution from the degeneration depicted in figure 11.4. Therefore, only terms with at most two poles in the field theory limit should appear in the closed string disk amplitude (8.59).

For simplicity, we solely consider the subset of terms in the amplitude (8.59) corresponding to external states from the NSNS sector, where the polarisation tensors have no contractions with momenta, i.e. the polarizations are only contracted among themselves. At leading order in the α'-expansion we get for (8.59)

$$
\begin{aligned}
&\lim_{\alpha'\to 0} \mathcal{A}\Big|_{\left.\begin{smallmatrix} e_i\cdot p_j \\ \bar{e}_i\cdot p_j \end{smallmatrix}\right\}\to 0} = \\
&= \frac{1}{2} g_c^3 T_p \Bigg[\Bigg(\frac{s_{12}s_{13} - s_{1\bar{3}}s_{2\bar{3}} - s_{1\bar{2}}(s_{1\bar{3}} + s_{2\bar{3}})}{s_{1\bar{1}}s_{23}} + \frac{s_{12}s_{13}}{s_{23}(s_{12}+s_{13}+s_{23})} \Bigg) \operatorname{Tr}(D\cdot\epsilon_1) \operatorname{Tr}(\epsilon_2^T\cdot\epsilon_3) \\
&\quad + \frac{s_{23} \operatorname{Tr}(\epsilon_1\cdot\epsilon_2^T\cdot D\cdot\epsilon_3^T)}{s_{12}+s_{13}+s_{23}} + \frac{s_{23} \operatorname{Tr}(\epsilon_1\cdot\epsilon_3^T\cdot D\cdot\epsilon_2^T)}{s_{12}+s_{13}+s_{23}} - \frac{(s_{1\bar{2}} + s_{1\bar{3}})}{s_{1\bar{1}}} \operatorname{Tr}(D\cdot\epsilon_1) \operatorname{Tr}(D\cdot\epsilon_2\cdot D\cdot\epsilon_3)
\end{aligned}
$$

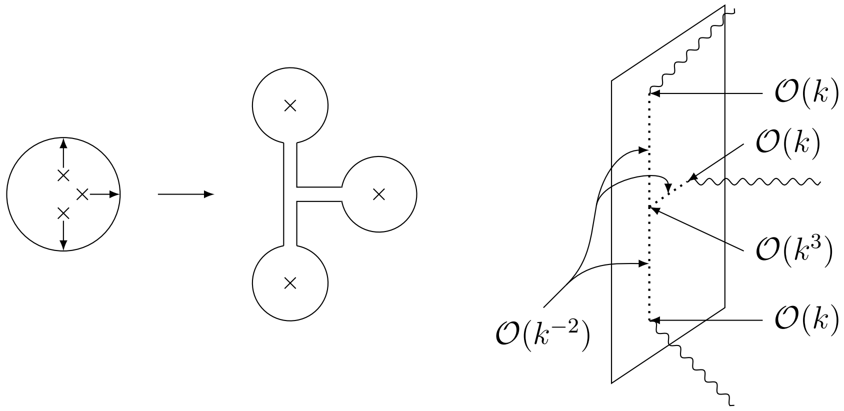

Figure 11.4: Degeneration limits with no vertex collisions.

$$
\begin{aligned}
&-\frac{1}{3}\operatorname{Tr}(\epsilon_1{\cdot}D{\cdot}\epsilon_2{\cdot}D{\cdot}\epsilon_3{\cdot}D)-\frac{1}{3}\operatorname{Tr}(\epsilon_1^T{\cdot}D{\cdot}\epsilon_2^T{\cdot}D{\cdot}\epsilon_3^T{\cdot}D)\\
&-\left(\frac{1}{3}+\frac{s_{12}+s_{13}-s_{23}}{2s_{1\bar{1}}}+\frac{s_{12}s_{13}}{s_{2\bar{2}}s_{3\bar{3}}}+\frac{s_{1\bar{1}}(s_{12}+s_{13}+s_{23})}{4s_{2\bar{2}}s_{3\bar{3}}}\right)\operatorname{Tr}(D{\cdot}\epsilon_1)\operatorname{Tr}(D{\cdot}\epsilon_2)\operatorname{Tr}(D{\cdot}\epsilon_3)\Bigg]\\
&+\{1\leftrightarrow 2\}+\{1\leftrightarrow 3\}\ , \qquad (11.22)
\end{aligned}
$$

where we have inserted the expansion for the hypergeometric integrals in (11.18) and explicitly carried out the zero mode integration in the SYM amplitudes. The expression for A_{SYM} in terms of polarizations and momenta can be found in [185]. The momenta p_j stand schematically for closed string momenta of the left- and right-movers similar as in (11.8). Note that (11.22) is manifestly invariant under exchanging of the external states.

We want to compare (11.22) with the field theory result derived from the DBI action (11.20), for the special case where all external states are graviton excitations. Also for the field theory calculation we again restrict the discussion to those terms where the graviton polarization tensors are completely contracted among themselves and take $e_i{\cdot}p_j \to 0$ and $\overline{e}_i{\cdot}p_j \to 0$. In addition, we assume that all gravitons have orthogonal polarizations so that $(\epsilon_i)^\mu{}_\nu(\epsilon_j)^\nu{}_\rho = 0$. Therefore, the Einstein-Hilbert term in the bulk does not contribute to scattering processes in this setup, i.e. in this case the only non-vanishing contributions from the diagrams shown in figure 11.1–11.3 come from the two diagrams in figure 11.1.

For the scattering of three gravitons we only require the part

$$
S_{\mathrm{DBI}}^{gravity} = -T_p\int \mathrm{d}^{p+1}\sigma\,\operatorname{Tr}\Big(\sqrt{-\det(\tilde{g}_{ab})}\Big) \qquad (11.23)
$$

of (11.20), which contains the gravitational interaction of Dp-branes. To expand the Lagrangian $\mathcal{L}$ of the action (11.23) around a flat background $g_{\mu\nu}=\eta_{\mu\nu}+2\kappa h_{\mu\nu}$ we utilize

$$
\begin{aligned}
\sqrt{-\det(\delta^a{}_b+M^a{}_b)} = 1&+\frac{1}{2}M^a{}_a-\frac{1}{4}M^a{}_bM^b{}_a+\frac{1}{8}(M^a{}_a)^2+\frac{1}{6}M^a{}_bM^b{}_cM^c{}_a\\
&-\frac{1}{8}M^a{}_aM^b{}_cM^c{}_b+\frac{1}{48}(M^a{}_a)^3+\dots\ . \qquad (11.24)
\end{aligned}
$$

Keeping only the relevant terms for the calculation of the diagrams in figure 11.1 the expansion of $\mathcal{L}$ results in

$$\begin{aligned}\mathcal{L} = -\kappa\Big[&T_p h^a{}_a + \sqrt{T_p}\lambda^i\partial_i h^a{}_a + \frac{1}{2}\lambda^i\lambda^j\partial_i\partial_j h^a{}_a + \frac{1}{2}(\partial\lambda)^2 h^a{}_a\Big]\\ -\kappa^2\Big[&-T_p h^a{}_b h^b{}_a + \frac{1}{2}\sqrt{T_p}\lambda^i\partial_i(h^a{}_a)^2 - \sqrt{T_p}\lambda^i\partial_i(h^a{}_b h^b{}_a)\Big]\\ -\kappa^3 T_p\Big[&\frac{1}{6}(h^a{}_a)^3 - h^a{}_a h^b{}_c h^c{}_b + \frac{4}{3}h^a{}_b h^b{}_c h^c{}_a\Big] + \dots\ ,\end{aligned} \tag{11.25}$$

where the open string modes along the Dp-brane were normalized as $X^i = \frac{1}{\sqrt{T_p}}\lambda^i$. This Lagrangian describes the interaction of one or two gravitons with open string excitations on the Dp-brane, whose vertices are given by

$$\begin{aligned}\tilde{V}^{\alpha\beta;i}_{h\lambda} &= \kappa\sqrt{T_p}V^{\alpha\beta}k^i_n\ ,\\ \tilde{V}^{\alpha\beta,\gamma\delta;i}_{hh\lambda} &= \kappa^2\sqrt{T_p}(k^i_{n_1}+k^i_{n_2})\Big(\frac{1}{2}V^{\alpha\beta}V^{\gamma\delta} - V^{\alpha\delta}V^{\beta\gamma}\Big)\ ,\\ \tilde{V}^{\alpha\beta;i,j}_{h\lambda\lambda} &= \frac{i}{2}\kappa\Big(k^i_{n_1}k^j_{n_1} - (k_{n_1}+k_{n_2})\cdot V\cdot k_{n_2}N^{ij}\Big)V^{\alpha\beta}\end{aligned} \tag{11.26}$$

and the corresponding Feynman diagrams are shown in figure 11.5. In (11.26) we have

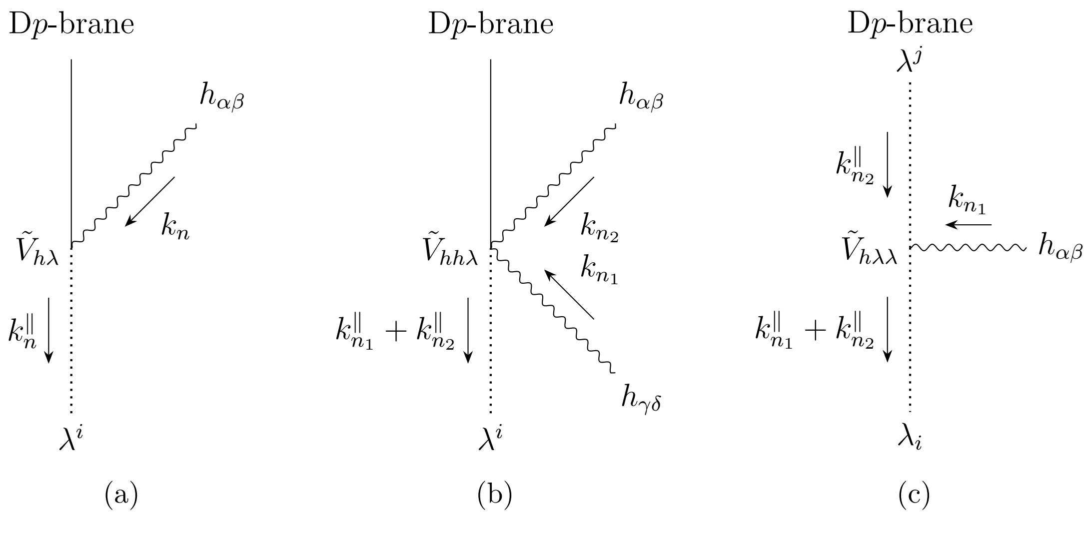

Figure 11.5: Interactions of one and two gravitons with open string excitations on the Dp-brane.

introduced the projector $V_{\mu\nu} = \frac{1}{2}(\eta_{\mu\nu} + D_{\mu\nu})$ into the subspace parallel to the Dp-brane. Moreover, one can also define the orthogonal projector into the subspace transverse to the Dp-brane $N_{\mu\nu} = \frac{1}{2}(\eta_{\mu\nu} - D_{\mu\nu})$. More details on these projectors can be found in appendix A of [92]. Furthermore, the contact term in the Lagrangian (11.25) generates an interaction

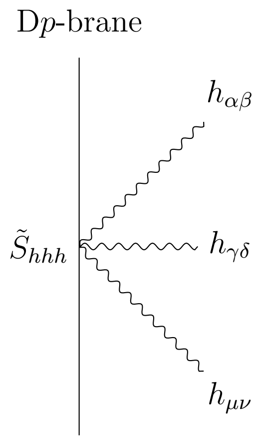

Figure 11.6: Three gravitons sourced from a Dp-brane.

of three gravitons on the brane, which gives rise to the diagram in figure 11.6. The source of this contact term can be read off the Lagrangian as

$$\tilde{S}_{hhh}^{\alpha\beta,\gamma\delta,\mu\nu} = -i\kappa^3 T_p \Big(\frac{1}{6} V^{\alpha\beta} V^{\gamma\delta} V^{\mu\nu} - V^{\alpha\beta} V^{\gamma\nu} V^{\mu\delta} + \frac{4}{3} V^{\alpha\nu} V^{\gamma\beta} V^{\mu\delta} \Big) \ . \tag{11.27}$$

With these preparations we can compute the Feynman diagrams in figure 11.1. In terms of the vertex operators (11.26) the first diagram can be written as

$$\begin{aligned} A_{hhh} &= \sum_{\sigma\in S_3} \epsilon^{\sigma(1)}_{\alpha\beta} \epsilon^{\sigma(2)}_{\gamma\delta} \epsilon^{\sigma(3)}_{\mu\nu} \tilde{V}^{\alpha\beta;i}_{h\lambda} G^{\lambda}_{ij} \tilde{V}^{\gamma\delta;j,m} G^{\lambda}_{mn} \tilde{V}^{\mu\nu;n} \\ &= -i\kappa^3 T_p \sum_{\sigma\in S_3} \Bigg(\frac{2k_{\sigma(1)}\cdot N\cdot k_{\sigma(2)} k_{\sigma(2)}\cdot N\cdot k_{\sigma(3)} + k_{\sigma(1)}\cdot N\cdot k_{\sigma(3)} k_{\sigma(2)}\cdot V\cdot k_{\sigma(2)}}{4k_{\sigma(1)}\cdot V\cdot k_{\sigma(1)} k_{\sigma(3)}\cdot V\cdot k_{\sigma(3)}} \\ &\quad - \frac{k_{\sigma(1)}\cdot N\cdot k_{\sigma(3)}}{4k_{\sigma(1)}\cdot V\cdot k_{\sigma(1)}} - \frac{k_{\sigma(1)}\cdot N\cdot k_{\sigma(3)}}{4k_{\sigma(3)}\cdot V\cdot k_{\sigma(3)}} \Bigg) \mathrm{Tr}(\epsilon_{\sigma(1)}\cdot V)\, \mathrm{Tr}\Big(\epsilon_{\sigma(2)}\cdot V\Big)\, \mathrm{Tr}(\epsilon_{\sigma(3)}\cdot V) \ , \end{aligned} \tag{11.28}$$

where the propagator of the open string scalars with momentum k is given by [102]

$$G^{\lambda}_{mn} = -i \frac{N^{mn}}{k\cdot V\cdot k} \ . \tag{11.29}$$

The second diagram can be evaluated as follows

$$\begin{aligned} A_{hh^2} &= \sum_{\sigma\in S_3} \epsilon^{\sigma(1)}_{\alpha\beta} \epsilon^{\sigma(2)}_{\gamma\delta} \epsilon^{\sigma(3)}_{\mu\nu} \tilde{V}^{\alpha\beta;i}_{h\lambda} G^{\lambda}_{ij} \tilde{V}^{\gamma\delta,\mu\nu;j} \\ &= -i\kappa^3 T_p \sum_{\sigma\in S_3} \Bigg(\frac{k_{\sigma(1)}\cdot N\cdot k_{\sigma(2)} + k_{\sigma(1)}\cdot N\cdot k_{\sigma(3)}}{2k_{\sigma(1)}\cdot V\cdot k_{\sigma(1)}} \mathrm{Tr}(\epsilon_{\sigma(1)}\cdot V)\, \mathrm{Tr}\Big(\epsilon_{\sigma(2)}\cdot V\Big)\, \mathrm{Tr}(\epsilon_{\sigma(3)}\cdot V) \\ &\quad - \frac{k_{\sigma(1)}\cdot N\cdot k_{\sigma(2)} + k_{\sigma(1)}\cdot N\cdot k_{\sigma(3)}}{k_{\sigma(1)}\cdot V\cdot k_{\sigma(1)}} \mathrm{Tr}(\epsilon_{\sigma(1)}\cdot V)\, \mathrm{Tr}(\epsilon_{\sigma(2)}\cdot V\cdot \epsilon_{\sigma(3)}\cdot V) \Bigg) \ . \end{aligned} \tag{11.30}$$

Furthermore, if we include the contribution coming from the contact term

$$\begin{aligned}
A_{h^3} &= \sum_{\sigma\in S_3} \epsilon^{\sigma(1)}_{\alpha\beta}\epsilon^{\sigma(2)}_{\gamma\delta}\epsilon^{\sigma(3)}_{\mu\nu}\tilde{S}^{\alpha\beta,\gamma\delta,\mu\nu}_{hhh} \\
&= -i\kappa^3 T_p\Big[\operatorname{Tr}(\epsilon_1\cdot V)\operatorname{Tr}(\epsilon_2\cdot V)\operatorname{Tr}(\epsilon_3\cdot V) - 2\operatorname{Tr}(\epsilon_1\cdot V)\operatorname{Tr}(\epsilon_2\cdot V\cdot\epsilon_3\cdot V) \\
&\quad -2\operatorname{Tr}(\epsilon_2\cdot V)\operatorname{Tr}(\epsilon_1\cdot V\cdot\epsilon_3\cdot V) - 2\operatorname{Tr}(\epsilon_3\cdot V)\operatorname{Tr}(\epsilon_1\cdot V\cdot\epsilon_2\cdot V) \\
&\quad +8\operatorname{Tr}(\epsilon_1\cdot V\cdot\epsilon_2\cdot V\cdot\epsilon_3\cdot V)\Big]\ , \qquad (11.31)
\end{aligned}$$

the field theory calculation for the interaction of three gravitons with orthogonal polarizations in the presence of a Dp-brane yields

$$\begin{aligned}
&A_{hhh} + A_{hh^2} + A_{h^3} = \\
&= -i\kappa^3 T_p\Big[-\Big(2 + \frac{2k_1\cdot N\cdot k_2}{k_1\cdot V\cdot k_1} + \frac{2k_1\cdot N\cdot k_3}{k_1\cdot V\cdot k_1}\Big)\operatorname{Tr}(\epsilon_1\cdot V)\operatorname{Tr}(\epsilon_2\cdot V\cdot\epsilon_3\cdot V) \\
&\quad +\Big(\frac{1}{3} + \frac{k_1\cdot N\cdot k_3 k_2\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1 k_2\cdot V\cdot k_2} + \frac{k_1\cdot N\cdot k_2 k_2\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1 k_3\cdot V\cdot k_3} + \frac{k_1\cdot N\cdot k_2 k_3\cdot V\cdot k_3}{4k_1\cdot V\cdot k_1 k_2\cdot V\cdot k_2} \\
&\quad +\frac{k_1\cdot N\cdot k_3 k_2\cdot V\cdot k_2}{4k_1\cdot V\cdot k_1 k_3\cdot V\cdot k_3} + \frac{k_1\cdot N\cdot k_2}{2k_1\cdot V\cdot k_1} + \frac{k_1\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1}\Big)\operatorname{Tr}(\epsilon_1\cdot V)\operatorname{Tr}(\epsilon_2\cdot V)\operatorname{Tr}(\epsilon_3\cdot V) \\
&\quad +\frac{8}{3}\operatorname{Tr}(\epsilon_1\cdot V\cdot\epsilon_2\cdot V\cdot\epsilon_3\cdot V)\Big] + \{1\leftrightarrow 2\} + \{1\leftrightarrow 3\}\ . \qquad (11.32)
\end{aligned}$$

In the limit $\alpha' \to 0$ the result (11.22) obtained from the scattering of three closed strings off a Dp-brane in (8.59) should reproduce the field theory result presented above. To compare these we rewrite the kinematic invariants

$$\begin{aligned}
s_{12} &= \frac{1}{2}k_3\cdot V\cdot k_3 - \frac{1}{2}k_1\cdot V\cdot k_1 - \frac{1}{2}k_2\cdot V\cdot k_2 + k_1\cdot N\cdot k_2\ , \\
s_{13} &= \frac{1}{2}k_2\cdot V\cdot k_2 - \frac{1}{2}k_1\cdot V\cdot k_1 - \frac{1}{2}k_3\cdot V\cdot k_3 + k_1\cdot N\cdot k_3\ , \\
s_{23} &= \frac{1}{2}k_1\cdot V\cdot k_1 - \frac{1}{2}k_2\cdot V\cdot k_2 - \frac{1}{2}k_3\cdot V\cdot k_3 + k_2\cdot N\cdot k_3\ . \qquad (11.33)
\end{aligned}$$

which follows from $D_{\mu\nu} = 2V_{\mu\nu} - \eta_{\mu\nu}$. If all external states are described by gravitons with orthogonal polarizations (11.22) becomes

$$\begin{aligned}
\lim_{\alpha'\to 0}\mathcal{A}\Big|_{\substack{e_i\cdot p_j \\ \bar{e}_i\cdot p_j}\}\to 0} &= -\frac{1}{2}g_c^3 T_p\Big[\frac{(s_{1\bar{2}} + s_{1\bar{3}})}{s_{1\bar{1}}}\operatorname{Tr}(D\cdot\epsilon_1)\operatorname{Tr}(D\cdot\epsilon_2\cdot D\cdot\epsilon_3) + \Big(\frac{1}{3} + \frac{s_{12}+s_{13}-s_{23}}{2s_{1\bar{1}}} \\
&\quad +\frac{s_{12}s_{13}}{s_{2\bar{2}}s_{3\bar{3}}} + \frac{(s_{12}+s_{13}+s_{23})s_{1\bar{1}}}{4s_{2\bar{2}}s_{3\bar{3}}}\Big)\operatorname{Tr}(D\cdot\epsilon_1)\operatorname{Tr}(D\cdot\epsilon_2)\operatorname{Tr}(D\cdot\epsilon_3) \\
&\quad +\frac{2}{3}\operatorname{Tr}(D\cdot\epsilon_1\cdot D\cdot\epsilon_2\cdot D\cdot\epsilon_3)\Big] + \{1\leftrightarrow 2\} + \{1\leftrightarrow 3\} \qquad (11.34)
\end{aligned}$$

$$
\begin{aligned}
&= -g_c^3 T_p \Bigg[-\left(2 + \frac{2k_1\cdot N\cdot k_2}{k_1\cdot V\cdot k_1} + \frac{2k_1\cdot N\cdot k_3}{k_1\cdot V\cdot k_1}\right) \mathrm{Tr}(\epsilon_1\cdot V)\,\mathrm{Tr}(\epsilon_2\cdot V\cdot\epsilon_3\cdot V) \\
&\quad + \left(\frac{1}{3} + \frac{k_1\cdot N\cdot k_3 k_2\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1 k_2\cdot V\cdot k_2} + \frac{k_1\cdot N\cdot k_2 k_2\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1 k_3\cdot V\cdot k_3} + \frac{k_1\cdot N\cdot k_2 k_3\cdot V\cdot k_3}{4k_1\cdot V\cdot k_1 k_2\cdot V\cdot k_2}\right. \\
&\quad \left. + \frac{k_1\cdot N\cdot k_3 k_2\cdot V\cdot k_2}{4k_1\cdot V\cdot k_1 k_3\cdot V\cdot k_3} + \frac{k_1\cdot N\cdot k_2}{2k_1\cdot V\cdot k_1} + \frac{k_1\cdot N\cdot k_3}{2k_1\cdot V\cdot k_1}\right) \mathrm{Tr}(\epsilon_1\cdot V)\,\mathrm{Tr}(\epsilon_2\cdot V)\,\mathrm{Tr}(\epsilon_3\cdot V) \\
&\quad + \frac{8}{3}\,\mathrm{Tr}(\epsilon_1\cdot V\cdot\epsilon_2\cdot V\cdot\epsilon_3\cdot V) \Bigg] + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\}\ . \qquad (11.35)
\end{aligned}
$$

After assuming that $g_c \sim \kappa$ and comparing the string result with the field theory computation we find that they are related via

$$
\mathcal{A} \sim i(A_{hhh} + A_{hh^2} + A_{h^3})\ , \qquad (11.36)
$$

which provides a non-trivial consistency check for (8.59).

To conclude this chapter we want to give some remarks about the α'-expansion of section 11.1: Relative to the field theory contribution, which is the lowest order in the expansion (11.18), the α'-corrections start at order $\mathcal{O}(\alpha'^2)$. This is in agreement with the results of the scattering of two closed strings on the disk [91, 92, 102]. Hence, α'-corrections to the DBI action involve terms with at least four derivatives, as was argued in [84] for R^2-terms. Especially, there is evidence for a contribution to the Einstein-Hilbert term, which is coming from the disk. According to [179] S-duality between heterotic and type I string theory provides indirect arguments for the existence of an $\epsilon_{10}\epsilon_{10}R^4$ term in the worldvolume theory of a D9-brane. Upon compactification on a Calabi-Yau manifold with non-vanishing Euler number it was anticipated by [186] that this term results in corrections of the Einstein-Hilbert term in four dimensions. As the scattering of three gravitons with four dimensional polarisations is agnostic about the shape of the additional six dimensions, one might assume that an Einstein-Hilbert term on the D9 worldvolume exists. Moreover, the scattering of two on-shell gravitons in the presence of a D9-brane vanishes [91, 92, 102] such that this 2-point function seems to degenerate in order to draw any conclusion about the existence of an Einstein-Hilbert term. Hence, the scattering of three closed string on the disk was the next logical candidate to provide the correct low energy contribution for such a term, but our result speaks against it.

In principal, there are further higher derivative corrections in addition to the predicted term $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ at disk level [179, 187]. Contrary to the $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$-term, which does not correct the dilaton equations of motion in ten dimensions[1] [186], the additional R^4 terms can lead to corrections of the ten dimensional dilation at disk level. Therefore, when we compactify the ten dimensional Einstein Hilbert term $e^{-2\Phi}R$ to four dimensions following the steps in [188], we would also have to account for the contribution from the

[1]Due to the epsilon tensors in $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ the flat non-compact space enters in at least one of the Riemann tensors.

additional terms. Thus, these two contributions from the disk to the four dimensional Einstein-Hilbert term might cancel against each other.

Chapter 12

Concluding remarks

In this thesis we have studied the interactions of closed strings with non-perturbative objects: We have computed the *complete* tree-level disk amplitude involving any three external closed string states in the NSNS, NSR, RNS or RR sector, which can be found in (8.59) and also provide an ansatz for the generalization to an arbitrary number of closed strings in (9.13). Therefore, these result are interesting both from a conceptual and physical point of view. We have demonstrated that the closed string amplitude on the disk can be expressed in terms of a basis of six-point open string subamplitudes. This allows us to connect the closed string amplitude on the disk to the scattering of open strings on the disk via KLT-like relations and a $PSL(2,\mathbb{R})$-transformation. Surprisingly, the final result (8.56) can be expressed in terms of only two six-point open string subamplitudes, whereas the basis of these subamplitudes contains six elements. Hence, one might have guessed that also the scattering of three closed strings is given in terms of six subamplitudes. We conjecture that this pattern persists for a closed string n-point function as shown in (9.13). In order to derive our result, we introduced monodromy relations for open strings in section 7.2, which contain n terms instead of only $n-1$ terms as in the open string subamplitude relations in [35].

The evaluation of the correlator in appendix C shows that we can express the correlator of three closed strings on the disk in terms of SYM amplitudes, c.f. (C.34). This might not be surprising, because similar relations have been found for open strings on the disk in [62, 162]. Nevertheless, it is an important and non-trivial consistency check, as it required some additional computational effort to show this. For example, we had to perform a $PSL(2,\mathbb{R})$-transformation, which required to use composite superfields still containing all BRST exact terms that are usually discarded from the beginning. These composite superfields in the correlator are commonly compute by using OPE contractions in the PSF [158, 160, 161]. We have chosen a different (more fundamental) approach and computed the correlator in the PSF by applying Wick's theorem.

In the low energy limit, where $\alpha' \to 0$, we verified that for a subset of terms the field theory result from the string theory calculation (11.35) matches the corresponding terms

(11.32) obtained from the DBI action. In addition, we could show that the result in (11.22) has the correct pole structure, which is expected to be found for the physical amplitude. In the literature, see for example [35, 189], the scattering of three closed strings was performed on the double cover of the disk, i.e. the sphere. In this case the s-channel poles in $s_{i\bar{\imath}}$ are not present in the low energy expansion. The absence of these terms on the double cover is not surprising and we presented an argument at the end of chapter 8 that should also hold for higher-point scattering amplitudes. More importantly, this argument shows that it is not possible to extend the integration over the upper half plane to the entire complex plane for a closed string amplitude on the disk, as it was done in [189].

Furthermore, we pioneer the scattering of closed strings on the real projective plane with more than two vertex operator insertions. We provide a prescription for the scattering of n closed strings off an Op-plane and explicitly apply this prescription for $n = 3$: Using parts of the previous calculation of three closed strings on the disk we arrive at an expression that can be written in terms of four open string partial amplitudes. Similar as for the disk amplitude, the final result contains less open string subamplitudes than the number of elements in the minimal basis. To achieve this we had to carry out the analytic continuation for an unoriented surface, which turned out to be a more involved calculation than anticipated in [189], where an attempt was made to derive the general expression for the n-point function of closed strings on the real projective plane after analytic continuation. Hence, our calculation leads a different result compared to [189].

There are a lot of possible directions to generalize the computations in this thesis: Performing the corresponding evaluations at one-loop would be interesting. For example, this would amount to closed string scattering on a cylinder worldsheet, which would give an explicit application of the techniques developed in [190, 191, 192, 193] to perform the cylinder worldsheet integrations. On the other hand, KLT or monodromy relations on genus one non-oriented surfaces are mostly uncharted territory, but an ansatz was proposed in [194]. Another direction could be to extend the scattering process on the disk and projective plane to massive external states in the spirit of [195, 196]. So far, we have only performed the computation of the scattering of three closed strings on the real projective plan, but it would be interesting to investigate the low energy effective description that arises from this amplitude. Maybe it would be possible to get further insight on a term like $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ from this amplitude. Finally, it would be important to understand why the disk calculation does not see any hints of an Einstein-Hilbert term on the disk, which should arise when compactifying an $e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$ term in the worldvolume of a D9-brane to four dimensions as discussed in chapter 11. In this case it might be helpful to analyse the corresponding three graviton amplitude on the projective plane or go beyond three external closed string states on the disk. In any case, it would be important to compute higher multiplicity closed string amplitudes on the disk starting with four closed strings scattered off a Dp-brane to see if the desired term is hidden somewhere in these amplitudes. Thereby, it could be possible to find the precise n-point generalization of our ansatz in (9.13).

Appendix A

$U(5)$ decomposition of $SO(10)$

To analyse the pure spinor constraint (3.25) in section 3.2.1 we have to decompose the Lorentz group $SO(1,9)$ after a Wick rotation in terms of $U(5)$. Hence, we will give the basics of the $U(5) = SU(5) \otimes U(1)$ decomposition of the Wick rotated Lorentz group $SO(10)$ in this appendix following the lines of [114].

A.1 The Wick rotated Clifford algebra in $\mathbb{R}^{10}$

Before we can start with the $U(5)$ decomposition of $SO(10)$ we have to establish the Clifford algebra of a Euclidean space with ten dimensions. Therefore, we can Wick rotate the Lorentz group $SO(1,9)$ and obtain $SO(10)$. The Clifford algebra in ten dimensional Euclidean space becomes

$$\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}, \quad m,n = 1,2,\ldots,10 \ . \tag{A.1}$$

This algebra admits a redefinition [197] for the Γ-matrices in terms of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

together with the unit matrix $\mathbb{1} = 1_{2\times 2}$. The Pauli matrices satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ for $i,j = 1,2,3$. According to [198] we can use the Kronecker product of Pauli sigma matrices to construct the $2^5 \times 2^5$ gamma matrices Γ^m in ten dimensions

$$\begin{aligned}
\Gamma^1 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^6 &= \sigma_2 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , \\
\Gamma^2 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^7 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \ , \\
\Gamma^3 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \ , & \Gamma^8 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \ , \\
\Gamma^4 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \ , & \Gamma^9 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \ , \\
\Gamma^5 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \ , & \Gamma^{10} &= -\sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ ,
\end{aligned} \tag{A.3}$$

which is the analogous representation of Γ^m for Minkowski spacetime $\mathbb{R}^{1,9}$, see (2.5). The properties of the Kronecker product

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) , \qquad (A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}} \tag{A.4}$$

ensure that the Clifford algebra (A.1) is satisfied by the matrices (A.3). In addition, we find the following symmetry properties of the Γ-matrices (A.3)

$$\Gamma_m^{\mathrm{T}} = \begin{cases} -\Gamma_m , & m = 1, \dots, 5 \\ \Gamma_m , & m = 6, \dots, 10 . \end{cases} \tag{A.5}$$

Moreover, this representation is hermitian: Because the gamma matrices in (A.3) are constructed from an even or odd number of σ_2, they are purely imaginary for $m = 1, \dots, 5$ and real for $m = 6, \dots 10$. Hence, they satisfy

$$\Gamma_m^{\dagger} = \Gamma_m . \tag{A.6}$$

We can define a charge conjugation matrix C, which has to obey

$$C\Gamma_m = -\Gamma_m^{\mathrm{T}} C \tag{A.7}$$

according to the above symmetry properties of Γ_m. Hence, the charge conjugation matrix C is given by the product of all antisymmetric gamma matrices [199] and can be written in the representation of the Pauli matrices as

$$C = \Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 = -\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 . \tag{A.8}$$

Therefore, the charge conjugation matrix itself is antisymmetric. Further, it is off-diagonal and satisfies $C^2 = 1_{32\times32}$.

The chirality matrix Γ_{11} can be written as a product of all ten gamma matrices

$$\Gamma_{11} = -i\Gamma_1 \cdots \Gamma_{10} = \begin{pmatrix} 1_{16\times16} & 0 \\ 0 & -1_{16\times16} \end{pmatrix} \tag{A.9}$$

and is the same as for a ten dimensional Lorentzian spacetime.

A.2 Vector and spinor representations and Lorentz generators of $SO(10)$

The vector V^p and spinor Ψ representation of the Wick rotated Lorentz group $SO(10)$ are defined by their transformation under $SO(10)$

$$[M^{m,n}, V^p] = \delta^{mp} V^n - \delta^{np} V^m ,$$

$$[M^{mn}, \Psi] = \frac{1}{2}\Gamma^{mn}\Psi \ , \tag{A.10}$$

where M^{mn} are the generators of the group $SO(10)$. They are antisymmetric $M^{mn} = -M^{nm}$ and satisfy the following Lie algebra

$$[M^{mn}, M^{pq}] = \delta^{mp}M^{nq} - \delta^{np}M^{mq} - \delta^{mq}M^{np} + \delta^{nq}M^{mp} \ . \tag{A.11}$$

We are interested in the $U(5)$ decomposition of the vectors V^p, spinors Ψ and the Lorentz generators M^{mn}. To decompose the vectorial representation form $SO(10) \to U(5)$ we have to split the ten components of the vector V^p with $p = 1, \dots, 10$ in two sets. For $a = 1, \dots, 5$ we can define two vectors v^a, v_a as a linear combination of the components V^p

$$v^a = \frac{1}{\sqrt{2}}\Big(V^a + iV^{a+5}\Big) \ , \qquad v_a = \frac{1}{\sqrt{2}}\Big(V^a - iV^{a+5}\Big) \ , \quad a = 1, \dots, 5 \ . \tag{A.12}$$

We split the components of tensors of $SO(10)$ following the same pattern as for the vector decomposition in (A.12) while preserving the symmetries of the corresponding tensor. Concretely, the Lorentz generators M^{mn} decompose as

$$\begin{aligned}
m^{ab} &= \frac{1}{2}\Big(M^{ab} + iM^{a(b+5)} + iM^{(a+5)b} - M^{(a+5)(b+5)}\Big) \ , \\
m_{ab} &= \frac{1}{2}\Big(M^{ab} - iM^{a(b+5)} - iM^{(a+5)b} - M^{(a+5)(b+5)}\Big) \ , \\
m^a_b &= \frac{1}{2}\Big(M^{ab} - iM^{a(b+5)} + iM^{(a+5)b} + M^{(a+5)(b+5)}\Big)
\end{aligned} \tag{A.13}$$

and the trace of m^a_b is given by

$$m = \sum_{a=1}^{5} m^a_a = i\sum_{a=1}^{5} M^{(a+5)a} \ . \tag{A.14}$$

We can invert (A.12) and (A.13) to write the $SO(10)$ components in terms of their $U(5)$ decomposition. Then, plugging them in (A.11) gives the $SO(10)$ Lie algebra decomposed to $U(5)$ as

$$\begin{aligned}
&[m_{ab}, m_{cd}] = 0 \ , && [m^{ab}, m^{cd}] = 0 \ , \\
&[m_{ab}, m^{cd}] = -\delta^c_a m^d_b + \delta^d_a m^c_b + \delta^c_b m^d_a - \delta^d_b m^c_a \ , && [m_{ab}, m^c_d] = -\delta^c_b m_{ab} + \delta^c_a m_{bd} \ , \\
&[m_{ab}, m^c_d] = -\delta^a_d m^{bc} + \delta^b_d m^{ac} \ , && [m^a_b, m^c_d] = -\delta^c_b m^a_d + \delta^a_d m^c_b \ , \\
&[m, m_{ab}] = 2m_{ab} \ , && [m, m^{ab}] = -2m^{ab} \ , \\
&[m, m^a_b] = 0 \ , && [m, m] = 0 \ .
\end{aligned} \tag{A.15}$$

Similarly, the commutators of the $SO(10)$ Lorentz generators with the vectors in (A.10) decompose under $U(5)$ as follows

$$[m^{ab}, v^c] = 0 \ , \qquad [m_{ab}, v_c] = 0 \ ,$$

$$
\begin{aligned}
[m_{ab}, v^c] &= \delta^c_a v_b - \delta^c_b v_a \ , \qquad & [m^{ab}, v_c] &= \delta^a_c v^b - \delta^b_c v^a \ , \\
[m^a_b, v^c] &= -\delta^c_b v^a \ , & [m^a_b, v_c] &= \delta^a_c v_b \ , \\
[m, v^c] &= -v^c \ , & [m, v_c] &= v_c \ .
\end{aligned} \tag{A.16}
$$

Hence, from (A.15) we can conclude that m^a_b are the generators of $U(5)$ embedded in $SO(10)$ and m^{ab} and m_{ab} transform as two forms under $U(5)$. Furthermore, in (A.16) the vectors v_a and v^a transform in the defining representations **5** and $\overline{\mathbf{5}}$ of $U(5)$.

From the commutator

$$
[m^a_b, m^c_d] = -\delta^c_b m^a_d + \delta^a_d m^c_b \tag{A.17}
$$

it follows that the $SO(10)$ Lie algebra has an $5^2 = 25$ dimensional subalgebra, which contains a $U(1)$ subgroup generated by m and the other $5^2 - 1 = 24$ generators are given by

$$
\tilde{m}^a_b = m^a_b - \frac{1}{5}\delta^a_b m \ , \tag{A.18}
$$

which are traceless and generate the $SU(5)$ algebra

$$
[\tilde{m}^a_b, \tilde{m}^c_d] = -\delta^c_b \tilde{m}^a_d + \delta^a_d \tilde{m}^c_b \ . \tag{A.19}
$$

Hence, we have shown that $U(5)$ decomposes into $SU(5) \otimes U(1)$ [141]. Each of these representations R carries a $U(1)$ charge q_R, which is defined by $[m, R] = q_R R$. We can denote this charge by a subscript $\mathbf{N}_{q_R}$ for an N-dimensional representation of $SU(5)$. For the components of the antisymmetric tensor M^{mn} and vector V^p we find the following transformations under the $SU(5) \otimes U(1)$ decomposition of $SO(10)$

$$
\begin{aligned}
V^m &\to v^a \oplus v_a \ , \qquad & M^{mn} &\to m^{ab} \oplus m_{ab} \oplus m^a_b \oplus m \ , \\
\mathbf{10} &\to \mathbf{5}_{-1} \oplus \overline{\mathbf{5}}_1 \ , & \mathbf{45} &\to \mathbf{10}_{-2} \oplus \overline{\mathbf{10}}_2 \oplus \mathbf{24}_0 \oplus \mathbf{1}_0 \ .
\end{aligned} \tag{A.20}
$$

To obtain the decomposition of the spinorial representation of $SO(10)$ under $SU(5) \otimes U(1)$ we will consider a linear combination of the Γ-matrices [200]

$$
b^a = \frac{1}{2}\Big(\Gamma^a + i\Gamma^{a+5}\Big) \ , \qquad b_a = \frac{1}{2}\Big(\Gamma^a - i\Gamma^{a+5}\Big) \ , \quad a = 1, \ldots, 5 \ . \tag{A.21}
$$

Following from (A.5) and (A.6) these matrices satisfy

$$
\begin{aligned}
b^\dagger_a &= b^a \ , \qquad & (b^a)^\dagger &= b_a \ , \\
b^{\mathrm{T}}_a &= -b^a \ , & (b^a)^{\mathrm{T}} &= -b_a \ ,
\end{aligned} \tag{A.22}
$$

for $a = 1 \ldots, 5$. The Clifford algebra (A.1) for the matrices b_a and b^a becomes

$$
\{b_a, b^b\} = \delta^b_a \ , \qquad \{b_a, b_b\} = \{b^a, b^b\} = 0 \ . \tag{A.23}
$$

Since (A.23) is a fermionic oscillator algebra, we can interpret the matrices b^a and b_a as creation and annihilation operators, respectively. Hence, we can define a vacuum state

$|0\rangle$ that is annihilated by all b_a operators, i.e. $b_a|0\rangle = 0$ and similarly $\langle 0|b^a = 0$, and normalized $\langle 0|0\rangle = 1$. We can create up 32 states by acting with the creation operators b^a on the vacuum. Moreover, for such a state we define $\langle\psi| = |\psi\rangle^{\mathrm{T}}$. In terms of the raising and lowering operators the charge conjugation and chirality matrices are given by

$$C = \prod_{a=1}^{5}(b_a + b^a)\ , \qquad \Gamma_{11} = \prod_{a=1}^{5}(b^a b_a - b_a b^a) = \prod_{a=1}^{5}(2b^a b_a - 1)\ . \tag{A.24}$$

The Lorentz generators M^{mn} for the spinorial representation of $SO(10)$ are given by

$$M^{mn} \to -\frac{1}{2}\Gamma^{mn} = -\frac{1}{4}[\Gamma^m, \Gamma^n]\ , \tag{A.25}$$

which can be checked to satisfy the Lie algebra (A.11) of the Lorentz generators. We use the (inverted) $U(5)$ decomposition (A.21) of $SO(10)$ for the Euclidean gamma matrices to write the $U(5)$ generators (A.13) and (A.14) in terms of the matrices b_a and b^a after plugging in the spinorial representation (A.25) as

$$\begin{aligned} m^a_b &= -\frac{1}{2}(b^a b_b - b_b b^a)\ , & m &= -\frac{1}{2}(b^a b_a - b_a b^a) = -b^a b_a + \frac{5}{2}\ , \\ m^{ab} &= -b^a b^b\ , & m_{ab} &= -b_a b_b\ . \end{aligned} \tag{A.26}$$

The spinorial $U(5)$ Lorentz generators satisfy the decomposition (A.15) of the $SO(10)$ Lie algebra, which can be verified using $[A, BC] = [A, B], C + B[A, C]$ and $[A, BC] = \{A, B\}C + B\{A, C\}$. Moreover, the equivalent of the $U(5)$ vector transformations is given by

$$\begin{aligned} [m^{ab}, b^c] &= 0\ , & [m_{ab}, b_c] &= 0\ , \\ [m_{ab}, b^c] &= \delta^c_a b_b - \delta^c_b b_a\ , & [m^{ab}, b_c] &= \delta^a_c b^b - \delta^b_c b^a\ , \\ [m^a_b, b^c] &= -\delta^c_b b^a\ , & [m^a_b, b_c] &= \delta^a_c b_b\ , \\ [m, b^c] &= -b^c\ , & [m, b_c] &= b_c\ . \end{aligned} \tag{A.27}$$

From these relations it follows that m^a_b are the generators of $U(5)$ and m is the generator of $U(1)$ in the decomposition $U(5) = SU(5) \otimes U(1)$ as before. Similarly, the creation operators b^a and annihilation operators b_a transform in the vector representations $\mathbf{5}_{-1}$ and $\overline{\mathbf{5}}_1$ of $SU(5) \otimes U(1)$, respectively.

Applications to pure spinors

For the fundamentals of the pure spinor formalism it is important to know how the $U(5)$ components of a pure spinor λ^α transform under $SO(10)$ rotations. For this purpose we use the notation [200] $O|\psi\rangle = |O\psi\rangle$ for an arbitrary operator O to read off how the different components transform under O. The spinorial Lorentz transformations (A.10), which in this case are generated by $M^{mn} = -\frac{1}{2}\Gamma^{mn}$, decompose as follows

$$
\begin{aligned}
m_{ab}\left|\lambda\right\rangle &= -\lambda_{ab}\left|0\right\rangle - \frac{1}{2}\epsilon_{abcde}\lambda^{e}b^{d}b^{c}\left|0\right\rangle \ , \\
m^{a}_{b}\left|\lambda\right\rangle &= \frac{1}{2}\delta^{a}_{b}\left|\lambda\right\rangle - \lambda_{cd}b^{a}b^{c}\left|0\right\rangle - \frac{1}{3!}\lambda^{c}\epsilon_{cdefb}b^{a}b^{f}b^{e}b^{d}\left|0\right\rangle \ , \\
m\left|\lambda\right\rangle &= \frac{5}{2}\lambda^{+}\left|0\right\rangle + \frac{1}{4}\lambda_{ab}b^{b}b^{a}\left|0\right\rangle - \frac{1}{16}\epsilon_{abcde}b^{e}b^{d}b^{c}b^{b}\left|0\right\rangle
\end{aligned}
\tag{A.28}
$$

when using the $U(5)$ decomposition (A.26) for the Lorentz generators. We can project these transformations onto the $U(5)$ components (3.29): For example, the **10** component of $SU(5)$ can be obtained from the projection

$$
\left|m\lambda\right\rangle_{ab} = \left\langle 0\right| b_{a}b_{b}m\left|\lambda\right\rangle = \frac{1}{4}\lambda_{cd}\left\langle 0\right| b_{a}b_{b}b^{c}b^{d}\left|0\right\rangle = \frac{1}{2}\lambda_{ab} \ . \tag{A.29}
$$

Consequently, the $SO(10)$-transformations imply

$$
\begin{aligned}
m_{ab}\lambda^{+} &= -\lambda_{ab} \ , & m_{ab}\lambda_{cd} &= -\epsilon_{abcde}\lambda^{e} \ , & m_{ab}\lambda^{c} &= 0 \ , \\
m^{ab}\lambda^{+} &= 0 \ , & m^{ab}\lambda_{cd} &= \delta^{a}_{[c}\delta^{a}_{d]}\lambda^{+} \ , & m^{ab}\lambda^{c} &= \frac{1}{2}\epsilon^{abcde}\lambda_{de} \ , \\
m^{a}_{b}\lambda^{+} &= \frac{1}{2}\delta^{a}_{b}\lambda^{+} \ , & m^{a}_{b}\lambda_{c}d &= \delta^{a}_{[c}\lambda_{d]b} + \frac{1}{2}\delta^{a}_{b}\lambda_{cd} \ , & m^{a}_{b}\lambda^{c} &= \delta^{c}_{b}\lambda^{a} - \frac{1}{2}\delta^{a}_{b}\lambda^{c} \ , \\
m\lambda^{+} &= \frac{5}{2}\lambda^{+} \ , & m\lambda_{cd} &= \frac{1}{2}\lambda_{cd} \ , & m\lambda^{c} &= -\frac{3}{2}\lambda^{c} \ .
\end{aligned}
\tag{A.30}
$$

After performing the identification (3.49) we find the $SO(10) \to SU(5) \otimes U(1)$ decompositions for the projections of the $SO(10)$-transformations

$$
\begin{aligned}
n_{ab}\lambda^{+} &= \lambda_{ab} \ , & n_{ab}\lambda_{cd} &= \epsilon_{abcde}\lambda^{e} \ , & n_{ab}\lambda^{c} &= 0 \ , \\
n^{ab}\lambda^{+} &= 0 \ , & n^{ab}\lambda_{cd} &= -\delta^{a}_{[c}\delta^{a}_{d]}\lambda^{+} \ , & n^{ab}\lambda^{c} &= -\frac{1}{2}\epsilon^{abcde}\lambda_{de} \ , \\
n^{a}_{b}\lambda^{+} &= 0 \ , & n^{a}_{b}\lambda_{c}d &= -\delta^{a}_{[c}\lambda_{d]b} - \frac{2}{5}\delta^{a}_{b}\lambda_{cd} \ , & n^{a}_{b}\lambda^{c} &= -\delta^{c}_{b}\lambda^{a} + \frac{1}{5}\delta^{a}_{b}\lambda^{c} \ , \\
n\lambda^{+} &= -\frac{\sqrt{5}}{2}\lambda^{+} \ , & n\lambda_{cd} &= -\frac{1}{2\sqrt{5}}\lambda_{cd} \ , & n\lambda^{c} &= \frac{3}{2\sqrt{5}}\lambda^{c} \ ,
\end{aligned}
\tag{A.31}
$$

which are the single pole coefficients in the OPE (3.52).

Appendix B

$PSL(2,\mathbb{R})$-transformations on the disk

In [130] the invariance of a correlator on the sphere under $SL(2,\mathbb{C})$-transformations is discussed to derive the general structure of OPEs between conformal primaries. Moreover, in [130] differential equations were derived, which have to be satisfied by each correlation function of conformal primaries. Following the steps in [109] we want to derive similar relations for a correlator on the disk, which will be used to change the $PSL(2,\mathbb{R})$ frame of the correlator in (C.10).

A correlation function on the disk can be interpreted as the vacuum expectation value of a radially ordered product of fields, where the ground state is invariant under $PSL(2,\mathbb{R})$-transformations. Therefore, a CFT correlator with n primary fields $\phi_i(z_i,\overline{z}_i)$ has to transform as

$$\langle\!\langle \phi'_1(z_1,\overline{z}_1)\cdots\phi'_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} = \langle\!\langle \phi_1(z_1,\overline{z}_1)\cdots\phi_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} \ , \tag{B.1}$$

where $\phi'_i(z_i,\overline{z}_i) = U\phi_i(z_i,\overline{z}_i)U^{-1}$ is the $PSL(2,\mathbb{R})$ transformed field $\phi_i(z_i,\overline{z}_i)$ and the transformation U leaves the in and out state invariant. As mentioned in section 8.1 due to the boundary of the disk the holomorphic and anthropomorphic fields interact with each other. Compared to the sphere we cannot separate left- and right-movers.

The globally defined generators L_{-1}, L_0 and L_1 of $PSL(2,\mathbb{R})$ act on conformal primaries $\phi(z,\overline{z})$ as

$$\begin{array}{rccl}
L_{-1}: & \text{translations} & U=e^{bL_{-1}} & \phi'(z,\overline{z}) = \phi(z+b,\overline{z}+b)\ , \\
L_0: & \text{dilatations and rotations} & U=e^{\ln a L_0} & \phi'(z,\overline{z}) = a^h a^{\overline{h}}\phi(az,a\overline{z})\ , \\
L_1: & \text{special conformal transformations} & U=e^{cL_1} & \phi'(z,\overline{z}) = \left(\frac{1}{1-cz}\right)^{2h}\left(\frac{\overline{z}}{1-c\overline{z}}\right)^{2\overline{h}}\phi\left(\frac{z}{1-cz},\frac{\overline{z}}{1-c\overline{z}}\right)\ ,
\end{array} \tag{B.2}$$

where $(h,\overline{h})$ is the conformal dimension of $\phi(z,\overline{z})$. The transformation parameters have to be real, because the conformal Killing group of the disk is $PSL(2,\mathbb{R})$.[1] Moreover, this

[1] Note that the CKG of the disk is actually $SU(1,1)$ and the upper half plane $\mathbb{H}_+$ has CKG $PSL(2,\mathbb{R})$. Since we can map the disk to the upper half plane, we use the two expressions synonymously.

implies that the infinitesimal conformal transformation for worldsheet coordinates are given by

$$z' = z + \varepsilon(z) \ , \qquad \overline{z}' = \overline{z} + \overline{\varepsilon}(\overline{z}) = \overline{z} + \varepsilon(\overline{z}) \ , \tag{B.3}$$

where also here the transformation parameters are real, i.e. $\overline{\varepsilon}(\overline{z}) = \varepsilon(\overline{z})$. Using (B.3) we proceed to calculate the infinitesimal transformation of a primary field $\phi(z,\overline{z})$

$$\delta_\varepsilon \phi(z,\overline{z}) = \phi'(z,\overline{z}) - \phi(z,\overline{z}) = \phi'(z' - \varepsilon(z), \overline{z}' - \varepsilon(\overline{z})) - \phi(z,\overline{z}) \ . \tag{B.4}$$

Then, we carry out a Taylor expansion of $\phi'(z'-\varepsilon(z), \overline{z}'-\varepsilon(\overline{z}))$ in the infinitesimal parameter ε up to linear order in ε so that (B.4) becomes

$$\delta_\varepsilon \phi(z,\overline{z}) = \phi'(z',\overline{z}') - \varepsilon(z)\partial\phi(z,\overline{z}) - \varepsilon(\overline{z})\overline{\partial}\phi(z,\overline{z}) - \phi(z,\overline{z}) + \mathcal{O}(\varepsilon^2) \ . \tag{B.5}$$

In general, the conformal transformation of a primary field with dimension $(h,\overline{h})$ is given by

$$\phi'(z',\overline{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{-\overline{h}} \phi(z,\overline{z}) \ . \tag{B.6}$$

Inserting this transformation in (B.5) yields

$$\begin{aligned} \delta_\varepsilon \phi(z,\overline{z}) &= \Big[(1+\partial\varepsilon(z))^{-h}(1+\overline{\partial}\varepsilon(\overline{z}))^{-\overline{h}} - \Big(1 + \varepsilon(z)\partial + \varepsilon(\overline{z})\overline{\partial}\Big)\Big]\phi(z,\overline{z}) + \mathcal{O}(\varepsilon^2) \\ &= -\Big(h\partial\varepsilon(z) + \overline{h}\overline{\partial}\varepsilon(\overline{z}) + \varepsilon(z)\partial + \varepsilon(\overline{z})\overline{\partial}\Big)\phi(z,\overline{z}) + \mathcal{O}(\varepsilon^2) \ . \end{aligned} \tag{B.7}$$

Then, writing the infinitesimal parameter ε of the transformation of the worldsheet fields as

$$\varepsilon(z) = \varepsilon_{-1} + \varepsilon_0 z + \varepsilon_1 z^2 \tag{B.8}$$

and plugging it into (B.7) gives after computing derivatives and rearranging some terms

$$\delta_\varepsilon \phi(z,\overline{z}) = -\Big[\varepsilon_{-1}\Big(\partial + \overline{\partial}\Big) + \varepsilon_0\Big(h + z\partial + \overline{h} + \overline{z}\overline{\partial}\Big) + \varepsilon_1\Big(2hz + z^2\partial + 2\overline{h}\overline{z} + \overline{z}^2\overline{\partial}\Big)\Big]\phi(z,\overline{z}) \ . \tag{B.9}$$

Because a CFT correlator is invariant under conformal transformations, i.e. satisfies (B.1), the corresponding infinitesimal transformation of the correlator has to vanish

$$\begin{aligned} 0 = \delta_\varepsilon \langle\!\langle \phi_1(z_1,\overline{z}_1)\cdots\phi_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} &= \sum_{i=1}^n \langle\!\langle \phi_1(z_1,\overline{z}_1)\cdots\delta_\varepsilon\phi_i(z_i,\overline{z}_i)\cdots\phi_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} \\ = \sum_{i=1}^n \Big[\varepsilon_{-1}\Big(\partial_i + \overline{\partial}_i\Big) &+ \varepsilon_0\Big(h_i + z_i\partial_i + \overline{h}_i + \overline{z}_i\overline{\partial}_i\Big) \\ +\varepsilon_1\Big(2h_i z_i + z_i^2\partial_i &+ 2\overline{h}_i\overline{z}_i + \overline{z}_i^2\overline{\partial}_i\Big)\Big]\langle\!\langle \phi_1(z_1,\overline{z}_1)\cdots\phi_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} \ , \end{aligned} \tag{B.10}$$

where we have used (B.9) for the fields ϕ_i. Because the parameters in the transformation (B.8) are in general not vanishing, by equating coefficient for $\varepsilon_{-1}, \varepsilon_0$ and ε_1 we can extract three independent equations from (B.10)

$$0 = \sum_{i=1}^n \Big(\partial_i + \overline{\partial}_i\Big)\langle\!\langle \phi_1(z_1,\overline{z}_1)\cdots\phi_n(z_n,\overline{z}_n)\rangle\!\rangle_{D_2} \ ,$$

$$0 = \sum_{i=1}^{n} \Big(h_i + z_i \partial_i + \overline{h}_i + \overline{z}_i \overline{\partial}_i \Big) \langle\!\langle \phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) \rangle\!\rangle_{D_2} \ ,$$
$$0 = \sum_{i=1}^{n} \Big(2h_i z_i + z_i^2 \partial_i + 2\overline{h}_i \overline{z}_i + \overline{z}_i^2 \overline{\partial}_i \Big) \langle\!\langle \phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) \rangle\!\rangle_{D_2} \ . \qquad \text{(B.11)}$$

Hence, if a correlator satisfies the equations (B.11), it is invariant under global conformal transformations and vice versa.

Splitting the primary fields into their holomorphic and antiholomorphic components, which we denote by $\phi_i(z_i)$ and $\overline{\phi}_i(\overline{z}_i)$ depending only on z_i and $\overline{z}_i$, respectively, and inserting them into (B.1) gives

$$\langle\!\langle \phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) \rangle\!\rangle_{D_2} \to \langle\!\langle \phi_1(z_1) \overline{\phi}_1(\overline{z}_1) \cdots \phi_n(z_n) \overline{\phi}_n(\overline{z}_n) \rangle\!\rangle_{D_2} \ . \qquad \text{(B.12)}$$

Because $\phi_i(z_i)$ and $\overline{\phi}_i(\overline{z}_i)$ have to be treated as independent fields, we have obtained a correlator of $2n$ primaries, where each field interacts with the $2n-1$ other fields. Thus, we can interpret (B.11) in the following way: They describe the conditions for a correlator of $2n$ primary fields with conformal weight h_i or $\overline{h}_i$ to be conformally invariant. In that sense, they are very similar to the equivalent relations on the sphere, c.f. for example [130].

Conformal invariance of the Koba-Nielsen factor

We use in appendix C that (B.11) is linear in the derivatives of the worldsheet coordinates. Hence, it is possible to check that the Koba-Nielsen factor and the correlator in (C.10) satisfy (B.11) separately. More importantly, the Koba-Nielsen factor is not involved in finding the relations (C.14) and those relations originate only from the zero modes correlators.

Here, we want to be more general and consider the Koba-Nielsen factor of n-closed strings scattering off a Dp-brane, which can be written as

$$\mathrm{KN}_n = \left\langle\!\!\left\langle \prod_{i=1}^{n} e^{ik_i \cdot X(z_i, \overline{z}_i)} \right\rangle\!\!\right\rangle_{D_2} = \prod_{i=1}^{n} |z_i - \overline{z}_i|^{k_i D k_i} \prod_{\substack{i,j=1 \\ i<j}}^{n} |z_i - z_j|^{2k_i \cdot k_j} |z_i - \overline{z}_j|^{2k_i \cdot D \cdot k_j} \ . \qquad \text{(B.13)}$$

For the first equation in (B.11) we need to compute the derivatives with respect to z_i and $\overline{z}_i$ of the Koba-Nielsen factor

$$\partial_i \, \mathrm{KN}_n = \frac{1}{2} \left[\frac{2k_i \cdot D \cdot k_i}{z_i - \overline{z}_i} + \sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{2k_i \cdot k_j}{z_i - z_j} + \frac{2k_i \cdot D \cdot k_j}{z_i - \overline{z}_j} \right) \right] \mathrm{KN}_n \ ,$$
$$\overline{\partial}_i \, \mathrm{KN}_n = \frac{1}{2} \left[-\frac{2k_i \cdot D \cdot k_i}{z_i - \overline{z}_i} + \sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{2k_i \cdot k_j}{\overline{z}_i - \overline{z}_j} + \frac{2k_i \cdot D \cdot k_j}{\overline{z}_i - z_j} \right) \right] \mathrm{KN}_n \ . \qquad \text{(B.14)}$$

Summing over all derivatives and rearranging terms yields

$$\sum_{i=1}^{n} \Big(\partial_i + \overline{\partial}_i \Big) \mathrm{KN}_n = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left[k_i \cdot k_j \left(\frac{1}{z_i - z_j} + \frac{1}{\overline{z}_i - \overline{z}_j} \right) + k_i \cdot D \cdot k_j \left(\frac{1}{z_i - \overline{z}_j} + \frac{1}{\overline{z}_i - z_j} \right) \right] \mathrm{KN}_n$$

$$= 0 \ , \tag{B.15}$$

which vanishes, because each term appears twice but with a different overall sign: For instance we find $\frac{1}{z_i - z_j}$ and $\frac{1}{z_j - z_i}$ for fixed i and j, which add up to zero.

The conformal weight of the plane wave factor $e^{ik_i \cdot X(z_i, \overline{z}_i)}$ is given by $(h, \overline{h}) = (\frac{k_i^2}{2}, \frac{k_i^2}{2})$ such that we get for the second equation in (B.11)

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{k_i^2}{2} + z_i \partial_i + \frac{k_i^2}{2} + \overline{z}_i \overline{\partial}_i \right) \mathrm{KN}_n &= \sum_{i=1}^{n} \left(k_i^2 + k_i \cdot D \cdot k_i \right) \mathrm{KN}_n \\
&+ \sum_{\substack{i,j=1 \\ i \neq j}} \left[k_i \cdot k_j \left(\frac{z_i}{z_i - z_j} + \frac{\overline{z}_i}{\overline{z}_i - \overline{z}_j} \right) + k_i \cdot D \cdot k_j \left(\frac{z_i}{z_i - \overline{z}_j} + \frac{\overline{z}_i}{\overline{z}_i - z_j} \right) \right] \mathrm{KN}_n \\
&= \sum_{i,j=1} \left(k_i \cdot k_j + k_i \cdot D \cdot k_j \right) \mathrm{KN}_n = 0 \ .
\end{aligned} \tag{B.16}$$

To get from the first to the second line we used that terms like $\frac{z_i}{z_i - z_j}$ and $\frac{z_j}{z_j - z_i}$ add up to one. After using momentum conservation the expression vanishes and also the second equation is satisfied by the n closed string Koba-Nielsen factor.

Furthermore, we find for the last equation in (B.11)

$$\sum_{i=1}^{n} \left(2 z_i \frac{k_i^2}{2} + z_i^2 \partial_i + 2 \overline{z}_i \frac{k_i^2}{2} + \overline{z}_i^2 \overline{\partial}_i \right) \mathrm{KN} = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} k_i \cdot (k_j + D \cdot k_j)(z_i + \overline{z}_i) \right] \mathrm{KN} = 0 \ . \tag{B.17}$$

Thus, the Koba-Nielsen factor of n closed strings scattered off a Dp-brane is conformally invariant.

Appendix C

The correlator of three closed strings on the disk

After using the doubling trick one would expect the correlation function of n closed strings on the disk to be similar to the correlation function of $2n$ open strings on the disk. But the gauge fixing of the path integral for closed strings on the disk results in a correlation function that has no vertex operator fixed at infinity. Therefore, the contractions in the correlator involve all vertex operators, which is different compared to [62]. For open strings on the disk one vertex operator position can be fixed at infinity using the invariance under Möbius transformations of the amplitude. This vertex operator contributes to the amplitude only via the zero mode correlator, because contractions with the vertex operator whose position is fixed at infinity vanish as they go as $\lim_{z\to\infty} \frac{1}{z} \to 0$.

In this appendix we want to connect these two approaches and explicitly write the correlator of three closed string on the disk as a correlator of six open strings. Therefore, we use the invariance of this correlator under $PSL(2,\mathbb{R})$-transformations to change the vertex operator fixing to $(0, 1, \infty)$.

Moreover, we want to explicitly execute the steps presented in the chapters 4–6 to obtain a simple form of the amplitude in terms of SYM amplitudes. The discussion in this appendix is based on [109].

C.1 The correlator of three closed strings expressed as the correlator of six open strings

According to section 4.1 the correlator of the amplitude in (8.26) can be expressed as a sum over all possible contractions of the integrated vertex operators among themselves and with the unintegrated vertex operators. Using the OPEs (3.76) we can organise the contractions in terms of the composite superfields $K_{\overline{2}i3j\overline{3}k}$ and thereby obtain[1]

$$\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle = \sum_{i,j,k} \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\frac{\langle K_{\overline{2}i3j\overline{3}k}\rangle}{z_{\overline{2}i}z_{3j}z_{\overline{3}k}} \tag{C.1}$$

for the disk correlator of three closed strings in (8.26) with position fixed vertex operators $(z_1, \overline{z}_1, z_2)$. Note that we use the notation $z_{\overline{\imath}} \equiv \overline{z}_i$. The sum runs over $i, j, k \in \{1, \overline{1}, 2, \overline{2}, 3, \overline{3}\}$ with the restriction $i \neq \overline{2}, j \neq 3$ and $k \neq \overline{3}$. After contracting the plane wave factors the Koba-Nielsen factor takes the following form

$$\begin{aligned} \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3) &= \left\langle\!\!\left\langle e^{ik_1\cdot X(z_1,\overline{z}_1)}e^{ik_2\cdot X(z_2,\overline{z}_2)}e^{ik_3\cdot X(z_3,\overline{z}_3)}\right\rangle\!\!\right\rangle \\ &= \prod_{i=1}^{3}|z_i - \overline{z}_i|^{s_{i\overline{\imath}}} \prod_{\substack{i,j=1\\ i<j}}^{3} |z_i - z_j|^{2s_{ij}}|z_i - \overline{z}_j|^{2s_{i\overline{\jmath}}} \ . \end{aligned} \tag{C.2}$$

Since we can contract each integrated vertex operator with five other vertex operators, the sum contains a total of $5 \times 5 \times 5 = 125$ different contractions $K_{\overline{2}i3j\overline{3}k}$. For two vertex operators the contractions are given by (4.11) and (4.12) and for contractions involving more than two vertex operators we find for example

$$\begin{aligned} K_{\overline{2}13\overline{2}} &= z_{\overline{2}1}z_{3\overline{2}}U_{\overline{2}}(\overline{z}_2)U_3(z_3)V_1(z_1) \\ &= \Big[(ik_1\cdot A_{\overline{2}})(ik_{\overline{2}}\cdot A_3) - ik^1_m(W_{\overline{2}}\gamma^m W_3) + s_{1\overline{2}}(A_2W_3)\Big]V_1 \\ &\quad +(\lambda\gamma^m W_{\overline{2}})(ik_{\overline{2}}\cdot A_3)A^1_m - \frac{1}{4}(\lambda\gamma^m\gamma^{pq}W_3)A^1_m\mathcal{F}^{\overline{2}}_{pq} - s_{\overline{2}3}(A_1W_{\overline{2}})V_3 \\ &\quad -Q\Big[(ik_{\overline{2}}\cdot A_3)(A_1W_{\overline{2}}) - \frac{1}{4}(A_1\gamma^{mn}W_3)\mathcal{F}^{\overline{2}}_{mn}\Big] \ , \end{aligned} \tag{C.3}$$

$$\begin{aligned} K_{\overline{2}13\overline{2}\overline{3}1} &= z_{\overline{2}1}z_{3\overline{2}}z_{\overline{3}1}U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)V_1(z_1) \\ &= \Big[(ik_1\cdot A_{\overline{2}})(ik_{\overline{2}}\cdot A_3) - ik^1_m(W_{\overline{2}}\gamma^m W_3) + s_{1\overline{2}}(A_{\overline{2}}W_3)\Big]K_{\overline{3}1} + \Big[(\lambda\gamma^m W_{\overline{2}})(ik_{\overline{2}}\cdot A_3) \\ &\quad -\frac{1}{4}(\lambda\gamma^m\gamma^{pq}W_3)\mathcal{F}^{\overline{2}}_{pq}\Big]\Big(-(ik_1\cdot A_{\overline{3}})A^1_m + (W_1\gamma^m W_{\overline{3}}) + k^1_m(A_1W_{\overline{3}})\Big) \\ &\quad +\frac{1}{8}\Big[(\lambda\gamma^{rs}\gamma^m W_{\overline{2}})(ik_{\overline{2}}\cdot A_3) - \frac{1}{4}(\lambda\gamma^{rs}\gamma^m\gamma^{pq}W_3)\Big]A^1_m\mathcal{F}^{\overline{2}}_{pq}\mathcal{F}^{\overline{3}}_{rs} \\ &\quad +s_{\overline{2}3}\Big[(ik_1\cdot A_{\overline{3}})(A_1W_{\overline{2}})V_3 + D_\alpha A^1_\beta W^\beta_{\overline{2}}W^\alpha_{\overline{3}}V_3\Big] \\ &\quad -s_{1\overline{3}}\Big[(ik_{\overline{2}}\cdot A_3)(A_1W_{\overline{2}}) + \frac{1}{4}(A_1\gamma^{mn}W_3)\mathcal{F}^{\overline{2}}_{mn}\Big]V_{\overline{3}} \ , \end{aligned} \tag{C.4}$$

$$K_{\overline{2}33\overline{2}3\overline{2}} = z_{\overline{2}3}z_{3\overline{2}}z_{\overline{3}\overline{2}}U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)$$

[1] We have left the (uncontracted) unintegrated vertex operators implicit here to obtain this compact expression, but they are still present in the correlator.

$$= (1 - s_{\overline{2}3})\Big\{(ik_{\overline{2}} \cdot A_{\overline{3}})[(A_{\overline{2}}W_3) + (A_3 W_{\overline{2}}) - (A_{\overline{2}} \cdot A_3)] - D_\alpha A^{\overline{2}}_\beta W^\beta_3 W^\alpha_{\overline{3}}$$
$$- \frac{1}{4}(A_3\gamma^{mn}W_{\overline{3}})\mathcal{F}^{\overline{2}}_{mn} + (W_{\overline{2}}\gamma^m W_{\overline{3}})A^3_m + (ik_{\overline{2}} \cdot A_3)(A_{\overline{2}}W_{\overline{3}})\Big\} \ . \qquad \text{(C.5)}$$

As explained before the arrow notation is defined such that for example in (C.3) the contraction $U_{\overline{2}}(\overline{z}_2)V_1(z_1)$ is given by (4.11). After the contraction fields with index $\overline{2}$ depend on $\overline{z}_2$ (and only those fields). The contraction between integrated vertex operators $U_{\overline{2}}(\overline{z}_2)U_3(z_3)$ corresponds to the contraction of $h = 1$ primaries in $U_3(z_3)$ with only those terms in $U_{\overline{2}}(\overline{z}_2)V_1(z_1)$ originating from $U_{\overline{2}}$ and depending on $\overline{z}_2$. Similar remarks hold for (C.4) and (C.5).

This comparably large number of terms can be reduced by relating the composite superfields K and $\tilde{L}$. Therefore, we can use partial fractioning

$$\frac{1}{z_{ji}z_{ki}} + \frac{1}{z_{jk}z_{ji}} = \frac{1}{z_{jk}z_{ki}} \ , \qquad \text{(C.6)}$$

to express the denominators in (C.1) in terms of the denominators obtained by computing the correlator using (5.1). For a particular denominator we can compare the numerators obtained from both methods for integrating out the $h = 1$ fields in the correlator, which leads to the set of relations[2]

$$\begin{aligned}
\tilde{L}_{\overline{2}i3i\overline{3}i} &= K_{\overline{2}i3i\overline{3}i} + K_{\overline{2}i3i\overline{3}\overline{2}} + K_{\overline{2}i3i\overline{3}3} + K_{\overline{2}i3\overline{2}\overline{3}i} + K_{\overline{2}i3\overline{2}\overline{3}\overline{2}} + K_{\overline{2}i3\overline{2}\overline{3}3} \ , \\
\tilde{L}_{\overline{2}33i\overline{3}i} &= K_{\overline{2}33i\overline{3}i} + K_{\overline{2}33i\overline{3}\overline{2}} + K_{\overline{2}33i\overline{3}3} - K_{\overline{2}i3\overline{2}\overline{3}i} - K_{\overline{2}i3\overline{2}\overline{3}\overline{2}} - K_{\overline{2}i3\overline{2}\overline{3}3} \ , \\
\tilde{L}_{\overline{2}33\overline{3}\overline{3}i} &= K_{\overline{2}33\overline{3}\overline{3}i} - K_{\overline{2}33i\overline{3}3} - K_{\overline{2}33i\overline{3}\overline{2}} + K_{\overline{2}i3\overline{2}\overline{3}\overline{2}} + K_{\overline{2}i3\overline{2}\overline{3}3} - K_{\overline{2}\overline{3}3\overline{2}\overline{3}i} \ , \\
\tilde{L}_{\overline{2}3\overline{2}33\overline{3}} &= K_{\overline{2}33\overline{2}\overline{3}\overline{2}} - K_{\overline{2}33\overline{3}\overline{3}\overline{2}} - K_{\overline{2}\overline{3}3\overline{2}\overline{3}3} - K_{\overline{2}\overline{3}3\overline{2}\overline{3}\overline{2}} + K_{\overline{2}\overline{3}3\overline{3}\overline{3}3} \ , \\
\tilde{L}_{\overline{2}i3\overline{3}\overline{3}i} &= K_{\overline{2}i3\overline{3}\overline{3}i} - K_{\overline{2}i3i\overline{3}3} - K_{\overline{2}i3\overline{2}\overline{3}3} + K_{\overline{2}i3\overline{3}\overline{3}\overline{2}} \ , \\
\tilde{L}_{\overline{2}i3\overline{3}\overline{3}j} &= K_{\overline{2}i}\Big(K_{3\overline{3}\overline{3}j} - K_{3j\overline{3}3}\Big) \ , \qquad \tilde{L}_{\overline{2}i3i\overline{3}j} = \Big(K_{\overline{2}i3i} + K_{\overline{2}i3\overline{2}}\Big)K_{\overline{3}j} \ , \\
\tilde{L}_{\overline{2}\overline{3}3\overline{3}3\overline{3}} &= K_{\overline{2}33\overline{2}\overline{3}3} + K_{\overline{2}33\overline{2}\overline{3}\overline{2}} \ , \qquad \tilde{L}_{\overline{2}i3\overline{3}3\overline{3}} = K_{\overline{2}i3\overline{3}\overline{3}3} \ , \\
\tilde{L}_{\overline{2}i3j\overline{3}k} &= K_{\overline{2}i}K_{3j}K_{\overline{3}k}
\end{aligned} \qquad \text{(C.7)}$$

and similar relations with permutations of vertex operator labels. Therefore, we can reorganize the correlator by using these relations between the kinematic terms and write the integrand of the scattering process of three closed strings on the disk as a sum over single and double poles

$$\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle =$$

[2]These relations can also be viewed as the definition of the composite superfields $\tilde{L}$, but they are only valid inside a correlator, because K depends on various vertex operator positions, whereas $\tilde{L}$ depends only on one worldsheet coordinate.

$$
\begin{aligned}
&= \mathrm{KN}(\bar{z}_2, z_3, \bar{z}_3)\Big\langle \sum_{i,j,k} \epsilon_{ijk} \Big(\frac{\tilde{L}_{\bar{2}i3j\bar{3}k}}{z_{\bar{2}i}z_{3j}z_{\bar{3}k}} + \frac{\tilde{L}_{\bar{2}i3i\bar{3}j}V_k}{z_{\bar{2}i}z_{3i}z_{\bar{3}j}} + \frac{\tilde{L}_{\bar{2}i3j\bar{3}i}V_k}{z_{\bar{2}i}z_{3j}z_{\bar{3}i}} + \frac{\tilde{L}_{\bar{2}j3i\bar{3}i}V_k}{z_{\bar{2}j}z_{3i}z_{\bar{3}i}} + \frac{\tilde{L}_{\bar{2}i3\overline{3\bar{3}}j}V_k}{z_{\bar{2}i}z_{3\bar{3}}z_{\bar{3}j}} \\
&\quad + \frac{\tilde{L}_{\overline{\bar{2}3}3i\bar{3}j}V_k}{z_{\overline{\bar{2}3}}z_{3i}z_{\bar{3}j}} + \frac{\tilde{L}_{\bar{2}33i\bar{3}j}V_k}{z_{\bar{2}3}z_{3i}z_{\bar{3}j}} + \frac{\tilde{L}_{\bar{2}i3i\bar{3}i}V_jV_k}{z_{\bar{2}i}z_{3i}z_{\bar{3}i}} + \frac{\tilde{L}_{\bar{2}i3\overline{3\bar{3}}i}V_jV_k}{z_{\bar{2}i}z_{3\bar{3}}z_{\bar{3}i}} + \frac{\tilde{L}_{\overline{\bar{2}3}3i\bar{3}i}V_jV_k}{z_{\overline{\bar{2}3}}z_{3i}z_{\bar{3}i}} + \frac{\tilde{L}_{\bar{2}33i\bar{3}i}V_jV_k}{z_{\bar{2}3}z_{3i}z_{\bar{3}i}} \\
&\quad + \frac{\tilde{L}_{\overline{\bar{2}3}3\bar{3}3i}V_jV_k}{z_{\overline{\bar{2}3}}z_{3\bar{3}}z_{3i}} + \frac{\tilde{L}_{\bar{2}3\overline{3\bar{3}}i}V_jV_k}{z_{\bar{2}3}z_{3\bar{3}}z_{\bar{3}i}} + \frac{\tilde{L}_{\bar{2}i3\bar{3}3\bar{3}}V_jV_k}{z_{\bar{2}i}z_{3\bar{3}}^2} + \frac{\tilde{L}_{\overline{\bar{2}3\bar{2}3}3i}V_jV_k}{z_{\overline{\bar{2}3}}^2z_{3i}} + \frac{\tilde{L}_{\bar{2}3\bar{2}3\bar{3}i}V_jV_k}{z_{\bar{2}3}^2z_{\bar{3}i}} \Big) \\
&\quad + \frac{\tilde{L}_{\bar{2}33\bar{3}3\bar{3}}V_{\bar{1}}V_1V_2}{z_{\bar{2}3}z_{3\bar{3}}^2} + \frac{\tilde{L}_{\overline{\bar{2}3}3\bar{3}3\bar{3}}V_{\bar{1}}V_1V_2}{z_{\overline{\bar{2}3}}z_{3\bar{3}}^2} + \frac{\tilde{L}_{\overline{\bar{2}3\bar{2}3}3\bar{3}}V_{\bar{1}}V_1V_2}{z_{\overline{\bar{2}3}}^2z_{3\bar{3}}} + \frac{\tilde{L}_{\bar{2}3\bar{2}33\bar{3}}V_{\bar{1}}V_1V_2}{z_{\bar{2}3}^2z_{3\bar{3}}} \Big\rangle \ , \qquad \text{(C.8)}
\end{aligned}
$$

where the sum runs over $i, j, k \in \{1, \bar{1}, 2\}$ with mutually different i, j, k and $\epsilon_{1\bar{1}2} = 1$.

For the computation of (C.1) and also (C.8) we have chosen a particular order to integrate out the conformal dimension one fields: We start with $\bar{2}$, continue with 3 and in the end contract $\bar{3}$. But in string theory the CFT correlator is independent of the order of contraction and we could instead start with 3 or $\bar{3}$.[3] Since all possibilities give the same result in the end, we can compare the different orders of contraction to find identities between the kinematic factors [160].

The different orders of contraction can be obtained by just relabelling the composite superfields $\tilde{L}$ and their worldsheet dependent denominators $\frac{1}{z_{ij}z_{mn}z_{rs}}$ according to the new order in which the vertex operators will be contracted. After relabelling z_{ij} in the denominators of (C.8) we recognize that we have introduced new poles in the correlator that are not present the original integrand (C.8). Hence, comparing (C.8) and the relabelled expression is not straightforward. However, utilizing partial fractioning (C.6) we can subtract correlators with different orders of vertex operator contractions and obtain identities of the form [158, 160]

$$
\begin{aligned}
\tilde{L}_{\bar{2}33i\bar{3}i} &= \tilde{L}_{3i\bar{2}i\bar{3}i} - \tilde{L}_{\bar{2}i3i\bar{3}i} \ , \\
\tilde{L}_{\bar{2}i3\overline{3\bar{3}}i} &= \tilde{L}_{\bar{2}i\bar{3}i3i} - \tilde{L}_{\bar{2}i3i\bar{3}i} \ , \\
\tilde{L}_{\bar{2}3\overline{3\bar{3}}i} &= \tilde{L}_{\bar{3}i3i\bar{2}i} - \tilde{L}_{\bar{3}i\bar{2}i3i} + \tilde{L}_{\bar{2}i3i\bar{3}i} - \tilde{L}_{3i\bar{2}i\bar{3}i} \ , \\
\tilde{L}_{\bar{2}33i\bar{3}j} &= \tilde{L}_{3i\bar{2}i\bar{3}j} - \tilde{L}_{\bar{2}i3i\bar{3}j} \ , \\
\tilde{L}_{\bar{2}j3\overline{3\bar{3}}i} &= \tilde{L}_{\bar{2}j\bar{3}i3i} - \tilde{L}_{\bar{2}j3i\bar{3}i} \ . \qquad \text{(C.9)}
\end{aligned}
$$

Furthermore, the identities (C.9) reduce the amount of superfield manipulations we have to perform, as they decrease the number of composite superfields, which we have to consider: For example, we do not have to compute $L_{\bar{2}3\overline{3\bar{3}}i}$ explicitly, which is rather tedious, because the contractions involve OPEs among unintegrated vertex operators. On the other hand, computing kinematic factors $\tilde{L}_{\bar{3}i3i\bar{2}i}, \tilde{L}_{\bar{3}i\bar{2}i3i}, \tilde{L}_{\bar{2}i3i\bar{3}i}$ and $\tilde{L}_{3i\bar{2}i\bar{3}i}$ is simpler, since we have to consider contractions between unintegrated and integrated vertex operators.

[3]Nevertheless, for a particular computation one has to chose an order for integrating out the $h = 1$ primaries and stick to it during this computation.

Using the relations (C.9) between the composite superfields the correlator (C.8) becomes

$$
\begin{aligned}
&\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle = \\
&= \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\Big\langle \sum_{\substack{i,j,k\\ \in\{1,\overline{1},2\}}} \epsilon_{ijk}\Big(\frac{\tilde{L}_{\overline{2}i3j\overline{3}k}}{z_{\overline{2}i}z_{3j}z_{\overline{3}k}} + \frac{\tilde{L}_{3i\overline{2}i\overline{3}j}V_k}{z_{\overline{2}3}z_{3i}z_{\overline{3}j}} + \frac{\tilde{L}_{\overline{2}j\overline{3}i3i}V_k}{z_{\overline{2}j}z_{3\overline{3}}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{3}i3j\overline{2}i}V_k}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{3}i}} - \frac{\tilde{L}_{\overline{2}i3i\overline{3}j}V_k}{z_{\overline{2}3}z_{\overline{2}i}z_{\overline{3}j}} \\
&\quad - \frac{\tilde{L}_{\overline{2}i3j\overline{3}i}V_k}{z_{\overline{23}}z_{\overline{2}i}z_{3j}} - \frac{\tilde{L}_{\overline{2}j3i\overline{3}i}V_k}{z_{3\overline{3}}z_{3i}z_{\overline{2}j}} + \frac{\tilde{L}_{\overline{2}i3i\overline{3}i}V_jV_k}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{2}i}} + \frac{\tilde{L}_{3i\overline{3}i\overline{2}i}V_jV_k}{z_{\overline{23}}z_{3\overline{3}}z_{3i}} + \frac{\tilde{L}_{\overline{3}i3i\overline{2}i}V_jV_k}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{3}i}} - \frac{\tilde{L}_{3i\overline{2}i\overline{3}i}V_jV_k}{z_{\overline{2}3}z_{\overline{23}}z_{3i}} \\
&\quad - \frac{\tilde{L}_{\overline{2}i\overline{3}i3i}V_jV_k}{z_{\overline{23}}z_{3\overline{3}}z_{\overline{2}i}} - \frac{\tilde{L}_{\overline{3}i\overline{2}i3i}V_jV_k}{z_{\overline{2}3}z_{\overline{23}}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{2}i3\overline{3}3\overline{3}}V_jV_k}{z_{\overline{2}i}z_{3\overline{3}}^2} + \frac{\tilde{L}_{\overline{2323}3i}V_jV_k}{z_{\overline{23}}^2 z_{3i}} + \frac{\tilde{L}_{\overline{2}3\overline{2}3\overline{3}i}V_jV_k}{z_{\overline{2}3}^2 z_{\overline{3}i}}\Big) \\
&\quad + \frac{\tilde{L}_{\overline{2}33\overline{3}3\overline{3}}V_{\overline{1}}V_1V_2}{z_{\overline{2}3}z_{3\overline{3}}^2} + \frac{\tilde{L}_{\overline{23}3\overline{3}3\overline{3}}V_{\overline{1}}V_1V_2}{z_{\overline{23}}z_{3\overline{3}}^2} + \frac{\tilde{L}_{\overline{2323}3\overline{3}}V_{\overline{1}}V_1V_2}{z_{\overline{23}}^2 z_{3\overline{3}}} + \frac{\tilde{L}_{\overline{2}3\overline{2}33\overline{3}}V_{\overline{1}}V_1V_2}{z_{\overline{2}3}^2 z_{3\overline{3}}}\Big\rangle\ .
\end{aligned}
\tag{C.10}
$$

A priori the composite superfields in (C.10) contains contractions with all vertex operators. By using the invariance of the amplitude under $PSL(2,\mathbb{R})$-transformations we map the fixed vertex operator positions $(\overline{z}_1, z_2, z_1)$ to $(0, 1, \infty)$. Thereby, the kinematic factors containing $V_1(z_1)$ should drop out of the correlator and we arrive at a similar expression as in [62, 158]. The corresponding Möbius transformation is given by

$$
F(z) = -\frac{(z_2 - z_1)(z - \overline{z}_1)}{(\overline{z}_1 - z_2)(z - z_1)}\ , \tag{C.11}
$$

which reduces to (8.44) for $(z_1, \overline{z}_1, z_2) = (1, -1, y)$. Just using the transformation (C.11) we obtain a correlation function that has a non-trivial dependence $\varphi(\{z_i\})$ on the vertex operator positions. But due to conformal invariance only differences $z_i - z_j$ of vertex operator positions can appear in a correlator after contracting conformal primaries [130]. Therefore, we have to ensure that (C.10) is manifestly invariant under global conformal transformations. According to appendix B this is equivalent to the correlator satisfying (B.11), which become here[4]

$$
\begin{aligned}
0 &= \sum_{i=1}^{3}\Big(\partial_i + \overline{\partial}_i\Big)\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle\ , \\
0 &= \sum_{i=1}^{3}\Big(h_i + z_i\partial_i + \overline{h}_i + \overline{z}_i\overline{\partial}_i\Big)\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle\ , \\
0 &= \sum_{i=1}^{3}\Big(2h_iz_i + z_i^2\partial_i + 2\overline{h}_i\overline{z}_i + \overline{z}_i^2\overline{\partial}_i\Big)\langle\!\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)\rangle\!\rangle\ .
\end{aligned}
\tag{C.12}
$$

[4]Using momentum conservation and that we are considering massless states the Koba-Nielsen factor satisfies these equations immediately, which is discussed in appendix B. Therefore, we can consider the vertex operators without a plane wave factor in (C.12), since these differential equations are linear in the derivates.

The unintegrated vertex operators V and the integrated vertex operators U have conformal dimension $h = 0$ and $h = 1$, respectively. Using this the first two equations are satisfied by (C.10) after applying momentum conservation. But the third equation implies the following condition for the composite superfields

$$
\begin{aligned}
0 &= \sum_{i=1}^{3}\Big(2h_i z_i + z_i^2\partial_i + 2\overline{h}_i\overline{z}_i + \overline{z}_i^2\overline{\partial}_i\Big)\langle V_1(z_1)V_{\overline{1}}(\overline{z}_1)V_2(z_2)U_{\overline{2}}(\overline{z}_2)U_3(z_4)U_{\overline{3}}(\overline{z}_3)\rangle = \\
&= \frac{1}{z_{\overline{12}}z_{\overline{2}3}}\langle \tilde{L}_{\overline{2}33\overline{131}}V_2V_1 + V_{\overline{1}}\tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}2}V_1 + V_{\overline{1}}V_2\tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}1}\rangle + \frac{1}{z_{\overline{12}}z_{\overline{1}3}}\langle \tilde{L}_{\overline{21}3\overline{131}}V_2V_1 + \tilde{L}_{\overline{21}3\overline{1}}\tilde{L}_{\overline{32}}V_1 \\
&\quad + \tilde{L}_{\overline{21}3\overline{1}}V_2\tilde{L}_{\overline{3}1} + \tilde{L}_{\overline{2}33\overline{131}}V_2V_1 + V_{\overline{1}}\tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}2}V_1 + V_{\overline{1}}V_2\tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}1}\rangle + \ldots \ .
\end{aligned} \tag{C.13}
$$

In general, the fractions $\frac{1}{z_{ij}z_{mn}}$ in (C.13) do not vanish for arbitrary z_i (and $z_{\overline{i}}$) and are independent. Hence, by equating coefficients (C.13) gives rise to relations between the building blocks $\tilde{L}$

$$
\begin{aligned}
0 &= \langle \tilde{L}_{\overline{21}3\overline{131}}V_2V_1 + \tilde{L}_{\overline{21}3\overline{331}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{131}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{331}}V_2V_1 + \tilde{L}_{\overline{233}\overline{131}}V_2V_1 \\
&\quad + \tilde{L}_{3\overline{131}}\tilde{L}_{\overline{2}2}V_1 + \tilde{L}_{3\overline{331}}\tilde{L}_{\overline{2}2}V_1 + \tilde{L}_{3\overline{131}}V_2\tilde{L}_{\overline{2}1} + \tilde{L}_{3\overline{331}}V_2\tilde{L}_{\overline{2}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}3\overline{131}}V_2V_1 + \tilde{L}_{\overline{21}3\overline{331}}V_2V_1 + \tilde{L}_{\overline{2131}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{2131}}V_2\tilde{L}_{31} + \tilde{L}_{\overline{233}\overline{131}}V_2V_1 \\
&\quad + \tilde{L}_{\overline{2331}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{2331}}V_2\tilde{L}_{31}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{2}33\overline{331}}V_2V_1 + V_{\overline{1}}\tilde{L}_{\overline{2}33\overline{33}2}V_1 + V_{\overline{1}}V_2\tilde{L}_{\overline{2}33\overline{33}1} - \tilde{L}_{\overline{23333}\overline{1}}V_2V_1 - V_{\overline{1}}\tilde{L}_{\overline{23333}2}V_1 - V_{\overline{1}}V_2\tilde{L}_{\overline{23333}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}3\overline{131}}V_2V_1 + \tilde{L}_{\overline{21}3\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{\overline{21}3\overline{1}}V_2\tilde{L}_{\overline{3}1} + \tilde{L}_{\overline{2}33\overline{131}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{\overline{2}33\overline{1}}V_2\tilde{L}_{\overline{3}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}3\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{\overline{2}33\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{\overline{233}\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{3\overline{1}}\tilde{L}_{\overline{2}2\overline{3}2}V_1 + \tilde{L}_{3\overline{1}}\tilde{L}_{\overline{3}2}\tilde{L}_{\overline{2}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}}\tilde{L}_{3\overline{3}3\overline{3}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{3}3\overline{3}}V_{\overline{1}}V_2V_1 + \tilde{L}_{\overline{233}\overline{3}3\overline{3}}V_{\overline{1}}V_2V_1 + V_{\overline{1}}\tilde{L}_{\overline{2}2}\tilde{L}_{3\overline{3}3\overline{3}}V_1 + V_{\overline{1}}V_2\tilde{L}_{\overline{2}1}\tilde{L}_{3\overline{3}3\overline{3}}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}3\overline{331}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{331}}V_2V_1 - \tilde{L}_{\overline{23333}\overline{1}}V_2V_1 + \tilde{L}_{3\overline{331}}\tilde{L}_{\overline{2}2}V_1 + \tilde{L}_{3\overline{331}}V_2\tilde{L}_{\overline{2}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{2131}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{2}3\overline{31}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{2331}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{31}}\tilde{L}_{\overline{2}232}V_1 + \tilde{L}_{\overline{31}}\tilde{L}_{32}\tilde{L}_{\overline{2}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{21}3\overline{1}}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{\overline{21}}\tilde{L}_{3\overline{33}2}V_1 + \tilde{L}_{\overline{21}}\tilde{L}_{32\overline{3}2}V_1 + \tilde{L}_{\overline{21}}\tilde{L}_{\overline{3}2}\tilde{L}_{31}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{2}3\overline{2}33\overline{3}}V_{\overline{1}}V_2V_1 - \tilde{L}_{\overline{2}3\overline{2}3}\tilde{L}_{\overline{31}}V_2V_1 - \tilde{L}_{\overline{2}3\overline{2}3}V_{\overline{1}}\tilde{L}_{\overline{3}2}V_1 - \tilde{L}_{\overline{2}3\overline{2}3}V_{\overline{1}}V_2\tilde{L}_{\overline{3}1}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{2}33\overline{131}}V_2V_1 + \tilde{L}_{\overline{2}33\overline{331}}V_2V_1 + \tilde{L}_{\overline{31}}\tilde{L}_{\overline{2}332}V_1 + \tilde{L}_{\overline{31}}V_2\tilde{L}_{\overline{2}331}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{2323}}\tilde{L}_{3\overline{1}}V_2V_1 + \tilde{L}_{\overline{232}33\overline{3}}V_{\overline{1}}V_2V_1 + \tilde{L}_{\overline{2323}}V_{\overline{1}}\tilde{L}_{32}V_1 + \tilde{L}_{\overline{2323}}V_{\overline{1}}V_2\tilde{L}_{31}\rangle \ , \\
0 &= \langle \tilde{L}_{\overline{233}\overline{131}}V_2V_1 + \tilde{L}_{\overline{233}2}\tilde{L}_{\overline{3}2}V_1 + \tilde{L}_{3\overline{1}}V_2\tilde{L}_{\overline{233}1} + \tilde{L}_{\overline{23333}\overline{1}}V_2V_1\rangle
\end{aligned} \tag{C.14}
$$

and permutations thereof. Finally, with (C.14) and the permutations of these relations we can perform the $PSL(2,\mathbb{R})$-transformation (C.11) of the correlator in (C.10) and obtain

$$
\begin{aligned}
&\langle\!\langle V_{\overline{1}}(0)U_{\overline{2}}(\overline{z}_2)U_3(z_3)U_{\overline{3}}(\overline{z}_3)V_2(1)V_1(\infty)\rangle\!\rangle = \\
&= \frac{1}{2}\det(\mathcal{J})^{-1}\,\mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\Big\langle \sum_{i,j\in\{\overline{1},2\}} \epsilon_{ij}\bigg(\frac{\tilde{L}_{\overline{2}i3i\overline{3}j}V_1}{z_{\overline{2}i}z_{3i}z_{\overline{3}j}} + \frac{\tilde{L}_{\overline{2}i3j\overline{3}i}V_1}{z_{\overline{2}i}z_{3j}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{2}j3i\overline{3}i}V_1}{z_{\overline{2}j}z_{3i}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{2}i3\overline{33}j}V_1}{z_{\overline{2}i}z_{3\overline{3}}z_{\overline{3}j}} \\
&\quad + \frac{\tilde{L}_{\overline{233}i\overline{3}j}V_1}{z_{\overline{23}}z_{3i}z_{\overline{3}j}} + \frac{\tilde{L}_{\overline{2}33i\overline{3}j}V_1}{z_{\overline{2}3}z_{3i}z_{\overline{3}j}} + \frac{\tilde{L}_{\overline{2}i3i\overline{3}i}V_jV_1}{z_{\overline{2}i}z_{3i}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{2}i3\overline{33}i}V_jV_1}{z_{\overline{2}i}z_{3\overline{3}}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{233}i\overline{3}i}V_jV_1}{z_{\overline{23}}z_{3i}z_{\overline{3}i}} + \frac{\tilde{L}_{\overline{2}33i\overline{3}i}V_jV_1}{z_{\overline{2}3}z_{3i}z_{\overline{3}i}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\tilde{L}_{\overline{\bar{2}33\bar{3}}3i}V_jV_1}{z_{\overline{\bar{2}3}}z_{3\bar{3}}z_{3i}}+\frac{\tilde{L}_{\bar{2}33\overline{3\bar{3}}i}V_jV_1}{z_{\bar{2}3}z_{3\bar{3}}z_{\bar{3}i}}+\frac{\tilde{L}_{\bar{2}i3\bar{3}3\bar{3}}V_jV_1}{z_{\bar{2}i}z^2_{3\bar{3}}}+\frac{\tilde{L}_{\overline{\overline{\bar{2}3}\bar{2}3}3i}V_jV_1}{z^2_{\overline{\bar{2}3}}z_{3i}}+\frac{\tilde{L}_{\bar{2}3\bar{2}3\bar{3}i}V_jV_1}{z^2_{\bar{2}3}z_{\bar{3}i}}\Bigg) \\
&+\frac{\tilde{L}_{\overline{\bar{2}33\bar{3}}3\bar{3}}V_{\bar{1}}V_2V_1}{z_{\bar{2}3}z^2_{3\bar{3}}}+\frac{\tilde{L}_{\overline{\bar{2}33\bar{3}3\bar{3}}}V_{\bar{1}}V_2V_1}{z_{\overline{\bar{2}3}}z^2_{3\bar{3}}}+\frac{\tilde{L}_{\overline{\overline{\bar{2}3}\bar{2}3}3\bar{3}}V_{\bar{1}}V_2V_1}{z^2_{\overline{\bar{2}3}}z_{3\bar{3}}}+\frac{\tilde{L}_{\bar{2}3\bar{2}33\bar{3}}V_{\bar{1}}V_2V_1}{z^2_{\bar{2}3}z_{3\bar{3}}}\Bigg\rangle\,,
\end{aligned}
\tag{C.15}
$$

where the Jacobi determinant of this transformation is given by

$$
\det(\mathcal{J}) = -\frac{(\bar{z}_1-z_2)^2(\bar{z}_1-z_1)(z_2-z_1)}{((\bar{z}_1-z_2)\bar{z}_2+z_2-z_1)^2((\bar{z}_1-z_2)z_3+z_2-z_1)^2((\bar{z}_1-z_2)\bar{z}_3+z_2-z_1)^2}\,. \tag{C.16}
$$

After the identification (8.28) and up to an overall factor (C.15) has the same form as the correlator of six open strings on the disk in [158]. The overall factor $\det(\mathcal{J})^{-1}$ will cancel against the Jacobian of the $PSL(2,\mathbb{R})$-transformation, when we take also the measure of the worldsheet integrals into account, i.e. when we consider the complete three point amplitude in the next section.

C.2 The three–point amplitude on the disk

In the previous section we have discussed the relation between the correlator of three closed strings and six open strings on the disk. Next, we can apply the procedure presented in chapter 5 and chapter 6, c.f. [62, 158, 162], to write the amplitude in terms of SYM amplitudes. We start by replacing the superfield expressions $\tilde{L}_{ji}, \tilde{L}_{jiki}$ and $\tilde{L}_{jikili}$ by their corresponding BRST building blocks T_{ij}, T_{ijk} and T_{ijkl}.

C.2.1 Double pole integrands and total derivative techniques

Contrary to [158] the correlator in (C.15) still contains BRST exact terms, i.e. we have $\tilde{L}$ instead of L, which are crucial for the invariance of the correlator under global conformal transformations. Hence, they were important to find (C.14) and moreover the relations between kinematic terms would not hold, even though the BRST exact terms do not contribute to the end result of the amplitude.

Since we have performed the $PSL(2,\mathbb{R})$-transformation, we can simplify the amplitude and drop these additional terms. In the end, we obtain the same correlator as in (C.15) with the substitution $\tilde{L}\to L$. In order to cancel the BRST exact terms we have to use partial fractioning and integration by parts. Therefore, we include the integration over worldsheet variables of the integrated vertex operators:[5]

$$
\mathcal{A} = \frac{1}{2}\int \mathrm{d}\bar{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\bar{3}}\,\langle\!\langle V_{\bar{1}}(0)U_{\bar{2}}(\bar{z}_2)U_3(z_3)U_{\bar{3}}(\bar{z}_3)V_2(1)V_1(\infty)\rangle\!\rangle
$$

[5]The Jacobian obtained by the $PSL(2,\mathbb{R})$-transformation of the measure cancels against the prefactor in (C.15).

$$
\begin{aligned}
&= \frac{1}{2}\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2,z_3,\overline{z}_3)\Bigg\{\sum_{\substack{i,j\in\{\overline{1},2\}\\ i\neq j}}\epsilon_{ij}\Bigg(\frac{\langle L_{\overline{2}i3i\overline{3}j}V_1\rangle}{z_{\overline{2}i}z_{3i}z_{\overline{3}j}}+\frac{\langle L_{\overline{2}i3j\overline{3}i}V_1\rangle}{z_{\overline{2}i}z_{3j}z_{\overline{3}i}}+\frac{\langle L_{\overline{2}j3i\overline{3}i}V_1\rangle}{z_{\overline{2}j}z_{3i}z_{\overline{3}i}}\\
&\quad+\frac{\langle L_{\overline{2}i3\overline{33}j}V_1\rangle}{z_{\overline{2}i}z_{3\overline{3}}z_{\overline{3}j}}+\frac{\langle L_{\overline{23}3i\overline{3}j}V_1\rangle}{z_{\overline{23}}z_{3i}z_{\overline{3}j}}+\frac{\langle L_{\overline{2}33i\overline{3}j}V_1\rangle}{z_{\overline{2}3}z_{3i}z_{\overline{3}j}}+\frac{\langle L_{\overline{2}i3i\overline{3}i}V_jV_1\rangle}{z_{\overline{2}i}z_{3i}z_{\overline{3}i}}\\
&\quad+\frac{\langle L_{\overline{2}i3\overline{33}i}V_jV_1\rangle}{z_{\overline{2}i}z_{3\overline{3}}z_{\overline{3}i}}+\frac{\langle L_{\overline{23}3i\overline{3}i}V_jV_1\rangle}{z_{\overline{23}}z_{3i}z_{\overline{3}i}}+\frac{\langle L_{\overline{2}33i\overline{3}i}V_jV_1\rangle}{z_{\overline{2}3}z_{3i}z_{\overline{3}i}}+\frac{\langle L_{\overline{23}3\overline{3}3i}V_jV_1\rangle}{z_{\overline{23}}z_{3\overline{3}}z_{3i}}\\
&\quad+\frac{\langle L_{\overline{2}33\overline{3}3i}V_jV_1\rangle}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{3}i}}+\frac{\langle L_{\overline{2}i3\overline{3}3\overline{3}}V_jV_1\rangle}{z_{\overline{2}i}z^2_{3\overline{3}}}+\frac{\langle L_{\overline{23}\overline{23}3i}V_jV_1\rangle}{z^2_{\overline{23}}z_{3i}}+\frac{\langle L_{\overline{2}3\overline{2}3\overline{3}i}V_jV_1\rangle}{z^2_{\overline{2}3}z_{\overline{3}i}}\Bigg)\\
&\quad+\frac{\langle L_{\overline{2}33\overline{3}3\overline{3}}V_{\overline{1}}V_2V_1\rangle}{z_{\overline{2}3}z^2_{3\overline{3}}}+\frac{\langle L_{\overline{23}3\overline{3}3\overline{3}}V_{\overline{1}}V_2V_1\rangle}{z_{\overline{23}}z^2_{3\overline{3}}}+\frac{\langle L_{\overline{23}\overline{23}3\overline{3}}V_{\overline{1}}V_2V_1\rangle}{z^2_{\overline{23}}z_{3\overline{3}}}+\frac{\langle L_{\overline{2}3\overline{2}33\overline{3}}V_{\overline{1}}V_2V_1\rangle}{z^2_{\overline{2}3}z_{3\overline{3}}}\Bigg\}\ . \qquad \text{(C.17)}
\end{aligned}
$$

In the following discussion, which is based on [109] and is an application of the procedure outlined in [114, 158], we want to cancel the tachyonic poles $\frac{1}{1-s_{ij}}$ in (C.17), which originate from double poles in the integrand (C.17), i.e. the terms proportional to $|z_{ij}|^{s_{ij}-2}$. Therefore, we are going to derive total derivative relations. With our choice of fixed vertex operator positions $(\overline{z}_1, z_2, z_1) = (0, 1, \infty)$ the Koba-Nielsen factor (C.2) takes the following form

$$
\begin{aligned}
\mathrm{KN}(\overline{z}_2,z_3,\overline{z}_3) &= |\overline{z}_2|^{s_{\overline{12}}}|z_3|^{s_{\overline{1}3}}|\overline{z}_3|^{s_{\overline{13}}}|1-\overline{z}_2|^{s_{2\overline{2}}}|1-z_3|^{s_{23}}|1-\overline{z}_3|^{s_{2\overline{3}}}\\
&\quad\times|\overline{z}_2-z_3|^{s_{\overline{2}3}}|\overline{z}_2-\overline{z}_3|^{s_{\overline{23}}}|z_3-\overline{z}_3|^{s_{3\overline{3}}}\ . \qquad \text{(C.18)}
\end{aligned}
$$

In addition to partial fractioning (C.6), we can then find further relations among different $\frac{1}{z_{ij}z_{mn}z_{rs}}$ appearing in (C.17), which are given by

$$
\begin{aligned}
&\int \mathrm{d}\overline{z}_2\int \mathrm{d}z_3\int \mathrm{d}\overline{z}_3\,\frac{\partial}{\partial z_I}\left(\frac{\mathrm{KN}(\overline{z}_2,z_3,\overline{z}_3)}{z_{ij}z_{kl}}\right)=0\ , \qquad i,j,k,l\neq I,\quad I\in\{\overline{2},3,\overline{3}\}\ ,\\
&\int \mathrm{d}z_{\overline{2}}\int \mathrm{d}z_3\int \mathrm{d}\overline{z}_3\,\frac{\partial}{\partial z_I}\left(\frac{\mathrm{KN}(\overline{z}_2,z_3,\overline{z}_3)}{z_{Ij}z_{kl}}\right)=0\ , \qquad i,j,k,l\neq I,\quad I\in\{\overline{2},3,\overline{3}\}\ . \qquad \text{(C.19)}
\end{aligned}
$$

The boundaries of the two integration regions in (8.56) correspond to vertex operator positions. Hence, for positive and sufficiently large (real part of) s_{ij} the Koba-Nielsen factor has zeros at all integration boundaries such that the boundary terms in (C.19) vanish. Moreover, we can analytically continue this result using the cancelled propagator argument (4.33) such that the validity of (C.19) can be extended for generic complex s_{ij}. Explicitly, the first equation in (C.19) leads to multiple equations of the following form

$$
\frac{s_{\overline{12}}}{z_{\overline{21}}z_{3\overline{1}}z_{\overline{31}}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}z_{3\overline{1}}z_{\overline{31}}}+\frac{s_{\overline{23}}}{z_{\overline{23}}z_{3\overline{1}}z_{\overline{31}}}+\frac{s_{1\overline{2}}}{z_{\overline{2}1}z_{3\overline{1}}z_{\overline{31}}}=0\ . \qquad \text{(C.20)}
$$

Similarly, the second equation in (C.19) gives rise to relations

$$
\frac{s_{\overline{12}}}{z_{\overline{21}}z_{\overline{2}3}z_{\overline{31}}}-\frac{1-s_{\overline{2}3}}{z^2_{\overline{2}3}z_{\overline{31}}}+\frac{s_{\overline{23}}}{z_{\overline{23}}z_{\overline{2}3}z_{\overline{31}}}+\frac{s_{1\overline{2}}}{z_{\overline{2}1}z_{\overline{2}3}z_{\overline{31}}}=0\ . \qquad \text{(C.21)}
$$

After dropping the BRST exact terms the OPE computations of the double poles in the last two lines of (C.17) lead to

$$L_{\overline{2}i3\overline{3}3\overline{3}} = -(1-s_{3\overline{3}})D_{3\overline{3}}L_{\overline{2}i} \ , \tag{C.22}$$

$$L_{\overline{23}\,\overline{2}3\overline{3}i} = -(1-s_{\overline{23}})D_{\overline{23}}L_{3i} \ , \tag{C.23}$$

$$L_{\overline{2}3\overline{2}3\overline{3}i} = -(1-s_{\overline{2}3})D_{\overline{2}3}L_{\overline{3}i} \ . \tag{C.24}$$

Applying total derivative relations in (C.19) the overall factors $(1-s_{ij})$ above cancel the tachyonic double poles that would otherwise appear in the integrand (C.17). The remaining OPE contractions of the double pole terms are given by

$$\begin{aligned}
L_{\overline{2}33\overline{3}3\overline{3}} &= (D_{3\overline{3}}(ik_3\cdot A_{\overline{2}}) - D_{\overline{23}}(ik_{\overline{2}}\cdot A_3))(1-s_{\overline{23}}-s_{3\overline{3}}) + D_{\overline{2}3}(ik_{\overline{2}}\cdot A_{\overline{3}})(1-s_{\overline{2}3}-s_{3\overline{3}}) \\
&\quad -(D_{\overline{23}}(ik_{\overline{3}}\cdot A_3) - D_{3\overline{3}}(ik_{\overline{3}}\cdot A_{\overline{2}}))s_{\overline{2}3} + D_{\overline{23}}(ik_3\cdot A_{\overline{3}})s_{\overline{23}} \ , \\
L_{\overline{23}3\overline{3}3\overline{3}} &= (D_{3\overline{3}}(ik_{\overline{3}}\cdot A_{\overline{2}}) - D_{\overline{2}3}(ik_{\overline{2}}\cdot A_{\overline{3}}))(1-s_{\overline{2}3}-s_{3\overline{3}}) + D_{\overline{23}}(ik_{\overline{2}}\cdot A_3)(1-s_{\overline{23}}-s_{3\overline{3}}) \\
&\quad -(D_{\overline{2}3}(ik_3\cdot A_{\overline{3}}) - D_{3\overline{3}}(ik_3\cdot A_{\overline{2}}))s_{\overline{23}} + D_{\overline{2}3}i(k_{\overline{3}}\cdot A_{\overline{3}})s_{\overline{2}3} \ , \\
L_{\overline{2323}3\overline{3}} &= (1-s_{\overline{23}})D_{\overline{23}}(A_3\cdot(ik_{\overline{2}}+ik_{\overline{3}})) \ , \\
L_{\overline{2}3\overline{2}33\overline{3}} &= -(1-s_{\overline{2}3})D_{\overline{2}3}(A_{\overline{3}}\cdot(ik_{\overline{2}}+ik_3)) \ .
\end{aligned} \tag{C.25}$$

The integrands in (C.17) of the first two superfield expressions above are proportional to $|z_{\overline{2}3}|^{s_{\overline{2}3}-1}|z_{3\overline{3}}|^{s_{3\overline{3}}-2}$ and $|z_{\overline{23}}|^{s_{\overline{2}3}-1}|z_{3\overline{3}}|^{s_{3\overline{3}}-2}$, which correspond to tachyonic poles in $\frac{1}{1+s_{ijk}}$. But also for these we can utilize the equations (C.19) to rewrite double pole integrals such that they become corrections to single pole integrals.

C.2.2 The three-point amplitude with BRST building blocks

Following the lines of [62, 158] we use these corrections, which were obtained by writing the ten double pole integrals as correction to the $(n-2)! = 24$ single pole integrals, to transform the composite superfield L_{ij}, L_{ijki} and L_{ijkili} into their corresponding BRST building blocks T_{ij}, T_{ijk} and T_{ijkl}. The exact form of the BRST building blocks and how to preform the substitution $L \to T$ can be found in chapter 5 and more details on the construction of T_{ij}, T_{ijk} and T_{ijkl} in section 5.2. After a long but straightforward computation involving the total derivative relations (C.19) one can check that the corrections are precisely of the correct form such that we arrive at a rather simple result

$$\begin{aligned}
\mathcal{A} = \frac{1}{2}\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3) \sum_{\substack{i,j\in\{\overline{1},2\}\\ i\neq j}} \epsilon_{ij}\Bigg(&\frac{\langle T_{i\overline{2}3\overline{3}}V_jV_1\rangle}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{2}i}} + \frac{\langle T_{i3\overline{32}}V_jV_1\rangle}{z_{\overline{23}}z_{\overline{3}3}z_{3i}} \\
+\frac{\langle T_{i\overline{3}3\overline{2}}V_jV_1\rangle}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{3}i}} - \frac{\langle T_{i3\overline{23}}V_jV_1\rangle}{z_{\overline{2}3}z_{\overline{23}}z_{3i}} - \frac{\langle T_{i\overline{23}3}V_jV_1\rangle}{z_{\overline{23}}z_{3\overline{3}}z_{\overline{2}i}} - \frac{\langle T_{i\overline{32}3}V_jV_1\rangle}{z_{\overline{2}3}z_{\overline{23}}z_{\overline{3}i}} + &\frac{\langle T_{i3\overline{2}}T_{j\overline{3}}V_1\rangle}{z_{\overline{2}3}z_{3i}z_{\overline{3}j}} \\
+\frac{\langle T_{j\overline{2}}T_{i\overline{3}3}V_1\rangle}{z_{\overline{2}j}z_{3\overline{3}}z_{\overline{3}i}} + \frac{\langle T_{i\overline{32}}T_{j3}V_1\rangle}{z_{\overline{2}3}z_{3\overline{3}}z_{\overline{3}i}} - \frac{\langle T_{i\overline{2}3}T_{j\overline{3}}V_1\rangle}{z_{\overline{2}3}z_{\overline{2}i}z_{\overline{3}j}} - \frac{\langle T_{i\overline{23}}T_{j3}V_1\rangle}{z_{\overline{23}}z_{\overline{2}i}z_{3j}} - &\frac{\langle T_{j\overline{2}}T_{i3\overline{3}}V_1\rangle}{z_{3\overline{3}}z_{3i}z_{\overline{2}j}}\Bigg) \ .
\end{aligned} \tag{C.26}$$

In (C.26) the denominators of the associated BRST building blocks follow a patter: For an unintegrated vertex operator $i \in \{\overline{1}, 2\}$ and integrated vertex operators $j, k, l \in \{\overline{2}, 3, \overline{3}\}$ we find that

$$T_{ij} \leftrightarrow \frac{1}{z_{ij}} \ , \qquad T_{ijk} \leftrightarrow \frac{1}{z_{ij} z_{jk}} \ , \qquad T_{ijkl} \leftrightarrow \frac{1}{z_{ij} z_{jk} z_{kl}} \ . \tag{C.27}$$

Moreover, the building blocks have always the same structure but with permutations in the labels of integrated vertex operators such that we can write the amplitude in a more compact form

$$\begin{aligned}\mathcal{A} = -\frac{1}{2} \int \mathrm{d}\overline{z}_2 \, \mathrm{d}z_3 \, \mathrm{d}z_{\overline{3}} \ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3) \bigg(& \frac{\langle T_{\overline{1}\overline{2}3\overline{3}} V_2 V_1 \rangle}{z_{\overline{1}\overline{2}} z_{\overline{2}3} z_{3\overline{3}}} + \frac{\langle T_{\overline{1}\overline{2}3} T_{2\overline{3}} V_1 \rangle}{z_{\overline{1}\overline{2}} z_{\overline{2}3} z_{2\overline{3}}} \\ & + \frac{\langle T_{\overline{1}\overline{2}} T_{23\overline{3}} V_1 \rangle}{z_{\overline{1}\overline{2}} z_{\overline{2}\overline{3}} z_{\overline{3}3}} + \frac{\langle V_{\overline{1}} T_{2\overline{3}3\overline{2}} V_1 \rangle}{z_{\overline{1}\overline{2}} z_{\overline{2}\overline{3}} z_{3\overline{3}}} + \mathcal{P}(\overline{2}, 3, \overline{3}) \bigg) \ ,\end{aligned} \tag{C.28}$$

where $\mathcal{P}(\overline{2}, 3, \overline{3})$ refers to the sum over all $(n-3)!$ permutations of the labels $(\overline{2}, 3, \overline{3})$.

C.2.3 The three-point amplitude in terms of SYM amplitudes

As discussed in section 6.2 the BRST building blocks are the components of the supersymmetric Berends-Giele currents and the corresponding definition of the supercurrents can be found in (6.8). Therefore, it is only natural that (C.28) can be written in terms of supersymmetric Berends-Giele currents. Based on the discussion in [62] we want to perform the conversion from T to M, which can be done by realising for example that

$$\begin{aligned}\frac{T_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}} &= -\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}} M_{\overline{1}\overline{2}} \ , \\ \frac{T_{\overline{1}\overline{2}3}}{z_{\overline{1}\overline{2}} z_{\overline{2}3}} + \mathcal{P}(\overline{2}, 3) &= \frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}} \left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}} + \frac{s_{\overline{2}3}}{z_{\overline{2}3}} \right) M_{\overline{1}\overline{2}3} + \mathcal{P}(\overline{2}, 3) \ , \\ \frac{T_{\overline{1}\overline{2}3\overline{3}}}{z_{\overline{1}\overline{2}} z_{\overline{2}3} z_{3\overline{3}}} + \mathcal{P}(\overline{2}, 3, \overline{3}) &= -\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}} \left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}} + \frac{s_{\overline{2}3}}{z_{\overline{2}3}} \right) \left(\frac{s_{\overline{1}\overline{3}}}{z_{\overline{1}\overline{3}}} + \frac{s_{\overline{2}\overline{3}}}{z_{\overline{2}\overline{3}}} + \frac{s_{3\overline{3}}}{z_{3\overline{3}}} \right) M_{\overline{1}\overline{2}3\overline{3}} + \mathcal{P}(\overline{2}, 3, \overline{3}) \ .\end{aligned} \tag{C.29}$$

For completeness we give the definition of the Berends-Giele currents above here once again:

$$\begin{aligned}M_{\overline{1}\overline{2}} &= -\frac{T_{\overline{1}\overline{2}}}{s_{\overline{1}\overline{2}}} \ , \\ M_{\overline{1}\overline{2}3} &= \frac{1}{s_{\overline{1}\overline{2}3}} \left(\frac{T_{\overline{1}\overline{2}3}}{s_{\overline{1}\overline{2}}} + \frac{T_{\overline{1}\overline{2}3} - T_{\overline{1}3\overline{2}}}{s_{\overline{2}3}} \right) \ , \\ M_{\overline{1}\overline{2}3\overline{3}} &= -\frac{1}{s_{\overline{1}\overline{2}3\overline{3}}} \bigg(\frac{T_{\overline{1}\overline{2}3\overline{3}}}{s_{\overline{1}\overline{2}} s_{\overline{1}\overline{2}3}} + \frac{T_{\overline{1}\overline{2}3\overline{3}} - T_{\overline{1}3\overline{2}\overline{3}}}{s_{\overline{2}3} s_{\overline{1}\overline{2}3}} + \frac{T_{\overline{1}\overline{2}3\overline{3}} - T_{\overline{1}\overline{2}\overline{3}3} + T_{\overline{1}\overline{3}3\overline{2}} - T_{\overline{1}3\overline{3}\overline{2}}}{s_{3\overline{3}} s_{\overline{2}3\overline{3}}} \\ &\quad + \frac{T_{\overline{1}\overline{3}3\overline{2}} - T_{\overline{1}\overline{3}\overline{2}3} + T_{\overline{1}\overline{2}\overline{3}3} - T_{\overline{1}3\overline{2}\overline{3}}}{s_{\overline{2}\overline{3}} s_{\overline{2}3\overline{3}}} + \frac{T_{\overline{1}\overline{2}3\overline{3}}}{s_{\overline{1}\overline{2}} s_{3\overline{3}}} - \frac{T_{\overline{1}\overline{2}\overline{3}3}}{s_{\overline{1}\overline{2}} s_{3\overline{3}}} \bigg) \ .\end{aligned} \tag{C.30}$$

After using (C.29) to transform the BRST building blocks into the corresponding Ms we obtain for (C.28) the following result

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\Bigg\{&\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\left(\frac{s_{\overline{1}\overline{3}}}{z_{\overline{1}\overline{3}}}+\frac{s_{\overline{2}\overline{3}}}{z_{\overline{2}\overline{3}}}+\frac{s_{3\overline{3}}}{z_{3\overline{3}}}\right)\langle M_{\overline{1}\overline{2}3\overline{3}}V_2V_1\rangle\\ &+\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\langle M_{\overline{1}\overline{2}3}M_{\overline{3}2}V_1\rangle+\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{3\overline{3}}}{z_{3\overline{3}}}+\frac{s_{32}}{z_{32}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\langle M_{\overline{1}\overline{2}}M_{2\overline{3}3}V_1\rangle\\ &+\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\left(\frac{s_{3\overline{3}}}{z_{3\overline{3}}}+\frac{s_{32}}{z_{32}}\right)\left(\frac{s_{2\overline{2}}}{z_{2\overline{2}}}+\frac{s_{\overline{3}\overline{2}}}{z_{\overline{3}\overline{2}}}+\frac{s_{3\overline{2}}}{z_{3\overline{2}}}\right)\langle V_{\overline{1}}M_{2\overline{3}3\overline{2}}V_1\rangle+\mathcal{P}(\overline{2},3,\overline{3})\Bigg\}\ ,\end{aligned}\tag{C.31}$$

which at first seems to be more complicated than (C.28). But we can simplify this expression utilizing integration by parts. For example, we can use that

$$\begin{aligned}&\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\left(\frac{s_{\overline{1}\overline{3}}}{z_{\overline{1}\overline{3}}}+\frac{s_{\overline{2}\overline{3}}}{z_{\overline{2}\overline{3}}}+\frac{s_{3\overline{3}}}{z_{3\overline{3}}}\right)\\ &=\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\end{aligned}\tag{C.32}$$

and thereby obtain

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\Bigg\{&\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\langle M_{\overline{1}\overline{2}3\overline{3}}V_2V_1\\ &+M_{\overline{1}\overline{2}3}M_{\overline{3}2}V_1+M_{\overline{1}\overline{2}}M_{3\overline{3}2}V_1+V_{\overline{1}}M_{\overline{2}3\overline{3}2}V_1\rangle+\mathcal{P}(\overline{2},3,\overline{3})\Bigg\}\ .\end{aligned}\tag{C.33}$$

In (C.33) we can identify the combination of Berends-Giele currents as $A_{\mathrm{SYM}}(\overline{1},\overline{2},3,\overline{3},2,1)$, see (8.67) and (6.18) for the general definition of SYM amplitudes in terms of Berends-Giele currents. Therefore, the final result for scattering of three closed strings on the disk is given by

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\Bigg\{\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}A_{\mathrm{SYM}}(\overline{1},\overline{2},3,\overline{3},2,1)\\ &\qquad+\mathcal{P}(\overline{2},3,\overline{3})\Bigg\}\\ &=-\frac{1}{2}A_{\mathrm{SYM}}(\overline{1},\overline{2},3,\overline{3},2,1)F^{(\overline{2},3,\overline{3})}+\mathcal{P}(\overline{2},3,\overline{3})\ .\end{aligned}\tag{C.34}$$

The integrals in (C.34) are the six hypergeometric basis integrals

$$\begin{aligned}F^{(\overline{2},3,\overline{3})} &= -\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}z_{\overline{3}}\ \mathrm{KN}(\overline{z}_2, z_3, \overline{z}_3)\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\\ &=-\int \mathrm{d}\overline{z}_2\,\mathrm{d}z_3\,\mathrm{d}\overline{z}_3\left(\prod_{i<j}|z_{ij}|^{s_{ij}}\right)\frac{s_{\overline{1}\overline{2}}}{z_{\overline{1}\overline{2}}}\frac{s_{\overline{3}2}}{z_{\overline{3}2}}\left(\frac{s_{\overline{1}3}}{z_{\overline{1}3}}+\frac{s_{\overline{2}3}}{z_{\overline{2}3}}\right)\ ,\end{aligned}\tag{C.35}$$

where the fixed positions are given by $(z_{\overline{1}} = 0, z_2 = 1, z_1 = \infty)$ and the product over $i < j$ also involves overlined indices. The basis of hypergeometric functions in (C.35) can be related to the previously found basis functions in (2.9) in [63] under the identification (8.28) and up to the integration region. Here, we integrate over the domains in (8.61) instead of $0 < z_2 < z_3 < z_4 < 1$.

Note that the polarization dependence of the amplitude is entirely carried by the linear combination of six field theory SYM amplitudes, whereas the α'-dependence of (C.34) resides in the integrals over parts of the real line. Therefore, all stringy corrections to SYM amplitudes are given by scalar integrals that are independent of the polarization [114].

By performing these steps all singularities in the correlator (C.17) have become logarithmic, since the z_{ij} in the individual denominators appear with power one. When interpreting the poles $\frac{1}{z_{ij}}$ as edges between vertices i and j the corresponding logarithmic singularities correspond to tree graphs. Moreover, the loop subdiagrams arising from for instance double poles like $\frac{1}{z_{ij}^2}$ or $\frac{1}{z_{ij}z_{jk}z_{ki}}$ are removed using integration by parts. This is possible, because the tachyon poles are accompanied by numerators proportional to $(1-s_{ij})$ and $(1-s_{ijk})$, which originate from contractions of integrated vertex operators. In general, it was shown in [201] that non-logarithmic singularities can be removed utilizing integration by parts. Nevertheless, it is a special feature of the superstring that any tree-level amplitude becomes free of tachyon poles and is homogenous in α', since this is not the case for bosonic of heterotic strings [114, 202, 203].

Appendix D

Complex integration and analytic continuation

In [34] a relation between open and closed string scattering amplitudes was proposed. Therefore, to account for the correct branch of integration a phase was introduced, which depends only on the kinematic invariants and the ordering of the worldsheet coordinates. In this appendix we want to present the derivation of the corresponding phase (8.34) for the scattering of three closed strings on the disk. This appendix is based on [109], which in turn follows the steps of [34] and is guided by the review [172].

The amplitude (8.26) can be split into the Koba-Nielsen factor $\mathrm{KN}(y, z, \overline{z})$ containing the branch cuts of the amplitude and the branch cut independent correlator $\langle \mathcal{K}(y, z, \overline{z}) \rangle$ originating from the vertex operator contractions. So far, we have a similar setup as in section 7.1 and can write the amplitude (8.26) as

$$
\begin{aligned}
\mathcal{A} \sim& \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z \, \langle\!\langle V_1(i) V_{\overline{1}}(-i) V_2(iy) U_{\overline{2}}(-iy) U_3(z) U_{\overline{3}}(\overline{z}) \rangle\!\rangle \\
=& \int_0^1 \mathrm{d}y \int_{\mathbb{H}_+} \mathrm{d}^2 z \, \langle \mathcal{K}(y, z, \overline{z}) \rangle 2^{s_{1\overline{1}}} |2y|^{s_{2\overline{2}}} |1-y|^{2s_{12}} |1+y|^{2s_{1\overline{2}}} |i-z|^{2s_{13}} |i+z|^{2s_{1\overline{3}}} \\
&\times |iy-z|^{2s_{23}} |iy+z|^{2s_{2\overline{3}}} |z-\overline{z}|^{s_{3\overline{3}}} \ . \qquad \text{(D.1)}
\end{aligned}
$$

Since we integrate $0 < y < 1$ and $(z, \overline{z})$ over the upper half plane, i.e. $\Im(z) \geq 0$, no branch cuts arise from $|2y|^{s_{2\overline{2}}}$, $|1-y|^{2s_{12}}$, $|1+y|^{2s_{1\overline{2}}}$ and $|z-\overline{z}|^{s_{3\overline{3}}} = |2\Im z|^{s_{3\overline{3}}}$ as they are all bigger or equal to zero in this integration domain. Hence, they will not contribute to the monodromy phase.

Then, the analytic continuation described in section 8.3.1 can be carried out by deforming the integration contour of $\Re(z) = z_1$ while avoiding all branch cuts as shown in figure 8.1. Formally, this can be achieved by

$$
z_1 \to i e^{-2i\varepsilon} z_1 \approx i(1-2i\varepsilon) z_1 = i z_1 + 2\varepsilon z_1 \ , \qquad \text{(D.2)}
$$

where ε is a small and positive constant. Thus, for $\lambda \in \mathbb{R}$ the terms in (D.1) containing branch cuts can be written after the analytic continuation as[1]

$$\begin{aligned}|i\lambda - z|^{2s} &= \left[z_1^2 + (\lambda - z_2)^2\right]^s \\ &\to \left[(iz_1 + 2\varepsilon z_1)^2 + (\lambda - z_2)^2\right]^s \\ &= \left[(\xi - \lambda - i\varepsilon\delta)(-\eta - \lambda + i\varepsilon\delta)\right]^s ,\end{aligned} \tag{D.3}$$

where we have introduced $\delta = 2z_1 = \xi + \eta$ and the variables

$$z \to iz_1 + iz_2 = i\xi , \qquad \overline{z} \to iz_1 - iz_2 = i\eta , \tag{D.4}$$

where $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$ have to satisfy $\xi - \eta \geq 0$ to preserve the integration over the upper half plane. Performing the coordinate transformation the amplitude (D.1) yields

$$\begin{aligned}\mathcal{A} \sim &\int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \langle \mathcal{K}(y,\xi,\eta)\rangle 2^{s_{1\bar{1}}} |2y|^{s_{2\bar{2}}} |1-y|^{2s_{12}} |1+y|^{2s_{1\bar{2}}} |\xi - \eta|^{s_{3\bar{3}}} \\ &\times [(\xi - 1 - i\varepsilon\delta)(-\eta - 1 + i\varepsilon\delta)]^{s_{13}} [(\xi + 1 - i\varepsilon\delta)(-\eta + 1 + i\varepsilon\delta)]^{s_{1\bar{3}}} \\ &\times [(\xi - y - i\varepsilon\delta)(-\eta - y - i\varepsilon\delta)]^{s_{23}} [(\xi + y - i\varepsilon\delta)(-\eta + y + i\varepsilon\delta)]^{s_{2\bar{3}}}\end{aligned} \tag{D.5}$$

$$\begin{aligned}= &\int_0^1 \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \langle \mathcal{K}(y,\xi,\eta)\rangle 2^{s_{1\bar{1}}} |2y|^{s_{2\bar{2}}} |1-y|^{2s_{12}} |1+y|^{2s_{1\bar{2}}} |\xi - \eta|^{s_{3\bar{3}}} \\ &\times (-\xi + 1 + i\varepsilon\delta)^{s_{13}} (\eta + 1 - i\varepsilon\delta)^{s_{13}} (-\xi - 1 + i\varepsilon\delta)^{s_{1\bar{3}}} (\eta - 1 - i\varepsilon\delta)^{s_{1\bar{3}}} \\ &\times (-\xi + y + i\varepsilon\delta)^{s_{23}} (\eta + y - i\varepsilon\delta)^{s_{23}} (-\xi - y + i\varepsilon\delta)^{s_{2\bar{3}}} (\eta - y - i\varepsilon\delta)^{s_{2\bar{3}}} .\end{aligned} \tag{D.6}$$

To get from the first to the second line we utilized that

$$(z_1 z_2)^c = (-z_1)^c (-z_2)^c \quad \text{for } \operatorname{sign}(\Im(z_1)) = -\operatorname{sign}(\Im(z_2)) . \tag{D.7}$$

As in section 7.1 we have chosen the branch cut of the power function z^c to lie on the negative real axis. Therefore, for the power function $z^c = |z|^c e^{ic\theta}$ with $-\pi < \theta < \pi$ we can again use (7.14), which together with

$$(z_1 z_2)^c = z_1^c z_2^c \quad \text{for } \operatorname{sign}(\Im(z_1)) = -\operatorname{sign}(\Im(z_2)) . \tag{D.8}$$

results in equation (D.7).

Next, we want to determine the η-integration contour by analysing the behaviour of the imaginary parts in the η-terms at the branch points. For $\xi < -1$ we find

$$\begin{aligned}\eta \approx -1 : &\quad \delta = \xi + \eta \approx \xi - 1 < 0 , \\ \eta \approx -y : &\quad \delta \approx \xi - y < 0 , \\ \eta \approx y : &\quad \delta \approx \xi + y < 0 , \\ \eta \approx 1 : &\quad \delta \approx \xi + 1 < 0\end{aligned} \tag{D.9}$$

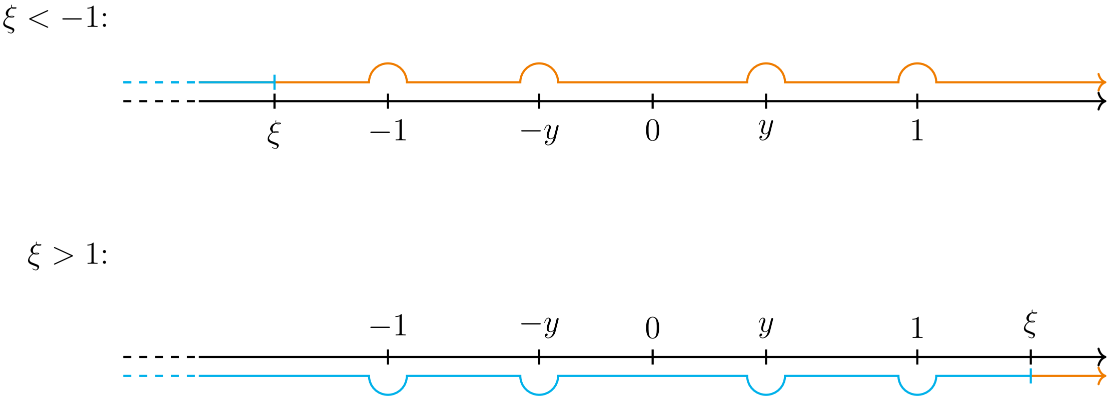

Figure D.1: η-integration contour ($\eta < \xi$ in blue and $\eta > \xi$ in orange) for $\xi < -1$ and $1 < \xi$.

and similar for $\xi > 1$ at all branch points we get $\delta > 0$, which yields the integration contours depicted in figure D.1. The η-integration in figure D.1 only ranges over the blue contour, because it has to end at ξ. Moreover, for $\xi \in]-1, 1[$ we get $\delta < 0$ for $\eta < -\xi$ and $\delta > 0$ for $\eta > -\xi$ and the corresponding η-integration contour is shown in figure D.2 for the particular case $y < \xi < 1$.

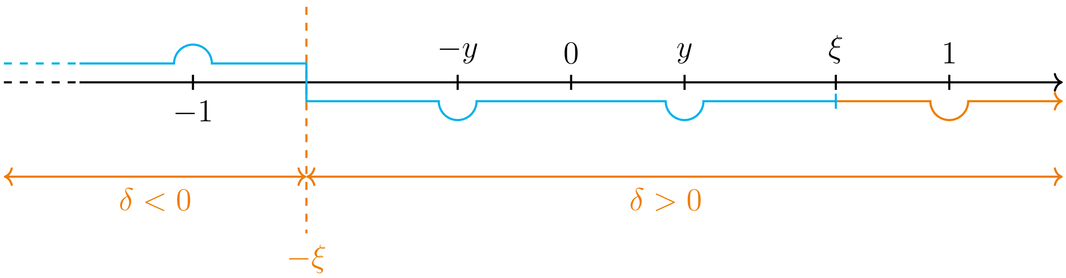

Figure D.2: η-integration contour ($\eta < \xi$ in blue and $\eta > \xi$ in orange) for $y < \xi < 1$.

Next, we want to demonstrate how to determine the correct monodromy phase for each integration branch. Therefore, we consider the integration region $0 < y < 1$, for which (D.6) has the following behaviour of the imaginary parts in the η-terms at the branch points

$$
\begin{aligned}
\eta \approx -1 : \quad & \delta = \xi + \eta \approx \xi - 1 < 0 \, , \\
\eta \approx -y : \quad & \delta \approx \xi - y > 0 \, , \\
\eta \approx y : \quad & \delta \approx \xi + y > 0 \, , \\
\eta \approx 1 : \quad & \delta \approx \xi + 1 > 0 \, .
\end{aligned}
\tag{D.10}
$$

[1] The definition of δ in this appendix deviates form δ in section 7.1.

For this particular example the real part of the ξ-dependent terms is negative except for $1-\xi+i\varepsilon\delta$, which has positive real part. We want all the real parts of ξ-dependent terms to be positive, which can be achieved by altering the signs in the former case using (D.7) while simultaneously changing the signs in the corresponding η-terms:

$$\begin{aligned}\mathcal{A}\Big|_{y<\xi<1} \sim \int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \langle \mathcal{K}(y,\xi,\eta)\rangle 2^{s_{1\bar1}} |2y|^{s_{2\bar2}} |1-y|^{2s_{12}} |1+y|^{2s_{1\bar2}} |\xi-\eta|^{s_{3\bar3}} \\ \times(-\xi+1+i\varepsilon\delta)^{s_{13}}(\eta+1-i\varepsilon\delta)^{s_{13}}(\xi+1-i\varepsilon\delta)^{s_{1\bar3}}(-\eta+1+i\varepsilon\delta)^{s_{1\bar3}} \\ \times(\xi-y-i\varepsilon\delta)^{s_{23}}(-\eta-y+i\varepsilon\delta)^{s_{23}}(\xi+y-i\varepsilon\delta)^{s_{2\bar3}}(-\eta+y+i\varepsilon\delta)^{s_{2\bar3}} \,. \end{aligned} \tag{D.11}$$

Furthermore, we want all the real parts of η-dependent terms in the Koba-Nielsen factor

	$\eta<-1$	$-1<\eta<-y$	$-y<\eta<y$	$y<\eta<\xi$
$1+\eta$:	<0	>0	>0	>0
$1-\eta$:	>0	>0	>0	>0
$-y-\eta$:	>0	>0	<0	<0
$y-\eta$:	>0	>0	>0	<0

Table D.1: Real part of the η-dependent terms in (D.11) for $y<\xi<1$.

to be positive as well: Each time one of the η-dependent terms in table D.1 has a negative sign, we use (7.14) to make it positive. Thereby, we pick up a monodromy phase $e^{i\pi s_{j\bar3}}$ or $e^{-i\pi s_{j\bar3}}$, depending on the sign of the imaginary part of $z_j-\eta$, i.e. the sign of δ at the branch point z_j given in figure D.2. Because we are now avoiding all branch points, we can take the limit $\varepsilon\to 0$ and write the amplitude (D.11) as

$$\begin{aligned}\mathcal{A}\Big|_{y<\xi<1} \sim \int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\eta \, \langle \mathcal{K}(y,\xi,\eta)\rangle \Pi(y,\xi,\eta) 2^{s_{1\bar1}} |2y|^{s_{2\bar2}} |1-y|^{2s_{12}} |1+y|^{2s_{1\bar2}} |\xi-\eta|^{s_{3\bar3}} \\ \times|1-\xi|^{s_{13}}|1+\eta|^{s_{13}}|1+\xi|^{s_{1\bar3}}|1-\eta|^{s_{1\bar3}}|y-\xi|^{s_{23}}|y+\eta|^{s_{23}}|y+\xi|^{s_{2\bar3}}|y-\eta|^{s_{2\bar3}} \end{aligned} \tag{D.12}$$

where the phase $\Pi(y,\xi,\eta)$ along the integration contour for η is shown in figure D.3. The

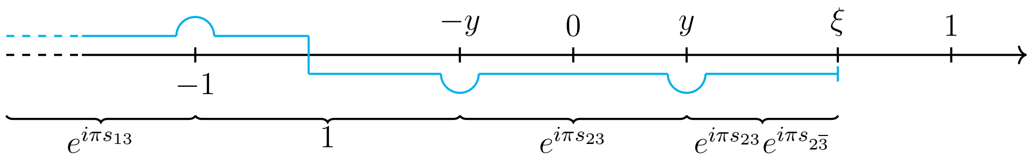

Figure D.3: Phase $\Pi(y,\xi,\eta)$ for $y<\xi<1$.

above procedure shows that for an integration region we get a monodromy phase from corresponding ξ- and η-dependent terms, if their real parts have opposite signs. In the end, the phase factor derived by this analysis is consistent with

$$\Pi(y,\xi,\eta) = e^{i\pi s_{13}\Theta(-(1-\xi)(1+\eta))} e^{i\pi s_{1\bar{3}}\Theta(-(1+\xi)(1-\eta))} e^{i\pi s_{23}\Theta(-(y-\xi)(y+\eta))} e^{i\pi s_{2\bar{3}}\Theta(-(y+\xi)(y-\eta))} \,. \tag{D.13}$$

In section 8.3.1 we have added $e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))}$ to the phase factor for completeness. This factor accounts for the contribution coming from $(\xi-\eta)^{s_{3\bar{3}}}$ for $\xi < \eta$ such that we can write

$$(\xi-\eta)^{s_{3\bar{3}}} = |\xi-\eta|^{s_{3\bar{3}}} e^{i\pi s_{3\bar{3}}\Theta(-(\xi-\eta))} \tag{D.14}$$

using (7.14) for all values of ξ and η.

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