On the $\mathbb{H}_0^{\mathbb{A}^1}$ of Classifying Spaces of Algebraic Groups

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Abstract

The homotopy sheaves in $\mathbb{A}^1$-homotopy theory, which was developed by Morel and Voevodsky [54], are, similarly to the homotopy groups in topology, notoriously difficult to calculate explicitly. Here it seems that in contrast to the latter the base case of $\pi_0^{\mathbb{A}^1}$ has a special quality, which we can see from the fact that Morel was able to derive $\mathbb{A}^1$-invariance for the higher homotopy sheaves [50], but not for $\pi_0^{\mathbb{A}^1}$. In special cases, Choudhury [12] and Elemanto, Kulkarni, and Wendt [17] were able to prove $\mathbb{A}^1$-invariance, but counterexamples due to Ayoub show that the general conjecture is not correct.

In the present work, we consider an abelian variant of the $\mathbb{A}^1$-homotopy theory, the $\mathbb{A}^1$-derived category, which was likewise introduced by Morel (cf. [50]). Using the spectrum of a field as a base scheme, it is already known that the zero$^{\text{th}}$ homology is strictly $\mathbb{A}^1$-invariant and it follows that $\mathbb{H}_0^{\mathbb{A}^1}$ has the quality of a free strictly $\mathbb{A}^1$-invariant functor. In the light of the recently published results of Elmanto, Kulkarni and Wendt on the determination of $\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$, for reductive algebraic groups G, as sheafified étale cohomology, we calculate the associated zero$^{\text{th}}$ $\mathbb{A}^1$-homology for classifying spaces of some algebraic groups. For this we first develop tools, in particular we extend theorems about unramified sheaves, which were introduced by Morel [50], and treat among others the cases of (special) orthogonal groups, unitary groups, split groups of type G_2, and spin groups of low dimension. The arguments used are based on the well-elaborated theory of cohomological invariants of these groups dissemenated by Garibaldi, Merkurjev and Serre (cf. [21] and [20]).

Zusammenfassung

Die Homotopiegarben der $\mathbb{A}^1$-Homotopietheorie, welche von Morel und Voevodsky [54] entwickelt wurde, sind, ähnlich wie die Homotopiegruppen in der Topologie, tendenziell schwer explizit zu bestimmen. Hierbei wirkt es so, dass der Basisfall von $\pi_0^{\mathbb{A}^1}$ eine besondere Qualität aufweist, was wir daran festmachen, dass Morel für die höheren Homotopiegarben $\mathbb{A}^1$-Invarianz zeigen konnte [50], jedoch nicht für $\pi_0^{\mathbb{A}^1}$. In Spezialfällen konnten Choudhury [12] und Elemanto, Kulkarni und Wendt [17] die $\mathbb{A}^1$-Invarianz nachweisen, jedoch zeigen Gegenbeispiele von Ayoub (cf. [5]) auf, dass die allgemeine Vermutung nicht korrekt ist.

In der vorliegenden Arbeit betrachten wir eine abelsche Variante der $\mathbb{A}^1$-Homotopietheorie, die $\mathbb{A}^1$-derivierte Kategorie, welche ebenfalls durch Morel eingeführt wurde (cf. [50]). Über dem Spektrum eines Körpers als Basisschema ist hier bereits bekannt, dass die nullte Homologie strikt $\mathbb{A}^1$-invariant ist und daraus resultiert, dass $\mathbb{H}_0^{\mathbb{A}^1}$ die Qualität eines freien strikt $\mathbb{A}^1$-invarianten Funktors aufweist. Im Lichte der kürzlich erschienenen Resultate von Elmanto, Kulkarni und Wendt [17] zur Bestimmung von $\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$, für reduktive algebraische Gruppen, als garbifizierte Étalekohomologie, berechnen wir die zugehörige Basishomologie für klassifizierende Räume ausgewählter algebraischer Gruppen. Hierzu entwickeln wir zunächst Hilfsmittel, insbesondere erweitern wir Sätze über unverzweigte Garben, welche durch Morel eingeführt wurden [50], und behandeln unter anderen die Fälle der (speziellen) orthogonalen Gruppen, unitären Gruppen, zerfallenden Gruppen von Typ G_2, und Spingruppen von niedriger Dimension. Die verwendeten Argumente basieren auf der wohl ausgearbeiteten Theorie der kohomologischen Invarianten dieser Gruppen nach Garibaldi, Merkurjev und Serre (cf. [21] und [20]).

1

Introduction

One of the simplest structures in topology are discrete spaces, for which every subset of some underlying set M is considered to be open. In that way, any map from M into some topological space X is continuous. Vice versa, any continuous map from X into M, factors through $\pi_0(X)$, the set of path-components. By adding the assumption that X is a CW-complex, the canonical surjection $X \twoheadrightarrow \pi_0(X)$ becomes continuous, where we equip $\pi_0(X)$ with the discrete topology, and thus the continuous maps from X to M are in one-to-one correspondence with the mappings $\pi_0(X) \to M$. Let us moreover identify the phenomenon that any homotopy $X \times [0,1] \to M$ is constant, so that this bijection factors through taking homotopy classes of maps. If we assert furthermore functoriality in both X and M, with the added amplification that the X-part factors again through taking homotopy classes, we may state that the functor π_0, from the homotopy category on some convenient category of topological spaces[1] to the category of sets, is left-adjoint to the standard inclusion functor. A more combinatorially inclined reader may have started with a set M, and some simplicial set X, and arrived at completely analogous results. We explained this trivial example, since the subject of the present thesis is concerned with a variation of this principle.

We would like to consider this paradigm in $\mathbb{A}^1$-algebraic topology. $\mathbb{A}^1$-algebraic topology is a collective term that stands for a transfer of structures, notions, and theorems from algebraic topology to the theory of schemes, by assigning the affine line $\mathbb{A}^1$ the role of the contractible unit interval $[0,1]$. The unstable version of a homotopy theory for schemes with this trait was introduced first by Morel and Voevodsky in their seminal article [54]. Their construction starts out by first enlarging the category of smooth schemes, to the category of sheaves with respect to the Nisnevich topology, similarly as the category of topological spaces was replaced by some convenient category. Here the Nisnevich topology is granted precedence over e.g. the Zariski, or étale topologies, since it combines positive properties of both of them, among which there are the cohomological dimension being bounded by the Krull dimension (Zariski), descent for K-theory (Zariski), exactness of the direct image functor for finite morphisms (étale), and the equivalence of smooth pairs to inclusions of affine spaces (étale). The aim of this replacement is moreover to make the working category complete and co-complete, equipped with internal hom-objects.

Next, a reasonable model category structure is built on top of that by considering the simplicial sheaves, and by identifying the canonical morphisms as weak equivalences. Inverting the latter leaves us with the simplicial homotopy category, and this is a good point to come back to our initial example. Recall, that for the category of simplicial sets (or equivalently *CGHaus*) the discrete objects were simply sets, and we will see below that one may analogously regard the sheaves as discrete objects in the homotopy category of simplicial sheaves. In this scenario the affine line plays no distinguished role, yet, which is changed by localizing with respect to the inclusion of zero $S \hookrightarrow \mathbb{A}^1_S$. So, carrying the paradigm further, one would like to consider

[1] E.g. the category of compactly generated Hausdorff spaces, see [70].

the $\mathbb{A}^1$-local sheaves, or equivalently the $\mathbb{A}^1$-invariant sheaves, as discrete. However, in the cases we explained above the associated zero$^{\text{th}}$ homotopy object was always of the discrete type, viz. a set for the case of topological spaces, or a sheaf for the case of simplicial sheaves. It is a long-standing conjecture due to Morel, that $\pi_0^{\mathbb{A}^1}(\mathcal{X}) := \pi_0^s(L_{\mathbb{A}^1}(\mathcal{X}))$ is an $\mathbb{A}^1$-invariant sheaf. Evidence in this direction comes from work e.g. of Morel himself (cf. [50]), who showed that the higher $\mathbb{A}^1$-homotopy sheaves fulfil (strong resp. strict) $\mathbb{A}^1$-invariance, or Choudhury [12], who showed that $\pi_0^{\mathbb{A}^1}$'s of H-spaces, in particular of algebraic groups, are $\mathbb{A}^1$-invariant. Another step in this direction is the article by Elmanto, Kulkarni, and Wendt [17], who showed that in the case that the étale cohomology over fields for some reductive algebraic group is $\mathbb{A}^1$-invariant, that $\pi_0^{\mathbb{A}^1}$ of the étale classifying space is isomorphic to the Nisnevich sheafification of the étale cohomology set of G, and thus $\mathbb{A}^1$-invariant. However, a recent preprint due to Ayoub [5] gives a counterexample to $\pi_0^{\mathbb{A}^1}$ being $\mathbb{A}^1$-invariant, for all spaces.

The avid or knowing reader may have noticed that we added the qualifier "étale" in front of classifying space, so let us first reassure that taking the fundamental homotopy sheaf $\pi_1^{\mathbb{A}^1}$ still recovers the underlying sheaf of groups. The signifier for the situs comes from the following construction: As one may reproduce the definition of the homotopy category for an arbitrary site, and the comparison morphisms between sites induce functors between the associated homotopy categories, we may thus restrict the classifying space over some sheaf of groups on the étale site to a space with respect to the $\mathbb{A}^1$-homotopy category over the Nisnevich site. There are some intricacies with this definition, though. In particular, the (étale) classifying spaces need not be connected any longer, in fact recently it was shown by Elmanto, Kulkarni, and Wendt (cf. [17]), that in reasonable cases $\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is given as sheafified étale cohomology $a_{\text{Nis}}(H^1_{\text{ét}}(-,G))$.

Our aim below is to analyse the étale classifying space in an abelian variant of the $\mathbb{A}^1$-homotopy category. Here we work in a homological setting, i.e. differentials of degree -1, and thus replace homotopy sheaves by homology sheaves. The respective homotopy category, called the $\mathbb{A}^1$-derived category, will be the localization of some derived category of sheaves, and due to the presence of a homological t-structure, we may show that strictly $\mathbb{A}^1$-invariant sheaves are discrete objects in that setting. Strict $\mathbb{A}^1$-invariance is a form of invariance for abelian sheaves M, which entails that all presheaves $H^i_{\text{Nis}}(-,M)$ are $\mathbb{A}^1$-invariant. They appear in a famous theorem by Voevodsky (cf. [45, Thm. 13.8]), asserting that over a perfect field $\mathbb{A}^1$-invariant abelian sheaves with transfers fulfil this criterion. As above by stating that strictly $\mathbb{A}^1$-invariant sheaves are discrete, we mean two assertions: On the one hand, we would like the homology sheaves to be strictly $\mathbb{A}^1$-invariant, i.e. discrete, and secondly, we would like morphisms up to homotopy into discrete objects to factor uniquely through the homology sheaves. Both of these assertions hold, and we expose a proof of this fact due to Asok [4] in our exposition. If one works consequently with transfers, i.e. considers Voevodsky's triangulated category of motives $\mathsf{DM}(k)$ (cf. [78]), then the homotopy invariant sheaves with transfers are the discrete objects, and the homology involved is Suslin homology (cf. [71]). It is well-known that $\mathbb{A}^1$-homology and Suslin-homology do not necessarily agree. An example computation close to our endavour is the determination of the first two Suslin-homology sheaves of split simply connected semi-simple groups by Gille [22].

Strictly $\mathbb{A}^1$-invariant sheaves have many nice traits, in particular it holds that for any open subscheme U of a scheme X that contains all points of codimension $< d$, the restriction along the inclusion induces an isomorphism for all $n \leq d-2$ (and a monomorphism for $n = d-1$)

$$H^n_{\text{Nis}}(X,M) \longrightarrow H^n_{\text{Nis}}(U,M). \tag{1.1}$$

Sheaves with this property, corresponding to values $d = 1, 2$, will be called *unramified*, and for our purposes it suffices to restrict to unramified sheaves. This notion was first codified by Morel [50], however its beginning can be traced back to Colliot-Thélène and Sansuc, who proved that the étale cohomology of reductive groups is unramified, if one restricts themselves to schemes of dimension 2 (cf. [14]). By rewriting (1.1) in a more diagrammatic form, one may agree with the sentiment that "[i]n some sense, being unramified is a weak properness statement." (cf. [17, Rem. 5.9]). When analysing unramified sheaves below, we will employ this picture, and show, by taking arguments of Morel [50], and generalising arguments of Gille and Hirsch [23], to find that over a perfect field, morphisms between (weakly) unramified sheaves are in one-to-one correspondence with natural transformations over fields (see propositions 3.25 and 3.28). Unfortunately, we could not get rid of the perfectness assumption in our compatibility statement.

So under this hypothesis, for any reductive algebraic group, morphisms

$$a_{\mathrm{Nis}}(H^1_{\text{ét}}(-,G)) \longrightarrow M,$$

into some unramified sheaf M, reduce to cohomological invariants with values in M by the compatibility statement. Tying this back to the initial paradigm, the calculation for $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ comes down to finding some strictly $\mathbb{A}^1$-invariant sheaf, which factors morphisms from the cohomology sheaves $a_{\mathrm{Nis}}(H^1_{\text{ét}}(-,G))$ to strictly $\mathbb{A}^1$-invariant sheaves uniquely. So, our upshot is to find universal invariants, with values in strictly $\mathbb{A}^1$-invariant sheaves. Therefore we seek to generalize and build upon the existing literature on cohomological invariants due to Garibaldi, Serre, and Merkurjev (cf. [21] and [20]).

Cohomological invariants are a useful tool in many fields. Starting from their simple definition as natural transformations of functors on fields, they turn the problem of knowing equivalence classes of torsors, into working with characteristic values living in well-known abelian groups. They have been applied in determining the essential dimension of groups, which is a measure for how many parameters are at least needed to describe torsors over that particular group (cf. [64]), counterexamples to the Noether problem, where one determines whether the field of invariants of a representation of some finite group is rational (cf. the exposition in [21, Sec. 33]), or to analyse the shape of quadratic forms in the third power of the fundamental ideal $I^3(k)$ (cf. originally [65], see also [20, Sec. 21]).

Moreover, the now standard arguments that are used to classify, e.g. the cohomological invariants of the orthogonal groups allow for generalization to the $\mathbb{A}^1$-topological situation. The main reason hereby stems from the nature of the reasoning: By finding group homomorphisms $H \to G$ that induce surjections on Galois cohomology for all fields, one may reduce the computation of the G-invariants to determining those H-invariants that come from G. Apart from these reduction steps, another ingredient is needed, and comes in the form of the analysis of $\mathbb{H}_0^{\mathbb{A}^1}$ of the spheres $\mathbb{G}_m^{\wedge i}$ by Morel, who determined them to be the unramified Milnor-Witt K-theory (cf. [50, Thm. 3.37]). From this we first derive $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2)$, and via the splitting principle $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n)$. This was already known to Morel (cf. [49, Rem. 5.4]), an explicit proof of this fact did not appear in print, yet. So, we check that there is an isomorphism (see lemma 4.15)

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n) \longrightarrow \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}} \oplus \cdots \oplus \underline{\mathbf{K}}_n^{\mathrm{W}},$$

which is induced by sending a diagonal quadratic form $\langle u_1, \dots, u_n \rangle$ to the i^{th}-elementary symmetric polynomial in the symbols $[u_1], \dots, [u_n]$. This makes these maps clearly eligible for the name *refined Stiefel-Whitney classes*. Next, we derive of this the case of $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}_n)$, essentially by doing some linear algebra. The result is, unsurprising to a reader knowledgeable in the cohomological invariants of $\mathbf{SO}_n$ (see proposition 4.27)

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}_n) \xrightarrow{\cong} \mathbb{Z} \oplus \bigoplus_{\substack{i=2 \\ i \text{ even}}}^{n-1} \underline{\mathbf{K}}_i^{\mathrm{W}} \oplus \begin{cases} \underline{\mathbf{K}}_{n-1}^{\mathrm{W}}, & n \text{ even, and} \\ 0, & \text{otherwise.} \end{cases}$$

Note, that in the theory of cohomological invariants, there are no degree 1 invariants for connected groups, and this difference can also be seen in the above two statements with the disappearance of $\underline{\mathbf{K}}_1^{\mathrm{W}}$. The case of the symmetric groups is also quite similar to the classical case, and the universal invariants one finds, may be described by sending a S_n-torsor, in the form of an isomorphism class of an n-dimensional étale algebra, to the value of the refined Stiefel-Whitney classes of its trace form (see theorem 4.33). This was also already announced in [49], and we provide an explicit proof for that. Finally, we generalize several arguments due to Garibaldi (see [20],and originally [66]) to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n)$ for $7 \leq n \leq 12$, where $\mathbf{Spin}'_n$ denotes the spin group associated with a split quadratic space of dimension n. The arguments deviate not much of what was done classically, and again analogously to the vanishing of degree 2 invariants for connected semi-simple simply connected groups, the term of least "degree" that appears in all our $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n)$ is $\underline{\mathbf{K}}_3^{\mathrm{W}}$. This motivates three questions that we were not able to answer generally:

Questions. *Given a reductive group G, is it true that $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is isomorphic to a sum of unramified Milnor-Witt K-sheaves, modulo ideals that are generated entirely by elements defined over the ground field? In the specific case that G is already defined over $\mathbb{Z}$, are these ideals necessarily generated by elements that are defined over $\mathbb{Z}$? Moreover, does the type of connectedness of the group reflect in the "degree" of the Milnor-Witt K-sheaves occurring?*

In tackling these our proximity to the algebraic approach to cohomological invariants, in the style advanced by Gariabaldi and Serre, could have been an Achilles' heel, and a more geometric approach, by means of the stratification method due to Vezossi (cf. [76]), which was already employed in the context of cohomological invariants by Guillot (cf. [28]), might prove more fruitful in the future.

Outline of the thesis

In chapter two we develop $\mathbb{A}^1$-algebraic topology insofar as it is needed to understand our results and the motivation(s) behind it. Therefore we dive first into the construction of the $\mathbb{A}^1$-derived category, where we give more details, as these are not readily available in the existing literature. We also put an emphasis on connecting the $\mathbb{A}^1$-topological language to the general theory of triangulated categories and their localizations. We follow that up with a short introduction into the $\mathbb{A}^1$-homotopy theory defined by Morel and Voevodsky, and finish its discussion with a quick glance at classifying spaces. After that we tie the $\mathbb{A}^1$-derived category and the $\mathbb{A}^1$-homotopy category together by describing a correspondence, which is motivated by the classical Dold-Kan correspondence. Finally, we put a tensor product on the category of strictly $\mathbb{A}^1$-invariant sheaves, as it is a useful tool for the computation of $\mathbb{H}_0^{\mathbb{A}^1}$.

The following chapter is devoted to more hands-on terms. With the general theory settled, we first discuss the notion of essentially smooth schemes that we intend to use, and describe the extension of sheaves onto this larger category. Next, we put this to work in the definition, and discussion of first examples, of unramified sheaves. We conclude the chapter by analysing morphisms between (weakly) unramified sheaves. In particular, we show the compatibility statement, a proof of which is based on arguments of [23], and a generalization of a principle due to Totaro, that determines cohomological invariants with values in unramified sheaves, as values associated to some tangible geometric classifying space that is the quotient of a free group action.

Lastly, we get to the computation of $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ for several (split) algebraic groups. Therefore, we hand the reader a crash course in Milnor-Witt K-theory, which, by a theorem of Morel, is $\mathbb{H}_0^{\mathbb{A}^1}$ of the spheres $\mathbb{G}_m^{\wedge i}$. Then we proceed to lay out the construction and identification of the refined Stiefel-Whitney classes as the universal invariants for the orthogonal groups, a program that was outlined in the ICM address [49]. Next we derive thereof the $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(-))$ of the special orthogonal groups, the symmetric groups, and the unitary groups, three cases in which the Stiefel-Whitney classes take on a prominent role. After that we consider split groups of type G_2, which in contrast to the above features a refinement of the Arason invariant as universal invariant. Via a small interlude, in which we lift some known invariants of the split groups of type F_4, we proceed to the case of split spin groups of small rank. Here we use arguments that go back to Rost [66] and Garibaldi [20], and our earlier generalizations of the classical theory.

Conventions

In this work the natural numbers $\mathbb{N}$ include 0, and all rings are commutative and unital, except if noted otherwise. In the same vein, all ring homomorphisms preserve the unit. Categories will be denoted by a sans-serif italic font, e.g. *Set*, $\mathit{Set}_\bullet$, or *Ab* mean respectively the category of sets, pointed sets, or abelian groups. Up to rare explicit exceptions the schemes in this work, and their morphisms, will be based over some scheme S, that is noetherian and finite dimensional for chapter two, the spectrum of a field starting with chapter three, and of a perfect field in chapter four. If no additional modifier is given, a sheaf is defined with respect to the Nisnevich topology (recalled in definition 2.1).

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2

$\mathbb{A}^1$-Algebraic Topology

In this chapter we begin by introducing the basics of the framework we will work in, namely $\mathbb{A}^1$-algebraic topology. So we try to shed some light on the definition of two categories, which distil the geometric information that we want to deal with. The first category we showcase will be the $\mathbb{A}^1$-derived category, first defined by Morel in [50], as the localisation of some derived category. It is moreover an abelian variant of the second category, the unstable $\mathbb{A}^1$-homotopy category, defined by Morel and Voevodsky in their seminal work [54] in an effort to introduce the methods of unstable homotopy theory into algebraic geometry. We chose this order, despite being achronological, since we will later on focus more on the $\mathbb{A}^1$-derived category. By the name $\mathbb{A}^1$-algebraic topology we mean both theories collectively, together with their Dold-Kan style comparison functors, which will come in the latter part of this chapter. Since we present the state of the art, we only give proofs to patch holes in the existing literature, or if it fits our exposition.

2.1 The $\mathbb{A}^1$-Derived Category

Our starting point for both theories is some category of geometric objects, and a Grothdendieck topos defined over it. To have some flexibility, we choose a noetherian, and finite dimensional base scheme S. Denoting by Sch_S the category of all S-schemes, together with the S-morphisms, we restrict ourselves to the full subcategory Sm_S of Sch_S that consists of smooth and finite type S-schemes. Although the majority of our results are concerned with the case $S = \mathrm{Spec}(k)$, for k being a perfect field, we feel it is more conceptual to introduce the theory in a general fashion.

2.1.1 Nisnevich Topology

The following gives a pretopology in the sense of [3, Déf. II.1.3].

Definition 2.1. Let $X \in \mathsf{Sm}_S$ be arbitrary. A finite family $\{X_i \to X\}_{i\in I}$ of étale morphisms is called a *Nisnevich covering* of X if and only if for every point $x \in X$ there is an index $i \in I$, and a point $x_i \in X_i$ such that the induced homomorphism on residue fields $\kappa(x) \to \kappa(x_i)$ is an isomorphism.

Below we will refer to this topology as *Nisnevich topology* (or *Nisnevich site*), and denote the associated sheaf topos by $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ and its corresponding abelian category by $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$. We recall some basic facts about the Nisnevich site:

2.2. The forgetful functor $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \mathsf{PSh}(\mathsf{Sm}_S)$ from sheaves to presheaves has a left adjoint $a_{\mathrm{Nis}} \colon \mathsf{PSh}(\mathsf{Sm}_S) \to \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ that preserves finite limits, and is called the *sheafification*. The Nisnevich topology is finer than the Zariski topology, i.e. open covers are Nisnevich covers, and is coarser than the étale topology, which handily implies that representable presheaves are in fact sheaves. We denote the Yoneda embedding by $h_{\dots}\colon \mathsf{Sm}_S \to \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, but whenever confusion cannot arise we will not write it explicitly. From [58, pp. 256–267] we learn how to define stalk-functors for the Nisnevich topology: Fixing a point x on a scheme $X \in \mathsf{Sm}_S$, one may define a Nisnevich neighbourhood of x by an étale morphism $U \to X$ such that there is a point $y \in U$ with the induced homomorphism $\kappa(x) \to \kappa(y)$ being an isomorphism. One can make the Nisnevich neighbourhoods of x a cofiltered category, and consider the functor

$$\begin{aligned} x^*\colon \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S) &\longrightarrow \mathsf{Set} \\ \mathcal{F} &\longmapsto \operatorname*{colim}_{U \text{ Nis. nbh. of } x} \mathcal{F}(U). \end{aligned}$$

It turns out that the family of functors of this type is jointly conservative on $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, by which we mean that a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if the induced morphism on all stalks is an isomorphism.[1]

Definition 2.3. We call a cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X \end{array} \tag{2.1}$$

elementary distinguished if and only if i is an open immersion, p is étale, and the induced morphism $p^{-1}(X \smallsetminus U)_{\mathrm{red}} \to (X \smallsetminus U)_{\mathrm{red}}$ is an isomorphism.

2.4. We note first that $\{\iota, p\}$ as in the above definition is a Nisnevich cover of X. Moreover, one can show that in order to check that a presheaf $\mathcal{F}$ on Sm_S is a sheaf, it is necessary and sufficient to check that for every elementary distinguished square (2.1) the induced square

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(i)} & \mathcal{F}(U) \\ {\scriptstyle \mathcal{F}(p)}\downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

is cartesian (cf. [54, Prop. 3.1.4]). Furthermore, if we apply the Yoneda embedding to (2.1), we obtain a cocartesian diagram in $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ (cf. [54, Lem. 3.1.6]), which readily yields a long exact sequence in Nisnevich cohomology of the Mayer-Vietoris type:

$$\cdots \to H^i_{\mathrm{Nis}}(X, \mathcal{F}) \to H^i_{\mathrm{Nis}}(U, \mathcal{F}) \oplus H^i_{\mathrm{Nis}}(V, \mathcal{F}) \to H^i_{\mathrm{Nis}}(U \times_X V, \mathcal{F}) \to H^{i+1}_{\mathrm{Nis}}(X, \mathcal{F}) \to \cdots$$

2.5. We have already indicated that the Nisnevich topology lies between the étale and the Zariski topology. So it shares some of the traits of both these, in particular the existence of a vanishing proposition, so that the Nisnevich cohomology of a scheme with Krull dimension d, vanishes in ranks $> d$ (cf. [54, Prop. 3.1.8]). For more details on the Nisnevich topology we refer the reader to the original source [58] by Nisnevich, the short note [32] by Hoyois and section 3.1 of [54] by Morel and Voevodsky.

Pointed Sheaves

We close this paragraph with some remarks on the case of pointed sheaves. We denote by $\bullet \in \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ the sheaf that is represented by the base scheme S itself. The *category of pointed sheaves* $\mathsf{Sh}^\bullet_{\mathrm{Nis}}(\mathsf{Sm}_S)$ is then the category of sheaves under $\bullet$, and it may be seen equivalently

[1] We note in passing that our notation alludes to x^* being the left adjoint part of a morphism of topoi $\mathsf{Set} \xrightarrow{x} \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ in the sense of [3, Définition IV.6.1]. Thus x^* preserves colimits and finite limits.

as the category of contravariant functors from Sm_S to pointed sets which are sheaves after forgetting basepoints.

The categories $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ resp. $\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ are both closed symmetric monoidal (see [41, VII.]), with respect to the product resp. smash product. We use this opportunity to recall the definition of the latter:

Let (A, a_0) and (B, b_0) be two pointed sets. The *smash product* $(A, a_0) \wedge (B, b_0)$ is defined as $A \times B/\sim$, where the equivalence relation $\sim$ makes the following identification:

$$(a_1, b_1) \sim (a_2, b_2) :\Longleftrightarrow (a_1 = a_2 \wedge b_1 = b_2) \vee ((a_1, b_1), (a_2, b_2) \in \{a_0\} \times B \cup A \times \{b_0\}).$$

If the basepoints are clear from the context, we will usually forget to write them. The equivalence class of $(a, b) \in A \times B$ will commonly be denoted as $a \wedge b$. We regard $A \wedge B$ again as a pointed set, with basepoint $a_0 \wedge b_0$. Now given two presheaves of pointed sets $\mathcal{F}$, $\mathcal{G}$, we may define the smash product of presheaves via

$$X \longmapsto \mathcal{F}(X) \wedge \mathcal{G}(X).$$

The smash product of sheaves is defined as the sheaf that is associated to this presheaf. The operator $\wedge$ defines a symmetric tensor product on $\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, with unit $\bullet \sqcup \bullet$, and with the usual coherence axioms in place. This structure is moreover closed, since we have the following right adjoint: Given pointed sheaves $\mathcal{F}$ and $\mathcal{G}$ one may define

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}, \mathcal{G})(X) := \{\varphi\colon \mathcal{F} \times h_X \to \mathcal{G} \mid \varphi \text{ morphism of sheaves, with } \varphi(*_U, f) = *_U, \\ \forall\, U \in \mathsf{Sm}_S, \forall\, f \in \mathrm{Hom}_{\mathsf{Sm}_S}(V, X)\}\end{aligned}$$

for every $X \in \mathsf{Sm}_S$, where by $*_U$ we mean the basepoint in the set $\mathcal{F}(U)$ resp. $\mathcal{G}(U)$.

2.1.2 The Derived Category

We still fix a noetherian and finite dimensional scheme S and consider the category $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ of abelian Nisnevich sheaves. Recall the following definition:

Definition 2.6. Let C be an abelian category. C is called a *Grothendieck abelian category* if and only if the following axioms hold:

AB 3) C possesses arbitrary coproducts.

AB 5) Filtered colimits of exact sequences are exact (cf. [27, Prop. 1.8]).

G) C admits a family of generators (cf. [27, 1.9]).

The conditions **AB 3)** and **AB 5)** are clearly met by $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ (cf. Tags 03CN and 03CO in [73]). To see the existence of a family of generators, we make the following point:

2.7. Consider the functor $\mathbb{Z}[-]\colon \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S) \to \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ that sends a sheaf $\mathcal{F}$ to the sheaf associated to $X \mapsto \bigoplus_{\mathcal{F}(X)} \mathbb{Z}$. We call this functor the *free sheaf functor*, and we note that it is left adjoint to the forgetful functor $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$. Before we proceed, we want to remark that there is also a basepointed version of this adjunction:

$$\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{Z}(\mathcal{F}), \mathcal{A}) \cong \mathrm{Hom}_{\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}, \mathcal{A}),$$

where $\mathbb{Z}(-)\colon \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S) \to \mathsf{Ab}_{Nis}(\mathsf{Sm}_S)$ sends a pointed sheaf $\mathcal{F}$ to the sheaf associated to $X \mapsto \bigoplus_{\mathcal{F}(X)\smallsetminus\{*\}} \mathbb{Z}$, and $\mathcal{A}$ is pointed by 0. Now we may embed a scheme X into $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ by concatenating the Yoneda embedding and the free sheaf functor:

$$\begin{aligned}\mathsf{Sm}_S &\longrightarrow \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)\\ X &\longmapsto \mathbb{Z}[X].\end{aligned}$$

To finally see that $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ fulfils **G)**, we may first remark that the category Sm_S is essentially small, so that there is a small category Sm'_S that it is equivalent to. Thus in particular the objects of Sm'_S form a set. Now, a generating family of $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ is given by $(\mathbb{Z}[X])_{X\in \mathit{Sm}'_S}$.

2.8. The category $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ is equipped with a tensor product: Let $\mathcal{F}, \mathcal{G}$ be abelian sheaves, then we define $\mathcal{F}\otimes\mathcal{G}$ to be the sheaf associated to the presheaf $X \mapsto \mathcal{F}(X)\otimes_{\mathbb{Z}}\mathcal{G}(X)$. One may check that $\otimes$ equips $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ with the structure of a symmetric monoidal category. Moreover, this monoidal structure is closed, i.e. one finds a right adjoint to every functor $-\otimes\mathcal{G}$, which is defined similarly to the internal Hom above. We denote it by $\underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathcal{G}, -)$.

In order to be able to do homology we furthermore embed $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ into the category of unbounded chain complexes[2] $\mathit{Comp}(\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S))$ by putting a given abelian sheaf in degree 0, and setting the remaining degrees to 0. Our shorthand notation for this latter category is $\mathit{Spc}_S^{\mathrm{ab}}$, and we refer to its objects simply as *complexes*.

2.9. The category $\mathit{Spc}_S^{\mathrm{ab}}$ is symmetric monoidal, with respect to the tensor product of complexes, whose definition we recall here, for reference. Let $C_* = (C_n, d_n^C)_{n\in\mathbb{Z}}$ and $D_* = (D_n, d_n^D)_{n\in\mathbb{Z}}$ be chain complexes, with differentials $C_n \xrightarrow{d_n^C} C_{n-1}$, $D_n \xrightarrow{d_n^D} D_{n-1}$, for $n\in\mathbb{Z}$. The tensor product $C_*\otimes D_*$ is defined by

$$(C_*\otimes D_*)_n := \bigoplus_{p+q=n} C_p\otimes D_q,$$

with the differential induced by $d_p^C\otimes \mathrm{id}_{D_q} + (-1)^p\,\mathrm{id}_{C_p}\otimes d_q^D$. This monoidal structure is also closed, as evidenced by the existence of the right adjoint internal Hom, given by:

$$\underline{\mathrm{Hom}}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*)_n := \prod_{p+q=n} \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(C_{-p}, D_q),$$

with the differential induced by $(d_q^D)_* - (-1)^n(d_{-p+1}^C)^*$, where we used the abbreviations

$$\left(d_q^D\right)_* := \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(C_{-p}, d_q^D) \quad\text{and}\quad \left(d_{-p+1}^C\right)^* := \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(d_{-p+1}^C, D_q).$$

2.10. The category $\mathit{Spc}_S^{\mathrm{ab}}$ is itself a Grothendieck category. Hereby the axioms **AB 3)** and **AB 5)** follow from the fact that colimits in $\mathit{Spc}_S^{\mathrm{ab}}$ can be computed degreewise. Denoting by $(G_i)_{i\in I}$ a set of generators for $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$, then we may give a set of generators of $\mathit{Spc}_S^{\mathrm{ab}}$ by the *disks* $(D^n(G_i))_{i\in I, n\in\mathbb{Z}}$

$$D^n(G_i)_m := \begin{cases} G_i, & \text{if } m=n-1 \vee m=n,\\ 0, & \text{otherwise,}\end{cases} \qquad d_m^{D^n(G_i)} := \begin{cases} \mathrm{id}_{G_i}, & \text{if } m=n,\\ 0, & \text{otherwise.}\end{cases}$$

The fact that these generate can be deduced from the the adjunction

$$\mathrm{Hom}_{\mathit{Spc}_S^{\mathrm{ab}}}(D^n(G_i), C_*) \cong \mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(G_i, C_n).$$

2.11. As a closed symmetric monoidal category, $\mathit{Spc}_S^{\mathrm{ab}}$ is enriched in itself. However, sometimes it might be useful to break statements down to a simpler case than to the case of complexes of sheaves. Therefore we note that by

$$\mathrm{Map}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*)_n := \prod_{p+q=n} \mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(C_{-p}, D_q),$$

[2]The differential has degree -1.

we also have the structure of a $\mathit{Comp}(\mathit{Ab})$-enrichment on $\mathit{Spc}_S^{\mathrm{ab}}$. There are the following compatibilities:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*)(S) &= \mathrm{Map}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*), \\ Z_0(\mathrm{Map}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*)) &= \mathrm{Hom}_{\mathit{Spc}^{\mathrm{ab}}}(C_*, D_*), \quad \text{and} \\ H_0(\mathrm{Map}_{\mathit{Spc}_S^{\mathrm{ab}}}(C_*, D_*)) &= \mathrm{Hom}_{\mathit{K}(S)}(C_*, D_*), \end{aligned}$$

where Z_0 gives the zero cycles, and by $\mathit{K}(S)$ we denote the homotopy category of chain complexes (in the sense of homological algebra). We may map $\mathit{Comp}(\mathit{Ab})$ back into $\mathit{Spc}_S^{\mathrm{ab}}$ by taking constant sheaves at each $n \in \mathbb{Z}$.

Next we discuss a first model category structure on $\mathit{Spc}_S^{\mathrm{ab}}$, which we will later localize, to reflect the contractibility of $\mathbb{A}_S^1$. The fact, that the particular choice of cofibrations, fibrations, and weak equivalences forms a model category structure on $\mathit{Spc}_S^{\mathrm{ab}}$, was checked independently by Hovey (cf. [30, Thm. 2.2]) and Beke (cf. [8, Prop. 3·13]). Later Cisinski and Déglise showed that this model category is moreover cellular and proper (cf. [13, Thm. 2.1]). Recall that a morphism of complexes is a *quasi-isomorphism* if and only if it induces isomorphisms on all homology sheaves.

Proposition 2.12. *The category $\mathit{Spc}_S^{\mathrm{ab}}$ admits the structure of a proper cellular model category, with weak equivalences the quasi-isomorphisms of chain complexes, and cofibrations all monomorphisms. This model category structure is called the* ***injective model category structure*** *on unbounded chain complexes.*

Note that in the above proposition the fibrations are implicitly defined as those morphisms that have the right lifting property with respect to all cofibrations that are also quasi-isomorphisms. We fix special names for the latter:

- cofibration + weak equivalence = *acyclic cofibration*, and
- fibration + weak equivalence = *acyclic fibration.*

The homotopy category of this model category is the derived category $\mathit{D}(S)$ of the abelian category $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$, which is clear from the usual characterisation of the derived category as the category of chain complexes localized with respect to the quasi-isomorphisms.
The category $\mathit{D}(S)$ is triangulated (see [56, Def. 1.3.13]), and admits a t-structure (see [7, Déf. 1.3.1]). For easier reference we recall the structure here: The (left) shifting functor on $\mathit{D}(S)$ is induced by the shifting of complexes

$$(-)[1] \colon \mathit{Spc}_S^{\mathrm{ab}} \to \mathit{Spc}_S^{\mathrm{ab}}, \quad (C_n, d_n)_{n\in\mathbb{Z}} \mapsto (C_{n-1}, d_{n-1})_{n\in\mathbb{Z}},$$

which respects quasi-isomorphisms, as $H_i(C_*[1]) = H_{i-1}(C_*)$, and thus descends to a functor on $\mathit{D}(S)$. The distinguished triangles in $\mathit{D}(S)$ are those triangles that are isomorphic to triangles of the form

$$C_* \xrightarrow{f} D_* \to \mathrm{cone}(f) \to C_*[1],$$

where $\mathrm{cone}(f)$ denotes the *mapping cone* of f, defined as

$$\mathrm{cone}(f)_n := D_n \oplus C_{n-1} \quad \text{and} \quad d_n^{\mathrm{cone}(f)} := \begin{pmatrix} d_n^D & f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix}.$$

As for the t-structure, we have the strictly full subcategories

$$\begin{aligned} \mathit{D}(S)_{\geq 0} &:= \{C_* \in \mathit{D}(S) \mid H_i(C_*) = 0, \ \forall i < 0\} \quad \text{and} \\ \mathit{D}(S)_{\leq 0} &:= \{C_* \in \mathit{D}(S) \mid H_i(C_*) = 0, \ \forall i > 0\}, \end{aligned}$$

which fulfil the axioms of a (homological) t-structure encompassing precisely the following axioms:

(t1) For all $C_* \in \mathsf{D}(S)_{\geq 0}$ and $D_* \in \mathsf{D}(S)_{\leq -1}$ we have $\mathrm{Hom}_{\mathsf{D}(S)}(C_*, D_*) = 0$.

(t2) $\mathsf{D}(S)_{\geq 1} \subseteq \mathsf{D}(S)_{\geq 0}$ and $\mathsf{D}(S)_{\leq 1} \supseteq \mathsf{D}(S)_{\leq 0}$.

(t3) For every complex $C_* \in \mathsf{D}(S)$, we may find $(C_*)_{\geq 0} \in \mathsf{D}(S)_{\geq 0}$ and $(C_*)_{\leq -1} \in \mathsf{D}(S)_{\leq -1}$ that fit into a triangle

$$(C_*)_{\geq 0} \to C_* \to (C_*)_{\leq -1} \to (C_*)_{\geq 0}[1].$$

We may define explicit truncation functors via:

$$\begin{aligned} \tau_{\geq 0} \colon \mathsf{D}(S) &\to \mathsf{D}(S)_{\geq 0} \\ C_* &\mapsto (\cdots \to C_2 \to C_1 \to \ker(d_0) \to 0 \to \cdots) \end{aligned} \quad \text{and}$$

$$\begin{aligned} \tau_{\leq 0} \colon \mathsf{D}(S) &\to \mathsf{D}(S)_{\leq 0} \\ C_* &\mapsto (\cdots \to 0 \to C_1/\ker(d_1) \to C_0 \to C_{-1} \to \cdots). \end{aligned}$$

By the homological version of [7, Prop. 1.3.3], we have that $\tau_{\geq 0}$ is right adjoint to the inclusion, and $\tau_{\leq 0}$ is left adjoint to the inclusion. We also make the definitions $\mathsf{D}(S)_{\geq n} := \mathsf{D}(S)_{\geq 0}[n]$ and $\mathsf{D}(S)_{\leq n} := \mathsf{D}(S)_{\leq 0}[n]$, and similarly for the truncation functors. This t-structure is called the *natural t-structure* on $\mathsf{D}(S)$, and its heart $\mathsf{D}(S)^{\heartsuit} := \mathsf{D}(S)_{\geq 0} \cap \mathsf{D}(S)_{\leq 0}$ is an abelian category (cf. [7, Thm. 1.3.6]), which is moreover equivalent to $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ via taking zero homology, and the canonical inclusion $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \mathsf{D}(S)^{\heartsuit}$ putting an abelian sheaf in degree zero. As a first application of the natural t-structure, we note the following lemma, which follows by the argument in [4, Lem. 3.3].

Lemma 2.13. *Let M be an abelian sheaf. Then we have a natural (in both entries) isomorphism*

$$\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(H_0(C_*), M) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{D}(S)}(C_*, M[0]),$$

for every complex C_*.

The above lemma is analogous to the well-known statement in algebraic topology that homotopies with target a discrete topological space are constant and that maps into discrete spaces are up to homotopy completely determined by the induced maps on path-components. So the slogan to the lemma is: Sheaves are discrete in $\mathsf{Spc}_S^{\mathrm{ab}}$.

A complex C_* such that the unique morphism to 0 is a fibration is called *fibrant*. From the factorization axiom (cf. [29, Def. 7.1.3]) we obtain a functor $\mathrm{Ex}^{\mathrm{ab}} \colon \mathsf{Spc}_S^{\mathrm{ab}} \to \mathsf{Spc}_S^{\mathrm{ab}}$, together with a natural transformation $\theta \colon \mathrm{id} \to \mathrm{Ex}^{\mathrm{ab}}$ such that $\mathrm{Ex}^{\mathrm{ab}}(C_*)$ is fibrant, $\theta(C_*)$ is an acyclic cofibration, and

$$C_* \to 0 = C_* \xrightarrow{\theta(C_*)} \mathrm{Ex}^{\mathrm{ab}}(C_*) \to 0.$$

This justifies the name *fibrant replacement functor* for $\mathrm{Ex}^{\mathrm{ab}}$. The following lemma goes under the grandiose name *fundamental lemma of homological algebra*. It describes a deep connection between the model categorical viewpoint on $\mathsf{D}(S)$, and the classical approach via chain homotopies.

Lemma 2.14. *Let D_* be a fibrant complex. Then for any complex C_* the natural homomorphism*

$$\mathrm{Hom}_{\mathsf{K}(S)}(C_*, D_*) \to \mathrm{Hom}_{\mathsf{D}(S)}(C_*, D_*)$$

is an isomorphism.

Remark 2.15. The above lemma seems to be well known to the experts. It follows from Quillen's homotopical algebra [62]. See also [50, Lem. 6.16] for more references.

2.1.3 $\mathbb{A}^1$-Localization

The main point of $\mathbb{A}^1$-algebraic topology is the additional requirement that the affine line is contractible, which in a sense identifies $\mathbb{A}^1_S$ with the unit interval from algebraic topology. However, such a relation, i.e. the triviality of the complex $\mathbb{Z}[\mathbb{A}^1_S]/\mathbb{Z}$, is not present in $\mathsf{D}(S)$, so that one has to add it via an additional step, which (again) takes the form of a localization of categories.

Definition 2.16. Let $E_* \in \mathsf{Spc}^{\mathrm{ab}}_S$ be a complex. E_* is called $\mathbb{A}^1$-*local* if and only if the homomorphism

$$\mathrm{Hom}_{\mathsf{D}(S)}(C_* \otimes Z[\mathbb{A}^1_S], E_*) \longrightarrow \mathrm{Hom}_{\mathsf{D}(S)}(C_*, E_*),$$

induced by the inclusion of zero $S \hookrightarrow \mathbb{A}^1_S$, is bijective, for all complexes C_*. We denote the full subcategory of the $\mathbb{A}^1$-local objects of $\mathsf{D}(S)$ by $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$.

Let us remark first that this definition is well-defined: Indeed, since $\mathbb{Z}[\mathbb{A}^1_S]$ is a torsion-free abelian sheaf, tensoring with it preserves cofibrations, and acyclic cofibrations. As such $(-) \otimes \mathbb{Z}[\mathbb{A}^1_S]$ is a left Quillen functor, and induces a well-defined functor on $\mathsf{D}(S)$ (see [29, Def. 8.5.2]).

2.17. The subcategory $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$ is a strictly full, saturated, triangulated subcategory of $\mathsf{D}(S)$, indeed:

- *Strictly full:* Every complex that is isomorphic to an $\mathbb{A}^1$-local complex, is itself $\mathbb{A}^1$-local. By definition $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$ is a full subcategory.

- The subcategory $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$ contains 0 and direct sums of $\mathbb{A}^1$-local complexes, and is thus an additive subcategory.

- Shifts of $\mathbb{A}^1$-local complexes are again $\mathbb{A}^1$-local.

- Given a triangle $E^1_* \to E^2_* \to E_* \to E^1_*[1]$, with E^1_* and E^2_* being $\mathbb{A}^1$-local, we see that E_* is $\mathbb{A}^1$-local, by applying the homology functors (see [56, Lem. 1.1.10]) $\mathrm{Hom}_{\mathsf{D}(S)}(C_* \otimes \mathbb{Z}[\mathbb{A}^1_S], -)$ resp. $\mathrm{Hom}_{\mathsf{D}(S)}(C_*, -)$ to the triangle, and using the five lemma. Thus $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$ is a triangulated subcategory.

- *Saturated:* For every two complexes E_* and F_* such that $E_* \oplus F_*$ is $\mathbb{A}^1$-local, E_* and F_* are individually $\mathbb{A}^1$-local: We have the split triangle $E_* \to E_* \oplus F_* \to F_* \to E_*[1]$, which induces the following diagram with short exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(C_* \otimes \mathbb{Z}[\mathbb{A}^1_S], E_*) & \longrightarrow & \mathrm{Hom}(C_* \otimes \mathbb{Z}[\mathbb{A}^1_S], E_* \oplus F_*) & \longrightarrow & \mathrm{Hom}(C_* \otimes \mathbb{Z}[\mathbb{A}^1_S], F_*) & \to 0 \\ & \downarrow & & \downarrow \cong & & \downarrow & \\ 0 \to & \mathrm{Hom}_{\mathsf{D}(S)}(C_*, E_*) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}(S)}(C_*, E_* \oplus F_*) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}(S)}(C_*, F_*) & \to 0, \end{array}$$

for any complex C_*. Since the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[\mathbb{A}^1_S]$ has a section, we know that the left and right vertical homomorphisms need to be surjective. By a diagram chase it follows that they need to be isomorphisms as well. This checks the claim.

Next we introduce the morphisms that we would like to see inverted in a potential $\mathbb{A}^1$-derived category:

Definition 2.18. A morphism $f\colon C_* \to D_*$ in $\mathsf{Spc}^{\mathrm{ab}}_S$ is called an $\mathbb{A}^1$-quasi-isomorphism if and only if for any $\mathbb{A}^1$-local complex E_*, the induced homomorphism

$$\mathrm{Hom}_{\mathsf{D}(S)}(D_*, E_*) \longrightarrow \mathrm{Hom}_{\mathsf{D}(S)}(C_*, E_*)$$

is an isomorphism.

So, we want to localize $\mathsf{D}(S)$ with respect to all $\mathbb{A}^1$-quasi-isomorphisms. Since $\mathsf{D}(S)$ is a triangulated category, by the method of forming the quotient of triangulated categories, this is always possible. Indeed, setting

$$\mathsf{T}_{\mathbb{A}^1} := {}^{\perp}\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) = \left\{ Q_* \in \mathsf{D}(S) \;\middle|\; \mathrm{Hom}_{\mathsf{D}(S)}(Q_*, E_*) = 0, \; \forall E_* \in \mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) \right\}$$

defines a strictly full, saturated, triangulated subcategory of $\mathsf{D}(S)$ (cf. [73, Tag 0FXC]). By [56, Thm. 2.1.8] (and [56, Prop. 2.1.24]) we obtain a (potentially not locally small) triangulated category $\mathsf{D}_{\mathbb{A}^1}(S) := \mathsf{D}(S)/\mathsf{T}_{\mathbb{A}^1}$, together with a triangulated quotient functor $\upsilon\colon \mathsf{D}(S) \to \mathsf{D}_{\mathbb{A}^1}(S)$ such that any functor $F\colon \mathsf{D}(S) \to \mathsf{C}$, into some category C, which inverts all $\mathbb{A}^1$-quasi-isomorphisms factors through υ. The triangles in $\mathsf{D}_{\mathbb{A}^1}(S)$ are all those triangles isomorphic to triangles of the form

$$\upsilon(C_*) \to \upsilon(D_*) \to \upsilon(E_*) \to \upsilon(C_*)[1],$$

where $C_* \to D_* \to E_* \to C_*[1]$ is a triangle in $\mathsf{D}(S)$. By [9, Prop. 1.6] we also have means to check that $\mathsf{D}_{\mathbb{A}^1}(S)$ is locally small: Assume that the inclusion of $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) \hookrightarrow \mathsf{D}(S)$ admits a left-adjoint

$$L^{\mathrm{ab}}_{\mathbb{A}^1}\colon \mathsf{D}(S) \longrightarrow \mathsf{D}_{\mathbb{A}^1\text{-loc}}(S).$$

The functor $L^{\mathrm{ab}}_{\mathbb{A}^1}$ gives rise to an equivalence of triangulated categories $\mathsf{D}_{\mathbb{A}^1}(S) \to \mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$, and the induced functor

$$\mathsf{D}_{\mathbb{A}^1}(S) \to \mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) \subseteq \mathsf{D}(S)$$

is left adjoint to the universal quotient functor υ. Thus $L^{\mathrm{ab}}_{\mathbb{A}^1}$ deserves the name $\mathbb{A}^1$-*localization functor*. The following existence theorem is due to Morel[3], in the context of stable $\mathbb{A}^1$-homotopy theory, but, as he remarks, it is true in the $\mathbb{A}^1$-derived setting as well.

Theorem 2.19. *The inclusion $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) \subseteq \mathsf{D}(S)$ admits a left adjoint $L^{\mathrm{ab}}_{\mathbb{A}^1}$, which is exact as a functor of triangulated categories.*

Remark 2.20. By the above we obtain that the localization of $\mathsf{D}(S)$ with respect to all $\mathbb{A}^1$-quasi-isomorphisms exists as a locally small category. Unfortunately, this gives *a priori* no statement about a potential model category structure on $\mathsf{Spc}^{\mathrm{ab}}_S$, with the cofibrations being the monomorphisms, and the weak equivalences being given by the $\mathbb{A}^1$-quasi-isomorphisms, whose associated homotopy category would also be $\mathsf{D}_{\mathbb{A}^1}(S)$. However, by either redoing the proof of [54, Thm. 2.2.21], where Morel and Voevodsky constructed the unstable $\mathbb{A}^1$-model category structure, in the derived setting, or by following [53, Rem. 4.2.5], one obtains the desired *abelian* $\mathbb{A}^1$-*model category structure* on $\mathsf{Spc}^{\mathrm{ab}}_S$.

Remark 2.21. We also remark that by the factorization axiom (cf. [29, Def. 7.1.3]) there is a model category analogue to the localization functor, namely a functor $\mathrm{Ex}^{\mathrm{ab}}_{\mathbb{A}^1}\colon \mathsf{Spc}^{\mathrm{ab}}_S \to \mathsf{Spc}^{\mathrm{ab}}_S$, which maps every complex C_* to one that factors the unique morphism $C_* \to 0$ into an $\mathbb{A}^1$-acyclic cofibration $C_* \to \mathrm{Ex}^{\mathrm{ab}}_{\mathbb{A}^1}(C_*)$, and an $\mathbb{A}^1$-fibration $\mathrm{Ex}^{\mathrm{ab}}_{\mathbb{A}^1}(C_*) \to 0$. In this context we also record, that one may prove as in [54, Prop. 2.2.28], that a fibrant complex C_* is $\mathbb{A}^1$-local if and only if C_* is $\mathbb{A}^1$-fibrant.

Corollary 2.22. *The abelian $\mathbb{A}^1$-model category structure is proper.*

[3] See [53], theorem 4.2.1 and remark 8. There is also a version for S being the spectrum of a perfect field k, given in [50, Cor. 6.19].

Proof. Since all objects in $\mathit{Spc}_S^{\mathrm{ab}}$ are cofibrant, left properness is formal (cf. [29, Cor. 13.1.3]). We will show right properness as in [13, Prop. 4.3]. Therefore consider a cartesian square

$$\begin{array}{ccc} C_* & \xrightarrow{f'} & E_* \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ D_* & \xrightarrow[f]{} & F_*, \end{array}$$

where p is an $\mathbb{A}^1$-fibration, and f is an $\mathbb{A}^1$-quasi-isomorphism. First note that, since p has in particular the right lifting property with respect to all acyclic cofibrations, that p is a fibration, as well. Thus, by [29, Cor. 13.3.8], the above square is homotopy cartesian. In triangulated category terms this means that

$$C_* \xrightarrow{\begin{pmatrix} p' \\ -f' \end{pmatrix}} D_* \oplus E_* \xrightarrow{\begin{pmatrix} f & p \end{pmatrix}} F_* \to C_*[1]$$

is a distinguished triangle for some morphism $F_* \to C_*[1]$ (see [56, Def. 1.4.1]). Since the quotient functor $v\colon \mathit{D}(S) \to \mathit{D}_{\mathbb{A}^1}(S)$ is triangulated, this is mapped to a distinguished triangle in the $\mathbb{A}^1$-derived category. In $\mathit{D}_{\mathbb{A}^1}(S)$ we have that f is an isomorphism, which implies that f' is an isomorphism as well. Thus f' has to be an $\mathbb{A}^1$-quasi-isomorphism (cf. [29, Thm. 8.3.10]). This completes the proof that the abelian $\mathbb{A}^1$-model category structure is proper. □

Strictly $\mathbb{A}^1$-Invariant Sheaves

In the following we briefly discuss one of the most important examples of $\mathbb{A}^1$-local complexes to us.

Definition 2.23. Let $M \in \mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ be a sheaf of abelian groups. We call M *strictly* $\mathbb{A}^1$-*invariant* if and only if for all $X \in \mathit{Sm}_S$ the homomorphism

$$H^i_{\mathrm{Nis}}(\mathbb{A}^1_X, M) \longrightarrow H^i_{\mathrm{Nis}}(X, M)$$

induced by the embedding of zero $X \hookrightarrow \mathbb{A}^1_X$ is an isomorphism for all $i \in \mathbb{N}$. We denote by $\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_S)$ the full subcategory of $\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)$ consisting of the strictly $\mathbb{A}^1$-invariant sheaves.

Example 2.24. If M denotes an abelian group, by [53, Ex. 3.2.4] we know that, regarding M as a constant Nisnevich sheaf,

$$H^i_{\mathrm{Nis}}(X, M) = \begin{cases} M(X), & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

holds. Since $M(X)$ are the continuous maps from X into the discrete topological space M, we see that M is strictly $\mathbb{A}^1$-invariant. An instance of this we use in abundance is $\mathbb{Z}$.

Example 2.25. In [77, Thm. 5.6] Voevodsky has shown that any homotopy invariant pretheory with transfers, over $S = \mathrm{Spec}(k)$, for k being a perfect field, is strictly $\mathbb{A}^1$-invariant. This specializes to the fact that any Nisnevich sheaf with transfers is strictly $\mathbb{A}^1$-invariant.

Below we give more involved examples of strictly $\mathbb{A}^1$-invariant sheaves, but before we proceed to those, let us sketch the reason for our interest in them. We start with the following lemma:

Lemma 2.26. *Denote by $\mathit{Sm}'_S \subseteq \mathit{Sm}_S$ a small equivalent subcategory. The set*

$$\{\mathbb{Z}[U] \mid U \in \mathit{Sm}'_S\}$$

is a set of generators for the triangulated category $\mathsf{D}(S)$, i.e. an object $C_* \in \mathsf{D}(S)$ is zero if and only if

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], C_*[n]) = 0,$$

for all $n \in \mathbb{Z}$ and $U \in \mathsf{Sm}'_S$.

Proof. Suppose we have $C_* \in \mathsf{D}(S)$ such that there is no nontrivial morphism from any $\mathbb{Z}[U]$ into any shift of C_*. We may replace C_* by $\mathrm{Ex}^{\mathrm{ab}}(C_*)$, since they are related by an acyclic cofibration. Thus we have

$$0 = \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], C_*[n]) = \mathrm{Hom}_{\mathsf{K}(S)}(\mathbb{Z}[U], C_*[n]) = H_{-n}(C_*(U)),\ \forall U \in \mathsf{Sm}'_S,\ n \in \mathbb{Z}.$$

This means that the complex $C_*(U)$ is exact. Since the points on the topos $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ are given by filtered colimits over objects of Sm'_S, we obtain $H_n(C_*) = 0$, for all $n \in \mathbb{Z}$. Thus C_* is exact.

Conversely, assume that C_* is an exact complex, and we are given $U \in \mathsf{Sm}'_S$ and $n \in \mathbb{Z}$ arbitrarily. Again we may assume that C_* is fibrant. By [30, Prop. 2.12] we know that C_* consists of injective sheaves, and is exact. Now set $N := \dim(U)$. N is finite by our assumptions on S, and thus we may consider the following truncation of C_*:

$$0 \to C_{N-n+2}/\ker(d_{N-n+2}) \to C_{N-n+1} \to \dots .$$

We think of this as an injective resolution of the sheaf $C_{N-n+2}/\ker(d_{N-n+2})$. By the vanishing theorem for Nisnevich cohomology, we see that

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], C_*[n]) = H_{-n}(C_*(U)) = H^{N+1}_{\mathrm{Nis}}(U, C_{N-n+2}/\ker(d_{N-n+2})) = 0.$$

This yields our claim. □

Proposition 2.27. *Let M be an abelian sheaf. The following are equivalent:*

(i) M, considered as a complex concentrated in degree 0, is $\mathbb{A}^1$-local.

(ii) M is strictly $\mathbb{A}^1$-invariant.

Proof. (i) $\Longrightarrow$ (ii): For every $U \in \mathsf{Sm}_S$, and $n \in \mathbb{Z}$, we have in particular that the homomorphism

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U] \otimes \mathbb{Z}[\mathbb{A}^1_S], M[n]) \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], M[n]),$$

is an isomorphism. Note that we have a natural isomorphism $\mathbb{Z}[U] \otimes \mathbb{Z}[\mathbb{A}^1_S] \cong \mathbb{Z}[\mathbb{A}^1_U]$. Let us fix an injective resolution of the sheaf M via

$$0 \to M \to I_0 \to I_{-1} \to \dots ,$$

which we equivalently regard as an acyclic cofibration $M \to I_*$. By [30, Proposition 2.12.] I_* is a fibrant complex. Thus we have the following natural chain of isomorphisms:

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], M[n]) \cong \mathrm{Hom}_{\mathsf{K}(S)}(\mathbb{Z}[U], I_*[n]) \cong H_{-n}(I_*(U)) = H^n_{\mathrm{Nis}}(U, M).$$

By the above, we thus see that M is strictly $\mathbb{A}^1$-invariant.

(ii) $\Longrightarrow$ (i): Since $\mathbb{Z}[\mathbb{A}^1_S]$ is a torsion-free sheaf, we have that $\otimes\mathbb{Z}[\mathbb{A}^1_S]$ is exact. Thus we have that the pair of adjoint functors

$$- \otimes \mathbb{Z}[\mathbb{A}^1_S] \dashv \underline{\mathrm{Hom}}_{\mathsf{Spc}^{\mathrm{ab}}_S}(\mathbb{Z}[\mathbb{A}^1_S], -)$$

forms a Quillen pair (see [29, Def. 8.5.2]). So they induce a pair of adjoint functors on $\mathsf{D}(S)$, namely

$$- \otimes_{\mathbf{L}} \mathbb{Z}[\mathbb{A}^1_S] \dashv \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(\mathbb{Z}[\mathbb{A}^1_S], -).$$

Using this adjointness, the Yoneda lemma, and [29, Thm. 8.3.10], one sees that the condition for M to be $\mathbb{A}^1$-local is equivalent to

$$h\colon \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(\mathbb{Z}[\mathbb{A}^1_S], M) \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(\mathbb{Z}, M)$$

being an isomorphism in $\mathsf{D}(S)$. Equivalently, one may demand that the cone of h is zero. By lemma 2.26 this is precisely the case if and only if

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], \mathrm{cone}(h)[n]) = 0, \quad \forall\, U \in \mathsf{Sm}_S,\ n \in \mathbb{Z}.$$

Since $\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], -)$ is a homology functor, we see by using adjointness once more that this is equivalent to

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U] \otimes \mathbb{Z}[\mathbb{A}^1_S], M[n]) \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], M[n])$$

being an isomorphism. By the above, statement (i) follows. □

2.1.4 $\mathbb{A}^1$-Connectivity and $\mathbb{A}^1$-Local t-Structure

In this section we are concerned with defining a t-structure on $\mathsf{D}_{\mathbb{A}^1}(S)$ with nice geometric properties. A helpful condition is the following, first given by Morel in the context of stable $\mathbb{A}^1$-homotopy theory in [53].

Definition 2.28. We say that the *abelian $\mathbb{A}^1$-connectivity property* holds over S if and only if for every $C_* \in \mathsf{D}(S)_{\geq 0}$ we have that $L^{\mathrm{ab}}_{\mathbb{A}^1}(C_*)$ also lies in $\mathsf{D}(S)_{\geq 0}$.

Morel has shown that the abelian $\mathbb{A}^1$-connectivity property holds for S being the spectrum of a field (cf. [53, Rem. 8]). Since this is the case we are primarily interested in, we proceed to discuss its consequences with respect to the above notions:

Lemma 2.29 ([53, Lem. 6.2.6])**.** *Assume the abelian $\mathbb{A}^1$-connectivity holds over S, and let E_* be an $\mathbb{A}^1$-local complex. Then the truncated complex $\tau_{\geq 0}(E_*)$ is also $\mathbb{A}^1$-local.*

Proof. We reproduce the proof here, because it is instructive: By functoriality we have a morphism

$$L^{\mathrm{ab}}_{\mathbb{A}^1}(\tau_{\geq 0}(E_*)) \longrightarrow L^{\mathrm{ab}}_{\mathbb{A}^1}(E_*) \cong E_*.$$

Since abelian $\mathbb{A}^1$-connectivity holds, we know that $L^{\mathrm{ab}}_{\mathbb{A}^1}(\tau_{\geq 0}(E_*))$ lies in $\mathsf{D}(S)_{\geq 0}$. Now, by right adjointness of $\tau_{\geq 0}$ the above morphism induces a splitting of the unit of the localization functor $\tau_{\geq 0}(E_*) \to L^{\mathrm{ab}}_{\mathbb{A}^1}(\tau_{\geq 0}(E_*))$. Since $\mathsf{D}_{\mathbb{A}^1\text{-loc}}(S)$ is saturated, we win. □

Lemma 2.30. *Let C_* be a complex, fix $n, i \in \mathbb{Z}$, and let $U \in \mathsf{Sm}_S$.*

(a) As $\tau_{\geq n}$ is right adjoint to the inclusion of $\mathsf{D}(S)_{\geq n}$, we have a natural morphism $\tau_{\geq n}(C_*) \to C_*$. This morphism induces the homomorphism

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\geq n}(C_*)) \longrightarrow \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], C_*),$$

which is a monomorphism, whenever $i + 1 \geq n$, and an isomorphism, whenever $i \geq n$.

(b) As $\tau_{\leq n}$ is left adjoint to the inclusion of $\mathsf{D}(S)_{\leq n}$, we have a natural morphism $C_* \to \tau_{\leq n}(C_*)$. This morphism induces the homomorphism

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], C_*) \longrightarrow \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\leq n}(C_*)),$$

which is an epimorphism, whenever $n \geq i+d$, and an isomorphism, whenever $n \geq i+d+1$, where $d := \dim(U)$.

Proof. (a) The natural t-structure gives us the canonical triangle

$$\tau_{\geq n}(C_*) \to C_* \to \tau_{\leq n-1}(C_*) \to \tau_{\geq n}(C_*)[1],$$

for every complex C_* in $\mathsf{D}(S)$. Applying the homology functor $\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U], -)$ to this triangle, we obtain an exact sequence

$$\begin{aligned} \ldots \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\leq n-1}(C_*)[-1]) \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\geq n}(C_*)) \to \ldots \\ \ldots \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], C_*) \to \ldots \to \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\leq n-1}(C_*)) \to \ldots \end{aligned}$$

Now the claim follows by axiom **(t1)**.

(b) By the above triangle, we have to deduce the vanishing of the abelian groups

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], \tau_{\geq n+1}(C_*)) \quad \text{resp.} \quad \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i-1], \tau_{\geq n+1}(C_*)).$$

In order to handle both cases at once, we set $j = i$ resp. $j = i-1$, and assume correspondingly the inequality $n \geq j+d+1$. Let $\tau_{\geq n+1}(C_*) \to I_*$ be an acyclic cofibration, and I_* a fibrant complex. By [30, Prop. 2.12] we know that the complex I_* consists of injective objects. Since $\tau_{\geq n+1}(C_*)$ has no homology in degrees $\leq n$, we see that for $\mathcal{F} := \mathrm{im}(I_{n+1} \to I_n)$ we have the exact sequence

$$0 \to \mathcal{F} \to I_n \to I_{n-1} \to \ldots,$$

which we regard as an injective resolution of $\mathcal{F}$. Coming back to the sets of morphisms in question, we have

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][j], \tau_{\geq n+1}(C_*)) = \mathrm{Hom}_{\mathsf{K}(S)}(\mathbb{Z}[U][j], I_*) \cong \mathrm{Hom}_{\mathsf{K}(S)}(\mathbb{Z}[U], I_*[-j]) = H_j(I_*(U)).$$

As we always have $n - j \geq d+1 > 0$, $H_j(I_*(U)) = H^{n-j}_{\mathrm{Nis}}(U, \mathcal{F})$ follows, which is zero by the vanishing theorem for Nisnevich cohomology. This yields the claim. ☐

Theorem 2.31 ([53, Thm. 6.2.7])**.** *Assume the abelian $\mathbb{A}^1$-connectivity holds over S, and let E_* be a complex. The following are equivalent:*

(i) E_* is $\mathbb{A}^1$-local.

(ii) $\tau_{\geq n}(E_*)$ is $\mathbb{A}^1$-local, for all $n \in \mathbb{Z}$.

(iii) The sheaves $H_n(E_*)$ are strictly $\mathbb{A}^1$-invariant, for all $n \in \mathbb{Z}$.

Proof. (i) $\Longrightarrow$ (ii): Consequence of Lemma 2.29. (ii) $\Longrightarrow$ (iii): From the diagram

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{d_{n+2}} & E_{n+1} & \xrightarrow{d_{n+1}} & \mathrm{im}(d_{n+1}) & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ \ldots & \xrightarrow{d_{n+2}} & E_{n+1} & \xrightarrow{d_{n+1}} & \ker(d_n) & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & 0 & \longrightarrow & H_n(E_*) & \longrightarrow & 0 & \longrightarrow & \ldots, \end{array}$$

we obtain a short exact sequence $0 \to F_* \hookrightarrow \tau_{\geq n}(E_*) \twoheadrightarrow H_n(E_*)[n] \to 0$, which in turn induces a triangle

$$\tau_{\geq n+1}(E_*) \to \tau_{\geq n}(E_*) \to H_n(E_*)[n] \to \tau_{\geq n+1}(E_*)[1], \tag{2.2}$$

since $\tau_{\geq n+1}(E_*)$ is quasi-isomorphic to F_*. Thus we see that the sheaves $H_n(E_*)$ are $\mathbb{A}^1$-local. By the characterization of $\mathbb{A}^1$-local sheaves in proposition 2.27, we see that $H_n(E_*)$ is strictly $\mathbb{A}^1$-invariant.

(iii) $\Longrightarrow$ (i): By (2.2) and the symmetric triangle

$$H_n(E_*)[n] \to \tau_{\leq n}(E_*) \to \tau_{\leq n-1}(E_*) \to H_n(E_*)[n+1],$$

we obtain by induction that for any choice $m \geq n$ in $\mathbb{Z}$ we have that

$$\tau_{\geq n}(\tau_{\leq m}(E_*)) \quad \text{and} \quad \tau_{\leq m}(\tau_{\geq n}(E_*))$$

are $\mathbb{A}^1$-local. By what we have used in the proof of proposition 2.27 to show that E_* is $\mathbb{A}^1$-local, it suffices to check that

$$\mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[\mathbb{A}^1_U][i], E_*) \longrightarrow \mathrm{Hom}_{\mathsf{D}(S)}(\mathbb{Z}[U][i], E_*)$$

is an isomorphism for all $U \in \mathsf{Sm}_S$ and $i \in \mathbb{Z}$. Now (i) follows with lemma 2.30. Indeed, by first choosing n small enough, we see that $\tau_{\leq m}(E_*)$ is $\mathbb{A}^1$-local for all $m \in \mathbb{Z}$. By part (b) of lemma 2.30, and choosing m large enough, we obtain from this that E_* is $\mathbb{A}^1$-local. □

The above theorem presents another justification for our attention to strictly $\mathbb{A}^1$-invariant sheaves. The main consequence of the above is the possibility to introduce the following interesting t-structure:

Definition 2.32. Assume that S has the abelian $\mathbb{A}^1$-connectivity property. For any complex $C_* \in \mathsf{D}(S)$ and integer $n \in \mathbb{Z}$ we define

$$\mathbb{H}_n^{\mathbb{A}^1}(C_*) := H_n(L_{\mathbb{A}^1}^{\mathrm{ab}}(C_*)) \in \mathsf{Ab}_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathsf{Sm}_S).$$

We also define the following strictly full subcategories of $\mathsf{D}_{\mathbb{A}^1}(S)$:

$$\mathsf{D}_{\mathbb{A}^1}(S)_{\geq 0} := \left\{ C_* \in \mathsf{D}_{\mathbb{A}^1}(S) \;\middle|\; \mathbb{H}_n^{\mathbb{A}^1}(C_*) = 0,\ \forall n < 0 \right\} \quad \text{and}$$
$$\mathsf{D}_{\mathbb{A}^1}(S)_{\leq 0} := \left\{ C_* \in \mathsf{D}_{\mathbb{A}^1}(S) \;\middle|\; \mathbb{H}_n^{\mathbb{A}^1}(C_*) = 0,\ \forall n > 0 \right\}.$$

(Note that the above is well-defined, since $\mathbb{A}^1$-quasi-isomorphic complexes have quasi-isomorphic $\mathbb{A}^1$-localizations.)

The following can be derived from the properties of the natural t-structure:

Lemma 2.33 ([53, Lem. 6.2.11])**.** *Assume that S satisfies the abelian $\mathbb{A}^1$-connectivity property. Then $(\mathsf{D}_{\mathbb{A}^1}(S)_{\geq 0}, \mathsf{D}(S)_{\mathbb{A}^1}(S)_{\leq 0})$ defines a t-structure on $\mathsf{D}_{\mathbb{A}^1}(S)$, which we call the* ***homology t-structure****. Moreover, the abelian $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}^{\mathrm{ab}}$ induces a functor*

$$\mathsf{D}_{\mathbb{A}^1}(S) \to \mathsf{D}_{\mathbb{A}^1\text{-loc}}(S) \subseteq \mathsf{D}(S)$$

that respects the corresponding t-structures (and is moreover exact).

The homology t-structure is not the only available t-structure on $\mathsf{D}_{\mathbb{A}^1}(S)$. It is even possible to define a t-structure without the abelian $\mathbb{A}^1$-connectivity property present (cf. [53, Rem. 6.2.12]), however there is no obvious way to relate them to the natural t-structure on $\mathsf{D}(S)$. The existence of the homology t-structure, implies in the same way as lemma 2.13:

Lemma 2.34. *Assume that S has the abelian $\mathbb{A}^1$-connectivity property. Let M be a strictly $\mathbb{A}^1$-invariant sheaf. Then we have a natural (in both entries) isomorphism*

$$\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(C_*), M) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{D}_{\mathbb{A}^1}(S)}(C_*, M[0]),$$

for every complex $C_* \in \mathsf{D}_{\mathbb{A}^1}(S)$.

Recall that this states that strictly $\mathbb{A}^1$-invariant sheaves behave like discrete spaces in $\mathsf{D}_{\mathbb{A}^1}(S)$.

2.35. Finally we analyse the heart of the homology t-structure, in the case that S has the abelian $\mathbb{A}^1$-connectivity property. Recall from above that H_0 and the inclusion $(-)[0]\colon \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \mathsf{D}(S)$, induce an equivalence of categories between $\mathsf{D}(S)^{\heartsuit}$ and $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$. This readily implies that

$$\mathbb{H}_0^{\mathbb{A}^1}\colon \mathsf{D}_{\mathbb{A}^1}(S) \to \mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S) \quad \text{and} \quad (-)[0]\colon \mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \mathsf{D}_{\mathbb{A}^1}(S)$$

induce an equivalence of categories between $\mathsf{D}_{\mathbb{A}^1}(S)^{\heartsuit}$ and $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)$. Elaborating on that, we even have the commutative diagram (up to natural isomorphism)

$$\begin{array}{ccc} \mathsf{D}_{\mathbb{A}^1}(S)^{\heartsuit} & \xrightarrow{L^{\mathrm{ab}}_{\mathbb{A}^1}} & \mathsf{D}(S)^{\heartsuit} \\ {\scriptstyle \mathbb{H}_0^{\mathbb{A}^1}}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle H_0} \\ \mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S) & \hookrightarrow & \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S), \end{array}$$

where the lower horizontal functor is the canonical inclusion, and where we took the liberty of denoting the induced functors by the inducing functors. Now, since $\mathbb{H}_0^{\mathbb{A}^1}$ and H_0 are exact by virtue of [7, Thm. 1.3.6], and $L^{\mathrm{ab}}_{\mathbb{A}^1}$ is exact as well, we see that the inclusion $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S) \subseteq \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ is exact as an inclusion of abelian categories.

Corollary 2.36. *Assuming the abelian $\mathbb{A}^1$-connectivity property on S, kernels and cokernels of morphisms between strictly $\mathbb{A}^1$-invariant sheaves, calculated in the category $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, are strictly $\mathbb{A}^1$-invariant.*

Applying lemma 2.34 in the case of C_* being $N[0]$, for N an abelian sheaf, we obtain the following useful adjunction

$$\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(N[0]), M) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(N, M),$$

which identifies $\mathbb{H}_0^{\mathbb{A}^1}(N[0])$ as a universal strictly $\mathbb{A}^1$-invariant sheaf mapping into the strictly $\mathbb{A}^1$-invariant sheaf M, given N.

2.1.5 Base Change

In this section we consider the base change of the above constructions along a fixed morphism $f\colon S' \to S$ between noetherian and finite dimensional schemes S and S'. On a first reading one may skip this part, as its statements will mainly shine in the computation of $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}} S_n)$. Our main reference is [53, 5.1]. Pulling back induces the following canonical functor on our geometric categories:

$$\begin{aligned} f^{\times}\colon \mathsf{Sm}_S &\to \mathsf{Sm}_{S'} \\ X &\mapsto X \times_S S'. \end{aligned}$$

We extend this to the category of abelian sheaves by pre-composing:

$$\begin{aligned} f_*\colon \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_{S'}) &\to \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \\ \mathcal{F} &\mapsto \mathcal{F} \circ f^{\times}. \end{aligned}$$

The existence of a left adjoint $f^*\colon \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \to \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_{S'})$ to f_* is a formal consequence of constructing a left adjoint for presheaves via [38, Thm. 2.3.3], using that Ab is cocomplete, and then sheafifying. This pair of adjoint functors f^* and f_* is also additive, which allows us to extend them to complexes degreewise:

$$\left(\begin{array}{rl} f^*\colon \mathsf{Spc}_S^{\mathrm{ab}} & \to \mathsf{Spc}_{S'}^{\mathrm{ab}} \\ (C_n, d_n)_{n\in\mathbb{Z}} & \mapsto (f^*(C_n), f^*(d_n))_{n\in\mathbb{Z}} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{rl} f_*\colon \mathsf{Spc}_{S'}^{\mathrm{ab}} & \to \mathsf{Spc}_S^{\mathrm{ab}} \\ (C'_n, d'_n)_{n\in\mathbb{Z}} & \mapsto (f_*(C'_n), f_*(d'_n))_{n\in\mathbb{Z}} \end{array}\right).$$

We do not distinguish these extensions notationally, but note that they are again adjoint to one another. We want to lift this adjointness relation to the homotopy categories $\mathsf{D}(S)$ and $\mathsf{D}(S')$, and even more so we want a comparison between the associated model category structures, for which the notion of Quillen pairs (see [29, Def. 8.5.2]) is well suited. We record the following obstruction to our endeavour, due to Hovey:

Proposition 2.37 ([30, Prop. 2.13])**.** *Suppose $F\colon \mathsf{A} \to \mathsf{B}$ and $G\colon \mathsf{B} \to \mathsf{A}$ are functors between Grothendieck abelian categories A and B that fulfil the adjointness $F \dashv G$. Then (F, G) induces a Quillen pair between the injective model categories on $\mathsf{Comp}(\mathsf{A})$ and $\mathsf{Comp}(\mathsf{B})$ if and only if F is exact.*

In the general case we consider right now, there is no obvious way to conclude that f^* is exact, so we specialize to the following case: Assume that f is smooth and of finite type. The presheaf version of f^* evaluates on a given $X' \in \mathsf{Sm}_{S'}$ and presheaf $\mathcal{F}$ to a colimit running over the opposite category of diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S. \end{array}$$

With our assumption on f, we can see that the digram

$$\begin{array}{ccc} X' & = & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

is final in the alluded category of diagrams. Thus we may conclude that f^* is explicitly given by:

$$\left(\begin{array}{rl} f^*\colon \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) & \to \mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_{S'}) \\ \mathcal{F} & \mapsto \mathcal{F} \circ \tilde{f} \end{array}\right), \quad \text{where} \quad \left(\begin{array}{rl} \tilde{f}\colon \mathsf{Sm}_{S'} & \to \mathsf{Sm}_S \\ X' \xrightarrow{x'} S' & \mapsto f \circ x' \end{array}\right).$$

Indeed, if $\mathcal{F}$ is an abelian sheaf on Sm_S, one may check that $\mathcal{F} \circ \tilde{f}$ fulfils again the sheaf condition, and so the above identification follows.

Now we can repeat the technique from above: By left Kan extending any abelian sheaf on $\mathsf{Sm}_{S'}$ along $\tilde{f}$, and then sheafifying, we obtain a left adjoint $f_\sharp$ to f^*. So f^* preserves limits *and* colimits, i.e. is exact.

Lemma 2.38. *The functor $f^*\colon \mathsf{Spc}_S^{\mathrm{ab}} \to \mathsf{Spc}_{S'}^{\mathrm{ab}}$ admits a left adjoint denoted by*

$$f_\sharp\colon \mathsf{Spc}_{S'}^{\mathrm{ab}} \to \mathsf{Spc}_S^{\mathrm{ab}},$$

which satisfies the following projection formula for all $C'_* \in \mathit{Spc}^{\mathrm{ab}}_{S'}$ and $X \in \mathit{Sm}_S$

$$f_\sharp(C'_* \otimes \mathbb{Z}[X \times_S S']) \cong f_\sharp(C'_*) \otimes \mathbb{Z}[X].$$

Consequently, by adjunction we have that

$$f^*(\underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_{S'})}(\mathbb{Z}[X], C'_*)) \to \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathbb{Z}[X \times_S S'], f^*(C'_*))$$

is an isomorphism.

Proof. As for the case of f^* and f_*, $f_\sharp$ is extended degreewise, and the claimed adjointness is derived from the case of abelian sheaves. The statement about the projection formula follows by employing the corresponding sheaf of sets version (cf. [53, Lem. 5.1.1]), and by using

$$\mathbb{Z}[f_\sharp^{\mathit{Set}}(\mathcal{F})] \cong f_\sharp(\mathbb{Z}[\mathcal{F}]),$$

for any $\mathcal{F} \in \mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_{S'})$, and $f_\sharp^{\mathit{Set}}$ being the canonical functor $\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_{S'}) \to \mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_S)$ left adjoint to $f^* \colon \mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_S) \to \mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_{S'})$. ☐

Now one may adapt a proof of Morel (cf. [53, Lem. 5.1.2]) to obtain:

Lemma 2.39. *Let $f \colon S' \to S$ be a morphism between noetherian schemes of finite dimension such that f is a filtered colimit of smooth and finite type S-schemes with affine transition morphisms. Then f^* is exact, and we have an isomorphism*

$$f^*(\underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_{S'})}(\mathbb{Z}[X], C'_*)) \to \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathbb{Z}[X \times_S S'], f^*(C'_*)), \tag{2.3}$$

for all $C'_* \in \mathit{Spc}^{\mathrm{ab}}_{S'}$ and $X \in \mathit{Sm}_S$.

The following appears in [53, Lem. 5.2.1] and [53, Lem. 5.2.3].

Theorem 2.40. *Let f be as in lemma 2.39. Then (f^*, f_*) forms a Quillen pair between $\mathit{Spc}^{\mathrm{ab}}_S$ and $\mathit{Spc}^{\mathrm{ab}}_{S'}$ equipped with the injective model category structures. In particular we have a pair of adjoint functors*

$$\mathbf{L}f^* \colon D(S) \to D(S') \quad \text{and} \quad \mathbf{R}f_* \colon D(S') \to D(S).$$

Moreover, considering the $\mathbb{A}^1$-localized model category structures, we find that $\mathbf{R}f_*$ maps $\mathbb{A}^1$-local to $\mathbb{A}^1$-local complexes, and thus (f^*, f_*) forms a Quillen pair for the $\mathbb{A}^1$-localized model category structures, and we denote the associated derived functors by

$$\mathbf{L}^{\mathbb{A}^1} f^* \colon D_{\mathbb{A}^1}(S) \to D_{\mathbb{A}^1}(S') \quad \text{and} \quad \mathbf{R}^{\mathbb{A}^1} f_* \colon D_{\mathbb{A}^1}(S') \to D_{\mathbb{A}^1}(S).$$

Proof. The statement about the derived category follows, since f^* is exact by proposition 2.37. Now given an $\mathbb{A}^1$-local complex $C'_* \in D(S')$, we want to check that $\mathbf{R}f_*(C'_*)$ is $\mathbb{A}^1$-local. For this we use formula (2.3) in the derived setting. Therefore we first note that, since $\mathbb{Z}[\mathbb{A}^1_S]$ resp. $\mathbb{Z}[\mathbb{A}^1_{S'}]$ is torsion-free, $(-) \otimes \mathbb{Z}[\mathbb{A}^1_S]$ resp. $(-) \otimes \mathbb{Z}[A^1_{S'}]$ is exact. Thus we have an associated Quillen pair

$$(-) \otimes \mathbb{Z}[\mathbb{A}^1_S] \dashv \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathbb{Z}[\mathbb{A}^1_S], -).$$

Recall that the construction of a total right derived functor of the right adjoint of a Quillen pair is fairly simple: One plugs the fibrant replacement instead of the original object into the functor (see [29, Lem. 8.5.9]). Since right Quillen functors preserve fibrations, we may use the isomorphism (2.3) in the case

$$f^*(\underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_{S'})}(\mathbb{Z}[X], \mathrm{Ex}^{\mathrm{ab}}(C'_*))) \to \underline{\mathrm{Hom}}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathbb{Z}[X \times_S S'], f^*(\mathrm{Ex}^{\mathrm{ab}}(C'_*)))$$

which induces an isomorphism in $D(S)$

$$\mathbf{R}f^*(\mathbf{R}\underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}(Sm_{S'})}(\mathbb{Z}[X], C'_*)) \to \mathbf{R}\underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{Z}[X \times_S S'], \mathbf{R}f^*(C'_*))$$

Now, checking that $\mathbf{R}f_*(C'_*)$ is $\mathbb{A}^1$-local follows by a quick calculation using adjunction, which we omit. In order to proof that (f^*, f_*) is a Quillen pair for the $\mathbb{A}^1$-model category structures, it suffices to check that $\mathbf{L}f^*$ preserves $\mathbb{A}^1$-quasi-isomorphisms (cf. [29, Prop. 8.5.3]). So take an $\mathbb{A}^1$-quasi-isomorphism $\varphi\colon C_* \to D_*$ in $D(S)$. For any $\mathbb{A}^1$-local complex E'_* in $D(S')$ we can now see by the derived adjunction that

$$\begin{array}{ccc}
\mathrm{Hom}_{D(S')}(\mathbf{L}f^*(D_*), E'_*) & \xrightarrow{\mathrm{Hom}(\mathbf{L}f^*(\varphi), E'_*)} & \mathrm{Hom}_{D(S')}(\mathbf{L}f^*(C_*), E'_*) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_{D(S)}(D_*, \mathbf{R}f_*(E'_*)) & \underset{\cong}{\xrightarrow{\mathrm{Hom}(\varphi, \mathbf{R}f_*(E'_*))}} & \mathrm{Hom}_{D(S)}(C_*, \mathbf{R}f_*(E'_*))
\end{array}$$

commutes. In particular $\mathbf{L}f^*(\varphi)$ is an $\mathbb{A}^1$-quasi-isomorphism, and by the above we are done. □

If one is more careful and employs an analogous theory to Morel's Brown-Gersten spectra, one can exploit the adjunction $f_\sharp \dashv f^*$ to obtain another Quillen pair (cf. [53, Lem. 5.2.6]). We only record the following consequence of this here, since it suits our needs:

Corollary 2.41 ([53, Ex. 6.2.5])**.** *Let $f\colon S' \to S$ be as in lemma 2.39, and let M be a strictly $\mathbb{A}^1$-invariant sheaf, then $f^*(M)$ is strictly $\mathbb{A}^1$-invariant.*

2.2 Unstable $\mathbb{A}^1$-Homotopy Theory

In the previous section we introduced the abelian variant of unstable $\mathbb{A}^1$-homotopy theory, since it stands in the main focus of our work. However, we will also want to compare it to its unstable predecessor, especially as in this theory the classifying spaces that we want to work with are defined. Other than in the abelian case, there is a definitive source, namely [54] due to Morel and Voevodsky, and this is the reason why this section is kept brief. We start by introducing a basic category, the *simplex category*.

Definition 2.42. Denote by Δ the category with objects $[n] := \{0, \ldots, n\}$ the finite totally ordered sets, and with morphisms the order preserving maps. For any category C we denote the category of functors from Δ^{op} into C by $\Delta^{\mathrm{op}}C$. The objects of this latter category are called the *simplicial objects* over C.

As a first example of the above definition, one may consider $\Delta^{\mathrm{op}}Set$. We call the representable object $\Delta^n := h_{[n]}$ of this category the *n-simplex*. Given any simplicial object $C_{\dots} \in \Delta^{\mathrm{op}}C$, we will write C_n instead of $C_{[n]}$ for the simplices of rank n. More details about simplicial sets in particular, and more generally simplicial objects, can be found in [44], or [24]. Coming back to the task at hand, we will consider the following category of *spaces*:

$$Spc_S := \Delta^{\mathrm{op}}Sh_{\mathrm{Nis}}(Sm_S).$$

2.43. Let us record some facts about the category of spaces:

(a) Since Spc_S is a functor category over $Sh_{\mathrm{Nis}}(Sm_S)$, it is complete and cocomplete, and the limits and colimits are calculated degreewise.

(b) For every point x on a scheme $X \in Sm_S$, we may extend the functor x^* to a functor $x^*\colon Spc_S \to \Delta^{\mathrm{op}}Set$ that still preserves colimits and finite limits.

(c) We have two sorts of embeddings: Given a sheaf $\mathcal{F}$, we may consider the *constant* simplicial sheaf, i.e. the simplicial sheaf

$$[n] \mapsto \mathcal{F} \quad \text{and} \quad ([n] \to [m]) \mapsto \mathrm{id}_{\mathcal{F}},$$

which we abusively denote by $\mathcal{F}$ as well. Similarly, we may regard any simplicial set $S \in \Delta^{\mathrm{op}}\mathsf{Set}$ as a simplicial sheaf, by mapping every $[n]$ to the constant sheaf with values S_n.

(d) The product of spaces defines a closed symmetric monoidal structure. Hereby, the required right adjoint is given by

$$\underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(\mathcal{X}, \mathcal{Y})_n(X) := \mathrm{Hom}_{\mathsf{Spc}_S}(\mathcal{X} \times h_X \times \Delta^n, \mathcal{Y}),$$

with $\mathcal{X}, \mathcal{Y} \in \mathsf{Spc}_S$, $n \in \mathbb{N}$, $X \in \mathsf{Sm}_S$, and we regard h_X and Δ^n as spaces by (c).

(e) The category of spaces is enriched in the category of simplicial sets via:

$$\mathrm{Map}_{\mathsf{Spc}_S}(\mathcal{X}, \mathcal{Y})_n := \mathrm{Hom}_{\mathsf{Spc}_S}(\mathcal{X} \times \Delta^n, \mathcal{Y}),$$

where $\mathcal{X}, \mathcal{Y}$ denote spaces, and n lies in $\mathbb{N}$.

After this preliminary discussion, we move on to the homotopy theoretic notions, we need in Spc_S:

Definition 2.44. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial sheaves; φ will be called

- a *cofibration* if and only if φ is a monomorphism,
- a *(simplicial) weak equivalence* if and only if $x^*(\varphi)$ for every point x on a scheme $X \in \mathsf{Sm}_S$ is a weak equivalence of simplicial sets, and
- a *(simplicial) fibration* if and only φ admits a dashed diagonal for every commutative solid arrow diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{X} \\ {\scriptstyle \iota}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle \varphi} \\ \mathcal{B} & \longrightarrow & \mathcal{Y}, \end{array}$$

where ι is a *trivial cofibration*, by which we mean that ι is both a cofibration and a weak equivalence. Of course we demand that the diagram with the diagonal is commutative, as well.

We denote the class of cofibrations by $\mathscr{C}$, the class of (simplicial) fibrations by $\mathscr{F}_s$, and the class of (simplicial) weak equivalences by $\mathscr{W}_s$.

The following statement is the starting point of [54]:

Theorem 2.45. *The category Spc_S is a proper, cofibrantly generated, simplicial model category, with the above choice of enrichment* $\mathrm{Map}(-,-)$*, and* $(\mathscr{W}_s, \mathscr{C}, \mathscr{F}_s)$*.*

Proof. Jardine showed in [36, Cor. 2.7] that the Spc_S together with $(\mathscr{W}_s, \mathscr{C}, \mathscr{F}_S)$ form a model category. As a corollary to his proof, one can observe that the model category structure is cofibrantly generated (cf. in particular lemma 2.4 and p. 68 in [36]). The fact that Spc_S is indeed a simplicial model category was observed in [54, Rem. 2.1.9]. Finally in [37, Prop. 1.4] it is shown that the category of simplicial presheaves is proper, however, by using sheafification, one may show that this transfers to simplicial sheaves. □

We denote the *simplicial homotopy category*, i.e. the category that is the localization of Spc_S with respect to all simplicial weak equivalences $\mathscr{W}_s$, by $\mathsf{H}_s(S)$.

As we noted before the category Spc_S is complete and cocomplete. As such we have an initial object, that we denote by $\varnothing$, and a terminal object, which is $\bullet$. Standard terminology in model categories declares spaces $\mathcal{X}$, whose unique morphism from the initial object is a cofibration, as *cofibrant*, and dually those as *fibrant*, whose unique morphism to the terminal objects is a fibration. Hence we see that every space is cofibrant. By the model category structure we also obtain a pair of functors $C, F\colon \mathsf{Spc}_S \to \mathsf{Arr}(\mathsf{Spc}_S)$ such that every space $\mathcal{X}$ is sent to a trivial cofibration $C(\mathcal{X}) = \mathcal{X} \to \mathrm{Ex}(\mathcal{X})$ and a fibration $F(\mathcal{X}) = \mathrm{Ex}(\mathcal{X}) \to \bullet$ with

$$\mathcal{X} \to \bullet = F(\mathcal{X}) \circ C(\mathcal{X}).$$

In the following we consider the functors C, F as implicit, and will usually only be concerned with the *fibrant replacement functor* $\mathrm{Ex}\colon \mathsf{Spc}_S \to \mathsf{Spc}_S$. Note that such a name makes sense, since the space $\mathrm{Ex}(\mathcal{X})$ is always fibrant, and the comparison morphism $C(\mathcal{X})$ is an isomorphism in the homotopy category.

Examples of fibrant objects include the constant spaces coming from a sheaf $\mathcal{F}$. Since this is instructive, we check it here. Therefore let $\iota\colon \mathcal{A} \hookrightarrow \mathcal{B}$ be a trivial cofibration, and let $\varphi\colon \mathcal{A} \to \mathcal{F}$ be given. So the situation is:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{F} \\ {\scriptstyle\iota}\downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \bullet \end{array}$$

We consider the sheaf $\pi_0^s(\mathcal{A})$ that is associated to the presheaf $U \mapsto \pi_0(\mathcal{A}(U))$. π_0^s is a functor, and there is a natural transformation $\mathrm{id}_{\mathsf{Spc}} \to \pi_0^s$, which is always an epimorphism. Since taking stalks is exact, we see that $\pi_0^s(\iota)$ needs to be an isomorphism, due to ι being a weak equivalence. Thus we may check that the following definition is well-defined

$$\begin{aligned} \psi_0\colon \mathcal{B}_0 &\longrightarrow \mathcal{F} \\ \text{on } U \in \mathsf{Sm}_S: \quad b &\longmapsto \pi_0^s(\varphi)(\tilde{a}), \text{ with } \pi_0^s(\iota)^{-1}([b]) = \tilde{a} \in \pi_0^s(\mathcal{A})(U), \end{aligned}$$

which follows from the fact that all face maps of $\mathcal{F}$ are the identity, and hence $\pi_0^s(\mathcal{F}) = \mathcal{F}$. For $n \in \mathbb{N}^+$, we may set

$$\begin{aligned} \psi_n\colon \mathcal{B}_n &\longrightarrow \mathcal{F} \\ \text{on } U \in \mathsf{Sm}_S: \quad b &\longmapsto \psi_0(\mathcal{B}_\sigma(b)), \text{ with any } \sigma\colon [0] \to [n], \end{aligned}$$

and the family $(\psi_n)_{[n] \in \Delta^{\mathrm{op}}}$ gives a simplicial morphism extending φ. Thus we have shown that $\mathcal{F} \to \bullet$ lifts against every trivial cofibration, and thus must be a fibration. The following lemma is an amplification of the above:

Lemma 2.46. *Let $\mathcal{X} \in \mathsf{Spc}_S$ be a space, and $\mathcal{F}$ a sheaf. We define $\pi_0^s(\mathcal{X})$ as the sheaf associated to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$. Then we have the following adjunction*

$$\mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}}(\pi_0^s(\mathcal{X}), \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{H}_s(S)}(\mathcal{X}, \mathcal{F}).$$

Proof. We will freely use the notation of [54, 2.1.11–2.1.14]. In particular for $\mathcal{X}$ (cofibrant), and $\mathcal{Y}$ fibrant, we denote by $\pi(\mathcal{X}, \mathcal{Y})$ the set of homotopy classes of morphisms between them. From the theory of model categories, this is in bijection with $\mathrm{Hom}_{\mathsf{H}_s(S)}(\mathcal{X}, \mathcal{Y})$. Moreover we have the natural bijection

$$\operatorname*{colim}_{\mathcal{X}' \to \mathcal{X} \in \pi Triv/\mathcal{X}} \pi(\mathcal{X}', \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathsf{H}_s(S)}(\mathcal{X}, \mathcal{F}),$$

since $\mathcal{F}$ is locally fibrant. Hereby $\pi Triv/\mathcal{X}$ denotes the category of trivial local fibrations to $\mathcal{X}$, i.e. those morphisms which become trivial fibrations of simplicial sets after applying the stalk functors. So we actually want to prove

$$\operatorname*{colim}_{\mathcal{X}'\to\mathcal{X}\in\pi Triv/\mathcal{X}} \pi(\mathcal{X}',\mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\pi_0^s(\mathcal{X}),\mathcal{F}) \tag{2.4}$$

is a natural bijection. Since $\mathcal{F}$ is fibrant, we know by [29, Prop. 9.5.24] that every homotopy identifying two elements of $\mathrm{Hom}_{\mathit{Spc}_S}(\mathcal{X}',\mathcal{F})$ in $\pi(\mathcal{X}',\mathcal{F})$, comes from a morphism of spaces $\mathcal{X}'\times\Delta^1\to\mathcal{F}$. However, since the face maps in $\mathcal{F}$ are the identity, it follows that every such homotopy factors through the projection $\mathcal{X}'\times\Delta^1\to\pi_0^s(\mathcal{X}')$. Thus we have $\pi(\mathcal{X}',\mathcal{F})\cong\mathrm{Hom}_{\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\pi_0^s(\mathcal{X}'),\mathcal{F})$, and the bijectivity of (2.4) follows from the fact that a trivial local fibration induces an isomorphism on π_0^s. Naturality in $\mathcal{F}$ is trivial, and naturality with respect to a morphism of spaces $\mathcal{X}\to\mathcal{Y}$ follows by replacing $\mathcal{Y}$ fibrantly, and arguing on homotopy classes of morphisms. □

We have seen adjunctions of the above type in two places already, namely in lemmas 2.13 and 2.34. So the above statement has to be seen in their context, in particular we may phrase it as *sheaves are discrete in the category of spaces.* Note that in comparing the above statement to lemma 2.13, we see that π_0^s takes the role of the zero$^{\text{th}}$ homology functor H_0. One may extend this correspondence even further and define higher simplicial homotopy sheaves by choosing basepoints. However, since we have no need for the higher homotopies in this work, we leave it at that remark.
We come to $\mathbb{A}^1$-localizing the category $\mathit{H}_s(S)$. The following is parallel to definitions 2.16 and 2.18.

Definition 2.47. Let $\mathcal{X}$ be a space. $\mathcal{X}$ is called $\mathbb{A}^1$-*local* if and only if

$$\mathrm{Hom}_{\mathit{H}_s(S)}(\mathcal{Y}\times\mathbb{A}^1_S,\mathcal{X})\to\mathrm{Hom}_{\mathit{H}_s(S)}(\mathcal{Y},\mathcal{X}),$$

induced by the inclusion of the zero section $S\hookrightarrow\mathbb{A}^1_S$ is a bijection for all spaces $\mathcal{Y}\in\mathit{Spc}$. We denote the full subcategory of $\mathit{H}_s(S)$ that is spanned by the $\mathbb{A}^1$-local spaces by $\mathit{H}_{s,\mathbb{A}^1\text{-loc}}(S)$.
Any morphism $f\colon\mathcal{Y}\to\mathcal{Z}$ such that $\mathrm{Hom}_{\mathit{H}_s(S)}(f,\mathcal{X})$ is a bijection for all $\mathcal{X}$ $\mathbb{A}^1$-local, is called an $\mathbb{A}^1$-*weak equivalence.*

As we have already hinted at, the construction of the abelian $\mathbb{A}^1$-homotopy category is modelled in parallel to the construction of the unstable $\mathbb{A}^1$-homotopy category. Since we gave more details above, we rush through the most important statements:[4]

Theorem 2.48. *There is a cofibrantly generated, proper model category structure on Spc_S, where the cofibrations are the monomorphisms, and the weak equivalences are the $\mathbb{A}^1$-weak equivalences. Moreover, the $\mathbb{A}^1$-fibrant spaces are those spaces that are simultaneously $\mathbb{A}^1$-local and simplicially fibrant. The $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}\colon\mathit{H}_s(S)\to\mathit{H}_{s,\mathbb{A}^1\text{-loc}}(S)$ induces an equivalence of categories between the $\mathbb{A}^1$-homotopy category $\mathit{H}_{\mathbb{A}^1}(S)$ and $\mathit{H}_{s,\mathbb{A}^1\text{-loc}}(S)$. The corresponding natural transformation $\theta\colon\mathrm{id}_{\mathit{H}_s(S)}\to L_{\mathbb{A}^1}$ induces an epimorphism on π_0^s.*

Let us add more context to the last statement of the theorem: Recall that with the introduction of the abelian $\mathbb{A}^1$-localization functor we defined $\mathbb{A}^1$-homology functors out of the ordinary homology functors. A similar definition is possible in the unstable case: Let $\mathcal{X}$ be a space, then we define the $\mathbb{A}^1$-path components by

$$\pi_0^{\mathbb{A}^1}(\mathcal{X}):=\pi_0^s(L_{\mathbb{A}^1}(\mathcal{X})).$$

With this definition the above statement implies the following *unstable $\mathbb{A}^1$-connectivity property*:

Corollary 2.49. *Any simplicially connected space (= trivial π_0^s), is $\mathbb{A}^1$-connected (= trivial $\pi_0^{\mathbb{A}^1}$).*

[4] For reference, see theorem 2.3.2 and corollary 2.3.22 in [54].

Base Change

In [54] Morel and Voevodsky introduce the simplicial homotopy category and its $\mathbb{A}^1$-counterpart in much more generality. In particular, they provide a proof that their constructions are valid not only over the Nisnevich topology, but also over the étale topology. Considering the continuous morphism of sites $\alpha\colon (\mathsf{Sm}_S)_{\text{ét}} \to (\mathsf{Sm}_S)_{\text{Nis}}$, together with its left and right adjoints

$$\alpha^*\colon \mathsf{Sh}_{\text{Nis}}(\mathsf{Sm}_S) \to \mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S) \quad \text{and} \quad \alpha_*\colon \mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S) \to \mathsf{Sh}_{\text{Nis}}(\mathsf{Sm}_S),$$

one obtains the following result:

Proposition 2.50 ([54, Prop. 2.1.47])**.** *The pair (α^*, α_*) is a Quillen pair for the simplicial model category structures on $\mathsf{Spc}_{S,\text{ét}}$ and Spc_S.*

Proof. This follows from the fact that α^* is the étale sheafification, and thus exact. $\square$

The following proposition is again parallel to the abelian case.

Proposition 2.51 ([54, Prop. 3.2.8])**.** *For any morphism $f\colon S' \to S$ between noetherian schemes of finite dimension, we have a pair of adjoint functors on the associated $\mathbb{A}^1$-homotopy categories*

$$\mathbf{L}^{\mathbb{A}^1} f^*\colon \mathsf{H}_{\mathbb{A}^1}(S) \to \mathsf{H}_{\mathbb{A}^1}(S') \quad \textit{and} \quad \mathbf{R}^{\mathbb{A}^1} f_*\colon \mathsf{H}_{\mathbb{A}^1}(S') \to \mathsf{H}_{\mathbb{A}^1}(S).$$

2.2.1 Example: Classifying Spaces

In this subsection we define principal homogeneous bundles in the category of spaces with respect to sheaves of groups. We recall a construction due to Morel and Voevodsky that yields a space such that homotopy classes of morphisms into that space classify torsors. This situation is analogous to the classifying spaces of algebraic topology. We quote from [54, Section 4].

Let us fix a group object G in the category of Nisnevich sheaves $\mathsf{Sh}_{\text{Nis}}(\mathsf{Sm}_S)$. As before we regard G a constant space. A (right) action of G on a space $\mathcal{X}$ is a morphism $\nu\colon \mathcal{X} \times G \to \mathcal{X}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{X} \times G \times G & \xrightarrow{\mathrm{id}_{\mathcal{X}} \times \mu} & \mathcal{X} \times G \\ {\scriptstyle \nu \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle \nu} \\ \mathcal{X} \times G & \xrightarrow{\nu} & \mathcal{X} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X} \times \bullet & \xrightarrow{\mathrm{id}_{\mathcal{X}} \times e} & \mathcal{X} \times G \\ & {\scriptstyle \mathrm{pr}_{\mathcal{X}}} \searrow \cong & \downarrow{\scriptstyle \nu} \\ & & \mathcal{X}, \end{array}$$

are commuting, where $\mu\colon G \times G \to G$, and $e\colon \bullet \to G$ are part of the group structure. The action ν is called *free* if and only if the diagonal of the action

$$\begin{gathered} \mathcal{X} \times G \to \mathcal{X} \times \mathcal{X} \\ \text{on } U \in \mathsf{Sm}_S\colon (x, g) \mapsto (\nu(x, g), x) \end{gathered}$$

is a monomorphism. On the other hand, we call an action ν trivial if and only if $\nu = \mathrm{pr}_{\mathcal{X}}$. If we consider two spaces $\mathcal{X}_1, \mathcal{X}_2$ with corresponding G-actions ν_1, ν_2, and a morphism $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$, we call φ *G-equivariant* if and only if

$$\begin{array}{ccc} \mathcal{X}_1 \times G & \xrightarrow{\varphi \times \mathrm{id}_G} & \mathcal{X}_2 \times G \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ \mathcal{X}_1 & \xrightarrow{\varphi} & \mathcal{X}_2 \end{array}$$

is commuting. We denote by Spc_S^G the category of *G-spaces*, with objects the spaces, equipped with a (right) G-action, and morphisms the G-equivariant morphisms. Note that equipping a space with the trivial G-action yields a faithful embedding $\mathsf{Spc}_S \to \mathsf{Spc}_S^G$.

The *orbit space* $\mathcal{X}/G$ of a G-action ν on a space $\mathcal{X}$ is defined as the coequaliser of ν and the trivial action, i.e. the following diagram is exact:

$$\mathcal{X} \times G \underset{\mathrm{pr}_{\mathcal{X}}}{\overset{\nu}{\rightrightarrows}} \mathcal{X} \xrightarrow{\ \rho\ } \mathcal{X}/G.$$

With the notation introduced above, we see that ρ is a G-equivariant morphism with trivial G-action on $\mathcal{X}/G$, and ρ is even the universal one with that property, i.e. we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Spc}_S}(\mathcal{X}/G, \mathcal{Y}) \cong \mathrm{Hom}_{\mathsf{Spc}_S^G}(\mathcal{X}, \mathcal{Y}),$$

for every space $\mathcal{Y}$ from it. We also note another adjunction, which comes from the existence of the internal Hom on spaces: Using the standard isomorphism

$$\underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \mathcal{X})) \cong \underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G \times G, \mathcal{X}),$$

and adjunction, one sees that μ induces a G-action

$$\underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \mathcal{X}) \times G \to \underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \mathcal{X})$$

on $\underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \mathcal{X})$. For this *wreath product*, we have the following adjunction relation involving the forgetful functor:

$$\mathrm{Hom}_{\mathsf{Spc}_S}(\mathcal{X}, \mathcal{Y}) \cong \mathrm{Hom}_{\mathsf{Spc}_S^G}(\mathcal{X}, \underline{\mathrm{Hom}}_{\mathsf{Spc}_S}(G, \mathcal{Y})),$$

where $\mathcal{X}$ is a G-space, and $\mathcal{Y}$ is any space.

Definition 2.52. Let G be a sheaf of groups in the Nisnevich topology on Sm_S, and let $\mathcal{X}$ be a space. A *G-torsor*[5] is a free G-space $\mathcal{Y}$ over the trivial G-space $\mathcal{X}$ such that the induced morphism $\mathcal{Y}/G \to \mathcal{X}$ is an isomorphism. We denote the set of isomorphism classes of G-torsors over $\mathcal{X}$ by $\mathrm{P}(\mathcal{X}, G)$.

We remark that the set $\mathrm{P}(\mathcal{X}, G)$ is canonically pointed: Indeed, consider the space $\mathcal{X} \times G$, equipped with the G-action induced by right multiplication on the G-entry. This action is obviously free, and has orbit space $\mathcal{X}$. The next lemma also follows by standard calculation.

Lemma 2.53. *Let G be a sheaf of groups in the Nisnevich topology, and let $\varphi\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of spaces. For every G-torsor $\mathcal{Y} \xrightarrow{\xi} \mathcal{X}$, the pullback $\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$ admits a canonical structure of a G-bundle over $\mathcal{X}'$. This makes $\mathrm{P}(-, G)$ a contravariant functor from the category of spaces to $\mathsf{Set}_\bullet$.*

Remark 2.54. Recall from [47, Prop. III.4.6] that, for $U \in \mathsf{Sm}_S$, the elements in $\mathrm{P}(U, G)$ are in one-to-one correspondence with the Čech cohomology classes, with respect to the Nisnevich topology. As those coincide with the cohomology classes, whenever G is commutative (cf. [47, Cor. III.2.10]), we will write $H^1_{\mathrm{Nis}}(U, G)$ instead of $\check{H}^1_{\mathrm{Nis}}(U, G)$ and $\mathrm{P}(U, G)$.

Note that aside from that discussion, one could also ask, in case G is representable, whether all G-torsors must be representable, too. In that generality this question is quite hard, but there are some known conditions of interest to us, for example, if G is a smooth linear algebraic group, all étale G-torsors will be represented by smooth and affine S-schemes (cf. [47, Rem. III.4.8]).

We will now define a canonical G-torsor, which will classify all other G-torsors. Therefore we introduce first, a canonical covering space $\mathrm{E}(\mathcal{F})$ with respect to any sheaf of sets $\mathcal{F}$. Let us

[5] Equivalent names: *Principal G-bundle*, *G-bundle*.

set $\mathrm{E}(\mathcal{F})_n$ to the sheaf $\mathcal{F}^{n+1}$, and equip the family $(\mathrm{E}(\mathcal{F})_n)_{n\in\mathbb{N}}$ with a simplicial structure by defining

$$\begin{aligned} \mathrm{E}(\mathcal{F})_n &\longrightarrow \mathrm{E}(\mathcal{F})_m \\ \text{on } U \in \mathit{Sm}_S\colon (f_0,\dots,f_n) &\longmapsto (f_{\vartheta(0)},\dots,f_{\vartheta(m)}) \end{aligned}$$

for some order-preserving map $[m] \xrightarrow{\vartheta} [n]$. We run through some trivial properties of the functor $\mathrm{E}(-)$:

(a) Given sheaves $\mathcal{F},\mathcal{G}$, we have a canonical isomorphism $\mathrm{E}(\mathcal{F}\times\mathcal{G}) \cong \mathrm{E}(\mathcal{F})\times\mathrm{E}(\mathcal{G})$.

(b) If G is a sheaf of groups, then $\mathrm{E}(G)$ is a group object in the category of spaces, where the multiplication is defined entry-wise. There is a free (right) G-action on $\mathrm{E}(G)$, which is defined entry-wise, as well.

(c) For any space $\mathcal{X} \in \mathit{Spc}_S$ we have the adjointness relation

$$\mathrm{Hom}_{\mathit{Spc}_S}(\mathcal{X},\mathrm{E}(\mathcal{F})) \cong \mathrm{Hom}_{\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_S)}(\mathcal{X}_0,\mathcal{F}).$$

(d) $\mathrm{E}(G)$ is contractible, for G any sheaf of groups, or even more generally a sheaf of pointed sets.

Next we define the classifying space $\mathrm{B}(G)$. Therefore recall first the definition of the nerve $\mathrm{B}(\mathit{C})$ of a small category C: For any $n\in\mathbb{N}$, regard the partially ordered finite set $[n]$ as a category. Then let $B(\mathit{C})_n$ be the set of all functors $[n]\to\mathit{C}$, and the face and degeneracy maps shall be given by precomposition. Now viewing any group $G(U)$ as a category with one object, whose morphisms are given by $G(U)$, and where composition of morphisms is defined by the group law, we obtain a simplicial set $\mathrm{B}(G(U))$. Since for any $n\in\mathbb{N}$ one essentially has $\mathrm{B}(G(U))_n \cong G(U)^n$, we see that $U\mapsto \mathrm{B}(G(U))$ defines a space $\mathrm{B}(G)$. Let us call $\mathrm{B}(G)$ the *classifying space of* G.

There is a morphism of spaces $\pi_G\colon\ \mathrm{E}(G)\to\mathrm{B}(G)$ given by

$$\text{on } U\in\mathit{Sm}_S,\ n\in\mathbb{N}\colon (g_0,\dots,g_n)\longmapsto (g_0g_1^{-1},g_1g_2^{-1},\dots,g_{n-1}g_n^{-1}),$$

where we identify the latter n-tuple with the canonical functor $[n]\to G(U)$ that sends each $i\in[n]$ to the canonical object, and each $i\le j$ in $[n]$ to

$$\prod_{t=i}^{j-1}(g_tg_{t+1}^{-1}) = g_ig_j^{-1}.$$

It is clear that π_G is G-equivariant, when we equip $\mathrm{E}(G)$ with the standard G-action by entrywise multiplication from the right, and $\mathrm{B}(G)$ with the trivial G-action. One may check that π_G induces an isomorphism $\mathrm{E}(G)/G\cong\mathrm{B}(G)$. We call the resulting G-torsor $\pi_G\colon\ \mathrm{E}(G)\to\mathrm{B}(G)$ the *universal G-torosr over* $\mathrm{B}(G)$. The following result by Morel and Voevodsky justifies this name:

Proposition 2.55 ([54, Prop. 4.1.15])**.** *Let G be a sheaf of groups, and $\mathcal{X}$ be a space. Then we have a natural bijection*

$$\mathrm{P}(\mathcal{X},G)\to\mathrm{Hom}_{\mathit{H}_s(S)}(\mathcal{X},\mathrm{B}G).$$

In [54] the above result is formulated site-agnostic, i.e. it does not depend on the Nisnevich topology. In particular, one could perform the above construction with respect to the étale topology. If we set (see proposition 2.50)

$$\mathrm{B}_{\text{ét}}(G) := \mathbf{R}\alpha_*\mathbf{L}\alpha^*\,\mathrm{B}(G),$$

we obtain the following corollary:

Corollary 2.56. *Let G be a group object in $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, and let us denote by $\mathcal{H}^1_{\text{ét}}(G)$ the Nisnevich sheaf associated to $U \mapsto H^1_{\text{ét}}(U,G)$. Then we have a natural bijection*

$$\pi_0^s(\mathrm{B}_{\text{ét}}G) \cong \mathcal{H}^1_{\text{ét}}(G).$$

Note that the corresponding statement for $\mathrm{B}(G)$ yields $\pi_0^s(\mathrm{B}G) = \bullet$, since G-torsors in the Nisnevich topology are Nisnevich-locally trivial.

2.3 The $\mathbb{A}^1$-Dold-Kan Correspondence

In this section we want to relate the $\mathbb{A}^1$-derived category to the unstable $\mathbb{A}^1$-homotopy category via an adjunction. We extend this from the usual Dold-Kan-correspondence, therefore let us begin by recalling what is known: We have a functor $N\colon \Delta^{\mathrm{op}}\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \to \mathsf{Comp}_{\geq 0}(\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S))$ that sends a simplicial abelian group $\mathcal{A}$ to the non-negative chain complex, with sheaf

$$N(\mathcal{A})_n := \bigcap_{i=0}^{n-1} \ker(d_i^{\mathcal{A}}) \subseteq \mathcal{A}_n,$$

in degree $n \in \mathbb{N}$, and 0 in negative degrees, with the differential set to the restriction of $(-1)^n d_n^{\mathcal{A}}$, where we denoted the face homomorphisms by $d_i^{\mathcal{A}}\colon \mathcal{A}_n \to \mathcal{A}_{n-1}$ for $n \in \mathbb{N}^+$ and $i \in \{0,\dots,n\}$. Note that hereby $N(\mathcal{A})_0 := \mathcal{A}_0$. N is called the *normalized chain complex functor* (cf. [24, Sec. III.2]), and is quasi-inverse to the *Eilenberg-MacLane space functor*

$$K_{\geq 0}\colon \mathsf{Comp}_{\geq 0}(\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)) \to \Delta^{\mathrm{op}}\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k),$$

whose definition we will not recall here (see e.g. [24, p. 149]). We have to make two modifications to the above equivalence, that will eventually lead us to considering an adjunction rather than an equivalence.

First, we replace simplicial abelian sheaves, by spaces, and in order to do that, we use the free sheaf functor $\mathbb{Z}[-]$, which is left adjoint to the inclusion

$$\Delta^{\mathrm{op}}\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S) \hookrightarrow \Delta^{\mathrm{op}}\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S),$$

that we will forget to write. Secondly, we need to consider unbounded chain complexes instead of non-negative ones. In the derived category, we handled this sort of business with the truncation functor $\tau_{\geq 0}$, and as a quick check shows, we may use the same formula to obtain an adjunction:

$$\iota_{\geq 0} : \ \mathsf{Comp}_{\geq 0}(\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)) \rightleftarrows \mathsf{Comp}(\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)) \ : \tau_{\geq 0}.$$

By again omitting the canonical inclusion functor $\iota_{\geq 0}$, we thus get an adjunction

$$C : \ \mathsf{Spc}_S \rightleftarrows \mathsf{Spc}_S^{\mathrm{ab}} \ : K,$$

with $C := N \circ \mathbb{Z}[-]$, and $K := K_{\geq 0} \circ \tau_{\geq 0}$.

Proposition 2.57. *The pair of functors (C,K) is a Quillen pair for the choice of*

(a) the injective model category structure on $\mathsf{Spc}_S^{\mathrm{ab}}$ and the simplicial one on Spc_S, and

(b) the $\mathbb{A}^1$-derived model category structure on $\mathsf{Spc}_S^{\mathrm{ab}}$ and the $\mathbb{A}^1$-local structure on Spc_S.

Proof. To show (a) it suffices by [29, Prop. 8.5.3] to show that C preserves monomorphisms, and maps weak equivalences to quasi-isomorphisms. However, this is already true for N and $\mathbb{Z}[-]$ individually by the classical case, see theorem III.2.5 and proposition III.2.16 in [24].

In order for (b) to hold true, it suffices to check that C sends $\mathbb{A}^1$-weak equivalences to $\mathbb{A}^1$-quasi-isomorphisms. This in turn will follow at once by adjointness of $\mathbf{L}C$ and $\mathbf{R}K$, as soon as we check that $\mathbf{R}K$ preserves $\mathbb{A}^1$-local objects. Therefore, it suffices to remark that there is a natural (in $\mathcal{X} \in \mathsf{Spc}_S$) isomorphism

$$C(\mathcal{X} \times \mathbb{A}^1_S)_* \cong C(\mathcal{X})_* \otimes \mathbb{Z}[\mathbb{A}^1_S].$$

By the above reductions, the claim follows. $\square$

By means of the above proposition we established a way to obtain complexes from spaces, namely by applying $\mathbf{L}C$. To not make the notation more overbearing, and since it is usually clear from the context whether we are dealing with complexes, or with spaces, we will omit the functor $\mathbf{L}C$. The next proposition will be used throughout chapter 4, as it helps us to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$:

Proposition 2.58. *Assume that S has the abelian $\mathbb{A}^1$-connectivity property, and let $\mathcal{X}$ be any space. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X}), M) \cong \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\pi_0^s(\mathcal{X}), M),$$

where M is any strictly $\mathbb{A}^1$-invariant sheaf. In the case $\mathcal{X} = \mathrm{B}_{\text{ét}}G$, for some group object G in $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, we have in particular the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G), M) \cong \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{H}^1_{\text{ét}}(G), M).$$

Proof. In this proposition we need to combine several of the above statements. Let us start with the left-hand side:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X}), M) &\overset{2.34}{\cong} \mathrm{Hom}_{\mathsf{D}_{\mathbb{A}^1}(S)}(\mathbf{L}C(\mathcal{X}))_*, M[0]) \cong \dots \\ \dots &\overset{2.27}{\cong} \mathrm{Hom}_{\mathsf{D}(S)}(\mathbf{L}C(\mathcal{X})_*, M[0]) \cong \dots \\ \dots &\overset{2.57}{\cong} \mathrm{Hom}_{\mathsf{H}_s(S)}(\mathcal{X}, \mathbf{R}K(M[0])),\end{aligned}$$

where for the second bijection we have used that morphisms in $\mathsf{D}_{\mathbb{A}^1}(S)$ into an $\mathbb{A}^1$-local object agree with morphisms in $\mathsf{D}(S)$. Let us analyse $\mathbf{R}K(M[0])$: By construction (recalled e.g. in [29, Prop. 8.4.4]), we have that $\mathbf{R}K(M[0])$ is given by the image in $\mathsf{H}_s(S)$ of K applied to a fibrant replacement of $M[0]$. As a fibrant replacement of $M[0]$, we may take an injective resolution I_* of M starting in degree 0. The truncation $\tau_{\geq 0}(I_*)$ is again $M[0]$, and $K_{\geq 0}(M[0])$ is the constant simplicial sheaf M. So we are left with

$$\mathrm{Hom}_{\mathsf{H}_s(S)}(\mathcal{X}, M) \overset{2.46}{\cong} \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\pi_0^s(\mathcal{X}), M),$$

and the particular case follows with corollary 2.56. $\square$

Using the Yoneda lemma in the category $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, one reduces the task of determining $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ to finding a strictly $\mathbb{A}^1$-invariant sheaf that corepresents $\mathcal{H}^1_{\text{ét}}(G)$.

Remark 2.59. Consider a pointed space $\mathcal{X}$, with pointing morphism $\bullet \to \mathcal{X}$, which clearly is a section to the unique morphism from $\mathcal{X}$ to the terminal object $\bullet$. Thus, by applying the functor C to this situation, we obtain a split monomorphism $\mathbb{Z} \hookrightarrow C(\mathcal{X})_*$. Applying moreover the functor $\mathbb{H}_0^{\mathbb{A}^1}$ to this monomorphism, we obtain a split exact sequence

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X}) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{X}) \to 0.$$

Bearing the situation of algebraic topology in mind, we call the quotient $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{X})$ the *reduced (zeroth) $\mathbb{A}^1$-homology sheaf of $\mathcal{X}$.* Applying the functor $\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(-, M)$ to this split exact sequence, and using proposition 2.58, we see that $\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{X}), M)$ is given by the kernel of

$$\mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\pi_0^s(\mathcal{X}), M) \longrightarrow M(S),$$

which is the evaluation at the basepoint. From this we deduce that there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{X}), M) \cong \mathrm{Hom}_{\mathsf{Sh}^\bullet_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\pi_0^s(\mathcal{X}), M).$$

Note that $\mathrm{B}(G)$, and thus $\mathrm{B}_{\text{ét}}(G)$, are canonically pointed, as $\mathrm{B}(G)_0 = \bullet$, so that the above applies.

2.3.1 The $\mathbb{A}^1$-Tensor Product

In this section we define a closed symmetric monoidal structure on the category $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_k)$. We use the notation in [41, Ch. VII], and briefly recall the relevant parts of the definition.

Given any category C, a symmetric monoidal structure consists of the datum of a bifunctor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, a *unit object* $\mathbb{1} \in \mathsf{C}$, and four natural isomorphisms

$$\begin{aligned} \alpha_{c_1,c_2,c_3}&\colon c_1 \otimes (c_2 \otimes c_3) \longrightarrow (c_1 \otimes c_2) \otimes c_3, \\ \lambda_c&\colon \mathbb{1} \otimes c \longrightarrow c, \\ \rho_c&\colon c \otimes \mathbb{1} \longrightarrow c, \quad \text{and} \\ \gamma_{c_1,c_2}&\colon c_1 \otimes c_2 \longrightarrow c_2 \otimes c_1, \end{aligned}$$

respectively called the *associator*, the *left unitor*, the *right unitor*, and the braiding. These are then subject to a list of coherence axioms. The aim is to mimic the structure that comes with a tensor product. The symmetric monoidal structure is moreover called *closed*, if there is an internal hom-functor $\underline{\mathrm{Hom}}_{\mathsf{C}}(c, -)$ that is right adjoint to $c \otimes (-)$ for every object $c \in \mathsf{C}$. We will utilize the following lemma, to descend the closed symmetric monoidal structure given on $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)$ by the tensor product down to $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_k)$:

Lemma 2.60. *Let C be a closed symmetric monoidal category, and suppose $\iota\colon \mathsf{D} \hookrightarrow \mathsf{C}$ is a full subcategory. Assume moreover that ι admits a left-adjoint $a\colon \mathsf{C} \to \mathsf{D}$, that the unit $\mathbb{1}$ lies in the essential image of ι, and that there is a bifunctor $\underline{\mathrm{Hom}}_{\mathsf{D}}\colon \mathsf{D} \times \mathsf{D} \to \mathsf{D}$ such that there is a natural isomorphism*

$$\iota(\underline{\mathrm{Hom}}_{\mathsf{D}}(d_1, d_2)) \cong \underline{\mathrm{Hom}}_{\mathsf{C}}(\iota(d_1), \iota(d_2)).$$

Then D is a closed symmetric monoidal category with tensor product given by

$$d_1 \otimes_{\mathsf{D}} d_2 := a\left(\iota(d_1) \otimes \iota(d_2)\right).$$

Proof. Let us briefly demonstrate how to construct the necessary datum for a closed symmetric monoidal structure on D: We begin by defining the braiding morphism $\gamma^{\mathsf{D}}_{d_1,d_2}$ as $a(\gamma_{\iota(d_1),\iota(d_2)})$. Then we fix a lifting $\mathbb{1}_{\mathsf{D}} \in \mathsf{D}$ of the unit element $\mathbb{1} \in \mathsf{C}$, together with a witnessing isomorphism $\varphi\colon \iota(\mathbb{1}_{\mathsf{D}}) \to \mathbb{1}$. The left resp. right unitor may then be defined via

$$\begin{aligned} \lambda^{\mathsf{D}}_d&\colon \mathbb{1}_{\mathsf{D}} \otimes_{\mathsf{D}} d = a(\iota(\mathbb{1}_{\mathsf{D}}) \otimes \iota(d)) \xrightarrow{a(\varphi \otimes \mathrm{id}_{\iota(d)})} a(\mathbb{1} \otimes \iota(d)) \xrightarrow{a(\lambda_{\iota(d)})} a(\iota(d)) \xrightarrow{\epsilon_d} d, \quad \text{resp.} \\ \rho^{\mathsf{D}}_d&\colon d \otimes_{\mathsf{D}} \mathbb{1}_{\mathsf{D}} = a(\iota(d) \otimes \iota(\mathbb{1}_{\mathsf{D}})) \xrightarrow{a(\mathrm{id}_{\iota(d)} \otimes \varphi)} a(\iota(d) \otimes \mathbb{1}) \xrightarrow{a(\rho_{\iota(d)})} a(\iota(d)) \xrightarrow{\epsilon_d} d, \end{aligned}$$

where $\epsilon\colon a \circ \iota \to \mathrm{id}_{\mathsf{D}}$ is the counit of the adjunction $a \dashv \iota$, which is a natural isomorphism, as ι is full and faithful. So, we come to the definition of the associator α^{D}. Here we would like to have

$$\begin{array}{ccc} a(\iota(d_1) \otimes \iota(a(\iota(d_2) \otimes \iota(d_3)))) & \xrightarrow{\alpha^{\mathsf{D}}_{d_1,d_2,d_3}} & a(\iota(a(\iota(d_1) \otimes \iota(d_2)) \otimes \iota(d_3)) \\ {\scriptstyle a(\mathrm{id}_{\iota(d_1)} \otimes \eta_{\iota(d_2)\otimes\iota(d_3)})}\uparrow & & \uparrow{\scriptstyle a(\eta_{\iota(d_1)\otimes\iota(d_2)} \otimes \mathrm{id}_{\iota(d_3)})} \\ a(\iota(d_1) \otimes (\iota(d_2) \otimes \iota(d_3))) & \xrightarrow[a(\alpha_{\iota(d_1),\iota(d_2),\iota(d_3)})]{} & a((\iota(d_1) \otimes \iota(d_2)) \otimes \iota(d_3)), \end{array}$$

and the obstruction therefore is to show that the vertical morphisms are isomorphisms. We address this by deriving that

$$a(\iota(d_1) \otimes c) \xrightarrow{a(\mathrm{id}_{\iota(d_1)} \otimes \eta_c)} a(\iota(d_1) \otimes \iota(a(c))) \tag{2.5}$$

is a natural isomorphism for all $d_1 \in \mathsf{D}$ and $c \in \mathsf{C}$. Note first that by the adjunction $a \dashv \iota$ and Yoneda's lemma, it suffices to check that there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(\iota(d_1) \otimes \iota(a(c)), \iota(d)) \cong \mathrm{Hom}_{\mathsf{C}}(\iota(d_1) \otimes c, \iota(d))$$

for all $d \in \mathsf{D}$, induced by the unit $\eta_c\colon c \to \iota(a(c))$. Let us now fix a name ψ for the natural isomorphism

$$\psi_{d_1,d_2}\colon \iota(\underline{\mathrm{Hom}}_{\mathsf{D}}(d_1, d_2)) \longrightarrow \underline{\mathrm{Hom}}_{\mathsf{C}}(\iota(d_1), \iota(d_2)),$$

then we obtain the following chain of natural isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{C}}(\iota(d_1) \otimes c, \iota(d)) \cong \dots \\ \dots \cong &\mathrm{Hom}_{\mathsf{C}}(c, \underline{\mathrm{Hom}}_{\mathsf{C}}(\iota(d_1), \iota(d))) \cong \dots && [\iota(d_1) \otimes (-) \dashv \mathrm{Hom}_{\mathsf{C}}(\iota(d_1), -)] \\ \dots \cong &\mathrm{Hom}_{\mathsf{C}}(c, \iota(\underline{\mathrm{Hom}}_{\mathsf{D}}(d_1, d))) \cong \dots && \left[\mathrm{Hom}_{\mathsf{C}}(c, \psi^{-1}_{d_1,d})\right] \\ \dots \cong &\mathrm{Hom}_{\mathsf{D}}(a(c), \underline{\mathrm{Hom}}_{\mathsf{D}}(d_1, d)) \cong \dots && [a \dashv \iota] \\ \dots \cong &\mathrm{Hom}_{\mathsf{C}}(\iota(a(c)), \iota(\underline{\mathrm{Hom}}_{\mathsf{D}}(d_1, d))) \cong \dots && [\iota \text{ fully faithful}] \\ \dots \cong &\mathrm{Hom}_{\mathsf{C}}(\iota(a(c)), \underline{\mathrm{Hom}}_{\mathsf{C}}(\iota(d_1), \iota(d))) \cong \dots && [\mathrm{Hom}_{\mathsf{C}}(\iota(a(c)), \psi_{d_1,d})] \\ \dots \cong &\mathrm{Hom}_{\mathsf{C}}(\iota(d_1) \otimes \iota(a(c)), \iota(d)), && [\iota(d_1) \otimes (-) \dashv \mathrm{Hom}_{\mathsf{C}}(\iota(d_1), -)], \end{aligned}$$

where we have given a reason on the right side in grey. It is an easy exercise to show that this is the morphism induced by η_c, and that the coherence axioms hold. The functor $\underline{\mathrm{Hom}}_{\mathsf{D}}(d, -)$ can then be shown to be a right adjoint to $d \otimes_{\mathsf{D}} (-)$. This concludes the proof of the lemma. □

After this technical excursion, we only need to check that the lemma applies. Suppose that the abelian $\mathbb{A}^1$-connectivity property holds for S. Above we have seen that $\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ is an exact and full subcategory of $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, and that $\mathbb{H}_0^{\mathbb{A}^1}$ may be regarded as a left adjoint to the embedding functor (cf. lemma 2.34)[6]. The tensor product on $\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ gives it a structure of a closed symmetric monoidal category, and by the lemma it remains to check that $\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(M, N)$ is a strictly $\mathbb{A}^1$-invariant sheaf, whenever M and N are themselves strictly $\mathbb{A}^1$-invariant. In fact, we are going to prove the following:

Lemma 2.61. *Assuming the abelian $\mathbb{A}^1$-connectivity property on S, then for all abelian sheaves M, and strictly $\mathbb{A}^1$-invariant sheaves N, we find that $\underline{\mathrm{Hom}}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(M, N)$ is strictly $\mathbb{A}^1$-invariant.*

[6] With our convention $\mathbb{H}_0^{\mathbb{A}^1}$ is the composition $H_0 \circ L^{\mathrm{ab}}_{\mathbb{A}^1}$ on abelian sheaves and $H_0 \circ L^{\mathrm{ab}}_{\mathbb{A}^1} \circ \mathbb{Z}[-]$ on sheaves of sets.

Proof. Before we addressed the definedness of $(-) \otimes^{\mathbf{L}} (-)$ and $\mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}$ in the case of some flat complex via the general theory of model categories. In the situation of the derived category however, it is possible to define the derived tensor product and internal hom more generally. As a convenience to the reader, we try to trace the construction by references to the stacks project [73].

For any two complexes C_*, D_* it is known that there exists a resolution by K-injective (cf. [73, Tag 070H]) objects (cf. [73, Tag 079P]), and that the definition

$$\mathbf{R}\mathrm{Hom}_{\mathsf{Spc}_S^{\mathrm{ab}}}(C_*, D_*) := \mathrm{Hom}_{\mathsf{D}(S)}(C_*, I_*),$$

is well-defined (cf. [73, Tag 0A95]), where $D_* \to I_*$ is a quasi-isomorphism, and I_* is a K-injective complex. Similarly, it is possible to define the derived tensor product as (see above [73, Tag 06YU])

$$C_* \otimes^{\mathbf{L}} D_* := C_* \otimes J_*,$$

where $J_* \to D_*$ is a quasi-isomorphism, and J_* is a K-flat complex (cf. [73, Tag 06YN]). For any third complex E_*, we have the desired adjointness relation (cf. [73, Tag 08J9])

$$\mathrm{Hom}_{\mathsf{D}}(C_*, \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(D_*, E_*)) \cong \mathrm{Hom}_{\mathsf{D}}(C_* \otimes^{\mathbf{L}} D_*, E_*).$$

Since by [73, Tag 06YQ] the complex $\mathbb{Z}[\mathbb{A}^1_S]$ is K-flat, we see that $C_* \otimes \mathbb{Z}[\mathbb{A}^1_S]$ agrees with $C_* \otimes^{\mathbf{L}} \mathbb{Z}[\mathbb{A}^1_S]$, and thus for any $\mathbb{A}^1$-local complex E_* and any complex D_* the internal hom $\mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(D_*, E_*)$ is $\mathbb{A}^1$-local. Now by [73, Tag 0BKV] we know that

$$a_{\mathrm{Nis}}\Big(U \mapsto H_0(\mathbf{R}\Gamma(U, \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(M[0], N[0])))\Big)$$

agrees with the sheaf homology $H_0(\mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(C_*, D_*))$, and that by theorem 2.31 this is strictly $\mathbb{A}^1$-invariant. For any $U \in \mathsf{Sm}_S$, we have (cf. [73, Tag 08JA])

$$H_0(\mathbf{R}\Gamma(U, \mathbf{R}\underline{\mathrm{Hom}}_{\mathsf{Spc}_S^{\mathrm{ab}}}(M[0], N[0]))) = \mathrm{Hom}_{\mathsf{D}(\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_{S,U}))}(M[0]_{\restriction U}, N[0]_{\restriction U}),$$

and from lemma 2.13 we learn, that this agrees with $\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_{S,U})}(M_{\restriction U}, N_{\restriction U})$, which is precisely $\underline{\mathrm{Hom}}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(M, N)(U)$. This completes the proof of the lemma. □

2.62. As a consequence of the above two lemmas we note that that there is a closed symmetric monoidal structure on the category of strictly $\mathbb{A}^1$-invariant sheaves, given by

$$M \otimes_{\mathbb{A}^1} N := H_0(L^{\mathrm{ab}}_{\mathbb{A}^1}(M \otimes N)).$$

We bring this into relation with zero$^{\mathrm{th}}$ $\mathbb{A}^1$-homology:

Proposition 2.63. *Assume that S has the abelian $\mathbb{A}^1$-connectivity property, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be sheaves of sets. Then there is a natural isomorphism of strictly $\mathbb{A}^1$-invariant sheaves*

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1 \times \mathcal{F}_2) \cong \mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1) \otimes_{\mathbb{A}^1} \mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_2).$$

Proof. Starting with lemma 2.34, the property of the free abelian sheaf, and the fact that $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ is cartesian closed, we obtain a natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1 \times \mathcal{F}_2), M) \cong& \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}_1 \times \mathcal{F}_2, M) \cong \dots \\ \dots \cong& \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}_1, \underline{\mathrm{Hom}}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}_2, M)),\end{aligned}$$

for any strictly $\mathbb{A}^1$-invariant sheaf M. Furthermore, it is well-known that the abelian sheaves $\underline{\mathrm{Hom}}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathcal{F}_2, M)$ and $\underline{\mathrm{Hom}}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_S)}(\mathbb{Z}[\mathcal{F}_2], M)$ agree, with the latter one being strictly

$\mathbb{A}^1$-invariant by the above lemma. So, we may continue by using the adjunction of $\mathbb{H}_0^{\mathbb{A}^1}$ and the inclusion of strictly $\mathbb{A}^1$-invariant sheaves in the abelian sheaves to see that this is naturally isomorphic to

$$\begin{aligned} &\mathrm{Hom}_{Ab_{\mathrm{Nis}}^{\mathbb{A}^1}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1), \underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}}(\mathbb{Z}[\mathcal{F}_2], M)) \cong \dots \\ \dots \cong &\mathrm{Hom}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1), \underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{Z}[\mathcal{F}_2], M)), \end{aligned}$$

as this embedding is full and faithful. Using the closed symmetric monoidal structure on the category of abelian sheaves, we find moreover that this is naturally isomorphic to

$$\mathrm{Hom}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{Z}[\mathcal{F}_2], \underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1), M)).$$

Now we may conclude with

$$\begin{aligned} &\mathrm{Hom}_{Ab_{\mathrm{Nis}}^{\mathbb{A}^1}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_2), \underline{\mathrm{Hom}}_{Ab_{\mathrm{Nis}}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1), M)) \cong \dots \\ \dots \cong &\mathrm{Hom}_{Ab_{\mathrm{Nis}}^{\mathbb{A}^1}(Sm_S)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_1) \otimes_{\mathbb{A}^1} \mathbb{H}_0^{\mathbb{A}^1}(\mathcal{F}_2), M), \end{aligned}$$

which comes from lemma 2.34, and the closed symmetric monoidal structure that was found above. □

Remark 2.64. Using reduced $\mathbb{A}^1$-homology we have an analogous statement for sheaves of pointed sets $\mathcal{F}_1$ and $\mathcal{F}_2$, still assuming abelian $\mathbb{A}^1$-connectivity on the base scheme:

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \wedge \mathcal{F}_2) \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1) \otimes_{\mathbb{A}^1} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_2).$$

3

Unramified Sheaves

This chapter is concerned with introducing the concept of unramified sheaves, which are a special kind of sheaves subject to a rationality condition. Examples of unramified sheaves include strictly $\mathbb{A}^1$-invariant sheaves, and our interest stems from the fact that these sheaves may be defined on fields and discrete valuation rings only. But before we go into their definition, their most important properties, and characterise morphisms between unramified sheaves, we discuss extending sheaves to essentially smooth schemes. To that end we set our base scheme S to be the spectrum of a field k, which transfers right away into our notation by changing Sm_S to Sm_k.

3.1 Essentially Smooth Schemes

The sheaf category that sits at the core of the $\mathbb{A}^1$-model category structure is $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)$. However, for most of our purposes the datum of a Nisnevich sheaf on Sm_k carries a lot of overhead, which is why we would like to boil it down a bit. For example: Instead of working with sections on an irreducible scheme $X \in \mathsf{Sm}_k$ we might want rational sections, i.e. sections over its function field $k(X)$. In order to achieve such a reduction, we replicate the construction of the stalks, and form the colimit

$$\operatorname*{colim}_{\varnothing \neq U \subseteq X} \mathcal{F}(U), \tag{3.1}$$

which we call $\check{\mathcal{F}}(\mathrm{Spec}(k(X)))$, or $\check{\mathcal{F}}(k(X))$ for short. In formalizing this approach we first seek to enlarge the category of values that we are allowed to plug into $\mathcal{F}$. This is done via a (left) Kan extension (cf. [38, Def. 2.3.1.(ii)]).

3.1.1 Definition and First Properties

Before concerning ourselves with the process of extending, we need to give some details about the category C we intend to extend towards. By our motivating example (3.1), the category C needs to contain F_k, i.e. the finitely generated (separable) extension fields of k. Moreover, we would like to include at least the following relevant types of local rings on smooth schemes, i.e. for some $X \in \mathsf{Sm}_k$, and $x \in X$, we would like our category C to admit the spectra of the rings

$$\mathcal{O}_{X,x} := \operatorname*{colim}_{\substack{x \in U \subseteq X \\ U \text{ open nbh.}}} \mathcal{O}_X(U), \quad \mathcal{O}^h_{X,x} = \operatorname*{colim}_{\substack{U \to X \\ U \text{ Nis. nbh. of } x}} \mathcal{O}_X(U) \quad \text{and} \quad \mathcal{O}^{sh}_{X,x} = \operatorname*{colim}_{\substack{U \to X \\ U \text{ ét. nbh. of } x}} \mathcal{O}_X(U),$$

where a superscript h resp. sh denotes the henselization resp. strict henselization of a local ring. This motivates the following definition (recall that noetherianity of X transfers to the localization resp. the henselizations [73, Tag 06LJ]):

Definition 3.1. A noetherian scheme X over k is called *essentially smooth* if and only if there is a small cofiltered category I, and a functor $Y_{\dots}\colon \mathsf{I} \to \mathsf{Sm}_k$, with $Y_i \to Y_j$ being affine and étale for every $i \to j$ in I, such that

$$X \cong \lim_{i \in \mathsf{I}} Y_i$$

holds. Denote by EssSm_k the full subcategory of the category of k-schemes, whose objects are the essentially smooth schemes.

From the above presentations, it can be seen that $\mathrm{Spec}(\mathcal{O}_{X,x})$, $\mathrm{Spec}(\mathcal{O}^h_{X,x})$, and $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ are examples of essentially smooth schemes (for $X \in \mathsf{Sm}_k$, and $x \in X$).

Remark 3.2. The above notion of an essentially smooth scheme is far from being universal. Depending on the use case, one may add or omit some hypotheses. Examples include:

- In [15, App. B] the noetherianity assumption is omitted.
- In [31, App. A] noetherianity is also dropped, and the transition morphisms are assumed to be affine and dominant.

The above definition is aligned with the one Morel gives in [50], which is our source for developing $\mathbb{A}^1$-algebraic topology.

Let us continue with a brief discussion of essentially smooth schemes:

Lemma 3.3. *Let F/k be a field extension such that* $\mathrm{Spec}(F) \to \mathrm{Spec}(k)$ *is essentially smooth, and let X be an essentially smooth F-scheme. Then X may be regarded as an essentially smooth k-scheme.*

Proof. As X is noetherian, this follows from the general discussion in [15, Prop. B.4]. □

Example 3.4. Due to the noetherian hypothesis the base change of an essentially smooth k-scheme X might not be essentially smooth, even if we base change through an essentially smooth morphism. As an example, consider $k = \mathbb{Q}$. The strict henselization of $\mathbb{Q}$ is its algebraic closure $\overline{\mathbb{Q}}$, whose spectrum is essentially smooth over $\mathbb{Q}$ by the above. However, $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not noetherian, e.g. by [75, Thm. 11], and the fact that $\overline{\mathbb{Q}}$ is not finitely generated over $\mathbb{Q}$.

If we restrict ourselves to finitely generated field extensions, stability under base change holds:

Lemma 3.5. *Let F/k be a finitely generated field extension, and let X be an essentially smooth k-scheme. Then $X_F := X \times_k \mathrm{Spec}(F)$ is an essentially smooth F-scheme.*

Proof. By the discussion in [15, Prop. B.4] we only have to check that X_F is noetherian. Since X is quasi-compact, the same is true for X_F, and by [73, Tag 038R] we see that X_F is also noetherian. □

Proposition 3.6. *Let X be an essentially smooth k-scheme. Then X is regular.*

Proof. We need to check that for every $x \in X$, the noetherian local ring $\mathcal{O}_{X,x}$ is regular. Let I be a filtered category, and let $X_{\dots}\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{Sm}_k$ a functor, whose transition morphisms $f_{ij}\colon X_i \to X_j$ are affine and etale, for $j \to i$ in I, such that $\lim_{i \in \mathsf{I}^{\mathrm{op}}} X_i \cong X$. By [73, Tag 0032], we may assume that I is a directed set, with order $\leq$. Denote the canonical projection morphisms $X \to X_i$ by pr_i, for $i \in \mathsf{I}$. Setting $x_i := \mathrm{pr}_i$, we have

$$\mathcal{O}_{X,x} \cong \operatorname*{colim}_{i \in \mathsf{I}} \mathcal{O}_{X_i,x_i}. \tag{3.2}$$

This follows from the way in which the limit X is constructed (cf. [73, Tag 01YX]): We may choose any $0 \in I$, and obtain $\lim_{i \in (I_{\geq 0})^{\mathrm{op}}} X_i \cong X$. So, for $x \in X$ we may choose an affine open neighbourhood U_0 of x_0 in X_0, and fix the preimages $U_i := f_{i0}^{-1}(U_0)$, for $i \geq 0$. Then by [73, Tag 01YX] we have that $\mathrm{pr}_0^{-1}(U_0) \cong \lim_{i \in (I_{\geq 0})^{\mathrm{op}}} U_i$ is an affine open neighbourhood of x. Now (3.2) follows from

$$\mathcal{O}_X(\mathrm{pr}_0^{-1}(U_0)) \cong \operatorname*{colim}_{i \in I_{\geq 0}} \mathcal{O}_{X_i}(U_i),$$

which in turn is a consequence of [73, Tag 01YX]. We conclude with [72, Lem. 1.4], which states that a noetherian, filtered colimit of regular local rings is again regular. □

As an easy corollary of the above proposition, we may note that all essentially smooth k-schemes are reduced and normal. The following proposition deals with the question, what kind of extension fields of k are essentially smooth. To that end, we recall the following two notions of separability:

Definition 3.7. Let F/k be a field extension.

(i) F/k is *separably generated* if and only if there is a transcendence basis $\Gamma \subseteq F$ such that $F/k(\Gamma)$ is separably algebraic.

(ii) F/k is *separable* if and only if $F \otimes_k K$ is reduced, for all field extensions K/k.

We also remind the reader that any separably generated field extension F/k is separable (cf. [43, Thm. 26.1]), but that the converse only holds in the case that F is finitely generated over k ([43, Thm. 26.2]). A counterexample may be provided by the perfect closure F of $\mathbb{F}_p(t)$ over $k := \mathbb{F}_p$.

Proposition 3.8. *Let F/k be an extension field of k.* $\mathrm{Spec}(F)$ *lies in* EssSm_k *if and only if F is separably generated over k, and of finite transcendence degree.*

Proof. Suppose first that F is separably generated over k, and of finite transcendence degree over k. Thus, we can find algebraically independent transcendental variables $t_1, \ldots, t_n \in F$, such that $F/k(t_1, \ldots, t_n)$ is separably algebraic. As $k(t_1, \ldots, t_n)$ is the function field of the smooth scheme $\mathbb{A}^n_k$, we see that $k(t_1, \ldots, t_n)$ is essentially smooth over k. Moreover, since $F/k(t_1, \ldots, t_n)$ is separable algebraic, we may write F as a filtered colimit of the finitely generated intermediary fields of $F/k(t_1, \ldots, t_n)$. Now by lemma 3.3, we see that $\mathrm{Spec}(F)$ is essentially smooth over k.

Conversely, assume that there is a filtered category I, and a functor $X_{\ldots} \colon I^{\mathrm{op}} \to \mathsf{Sm}_k$ such that $\mathrm{Spec}(F) \cong \lim_{i \in I^{\mathrm{op}}} X_i$. As in the proof of Proposition 3.6, we may assume that I is a directed set with minimal element, and that all X_i are affine. Assume that $X_i \cong \mathrm{Spec}(A_i)$, where A_i is a smooth and finitely generated k-algebra. So we have

$$F \cong \operatorname*{colim}_{i \in I} A_i,$$

and let us denote the étale transition homomorphisms by $\varphi_{ij} \colon A_i \to A_j$, for $i \leq j$, and the canonical embeddings by $\mathrm{in}_i \colon A_i \to F$, for every $i \in I$. The kernels $\mathfrak{p}_i := \ker(\mathrm{in}_i)$ are prime ideals, and as F is a field, we have induced embeddings

$$\overline{\mathrm{in}}_i \colon \kappa(\mathfrak{p}_i) \longrightarrow F.$$

One may check by hand that the induced homomorphism $\mathrm{colim}_{i \in I} \kappa(\mathfrak{p}_i) \to F$ is an isomorphism. Since $A_i \to A_j$ is étale, for $i \to j$ in I, we know by [73, Tag 00U4] that $\kappa(\mathfrak{p}_j)/\kappa(\mathfrak{p}_i)$ is finite and separable. Thus $F/\kappa(\mathfrak{p}_0)$ is separably algebraic. We may also deduce that F/k is separable (viz. geometrically reduced): Indeed, let K/k be any field extension, then we have

$$F \otimes_k K \cong \operatorname*{colim}_{i \in I} \left(A_i \otimes_k K \right),$$

and since smoothness is stable under base-change, the right-hand side is a filtered colimit of reduced K-algebras, and thus reduced. Therefore F is separable. This implies in particular that the intermediate extension $\kappa(\mathfrak{p}_0)/k$ is separable. Since $\kappa(\mathfrak{p}_0)$ is finitely generated over k, it is also separably generated (cf. [43, Thm. 26.2]). So it follows that F/k is separably generated. The statement about the transcendence degree follows, since $\kappa(\mathfrak{p}_0)/k$ has finite transcendence degree, and $F/\kappa(\mathfrak{p}_0)$ is separably algebraic. □

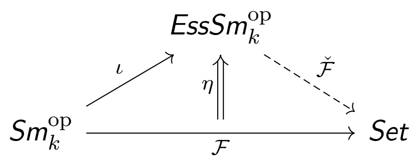
In the following we will often be concerned with essentially smooth discrete valuation rings, i.e. with with a discrete valuation ring whose spectrum is essentially smooth. The fraction fields of such rings are naturally also essentially smooth, however their residue fields might not be essentially smooth, as one learns from the case of $A := k[T]_{(T^p - t)}$, with $k = \mathbb{F}_p(t)$. In the case of a perfect ground field k such an example cannot happen: Indeed, for k perfect, all extension fields of k are separable (cf. [43, Thm. 26.3]), and thus, if they are finitely generated, they will also be separably generated. As it will be sufficient to confine ourselves to the finitely generated case, we make the following definition, with the case k being perfect in mind:

Definition 3.9. Denote by F_k the full subcategory of the category of all field extensions of k (so that the morphisms are k-homomorphisms) which are finitely generated, and whose spectrum appears in EssSm_k.

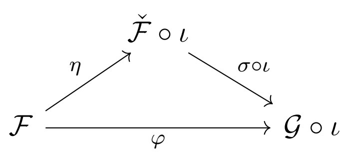
We note in closing that the objects of F_k are exactly the function fields of irreducible smooth k-schemes.

3.1.2 Extending to EssSm_k

Let us consider a (pre-)sheaf $\mathcal{F}$ on Sm_k, i.e. a functor $(\mathsf{Sm}_k)^{\mathrm{op}} \to \mathsf{Set}$. Denote by ι the functor $\mathsf{Sm}_k \to \mathsf{EssSm}_k$ that embeds smooth schemes into essentially smooth schemes, and which, by definition, is full and faithful. A left Kan extension of $\mathcal{F}$ along ι is a functor $\check{\mathcal{F}} \colon (\mathsf{EssSm}_k)^{\mathrm{op}} \to \mathsf{Set}$, together with a natural transformation $\eta \colon \mathcal{F} \to \check{\mathcal{F}} \circ \iota$, such that the following diagram commutes:



Moreover, one demands the following universal property: For every functor $\mathcal{G} \colon (\mathsf{EssSm}_k)^{\mathrm{op}} \to \mathsf{Set}$, and natural transformation $\varphi \colon \mathcal{F} \to \mathcal{G} \circ \iota$, one finds a unique natural transformation $\sigma \colon \check{\mathcal{F}} \to \mathcal{G}$ such that the diagram of functors



commutes. One might express this more concisely by stating

$$\mathrm{Hom}_{\mathsf{PSh}(\mathsf{EssSm}_k)}(\check{\mathcal{F}}, \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Sm}_k)}(\mathcal{F}, \mathcal{G} \circ \iota),$$

or that $\check{(-)}$ is left adjoiont to $(-) \circ \iota$, i.e. that extending is left-adjoint to restricting. In particular, η is the unit of the adjunction $\check{(-)} \dashv (-) \circ \iota$. By [38, Thm. 2.3.3.(i)], the existence of small colimits in Set (cf. [38, Thm. 2.4.1]), and the fact that the category Sm_k is essentially small, we see that for any $\mathcal{F} \in \mathsf{PSh}(\mathsf{Sm}_k)$ the extension $\check{\mathcal{F}}$ exists and for $X \in \mathsf{EssSm}_k$ is given by the formula

$$\check{\mathcal{F}}(X) = \operatorname*{colim}_{\substack{X \to \iota(Y) \\ Y \in \mathsf{Sm}_k}} \mathcal{F}(Y).$$

This agrees with our intuition (3.1). Indeed, let X be an essentially smooth k-scheme. We may fix a cofiltered category I, and a functor $X_{(-)}\colon \mathsf{I} \to \mathsf{Sm}_k$, whose transition morphisms are affine and étale. Thus we have a canonical comparison functor

$$\begin{aligned} \lambda\colon \mathsf{I} &\longrightarrow (X \downarrow \mathsf{Sm}_k) \\ i &\longmapsto \left(X \xrightarrow{\mathrm{pr}_i} X_i\right), \end{aligned}$$

where $X \downarrow \mathsf{Sm}_k$ denotes the comma category of k-morphisms from X to smooth k-schemes. Any $Y \in \mathsf{Sm}_k$ is finitely presented over $\mathrm{Spec}(k)$, and thus, by our assumptions on I and the functor $X_{(-)}$, we have that

$$\operatorname*{colim}_{i\in \mathsf{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sm}_k}(X_i, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Sch}_k}(\lim_{i\in \mathsf{I}} X_i, Y) = \mathrm{Hom}_{\mathsf{Sch}_k}(X, Y)$$

is bijective (cf. [73, Tag 01ZC]). This, together with the fact that I^{op} is filtered, implies that λ is cofinal (see [73, Tag 04E6]). Thus we have

$$\check{\mathcal{F}}(X) = \operatorname*{colim}_{i\in \mathsf{I}^{\mathrm{op}}} \mathcal{F}(X_i),$$

agreeing with our intuition coming from (3.1).

Remark 3.10. Let $\mathcal{G}$ be a sheaf of abelian groups on Sm_k. By replacing the category Set with Ab in the above, we may construct an essentially smooth extension $\check{\mathcal{G}}$ of $\mathcal{G}$, with values in abelian groups. Since the defining colimits may be chosen to be over filtered categories, this does not differ from our first definition, and we only need to keep one symbol for the essentially smooth extension.

Later on, the following mixed situation will also occur: Given a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves of sets, we ought to determine the essentially smooth extension, by also keeping the group structure in the codomain. This is possible, since there is always a natural map ($X \in \mathsf{EssSm}_k$)

$$\check{\mathcal{F}}(X) = \operatorname*{colim}_{\substack{X\to\iota(Y)\\ Y\in \mathsf{Sm}_k}} \mathcal{F}(Y) \longrightarrow \operatorname*{colim}_{\substack{X\to\iota(Y)\\ Y\in \mathsf{Sm}_k}} \mathcal{G}(Y) = \check{\mathcal{G}}(X),$$

where the colimit on the left-hand side is taken in Set and the colimit on the right-hand side is taken in Ab. Another view on that matter could be, that the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves and reflects filtered colimits, reducing this mixed case to the one where we consider all sheaves as sheaves of sets.

3.1.3 Cocontinuity

In this subsection we discuss situations in which cofiltered limits are preserved by presheaves on Sm_k resp. by their essentially smooth extensions. Therefore we first note:

Lemma 3.11. *Let $\mathcal{F}$ be a presheaf of sets on Sm_k, let I be a filtered category, and $X_{(-)}\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{Sm}_k$ be a functor with affine transition morphisms such that its limit*[1] $\lim_{i\in \mathsf{I}} X_i$ *lies in Sm_k. Then we have that the canonical map*

$$\operatorname*{colim}_{i\in \mathsf{I}} \mathcal{F}(X_i) \longrightarrow \mathcal{F}\left(\lim_{i\in \mathsf{I}^{\mathrm{op}}} X_i\right) \tag{3.3}$$

is bijective.

[1] The limit exists as a scheme by [73, Tag 01YX].

Proof. As above we may assume that I is a directed set, and that the transition morphisms are $f_{ij}\colon X_i \to X_j$, for $j \leq i$. Moreover, we fix the names $X := \lim_{i\in \mathsf{I}^{\mathrm{op}}} X_i$ for the limit, and $\mathrm{pr}_i\colon X \to X_i$ for the canonical projections. For every $Y \in \mathsf{Sm}_k$ we know by [73, Tag 01ZC] that

$$\begin{aligned} \operatorname*{colim}_{i\in \mathsf{I}} \mathrm{Hom}_{\mathsf{Sm}_k}(X_i, Y) &\longrightarrow \mathrm{Hom}_{\mathsf{Sm}_k}(X, Y) \\ [i, f_i\colon X_i \to Y] &\longmapsto f_i \circ \mathrm{pr}_i \end{aligned} \tag{3.4}$$

is bijective. Let us apply (3.4) in the case $Y = X$: Since I is filtered, we can thus find some $i_0 \in \mathsf{I}$ such that for all $i \geq i_0$, there is a morphism $f_i\colon X_i \to X$ with $f_i \circ \mathrm{pr}_i = \mathrm{id}_X$. Applying (3.4) once more in the case $Y = X_i$ (for $i \geq i_0$), we find

$$\mathrm{pr}_i \circ f_i \circ f_{j(i),i} = f_{j(i),i} \tag{3.5}$$

for some $j(i) \geq i$. Armed with this, we may check bijectivity of (3.3). Since all $\mathcal{F}(\mathrm{pr}_i)$ admit a section, for $i \geq i_0$, we see the surjectivity of (3.3). To check injectivity fix two elements $[i_1, \sigma_1]$, $[i_2, \sigma_2]$, with $\sigma_1 \in \mathcal{F}(X_{i_1})$ and $\sigma_2 \in \mathcal{F}(X_{i_2})$, such that

$$\mathcal{F}(\mathrm{pr}_{i_1})(\sigma_1) = \mathcal{F}(\mathrm{pr}_{i_2})(\sigma_2)$$

holds. We may assume that i_1, i_2 are $\geq i_0$. Moreover, by (3.5) and the assumption that I is directed, we may find some $i_3 \geq i_1, i_2$ such that

$$\mathcal{F}(f_{i_3,i_1})(\sigma_1) = \mathcal{F}(f_{i_1} \circ f_{i_3,i_1})\left(\mathcal{F}(\mathrm{pr}_{i_1})(\sigma_1)\right) = \mathcal{F}(f_{i_2} \circ f_{i_3,i_2})\left(\mathcal{F}(\mathrm{pr}_{i_2})(\sigma_2)\right) = \mathcal{F}(f_{i_3,i_2})(\sigma_2)$$

holds. Thus we have $[i_1, \sigma_1] = [i_2, \sigma_2]$ in $\mathrm{colim}_{i\in \mathsf{I}} \mathcal{F}(X_i)$. □

We expect to generalize this result to the essentially smooth extension. Therefore we note the following lemma about (left) Kan extensions, which we learned from John Bourke[2]:

Lemma 3.12. *Let C, $\tilde{\mathsf{C}}$, and D be categories, with C being essentially small, and D cocomplete. Let $\iota\colon \mathsf{C} \to \tilde{\mathsf{C}}$ and $F\colon \mathsf{C} \to \mathsf{D}$ be functors, and denote by $\mathrm{Lan}_\iota(F)$ the left Kan extension of F along ι. Let $C_{(-)}\colon \mathsf{I} \to \tilde{\mathsf{C}}$ be a functor, whose colimit exists in $\tilde{\mathsf{C}}$. If for all $C \in \mathsf{C}$ the canonical map*

$$\operatorname*{colim}_{i\in \mathsf{I}} \mathrm{Hom}_{\tilde{\mathsf{C}}}(\iota(C), C_i) \longrightarrow \mathrm{Hom}_{\tilde{\mathsf{C}}}(\iota(C), \operatorname*{colim}_{i\in \mathsf{I}} C_i)$$

is a bijection, then the canonical morphism

$$\operatorname*{colim}_{i\in \mathsf{I}} \mathrm{Lan}_\iota(F)(C_i) \longrightarrow \mathrm{Lan}_\iota(F)\left(\operatorname*{colim}_{i\in \mathsf{I}} C_i\right)$$

is an isomorphism in D.

Proof. Since C is essentially small, we have the following formula for the left Kan extension using coends (cf. [41, Thm. X.4.1])

$$\mathrm{Lan}_\iota(F)(X) = \int^{C\in \mathsf{C}} \pi_{F(C)}\left(\mathrm{Hom}_{\tilde{\mathsf{C}}}(\iota(C), X)\right),$$

where the functor $\pi_D\colon \mathsf{Set} \to \mathsf{D}$ is the *copower functor* with respect to some $D \in \mathsf{D}$ and $S \in \mathsf{Set}$

$$\pi_D(S) := \bigsqcup_S D.$$

[2] See `https://mathoverflow.net/questions/112137`.

For the copower functor, we have the following adjunction

$$\mathrm{Hom}_{\mathsf{D}}(\pi_D(S), D') \cong \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Hom}_{\mathsf{D}}(D, D')), \text{ i.e. } \quad \pi_D \dashv \mathrm{Hom}_{\mathsf{D}}(D, -),$$

guaranteeing us that π_D commutes with colimits for all $D \in \mathsf{D}$. This fact, together with our assumption, implies that the canonical morphism

$$\operatorname*{colim}_{i \in \mathsf{I}} \pi_{F(C')} \left(\mathrm{Hom}_{\check{\mathsf{C}}}(\iota(C), C_i)\right) \longrightarrow \pi_{F(C')} \left(\mathrm{Hom}_{\check{\mathsf{C}}}(\iota(C), \operatorname*{colim}_{i \in \mathsf{I}} C_i)\right)$$

is an isomorphism in D, for all $C, C' \in \mathsf{C}$. Now, consider the functor

$$\begin{aligned} G \colon \mathsf{I}^{\mathrm{op}} \times \mathsf{I} \times \mathsf{C}^{\mathrm{op}} \times \mathsf{C} &\to \mathsf{D} \\ (i_1, i_2, C, C') &\mapsto \pi_{F(C')} \left(\mathrm{Hom}_{\check{\mathsf{C}}}(\iota(C), C_{i_2})\right). \end{aligned}$$

[41, Prop. IX.5.3], together with Fubini's theorem (cf. [41, Prop. IX.8]), imply:

$$\begin{aligned} \operatorname*{colim}_{i \in \mathsf{I}} \int^{C \in \mathsf{C}} \pi_{F(C)} \left(\mathrm{Hom}_{\mathsf{C}}(\iota(C), C_i)\right) &= \int^{i \in \mathsf{I}} \int^{C \in \mathsf{C}} G(i, i, C, C) = \dots \\ &= \int^{C \in \mathsf{C}} \int^{i \in \mathsf{I}} G(i, i, C, C) = \dots \\ &= \int^{C \in \mathsf{C}} \operatorname*{colim}_{i \in \mathsf{I}} \pi_{F(C)} \left(\mathrm{Hom}_{\check{\mathsf{C}}}(\iota(C), C_i)\right) = \dots \\ &= \int^{C \in \mathsf{C}} \pi_{F(C)} \left(\mathrm{Hom}_{\check{\mathsf{C}}}(\iota(C), \operatorname*{colim}_{i \in \mathsf{I}} C_i)\right). \end{aligned}$$

By using the formula for the left Kan extension this calculation yields the claim. □

Corollary 3.13. *Let $\mathcal{F} \colon \mathsf{Sm}_k \to \mathsf{Set}$ be a presheaf, let I be a filtered category, and let $X_{(-)} \colon \mathsf{I} \to \mathsf{EssSm}_k$ be a functor with affine transition morphisms, whose limit exists in EssSm_k. Then we know that the canonical map*

$$\operatorname*{colim}_{i \in \mathsf{I}} \check{\mathcal{F}}(X_i) \longrightarrow \check{\mathcal{F}}\left(\lim_{i \in \mathsf{I}^{\mathrm{op}}} X_i\right)$$

is bijective.

Proof. The statement follows from Lemma 3.12 and [73, Tag 01ZC]. □

The above gives a justification for restricting ourselves to finitely generated fields in the definition of F_k, since any essentially smooth field may be written as a filtered colimit of its finitely generated subfields. Moreover, on the category F_k itself colimits are not doing much. Indeed, any functor from a filtered category into F_k such that its colimit lies again in F_k, needs to be constant up to restriction to a cofinal subcategory.

3.2 Unramified Sheaves

In this section we introduce unramified sheaves, and discuss how to reduce the structure that is needed to define one. Therefore passing to the essentially smooth extension will be crucial.

3.2.1 Definition and Examples

We start by defining unramifiedness for presheaves:

Definition 3.14. Let $\mathcal{F}$ be a presheaf (of sets, pointed sets, abelian groups, etc.) on Sm_k. $\mathcal{F}$ is called *unramified* if and only if the following conditions are met:

(0) Let $X \in \mathit{Sm}_k$ be arbitrary, and denote by $X_1, \ldots, X_n$ the distinct irreducible components of X.[3] The natural map

$$\mathcal{F}(X) \longrightarrow \prod_{i=1}^{n} \mathcal{F}(X_i)$$

is bijective.

(1) Let $X \in \mathit{Sm}_k$ be arbitrary, and $U \subseteq X$ an open and dense subscheme. The restriction map

$$\mathcal{F}(X) \hookrightarrow \mathcal{F}(U)$$

is injective.

(2) $\mathcal{F}$ is a sheaf with respect to the Zariski site on Sm_k, and for every $X \in \mathit{Sm}_k$ and $U \subseteq X$ open, with $X^{(1)} \subseteq U$, we have that the restriction map

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$$

is bijective.

If $\mathcal{F}$ only fulfils axioms **(0)** and **(1)**, we call $\mathcal{F}$ *weakly unramified.*

Remark 3.15. The notion of unramified presheaves is due to Morel (cf. [50, Def. 2.1]), namely let $\mathcal{F}$ be a weakly unramified presheaf in our sense, then $\mathcal{F}$ is unramified if and only if for all irreducible $X \in \mathit{Sm}_k$, with function field F, the induced map

$$\mathcal{F}(X) \to \bigcap_{x \in X^{(1)}} \check{\mathcal{F}}(\operatorname{Spec}(\mathcal{O}_{X,x})) \subseteq \check{\mathcal{F}}(\operatorname{Spec}(F)) \qquad \textbf{(2')}$$

is bijective. Morel notes in [50, Rem. 2.4] that this is equivalent to our definition, and a proof of this fact may be found in [33, Lem. 4.15]. One motivation of this notion comes from initial research in this direction by Colliot-Thélène and Sansuc (cf. [14, §6]).

We start by listing several easy

Examples 3.16. (a) Let Y be a separated k-scheme. The irreducible components $X_1, \ldots, X_n$ of a scheme $X \in \mathit{Sm}_k$ are disjoint and open, and hence there is a canonical bijection

$$\operatorname{Hom}_{\mathit{Sch}_k}\left(\coprod_{i=1}^{n} X_i, Y\right) \cong \prod_{i=1}^{n} \operatorname{Hom}_{\mathit{Sch}_k}(X_i, Y).$$

Moreover, for any dense $U \subseteq X$, and k-morphisms $f, g \colon X \to Y$ agreeing on U, we have that $f_{\restriction X_{\mathrm{red}}} = g_{\restriction X_{\mathrm{red}}}$ by [25, Cor. 9.9]. Since X is smooth, and thus reduced, one even finds $f = g$. So we have shown that the presheaf

$$\operatorname{Hom}_{\mathit{Sch}_k}(-, Y) \colon \mathit{Sm}_k \to \mathit{Set}$$

is weakly unramified, for any separated k-scheme Y. We remark that $\operatorname{Hom}_{\mathit{Sch}_k}(-, Y)$ is even a sheaf with respect to the Nisnevich site.

[3]As X is noetherian, there are only finitely many irreducible components. As X is moreover smooth, and thus locally integral, the $(X_i)_{1 \le i \le n}$ are pairwise disjoint. This implies that the X_i are also open.

(b) Let $Y = \mathrm{Spec}(A)$ be an affine k-scheme. We claim that $\mathrm{Hom}_{\mathit{Sch}_k}(-, Y)$ is an unramified presheaf. So let $X \in \mathit{Sm}_k$ be a scheme, and let $U \subseteq X$ be an open subset that contains all points of codimension 1 in X. To check **(2)**, we have to extend any k-morphism $U \to Y$ to a k-morphism $X \to Y$. Uniqueness of such an extension follows by the arguments in (a) above. Since X is noetherian, and the codimension of $X \setminus U$ in X is ≥ 2, we know, by the algebraic Hartog's theorem [25, Thm. 6.45], that the restriction homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is an isomorphism. By the adjunction of the global section functor and the Spec-functor, we have:

$$\mathrm{Hom}_{\mathit{Sch}_k}(X, Y) \cong \mathrm{Hom}_{\mathit{Alg}_k}(A, \mathcal{O}_X(X)) \cong \mathrm{Hom}_{\mathit{Alg}_k}(A, \mathcal{O}_X(U)) \cong \mathrm{Hom}_{\mathit{Sch}_k}(U, Y).$$

(c) Any finite limit of unramified presheaves will be a unramified. Note first that limits in the presheaf category $\mathit{PSh}(\mathit{Sm}_k)$ can be computed locally for each $X \in \mathit{Sm}_k$ (and even so in the sheaf category $\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_k)$). Using this and the fact that limits commute with each other (cf. [38, Prop. 2.1.7]), gives **(0)** and **(2)**. The commutativity of limits also helps with axiom **(1)**, since a map of sets $S_1 \xrightarrow{f} S_2$ is injective if and only if the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\mathrm{id}_{S_1}} & S_1 \\ {\scriptstyle \mathrm{id}_{S_1}}\downarrow & & \downarrow{\scriptstyle f} \\ S_1 & \xrightarrow[f]{} & S_2 \end{array}$$

is cartesian.

(d) In [50, Ex. 2.3] Morel remarks that Rost cycle modules and the sheaf associated to the Witt presheaf $X \mapsto W(X)$ is unramified, for k a perfect field.

The above examples all had in common that the underlying (weakly) unramified presheaf fulfilled the sheaf condition. Since this situation will occur more often below we make the following

Definition 3.17. If $\mathcal{F}$ is a sheaf (of sets, pointed sets, abelian groups, etc.) on Sm_k with respect to the Nisnevich site, and unramified as a presheaf, we say $\mathcal{F}$ is an *unramified sheaf* (of sets, pointed sets, abelian groups, etc.).

Let us study the relation of unramified presheaves of pointed sets and the smash product in the category of presheaves of pointed sets:

Lemma 3.18. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be unramified presheaves of pointed sets. Denote by $\mathcal{F}$ the presheaf ($X \in \mathit{Sm}_k$)*

$$X \mapsto \mathcal{F}_1(X) \wedge \cdots \wedge \mathcal{F}_n(X).$$

Then $a_{\mathrm{Zar}}(\mathcal{F})$ is an unramified presheaf of pointed sets, and moreover we have

$$a_{\mathrm{Zar}}(\mathcal{F})(X) = \prod_{i=1}^{n} \mathcal{F}(X_i),$$

where $X \in \mathit{Sm}_k$ and $X_1, \ldots, X_n$ denote the distinct irreducible components of X.

Proof. Since major parts of this proof are rather trivial checks, we will only give a sketch. Firstly, one starts out by verifying the moreover part of the lemma. And to do that, one proves first that the presheaf $\mathcal{G}$ given by

$$X \mapsto \prod_{i=1}^{r} \mathcal{F}(X_i),$$

where the $X_1, \dots, X_r$ are the distinct irreducible components of X, is a Zariski sheaf. This will follow from axioms **(0)**, **(1)**, and **(2)**, which are available for all $\mathcal{F}_1, \dots, \mathcal{F}_n$. Next one notices that there is a canonical morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, sending a section $\sigma \in \mathcal{F}(X)$ to the product of its restrictions to the irreducible components. Moreover, every other morphism $\mathcal{F} \to \mathcal{H}$, for $\mathcal{H}$ a Zariski sheaf, factors uniquely through φ. Thus $a_{\mathrm{Zar}}(\mathcal{F})$ has the particular form we claimed. Since $\mathcal{G}$ is a Zariski sheaf axiom **(0)** follows. To check axiom **(1)** (resp. **(2)**) one may reduce to the case of $X \in \mathsf{Sm}_k$ being irreducible, and $U \subseteq X$ being a nonempty open subset (resp. containing $X^{(1)}$). Thus every restriction map $\mathcal{F}_j(X) \to \mathcal{F}_j(U)$ is injective (resp. bijective), which implies that $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective (resp. bijective), as well. This yields the claim. □

Since we will need it later on, we note the following special case:

Corollary 3.19. *If we have $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbb{G}_m$, then, with the notation of lemma 3.18, $a_{\mathrm{Zar}}(\mathcal{F})$ is an unramified sheaf of pointed sets.*

Proof. Again, we will only sketch the arguments. Let $X \in \mathsf{Sm}_k$ be given, and suppose $(f_s\colon U_s \to X)_{s\in S}$ is a Nisnevich cover. We may assume, since for smooth schemes irreducible components are open, that the U_s are irreducible. Moreover, since f_s is étale (thus open), and $f_s(U_s)$ is irreducible, we may also assume that X is irreducible. Now, since f_s is open, hence dominant, $\mathbb{G}_m$ is separated, and X is reduced, we have that

$$\mathbb{G}_m(X) \to \mathbb{G}_m(U_s)$$

is injective, by [25, Cor. 9.9]. Thus $\mathcal{F}(X) \to \mathcal{F}(U_s)$ is injective, i.e. $a_{\mathrm{Zar}}(\mathcal{F})$ is a separated sheaf.

Suppose we are given sections $\sigma_1^{(s)} \wedge \cdots \wedge \sigma_n^{(s)} \in \mathcal{F}(U_s)$ which are compatible with restricting to the intersections $U_s \times_X U_t$ on $a_{\mathrm{Zar}}(\mathcal{F})$. This means that

$$\sigma_1^{(s)} \wedge \cdots \wedge \sigma^{(s)}_{n\restriction (U_s \times_X U_t)_h} = \sigma_1^{(t)} \wedge \cdots \wedge \sigma^{(t)}_{n\restriction (U_s \times_X U_t)_h}$$

holds, where h runs over all irreducible components of $U_s \times_X U_t$, and $s, t \in S$. If $\sigma_1^{(s)} \wedge \cdots \wedge \sigma_n^{(s)}$ is the basepoint for one $s \in S$, we may choose the basepoint as glued section. Conversely, if that is not the case, we may use the injectivity of

$$\mathbb{G}_m(U_s) \to \mathbb{G}_m\left((U_s \times_X U_t)_h\right),$$

and the sheaf property of $\mathbb{G}_m$ to obtain a glued section. □

Note that, contrary to what is claimed in [50, Lem. 3.36], the presheaf

$$X \longmapsto \left(\mathcal{O}_X(X)^\times\right)^{\wedge n}$$

is not a sheaf in the Zariski topology, and thus is not unramified. To see this one may look at $X = \operatorname{Spec}(k \times k)$. Here the canonical map $\mathcal{O}_X(X)^\times \wedge \mathcal{O}_X(X)^\times \to \prod_{i=1}^2 \mathcal{O}_{X_i}(X_i)^\times \wedge \mathcal{O}_{X_i}(X_i)^\times$ is given by

$$\begin{aligned}(k\times k)^\times \wedge (k \times k)^\times &\longrightarrow (k^\times \wedge k^\times) \times (k^\times \wedge k^\times)\\ (\lambda_1, \mu_1) \wedge (\lambda_2, \mu_2) &\longmapsto (\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2),\end{aligned}$$

which is not injective, since $(\lambda, 1) \wedge (1, \lambda)$ is sent to the basepoint for $\lambda \in k \smallsetminus \{0, 1\}$. This violates axiom **(0)**. However, since by the above, the Zariski sheafification of this presheaf is an unramified sheaf of pointed sets, there is no real change to be made in [50, Sec. 3.3]. The spaces $\mathbb{G}_m^{\wedge n}$ are interesting to us, as they form one type of spheres in $\mathbb{A}^1$-algebraic topology.
From here onwards we will denote by $\mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_n$ the Nisnevich sheaf associated to $X \mapsto \mathcal{F}_1(X) \wedge \cdots \wedge \mathcal{F}_n(X)$, for presheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ on Sm_k.
The following proposition provides many examples, which become central to our analysis below.

Proposition 3.20. *Let G be a smooth algebraic group over k.*

(a) If G has the strong Grothendieck-Serre property, i.e. if for any essentially smooth local k-algebra A the map

$$H^1_{\text{ét}}(\operatorname{Spec}(A), G) \longrightarrow H^1_{\text{ét}}(\operatorname{Spec}(\operatorname{Frac}(A)), G)$$

is injective, then $\mathcal{H}^1_{\text{ét}}(G)$ is weakly unramified.[4]

(b) If G is reductive and satisfies purity for henselization, i.e. if for all $X \in \mathsf{Sm}_k$ and $x \in X$ the map

$$\begin{aligned} &\operatorname{im}\Big(H^1_{\text{ét}}(Y, G) \to H^1_{\text{ét}}(\operatorname{Spec}(k(Y)), G)\Big) \longrightarrow \dots \\ &\qquad \dots \longrightarrow \bigcap_{y \in Y^{(1)}} \operatorname{im}\Big(H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_{Y,y}), G) \to H^1_{\text{ét}}(\operatorname{Spec}(k(Y)), G)\Big) \end{aligned}$$

is surjective, where $Y := \operatorname{Spec}(\mathcal{O}^h_{X,x})$, then $\mathcal{H}^1_{\text{ét}}(G)$ is unramified.

(c) If G is semi-simple and simply-connected, then $\mathcal{H}^1_{\text{ét}}(G)$ is $\mathbb{A}^1$-invariant.

Proof. (a) Axiom **(0)** follows from the fact that every Nisnevich sheaf is also a Zariski sheaf, and that the irreducible components in some $X \in \mathsf{Sm}_k$ are finite, disjoint and open. To address axiom **(1)**, we first reduce to the case of an irreducible $X \in \mathsf{Sm}_k$ and some nonempty open $U \subseteq X$. Next, we would like to apply [17, Prop. 3.1] to the presheaf $\mathcal{F}\colon \mathsf{EssSm}_k \to \mathsf{Set}_\bullet$, with $X \mapsto H^1_{\text{ét}}(X, G)$. To that end, one has to check that $\mathcal{F}$ sends cofiltered limits to filtered colimits, which is a consequence of [42, Thm. 2.1], and use the strong Grothendieck-Serre property. We thus obtain a commuting diagram

$$\begin{array}{ccc} \mathcal{H}^1_{\text{ét}}(G)(X) & \xrightarrow{\operatorname{res}^X_U} & \mathcal{H}^1_{\text{ét}}(G)(U) \\ & \searrow & \downarrow \\ & & H^1_{\text{ét}}(\operatorname{Spec}(K), G), \end{array}$$

in which the downward arrows are injective. Thus res^X_U is injective.

(b) This is [17, Lem. 4.7].

(c) This statement is an amplification of the result [17, Prop. 3.7] in characteristic 0 by Elmanto-Kulkarni-Wendt, due to Balwe-Hogadi-Sawant [6, Thm. 3.9]. □

Remark 3.21. For all groups we consider in chapter 4, it is known that the strong Grothendieck-Serre property holds: For the subgroups $\boldsymbol{\mu}_\ell$ of $\mathbb{G}_m$, with ℓ being prime to the characteristic, this follows from a quick calculation using that regular (see proposition 3.6) local rings are integrally closed. Similarly, one may handle the case of constant finite groups, of which we especially consider the symmetric groups S_n, and the reader can find an explicit proof in Moser's diploma thesis (cf. [55, Satz 2.4.2]). The orthogonal and unitary groups posses the strong Grothendieck-Serre property by theorems of Ojanguren and Panin (cf. 5.1 and 9.2 in [59]), and a standard twisting argument.

Moreover, we may include all smooth reductive algebraic groups G in our considerations. Indeed, as any essentially smooth local k-algebra is regular, we may use [19, Cor. 1] in the case that k is infinite, and [60, Cor. 1.2] for k finite.

As an example for part (b) one may consider $\mathbf{G}_2$. Indeed, we already know that $\mathcal{H}^1_{\text{ét}}(\mathbf{G}_2)$ is weakly unramified, and adding to that the results of Panin and Pimenov (specifically [61, Rem. 3.2]), and Chernousov and Panin (cf. [11, Thm. 1]), we find that $\mathcal{H}^1_{\text{ét}}(\mathbf{G}_2)$ is unramified, if the ground field k is additionally infinite.

[4]This sheaf was defined in corollary 2.56.

Lastly, we come to a more general kind of examples of unramified sheaves, namely strictly $\mathbb{A}^1$-invariant sheaves. Their importance to us comes from the central proposition 2.58. Although the statement is given in [50, Ex. 2.3], we recall part of its proof, to sketch in which way being a strictly $\mathbb{A}^1$-invariant sheaf is a stronger property than being unramified.

Proposition 3.22. *Let M be a strictly $\mathbb{A}^1$-invariant sheaf. Then M is unramified.*

Proof. Axiom **(0)** follows, since any Nisnevich sheaf is also a Zariski sheaf, and the irreducible components in a smooth k-scheme are disjoint and open. The axioms **(1)** and **(2)** can then be deduced from the following lemma due to Morel.

Lemma ([53, Lem. 6.4.4])**.** *Let $U \subseteq X$ be an open subscheme of a smooth k-scheme X such that the codimension of the closed complement $X \smallsetminus U$ in X is at least d. Then for any strictly $\mathbb{A}^1$-invariant sheaf on Sm_k the morphism*

$$H^n_{\mathrm{Nis}}(X, M) \longrightarrow H^n_{\mathrm{Nis}}(U, M)$$

is an isomorphism for $n \le d-2$ and a monomorphism for $n = d-1$. □

3.2.2 Reconstructing Unramified Sheaves from $\dim \le 1$-Data

In [50, Sec. 2.1] Morel introduced the notion of unramified presheaves as a technical device, since they can be reconstructed by significantly less data, than one put in. Specifically, it is sufficient to define unramified (pre-)sheaves on fields and discrete valuation rings. Since we will use this to some extent, we provide some insight to how these reductions are derived.

As noted above, any $F \in \mathsf{F}_k$ may be realized as the function field of an irreducible smooth scheme. For any such $F \in \mathsf{F}_k$, a *geometric discrete valuation ring* will be a discrete valuation ring A, which is isomorphic to $\mathcal{O}_{X,x}$ for some irreducible $X \in \mathsf{Sm}_k$ and $x \in X^{(1)}$, with $k(X) \cong F$. If k were perfect, the residue field of such an A would lie again in F_k.

Proposition 3.23. *Let $\mathcal{F}$ be an unramified sheaf. Then $\mathcal{F}$ gives rise to the following data:*

(D1) *A functor $\check{\mathcal{F}} \colon \mathsf{F}_k \to \mathsf{Set}$;*

(D2) *for any geometric discrete valuation ring A, a subset $\check{\mathcal{F}}(A) \subseteq \check{\mathcal{F}}(\mathrm{Frac}(A))$; and*

(D3) *for any geometric discrete valuation ring A, with separable (over k) residue field κ, a specialization map*

$$s_A \colon \check{\mathcal{F}}(A) \longrightarrow \check{\mathcal{F}}(\kappa),$$

which fulfil the following properties:

(A1) *Let $\iota \colon E \to F$ be an extension in F_k, and let there be a geometric discrete valuation ring B, with fraction field F, residue field λ, and valuation w, such that w restricts to a valuation v on E, with geometric discrete valuation ring A, and residue field κ.*[5] *Then we have*

$$\check{\mathcal{F}}(\iota)\left(\check{\mathcal{F}}(A)\right) \subseteq \check{\mathcal{F}}(B).$$

If moreover F/E is separably algebraic, the ramification index of w/v is 1, and the extension of residue fields $\bar{\iota} \colon \kappa \to \lambda$ is an isomorphism, then

$$\begin{array}{ccc} \check{\mathcal{F}}(A) & \xrightarrow{\check{\mathcal{F}}(\iota)\restriction_{\check{\mathcal{F}}(A)}} & \check{\mathcal{F}}(B) \\ \cap & & \cap \\ \check{\mathcal{F}}(E) & \xrightarrow[\check{\mathcal{F}}(\iota)]{} & \check{\mathcal{F}}(F) \end{array} \tag{3.6}$$

is cartesian.

[5] This is the *situation of **(A1)*** in the following.

(A2) *Let $X \in \mathsf{Sm}_k$ be irreducible, and let $f \in \check{\mathcal{F}}(k(X))$. Then f lies in all but finitely many $\check{\mathcal{F}}(\mathcal{O}_{X,x})$, for $x \in X^{(1)}$.*

(A3) ***(i)*** *In the situation of **(A1)** with κ and λ separable, we have the following commuting diagram:*

$$\begin{array}{ccc} \check{\mathcal{F}}(A) & \xrightarrow{\check{\mathcal{F}}(\iota)_{\restriction \check{\mathcal{F}}(A)}} & \check{\mathcal{F}}(B) \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ \check{\mathcal{F}}(\kappa) & \xrightarrow[\check{\mathcal{F}}(\bar{\iota})]{} & \check{\mathcal{F}}(\lambda). \end{array}$$

(ii) *In the situation of **(A1)**, with the added assumptions that the valuation v is trivial and λ is separable, we have an induced extension of residue fields $\bar{\iota}\colon E \to \lambda$, and the following diagram commutes:*

$$\begin{array}{ccc} \check{\mathcal{F}}(E) & \xrightarrow{\check{\mathcal{F}}(\iota)} & \check{\mathcal{F}}(B) \\ & \searrow{\scriptstyle \check{\mathcal{F}}(\bar{\iota})} & \downarrow{\scriptstyle s_B} \\ & & \check{\mathcal{F}}(\lambda) \end{array}$$

Proof. **(D1)** As already suggested by the notation we set $\check{\mathcal{F}}$ to be the restriction of the essentially smooth extension of $\mathcal{F}$, where we suppress the functor Spec.

(D2) Let $F \in \mathsf{F}_k$ be given, and let $A := \mathcal{O}_{X,x}$ be a geometric discrete valuation ring on F, with $X \in \mathsf{Sm}_k$ irreducible, and $x \in X^{(1)}$. From the construction of the essentially smooth extension, we have a map

$$\check{\mathcal{F}}(A) := \operatorname*{colim}_{\substack{x \in U \subseteq X \\ U \text{ open}}} \mathcal{F}(U) \longrightarrow \operatorname*{colim}_{\substack{\varnothing \neq U \subseteq X \\ U \text{ open}}} = \check{\mathcal{F}}(F).$$

Axiom **(1)** implies that this map is injective: Indeed, let $[U_1, f_1], [U_2, f_2]$ be elements in $\check{\mathcal{F}}(A)$, given by open neighbourhoods $U_1, U_2 \subseteq X$ of x, and sections $f_1 \in \mathcal{F}(U_1)$ resp. $f_2 \in \mathcal{F}(U_2)$, such that these agree in $\check{\mathcal{F}}(F)$. Hence there is some $\varnothing \neq V \subseteq U_1 \cap U_2$ open, such that

$$f_{1 \restriction V} = f_{2 \restriction V}$$

holds. This implies, since $U_1 \cap U_2$ is irreducible and $\mathcal{F}(U_1 \cap U_2) \to \mathcal{F}(V)$ injective, that $f_{1 \restriction U_1 \cap U_2} = f_{2 \restriction U_1 \cap U_2}$ holds, but that implies that $[U_1, f_1] = [U_2, f_2]$ holds in $\check{\mathcal{F}}(A)$. So we have found a subset $\check{\mathcal{F}}(A) \subseteq \check{\mathcal{F}}(F)$ up to the choice of a witnessing k-scheme X.

(D3) The specialization map can again be derived from the essentially smooth extension.

(A1) The commutativity of the diagram follows from the functoriality of the essentially smooth extension, and the corresponding diagram of essentially smooth schemes. So we need to address the "moreover"-part. Assume that we are given affine irreducible smooth schemes $X = \operatorname{Spec}(B')$ and $Y = \operatorname{Spec}(A')$ of finite type over k such that $A = A'_{\mathfrak{p}}$ and $B = B'_{\mathfrak{q}}$ hold, for suitable prime ideals $\mathfrak{p} \subseteq A'$ and $\mathfrak{q} \subseteq B'$. By restricting X to a principal open subscheme if necessary, the extension of valued fields E/F induces an étale homomorphism $\varphi\colon A' \to B'$ such that $\mathfrak{q}$ lies over $\mathfrak{p}$, $\varphi(\mathfrak{p})B' = \mathfrak{q}$, and the induced homomorphism of residue fields $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism. We denote the corresponding morphism of schemes by $\phi\colon X \to Y$. It remains to check that (3.6) is cartesian. Since $\check{\mathcal{F}}(A) \hookrightarrow \check{\mathcal{F}}(E)$ is injective, we only have to show that given elements $[U_E, f_E] \in \check{\mathcal{F}}(E)$ and $[U_B, f_B] \in \check{\mathcal{F}}(B)$, with $\mathfrak{q} \in U_B \subseteq X$, $\varnothing \neq U_E \subseteq Y$ and $f_B \in \mathcal{F}(U_B)$, $f_E \in \mathcal{F}(U_E)$, agreeing in $\check{\mathcal{F}}(F)$, that there is an element in $\check{\mathcal{F}}(A)$ mapping to both. If U_E were to contain $\mathfrak{p}$, we already would have $[U_E, f_E] \in \check{\mathcal{F}}(A)$, so we may assume that $\mathfrak{p}$ does not lie in U_E. We intend to use that $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology, and thus we want to modify the cartesian square

$$\begin{array}{ccc} \phi^{-1}(U_E) \cap U_B & \longrightarrow & U_E \\ \downarrow & & \downarrow \\ U_B & \xrightarrow{\phi_{\restriction U_B}} & \phi(U_B) \cup U_E \end{array}$$

into an elementary distinguished one (see definition 2.3). As tools in such a construction, one may use axiom **(2)** to extend f_E and enlarge U_E, or restrict U_B to a possibly smaller neighbourhood of $\mathfrak{q}$ in X. To find that the resulting diagram is completely decomposed, one could translate the isomorphism on residue fields to a birational map $V(\mathfrak{q}) \dashrightarrow V(\mathfrak{p})$. The precise details are left to the reader.

(A2) This follows with the usual argument: Any $f \in \check{\mathcal{F}}(k(X))$ is given as an equivalence class $[U, f']$, with $U \subseteq X$ open and nonempty and $f' \in \mathcal{F}(U)$. Thus f will already lie in all $\check{\mathcal{F}}(\mathcal{O}_{X,x})$, with $x \in U \cap X^{(1)}$, since $[U, f']$ appears in that set as well. We conclude because $X \smallsetminus U$ contains only finitely many points of $X^{(1)}$, as X is noetherian.

(A3) The diagrams under this axiom are implied by the functoriality of the essentially smooth extension, and corresponding commutative diagrams of essentially smooth schemes.

Finally we remark, since we will refer to this later, that everything, except the cartesianess of (3.6), can already be deduced for a weakly unramified presheaf $\mathcal{F}$. □

Remark 3.24. In [50, p. 18] it is shown, that one can derive an extra axiom **(A4)**. Moreover, the condition that $\check{\mathcal{F}}(\iota)$ maps $\check{\mathcal{F}}(A)$ to $\check{\mathcal{F}}(B)$ is split over **(A1)** and **(A3)(i)**, to obtain the intermediary notion of $\tilde{F}_k$-data. Here we adapted Morel's axioms to our specific needs.

Under the assumption that k is a perfect field, Morel proved in [50, Thm. 2.11] that to each F_k-datum, consisting of **(D1)**-**(D3)**, and fulfilling **(A1)**-**(A4)**, one can assign an unramified sheaf, and even stronger that the full subcategory of unramified sheaves (of the category of all sheaves) is equivalent to a category of F_k-data.

The notion of an F_k-datum can also be extended to pointed sets (resp. abelian groups), so that the above characterisation of unramified sheaves of pointed sets (resp. abelian groups) still holds (cf. [50, Rem. 2.14]).

3.3 Morphisms of Unramified Sheaves

Above we recalled a statement of Morel about reconstructing unramified sheaves from data on fields and discrete valuation rings. In order to use this in the context of proposition 2.58, we need to analyse how morphisms behave with respect to this construction.

3.3.1 Morphisms of Sheaves in Terms of Morphisms of Data

In [50, Rem. 2.15] Morel left the proof of the following proposition to the reader. Since we intend to use this statement, we give a proof here:

Proposition 3.25. *Let $\mathcal{E}$ be a weakly unramified presheaf, and $\mathcal{F}$ an unramified presheaf. Any morphism of presheaves $\varphi\colon \mathcal{E} \to \mathcal{F}$ is uniquely determined by a natural transformation*

$$\phi\colon \check{\mathcal{E}}_{\restriction F_k} \to \check{\mathcal{F}}_{\restriction F_k} \tag{MD}$$

such that the following two conditions are met, in which A denotes a geometric discrete valuation ring, with fraction field F and residue field κ:

(MA1) *ϕ_F maps $\check{\mathcal{E}}(A)$ to $\check{\mathcal{F}}(A)$.*

(MA2) *If κ is separable, the following square commutes:*

$$\begin{array}{ccc} \check{\mathcal{E}}(A) & \xrightarrow{(\phi_F)_{\restriction \check{\mathcal{E}}(A)}} & \check{\mathcal{F}}(A) \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_A} \\ \check{\mathcal{E}}(\kappa) & \xrightarrow[\phi_\kappa]{} & \check{\mathcal{F}}(\kappa). \end{array}$$

Any such ϕ will be called a ***morphism of F_k-data.***

Proof. As suggested by the notation, the correspondence that sends a morphism of presheaves $\varphi\colon \mathcal{E} \to \mathcal{F}$ to a morphism of F_k-data, is given as a restriction of the morphism $\check{\varphi}\colon \check{\mathcal{E}} \to \check{\mathcal{F}}$ of the essentially smooth extension. We have to check that the axioms **(MA1)** and **(MA2)** hold. From the proof of proposition 3.23 we know that $\check{\mathcal{E}}(A) \subseteq \check{\mathcal{E}}(\mathrm{Frac}(A))$ holds for every geometric discrete valuation ring A. Since $\check{\varphi}$ is a natural transformation, we have the commuting diagram

$$\begin{array}{ccc} \check{\mathcal{E}}(A) & \xrightarrow{\check{\varphi}_{\mathrm{Spec}(A)}} & \check{\mathcal{F}}(A) \\ \cap & & \cap \\ \check{\mathcal{E}}(F) & \xrightarrow{\check{\varphi}_{\mathrm{Spec}(F)}} & \check{\mathcal{F}}(F), \end{array}$$

implying **(MA1)**. The remaining axiom follows by a similar argument, which we omit.

In order to prove the claim, we need to check that this correspondence is a bijection. Therefore, let there be morphisms of presheaves $\varphi_1, \varphi_2\colon \mathcal{E} \to \mathcal{F}$ such that the corresponding morphisms of F_k-data agree, and let us denote this morphism of F_k-data by ϕ. We need to show that $\varphi_{1,X} = \varphi_{2,X}$ holds for all $X \in \mathsf{Sm}_k$. By axiom **(0)**, we may assume that X is irreducible. Note that by construction of the essentially smooth extension, we have the commuting diagram (with $i = 1, 2$)

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{\varphi_{i,X}} & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ \check{\mathcal{E}}(k(X)) & \xrightarrow{\phi_{k(X)}} & \check{\mathcal{F}}(k(X)). \end{array}$$

We can conclude that the vertical maps in this diagram are injective, by employing an argument (and thereby using **(1)**), similar to the one we used to obtain **(D2)**. This yields $\varphi_1 = \varphi_2$.

So let there be a morphism of F_k-data $\phi\colon \check{\mathcal{E}}_{\restriction \mathsf{F}_k} \to \check{\mathcal{F}}_{\restriction \mathsf{F}_k}$. We construct a morphism of presheaves $\varphi\colon \mathcal{E} \to \mathcal{F}$ that induces ϕ. Hence fix $X \in \mathsf{Sm}_k$. By axiom **(0)**, one may assume that X is irreducible. We note that by **(MA1)** the following diagram commutes:

$$\begin{array}{ccc} \bigcap_{x \in X^{(1)}} \check{\mathcal{E}}(\mathcal{O}_{X,x}) & \longrightarrow & \bigcap_{x \in X^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{X,x}) \\ \cap & & \cap \\ \check{\mathcal{E}}(k(X)) & \xrightarrow{\phi_{k(X)}} & \check{\mathcal{F}}(k(X)), \end{array}$$

i.e. the upper horizontal map is a restriction of the lower one. By using (2') we may define a map φ_X such that

$$\begin{array}{ccc} \mathcal{E}(X) & \overset{\varphi_X}{\dashrightarrow} & \mathcal{F}(X) \\ \downarrow & & \downarrow{\scriptstyle \cong} \\ \bigcap_{x \in X^{(1)}} \check{\mathcal{E}}(\mathcal{O}_{X,x}) & \longrightarrow & \bigcap_{x \in X^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{X,x}) \end{array}$$

commutes. The hard part is to check the naturality of this definition. In a first step, we restrict ourselves to the wide subcategory $\tilde{\mathsf{Sm}}_k$ with only flat morphisms. Given a flat morphism $f\colon X \to Y$ between irreducible (use **(0)**) schemes $X, Y \in \mathsf{Sm}_k$, we know by the going-down property, that $f(x)$, for $x \in X^{(1)}$, is either of codimension 1 or the generic point of Y. Hence we see that

$$\begin{array}{ccccc} \mathcal{F}(Y) & = & \bigcap_{y\in Y^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{Y,y}) & \hookrightarrow & \check{\mathcal{F}}(k(Y)) \\ \downarrow{\scriptstyle \mathcal{F}(f)} & & \downarrow & & \downarrow{\scriptstyle \phi_{k(X)/k(Y)}} \\ \mathcal{F}(X) & = & \bigcap_{x\in X^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{X,x}) & \hookrightarrow & \check{\mathcal{F}}(k(X)). \end{array}$$

commutes. Here we have used the first part of **(A1)** to obtain the dashed map, and (2') to obtain the equalities on the left-hand side. Since we have only used the first part of **(A1)**, we see that there is a similar diagram for $\mathcal{E}$, when we replace the equalities by injective maps. This latter injectivity is a consequence of **(1)**. Thus we see that axiom **(MA1)** implies commutativity of the following diagram.

$$\begin{array}{ccccccc} \mathcal{E}(Y) & \hookrightarrow & \bigcap_{y\in Y^{(1)}} \check{\mathcal{E}}(\mathcal{O}_{Y,y}) & \longrightarrow & \bigcap_{y\in Y^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{Y,y}) & = & \mathcal{F}(Y) \\ \downarrow{\scriptstyle \mathcal{E}(f)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathcal{E}(X) & \hookrightarrow & \bigcap_{x\in X^{(1)}} \check{\mathcal{E}}(\mathcal{O}_{X,x}) & \longrightarrow & \bigcap_{x\in X^{(1)}} \check{\mathcal{F}}(\mathcal{O}_{X,x}) & = & \mathcal{F}(X). \end{array}$$

This yields the naturality of φ on $\tilde{\mathsf{Sm}}_k$.

So we proceed to the case of Sm_k. Let $f\colon X \to Y$ be a morphism of irreducible (use **(0)**) k-schemes. First we note that f may be factored as $p \circ \iota$, where $\iota\colon X \to Z$ is a closed immersion, $p\colon Z \to Y$ is flat, and Z is an irreducible smooth k-scheme. To obtain such a decomposition, one factors f via its graph morphism, i.e. as

$$X \xrightarrow{\Gamma_f} X \times_k Y \xrightarrow{p_2} Y.$$

Γ_f is always an immersion (cf. [73, Tag 01KJ]), so it may be factored as $j \circ \iota'$, with $\iota'\colon X \to Z'$ being a closed immersion, and $j\colon Z' \to X \times_k Y$ being an open immersion. Denote by Z the irreducible component of Z', in which the image of X lies in. Then we find $\iota\colon X \to Z$, a closed immersion, and setting $p := p_2 \circ j_{\restriction Z}$ does the trick, with p being even smooth.

Next we use a trick of Morel (cf. [50, pp. 19–21]), to handle the case of showing naturality of φ with respect to a closed immersion $\iota\colon X \to Z$ of irreducible smooth schemes. Recall firstly, that a closed immersion between smooth k-schemes is a regular immersion (cf. [73, Tag 0E9J]), which implies that (Zariski-)locally it can be factored as a composition of codimension 1 closed immersions. This comes from the fact that for the associated quasi-coherent ideal, (Zariski-)locally regular sequences exist (cf. [73, Tag 063D]). However, [26, Cor. (17.12.2)] provides an even stronger assertion: We find a cover $(U_i)_{i\in I}$, consisting of open subsets of Z, with the property $\iota(X) \subseteq \bigcup_{i\in I} U_i$, and such that for all $i \in I$ we have a cartesian diagram

$$\begin{array}{ccc} \iota^{-1}(U_i) & \xrightarrow{\iota} & U_i \\ \downarrow & & \downarrow{\scriptstyle \xi_i} \\ \mathbb{A}_k^{n-d} & \longrightarrow & \mathbb{A}_k^n, \end{array}$$

where d is the codimension of X in Z, ξ_i is étale, and $\mathbb{A}_k^{n-d} \to \mathbb{A}_k^n$ is the canonical closed immersion of the first few coordinates. Fixing an index $i_0 \in I$, we may find, by pulling the canonical closed immersions $\mathbb{A}_k^{n-d} \to \mathbb{A}_k^{n-d+1} \to \cdots \to \mathbb{A}_k^n$ back along ξ_{i_0}, a flag

$$\underbrace{Z_0}_{\iota^{-1}(U_{i_0})=} \xrightarrow{\iota_1} Z_1 \xrightarrow{\iota_2} Z_2 \xrightarrow{\iota_3} \dots \xrightarrow{\iota_d} \underbrace{Z_d}_{U_{i_0}=},$$

where the ι_j are closed immersions of codimension 1, and the Z_j are smooth over k. So we analyse the case of such a closed immersion of codimension 1 first. Therefore let $j \in \{1, \dots, d\}$ be fixed, and assume moreover that Z_{j-1} and Z_j are irreducible (by usage of **(0)**). If η_{j-1} denotes the generic point of Z_{j-1} in Z_j, we find the following commuting diagram of essentially smooth schemes:

$$\begin{array}{ccc} \operatorname{Spec}(\kappa(\eta_{j-1})) & \longrightarrow & Z_{j-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota_j} \\ \operatorname{Spec}(\mathcal{O}_{Z_j,\eta_{j-1}}) & \longrightarrow & Z_j. \end{array}$$

Using axiom **(1)** and the definition of the specialization map, this induces the commuting diagram:

$$\begin{array}{ccc} \mathcal{E}(Z_j) & \subseteq & \check{\mathcal{E}}(\mathcal{O}_{Z_j,\eta_{j-1}}) \\ {\scriptstyle \mathcal{E}(\iota_j)}\big\downarrow & & \big\downarrow{\scriptstyle s_{\eta_{j-1}}} \\ \mathcal{E}(Z_{j-1}) & \subseteq & \check{\mathcal{E}}(\kappa(\eta_{j-1})). \end{array}$$

From this we deduce that the restriction map that is induced by ι_j, is just a particular restriction of $s_{\eta_{j-1}}$. The naturality of φ in the case of ι_j, follows from the following commuting diagram, where the inner square commutes by axiom **(MA2)**, the inclusions are consequences of **(1)**, and the outer commuting parts follow from commuting diagrams of essentially smooth schemes:

$$\begin{array}{ccccccc} & & \check{\mathcal{E}}(k(Z_j)) & \xrightarrow{\phi_{k(Z_j)}} & \check{\mathcal{F}}(k(Z_j)) & & \\ & \nearrow & \cup & & \cup & \nwarrow & \\ \mathcal{E}(Z_j) & \subseteq & \check{\mathcal{E}}(\mathcal{O}_{Z_j,\eta_{j-1}}) & \longrightarrow & \check{\mathcal{F}}(\mathcal{O}_{Z_j,\eta_{j-1}}) & \supseteq & \mathcal{F}(Z_j) \\ \big\downarrow{\scriptstyle \mathcal{E}(\iota_j)} & & \big\downarrow{\scriptstyle s_{\eta_{j-1}}} & & \big\downarrow{\scriptstyle s_{\eta_{j-1}}} & & \big\downarrow{\scriptstyle \mathcal{F}(\iota_j)} \\ \mathcal{E}(Z_{j-1}) & \subseteq & \check{\mathcal{E}}(\kappa(\eta_{j-1})) & \xrightarrow{\phi_{\kappa(\eta_{j-1})}} & \check{\mathcal{F}}(\kappa(\eta_{j-1})) & \supseteq & \mathcal{F}(Z_{j-1}). \end{array}$$

By composing these results, we obtain the naturality of φ for $\iota_{\restriction \iota^{-1}(U_{i_0})}$. Since $i_0 \in I$ was chosen arbitrarily, we may conclude the naturality of φ with respect to ι from the property that $\mathcal{F}$ is a sheaf on the Zariski site (cf. **(2)**). So we have shown that φ is indeed a morphism of presheaves on Sm_k. The fact that one obtains ϕ back by restricting the essentially smooth extension of φ, follows by construction. □

Remark 3.26. Analogously to the proposition, one can give a characterization of the homomorphisms between a weakly unramified presheaf of abelian groups $\mathcal{E}$, and some unramified sheaf of abelian groups $\mathcal{F}$, in terms of natural homomorphisms

$$\check{\mathcal{E}}_{\restriction \mathsf{F}_k} \longrightarrow \check{\mathcal{F}}_{\restriction \mathsf{F}_k},$$

and axioms **(MA1)**, **(MA2)**.

We record the following corollary, of which we later give an amplification.

Corollary 3.27. *Let G be a smooth algebraic group over k, fulfilling the strong Grothendieck-Serre property, and let $\mathcal{F}$ be an unramified sheaf. Any morphism of sheaves $\mathcal{H}^1_{\text{ét}}(G) \to \mathcal{F}$ is uniquely determined by a natural transformation*

$$\phi\colon H^1_{\text{ét}}(\operatorname{Spec}(-), G) \longrightarrow \check{\mathcal{F}}_{\restriction \mathsf{F}_k}$$

such that the following two conditions are met, in which A denotes a geometric discrete valuation ring, with fraction field F and residue field κ:

- $\phi_F(H^1_{\text{ét}}(\operatorname{Spec}(A), G)) \subseteq \check{\mathcal{F}}(A)$.
- *If κ is separable over k, one has the following commuting diagram:*

$$\begin{array}{ccc} H^1_{\text{ét}}(\operatorname{Spec}(A), G) & \xrightarrow{\phi_{F\restriction H^1_{\text{ét}}(\operatorname{Spec}(A),G)}} & \check{\mathcal{F}}(A) \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_A} \\ H^1_{\text{ét}}(\operatorname{Spec}(\kappa), G) & \xrightarrow[\phi_\kappa]{} & \check{\mathcal{F}}(\kappa). \end{array}$$

Proof. By proposition 3.20 we may apply the above proposition 3.25. Since $\mathcal{H}^1_{\text{ét}}(G)$ is a sheaf in the Nisnevich topology, and fields are points on the Nisnevich site, we have

$$\mathcal{H}^1_{\text{ét}}(G)(F) = H^1_{\text{ét}}(\operatorname{Spec}(F), G)$$

for every field $F \in \mathsf{F}_k$. Moreover, one has $H^1_{\text{ét}}(\operatorname{Spec}(A), G) \subseteq H^1_{\text{ét}}(\operatorname{Spec}(F), G)$ by the strong Grothendieck-Serre property, for every geometric discrete valuation ring A, with fraction field F. The essential simplifications of the axioms **(MA1)** and **(MA2)** now stem from the fact that

$$H^1_{\text{ét}}(\operatorname{Spec}(A), G) \twoheadrightarrow \mathcal{H}^1_{\text{ét}}(G)(A)$$

is surjective, which follows from [17, Prop. 3.3] once we check that for every Nisnevich distinguished square

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \iota} \\ Z & \xrightarrow[p]{} & X, \end{array}$$

where ι is an open immersion, p is étale, the diagram is cartesian, and $(Z \setminus W)_{\text{red}} \xrightarrow{(p_{\restriction Z\setminus W})_{\text{red}}} (X \setminus U)_{\text{red}}$ is an isomorphism, we have that

$$H^1_{\text{ét}}(X, G) \longrightarrow H^1_{\text{ét}}(Z, G) \times_{H^1_{\text{ét}}(W,G)} H^1_{\text{ét}}(U, G)$$

is surjective. This will follow from descent with respect to the étale (or Nisnevich) topology: Indeed, pick $T_Z \to Z$ and $T_U \to U$ étale G-torsors that agree on W. Since G is affine, these morphisms are affine, and using that they are isomorphic on W, we obtain a W-isomorphism

$$\varphi \colon \underbrace{T_Z \times_X U}_{T_Z \times_Z U \cong} \longrightarrow \underbrace{Z \times_X T_U}_{W \times_U T_U \cong}.$$

Thus (T_U, T_V, φ) forms a descent datum with respect to the covering $\{p, \iota\}$ in the sense of [73, Tag 023W], which is effective by [73, Tag 0245]. So we obtain an affine morphism $T \to X$, and we omit the verification that it is indeed a G-torsor in the étale topology. ☐

With the conclusion of this subsection we will stop writing the check to signalise the transition from an unramified sheaf to an unramified F_k-datum.

3.3.2 A Compatibility Statement

In this subsection we show that the compatibility conditions **(MA1)** and **(MA2)** are already fulfilled, in the case that the residue field of the geometric discrete valuation ring is separable. So, if the ground field k is perfect, which is the case that we consider in chapter 4, these two conditions are obsolete.

Proposition 3.28. *Let $X \in \mathsf{Sm}_k$ be irreducible, with function field $K := k(X)$, let N be a weakly unramified presheaf and M be an unramified sheaf, and let $a\colon N_{\restriction \mathsf{F}_k} \to M_{\restriction \mathsf{F}_k}$ be a natural transformation. Fix an element $T \in N(X)$, then we have for every point $x \in X^{(1)}$ with separable $\kappa(x)/k$:*

$$a_K(T_K) \in M(\mathcal{O}_{X,x}) \quad \textit{and} \quad s_x(a_K(T_K)) = a_{\kappa(x)}(T_{\kappa(x)}).$$

Proof. We follow the proof [23, Thm. 3.2.1] closely, however we reproduce it here for the convenience of the reader. Set $A := \mathcal{O}_{X,x}$, κ to be its residue field, and $q\colon A \to \kappa$ the canonical projection. In the following, we will denote the restriction of T to some essentially smooth affine scheme by an index, e.g. T_A or T_κ.

We need to work with the henselization A^h of A. By the theory in [63, Ch. VIII] there is a set Λ (consisting of isomorphism classes of local-étale A-algebras with residue field κ) that is partially ordered and filtered, together with an inductive system of local-étale A-algebras $(A_\lambda)_{\lambda\in\Lambda}$ with residue field κ and local transition homomorphisms,[6] such that

$$\operatorname*{colim}_{\lambda\in\Lambda} A_\lambda = A^h$$

holds. Since A is a (geometric) discrete valuation ring (with valuation v), we know by [73, Tag 0AH0] and [73, Tag 00TV] that each A_λ is a (geometric) discrete valuation ring as well (with valuation v_λ). So, the involved rings are all essentially smooth, and corollary 3.13 is applicable. Let us fix some abbreviations and names for homomorphisms via the following diagram:

$$\begin{array}{ccccc}
\overbrace{K}^{Q(A)=:} & \xrightarrow{\varphi'_\lambda} & \overbrace{K_\lambda}^{Q(A_\lambda)=:} & \xrightarrow{\psi'_\lambda} & \overbrace{K^h}^{Q(A^h)=:} \\
\eta\uparrow & & \eta_\lambda\uparrow & & \eta^h\uparrow \\
A & \xrightarrow{\varphi_\lambda} & A_\lambda & \xrightarrow{\psi_\lambda} & A^h \\
& \searrow^{q} & \downarrow q_\lambda & \swarrow^{q^h} & \\
& & \kappa & &
\end{array}$$

Since K^h/K is separable algebraic (e.g. by [73, Tag 07QQ]), φ_λ is unramified at the maximal ideal, and since the residue fields of A and A_λ are isomorphic, we know by **(A1)** that the following diagram is cartesian, for all $\lambda \in \Lambda$:

$$\begin{array}{ccc}
M(A) & \xhookrightarrow{(\eta)_M} & M(K) \\
(\varphi_\lambda)_M \downarrow & & \downarrow (\varphi'_\lambda)_M \\
M(A_\lambda) & \xhookrightarrow[(\eta_\lambda)_M]{} & M(K_\lambda).
\end{array}$$

So, since a is natural, we have the commutative diagram

$$\begin{array}{ccc}
N(K) & \xrightarrow{a_K} & M(K) \\
(\varphi'_\lambda)_N \downarrow & & \downarrow (\varphi'_\lambda)_M \\
N(K_\lambda) & \xrightarrow[a_{K_\lambda}]{} & M(K_\lambda),
\end{array}$$

and it suffices to find a single $\lambda \in \Lambda$ such that $a_{K_\lambda}(T_{K_\lambda})$ lies in $M(A_\lambda)$.

Before we begin to find such a λ, we want to obtain a splitting $j^h\colon \kappa \to A^h$ to the projection $q^h\colon A^h \twoheadrightarrow \kappa$. By [43, Thm. 28.3] we obtain a splitting $\hat{j}\colon \kappa \to \hat{A}$ for the completion $\hat{q}\colon \hat{A} \twoheadrightarrow \kappa$,

[6] *Local-étale* A-algebras are localizations of étale A-algebras at a prime ideal over the maximal ideal of A.

which, since κ is separable over k, can even be chosen to be a k-homomorphism. We intend to lift this to a splitting for A^h by use of the *approximation property* for excellent henselian discrete valuation rings [10, Cor. 9 in 3.6], with which we can obtain a dashed morphism in the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\hat{A}) & \xleftarrow{\hat{q}^*} & \operatorname{Spec}(\kappa) \\ {\scriptstyle \hat{j}}\downarrow & & \downarrow{\scriptstyle (q^h)^*} \\ \operatorname{Spec}(\kappa) & \overset{(j^h)^*}{\dashleftarrow} & \operatorname{Spec}(A^h) \\ {\scriptstyle (q^h)^*}\downarrow & \swarrow{\scriptstyle =} & \\ \operatorname{Spec}(A^h). & & \end{array}$$

Since A^h is excellent, if A is excellent [26, Cor. (18.7.6)], the excellence of A remains to be checked. But this can be concluded, as A is the localization of a finite type k-algebra (cf. [73] Tags 07QU and 07QW). Thus we obtained a k-section $j^h\colon \kappa \to A^h$ for $q^h\colon A^h \to \kappa$.

For all $\lambda \in \Lambda$ set κ_λ to be the preimage of $j^h(\kappa) \subseteq A^h$ under the homomorphism ψ_λ. The κ_λ are subfields of the discrete valuation rings A_λ: Indeed, since the transition morphisms $A_\lambda \to A_\mu$ are local, the ψ_λ are local homomorphisms as well. Hence κ_λ consists of units of A_λ and thus must be a subfield. As it is also a subfield of κ, it even lies in F_k. Moreover, we have

$$\operatorname*{colim}_{\lambda\in\Lambda} \kappa_\lambda = \kappa,$$

with induced canonical injections $\overline{\psi}_\lambda\colon \kappa_\lambda \hookrightarrow \kappa$. Since κ is finitely generated as a field and Λ is filtered, we have

$$\operatorname*{colim}_{\lambda\in\Lambda} N(\kappa_\lambda) \cong N(\kappa).$$

From this we see that we may find some $\lambda_0 \in \Lambda$ such that

$$\overline{\psi}_{\lambda_0 N}(T_{\kappa_{\lambda_0}}) = T_\kappa \tag{3.7}$$

holds in $N(\kappa)$. The commutative diagram

$$\begin{array}{ccc} \kappa_{\lambda_0} & \xhookrightarrow{j_{\lambda_0}} & A_{\lambda_0} \\ {\scriptstyle \overline{\psi}_{\lambda_0}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{\lambda_0}} \\ \kappa & \xrightarrow[j^h]{} & A^h \end{array}$$

implies

$$\psi_{\lambda_0 N}(j_{\lambda_0 N}(T_{\kappa_{\lambda_0}})) = j^h_N(\overline{\psi}_{\lambda_0 N}(T_{\kappa_{\lambda_0}})) = j^h_N(T_\kappa) = j^h_N(q^h_N(T_{A^h})) = T_{A^h} = \psi_N(T_A) = \psi_{\lambda_0 N}(T_{A_{\lambda_0}}).$$

Again, since Λ is filtered we have $\operatorname{colim}_{\lambda\in\Lambda} N(A_\lambda) = N(A^h)$, by corollary 3.13, and thus we may find some $\lambda_1 \in \Lambda$, with $\lambda_0 \leq \lambda_1$ such that

$$j_{\lambda_1 N}(T_{\kappa_{\lambda_1}}) = T_{A_{\lambda_1}} \tag{3.8}$$

holds on the nose. We will now check that λ_1 is the element in Λ we were looking for. Note first, that the following diagram is commuting, since a is natural and the participating fields lie in F_k:

$$\begin{array}{ccc} N(\kappa_{\lambda_1}) & \xrightarrow{a_{\kappa_{\lambda_1}}} & M(\kappa_{\lambda_1}) \\ {\scriptstyle (\eta_{\lambda_1}\circ j_{\lambda_1})_N}\big\downarrow & & \big\downarrow{\scriptstyle (\eta_{\lambda_1}\circ j_{\lambda_1})_M} \\ N(K_{\lambda_1}) & \xrightarrow[a_{K_{\lambda_1}}]{} & M(K_{\lambda_1}) \end{array}$$

So we may deduce

$$a_{K_{\lambda_1}}(T_{K_{\lambda_1}}) = a_{K_{\lambda_1}}(\eta_{\lambda_1 N}(T_{A_{\lambda_1}})) = a_{K_{\lambda_1}}((\eta_{\lambda_1} \circ j_{\lambda_1})_N(T_{\kappa_{\lambda_1}})) = (\eta_{\lambda_1} \circ j_{\lambda_1})_M(a_{\kappa_{\lambda_1}}(T_{\kappa_{\lambda_1}})),$$

by using the definition of $T_{K_{\lambda_1}}$, (3.8) and the fact that a is natural. By the above reduction using axiom **(A1)** $a_K(T_K)$ lies in $M(A)$. This completes the proof of the first part.

Since s_v is a map $M(A) \to M(\kappa)$, we know by the above that the expression $s_v(a_K(T_K))$ is well-defined and lies in $M(\kappa)$. Note that we have dubbed the valuation corresponding to the point x in codimension one v. The outstanding claim follows by the following small calculation:

$$\begin{aligned}
s_v(a_K(T_K)) &= s_{v_{\lambda_1}}((\varphi'_{\lambda_1})_M(a_K(T_K))) && \textbf{(A3)(i)}\\
&= s_{v_{\lambda_1}}(a_{K_{\lambda_1}}(\varphi'_{\lambda_1 N}(T_K))) && a \text{ is natural}\\
&= s_{v_{\lambda_1}}(a_{K_{\lambda_1}}(\varphi'_{\lambda_1 N}(\eta_N(T_A)))) && \\
&= s_{v_{\lambda_1}}(a_{K_{\lambda_1}}(\eta_{\lambda_1 N}(\varphi_{\lambda_1 N}(T_A)))) && \eta_\lambda \circ \varphi_\lambda = \varphi'_\lambda \circ \eta\\
&= s_{v_{\lambda_1}}(a_{K_{\lambda_1}}(\eta_{\lambda_1 N}(T_{A_{\lambda_1}}))) && \\
&= s_{v_{\lambda_1}}(a_{K_{\lambda_1}}((\eta_{\lambda_1} \circ j_{\lambda_1})_N(T_{\kappa_{\lambda_1}}))) && (3.8)\\
&= s_{v_{\lambda_1}}((\eta_{\lambda_1} \circ j_{\lambda_1})_M(a_{\kappa_{\lambda_1}}(T_{\kappa_{\lambda_1}}))) && a \text{ is natural}\\
&= (\overline{\psi}_{\lambda_1})_M(a_{\kappa_{\lambda_1}}(T_{\kappa_{\lambda_1}})) && \textbf{(A3)(ii)}\\
&= a_\kappa(\overline{\psi}_{\lambda_1 N}(T_{\kappa_{\lambda_1}})) && a \text{ is natural}\\
&= a_\kappa(T_\kappa) && (3.7).
\end{aligned}$$

This concludes the proof of the compatibility statement. □

3.3.3 Totaro's Geometric Description

There is the following recognition principle generalising [21, Thm. 12.3][7] and [23, Thm. 3.2.3].

Lemma 3.29. *Let G be a smooth linear algebraic group over k, and let P be a versal torsor (cf. [21, Def. 5.1]), defined over the field $K \in \mathsf{F}_k$. Let M be an umramified presheaf such that $M(F) \to M(F(t))$ is injective for all $F \in \mathsf{F}_k$ and t transcendental over F. Then we have*

$$\forall \varphi, \psi \in \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(G), M), \ \textit{with } \varphi([P]) = \psi([P]) \implies \varphi = \psi.$$

Proof. We adapt the arguments of the above cited sources, and start by remarking that by proposition 3.25 it is sufficient to check that (the essentially smooth extensions of) φ and ψ agree on F_k. So, let $F \in \mathsf{F}_k$ be given. We briefly check that our assumption on M allows us to assume that F is an infinite field. Indeed, for any variable t transcendental over F, and $\chi \in \{\varphi, \psi\}$ we have a commuting diagram

$$\begin{array}{ccccc}
H^1_{\text{ét}}(\mathrm{Spec}(F), G) & = & \mathcal{H}^1_{\text{ét}}(G)(F) & \xrightarrow{\chi} & M(F)\\
\downarrow & & \downarrow & & \downarrow\\
H^1_{\text{ét}}(\mathrm{Spec}(F(t)), G) & = & \mathcal{H}^1_{\text{ét}}(G)(F(t)) & \xrightarrow{\chi} & M(F(t)),
\end{array}$$

in which the right-hand vertical map is injective. Let us proceed by unpacking the definition of a versal G-torsor, which also gives a reason for reducing to the case of an infinite field: Namely, one finds a smooth, irreducible scheme X over k with function field K and a G-torsor $Q \to X$ such that the following axioms hold:

[7] Whenever we cite [21] like this, we refer to its first part due to Garibaldi and Serre. If we point to some statement from Merkurjev's part, we preface its number by an "M".

(Ve1) The generic fibre of $Q \to X$ is P.

(Ve2) For every extension field F of k with F infinite, every G-torsor T over F, and every nonempty open subscheme U of X, there exists an F-rational point $x \in U(F)$ whose fibre Q_x is isomorphic to T.

So, let us fix a G-torsor T over F, and by **(Ve2)** a corresponding F-rational point x in X, such that we have a cartesian diagram

$$\begin{array}{ccc} T & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \xrightarrow{x} & X. \end{array}$$

Since M is unramified, we have $M(X) \subseteq M(K)$. Using this and the assumption $\varphi([P]) = \psi([P])$, we obtain $\varphi([Q]) = \psi([Q])$, where we regard the G-torsor Q as an element of $\mathcal{H}^1_{\text{ét}}(G)(X)$. Now plugging $[Q]$ into the commuting diagram (with $\chi \in \{\varphi, \psi\}$)

$$\begin{array}{ccc} \mathcal{H}^1_{\text{ét}}(G)(X) & \xrightarrow{\chi} & M(X) \\ \downarrow & & \downarrow \\ \mathcal{H}^1_{\text{ét}}(G)(F) & \xrightarrow{\chi} & M(F) \end{array}$$

leads to $\varphi([T]) = \psi([T])$, and with the above reductions we arrive at the claim. □

We will now consider unramified sheaves M that are also $\mathbb{A}^1$-invariant. This entails that for all $X \in \mathsf{Sm}_k$ we have that the map induced by the zero section $M(\mathbb{A}^1_X) \to M(X)$ is a bijection. We record the following characterisation yielding in particular the hypothesis of lemma 3.29.

Proposition 3.30 ([50, Lem. 2.16])**.** *Let M be an unramified sheaf. The following statements are equivalent:*

(i) M is $\mathbb{A}^1$-invariant.

(ii) For all $F \in \mathsf{F}_k$, the canonical map $M(\mathrm{Spec}(F)) \to M(\mathbb{A}^1_F)$ is bijective.

From the above proposition we gather in particular that any $\mathbb{A}^1$-invariant unramified sheaf fulfils the hypothesis of lemma 3.29. We will use this in the following proposition which elaborates on a principle of Totaro in the setting of cohomological invariants (cf. [21, App. C]). We recall the proof as a convenience to the reader.

Proposition 3.31 (Totaro)**.** *Let M be an $\mathbb{A}^1$-invariant umramified sheaf, and let G be a smooth linear algebraic group over k. Let $G \to \mathbf{GL}(V)$ be a representation of G over k, such that V admits a G-invariant Zariski-open subset U, with $V \smallsetminus U$ of codimension ≥ 2, such that U is a G-torsor. Then we have that the following map is bijective:*

$$\phi\colon \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(G), M) \longrightarrow M(U/G)$$
$$\varphi \longmapsto \varphi([U \to U/G]).$$

If M is an abelian sheaf, then ϕ is also a homomorphism of abelian groups.

Proof. From [21, Ex. 5.4] we see that the generic fibre of $U \to U/G$ forms a versal torsor. Thus by lemma 3.29 we see that ϕ is injective. The fact that ϕ is a homomorphism of abelian groups if M is an abelian sheaf, can be deduced quickly, since the group structures on both sides will be induced by the one on M.

So, we check that ϕ is also surjective: Let $m \in M(U/G)$ be given arbitrarily, and our aim is to construct a morphism $\varphi' \colon H^1_{\text{ét}}(-,G) \to M$ of presheaves such that the induced morphism of sheaves $\varphi \colon \mathcal{H}^1_{\text{ét}}(G) \to M$ is mapped to M via ϕ. Therefore fix some $X \in \mathsf{Sm}_k$ and a G-torsor T over X. We consider the diagonal (right) action of G on $T \times_k U$ resp. $T \times_k V$

$$\begin{aligned} T \times_k U \times_k G &\longrightarrow T \times_k U \\ (t,u,g) &\longmapsto (t.g, g^{-1}.u). \end{aligned}$$

An argument using faithfully flat descent of quasi-affine morphisms (cf. [73, Tag 0247]) along $T \to X$ demonstrates the existence of the smooth k-schemes $(T \times_k U)/G$ and $(T \times_k V)/G$.

We first notice that $(T \times_k U)/G$ is an open subset of $(T \times_k V)/G$ that already contains all points that are of codimension 1. Thus we have a bijection

$$M(T \times_k V/G) \to M(T \times_k U/G),$$

by axiom **(2)**.

Moreover by the same axiom, M is a Zariski sheaf, and since the first projection induces the structure of a vector bundle $p_1 \colon (T \times_k V)/G \to T/G \cong X$, we have another bijection

$$M(X) \to M(T \times_k V/G)$$

by $\mathbb{A}^1$-invariance of M. Composing these results, we obtain a map

$$M(U/G) \xrightarrow{p_2} M(T \times_k U/G) \underset{\cong}{\leftarrow} M(T \times_k V/G) \underset{\cong}{\xleftarrow{p_1}} M(X),$$

where $p_2 \colon T \times_k U/G \to U/G$ denotes the morphism that is induced by the second projection. If we consider a morphism $X' \to X$, with $X' \in \mathsf{Sm}_k$, pull the G-torsor T back to X', and repeat the above construction, we see that this process is natural. To obtain a morphism of presheaves $\varphi' \colon H^1_{\text{ét}}(-,G) \to M$, we simply plug m in. Now, since M is unramified, fields are points in the Nisnevich topology, and U/G is an integral scheme, we may check that φ' induces a preimage of m along ϕ by checking that

$$M(U/G) \xrightarrow{p_2} M(U \times_k U/G) \underset{\cong}{\leftarrow} M(U \times_k V/G) \underset{\cong}{\xleftarrow{p_1}} M(U/G)$$

is the identity. This will follow, once we verify that the two projections

$$(U \times_k U)/G \underset{p_2}{\overset{p_1}{\rightrightarrows}} U/G$$

agree, after applying M, but this can be done as in the original [21, App. C]. □

Remark 3.32. A similar statement may be obtained by using an embedding of G into $\mathbf{SL}_n$ instead of a representation, when additionally assuming that $\mathcal{H}^1_{\text{ét}}(G)$ is weakly unramified: Note first that the generic fibre of $\mathbf{SL}_n \to \mathbf{SL}_n/G$ is a versal torsor (cf. [21, 5.3]), and thus by the recognition principle we have an injection:

$$\operatorname{Hom}_{\mathsf{Sh}_{\text{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(G), M) \hookrightarrow M(\mathbf{SL}_n/G).$$

We address surjectivity by using corollary 3.27. Therefore, fix a field $F \in \mathsf{F}_k$ and consider the exact sequence in étale cohomology:

$$\mathbf{SL}_n(F) \to (\mathbf{SL}_n/G)(F) \twoheadrightarrow H^1_{\text{ét}}(F,G) \to \underbrace{H^1_{\text{ét}}(F,\mathbf{SL}_n)}_{=*}.$$

By standard Galois cohomology theory (cf. [68, Cor. 1, Prop. 36]) we know that the elements in $H^1_{\text{ét}}(F,G)$ may be identified with the orbits in $(\mathbf{SL}_n/G)(F)$ by the action of $\mathbf{SL}_n(F)$. We would like to map a torsor T to

$$\operatorname{Spec}(F) \xrightarrow{x} \mathbf{SL}_n/G \to M,$$

where $x \in (\mathbf{SL}_n/G)(F)$ is an F-point corresponding to the cohomology class $[T] \in H^1_{\text{ét}}(F,G)$. To handle well-definedness, assume we are given $x_1, x_2 \in (\mathbf{SL}_n/G)(F)$ that are related by a matrix $A \in \mathbf{SL}_n(F)$. Since A may be written as a product of elementary matrices, one may write down a chain of naïve $\mathbb{A}^1$-homotopies relating

$$\operatorname{Spec}(F) \xrightarrow{x_1} \mathbf{SL}_n/G \to M \quad \text{and} \quad \operatorname{Spec}(F) \xrightarrow{x_2} \mathbf{SL}_n/G \to M.$$

Since M is $\mathbb{A}^1$-invariant these homotopies need to be constant, and thus well-definedness follows. To check the compatibility as demanded by corollary 3.27, we remark that the above argument still works, if we replace F by a geometric discrete valuation ring.

Remark 3.33. Inspired by Totaro's work on the Chow ring of a classifying space of a linear algebraic group G (cf. [74]), Morel and Voevodsky defined an ind-scheme $\mathrm{B}_{\mathrm{gm}}G$ (cf. [54, Sec. 4.2]), that is $\mathbb{A}^1$-weakly equivalent to $\mathrm{B}_{\text{ét}}G$. In the context of this geometric model, proposition 3.31 tells us, that for the computation of $\mathbb{H}_0^{\mathbb{A}^1}$ only the first two non-trivial terms are relevant.

Example 3.34. Let us finish this chapter with an example that demonstrates, why the complement of the G-invariant open subset has to be of codimension ≥ 2, and not merely ≥ 1 in Totaro's geometric description 3.31. A reader unfamiliar with parts of the notation, may skip it in a first reading and go back to it, after checking chapter 4 out.

Assume $\operatorname{char}(k) \neq 2$ and set $G = \boldsymbol{\mu}_2$. Let M be a strictly $\mathbb{A}^1$-invariant sheaf. Consider the diagonal action of G on the affine space $\mathbb{A}^n_k$. For $n = 1$ the group G acts freely on $\mathbb{G}_m$, and the quotient $\mathbb{G}_m/G$ is again given by $\mathbb{G}_m$. By Morel's theorem [50, Thm. 3.37], we have $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) = \underline{\mathbf{K}}_1^{\mathrm{MW}}$. Note that the codimension of $V(T) \subseteq \mathbb{A}^1_k$ is one.

For $n = 2$, we have that G acts freely on the space $\mathbb{A}^2_k \smallsetminus \{0\}$ and $V(T_1, T_2)$ is of codimension 2 in $\mathbb{A}^2_k$. The quotient of $\mathbb{A}^2_k \smallsetminus \{0\}$ by G is given by the cone of the image of the Veronese embedding $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k$ in $\mathbb{A}^3_k \smallsetminus \{0\}$, and its $\mathbb{H}_0^{\mathbb{A}^1}$ is given by $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}}$. Indeed the quotient $\mathbb{A}^2_k \smallsetminus \{0\}/G$ may be written as a pushout

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{(x,y)\mapsto(x,xy)} & \mathbb{A}^1_k \times \mathbb{G}_m \\ {\scriptstyle (x,y)\mapsto(xy^{-1},x)}\big\downarrow & & \\ \mathbb{G}_m \times \mathbb{A}^1_k & & \end{array},$$

and thus the $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$ of it will be a pushout

$$\begin{array}{ccc} \underline{\mathbf{K}}_1^{\mathrm{MW}} \oplus \underline{\mathbf{K}}_1^{\mathrm{MW}} \oplus \underline{\mathbf{K}}_2^{\mathrm{MW}} & \xrightarrow{\begin{pmatrix} 1 & 1 & \eta \end{pmatrix}} & \underline{\mathbf{K}}_1^{\mathrm{MW}} \\ {\scriptstyle \begin{pmatrix} 1 & \epsilon & \eta \end{pmatrix}}\big\downarrow & & \\ \underline{\mathbf{K}}_1^{\mathrm{MW}} & & \end{array}.$$

It is an easy check that this pushout comes out as $\underline{\mathbf{K}}_1^{\mathrm{W}}$. Later we derive this by another reasoning.

4

On $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\, G)$ for some affine algebraic groups G

In the present chapter we want to deduce the zero$^{\text{th}}$ $\mathbb{A}^1$-homology for a few smooth linear algebraic groups. Therefore we base our case on the strongest of the above introduced assumptions, and henceforth demand that k is a perfect field.

The vantage point of our method forms proposition 2.58, which characterises $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$, for some smooth linear algebraic group G, as the universal strictly $\mathbb{A}^1$-invariant sheaf to which sheafified étale cohomology $\mathcal{H}^1_{\text{ét}}(G)$ maps to. For the second important reduction step we leverage our analysis of morphisms between (weakly) unramified sheaves above (cf. proposition 3.25 and remark 3.21), together with the compatibility statement 3.28, to see that morphisms $\mathcal{H}^1_{\text{ét}}(G) \to M$ are precisely determined as natural transformations of functors $\mathsf{F}_k \to \mathsf{Set}$, i.e. as cohomological invariants in the sense of Serre.

Hence, in order to determine the zero$^{\text{th}}$ $\mathbb{A}^1$-homology for the orthogonal, special orthogonal, and symmetric groups, we may resort to the classical theory developed by Garibaldi, Merkurjev, and Serre (cf. [21]). What we find in particular is, that the universal morphisms $\mathcal{H}^1_{\text{ét}}(G) \to \mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ for those groups, are essentially generalizations of the well-known Stiefel-Whitney classes. We proceed from those examples to the cases of the unitary groups, and the exceptional group $\mathbf{G}_2$. As a highlight in the end, we lift some arguments due to Garibaldi (cf. [20]), to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(-))$ for spin groups associated with quadratic spaces of dimensions ≤ 12.

However, before we begin, we need to recall the form of the zero$^{\text{th}}$ $\mathbb{A}^1$-homology of spheres of the form $\mathbb{G}_m^{\wedge n}$, which is due to Morel, to get us off the ground.

Background: Milnor-Witt K-theory

We recall the definition of the following generalization of Milnor K-theory, which was first introduced by Morel, and subsequently simplified together with Hopkins.[1] The motivation for its introduction was its relation to the motivic π_0 of the sphere spectrum (cf. [51]).

Definition 4.1. Let F be a field. Denote by $K_*^{\text{MW}}(F)$ the (not necessarily) commutative, $\mathbb{Z}$-graded, associative, and unital ring that is freely generated by the symbols $[u]$, for all $u \in F^\times$, in degree 1, and by η in degree -1, subject to the relations:

(MW1) $[u][1-u] = 0$, for all $u \in F \setminus \{0,1\}$ (*Steinberg relation*).

(MW2) $[uv] = [u] + [v] + \eta[u][v]$, for all $u, v \in F^\times$.

(MW3) $[u]\eta = \eta[u]$, for all $u \in F^\times$.

(MW4) $\eta^2[-1] + 2\eta = 0$, or equivalently $\eta h = 0$, with $h := 2 + \eta[-1]$.

[1] Although already appearing earlier, we mainly reference [50, Ch. 3], since it contains a variety of introductory information about Milnor-Witt K-theory.

The graded ring $K_*^{\mathrm{MW}}(F)$ is called the *Milnor-Witt K-theory* over F. Important quotients of Milnor-Witt K-theory can be obtained by forming the quotients with respect to the graded ideals (h), (η), and (h, η):

$$K_*^{\mathrm{MW}}(F)/(h) =: K_*^{\mathrm{W}}(F), \quad K_*^{\mathrm{MW}}(F)/(\eta) = K_*^{\mathrm{M}}(F), \quad \text{and} \quad K_*^{\mathrm{MW}}(F) = k_*^{\mathrm{M}}(F),$$

where $K_*^{\mathrm{W}}(F)$ is called the *Witt K-theory* over F, and $K_*^{\mathrm{M}}(F)$ resp. $k_*^{\mathrm{M}}(F)$ denote Milnor K-theory resp. Milnor K-theory modulo 2.

Since a considerable part of this chapter is involved with calculations in and around Milnor-Witt K-theory, we would like to bring the reader up to speed with some consequences of the above abstract definition. First we start with some computational results:

Lemma 4.2 ([50, 3.5–3.8])**.** *Let F be a field. Then one finds for all units $u \in F^\times$ the identities*

$$[u][-u] = 0, \quad [u][u] = [u][-1] = [-1][u], \quad \textit{and} \quad [1] = 0. \tag{4.1}$$

Setting $\langle u \rangle := 1 + \eta[u] \in K_0^{\mathrm{MW}}(F)$, one obtains moreover for all $u, v \in F^\times$

$$\langle uv \rangle = \langle u \rangle \langle v \rangle, \quad \langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle = h, \quad \textit{and} \quad \langle 1 \rangle = 1.$$

Let there be $\alpha \in K_m^{\mathrm{MW}}(F)$ and $\beta \in K_n^{\mathrm{MW}}(F)$, then we have the graded commutativity relation

$$\alpha\beta = (\underbrace{-\langle -1 \rangle}_{\epsilon :=})^{mn} \beta\alpha.$$

Finally, given $u \in F^\times$ and $n \in \mathbb{Z}$, one finds $[u^n] = n_\epsilon [u]$, where (cf. [50, Lem. 3.14])

$$n_\epsilon := \begin{cases} \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle, & \textit{if } n \geq 0, \\ \epsilon(-n)_\epsilon, & \textit{if } n < 0. \end{cases}$$

Although we introduced the abbreviations $\langle - \rangle$, $h = 2_\epsilon$, ϵ, and n_ϵ only in passing, we want to stress the importance of these symbols and keep them fixed below. Next we proceed to recall several statements regarding the presentation of Milnor-Witt K-theory:

Lemma 4.3 ([50, 3.6–3.13])**.** *Let there be $n \in \mathbb{Z}$. The group $K_n^{\mathrm{MW}}(F)$ is (additively) generated by terms of the form*

$$\begin{cases} \eta^n \langle u \rangle, & \textit{with } u \in F^\times, & \textit{if } n \leq 0, \quad \textit{and} \\ [u_1] \cdots [u_n], & \textit{with } u_1, \ldots, u_n \in F^\times, & \textit{if } n \geq 1. \end{cases}$$

More specifically, denoting by $GW(F)$ the Grothendieck-Witt ring of isomorphism classes of symmetric bilinear forms, and setting $\langle u \rangle_B$ to be the symmetric bilinear form $F^2 \to F$, $(x, y) \mapsto xuy$, there is a (natural in F) induced isomorphism of rings

$$\begin{aligned} \phi \colon K_0^{\mathrm{MW}}(F) &\longrightarrow GW(F) \\ \langle u \rangle &\longmapsto \langle u \rangle_B. \end{aligned}$$

The annihilator of η in $K_0^{\mathrm{MW}}(F)$ is (h). Moreover, for any $r \in \mathbb{N}^+$, we find a (natural) isomorphism ϕ_{-r} such that the following diagram commutes

$$\begin{array}{ccc} K_0^{\mathrm{MW}}(F) & \xrightarrow{\eta^r \cdot (-)} & K_{-r}^{\mathrm{MW}}(F) \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{-r}} \\ GW(F) & \xrightarrow{\text{mod } (h)} & W(F). \end{array}$$

In the seminal article [48] Milnor defined his flavour of K-theory, and showed the existence of a split exact sequence that in the case of algebraic K-theory was derived by Tate. Hence, in order to conclude our statements about Milnor-Witt K-theory, we add Morel's account of such a sequence for Milnor-Witt K-theory (cf. theorems 3.15+3.24 in [50]):

Proposition 4.4. *Let F be a field, with discrete valuation v, valuation ring $\mathcal{O}_v$, uniformizer $\pi \in \mathcal{O}_v$, and residue field $\kappa(v)$. Then there is only one homomorphism of graded abelian groups*

$$\partial_v^\pi \colon K_*^{\mathrm{MW}}(F) \longrightarrow K_{*-1}^{\mathrm{MW}}(\kappa(v)),$$

which commutes with multiplication by η, and satisfies the formulae

$$\partial_v^\pi([\pi][u_1]\dots[u_n]) = [\overline{u_1}]\dots[\overline{u_n}] \quad \text{and} \quad \partial_v^\pi([u_1]\dots[u_n]) = 0,$$

with $u_1, \dots, u_n \in \mathcal{O}_v^\times \subseteq F^\times$. Moreover, one finds the following (split) short exact sequence of $K_*^{\mathrm{MW}}(F)$-modules:

$$0 \to K_*^{\mathrm{MW}}(F) \hookrightarrow K_*^{\mathrm{MW}}(F(T)) \xrightarrow{\sum_P \partial^P_{(P)}} \bigoplus_P K_{*-1}^{\mathrm{MW}}(F[T]/(P)) \to 0,$$

where P runs over the irreducible and monic polynomials in $F[T]$.

One may also introduce specialization homomorphisms $s_v^\pi \colon K_*^{\mathrm{MW}}(F) \to K_*^{\mathrm{MW}}(\kappa(v))$, which are indeed homomorphisms of rings, and which fulfil $s_v^\pi(\alpha) = \langle -1 \rangle \partial_v^\pi([-\pi]\alpha)$. We follow Morel to obtain a sheaf version of Milnor-Witt K-theory, by using the theory highlighted in remark 3.24: First note, that for any discrete valuation v on a field F the kernel of ∂_v^π does not depend on the choice of a particular uniformizer π so that the notation $\ker(\partial_v^?)$ makes sense. Now for any irreducible $X \in \mathsf{Sm}_k$ (k is a perfect field), we set

$$\underline{\mathbf{K}}_*^{\mathrm{MW}}(X) := \bigcap_{x \in X^{(1)}} \ker(\partial_{v_x}^?) \subseteq K_*^{\mathrm{MW}}(k(X)),$$

where v_x denotes the valuation defined by a codimension 1 point $x \in X$, and extend this to any non-irreducible X by taking the product of the above applied to each irreducible component. The following is due to Morel (cf. [50, Sec. 3.2]), and uses in particular the above Bass-Tate sequence:

Theorem 4.5. *$\underline{\mathbf{K}}_*^{\mathrm{MW}}$ is a $\mathbb{Z}$-graded Nisnevich sheaf of rings on Sm_k, with restriction homomorphisms induced by injections of fields and specialiaztion maps. Moreover, $\underline{\mathbf{K}}_n^{\mathrm{MW}}$ is strictly $\mathbb{A}^1$-invariant, for all $n \in \mathbb{Z}$.*

Remark 4.6. On a field $F \in \mathsf{F}_k$ the sheaf $\underline{\mathbf{K}}_*^{\mathrm{MW}}$ evaluates to the graded ring $K_*^{\mathrm{MW}}(F)$, and Morel's arguments show moreover (cf. [50, Lem. 3.32]), that for any graded ideal $R \subseteq K_*^{\mathrm{MW}}(k)$, one obtains a sheaf of graded rings $\underline{\mathbf{K}}_*^R$ that evaluates to $K_*^{\mathrm{MW}}(F)/R^e$ on F, where we extended the ideal R along $K_*^{\mathrm{MW}}(k) \to K_*^{\mathrm{MW}}(F)$. In this way we obtain the strictly $\mathbb{A}^1$-invariant sheaves of (unramified)[2] Witt K-theory and Milnor K-theory,

$$\underline{\mathbf{K}}_*^{\mathrm{W}}, \quad \text{for } R = (h), \qquad \text{and} \qquad \underline{\mathbf{K}}_*^{\mathrm{M}}, \quad \text{for } R = (\eta).$$

We link the above to what we have learned in chapter 3: By proposition 3.22 the sheaf $\underline{\mathbf{K}}_*^{\mathrm{MW}}$, and its descendants $\underline{\mathbf{K}}_*^{\mathrm{W}}$ and $\underline{\mathbf{K}}_*^{\mathrm{M}}$, is unramified. From corollary 3.19 we learn similarly that $\mathbb{G}_m^{\wedge n}$ is unramified, and thus by the characterisation of morphisms between unramified sheaves 3.25

[2] In [50] the sheaves $\underline{\mathbf{K}}_*^{\mathrm{MW}}$, $\underline{\mathbf{K}}_*^{\mathrm{W}}$, etc. are decorated with the prefix *unramified*, which we omit here, because it is unlikely to cause confusion.

and the compatibility statement 3.28 (using that k is perfect), a morphism $\sigma_n\colon \mathbb{G}_m^{\wedge n} \to \underline{\mathbf{K}}_n^{\mathrm{MW}}$, for all $n \in \mathbb{N}^+$, is induced by the natural transformation

$$\text{on } F \in \mathcal{F}_k\colon u_1 \wedge \cdots \wedge u_n \longmapsto [u_1]\cdots[u_n],$$

with $u_1, \ldots, u_n \in F^\times$. With this definition, we can finally add an account of Morel's characterization of the homology of the spheres $\mathbb{G}_m^{\wedge n}$ in terms of Milnor-Witt K-theory.

Theorem 4.7 ([50, Thm. 3.37])**.** *Let k be a perfect field. Then for all $n \in \mathbb{N}^+$, there is an isomorphism of strictly $\mathbb{A}^1$-invariant sheaves $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge n}) \cong \underline{\mathbf{K}}_n^{\mathrm{MW}}$ induced by $\sigma_n\colon \mathbb{G}_m^{\wedge n} \to \underline{\mathbf{K}}_n^{\mathrm{MW}}$.*

The following is an immediate consequence of the theorem and the symmetric monoidal structure on $\mathcal{A}b_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathcal{S}m_k)$ (see 2.62).

Corollary 4.8. *Let there be $m, n \in \mathbb{N}^+$, and $R_1, R_2 \subseteq K_*^{\mathrm{MW}}(k)$ graded ideals. Then we have an isomorphism of strictly $\mathbb{A}^1$-invariant sheaves*

$$\underline{\mathbf{K}}_m^{R_1} \otimes_{\mathbb{A}^1} \underline{\mathbf{K}}_n^{R_2} \cong \underline{\mathbf{K}}_{m+n}^{R_1+R_2},$$

that is induced by the product on $\underline{\mathbf{K}}_*^{\mathrm{MW}}$.

Proof. It suffices to check the claim in the special case $R_1 = R_2 = (0)$, since the general one will follow from the closed symmetric monoidal structure provided by $(-) \otimes_{\mathbb{A}^1} (-)$. We start by considering the following chain of natural isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{A}b_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_m^{\mathrm{MW}} \otimes_{\mathbb{A}^1} \underline{\mathbf{K}}_n^{\mathrm{MW}}, M) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{A}b_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_m^{\mathrm{MW}}, \underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_n^{\mathrm{MW}}, M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathbb{G}_m^{\wedge m}, \underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_n^{\mathrm{MW}}, M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\mathbb{Z}(\mathbb{G}_m^{\wedge m}), \underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_n^{\mathrm{MW}}, M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{A}b_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_n^{\mathrm{MW}}, \underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\mathbb{Z}(\mathbb{G}_m^{\wedge m}), M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathbb{G}_m^{\wedge n}, \underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\mathbb{Z}(\mathbb{G}_m^{\wedge m}), M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathbb{G}_m^{\wedge n}, \underline{\mathrm{Hom}}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathbb{G}_m^{\wedge m}, M)) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathbb{G}_m^{\wedge (m+n)}, M) \cong \ldots\\
&\mathrm{Hom}_{\mathcal{A}b_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathcal{S}m_k)}(\underline{\mathbf{K}}_{m+n}^{\mathrm{MW}}, M).
\end{aligned}$$

Here we used in particular that the internal hom-functor for strictly $\mathbb{A}^1$-invariant sheaves agrees with the internal hom-functor on abelian sheaves (see lemmas 2.60 and 2.61), theorem 4.7, and the fact that there is a natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{A}b_{\mathrm{Nis}}(\mathcal{S}m_k)}(\mathbb{Z}(\mathcal{F}), M) \cong \underline{\mathrm{Hom}}_{\mathcal{S}h_{\mathrm{Nis}}^{\bullet}(\mathcal{S}m_k)}(\mathcal{F}, M),$$

for all sheaves of pointed sets $\mathcal{F}$. As the above chain of isomorphisms is also natural in M, and induced by the product on $\underline{\mathbf{K}}_*^{\mathrm{MW}}$, the corollary follows. ☐

4.1 Orthogonal Groups

In this section we expose the proof of a fact that has been stated by Morel in [49, Rem. 5.4], namely that $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n)$ is determined by generalizations of the Stiefel-Whitney classes to Witt K-theory. Before we get to those, however, we will introduce a set of lemmas analysing products, as we will have to look at $\boldsymbol{\mu}_2^n \subseteq \mathbf{O}_n$.

Lemma 4.9. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be pointed sheaves. Then we have a natural isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n) \cong \bigoplus_{i=1}^{n} \bigoplus_{1 \le j_1 < \cdots < j_i \le n} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_{j_1} \wedge \cdots \wedge \mathcal{F}_{j_i}),$$

and the universal morphism $\theta^i_{j_1,\dots,j_i} \colon \mathcal{F}_1 \times \cdots \times \mathcal{F}_n \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_{j_1} \wedge \cdots \wedge \mathcal{F}_{j_i})$ is induced by the projection to $\mathcal{F}_{j_1} \wedge \cdots \wedge \mathcal{F}_{j_i}$ and the universal morphism

$$\mathcal{F}_{j_1} \wedge \cdots \wedge \mathcal{F}_{j_i} \longrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_{j_1} \wedge \cdots \wedge \mathcal{F}_{j_i}).$$

Proof. We only give the proof for $n = 2$, since it is instructive, and leave the rest to the reader: First, note that we have a cofiber sequence

$$\mathcal{F}_1 \vee \mathcal{F}_2 \to \mathcal{F}_1 \times \mathcal{F}_2 \twoheadrightarrow \mathcal{F}_1 \wedge \mathcal{F}_2$$

in $\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_k)$, where $\mathcal{F}_1 \vee \mathcal{F}_2$ is the pushout (in $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)$) of the diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \mathcal{F}_2 \\ \downarrow & & \\ \mathcal{F}_1 & & \end{array},$$

with canonical basepoint $\bullet \to \mathcal{F}_1 \vee \mathcal{F}_2$. Since $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$ is left adjoint, it preserves colimits, from which we deduce that

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \vee \mathcal{F}_2) \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \times \mathcal{F}_2) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \wedge \mathcal{F}_2) \to 0 \tag{4.2}$$

is exact. We intend to construct a splitting to the left-hand side homomorphism, for which we first need to bring $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \vee \mathcal{F}_2)$ into a more useful form. Therefore we use a similar line of reasoning: We may regard $\mathcal{F}_1 \vee \mathcal{F}_2$ as the sum of $\mathcal{F}_1$ and $\mathcal{F}_2$ in the category $\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_k)$. Again, since $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$ preserves colimits, we learn that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1) \oplus \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_2)$ is isomorphic to $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_1 \vee \mathcal{F}_2)$. From this, we see that the projections

$$\mathrm{pr}_1 \colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_1 \quad \text{and} \quad \mathrm{pr}_2 \colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_2$$

induce a splitting of the left-hand homomorphism in (4.2). This concludes the proof of the lemma in the case $n = 2$, and as hinted above, the higher cases can be deduced inductively by using similar ideas. □

We may apply this lemma in the case $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathcal{F}$, for some pointed sheaf of sets $\mathcal{F} \in \mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, to obtain the natural (in $\mathcal{F}$) isomorphism:

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n) \cong \bigoplus_{i=1}^{n} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i})^{\oplus \binom{n}{i}}$$

Let us analyse this situation further. We keep $n \in \mathbb{N}^+$ fixed, and may consider the action of the n^{th} symmetric group S_n on product sheaf $\mathcal{F}^n$ resp. the smash product $\mathcal{F}^{\wedge n}$. So, for any permutation $\sigma \in S_n$, we have morphisms

$$\mathcal{F}^n \to \mathcal{F}^n \quad \text{and} \quad \mathcal{F}^{\wedge n} \to \mathcal{F}^{\wedge n},$$

that permute the entries via σ, and which we denote again by σ for simplicity. When we apply the functor $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(-)$ to the diagram of pointed sheaves that is induced by this action, we may form its colimit $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n)_{S_n}$ resp. $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge n})_{S_n}$, i.e. we want diagrams like

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n) \xrightarrow[\tau]{\overset{\sigma}{\vdots}\ \mathrm{id}\ \vdots} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n) \longrightarrow\!\!\!\!\to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n)_{S_n}$$

to be exact. Note that this implies, since $\mathrm{Hom}_{\mathcal{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathcal{Sm}_k)}(-, M)$ maps colimits to limits, that we have a natural isomorphism (in $\mathcal{F}$ and M)

$$\mathrm{Hom}_{\mathcal{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathcal{Sm}_k)}(\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^n)_{S_n}, M) \cong \mathrm{Hom}_{\mathcal{Sh}^{\bullet}_{\mathrm{Nis}}(\mathcal{Sm}_k)}(\mathcal{F}^n, M)^{S_n},$$

where on the right-hand side we consider S_n-invariant morphisms. Now for every $i \in \{1, \dots, n\}$, and $1 \leq j_1 < \cdots < j_i \leq n$, we have a canonical morphism

$$\mathcal{F}^n \xrightarrow{\theta^i_{j_1,\dots,j_i}} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i}) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i})_{S_i},$$

where the second morphism is just the canonical projection to the quotient. We sum all those morphisms to a single one $\theta^i \colon \mathcal{F}^n \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i})_{S_i}$, and immediately see that those are S_n-invariant. The following lemma states that they are also universal.

Lemma 4.10. *Let $\mathcal{F}$ be a pointed sheaf. Then we have a natural (in $\mathcal{F}$) isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F} \times \cdots \times \mathcal{F})_{S_n} \xrightarrow{\cong} \bigoplus_{i=1}^{n} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\underbrace{\mathcal{F} \wedge \cdots \wedge \mathcal{F}}_{i-fold})_{S_i},$$

which is induced by the $(\theta^i)_{1 \leq i \leq n}$.

Proof. A proof may be found by defining a suitable inverse. We omit the details. □

We proceed to calculate the base $\mathbb{A}^1$-homology of the geometric classifying space of $\boldsymbol{\mu}_2^n$ for any positive natural number $n \in \mathbb{N}^+$. The motivation for this is classical: In order to determine the cohomological invariants of $\mathbf{O}_n$, one first determines the invariants of $\boldsymbol{\mu}_2^n$, and then employs Serre's splitting principle (cf. [21, Thm. 17.3]). So we start with:

Lemma 4.11. *Let k be a perfect field ,with $p := \mathrm{char}(k)$, and fix a natural number $\ell > 1$ prime to p. Then we have:*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_\ell) \cong \underline{\mathbf{K}}_1^{\mathrm{MW}}/(\ell_\epsilon) \cong \begin{cases} \underline{\mathbf{K}}_1^{\mathrm{M}}/(\ell), & \text{if } \ell \text{ is odd,} \\ \underline{\mathbf{K}}_1^{\mathrm{MW}}/(\frac{\ell}{2}h), & \text{if } \ell \text{ is even.} \end{cases}$$

Proof. Let us start with the Kummer exact sequence

$$0 \to \boldsymbol{\mu}_\ell \to \mathbb{G}_m \xrightarrow{(-)^\ell} \mathbb{G}_m \to 0,$$

which is an exact sequence of étale sheaves. Now, for any $U \in \mathcal{Sm}_k$, we have a long exact sequence of cohomology groups, of which the following part is important to us:

$$\mathbb{G}_m(U) \xrightarrow{(-)^\ell} \mathbb{G}_m(U) \to H^1_{\text{ét}}(U, \boldsymbol{\mu}_\ell) \to H^1_{\text{ét}}(U, \mathbb{G}_m).$$

By varying U, we may think of the above sequence as an exact sequence of presheaves, so sheafifying (with respect to the Nisnevich topology) yields an exact sequence of sheaves:

$$\mathbb{G}_m \xrightarrow{(-)^\ell} \mathbb{G}_m \to \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_\ell) \to \mathcal{H}^1_{\text{ét}}(\mathbb{G}_m).$$

Note that by Hilbert's theorem 90, we have $\mathcal{H}^1_{\text{ét}}(\mathbb{G}_m) = 0$, and thus $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_\ell)$ is the cokernel of the ℓ-power map in the category of abelian sheaves. We may translate this into a coequaliser diagram in the category of pointed sheaves of the following morphisms:

$$\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mathrm{pr}_2} \mathbb{G}_m \quad \text{and} \quad \begin{pmatrix} \varphi_\ell \colon \mathbb{G}_m \times \mathbb{G}_m & \to \mathbb{G}_m \\ \text{on } U \in \mathcal{Sm}_k \colon (u, v) & \mapsto u^\ell v \end{pmatrix}.$$

Since $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$ is a left adjoint, it preserves colimits, and thus in particular coequaliser diagrams. From this we see that

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\varphi_\ell) - \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{pr}_2)} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_\ell))$$

is exact. We use theorem 4.7 in conjunction with lemma 4.9 to obtain

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m \times \mathbb{G}_m) \cong \underline{\mathbf{K}}_1^{\mathrm{MW}} \oplus \underline{\mathbf{K}}_1^{\mathrm{MW}} \oplus \underline{\mathbf{K}}_2^{\mathrm{MW}} \quad \text{and} \quad \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \cong \underline{\mathbf{K}}_1^{\mathrm{MW}},$$

and we focus on determining the induced homomorphisms between those. It is clear that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{pr}_2)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

and to obtain $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\varphi_\ell)$, we express the symbol $[u^\ell v]$ in terms of $[u]$, $[v]$, and $[u][v]$. This can be achieved by using **(MW2)**, and lemma 4.2:

$$[u^\ell v] = \ell_\epsilon[u] + [v] + \eta\ell_\epsilon[u][v].$$

Thus we find the matrix

$$\begin{pmatrix} \ell_\epsilon & 1 & \eta\ell_\epsilon \end{pmatrix}$$

for the homomorphism $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\varphi_\ell)$. Note that the homomorphism $\underline{\mathbf{K}}_1^{\mathrm{MW}} \to \underline{\mathbf{K}}_1^{\mathrm{MW}}$ defined by ℓ_ϵ is given by multiplication with ℓ_ϵ. So we find that the cokernel of $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\varphi_\ell) - \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{pr}_2)$ is given by

$$\underline{\mathbf{K}}_1^{\mathrm{MW}}/(\ell_\epsilon).$$

From this, the statement of the lemma can be seen quickly, as we have

$$\ell_\epsilon = \begin{cases} \frac{l-1}{2}h + 1, & \text{if } \ell \text{ is odd, and} \\ \frac{l}{2}h, & \text{if } \ell \text{ is even.} \end{cases}$$

To precisely arrive at the claim in the odd case, we note, using also $\eta h = 0$ **(MW4)**, that the principal ideal $(\ell_\epsilon) \subseteq K_*^{\mathrm{MW}}(k)$ contains the principal ideal (η), and that $\ell_\epsilon \equiv \ell \mod \eta$. Recall that $\underline{\mathbf{K}}_*^{\mathrm{MW}}/(\eta)$ yields unramified Milnor K-theory.

In closing, we note that the universal morphism inducing $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_\ell) \cong \underline{\mathbf{K}}_1^{\mathrm{MW}}/(\ell_\epsilon)$ may be defined by

$$\begin{aligned} \sigma^{(\ell)} \colon \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_\ell) &\longrightarrow \underline{\mathbf{K}}_1^{\mathrm{MW}}/(\ell_\epsilon) \\ u(F^\times)^\ell &\longmapsto [u] + (\ell_\epsilon). \end{aligned}$$

□

Remark 4.12. If k is of characteristic p, and we consider p^r, for some $r \in \mathbb{N}^+$, we obtain that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r))$ is trivial, since by [54, Prop. 4.3.3] $\mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r)$ is $\mathbb{A}^1$-weakly equivalent to the point.

Note that this allows us to derive the case of $\mathbb{Z}/(n)$: Let $\ell \in \mathbb{N}^+$ be such that $(\ell, p) = 1$, and that we have a primitive ℓ^{th} root of unity in k. We consider the abelian group $\mathbb{Z}/(p^r\ell)$. By the Chinese remainder theorem, we have $\mathbb{Z}/(p^r\ell) \cong \mathbb{Z}/(p^r) \times \mathbb{Z}/(\ell)$, and this leads to an isomorphism (in $\mathcal{H}_s(k)$)

$$\mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r\ell) \cong \mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r) \times \mathrm{B}_{\text{ét}}\mathbb{Z}/(\ell).$$

Indeed, on the level of the model category $\mathcal{Spc}_k$, the étale classifying space is given by applying the functors B, α^*, Ex, and $\mathbf{R}\alpha_*$, which all preserve finite products (cf. [54, Thm. 2.1.66]). Now, one only has to use that the $\mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r)$-factor is weakly $\mathbb{A}^1$-contractible, and that this extends suitably to the product (cf. [54, Lem. 2.2.15]), so that we have an isomorphism (in $\mathcal{H}_{\mathbb{A}^1}(k)$)

$$\mathrm{B}_{\text{ét}}\mathbb{Z}/(p^r\ell) \cong \mathrm{B}_{\text{ét}}\mathbb{Z}/(\ell) \cong \mathrm{B}_{\text{ét}}\boldsymbol{\mu}_\ell,$$

and the above lemma gives $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$.

A similar approach as above would also yield the following generalization. However, we will use the opportunity to demonstrate the strength of the $\mathbb{A}^1$-tensor product in this context.

Lemma 4.13. *Let k be a perfect field of characteristic $\neq 2$, and fix $n \in \mathbb{N}^+$. Then there is an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n) \cong \bigoplus_{i=1}^{n} \left(\underline{\mathbf{K}}_i^{\mathrm{W}}\right)^{\oplus\binom{n}{i}},$$

that is induced by all possible ways to form products of n symbols.

Proof. By lemma 4.11 $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2)$ is given by $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}}$. Using this, together with proposition 2.63 and the fact that $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n) \cong \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2)^n$, we see that there is an isomorphism

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n) \cong \left(\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}}\right)^{\otimes_{\mathbb{A}^1} n}.$$

The claim then follows with corollary 4.8, by expanding the $\mathbb{A}^1$-tensor product on the right. For ease of reference, we remark that the universal morphism $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n) \to \underline{\mathbf{K}}_i^{\mathrm{W}}$, for $i \in \{1, \ldots, n\}$, that is associated to a choice of indices $1 \leq j_1 < \cdots < j_i \leq n$ is hence induced by the following natural map on fields $F \in \mathsf{F}_k$:

$$\begin{aligned} \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n)(F) &\longrightarrow \underline{\mathbf{K}}_i^{\mathrm{W}}(F) \\ (u_1(F^\times)^2, \ldots, u_n(F^\times)^2) &\longmapsto [u_{j_1}] \cdots [u_{j_i}]. \end{aligned}$$

□

We would like to combine this statement about $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n)$ with lemma 4.10, in the case $\mathcal{F} = \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2)$. Again this is possible, as we have in particular $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n) \cong \mathcal{F}^n$. However, before we are able to proceed, we have to calculate $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$ of the smash products $\mathcal{F}^{\wedge i}$. Here we may again employ the $\mathbb{A}^1$-tensor product by referring to remark 2.64 instead of proposition 2.63. Thus we arrive at $\underline{\mathbf{K}}_i^{\mathrm{W}}$ for $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i})$. Furthermore, since $\underline{\mathbf{K}}_*^{\mathrm{W}}$ is commutative, with respect to the product of symbols, one obtains the identity:

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}^{\wedge i})_{S_i} = (\underline{\mathbf{K}}_i^{\mathrm{W}})_{S_i} = \underline{\mathbf{K}}_i^{\mathrm{W}}, \quad \forall i \in \{1, \ldots, n\}.$$

Now, lemma 4.10 implies:

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n)_{S_n} \cong \bigoplus_{i=1}^{n} \underline{\mathbf{K}}_i^{\mathrm{W}}.$$

Hereby the universal morphisms $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n) \to \underline{\mathbf{K}}_i^{\mathrm{W}}$ are given on fields $F \in \mathsf{F}_k$ by

$$\begin{aligned} \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n)(F) &\longrightarrow \underline{\mathbf{K}}_i^{\mathrm{W}}(F) \\ (u_1(F^\times)^2, \ldots, u_n(F^\times)^2) &\longmapsto e_i([u_1], \ldots, [u_n]), \end{aligned}$$

where $e_i \in \mathbb{Z}[T_1, \ldots, T_n]$ is the i^{th}-elementary symmetric polynomial in n variables.

Remark 4.14. If we consider the cofiber sequence $\mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \to \mathcal{F}$ in $\mathsf{Sh}^{\bullet}_{\mathrm{Nis}}(\mathsf{Sm}_k)$, under the assumption $\mathrm{char}(k) \neq 2$, and calculate $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F})$, we obtain $\underline{\mathbf{K}}_1^{\mathrm{W}}$. However, due to the formula $h^n = 2^{n-1}h = (2^n)_\epsilon$, we obtain

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathcal{F}_n) \cong \bigoplus_{i=1}^{n} \left(\underline{\mathbf{K}}_i^{\mathrm{MW}}/(2^{i-1}h)\right)^{\oplus\binom{n}{i}},$$

for $\mathcal{F}_n$ fitting into the cofiber sequence $\mathbb{G}_m^n \xrightarrow{(-)^2} \mathbb{G}_m^n \to \mathcal{F}_n$. So, we see that for $n > 1$ these cofibers do not yield the correct $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}$.

In the following we are concerned with the euclidean quadratic form $q_n = n \cdot \langle 1 \rangle$, and its associated orthogonal group $\mathbf{O}_n = \mathbf{O}(k^n, q_n)$, in characteristic $\neq 2$. In this particular case, there is an embedding

$$\begin{aligned} \iota_n \colon \boldsymbol{\mu}_2^n &\hookrightarrow \mathbf{O}_n \\ (\epsilon_1, \dots, \epsilon_n) &\longmapsto \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix}, \end{aligned}$$

which induces a surjection on torsors over any field $F \in \mathsf{F}_k$:

$$\begin{aligned} H^1_{\text{ét}}(F, \boldsymbol{\mu}_2^n) &\longrightarrow H^1_{\text{ét}}(F, \mathbf{O}_n) \\ (u_1(F^\times)^2, \dots, u_n(F^\times)^2) &\longmapsto \langle u_1, \dots, u_n \rangle. \end{aligned}$$

To see this, one may use that, for $\operatorname{char}(k) \neq 2$, the $\mathbf{O}_n$-torsors over F are given by the isometry classes of n-dimensional nonsingular quadratic forms (cf. [39, (29.28)]). Using this, in combination with corollary 3.27 and proposition 3.28, we obtain a monomorphism

$$\operatorname{Hom}_{\mathsf{Sh}^\bullet_{\text{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\mathbf{O}_n), M) \hookrightarrow \operatorname{Hom}_{\mathsf{Sh}^\bullet_{\text{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2^n), M),$$

for all strictly $\mathbb{A}^1$-invariant sheaves M, which in turn, by remark 2.59, implies that the induced homomorphism

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n), \tag{4.3}$$

is an epimorphism.

Now, since $\mathbf{O}_n$ is associated to the euclidean quadratic form, one sees that the permutation matrices are k-rational points on $\mathbf{O}_n$. Thus by conjugating with these permutation matrices, one obtains an action of S_n on $\mathbf{O}_n$. The actions of S_n on $\boldsymbol{\mu}_2^n$ and on $\mathbf{O}_n$ are compatible under the homomorphism ι_n, as conjugating a diagonal matrix by a permutation matrix permutes the diagonal entries, effectively. Moreover, the induced action of S_n on $H^1_{\text{ét}}(F, \mathbf{O}_n)$ is trivial, as one may see by using the 1-cocycle representation of the cohomology set $H^1_{\text{ét}}(F, \mathbf{O}_n)$. So, by taking S_n-coinvariants on both sides of (4.3), we obtain an epimorphism

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\iota_n)_{S_n} \colon \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^n)_{S_n} \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n)$$

We intend to show that the above homomorphism is an isomorphism by finding a suitable left inverse, which, as $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\iota_n)_{S_n}$ is an epimorphism, will also be an inverse. So, the goal is to demonstrate the following, which was announced in [49, Rem. 5.4].

Lemma 4.15. *Let k be a perfect field of characteristic $\neq 2$, and $n \in \mathbb{N}^+$. Then there is an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n) \cong \bigoplus_{i=1}^{n} \underline{\mathbf{K}}_i^{\mathrm{W}},$$

which is induced by the ***refined Stiefel-Whitney classes*** $W_i \colon \mathcal{H}^1_{\text{ét}}(\mathbf{O}_n) \to \underline{\mathbf{K}}_i^{\mathrm{W}}$.

Definition of the Refined Stiefel-Whitney Classes

In this paragraph we mimic the construction of the Stiefel-Whitney classes as presented in [48], with the aim to introduce their corresponding refined versions. This kind of generalization was outlined in [49, §5], and we provide the details explicitly. We assume that the base field k is perfect and of characteristic $\neq 2$.

Recall that the argument in [48] amounted to the following: To the $\mathbb{N}$-graded ring $k_*^{\mathrm{M}}(F)$, of Milnor K-theory modulo 2, one associates the ring $k_\Pi^{\mathrm{M}}(F)$ of formal series, with coefficients in $k_*^{\mathrm{M}}(F)$, where $F \in \mathsf{F}_k$. Then a homomorphism

$$\mathbb{Z}[F^\times] \longrightarrow k_\Pi^{\mathrm{M}}(F)^\times$$

is defined, which sends a unit $u \in F^\times$ to the unit $1 + \{u\}$. Finally the relations of the Grothendieck-Witt ring are checked, to obtain a homomorphism

$$w\colon GW(F) \longrightarrow k_\Pi(F)^\times.$$

The Stiefel-Whitney classes $w_i\colon GW(F) \cap H^1_{\text{ét}}(F, \mathbf{O}_n) \to k_i(F)$ can then be peeled of for $n \in \mathbb{N}^+$ and $1 \le i \le n$. We intend to reproduce this construction, where we exchange $k_*^{\mathrm{M}}(F)$ for $K_{\ge 0}^{\mathrm{W}}(F)$, i.e. we will construct a natural (in $F \in \mathsf{F}_k$) homomorphism

$$W\colon GW(F) \longrightarrow K_\Pi^{\mathrm{W}}(F)^\times$$

that yields w, after composition with the canonical projection $K_\Pi^{\mathrm{W}}(F) \twoheadrightarrow K_\Pi^{\mathrm{W}}(F)/(\eta) \cong k_\Pi^{\mathrm{M}}(F)$. Therefore, let us briefly recall the following presentation of the Grothendieck-Witt ring of quadratic forms $GW(F)$.

Proposition 4.16 ([50, Lem. 3.9])**.** *The group $GW(F)$ is generated by elements $\langle u\rangle$, for $u \in F^\times$, subject to the following exhausting relations:*

$$\forall (u,v) \in \left(F^\times\right)^2 : \langle u(v^2)\rangle = \langle u\rangle \quad \textit{and} \tag{4.4}$$

$$\forall (u,v) \in \left(F^\times\right)^2,\ u+v \in F^\times : \langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle uv(u+v)\rangle. \tag{4.5}$$

Remark 4.17. The assumption $\mathrm{char}(k) \neq 2$ is crucial in the above. Indeed, if we assume $\mathrm{char}(k) = 2$, one obtains a presentation of the Grothendieck-Witt ring of symmetric bilinear forms, by adding the additional relation

$$\forall u \in F^\times :\ \langle u\rangle + \langle -u\rangle = \langle 1\rangle + \langle -1\rangle. \tag{4.6}$$

However, the generators and relations for the Grotendieck-Witt ring of quadratic forms in characteristic 2 are different, as evidenced for example by the way that quadratic forms in characteristic 2 decompose (cf. [16, Proposition 7.31]). Note that for $\mathrm{char}(k) \neq 2$, (4.4) and (4.5) imply (4.6), and the Grothendieck-Witt rings of symmetric bilinear forms, and quadratic forms are isomorphic.

Now we are in a position to construct the refined Stiefel-Whitney classes. As we explained above, we first write down a homomorphism of abelian groups

$$\begin{aligned} \mathbb{Z}[F^\times] &\longrightarrow K_\Pi^{\mathrm{W}}(F)^\times \\ \langle u\rangle &\longmapsto (1 + [u]). \end{aligned}$$

Let us check the relations (4.4) and (4.5). To that end we will only use the axioms **(MW1)**-**(MW4)**, and their immediate consequences (cf. lemma 4.2). To check relation (4.4) holds, we calculate first:

$$[v^2] \overset{\textbf{(MW2)}}{=} [v] + [v] + \eta[v][v] \overset{(4.1)}{=} 2[v] + \eta[v][-1] = h[v] = 0.$$

Above we used implicitly that $K_*^{\mathrm{W}}(F)$ is commutative, which follows easily from $K_*^{\mathrm{MW}}(F)$ being ϵ-graded commutative. So we have $[uv^2] = [u]$ in $K_1^{\mathrm{W}}(F)$, for another unit $u \in F^\times$, by axiom **(MW2)**. Hence it only remains to check (4.5). For $(u,v) \in (F^\times)^2$ with $u + v \neq 0$ this means, that we have to obtain the equality

$$(1 + [u])(1 + [v]) = (1 + [u+v])(1 + [uv(u+v)]) \in K_\Pi^{\mathrm{W}}(F),$$

or equivalently the equations

$$[u] + [v] = [u+v] + [uv(u+v)] \in K_1^{\mathrm{W}}(F) \quad \text{and} \tag{4.7}$$

$$[u][v] = [u+v][uv(u+v)] \in K_2^{\mathrm{W}}(F). \tag{4.8}$$

Actually, both equations imply each other. Let us show (4.8) $\Longrightarrow$ (4.7) first. We start by using the independence of symbols of squares:

$$[uv] = [uv(u+v)^2] \overset{\textbf{(MW2)}}{=} [uv(u+v)] + [u+v] + \eta[uv(u+v)][u+v].$$

Expanding $[uv]$ according to axiom **(MW2)** yields

$$[u] + [v] + \eta[u][v] = [uv].$$

If we subtract η times (4.8) on their respective sides we obtain (4.7). Let us now show that (4.7) implies (4.8). The Steinberg relation **(MW1)** implies

$$\left[\frac{u}{u+v}\right]\left[\frac{v}{u+v}\right] = 0,$$

which we may rewrite as $[u(u+v)][v(u+v)] = 0$ in $K_2^{\mathrm{W}}(F)$ by the independence of squares. A trivial application of **(MW2)** lets us expand this expression to

$$([u] + [u+v] + \eta[u][u+v])\,([v] + [u+v] + \eta[v][u+v]) = 0.$$

Next we employ the following two side calculations

$$2\eta[u][v][u+v] \overset{\textbf{(MW4)}}{=} -\eta^2.[-1][u][v][u+v] \overset{(4.1)}{=} -\eta^2[u][v][u+v]^2, \qquad \text{and}$$
$$\eta[x][u+v]^2 \overset{(4.1)}{=} \eta[x][u+v][-1] \overset{h=0}{=} -2[x][u+v] = -2[u+v][x],$$

for $x \in \{u, v\}$, to obtain

$$[u][v] - [u][u+v] - [v][u+v] + [u+v]^2 = 0$$

or equivalently

$$[u][v] = [u+v]([u] + [v] - [u+v]).$$

By using (4.7) now, we immediately obtain (4.8). So, let us focus on showing equation (4.7), and let us start by addressing the special case $u + v = 1$. By $[1] = 0$, this simplifies the task to checking:

$$[u] + [1-u] = [u(1-u)], \quad \forall\, u \in F^{\times}.$$

However, we know that this is always true by using **(MW2)**, and the Steinberg relation **(MW1)**. Coming back to the case of $u, v \in F^{\times}$, with $u + v \neq 0$, we have

$$\langle u+v\rangle \left(\left[\frac{u}{u+v}\right] + \left[\frac{v}{u+v}\right] - \left[\frac{uv}{(u+v)^2}\right]\right) = 0,$$

from our special case. Using axiom **(MW2)**, and the independence of symbols in $K_*^{\mathrm{W}}(F)$ of squares, we immediately expand this to (4.7). Thus we obtain a homomorphism of abelian groups

$$W\colon GW(F) \to K_{\Pi}^{\mathrm{W}}(F),$$

and the projections to the individual terms $W_i\colon GW(F) \to K_i^{\mathrm{W}}(F)$ constitute the *refined Stiefel-Whitney classes.*

Remark 4.18. If we think of $GW(F)$ as the Grothendieck-Witt ring of symmetric bilinear forms, we may extend the above definition to characteristic 2: Indeed, from the above, we only have to check the additional relation (derived of (4.6))

$$(1+[u])(1+[u]) = 1 \in K_{\Pi}^{\mathrm{W}}(F),$$

which is equivalent to $[u][u] = 0$ and $2[u] = 0$. The former follows from (4.1), since $[-1] = [1] = 0$, and the latter is a consequence of the independence of squares, axiom **(MW2)**, and the former:

$$0 = [1] = [u^2] = 2[u] + \eta[u][u] = 2[u].$$

However, we will have not much use for this.

To come back to the proof of lemma 4.15, we note first that the above construction is functorial in the field F. Thus by propositions 3.28 and corollary 3.27, we obtain sheaf morphisms $W_i\colon \mathcal{H}^1_{\mathrm{\acute{e}t}}(\mathbf{O}_n) \to \underline{\mathbf{K}}_i^{\mathrm{W}}$, for $1 \leq i \leq n$. These induce a homomorphism $\Omega_n\colon \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}_n) \to \bigoplus_{i=1}^n \underline{\mathbf{K}}_i^{\mathrm{W}}$, and it remains to check that

$$\bigoplus_{i=1}^n \underline{\mathbf{K}}_i^{\mathrm{W}} \xrightarrow{\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\iota_n)_{S_n}} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}_n) \xrightarrow{\Omega_n} \bigoplus_{i=1}^n \underline{\mathbf{K}}_i^{\mathrm{W}}$$

is the identity. To see this, we may first precompose with the canonical epimorphism

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\boldsymbol{\mu}_2^n) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\boldsymbol{\mu}_2^n)_{S_n} \cong \bigoplus_{i=1}^n \underline{\mathbf{K}}_i^{\mathrm{W}},$$

and then in turn with the universal morphism $\mathcal{H}^1_{\mathrm{\acute{e}t}}(\boldsymbol{\mu}_2^n) \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\boldsymbol{\mu}_2^n)$. The resulting composite is nothing else than the universal morphism

$$\mathcal{H}^1_{\mathrm{\acute{e}t}}(\boldsymbol{\mu}_2^n) \longrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\boldsymbol{\mu}_2^n)_{S_n},$$

that we derived earlier by use of lemma 4.10, and this completes the proof of the lemma.

Remark 4.19. From proposition 2.58, together with proposition 3.28, we know that the abelian group $\mathrm{Hom}_{\mathsf{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}_n), M)$, for any strictly $\mathbb{A}^1$-invariant sheaf M, is given by the natural transformations between functors $\mathsf{F}_k \to \mathsf{Set}$

$$a\colon \mathcal{H}^1_{\mathrm{\acute{e}t}}(\mathbf{O}_n) \longrightarrow M,$$

i.e. by cohomological invariants in the sense of [21, Def. 1.1]. Using moreover [39, (29.28)], we see that $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}_n)$ only depends on the functor

$$\begin{aligned} \mathsf{Quad}_n\colon \mathsf{F}_k &\longrightarrow \mathsf{Set} \\ F &\longmapsto \left\{ \begin{array}{c} \text{isometry classes of non-deg.} \\ \text{quadratic forms of rank } n \text{ over } F \end{array} \right\}. \end{aligned}$$

As such we have, again by [39, (29.28)], that $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}(V,q)) \cong \mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{O}_n)$, where (V,q) is another non-degenerate quadratic space of rank n, however the basepoint is shifted to q. We note that such an isomorphism is more generally always present, when one considers an inner form of a smooth algebraic group.

Remark 4.20 (Embedding of $\pi_0^{\mathbb{A}^1}$). Before we analyse the interdependence of the Stiefel-Whitney classes, we would like use the opportunity to address the following homotopy-theoretic situation: Given a topological space X, recall that $H_0(X)$ is a free abelian group, for which the path components $\pi_0(X)$ describe an explicit generating set. A similar set-up can be found for $\mathbb{A}^1$-algebraic topology. Using proposition 2.58, we may consider the functor $\mathbb{H}_0^{\mathbb{A}^1}$ from sheaves of sets to strictly $\mathbb{A}^1$-invariant sheaves as a *free strictly $\mathbb{A}^1$-invariant functor*. Moreover, employing the canonical isomorphism $\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X}) \cong \mathbb{H}_0^{\mathbb{A}^1}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ (cf. [4, Prop. 3.5]), one may construct a canonical morphism ρ

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \swarrow & & \searrow & \\ \pi_0^{\mathbb{A}^1}(\mathcal{X}) & & \xrightarrow{\quad\rho\quad} & & \mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X}), \\ & \searrow & & \nwarrow{\scriptstyle\cong} & \\ & & \mathbb{H}_0^{\mathbb{A}^1}(\pi_0^{\mathbb{A}^1}(\mathcal{X})) & & \end{array}$$

given some space $\mathcal{X} \in \mathcal{S}pc_k$. As in the classical case the path components not only generate, but also embed, this raises the question whether ρ is a monomorphism. We will address this problem throughout chapter 4 anecdotally and start with the case of $\mathcal{X} = \mathrm{B}_{\text{ét}}\mathbf{O}_n$.

Since ρ is a morphism from an unramified sheaf $\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n) \cong \mathcal{H}_{\text{ét}}^1(\mathbf{O}_n)$ (cf. [17, Prop. 6.1]) to an unramified sheaf, it suffices to check that for all fields $F \in \mathcal{F}_k$ the Stiefel-Whitney classes determine the Galois cohomology classes $H_{\text{ét}}^1(F, \mathbf{O}_n)$, in order to see that ρ is a monomorphism. Now, given two quadratic forms $q_1 = \langle u_1, \dots, u_n\rangle$ and $q_2 = \langle v_1, \dots, v_n\rangle$ over F, with $W_1(q_1) = W_1(q_2)$, we obtain the equality

$$\langle\langle u_1\rangle\rangle + \cdots + \langle\langle u_n\rangle\rangle = \langle\langle v_1\rangle\rangle + \cdots + \langle\langle v_n\rangle\rangle$$

in $I(F)$ by using the isomorphism $\underline{\mathbf{K}}_1^{\mathrm{W}}(F) \cong I(F)$ (cf. [50, Cor. 3.47]). The desired isometry of q_1 and q_2 can then be deduced by Witt's cancellation theorem.

4.1.1 Some Combinatorial Identities

The refined Stiefel-Whitney classes are not completely independent of each other. There are several combinatorial identities that relate them, most of which were also known classically. Here we prove some of those that are useful for our later calculations. Therefore we will keep a positive natural number $n \geq 1$ fixed, as well as our assumption that k is a perfect field of characteristic $\neq 2$. We start with a simple statement that describes what happens, when we plug padded quadratic forms into the refined Stiefel-Whitney classes.

Lemma 4.21. *Let $F \in \mathcal{F}_k$, $q \in H_{\text{ét}}^1(F, \mathbf{O}_n)$, and $\delta \in F^\times$ be arbitrary. Setting $W_0 = 1$, and $W_m = 0$, for all $m > n$, then we find for all $i \in \{0, \dots, n+n'\}$ and $n' \in \mathbb{N}$*

$$W_i(q \oplus n' \cdot \langle\delta\rangle) = \sum_{j=\max\{0,i-n'\}}^{i} \binom{n'}{i-j} [\delta]^{i-j} W_j(q),$$

where on the left-hand side we have rank $n + n'$ SW-classes, and on the right-hand side rank n SW-classes. Moreover, we also have the following identity for all $i \in \{0, \dots, n\}$

$$W_i(q \cdot \langle\delta\rangle) = \sum_{j=0}^{i} \binom{n-j}{i-j} [\delta]^{i-j} \langle\delta\rangle^j W_j(q).$$

Proof. We may assume that q is given in diagonal form by $\langle u_1, \dots, u_n\rangle$. From this the identities follow quickly by corresponding identities of elementary symmetric polynomials. □

The next formula is useful, whenever we encounter a product of SW-classes. It allows us to expand the product into a sum of SW-classes with combinatorial coefficients. These formulae are not new, since they also hold in the classical case (cf. [48, pp. 330f.]).

Lemma 4.22. *Let $F \in \mathsf{F}_k$, and $q \in H^1_{\text{ét}}(F, \mathbf{O}_n)$ be arbitrary. Then we have for all $0 \le i \le j \le n$:*

$$W_i(q)W_j(q) = \sum_{l=0}^{i} \binom{i+j-l}{l, i-l, j-l} [-1]^l W_{i+j-l}(q),$$

where we have used the ***trinomial coefficient****, which is defined for a natural number $m \in \mathbb{N}$, and indices $a, b, c \in \{0, \dots, m\}$ summing to m, via:*

$$\binom{m}{a, b, c} = \frac{m!}{a!b!c!}.$$

Proof. We proceed as before, and assume q to be the diagonal form $\langle u_1, \dots, u_n\rangle$. Define a homomorphism of rings $\psi\colon \mathbb{Z}[X_1, \dots, X_n, t] \to K_*^{\mathrm{W}}(F)$ that sends X_i to $[u_i]$, and t to $[-1]$. From lemma 4.2 we know that the ideal

$$\mathfrak{a} := \left(X_i^2 - X_i t \;\middle|\; i = 1, \dots, n \right)$$

lies in the kernel of ψ. To establish the claim, it will be sufficient to check that

$$e_i e_j \equiv \sum_{l=0}^{i} \binom{i+j-l}{l, i-l, j-l} t^l e_{i+j-l} \mod \mathfrak{a}$$

holds for all $0 \le i \le j \le n$, where $e_0, \dots, e_n \in \mathbb{Z}[X_1, \dots, X_n]$ are the elementary symmetric polynomials in n variables, with e_i being homogeneous of degree i, and where we set $e_m = 0$, for all $m > n$. We omit giving more details. □

We observe, that in the case that -1 is a square the refined SW-classes become multiplicative.

The next formula we state is rather complicated, so we first provide some motivation. Note that there are several ways to embed $\mathbf{O}_n$ into $\mathbf{O}_{n+1}$ (for $n \ge 1$), with the obvious embedding being given by

$$\mathbf{O}_n \to \mathbf{O}_{n+1}, \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Below, our aim will be to determine $\mathbb{H}_0^{\mathbb{A}^1}$ for the special orthogonal groups, so we will have particular use for the embedding

$$\mathbf{O}_n \to \mathbf{O}_{n+1}, \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix},$$

as it factors through $\mathbf{SO}_{n+1}$. On Galois cohomology this later embedding sends a quadratic form q to $q \oplus \langle \det(q)\rangle$. We analyse this situation with an additional parameter in the following

Lemma 4.23. *Let $F \in \mathsf{F}_k$, $q \in H^1_{\text{ét}}(F, \mathbf{O}_n)$, and $\delta \in F^\times$ be arbitrary. Then we have for all $1 \le i \le n$:*

$$W_i(q \oplus \langle \det(q)\delta\rangle) = \left([\delta] + \langle\delta\rangle[-1]\frac{1 + (-1)^i}{2} \right) W_{i-1}(q) + \left(1 + (-1)^{i-1} i \langle\delta\rangle\right) W_i(q) + \dots$$
$$\dots + (-1)^{i-1}\langle\delta\rangle \sum_{j=i+1}^{n} \binom{j}{i-1} \eta^{j-i} W_j(q).$$

Proof. We proceed by employing lemma 4.21, together with axiom **(MW2)**:

$$W_i(q \oplus \langle\det(q)\delta\rangle) = W_{i-1}(q)[\det(q)\delta] + W_i(q) = W_{i-1}(q)([\delta] + \langle\delta\rangle[\det(q)]) + W_i(q).$$

Since symbols in $K_1^{\mathrm{W}}(F)$ are independent of squares, we may assume, when expanding $[\det(q)]$, that q is in diagonal form. Then we find inductively, using lemmas 4.21 and **(MW2)**,

$$[\det(q)] = \sum_{j=1}^{n} \eta^{j-1} W_j(q).$$

One concludes by lemma 4.22 and a lengthy (but straightforward) calculation. □

4.2 Special Orthogonal Groups

We continue assuming k to be perfect and of $\mathrm{char}(k) \neq 2$. The *special orthogonal group* with respect to a quadratic space (V, q), where V is a finite dimensional k-vector space, and q is a non-singular quadratic form on V over k is defined as the group of isometries of (V, q) of determinant one:

$$\mathbf{SO}(V, q) := \ker\left(\mathbf{O}(V, q) \xrightarrow{\det} \boldsymbol{\mu}_2\right).$$

Again we set $\mathbf{SO}_n := \mathbf{SO}(k^n, q_n)$, where q_n is the rank n euclidean quadratic form. From the arguments in [39, (29.29)], one obtains that on the category F_k the cohomology sheaf $\mathcal{H}^1_{\text{ét}}(\mathbf{SO}(V, q))$ is given by the functor

$$\begin{aligned}\mathsf{Quad}_{n,\delta}\colon \mathsf{F}_k &\longrightarrow \mathsf{Set}\\ F &\longmapsto \left\{\begin{array}{c}\text{isometry classes of}\\ \text{non-deg. quadratic spaces } (V', q'), \text{ with}\\ \dim_F(V') = n \text{ and } \det(q') = \delta\end{array}\right\},\end{aligned}$$

where $\delta := \det(q)$. So, in order to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}(V, q))$ for all non-degenerated quadratic spaces (V, q), it suffices to determine the cohomological invariants of $\mathsf{Quad}_{n,\delta}$ in any strictly $\mathbb{A}^1$-invariant sheaf M, for all $n > 1$ and $\delta \in k^\times$. Our approach is motivated by the discussion in [21, Ch. VI].

We postpone the case $n = 2$, and fix $n > 2$ and $\delta \in k^\times$. In addition to the euclidean form $q_n = n \cdot \langle 1 \rangle$, we also define the form $q_{n,\delta} := (n-1)\langle 1 \rangle \oplus \langle \delta \rangle$ and associated groups $\mathbf{O}_{n,\delta}$ resp. $\mathbf{SO}_{n,\delta}$. By setting $\iota_{n,\delta}\colon \mathbf{O}_{n-1} \to \mathbf{SO}_{n,\delta}$ to be the homomorphism

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix}$$

of algebraic groups, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{O}_{n-1} & \xrightarrow{\iota_{n,\delta}} & \mathbf{SO}_{n,\delta} \\ & \searrow & \downarrow \\ & & \mathbf{O}_{n,\delta}, \end{array}$$

leading to the induced diagram of $\mathsf{F}_k \to \mathsf{Set}$ functors

$$\begin{array}{ccc} \mathsf{Quad}_{n-1} & \twoheadrightarrow & \mathsf{Quad}_{n,\delta} \\ & \searrow & \downarrow \\ & & \mathsf{Quad}_n. \end{array}$$

Hereby the horizontal arrow sends a non-degenerate quadratic form q of rank $n-1$ to the form $q \oplus \langle \det(q)\delta \rangle$. As such, this transformation is surjective on all $F \in \mathsf{F}_k$, since any quadratic form $q' \in \mathsf{Quad}_{n,\delta}$ can be diagonalised, so we may assume $q' \simeq \langle u_1, \ldots, u_n \rangle$. As we have $\det(q') = \delta$, we find in particular $u_n \equiv u_1 \cdots u_{n-1}\delta$ modulo $(F^\times)^2$, from which surjectivity is evident. This leads to an induced injection

$$\underbrace{\mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\mathbf{SO}_{n,\delta}), M)}_{\cong \mathrm{Inv}_k(\mathsf{Quad}_{n,\delta}, M)} \hookrightarrow \underbrace{\mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\mathbf{O}_{n-1}), M)}_{\cong \mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M)}.$$

So, we would like to determine those invariants in $\mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M)$ that actually come from invariants on $\mathsf{Quad}_{n,\delta}$. Moreover, by restricting invariants from Quad_n, we also have a method of generating new invariants on $\mathsf{Quad}_{n,\delta}$.

Example 4.24. Set $n = 5$, and $\delta = 1$. Let q' be a quadratic form of dimension four. When we evaluate the SW-classes defined on $\mathcal{H}^1_{\text{ét}}(\mathbf{O}_5)$ on the form $q := q' \oplus \langle \det(q') \rangle$, we obtain by lemma 4.23, where on the right-hand side we have SW-classes on $\mathcal{H}^1_{\text{ét}}(\mathbf{O}_4)$:

$$\begin{aligned}
W_1(q) &= 2W_1(q') + \eta W_2(q') + \eta^2 W_3(q') + \eta^3 W_4(q'),\\
W_2(q) &= [-1]W_1(q') - W_2(q') - 3\eta W_3(q') - 4\eta^2 W_4(q'),\\
W_3(q) &= 4W_3(q') + 6\eta W_4(q'),\\
W_4(q) &= [-1]W_3(q') - 3W_4(q'), \quad \text{and} \quad W_5(q) = 0.
\end{aligned}$$

From this, we immediately see that the SW-classes on $\mathcal{H}^1_{\text{ét}}(\mathbf{O}_5)$ restricted to $\mathsf{Quad}_{5,1}$ are not independent, as we have at least the following relations:

$$W_1(q) = -\eta W_2(q) + \eta^3 W_4(q), \quad W_3(q) = -2\eta W_4(q), \quad \text{and} \quad W_5(q) = 0.$$

In particular, we see that the odd SW-classes are $K^{\mathrm{W}}_*(k)$-dependent on the even ones. We would like to use this opportunity to introduce another perspective. Any normed[3] invariant $a \in \mathrm{Inv}^{\text{norm.}}_k(\mathsf{Quad}_5, M)$ may be given by finding $\lambda_i \in \mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}^{\mathrm{W}}_i, M)$, for all $i = 1, \ldots, 5$ such that

$$a = \lambda_1 \circ W_1 + \cdots + \lambda_5 \circ W_5$$

holds. In the following we omit denoting the composition explicitly by $\circ$. Now restricting along $\iota_{5,1}$, we obtain another invariant, which we may write as

$$\lambda'_1 W_1 + \ldots \lambda'_4 W_4,$$

with SW-classes of rank 4, and where the λ'_i lie in $\mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}^{\mathrm{W}}_i, M)$. Any element $x \in K^{\mathrm{W}}_i(k)$ induces a homomorphism $x \cdot (-) \colon \underline{\mathbf{K}}^{\mathrm{W}}_* \to \underline{\mathbf{K}}^{\mathrm{W}}_{*+i}$, by proposition 3.28, which we abusively denote by x as well. By precomposing, this gives an action of $K^{\mathrm{W}}_*(k)$ on the graded abelian group $\mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}^{\mathrm{W}}_*, M)$. We use this to describe the λ'_i:

$$\begin{aligned}
\lambda'_1 &= 2\lambda_1 + \lambda_2[-1],\\
\lambda'_2 &= \lambda_1\eta - \lambda_2,\\
\lambda'_3 &= \lambda_1\eta^2 - 3\lambda_2\eta + 4\lambda_3 + \lambda_4[-1], \quad \text{and}\\
\lambda'_4 &= \lambda_1\eta^3 - 4\lambda_2\eta^2 + 6\lambda_3\eta - 3\lambda_4.
\end{aligned} \tag{4.9}$$

Moreover, we see that with respect to this action, the λ'_i are not independent, as we have the relations

$$\lambda'_1 + \lambda'_2[-1] = 0, \quad \text{and} \quad \lambda'_1\eta^2 - 3\lambda'_2\eta + 3\lambda'_3 + \lambda'_4[-1] = 0. \tag{4.10}$$

We briefly show the converse, namely, that any invariant $\lambda'_1 W_1 + \cdots + \lambda'_4 W_4$ in $\mathrm{Inv}^{\text{norm.}}_k(\mathsf{Quad}_4, M)$ comes from an invariant of $\mathsf{Quad}_{5,1}$, when the relations (4.10) hold. In order to do that, we have to construct an invariant of $\mathsf{Quad}_{5,1}$, which we do by restricting a suitable invariant of Quad_5. Since we already know that the odd SW-classes are dependent, we start with an ansatz in which λ_1, λ_3 and λ_5 are zero (cf. (4.9)). This allows us to derive $\lambda_2 = -\lambda'_2$ and

$$\lambda'_3\eta - \lambda'_4 = \lambda_4 + \lambda_2\eta^2,$$

[3] The functor Quad_5 maps canonically to $\mathit{Set}_\bullet$, by taking q_5 as the distinguished point. As such, being a *normed invariant* means sending q_5 to zero. In the morphism of sheaves point of view, the normed invariants correspond to morphisms of pointed sheaves.

or equivalently $\lambda_4 = \lambda_2'\eta^2 + \lambda_3'\eta - \lambda_4'$. With this choice of λ_2 and λ_4, together with the relations (4.10), we obtain $\lambda_1', \dots, \lambda_4'$ back from (4.9). In particular the map

$$\begin{aligned}\{(\lambda_1', \dots, \lambda_4') \mid \text{fulfilling (4.10)}\} &\longrightarrow \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_2^{\mathrm{W}}, M) \oplus \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_4^{\mathrm{W}}, M)\\ (\lambda_1', \dots, \lambda_4') &\longmapsto (-\lambda_2', \lambda_2'\eta^2 + \lambda_3'\eta - \lambda_4')\end{aligned}$$

is bijective, with inverse induced by (4.9). In this way, we obtain a twofold description of the normed invariants of $\mathsf{Quad}_{5,1}$: Namely, as those Quad_4-invariants that are subject to the relations (4.10), or equivalently as those invariants of Quad_5 that are spanned by W_2 and W_4. The latter point of view immediately yields an isomorphism

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}_5) \cong \underline{\mathbf{K}}_2^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}},$$

induced by W_2 and W_4.

The above example is the blueprint for our discussion of $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}(V, q))$, for (V, q) a non-degenerate quadratic space. Therefore we first note the following corollary readily employing the notation introduced in the example:

Corollary 4.25. *Let $n \geq 1$, and $\delta \in k^\times$ be arbitrary. Given an invariant a of Quad_n, defined by $\lambda_i \in \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$, for $i = 1, \dots, n$, and $\lambda_0 \in M(k)$ such that*

$$a = \lambda_0 + \lambda_1 W_1 + \cdots + \lambda_n W_n$$

holds, then its restriction along $\iota_{n,\delta}$ is given as a combination $\lambda_0' + \lambda_1' W_1 + \cdots + \lambda_{n-1}' W_{n-1}$, with

$$\lambda_i' = \sum_{l=1}^{i-1} (-1)^{l-1} \binom{i}{l-1} \lambda_l \langle\delta\rangle \eta^{i-l} + \lambda_i \left(1 + (-1)^{i-1}\langle\delta\rangle i\right) + \lambda_{i+1} \left([\delta] + \langle\delta\rangle[-1]\frac{1 + (-1)^{i+1}}{2}\right),$$

for all $i \in \{0, \dots, n-1\}$, where λ_j with j outside $\{0, \dots, n\}$ is set to zero.

Proof. Follows from lemma 4.23 by elementary sum transformations. □

An important feature we would like to highlight about the preceding statement is, that the λ_i' are independent of n, a fact which allows us to employ inductive arguments later on. Let us derive some relations in the style of example 4.24. Later we will see that these are all necessary relations. As an “a priori” motivation of those, we stress that in the classical case Garibaldi and Serre (cf. [21, Sec. 19]) proceed similarly.

Let $n > 2$, and $\delta \in k^\times$ be arbitrary. We also fix a strictly $\mathbb{A}^1$-invariant sheaf M, and some field $F \in \mathsf{F}_k$. Now given an invariant $a \in \mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M)$, which we assume comes from an invariant $a' \in \mathrm{Inv}_k(\mathsf{Quad}_{n,\delta}, M)$, then for any choice $u_1, \dots, u_{n-1} \in F^\times$, we have

$$a'(\langle u_1, \dots, u_{n-1}, \delta u_1 \cdots u_{n-1}\rangle) = a'(\langle u_1, \dots, u_{n-2}, \delta u_1 \cdots u_{n-1}, u_{n-1}\rangle).$$

Since a is the restriction of a', the above equality implies

$$a(\langle u_1, \dots, u_{n-1}\rangle) = a(\langle u_1, \dots, u_{n-2}, \delta u_1 \cdots u_{n-2}\rangle).$$

In order to derive conditions on a, we want to read the latter equality as an equality of invariants on Quad_{n-2}, so assume we are given some quadratic form $q \simeq \langle u_1, \dots, u_{n-2}\rangle$, and take a parameter $u_{n-1} \in k^\times$, then we consider the homomorphism

$$\begin{aligned}\Delta^{(n)}_{u_{n-1},\delta} \colon \mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M) &\longrightarrow \mathrm{Inv}_k(\mathsf{Quad}_{n-2}, M)\\ a &\longmapsto \begin{pmatrix} \mathsf{Quad}_{n-2} & \to M \\ \text{on } F\colon q & \mapsto a(q \oplus \langle u_{n-1}\delta \det(q)\rangle) - a(q \oplus \langle u_{n-1}\rangle)\end{pmatrix}.\end{aligned}$$

Any $\mathsf{Quad}_{n,\delta}$-invariant is mapped to zero, so we proceed by analysing conditions for an invariant $a \in \mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M)$ to be in the kernel of $\Delta^{(n)}_{u_{n-1},\delta}$. Assume that a is given by $\lambda_0 \in M(k)$, and $\lambda_i \in \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$. Our aim is to derive conditions that are expressions in the coefficients λ_i. By lemma 4.21, we have

$$a(q \oplus \langle u_{n-1}\rangle) = \lambda_0 + \lambda_1[u_{n-1}] + \sum_{i=1}^{n-2} (\lambda_{i+1}[u_{n-1}] + \lambda_i)\, W_i(q).$$

If we write $\Delta^{(n)}_{u_{n-1},\delta}(a)$ in the form $\lambda_0' + \sum_{i=1}^{n-2} \lambda_i' W_i$, with $\lambda_0' \in M(k)$ and the λ_i' lying in $\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$, then the above, together with corollary 4.25, yields

$$\lambda_i' = \sum_{l=1}^{i} (-1)^{l-1} \binom{i}{l-1} \lambda_l \eta^{i-l} \langle u_{n-1}\delta\rangle + \lambda_{i+1}\left([\delta] + \langle\delta\rangle[-1]\frac{1+(-1)^{i+1}}{2}\right)\langle u_{n-1}\rangle,$$

for all $i = 0, \ldots, n-2$. Since $\langle u_{n-1}\rangle$ squares to one in $K_*^{\mathrm{MW}}(k)$ (cf. lemma 4.2), we see that being in the kernel of $\Delta^{(n)}_{u_{n-1},\delta}$ is independent of a particular choice of u_{n-1}. By multiplying with $\langle\delta\rangle$, we obtain the following set of relations

$$\sum_{l=1}^{i} (-1)^{l-1} \binom{i}{l-1} \lambda_l \eta^{i-l} + \frac{1+(-1)^{i+1}}{2} \lambda_{i+1}[-1] = \lambda_{i+1}[\delta], \tag{R}$$

numbered by $i \in \{0, \ldots, n-2\}$, holding in $\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$ resp. $M(k)$. The form of these conditions has the striking advantage that they strictly contain each other, i.e. by increasing n, one only has to add relations, keeping earlier ones. Hence, let us give a list of the first four, among which we may recognize some of the relations that we found in example 4.24:

$$\begin{aligned} 0 &= \lambda_1[\delta], \\ \lambda_1 + \lambda_2[-1] &= \lambda_2[\delta], \\ \lambda_1\eta - 2\lambda_2 &= \lambda_3[\delta], \\ \lambda_1\eta^2 - 3\lambda_2\eta + 3\lambda_3 + \lambda_4[-1] &= \lambda_4[\delta], \quad \text{and so forth.} \end{aligned}$$

We see explicitly that these relations are far from being independent of each other, since e.g. (R)(1) implies (R)(0) and (R)(2). However, they turn out to contain all the necessary ones. In particular, when we fix the name $R^M_{n,\delta}$ for the kernel of $\Delta^{(n)}_{x,\delta}$, with $x \in k^\times$ arbitrary, we obtain

Proposition 4.26. *The restriction along $\iota_{n,\delta}\colon \mathsf{Quad}_{n-1} \twoheadrightarrow \mathsf{Quad}_{n,\delta}$ induces a (natural in M) isomorphism*

$$\mathrm{Inv}_k(\mathsf{Quad}_{n,\delta}, M) \xrightarrow{\cong} R^M_{n,\delta} \subseteq \mathrm{Inv}_k(\mathsf{Quad}_{n-1}, M).$$

Proof. Clearly, by construction of $R^M_{n,\delta}$, and the surjectivity of $\iota_{n,\delta}$, we obtain an injective homomorphism

$$r\colon \mathrm{Inv}_k(\mathsf{Quad}_{n,\delta}, M) \longrightarrow R^M_{n,\delta},$$

as claimed. So, it remains to check the surjectivity of r. To that end, fix an invariant $a \in R^M_{n,\delta}$, and our aim is to check that it comes from an invariant on $\mathsf{Quad}_{n,\delta}$. Therefore we define, for every $F \in \mathsf{F}_k$ and diagonal quadratic form $\langle u_1, \ldots, u_n\rangle \in \mathsf{Quad}_{n,\delta}(F)$,

$$a'(\langle u_1, \ldots, u_n\rangle) = a(\langle u_1, \ldots, u_{n-1}\rangle).$$

We have to check that this is well-defined. So, given an isometry $\langle u_1, \ldots, u_n\rangle \simeq \langle v_1, \ldots, v_n\rangle$ of diagonal forms having determinant δ, we know by Witt's chain equivalence theorem [40, Thm.

I.5.2] that there is a chain equivalence between those two presentations, which we may assume to be simple. Thus we find two distinct indices $i, j \in \{1, \dots, n\}$, with $i < j$, such that

$$\langle u_i, u_j \rangle \simeq \langle v_i, v_j \rangle,$$

holds, as well as $u_{i'} = v_{i'}$ for the remaining indices $i' \in \{1, \dots, n\} \smallsetminus \{i, j\}$. When we have $i, j < n$, we immediately obtain the desired equality

$$a(\langle u_1, \dots, u_{n-1} \rangle) = a(\langle v_1, \dots, v_{n-1} \rangle),$$

by Witt's extension theorem (cf. [40, Exc. I.11]). For the remaining cases, it suffices without loss of generality to consider $i < n-1$ and $j = n$. Note that we may derive

$$a(\langle u_1, \dots, u_{n-2}, \delta u_1 \cdots u_{n-2} u_n \rangle) = a(\langle u_1, \dots, u_{n-2}, u_n \rangle),$$

by using the above combinatorial identities 4.21 and 4.23, and employing the relations (R). Now we conclude well-definedness by the following calculation:

$$\begin{aligned} a(\langle u_1, \dots, u_{n-1} \rangle) = a(\langle u_1, \dots, u_{n-2}, \delta u_1 \cdots u_{n-2} u_n \rangle) = a(\langle u_1, \dots, u_{n-2}, u_n \rangle) = \\ = a(\langle v_1, \dots, v_{n-2}, v_n \rangle) = a(\langle v_1, \dots, v_{n-1} \rangle). \end{aligned}$$

From this it is clear that a' defines an invariant that restricts to a on Quad_{n-1}. $\square$

The above proposition determines the invariants of $\mathsf{Quad}_{n,\delta}$ completely, however, in order to obtain a neat formula for $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}_{n,\delta})$, we need to find a strictly $\mathbb{A}^1$-invariant sheaf that corepresents $R^{(-)}_{n,\delta}$. We reduce the problem to linear algebra, in the following way:
The homomorphism $\Delta^{(n)}_{1,\delta}$ induces a homomorphism on normed invariants, which we denote by the same name. We may represent $\Delta^{(n)}_{1,\delta}$ by the matrix

$$(A(n,\delta))_{\substack{1 \le i \le n-2 \\ 1 \le j \le n-1}} := \begin{cases} (-1)^{j-1} \binom{i}{j-1} \eta^{i-j}, & \text{if } j \le i, \\ \frac{1+(-1)^{i+1}}{2}[-1] - [\delta], & \text{if } j = i+1, \quad \text{and} \\ 0, & \text{else }, \end{cases}$$

in such a way that a normed invariant $\lambda_1 W_1 + \cdots + \lambda_{n-1} W_{n-1}$ in $\mathrm{Inv}_k^{\text{norm.}}(\mathsf{Quad}_{n-1}, M)$, with $\lambda_i \in \mathrm{Hom}_{\mathit{Ab}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$ for $i = 1, \dots, n$, is sent to the normed invariant

$$\sum_{i=1}^{n-2} \left(\sum_{j=1}^{n-1} \lambda_j A(n,\delta)_{i,j} \right) W_i.$$

We now define transformations of $\mathrm{Inv}_k^{\text{norm.}}(\mathsf{Quad}_{n-1}, M)$, and $\mathrm{Inv}_k^{\text{norm.}}(\mathsf{Quad}_{n-2}, M)$, again in matrix notation, so that the kernel of $\Delta^{(n)}_{1,\delta}$ can easily be read off. We begin by defining an invertible $(n-1) \times (n-1)$-matrix $T(n,\delta)$ with entries in $K^{\mathrm{W}}_*(k)$ via

$$(T(n,\delta))_{1 \le i,j \le n-1} := \begin{cases} [\delta] - [-1], & \text{if } j \text{ is even} \wedge i = j-1, \\ 1 + \frac{i-2}{2}\eta([\delta] - [-1]), & \text{if } j \text{ is even} \wedge i = j, \\ \eta^{i-j}\left(\binom{i-1}{j-2} - \binom{i-1}{j-1} \frac{j-2}{2} \right), & \text{if } j \text{ is even} \wedge i > j, \\ 1, & \text{if } j \text{ is odd} \wedge i = j, \\ \frac{j-1}{2}\eta, & \text{if } j \text{ is odd} \wedge i = j+1, \quad \text{and} \\ 0, & \text{else.} \end{cases}$$

It is a straightforward exercise to show that the product $A(n,\delta)T(n,\delta)$ fulfils

$$(A(n,\delta)T(n,\delta))_{\substack{1\le i\le n-2\\ 1\le j\le n-1}} := \begin{cases} -[\delta], & \text{if } j \text{ is odd} \wedge i = j-1,\\ 1-\frac{i-1}{2}\eta[\delta], & \text{if } j \text{ is odd} \wedge i = j,\\ \eta^{i-j}\left(\binom{i}{j-1}-\binom{i}{j}\frac{j-1}{2}\right), & \text{if } j \text{ is odd} \wedge i > j,\\ 0, & \text{else,}\end{cases},$$

so that by multiplying on the left with a product of elementary matrices, one obtains a matrix

$$(D(n,\delta))_{\substack{1\le i\le n-2\\ 1\le j\le n-1}} := \begin{cases} -[\delta], & \text{if } j \text{ is odd} \wedge i = j-1,\\ 1, & \text{if } j \text{ is odd} \wedge i = j, \quad \text{and}\\ 0, & \text{else.}\end{cases}$$

From this, the following is an easy corollary:

Proposition 4.27. *Let k be a perfect field of characteristic $\neq 2$, $n \geq 3$, and $\delta \in k^\times$. If n is odd, then there is an isomorphism*

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{ét}}\mathbf{SO}_{n,\delta}) \cong \mathbb{Z} \oplus \bigoplus_{\substack{i=2\\ i \text{ even}}}^{n-1} \underline{\mathbf{K}}_i^{\mathrm{W}}.$$

In the case that n is even, we consider the graded ideal $R := ([\delta], h) \subseteq K_*^{\mathrm{MW}}(k)$, so that by remark 4.6 the sheaf $\underline{\mathbf{K}}_{n-1}^R$ is strictly $\mathbb{A}^1$-invariant. Then there is an isomorphism

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{ét}}\mathbf{SO}_{n,\delta}) \cong \mathbb{Z} \oplus \bigoplus_{\substack{i=2\\ i \text{ even}}}^{n-2} \underline{\mathbf{K}}_i^{\mathrm{W}} \oplus \underline{\mathbf{K}}_{n-1}^R.$$

Proof. We only treat the case of n being even and ≥ 4, since the complementary case is simpler and can be handled analogously. For any M strictly $\mathbb{A}^1$-invariant, the above constructed similarity transform of $A(n,\delta)$ comes with two isomorphisms, one of $\mathrm{Inv}_k^{\mathrm{norm.}}(\mathsf{Quad}_{n-1}, M)$, and one of $\mathrm{Inv}_k^{\mathrm{norm.}}(\mathsf{Quad}_{n-2}, M)$, such that the kernel $R_{n,\delta}^M$ of $\Delta_{1,\delta}^{(n)}$ is isomorphic to

$$M \oplus \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_2^{\mathrm{W}}, M) \oplus \cdots \oplus \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_{n-2}^{\mathrm{W}}, M) \oplus U_{n,\delta}^M,$$

where $U_{n,\delta}^M$ is the subgroup of $\mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_{n-1}^{\mathrm{W}}, M)$ given by homomorphisms annihilating the subgroup $[\delta]K_{n-2}^{\mathrm{W}}(F)$, for all $F \in \mathsf{F}_k$. Note that from the construction of the similarity transform it is clear, that this isomorphism is natural with respect to M.
Using the theory of [50, Lem. 3.32], we see that $\underline{\mathbf{K}}_{n-1}^R$ yields a strictly $\mathbb{A}^1$-invariant sheaf, that evaluates to the abelian group $K_{n-1}^{\mathrm{W}}(F)/[\delta]K_{n-2}^{\mathrm{W}}(F)$, for every $F \in \mathsf{F}_k$. By propositions 3.25 and 3.28 we get that $\underline{\mathbf{K}}_{n-1}^R$ corepresents $U_{n,\delta}^{(-)}$, concluding the proof of the proposition. ☐

Remark 4.28. From the above proposition, we may deduce the form of $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B_{ét}}\mathbf{SO}(V,q))$, for any non-degenerate quadratic space (V,q) of dimension $n \geq 3$ and determinant δ, by considering $\mathbf{SO}(V,q)$ an inner form of $\mathbf{SO}_{n,\delta}$, with $\delta := \det(q)$, similarly as in remark 4.19.

Remark 4.29 (Embedding of $\pi_0^{\mathbb{A}^1}$). The invariants of proposition 4.27 determine the elements of the Galois cohomology set $H^1_{\text{ét}}(-, \mathbf{SO}_{n,\delta})$ completely. Indeed, for quadratic forms of rank n and determinant δ with agreeing $\mathbf{SO}_{n,\delta}$-invariants, one may derive that all their $\mathbf{O}_n$-invariants agree as well, by restricting the $\mathbf{O}_n$-invariants and rewriting them in terms of the universal $\mathbf{SO}_{n,\delta}$-invariants.[4] One then finishes by employing remark 4.20.

[4] The explicit formulae relating $\mathbf{O}_n$- and $\mathbf{SO}_{n,\delta}$-invariants are rather lengthy, and since they bring little additional benefit, we omit them here.

4.3 The Symmetric Groups

In this section our aim is to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}} S_n)$, where S_n denotes the symmetric group in n symbols. We proceed, as laid out by Morel in his ICM address (cf. [49, Thm. 5.3]), and thus, since we intend to use the case of the orthogonal groups above, we still assume k to be perfect and of characteristic $\neq 2$. One may also compare our course of action with the classical case of [21, Thm. 24.11], i.e. the determination of the invariants of S_n, with values in the Galois cohomology of $\mathbb{Z}/(2)$, in light of the characterisation of morphisms of unramified sheaves.

Let us first define the functor $\mathsf{Et}_n \colon \mathsf{F}_k \to \mathsf{Set}$, which maps a field $F \in \mathsf{F}_k$ to the set of F-isomorphism classes of rank n étale F-algebras. By [39, (29.9)] we have a natural (in $F \in \mathsf{F}_k$) bijection

$$\mathsf{Et}_n(F) \cong H^1_{\text{ét}}(F, S_n),$$

that works in the following way: Given such an étale F-algebra, postcomposition by elements of the Galois group $\mathrm{Gal}(F_{\text{sep}}/F)$ defines an action on the set

$$\mathrm{Hom}_{\mathsf{Alg}_F}(E, F_{\text{sep}})$$

containing precisely n elements, effectively defining a 1-cocycle. Conversely, any continuous homomorphism $\gamma \colon \mathrm{Gal}(F_{\text{sep}}/F) \to S_n$ determines the étale F-algebra of rank n

$$\left\{(x_1, \dots, x_n) \in (F_{\text{sep}})^n \;\middle|\; \sigma(x_i) = x_{\gamma(\sigma)(i)},\ \forall \sigma \in \mathrm{Gal}(F_{\text{sep}}/F)\right\}.$$

We also recall the following piece of notation: Any étale F-algebra E is called *multiquadratic* if and only if it is isomorphic to a product of étale F-algebras of rank 1 or 2. In the following proposition we relax the assumption that k is perfect, in order to salvage an inductive argument (cf. [21, Thm. 24.9]) due to Garibaldi and Serre over to our setting.

Proposition 4.30 (Splitting principle)**.** *Let M be a strictly $\mathbb{A}^1$-invariant sheaf, and let us fix a morphism*

$$a \in \mathrm{Hom}_{\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(S_n), M)$$

such that its restriction to F_k vanishes for all multiquadratic étale algebras. Then a is trivial as a morphism of pointed sheaves.

Proof. We proceed by induction over $n \in \mathbb{N}^+$. The cases of $n = 1$ or $n = 2$ are trivial by assumption, using the fact that morphisms from a weakly unramified sheaf to an unramified sheaf are determined on fields (cf. proposition 3.25 and remark 3.21).

So assume $n \geq 3$. We first derive from the inductive hypothesis, that given an étale F-algebra E of rank n that decomposes as $E' \times E''$, where E' resp. E'' are étale F-algebras of rank i resp. $n - i$, with $i \in \{1, 2\}$, we have $a(E) = 0$.[5] Let us denote the structural morphism $\mathrm{Spec}(F) \to \mathrm{Spec}(k)$ by f, and consider the morphism $f^*(a)$. To be precise, we apply the pullback functor f^* on sheaves of sets, and as Sm_k has fiber products $f^*(M)$ agrees with the pullback on sheaves of abelian groups (cf. [47, Prop. II.2.3]). By corollary 2.41 we thus have that $f^*(M)$ is again strictly $\mathbb{A}^1$-invariant. Moreover, connecting the definition of f^* (e.g. [47, Prop. II.2.2]) and of the essentially smooth extension, we see that

$$f^*(M) \cong a_{\mathrm{Nis}}\left(\check{M}_{\restriction \mathsf{Sm}_F}\right)$$

[5] Note that we refrain from distinguishing étale algebras and their associated isomorphism classes, in an effort increase readability.

holds. Applying a similar reasoning to the sheaf $\mathcal{H}^1_{\text{ét}}(S_n)$ and using the continuity of $H^1_{\text{ét}}(-, S_n)$ (cf. [42, Thm. 2.1]), we obtain

$$f^*(\mathcal{H}^1_{\text{ét}}(S_n)) \cong \mathcal{H}^1_{\text{ét}}(S_n)$$

as sheaves of sets over Sm_F. Thus the restrictions of $f^*(a)$ and a to F_F agree. So we may construct, using the product induced by the homomorphism $S_i \times S_{n-i} \to S_n$, a morphism

$$\begin{aligned} a' \colon \mathcal{H}^1_{\text{ét}}(S_{n-i}) &\longrightarrow f^*(M) \\ \text{on } X \in \mathsf{Sm}_F \colon T &\longmapsto f^*(a)\left(E'_X \cdot T\right). \end{aligned}$$

By the above arguments the morphism a' inherits the property of a to vanish on multiquadratic étale algebras, and thus by the inductive hypothesis, a' needs to be trivial. Hence we obtain our claim $a(E) = 0$, of which we intend to make use of below.

To proceed, we recall the definition of a versal torsor (cf. 5.1 and 24.6 in [21]) associated to S_n in order to apply the recognition principle 3.29: Denote by Aff_x and Aff_e the affine spaces

$$\mathrm{Spec}(k[x_1, \dots, x_n]) \quad \text{and} \quad \mathrm{Spec}(k[e_1, \dots, e_n]),$$

where the $x_1, \dots, x_n$ resp. $e_1, \dots, e_n$ are variables. Sending the variable e_i to the elementary symmetric polynomial of degree $i \in \{1, \dots, n\}$ in the variables $x_1, \dots, x_n$, we obtain a ring homomorphism

$$k[e_1, \dots, e_n] \longrightarrow k[x_1, \dots, x_n],$$

which induces a morphism $\varphi\colon \mathrm{Aff}_x \to \mathrm{Aff}_e$. By the fundamental theorem for symmetric polynomials we may identify the ring $k[e_1, \dots, e_n]$ with the ring of invariants $k[x_1, \dots, x_n]^{S_n}$, where the action of S_n works by permuting the x_i. This action is free outside of

$$\bigcup_{1 \le i < j \le n} V(x_i - x_j),$$

which is the preimage (under φ) of the divisor[6] $V(\delta) \subseteq \mathrm{Aff}_e$, where δ denotes the discriminant of the polynomial

$$\prod_{i=1}^{n} (T - x_i) = T^n - e_1 T^{n-1} + e_2 T^{n-2} - \dots \pm e_n,$$

and thus is given as

$$\prod_{1 \le i < j \le n} (x_i - x_j)^2,$$

or by some irreducible polynomial in the $e_1, \dots, e_n$, as $\mathrm{char}(k) \neq 2$. With the notation of [21, Def. 5.1] in mind, we set $X := D(\delta)$, and Q its preimage under φ, so that $Q \xrightarrow{\varphi} X$ becomes an étale S_n-torsor over X. The generic fibre of φ, yields the field extension

$$\tilde{E}^{\text{gen}} := k(x_1, \dots, x_n) \quad \text{over} \quad F := k(e_1, \dots, e_n),$$

which is a Galois extension, with Galois group S_n, i.e. a S_n-torsor over F. An associated étale F-algebra of rank n may be given by

$$F(x_1), \dots, F(x_n), \quad \text{or} \quad E^{\text{gen}} := F[T]/(T^n - e_1 T^{n-1} + e_2 T^{n-2} - \dots \pm e_n).$$

[6] Here we are using $\mathrm{char}(k) \neq 2$, since otherwise δ would not be irreducible.

The possible choice $F(x_i)$ is in Galois correspondence to the embedding of S_{n-1} into S_n fixing $i = 1, \dots, n$, and all of these options are conjugate.

We want to check that $a(E^{\mathrm{gen}})$ lies in the image of $M(k)$ in $M(F)$. Therefore we note first, that by the construction of $\tilde{E}^{\mathrm{gen}}$ as the generic fibre of $Q \to X$, we have

$$a(E^{\mathrm{gen}}) \in M(X) = \bigcap_{x \in X^{(1)}} M(\mathcal{O}_{X,x}).$$

We check that it also resides in $M(A)$, with $A := \mathcal{O}_{\mathrm{Aff}_e,(\delta)}$, associated to the remaining codimension 1 point in Aff_e. In order to proceed, consider the decomposition field of $\tilde{E}^{\mathrm{gen}}/F$ (cf. [57, I.9]): So, we fix a prime ideal (of $k[x_1, \dots, x_n]_{\mathfrak{p}}$) lying over $\mathfrak{p} := (\delta) \subseteq A$, and out of convenience we choose $\mathfrak{P} := (x_1 - x_2)$. The associated decomposition group is given by

$$S_2 \times S_{n-2} \subseteq S_n,$$

and the corresponding subfield of $\tilde{E}^{\mathrm{gen}}$ will be denoted by D. It is well known that D/F is an immediate extension, i.e. of ramification index 1, and with trivial extension of residue fields. Moreover, from the explicit description of the decomposition group, we see that the polynomial defining E^{gen} splits over D, at least as:

$$\prod_{i=1}^{n}(T - x_i) = (T - T(x_1 + x_2) + x_1x_2) \cdot \prod_{i=3}^{n}(T - x_i).$$

This implies that $E^{\mathrm{gen}} \otimes_F D$ is the product of a quadratic field extension, and some rest. So, from the case treated earlier, we derive:

$$a(E^{\mathrm{gen}})_D = 0.$$

As D/F is immediate, we may apply **(A1)** to obtain a cartesian diagram[7]

$$\begin{array}{ccc} M(A) & \longrightarrow & M(B) \\ \cap & & \cap \\ M(F) & \longrightarrow & M(D) \end{array},$$

where B denotes the integral closure of A in D, localized with respect to $(x_1 - x_2)$. As $M(B)$ is a subgroup of $M(D)$, we find that $a(E^{\mathrm{gen}})$ lies in $M(A)$, and since M is an unramified sheaf, we conclude

$$a(E^{\mathrm{gen}}) \in \bigcap_{x \in \mathrm{Aff}_e^{(1)}} M(\mathcal{O}_{\mathrm{Aff}_e,x}) \cong M(\mathrm{Aff}_e).$$

Even more so, since M is $\mathbb{A}^1$-invariant, we have $M(\mathrm{Aff}_e) \cong M(k)$. This checks the above claim.

By the recognition principle 3.29, we see that a and the constant morphism with value $a(E^{\mathrm{gen}}) \in M(k)$ agree. So we find that a is constant. Moreover, since a evaluates, at least on multiquadratic étale algebras, to zero, it must be zero. This concludes the proof of the proposition. □

Remark 4.31. In [23] Gille and Hirsch provide a splitting principle for Weyl groups, with values in cycle modules. It seems likely that their arguments generalize to the case of strictly $\mathbb{A}^1$-invariant sheaves, however, since we were unable to obtain more $\mathbb{H}_0^{\mathbb{A}^1}$ by using their splitting principle, we did not explore this direction further.

[7] Here we use **(A1)** in the case of abelian groups, not merely of sets.

Let $n \in \mathbb{N}^+$ be a positive integer, and denote by $m = \lfloor \frac{n}{2} \rfloor$ the smallest integer less or equal to $\frac{n}{2}$. Then we have an injection

$$c_n\colon \boldsymbol{\mu}_2^m \longrightarrow S_n$$
$$(\epsilon_1, \dots, \epsilon_m) \longmapsto (1\ 2)^{\frac{1-\epsilon_1}{2}} \cdots (2m-1\ 2m)^{\frac{1-\epsilon_m}{2}}.$$

As in [21, Thm. 24.11], one deduces from the splitting principle that the restriction of invariants along c_n is injective. Let us reformulate this in the language of $\mathbb{A}^1$-algebraic topology:

Corollary 4.32. *The homomorphism of groups c_n induces an epimorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^m) \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}S_n)$$

of strictly $\mathbb{A}^1$-invariant sheaves.

Consider S_m as the subgroup of S_n that permutes the pairs $\{(1,2),(3,4),\dots,(2m-1,2m)\}$, which nets an action of S_m on S_n by conjugation. This action is compatible, via the homomorphism c_n, with the action that permutes the entries in $\boldsymbol{\mu}_2^m$. Considering the induced action on $\mathcal{H}^1_{\text{ét}}$, we see by using the 1-cocycle definition of cohomology, as in the case of $\mathbf{O}_n$ above, that the induced action of S_m on $H^1_{\text{ét}}(F, S_n)$ is trivial, for all $F \in \mathsf{F}_k$. Thus we have an induced epimorphism of coinvariants

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^m)_{S_m} \twoheadrightarrow \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}S_n)_{S_m} \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}S_n).$$

By defining a suitable left inverse, we will obtain that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(c_n))_{S_m}$ is actually an isomorphism. Therefore we use the explicit description of the left-hand side as

$$\underline{\mathbf{K}}_1^{\mathrm{W}} \oplus \cdots \oplus \underline{\mathbf{K}}_m^{\mathrm{W}}, \tag{4.11}$$

available by lemma 4.9. We compose c_n with the embedding of S_n into $\mathbf{O}_n$ via the permutation matrices, and consider the induced situation on $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(-))$ and their associated S_m-coinvariants:

$$\begin{array}{ccccc}
\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^m) & \xrightarrow{\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}c_n)} & \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}S_n) & \longrightarrow & \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n) \\
\downarrow & & \downarrow\cong & & \downarrow\cong \\
\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_2^m)_{S_m} & \longrightarrow & \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}S_n)_{S_m} & \longrightarrow & \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{O}_n)_{S_m}.
\end{array}$$

By means of the refined Stiefel-Whitney classes, the first row leads to a homomorphism to (4.11), and we shall focus on unwrapping how it is defined: Fix a field $F \in \mathsf{F}_k$, and a tuple of units up to squares $(u_1(F^\times)^2, \dots, u_m(F^\times)^2)$ representing a 1-cocycle in $H^1_{\text{ét}}(F, \boldsymbol{\mu}_2^m)$, then by running through the definitions, we arrive at

$$(W_1(q), \dots, W_n(q)), \quad \text{with} \quad q = m \cdot \langle 2 \rangle \oplus \langle 2 \rangle \cdot \langle u_1, \dots, u_m \rangle \,\{\oplus\langle 1\rangle, \text{ if } n \text{ is odd.}$$

We remark that the form q is the trace form of the multiquadratic étale F-algebra

$$F[T]/(T^2 - u_1) \oplus \cdots \oplus F[T]/(T^2 - u_m) \,\{\oplus F, \text{ if } n \text{ is odd.}$$

Using lemma 4.21, we obtain

$$W_i(q) = (i-1)[2]W_{i-1}(q') + \langle 2 \rangle^i W_i(q'), \quad \forall i = 0, \dots, m, \tag{4.12}$$

where $q' = \langle u_1, \dots, u_m \rangle$. Here we have used that, by lemma 4.2 and the Steinberg relation **(MW1)**, we have

$$[2]^i = [2][-1][2]^{i-2} = 0,$$

for all $i \geq 2$, and resulting from that $2[2] = [4] = 0$ by **(MW2)**, and the independence of squares in Witt K-theory. It is easy to see that formula (4.12) may be inverted, i.e. we obtain:

$$W_i(q') = (i-1)[2]W_{i-1}(q) + \langle 2 \rangle^i W_i(q), \quad \forall i = 0, \dots, m. \tag{4.13}$$

This formula motivates the following definition of a left-inverse: For any $F \in \mathsf{F}_k$, and E étale F-algebra of rank n, representing an element of the cohomology set $H^1_{\text{ét}}(F, S_n)$, we define the i^{th} *refined Galois-Stiefel-Whitney class* by $(1 \leq i \leq m)$

$$W_i^{\text{gal}}([E]) := (i-1)[2]W_{i-1}(q_E) + \langle 2 \rangle^i W_i(q_E),$$

where q_E is a quadratic form representing the element to which E is mapped under the homomorphism induced by $S_n \hookrightarrow \mathbf{O}_n$.[8] From (4.13) we obtain the following

Theorem 4.33 ([49, Thm. 5.3])**.** *Let k be a perfect field of characteristic $\neq 2$. Then we have an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}} S_n) \xrightarrow{\cong} \bigoplus_{i=1}^{m} \underline{\mathbf{K}}_i^{\mathrm{W}},$$

which is induced by the refined Galois-Stiefel-Whitney classes $W_i^{\text{gal}} \colon \mathcal{H}^1_{\text{ét}}(S_n) \to \underline{\mathbf{K}}_i^{\mathrm{W}}$.

Remark 4.34 (Embedding of $\pi_0^{\mathbb{A}^1}$). The above invariants do not determine the étale algebras completely, and let us briefly describe a counterexample for $k = \mathbb{Q}$ and $n = 3$. By a result of Epkenhans and Krüskemper, one finds a cubic field extension $E/\mathbb{Q}$ with its trace form isometric to the trace form of the split étale $\mathbb{Q}$-algebra $\mathbb{Q}^3$ (cf. [18, Thm. 1]). Naturally, the Galois-Stiefel-Whitney classes of E and $\mathbb{Q}^3$ agree, however these $\mathbb{Q}$-algebras cannot be isomorphic.

Interlude: Small Tori

As a respite leading up to the case of the unitary groups, we treat the afore omitted special orthogonal groups with respect to non-degenerate quadratic spaces (V, q) of rank 2. As above we assume our field k to be perfect and of characteristic $\neq 2$. When considering $\mathbf{SO}(V, q)$, similar quadratic spaces lead to isomorphic groups, so that in the case of dimension 2, it suffices to consider $(k^2, \langle\langle \delta \rangle\rangle)$. It is well-known that $\mathbf{SO}_{2,-\delta} = \mathbf{SO}(k^2, \langle\langle \delta \rangle\rangle)$ is isomorphic to the unitary group $\mathbf{U}_{1,\delta}$, with respect to the hermitian form $\langle 1 \rangle_H$, and the étale extension $k[X]/(X^2 - \delta)/k$, therefore linking this paragraph to the following section.

By using [39, (29.29)] as above, we see that the Galois cohomology group of $\mathbf{SO}_{2,-\delta}$ can be represented as the isometry classes of rank 2 quadratic spaces, having determinant $-\delta$. As noted previously, we have a surjection $\mathsf{Quad}_1 \twoheadrightarrow \mathsf{Quad}_{2,-\delta}$, so that we may identify the M-invariants of $\mathbf{SO}_{2,-\delta}$ with a certain subgroup of

$$\mathrm{Inv}_k(\mathsf{Quad}_1, M) \cong M(k) \oplus \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_1^{\mathrm{W}}, M),$$

where M denotes some strictly $\mathbb{A}^1$-invariant sheaf. Given an invariant $a = \lambda_0 + \lambda_1 W_1$ of Quad_1, we may derive the condition for a to be an invariant of $\mathsf{Quad}_{2,-\delta}$. For the subsequent arguments we fix a field $F \in \mathsf{F}_k$. Suppose we are given a quadratic form q, having determinant $-\delta$, and being represented by $\langle u_1, u_2 \rangle$ and $\langle v_1, v_2 \rangle$, where u_1, u_2 and v_1, v_2 are units in F. By [40, Prop. I.5.1] we know that v_1 is represented by $\langle u_1, u_2 \rangle$, and since $\langle u_1, u_2 \rangle \simeq \langle u_1, -\delta u_1 \rangle$, we find $(x, y) \in F^2 \smallsetminus \{0\}$ such that

$$u_1 x^2 - \delta u_1 y^2 = v_1$$

[8] Note that we may take q_E to be the *trace form* $x \mapsto \mathrm{Tr}_{E/F}(x^2)$ of E/F.

holds. From this we read off, that equivalently $u := \frac{v_1}{u_1}$ is represented by the Pfister form $\langle\langle\delta\rangle\rangle$. Now, for

$$\lambda_1([v_1]) \overset{\textbf{(MW2)}}{=} \lambda_1([u_1] + [u]\langle u_1\rangle)$$

to agree with $\lambda_1([u_1])$, we have to demand that $\lambda_1([u]\langle u_1\rangle) = 0$. So, in total the condition that the invariant a defines an invariant for $\mathsf{Quad}_{2,-\delta}$ is that λ_1 annihilates the $K_0^{\mathrm{W}}(F)$-submodule $A(F)$ of $K_1^{\mathrm{W}}(F)$ generated by the symbols $[u]$, with $u \in D_F(\langle\langle\delta\rangle\rangle)$, and does so for all choices $F \in \mathsf{F}_k$.

We claim that $A(F)$ is given by the annihilator of $[\delta]$, i.e. by the kernel of the homomorphism

$$K_1^{\mathrm{W}}(F) \xrightarrow{[\delta]\cdot(-)} K_2^{\mathrm{W}}(F).$$

It is a theorem due to Morel (cf. [52, Thm. 3.4.]) that $[u] \mapsto -\langle\langle u\rangle\rangle$ induces an isomorphism of graded rings from $K_*^{\mathrm{W}}(F)$ to $I^*(F)$, i.e. the powers of the fundamental ideal in $W(F)$. So we transfer the problem to $I^*(F)$. By work of Witt and Pfister (exposed in [67, 2.10.13]) and its generalization due to Arason and Elman (cf. [2, Thm. 2.3]), it is known that the annihilator of $\langle\langle\delta\rangle\rangle$ in $I^n(F)$ is composed of elements of the form $\langle\langle u\rangle\rangle q'$, with $u \in D_F(\langle\langle\delta\rangle\rangle)$ and $q' \in I^{n-1}(F)$. Transferring this back to Witt K-theory, we obtain our claim.

Now since $A(F)$ is the kernel of the product with the symbol $[\delta]$, we see that $K_1^{\mathrm{W}}(F)/A(F)$ is the "principal ideal" generated by $[\delta]$ in $K_2^{\mathrm{W}}(F)$. Hence we have that

$$\mathbb{Z} \oplus [\delta]\underline{\mathbf{K}}_1^{\mathrm{W}} \tag{4.14}$$

corepresents the invariants of $\mathbf{SO}_{2,-\delta}$. Since $[\delta]\underline{\mathbf{K}}_1^{\mathrm{W}}$ is the image of a homomorphism between the strictly $\mathbb{A}^1$-invariant sheaves $\underline{\mathbf{K}}_1^{\mathrm{W}}$ and $\underline{\mathbf{K}}_2^{\mathrm{W}}$, we see that (4.14) is isomorphic to $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{SO}_{2,-\delta})$. If we carefully run through the above arguments and identifications, we obtain that

$$\begin{aligned} \mathcal{H}^1_{\text{ét}}(\mathbf{SO}_{2,-\delta}) &\to [\delta]\underline{\mathbf{K}}_1^{\mathrm{W}} \\ \text{on } F \in \mathsf{F}_k\colon \langle u_1, u_2\rangle &\mapsto [\delta][u_1] \end{aligned}$$

is the universal morphism of pointed sheaves inducing this isomorphism. We are going to generalize the statements of this section in proposition 4.36.

4.4 Unitary Groups

In this section we handle the case of unitary groups, still in the case that the perfect base field k is of characteristic $\neq 2$. The definition of the unitary groups is analogous to the definition of the orthogonal groups: For any $n \in \mathbb{N}^+$ we consider the standard hermitian form $q_{n,H} := n\langle 1\rangle_H$ with respect to the étale extension k_1/k, where $k_1 := k[X]/(X^2 - \delta)$, and set the group of associated isometries to be $\mathbf{U}_{n,\delta}$ (cf. [39, p. 346]). By [39, Ex. (29.19)] we know that the Galois cohomology set of $\mathbf{U}_{n,\delta}$ can be functorially presented by

$$\begin{aligned} \mathsf{Herm}_{n,\delta}\colon \mathsf{F}_k &\longrightarrow \mathit{Set} \\ F &\longmapsto \left\{ \begin{array}{c} F\text{-isometry classes of} \\ \text{hermitian forms w.r.t. } F \otimes_k k_1/k \end{array} \right\}. \end{aligned}$$

Since every hermitian form is diagonalizable, and the map on Galois cohomology induced by the standard embedding $\mathbf{O}_n \hookrightarrow \mathbf{U}_{n,\delta}$ is given by

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{O}_n) &\longrightarrow H^1_{\text{ét}}(F, \mathbf{U}_{n,\delta}) \\ \langle u_1, \ldots, u_n\rangle &\longmapsto \langle u_1, \ldots, u_n\rangle_H, \end{aligned}$$

we see that this map needs to be surjective. Moreover, by embedding $\mathbf{U}_{n,\delta}$ into the orthogonal group $\mathbf{O}(n\langle\langle\delta\rangle\rangle)$, one obtains an induced injection on Galois cohomology

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{U}_{n,\delta}) &\longhookrightarrow H^1_{\text{ét}}(F, \mathbf{O}(n\langle\langle\delta\rangle\rangle)) \\ \langle u_1, \dots, u_n\rangle_H &\longmapsto \langle\langle\delta\rangle\rangle \cdot \langle u_1, \dots, u_n\rangle. \end{aligned} \tag{4.15}$$

We define invariants of $\mathsf{Herm}_{n,\delta}$ analogous to [21, Lem. 21.4]:

Lemma 4.35. *For every $1 \leq i \leq n$ we have well-defined invariants*

$$\begin{aligned} W_i^H \colon \mathsf{Herm}_{n,\delta} &\longrightarrow [\delta]\underline{\mathbf{K}}_i^{\mathrm{W}} \\ \text{on } F \in \mathsf{F}_k \colon \langle u_1, \dots, u_n\rangle_H &\longmapsto [\delta] W_i(\langle u_1, \dots, u_n\rangle). \end{aligned}$$

Proof. The case of $n = 1$ has been checked in the above interluding section. We come to $n = 2$: Suppose we are given a field $F \in \mathsf{F}_k$ and an hermitian form h with respect to $F \otimes_k k_1/k$ that may be presented in diagonal form by $\langle u_1, u_2\rangle_H$ and $\langle v_1, v_2\rangle_H$. Thus we have an F-isometry

$$\langle u_1, u_2, -\delta u_1, -\delta u_2\rangle \simeq \langle\langle\delta\rangle\rangle \otimes \langle u_1, u_2\rangle \simeq \langle\langle\delta\rangle\rangle \otimes \langle v_1, v_2\rangle \simeq \langle v_1, v_2, -\delta v_1, -\delta v_2\rangle$$

of quadratic forms, which are of determinant 1. So, we may apply the $\mathbf{SO}_4$ invariant W_3 to it, to obtain the equality

$$[u_1][u_2][u_1(-\delta)] = [v_1][v_2][v_1(-\delta)]$$

in $K_3^{\mathrm{W}}(F)$. Using $[u_1][-u_1] = 0$ from lemma 4.2, together with axiom **(MW2)**, we arrive at

$$[u_1][u_2][\delta] = [v_1][v_2][\delta],$$

checking the well-definedness of W_2^H in the case $n = 2$. So we consider W_1^H in $n = 2$ next. Note therefore that the determinant of the hermitian form h is given by u_1u_2, and is determined up to a factor from $D_F(\langle\langle\delta\rangle\rangle)$ (cf. [39, p. 114]). Hence by the case of small tori considered earlier, we have

$$[\delta][u_1u_2] = [\delta][v_1v_2].$$

Expanding both sides according to **(MW2)**, and using the case of W_2^H, we obtain

$$[\delta]([u_1] + [u_2]) = [\delta]([v_1] + [v_2]).$$

This checks well-definedness in the cases $n = 1, 2$. For all higher n, we appeal to Witt's chain equivalence for hermitian forms (cf. [79, p. 36]). □

We call the maps W_i^H the *refined hermitian Stiefel-Whitney classes.* The following is completely in parallel to [21, Thm. 21.6].

Proposition 4.36. *Let k be a perfect field of* $\operatorname{char}(k) \neq 2$*, and let $\delta \in k^\times$ be any unit. Then we have an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{U}_{n,\delta}) \cong [\delta]\underline{\mathbf{K}}_1^{\mathrm{W}} \oplus \cdots \oplus [\delta]\underline{\mathbf{K}}_n^{\mathrm{W}},$$

which is induced by the refined hermitian Stiefel-Whitney classes.

Proof. Let $a \colon \mathcal{H}^1_{\text{ét}}(\mathbf{U}_{n,\delta}) \to M$ be a morphism of pointed sheaves, where M denotes a strictly $\mathbb{A}^1$-invariant sheaf. We think of a as an invariant of $\mathsf{Herm}_{n,\delta}$. From the surjection $\mathsf{Quad}_n \twoheadrightarrow \mathsf{Herm}_{n,\delta}$, we obtain a unique normed invariant of $\mathbf{O}_n$, that we know is given as

$$\lambda_1 W_1 + \cdots + \lambda_n W_n,$$

where $\lambda_i \in \operatorname{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$. One checks inductively over $i = 1, \dots, n$ that each λ_i annihilates the subgroup $D_F(\langle\langle\delta\rangle\rangle)K_{i-1}^{\mathrm{W}}(F)$, for every $F \in \mathsf{F}_k$. Thus, by [2, Thm. 2.3], we know that each λ_i vanishes on the annihilator of $[\delta]$. As above, this allows us to regard λ_i as a homomorphism $[\delta]\underline{\mathbf{K}}_i^{\mathrm{W}} \to M$, and replace the W_i by W_i^H. Then one concludes by remark 2.59. □

Note that the above statement is trivial in the case that δ is a square, as $U_{n,\delta}$ is then isomorphic to the special group $\mathbf{GL}_n$.

Remark 4.37 (Embedding of $\pi_0^{\mathbb{A}^1}$). We argue as in remark 4.20 to see that the refined hermitian Stiefel-Whitney classes determine the elements of $H^1_{\text{ét}}(F, \mathbf{U}_{n,\delta})$, for all fields $F \in \mathsf{F}_k$: To that end, fix two diagonal hermitian forms $\langle u_1, \dots, u_n\rangle_H$ and $\langle v_1, \dots, v_n\rangle_H$ with agreeing W_1^H over F. Using the isomorphism $\underline{\mathbf{K}}_2^{\mathrm{W}}(F) \cong I^2(F)$, we obtain the equality

$$\langle\langle\delta\rangle\rangle\,(\langle\langle u_1\rangle\rangle + \dots + \langle\langle u_n\rangle\rangle) = \langle\langle\delta\rangle\rangle\,(\langle\langle v_1\rangle\rangle + \dots + \langle\langle v_n\rangle\rangle) \in I^2(F).$$

Then employing Witt's cancellation theorem, we arrive at the isometry $\langle\langle\delta\rangle\rangle\langle u_1, \dots, u_n\rangle \cong \langle\langle\delta\rangle\rangle\langle v_1, \dots, v_n\rangle$, from which we conclude by the injectivity of (4.15).

4.5 The Split Group of Type G_2

We still assume our ground field k to be perfect and of characteristic $\neq 2$, and turn our attention towards Pfister forms, first in the form of a functor:

Definition 4.38. Let $n \in \mathbb{N}^+$ be a positive natural number. We define a functor $\mathsf{Pfister}_n \colon \mathsf{F}_k \to \mathsf{Set}$ that is given by

$$\mathsf{Pfister}_n(F) := \{\text{isomorphism classes of } n\text{-fold Pfister forms over } F\},$$

for every choice of a field $F \in \mathsf{F}_k$. The isomorphism class of the isotropic n-fold Pfister form will be our basepoint, in any situation where one is needed.

There are some cases, in which this functor can be realized as the Galois cohomology of some algebraic group, and we may realize that the two initial ones have already been dealt with: Setting $n = 1$, then we have a natural (in $F \in \mathsf{F}_k$) bijection

$$\begin{aligned} H^1_{\text{ét}}(F, \boldsymbol{\mu}_2) &\longrightarrow \mathsf{Pfister}_1(F) \\ u(F^\times)^2 &\longmapsto \langle 1, -u\rangle = \langle\langle u\rangle\rangle, \end{aligned}$$

and $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{ét}}\boldsymbol{\mu}_2) \cong \underline{\mathbf{K}}_1^{\mathrm{W}}$ (cf. lemma 4.11). The universal invariant $E_1 \colon \mathsf{Pfister}_1 \to \underline{\mathbf{K}}_1^{\mathrm{W}}$ can thus be codified as $\langle\langle u\rangle\rangle \mapsto [u]$. Proceeding to $n = 2$, we denote by $\mathbf{SO}_3'$ the special orthogonal group with respect to the split quadratic form $h \oplus \langle 1\rangle$. Then one has a functorial map

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{SO}_3') &\longrightarrow \mathsf{Pfister}_2(F) \\ q &\longmapsto \langle 1\rangle \oplus \langle -1\rangle \cdot q, \end{aligned}$$

which is injective by Witt cancellation (cf. [40, I.4.2]), and whose surjectivity may be checked by regarding $H^1_{\text{ét}}(F, \mathbf{SO}_3')$ as the isometry classes of rank 3 quadratic forms with determinant -1. The universal invariant $E_2 \colon \mathsf{Pfister}_2 \to \underline{\mathbf{K}}_2^{\mathrm{W}}$ can then be identified with W_2 (for $\mathbf{SO}_3'$ cf. proposition 4.26) which maps $\langle\langle u, v\rangle\rangle \mapsto [u][v]$.

There is also an interpretation of the case $n = 3$ in terms of Galois cohomology: Recall first that the iterated Cayley-Dickson construction introduces a one-to-one correspondence between the isometry classes of 3-fold Pfister forms and the isomorphism classes of Cayley algebras[9] (cf. [39, Thm. (33.19)]). Fixing some Cayley algebra $\mathfrak{C}_s$ over k, whose norm form is isotropic, one finds that its automorphism group $\mathbf{G}_2 := \mathbf{Aut}(\mathfrak{C}_s)$ is a split simple algebraic group of type G_2 (cf. [39, Thm. (25.14)]), whose Galois cohomology set (pointed by $\mathfrak{C}_s$) can be classified by the

[9] A *Cayley algebra* is an eight-dimensional unital composition algebra, i.e. an algebra whose multiplication is compatible with a quadratic form, called the norm form of the algebra.

Cayley algebras (cf. [39, Prop. (33.24)]). So in summary, we have a bijective and functorial map (in $F \in \mathsf{F}_k$)

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{G}_2) &\longrightarrow \mathsf{Pfister}_3 \\ \mathfrak{C} &\longmapsto q_{\mathfrak{C}}, \end{aligned}$$

where $q_{\mathfrak{C}}$ denotes the norm form (and by abuse of notation, its equivalence class) of the Cayley algebra $\mathfrak{C}$. For the case $n > 3$, we remark that it is known that the functors $\mathsf{Pfister}_n$ are not representable as the Galois cohomology of an algebraic group over k (cf. [21, p. 10]).

We keep mimicking the procedures of [21, VI.18], so our aim is to define invariants of $\mathsf{Pfister}_n$ with values in $\underline{\mathbf{K}}^{\mathrm{W}}_n$. This is done in the following

Lemma 4.39. *We have a well-defined and natural (in $F \in \mathsf{F}_k$) map*

$$\begin{aligned} E_n \colon \mathsf{Pfister}_n(F) &\longrightarrow K^{\mathrm{W}}_n(F) \\ \langle\langle u_1, \dots, u_n \rangle\rangle &\longmapsto [u_1] \cdots [u_n]. \end{aligned}$$

Proof. We check that in the case we are given F-isometric n-fold Pfister forms $\langle\langle u_1, \dots, u_n \rangle\rangle$ and $\langle\langle v_1, \dots, v_n \rangle\rangle$, that

$$[u_1] \cdots [u_n] = [v_1] \cdots [v_n]$$

holds. Therefore we use that two isometric Pfister forms are chain P-equivalent (cf. [40, Thm. X.1.12]), and thus by induction we only have to check that if $\langle\langle u_1, \dots, u_n \rangle\rangle$ and $\langle\langle v_1, \dots, v_n \rangle\rangle$ are simply P-equivalent, E_n yields the same value in $K^{\mathrm{W}}_n(F)$. By commutativity of $K^{\mathrm{W}}_n(F)$, we immediately reduce to the case that $\langle\langle u_1, u_2 \rangle\rangle$ and $\langle\langle v_1, v_2 \rangle\rangle$ are isometric, and $u_i = v_i$ holds for all $i = 3, \dots, n$. We ought to check that $[u_1][u_2] = [v_1][v_2]$ holds. However, this is the case of $n = 2$ above, concluding the proof of the lemma. □

Remark 4.40 (Embedding of $\pi_0^{\mathbb{A}^1}$). The map E_n is injective on all fields $F \in \mathsf{F}$: Indeed, the composition of E_n and the refined Milnor epimorphism

$$\begin{aligned} S_n \colon K^{\mathrm{W}}_n(F) &\twoheadrightarrow I^n(F) \\ [u_1] \cdots [u_n] &\mapsto (-1)^n \langle\langle u_1, \dots, u_n \rangle\rangle, \end{aligned}$$

defined[10] for example in [50, Rem. 3.12], maps a Pfister form $\langle\langle u_1, \dots, u_n \rangle\rangle$ to its image in $I^n(F)$ times a unit. By [40, Prop. II.1.4] this mapping has to be injective, and thus E_n is injective as well. Note that this statement does not use the Milnor conjecture.

On the other hand by using the Milnor conjecture, Morel showed in [52, Thm. 2.4] that $S_n(F)$ is an isomorphism for $\mathrm{char}(F) \neq 2$, which implies that the invariants E_n may be extended to natural homomorphisms (in $F \in \mathsf{F}_k$)

$$I^n(-) \longrightarrow \underline{\mathbf{K}}^{\mathrm{W}}_n,$$

that we denote by E_n as well. Comparing with [1, Satz 5.7], where it is shown that the classical invariant e_3 extends to a natural homomorphism (the *Arason invariant*), it seems at least plausible that Arason's arguments can be transferred to the case of Witt K-theory in degree 3.

As before with the refined Stiefel-Whitney classes, we denote the invariant $\mathsf{Pfister}_3 \to \underline{\mathbf{K}}^{\mathrm{W}}_3$, and its induced morphism $\mathcal{H}^1_{\text{ét}}(\mathbf{G}_2) \to \underline{\mathbf{K}}^{\mathrm{W}}_3$, by the same symbol E_3 (cf. the notation in [21, VI.18]). The following proposition checks that E_3 is the universal pointed morphism from $\mathcal{H}^1_{\text{ét}}(\mathbf{G}_2)$ into a strictly $\mathbb{A}^1$-invariant sheaf:

[10] The epimorphism is defined in accordance with the original [48, Thm. 4.1] and with respect to the usual definition of Pfister forms as $\langle\langle u_1, \dots, u_n \rangle\rangle = \langle 1, -u_1 \rangle \otimes \cdots \otimes \langle 1, -u_n \rangle$ (cf. [40, Def. X.1.1] or [16] pp. 24, 53).

Proposition 4.41. *Let k be a perfect field of characteristic $\neq 2$. Then there is an isomorphism*

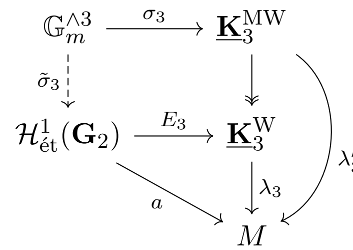
$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{G}_2) \xrightarrow{\cong} \underline{\mathbf{K}}_3^{\mathrm{W}},$$

induced by the ***refined Arason invariant*** *E_3.*

Proof. We check the universal property of $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{G}_2)$: Suppose we are given a map of pointed sheaves $a\colon \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2) \to M$, where M is any strictly $\mathbb{A}^1$-invariant sheaf, then we need to construct a homomorphism $\lambda_3\colon \underline{\mathbf{K}}_3^{\mathrm{W}} \to M$ factoring a via E_3 uniquely. To that end, let us consider the natural transformation

$$\begin{aligned} \tilde{\sigma}_3\colon \mathbb{G}_{m\restriction \mathsf{F}_k}^{\wedge 3} &\longrightarrow \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2)_{\restriction \mathsf{F}_k} \\ \text{on } F \in \mathsf{F}_k\colon u_1 \wedge u_2 \wedge u_3 &\longmapsto \langle\!\langle u_1, u_2, u_3 \rangle\!\rangle, \end{aligned}$$

which is surjective and well-defined. Moreover, if any of u_1, u_2 or u_3 happens to be a square, the image of $u_1 \wedge u_2 \wedge u_3$ will be the isometry class of the isotropic 3-fold Pfister form. The composition $a_{\restriction \mathsf{F}_k} \circ \tilde{\sigma}_3$ maps from $\mathbb{G}_m^{\wedge 3}$ to M, and as the latter sheaf is unramified, this defines a morphism of pointed sheaves $\mathbb{G}_m^{\wedge 3} \to M$ by propositions 3.25 and 3.28, and considering corollary 3.19. By theorem 4.7 this morphism induces a homomorphism $\lambda_3'\colon \underline{\mathbf{K}}_3^{\mathrm{MW}} \to M$. Since $\tilde{\sigma}_3$ annihilates squares, we have that λ_3' factors moreover through the canonical epimorphism $\underline{\mathbf{K}}_3^{\mathrm{MW}} \twoheadrightarrow \underline{\mathbf{K}}_3^{\mathrm{W}}$ setting (h) to zero, so that we have an induced homomorphism $\lambda_3\colon \underline{\mathbf{K}}_3^{\mathrm{W}} \to M$. We display the situation in the following diagram



,

where σ_3 is notation coming from 4.7. The square in the diagram commutes on F_k, and this is clear by the definitions of the morphisms at play. Moreover, the triangle spanned by $\mathbb{G}_m^{\wedge 3}$, $\underline{\mathbf{K}}_3^{\mathrm{MW}}$ and M commutes as morphisms of pointed sheaves, and thus also on F_k. As $\tilde{\sigma}_3$ is surjective, and morphisms from weakly unramified sheaves into unramified ones are determined on fields, we see that $a = \lambda_3 \circ E_3$ holds. The homomorphism λ_3 is uniquely determined, since for every $F \in \mathsf{F}_k$ the image of E_3 generates $K_3^{\mathrm{W}}(F)$ (cf. lemma 4.3), and λ_3 is already determined on F_k. □

4.6 The Split Group of Type F_4

We still keep the assumption that our base field k is perfect, and of characteristic $\neq 2$. Let us recall first, how to define a split group of type F_4, as the automorphism group of an Albert algebra, which is a simple exceptional Jordan algebra of dimension 27. We begin by defining such an algebra explicitly: Let us reuse the split k-Cayley algebra $\mathfrak{C}_s$ from $\mathbf{G}_2$, and fix two parameters $\alpha, \beta \in k^\times$. It turns out that the choice of the latter parameters is immaterial to the isomorphism class of the resulting group (cf. [39, Cor. (37.18)]), and hence whenever we get hands on, we set $\alpha = \beta = 1$. The underlying set $\mathcal{H}_3(\mathfrak{C}_s, \alpha, \beta)$ of the algebra will be the 3×3-matrices with values in $\mathfrak{C}_s$ that are invariant under the involution

$$\begin{pmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^{\dagger} \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix}^{-1}$$

where † transposes the matrix and conjugates its entries $c_{ij} \in \mathfrak{C}_s$. Explicitly, the elements of $\mathcal{H}_3(\mathfrak{C}_s, \alpha, \beta)$ are given by

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ \alpha^{-1}\beta\overline{c_{12}} & c_{22} & c_{23} \\ -\beta\overline{c_{13}} & -\alpha\overline{c_{23}} & c_{33} \end{pmatrix}, \text{with } c_{11},\ c_{22},\ c_{33} \in k^\times, \quad c_{12},\ c_{13}\ , c_{23} \in \mathfrak{C}_s,$$

with the overline referring to the conjugation in $\mathfrak{C}_s$. By defining a product via $c \diamond d := \frac{1}{2}(cd+dc)$, where on the right-hand side the matrix product is used, $\mathcal{H}_3(\mathfrak{C}_s, \alpha, \beta)$ admits the structure of a split Albert algebra over k, and we set $\mathbf{F}_4 := \mathbf{Aut}(\mathcal{H}_3(\mathfrak{C}_s, 1, 1))$.

Let us recall that $\mathbf{F}_4$ is a split simple algebraic group over k (cf. [39, Thm. (25.13)]), and that torsors over $\mathbf{F}_4$ are classified by Albert algebras (cf. [39, Prop. (37.11)]). For more information on Jordan and Albert algebras we refer the reader to [39, IX.37] and [69]. As in the case of G_2, we give a name to the functor represented by $H^1_{\text{ét}}(-, \mathbf{F}_4)$.

Definition 4.42. For every field $F \in \mathsf{F}_k$, we define

$$\mathsf{Alb}(F) := \{\text{isomorphism classes of Albert algebras over } F\},$$

which we consider to be pointed by the class of $\mathcal{H}_3(\mathfrak{C}_s, 1, 1)$.

Unfortunately, we have found no way to deduce the structure of the invariants of $\mathbf{F}_4$ with values in an arbitrary strictly $\mathbb{A}^1$-invariant sheaf from the arguments in [21, Thm. 22.5]. However, we can at least lift some of the known invariants:

Proposition 4.43. *There are well-defined and natural (in $F \in \mathsf{F}_k$) maps*

$$F_3 \colon H^1_{\text{ét}}(F, \mathbf{F}_4) \to K^{\mathrm{W}}_3(F) \quad \textit{and} \quad F_5 \colon H^1_{\text{ét}}(F, \mathbf{F}_4) \to K^{\mathrm{W}}_5(F)$$

lifting the well-known invariants $f_3 \colon \mathcal{H}^1_{\text{ét}}(\mathbf{F}_4) \to \underline{\mathbf{k}}^{\mathrm{M}}_3$ and $f_5 \colon \mathcal{H}^1_{\text{ét}}(\mathbf{F}_4) \to \underline{\mathbf{k}}^{\mathrm{M}}_5$ (cf. [21, VI.22]).

Proof. For any field $F \in \mathsf{F}_k$ and Albert algebra $J \in \mathsf{Alb}(F)$, one finds by [21, Thm. 22.4] a 3-fold resp. 5-fold Pfister form p_3 resp. p_5, which are unique up to isomorphism, such that

$$q_J \oplus p_3 \simeq \langle 2, 2, 2 \rangle \oplus p_5$$

holds, where $q_J(c) := \frac{1}{2}\operatorname{Tr}(c \diamond c)$ is the trace form on J halved. The claimed invariants can hence be defined as $F_3(J) = E_3(p_3)$ and $F_5(J) = E_5(p_5)$. □

4.7 Spin Groups

We continue to assume that k is perfect, and of characteristic $\neq 2$. The purpose of this section is to generalize some arguments due to Garibaldi [20] and Rost [66] concerning the cohomological invariants of spin groups, in order to determine $\mathbb{H}^{\mathbb{A}^1}_0(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n)$, for $7 \leq n \leq 12$. By $\mathbf{Spin}'_n$ we mean here the split spin group asscoiated to a non-degenerate quadratic space of dimension n, and maximal Witt index.

In its essence, Garibaldi's method boils down to the following: Given some algebraic group G with unknown structure of cohomological invariants, find a homomorphism $H \to G$, from a group H with known invariants, that induces a surjection $H^1_{\text{ét}}(F, H) \to H^1_{\text{ét}}(F, G)$, for all $F \in \mathsf{F}_k$. This implies that the G-invariants form a subset of the H-invariants. As usually one knows a generating set for the H-invariants, one is then tasked to construct corresponding G-invariants or disprove that such lifts exist. We mimic this procedure for $G = \mathbf{Spin}'_n$, in our setting, meaning that we replace $\mathbb{Z}/(2)$-Galois cohomology as values for the invariants by some strictly $\mathbb{A}^1$-invariant sheaf M, and conclude the corresponding $\mathbb{H}^{\mathbb{A}^1}_0(\mathrm{B}_{\text{ét}}(-))$ from that.

Before we start with the individual cases, we first give a generalization of the Rost invariant, which is present for all spin groups, and describe a method of constructing new invariants.

4.7.1 Establishing a Basic Invariant

In the classical theory of cohomological invariants, the group of invariants of degree 3 of some absolutely almost simple simply connected algebraic group G with values in the Galois cohomology of $\mathbb{Q}/\mathbb{Z}$ is finite cyclic (cf. [21, Thm. M.9.11]), and a canonical generator is given by the *Rost invariant*. In the case of spin groups, the Rost invariant has an accessible description, which allows for a generalization to Witt K-theory (cf. [39, p. 436f]):

Let (V, q) be a quadratic space, with q being non-degenerate. Since the characteristic of k is not 2, we may start with the short exact sequence of algebraic groups

$$0 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(V,q) \xrightarrow{\chi^{(0)}} \mathbf{SO}(V,q) \to 0,$$

where the homomorphism $\chi^{(0)}$ is the vector representation of the spin group. From the associated long exact sequence in Galois cohomology, one may deduce that the image of $\chi_*^{(0)}(\gamma) - q$ in the Witt ring, for some cohomology class $\gamma \in H^1_{\text{ét}}(F, \mathbf{Spin}(V,q))$ and $F \in \mathsf{F}_k$, has even dimension, trivial discriminant and trivial Hasse-Witt invariant. By a theorem of Merkurjev (cf. [16, Thm. 44.1]) it thus has to lie in $I^3(F)$. So, we may define an invariant by

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{Spin}(V,q)) &\longrightarrow K_3^{\mathrm{W}}(F) \\ \gamma &\longmapsto E_3(\chi_*^{(0)}(\gamma) - q), \end{aligned}$$

for which we write $\tilde{E}_3$, and which we will call the *refined Rost invariant*. Note that in the above formula, we omitted the projection to the Witt ring. In all the cases we see below, the refined Rost invariant $\tilde{E}_3$ will be the nonconstant invariant of least degree.

4.7.2 Extending Invariants by a Method due to M. Rost

Suppose G is a smooth algebraic k-group, and that there is some representation $G \to \mathbf{GL}(V)$. A nonzero element in the d-part of the symmetric space $S^d(V^*)$ will be called a d-form. Given any 1-cocycle $\gamma \in Z^1(F, G)$, for $F \in \mathsf{F}_k$, we may find some $g \in \mathbf{GL}(V)(F_{\text{sep}})$ such that

$$\gamma_\sigma = g^{-1}\sigma(g)$$

holds for all $\sigma \in \mathrm{Gal}(F_{\text{sep}}/F)$, by Hilbert's theorem 90. For any d-form $q \in S^d(V^*)$ that is invariant under the action of G, one may show that

$$q(g^{-1} \cdot (-))$$

is a $\mathrm{Gal}(F_{\text{sep}}/F)$-invariant d-form, and thus defines some *twisted form* q_γ over F. One may check that a cohomologous cocycle yields an isometric d-form, and thus by abuse of notation we will also twist by cohomology classes.

Recall also the following construction of a natural $\underline{\mathbf{K}}_*^{\mathrm{MW}}$-structure on the contracted sheaves M_{-*} due to Morel [50, Lem. 3.48]. To that end, we first remind the reader about their definition in terms of the exact sequence ($X \in \mathsf{Sm}_k$)

$$0 \longrightarrow M(X) \underset{\mathrm{ev}_1^*}{\overset{\mathrm{pr}_2^*}{\rightleftarrows}} M(\mathbb{G}_m \times_k X) \longrightarrow\!\!\!\!\!\rightarrow M_{-1}(X) \longrightarrow 0.$$

So given $\varphi \in \mathbb{G}_m(X)$ and $\lambda \in M_{-1}(X) = \ker(M(\mathbb{G}_{m,X}) \to M(X))$, one defines the action of φ on λ as the composition

$$X \xrightarrow{(\varphi, \mathrm{id}_X)} \mathbb{G}_m \times_k X \xrightarrow{\lambda} M.$$

Starting from this, Morel checked that the relations of $\underline{\mathbf{K}}_1^{\mathrm{MW}}$ are met.

Compatibility of the $\underline{\mathbf{K}}_1^{\mathrm{MW}}$-Action and the Residue Homomorphism

Consider a geometric discrete valuation ring $\mathcal{O}_v \subseteq F$, for some $F \in \mathcal{F}_k$. We remind the reader of a statement that we derived in the proof of 3.28, namely that for $\mathcal{O}_v$ henselian, we may assume that there is a subfield in $\mathcal{O}_v$, isomorphic to $\kappa(v)$. Hence in the following arguments we assume $\mathcal{O}_v$ to be henselian, and give pointers on how to handle the general case. Let us also fix a uniformizing element $\pi \in \mathcal{O}_v$. As in [50, p. 36] we define the *residue homomorphism* ∂_v^π to be the projection

$$M(F) \twoheadrightarrow M(F)/M(\mathcal{O}_v) \cong M_{-1}(\kappa(v)),$$

where the latter identification is derived from the elementary distinguished square (consisting of essentially smooth schemes)

$$\begin{array}{ccc} \mathrm{Spec}(F) & \hookrightarrow & \mathrm{Spec}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathbb{G}_{m,\kappa(v)} & \hookrightarrow & \mathbb{A}^1_{\kappa(v)} \end{array} ,$$

with the vertical morphisms being induced by $\kappa(v)[T] \to \mathcal{O}_v,\ T \mapsto \pi$. Indeed, the diagram leads to a monomorphism $M_{-1}(\kappa(v)) \hookrightarrow M(F)/M(\mathcal{O}_v)$, and Morel showed that it is an isomorphism (cf. [50, Lem. 2.24]). With the notion of a residue homomorphism defined, we can also extend the specialization maps, via

$$\begin{aligned} s_v^\pi \colon M(F) &\longrightarrow M(\kappa(v)) \\ m &\longmapsto s_v\left(m - [\pi].\partial_v^\pi(m)_F\right). \end{aligned}$$

Here we again used crucially that $\kappa(v)$ has an embedding into $\mathcal{O}_v$. In the non-henselian case, we may still construct specialization maps, by going to the henselization, and using the cartesian diagram (cf. **(A1)**)

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_v^h) \\ \downarrow & \lrcorner & \downarrow \\ M(F) & \longrightarrow & M(F^h). \end{array}$$

With this background, we may obtain the following compatibility between the action of $\underline{\mathbf{K}}_1^{\mathrm{MW}}$ on M_{-1}, and the residue homomorphism ($\alpha \in K_1^{\mathrm{MW}}(F)$, $m \in M_{-1}(F)$):

$$\partial_v^\pi(\alpha.m) = \partial_v^\pi(\alpha).s_v^\pi(m) + \left(\epsilon s_v^\pi(\alpha)\right).\partial_v^\pi(m) + \left([-1]\partial_v^\pi(\alpha)\right).\partial_v^\pi(m). \tag{4.16}$$

Note that we care about the residue homomorphism ∂_v^π, since $M(\mathcal{O}_v)$ is its kernel, and thus it gives us a computational method of checking membership. The statement (and proof) of this compatibility statement is similar to the one given in [21, Ex. 7.12]. More specifically, one first checks that in the henselian case, the homomorphism

$$M_{-1}(\kappa(v)) \to M_{-1}(F) \xrightarrow{[\pi].(-)} M(F)$$

is a splitting to the residue homomorphism ∂_v^π. Thus every $m \in M(F)$ can be written uniquely as

$$m = u_F + [\pi].r_F,$$

where u lies in $M(\mathcal{O}_v)$, and r lies in $M_{-1}(\kappa(v))$. Then the identity (4.16) follows from the uniqueness statement after some quick expansion. In the general (non-henselian) case the statement follows from comparison with the henselization, since it suffices to calculate the value of the residue homomorphism after henselization to already know its value in the general case:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\mathcal{O}_v) & \hookrightarrow & M(F) & \xrightarrow{\partial_v^\pi} & M_{-1}(\kappa(v)) & \longrightarrow & 0 \\ & & \downarrow & \lrcorner & \downarrow & & \| & & \\ 0 & \longrightarrow & M(\mathcal{O}_v^h) & \hookrightarrow & M(F^h) & \xrightarrow{\partial_{v^h}^\pi} & M_{-1}(\kappa(v)) & \longrightarrow & 0. \\ & & & & & \xleftarrow{[\pi].(-)_{F^h}} & & & \end{array}$$

The following proposition is originally due to Rost [66, Prop. 5.2], but in form closer to its generalization due to Garibaldi [20, Prop. 10.2].

Proposition 4.44. *Let G be a smooth algebraic k-group, admitting a representation $G \to \mathbf{GL}(V)$, and some G-invariant d-form $q \in S^d(V^*)$. Let M be some strictly $\mathbb{A}^1$-invariant sheaf such that the natural action of $\underline{\mathbf{K}}_1^{\mathrm{MW}}$ on the contracted sheaf M_{-1} factors through $\underline{\mathbf{K}}_1^{\mathrm{MW}}/(d_\epsilon)$, and let an invariant $a\colon H^1_{\text{ét}}(-,G) \to M_{-1}$ be given. Assume that a evaluates to zero, for every class $\gamma \in H^1_{\text{ét}}(F,G)$ such that q_γ is isotropic. Then the following defines an invariant for G in M:*

$$\begin{aligned} H^1_{\text{ét}}(-,G) &\longrightarrow M \\ \text{on } F \in \mathsf{F}_k\colon \gamma &\longmapsto [q_\gamma(v)].a(\gamma), \end{aligned}$$

where $v \in V \otimes_k F$ is any vector such that q_γ evaluates to a unit.

Proof. Using (4.16), Garibaldi's proof [20, Prop. 10.2] goes through verbatim. □

In the remaining part of this section we determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n)$, for $7 \leq n \leq 12$, by integrating the arguments of Garibaldi [20, Part III] about Galois cohomology with $\mathbb{A}^1$-algebraic topology. In the interest of not reproducing Garibaldi's work entirely, we refrain from being self-contained but suggest the reader to check the classical theory out.

4.7.3 $\mathbf{Spin}_7$ and $\mathbf{Spin}_8$

As we explained before, the method is to use homomorphisms $i_n\colon H_n \to \mathbf{Spin}'_n$ with a known structure of invariants on H_n, and such that i_n induces a surjection on Galois cohomology. Hence we begin by grazing over their construction due to Garibaldi, and then add a few arguments to arrive at $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n)$.

We start with the case of $\mathbf{Spin}'_8$, which we shall identify explicitly with $\mathbf{Spin}(\mathfrak{C}_s, q_{\mathfrak{C}_s})$, i.e. the spin group over the quadratic space given by a split Cayley algebra $\mathfrak{C}_s$ and its norm form $q_{\mathfrak{C}_s}$. Recall that it is part of the triality phenomenon on $\mathbf{Spin}'_8$ that there is an embedding

$$i_{8\upharpoonright\mathbf{G}_2}\colon \mathbf{G}_2 \longrightarrow \mathbf{Spin}'_8,$$

which, when composed with the vector representation $\chi_8^{(0)}\colon \mathbf{Spin}'_8 \to \mathbf{SO}'_8$, or any spin representation $\chi_8^{(+)}, \chi_8^{(-)}\colon \mathbf{Spin}'_8 \to \mathbf{SO}'_8$, yields precisely the standard representation of $\mathbf{G}_2$ in $\mathbf{SO}'_8$, which regards an automorphism of $\mathfrak{C}_s$ as a norm preserving automorphism of the underlying vector space. On the topic of these representations, we note that one may choose them in such a way that they induce homomorphisms of the centers (which we denote by the same name)

$$\chi_8^{(\sharp)}\colon \mathbf{Z}(\mathbf{Spin}'_8) \longrightarrow \mathbf{Z}(\mathbf{SO}'_8), \quad \text{with} \quad \sharp \in \{0,+,-\},$$

and such that these run through all three possible quotient homomorphisms. To fix notation, we regard $\chi_8^{(+)}$ as the projection on the first factor, $\chi_8^{(0)}$ as the projection on the second factor, and $\chi_8^{(-)}$ as the multiplication of both factors, where we use the identification

$$\chi_8^{(\sharp)}\colon \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \longrightarrow \boldsymbol{\mu}_2, \quad \text{with} \quad \sharp \in \{0, +, -\}.$$

In [20, 18.1] Garibaldi deduced that the product of the inclusion of $\mathbf{G}_2$ and $\mathbf{Z}(\mathbf{Spin}_8')$ in $\mathbf{Spin}_8'$ yields an embedding

$$i_8\colon \mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8') \longrightarrow \mathbf{Spin}_8',$$

which induces surjections on Galois cohomology. This surjection in turn leads to an injection on cohomological invariants, and by an application of the characterisation of morphisms 3.25 and the compatibility statement 3.28, we thus obtain an injection

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\mathbf{Spin}_8'), M) & \xhookrightarrow{i_8^*} & \operatorname{Hom}_{\mathit{Sh}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\mathcal{H}^1_{\text{ét}}(\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')), M) \\
\cong\uparrow & & \cong\uparrow \\
\operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\,\mathbf{Spin}_8'), M) & \xhookrightarrow{i_8^*} & \operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8'))), M),
\end{array}$$

where the comparison on the lower line comes from proposition 2.58, and where M denotes some strictly $\mathbb{A}^1$-invariant sheaf. We may use the $\mathbb{A}^1$-tensor product to analyse the lower right corner of this diagram, which leads to an isomorphism

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8'))) \longrightarrow \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}} \oplus \underline{\mathbf{K}}_1^{\mathrm{W}} \oplus \underline{\mathbf{K}}_2^{\mathrm{W}} \oplus \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}},$$

that is induced by the morphisms

$$\begin{array}{cccc}
 & \chi_8^{(+)}, & \chi_8^{(0)}, & \chi_8^{(+)} \otimes_{\mathbb{A}^1} \chi_8^{(0)}, \\
E_3, & E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)}, & E_3 \otimes_{\mathbb{A}^1} \chi_8^{(0)}, \quad \text{and} & E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)} \otimes_{\mathbb{A}^1} \chi_8^{(0)}.
\end{array}$$

Here we abused the notation in such a way that we identify a homomorphism $\chi_8^{(\sharp)}\colon \mathbf{Z}(\mathbf{Spin}_8') \to \mathbf{Z}(\mathbf{SO}_8') \cong \boldsymbol{\mu}_2$ with its induced map on Galois cohomology composed with $\sigma^{(2)}\colon \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2) \to \underline{\mathbf{K}}_1^{\mathrm{W}}$ (cf. 4.11).

So, to determine $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\,\mathbf{Spin}_8')$ we would like to find those combinations of the above invariants that are restrictions of $\mathbf{Spin}_8'$-invariants. To start out, we remark that by the construction of i_8 the group $\mathbf{Z}(\mathbf{Spin}_8')$ is the center of a semisimple split group, and thus its embedding factors through some split torus $\mathbb{G}_m^8$ (cf. [46, Prop. 21.7]). By Hilbert's theorem 90, it follows thus that

$$H^1_{\text{ét}}(F, \mathbf{Z}(\mathbf{Spin}_8')) \longrightarrow H^1_{\text{ét}}(F, \mathbf{Spin}_8')$$

is trivial, for all $F \in \mathit{F}_k$, or dually that all $\mathbf{Spin}_8'$-invariants are trivial, when restricted to $\mathbf{Z}(\mathbf{Spin}_8')$. Thus certainly, given a normed $\mathbf{Spin}_8'$-invariant a with values in some strictly $\mathbb{A}^1$-invariant M, we can find homomorphisms

$$\begin{aligned}
\lambda_3 &\in \operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}_3^{\mathrm{W}}, M) \\
\lambda_{4,1} &\in \operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}_4^{\mathrm{W}}, M) \\
\lambda_{4,2} &\in \operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}_4^{\mathrm{W}}, M) \\
\lambda_5 &\in \operatorname{Hom}_{\mathit{Ab}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathit{Sm}_k)}(\underline{\mathbf{K}}_5^{\mathrm{W}}, M), \quad \text{such that} \quad a_{\restriction \mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')} = \dots
\end{aligned}$$

$$\dots = \lambda_3 \circ E_3 + \lambda_{4,1} \circ (E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)}) + \lambda_{4,2} \circ (E_3 \circ_{\mathbb{A}^1} \chi_8^{(0)}) + \lambda_5 \circ (E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)} \otimes_{\mathbb{A}^1} \chi_8^{(0)}).$$

From this we may conclude at once that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\,\mathbf{Spin}_8')$ is given by $\underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}}$, as soon as we find lifts for the remaining $\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')$-invariants. In order to have a better understanding of the situation, we note that together with the above remark about the relation of the embedding of $\mathbf{G}_2$ and the triality phenomenon, we may derive explicitly ($\sharp \in \{+, 0, -\}$)

$$\begin{array}{ccccc} H^1_{\acute{e}t}(F, \mathbf{G}_2) \times H^1_{\acute{e}t}(F, \mathbf{Z}(\mathbf{Spin}_8')) & \xrightarrow{i_{8,*}} & H^1_{\acute{e}t}(F, \mathbf{Spin}_8') & \xrightarrow{\chi^{(\sharp)}_{8,*}} & H^1_{\acute{e}t}(F, \mathbf{SO}_8') \\ (\mathfrak{C}, z) & \longmapsto & & & \langle \chi^{(\sharp)}_{8,*}(z)\rangle q_{\mathfrak{C}}. \end{array}$$

From this we see that the refined Rost invariant $\tilde{E}_3$ restricted to $\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')$ maps $(\mathfrak{C}, z)$ to the quadratic form $\langle \chi^{(0)}_{8,*}(z)\rangle q_{\mathfrak{C}}$. Unfortunately, this does not agree right away with the value obtained from the invariant E_3 on $\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')$. However, we may employ the extension lemma 4.44 to the invariant $\tilde{E}_3$, the representation $\chi_8^{(0)}$, and the quadratic form $q_{\mathfrak{C}_s}$, to see that the invariant

$$\begin{aligned} H^1_{\acute{e}t}(F, \mathbf{G}_2) \times H^1_{\acute{e}t}(F, \mathbf{Z}(\mathbf{Spin}_8')) &\longrightarrow K_4^{\mathrm{W}}(F) \\ (\mathfrak{C}, z) &\longmapsto \langle \chi^{(0)}_{8,*}(z)\rangle [\chi^{(0)}_{8,*}(z)] E_3(q_{\mathfrak{C}}) \overset{(4.1)}{=} -[\chi^{(0)}_{8,*}(z)] E_3(q_{\mathfrak{C}}) \end{aligned}$$

admits a lift as a $\mathbf{Spin}_8'$-invariant that we will call $-A_4^{(0)}$. Indeed, we only have to check that, whenever the twist $\langle \chi^{(0)}_{8,*}(z)\rangle q_{\mathfrak{C}}$ is isotropic, then $\langle \chi^{(0)}_{8,*}(z)\rangle E_3(q_{\mathfrak{C}})$ vanishes, which is a standard application of the Arason-Pfister Hauptsatz (cf. [40, Thm. X.5.1]). This yields firstly a lift of the invariant $E_3 \otimes_{\mathbb{A}^1} \chi_8^{(0)}$, and secondly by a bit of puzzling, namely

$$\tilde{E}_3 - \eta A_4^{(0)},$$

also a lift of E_3. Hence we may apply the extension lemma to this invariant (the lift of E_3), the representation $\chi_8^{(+)}$, and the quadratic form $q_{\mathfrak{C}_s}$, to obtain an invariant $A_4^{(+)}$ that lifts $E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)}$. Again the necessary condition is easy to check. Using the extension lemma one last time, we can extend $A_4^{(0)}$ (resp. $A_4^{(+)}$) along the representation $\chi_8^{(+)}$ (resp. $\chi_8^{(0)}$) to obtain an invariant that lifts $E_3 \otimes_{\mathbb{A}^1} \chi_8^{(+)} \otimes_{\mathbb{A}^1} \chi_8^{(0)}$, and which we will call A_5. Hence we have shown the following

Proposition 4.45. *Let k be a perfect field of characteristic $\neq 2$. Then we have an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\,\mathbf{Spin}_8') \cong \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}},$$

that is induced by the refined Rost invariant, $A_4^{(+)}$, $A_4^{(0)}$ and A_5 constructed above.

Next we address the case of $\mathbf{Spin}_7'$, which we regard as being associated to the orthogonal complement of $1 \in \mathfrak{C}_s$, and accordingly embedded into $\mathbf{Spin}_8'$. We may immediately notice that $i_{8\restriction \mathbf{G}_2}$ factors through $\mathbf{Spin}_7'$, and Garibaldi showed that this induces a homomorphism $i_7 \colon \mathbf{G}_2 \times \boldsymbol{\mu}_2 \to \mathbf{Spin}_7'$, where $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{Spin}_7'$ is the embedding of the center of $\mathbf{Spin}_7'$, and that i_7 induces a surjection on Galois cohomology $H^1_{\acute{e}t}(F, -)$ for every $F \in \mathcal{F}_k$ (cf. [20, Ex. 17.5]). As above, we may deduce from this surjection, that the normed $\mathbf{Spin}_7'$-invariants form a subset of the normed invariants of $\mathbf{G}_2 \times \boldsymbol{\mu}_2$, which are spanned by the Arason invariant E_3 and the invariant $E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}$. By restricting the $\mathbf{Spin}_8'$-invariants $\tilde{E}_3$ and $A_4^{(+)}$ to $\mathbf{Spin}_7'$, we see that lifts for these $\mathbf{G}_2 \times \boldsymbol{\mu}_2$-invariants exist. Note that $\tilde{E}_{3\restriction \mathbf{Spin}_7'}$ is the refined Rost invariant of $\mathbf{Spin}_7'$. We thus have:

Proposition 4.46. *Let k be a perfect field of characteristic $\neq 2$. Then we have an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B_{\acute{e}t}}\mathbf{Spin}_7') \cong \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}},$$

induced by the refined Rost invariant, and the restriction of $A_4^{(+)}$.

4.7.4 $\mathbf{Spin}_9$ and $\mathbf{Spin}_{10}$

Before we begin outlining the definition of the embeddings i_9 and i_{10}, we recall from [20, Ex. 17.1] that for all $n \geq 4$ the embedding

$$\mathbf{Spin}'_{2n-1} \times \mathbf{Z}(\mathbf{Spin}'_{2n}) \longrightarrow \mathbf{Spin}'_{2n}$$

induced by the product, with the embedding $\mathbf{Spin}'_{2n-1} \hookrightarrow \mathbf{Spin}'_{2n}$ realizing $\mathbf{Spin}'_{2n-1}$ as the stabilizer of some anisotropic vector, yields a surjection on Galois cohomology. This is deduced by considering the spin representation of an encompassing group $\mathbf{Spin}'_{2n+2}$.

One may apply this to our current situation, for which we fix the identification

$$\begin{array}{ccccc} \mathbf{Spin}'_8 & \hookrightarrow & \mathbf{Spin}'_9 & \hookrightarrow & \mathbf{Spin}'_{10} \\ \| & & \| & & \| \\ \mathbf{Spin}(\mathfrak{C}_s, q_{\mathfrak{C}_s}) & \hookrightarrow & \mathbf{Spin}(k \oplus \mathfrak{C}_s, \langle 1 \rangle \oplus q_{\mathfrak{C}_s}) & \hookrightarrow & \mathbf{Spin}(k^2 \oplus \mathfrak{C}_s, h \oplus q_{\mathfrak{C}_s}), \end{array}$$

and our explicit choice for $\mathbf{Spin}'_9$ is in alignment with Igusa (see [34, p. 1014]). So, we start with i_9, and see for i_{10} afterwards. For the first case, Garibaldi combined explicit calculations by Igusa [34] about stabilizers of spinors, together with his machinery concerning Galois cohomology and projective spin representations, to deduce that (cf. [20, Ex. 17.7])

$$i_9 \colon \mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}'_8) \longrightarrow \mathbf{Spin}'_9$$

induced by the embedding of $\mathbf{G}_2 \subseteq \mathbf{Spin}'_8$ in $\mathbf{Spin}'_9$ and then taking the product, leads to a surjection on Galois cohomology for all fields $F \in \mathsf{F}_k$. Going further by using the above alluded comparison of spin groups with odd and even ranks, Garibaldi was able to show that (cf. [20, Ex. 17.8])

$$i_{10} \colon \mathbf{G}_2 \times \underbrace{\mathbf{Z}(\mathbf{Spin}'_{10})}_{\cong \boldsymbol{\mu}_4} \longrightarrow \mathbf{Spin}'_{10}$$

induces surjective maps on Galois cohomology. This concludes our preparation, and we move on to determine the cohomological invariants and $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(-))$. In that we will reuse the names A_4, and A_5.

Let us start with the case of $\mathbf{Spin}'_{10}$. By using the $\mathbb{A}^1$-tensor product, especially bilinearity and corollary 4.8, $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{G}_2) \cong \mathbb{Z} \oplus \underline{\mathbf{K}}_3^{\mathrm{W}}$ (cf. 4.41), and $\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\boldsymbol{\mu}_4) \cong \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{MW}}/(2h)$ (cf. lemma 4.11), we obtain

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}(\mathbf{G}_2 \times \boldsymbol{\mu}_4)) \cong \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{\mathrm{MW}}/(2h) \oplus \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}},$$

which is induced by the universal morphisms

$$\begin{aligned} \sigma^{(4)} \colon \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2 \times \boldsymbol{\mu}_4) &\longrightarrow \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_4) \longrightarrow \underline{\mathbf{K}}_1^{\mathrm{MW}}/(2h), \\ E_3 \colon \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2 \times \boldsymbol{\mu}_4) &\longrightarrow \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2) \longrightarrow \underline{\mathbf{K}}_3^{\mathrm{W}}, \quad \text{and} \\ E_3 \otimes_{\mathbb{A}^1} \sigma^{(4)} \colon \mathcal{H}^1_{\text{ét}}(\mathbf{G}_2 \times \boldsymbol{\mu}_4) &\longrightarrow \underline{\mathbf{K}}_4^{\mathrm{W}}. \end{aligned}$$

As before, since the factor $\boldsymbol{\mu}_4$ is central, we cannot have a contribution of $\sigma^{(4)}$ for any restriction of an $\mathbf{Spin}'_{10}$-invariant. This leaves us with constructing lifts of E_3 and $E_3 \otimes_{\mathbb{A}^1} \sigma^{(4)}$.

Composing the homomorphism i_{10} with the vector representation $\chi_{10}^{(0)} \colon \mathbf{Spin}'_{10} \to \mathbf{SO}'_{10}$ one finds the induced map on Galois cohomology to be given by ($F \in \mathsf{F}_k$)

$$\begin{aligned} (\chi_{10}^{(0)} \circ i_{10})_* \colon H^1_{\text{ét}}(F, \mathbf{G}_2) \times H^1_{\text{ét}}(F, \boldsymbol{\mu}_4) &\longrightarrow H^1_{\text{ét}}(F, \mathbf{SO}'_{10}) \\ (\mathfrak{C}, u(F^\times)^4) &\longmapsto \langle 1, -1 \rangle \oplus \langle u \rangle q_{\mathfrak{C}}. \end{aligned}$$

Armed with this knowledge, we may construct the desired lifts quite explicitly: Suppose we are given an element in $H^1_{\text{ét}}(F, \mathbf{Spin}'_{10})$ admitting two lifts to $H^1_{\text{ét}}(F, \mathbf{G}_2 \times \boldsymbol{\mu}_4)$ denoted by $(\mathfrak{C}, u(F^\times)^4)$ and $(\mathfrak{C}', v(F^\times)^4)$. Applying Witt cancellation (cf. [40, Thm. I.4.2]) to $(\chi^{(0)}_{10} \circ i_{10})_*(\mathfrak{C}, u(F^\times)^4)$ and $(\chi^{(0)}_{10} \circ i_{10})_*(\mathfrak{C}', v(F^\times)^4)$, we obtain an isometry

$$\langle u\rangle q_{\mathfrak{C}} \cong \langle v\rangle q_{\mathfrak{C}'}.$$

Using [39, Thm. (33.19)] we obtain from this similarity relation that the F-algebras $\mathfrak{C}$ and $\mathfrak{C}'$ are isomorphic, thus allowing us to lift E_3. Moreover, we can see that $\frac{u}{v}$ is a similarity factor for $q_{\mathfrak{C}}$, and, since $q_{\mathfrak{C}}$ is a 3-fold Pfister form, it is also represented by it (cf. [40, Thm. X.1.8]). Transferring the statement of [67, Thm. 2.10.13] to Witt K-theory, we see that

$$\left[\frac{u}{v}\right] E_3(\mathfrak{C}) = 0$$

holds. We remark in passing that we may check this latter identity also elementarily, using the pure subform theorem (cf. [16, Lemma I.6.11]). Utilizing axiom **(MW2)**, as well as $\langle u\rangle[u] = -[u]$ (cf. (4.1)), we may convert this into

$$[u]E_3(\mathfrak{C}) = [v]E_3(\mathfrak{C}) = [v]E_3(\mathfrak{C}'),$$

netting us the well-definedness of a lift A_4 of $E_3 \otimes_{\mathbb{A}^1} \sigma^{(4)}$. Similarly to the situation in the previous section, we see that the refined Rost invariant $\tilde{E}_3$ may be viewed as a lift of $E_3 + \eta E_3 \otimes_{\mathbb{A}^1} \sigma^{(4)}$. Moreover, one may show that the restriction of A_4 to $\mathbf{Spin}'_8$ is the invariant $A^{(0)}_4$. We summarize our findings:

Proposition 4.47. *Let k be a perfect field of characteristic $\neq 2$. Then there is an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_{10}) \cong \underline{\mathbf{K}}^{\mathrm{W}}_3 \oplus \underline{\mathbf{K}}^{\mathrm{W}}_4,$$

which is induced by the refined Rost invariant, and A_4.

We shift our focus towards $\mathbf{Spin}'_9$, and base our discussion on [20, III.18.9]. The usual argument using the $\mathbb{A}^1$-tensor product yields that the nonconstant $\mathbf{G}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$-invariants are spanned by

$$\begin{array}{cccc} \sigma^{(2)}_1, & \sigma^{(2)}_2, & & \sigma^{(2)}_1 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2, \\ E_3, & E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_1, & E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2, & \text{and} \quad E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_1 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2, \end{array}$$

where the index at $\sigma^{(2)}_i$ indicates which projection $\mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \to \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2)$ precedes the universal morphism $\sigma^{(2)} \colon \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_2) \to \underline{\mathbf{K}}^{\mathrm{W}}_1$. We note that, since $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ factors through $\mathbf{Spin}'_8$ and is central there, there are no contributions from the first row.

Restricting the invariants of $\mathbf{Spin}'_{10}$, we readily obtain lifts for the invariants E_3 and $E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2$. In order to see that the invariant $E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_1 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2$ can also be lifted, we take a detour to $\mathbf{F}_4$. Note that there is a well-known embedding of $\mathbf{Spin}'_9$ into $\mathbf{F}_4$ (cf. [35, Sec. XI.3]), and by going through our definitions, we find that computing the trace form of the induced Albert algebra, we get

$$\begin{array}{c} H^1_{\text{ét}}(F, \mathbf{G}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \to H^1_{\text{ét}}(F, \mathbf{Spin}'_9) \to H^1_{\text{ét}}(F, \mathbf{F}_4) \longrightarrow H^1_{\text{ét}}(F, \mathbf{SO}(J_s, q_{J_s})) \\ (\mathfrak{C}, u_1, u_2) \longmapsto \langle 2,2,2\rangle \oplus \langle -u_1, -u_2, u_1u_2\rangle q_{\mathfrak{C}}, \end{array}$$

where J_s denotes the split Albert algebra $\mathcal{H}_3(\mathfrak{C}_s, 1, 1)$ and q_{J_s} its trace form halved.[11] Hence we see that the restriction of the $\mathbf{F}_4$-invariant F_5 (cf. proposition 4.43) yields a lifting of $E_3 \otimes_{\mathbb{A}^1} \sigma^{(2)}_1 \otimes_{\mathbb{A}^1} \sigma^{(2)}_2$.

[11] By the particular way, in which we set up the groups $\mathbf{G}_2$, $\mathbf{F}_4$, and $\mathbf{Spin}'_9$, we are able to lift the cosmetic assumption that -1 is a square in [20, III.18.9].

Finally, we conclude that $E_3 \otimes_{\mathbb{A}^1} \sigma_1^{(2)}$ cannot contribute: Suppose we were given some homomorphism $\lambda \in \mathrm{Hom}_{\mathsf{Ab}_{\mathrm{Nis}}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_4^{\mathrm{W}}, M)$ such that $\lambda \circ (E_3 \otimes_{\mathbb{A}^1} \sigma_1^{(2)})$ is the restriction of a $\mathbf{Spin}_9'$-invariant. We check that λ evaluates to zero, on all possible choices of symbols $[u_1]\cdots[u_4]$, with $u_1, \ldots, u_4 \in F^\times$. By lemma 4.3 and proposition 3.25, we know that this implies $\lambda = 0$. So, suppose that $\mathfrak{C}$ is a Cayley algebra with norm form $\langle\langle u_1, u_2, u_3\rangle\rangle$, then we may construct a spin torsor, precisely as in [20, p. 55], to which both $\mathbf{G}_2 \times \mathbf{Z}(\mathbf{Spin}_8')$-torsors

$$(\mathfrak{C}, u_4(F^\times)^2, (F^\times)^2) \quad \text{and} \quad (\mathfrak{C}, (F^\times)^2, (F^\times)^2)$$

map to. From this we see $\lambda([u_1]\cdots[u_4]) = 0$. Hence we have checked the following

Proposition 4.48. *Let k be a perfect field of characteristic $\neq 2$. Then we have an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}_9') \cong \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_4^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}},$$

which is induced by the refined Rost invariant, the restriction of A_4, and the restriction of F_5.

4.7.5 $\mathbf{Spin}_{11}$ and $\mathbf{Spin}_{12}$

In this section, we add to the assumption that k is a perfect field of characteristic $\neq 2$ that -1 is square. This further hypothesis greatly simplifies the formulas that occur.

Let us start with the case of $\mathbf{Spin}_{12}'$, which we shall think of as being associated to the quadratic space (k^{12}, q_{12}'), with

$$q_{12}'(x) := \frac{1}{2}x^t \begin{pmatrix} \mathbf{0} & 6_\epsilon \\ 6_\epsilon & \mathbf{0} \end{pmatrix} x, \quad \text{with} \quad 6_\epsilon := ((-1)^i \delta_{ij})_{1 \leq i,j \leq 6},$$

to align ourselves with the conventions that Garibaldi [20], and Igusa [34] use, whose explicit calculations form the basis of the argument. For future reference, we add that $e_1, \ldots, e_{12}$ denotes the standard basis of k^{12}.

In [20, Ex. 17.12], Garibaldi has shown that there is an inclusion $i_{12}\colon \mathbf{SO}_6' \times \boldsymbol{\mu}_4 \to \mathbf{Spin}_{12}'$ which is surjective on Galois cohomology, and whose concatenation with the vector representation induces the natural map (in $F \in \mathsf{F}_k$)

$$\begin{aligned} H^1_{\text{ét}}(F, \mathbf{SO}_6' \times \boldsymbol{\mu}_4) &\longrightarrow H^1_{\text{ét}}(F, \mathbf{SO}_{12}') \\ (q, u(F^\times)^4) &\longmapsto \langle -1\rangle\langle\langle u\rangle\rangle q \simeq \langle\langle u\rangle\rangle q. \end{aligned}$$

This is shown by finding a suitable element of the spin representation of $\mathbf{Spin}_{12}'$, with stabilizer $\mathbf{SL}_6$, in which the group $\mathbf{SO}_6'$ resides in. The embedding of $\boldsymbol{\mu}_4$ on the other hand corresponds to the element

$$i(e_1 + e_7)\cdots(e_6 + e_{12}) \in \mathbf{Spin}_{12}'(k),$$

where i denotes some square root of -1 in k.

We now repeat the definition of some of the $\mathbf{Spin}_{12}'$-invariants. The construction goes back to a preprint of Rost [66], and has been expanded by Garibaldi in [20, III.20]: Under the assumption that -1 is a square, one may define natural divided power operations ($F \in \mathsf{F}_k$)

$$\begin{aligned} P_n\colon I^n(F) &\longrightarrow I^{2n}(F) \\ \sum_i \langle u_i\rangle p_i &\longmapsto \sum_{i<j} \langle u_i u_j\rangle p_i p_j, \end{aligned}$$

where $u_i \in F^\times$, and p_i are n-fold Pfister forms. Given some cohomology class $\gamma \in H^1_{\text{ét}}(F, \mathbf{Spin}_{12}')$, we know by [20, Thm. 17.13] that its image under map induced by the vector representation is of the form $\langle\langle u\rangle\rangle q$, for some $u \in F^\times$ and six dimensional form $q \in I^2(F)$. Using [20, Cor. 20.7], we may conclude that the map $\gamma \mapsto \langle\langle u\rangle\rangle P_2(q) \in I^5(F)$ is well-defined, and thus so is

$$\gamma \longmapsto E_5(\langle\langle u\rangle\rangle P_2(q)) \in K_5^{\mathrm{W}}(F).$$

We denote the resulting invariant by $A_5\colon \mathcal{H}^1_{\text{ét}}(\mathbf{Spin}'_{12}) \to \underline{\mathbf{K}}_5^{\mathrm{W}}$.[12] By applying the extension proposition 4.44 to the vector representation of $\mathbf{Spin}'_{12}$, the quadratic form q'_{12} and the invariant A_5, we obtain an invariant $A_6\colon \mathcal{H}^1_{\text{ét}}(\mathbf{Spin}'_{12}) \to \underline{\mathbf{K}}_6^{\mathrm{W}}$. Therefore we only have to check the hypothesis that, whenever $\langle\langle u\rangle\rangle q$ as image of some class $\gamma \in H^1(F, \mathbf{Spin}'_{12})$ is isotropic, the value $A_5(\gamma)$ vanishes, for which we reuse the argument of [20, Lemma 20.12]: Namely, whenever $\langle\langle u\rangle\rangle q$ is isotropic, one may show (cf. [20, Ex. 17.8]) that q is of the form $\langle v\rangle(p - 2h)$, when regarded as an element of $I^2(F)$, with $v \in F^\times$ and p some 2-fold Pfister form, and thus

$$A_5(\gamma) = E_5(\langle\langle u\rangle\rangle P_2(\langle v\rangle p + \langle -v\rangle\langle\langle 1,1\rangle\rangle)) = E_5(\langle -1\rangle\langle\langle u\rangle\rangle p\langle\langle 1,1\rangle\rangle) = 0.$$

This finishes the preparation for

Proposition 4.49. *Let k be a perfect field of characteristic $\neq 2$, and admitting a primitive 4^{th} root of unity. Then there is an isomorphism*

$$\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_{12}) \cong \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}} \oplus \underline{\mathbf{K}}_6^{\mathrm{W}}$$

that is induced by the refined Rost invariant $\tilde{E}_3$, A_5, and A_6.

Proof. Let M be a strictly $\mathbb{A}^1$-invariant sheaf. From the surjectivity of the map that is induced by i_{12} on Galois cohomology, we obtain, as in the cases above, an injection

$$\mathrm{Inv}_k^{\text{norm.}}(\mathbf{Spin}'_{12}, M) \hookrightarrow \mathrm{Inv}_k^{\text{norm.}}(\mathbf{SO}'_6 \times \boldsymbol{\mu}_4, M)$$

of groups of normed invariants. So, before we proceed, we take a detour and determine a nice form for the invariants of $\mathbf{SO}'_6$ under the additional assumption that -1 is a square. From proposition 4.26 we know that any $\mathbf{SO}'_6$-invariant with values in M may be written as

$$\lambda'_0 + \lambda'_1 W_1 + \lambda'_2 W_2 + \lambda'_3 W_3 + \lambda'_4 W_4 + \lambda'_5 W_5,$$

where the $W_1, \ldots, W_5$ are $\mathbf{O}_5$-invariants, and $\lambda'_i \in \mathrm{Hom}_{Ab_{\mathrm{Nis}}(Sm_k)}(\underline{\mathbf{K}}_i^{\mathrm{W}}, M)$ resp. $\lambda'_0 \in M(k)$ are subject to the relations

$$0 = \lambda'_1 \quad \text{and} \quad 0 = \lambda'_1\eta^2 + \lambda'_2\eta + \lambda'_3.$$

Hence any $\mathbf{SO}'_6$-invariant in this case may be written as $\lambda'_0 + \lambda'_2(W_2 + \eta W_3) + \lambda'_4 W_4 + \lambda'_5 W_5$, where the refined Stiefel-Whitney classes are the ones for 5-dimensional quadratic forms. However, in the present case we are able to obtain a description, at least partially, in terms of $\mathbf{O}_6$-invariants that is also closer to the classical result [21, Thm. 20.6]. Therefore, we note first that the usual argument using chain equivalences of diagonalizations allows us to define an invariant

$$\begin{aligned} B\colon \mathcal{H}^1_{\text{ét}}(\mathbf{SO}'_6) &\longrightarrow \underline{\mathbf{K}}_5^{\mathrm{W}} \\ \underbrace{q}_{\simeq\langle u_1,\ldots,u_6\rangle} &\longmapsto W_5(\langle u_1, \ldots, u_5\rangle). \end{aligned}$$

Now starting with an ansatz invariant $\lambda_0 + \lambda_2 W_2 + \lambda_4 W_4 + \tilde{\lambda} B$, where the Stiefel-Whitney classes are the ones for 6-dimensional forms, we find using the formulae of corollary 4.25 that its restriction is given by

$$\begin{aligned} \lambda'_0 &= \lambda_0, & \lambda'_1 &= 0, & \lambda'_2 &= \lambda_2, \\ \lambda'_3 &= \lambda_2\eta, & \lambda'_4 &= \lambda_4, \quad \text{and} & \lambda'_5 &= \lambda_2\eta^3 + \tilde{\lambda}. \end{aligned}$$

[12] Since we will not try to compare the $\mathbf{Spin}'_{12}$-invariants to any of the invariants that came before, we hope that this abuse of notation does not confuse the reader.

From this we see that W_2, W_4 and B are universal $\mathbf{SO}'_6$-invariants, at least in the case that -1 is a square. Now an argument using the $\mathbb{A}^1$-tensor product yields, that the group of invariants $\mathrm{Inv}_k^{\mathrm{norm.}}(\mathbf{SO}'_6 \times \boldsymbol{\mu}_4, M)$ is spanned by

$$\begin{array}{ccccc} & W_2, & W_4, & & B, \\ \sigma^{(4)}, & W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)}, & W_4 \otimes_{\mathbb{A}^1} \sigma^{(4)}, & \text{and} & B \otimes_{\mathbb{A}^1} \sigma^{(4)}, \end{array}$$

where $\sigma^{(4)} \colon \mathcal{H}^1_{\text{ét}}(\boldsymbol{\mu}_4) \to \underline{\mathbf{K}}_1^{\mathrm{MW}}/(2h)$ denotes the universal invariant of $\boldsymbol{\mu}_4$ of lemma 4.11. Of this list, we may immediately cross out the first line, since the embedding of $\mathbf{SO}'_6$ factors through a copy of $\mathbf{SL}_6$. Next we try to identify lifts of the remaining invariants, except in the case of $\sigma^{(4)}$.

To get a better understanding of the situation we derive a few identities, and to explain those, we fix a cohomology class $\gamma \in H^1(F, \mathbf{Spin}'_{12})$ and a particular lift $(q, u(F^\times)^4) \in H^1_{\text{ét}}(F, \mathbf{SO}'_6 \times \boldsymbol{\mu}_4)$. Using the classification of the $\mathbf{SO}'_6$-torsors as quadratic forms of the kind $\langle v\rangle(p'_1 \oplus \langle -1\rangle p'_2)$ (cf. [40, Cor. XII.2.13]), where $v \in F^\times$, and p_1, p_2 are 2-fold Pfister forms, with the primed versions denoting the pure subforms, we may explicitly show the following (recall that the image of q in the Witt ring lies in $I^2(F)$):

$$W_2(q) = E_2(q) + \eta^2 W_4(q), \tag{4.17}$$
$$W_4(q) = E_2(p_1)E_2(p_2) = E_4(P_2(q)), \quad \text{and} \tag{4.18}$$
$$B(q) = [v]W_4(q), \tag{4.19}$$

similarly as [20, p. 58]. If we apply the invariant A_5 to γ, we get the value

$$A_5(\gamma) = E_5(\langle\langle u\rangle\rangle P_2(q)) = [u]E_4(P_2(q)) = (W_4 \otimes_{\mathbb{A}^1} \sigma^{(4)})(q, u(F^\times)^4),$$

allowing us to see A_5 as a lift of $W_4 \otimes_{\mathbb{A}^1} \sigma^{(4)}$. Similarly, we see that $W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)}$ can be lifted by forming a combination of the refined Rost invariant and A_5:

$$(W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)})(q, u(F^\times)^4) = [u]E_2(q) + \eta^2[u]W_4(q) = \tilde{E}_3(\gamma) + \eta^2 A_5(\gamma),$$

because $\chi^{(0)}_{12,*}(\gamma) = \langle\langle u\rangle\rangle q$. Finally, (4.19) implies that we may lift $B \otimes \sigma^{(4)}$ as well, since we get

$$(B \otimes \sigma^{(4)})(q, u(F^\times)^4) = [u][v]W_4(q) = [u][v]E_4(P_2(q)) = [v]A_5(\gamma),$$

which is A_6 evaluated on γ. Lastly, we check that for every $\lambda \in \mathrm{Hom}_{Ab_{\mathrm{Nis}}(Sm_k)}(\underline{\mathbf{K}}_1^{\mathrm{W}}, M)$ the only invariant of the form $\lambda \circ \sigma^{(4)}$ coming from an invariant of $\mathbf{Spin}'_{12}$ is the trivial one. Since $K_1^{\mathrm{W}}(F)$ is additively generated by symbols of the form $[u]$, with $u \in F^\times$ (cf. lemma 4.3), it remains to show $\lambda([u]) = 0$. The 12-dimensional form $\langle\langle u\rangle\rangle(3h) \simeq \langle\langle 1\rangle\rangle(3h)$ is isotropic, and thus both $(3h, u(F^\times)^4)$ and $(3h, (F^\times)^4)$ map to the same $\mathbf{Spin}'_{12}$-torsor under $i_{12,*}$, from which we see that

$$\lambda([u]) = (\lambda \circ \sigma^{(4)})(3h, u(F^\times)^4) = (\lambda \circ \sigma^{(4)})(3h, (F^\times)^4) = 0,$$

as long as $\lambda \circ \sigma^{(4)}$ comes from some $\mathbf{Spin}'_{12}$-invariant. This finishes the proof. ☐

So, we turn our attention towards $\mathbf{Spin}'_{11}$ which we consider to be the subgroup of $\mathbf{Spin}'_{12}$ stabilizing $e_6 - e_{12}$. Garibaldi shows that the standard embedding

$$\begin{aligned} \mathbf{SO}'_5 &\hookrightarrow \mathbf{SO}'_6 \\ g &\longmapsto \begin{pmatrix} g & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \end{aligned}$$

gives rise to a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(F,\mathbf{SO}'_5)\times H^1_{\text{ét}}(F,\boldsymbol{\mu}_4) & \longrightarrow & H^1_{\text{ét}}(F,\mathbf{SO}'_6)\times H^1_{\text{ét}}(F,\boldsymbol{\mu}_4) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(F,\mathbf{Spin}'_{11}) & \longrightarrow & H^1_{\text{ét}}(F,\mathbf{Spin}'_{12}), \end{array}$$

in which the vertical arrows are surjective, for all $F \in \mathsf{F}_k$ (cf. [20, Ex. 17.14]). Using this diagram, we deduce the case of $\mathbf{Spin}'_{11}$ by restricting the one of $\mathbf{Spin}'_{12}$. So, denote the restriction of the $\mathbf{Spin}'_{12}$-invariant A_5 to $\mathbf{Spin}'_{11}$ by the same name. Note that the restriction of A_6 to $\mathbf{Spin}'_{11}$ is trivial, as one sees classically.

Proposition 4.50. *Let k be a perfect field of characteristic $\neq 2$, and admitting a primitive 4^{th} root of unity. Then there is an isomorphism*

$$\mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_{11}) \cong \mathbb{Z} \oplus \underline{\mathbf{K}}_3^{\mathrm{W}} \oplus \underline{\mathbf{K}}_5^{\mathrm{W}}$$

that is induced by E_3 and (the restriction of) A_5.

Proof. Under the assumption that -1 is a square, we see from example 4.24 that the universal invariants of the split $\mathbf{SO}'_5$ are given by the Stiefel-Whitney classes W_2 and W_4 on 5-dimensional forms. By the usual argument involving the $\mathbb{A}^1$-tensor product we have to consider the invariants

$$\begin{array}{cccc} & W_2, & & W_4, \\ \sigma^{(4)}, & W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)}, & \text{and} & W_4 \otimes_{\mathbb{A}^1} \sigma^{(4)}. \end{array}$$

The first line may be handled, again as above, by noting that the embedding $\mathbf{SO}'_5 \hookrightarrow \mathbf{Spin}'_{11}$ factors through $\mathbf{SL}_5$. The composition of the homomorphism $\mathbf{SO}'_5 \times \boldsymbol{\mu}_4 \to \mathbf{Spin}'_{11}$ with the vector representation induces on Galois cohomology the map

$$\begin{aligned} H^1_{\text{ét}}(F,\mathbf{SO}'_5)\times H^1_{\text{ét}}(F,\boldsymbol{\mu}_4) &\longrightarrow H^1_{\text{ét}}(F,\mathbf{SO}'_{11}) \\ (q,u(F^\times)^4) &\longmapsto \langle u,-1\rangle q \oplus \langle u\rangle \simeq \langle\langle u\rangle\rangle q \oplus \langle u\rangle, \end{aligned}$$

which implies that the restriction of the $\mathbf{Spin}'_{12}$ refined Rost invariant is precisely the $\mathbf{Spin}'_{11}$ refined Rost invariant. Now suppose that $\gamma \in H^1_{\text{ét}}(F,\mathbf{Spin}'_{11})$ is any torsor, with a lift $(q,u(F^\times)^4)$. Then we see

$$(W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)})(q,u(F^\times)^4) = (W_2 \otimes_{\mathbb{A}^1} \sigma^{(4)})(q \oplus \langle 1\rangle, u(F^\times)^4) = \tilde{E}_3(\gamma) + \eta^2 A_5(\gamma),$$

from (4.17). Moreover, (4.18) implies that a lift of $W_4 \otimes_{\mathbb{A}^1} \sigma^{(4)}$ is given by the $\mathbf{Spin}'_{11}$-version of A_5. The fact that for any $\lambda \in \mathrm{Hom}_{\mathsf{Ab}_{Nis}(\mathsf{Sm}_k)}(\underline{\mathbf{K}}_1^{\mathrm{W}}, M)$ the invariant $\lambda \circ \sigma^{(4)}$ does not come from a $\mathbf{Spin}'_{11}$-invariant can be checked as in the case of $\mathbf{Spin}'_{12}$ above. From this the proposition readily follows. □

Remark 4.51 (Embedding of $\pi_0^{\mathbb{A}^1}$). In the cases of $\mathbf{Spin}'_8$ and $\mathbf{Spin}'_{10}$ Garibaldi has shown that the classical versions of the above invariants completely determine the corresponding Galois cohomology sets (see 18.5 and 18.8.(2) in [20]), and thus one obtains monomorphisms

$$\rho\colon \pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n) \longrightarrow \mathbb{H}_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathbf{Spin}'_n),$$

for $n = 8, 10$. The remaining cases are open (or trivial).

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Eidesstattliche Versicherung
(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2, Pkt. 5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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