On the Simple Random Walk on Supercritical Galton–Watson Trees

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Jakob Linus Stern
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1. Gutachter: Prof. Dr. Peter Müller
2. Gutachter: Prof. Dr. Nina Gantert

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.2011, §8, Abs. 2, Pkt. 5.)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Jakob Linus Stern
München, 29.02.2024

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Zusammenfassung

Wir betrachten den einfachen Zufallslauf auf Galton-Watson Bäumen mit einer superkritischen Nachkommensverteilung $\{p_j\}_{j\in\mathbb{N}_0}$ bedingt auf Nicht-Aussterben. Darüber hinaus nehmen wir einen super-gaußschen Abfall von $\{p_j\}_{j\in\mathbb{N}_0}$ an, also einen exponentiellen Abfall in j^k für einen Exponenten $k > 2$. Obwohl dies eines der klassischen und bekanntesten Beispiele für Zufallsläufe in zufälligen Umgebungen darstellt, führt die Existenz von Blättern zu einigen interessanten, noch ungelösten Problemen.

Zuerst betrachten wir die Zeit τ_t' im über die Bäume gemittelten ("annealed") Maß, die der Zufallslauf aus insgesamt t Schritten auf dem Rückgrat des Baumes verbringt. Dies kann dazu verwendet werden, den Zufallslauf von Galton-Watson Bäumen auf seine technisch einfachere Version auf dem Rückgrat zu reduzieren. Durch ein Abschneideargument können wir ungeeignete Bäume bis auf einen vernachlässigbaren Fehler ausschließen und somit folgern, dass die gemittelte Wahrscheinlichkeit des Ereignisses $\tau_t' \leqslant t^{1-\vartheta-\varepsilon}$ für alle $\vartheta > 0$ exponentiell klein in t^ϑ ist, wobei $\varepsilon \geqslant \frac{2}{k}$ und $2\vartheta + \varepsilon \leqslant 1$.

Der Hauptteil dieser Dissertation beschäftigt sich mit der gemittelten Rückkehrwahrscheinlichkeit zur Zeit t. Wir beginnen mit dem Beweis für die behauptete untere Schranke, welche für gerade t exponentiell in $t^{\frac{1}{3}}$ abfällt. Dann zeigen wir für Nachkommensverteilungen mit unbeschränktem Träger und super-gaußschem Abfall mit $k > 8$ eine obere Schranke an die gemittelte Rückkehrwahrscheinlichkeit, welche mindestens exponentiell in $t^{\frac{1}{3}-\frac{8}{3k}}$ abfällt. Des Weiteren beweisen wir für $\{p_j\}_{j\in\mathbb{N}_0}$ mit beschränktem Träger den exponentiellen Abfall mit optimalem Exponenten $t^{\frac{1}{3}}$. Dies verbessert das bislang beste Resultat von Piau [Ann. Probab. **26**, 1016–1040 (1998)] und etabliert den exakten Skalierungsexponenten $\frac{1}{3}$ für den Fall, in dem $\{p_j\}_{j\in\mathbb{N}_0}$ für alle $k \in \mathbb{N}_0$ exponentiell in j^k abfällt.

Abstract

We consider the simple random walk on Galton-Watson trees with supercritical offspring distribution $\{p_j\}_{j\in\mathbb{N}_0}$ conditioned on non-extinction. In addition, we require a super-Gaussian decay of $\{p_j\}_{j\in\mathbb{N}_0}$ in the sense that it decays exponentially in j^k for some exponent $k > 2$. Despite being one of the most classical and most studied examples of random walks in random environments, the existence of leaves generates problems which yield several interesting questions that are still open.

First we study the time τ'_t the random walk spends on the backbone out of a total of t steps under the annealed measure. This is relevant when reducing the random walk on Galton-Watson trees to its technically simpler version on the backbone. A cut-off argument allows to exclude unfavourable trees at the cost of a negligible error term and we conclude that the annealed probability of the event $\tau'_t \leqslant t^{1-\vartheta-\varepsilon}$ is exponentially small in t^{ϑ} for any $\vartheta > 0$, where $\varepsilon \geqslant \frac{2}{k}$ and $2\vartheta + \varepsilon \leqslant 1$.

The main part of this dissertation investigates the annealed return probability at time t. We start by giving a proof for the proclaimed lower bound decaying exponentially in $t^{\frac{1}{3}}$ for t even. Next, we show an upper bound for the annealed return probability which decays at least exponentially in $t^{\frac{1}{3}-\frac{8}{3k}}$, and holds for offspring distributions with unbounded support and super-Gaussian decay for $k > 8$. Further, if $\{p_j\}_{j\in\mathbb{N}_0}$ has even bounded support we obtain an exponential decay in the optimal power $t^{\frac{1}{3}}$. This improves the best known results of Piau [Ann. Probab. **26**, 1016–1040 (1998)] and also establishes the exact scaling exponent $\frac{1}{3}$ in the case where $\{p_j\}_{j\in\mathbb{N}_0}$ decays exponentially in j^k for any $k \in \mathbb{N}_0$.

Published content

Some of the results presented in this thesis were obtained in scientific collaboration with P. Müller, which resulted in the publication of the following article.

[47] P. Müller and J. Stern, *On the return probability of the simple random walk on Galton-Watson trees*, arXiv: 2402.01600 [math.PR] (2024).

We note that the proofs in this article as well as its first draft were created by J. Stern. The relation to this published material is highlighted at the beginning of the respective sections, in especially chapter three section Upper Bound for the Annealed Return Probability and the appendix. Moreover, parts of the introduction, in particular the section Main Results, coincide both in content and writing with material from this publication.

Contents

Chapter 1

Introduction

1.1 Background

1.1.1 Graphs

We start with the basic notions of graphs and focus on random graphs and on Erdős-Rényi random graph in particular, as can be found in the following books [9, 23, 50, 51, 53].

Graph terminologies

First, we introduce some notations that will be used throughout this thesis. A graph $\mathbf{G} = (V, E)$ consists of a countable set of vertices V, called the vertex set, and a collection of edges E, called the edge set. The edge set is a subset of $V \times V$. If $(x, y) \in E$, we say that x is *adjacent* to y, and write $x \sim y$. In this case the edge (x, y) is called *incident* to x and y, and x, y are *neighbours* of each other. Similarly, two edges which share a common vertex are also called *adjacent*. A graph $\mathbf{G} = (V, E)$ is called *undirected*, if $(x, y) = (y, x)$ for all $(x, y) \in E$. To stress this, we will sometimes write $\{x, y\}$ instead of (x, y) for $(x, y) \in E$. Otherwise if $(x, y) \neq (y, x)$ for some x and y, then the graph is *directed*. An edge (x, y) is called a self-loop, if $x = y$, and a graph without any self-loops is called *irreflexive*. $\mathbf{G}$ is a *complete* graph, if $(x, y) \in E$ for all $x \neq y$. A *subgraph* $\mathbf{G}' = (V', E')$ of $\mathbf{G} = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$. In this case, we will write $\mathbf{G}' \subseteq \mathbf{G}$ for simplicity. For $S \subseteq V$, the *induced subgraph* is the graph with vertex set S and edge set given by $\{(x, y) \in E : x, y \in S\}$. Throughout this dissertation, unless specifically mentioned otherwise, we always refer to *undirected*, but not necessarily *irreflexive* graphs.

The *degree* d_x of a vertex x in the graph $\mathbf{G} = (V, E)$ is defined as the number of edges incident to it, i.e.,

$$d_x \coloneqq \#\{y \in V : x \sim y\} = \sum_{y \in V} \mathbf{1}_{\{x \sim y\}}. \tag{1.1}$$

A graph $\mathbf{G} = (V, E)$ is called *locally finite*, if $d_x < \infty$ for all $x \in V$. If the degree of every vertex is the same number d, the graph is called *(d-)regular*. Moreover, $\mathbf{G}$ has *maximal vertex degree* d, if $d_x \leqslant d$ for every $x \in V$.

A *path* π in a graph $\mathbf{G} = (V, E)$ is a sequence of vertices $(x_i)_{i=0,1,\dots,n} \subseteq V$ such that $(x_i, x_{i+1}) \in E$ for $i = 0, 1, \dots, n-1$, and π is said to *join* x_0 and x_n. If the path does not contain any vertex more than once or does not pass through any edge more than once, it is called *vertex simple*, respectively, *edge simple*. $S \subseteq V$ is called *connected*, if the induced subgraph is connected, that is, if for every pair (x, y) with $x \neq y$ there exists a path joining x and y. A subgraph $\mathcal{C}$ of $\mathbf{G}$ is called a *cluster*, if $\mathcal{C}$ is connected and is not connected to any other vertex that is not in $\mathcal{C}$. The *graph distance* $\operatorname{dist}(x, y)$ between x, y equals 1 plus the number, not counting x, y themselves, of vertices of the shortest path joining x, y if $x \neq y$ and 0 if $x = y$. If such a path does not exist, i.e. x and y are in different clusters, then we set $\operatorname{dist}(x, y) = \infty$.

If there are weights $c((x, y))$ assigned to the edges (x, y) of a graph, the resulting object is the *weighted graph* $\mathbf{G} = (V, w)$ which is sometimes also referred to as a *network*. We have $c\colon V \times V \to [0, \infty)$ with $c((x, y)) = 0$ if and only if $(x, y) \notin E$. If not mentioned otherwise, we always consider the weighted graph obtained by setting $c((x, y)) \equiv 1$ for all $(x, y) \in E$. We define the *weight of a vertex* x as the sum of the weights over all incident edges to x, i.e.,

$$w(x) \coloneqq \sum_{x \sim y} c((x, y)). \tag{1.2}$$

So, in our standard setting we get $w(x) = d_x$.

For a vertex set $S \subseteq V$, we define the *edge boundary* ∂S to be the set of edges connecting S to its complementary set of vertices in V, or short its *complement*. The *outer vertex boundary* of S is the set of vertices not in S but with a neighbour in S and the *inner vertex boundary* of S is the set of vertices in S with a neighbour not in S. The *number* of vertices or edges in the set Σ will be denoted by $\#\Sigma$ whereas the *(w-weighted vertex) volume* of the vertex set V_1 is given by $\|V_1\|_w \coloneqq \sum_{x \in V_1} w(x)$ and the *(c-weighted edge) volume* of the edge set E_1 is $\|E_1\|_c \coloneqq \sum_{(x,y) \in E_1} c((x, y))$. We note that the volume always requires and depends on the weights c, compare with (1.2) for the vertex volume.

The *Cheeger constant* of a graph $\mathbf{G} = (V, E)$ is given by

$$\Phi = \Phi_{\#}(\mathbf{G}) := \inf\Big\{ \frac{\#\partial S}{\#S} : \varnothing \neq S \subseteq V \text{ finite} \Big\} \geqslant 0. \tag{1.3}$$

The same quantity can also be considered for weighted graphs $\mathbf{G} = (V, w)$ with the volume of the sets instead of the number of their elements. This will be called the *volume Cheeger constant* and be denoted by Φ_w, i.e.,

$$\Phi_w = \Phi_w(\mathbf{G}) := \inf\Big\{ \frac{\|\partial S\|_c}{\|S\|_w} : \varnothing \neq S \subseteq V \text{ finite} \Big\} \geqslant 0. \tag{1.4}$$

We say that $\mathbf{G}$ satisfies the *strong isoperimetric inequality* if $\Phi > 0$ and that $\mathbf{G}$ satisfies the *strong volume isoperimetric inequality* if $\Phi_w > 0$. We note that $\Phi_w \leqslant 1$, by the definition of the weight function in (1.2).

The *anchored expansion constant* of a graph $\mathbf{G} = (V, E)$ at the vertex $x \in V$ is given by

$$\mathbf{i}(\mathbf{G}) = \mathbf{i}_{\#}(\mathbf{G}) := \lim_{n\to\infty} \inf\Big\{ \frac{\#\partial K}{\#K} : x \in K \subseteq V,\ K \text{ is connected},\ n \leqslant \#K < \infty \Big\} \tag{1.5}$$

and was introduced explicitly in [37]. Again, the same quantity can be considered for the volume of the sets instead of the number of their elements. This will then be called the *volume anchored expansion constant* and be denoted by $\mathbf{i}_w(\mathbf{G})$, i.e.,

$$\mathbf{i}_w(\mathbf{G}) := \lim_{n\to\infty} \inf\Big\{ \frac{\|\partial K\|_c}{\|K\|_w} : x \in K \subseteq V,\ K \text{ is connected},\ n \leqslant \#K < \infty \Big\}. \tag{1.6}$$

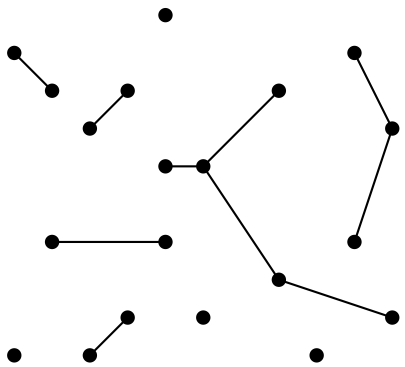
We remark that the anchored expansion constant is independent of the chosen vertex x [45, p. 214]. $\mathbf{G}$ satisfies the *(volume) anchored expansion property* if $\mathbf{i}_{(w)}(\mathbf{G}) > 0$.

Random graphs

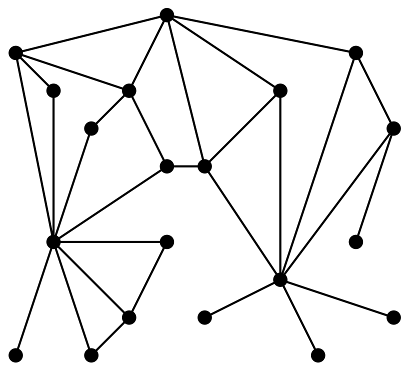
The concept "random graph" refers to a probability distribution with law G on a family of graphs which will be denoted by $\mathbb{G}$ to distinguish from the deterministic graph $\mathbf{G}$. Typically, in a random graph the edges are randomly generated. Random graphs were first introduced by Erdős and Rényi [24, 25, 26, 27] with ample results already given in [25]. Since then random graph theory was broadly extended and varieties of random graph models have been proposed and analysed. Alon and Spencer [3] as well as Bollobás [8] give more details about the early literature on random graphs. However, the Erdős-Rényi random graph is still one of the most instructive and investigated random graph models. Here we focus on this model because of its interesting properties and consequences as can be seen in the following subsection and Section 1.4.

Erdős-Rényi random graph

The Erdős-Rényi random graph model was first introduced by Erdős and Rényi [25], Gilbert [30], respectively Austin et al. [4] with slight differences. In this model one considers $[n] := \{1, 2, \dots, n\}$ as the vertex set V. For each pair of vertices $i \neq j$, the undirected edge (i, j) between the two vertices exists with probability p, and there are no self-loops. As a consequence we obtain a graph, which we denote by $\mathrm{ER}(n, p)$, with a deterministic vertex set but a random edge set with probability measure $\mathbb{P}$ and expectation $\mathbb{E}$. Two examples of Erdős-Rényi random graphs with different edge probability are illustrated in Figure 1.1.



(a) $\lambda = 0.5$.



(b) $\lambda = 1.5$.

Figure 1.1: Two realizations of Erdős-Rényi random graphs $\mathrm{ER}(n, \lambda/n)$ with $n = 20, \lambda = 0.5$ in (a) and $n = 20, \lambda = 1.5$ in (b) respectively.

We note that the degree of a vertex i in $\mathrm{ER}(n, p)$ follows a binomial distribution, that is, for any $k \in \mathbb{N}_0$

$$\mathbb{P}[d_i = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \tag{1.7}$$

since each vertex $i \in [n]$ can have up to $n - 1$ neighbours. The *typical degree* D_n is the random variable given by the degree of a vertex chosen uniformly at random, so $D_n = \frac{1}{n} \sum_{i \in [n]} d_i$. We obtain that $\mathbb{E}[D_n] = (n-1)p$. Since we are mainly interested in limit theorems for $\mathrm{ER}(n, p)$ which will be discussed in subsection "Limits of Erdős-Rényi random graphs" of 1.1.2, we refer to [34] for more details about the degree distribution or the connectivity of $\mathrm{ER}(n, p)$.

Before discussing limits of $\mathrm{ER}(n,p)$, we first need to recall the following different notions of probabilistic convergence. Consider a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, X_n converges to the random variable X *in distribution* as $n \to \infty$ if and only if $\lim_{n\to\infty} \int_\Omega f(X_n)\, \mathrm{d}\mathbb{P} = \int_\Omega f(X)\, \mathrm{d}\mathbb{P}$ for every bounded, continuous function $f\colon \mathbb{R} \to \mathbb{C}$, and we write $X_n \xrightarrow{d} X$ as $n \to \infty$. X_n converges to X *in probability* as $n \to \infty$ if and only if $\lim_{n\to\infty} \mathbb{P}[|X_n - X| > \varepsilon] = 0$ for all $\varepsilon > 0$, which will be denoted by $X_n \xrightarrow{\mathbb{P}} X$ as $n \to \infty$. Further, we say X_n converges to the random variable X *almost surely* as $n \to \infty$ if and only if $\mathbb{P}[\lim_{n\to\infty} X_n = X] = 1$ and write $X_n \xrightarrow{\text{a.s.}} X$ as $n \to \infty$. We recall that almost sure convergence, by Fatou's lemma [28], implies convergence in probability which, in turn, implies convergence in distribution.

With this at hand, we turn to the scaling limit of $\mathrm{ER}(n,p)$ as $n \to \infty$ which is broadly discussed, for example in [11, 35]. In particular, we consider the *adjacency matrix* A, which is given by setting the matrix element $a_{ij} = 1$ if the edge (i,j) is present and zero otherwise, and the *graph Laplacian* L given by

$$L \coloneqq D - A, \tag{1.8}$$

where the *degree matrix* D has diagonal elements equal to the degrees d_i, $i \in [n]$, and off-diagonal elements equal to zero.

For the last decades there has been a growing interest in the spectral properties of the graph Laplacian which was analysed, among others, in [16, 17, 20, 41, 46]. The spectral information on the Erdős-Rényi random graph Laplacian in the scaling limit $n \to \infty$ shares similarities to the spectral theory of large random matrices, which was originated by Wigner [54, 55], and is still to some extent an open question [7, 39, 42, 43]. However, it is easy to see that this will largely depend on the choice of the edge probability p.

We focus on the *sparse* case with $p = \frac{\lambda}{n}$, $\lambda > 0$ a constant independent of n. In general, for the sparse regime it is assumed that the expectation of the typical degree stays finite in the limes superior as $n \to \infty$, i.e.,

$$\limsup_{n\to\infty} \mathbb{E}\left[\frac{1}{n} \sum_{i\in[n]} d_i\right] < \infty. \tag{1.9}$$

Now, the sparse case can be divided into the *subcritical regime* with $\lambda \in \,]0,1[\,$, the *critical regime* with $\lambda = 1$, and the *supercritical regime* with $\lambda \in \,]1,\infty[$. In the supercritical regime, the spectral properties of the graph Laplacian in the scaling limit are among the aforementioned open questions.

To understand a basic structural difference of the subcritical and the supercritical regime, we consider the size of the largest cluster of $\mathrm{ER}(n, \frac{\lambda}{n})$, with $\lambda > 0$ constant. By $\mathcal{C}(i)$ we denote the cluster containing the vertex i and $\mathcal{C}_{\max}$ is the cluster such that

$$\#\mathcal{C}_{\max} = \max_{i\in[n]} \#\mathcal{C}(i). \tag{1.10}$$

However, this enables us only to identify the size of $\mathcal{C}_{\max}$, but not $\mathcal{C}_{\max}$ itself uniquely. If several clusters are of maximal size, we choose the one that contains the vertex with smallest index as $\mathcal{C}_{\max}$. Varying λ, the size of $\mathcal{C}_{\max}$ differs significantly, and thus exhibits a phase transition. In the subcritical phase, it was shown that

$$\frac{\#\mathcal{C}_{\max}}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\lambda - 1 - \log\lambda} \tag{1.11}$$

as $n \to \infty$, see for example [3, 13]. Further, the fraction of vertices which are either isolated or belong to tree clusters tends to one as $n \to \infty$ [8, 25]. On the other hand in the supercritical phase, there is a positive constant ζ such that for every $\nu \in (\frac{1}{2}, 1)$ there exists $\delta = \delta(\nu, \lambda) > 0$ with

$$\mathbb{P}\left[\left|\#\mathcal{C}_{\max} - \zeta n\right| \geqslant n^{\nu}\right] = O\left(n^{-\delta}\right), \tag{1.12}$$

as $n \to \infty$, compare with [3]. Around the critical value $\lambda = 1$, $\mathcal{C}_{\max}$ is asymptotically of size $n^{\frac{2}{3}}$ as $n \to \infty$ [38, 49]. Therefore, the Erdős-Rényi random graph Laplacian can be broken down into blocks of size $\log(n)$ in the subcritical case. This was used in [41] to prove the Lifshitz tail behaviour of its integrated density of states at the lower spectral edge $E = 0$. Unfortunately, the emerging giant cluster $\mathcal{C}_{\max}$ prevents this approach in the supercritical case.

Back to the scaling limit $n \to \infty$ of the vertex degree d_i, $i \in [n]$, a short calculation shows that in the sparse regime with $p = \frac{\lambda}{n}$, $\lambda > 0$, the probability of a vertex degree equal $k \in \mathbb{N}_0$ converges to the mass function of a Poisson random variable with parameter λ, i.e.,

$$\mathbb{P}[d_i = k] = \binom{n-1}{k}\frac{\lambda^k}{n^k}\Big(1 - \frac{\lambda}{n}\Big)^{n-1-k} \to e^{-\lambda}\frac{\lambda^k}{k!}, \tag{1.13}$$

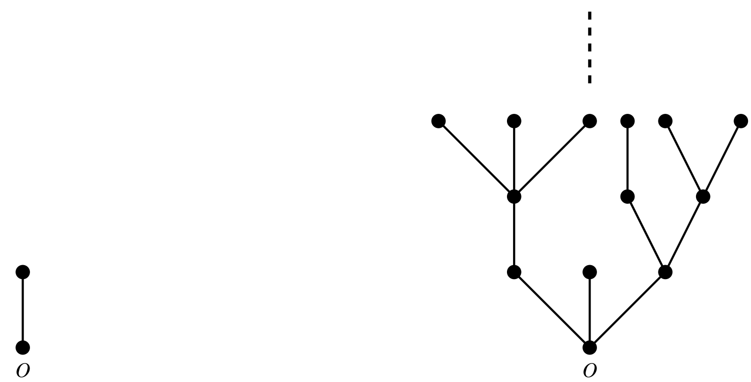
as $n \to \infty$, [34, Sect. 5.4]. This will be considered in subsection “Limits of Erdős-Rényi random graphs” of 1.1.2 in more detail.

1.1.2 Trees

A graph with no cycles is called a *forest* and a connected forest is a *tree*. Historically, the first random trees to be considered were a model of genealogical (family) trees. In this thesis we focus on one of these, namely Galton-Watson trees which were introduced in [29] by Henry W. Watson and Francis Galton.

Galton-Watson trees

Galton-Watson trees are locally finite, *rooted* trees, meaning that some vertex is designated as the root, denoted o. We imagine the tree as growing (upward) away from its root. Each vertex then has edges, called *branches*, leading to its *children*, which are its neighbours that are one step farther from the root. Their number is distributed according to the *offspring distribution* $\{p_j\}_{j\in\mathbb{N}_0} \in [0,1]^{\mathbb{N}_0}$ with $\sum_{j\in\mathbb{N}_0} p_j = 1$. To fix notation, we consider a Galton-Watson (family) tree $\mathbb{T}$ as the probability space $(\mathbb{T}, \mathcal{F}, G^*)$ of random tree graphs $\mathbf{T} \in \mathbb{T}$. $\mathbf{T}$ is sometimes called a realisation of $\mathbb{T}$. We do allow for the possibility of *leaves*, so there might be vertices without children, i.e., $p_0 > 0$. For $\mathbf{T} \in \mathbb{T}$ and a vertex $x \in \mathbf{T}$, the number of children of x is denoted by $Z(x)$. Here and throughout this thesis, we write $x \in \mathbf{T}$ for a vertex x of (the vertex set $V(\mathbf{T})$ of) the tree $\mathbf{T}$ and simply $\mathbf{T}$ for the vertex set of $\mathbf{T}$. Upon viewing $\mathbb{T}$ as a Galton-Watson branching process, $Z(x)$ is an i.i.d.-copy of a random variable Z with distribution $G^*[Z = j] = p_j$ for every $j \in \mathbb{N}_0$. Two examples of Galton-Watson trees with the offspring distribution $p_0 = 0.25$, $p_1 = 0.25$, $p_2 = 0.25$, and $p_3 = 0.25$ are illustrated in Figure 1.2.



(a) (b)

Figure 1.2: Two Galton-Watson trees with $p_0 = 0.25$, $p_1 = 0.25$, $p_2 = 0.25$, $p_3 = 0.25$ in (a) and (b) both, depicted up to generation 3.

For $o \neq x \in \mathbf{T}$, its neighbour closer to the root is called *ancestor* of x. There is no ancestor of the root o. The *ancestors* of the vertex $o \neq x \in \mathbf{T}$ are the vertices of the shortest path connecting x to the root without the vertex x itself, so the ancestor of x and then its ancestor and so on till the root. The *descendants* of the vertex $x \in \mathbf{T}$ are its children and then their children and so on, so every vertex is a descendant of the root. Furthermore, for any $x, y \in \mathbf{T}$ with $x \neq y$ there is a unique neighbour of x in the shortest path joining x and y, the last vertex before x. Such a path always exists since trees are connected and is unique since there are no cycles in trees.

For $n \in \mathbb{N}_0$, *generation* n, denoted by g_n, of the tree $\mathbf{T}$ are those vertices which are at graph distance exactly n to the root o, i.e., $g_n \coloneqq \{x \in \mathbf{T} : \ \mathrm{dist}(x, o) = n\}$. We denote the subtree up to generation n of the tree $\mathbf{T}$ by

$$\mathbf{T}_{0n} \coloneqq \{x \in \mathbf{T} : \ \mathrm{dist}(x, o) \leqslant n\} \tag{1.14}$$

and write $\mathbf{T}_{0n} \subseteq \mathbf{T}$ to clarify that it is a subtree. We consider a vertex $v \in \mathbf{T}$ of generation n, $n \in \mathbb{N}_0$, as part of the *backbone* (of the tree $\mathbf{T}$) if v possesses a descendent in every generation m for $m > n$, and as part of the *finite subtrees* otherwise. Throughout we assume $\mathbb{T}$ to be *supercritical*, i.e.,

$$\lambda \coloneqq \sum_{j \in \mathbb{N}_0} j p_j > 1. \tag{1.15}$$

Furthermore, we write

$$G \coloneqq G^*[\,\cdot \mid \#\mathbf{T} = \infty] \tag{1.16}$$

for the conditional probability measure conditioned on non-extinction. The event of extinction is given by $\{\exists n \in \mathbb{N} : \ \#g_n = 0\}$ and its probability, denoted by r, is the smallest root of $s = \sum_{j \in \mathbb{N}_0} p_j s^j$ with $s \in [0, 1]$. Then, we recall that $0 < G^*[\exists n \in \mathbb{N} : \ \#g_n = 0] < 1$ for any supercritical offspring distribution with $p_0 > 0$ by [45, Prop. 5.4]. Further results about the offspring distributions of Galton-Watson trees after conditioning on non-extinction can be found in [1].

Next, we extend the definition of the anchored expansion constant from (deterministic) graphs, in (1.5), to Galton-Watson family trees.

Definition 1.1. *For a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction, we define the anchored expansion constant by*

$$\mathbf{i}(\mathbb{T}) \coloneqq G\text{-}\operatorname*{essinf}_{\mathbf{T} \in \mathbb{T}} \lim_{n \to \infty} \inf \Big\{ \frac{\#\partial K}{\#K} : \ o \in K \subset \mathbf{T} \ \textit{connected}, n \leqslant \#K < \infty \Big\}. \tag{1.17}$$

Chen and Peres [14, Cor. 1.3], see also [45, Thm. 6.52], proved strict positivity of the limit inside the G-essinf in (1.17) for G-a.e. $\mathbf{T} \in \mathbb{T}$ without any further assumptions on the offspring distribution besides being supercritical. Assuming in addition that $\mathbb{T}$ has leaves, i.e. $p_0 > 0$, this can be strengthened.

Theorem 1.2. *For a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction, with an offspring distribution satisfying $p_0 > 0$ we have*

$$\mathbf{i}(\mathbb{T}) > 0. \tag{1.18}$$

Proof. We build upon the proof of [45, Thm. 6.52] for $p_0 > 0$. It is shown there that, given any $h > 0$ sufficiently small, the probability of the events

$$A(h,n) := \Big\{ \mathbf{T} \in \mathbb{T} : \exists\, K \subset \mathbf{T} \text{ connected with } o \in K, \#K = n, \#\partial K \leqslant hn \Big\} \tag{1.19}$$

decays exponentially

$$G^*[A(h,n)] \leqslant \exp(-c_h n) \tag{1.20}$$

for $n \in \mathbb{N}$, where $c_h > 0$ is a constant depending on h (but not on n). Hence, we have $\sum_{n\in\mathbb{N}} G[A(h,n)] \leqslant \frac{1}{1-r} \sum_{n\in\mathbb{N}} G^*[A(h,n)] < \infty$, and the Borel-Cantelli lemma implies that the event $A(h) := \limsup_{n\to\infty} A(h,n)$ is a G-null set. We conclude that

$$\mathbf{i}(\mathbb{T}) \geqslant \inf_{\mathbf{T}\in\mathbb{T}\setminus A(h)} \lim_{n\to\infty} \inf \Big\{ \frac{\#\partial K}{\#K} : \; o \in K \subset \mathbf{T} \in \mathbb{T} \text{ connected}, n \leqslant \#K < \infty \Big\} \geqslant h > 0. \tag{1.21}$$

□

Galton-Watson trees and their spectral properties are extensively studied as one of the classic examples of trees. We refer to [11, 12, 40] for results on their spectral measure, its continuous part, and the adjacency matrix.

Limits of Erdős-Rényi random graphs

In this subsection we will see that Galton-Watson trees also arise naturally as the local weak limit of sparse Erdős-Rényi random graphs $\mathrm{ER}(n, \frac{\lambda}{n})$, $\lambda > 0$. To this end, we cite Thm. 2.11 of [35].

Theorem 1.3. *$ER(n, \frac{\lambda}{n})$, $\lambda > 0$, converges in probability in the local weak sense to a Galton-Watson tree $\mathbb{T}$ with Poisson offspring distribution with parameter λ, i.e., for every $\mathbf{T} \in \mathbb{T}$ and every $r \in \mathbb{N}$*

$$\frac{1}{n} \sum_{u\in[n]} \mathbf{1}_{\big\{ B_r^{(ER(n,\frac{\lambda}{n}))}(u) \simeq \mathbf{T} \big\}} \xrightarrow{\mathbb{P}} \prod_{x\in\mathbf{T}_{0r-1}} e^{-\lambda} \frac{\lambda^{Z(x)}}{Z(x)!}, \qquad \text{as } n \to \infty, \tag{1.22}$$

where $Z(x)$ is the (deterministic) number of children of $x \in \mathbf{T}$, $B_r^{(ER(n,\frac{\lambda}{n}))}(u) := \{x \in [n] : \operatorname{dist}(u,x) \leqslant r\}$ for $u \in [n]$, and $B_r^{(ER(n,\frac{\lambda}{n}))}(u) \simeq \mathbf{T}$ if and only if $B_r^{(ER(n,\frac{\lambda}{n}))}(u)$ can be viewed as subgraph of $\mathbf{T}$ with $u = o$.

The convergence in distribution follows. Therefore, sparse supercritical Erdős-Rényi random graphs $\mathrm{ER}(n, \frac{\lambda}{n})$, $\lambda > 1$, converge locally weakly in distribution and in probability to Galton-Watson trees with a Poisson offspring distribution with parameter λ, which are in turn supercritical according to (1.15), since $\sum_{j \in \mathbb{N}_0} j e^{-\lambda} \frac{\lambda^j}{j!} = \lambda > 1$. We note that $p_0 > 0$ for the Poisson offspring distribution with parameter λ.

Furthermore, by [11, Prop. 1.14] the empirical distributions of the eigenvalues of the adjacency matrices A_n of $\mathrm{ER}(n, \frac{\lambda}{n})$, $\lambda > 0$, converge weakly to the expected spectral measure at the root of the Galton-Watson tree $\mathbb{T}$ with Poisson offspring distribution with parameter λ, as $n \to \infty$. For a more detailed analysis, we refer to [11].

1.2 Random Walks in Random Environments

Random walks in random environments are considered either in the *quenched regime* or the *annealed regime*. A quenched result for such a random walk is one that holds almost surely with respect to the choice of the environment. Whereas an annealed result is concerned with the random walk yielding this result in the expectation over the environments. We note that in general neither one implies the other, since the quenched result may depend on the environment.

1.2.1 Random Walks on Graphs

The (standard) discrete-time random walk $(X_t)_{t \in \mathbb{N}_0}$ on the w-weighted graph $\mathbf{G} = (V, w)$, with w defined in (1.2), is is a sequence of random variables on the probability space (Ω, Σ, P). It is characterized in terms of the transition probabilities $p(x,y) := \frac{c((x,y))}{w(x)}$ for every $x, y \in V$. The random walk is adapted to the natural filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{N}_0}$, where $\mathcal{F}_t \subseteq \Sigma$ is the σ-algebra generated by the random variables $X_0, X_1, ..., X_t$ up to time t. Moreover, the stopping times are given as the random variables $\tau \colon \Omega \to \mathbb{N}_0 \cup \{\infty\}$ such that $\{\tau = t\} \in \mathcal{F}_t$ for every $t \in \mathbb{N}_0$. By [45, Sect. 2.1] this Markov chain $(X_t)_{t \in \mathbb{N}_0}$ satisfies the *strong Markov property*, i.e., for every finite stopping time τ and for every $x \in V$ we know that

$$\mathcal{D}_x[(X_{\tau+t})_{t \in \mathbb{N}_0} | \mathcal{F}_\tau] = \mathcal{D}_{X_\tau}[(X_t)_{t \in \mathbb{N}_0}], \tag{1.23}$$

where $\mathcal{D}_y$ denotes the distribution of the Markov chain started at $X_0 = y \in V$. Further, the corresponding symmetric Markov kernel $\mathbf{P}$ on the weighted Hilbert space $l^2(V,w)$ of square summable real-valued functions on V is given by $(\mathbf{P}\psi)(x) := \sum_{y\in V} p(x,y)\psi(y)$ for $\psi \in l^2(V,w)$ and $x \in V$. Here, the w-weighted inner product on $l^2(V,w)$ is denoted by $\langle\cdot|\cdot\rangle_{V,w}$, i.e., $\langle\psi|\varphi\rangle_{V,w} := \sum_{x\in V} w(x)\psi(x)\varphi(x)$ for any $\psi,\varphi \in l^2(V,w)$. Then, the probability of the random walker to reach y from x in one step is $\langle \mathbf{1}_{\{x\}}|\mathbf{P}\mathbf{1}_{\{y\}}\rangle_V$, where $\mathbf{1}_S$ is the indicator function of the set $S \subseteq V$ and $\langle\cdot|\cdot\rangle_V$ is the (unweighted) l^2-inner product, i.e., $\langle\psi|\varphi\rangle_V := \sum_{x\in V}\psi(x)\varphi(x)$ for any $\psi,\varphi \in l^2(V,w)$.

It is well known, see e.g. [22, 45, 56], that isoperimetric inequalities on a graph imply bounds on the Markov kernel. We cite the following estimate on its operator norm.

Theorem 1.4 ([45], Thm. 6.7)**.** *We consider the standard discrete-time random walk on the connected, infinite, weighted graph $\mathbf{G} = (V,w)$ with volume Cheeger constant $\Phi_w > 0$, defined as the w-weighted version of* (1.3)*. Then, the Markov kernel $\mathbf{P}$ of the random walk on $\mathbf{G}$ fulfils*

$$\|\mathbf{P}\|_{BL(V,w)} \leqslant \sqrt{1-\Phi_w^2} \leqslant 1 - \frac{\Phi_w^2}{2}, \tag{1.24}$$

where $\|\cdot\|_{BL(V,w)}$ denotes the operator norm on the Banach space of bounded linear operators on $l^2(V,w)$.

Random walks on graphs, in general, and their return probability $P_x[X_{2t} = x]$ to the vertex x, in particular, have been studied in countless works, see e.g. [2, 10, 19, 6, 44]. However, none of these is sufficient to imply a sharp result on the annealed average of the return probability $P_o[X_{2t} = o]$ of the simple random walk $\{X_t\}_{t\in\mathbb{N}_0}$, starting at the root o, on a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction, which will be discuss in the next section.

1.2.2 Random Walks on Trees

Since every realisation $\mathbf{T}$ of the family tree $\mathbb{T}$ is an undirected, irreflexive, locally finite graph, the previous Section 1.23 also defines the random walk on a weighted tree $(\mathbf{T},w)$. Next, we focus on the simple random walk on the (deterministic) tree $\mathbf{T}$. First, we choose edge weights c_{SRW} equal 1 for all existing edges of $\mathbf{T}$ and 0 otherwise. Then, the vertex weight $w_{SRW}(x)$, obtained in this manner, is equal to the vertex degree of $x \in \mathbf{T}$ according to (1.2) and we get the symmetric weight function w_{SRW}. Considering the random walk on this weighted tree, and thereby weighted graph, $(\mathbf{T}, w_{SRW})$ leads precisely to the transition probabilities of the simple random walk on $\mathbf{T}$. In this thesis, if not explicitly stated otherwise, we will always consider the random walk started at the root o and, thus, write simply P for P_o.

We start by presenting the following simple example.

Lemma 1.5. *Let $\mathbb{T} = \mathbf{T} = \mathbb{N}_0$ be a Galton-Watson tree with offspring distribution $p_1 = 1$, depicted in Figure 1.3. Then, for $t \in \mathbb{N}$ and $k \in \mathbb{N}_0$, with $k \leqslant t$, the number of paths leading from the root to generation $2t - 2k$ after $2t$ steps is given by*

$$\binom{2t}{k} - \binom{2t}{k-1}, \tag{1.25}$$

where we set $\binom{2t}{-1} := 0$ for all $t \in \mathbb{N}$. Here, each generation consists only of one vertex. In particular, for $t \in \mathbb{N}$ the number of paths leading back to the root is

$$\binom{2t}{t} - \binom{2t}{t-1}. \tag{1.26}$$

The proof of this lemma is deferred to the appendix.

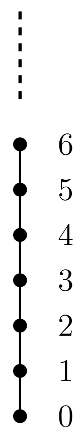


Figure 1.3: The Galton-Watson tree $\mathbb{N}_0$.

Now, we notice that after dividing by the total number of paths after $2t$, $t \in \mathbb{N}$, steps this corresponds to the probability of the simple random walk $(X_t)_{t\in\mathbb{N}}$, starting at the root 0, on this specific Galton-Watson tree $\mathbb{N}_0$ to end up in generation g_{2t-2k} for $k \in \mathbb{N}_0$ with $k \leqslant t$. In especially, since $\sum_{k=0}^{t}\left[\binom{2t}{k} - \binom{2t}{k-1}\right] = \binom{2t}{t}$, we have for every $t \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ with $k \leqslant t$ that

$$P[X_{2t} \in g_{2t-2k}] = \frac{\binom{2t}{k} - \binom{2t}{k-1}}{\binom{2t}{t}}. \tag{1.27}$$

In particular, for $k = t$ we obtain the return probability $P[X_{2t} = 0] = \frac{1}{t+1}$. But, as can be seen from the proof of Lemma 1.5, even for this simplest example of a Galton-Watson tree the direct computation of the return probability of the simple random walk becomes quite complicated. This calls for more sophisticated approaches.

They are given in numerous works, see e.g. [5, 15, 31, 36, 44]. Furthermore, also random walks on trees specifically, compared to graphs more generally, have been investigated thoroughly. But despite these efforts, there still exists no comprehensive, sharp result on the annealed average of the return probability $P[X_{2t} = o]$ of the simple random walk $(X_t)_{t\in\mathbb{N}_0}$, starting at the root o, on a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction. Again, this is not only a natural question, but also links to other interesting quantities like spectral properties of the random walk's generator. Now, we will, first, give the qualitative reasons for this and, then, summarise the best known results.

The above results together with those by Piau and Virág in [48, 52] indicate the behaviour of $e^{-ct^{\frac{1}{3}}}$ for the annealed average of the return probability after t steps, $c > 0$ constant. However, to show the underlying upper bound all of these results require either the absence of leaves, so $p_0 = 0$, or a strong (volume) isoperimetric inequality, as in (1.3). Unfortunately, neither one holds in the case of a supercritical Galton-Watson tree with general offspring distribution. Without these two assumptions there is no upper bound on the number of leaves close to the root with the resulting push-backs of the simple random walk, in turn, increasing its return probability uncontrollably, compare with Figure 1.4.

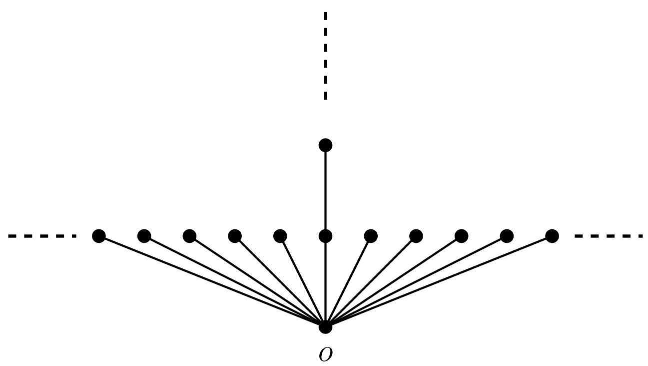


Figure 1.4: A Galton-Watson tree with multiple leaves close to the root o.

We denote the annealed return probability of the simple random walk, starting at the root o, by

$$R_t := GP[X_{2t} = o] := \int_{\mathbb{T}} P_o^{\mathbf{T}}[X_{2t} = o] \,\mathrm{d}G(\mathbf{T}), \qquad t \in \mathbb{N}_0, \tag{1.28}$$

and summarise the known results depending on the offspring distribution $\{p_j\}_{j\in\mathbb{N}_0}$

of the supercritical Galton-Watson family tree in Figure 1.5. Here, $c, c' > 0$ are constants (independent of t), which may differ from line to line, and the bounds hold for all $t \in \mathbb{N}_0$.

(a)	$p_0 = p_1 = 0$:	$\exp(-c't) \leqslant$	R_t	$\leqslant \exp(-ct)$
(b)	$p_0 > 0 \vee p_1 > 0$:	$\exp(-c't^{\frac{1}{3}}) \leqslant$	R_t	
(c)	$p_0 = 0$:		R_t	$\leqslant \exp(-ct^{\frac{1}{3}})$
(d)	$\{p_j\}_{j\in\mathbb{N}_0}$ finitely supported:		R_t	$\leqslant \exp(-ct^{\frac{1}{5}})$
(e)	general $\{p_j\}_{j\in\mathbb{N}_0}$:		R_t	$\leqslant \exp(-ct^{\frac{1}{6}})$.

Figure 1.5: Bounds on the annealed return probability.

Remark. *1. The statements (a) – (e) follow from [48, Thm. 2], who proves corresponding results for the tail of the annealed distribution* $GP[\tau_R \geqslant t]$ *of the first regeneration time* τ_R. *More precisely, concerning the upper bounds, this is a direct consequence of* $\{X_t = o\} \subset \{\tau_R \geqslant t\}$. *The lower bounds on* R_t *follow from analogous ideas as used for the lower bounds on* $GP[\tau_R \geqslant t]$ *in [48].*

2. *We refrained from introducing multiplicative constants in front of the exponentials because they can be absorbed in the constants* c *and* c' *in the exponent. This is always possible because* $R_t < 1$ *for every* $t \in \mathbb{N}$, *which follows, e.g., from the random walk having positive speed [45, p. 569] or being transient [18], see also [33, Lemma 2] which was announced in [32].*

3. *Case (a) differs from the other cases and is quite well understood: The exponential decay in time results from the random walk getting lost in a tree where the number of vertices at least doubles in each generation and where there are no deterministic push-backs due to the absence of leaves. Using the inclusion of events*
$$\{X_{2t} = 0\} \subseteq \Big\{\frac{\mathrm{dist}(o, X_{2t})}{2t} \in [0, \varepsilon[\,\Big\} \tag{1.29}$$
for any $\varepsilon > 0$, *the bound* $c' \geqslant \ln\frac{9}{8}$ *for the constant in the lower bound of (a) follows from large-deviation estimates of the speed in [21, Thm. 1.2].*

4. *The stretched-exponential behaviour* $\exp(-c't^{\frac{1}{3}})$ *in the lower bound of case (b) is believed to capture the exact long-time asymptotics of the annealed return probability in the entire parameter regime of (b), which is complementary to (a). Unfortunately, corresponding upper bounds are not known in such generality but only in the absence of leaves as specified in (c). In that case, the*

annealed return probability can be bound from above by the return probability on the deterministic tree with $p_1 = 1$*, which is a sufficient simplification to yield the exponential decay in* $t^{\frac{1}{3}}$*. However, in the case* $p_0 > 0$ *this is not possible and the best upper bound valid for all offspring distributions allowed in (b) - and also in (a) - exhibits the slower decay as* $\exp(-ct^{\frac{1}{6}})$*.*

5. *Virág [52, Ex. 6.2] proved a quenched stretched-exponential upper bound for the return probability with exponent* $\frac{1}{3}$*. It is valid for finitely supported offspring distributions and times larger than some initial time. However, the initial time depends on the realisation* $\mathbf{T} \in \mathbb{T}$ *and, thus, the result does not translate to the annealed regime.*

1.3 Main Results

In this thesis we investigate the simple random walk in the annealed regime on supercritical Galton-Watson trees conditioned on non-extinction. Since the case $p_0 = 0$ is already covered in [48, Thm. 2], cf. Remark 4, we will focus on fast-decaying offspring distributions with

$$p_0 > 0 \quad \text{and} \quad p_j \leqslant c_1 \exp(-c_2 j^k) \qquad \text{for every } j \in \mathbb{N}_0, \tag{1.30}$$

where $c_1, c_2 > 0$ and $k > 2$ are constants (all independent of j).

In the second chapter we analyse the time τ'_t the simple random walk spends on the backbone out of a total of t steps compared to the time spent in the finite subtrees. Expanding on the approach by Piau in [48] and using the fast decay of the offspring distribution, cf. (1.30), a cut-off argument allows to exclude unfavourable trees at the cost of a negligible error term and we obtain the following result.

Theorem 1.6. *Let* $\vartheta, \varepsilon > 0$ *with* $2\vartheta + \varepsilon \leqslant 1$*. Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying* (1.30) *for some constants* $c_1, c_2 > 0$ *and* $k > 2$*. Further, assume that* $\varepsilon \geqslant \frac{2}{k}$*. Then, there exist constants* $C, c > 0$ *such that for all* $t \in \mathbb{N}$

$$GP[\tau'_t \leqslant t^{1-\vartheta-\varepsilon}] \leqslant C \exp(-ct^{\vartheta}). \tag{1.31}$$

This is relevant when reducing the simple random walk on Galton-Watson trees to its technically simpler version on the backbone, which is done in Corollary 2.13 for the annealed return probability. Thereby, it solves the difficulties posed by the leaves and enables us to instead deal with a modified version of the random walk on trees with $p_0 = 0$.

In the third chapter we study the annealed return probability R_t at time t of the simple random walk, defined in (1.28). We start by giving an explicit proof for its proclaimed lower bound.

Theorem 1.7. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying* $p_1 > 0$. *Then, there is a constant* $c > 0$ *such that for all* $t \in \mathbb{N}_0$ *we have*

$$R_t \geqslant \exp(-ct^{\frac{1}{3}}). \tag{1.32}$$

Next, we turn to the upper bounds. We expand on the strategy of Virág in [52], with an approximation argument over the realisations of the Galton-Watson family tree, to conclude the following for the annealed regime.

Theorem 1.8. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution of bounded support. Then, there exists a constant* $c > 0$ *such that for all* $t \in \mathbb{N}_0$ *we have*

$$R_t \leqslant \exp(-ct^{\frac{1}{3}}). \tag{1.33}$$

Moreover, we extend Theorem 1.8 to general offspring distributions with a super-Gaussian decay at the expense of weakening the decay exponent.

Theorem 1.9. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying* (1.30) *for some constants* $c_1, c_2 > 0$ *and* $k > 8$. *Then, there exists a constant* $c > 0$, *independent of* k, *such that for all* $t \in \mathbb{N}_0$ *we obtain*

$$R_t \leqslant \exp(-ct^{\frac{1}{3}-\frac{8}{3k}}). \tag{1.34}$$

For even faster decaying offspring distributions, the annealed return probability decays almost as fast as $\exp(-ct^{\frac{1}{3}})$.

Corollary 1.10. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution decaying according to*

$$p_j \leqslant \exp\big(-\xi(j)\big) \qquad \textit{for every } j \in \mathbb{N}_0, \tag{1.35}$$

where $\xi : \mathbb{N}_0 \to]0, \infty[$ *(independent of* j*) grows faster than any polynomial. Then, for every* $\varepsilon > 0$, *the annealed return probability at time* t *decays at least exponentially* *in* $t^{\frac{1}{3}-\varepsilon}$*, i.e., there is a constant* $c > 0$ *such that for all* $t \in \mathbb{N}_0$ *we have*

$$R_t \leqslant \exp(-ct^{\frac{1}{3}-\varepsilon}). \tag{1.36}$$

Proof. Fix $\varepsilon > 0$. Then, there is $k > 8$ such that $\varepsilon \geqslant \frac{8}{3k}$. Now, we choose $c_1, c_2 > 0$ such that $p_j \leqslant c_1 \exp(-c_2 j^k)$ for every $j \in \mathbb{N}_0$. This is possible, since ξ grows faster than any polynomial. Then, we can apply Theorem 1.9 and obtain for all $t \in \mathbb{N}_0$

$$R_t \leqslant \exp(-ct^{\frac{1}{3}-\frac{8}{3k}}) \leqslant \exp(-ct^{\frac{1}{3}-\varepsilon}), \tag{1.37}$$

with a constant $c > 0$. □

Whereas Theorem 1.8 improves the exponent $\frac{1}{5}$ of (d) in Figure 1.5, displaying the best previous results, to the optimal value $\frac{1}{3}$, Theorem 1.9 yields an improvement to the exponent $\frac{1}{6}$ of (e) in Figure 1.5 for $k > 16$. In the case of Corollary 1.10 we get arbitrarily close to the optimal exponent.

1.4 Outlook on Lifshitz Tails for Spectra of Erdős-Rényi Random Graphs in the Supercritical Regime

For a future endeavour it comes to mind to find the counterpart to [41], so proving the Lifshitz tail behaviour of the integrated density of states at the lower spectral edge $E = 0$ for the Erdős-Rényi random graph Laplacian in the supercritical case.

In the subsection "Limits of Erdős-Rényi random graphs" of 1.1.2, we have already seen that supercritical Erdős-Rényi random graphs $\mathrm{ER}(n, \frac{\lambda}{n})$, $\lambda > 1$, converge in probability in the local weak sense to a Galton-Watson tree $\mathbb{T}$ with Poisson offspring distribution with parameter λ. Furthermore, by [11, Prop. 1.14] the empirical distributions of the eigenvalues of the adjacency matrices A_n of $\mathrm{ER}(n, \frac{\lambda}{n})$, $\lambda > 1$, converge weakly to the expected spectral measure at the root of this Galton-Watson family tree $\mathbb{T}$.

Therefore, it seems fruitful to analyse the annealed return probability of the simple random walk, starting at the root, of such a Galton-Watson family tree with Poisson offspring distribution with parameter $\lambda > 1$. However, the results of this thesis, presented in the Section 1.3, do not apply to this scenario. Despite being supercritical according to (1.15), the Poisson offspring distribution with parameter $\lambda > 1$ is neither bounded nor of super-Gaussian decay nor does it satisfy $p_0 = 0$.

Hence, in this case further work is needed to show a matching upper bound for the annealed return probability $R_t \leqslant \exp(-ct^{\frac{1}{3}})$ for every $t \in \mathbb{N}$ with some (t-independent) constant $c > 0$.

Chapter 2

Time Spent on the Backbone

For convenience we start by recalling the main result of this chapter from the Section 1.3.

Theorem 2.1. *Let $\vartheta, \varepsilon > 0$ with $2\vartheta+\varepsilon \leqslant 1$. Consider a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction, with offspring distribution $\{p_j\}_{j\in\mathbb{N}_0}$ satisfying (1.30) for some constants $c_1, c_2 > 0$ and $k > 2$. Further, assume that $\varepsilon \geqslant \frac{2}{k}$. Then, the annealed probability of the event that at most $t^{1-\vartheta-\varepsilon}$ steps, out of a total of t steps, of the simple random walk are on the backbone is exponentially small in t^{ϑ}, i.e., there exist (t-independent) constants $C, c > 0$ such that for all $t \in \mathbb{N}$*

$$GP[\tau'_t \leqslant t^{1-\vartheta-\varepsilon}] \leqslant C \exp(-ct^{\vartheta}). \tag{2.1}$$

Here, $\tau'_t \in \mathbb{N}_0$ denotes the time (or number of steps) the simple random walk spends on the backbone out of a total of t steps.

Remark. 1. *Like mentioned in Remark 2 in Subsection 1.2.2, also here we can refrain from the multiplicative constant C in front of the exponential function in (2.1), because it can be absorbed in the constant c in the exponent. This is possible because $1 > \vartheta + \varepsilon > 0$ by the initial assumption and, thus, $GP[\tau'_t \leqslant t^{1-\vartheta-\varepsilon}] \leqslant GP[\tau'_t \leqslant t-1] < 1$ for every $t \in \mathbb{N}\setminus\{1\}$ large enough, depending on ϑ and ε. Here, for $t \in \mathbb{N}\setminus\{1\}$ the inequality $GP[\tau'_t \leqslant t-1] < 1$ holds, since we conditioned on non-extinction.*

Therefore, under the same assumptions as in Theorem 2.1 we conclude that there exists a constant $c > 0$ and an initial time $t_1 \in \mathbb{N}\setminus\{1\}$ such that for all $t \geqslant t_1$

$$GP[\tau'_t \leqslant t^{1-\vartheta-\varepsilon}] \leqslant \exp(-ct^{\vartheta}). \tag{2.2}$$

2. *Theorem 2.1 indicates that the time which the simple random walk stays on*

the backbone as well as the time which the simple random walk stays in the finite subtrees out of t steps total is in the annealed average of order t each, up to logarithmic corrections.

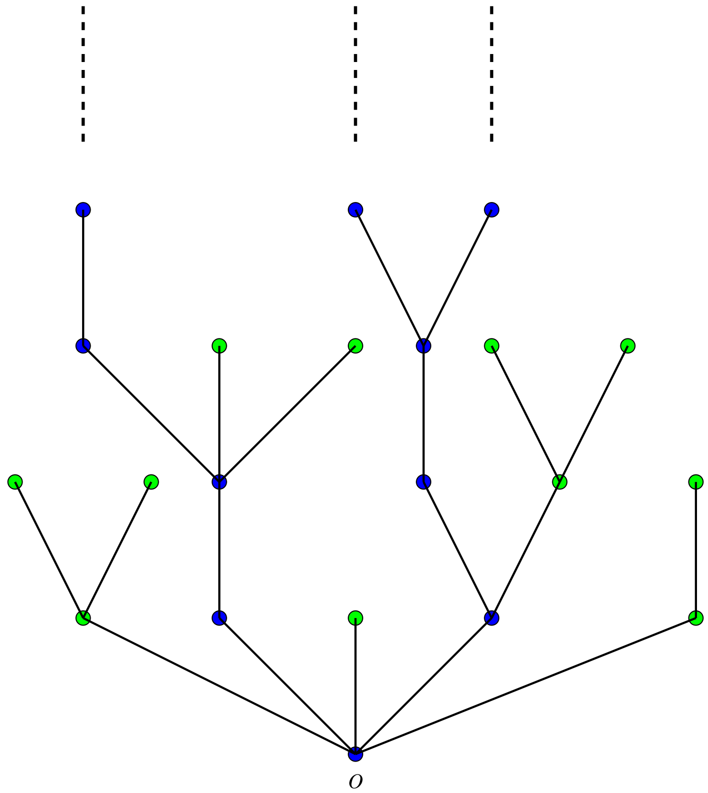


Figure 2.1: A Galton-Watson tree with its finite subtrees highlighted in green and its backbone highlighted in blue.

2.1 Proof of Theorem 2.1

The general ideas of the proof are along the lines of [48].

Let $\mathbf{T} \in \mathbb{T}$. For $x \in \mathbf{T}$, let $Z'(x)$ denote the number of children of x which are part of the backbone and let $Z''(x)$ denote the number of children of x which are part of the finite subtrees. Then, the number of children of x equals the sum of those on the backbone and those in the finite subtrees, i.e., $Z(x) = Z'(x) + Z''(x)$. Furthermore, let $\mathbf{T}' \subseteq \mathbf{T}$ denote the connected subgraph containing all the vertices of the backbone and let $\mathbf{T}'' \subseteq \mathbf{T}$ denote the, in general, not connected subgraph containing all the vertices of the finite subtrees. This at hand, we recall for the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ on $\mathbf{T}$

$$\tau'_t = \#\{n \in \mathbb{N} : \; X_n \in \mathbf{T}', n \leqslant t\} \tag{2.3}$$

with $t \in \mathbb{N}$. We also recall the definition of the subtree up to generation t, $t \in \mathbb{N}$, in (1.14) and extend it to subtrees up to height m, for $m > 0$, which we denote by

$$\mathbf{T}_{0m} \coloneqq \{x \in \mathbf{T} : \ \mathrm{dist}(x, o) \leqslant m\}. \tag{2.4}$$

First, we introduce classes of "bad" trees as subsets of $\mathbb{T}$, which we want to exclude and the G-probability of which is exponentially small in t^{ϑ}, $t \in \mathbb{N}$. Their negligence will result only in an in t^{ϑ} exponentially small error term in the annealed regime.

Definition 2.2. *Let $t \in \mathbb{N}$ and let $m > 0$. Further, let $c_4 > \ln \lambda$, where λ is defined in (1.15), $c_3 > 3 + \frac{c_4}{c_2}$, where c_2 is specified in (1.30), and $c_5 > \frac{c_4 2(2k)^{k-1}}{c_2} + 1$ be constants. We define the events*

$$\begin{aligned}
A_m &\coloneqq \{\#\mathbf{T}_{0m} \geqslant \exp(c_4 m)\},\\
B_t &\coloneqq \{\exists x \in \mathbf{T}_{0t} \cap \mathbf{T}'' : \ c_3 t^{\frac{1}{k}} \leqslant Z''(x)\} \setminus A_t,\\
C_m = C_{m,t} &\coloneqq \{\exists x \in \mathbf{T}_{0m} \cap \mathbf{T}' : \ c_5 \frac{t^{\frac{1}{k}}}{\ln t} Z'(x) \leqslant Z(x)\} \setminus A_m,\\
D_t &\coloneqq A_t \cup B_t \cup A_{t^{1-\frac{2}{k}}} \cup C_{t^{1-\frac{2}{k}}}.
\end{aligned} \tag{2.5}$$

Lemma 2.3. *Let $\vartheta > 0$ be as in the assumption of Theorem 2.1. Then there are constants $C_1', c_1' > 0$ such that for every $t \in \mathbb{N}$ with $t \geqslant \exp(4k^2)$ we have*

$$G[D_t] \leqslant C_1' \exp(-c_1' t^{\vartheta}). \tag{2.6}$$

For the proof of this lemma, we need some auxiliary results. We start with the following version of Chebychev's inequality.

Lemma 2.4. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\mathcal{E} \in \Sigma$ an event. Let $Y : \Omega \to [0, \infty]$ be measurable and $\varrho > 0$ a parameter. Then,*

$$\mathbb{P}[\{Y \geqslant \varrho\} \cap \mathcal{E}] \leqslant \exp(-\varrho) \int_{\mathcal{E}} \exp(Y) \,\mathrm{d}\mathbb{P}. \tag{2.7}$$

Proof. Denoting by $\mathbf{1}_M$ the indicator function of an event M, the claim follows from

$$\begin{aligned}\exp(\varrho)\mathbb{P}[\{Y \geqslant \varrho\} \cap \mathcal{E}] &= \exp(\varrho)\mathbb{P}[\{\exp(Y) \geqslant \exp(\varrho)\} \cap \mathcal{E}] \\ &= \int_{\mathcal{E}} \exp(\varrho)\mathbf{1}_{\{\exp(Y)\geqslant\exp(\varrho)\}}\, \mathrm{d}\mathbb{P} \\ &\leqslant \int_{\mathcal{E}} \exp(Y)\mathbf{1}_{\{\exp(Y)\geqslant\exp(\varrho)\}}\, \mathrm{d}\mathbb{P} \\ &\leqslant \int_{\mathcal{E}} \exp(Y)\, \mathrm{d}\mathbb{P}. \end{aligned} \tag{2.8}$$

□

Moreover, we will use the following three estimates on the real natural logarithm ln.

Lemma 2.5. *Let $k > 2$ be constant and $n \in \mathbb{N}$ with $n \geqslant \exp(4k^2)$. Then, $\frac{\ln n}{k} - \ln\ln n \geqslant \frac{\ln n}{2k}$.*

Proof. We notice that $0 \leqslant \frac{\ln n}{2k} - \ln\ln n = \ln \frac{n^{\frac{1}{2k}}}{\ln n}$ is equivalent to $2k \leqslant \frac{n^{\frac{1}{2k}}}{\ln n^{\frac{1}{2k}}}$. This follows from the monotonicity of the exponential function and logarithm, respectively. Since $\frac{x}{\ln x}$ is monotone increasing for every $x > e$, we get $\frac{n^{\frac{1}{2k}}}{\ln n^{\frac{1}{2k}}} \geqslant \frac{\exp(4k^2)^{\frac{1}{2k}}}{\ln(\exp(4k^2)^{\frac{1}{2k}})} = \frac{\exp(2k)}{2k} \geqslant 2k$. □

Lemma 2.6. *Let $k > 2$ be constant and $n \in \mathbb{N}$ with $n \geqslant \exp(4k^3)$. Then, $\frac{n^{\frac{1}{k}}}{\ln n} \geqslant \exp(2k^2) + 5$.*

Proof. Since $\frac{x}{k \ln x}$ is monotone increasing for $x > e$, we obtain $\frac{n^{\frac{1}{k}}}{\ln n} \geqslant \frac{\exp(4k^2)}{4k^3}$. But now, since $\exp(x) \geqslant x^2$ for every $x > 0$, we see that $\frac{\exp(4k^2)}{4k^3} \geqslant k\exp(2k^2)$. Finally, since $k > 2$, we have $k\exp(2k^2) \geqslant \exp(2k^2) + 5$. This yields the claim. □

Lemma 2.7. *Let $k > 2$ be constant and $x \in \mathbb{R}$ with $x \geqslant \exp(2k^2)$. Then, $x \geqslant (\ln x)^{k-1}$.*

Proof. Since $\frac{x}{(\ln x)^{k-1}}$ is monotone increasing for $x > \exp(k-1)$, we get $\frac{x}{(\ln x)^{k-1}} \geqslant \frac{\exp(2k^2)}{(2k^2)^{k-1}} = \frac{\exp(2k^2)}{\exp((k-1)(2\ln k + \ln 2))} \geqslant \frac{\exp(2k^2)}{\exp(3k \ln k)}$. Since $k > 2$ and thus $2k \geqslant 3\ln k$, we conclude that $\frac{x}{(\ln x)^{k-1}} \geqslant 1$. □

Proof of Lemma 2.3. Let $t \in \mathbb{N}$ with $t \geqslant \exp(4k^3)$ be fixed. Then, $t^{1-\frac{2}{k}} \geqslant 1$ because of the initial assumption $k > 2$.

Let $m \geqslant 1$. In the following arguments, $\lfloor x \rfloor$ denotes the largest integer not exceeding $x \in \mathbb{R}$. As to the decay of the probability of A_m, we notice that

$$\mathbb{G}^*[\#\mathbf{T}_{0m}] = \sum_{j=0}^{\lfloor m \rfloor} \lambda^j \leqslant \frac{\lambda^{m+1} - 1}{\lambda - 1}, \tag{2.9}$$

where $\mathbb{G}^*$ denotes the expectation under the probability measure G^* and we applied [45, Prop. 5.5] for the equality. We obtain from the definition of A_m, Chebyshev's inequality (in the form of Lemma 2.4) and (2.9)

$$G[A_m] \leqslant \frac{1}{1-r} G^*[A_m] \leqslant \frac{1}{1-r} \frac{\lambda^{m+1}-1}{\lambda - 1} \exp(-c_4 m), \tag{2.10}$$

where r denotes the extinction probability. Due to $c_4 > \ln \lambda$, the right-hand side of (2.10) decays exponentially in m.

Next, we turn to the probability of B_t. For every $\mathbf{T} \in \mathbb{T}$ and every $x \in \mathbf{T}$ we have $Z''(x) \leqslant Z(x)$ and, by the definition of B_t, we are in the complement of the event A_t. Therefore, we obtain

$$G[B_t] \leqslant \frac{1}{1-r} \exp(c_4 t) \sum_{j=\lfloor c_3 t^{\frac{1}{k}} \rfloor}^{\infty} G^*[Z = j]. \tag{2.11}$$

Inserting the decay (1.30) of the offspring distribution $p_j = G^*[Z = j]$ and estimating the resulting sum by an integral, we obtain

$$G[B_t] \leqslant \frac{c_1}{c_2(1-r)} \exp(c_4 t) \exp\big(-c_2(c_3-2)^k t\big) \leqslant \frac{c_1}{c_2(1-r)} \exp(-c_2 t). \tag{2.12}$$

Here, $c_3 > 3 + \frac{c_4}{c_2}$ yields the second inequality.

Now, we are left with estimating the probability of $C_{t^{1-\frac{2}{k}}}$. Let $x \in C_{t^{1-\frac{2}{k}}}$. We notice that $x \in C_{t^{1-\frac{2}{k}}} \subseteq \mathbf{T}'$ implies $Z'(x) \geqslant 1$ and, further, yields $c_5 \frac{t^{\frac{1}{k}}}{\ln t} Z'(x) \leqslant Z(x)$. Hence, we conclude that

$$1 \leqslant \frac{Z(x)}{Z'(x)} \frac{\ln t}{c_5 t^{\frac{1}{k}}} \leqslant \frac{y \ln t}{t^{\frac{1}{k}}}, \tag{2.13}$$

with $y := \frac{Z(x)}{Z'(x)} \geqslant 1$. Here, for the second inequality we used that $c_5 > 1$. By taking (2.13) to the power $k-2$ and inserting this back into (2.13), we obtain

$$1 \leqslant \frac{(y \ln t)^{k-1}}{c_5 t^{1-\frac{1}{k}}} = \frac{\rho (y \ln t)^{k-1}}{\rho c_5 t^{1-\frac{1}{k}}}, \tag{2.14}$$

with $\rho > 0$. Furthermore, by considering the logarithm of the defining equation of $C_{t^{1-\frac{2}{k}}}$, we have

$$\ln \frac{y}{c_5} \geqslant \ln \frac{t^{\frac{1}{k}}}{\ln t} = \frac{\ln t}{k} - \ln \ln t. \tag{2.15}$$

Here, we used the monotonicity of the natural logarithm to maintain the inequality. By Lemma 2.5, the monotonicity of the logarithm, and since $c_5 > 1$, $t \geqslant \exp(4k^3) \geqslant$

$\exp(4k^2)$, $k > 2$, we also get

$$\ln y \geqslant \ln \frac{y}{c_5} \geqslant \frac{\ln t}{k} - \ln\ln t \geqslant \frac{\ln t}{2k}. \tag{2.16}$$

Inserting (2.16) into (2.14), we obtain

$$1 \leqslant \frac{\rho(2ky\ln y)^{k-1}}{\rho c_5 t^{1-\frac{1}{k}}} = \frac{\rho(2k\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)})^{k-1}}{\rho c_5 t^{1-\frac{1}{k}}}. \tag{2.17}$$

We note that this inequality holds for every $x \in C_{t^{1-\frac{2}{k}}}$, since x was arbitrary.

By Lemma 2.6 and since $t \geqslant \exp(4k^3)$, $k > 2$, $x \in C_{t^{1-\frac{2}{k}}} \subseteq \mathbf{T}'$, we know that $Z(x) \geqslant c_5 \frac{t^{\frac{1}{k}}}{\ln t} Z'(x) \geqslant \frac{t^{\frac{1}{k}}}{\ln t} \geqslant \zeta + 4 \geqslant 12$ with $\zeta := \lceil \exp(2k^2) \rceil$. Hence, we have

$$\mathbb{G}^*\Big[\exp\Big(\rho\Big(\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)}\Big)^{k-1}\Big)\Big] = \mathbb{G}^*\Big[\exp\Big(\rho\Big(\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)}\Big)^{k-1}\Big)\Big|Z(x) \geqslant \zeta + 4\Big], \tag{2.18}$$

where we consider the expectation $\mathbb{G}^*$ conditioned on the event $\{Z(x) \geqslant \zeta + 4\}$ under the probability measure G^*. We conclude that

$$\begin{aligned}
&\mathbb{G}^*\Big[\exp\Big(\rho\Big(\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)}\Big)^{k-1}\Big)\Big|Z(x) \geqslant \zeta + 4\Big] \\
&\leqslant \mathbb{G}^*[\exp(\rho(Z(x)\ln Z(x))^{k-1})|Z(x) \geqslant \zeta + 4] \\
&\leqslant \mathbb{G}^*[\exp(\rho Z(x)^k)|Z(x) \geqslant \zeta + 4] \\
&\leqslant \sum_{j=12}^{\infty} \exp(\rho j^k) G^*[Z = j].
\end{aligned} \tag{2.19}$$

Here, for the first inequality we used that $x \in C_{t^{1-\frac{2}{k}}} \subseteq \mathbf{T}'$ and, thus, $Z'(x) \geqslant 1$, for the second inequality we applied Lemma 2.7, and the third inequality follows from $\zeta \geqslant 8$.

Now, choose $\rho := \frac{c_2}{2} > 0$. Then, inserting the decay (1.30) of the offspring distribution $p_j = G^*[Z = j]$ into (2.19) and estimating the resulting sum by an integral, we obtain

$$\mathbb{G}^*\Big[\exp\Big(\frac{c_2}{2}\Big(\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)}\Big)^{k-1}\Big)\Big] \leqslant \frac{2c_1}{c_2}\exp\big(-\frac{c_2}{2}11^k\big). \tag{2.20}$$

Again, we note that this inequality holds for every $x \in C_{t^{1-\frac{2}{k}}}$, since x was arbitrary.

By the definition of $C_{t^{1-\frac{2}{k}}}$, we are in the complement of the event $A_{t^{1-\frac{2}{k}}}$ and, thus, by (2.17) we have

$$G\big[C_{t^{1-\frac{2}{k}}}\big] \leqslant \frac{1}{1-r}\exp(c_4 t^{1-\frac{2}{k}}) G^*\Bigg[\frac{\frac{c_2}{2}\big(2k\frac{Z(x)}{Z'(x)}\ln\frac{Z(x)}{Z'(x)}\big)^{k-1}}{\frac{c_2}{2}c_5 t^{1-\frac{1}{k}}} \geqslant 1\Bigg], \tag{2.21}$$

for some $x \in C_{t^{1-\frac{2}{k}}}$. We obtain from Chebyshev's inequality (in the form of Lemma 2.4) and (2.20) that

$$G\big[C_{t^{1-\frac{2}{k}}}\big] \leqslant \frac{1}{1-r}\exp(c_4 t^{1-\frac{2}{k}})\frac{2c_1}{c_2}\exp\big(-\frac{c_2}{2}11^k\big)\exp\Big(-\frac{c_2 c_5 t^{1-\frac{1}{k}}}{2(2k)^{k-1}}\Big). \tag{2.22}$$

Due to $c_5 > \frac{c_4 2(2k)^{k-1}}{c_2} + 1$ and $1-\frac{2}{k} < 1-\frac{1}{k}$, the right-hand side of (2.22) decays exponentially in $t^{1-\frac{1}{k}}$.

Since we assumed $\vartheta, \varepsilon > 0$ with $2\vartheta+\varepsilon \leqslant 1$, $\varepsilon \geqslant \frac{2}{k}$ and, thus, $\vartheta < 1-\frac{2}{k} < 1-\frac{1}{k} < 1$, combining (2.10), (2.12), and (2.22), the claim follows. □

Let $\mathbf{T} \in \mathbb{T}$ be fixed. Next, we turn to the simple random walk $(X_t)_{t\in\mathbb{N}_0}$, starting at the root o, on $\mathbf{T}$. Let $\nu_i \in \mathbb{N}_0$, $i \in \mathbb{N}$, denote the number of excursions to the finite subtrees, so $\mathbf{T}''$, $(X_t)_{t\in\mathbb{N}_0}$ takes at position i before taking a step on the backbone $\mathbf{T}'$, to reach the distinct position $i+1$. So, position 1 is always the root o and ν_1 is the number of excursions of $(X_t)_{t\in\mathbb{N}_0}$ to $\mathbf{T}''$ before taking a first step on $\mathbf{T}'$; after this first step on $\mathbf{T}'$ the walk reaches position 2 and undertakes ν_2 excursions to $\mathbf{T}''$ before taking another step on $\mathbf{T}'$, and so on. We note that, by this construction, it is indeed possible that the position 3 is again the root. Further, let $d_l \in \mathbb{N}\setminus\{1\}$, $l = 1, ..., \sum_{i\in\mathbb{N}} \nu_i$, denote the time (or, more precisely, the number of steps) of the l-th excursion, including the step into and out of $\mathbf{T}''$, respectively. We write

$$F_t \coloneqq \sum_{i=1}^{\tau'_t} \nu_i, \tag{2.23}$$

for the number of excursions up to a total of t steps. There, we have exactly τ'_t positions, since τ'_t was defined, in (2.3), as the number of steps on the backbone $\mathbf{T}'$, out of t steps total on $\mathbf{T}$, and positions are defined as the vertex $x \in \mathbf{T}'$ before the i-th step on the backbone.

Definition 2.8. *Let $\mathbf{T} \in \mathbb{T}$, $t \in \mathbb{N}$, and $c_6 > 2c_5 \ln 2$. For $\mathbf{T}$, we define the events*

$$\begin{aligned} H_t &\coloneqq \{F_t \geqslant c_6\frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t},\ \tau'_t \leqslant t^{1-\vartheta-\frac{2}{k}}\}, \\ K_t &\coloneqq \Big\{\sum_{l=1}^{F_t} d_l \geqslant c_6\frac{t}{\ln t},\ F_t \leqslant c_6\frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\Big\}. \end{aligned} \tag{2.24}$$

Lemma 2.9. *There are constants* $C_2', C_3', c_2', c_3' > 0$ *such that for every* $t \in \mathbb{N}$ *with* $t \geqslant \exp(4k^3)$ *we have*

1.

$$GP[H_t, D_t^c] \leqslant C_2' \exp(-c_2' t^{\vartheta}), \tag{2.25}$$

2.

$$GP[K_t, D_t^c] \leqslant C_3' \exp(-c_3' t^{\vartheta}). \tag{2.26}$$

Here, we write $GP[\,\cdot\,, \mathcal{E}] \coloneqq G[\mathcal{E}]\ G_{\mathcal{E}}P[\,\cdot\,]$ *for* $\mathcal{E} \subseteq \mathbb{T}$*, where* $G_{\mathcal{E}}[\,\cdot\,]$ *is the probability measure conditioned on the event* $\mathcal{E}$*.*

For the proof of this lemma, we need some more auxiliary results and notations.

For $\mathbf{T} \in \mathbb{T}$ and the indices $i = 1, ..., \tau_t'$, let $x_i' \in \mathbf{T}$ denote the vertex at position i, where we add the prime to emphasize that x_i' is on the backbone, i.e., $x_i' \in \mathbf{T}'$. For $\tau_t' \leqslant t^{1-\frac{2}{k}}$, we notice that $x_i' \in \mathbf{T}_{0t^{1-\frac{2}{k}}} \cap \mathbf{T}'$ for every $i \in [1, \tau_t']$, since the graph distance satisfies $\operatorname{dist}(o, x_i') \leqslant \tau_t' \leqslant t^{1-\frac{2}{k}}$, for every $i \in [1, \tau_t']$, by the definition of x_i' and of the i-th position respectively. Here, we recall the notation $[m, n] = \{j \in \mathbb{N} :\ m \leqslant j \leqslant n\}$ for $m, n \in \mathbb{N}$. Let $\mathbf{T} \notin D_t$ and $\tau_t' \leqslant t^{1-\frac{2}{k}}$. Then, in particular, we know $\mathbf{T} \notin C_{t^{1-\frac{2}{k}}}$ and, thus, by Definition 2.2 we have

$$\frac{Z'(x_i')}{Z(x_i')} \geqslant \frac{\ln t}{c_5 t^{\frac{1}{k}}}, \qquad \text{for every } i \in [1, \tau_t']. \tag{2.27}$$

Here, $1 - \frac{Z'(x_i')}{Z(x_i')}$ describes the probability of the simple random walk stepping into $\mathbf{T}''$ at least once at position i.

Furthermore, we notice that $0 < \frac{\ln t}{c_5 t^{\frac{1}{k}}} < 1$ for every $t \geqslant \exp(4k^3)$ by Lemma 2.6 and since $c_5 > 1$. Therefore, we can construct a sequence of i.i.d. random variables $(\nu_i^*)_{i \in \mathbb{N}}$ with geometric distribution and parameter $\frac{\ln t}{c_5 t^{\frac{1}{k}}}$.

Lemma 2.10. *Let* $t \in \mathbb{N}$ *with* $t \geqslant \exp(4k^3)$*,* $\mathbf{T} \in \mathbb{T} \setminus D_t$*, and* $\tau_t' \leqslant t^{1-\frac{2}{k}}$*. Then, by coupling we obtain* $\nu_i \leqslant \nu_i^*$ *for every* $i \in [1, \tau_t']$*.*

Proof. We show this by constructing the coupling explicitly. Let $i \in [1, \tau_t']$. First, consider the random variable ξ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\xi \colon \Omega \to \{0, 1\}, \quad \omega \mapsto \xi(\omega), \tag{2.28}$$

where $\Omega = \Omega_0 \ \dot{\cup}\ \Omega_1$ with $\xi(\Omega_0) = 0$, $\xi(\Omega_1) = 1$, $\mathbb{P}[\Omega_0] = \frac{Z'(x_i')}{Z(x_i')}$. Here, we write $A \,\dot{\cup}\, B$ for the union of the disjoint sets A and B, so $A \cap B = \varnothing$.

Now, ν_i is given by

$$\nu_i\colon \Omega^{\mathbb{N}} \to \mathbb{N}_0, \quad \text{with } \{\nu_i = j\} \Leftrightarrow \underbrace{\Omega_1 \times \ldots \times \Omega_1}_{j} \times \Omega_0 \times \Omega \times \ldots, \text{ for every } j \in \mathbb{N}_0. \tag{2.29}$$

Next, we consider the random variable ξ^* on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\xi^*\colon \Omega \to \{0,1\}, \quad \omega \mapsto \xi^*(\omega). \tag{2.30}$$

Here, $\Omega = \Omega_0^* \dot\cup \Omega_1^*$ with $\xi^*(\Omega_0^*) = 0$, $\xi^*(\Omega_1^*) = 1$, and $\mathbb{P}[\Omega_0^*] = \frac{\ln t}{c_5 t^{\frac{1}{k}}}$.

Now, is given by

$$\nu_i^*\colon \Omega^{\mathbb{N}} \to \mathbb{N}_0, \quad \text{with } \{\nu_i^* = j\} \Leftrightarrow \underbrace{\Omega_1^* \times \ldots \times \Omega_1^*}_{j} \times \Omega_0^* \times \Omega \times \ldots, \text{ for every } j \in \mathbb{N}_0. \tag{2.31}$$

We notice $\{\nu_i^* \geqslant j\} \Leftrightarrow \underbrace{\Omega_1^* \times \ldots \times \Omega_1^*}_{j} \times \Omega \times \Omega \times \ldots$, for every $j \in \mathbb{N}_0$.

Since $\frac{Z'(x_i')}{Z(x_i')} \geqslant \frac{\ln t}{c_5 t^{\frac{1}{k}}}$ according to (2.27), this can be constructed such that $\Omega_0 \supseteq \Omega_0^*$ and, thus, $\Omega_1^* = (\Omega_0^*)^c \supseteq (\Omega_0)^c = \Omega_1$. Therefore, we obtain

$$\underbrace{\Omega_1 \times \ldots \times \Omega_1}_{j} \times \Omega_0 \times \Omega \times \ldots \subseteq \underbrace{\Omega_1^* \times \ldots \times \Omega_1^*}_{j} \times \Omega \times \Omega \times \ldots, \text{ for every } j \in \mathbb{N}_0. \tag{2.32}$$

This yields $\nu_i \leqslant \nu_i^*$ for all $i \in [1, \tau_t']$. □

To prove Lemma 2.9, we also need the following estimate on the exponential function.

Lemma 2.11. *Let $u \geqslant 0$. Then, $(1-2u)\exp(u) \leqslant 1-u$.*

Proof. This is obviously true for $u = 0$. So now, assume that $u > 0$. Here, we consider three cases separately.

First, assume that $\frac{1}{2} > u > 0$. Then, we consider the function $f\colon [0, \frac{1}{2}] \to \mathbb{R}$, $f(u) := \frac{1-2u}{1-u}\exp(u) - 1$ with derivative $f'(u) = \frac{\exp(u)}{(1-u)^2}(2u^2 - 3u)$. Since $f'(u) \leqslant 0$ for every $u \in [0, \frac{1}{2}]$ and $f(0) = 0$, we have $f(u) \leqslant 0$ for every $u \in (0, \frac{1}{2})$ and, thus, the inequality holds true in this case.

Second, let $1 \geqslant u \geqslant \frac{1}{2}$. Then, the inequality holds after comparing the signs of both sides.

Third, we assume that $u > 1$. Then, $\frac{u-1}{2u-1}\exp(-u) \leqslant (u-1)\exp(-u) \leqslant u\exp(-u) < 1$ and the inequality follows in this case as well. □

For $\mathbf{T} \in \mathbb{T}$, we recall the definition of generation $g_n = \{x \in \mathbf{T} : \operatorname{dist}(x, o) = n\}$, $n \in \mathbb{N}_0$, and introduce

$$G'' \coloneqq G^*[\,\cdot \mid \exists n \in \mathbb{N} : \#g_n = 0] \tag{2.33}$$

for the conditional probability measure conditioned on extinction. Furthermore, for $b \geqslant 0$ let

$$G''_b \coloneqq G''[\,\cdot \mid \forall x \in \mathbf{T} : Z(x) \leqslant b] \tag{2.34}$$

denote the conditional probability measure conditioned on at most b children. Further, let $\mathbb{G}''_b$ denote the expectation of G''_b.

Let $\mathbf{T} \in \mathbb{T}$ and let $(X_t^{(v)})_{t \in \mathbb{N}_0}$ denote the simple random walk on $\mathbf{T}$, starting at an arbitrary vertex $v \in \mathbf{T}$. Further, let $\tau_0 = \tau_0^{(v,u)}$ denote the first positive hitting time of the vertex $u \in \mathbf{T}$, i.e.,

$$\tau_0 \coloneqq \begin{cases} \infty, & \text{if there does not exist } n \in \mathbb{N} \text{ such that } X_n^{(v)} = u, \\ n, & \text{if } X_n^{(v)} = u \text{ and } \forall j \in [1, n-1] : X_j^{(v)} \neq u. \end{cases} \tag{2.35}$$

Next, in Lemma 2.12 we recall a result by Piau [48, Lemma 3] and give a more detailed proof. This will be used in the proof of Lemma 2.9 to estimate the time (or, more precisely, the number of steps) the simple random walk spends in the finite subtree, after having stepped into it, which corresponds to $d_l - 1$ for some $l \in \mathbb{N}$. Here, we notice that we need to consider a tree, conditioned on extinction, with a stump added to the root, since the walk already entered the finite subtree and, thus, has to revert this step to exit the finite subtree. Therefore, we introduce the trees $\mathbb{T}_{-1}$ which are constructed by adding the vertex o^- below the root o for every $\mathbf{T} \in \mathbb{T}$. The probability measure of the simple random walk on $\mathbf{T}_{-1} \in \mathbb{T}_{-1}$ will be denoted by P_{-1} and its expectation by $\mathbb{E}_{-1}$.

Lemma 2.12. *Let $b \geqslant 1$. Then, every constant $\frac{(1-\mathbb{G}''_b[Z])^2}{16} > c_7 > 0$ satisfies*

$$G''_b \mathbb{E}_{-1}\left[\exp\left(\frac{c_7 \tau_0^{(o,o^-)}}{b}\right)\right] \leqslant 2. \tag{2.36}$$

Here, we consider the expectation $\mathbb{E}_{-1}$ in the annealed average over G''_b.

Proof. Let $\mathbf{T} \in \mathbb{T}$ be a supercritical Galton-Watson tree, which goes extinct and satisfies $Z(x) \leqslant b$ for every $x \in \mathbf{T}$. Let $\mathbf{T}_{-1}$ be the corresponding tree with stump o^-. Further, let

$$\lambda_b \coloneqq \mathbb{G}''_b[Z] < 1 \tag{2.37}$$

by [45, Prop. 5.4], [1, Cor. 2.5]. First, we note that $\tau_0 := \tau_0^{(o,o^-)}$ will depend on the number of generations g_n, $n \in \mathbb{N}_0 \cup \{-1\}$, the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ on $\mathbf{T}_{-1}$, starting at the root o, explores before reaching o^- for the first time. Here, we introduced $g_{-1} := \{o^-\}$ and stick to $g_n = \{x \in \mathbf{T} : \operatorname{dist}(x, o) = n\}$, $n \in \mathbb{N}_0$. Hence, for $n \in \mathbb{N}_0$ let τ^n denote the first hitting time of o^-, conditioned on that the simple random walk never exceeds generation g_n or, equivalently, graph distance n to the root o, i.e., for all $m \in \mathbb{N}$ we have

$$P_{-1}[\tau_0 = m \,|\, \forall j \in [1, m-1] : \operatorname{dist}(X_j, o) \leqslant n] = P_{-1}[\tau^n = m]. \tag{2.38}$$

Here, we know $\tau_0, \tau^n < \infty$ for every $n \in \mathbb{N}_0$, since we only consider the simple random walk on trees conditioned on extinction, thus trees with a finite vertex set.

Now, we note that $P_{-1}[\tau^0 = 1] = 1$, since, here, the simple random walk can not exceed generation $g_0 = \{o\}$ and, thus, has to move to o^- in the first step. Further, let $z := Z(o)$, where in the case $z = 0$ we have $\tau_0 = \tau^n = 1$, for every $n \in \mathbb{N}_0$, and the claim follows from $c_7 \leqslant \frac{1}{2}$, cf. (2.37). Then, for $n \in \mathbb{N}_0$, $P_{-1}[\tau^{n+1} = 1] = \frac{1}{z+1}$ and $P_{-1}[\tau^{n+1} = 1 + \tau_*^n + \tau_*^{n+1}] = \frac{z}{z+1}$. Here, τ_*^n denotes the the first recurrence time to o of the simple random walk, starting at the root's child $x_1 = X_1 \neq o^-$, where the walk never exceeds graph distance n to x_1; and τ_*^{n+1} denotes an i.i.d.-copy of τ^{n+1}. Let $P_{x_1^+}$ denote the probability measure of the simple random walk, starting at the vertex x_1, on the subtree of $\mathbf{T}_{-1}$ given by x_1 and its descendants with the root o as stump. Furthermore, let the corresponding expectation be denoted by $\mathbb{E}_{x_1^+}$. Then, for $n \in \mathbb{N}_0$ and $\rho > 0$, we obtain

$$\mathbb{E}_{-1}[\exp(\rho\tau^{n+1})] = \frac{\exp(\rho)}{z+1} + \frac{\exp(\rho)z}{z+1}\,\mathbb{E}_{-1}[\exp(\rho\tau_*^{n+1})]\,\mathbb{E}_{x_1^+}[\exp(\rho\tau_*^n)], \tag{2.39}$$

which can be written as

$$\exp(\rho) = \big(1 - z(\exp(\rho)\mathbb{E}_{x_1^+}[\exp(\rho\tau_*^n)] - 1)\big)\mathbb{E}_{-1}[\exp(\rho\tau^{n+1})]. \tag{2.40}$$

Form now on, we will always consider

$$\frac{(1-\lambda_b)^2}{16\lfloor b\rfloor} > \rho > 0, \tag{2.41}$$

where λ_b was defined in (2.37). Since $Z(o)$ is an i.i.d.-copy of a random variable Z, for $z = 0, ..., \lfloor b\rfloor$, we conclude from (2.40) that

$$\exp(\rho) = \big(1 - z(\exp(\rho)G_b''\mathbb{E}_{-1}[\exp(\rho\tau^n)] - 1)\big)G_b''\mathbb{E}_{-1}[\exp(\rho\tau^{n+1})|Z(o) = z]. \tag{2.42}$$

Now, for $u \geqslant 1$ with $u \exp(\rho) < 1 + (\lfloor b \rfloor)^{-1}$, we define

$$g(u) := \mathbb{G}_b''\Big[\frac{\exp(\rho)}{1 - Z(u\exp(\rho) - 1)}\Big]. \tag{2.43}$$

Moreover, for $n \in \mathbb{N}_0$, we introduce

$$t_n := G_b''\mathbb{E}_{-1}[\exp(\rho\tau^n)]. \tag{2.44}$$

Then, we see that $t_0 = G_b''\mathbb{E}_{-1}[\exp(\rho\tau^0)] = \exp(\rho)$ and, by (2.42), that $t_{n+1} = G_b''\mathbb{E}_{-1}[\exp(\rho\tau^{n+1})] = g(G_b''\mathbb{E}_{-1}[\exp(\rho\tau^n)]) = g(t_n)$ for every $n \in \mathbb{N}_0$ with $t_n \exp(\rho) < 1 + (\lfloor b \rfloor)^{-1}$, so, in particular, for $n = 0$ and $\rho < \frac{1}{2}\ln(1 + (\lfloor b \rfloor)^{-1})$ which is guaranteed by (2.41).

Next, let

$$\varepsilon := \frac{1}{2}(1 - \lambda_b) > 0, \tag{2.45}$$

where λ_b was defined in (2.37). Now, let $u \geqslant 1$ with $u\exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1}$. Then $u\exp(\rho) < 1 + (\lfloor b \rfloor)^{-1}$, since $\varepsilon < 1$ by its definition (2.45). Further, the constant

$$c(\varepsilon) := (1 - \varepsilon)^{-1} > 0 \tag{2.46}$$

yields

$$(1 - z(u\exp(\rho) - 1))^{-1} \leqslant 1 + c(\varepsilon)z(u\exp(\rho) - 1), \tag{2.47}$$

for every $z = 0, ..., \lfloor b \rfloor$. Here, we estimated

$$\begin{aligned}
&(1 - z(u\exp(\rho) - 1))(1 + c(\varepsilon)z(u\exp(\rho) - 1)) \\
&= 1 - z(u\exp(\rho) - 1) + c(\varepsilon)z(u\exp(\rho) - 1) - c(\varepsilon)z^2(u\exp(\rho) - 1)^2 \\
&\geqslant 1 - z(u\exp(\rho) - 1) + c(\varepsilon)z(u\exp(\rho) - 1) - \varepsilon c(\varepsilon)z(u\exp(\rho) - 1) \\
&= 1,
\end{aligned} \tag{2.48}$$

where the inequality follows from $u\exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1}$ together with $z \leqslant \lfloor b \rfloor$, for every $z \in [0, \lfloor b \rfloor]$, and the last equality holds by the choice of $c(\varepsilon)$ in (2.46).

Furthermore, for $u \geqslant 1$ with $u\exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1}$, we define

$$g_\varepsilon(u) := \exp(\rho) + \exp(\rho)c(\varepsilon)(u\exp(\rho) - 1)\lambda_b. \tag{2.49}$$

Then, for every $n \in \mathbb{N}_0$, we obtain, by (2.47), that

$$t_n \exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1} \implies t_{n+1} = g(t_n) \leqslant g_\varepsilon(t_n). \tag{2.50}$$

We notice that g_ε is increasing and claim that, for every ρ satisfying (2.41), g_ε has a fixed-point $s = s(\rho) \geqslant \exp(\rho)$ with

$$s\exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1}, \tag{2.51}$$

which, by the definition of g_ε in (2.49), corresponds to the existence of $s \geqslant \exp(\rho)$ with

$$s\exp(\rho) < 1 + \varepsilon(\lfloor b \rfloor)^{-1} \quad \text{and} \quad s = \exp(\rho) + \exp(\rho)c(\varepsilon)(s\exp(\rho) - 1)\lambda_b. \tag{2.52}$$

To prove this, we first notice that

$$c(\varepsilon)\lambda_b < 1, \tag{2.53}$$

by the choice of $c(\varepsilon)$ in (2.46) and of ε in (2.45) together with the estimate (2.37). Furthermore, all of our choices of ρ in (2.41) yield

$$\exp(2\rho) < \frac{\varepsilon + \lfloor b \rfloor}{c(\varepsilon)\lambda_b\varepsilon + \lfloor b \rfloor}. \tag{2.54}$$

Here, we estimated as follows

$$2\rho < \frac{(1-\lambda_b)^2}{8\lfloor b \rfloor} < \frac{(1-\lambda_b)^2}{2(\lambda_b + 1)(\lfloor b \rfloor + \varepsilon)} = \frac{\varepsilon(1 - c(\varepsilon)\lambda_b)}{(\lfloor b \rfloor + \varepsilon)} < 1, \tag{2.55}$$

where we used (2.41) for the first inequality, $\varepsilon < 1 \leqslant \lfloor b \rfloor$ by (2.45) and by initial assumption for the second inequality, $\varepsilon c(\varepsilon)\lambda_b = \frac{(1-\lambda_b)\lambda_b}{1+\lambda_b}$, thus, $2\varepsilon - 2\varepsilon c(\varepsilon)\lambda_b = \frac{(1-\lambda_b)^2}{1+\lambda_b}$ by (2.45) and (2.46) for the equality, and once more the choices of $\varepsilon, c(\varepsilon)$ in (2.45), respectively, (2.46) for the last inequality. Then, we apply the elementary inequality $\ln(1+x) \leqslant x$, for $x \in \,]-1, 1[$, in the form of $x \leqslant \ln(\frac{1}{1-x})$ to $x = \frac{\varepsilon(1-c(\varepsilon)\lambda_b)}{(\lfloor b \rfloor + \varepsilon)}$ and conclude from (2.55) that

$$2\rho < \ln\Big(\frac{\varepsilon + \lfloor b \rfloor}{c(\varepsilon)\lambda_b\varepsilon + \lfloor b \rfloor}\Big), \tag{2.56}$$

which yields (2.54).

Now, considering the derivative g'_ε of g_ε, which we defined in (2.49), we obtain, by (2.54), that

$$g'_\varepsilon \equiv \exp(2\rho)c(\varepsilon)\lambda_b < c(\varepsilon)\lambda_b\frac{\varepsilon + \lfloor b \rfloor}{c(\varepsilon)\lambda_b\varepsilon + \lfloor b \rfloor}. \tag{2.57}$$

By (2.53) we conclude that $0 \leqslant g'_\varepsilon(u) < 1$ for every $u \geqslant 1$ and $g_\varepsilon(1) \geqslant \exp(\rho)$ with strict inequality for $\lambda_b > 0$. Therefore, by the Banach fixed point theorem g_ε has a fixed-point $s = s(\rho) \geqslant \exp(\rho)$ for every $0 < \rho < \frac{(1-\lambda_b)^2}{16\lfloor b\rfloor}$.

We are left to show that s satisfies (2.51). To this end, we assume, for contradiction, that $s\exp(\rho) \geqslant 1 + \varepsilon(\lfloor b\rfloor)^{-1}$. Since s is a fixed point of g_ε and, thus, in particular satisfies the second equation of (2.52), we have

$$\exp(2\rho) = \frac{s\exp(\rho)}{1 + c(\varepsilon)\lambda_b(s\exp(\rho) - 1)}. \tag{2.58}$$

Now, we apply the elementary estimate $\frac{d+a}{d+ba} \geqslant \frac{d+f}{d+bf}$, for $0 < f \leqslant a$ and $0 \leqslant b < 1 \leqslant d$, with $a = s\exp(\rho) - 1$, $b = c(\varepsilon)\lambda_b$, $d = 1$, and $f = \varepsilon(\lfloor b\rfloor)^{-1}$ and obtain, from (2.58) by (2.53) and by the assumption $s\exp(\rho) \geqslant 1 + \varepsilon(\lfloor b\rfloor)^{-1}$, that

$$\exp(2\rho) \geqslant \frac{1 + \varepsilon(\lfloor b\rfloor)^{-1}}{1 + c(\varepsilon)\lambda_b\varepsilon(\lfloor b\rfloor)^{-1}} = \frac{\varepsilon + \lfloor b\rfloor}{c(\varepsilon)\lambda_b\varepsilon + \lfloor b\rfloor}, \tag{2.59}$$

which contradicts (2.54). Hence, we conclude that s indeed satisfies (2.51), i.e., $s\exp(\rho) < 1 + \varepsilon(\lfloor b\rfloor)^{-1}$.

Therefore, we have shown that, for every ρ satisfying (2.41), there exists $s = s(\rho) \geqslant \exp(\rho)$ satisfying (2.52).

Finally, we recall that $t_0 = \exp(\rho) \leqslant s$ according to (2.44) and, thus, by the inequality in (2.52) we have $t_0\exp(\rho) < 1 + \varepsilon(\lfloor b\rfloor)^{-1}$. Since s is a fixed point of the increasing function g_ε satisfying (2.52), (2.50) iteratively implies $t_n \leqslant s$ for every $n \in \mathbb{N}_0$. Furthermore, for every tree $\mathbf{T} \in \mathbb{T}$, which goes extinct, there exists $j \in \mathbb{N}_0$ such that generation $j+1$ satisfies $\#g_{j+1} = 0$. Then, $\tau_0 = \tau^j$, i.e., for all $m \in \mathbb{N}$ we have

$$P_{-1}[\tau_0 = m] = P_{-1}[\tau^j = m]. \tag{2.60}$$

Hence, by (2.44), we notice, for every $\frac{(1-\lambda_b)^2}{16\lfloor b\rfloor} > \rho > 0$, that

$$G_b''\mathbb{E}_{-1}[\exp(\rho\tau_0)] \leqslant s. \tag{2.61}$$

We choose $c_7 := \rho\lfloor b\rfloor$ and the claim follows from $s < 1 + \varepsilon(\lfloor b\rfloor)^{-1} \leqslant 2$, where (2.52) yields the first and $\varepsilon < 1 \leqslant \lfloor b\rfloor$, by (2.45) and the initial assumption, yield the second inequality. $\square$

Proof of Lemma 2.9. Let $t \in \mathbb{N}$ with $t \geqslant \exp(4k^3)$.

1. First, we notice, by Definition 2.8 of H_t and the definition of F_t in (2.23), that

$$GP[H_t, D_t^c] \leqslant G[D_t^c]\, G_{D_t^c} P\Big[\sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i \geqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t},\ \tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\Big], \qquad (2.62)$$

where we take i.i.d. copies of ν_1 for $i > \tau_t'$.

Now, we recall the definition of $\nu_i + 1$, $i \in [1, \tau_t']$, as the first successful forward move at the position i. Furthermore, we recall the construction of $(\nu_i^*)_{i\in\mathbb{N}}$ as sequence of i.i.d. random variables with geometric distribution and parameter $\frac{\ln t}{c_5 t^{\frac{1}{k}}}$. Further, let $(\Omega, \mathcal{F}, \mathbb{P})$ be their joint probability space with expectation $\mathbb{E}$ from Lemma 2.10.

Let $\iota \in \mathbb{N}$ be fixed. Then, we see that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\frac{\nu_\iota^* \ln t}{2c_5 t^{\frac{1}{k}}}\Big)\Big] &= \sum_{j=0}^{\infty} \exp\Big(\frac{j\ln t}{2c_5 t^{\frac{1}{k}}}\Big)\Big(1-\frac{\ln t}{c_5 t^{\frac{1}{k}}}\Big)^j \frac{\ln t}{c_5 t^{\frac{1}{k}}} \\
&= \sum_{j=0}^{\infty}\Big(\Big(1-\frac{\ln t}{c_5 t^{\frac{1}{k}}}\Big)\exp\Big(\frac{\ln t}{2c_5 t^{\frac{1}{k}}}\Big)\Big)^j \frac{\ln t}{c_5 t^{\frac{1}{k}}} \\
&\leqslant \sum_{j=0}^{\infty}\Big(1-\frac{\ln t}{2c_5 t^{\frac{1}{k}}}\Big)^j \frac{\ln t}{c_5 t^{\frac{1}{k}}} = 2.
\end{aligned} \qquad (2.63)$$

Here, we applied Lemma 2.11 with $u = \frac{\ln t}{2c_5 t^{\frac{1}{k}}}$ to show the inequality.

This at hand, we notice that $\tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}$ implies $\tau_t' \leqslant t^{1-\frac{2}{k}}$, since $\vartheta > 0$ by the initial assumption of Theorem 2.1. Therefore, we conclude, by Chebyshev's inequality (in the form of Lemma 2.4), that

$$\begin{aligned}
&G[D_t^c]\, G_{D_t^c} P\Big[\sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i \geqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t},\ \tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\Big] \\
&\leqslant \mathbb{E}\Big[\exp\Big(\frac{\ln t}{2c_5 t^{\frac{1}{k}}} \sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i\Big)\Big] \exp\Big(-\frac{c_6 t^{1-\vartheta-\frac{2}{k}}}{2c_5}\Big).
\end{aligned} \qquad (2.64)$$

By Lemma 2.10 we have $\nu_i \leqslant \nu_i^*$ for every $i \in [1, \tau_t']$ and, thus,

$$\begin{aligned}
&G[D_t^c]\; G_{D_t^c} P\Big[\sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i \geqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t},\ \tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\Big] \\
&\leqslant \mathbb{E}\Big[\exp\Big(\frac{\ln t}{2c_5 t^{\frac{1}{k}}} \sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i^*\Big)\Big] \exp\Big(-\frac{c_6 t^{1-\vartheta-\frac{2}{k}}}{2c_5}\Big) \\
&\leqslant \mathbb{E}\Big[\exp\Big(\frac{\nu_\iota^* \ln t}{2c_5 t^{\frac{1}{k}}}\Big)\Big]^{t^{1-\vartheta-\frac{2}{k}}} \exp\Big(-\frac{c_6 t^{1-\vartheta-\frac{2}{k}}}{2c_5}\Big). \qquad (2.65)
\end{aligned}$$

Here, we used that $(\nu_i^*)_{i\in\mathbb{N}}$ are i.i.d. random variables for the second inequality. Finally, by (2.63) we have

$$G[D_t^c]\; G_{D_t^c} P\Big[\sum_{i=1}^{\lfloor t^{1-\vartheta-\frac{2}{k}}\rfloor} \nu_i \geqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t},\ \tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\Big] \leqslant 2^{t^{1-\vartheta-\frac{2}{k}}} \exp\Big(-\frac{c_6 t^{1-\vartheta-\frac{2}{k}}}{2c_5}\Big). \qquad (2.66)$$

Due to $c_6 > 2c_5 \ln 2$, the right-hand side of (2.66) decays exponentially in $t^{1-\vartheta-\frac{2}{k}}$. Since $\vartheta \leqslant 1-\vartheta-\frac{2}{k}$ by the initial assumptions of Theorem 2.1, we also have the exponentially decay in t^{ϑ} of the right-hand side of (2.66). Together with (2.62), this yields the claim for H_t.

2. We first notice, by Definition 2.8 of K_t, that

$$GP[K_t, D_t^c] \leqslant G[D_t^c]\; G_{D_t^c} P\Bigg[\sum_{l=1}^{\lfloor c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\rfloor} d_l \geqslant c_6 \frac{t}{\ln t}\Bigg], \qquad (2.67)$$

where we take i.i.d. copies of d_1 for $l > F_t$.

Now, we recall the definition of $d_l \in \mathbb{N} \setminus \{1\}$, $l = 1, ..., \sum_{i\in\mathbb{N}} \nu_i$, as the time (or, more precisely, the number of steps) of the simple random walk's excursions, including the step into and out of the finite subtree. Furthermore, we recall the definition of the first recurrence time τ_0 in (2.35). Further, let $j \in \big[1, \lfloor c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\rfloor\big]$ and choose $b := c_3 t^{\frac{1}{k}}$. Then, we see that

$$G_{D_t^c} P\big[d_j \geqslant t^{\vartheta+\frac{1}{k}}\big] \leqslant \frac{1}{G[D_t^c]}\; G_b'' P_{-1}\big[2\tau_0^{(o,o^-)} \geqslant t^{\vartheta+\frac{1}{k}}\big]. \qquad (2.68)$$

Here, every finite subtree is part of $\mathbf{T}''$ and, thus, on the r.h.s. of (2.68) we condition on extinction. Moreover, by Definition 2.2 we have $D_t^c \subseteq \{\forall x \in \mathbf{T}_{0t} \cap \mathbf{T}'' :\ c_3 t^{\frac{1}{k}} > Z''(x)\}$ and since we consider the times in (2.68) only up to a total of t steps, we know that the simple random walk has to stay in $\mathbf{T}_{0t}$. Therefore, we also condition the r.h.s. of (2.68) on the event $\{\forall x \in \mathbf{T} : \ Z(x) \leqslant b\}$.

Next, let $c_7 := \frac{(1-\mathbb{G}_b''[Z])^2}{32}$. Then, by Chebyshev's inequality (in the form of Lemma 2.4) and by the choice of $b = c_3 t^{\frac{1}{k}}$, we conclude that

$$G[D_t^c]\; G_{D_t^c} P\big[d_j \geqslant t^{\vartheta+\frac{1}{k}}\big] \leqslant G_b''\mathbb{E}_{-1}\left[\exp\left(\frac{c_7 \tau_0^{(o,o^-)}}{b}\right)\right] \exp\Big(-\frac{c_7 t^{\vartheta}}{2c_3}\Big). \qquad (2.69)$$

Since we chose $c_3 > 3$, see Definition 2.2, we have $b > 1$. Applying Lemma 2.12 yields

$$G[D_t^c]\; G_{D_t^c} P\big[d_j \geqslant t^{\vartheta+\frac{1}{k}}\big] \leqslant 2\exp\Big(-\frac{c_7 t^{\vartheta}}{2c_3}\Big). \qquad (2.70)$$

Since $\Big\{\sum_{l=1}^{\lfloor c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\rfloor} d_l \geqslant c_6 \frac{t}{\ln t}\Big\} \subseteq \Big\{\exists\; j \in \big[1, \lfloor c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\rfloor\big] \;:\; d_j \geqslant t^{\vartheta+\frac{1}{k}}\Big\}$, we conclude, by (2.70), that

$$G[D_t^c]\; G_{D_t^c} P\left[\sum_{l=1}^{\lfloor c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\rfloor} d_l \geqslant c_6 \frac{t}{\ln t}\right] \leqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\; 2\exp\Big(-\frac{c_7 t^{\vartheta}}{2c_3}\Big), \qquad (2.71)$$

where the r.h.s. decays exponentially in t^{ϑ}. Together with (2.67), this yields the claim for K_t.

□

Finally, we are ready to proof Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{T} \in \mathbb{T}$ and let $N := \max(\exp(4k^3), \exp(c_6 + 1))$. We first note for the simple random walk on $\mathbf{T}$ starting at the root that, for $t \in \mathbb{N}$ with $t \geqslant N$,

$$\{\tau_t' \leqslant t^{1-\vartheta-\varepsilon}\} \subseteq H_t \cup K_t. \qquad (2.72)$$

Here, we recall the definitions of H_t and K_t from Definition 2.8.

To this end, let $t \geqslant N$. Then, we conclude, by the initial assumption $\varepsilon \geqslant \frac{2}{k}$, that

$$\begin{aligned}\{\tau_t' \leqslant t^{1-\vartheta-\varepsilon}\} &\subseteq \{\tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\}\\ &\subseteq \left(\{\tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\} \cap \left\{F_t \geqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\right\}\right)\\ &\quad \cup \left(\{\tau_t' \leqslant t^{1-\vartheta-\frac{2}{k}}\} \cap \left\{F_t \leqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t}\right\}\right).\end{aligned} \qquad (2.73)$$

Furthermore, we know that $\tau_t' = t - \tau_t''$, where τ_t'' denotes the time (or number of steps) the simple random walk spends in the finite subtrees $\mathbf{T}''$ out of a total of t steps, i.e.,

$$\tau_t'' := \#\{n \in \mathbb{N} : \; X_n \in \mathbf{T}'', \; n \leqslant t\}. \tag{2.74}$$

Therefore, we obtain, by the definition of H_t, that

$$\{\tau_t' \leqslant t^{1-\vartheta-\varepsilon}\} \subseteq H_t \cup \left(\{\tau_t'' \geqslant t - t^{1-\vartheta-\frac{2}{k}}\} \cap \left\{ F_t \leqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t} \right\} \right). \tag{2.75}$$

Since $t \geqslant N$, we have, by Lemma 2.6, that $\frac{\ln(t)}{t^{\vartheta+\frac{2}{k}}} \leqslant 1$ and, thus, $t - t^{1-\vartheta-\frac{2}{k}} \geqslant c_6 \frac{t}{\ln t}$. Hence, we get

$$\{\tau_t' \leqslant t^{1-\vartheta-\varepsilon}\} \subseteq H_t \cup \left(\left\{ \tau_t'' \geqslant c_6 \frac{t}{\ln t} \right\} \cap \left\{ F_t \leqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t} \right\} \right). \tag{2.76}$$

Now, we recall the definition of F_t in (2.23) and of d_l, $l = 1, ..., F_t$, as the time (or, more precisely, the number of steps) of the simple random walk's excursions to the finite subtrees, respectively. Then, by the definition of τ_t'', see (2.74), we have $\tau_t'' \leqslant \sum_{l=1}^{F_t} d_l$, since each d_l also accounts for the step out of the finite subtree which is not included in τ_t''. Together with the definition of K_t we conclude that

$$\{\tau_t' \leqslant t^{1-\vartheta-\varepsilon}\} \subseteq H_t \cup \left(\left\{ \sum_{l=1}^{F_t} d_l \geqslant c_6 \frac{t}{\ln t} \right\} \cap \left\{ F_t \leqslant c_6 \frac{t^{1-\vartheta-\frac{1}{k}}}{\ln t} \right\} \right) = H_t \cup K_t, \tag{2.77}$$

which proves (2.72).

Next, by (2.72) we obtain, for $t \geqslant N$,

$$GP[\tau_t' \leqslant t^{1-\vartheta-\varepsilon}] \leqslant G[D_t] + GP[H_t, D_t^c] + GP[K_t, D_t^c]. \tag{2.78}$$

Hence, by Lemma 2.3 and Lemma 2.9 we conclude from (2.78) that, for $t \geqslant N$,

$$GP[\tau_t' \leqslant t^{1-\vartheta-\varepsilon}] \leqslant C_1' \exp(-c_1' t^{\vartheta}) + C_2' \exp(-c_2' t^{\vartheta}) + C_3' \exp(-c_3' t^{\vartheta}), \tag{2.79}$$

with constants $C_1', c_1', C_2', c_2', C_3', c_3' > 0$.

Now, let $C' := 3 \max(C_1', C_2', C_3', 1) > 0$ and let $c' := \min(c_1', c_2', c_3') > 0$. By (2.79) we have, for $t \geqslant N$,

$$GP[\tau_t' \leqslant t^{1-\vartheta-\varepsilon}] \leqslant C' \exp(-c' t^{\vartheta}). \tag{2.80}$$

Finally, let $t \in \mathbb{N}$ be arbitrary. Since $C' > 1$ by its definition, we obtain

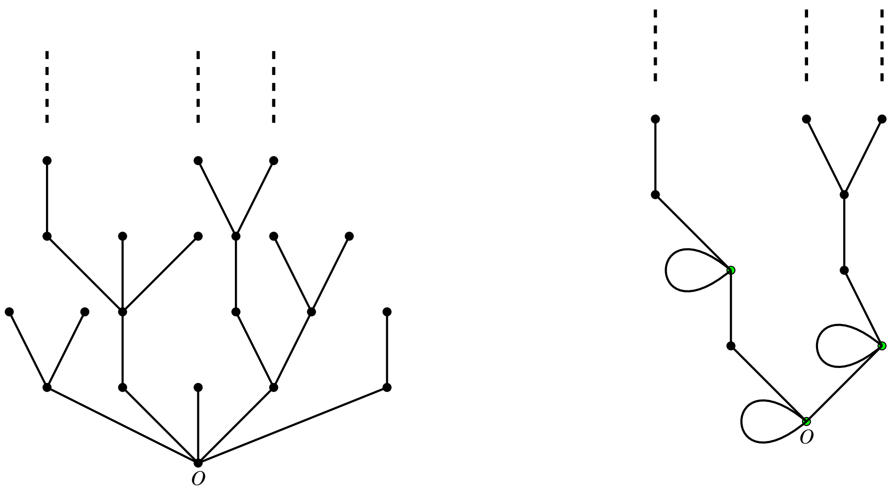
$$GP[\tau_t' \leqslant t^{1-\vartheta-\varepsilon}] \leqslant \exp(c'N^\vartheta)\, C' \exp(-c't^\vartheta). \tag{2.81}$$

Choosing $C := \exp(c'N^\vartheta)\, C' > 0$ and $c := c' > 0$ yields the claim. □

2.2 Application to the Annealed Return Probability

In this section we give one possible application of Theorem 2.1. By reducing the simple random walk on a Galton-Watson family tree with general offspring distribution to the steps taken on the backbone, we obtain an estimate for the annealed return probability by its version restricted to the backbone, so, in particular, for a random walk on trees with $p_0 = 0$. To this end, we first need to discuss how we reduce the simple random walk to the backbone.

Let $\mathbb{T}$ be a supercritical Galton-Watson family tree conditioned on non-extinction. Let $\mathbf{T} \in \mathbb{T}$ and let $\mathbf{T}'$ denote its backbone and $\mathbf{T}''$ its finite subtrees. Next, consider the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ on $\mathbf{T}$ starting at the root o. Now, we introduce the simple random walk reduced to the backbone $\mathbf{T}'$ which we will denote by $(X_m')_{m\in\mathbb{N}_0}$. This is a subsequence of the simple random walk on $\mathbf{T}$, i.e., for every $m \in \mathbb{N}_0$ we have $X_m' = X_{t_m}$ with some $t_m \in \mathbb{N}_0$. Here, we set $t_0 := 0$, thus $X_0' = X_0 = o$, and iteratively define $t_{m+1} := \inf\{n \in \mathbb{N} : \ X_n \in \mathbf{T}',\ n > t_m\}$ for every $m \in \mathbb{N}_0$. Then, this is well defined, since $\mathbb{T}$ is conditioned on non-extinction, and we have $X_m' \in \mathbf{T}'$ for every $m \in \mathbb{N}_0$. We note that, by this definition, $X_m' = X_{m+1}'$ whenever $X_{1+t_m} \in \mathbf{T}''$, $m \in \mathbb{N}_0$. The G^*-probability of $\{X_m' = X_{m+1}'\}$ can be bounded by the extinction probability r, compare with [1].



(a) (b)

Figure 2.2: The realisation of a Galton-Watson tree in (a) and its backbone with loops at the positions, highlighted in green, which previously led to finite subtrees in (b). Then we can visualise $(X'_m)_{m\in\mathbb{N}_0}$ as the random walk on a weighted version of the graph in (b).

Furthermore, we recall the definition of the annealed return probability $R_t = GP[X_{2t} = o]$, $t \in \mathbb{N}_0$, in (1.28). We introduce the annealed return probability on the backbone

$$R'_m := GP[X'_m = o], \qquad m \in \mathbb{N}_0. \tag{2.82}$$

Again, this is well defined, since $\mathbb{T}$ is conditioned on non-extinction, so, in particular, $o \in \mathbf{T}'$.

The following corollary of Theorem 2.1 enables us to quantify the error of reducing the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ to the backbone with respect to the annealed return probability.

Corollary 2.13. *Let $\vartheta, \varepsilon > 0$ with $2\vartheta + \varepsilon \leqslant 1$. Consider a supercritical Galton-Watson family tree $\mathbb{T}$, conditioned on non-extinction, with offspring distribution $\{p_j\}_{j\in\mathbb{N}_0}$ satisfying (1.30) for some constants $c_1, c_2 > 0$ and $k > 2$. Further, assume that $\varepsilon \geqslant \frac{2}{k}$. Then, there exist constants $C, c > 0$ such that for all $t \in \mathbb{N}$*

$$R_t \leqslant \sum_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} R'_m + C\exp(-ct^{\vartheta}). \tag{2.83}$$

Here, we recall that $\lceil x\rceil = \inf\{n \in \mathbb{N} :\ n \geqslant x\}$ for $x \in \mathbb{R}$.

Remark. *If the decay of R'_m can be bounded by a monotone decreasing function in m, we can always estimate the sum in (2.83) by $2t$ times $R'_{\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}$ and obtain, the* *more aesthetic,*

$$R_t \leqslant 2tR'_{\lceil (2t)^{1-\vartheta-\varepsilon}\rceil} + C\exp(-ct^{\vartheta}). \tag{2.84}$$

Proof. Let $\mathbf{T} \in \mathbb{T}$ and let $(X_t)_{t\in\mathbb{N}_0}$ denote the simple random walk on $\mathbf{T}$, starting at the root o, and let $(X'_m)_{m\in\mathbb{N}_0}$ denote its version reduced to the backbone, i.e., for every $m \in \mathbb{N}_0$ we have $X'_m = X_{t_m}$ with some $t_m \in \mathbb{N}_0$. Now, let $t \in \mathbb{N}$ be fixed.

We recall the definition of τ''_{2t}, in (2.74), as the time (or number of steps) the simple random walk spends in the finite subtrees $\mathbf{T}''$ out of a total of $2t$ steps. Contrarily, τ'_{2t}, defined in (2.3), denotes the time (or number of steps) the simple random walk spends on the backbone $\mathbf{T}'$ out of a total of $2t$ steps. Then, we have $\tau''_{2t} + \tau'_{2t} = 2t$.

This at hand, we notice that

$$\begin{aligned}
\{X_{2t} = o\} &= \bigcup_{m=1}^{2t} \Big(\{X'_m = o\} \cap \{\tau''_{2t} = 2t - m\} \Big) \\
&\subseteq \bigcup_{m=1}^{\lceil (2t)^{1-\vartheta-\varepsilon}\rceil - 1} \{\tau''_{2t} = 2t - m\} \cup \bigcup_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} \{X'_m = o\} \\
&\subseteq \{\tau''_{2t} \geqslant 2t - (2t)^{1-\vartheta-\varepsilon}\} \cup \bigcup_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} \{X'_m = o\} \\
&= \{\tau'_{2t} \leqslant (2t)^{1-\vartheta-\varepsilon}\} \cup \bigcup_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} \{X'_m = o\}.
\end{aligned} \tag{2.85}$$

Now, considering the annealed probability of the events in (2.85), we conclude that

$$GP[X_{2t} = o] \leqslant GP[\tau'_{2t} \leqslant (2t)^{1-\vartheta-\varepsilon}] + \sum_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} GP[X'_m = o]. \tag{2.86}$$

After recalling the definition of the annealed return probability R_t in (1.28) and of the annealed return probability on the backbone R'_m in (2.82), we have

$$R_t \leqslant GP[\tau'_{2t} \leqslant (2t)^{1-\vartheta-\varepsilon}] + \sum_{m=\lceil (2t)^{1-\vartheta-\varepsilon}\rceil}^{2t} R'_m. \tag{2.87}$$

Finally, applying Theorem 2.1 to the first term in (2.87) yields the claim. □

Remark. *It is easy to see from the proof of Corollary 2.13 that the error of reducing the simple random walk to the backbone can always be estimated in this way, and* *not only for the annealed return probability to the root, as long as the walk starts at a vertex* $v \in \mathbf{T}'$. *Thus, the (annealed) cost of reducing the simple random walk to the backbone can be bounded from above by an error term decaying exponentially in* t^{ϑ}.

Chapter 3

Annealed Return Probability

3.1 Lower Bound for the Annealed Return Probability

In this section, we first consider the simple random walk $(X_t)_{t\in\mathbb{N}_0}$, starting at the root o, on the Galton-Watson tree with offspring distribution $p_1 = 1$, see also Lemma 1.5. Let τ_m, $m > 0$, denote the first time $(X_t)_{t\in\mathbb{N}_0}$ hits height m, i.e.,

$$\tau_m := \inf\{t \in \mathbb{N}_0 : \ \mathrm{dist}(o, X_t) \geqslant m\}. \tag{3.1}$$

Then we recall the following well-known result for the random walk on $\mathbb{N}_0$, starting at 0, with up and down step probability $\frac{1}{2}$ each, reflected at 0, which corresponds to the simple random walk on a Galton-Watson tree with offspring distribution $p_1 = 1$; see, for instance, [48, Lemma 9].

Lemma 3.1. *Let $(X_t)_{t\in\mathbb{N}_0}$ denote the simple random walk on $\mathbb{N}_0$, starting at 0. Then, there is a constant c_0 such that for all $t \in \mathbb{N}$*

$$P\big[\tau_{t^{\frac{1}{3}}} \geqslant t\big] \geqslant \exp(-c_0 t^{\frac{1}{3}}). \tag{3.2}$$

With this in mind, we state the following lower bound for the annealed return probability R_t, defined in (1.28), of the simple random walk $(X_t)_{t\in\mathbb{N}_0}$, starting at the root o, on a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying $p_1 > 0$, which was, among others, already suggested by Piau. In [48, Thm. 2] Piau states for any $n \in \mathbb{N}$ that $GP[\tau_R \geqslant n] \geqslant \exp(-cn^{\frac{1}{3}})$, $c > 0$, where τ_R denotes the first regeneration time. However, a lower bound on the annealed probability of the event $\{\tau_R \geqslant n\}$ does not imply a lower bound for the annealed return probability. On that matter, an upper bound would transfer to an upper bound for the annealed return probability because of the inclusion $\{X_n = o\} \subset \{\tau_R \geqslant n\}$ for every $n \in \mathbb{N}$. Next, we follow the ideas outlined in [48] for the lower bound on $GP[\tau_R \geqslant t]$ to obtain a lower bound on R_t.

Theorem 3.2. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying $p_1 > 0$. Then, there is a constant $c > 0$ such that for all $t \in \mathbb{N}_0$ we have*

$$R_t \geqslant \exp(-ct^{\frac{1}{3}}). \tag{3.3}$$

Proof of Theorem 3.2. Let $n \in \mathbb{N}$ and set $t := 2n$.

We start by restricting ourselves to a certain class of trees, namely trees with exactly one child at each vertex up to height m, with $m > 0$. To this end, we define

$$\mathbb{T}^1_m := \{\forall x \in \mathbf{T}_{0(m+1)} : \; Z(x) = 1\}, \tag{3.4}$$

where the subtree $\mathbf{T}_{0(m+1)}$ up to height $m+1$, for $m > 0$, was defined in (2.4).

We notice that $G[\mathbb{T}^1_m] = p_1^{\lfloor m \rfloor + 1} \geqslant p_1^{m+1}$ according to [1] and that

$$GP[X_t = 0] \geqslant GP[X_t = 0, \mathbb{T}^1_m] = G_{\mathbb{T}^1_m} P[X_t = 0] G[\mathbb{T}^1_m]. \tag{3.5}$$

Here as before, we have $GP[\,\cdot\,, \mathcal{E}] = G[\mathcal{E}]\; G_{\mathcal{E}} P[\,\cdot\,]$ for $\mathcal{E} \subseteq \mathbb{T}$, where $G_{\mathcal{E}}[\,\cdot\,]$ is the probability measure conditioned on the event $\mathcal{E}$.

Next, we are interested in a certain class of simple random walks on these trees, namely walks which never exceed the height $m+1$. We define

$$B^m_t := \{\mathrm{dist}(o, X_s) \leqslant m + 1 \text{ for every } 0 \leqslant s \leqslant t\} \tag{3.6}$$

and obtain that

$$G_{\mathbb{T}^1_m} P[X_t = 0] \geqslant G_{\mathbb{T}^1_m} P[X_t = 0, B^m_t] \geqslant G_{\mathbb{T}^1_m} P_{B^m_t}[X_t = 0] G_{\mathbb{T}^1_m} P[B^m_t]. \tag{3.7}$$

Here, we write $G_{\mathbb{T}^1_m} P[X_t = 0, B^m_t] = \int_{\mathbb{T}^1_m} P^{\mathbf{T}}[X_t = 0, B^m_t] \,\mathrm{d}G_{\mathbb{T}^1_m}(\mathbf{T})$ and apply the reverse Hölder inequality to see

$$\begin{aligned} &\int_{\mathbb{T}^1_m} P^{\mathbf{T}}[X_t = 0, B^m_t] \,\mathrm{d}G_{\mathbb{T}^1_m}(\mathbf{T}) \\ &\geqslant \left(\int_{\mathbb{T}^1_m} \left(P^{\mathbf{T}}_{B^m_t}[X_t = 0]\right)^{\frac{1}{2}} \mathrm{d}G_{\mathbb{T}^1_m}(\mathbf{T}) \right)^2 \left(\int_{\mathbb{T}^1_m} \left(P^{\mathbf{T}}[B^m_t]\right)^{-1} \mathrm{d}G_{\mathbb{T}^1_m}(\mathbf{T}) \right)^{-1}. \end{aligned} \tag{3.8}$$

Since both probabilities on the r.h.s. are independent of $\mathbf{T} \in \mathbb{T}_m^1$, the last inequality of (3.7) follows.

From now on, we set $m := t^{\frac{1}{3}}$. Then by Lemma 3.1, we have

$$G_{\mathbb{T}_m^1} P[B_t^m] \geqslant \exp(-c_0(t+1)^{\frac{1}{3}}), \tag{3.9}$$

with some constant $c_0 > 0$.

To summarize so far, we have seen that the annealed return probability at time t can be bounded from below by the simple random walk on trees with exactly one child at each vertex up to height $t^{\frac{1}{3}} + 1$ which also never exceeds the height $t^{\frac{1}{3}} + 1$, namely $G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 0]$ with $m = t^{\frac{1}{3}}$.

Next, we notice that for these random walks on this set of trees the return probability to the root at time t is bounded from below by one half the probability to be at another vertex, i.e.,

$$G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 0] \geqslant \frac{1}{2} G_{\mathbb{T}_m^1} P_{B_t^m}[\mathrm{dist}(o, X_t) = 2l]\ G_{\mathbb{T}_m^1} P[B_t^m], \tag{3.10}$$

with $l = 1, ..., \lfloor \frac{m+1}{2} \rfloor$. Here, we first notice that this particular setting equals the simple random walk on $\mathbb{N}_0$, which never exceeds $m+1$. By mirroring, it can be identified with the simple random walk on $\mathbb{Z}$ which never exceeds the distance $m+1$ to 0; where we identify each level $i \in \mathbb{N}_0$ with the levels $\iota \in \mathbb{Z}$ such that $|\iota| = i$. The return probability at time t of the simple random walk on $\mathbb{Z}$ which never exceeds the distance $m+1$ to 0 will be denoted by $G_{\mathbb{Z}} P_{B_t^m}[X_t = 0]$ and we compute

$$\begin{aligned} G_{\mathbb{Z}} P_{B_t^m}[X_t = 0] \geqslant G_{\mathbb{Z}} P[X_t = 0] = \frac{1}{2^t} \binom{t}{\frac{t}{2}} = \frac{1}{2^t} \binom{t}{\frac{t}{2} \pm l} \prod_{j=0}^{l-1} \frac{\frac{t}{2} + 1 + j}{\frac{t}{2} - j} \\ = G_{\mathbb{Z}} P[X_t = \pm 2l] \prod_{j=0}^{l-1} \frac{\frac{t}{2} + 1 + j}{\frac{t}{2} - j} \geqslant G_{\mathbb{Z}} P[X_t = \pm 2l], \end{aligned} \tag{3.11}$$

with $l = 1, ..., \lfloor \frac{m+1}{2} \rfloor$. Thereby, we conclude that (3.10).

By (3.10), we now see that

$$\begin{aligned} G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 0] &= 1 - G_{\mathbb{T}_m^1} P_{B_t^m}[X_t \neq 0] = 1 - \sum_{l=1}^{\lfloor \frac{m+1}{2} \rfloor} G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 2l] \\ &\geqslant 1 - \sum_{l=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{2}{G_{\mathbb{T}_m^1} P[B_t^m]} G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 0] = 1 - \frac{\lfloor m+1 \rfloor}{G_{\mathbb{T}_m^1} P[B_t^m]} G_{\mathbb{T}_m^1} P_{B_t^m}[X_t = 0], \end{aligned} \tag{3.12}$$

and thus,

$$G_{\mathbb{T}^1_m} P_{B^m_t}[X_t = 0] \geqslant \frac{1}{1 + \frac{\lfloor m+1 \rfloor}{G_{\mathbb{T}^1_m} P[B^m_t]}} \geqslant \frac{G_{\mathbb{T}^1_m} P[B^m_t]}{m+2}. \tag{3.13}$$

Finally, combining (3.5), (3.7), and (3.9) with (3.13) and setting $m := t^{\frac{1}{3}}$ yields

$$GP[X_t = 0] \geqslant \frac{1}{t^{\frac{1}{3}} + 2} \exp(-2c_0(t+1)^{\frac{1}{3}}) p_1^{t^{\frac{1}{3}}+1} \geqslant \exp(-ct^{\frac{1}{3}}), \tag{3.14}$$

with some constant $c > 0$. Here, the last inequality holds for all large t.

But since $GP[X_t = 0] > 0$ for every $t \in \mathbb{N}$ even, there also is a constant $c > 0$ such that $GP[X_{2t} = 0] \geqslant \exp(-ct^{\frac{1}{3}})$ for all $t \in \mathbb{N}$. This completes the proof. □

Remark. *1. By* (3.11)*, the probability of the simple random walk on* $\mathbb{N}_0$ *starting at* 0 *to be at* 0 *after* t *steps, for large, even* t*, is one half the probability to be at any other point, since* $\prod_{j=0}^{l-1} \frac{\frac{t}{2}+1+j}{\frac{t}{2}-j} \to 1$ *as* $t \to \infty$*, for every* $l \in \mathbb{N}$*. In especially,* $G_{\mathbb{N}_0} P[X_t = 0] = \frac{1}{2} G_{\mathbb{N}_0} P[X_t = 2l] \prod_{j=0}^{l-1} \frac{\frac{t}{2}+1+j}{\frac{t}{2}-j}$ *for every* $l \in \mathbb{N}$*.*

2. Theorem 3.2 holds true for an offspring distribution with $p_0 > 0 \wedge p_1 = 0$*. This can be seen by considering linear pieces similar to* $\mathbb{T}^1_m$ *in the proof of Theorem 3.2 and proceeding in the same way. Here, we notice that a supercritical offspring distribution with* $p_0 > 0 \wedge p_1 = 0$ *implies that there is at least one* $k \in \mathbb{N} \setminus \{1\}$ *such that* $p_k > 0$*. Now, we can construct the 'linear' pieces of length* $\lfloor m+2 \rfloor$ *from the root,* $\mathbb{T}^k_m$*, as the set of trees with exactly k children at each vertex of which* $k-1$ *die out immediately, i.e. have no children of their own, up to height* $m+1$*, with* $m > 0$*.*

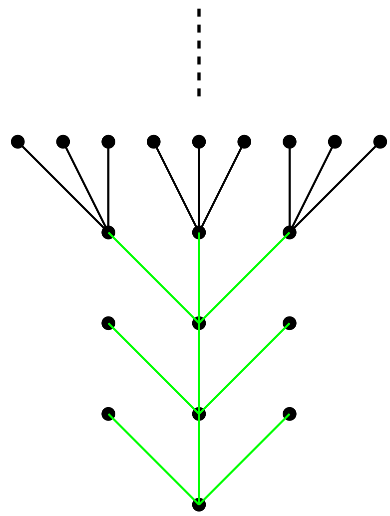


Figure 3.1: For an offspring with $p_0, p_3 > 0 \wedge p_1 = 0$, this illustrates a realisation $\mathbf{T} \in \mathbb{T}_1^3$, so a tree with a 'linear' pieces of length 3 which is highlighted in green.

3.2 Upper Bound for the Annealed Return Probability

The majority of this section coincides both in content and writing with [47], which was written in collaboration with P. Müller.

We start this section by recalling our main results on the upper bound for the annealed return probability to the root o at time t, which is denoted by R_t and defined in (1.28), from the Section 1.3.

Theorem 3.3. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution of bounded support. Then, there exists a constant $c > 0$ such that for all $t \in \mathbb{N}_0$ we have*

$$R_t \leqslant \exp(-ct^{\frac{1}{3}}). \tag{3.15}$$

Theorem 3.4. *Consider a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying* (1.30) *for some constants $c_1, c_2 > 0$ and $k > 8$. Then, there exists a constant $c > 0$ such that for all $t \in \mathbb{N}_0$ we obtain*

$$R_t \leqslant \exp(-ct^{\frac{1}{3}-\frac{8}{3k}}). \tag{3.16}$$

The remainder of this section is devoted to the proofs of Theorem 3.3 and Theorem 3.4.

Let $\mathbb{T}$ be a supercritical Galton-Watson family tree, conditioned on non-extinction, with an offspring distribution satisfying (1.30) for some constants $c_1, c_2 > 0$ and $k > 8$. Then, $\mathbb{T}$ fulfils the initial assumption of both theorems. Furthermore, we always consider $t \in \mathbb{N}$, even if not explicitly mentioned.

As in the proof of Theorem 2.1, we first introduce the following events of "bad" trees as subsets of $\mathbb{T}$, which we later want to exclude. Their negligence will result only in an error term exponentially small in $t^{\frac{1}{3}}$ in the annealed regime.

We start by recalling the definition of A_t from Definition 2.2. So, let $c_4 > \ln \lambda$ be a constant, where λ is defined in (1.15), then,

$$A_t = \{\#\mathbf{T}_{0t} \geqslant \exp(c_4 t)\}, \tag{3.17}$$

where $\mathbf{T}_{0t}$ denotes the subtree of $\mathbf{T} \in \mathbb{T}$ up to generation t, see (1.14). Furthermore, let $c_3 > 3 + \frac{c_4}{c_2}$ be a constant, where c_2 is specified in (1.30). We define the new sets of trees

$$\begin{aligned} I_t &:= \{\exists x \in \mathbf{T}_{0t} : \ c_3 t^{\frac{1}{k}} \leqslant Z(x)\} \setminus A_t, \\ F_t &:= A_t \cup I_t. \end{aligned} \tag{3.18}$$

In the proof of Lemma 2.3 we have already seen for every $t \in \mathbb{N}$ that $G[A_t] \leqslant \frac{1}{1-r} \frac{\lambda^{t+1}-1}{\lambda-1} \exp(-c_4 t)$, where r denotes the extinction probability. Since $c_4 > \ln \lambda$, $G[A_t]$ decays exponentially in t. Moreover, exactly as for the event B_t in the the proof of Lemma 2.3, we conclude that

$$G[I_t] \leqslant \frac{1}{1-r} \exp(c_4 t) \sum_{j=\lfloor c_3 t^{\frac{1}{k}} \rfloor}^{\infty} G^*[Z = j] \tag{3.19}$$

and, thus, inserting the decay (1.30) of the offspring distribution $p_j = G^*[Z = j]$ and estimating the resulting sum by an integral, we obtain

$$G[I_t] \leqslant \frac{c_1}{c_2(1-r)} \exp(c_4 t) \exp\big(- c_2 (c_3 - 2)^k t \big) \leqslant \frac{c_1}{c_2(1-r)} \exp(-c_2 t). \tag{3.20}$$

Here, again $c_3 > 3 + \frac{c_4}{c_2}$ yields the second inequality. Therefore, combining the exponential decay of A_t and of I_t, there are constants $C, c_5 > 0$ such that, for every $t \in \mathbb{N}$, we have

$$G[F_t] \leqslant C \exp(-c_5 t). \tag{3.21}$$

Furthermore, we also need to define

$$D_t := \Big\{ \mathbf{T} \in \mathbb{T} : \ \exists o \in K \subset \mathbf{T} \text{ connected}, t \leqslant \#K < \infty, \frac{\partial K}{\#K} \leqslant h \Big\} \tag{3.22}$$

for $t \in \mathbb{N}$ and $h > 0$. Then, by [45, Proof of Thm. 6.52], we know that

Lemma 3.5. *There exists a constant $c_6 > 0$ such that for all $h \in]0, h_{\max}[$ and every $t \in \mathbb{N}$*

$$G[D_t] \leqslant \exp(-c_6 t), \tag{3.23}$$

with $h_{\max} \in]0, \mathbf{i}(\mathbb{T})]$.

W.l.o.g. we will choose the constant $h \in]0, \min\{1, h_{\max}\}[$ in the sequel.

Next, we will introduce some basic notions as in [52, Sect. 3] to exploit the consequences of a positive anchored expansion constant. But whereas Virág works with weighted volumes, ours refer to the cardinality of the sets in accordance with our previous definitions. For this next definition, let $\mathbf{T}$ be any fixed graph with locally bounded vertex degrees. In particular, it does not need to be a realisation of a Galton-Watson tree here.

Definition 3.6. *Let $q > 0$ and let $\mathbf{T}$ be any infinite, connected, locally finite graph.*

*(a) The q-*isolation *of a (possibly empty) finite vertex subset $S \subseteq \mathbf{T}$ is given by*

$$\Delta_q S := \Delta_q^{\mathbf{T}} S := q\#S - \#\partial S. \tag{3.24}$$

We will omit the superscript $\mathbf{T}$ when there is no danger of confusion.

*(b) We say that a finite vertex subset $S \subseteq \mathbf{T}$ is q-*isolated *whenever*

$$\Delta_q S > 0. \tag{3.25}$$

*(c) A (possibly empty) finite vertex set $S \subseteq \mathbf{T}$ is called a q-*isolated core *of $\mathbf{T}$ whenever*

$$\Delta_q S > \Delta_q A \quad \text{for every } A \subsetneqq S. \tag{3.26}$$

*(d) We write $A_q := A_q(\mathbf{T})$ for the union of all q-isolated cores of $\mathbf{T}$ and call any connected component of A_q a (q-)*island. *The complement $\mathbf{T} \setminus A_q$ is called the (q-)*oceans.

Remark. *1. A non-empty q-isolated core is itself q-isolated because in this case the subset A can be chosen as the empty set with q-isolation $\Delta_q\varnothing = 0$.*

2. *Every connected component of a non-connected q-isolated core is itself a q-isolated core. This follows from the additivity of the q-isolation w.r.t. the connected components and by choosing the subset A to be the union of a proper subset of one connected component together with all other connected components.*

3. *It will turn out the q-islands act as traps for the random walk and thus prevent us from obtaining suitable heat-kernel bounds. Restricting the random walk to the q-oceans will allow us to benefit from non-anchored, i.e. global, isoperimetric constants.*

The definition (1.17) of the anchored expansion constant for Galton-Watson trees $\mathbf{i}(\mathbb{T})$ and Lemma A.3 directly imply

Corollary 3.7. *Let $q \in]0, \mathbf{i}(\mathbb{T})[$. Then, for G-almost every $\mathbf{T} \in \mathbb{T}$, every q-island of $\mathbf{T}$ has only finitely many vertices and thus is itself a q-isolated core of $\mathbf{T}$.*

Let $\mathbf{T} \in \mathbb{T}$ and $q > 0$ be fixed. A *bridge structure* interconnecting a vertex set $S \subset \mathbf{T}$ is a set of vertices $B \subset \mathbf{T}$ such that $B \cup S$ is a connected set. A *bridge* connecting two vertex sets $S_1, S_2 \subset \mathbf{T}$ is a vertex set $B \subset \mathbf{T}$ such that $B \cup S_1 \cup S_2$ has a connected component intersecting both S_1 and S_2. We define the *q-length* of a bridge $B \subset \mathbf{T}$ by

$$q\text{-length}(B) := \#(B \setminus A_q), \tag{3.27}$$

that is, the number of vertices of B belonging to the q-oceans of $\mathbf{T}$. Given a vertex set $S \subset \mathbf{T}$ and a vertex $v \in \mathbf{T}$, we define their *q-distance* by

$$\mathrm{dist}_q(v, S) := \begin{cases} 0\,, & v \in S, \\ 1 + \min\limits_{\substack{\text{bridges } B \subset \mathbf{T} \\ \text{connecting } \{v\} \text{ and } S}} \big\{q\text{-length}(B)\big\}, & v \notin S. \end{cases} \tag{3.28}$$

As noted before, q-islands pose a problem for obtaining heat-kernel bounds. Given $t \in \mathbb{N}$, the event

$$\begin{aligned} H^0_{q,t} := \Big\{ \mathbf{T} \in \mathbb{T} : \exists \text{ a finite union of } q\text{-islands } U_{q,t} = U_{q,t}(\mathbf{T}) \subseteq \mathbf{T} \text{ with} \\ 2^{\frac{5}{6}} t^{\frac{1}{3}} \leqslant q \# U_{q,t} < \infty \text{ and } \exists \text{ a bridge structure } B_{q,t} \\ \text{interconnecting } \{o\} \cup U_{q,t} \text{ with } \max_{v \in B_{q,t}} \mathrm{dist}(o, v) \leqslant t \Big\} \end{aligned} \tag{3.29}$$

describes trees where these islands are too dominant and situated too close to the root, that is, reachable for the random walk in t steps. The next lemma allows to exclude the particularly bad situation, where these islands are too close together and too close to the root with respect to the q-length. However, such control is only possible with a restriction on the growth in the relevant part of the tree.

Lemma 3.8. *For $t \in \mathbb{N}$ and $z_t \in \mathbb{N} \setminus \{1\}$ let*

$$M_{z_t} := \big\{\mathbf{T} \in \mathbb{T} : \; Z_{\mathbf{T}}(x) \leqslant z_t - 1 \; \forall x \in \mathbf{T}_{0t}\big\} \tag{3.30}$$

be the event of trees whose numbers of offsprings are bounded by the same constant $z_t - 1$ for every vertex up to generation t. Furthermore, we set

$$q := \frac{2}{3}h, \tag{3.31}$$

where h is given in Lemma 3.5, and define the subset

$$H_t := \left\{\mathbf{T} \in H^0_{q,t} : \; z_t \frac{\#\big((B_{q,t} \cup \{o\}) \setminus A_q\big)}{\#U_{q,t}} \leqslant \frac{h}{3}\right\} \tag{3.32}$$

of trees from $H^0_{q,t}$, for which there exists a small (w.r.t. the q-length) bridge structure connecting the bad q-islands with the root and among each other. Then, we have

$$G[M_{z_t} \cap H_t] \leqslant \exp(-c_6 t^{\frac{1}{3}}), \tag{3.33}$$

where $c_6 > 0$ is the constant from Lemma 3.5.

Proof. We fix $t \in \mathbb{N}$ and $q := \frac{2}{3}h < \frac{2}{3}$. Let $\mathbf{T} \in M_{z_t} \cap H_t \subseteq M_{z_t} \cap H^0_{q,t}$. Let A be the union of all q-islands of $\mathbf{T}$ intersecting $\{o\} \cup B_{q,t} \cup U_{q,t}$. Thus, $U_{q,t} \subseteq A \subseteq A_q$ and A is itself a q-isolated core so that $\frac{\#\partial A}{\#A} < q$. We define the part

$$S := (\{o\} \cup B_{q,t}) \setminus A_q = (\{o\} \cup B_{q,t} \cup U_{q,t}) \setminus A_q = (\{o\} \cup B_{q,t} \cup U_{q,t}) \setminus A \tag{3.34}$$

of the bridge structure and the root not belonging to any q-island.

Since we assume $\mathbf{T} \in H_t$, the definition (3.32) implies that

$$z_t \frac{\#S}{\#A} \leqslant z_t \frac{\#S}{\#U_{q,t}} \leqslant \frac{h}{3}. \tag{3.35}$$

Furthermore, we conclude from $A \cap S = \varnothing$ that

$$\frac{\#\partial(A \cup S)}{\#(A \cup S)} \leqslant \frac{\#\partial A + \#\partial S}{\#A + \#S} \leqslant \frac{q + z_t \frac{\#S}{\#A}}{1 + \frac{\#S}{\#A}}. \tag{3.36}$$

For the last inequality we used $\#\partial A < q\#A$ and $\#\partial S \leqslant z_t\#S$, which follows from $\mathbf{T} \in M_{z_t}$ and that the bridge structure has maximal graph distance t to the root.

For $0 < q < 1 < z_t$, the elementary estimate

$$\frac{q + z_t a}{1 + a} \leqslant \frac{q + z_t b}{1 + b} \tag{3.37}$$

holds for any $0 \leqslant a \leqslant b$. The inequality (3.35) allows to apply (3.37) to (3.36) with $a = \frac{\#S}{\#A}$ and $b = \frac{h}{3z_t}$, yielding

$$\frac{\#\partial(A \cup S)}{\#(A \cup S)} \leqslant \frac{q + \frac{h}{3}}{1 + \frac{h}{3z_t}} < h \tag{3.38}$$

by the definition (3.31) of q. To summarise we note that

$$K := A \cup S = A \cup \{o\} \cup B_{q,t} \cup U_{q,t} \tag{3.39}$$

contains the root o, is connected, has finite volume $\#K \geqslant \#U_{q,t} \geqslant 2^{\frac{5}{6}} t^{\frac{1}{3}}/q$ and satisfies (3.38). Hence, $\mathbf{T} \in D_{\lfloor 2^{\frac{5}{6}} t^{\frac{1}{3}}/q \rfloor}$ and the claim follows from (3.23) and $\lfloor 2^{\frac{5}{6}} t^{\frac{1}{3}}/q \rfloor \geqslant t^{\frac{1}{3}}$ for $t \in \mathbb{N}$. $\square$

Next, we consider a undirected, locally finite, connected, infinite graph (V, E) with vertex set V and edge set E. Let $\mathbf{G} = (V, w)$ denote its w-weighted version. Here, the weights w are given by (1.2) with edge weights $c\colon V \times V \to [0, \infty)$, where $c(\{x, y\}) = 0$ if and only if $\{x, y\} \notin E$. To stress the symmetry, since (V, E) is undirected, we write $\{x, y\}$ instead of (x, y). From now on, this will be interpreted by the symmetric weight function $c\colon V \times V \to [0, \infty)$, i.e., $c(x, y) = c(y, x)$ for every $x, y \in V$.

We recall the definition of the volume Cheeger constant of the weighted graph (V, w), compare with (1.3), and write

$$Q_w \coloneqq \inf\left\{ \frac{\|\partial S\|_c}{\|S\|_w} : \varnothing \neq S \subset V \text{ finite} \right\} \leqslant 1. \tag{3.40}$$

Here, we recall that $\|S\|_w \coloneqq \sum_{x \in S} w(x)$ and $\|\partial S\|_c \coloneqq \sum_{\{x,y\} \in \partial S} c(x, y)$.

Remark. *The volume Cheeger constant* (3.40) *is also called an edge-isoperimetric constant. It resembles the definition of the anchored expansion constant* (1.5) *but for weighted graphs, without the anchor, and without the connectivity. However, in the absence of the anchor, the edge-isoperimetric constant is typically zero for the realisations of a Galton-Watson tree.*

From now on, we consider an infinite rooted tree $\mathbf{T}$ for which every vertex $x \in \mathbf{T}$ has a finite vertex degree d_x. We compare with Subsection 1.2.1 and see that the simple random walk $\{X_t\}_{t\in\mathbb{N}_0}$ on $\mathbf{T}$ coincides with the standard random walk on the weighted graph $(\mathbf{T}, w_{\text{SRW}})$ with edge weights equal to

$$c_{\text{SRW}}(x,y) := \begin{cases} 1, & \text{if } \{x,y\} \text{ is an edge of } \mathbf{T}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.41}$$

Thus, the vertex weight is given by $w_{\text{SRW}}(x) := d_x$ for every $x \in \mathbf{T}$ according to (1.2). We write $\mathbf{P} := \mathbf{P_T}$ for the associated Markov operator on $l^2(\mathbf{T})$, which is symmetric on $l^2(\mathbf{T}, w_{\text{SRW}})$. Furthermore, we will sometimes add the superscript $\mathbf{T}$ to stress the specific graph $\mathbf{T}$ which we consider, e.g. for the probability measure this leads to $P^{\mathbf{T}}$.

In Lemma 3.10 we will establish that the q-oceans exhibit an isoperimetric inequality. Therefore we would like to apply Theorem 1.4 to the standard random walk on the weighted graph $(\mathbf{T} \setminus A_q, w_{\text{SRW}})$. But this requires a connected graph. Therefore we will construct a connected weighted graph $(\mathbf{T}_q, w_q)$ whose vertex set coincides with that of $\mathbf{T} \setminus A_q$ and such that the standard random walk $\{W_t\}_{t\in\mathbb{N}_0}$ on $(\mathbf{T}_q, w_q)$ behaves like the simple random walk $\{X_t\}_{t\in\mathbb{N}_0}$ on $\mathbf{T}$ if the latter is only observed on the q-oceans $\mathbf{T}\setminus A_q$. This standard random walk $\{W_t\}_{t\in\mathbb{N}_0}$ on $(\mathbf{T}_q, w_q)$ is often referred to as the induced Markov chain of $\{X_t\}_{t\in\mathbb{N}_0}$ on $\mathbf{T} \setminus A_q$ and is specified in the following definition.

Definition 3.9. *Let $q > 0$.*

1. *We write*
$$\tau_S := \inf\big\{t \in \mathbb{N} : X_t \in S\big\} \in \mathbb{N} \cup \{\infty\} \tag{3.42}$$
for the first hitting time *after zero of a vertex subset $S \subseteq \mathbf{T}$ by the simple random walk $\{X_t\}_{t\in\mathbb{N}_0}$ on $\mathbf{T}$. We also introduce the abbreviation $\tau_{\text{oc}} := \tau_{\mathbf{T}\setminus A_q}$ for the first hitting time of the q-oceans.*

2. *The edge weights of the weighted graph $(\mathbf{T}_q, w_q)$ with vertex set $\mathbf{T} \setminus A_q$ are given by*
$$c_q(x,y) := w_{\text{SRW}}(x) P_x^{\mathbf{T}}[X_{\tau_{\text{oc}}} = y] \tag{3.43}$$
for all vertices $x, y \in \mathbf{T}_q$. Accordingly, $P_x^{\mathbf{T}}[X_{\tau_{\text{oc}}} = y]$ is the probability that the simple random walk on $\mathbf{T}$ ends up at y when it first returns to the oceans $\mathbf{T} \setminus A_q$ after having left its starting point x.

3. *We write $P_x^{\mathbf{T}_q}$ for the probability measure of the standard random walk $\{W_t\}_{t\in\mathbb{N}_0}$ on $(\mathbf{T}_q, w_q)$, which starts at $x \in \mathbf{T}_q$. We use the symbol $\mathbf{P}_q := \mathbf{P}_{\mathbf{T}_q}$ for the associated Markov operator on $l^2(\mathbf{T}_q)$, which is symmetric on $l^2(\mathbf{T}_q, w_q)$.*

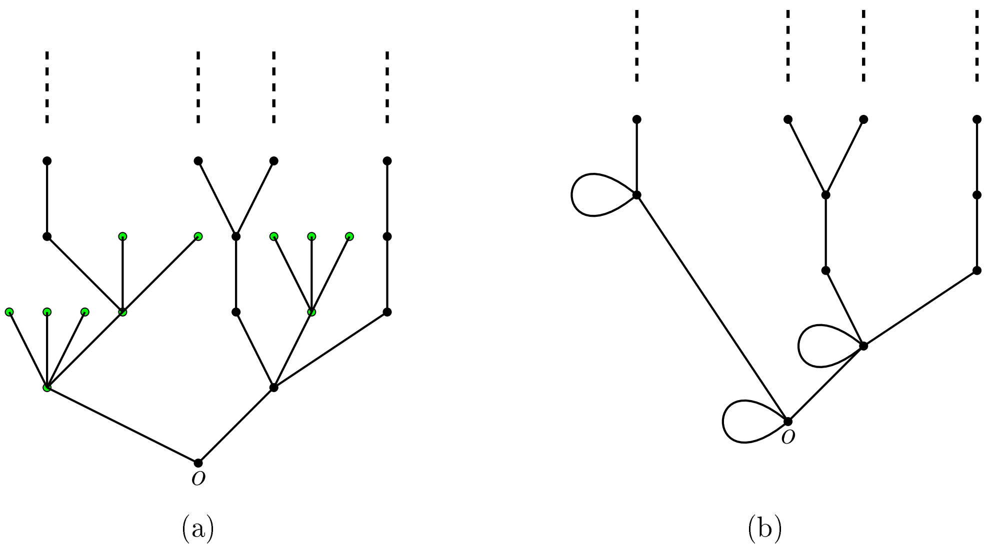


Figure 3.2: The realisation of a Galton-Watson tree $\mathbf{T} \in \mathbb{T}$ with the vertices in A_q highlighted in green in (a) and $\mathbf{T}_q$ in (b).

The following properties hold.

Remark. 1. *For every $x, y \in \mathbf{T}_q$ we have*

$$c_q(x,y) = c_q(y,x), \tag{3.44}$$

that is, c_q is symmetric and, hence, it is indeed an edge-weight function. This follows from time reversibility of the simple random walk on $\mathbf{T}$: Consider a path $X_0 = x, X_1 = x_1, \ldots, X_n = x_n, X_{n+1} = y$, where $n \in \mathbb{N}$ and $x_j \in \mathbf{T}$ for $j = 1, \ldots, n$, which contributes to the probability $P_x^{\mathbf{T}}[X_{\tau_{\mathrm{oc}}} = y]$. This path has probability $\frac{1}{w_{\mathrm{SRW}}(x)} \prod_{j=1}^n \frac{1}{w_{\mathrm{SRW}}(x_j)}$. It corresponds uniquely to a time-reversed path $X_0 = y, X_1 = x_n, \ldots, X_n = x_1, X_{n+1} = x$ contributing to $P_y^{\mathbf{T}}[X_{\tau_{\mathrm{oc}}} = x]$ with probability $\frac{1}{w_{\mathrm{SRW}}(y)} \prod_{j=1}^n \frac{1}{w_{\mathrm{SRW}}(x_{n-j+1})}$. The same holds vice versa and proves (3.44).

2. *For every $x, y \in \mathbf{T}_q$ we have*

$$c_q(x,y) \geqslant c_{\mathrm{SRW}}(x,y), \tag{3.45}$$

where strict inequality can only occur if both x and y belong to the outer vertex boundary $\partial_{\mathrm{out}} C := \{\widetilde{x} \in \mathbf{T} : \mathrm{dist}(\tilde{x}, C) = 1\}$ of the same q-island $C \in A_q$.

Indeed, if there is no edge between x and y, then $w_{\text{SRW}}(x, y) = 0$, and the inequality is trivial. If there is an edge between x and y then there exists a one-step path from x to y with $\tau_{\text{oc}} = 1$ and $w_{\text{SRW}}(x)P_x^{\mathbf{T}}[X_1 = y] = 1$. If both $x, y \in \partial_{\text{out}}C$ there may exist a path from x to y lying entirely in the q-island C except for the two endpoints x and y. In this case, $P_x^{\mathbf{T}}[X_{\tau_{\text{oc}}} = y$ and $\tau_{\text{oc}} > 1] > 0$ gives rise to an additional contribution to $c_q(x, y)$ beyond $c_{\text{SRW}}(x, y)$.

3. *Let $C \in A_q$ be a q-island and assume that $x \in \partial_{\text{out}}C$. Then*

$$c_q(x, x) > 0. \tag{3.46}$$

If $x \in \mathbf{T} \setminus A_q$ is not adjacent to any q-island, then $c_q(x, x) = 0$, as follows from (3.43). In other words, $(\mathbf{T}_q, w_q)$ is not a simple graph, but one with loops at the outer vertex boundaries of all q-islands.

4. *For every $x \in \mathbf{T} \setminus A_q$ we have $\sum_{y \in \mathbf{T} \setminus A_q} P_x^{\mathbf{T}}[X_{\tau_{\text{oc}}} = y] = 1$ and thus*

$$w_q(x) := \sum_{y \in \mathbf{T}_q} c_q(x, y) = w_{\text{SRW}}(x). \tag{3.47}$$

5. *We claim that the probability for two arbitrary vertices $x, y \in \mathbf{T} \setminus A_q$ to be connected by a path of the simple random walk $\{X_t\}_{t \in \mathbb{N}_0}$ on $\mathbf{T}$ is the same as for the standard random walk $\{W_t\}_{t \in \mathbb{N}_0}$ on $(\mathbf{T}_q, w_q)$, that is,*

$$P_x^{\mathbf{T}}[\exists\, t \in \mathbb{N} : X_t = y] = P_x^{\mathbf{T}_q}[\exists\, t \in \mathbb{N} : W_t = y]. \tag{3.48}$$

Before we prove (3.48) we note that it immediately implies

$$P_x^{\mathbf{T}}[\exists\, t \in \mathbb{N}_0 : X_t = y] = P_x^{\mathbf{T}_q}[\exists\, t \in \mathbb{N}_0 : W_t = y]. \tag{3.49}$$

To prove (3.48) let $\{\sigma_n\}_{n \in \mathbb{N}_0}$ be the strictly increasing sequence of stopping times which are uniquely defined by $\sigma_0 := 0$, $\sigma_n := \sigma_{n-1} + 1$ if both $X_{\sigma_{n-1}}$, $X_{\sigma_{n-1}+1} \in \mathbf{T} \setminus A_q$ for $n \in \mathbb{N}$ and the property that $X_t \in A_q$ if and only if

$\sigma_{n-1} < t < \sigma_n$ for some $n \in \mathbb{N}$. We infer that

$$\begin{aligned}
&P_x^{\mathbf{T}}\big[\exists\, t \in \mathbb{N} : X_t = y\big] \\
&\qquad = P_x^{\mathbf{T}}\big[\exists\, t \in \mathbb{N} : X_{\sigma_t} = y\big] \\
&\qquad = \sum_{t\in\mathbb{N}} P_x^{\mathbf{T}}\Big[X_{\sigma_t} = y \quad \text{and} \quad X_{\sigma_n} \in \mathbf{T} \setminus (A_q \cup \{y\})\ \forall n = 1,\dots,t-1\Big] \\
&\qquad = \sum_{t\in\mathbb{N}} \sum_{\substack{y_1,\dots,y_{t-1} \\ \in \mathbf{T}\setminus(A_q\cup\{y\})}} \prod_{n=1}^{t} P_{y_{n-1}}^{\mathbf{T}}\big[X_{\sigma_1} = y_n\big],
\end{aligned} \tag{3.50}$$

where $y_0 := x$, $y_t := y$ and we used the strong Markov property for the last equality. Now $\sigma_1 = \tau_{\mathrm{oc}}$ and the probability in the last line of (3.50) *is equal to*

$$P_{y_{n-1}}^{\mathbf{T}}[X_{\tau_{\mathrm{oc}}} = y_n] = \frac{c_q(y_{n-1}, y_n)}{w_q(y_{n-1})} = P_{y_{n-1}}^{\mathbf{T}_q}[W_1 = y_n], \tag{3.51}$$

where the first equality follows from (3.43) *and* (3.47)*, and the second equality from the definition of the standard random walk. Inserting* (3.51) *into* (3.50) *and using the Markov property for $\{W_t\}_{t\in\mathbb{N}_0}$, we infer*

$$P_x^{\mathbf{T}}\big[\exists\, t \in \mathbb{N} : X_t = y\big] = \sum_{t\in\mathbb{N}} P_x^{\mathbf{T}_q}\Big[W_t = y \quad \text{and} \quad W_n \in \mathbf{T}_q\setminus\{y\}\ \forall n = 1,\dots,t-1\Big] \tag{3.52}$$

so that (3.48) *follows.*

The next lemma requires a growth condition on the tree that must hold throughout the entire oceans in order to obtain the desired lower bound on the edge-isoperimetric constant. Later on, when applying this lemma to bound the return probability to the root, this growth condition can be satisfied at no additional cost for trees in the event M_{z_t} from (3.30) because the random walk cannot explore the parts of the tree at distance larger than t to the root.

Lemma 3.10. *Let $q > 0$ and let $\mathbf{T}$ be a rooted tree for which there exists $z \in \mathbb{N}\setminus\{1\}$ such that $d_x \leqslant z$ for every $x \in \mathbf{T} \setminus A_q$. Then the weighted graph $(\mathbf{T}_q, w_q)$ has edge-isoperimetric constant*

$$Q_{w_q} \geqslant \frac{q}{z}. \tag{3.53}$$

Proof. Let $\varnothing \neq S \subseteq \mathbf{T}_q = \mathbf{T} \setminus A_q$ be a finite vertex subset. First, we infer from (3.47) and the growth assumption that $\|S\|_{w_q} = \|S\|_{w_{\mathrm{SRW}}} \leqslant z\#S$. The inequality (3.45) implies that $\|\partial^{\mathbf{T}_q} S\|_{c_q} \geqslant \|\partial^{\mathbf{T}_q} S\|_{c_{\mathrm{SRW}}} = \#\partial^{\mathbf{T}\setminus A_q} S$. Let C be a (possibly empty) q-isolated core containing all vertices in A_q which are adjacent to S. When A_q is removed from $\mathbf{T}$, the number of edges $\#\partial S := \#\partial^{\mathbf{T}} S$ of the edge boundary of S in $\mathbf{T}$ decreases by the number of edges connecting S with C. Hence, we obtain $\#\partial^{\mathbf{T}\setminus A_q} S = \#\partial S - \#(\partial S \cap \partial C)$. Altogether, we arrive at the estimate

$$\frac{q}{z}\,\|S\|_{w_q} - \|\partial^{\mathbf{T}_q} S\|_{c_q} \leqslant \Delta_q S + \#(\partial S \cap \partial C) = \Delta_q(S\cup C) - \Delta_q C - \#(\partial S\cap\partial C), \quad (3.54)$$

where the equality results from an application of (A.6).

Since $S \subseteq \mathbf{T} \setminus A_q$, the vertex subset $S \cup C$ cannot be a q-isolated core. By definition, there must exist a (possibly empty) vertex subset $B \subsetneq S \cup C$ with

$$\Delta_q(S \cup C) \leqslant \Delta_q B. \quad (3.55)$$

W.l.o.g. we choose this vertex subset B to be minimal in the sense that no proper subset of B has the property (3.55). In other words, for every $\widetilde{B} \subsetneq B$, we must have $\Delta_q(S \cup C) > \Delta_q \widetilde{B}$. Together with (3.55), this means that B is a q-isolated core, whence $B \subseteq C$. Applying Lemma A.1 with $A = B$ and the q-isolated core $S = C$, yields $\Delta_q(B) \leqslant \Delta_q(B \cup C) = \Delta_q(C)$. Combining this inequality with (3.55), yields

$$\Delta_q(S \cup C) \leqslant \Delta_q C. \quad (3.56)$$

Now, (3.56) and (3.54) imply

$$\frac{q}{z}\,\|S\|_{w_q} - \|\partial^{\mathbf{T}_q} S\|_{c_q} \leqslant 0, \quad (3.57)$$

and the claim follows. □

Switching between the graphs $(\mathbf{T}, w_{\text{SRW}})$ and $(\mathbf{T}_q, w_q)$ will not only be done with the help of (3.48) but also on the level of the Hilbert spaces.

Definition 3.11. *Let $q > 0$ and let $\mathbf{T}$ be an infinite rooted tree with $d_x < \infty$ for every $x \in \mathbf{T}$. We introduce the restriction map*

$$\rho_{\mathbf{T}} : \begin{array}{rcl} l^2(\mathbf{T}, w_{\text{SRW}}) & \to & l^2(\mathbf{T}_q, w_q) \\ (\psi_x)_{x\in\mathbf{T}} & \mapsto & (\psi_x)_{x\in\mathbf{T}\setminus A_q} \end{array} \quad (3.58)$$

and its adjoint, the embedding

$$\rho_{\mathbf{T}}^* : \begin{array}{rcl} l^2(\mathbf{T}_q, w_q) & \to & l^2(\mathbf{T}, w_{\text{SRW}}) \\ (\varphi_x)_{x\in\mathbf{T}\setminus A_q} & \mapsto & (\widetilde{\varphi}_x)_{x\in\mathbf{T}} \end{array}, \quad \textit{where} \quad \widetilde{\varphi}_x := \begin{cases} \varphi_x, & \textit{if } x \in \mathbf{T}\setminus A_q, \\ 0, & \textit{if } x \in A_q. \end{cases} \quad (3.59)$$

We drop the index $\mathbf{T}$ *in our notation for both maps, if the underlying tree is clear. Both* ρ *and* ρ^* *have operator norm* 1 *due to* (3.47).

The next lemma estimates the probability for the random walk to enter a bad geometric region consisting of several q-islands. Since we are on a tree we are able to obtain an estimate which scales with the square root of the number of involved q-islands. Without the tree property, one would get a scaling with the square root of the volume of the edge boundaries of the involved q-islands as in [52]. The improved scaling for trees will be crucial when applying the lemma in the proof of Theorem 3.13.

Lemma 3.12. *Let* $q \in]0,1[$ *and let* $\mathbf{T}$ *be an infinite rooted tree with infinite* q*-oceans* $\#(\mathbf{T} \setminus A_q) = \infty$. *Furthermore, we assume a maximal vertex degree on* $\mathbf{T} \setminus A_q$, *i.e. the existence of* $z \in \mathbb{N} \setminus \{1\}$ *such that* $d_{x'} \leqslant z$ *holds for every* $x' \in \mathbf{T} \setminus A_q$. *Let* $\mathcal{C} := \bigcup_{j=1}^J C_j \subseteq \mathbf{T}$ *be a union of* $J \in \mathbb{N}$ q*-islands* $C_j \in A_q$, $j \in \{1, ..., J\}$. *We also fix a vertex* $x \in \mathbf{T}$ *with* $\operatorname{dist}_q(x, \mathcal{C}) \geqslant n$ *for some* $n \in \mathbb{N}$. *Then, the simple random walk* $(X_t)_{t\in\mathbb{N}_0}$ *on* $\mathbf{T}$ *satisfies*

$$P_x[\tau_{\mathcal{C}} < \infty] \leqslant 2\Big(1 - \frac{q^2}{z^2}\Big)^{\frac{n}{2}-1} \frac{z^{\frac{5}{2}}}{q^2} J^{\frac{1}{2}}. \tag{3.60}$$

Proof. To begin with we will argue that we may conduct the proof assuming w.l.o.g. that $x \in \mathbf{T} \setminus A_q$. Indeed, since $\operatorname{dist}_q(x, \mathcal{C}) \geqslant n$, we have $x \notin \mathcal{C}$. So suppose that $x \in A_q \setminus \mathcal{C}$. Then there exists a q-island $C' \subseteq A_q \setminus \mathcal{C}$ such that $x \in C'$ and we must have $\operatorname{dist}_q(x, \mathcal{C}) \geqslant \max\{n, 2\}$. In order to reach $\mathcal{C}$, the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ on $\mathbf{T}$ has to hit the outer vertex boundary $\partial_{\text{out}} C'$ before hitting $\mathcal{C}$. Therefore the strong Markov property of $(X_t)_{t\in\mathbb{N}_0}$ at the hitting time of $\partial_{\text{out}} C'$ implies

$$P_x[\tau_{\mathcal{C}} < \infty] = E_x\Big[P_{X_{\tau_{\partial_{\text{out}} C'}}}[\tau_{\mathcal{C}} < \infty]\Big] \leqslant \sup_{y \in \partial_{\text{out}} C'} P_y[\tau_{\mathcal{C}} < \infty], \tag{3.61}$$

where $E_x := \int \mathrm{d}P_x$ is the expectation associated with P_x. Because of (3.61), $\partial_{\text{out}} C' \subseteq \mathbf{T} \setminus A_q$ and $\operatorname{dist}_q(y, \mathcal{C}) \geqslant \operatorname{dist}_q(x, \mathcal{C}) - 1$ for all $y \in \partial_{\text{out}} C'$, which holds due to $C' \subseteq A_q$, it is sufficient to consider $x \in \mathbf{T} \setminus A_q$ with $\operatorname{dist}_q(x, \mathcal{C}) \geqslant \max\{n-1, 1\}$ in the rest of this proof.

So, let us fix $x \in \mathbf{T} \setminus A_q$ with $\operatorname{dist}_q(x, \mathcal{C}) \geqslant \max\{n-1, 1\}$. Since $\mathbf{T}$ is a tree and $\mathcal{C}$ consists of J connected components there exists a subset $V \subseteq \partial_{\text{out}} \mathcal{C} \subseteq \mathbf{T} \setminus A_q$ of the outer vertex boundary of $\mathcal{C}$ with $\#V \leqslant J$ and such that the simple random walk $(X_t)_{t\in\mathbb{N}_0}$ has to pass a vertex from V in the last step on his way from x before hitting $\mathcal{C}$ for the first time. Thus, we infer that

$$P_x[\tau_{\mathcal{C}} < \infty] \leqslant \sum_{y \in V} P_x\big[\exists\, t \in \mathbb{N}_0 :\ X_t = y\big]. \tag{3.62}$$

Applying (3.49) and the union bound to the probability on the right-hand side of (3.62), rewriting it in terms of the Markov operator $\mathbf{P}_q$ and then switching first from the unweighted Hilbert space $l^2(\mathbf{T}_q)$ to the weighted Hilbert space $l^2(\mathbf{T}_q, w_q)$ and finally to $l^2(\mathbf{T}, w_{\mathrm{srw}})$ with the embedding ρ^* and using (3.47), we obtain

$$P_x\big[\exists\, t \in \mathbb{N}_0 :\ X_t = y\big] \leqslant \frac{1}{w_{\mathrm{srw}}(x)} \sum_{t \in \mathbb{N}_0} \langle 1_{\{x\}} | \rho^* \mathbf{P}_q^t \rho 1_{\{y\}} \rangle_{\mathbf{T}, w_{\mathrm{srw}}}. \tag{3.63}$$

We deduce from (3.62) and (3.63) that

$$P_x[\tau_{\mathcal{C}} < \infty] \leqslant \frac{1}{w_{\mathrm{srw}}(x)} \sum_{t \in \mathbb{N}_0} \langle 1_{\{x\}} | \rho^* \mathbf{P}_q^t \rho 1_V \rangle_{\mathbf{T}, w_{\mathrm{srw}}}. \tag{3.64}$$

Since $\mathrm{dist}_q(x, \mathcal{C}) \geqslant \max\{n-1, 1\}$, the random walk needs at least $\nu := \max\{n-2, 0\}$ steps on the infinite connected weighted graph $(\mathbf{T}_q, w_q)$ to reach $V \subseteq \partial_{\mathrm{out}}\mathcal{C}$ from x and every term in the t-series in (3.64) with $t < \nu$ vanishes. We note that $\sum_{t=\nu}^{\infty} \mathbf{P}_q^t = \mathbf{P}_q^\nu \mathbf{K}_q$, where the Green kernel $\mathbf{K}_q := \sum_{t \in \mathbb{N}_0} \mathbf{P}_q^t$ exists in operator norm in the space of bounded operators on $l^2(\mathbf{T}_q, w_q)$ and satisfies the norm estimate

$$\|\mathbf{K}_q\|_{BL(\mathbf{T}_q, w_q)} \leqslant \frac{1}{1 - \|\mathbf{P}_q\|_{BL(\mathbf{T}_q, w_q)}} \leqslant \frac{2z^2}{q^2} \tag{3.65}$$

because $\|\mathbf{P}_q\|_{BL(\mathbf{T}_q, w_q)} \leqslant (1 - q^2/z^2)^{1/2} \leqslant 1 - q^2/(2z^2)$ due to Theorem 1.4 and Lemma 3.10. Accordingly, the t-series in (3.64) can be written as

$$\begin{aligned} \langle 1_{\{x\}} | \rho^* \mathbf{P}_q^\nu \mathbf{K}_q \rho 1_V \rangle_{\mathbf{T}, w_{\mathrm{SRW}}} &\leqslant \|1_{\{x\}}\|_{\mathbf{T}, w_{\mathrm{SRW}}} \|\mathbf{P}_q\|_{BL(\mathbf{T}_q, w_q)}^\nu \|\mathbf{K}_q\|_{BL(\mathbf{T}_q, w_q)} \|1_V\|_{\mathbf{T}, w_{\mathrm{SRW}}} \\ &\leqslant w_{\mathrm{SRW}}(x)^{\frac{1}{2}} \Big(1 - \frac{q^2}{z^2}\Big)^{\frac{\nu}{2}} \frac{2z^2}{q^2} J^{\frac{1}{2}} z^{\frac{1}{2}}, \end{aligned} \tag{3.66}$$

where the first inequality relies on the Cauchy-Schwarz inequality and the fact that the operator norms of ρ and ρ^* equal 1. Now, the lemma follows from (3.64) and (3.66). □

The next theorem is the main technical result used for the proofs of Theorem 3.3 and Theorem 3.4.

Theorem 3.13. *Let $(z_t)_{t \in \mathbb{N}} \subseteq \mathbb{N} \setminus \{1, 2\}$ be a sequence of constants with $z_t = \mathcal{O}(t^{\frac{1}{8}})$*

as $t \to \infty$. Then, there exists an initial time $t_0 \in \mathbb{N}$ such that

$$P_o^{\mathbf{T}}[X_t = o] \leqslant \exp\left[-\frac{h^2}{16}\left(\frac{t}{z_t^8}\right)^{\frac{1}{3}}\right] \tag{3.67}$$

for every $\mathbf{T} \in (M_{z_t} \cap H_t^c) \setminus \mathcal{N}$ and every $t \geqslant t_0$. Here, $h > 0$ is given by Lemma 3.5, and the G-null set $\mathcal{N}$ is the union of the G-null set where $|\mathbf{T}| = \infty$ fails with the G-null set where Corollary 3.7 fails. We note that the initial time $t_0 \in \mathbb{N}$ depends only on the given sequence $(z_t)_{t\in\mathbb{N}}$ and on h.

Before we can prove Theorem 3.13, we have to to deal with the possibly unbounded offsprings in the oceans of the tree $\mathbf{T} \in M_{z_t}$ beyond the height t.

Definition 3.14. *Let $q \in]0,1[$, $t \in \mathbb{N}$, $z_t \in \mathbb{N} \setminus \{1\}$ and consider a tree $\mathbf{T} \in M_{z_t}$.*

1. *We construct recursively, starting from the root, an associated q-*regularised tree $\mathbf{T}^q$ *– not to be confused with $\mathbf{T}_q$ from Definition 3.9 – by*

$$Z^{\mathbf{T}^q}(x) := \begin{cases} Z^{\mathbf{T}}(x), & \text{if } x \in \mathbf{T}_{0t} \text{ or if } x \in C \\ & \text{for some } q\text{-island } C \subseteq A_q(\mathbf{T}) \text{ with } \operatorname{dist}(o, C) \leqslant t, \\ z_t - 1, & \text{otherwise.} \end{cases} \tag{3.68}$$

   This means that $\mathbf{T}_{0t} = \mathbf{T}^q_{0t}$ and that the regularised tree $\mathbf{T}^q$ is homogenous from height $t+1$ onwards except at the vertices of those q-islands of $\mathbf{T}$ which have non-trivial intersection with $\mathbf{T}_{0t}$ and extend also beyond the height t.

2. *We write $\{X_t^{(q)}\}_{t\in\mathbb{N}_0}$ for the simple random walk on $\mathbf{T}^q$. The* regularised weighted graph $(\mathbf{T}^q_q, w_q^{(q)})$ *is given as in Definition 3.9 but with every reference to $\mathbf{T}$ there replaced by $\mathbf{T}^q$, that is,*

$$c_q^{(q)}(x,y) := w^{(q)}_{\text{SRW}}(x) P_x^{\mathbf{T}^q}[X^{(q)}_{\tau_{\mathbf{T}^q \setminus A_q(\mathbf{T}^q)}} = y] \tag{3.69}$$

   for every $x, y \in \mathbf{T}^q_q := \mathbf{T}^q \setminus A_q(\mathbf{T}^q)$ with $w^{(q)}_{\text{SRW}}(x) := d_x^{\mathbf{T}^q}$ being the vertex degree of x in $\mathbf{T}^q$. The standard random walk on $(\mathbf{T}^q_q, w_q^{(q)})$ will be denoted by $\{W_t^{(q)}\}_{t\in\mathbb{N}_0}$.

Lemma 3.15. *Let $0 < q' \leqslant q < \min\{\mathbf{i}(\mathbb{T}), 1\}$, let $z_t \in \mathbb{N} \setminus \{1, 2\}$ and consider an infinite tree $\mathbf{T} \in M_{z_t} \setminus \mathcal{N}$, with $\mathcal{N}$ being the null set from Theorem 3.13. Then*

1. *$Z^{\mathbf{T}^q}(x) \leqslant z_t - 1$ for all $x \in \mathbf{T}^q \setminus A_q(\mathbf{T}^q)$.*

2. *The representation*

$$A_{q'}(\mathbf{T}^q) = \bigcup_{q'\text{-islands } C\subseteq A_{q'}(\mathbf{T}):\ \mathrm{dist}(o,C)\leqslant t} \tag{3.70}$$

holds, and we have $\#(\mathbf{T}^q \setminus A_{q'}(\mathbf{T}^q)) = \infty$.

3. $c_q^{(q)}(x,y) = c_q(x,y)$ *for every* $x, y \in \mathbf{T}_{0t} \setminus A_q(\mathbf{T}) = \mathbf{T}_{0t}^q \setminus A_q(\mathbf{T}^q)$.

Proof. Part 1 holds by construction of $\mathbf{T}^q$ and because $\mathbf{T} \in M_{z_t}$.

As to Part 2 we define

$$\widetilde{A} := \bigcup_{q'\text{-islands } C\subseteq A_{q'}(\mathbf{T}):\ \mathrm{dist}(o,C)\leqslant t} C \tag{3.71}$$

and show the two inclusions to obtain equality of the sets $\widetilde{A}$ and $A_{q'}(\mathbf{T}^q)$. But first, we notice that the vertices of $\mathbf{T}_{0t} \cup \widetilde{A}$ also belong to $\mathbf{T}^q$ and have the same degree in $\mathbf{T}^q$ as in $\mathbf{T}$. Therefore and since $\mathrm{dist}(o, C) \leqslant t$ for every q'-island $C \subseteq \widetilde{A}$, we infer that

$$\Delta_{q'}^{\mathbf{T}} S = \Delta_{q'}^{\mathbf{T}^q} S \tag{3.72}$$

for every finite vertex subset $S \subseteq \mathbf{T}_{0t} \cup \widetilde{A}$.

"$\widetilde{A} \subseteq A_{q'}(\mathbf{T}^q)$" As each q'-island $C \subseteq \widetilde{A}$ is finite by Corollary 3.7 and thus a q'-isolated core in $\mathbf{T}$, the identity (3.72) implies that C is also a q'-isolated core in $\mathbf{T}^q$ and, hence, $C \subseteq A_{q'}(\mathbf{T}^q)$.

"$A_{q'}(\mathbf{T}^q) \subseteq \widetilde{A}$" Let $\varnothing \neq C' \subseteq A_{q'}(\mathbf{T}^q)$ be a q'-isolated core in $\mathbf{T}^q$. In particular, C' is finite. First, we consider the case where $C' \subseteq \mathbf{T}_{0t}^q \cup \widetilde{A}$. In this case, the identity (3.72) implies that C' is also a q'-isolated core in $\mathbf{T}$, i.e. $C' \subseteq A_{q'}(\mathbf{T})$ and, hence, $C' \subseteq \widetilde{A}$. We now show that the complementary case in which there exists a vertex $x \in C' \cap [\mathbf{T}^q \setminus (\mathbf{T}_{0t}^q \cup \widetilde{A})]$ cannot occur. Indeed, since $\mathbf{T}^q$ is a tree and C' is finite, it follows that there exists $x' \in C' \cap [\mathbf{T}^q \setminus (\mathbf{T}_{0t}^q \cup \widetilde{A})]$ with $d_{x'}^{\mathbf{T}^q} = z_t$ and $d_{x'}^{C'} = 1$. By the definition of C' being a q'-isolated core of $\mathbf{T}^q$ we have

$$0 < \Delta_{q'}^{\mathbf{T}^q} C' - \Delta_{q'}^{\mathbf{T}^q}(C' \setminus \{x'\}) = q' - \left(d_{x'}^{\mathbf{T}^q} - 2d_{x'}^{C'}\right) = q' - z_t + 2. \tag{3.73}$$

But this is a contradiction, because $q' < 1$ and $z_t \geqslant 3$. This finishes the proof of (3.70). The equality (3.70) implies in particular that $\#A_{q'}(\mathbf{T}^q) < \infty$, because Corollary 3.7 applied to $\mathbf{T} \notin \mathcal{N}$ guarantees the finiteness of each q'-island $C \subseteq A_{q'}(\mathbf{T})$. Moreover, $\#\mathbf{T} = \infty$ because $\mathbf{T} \notin \mathcal{N}$ so that $\mathbf{T} \setminus \widetilde{A} \neq \varnothing$, and the construction of $\mathbf{T}^q$ implies that $\#\mathbf{T}^q = \infty$. This finishes the proof of Part 2.

Finally, we prove Part 3. We recall that by the construction of $\mathbf{T}^q$ and (3.70), the tree $\mathbf{T}_{0t} \cup A_q(\mathbf{T}^q)$ is an identical subtree of both $\mathbf{T}$ and $\mathbf{T}^q$. Let $x, y \in \mathbf{T}_{0t} \setminus A_q(\mathbf{T}) = \mathbf{T}^q_{0t} \setminus A_q(\mathbf{T}^q)$. In particular, we have

$$w^{(q)}_{\text{SRW}}(x) = d^{\mathbf{T}^q}_x = d^{\mathbf{T}}_x = w_{\text{SRW}}(x). \tag{3.74}$$

Moreover, the simple random walk $\{X_s\}_{s\in\mathbb{N}_0}$ on $\mathbf{T}$ when restricted to $\mathbf{T}_{0t} \cup A_q(\mathbf{T}^q) \subseteq \mathbf{T}$ coincides with the simple random walk $\{X^{(q)}_s\}_{s\in\mathbb{N}_0}$ on $\mathbf{T}^q$ when restricted to $\mathbf{T}_{0t} \cup A_q(\mathbf{T}^q) \subseteq \mathbf{T}^q$. This implies

$$P^{\mathbf{T}^q}_x[X^{(q)}_{\tau_{\mathbf{T}^q\setminus A_q(\mathbf{T}^q)}} = y] = P^{\mathbf{T}}_x[X_{\tau_{\mathbf{T}\setminus A_q(\mathbf{T})}} = y] \tag{3.75}$$

and the assertion follows from (3.74) and (3.75). □

Proof of Theorem 3.13. We fix $t \in \mathbb{N}$, $\mathbf{T} \in (M_{z_t} \cap H^c_t) \setminus \mathcal{N}$, $q := \frac{2}{3}h$, and

$$q_t := \frac{h}{2\sqrt{2}\,(tz_t)^{\frac{1}{3}}} \tag{3.76}$$

so that $q_t < q$ and, hence,

$$A_{q_t}(\mathbf{T}) \subseteq A_q(\mathbf{T}) \tag{3.77}$$

by Lemma A.4.

We decompose the return probability of the simple random walk on $\mathbf{T}$ according to

$$\begin{aligned} P^{\mathbf{T}}_o[X_t = o] &= P^{\mathbf{T}}_o\big[X_t = o \wedge \forall s \in \{1,...,t\} :\ X_s \in \mathbf{T}_{0t} \setminus A_{q_t}(\mathbf{T})\big] \\ &\quad + P^{\mathbf{T}}_o\big[X_t = o \wedge \exists\, s \in \{1,...,t\} :\ X_s \in \mathbf{T}_{0t} \cap A_{q_t}(\mathbf{T})\big] \\ &= P^{\mathbf{T}^{q_t}}_o\big[X^{(q_t)}_t = o \wedge \forall s \in \{1,...,t\} :\ X^{(q_t)}_s \in \mathbf{T}^{q_t}_{0t} \setminus A_{q_t}(\mathbf{T}^{q_t})\big] \\ &\quad + P^{\mathbf{T}^q}_o\big[X^{(q)}_t = o \wedge \exists\, s \in \{1,...,t\} :\ X^{(q)}_s \in \mathbf{T}^q_{0t} \cap A_{q_t}(\mathbf{T}^q)\big]. \end{aligned} \tag{3.78}$$

As for the second equality, we note that the regularised trees satisfy $\mathbf{T}_{0t} = \mathbf{T}^{q'}_{0t}$ and $\mathbf{T}_{0t} \cap A_{q_t}(\mathbf{T}) = \mathbf{T}^{q'}_{0t} \cap A_{q_t}(\mathbf{T}^{q'})$, which follows from (3.70), for both $q' = q_t$ and $q' = q$.

Next, we estimate the probability in the third line of (3.78). The fact that $\mathbf{T}^{q_t}$ is a tree implies that the simple random walk in this probability jumps only between vertices in $\mathbf{T}^{q_t} \setminus A_{q_t}(\mathbf{T}^{q_t})$ and such that no two consecutive vertices x, y in any path belong to the outer vertex boundary of the same q_t-island of $\mathbf{T}^{q_t}$. This means that for each jump, we have the equality $c^{(q_t)}_{\text{SRW}}(x, y) = c^{(q_t)}_{q_t}(x, y)$, cf. (3.45). Therefore, the estimate

$$P_o^{\mathbf{T}^{q_t}}\big[X_t^{(q_t)} = o \wedge \forall s \in \{1, ..., t\}:\ X_s^{(q_t)} \in \mathbf{T}_{0t}^{q_t} \setminus A_{q_t}(\mathbf{T}^{q_t})\big] \leqslant P_o^{\mathbf{T}_{q_t}^{q_t}}[W_t^{(q_t)} = o] \quad (3.79)$$

holds, where the inequality arises because the requirements that $\{W_s^{(q_t)}\}_{s\in\{1,...,t\}}$ must not jump over q_t-islands or is forbidden to stay at a vertex have been dropped. Rewriting the right-hand side in terms of the associated Markov operator $\mathbf{P}_{\mathbf{T}_{q_t}^{q_t}}$ on the weighted Hilbert space $l^2(\mathbf{T}_{q_t}^{q_t}, w_{q_t}^{(q_t)})$, we obtain

$$\begin{aligned} P_o^{\mathbf{T}^{q_t}}\big[X_t^{(q_t)} &= o \wedge \forall s \in \{1, ..., t\}:\ X_s^{(q_t)} \in \mathbf{T}_{0t}^{q_t} \setminus A_{q_t}(\mathbf{T}^{q_t})\big] \\ &\leqslant \frac{1}{w_{q_t}^{(q_t)}(o)} \langle 1_{\{o\}} | \mathbf{P}_{\mathbf{T}_{q_t}^{q_t}}^t 1_{\{o\}} \rangle_{\mathbf{T}_{q_t}^{q_t}, w_{q_t}^{(q_t)}} \leqslant \|\mathbf{P}_{\mathbf{T}_{q_t}^{q_t}}\|_{\mathbf{T}_{q_t}^{q_t}, w_{q_t}^{(q_t)}}^t \leqslant \Big(1 - \frac{q_t^2}{z_t^2}\Big)^{\frac{t}{2}} \\ &\leqslant \exp\Big[-\frac{q_t^2 t}{2z_t^2}\Big] = \exp\Big[-\frac{h^2}{16}\Big(\frac{t}{z_t^8}\Big)^{\frac{1}{3}}\Big], \end{aligned} \quad (3.80)$$

where the last inequality in the second line follows from an application of Theorem 1.4 and Lemma 3.10 to the weighted graph $(\mathbf{T}_{q_t}^{q_t}, w_{q_t}^{(q_t)})$. This is justified because of Lemma 3.15 Part 1 and because $\#\mathbf{T}_{q_t}^{q_t} = \infty$, see Lemma 3.15 Part 2. The inequality in the last line follows from $\ln(1+u) \leqslant u$, $|u| < 1$, which is applicable by the definitions of q_t and z_t and due to $h < 1$.

Before we estimate the probability in the last line of (3.78), we need to introduce one more notion. Let

$$A_{q,t} := \bigcup_{q\text{-islands } C \subseteq A_q(\mathbf{T}^q) \text{ with } \#C > \frac{1}{q_t}} C \subseteq A_q(\mathbf{T}^q) \quad (3.81)$$

be the union of all q-islands C in $\mathbf{T}^q$ with volume $\#C > \frac{1}{q_t}$. We remark that by construction of $\mathbf{T}^q$, all such q-islands C obey $\mathrm{dist}(o, C) \leqslant t$. Applying Lemma A.5 with $q' = q_t$ to any of the remaining q-islands $S \subseteq A_q(\mathbf{T}^q) \setminus A_{q,t}$, gives $S \subseteq \mathbf{T}^q \setminus A_{q_t}(\mathbf{T}^q)$ so that (3.77) with $\mathbf{T}$ replaced by $\mathbf{T}^q$ can be sharpened to

$$A_{q_t}(\mathbf{T}^q) \subseteq A_{q,t}. \quad (3.82)$$

Thus, the probability in the last line of (3.78) can be estimated as

$$\begin{aligned} P_o^{\mathbf{T}^q}\big[X_t^{(q)} = o \wedge \exists\, s \in \{1, ..., t\}:\ X_s^{(q)} \in \mathbf{T}_{0t}^q \cap A_{q_t}(\mathbf{T}^q)\big] \\ \leqslant P_o^{\mathbf{T}^q}\big[\exists\, s \in \{1, ..., t\}:\ X_s^{(q)} \in A_{q,t}\big]. \end{aligned} \quad (3.83)$$

In order to proceed further, we define the *q-territory* of a q-island $C \subseteq A_{q,t}$ by

$$D_C := \Big\{ x \in \mathbf{T}^q : \ \mathrm{dist}_q(x, C) \leqslant \frac{q}{4q_t z_t} \Big\} \tag{3.84}$$

and assert two claims.

Claim 1: $o \notin D_C$ for any q-island $C \subseteq A_{q,t}$.
In view of (3.70), Claim 1 will be obtained from the following argument: We assume that $o \in D_C$ for some q-island $C \subseteq A_q(\mathbf{T})$ with $\mathrm{dist}(o, C) \leqslant t$ and $\#C > \frac{1}{q_t}$ and strive for a contradiction. In fact, given these assumptions we conclude $\mathbf{T} \in H^0_{q,t}$ because C qualifies as $U_{q,t}$ in the definition (3.29). Indeed, since $z_t \geqslant 2$ we have $q\#C > \frac{q}{q_t} \geqslant 2^{\frac{5}{6}} t^{\frac{1}{3}}$. Furthermore, since $\mathbf{T} \notin \mathcal{N}$ we have $\#C < \infty$ by Corollary 3.7 and, finally, since $\mathrm{dist}(o, C) \leqslant t$, there exists a bridge $B_{q,t}$ connecting the root o with C and satisfying $\max_{v \in B_{q,t}} \mathrm{dist}(o, v) \leqslant t$. Without loss of generality we assume that $B_{q,t}$ is the bridge with the shortest q-length among all such bridges. Then,

$$z_t \frac{\#((B_{q,t} \cup \{o\}) \setminus A_q)}{\#C} = z_t \frac{\mathrm{dist}_q(o, C)}{\#C} \leqslant z_t \frac{q q_t}{4 q_t z_t} < \frac{h}{3} \tag{3.85}$$

where we used $o \in D_C$ and $\#C > \frac{1}{q_t}$ for the first inequality. It follows that $\mathbf{T} \in H_t$ according to the definition (3.32). This contradicts the initial assumption $\mathbf{T} \in H_t^c$ and completes the proof of Claim 1.

Claim 2. There exist at most t-many (distinct) q-islands $C_j \subseteq A_{q,t}$, $j \in \{1, \ldots, t\}$, such that their territories form a connected set $\bigcup_{j=1}^t D_{C_j}$ of vertices.
We prove Claim 2 by contradiction and assume, again in view of (3.70), that there exists a union $U_{q,t} := \bigcup_{j=0}^t C_j$ of $(t+1)$-many q-islands in $\mathbf{T}$ with $\mathrm{dist}(o, C) \leqslant t$ and $\#C > \frac{1}{q_t}$. Then, there is a bridge structure $B_{q,t}$ interconnecting $U_{q,t} \cup \{o\}$ with $q\#U_{q,t} > (t+1)\frac{q}{q_t} > \frac{q}{q_t} \geqslant 2^{\frac{5}{6}} t^{\frac{1}{3}}$, $\max_{v \in B_{q,t}} \mathrm{dist}(o, v) \leqslant t$, for which we used that $\mathbf{T}$ is a tree, and

$$\#((B_{q,t} \cup \{o\}) \setminus A_q) \leqslant t + t\Big(\frac{q}{2q_t z_t} - 1\Big). \tag{3.86}$$

The bound (3.86) holds because the bridge structure $B_{q,t}$ requires at most t vertices to connect the root o with one of the islands, and in order to connect this island with the remaining t islands it requires t further bridges B_j, $j \in \{1, ..., t\}$, between two islands. Each such bridge B_j will be chosen to pass through a common vertex v_j of the territories of the two islands $C_1^{(j)}$ and $C_2^{(j)}$ which it connects so that

$$\#(B_j \setminus A_q) \leqslant \mathrm{dist}_q(v_j, C_1^{(j)}) + \mathrm{dist}_q(v_j, C_2^{(j)}) - 1 \leqslant \frac{q}{2q_t z_t} - 1. \tag{3.87}$$

Here we used (3.84) for the second bound. This justifies (3.86). Finally, we infer from (3.86) that

$$z_t \frac{\#((B_{q,t} \cup \{o\}) \setminus A_q)}{\#U_{q,t}} \leqslant t \frac{q}{2q_t \#U_{q,t}} \leqslant \frac{tq}{2(t+1)} \leqslant \frac{h}{3}. \tag{3.88}$$

It follows from (3.32) that $\mathbf{T} \in H_t$ which again contradicts the initial assumption $\mathbf{T} \in H_t^c$. The proof of Claim 2 is complete.

Now, there are finitely many "groups"

$$\mathcal{C}_r := \bigcup_{j=1}^{J_r} C_j^{(r)}, \tag{3.89}$$

of q-islands in $\mathbf{T}^q$, where $r \in \{1, \ldots, R\}$ for some $R \in \mathbb{N}$, each group – according to Claim 2 – consisting of at most t-many q-islands $C_j^{(r)} \subseteq A_{q,t}$ with $j \in \{1, \ldots, J_r\}$, $J_r \in \{1, \ldots, t\}$, and such that the territories of q-islands from different groups are disjoint. Moreover, for every $r \in \{1, \ldots, R\}$, the union of territories $D_{\mathcal{C}_r} := \bigcup_{j=1}^{J_r} D_{C_j^{(r)}}$ within each group is connected and possesses a unique vertex $y_r \in D_{\mathcal{C}_r}$ which is closest to the root because $\mathbf{T}^q$ is a tree. It follows from Claim 1 that y_r belongs to the inner vertex boundary of $D_{\mathcal{C}_r}$ and therefore

$$\operatorname{dist}_q(y_r, \mathcal{C}_r) = \lfloor \frac{q}{4q_t z_t} \rfloor. \tag{3.90}$$

Hence, the probability on the right-hand side of (3.83) can be estimated as

$$\begin{aligned}
&P_o^{\mathbf{T}^q}\big[\exists\, s \in \{1, ..., t\} :\ X_s^{(q)} \in A_{q,t}\big] \\
&\quad\leqslant P_o^{\mathbf{T}^q}\big[\exists\, r \in \{1, \ldots, R\}\ \exists\, s_0 \in \{1, ..., t\}\ \exists\, s \in \{s_0 + 1, \ldots, t\} \\
&\qquad\qquad :\ X_{s_0}^{(q)} = y_r \text{ and } X_s^{(q)} \in \mathcal{C}_r\big] \\
&\quad\leqslant \sum_{s_0=1}^{t} \sum_{r=1}^{R} P_o^{\mathbf{T}^q}\big[\exists\, s \in \mathbb{N} \setminus \{1, \ldots, s_0\} :\ X_{s_0}^{(q)} = y_r \text{ and } X_s^{(q)} \in \mathcal{C}_r\big] \\
&\quad= \sum_{s_0=1}^{t} \sum_{r=1}^{R} E_o^{\mathbf{T}^q}\Big[1_{\{y_r\}}(X_{s_0}^{(q)}) P_{y_r}^{\mathbf{T}^q}[\tau_{\mathcal{C}_r} < \infty]\Big],
\end{aligned} \tag{3.91}$$

where the equality rests on the Markov property and $E_o^{\mathbf{T}^q}$ is the expectation corresponding to $P_o^{\mathbf{T}^q}$. Abbreviating $\mathcal{S}_q := \sup_{r \in \{1, \ldots, R\}} P_{y_r}^{\mathbf{T}^q}[\tau_{\mathcal{C}_r} < \infty]$ and noting that the y_r's are pairwise distinct, we conclude from (3.91)

$$P_o^{\mathbf{T}^q}\big[\exists\, s \in \{1, ..., t\} :\ X_s^{(q)} \in A_{q,t}\big] \leqslant \mathcal{S}_q \sum_{s_0=1}^{t} P_o^{\mathbf{T}^q}\Big[X_{s_0}^{(q)} \in \bigcup_{r=1}^{R} \{y_r\}\Big] \leqslant t\mathcal{S}_q. \tag{3.92}$$

The supremum $\mathcal{S}_q$ can be estimated with Lemma 3.12, choosing $\mathbf{T}$ there as the regularised tree $\mathbf{T}^q$. This is possible because of Lemma 3.15 and gives

$$\mathcal{S}_q \leqslant 2\Big(1 - \frac{q^2}{z_t^2}\Big)^{\frac{q}{8q_t z_t} - \frac{3}{2}} \frac{z_t^{\frac{5}{2}}}{q^2}\, t^{\frac{1}{2}}. \tag{3.93}$$

Combining (3.92) and (3.93), we infer that there exists $t_0 \in \mathbb{N}$, which depends only on h and on the sequence $(z_t)_{t\in\mathbb{N}}$, such that

$$P_o^{\mathbf{T}^q}\big[\exists\, s \in \{1, ..., t\} :\ X_s^{(q)} \in A_{q,t}\big] \leqslant \exp\bigg[- \frac{h^2}{10}\Big(\frac{t}{z_t^8}\Big)^{\frac{1}{3}}\bigg] \tag{3.94}$$

holds, provided $t \geqslant t_0$. Thus, the theorem follows from (3.78), (3.80), (3.83) and (3.94). □

Finally, we will prove Theorems 3.3 and 3.4.

Proof of Theorem 3.3. The offspring distribution has bounded support by hypothesis of the theorem. Thus, there is $z \in \mathbb{N}$ such that $p_j = 0$ for all $j \geqslant z$, and we choose $z_t := \max\{3, z\}$ for every $t \in \mathbb{N}$. Moreover, $\mathbb{T} \setminus M_{z_t}$ is a G-null set for every $t \in \mathbb{N}$ so that Theorem 3.13 and Lemma 3.8 imply

$$\begin{aligned} GP[X_t = o] &\leqslant \int_{M_{z_t}\cap H_t^c} P_o^{\mathbf{T}}[X_t = o]\, \mathrm{d}G(\mathbf{T}) + G[M_{z_t} \cap H_t] \\ &\leqslant \exp\bigg[- \frac{h^2}{16 z^{\frac{8}{3}}}\, t^{\frac{1}{3}}\bigg] + \exp\big[- c_6 t^{\frac{1}{3}}\big] \end{aligned} \tag{3.95}$$

for all $t \geqslant t_0$, where t_0 depends only on h and z, and the constant $c_6 > 0$ is defined in Lemma 3.5. By the same argument as in Subsection 1.2.2 Remark 2 we obtain the claim of the theorem. □

Proof of Theorem 3.4. We consider a fast-decaying offspring distribution as in (1.30) for some constants $c_1, c_2 > 0$ and $k > 8$. Let $z_t := 3 + c_3 t^{\frac{1}{k}}$ for every $t \in \mathbb{N}$ with $c_3 > 0$ as required in (3.18). In particular, we then have $z_t \in o(t^{\frac{1}{8}})$ as $t \to \infty$ and $F_t^c \subseteq M_{z_t}$, where the former is defined in (3.18) and the latter in (3.30). We conclude

$$\begin{aligned} GP[X_t = o] &\leqslant \int_{F_t^c\cap H_t^c} P_o^{\mathbf{T}}[X_t = o]\, \mathrm{d}G(\mathbf{T}) + G[F_t] + G[M_{z_t} \cap H_t] \\ &\leqslant \exp\bigg[- \frac{h^2}{16(3 + c_3)^{\frac{8}{3}}}\, t^{\frac{1}{3} - \frac{8}{3k}}\bigg] + C\exp\big[- c_5 t\big] + \exp\big[- c_6 t^{\frac{1}{3}}\big], \end{aligned} \tag{3.96}$$

where the second inequality follows from Theorem 3.13, (3.21), and Lemma 3.8 and holds for all $t \geqslant t_0$ which arises from Theorem 3.13. By the same argument as in Subsection 1.2.2 Remark 2 we infer the claim of the theorem. $\square$

Appendix A

The majority of this chapter, excluding the proof of Lemma 1.5, coincides both in content and writing with [47], which was written in collaboration with P. Müller.

Proof of Lemma 1.5. This can be seen by induction over t and k.

First, we assume that $t = 1$. Then, there are two cases for k. We start with $k = 0$. In this case,

$$\binom{2}{0} - \binom{2}{-1} = 1 - 0 = 1, \tag{A.1}$$

which equals the number of paths leading from the root to generation 2 after 2 steps, namely the path consisting of two consecutive steps away from the root. Next, there is $k = 1$. In this case,

$$\binom{2}{1} - \binom{2}{0} = 2 - 1 = 1, \tag{A.2}$$

which equals the number of paths leading from the root to generation 0, so back to the root, after 2 steps, namely the path consisting of one step away followed by one step back to the root.

Now, we assume that the claim holds for some $t \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ with $k \leqslant t$.

Next, we aim to verify it for $t+1$ and every $k \in \mathbb{N}_0$ with $k \leqslant t+1$. We do so by considering three cases for k. First, we assume that $k = 0$. In this case,

$$\binom{2(t+1)}{0} - \binom{2(t+1)}{-1} = 1 - 0 = 1, \tag{A.3}$$

which exactly equals the number of paths leading from the root to generation $2(t+1)$ after $2(t+1)$ steps, namely the path consisting of $2(t+1)$ consecutive steps away from the root. Second, we assume that $0 < k < t+1$. In this case, the number of paths leading from the root to generation $2(t+1) - 2(k+1) = 2t - 2k$ after $2(t+1)$ steps is given by twice the number of paths leading from the root to generation $2t - 2k$ after $2t$ steps plus the number of paths leading from the root to generation

$2t-2(k-1)$ after $2t$ steps plus the number of paths leading from the root to generation $2t-2(k+1)$ after $2t$ steps, since these are exactly the paths after $2t$ steps which can reach the generation $2(t+1)-2(k+1)$ after $2(t+1)$ steps. For the first term this can be achieved from generation $2t-2k$ by either a step away from the root followed by a step back to the root, or a step back to the root followed by a step away from the root, thus the factor two. The second term originates from generation $2t-2(k-1)$ by two steps back to the root. For the third and last term this can be achieved by two consecutive steps away from the root from generation $2t-2(k+1)$. Therefore, by the induction assumption we obtain

$$\begin{aligned}
&2\left[\binom{2t}{k}-\binom{2t}{k-1}\right]+\left[\binom{2t}{k-1}-\binom{2t}{k-2}\right]+\left[\binom{2t}{k+1}-\binom{2t}{k}\right]\\
&=\left[\binom{2t}{k}+\binom{2t}{k+1}\right]-\left[\binom{2t}{k-1}+\binom{2t}{k-2}\right]\\
&=\binom{2t+1}{k+1}-\binom{2t+1}{k-1}+\left[\binom{2t+1}{k}-\binom{2t+1}{k}\right]=\binom{2(t+1)}{k+1}-\binom{2(t+1)}{k},
\end{aligned} \tag{A.4}$$

which was claimed for the number of paths leading from the root to generation $2(t+1)-2(k+1)$ after $2(t+1)$ steps. Here, for the second and third equality we used that $\binom{n}{j}+\binom{n}{j+1}=\binom{n+1}{j+1}$ for every $n\in\mathbb{N}$ and $j\in\mathbb{N}_0$ with $j\leqslant n-1$. Third, we assume that $k=t+1$. In this case, the number of paths leading from the root to generation $2(t+1)-2(t+1)=0$, so back to the root, after $2(t+1)$ steps is given by the number of paths at the root after $2t$ steps plus the number of paths at generation 2 after $2t$ steps, since these are the only paths to reach the root after $2(t+1)$ steps. For the first term this can be achieved, exactly, by a step away from the root followed by a step back to the root and for the second term, exactly, by two consecutive steps back to the root. Therefore, by the induction assumption we obtain

$$\begin{aligned}
&\left[\binom{2t}{t}-\binom{2t}{t-1}\right]+\left[\binom{2t}{t-1}-\binom{2t}{t-2}\right]\\
&=\left[\binom{2t}{t}+\binom{2t}{t-1}\right]-\left[\binom{2t}{t-1}+\binom{2t}{t-2}\right]\\
&=\binom{2t+1}{t}-\binom{2t+1}{t-1}+\left[\binom{2t+1}{t+1}-\binom{2t+1}{t}\right]=\binom{2(t+1)}{t+1}-\binom{2(t+1)}{t},
\end{aligned} \tag{A.5}$$

which was claimed for the number of paths leading from the root back to the root after $2(t+1)$ steps. Here, we for the second equality we used that $\binom{2t+1}{t+1}=\binom{2t+1}{t}$, for every $t \in \mathbb{N}$, by the symmetry of the binomial coefficients. Further, for the second and third equality we used that $\binom{n}{j} + \binom{n}{j+1} = \binom{n+1}{j+1}$ for every $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ with $j \leqslant n-1$.

Hence, we conclude the claim by induction. $\square$

Next, for the Section 3.2 we recall some basic properties of q-islands and q-oceans, which are, up to some minor modifications, taken from [52, Sect. 3]. Throughout, $\mathbf{T}$ can be any infinite, connected and locally finite graph. It does not need to be a realisation of a Galton-Watson tree here.

Lemma A.1. *Let $q > 0$, let $A \subseteq \mathbf{T}$ be a finite vertex subset and let $S \subseteq \mathbf{T}$ be a q-isolated core. Then, we have $\Delta_q A \leqslant \Delta_q(A \cup S)$ with equality if and only if $S \subseteq A$.*

Proof. If $S \subseteq A$, then the claim trivially holds with equality. So let us now suppose that S is not a subset of A.

We note that if B and C are finite disjoint vertex subsets of $\mathbf{T}$, then

$$\Delta_q(B \cup C) = \Delta_q B + \Delta_q C + 2\#(\partial B \cap \partial C). \tag{A.6}$$

The factor 2 in the above expression appears since common boundary edges of B and C are not boundary edges of their union, i.e., $2\#(\partial B \cap \partial C) = \#(\partial B) + \#(\partial C) - \#(\partial B \cup \partial C)$.

We conclude from (A.6) that $\Delta_q(A \cup S) = \Delta_q(A \setminus S) + \Delta_q S + 2\#(\partial(A \setminus S) \cap \partial S)$. Since we assumed that S is a q-isolated core, we have $\Delta_q S > \Delta_q(A \cap S)$ by definition because $A \cap S \subsetneq S$ due to S not being a subset of A. Also, $\partial(A \setminus S) \cap \partial S \supseteq \partial(A \setminus S) \cap \partial(A \cap S)$, since every edge in the intersection of sets has to connect S with its complement and is thus in ∂S. Therefore, another application of (A.6) yields

$$\Delta_q(A \cup S) > \Delta_q(A \setminus S) + \Delta_q(A \cap S) + 2\#(\partial(A \setminus S) \cap \partial(A \cap S)) = \Delta_q A. \tag{A.7}$$

$\square$

Corollary A.2. *Let $q > 0$. Then, the union of finitely many q-isolated cores of $\mathbf{T}$ is a q-isolated core of $\mathbf{T}$.*

Proof. It suffices to prove the claim for two q-isolated cores S and S' of $\mathbf{T}$. Let $A \subsetneq S \cup S'$ be arbitrary. Then, at least one of the sets S and S' must not be a subset of A. W.l.o.g. suppose that S is not a subset of A. Then, applying Lemma A.1 with A and S, followed by another application with $A \cup S$ and $A \cup S \cup S'$, yields

$\Delta_q A < \Delta_q(A \cup S) \leqslant \Delta_q(A \cup S \cup S') = \Delta_q(S \cup S')$. The last equality holds because of $A \subset S \cup S'$, and the claim follows. □

The following lemma relates to a statement in [52, Sect. 3] which is given there without proof.

Lemma A.3. *Let $q \in]0, \mathbf{i}(\mathbf{T})[$. Then, every q-island of $\mathbf{T}$ has only finitely many vertices and thus is itself a q-isolated core of $\mathbf{T}$.*

Proof. Suppose that there exists a q-island $S \subseteq \mathbf{T}$ with $\#S = \infty$. Thus, S must be formed by a countably infinite union $S = \bigcup_{j\in\mathbb{N}} S_j$ of q-isolated cores S_j of $\mathbf{T}$. Then, $A_n := \bigcup_{j=1}^{n} S_j$ is a q-isolated core for every $n \in \mathbb{N}$ by Corollary A.2. Hence, we have

$$\frac{\#\partial A_n}{\#A_n} < q \tag{A.8}$$

for every $n \in \mathbb{N}$ by Remark 1. W.l.o.g. it can be assumed that each S_j is not empty and, due to Remark 2, connected. Since S is connected by hypothesis a suitable renumbering of the S_j's will guarantee that A_n is connected for every $n \in \mathbb{N}$. Furthermore, we can assume w.l.o.g. that $S_{j+1} \setminus A_j \neq \varnothing$ for every $j \in \mathbb{N}$. Thus, $\#A_n \geqslant n$ for every $n \in \mathbb{N}$. Finally, we connect A_n with the root o for every $n \in \mathbb{N}$ by attaching a suitable linear path $P_n \subset \mathbf{T}$ to it. If $o \in A_n$ already, we set $P_n = \varnothing$. Since $A_n \subseteq A_{n+1}$, we have $P_n \supseteq P_{n+1}$, and because of the linear structure of P_n this implies $\#\partial P_n \geqslant \#\partial P_{n+1}$ for every $n \in \mathbb{N}$. Defining $K_n := P_n \cup A_n$ for $n \in \mathbb{N}$, we conclude that $o \in K_n \subseteq \mathbf{T}$ is connected, $\#K_n \geqslant \#A_n \geqslant n$ and $\#\partial K_n \leqslant \#\partial A_n + \#\partial P_n \leqslant \#\partial A_n + \#\partial P_1$ for every $n \in \mathbb{N}$. We thus infer a contradiction in that

$$\mathbf{i}(\mathbf{T}) \leqslant \lim_{n\to\infty} \frac{\#\partial K_n}{\#K_n} \leqslant q, \tag{A.9}$$

where we used (A.8) for the last estimate. Hence, every q-island of $\mathbf{T}$ is finite, therefore a finite union of q-isolated cores and therefore itself a q-isolated core by Corollary A.2. □

Next, we argue that decreasing q raises the sea level of the oceans.

Lemma A.4. *Let $0 < q' < q$. Then, $A_{q'} \subseteq A_q$.*

Proof. We have $\Delta_q S = (q - q')\#S + \Delta_{q'} S \geqslant \Delta_{q'} S$ for any finite vertex subset $S \subseteq \mathbf{T}$. So any q'-isolated sets are also q-isolated. Moreover, if $A \subsetneqq S$ with $\Delta_{q'} A < \Delta_{q'} S$, then also $\Delta_q A < \Delta_q S$. Therefore, q'-isolated cores are q-isolated cores as well, giving $A_{q'} \subseteq A_q$. □

In the next lemma, we quantify the preceding statement in that too small q-islands sink into the oceans when lowering q.

Lemma A.5. *Let* $0 < q' < q$ *and* $S \subseteq \mathbf{T}$ *be a union of* q*-islands with* $\#S \leqslant \frac{1}{q'}$*. Then,* $S \subseteq \mathbf{T} \setminus A_{q'}$*.*

Proof. We argue by contradiction and assume that there exists $\varnothing \neq S' \subseteq S$ with $S' \subseteq A_{q'}$. Since S is a finite union of q-islands and $A_{q'} \subseteq A_q$ by Lemma A.4, it follows that S' is a finite union of q'-islands and, thus, a q'-isolated core, i.e. $\Delta_{q'} S' > 0$. On the other hand,

$$\Delta_{q'} S' \leqslant q' \#S - \#\partial S' \leqslant 1 - \#\partial S' \leqslant 0, \tag{A.10}$$

where we used the volume assumption for S in the second inequality and $\#\partial S' \geqslant 1$ in the last inequality. This holds because $\mathbf{T}$ is infinite and connected. $\square$

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