Intertwining for particle systems in the continuum

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität München

eingereicht von
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19. Dezember 2023

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Tag der mündlichen Prüfung: 8. Mai 2024

Eidesstattliche Versicherung

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München, den 19. Dezember 2023

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Zusammenfassung

Stochastische Dualität ist ein mächtiges Werkzeug, das zwei Markov-Prozesse mithilfe einer gemeinsamen Observablen, der Dualitätsfunktion, verknüpft. Dieses Werkzeug ist nützlich, wenn einer der Prozesse einfacher zu untersuchen ist. Im Fall der Selbst-Dualität können Vereinfachungen erzielt werden, indem eine einfachere Anfangsbedingung im dualen Prozess verwendet wird. In verschiedenen Bereichen, darunter Modelle der Populationsgenetik und (stochastische) partielle Differentialgleichungen, findet stochastische Dualität weitreichende Anwendung. Selbst-Dualität ist besonders nützlich bei der Untersuchung von *fluctuation fields*, dem Boltzmann-Gibbs-Prinzip und Kumulanten in *non-equilibrium steady states*. In den letzten Jahren wurden Dualitäten für interagierende Teilchensysteme auf Gittern wie dem *symmetric inclusion process* (SIP), *symmetric exclusion process* (SEP) und unabhängigen zufälligen Irrläufern (IRW) untersucht. Diese Systeme sind selbst-dual, entweder ausgedrückt mit fallenden Faktoriellen oder mit orthogonalen Polynome bezüglich einem reversiblem Maß. Diese Dualitäten ermöglichen es, die zeitliche Entwicklung von n-Punkt-Korrelationsfunktionen eines Vielteilchensystems mit der zeitlichen Entwicklung eines n-Teilchensystems auszudrücken, was eine erhebliche Vereinfachung darstellt.

Diese Arbeit untersucht Verallgemeinerungen für Teilchen in allgemeinen Räumen, zum Beispiel $\mathbb{R}^d$, anstelle von Gittern. Wir verwenden die Sprache der Punktprozesstheorie, um einen natürlichen Rahmen zu etablieren, in dem der Begriff der Dualität zu *intertwining*-Beziehungen verallgemeinert wird. Beispiele hierfür sind unabhängige Diffusionen und freie Kawasaki-Dynamik, die bereits zuvor untersucht wurden, sowie eine neue Version des SIP im Kontinuum, die mit dem maßwertigen Moran-Modell in der Populationsgenetik zusammenhängt. Diese Modelle teilen eine Konsistenz-Eigenschaft, die wir für allgemeine Räume näher erläutern. Konsistenz bedeutet grob gesagt, dass das zufällige Entfernen eines Teilchens mit der zeitlichen Entwicklung des Prozesses kommutiert. Dieses Konzept gilt für eine breites Spektrum von Teilchensystemen im Diskreten und Kontinuum. Wir zeigen, dass *compatible families*, die von Le Jan und Raimond im Jahr 2004 definiert wurden, ebenfalls in diese Kategorie fallen. Diese Familien umfassen *sticky Brownian motions*, die mit dem Howitt-Warren-Martingalproblem zusammenhängen, und korrelierten Brownschen Bewegungen.

Die wichtigsten Ergebnisse sind: (1) Selbst-Dualitäten, ausgedrückt durch fallende Faktorielle, verallgemeinern sich zu *self-intertwining*-Beziehungen in Form von Lenards K-Transformation. (2) Teilchensysteme, die reversibel und konsistent sind, erfüllen eine *self-intertwining*-Beziehung in Form von unendlich dimensionalen orthogonalen Polynomen, die in der Theorie der Chaoszerlegungen, Lévy-Zufallsfeldern und mehrdimensionalen stochastischen Integralen eingeführt werden. (3) Für die Kontinuumsversion des SIP geben wir eine *self-intertwining*-Beziehung in Form von unendlich dimensio-

nalen Meixner-Polynomen an. Diese Polynome sind orthogonal bezüglich der Verteilung des Pascal-Prozesses. Die Beziehung verallgemeinert und rekonstruiert die Dualitätsbeziehung des SIP. (4) Wir untersuchen Systeme mit unendlich vielen Teilchen. Wir konzentrieren uns auf ein System bestehend aus *sticky Brownian motions* und zeigen, dass unendlich dimensionale Meixner-Polynome die Dynamik von unendlich vielen Teilchen und deren n-Teilchen-Evolution verflechten. Hierzu leiten wir neue explizite Formeln für diese Polynome her. Als Ergebnis stellen wir fest, dass die Verteilung des Pascal-Punktprozesses für unendlich viele *sticky Brownian motions* reversibel ist. (5) Wir verallgemeinern die bekannte Dualität des SIP und des *Brownschen energy process*, der eine Vielteilchen-Skalierung des SIP darstellt, auf das Kontinuum unter Verwendung unendlich dimensionaler Laguerre-Polynome. (6) Der algebraische Ansatz zur Dualität verknüpft Erzeuger von Markov-Prozessen und Darstellungen von Lie-Algebren, wobei Dualitäten Wechseln von Darstellungen entsprechen. Wir entwickeln den Lie-algebraischen Ansatz zu *intertwining*-Beziehungen im Kontext von Teilchen, die sich in allgemeinen Räumen bewegen. Wir konzentrieren uns auf die $su(1,1)$-Algebra und Meixner-Polynome.

Abstract

Stochastic duality is a powerful tool that links two Markov processes using a common observable, the duality function. This tool is useful if one of the processes is easier to study. In the case of self-duality, simplification can often be achieved by using a simpler initial condition in the dual process. Stochastic duality finds widespread application in diverse fields, including population genetics models and (stochastic) partial differential equations. Self-duality is particularly valuable in studying fluctuation fields, the Boltzmann Gibbs principle and cumulants in non-equilibrium steady states. In recent years, dualities for interacting particle systems on lattices, such as the symmetric inclusion process (SIP), symmetric exclusion processes (SEP) and independent random walks (IRW), have been studied. These systems are self-dual, either expressed in terms of falling factorials or of orthogonal polynomials with respect to a reversible measure. These dualities allow to map the time evolution of n-point correlation functions of a many-particle system to the time evolution of an n-particle system, a considerable simplification.

The thesis explores generalizations for particles in general spaces, e.g., $\mathbb{R}^d$, rather than on the lattice. We use the language of point process theory to establish a natural framework in which the notion of duality generalizes to intertwining relations. Examples of this include independent diffusions and free Kawasaki dynamics, which have been previously studied, as well as a new version of the SIP in the continuum, related to the measure-valued Moran model in population genetics. These models share a consistency property that we elaborate on for general spaces. Roughly speaking, consistency means that removing a particle uniformly at random commutes with the time evolution of the process. This concept applies to a wide range of particle systems in both discrete and continuum settings. We demonstrate that compatible families which are defined by Le Jan and Raimond in 2004 also fall within this category. These families include sticky Brownian motions related to the Howitt-Warren martingale problem and correlated Brownian motions.

The main results are: (1) Falling factorial self-dualities generalize to self-intertwining relations in terms of Lenard's K-transform. (2) Particle systems that are reversible and consistent satisfy a self-intertwining relation in terms of infinite-dimensional orthogonal polynomials which are introduced in chaos decompositions, Lévy random fields and multiple stochastic integrals. (3) For the continuum version of the SIP, we state a self-intertwining relation with infinite-dimensional Meixner polynomials. These polynomials are orthogonal with respect to the distribution of the Pascal process. The relation both generalizes and recovers the duality relation of the SIP. (4) We investigate systems with infinitely many particles. We focus on a system of sticky Brownian motions and prove that infinite-dimensional Meixner polynomials intertwine the dynamics of infinitely many particles and the n-particle evolution. To show this, we deduce new explicit

formulas for these polynomials. As a result, we find that the distribution of the Pascal process is reversible for infinitely many sticky Brownian motions. (5) We generalize the known duality of the SIP and the Brownian energy process, which is a many-particle scaling of the SIP, to the continuum using infinite-dimensional Laguerre polynomials. (6) The algebraic approach to duality connects generators of Markov processes and representations of Lie algebras with dualities corresponding to changes of representations. We develop the Lie algebraic approach to intertwining relations in the context of particles evolving in general spaces. We focus on the $su(1,1)$ algebra and Meixner polynomials.

Acknowledgments

I would like to express my gratitude to my supervisor Sabine Jansen. Her deep understanding in various fields in mathematics and physics consistently impressed me. She gave me great advice but also let me do my own research which I really appreciate. Working with her over the past few years has always been a pleasure.

I extend my sincere thanks to Cristian Giardinà and Dario Spanò for reviewing this thesis. Particularly, I am grateful to Dario for engaging in pleasant and fruitful discussions in our ongoing project.

A huge thanks goes to my collaborators, Frank Redig and Simone Floreani. Thanks for your hard work on our project(s). Simone helped me connect within the community by inviting me to various conferences and workshops. Our math collaboration turned into a great friendship.

I couldn't have done my PhD without the support from the Munich Center for Quantum Science and Technology (MCQST) funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's excellence strategy EXC-2111-390814868. I express my gratitude to the Hausdorff Institute for Mathematics (Bonn) for hosting me during the Junior Trimester Program on *Stochastic Modelling in Life Sciences* funded under Germany's Excellence Strategy EXC-2047/1-390685813.

Lastly, thanks to all my colleagues—Alejandro, Jakob, Julian, Kilian, Leon, Leonid, Matija, Matthew, Nannan, Philipp, Simon, Tamás, Thomas, Umberto and Vincent—for making the institute a great place to work. And a special thanks to Markus Heydenreich for his great contributions to the institute's life.

Thanks to everyone for being part of this journey!

Prior publications

Essential parts of this thesis are based on the published articles [FJRW24] and [Wag24]. Further substantial parts are based on prior preprints [FJW23], [FJW24] available on arXiv and submitted to peer-reviewed journals.

[FJRW24] S. Floreani, S. Jansen, F. Redig, and S. Wagner, *Intertwining and duality for consistent Markov processes*, Electron. J. Probab. **29** (2024), Paper No. 1.

[FJW23] S. Floreani, S. Jansen, and S. Wagner, *Intertwinings for Continuum Particle Systems: An Algebraic Approach*, Preprint. arXiv:2311.08763, 2023.

[FJW24] ———, *Representations of the* $su(1,1)$ *current algebra and probabilistic perspectives*, Preprint. arXiv:2402.07493, 2024.

[Wag24] S. Wagner, *Orthogonal intertwiners for infinite particle systems in the continuum*, Stochastic Process. Appl. **168** (2024), 104269.

The original research question, aiming to generalize duality to more general spaces, was posed by Sabine Jansen in the context of independent Markov processes. Frank Redig contributed by posing the question for the class of consistent particle systems, resulting in the article [FJRW24]. I discovered all the results and their corresponding proofs. With consultation from the other authors, I, along with Simone Floreani, took charge of the final implementation in LaTeX. Frank Redig elaborated fully on [FJRW24, Appendix A].

Sabine Jansen and Frank Redig proposed the idea of generalizing the algebraic approach to duality and suggested writing the papers [FJW23] and [FJW24]. Under the guidance of Sabine Jansen and Simone Floreani, I found all the results from [FJW23] along with their proofs. Notably, I provided a proof for [FJW23, Theorem 2.2], which is included in this thesis. An alternative proof was discovered by Sabine Jansen and is detailed in [FJW23]. Moreover, with input from Sabine Jansen and Simone Floreani, I was able to prove the main results from [FJW24]. Particularly, I discovered the representation [FJW24, Section 3.3] on my own.

The following provides an overview that aligns the results of this thesis with the corresponding sections in the articles.

- *Chapter 2*: Section 2.1 corresponds to [FJRW24, Section 2.1 and Section 2.2] and includes [FJW23, Proposition 4.1]. Section 2.2 corresponds to [Wag24, Section 2.2 and Section 4.1].

- *Chapter 3*: Section 3.1 corresponds to [FJRW24, Section 2.3]. Section 3.2, more precisely, Subsection 3.2.1 is a recapitulation, which can be found in [Wag24, Section 2.3], of known facts from other references. Subsection 3.2.2 aligns with ideas from [Wag24, Corollary 2.6]. Section 3.3 covers results from [FJRW24, Section 4.2], [Wag24, Section 3.2], [FJW23, Section 3]. Section 3.4 corresponds to [Wag24, Section 4.2].

- *Chapter 4:* Section 4.1 corresponds to the findings in [FJRW24, Section 3.1]. Section 4.2 covers [FJRW24, Section 3.2] and [Wag24, Remark 4.7]. Section 4.3 has its counterpart in [FJRW24, Section 4] and [FJW23, Section 2.1]. Sections 4.4 and 4.5 correspond to [Wag24, Sections 2 and 3].

- *Chapter 6* unifies [FJW23] and [FJW24]. More precisely, Sections 6.1, 6.2, 6.3 align with the article [FJW24], while [FJW23] covers Sections 6.4 and 6.5 and also delves into Subsection 6.2.1.

Occasionally, intentional variations from the approach in the articles are made, such as proofs being conducted differently. This is explained in the text at the respective locations.

Chapter 5 is not part of the articles [FJW23], [FJRW24], [FJW24], [Wag24] and thus appears for the first time in this thesis. Furthermore, this work includes additional supplements: Proposition 3.3.9, Remarks 4.3.2, 4.3.4, 4.5.1, Proposition 6.1.3, Corollary 6.1.4, Proposition 6.2.5, Proposition 6.2.7 and numerous other minor details.

Contents

1 Introduction

Stochastic duality is a powerful tool that establishes a connection between two Markov processes using a common observable, known as the duality function (see, e.g., [JK14]). More precisely, two Markov processes $(X_t)_{t\geq 0}$ (with state space $\mathbb{X}$) and $(Y_t)_{t\geq 0}$ (with state space $\mathbb{Y}$) are called dual with respect to a function $\mathcal{D} : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ if

$$\mathbb{E}^x \left[\mathcal{D}(X_t, y)\right] = \mathbb{E}_y \left[\mathcal{D}(x, Y_t)\right] \tag{1.1}$$

holds for all $x \in \mathbb{X}$, $y \in \mathbb{Y}$ and all $t \geq 0$ where $\mathbb{E}^x$ (or $\mathbb{E}_y$) denotes the expectation where $(X_t)_{t\geq 0}$ (or $(Y_t)_{t\geq 0}$) is starting at x (or y). When the two processes are identical, we speak of self-duality. Duality is particularly useful when one of the processes is easier to study and the duality function is a meaningful observable. In the case of self-duality, simplification may arise from using a simpler initial condition in the dual process.

Stochastic duality has found widespread application in diverse fields, including interacting particle systems (see, e.g., [Lig05]), population genetics models (see, e.g., [EK86, Chapter 10], [Mö99], [Eth06], [DG11]) and (stochastic) partial differential equations (see, e.g., [Mue15]). Self-duality, specifically with orthogonal polynomials, is particularly valuable in studying fluctuation fields, the Boltzmann Gibbs principle and cumulants in non-equilibrium steady states. This approach is elaborated upon in [DP91], [ACR18], [ACR21b], [CS21] and [FRS22]. Furthermore, there are applications and deep connections of duality with other areas, such as stochastic monotonicity [Che04], [Wan09], exit-entrance laws [CR84], ruin probabilities in financial modeling [Dje93], birth-death processes [vD80], [And91] and the well-posedness of the martingale Problem [DGP23].

Furthermore, stochastic dualities of Markov processes can be further characterized by examining infinitesimal generators, see [JK14, Proposition 1.2] (see also [KL13]) and their eigenfunctions (see [RS19]). Besides stochastic duality, there is also a generalized concept, the *Feynman-Kac duality*, see, e.g., [EK86, page 189, Equation (4.36)] or [DG99].

1.1 Duality for interacting particle systems

In the realm of interacting particle systems, dualities and self-dualities are typically studied within the context of lattices. The motivation behind this work is to extend these dualities to more general spaces, e.g., $\mathbb{R}^d$. More specifically, our aim is to investigate whether dualities demonstrated in three prominent discrete interacting particle systems—the *exclusion process*, *inclusion process* and *independent random walks*, as explored in [GKR07], [GKRV09], [RS18], [FG19], [Gro19], [CFG$^+$19] and [FRS22]—can be generalized. In this section, we provide a brief overview of these processes and their corresponding dualities.

Let E be a countable set, either finite or infinite with at least two elements. Let $c_{i,j}$ be non-negative real numbers for $i, j \in E$ that are symmetric, i.e., $c_{i,j} = c_{j,i}$. Additionally, let α_i, $i \in E$ be positive real numbers and $\sigma \in \{-1, 0, 1\}$. We examine the Markov process with state space $\mathbb{N}_0^E$, where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and the formal generator

$$\mathcal{L}f(\mu) = \sum_{i,j\in E} c_{i,j} \left(f(\mu - \delta_i + \delta_j) - f(\mu)\right)(\alpha_j + \sigma\mu_j)\mu_i, \quad \mu = (\mu_i)_{i\in E} \in \mathbb{N}_0^E. \tag{1.2}$$

Thereby, $\mu = (\mu_j)_{i\in E}$ represents a configuration of particles with μ_i denoting the number of particles at location $i \in E$. The symbol $\delta_j := (\mathbb{1}_{\{i=j\}})_{i\in E} \in \mathbb{N}_0^E$ denotes the configuration with a single particle at location i and no particles at other locations.

- When $\sigma = -1$, we arrive at the (inhomogeneous) *symmetric exclusion process* (SEP) as described, for example, in [FRS21, Equation (1.3)]). In this scenario, we restrict α_i to be natural numbers and the state space is limited to the set of μ with a maximum of α_i particles at each site i, that is, $\mu_i \in \{0, \ldots, \alpha_i\}$ for all $i \in E$.
- For $\sigma = 0$, we obtain a system of *independent random walks* (IRW), studied, e.g., in [DP91, Section 2.1].
- For $\sigma = 1$, we arrive at the (inhomogeneous) *symmetric inclusion process* (SIP) as described, for example, in [FRS22, Equation (2.2)].

Self-duality These three models share a self-duality property which we briefly recapitulate in this section. We denote by $(\mathbb{N}_0^E)_{<\infty}$ the set of configurations $(\mu_i)_{i\in E} = \mu \in \mathbb{N}_0^E$ with a finite number of particles which means $|\mu| := \sum_{i\in E} \mu_i < \infty$. The *rising factorial*, also known as *Pochhammer symbol*, is defined by $(a)^{(0)} := 1$ and $(a)^{(k)} := a(a+1)\cdots(a+k-1)$ for $a \in \mathbb{R}$ and $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Similarly, the *falling factorial* is given by $(a)_k := a(a-1)\cdots(a-k+1)$, $(a)_0 := 1$. Then, we define $\mathcal{D}^{\text{cl}} : (\mathbb{N}_0^E)_{<\infty} \times \mathbb{N}_0^E$ as follows:

$$\mathcal{D}^{\text{cl}} : (\xi, \eta) \mapsto \prod_{i\in E} \frac{1}{w_i(\xi_i)} (\eta_i)_{\xi_i}, \quad w_i(k) := \begin{cases} (\alpha_i)_k & \text{SEP} \\ \alpha_i^k & \text{IRW} \\ (\alpha_i)^{(k)} & \text{SIP} \end{cases}, \quad k \in \mathbb{N}_0 \tag{1.3}$$

The function $\mathcal{D}^{\text{cl}}$ is a self-duality function which means it satisfies (1.1). In the terminology of [RS18], it is referred to as the *classical* duality function. Both processes involved here are instances of the SEP (or IRW or SIP), (see, e.g., [Lig05, Theorem 1.1] or [FRS22, Proposition 2.3]). The first of the two processes starts at a possibly infinite configuration, while the other starts with a finite configuration. Here, self-duality means that the time evolution of a falling factorial polynomial of degree n weighted by w_i can be expressed by the time evolution of n dual particles. This property is crucial because it allows us to characterize properties of a system of many particles (or even an infinite number) by analyzing only n-particles. In its simplest form ($n = 1$), it is the property that the expected number of particles at a given location i at time $t > 0$ can be expressed in terms of the initial configuration and the location at time $t > 0$ of a single particle starting at position i.

In addition to the self-duality functions in terms of falling factorials, there are also orthogonalized versions of them, namely self-duality functions in terms of orthogonal polynomials which we delve into in the following. Through a straightforward detailed balance computation, it can be proved that, for the SIP, the IRW and the SEP, there exist one-parameter families indexed by p (or θ) of reversible measures $\rho = \bigotimes_{i\in E} \rho_i$. For a similar statement when $\alpha_1, \dots, \alpha_N$ are all equal, we refer to [CGGR13, Section 3.1].

- In the case of the SEP, each ρ_i is defined by a binomial distribution with parameters p and α_i for $i \in E$ where $p \in [0, 1]$.

- For IRW, each ρ_i is the Poisson distribution with parameter $\theta\alpha_i$ for $i \in E$ where $\theta > 0$.

- In the context of the SIP, each ρ_i is a negative binomial distribution with parameters p and α_i, i.e.,

$$\rho_i(\{k\}) = \text{NegativeBinomial}(\alpha_i, p)(\{k\}) = (1-p)^{\alpha_i} \frac{p^k}{k!} (\alpha_i)^{(k)}, \quad k \in \mathbb{N}_0 \tag{1.4}$$

  for $i \in E$ with $p \in [0, 1)$.

If $\sum_{i\in E} \alpha_i < \infty$, then ρ is concentrated on the set of finite configurations $(\mathbb{N}_0)_{<\infty}$. In this scenario, the measures ρ restricted to the n-particle sector $\left\{\mu \in \mathbb{N}_0^E : \sum_{i\in E} \mu_i\right\}$, are also reversible. This is because each of the three models conserves the total number of particles. The parameterization by p (or θ) corresponds to different weightings of the respective restrictions on the n-particle sectors.

Next, we recall the definition of the *Charlier* and *Meixner polynomials*, see, e.g., [KLS10, Sections 9.14 and 9.10]. These polynomials are the orthogonal polynomials associated with the Poisson and the negative binomial distribution, respectively. In contrast to the conventional definition found in the literature, we normalize these orthogonal polynomials to be monic, meaning they have a leading coefficient of one. The monic Charlier polynomials are given by

$$\mathscr{C}_n(l; a) := \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} (l)_k, \quad l \in \mathbb{N}_0 \tag{1.5}$$

for $n \in \mathbb{N}_0$ and $a > 0$. They satisfy the orthogonality relation

$$\sum_{l=0}^{\infty} \mathscr{C}_n(l; a) \mathscr{C}_m(l; a) \text{Poi}(a)(\{l\}) = \mathbb{1}_{\{n=m\}} a^n n! \tag{1.6}$$

for $n, m \in \mathbb{N}_0$ where $\mathrm{Poi}(a)(\{l\}) = e^{-a}\frac{a^l}{l!}$ denotes the Poisson distribution. Moving on to the monic Meixner polynomials:

$$\mathscr{M}_n(l; a; p) := \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{1}{p}\right)^{k-n} (a+k)^{(n-k)} (l)_k, \quad l \in \mathbb{N}_0 \tag{1.7}$$

for $n \in \mathbb{N}_0$, $a > 0$, $p \in (0, 1)$. They satisfy the orthogonality relation

$$\sum_{l=0}^{\infty} \mathscr{M}_n(l; a; p)\mathscr{M}_m(l; a; p)\mathrm{NegativeBinomial}(a, p)(\{l\}) = \mathbb{1}_{\{n=m\}} \frac{p^n n! (a)^{(n)}}{(1-p)^{2n}} \tag{1.8}$$

for $n, m \in \mathbb{N}_0$. It is worth noting that both the Charlier and Meixner polynomials can be expressed in terms of the generalized hypergeometric function, see, e.g., [KLS10, Equations (9.14.1) and (9.10.1)].

The function $\mathcal{D}^{\mathrm{ort}} : (\mathbb{N}_0^E)_{<\infty} \times \mathbb{N}_0^E \to \mathbb{R}$ is defined as follows:

$$\mathcal{D}^{\mathrm{ort}} : (\xi, \eta) \mapsto \prod_{i \in E} \frac{1}{w_i(\xi_i)} \mathscr{P}_{\xi_i}(\eta_i; \alpha_i) \tag{1.9}$$

Here, $\mathscr{P}_n(l; a)$ represents the Charlier polynomials for IRW (or Meixner polynomials with arbitrary p that does not depend on i for the SIP). This function serves as a self-duality function for IRW (or SIP). Similarly, (1.9) is a self-duality function for the SEP when $\mathscr{P}_n(l; a)$ is given by the *Krawtchouk polynomials* that are the orthogonal polynomials with respect to the binomial distribution, see, e.g., [KLS10, Section 9.11]). Until now, self-duality in terms of orthogonal polynomials for classical discrete systems, such as the SEP, the SIP and the IRW, have been established through various methods, including the use of three-term recurrence relations [FG19], Lie algebra representation theory [Gro19], unitary symmetries [CFG$^+$19] and a direct connection between the self-duality in terms of falling factorials (1.3) and orthogonal duality functions [FRS22].

Duality Duality and self-duality are closely related properties. In fact, for many systems such as IRW and the SIP, duality can deduced from self-duality. This is achieved by, for example, taking a many-particle limit in the original particle number variables while keeping the dual variables fixed. This procedure, often referred to as the *diffusion limit*, results, e.g., in duality between IRW and a deterministic system of coupled ordinary differential equations as well as in duality between SIP and the Brownian energy process (BEP) introduced in [GKRV09], see [CGGR13]. The BEP can be deduced from the Brownian momentum process, see [GKRV09].

More precisely, if we have $\nu = (\nu_i)_{i \in E} \in [0, \infty)^E$ and $\mu_k \in \mathbb{N}_0^E$ such that $\epsilon_k \mu_k \to \nu$ as $k \to \infty$ for some $\epsilon_k > 0$ with $\epsilon_k \to 0$, then, according to [CGGR13, Proposition 2.5], the SIP (generated by (1.2) for $\sigma = 1$) with the initial condition μ_k and subsequent scaling by ϵ_k converges to the BEP with the initial data ν as $k \to \infty$. The BEP is a Markov process with state space $[0, \infty)^E$ and formal generator

$$\hat{\mathcal{L}}g(\nu) = \frac{1}{2} \sum_{i,j \in E} c_{i,j} \nu_i \nu_j \left(\frac{\partial}{\partial \nu_j} - \frac{\partial}{\partial \nu_i}\right)^2 g(\nu) + \sum_{i,j \in E} c_{i,j} \nu_i \alpha_j \left(\frac{\partial}{\partial \nu_j} - \frac{\partial}{\partial \nu_i}\right) g(\nu), \tag{1.10}$$

see, e.g., [GKRV09, Theorem 6.1] or [FG19, Equation (112)] in the homogeneous context, i.e., all α_k are equal.

In the field of population genetics, this limit is closely related to *Fleming-Viot limits*. For instance, the SIP can be interpreted, in the context of population genetics, see, e.g., [CGGR15, Proposition 5.4], describing a Moran-type population (see [Mor58]). When we scale this system, we arrive at the Fleming-Viot process, as discussed in [FV79], [EK93]. This makes it closely related to the BEP.

A straightforward calculation shows that $\bigotimes_{i\in E} \mathrm{Gamma}(\alpha_k, \beta)$ is a reversible measure for the BEP for all $\beta > 0$, see, e.g., [CGGR13, Proposition 3.1] where $\mathrm{Gamma}(a, \beta)$ denotes the Gamma distribution with shape $a > 0$ and rate β, i.e., $\mathrm{Gamma}(a, \beta)(\mathrm{d}z) = \frac{\beta^a}{\Gamma(a)} z^{a-1} \mathrm{e}^{-\beta z} \,\mathrm{d}z$. Both of the following functions

$$\mathbb{N}_0^E \times [0,\infty)^E \ni (\xi, \zeta) \mapsto \prod_{i\in E} \frac{1}{(\alpha_i)^{(\xi_i)}} \zeta_i^{\xi_i} \tag{1.11}$$

$$\mathbb{N}_0^E \times [0,\infty)^E \ni (\xi, \zeta) \mapsto \prod_{i\in E} \frac{1}{(\alpha_i)^{(\xi_i)}} \mathscr{L}_{\xi_i}^{(\alpha_i-1)}(\zeta_i) \tag{1.12}$$

serve as duality functions for the SIP and the BEP. In other words, they satisfy (1.1), see [GKRV09, Theorem 6.2] and [FG19, Theorem 6]. Here, $\mathscr{L}_n^{(a-1)}$ for $n \in \mathbb{N}_0$, $a > 0$ denotes the *Laguerre polynomial* of degree n, sometimes referred to as the *generalized* or *associated Laguerre polynomial*. It is given by

$$\mathscr{L}_n^{(a-1)}(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k+a)^{(n-k)} z^k, \quad z \geq 0, \tag{1.13}$$

see, e.g., [KLS10, Section 9.12]. The Laguerre polynomials are orthogonal with respect to the Gamma distribution with shape a and rate 1, specifically

$$\int \mathscr{L}_n^{(a-1)}(z) \mathscr{L}_m^{(a-1)}(z) \,\mathrm{Gamma}(\alpha, 1)(\mathrm{d}z) = \mathbb{1}_{\{n=m\}}\, n!(a)^{(n)} \tag{1.14}$$

for all $n, m \in \mathbb{N}_0$. Other then usually done in the literature, we use the scaling such that $\mathscr{L}_d^{(a-1)}$ is monic.

The concept of self-duality also proves to be useful when studying other scaling limits, such as in the analysis of condensation, hydrodynamic limits and the associated fluctuation fields, as demonstrated in [GRV11], [GRV13], [OR15], [CGR20], [ACR21a], [CGG23].

The algebraic approach A common saddle point in studying duality is that finding duality relations is not easy. The algebraic approach to duality provides a structured way to find duality relations and duality functions (see, e.g., [SSV20] for a review). In particular, we note that duality relations have been found only via the algebraic approach for several processes (see, e.g., [CGRS16a], [CGRS16b], [Kua16] and [Kua18]). Inspired by [SS94], such approach has been developed in [GKRV09] and [CGRS16a] for particle systems in countable graphs, whose generator $\mathcal{L}$ can be written as a sum of *single edge-generators* $\mathcal{L}_{i,j}$ describing the dynamics of particles among the edge $\{i, j\}$ of the underlying graph. The first step of the algebraic approach consists in identifying the underlying Lie algebra associated to the Markov process, such that $\mathcal{L}_{i,j}$ is an element of the universal enveloping algebra in a given representation. One then can exploit the commutation relations of the Lie algebra to find symmetries of the generator. More precisely, the discovery of a linear operator Λ that intertwines two Markov semigroups $(P_t)_{t\geq 0}$ (of a Markov process $(X_t)_{t\geq 0}$) and $(Q_t)_{t\geq 0}$ (of $(Y_t)_{t\geq 0}$)—meaning that $\Lambda P_t = Q_t\Lambda$ for all $t \geq 0$—enables us to enhance other duality functions. For instance, if $(X_t)_{t\geq 0}$ is a Markov process with a countable state space $\mathbb{X}$ and a reversible measure ρ such that $\rho(\{x\}) > 0$ for all $x \in \mathbb{X}$, it can readily be verified that the function

$$\mathcal{D}^{\text{cheap}} : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}, \quad (x, y) \mapsto \mathbb{1}_{\{x=y\}} \frac{1}{\rho(\{x\})} \tag{1.15}$$

always serves as a self-duality function for $(\eta_t)_{t\geq 0}$, known as the cheap duality function, see, e.g., [GKRV09]. This function can be employed as a starting point. By applying Λ, we obtain that

$$\mathbb{X} \times \mathbb{Y} \ni (x, y) \mapsto \Lambda\mathcal{D}^{\text{cheap}}(x, \cdot)(y) \tag{1.16}$$

is a duality function for $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, see, e.g., [CFG$^+$19, Theorem 2.5]. In the case that the two processes are equal, meaning that Λ commutes with P_t, i.e., $\Lambda P_t = P_t\Lambda$, the newly generated self-duality function may represent an improvement over the cheap duality function. In [GKRV09], Λ is referred to as *symmetry* in this case.

We briefly explain how to find such an operator Λ, using the example of the SIP. Fix $p \in (0, 1)$ and consider for $a > 0$ the operators $k^\pm, k^0$ defined by

$$k^+h(n) = \frac{1}{\sqrt{p}}nh(n-1), \tag{1.17}$$

$$k^0h(n) = \left(n + \frac{a}{2}\right)h(n), \tag{1.18}$$

$$k^-h(n) = \sqrt{p}(a + n)h(n + 1) \tag{1.19}$$

for $n \in \mathbb{N}_0$ where $0h(-1) := 0$. These operators act on functions $h : \mathbb{N}_0 \to \mathbb{C}$ with at most finitely many non-zero values and are a representation of the $su(1, 1)$ algebra, i.e., they satisfy the commutation relations

$$[k^0, k^\pm] = \pm k^\pm, \quad [k^-, k^+] = 2k^0, \tag{1.20}$$

see [CFG$^+$19, Equations (2.3)]. We refer to [CDP06] for more information on the $su(1, 1)$ algebra and [FH13] for a general discussion on representation theory.

The formal generator $\mathcal{L}$, given by (1.2) with $\sigma = 1$, rewrites as, see [CFG$^+$19, last equation of Section 2.2.1] or [Gro19, Lemma 4.2],

$$\mathcal{L} = \sum_{\substack{\{i,j\}\subset E^2\\ i\neq j}} c_{i,j}\mathcal{L}_{i,j} \text{ with } \mathcal{L}_{i,j} = k_i^+k_j^- + k_i^-k_j^+ - 2k_i^0k_j^0 + \frac{\alpha_i\alpha_j}{2}\text{id}. \tag{1.21}$$

Here, $k_j^{\#}$ denotes the operator $k^{\#}$ with $a = \alpha_j$ acting on the variable indexed by j for $\# \in \{+, 0, -\}$. The identity operator is denoted by id.

Exploiting (1.21) and the commutation relations (1.20), we can show that $\sum_{j \in E} k_j^{\#}$, and therefore also $\Lambda := \mathrm{e}^{\xi \sum_{j \in E}(k_j^+ - k_j^-)} \mathrm{e}^{2\mathrm{i}\phi \sum_{j\in E} k_j^0}$, commutes with $\mathcal{L}$ for all $\xi \in \mathbb{C}$ and $\phi \in \mathbb{R}$. Consequently, Λ also commutes with the corresponding Markov semigroup P_t for $t \geq 0$. By considering an appropriate Hilbert space L^2, it can be showed that k_j^+ and k_j^- are adjoint to each other and k_j^0 is self-adjoint which implies that Λ is a unitary operator. Let $\mathcal{D}^{\mathrm{cheap}}$ be the cheap duality function (1.15) for the SIP in terms of the reversible measure ρ which is the product of negative binomial distributions with parameters p and α_k. A rephrasing of [CFG+19, Theorem 3.1] shows

$$\Lambda \mathcal{D}^{\mathrm{cheap}}(\xi, \cdot)(\eta) = (1-p)^{-\frac{\sum_{j\in E} \alpha_j}{2}} \left(\frac{1-p}{p}\right)^{\sum_{j \in E} \xi_j} \prod_{j \in E} \frac{1}{(\alpha_j)^{(\xi_j)}} \mathscr{M}_{\xi_j}(\eta_j; \alpha_j; p)$$

for a special choice of ξ and ϕ, more precisely, $\tanh \xi = \sqrt{p}$ and $\phi = 0$. In other words, improving the cheap duality function by Λ recovers the orthogonal duality function (1.9), up to a multiplicative constant that depends on the total number of particles $\sum_{j\in E} \xi_j$ which is a conserved quantity.

Another representation of $su(1,1)$ is

$$K^+ h(l) = \frac{1}{1-p}\left(\frac{1}{\sqrt{p}} l h(l-1) - 2\sqrt{p} h(l)\left(l + \frac{a}{2}\right) + p\sqrt{p} h(l+1)(l+a)\right) \tag{1.22}$$

$$K^0 h(l) = \frac{1}{1-p}\left(-l h(l-1) + (p+1) h(l)\left(l + \frac{a}{2}\right) - p h(l+1)(l+a)\right) \tag{1.23}$$

$$K^- h(l) = \frac{1}{1-p}\left(\sqrt{p} l h(l-1) - 2\sqrt{p} h(l)\left(l + \frac{a}{2}\right) + \sqrt{p} h(l+1)(l+a)\right) \tag{1.24}$$

for $l \in \mathbb{N}_0$, see, e.g., [Gro19, Lemma 4.5]. These operators act on polynomials $h : \mathbb{N}_0 \to \mathbb{C}$ contained in another appropriate Hilbert space. Additionally, the relation $K_j^{\#} = \Lambda k_j^{\#} \Lambda^{-1}$ holds true (see [Gro19, Proposition 4.7]) where $K_j^{\#}$ denotes the operator $K^{\#}$ with $a = \alpha_j$ acting on the variable indexed by j. Moreover, (1.21) holds true in terms of K^+, K^0, K^- instead of k^+, k^0, k^- as well, thus indicating that Λ and, therefore, the duality, correspond to a change of representation.

Similarly, the operator $\mathrm{e}^{\sqrt{p} \sum_{j \in E} k_j^+}$ commutes with P_t. When we apply it to the cheap duality function, we recover (up to a multiplicative constant) the duality function (1.3) in terms of falling factorials, see [CFG+19, Proposition 2.8].

On the other hand, as a consequence of [Gro19, Lemma 4.12], another representation of $su(1,1)$ is given by the operators

$$\mathscr{K}^+ h(z) = (z-a) h(z) + (a - 2z) h'(z) + z h''(z) \tag{1.25}$$

$$\mathscr{K}^0 h(z) = \frac{a}{2} h(z) + (z-a) h'(z) - z h''(z) \tag{1.26}$$

$$\mathscr{K}^- h(z) = a h'(z) + z h''(z) \tag{1.27}$$

acting on polynomials $h : [0,\infty) \to \mathbb{C}$ where $z \in [0,\infty)$. The formal generator (1.10) of the BEP can be rewritten as

$$\hat{\mathcal{L}} = \sum_{\substack{\{i,j\}\in E^2 \\ i\neq j}} c_{i,j}\hat{\mathcal{L}}_{i,j} \text{ with } \hat{\mathcal{L}}_{i,j} = \mathscr{K}_i^+\mathscr{K}_j^- + \mathscr{K}_i^-\mathscr{K}_j^+ - 2\mathscr{K}_i^0\mathscr{K}_j^0 + \frac{\alpha_i\alpha_j}{2}\mathrm{id}$$

where

$$\hat{\mathcal{L}}_{i,j}g(\mu) = \mu_i\mu_j\left(\frac{\partial}{\partial\mu_j} - \frac{\partial}{\partial\mu_i}\right)^2 g(\mu) + (\mu_i\alpha_j - \mu_j\alpha_i)\left(\frac{\partial}{\partial\mu_j} - \frac{\partial}{\partial\mu_i}\right)g(\mu).$$

The operator that switches from the operator $k_j^{\#}$ to $\mathscr{K}_j^{\#}$ is related to Laguerre polynomials and recovers the duality function from (1.12) when applied to the cheap duality function (1.15). Once again, this operator change can be interpreted as a change of representation.

An application of self-duality As a next step, to illustrate the benefit of duality, particularly self-duality for the SIP, we delve into its ergodic theory. For a comprehensive introduction to ergodic theory in general, we refer to [EFHN15]. The following results can be found in [KR16], or in a more abstract form in [RvW23].

Let $d, m \in \mathbb{N}$. We consider the SIP with the choice of parameters $E = \mathbb{Z}^d$, $\alpha_j = \frac{m}{2}$ for all $j \in \mathbb{Z}^d$ and $p(i,j) = \frac{1}{2d}$ if i, j are neighbors in the lattice $\mathbb{Z}^d$, and zero otherwise. We have already observed that there exists a family of reversible measures ρ indexed by $p \in [0,1)$ where ρ is an infinite product of negative binomial distributions. Two questions arise. First, are these measures ergodic? If so, are they the only ergodic measures?

To address these questions, the concept of duality is a useful instrument, specifically, the duality function (1.3), is the key element. It establishes a connection between the SIP and the process describing the evolution of a finite number of particles. The answer to the first question is positive. The second question is also answered positively, at least when restricting to the set of so-called *tempered* probability measures. In the following, we briefly summarize the key ideas in answering the first question.

Recall that a measure ρ is called ergodic if all invariant $f \in L^2(\rho)$ are ρ-almost surely equal to $\int f \,\mathrm{d}\rho$. A function f is called invariant if $P_t f = f$ holds for all $t > 0$ where P_t is the Markov semigroup associated with the SIP. The ergodicity of ρ follows by the fact that ρ is mixing, i.e.,

$$\int f P_t g \,\mathrm{d}\rho \to \int f \,\mathrm{d}\rho \int g \,\mathrm{d}\rho, \quad t \to \infty \tag{1.28}$$

for all $f, g \in L^2(\rho)$. This property is deduced from the behavior of the SIP with finitely many particles where particles tend to spread out over time: for all $\xi, \xi' \in (\mathbb{N}_0^E)_{<\infty}$, we have $\mathbb{P}_\xi[\xi_t \perp \xi'] \to 1$ as $t \to \infty$ for all $\xi, \xi' \in (\mathbb{N}_0^E)_{<\infty}$. Here, the Markov process $\xi_t = ((\xi_t)_i)_{i\in E}, t \geq 0$, together with the probability measure $\mathbb{P}_\xi$ (and expectation $\mathbb{E}_\xi$), represents the SIP starting at the configuration $\xi \in (\mathbb{N}_0^E)_{<\infty}$. The notation $\xi_t \perp \xi'$ means that ξ_t and ξ' have no particles in common, i.e., $\sum_{i\in E}(\xi_t)_i(\xi')_i = 0$.

Then, the convergence (1.28) follows when considering specific choices of f and g, particularly $f = \mathcal{D}^{\mathrm{cl}}(\xi', \cdot)$, $g = \mathcal{D}^{\mathrm{cl}}(\xi, \cdot)$. Indeed, we obtain

$$\begin{aligned}
&\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta) P_t \mathcal{D}^{\mathrm{cl}}(\xi, \cdot)(\eta)\rho(\mathrm{d}\eta) = \int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\mathbb{E}_\xi\left[\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\right]\rho(\mathrm{d}\eta) \\
&= \mathbb{E}_\xi\left[\mathbb{1}_{\{\xi_t \not\perp \xi'\}}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\rho(\mathrm{d}\eta)\right] + \mathbb{E}_\xi\left[\mathbb{1}_{\{\xi_t \perp \xi'\}}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\rho(\mathrm{d}\eta)\right]
\end{aligned}$$

using that $\mathcal{D}^{\mathrm{cl}}$ is a self-duality function. For the first term, applying the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\mathbb{E}_\xi\left[\mathbb{1}_{\{\xi_t \not\perp \xi'\}}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\rho(\mathrm{d}\eta)\right] \\
&\le \left(1 - \mathbb{P}_\xi\left[\xi_t \perp \xi'\right]\right)\left(\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)^2\rho(\mathrm{d}\eta)\right)^{\frac{1}{2}} \sup_{\xi'' \in \mathbb{N}_0^E : |\xi''| = |\xi|}\left(\int \mathcal{D}^{\mathrm{cl}}(\xi'',\eta)^2\rho(\mathrm{d}\eta)\right)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $t \to \infty$ with the observation that the supremum is finite. Exploiting the product structure of both $\mathcal{D}^{\mathrm{cl}}$ and ρ, along with analogous arguments as in the first term, duality and the invariance of ρ, we conclude

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_\xi\left[\mathbb{1}_{\{\xi_t \perp \xi'\}}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\rho(\mathrm{d}\eta)\right] \\
&\qquad = \lim_{t\to\infty}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\rho(\mathrm{d}\eta)\int \mathbb{E}_\xi\left[\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\mathbb{1}_{\{\xi_t \perp \xi'\}}\right]\rho(\mathrm{d}\eta) \\
&\qquad = \lim_{t\to\infty}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\rho(\mathrm{d}\eta)\int \mathbb{E}_\xi\left[\mathcal{D}^{\mathrm{cl}}(\xi_t,\eta)\right]\rho(\mathrm{d}\eta) \\
&\qquad = \lim_{t\to\infty}\int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\rho(\mathrm{d}\eta)\int P_t\mathcal{D}^{\mathrm{cl}}(\xi, \cdot)(\eta)\rho(\mathrm{d}\eta) \\
&\qquad = \int \mathcal{D}^{\mathrm{cl}}(\xi',\eta)\rho(\mathrm{d}\eta)\int \mathcal{D}^{\mathrm{cl}}(\xi,\eta)\rho(\mathrm{d}\eta).
\end{aligned}$$

The linear hull of $\mathcal{D}^{\mathrm{cl}}(\xi, \cdot)$, $\xi \in (\mathbb{N}_0^E)_{<\infty}$ is dense in $L^2(\theta)$. Consequently, by an approximation argument, (1.28) holds for $f, g \in L^2(\rho)$.

1.2 Intertwining for particle systems in the continuum

Having collected several known facts about self-dualities and dualities of interacting particle systems in discrete environments, we are now able to formulate the central research question for this thesis:

What happens if we relax the assumption that the space E, where the particles are moving, is countable and extend our consideration to an uncountable space, e.g., $\mathbb{R}^d$?

In many natural scenarios, the language and formulation of duality using occupation variables at discrete lattice sites breaks down. For instance, when dealing with particles moving in $\mathbb{R}$, like independent Brownian motions, it is not immediately clear how to establish and achieve self-duality. Nevertheless, it is quite intuitive to expect that discrete systems with self-duality properties should have counterparts in uncountable spaces. Consequently, this leads us to the following specific challenges:

(i) We need to find a suitable configuration space, i.e., a suitable way to model particle configurations.

(ii) We need to find a framework that allows the generalization of dualities while preserving the discrete dualities.

(iii) We need to understand the conditions under which these generalized relations remain valid.

(iv) Does the theory remain applicable for systems with an infinite number of particles?

(v) Are there generalizations of the symmetric inclusion process, symmetric exclusion process and independent random walks? If so, do the generalized models satisfy self-duality relations?

(vi) Is there a generalization of the Brownian energy process in uncountable spaces? Can we establish duality between the generalized symmetric inclusion process and the generalized Brownian energy process?

(vii) How does the algebraic approach fit into this framework?

(viii) Do duality relations for the generalized models have the same applications as their discrete counterparts, such as the ergodic theory of the SIP?

In the upcoming paragraphs, we delve into Challenges (i)–(viii) in more detail and present an overview of our approach to each respective challenge.

The configuration space To study the evolution of particle configurations, we are naturally led to the framework of point processes (see [LP17]). A particle configuration denoted by μ is modeled using counting measures. Specifically, it is a measure on the set E and takes the form $\mu = \sum_{k=1}^{N} \delta_{x_k}$, where N represents the total number of particles and $x_k \in E$ represents their positions. It is an unlabeled notation, meaning that any permutation of the particles results in the same configuration μ.

From dualities to intertwining It turns out that employing the concept of stochastic duality and, and specifically using duality functions, is not practicable in uncountable spaces. However, we can indirectly achieve generalization by introducing the notion of intertwiners. We say that a linear operator Λ intertwines two Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with Markov semigroups $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ if

$$\Lambda P_t = Q_t \Lambda \tag{1.29}$$

holds for all $t \geq 0$. Λ is called an *intertwiner*. If the two processes are equal, we refer to Λ as *self-intertwiner*.

The concept of *intertwining relations* in the context of Markov processes goes back to Dynkin [Dyn65]. He employed them for the construction of new Markov semigroups based on existing ones. Building upon Dynkin's work, Rogers and Pitman expanded these ideas in [RP81] which led to the characterization of Markov functions—measurable maps that preserve the Markov property. Furthermore, intertwining relations play a crucial role in the analysis of the convergence to equilibrium in the top-to-random shuffle, see [AD86]. In [Mic18], intertwining relations are discussed in the context of the algebraic concept of similarity transformation. Moreover, they are used in [BO13] for the construction of Markov processes on infinite-dimensional spaces and in [DF90] in relation to strong stationary times. Additionally, [PS12] explores their connection with fractional operators, while [Dub04], [PS11], [AOW19] delve into their application in the context of diffusions. Intertwining relations for Ehrenfest, Yule and Ornstein-Uhlenbeck processes are studied in [MP22]. For additional examples of intertwining relations, we refer to [Bia95] and [JK14]. In particular, in [CPY98] the connection between intertwining and various notions of duality is explored. Lastly, in [MP21], the notion of interweaving relations was introduced.

In the following, we recall the relationship between duality functions and intertwiners. Intertwiners can be useful for improving duality functions, see, e.g., [GKRV09, Remark 2.7] or [CFG$^+$19, Theorem 2.5], as we have already seen in the case of the cheap duality function (1.15). Moreover, self-duality functions are related to intertwiners for reversible systems. It is a well-known fact in a discrete setup (see, e.g., [RS18, Section 5.2] or [Gro19, Lemma 2.1]) that if Λ is an integral operator

$$\Lambda f(x) = \int \mathcal{D}(x,y) f(y) \rho(\mathrm{d}y) \tag{1.30}$$

where ρ, with $\rho(\{y\}) > 0$ for all $y \in \mathbb{Y}$, is a reversible measure of $(Q_t)_{t\geq 0}$, then $\mathcal{D}$ is a self-duality function if and only if Λ is an intertwiner. In this case,

$$\Lambda \mathcal{D}^{\mathrm{cheap}}(x, \cdot)(y) = \mathcal{D}(x,y)$$

with $\mathcal{D}^{\mathrm{cheap}}$, the cheap duality function (1.15). The intertwiner associated with the cheap duality function is the identity operator. Moreover, if Λ is a unitary operator from $L^2(\rho)$ to another L^2 space, it follows that $(\mathcal{D}(x, \cdot))_{x \in E}$ forms an orthogonal family.

We employ precisely this approach: We do not generalize the duality functions (1.3) and (1.9) but rather the integral operator (1.30). The generalized operators are not new. Generalizing the duality in terms of falling factorials leads us to the concept of *factorial measures*, well-known in the context of point processes (see, e.g., [LP17]), and to *Lenard's K-transform* (see [Len73], [Len75]).

When generalizing orthogonal duality, we delve into the theory of *infinite-dimensional polynomials*. Infinite-dimensional orthogonal polynomials (see, e.g., [Sch00, Chapter 5], [Yab08, page 678]) naturally arise in the study of non-Gaussian white noise [Ber96, Ber02]. They are used to prove chaos decompositions which are related to Fock spaces [Mey95], [Lyt03a], [Las16].

Consistency A fundamental property shared by the SIP, the SEP and the IRW is *consistency* which means—roughly speaking—that the system's time evolution commutes with the action of *randomly* removing a single particle from the system. In [CGR21], the connection between self-duality and consistency in reversible particle systems in a countable set was investigated. Moreover, a characterization of consistent particle system was proved (see [CGR21, Theorem 3.3]).

Within this thesis, we extend the notion of consistency to systems of particles moving in uncountable spaces. We prove the equivalence of consistency and the intertwining relation in terms factorial measures. Moreover, we develop a general method that reveals how the orthogonal dualities in these systems rely solely on the conditions of consistency and reversibility. This method does not require an explicit formula of the polynomials and is applicable not only for particle systems in countable spaces but also on more general spaces. This approach produces then self-intertwiners in terms of infinite-dimensional orthogonal polynomials.

Infinite particle systems For particle systems with infinitely many particles, we opt for an alternative approach. Firstly, it is essential to properly formulate the concept of consistency since removing a *single* particle from an infinite system does not establish a link between infinite and finite dynamics. Therefore, we define consistency for an infinite number of particles in terms of the intertwining relation which, in terms of factorial measures, is equivalent to consistency already observed in finite particle cases. This intertwining relation connects the dynamics of infinitely many particles with the dynamics of finitely many particles. To be more precise, the *generalized falling factorial polynomial* of degree n intertwines the dynamics of infinitely many particles with the dynamics of n particles.

The primary focus lies on intertwining relations involving orthogonal polynomials. The discovery of reversible measures for n-particle dynamics, which are much easier to obtain than reversible measures for infinite dynamics, coupled with consistency, implies that the infinite-dimensional orthogonal polynomial of degree n intertwines the dynamics of infinitely many particles with the dynamics of n particles where $n < \infty$. This represents a significant simplification of the system. We particularly concentrate on intertwining relations related to infinite-dimensional orthogonal polynomials with respect to the distribution of the Poisson process or the Pascal process. The Pascal process serves as the continuum counterpart to the product of negative binomial laws. Both families of infinite-dimensional orthogonal polynomials fall under the Meixner class [Mei34], [Lyt03b].

Models Our approach recovers the known self-duality functions for the IRW, the SIP and the SEP and avoids the need of ad hoc computations for each system when proving duality. This makes the process much simpler compared to previous proofs.

The next question to address is whether these three models have analogs in uncountable spaces. For IRW, the generalization is immediately clear: The IRW generalize to independent particles moving in uncountable spaces, such as independent Brownian motions, and this model is consistent. We obtain intertwining relations in terms of infinite-dimensional orthogonal polynomials with respect to the distribution of the Poisson process. Their connection to multiple Wiener-Itô integrals has been studied in [Ogu72] or [Sur83].

We introduce and study a new process in uncountable spaces; we call it the *generalized symmetric inclusion process* (gSIP) which generalizes the SIP and remains consistent. Our self-intertwining results apply to the gSIP. The gSIP is closely related to the measure-valued Moran model. The *measure-valued Moran model* (see, e.g., [Daw93, Section 2.5] or see, e.g., [Eth00, Section 5.4] and the references therein) is an extension of the classical Moran model [Mor58] in population genetics. It turns out that the distribution of the Pascal processes is a reversible measure of the gSIP. We prove that infinite-dimensional Meixner polynomials, the orthogonal polynomials with respect to the distribution of the Pascal process, are self-intertwiners for the gSIP and present properties of these orthogonal polynomials. As for the SEP, there is no meaningful direct generalization that is analogous to the gSIP.

The class of consistent particle systems is much broader. It also includes the *compatible systems* according to [LR04a] which we refer to as *strongly consistent.* These systems have a one-to-one correspondence with *stochastic flows.* Compatibility, in this context, means that the system's time evolution commutes with the action of any *deterministic* removal of a particle from the system. Examples are the Brownian web or interacting Brownian motions as the Howitt-Warren flow, see, e.g., [HW09a], [SSS17]. In this class of models, we particularly delve into the context of infinitely many particles and illustrate our procedure using two strongly consistent examples. The first model is a system of correlated Brownian motions. We obtain intertwining relations in terms of infinite-dimensional orthogonal polynomials with respect to the distribution of the Poisson process.

The second example refers to a system of sticky Brownian motions which were introduced by Howitt and Warren through a martingale problem (see [HW09a]). They are observed as a scaling limit of random walks in random environments, a generalized exclusion process, or via a condensation rescaling of symmetric inclusion processes (see, e.g., [RS15], [CGR20] and [ACR21a]). Sticky interactions are used to model colloids in materials science (see [HC17]). In addition, these interactions have recently received attention in [DDP23]. We focus on a special case of the Howitt-Warren martingale problem called *uniform sticky Brownian motions* and studied in [BR20] and [BW23]. For this model, we investigate intertwining relations in terms of infinite-dimensional Meixner polynomials. For this latter family, we prove a new explicit formula.

The intertwining relation in terms of orthogonal polynomials has a fruitful application: we obtain a reversible measure for infinite dynamics. Specifically, this leads to a new result in the case of uniform sticky Brownian motions: the distribution of the Pascal process is reversible for a system of infinitely many uniform sticky Brownian motions.

The inclusion process and the Brownian energy process in the continuum As a generalization of the Brownian energy process, it turns out to be the well-known measure-valued Fleming-Viot process [FV79] (see also [Shi90], [EK93], [EG93a]) in population genetics. Duality plays an important role in population genetics. In addition to the duality relation between Fleming-Viot processes and partition-valued Markov processes, in which the evolution involves, for example, the coalescence of sets, as seen in, e.g., [DG14, Chapter 5], there exists another well-established intertwining relation in terms of moment measures: It connects the measure-valued Fleming-Viot processes and the measure-valued Moran processes (see [Daw93, Chapter 2] which has a close connection to the gSIP.

In this thesis, we revisit this well-known intertwining relation and show that it is a generalization of the duality function (1.11) between the SIP and the BEP. Moreover, we recall its connection with de Finetti's theorem and the many-particle limit, also known as the *Fleming-Viot limit*, of the Moran model as the number of particles tends to infinity (see, e.g., [DK96]). Building upon this foundational understanding, we are able to formulate our main result. Specifically, we show that infinite-dimensional Laguerre polynomials of degree n intertwine the dynamics of the generalized BEP and the n-particle dynamics of the gSIP. This result is particularly useful as it allows us to recover the duality function (1.12) in terms of Laguerre polynomials with the methods we have developed for self-intertwiners.

An algebraic approach So far, the algebraic approach to duality has been developed only in the case of particles hopping in countable spaces, relying on the possibility of rewriting the generator of the process under consideration as a sum of edge-generators: this, however, is not possible in uncountable spaces. Our aim is to extend the algebraic approach to duality in uncountable spaces, in the context of the $su(1,1)$ algebra, aiming at recovering the intertwining relations.

The first challenge that we face is how to find the right infinite-dimensional analogue of the Lie algebra under consideration to be able to express dynamics in uncountable spaces via an algebraic description. The same problem was addressed in a different context, namely in the literature of higher power of (quantum) white noise (see, e.g., [AFS02, AB09] and references therein) and the so-called *renormalization problem.* They were led to the concept of *current algebras* from quantum field theory (see, e.g., [ABM10]), i.e., roughly, operator-valued distributions [Tal22, Chapter 3.7], or in other words, Lie algebras whose generators are indexed by functions. Finding representations of such infinite-dimensional algebras is non-trivial and interesting in its own. In many-body quantum mechanics literature, representations of the canonical commutation relations in terms families of operators indexed by functions are standard and well-known (see, e.g., [RS75] and [BR13]). Current algebras and their representations have been widely studied in quantum field theory and we refer the reader to, e.g., [Kac90, Chapter 7].

In this thesis, we construct three representations of the $su(1,1)$ current algebra which generalize the three representations $k^{\#}$, $K^{\#}$ and $\mathscr{K}^{\#}$, $\# \in \{+,0,-\}$ from Section 1.1 to uncountable spaces. The first representation is employed to provide a generalization of [CFG$^+$19, Theorem 3.1 1] that enables us to obtain intertwining relations through algebraic methods. Furthermore, we present a Baker-Campbell-Hausdorff formula that generalizes the Weyl relation for the free field (see, e.g., [RS75, page 231]) to our context.

Future perspectives In this thesis, we do not delve into potential applications of our intertwining relations. The primary motivation of this thesis is conceptual and establishes the theoretical foundations, by building a bridge between various mathematical disciplines. Our research serves as a starting point for future applications, specifically the study of properties of particle systems in general state spaces. This includes the characterization of the stationary measures and their attractors (see, e.g., [Lig05, Chapter 8]), hydrodynamic limits (see, e.g., [DP91, Chapter 2]) and fluctuations (see, e.g., [ACR21b]).

Moreover, as we have outlined previously, the product of the negative binomial distribution serves as an ergodic measure for the symmetric inclusion process in the discrete case, see [KR16], [RvW23]. It is natural to explore the application of intertwining relations in the investigation of ergodic theory for particle systems in general state spaces, aiming to extend the obtained results beyond discrete spaces. More specifically, one could examine whether the distribution of the Pascal process is an ergodic measure for our generalized symmetric inclusion process by adopting a similar approach as outlined [KR16], [RvW23]. If this question is answered positively, the subsequent investigation could explore whether these measures are already the only ergodic measures.

Furthermore, intertwining relations could potentially be discovered in the context of non-equilibrium systems, such as systems driven by boundary reservoirs where particles can enter and leave the system. In discrete settings, we find duality with a system in which the reservoirs are replaced by absorbing boundaries, a very useful and powerful tool (see [KMP82], [DLS01], [CGGR13], [FG21], [FGdH$^+$22], [FRS22], [FC23]).

Another natural open question is the following: In addition to the self-dualities of the SIP, the SEP, the IRW and the duality of the SIP and the BEP, there are also other dualities in terms of orthogonal polynomials (see, e.g., [FG19, Theorem 1], namely the duality of the Brownian Momentum process to the SIP and the Kipnis-Marchioro-Presutti (KMP) to the dual-KMP. Here, the question arises as to whether these models have continuum counterparts and whether duality generalizes there.

Furthermore, this thesis extensively examines the infinite-dimensional counterparts of Charlier, Meixner, and Laguerre polynomials. These belong to a broader class, namely *hypergeometric polynomials*, which also include, among others, Wilson, Hahn, Jacobi, and Bessel polynomials. An overview of hypergeometric polynomials is provided by the *Askey scheme*, see [KLS10, Page 183]. A future task could involve, beyond the scope of intertwining relations, creating a comprehensive review of all polynomials in the Askey scheme focusing on possible meaningful infinite-dimensional generalizations and pointing out their connections to chaos decompositions.

1.3 Organization of the thesis

In the first chapter (Chapter 2), we investigate intertwining relations in terms of generalized falling factorial polynomials. The first main theorem in Section 2.1 proves its equivalence to consistency for the finite particle case. We appropriately define consistency for infinite particle systems in Section 2.2 and prove that strongly consistent systems are consistent.

Chapter 3 is dedicated to intertwining relations in terms of infinite-dimensional orthogonal polynomials. In particular, the main theorem in Section 3.1 demonstrates that this intertwining relation holds true for any reversible and consistent system without the need of an explicit formula for the orthogonal polynomials. We also explore some properties of infinite-dimensional orthogonal polynomials which are of independent interest. In Section 3.2 and Section 3.3, we focus on intertwining in terms of orthogonal polynomials of the Poisson and Pascal processes with special emphasis on the Pascal process. We provide properties and an explicit formula for the infinite-dimensional Meixner polynomials. Finally, in Section 3.4, we present an alternative method to obtain intertwining relations, particularly suited to infinite particle systems.

We apply the abstract theory to various models in Chapter 4. Firstly, we use our theory to recover self-duality functions for the SIP, the SEP and the IRW. Secondly, we consider generalizations of the IRW and the SIP, first by examining independent particle systems in general spaces and then by introducing the generalized symmetric inclusion process where we investigate reversibility, consistency and intertwining relations. We apply the theory for infinite particle systems to correlated and sticky Brownian motions. We also address the SEP, for which no canonical generalization to uncountable spaces exists.

In Chapter 5, we generalize the duality between the SIP and the BEP to uncountable spaces. In Chapter 6, we delve into the algebraic approach for particle systems in general state spaces.

2 Generalized falling factorial polynomials

In this chapter, we introduce the setting and the class of processes under consideration, specifically consistent Markov processes. To begin, in Section 2.1, we focus on finite particle systems. Our main contribution is to prove that consistency is equivalent to a self-intertwining relation in terms of factorial measures which serves a generalization of the self-duality function (1.3) in terms of falling factorials. Subsequently, in Section 2.2, we delve into infinite particle systems, provide a suitable definition of consistency and address strongly consistent systems.

Throughout this thesis we investigate Markov processes whose state space consists of configurations of non-labeled particles in some general measurable space $(E, \mathcal{E})$. We follow modern point process notation in modeling such configurations as finite counting measures on $(E, \mathcal{E})$, see, e.g., [LP17]. Thus, let $\mathbf{N}$ denote the space of *counting measures*, i.e., the space of countable sums of measures that assign values in $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ to every $B \in \mathcal{E}$. We equip $\mathbf{N}$ with the smallest σ-algebra $\mathcal{N}$ such that $\mathbf{N} \ni \mu \mapsto \mu(B)$ is measurable for each $B \in \mathcal{E}$. Assumptions on $(E, \mathcal{E})$ are needed to ensure that every counting measure is a sum of Dirac measures, therefore we assume throughout the thesis that $(E, \mathcal{E})$ is a Borel space (see [LP17, Definition 6.1]). The reader may think of a Polish space or $\mathbb{R}^d$ endowed with the Borel σ–algebra. It is well-known (see, e.g., [LP17, Chapter 6] or [Kal17, Section 1.1]) that for a Borel space, every counting measure $\mu \in \mathbf{N}$ is either zero or of the form $\mu = \sum_{k=1}^{N} \delta_{x_k}$ for some $N \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} := \{1, 2, 3, \ldots\}$, and $x_k \in E$. In particular, $\mu(E) = N$ corresponds to the total number of particles. We denote by $\mathbf{N}_{<\infty} := \{\mu \in \mathbf{N} : \mu(E) < \infty\}$ the set of *finite configurations* and by $\mathbf{N}_n := \{\mu \in \mathbf{N} : \mu(E) = n\}$ the set of configurations consisting of exactly $n \in \mathbb{N}_0 \cup \{\infty\}$ particles. We equip $\mathbf{N}_{<\infty}$ and $\mathbf{N}_n$ with their respective trace σ-algebras. In the following, Cartesian products are always equipped with product σ-algebras.

For our purpose, a *Markov family with state space* $\mathbf{N}_{<\infty}$ is a collection $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu \in \mathbf{N}_{<\infty}})$ consisting of a measurable space $(\Omega, \mathcal{F})$, measure maps $\eta_t : \Omega \to \mathbf{N}_{<\infty}$, $t \geq 0$ and probability measures $\mathbb{P}_\mu$, $\mu \in \mathbf{N}_{<\infty}$ on $(\Omega, \mathcal{F})$ such that

(i) $\mathbb{P}_\mu [\eta_0 = \mu] = 1$ for each $\mu \in \mathbf{N}_{<\infty}$;

(ii) the map $\mu \to \mathbb{P}_\mu [\eta_t \in B]$ is measurable for each $B \in \mathcal{N}_{<\infty}$ and $t \geq 0$;

(iii) the Markov property is satisfied with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : 0 \leq s \leq t)$.

We denote by $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$. Note that (ii) is equivalent to the fact that the map

$$E^n \ni x \mapsto \mathbb{P}_{\iota_n(x)}[\eta_t \in B] \tag{2.1}$$

is measurable for all $B \in \mathcal{N}_{<\infty}$, $n \in \mathbb{N}$ and $t \geq 0$ where

$$\iota_n(x) := \sum_{k=1}^{n} \delta_{x_k}, \quad x = (x_1, \dots, x_n) \in E^n. \tag{2.2}$$

When considering an infinite number of particles, the requirement that the map defined in (2.1) is measurable for $n = \infty$ is in general weaker compared to the measurability condition (ii) on $\mathbf{N}_\infty$. Here, we define

$$\iota_\infty : E^\infty := \{x = (x_k)_{k\in\mathbb{N}}, x_k \in E\} \to \mathbf{N}_\infty, \quad x \mapsto \sum_{k=1}^{\infty} \delta_{x_k} \tag{2.3}$$

and equip E^∞ with the cylindrical σ-algebra denoted by $\mathcal{E}^{\otimes\infty}$.

We define a *Markov family with state space* $\mathbf{N}$ to describe the evolution of infinite configurations in a way analogous to the case of a finite number of particles but with the difference that we assume the weaker form of measurability of (2.1) for all $n \in \mathbb{N} \cup \{\infty\}$. More precisely, we say that $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ is a *Markov family with state space* $\mathbf{N}$ if $(\Omega, \mathcal{F})$ is a measurable space, $\eta_t : \Omega \to \mathbf{N}$, $t \geq 0$ are measure maps, $\mathbb{P}_\mu$, $\mu \in \mathbf{N}$ are probability measures on $(\Omega, \mathcal{F})$ such that (i) holds for each $\mu \in \mathbf{N}$, (iii) holds and (2.1) is measurable for all $B \in \mathcal{N}$, $t \geq 0$ and $n \in \mathbb{N} \cup \{\infty\}$. With this approach, we can easily convert the concept of labeled particles into that of unlabeled particles without requiring a detailed analysis of the intricate issue of measurability in (ii) for infinitely many particles.

Throughout this thesis, for the sake of simplicity in the exposition, we frequently abuse notation and consider $(\eta_t)_{t\geq 0}$ to already represent the entire Markov family.

2.1 Consistency for finite particle systems

In this section, we focus on systems with a finite number of particles. The treatment of the infinite case is addressed in Section 2.2 below. We concentrate on a special class of Markov processes known as consistent Markov processes. This concept has been explored, e.g., in [CGR21]. Intuitively speaking, consistency refers to the fact that the removal of a particle uniformly at random commutes with the time evolution of the process. The following definition is a straightforward generalization of the established concept of consistency for particle systems in countable spaces, as presented in [CGR21, Definition 3.1 b)].

Definition 2.1.1. A Markov family $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ of finitely many particles is said to be *consistent* if

$$\mathbb{E}_\mu\left(\int F(\eta_t - \delta_x)\eta_t(\mathrm{d}x)\right) = \int \mathbb{E}_{\mu-\delta_x}(F(\eta_t))\mu(\mathrm{d}x) \tag{2.4}$$

holds for all $\mu \in \mathbf{N}_{<\infty}$, $t \geq 0$ and measurable functions $F : \mathbf{N}_{<\infty} \to [0, \infty)$.

We remark that, in general, integrability is not guaranteed on either the left-hand side or the right-hand side in (2.4). In the case of non-integrability, we interpret the equation as $\infty = \infty$.

We define a Markov family with state space $\mathbf{N}_{<\infty}$ (or $\mathbf{N}$) as being *conservative* if the condition $\eta_t(E) = \mu(E)$ holds $\mathbb{P}_\mu$-almost surely for all $\mu \in \mathbf{N}_{<\infty}$ (or $\mathbf{N}$) and $t \geq 0$. In other words, this means that no particles are created or annihilated over time.

Proposition 2.1.2. *A consistent Markov family of finitely many particles is conservative.*

Thus, if consistency holds, we obtain

$$\mathbb{E}_\mu \left(\frac{1}{\eta_t(E)} \int F(\eta_t - \delta_x)\eta_t(\mathrm{d}x) \right) = \frac{1}{\mu(E)} \int \mathbb{E}_{\mu-\delta_x}(F(\eta_t))\mu(\mathrm{d}x) \tag{2.5}$$

for all non-zero $\mu \in \mathbf{N}_{<\infty}$, $t \geq 0$ and measurable functions $F : \mathbf{N}_{<\infty} \to [0, \infty)$ which has the following probabilistic interpretation: On the left-hand side of (2.5) we first evolve the system and after we remove uniformly at random a particle, while on the right-hand side we first remove uniformly at random a particle from the initial configuration and then we let evolve the process from the new initial state. In Chapter 4, we present examples of consistent Markov processes.

In terms of operators, the consistency property (2.4) can be rewritten as the commutation property

$$P_t \mathcal{A} F(\mu) = \mathcal{A} P_t F(\mu), \quad \mu \in \mathbf{N}_{<\infty} \tag{2.6}$$

for measurable $F : \mathbf{N}_{<\infty} \to [0, \infty)$. Here, $\mathcal{A}$ is the so-called *lowering operator*, defined as

$$\mathcal{A} F(\mu) := \int F(\mu - \delta_x)\mu(\mathrm{d}x), \quad \mu \in \mathbf{N}_{<\infty}, \tag{2.7}$$

and $(P_t)_{t\geq 0}$, $P_t F(\mu) = \mathbb{E}_\mu [F(\eta_t)]$, $\mu \in \mathbf{N}_{<\infty}$ is the Markov semigroup. Our definition of consistency is a direct generalization of [CGR21, Equation (2.13)] where $\mathcal{A}$ is examined for countable spaces E. For further insights into the characterization of consistency in terms of the infinitesimal generator $\mathcal{L}$, specifically $\mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}$, we refer to [CGR21, Theorem 2.7].

We remind the reader that the *falling factorial* is defined by $(a)_k := a(a-1)\cdots(a-k+1)$, $(a)_0 := 1$ for $a \in \mathbb{R}$, $k \in \mathbb{N}_0$.

Proof of Proposition 2.1.2. We claim $\mathbb{E}_\mu [(\eta_t(E))_k] = \mu(E)_k$ for all $\mu \in \mathbf{N}_{<\infty}$ and $k \in \mathbb{N}_0$. In other words, the factorial moments of $\eta_t(E)$, where $(\eta_t)_{t\geq 0}$ is starting at μ, are given by $\mu(E)_k$. From that claim, since the moment problem is uniquely solvable for a deterministic random variable, we obtain $\eta_t(E) = \mu(E)$ $\mathbb{P}_\mu$-almost surely. Indeed, let $F_j = \mathbb{1}_{\mathbf{N}_j}$. Denote by $\mathcal{A}^0$ the identity operator and by $\mathcal{A}^k$ the k-fold application of $\mathcal{A}$,

$k \in \mathbb{N}$. A direct computation provides $\mathcal{A}F_j = (j+1)F_{j+1}$ which yields together with consistency

$$\begin{aligned}\mathcal{A}^k \mathbb{1}_{\mathbf{N}_{<\infty}}(\mu) &= \mathcal{A}^k P_t \mathbb{1}_{\mathbf{N}_{<\infty}}(\mu) = P_t \mathcal{A}^k \mathbb{1}_{\mathbf{N}_{<\infty}}(\mu) \\ &= \sum_{j=0}^{\infty} P_t \mathcal{A}^k F_j(\mu) = \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} P_t F_{j+k}(\mu) = \sum_{j=0}^{\infty} (j)_k P_t F_j(\mu).\end{aligned}$$

On the one hand,

$$\begin{aligned}\mathcal{A}^k \mathbb{1}_{\mathbf{N}_{<\infty}}(\mu) &= \int \mathbb{1}_{\mathbf{N}_{<\infty}}(\mu - \delta_{x_1} + \ldots + \delta_{x_k}) \\ &\qquad (\mu - \delta_{x_1} - \ldots - \delta_{x_{k-1}})(\mathrm{d}x_k) \cdots (\mu - \delta_{x_1})(\mathrm{d}x_2)\mu(\mathrm{d}x_1) \\ &= \mu(E)_k.\end{aligned}$$

On the other hand, $\sum_{j=0}^{\infty} (j)_k P_t F_j(\mu) = \mathbb{E}_\mu[\eta_t(E)_k]$ for all $\mu \in \mathbf{N}_{<\infty}$. □

A function f_n is called *symmetric* if $f_n(x_1, \ldots, x_n) = f_n(x_{s(1)}, \ldots, x_{s(n)})$ holds for all $x_1, \ldots, x_n \in E$ and $s \in \mathfrak{S}_n$ where $\mathfrak{S}_n$ denotes the set of permutations of the numbers $\{1, \ldots, n\}$. There is a one-to-one correspondence between measurable functions $F : \mathbf{N}_n \to \mathbb{R}$ and measurable symmetric functions $f_n : E^n \to \mathbb{R}$ through the relationship

$$f_n = F \circ \iota_n \tag{2.8}$$

where $\iota_n(x_1, \ldots, x_n) = \delta_{x_1} + \ldots + \delta_{x_n}$, $x_1, \ldots, x_n \in E$. This one-to-one correspondence extends to Markov semigroups as well. More precisely, any Markov semigroup $(P_t)_{t \geq 0}$ of a conservative Markov family with state space $\mathbf{N}_{<\infty}$ is in one-to-one correspondence in a family of Markov semigroups $(P_t^{[n]})_{t \geq 0}$, $n \in \mathbb{N}$. This relationship is described by the equation

$$P_t^{[n]}(F \circ \iota_n) = (P_t F) \circ \iota_n \tag{2.9}$$

for non-negative (or bounded) measurable functions $F : \mathbf{N}_{<\infty} \to \mathbb{R}$. Each $P_t^{[n]}$ acts on non-negative (or bounded) measurable symmetric $f_n : E^n \to \mathbb{R}$ and it models the evolution of n unlabeled particles. We refer to it as the *n-particle semigroup*. Notice that the semigroup property of $(P_t^{[n]})_{t \geq 0}$ is a direct consequence of the semigroup property of $(P_t)_{t \geq 0}$.

Remark 2.1.3. We remark that when starting with the semigroup $(P_t)_{t \geq 0}$, we are dealing with unlabeled particles only. As a result, $P_t^{[n]}$ is defined only for symmetric functions. Extending the definition of $(P_t^{[n]})_{t \geq 0}$ to non-symmetric functions would demand specifying a labeling of the particles which is not provided by $(P_t)_{t \geq 0}$. In general, there are multiple ways to label the particles leading to the same unlabeled process, see Section 4.3 below. For this reason, in the literature (see, e.g., [CGR21]), a distinction is often made between the so-called *configuration process*, representing the unlabeled notation, and the *coordinate process* which use labeled notation.

Consistency, in terms of the $P_t^{[n]}$, can be expressed as

$$P_t^{[n]}(f_{n-1} \otimes_{\mathrm{s}} \mathbb{1}_E) = (P_t^{[n-1]} f_{n-1}) \otimes_{\mathrm{s}} \mathbb{1}_E \tag{2.10}$$

for non-negative (or bounded) measurable $f_{n-1} : E^{n-1} \to \mathbb{R}$, $n \geq 2$. Here, $f_{n-1} \otimes_{\mathrm{s}} \mathbb{1}_E$ denotes the symmetrization of the function $f_{n-1} \otimes \mathbb{1}_E : (x_1, \ldots, x_n) \mapsto f_n(x_1, \ldots, x_{n-1})$. The *symmetrization* of a function $g_n : E^n \to \mathbb{R}$ is defined by taking the average of g_n over all permutations of the coordinates (see, e.g., [Las16, Equation (27)]). In other words, the symmetrization of f_n is given by

$$\widetilde{g_n}(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} g_n(x_{s(1)}, \ldots, x_{s(n)}), \quad x_1, \ldots, x_n \in E. \tag{2.11}$$

For $\mu = \sum_{i=1}^N \delta_{x_i} \in \mathbf{N}$ and $N \in \mathbb{N} \cup \{\infty\}$, we recall the *n-th factorial measure of* μ (see, e.g., [LP17, Equation (4.5)]), a measure on $(E^n, \mathcal{E}^{\otimes n})$ given by

$$\mu^{(n)} := \sum_{1 \leq i_1, \ldots, i_n \leq N}^{\neq} \delta_{(x_{i_1}, \ldots, x_{i_n})}. \tag{2.12}$$

For $\mu = 0$, we set $\mu^{(n)} = 0$. Using the notation adopted in [LP17], the superscript $\neq$ indicates summation over n-tuples $(i_1, \ldots, i_n)$ with pairwise different entries where, for N equal to infinity, only integer-valued indices are involved. Intuitively, that means that all subconfigurations of length n are chosen from a configuration μ, considering permutations as well. In this context, when two or more particles occupy the same position, they are treated as if they are distinguishable. Another representation of the factorial measure is given by, see, e.g., [LP17, Exercise 4.2],

$$\int f_n \, \mathrm{d}\mu^{(n)} = \int \cdots \int f_n(x_1, \ldots, x_n)(\mu - \delta_{x_1} - \cdots - \delta_{x_{n-1}})(\mathrm{d}x_n) \\ \cdots (\mu - \delta_{x_1} - \delta_{x_2})(\mathrm{d}x_3)(\mu - \delta_{x_1})(\mathrm{d}x_2)\mu(\mathrm{d}x_1). \tag{2.13}$$

Definition 2.1.4. For $n \in \mathbb{N}$ we put the *generalized falling factorial polynomial*

$$J_n f_n(\mu) := \int f_n(x_1, \ldots, x_n) \mu^{(n)}(\mathrm{d}(x_1, \ldots, x_n)) \tag{2.14}$$

for all measurable $f_n : E^n \to \mathbb{R}$ and $\mu \in \mathbf{N}$ for which the integral exists. For $n = 0$ and $f_0 \in \mathbb{R}$ we set $J_0 f_0(\mu) := \int f_0 \, \mathrm{d}\mu^{(0)} := f_0$.

From (2.13), it follows that J_n generalizes falling factorial polynomials $(a)_k = a(a-1) \cdots (a-k+1)$, $(a)_0 = 1$: If $f_n = \mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}$ for pairwise disjoint sets $B_1, \ldots, B_N \in \mathcal{E}$ where $N \in \mathbb{N}$, $d_1 \ldots, d_N \in \mathbb{N}_0$ and $n := d_1 + \ldots + d_N$, then

$$J_n f_n(\mu) = (\mu(B_1))_{d_1} \cdots (\mu(B_N))_{d_N}, \quad \mu \in \mathbf{N}_{<\infty} \tag{2.15}$$

holds, see, e.g., [Len73, Equation (2.3)]. Equation (2.15) is used in Section 4.1 below to recover the self-duality functions (1.3) for particle systems in finite sets using Theorem 2.1.5 below. For additional properties of the generalized falling factorial polynomials, we refer to [FKLO21].

The main result in this section is Theorem 2.1.5 below that provides a characterization of consistency in terms of an intertwining relation of $(P_t)_{t\geq 0}$ and $(P_t^{[n]})_{t\geq 0}$ in terms of the generalized falling factorial polynomials J_n. We say that the *factorial measure intertwining relation* is satisfied if

$$\mathbb{E}_\mu\left[\int F(\delta_{x_1}+\ldots+\delta_{x_n})\eta_t^{(n)}(\mathrm{d}(x_1,\ldots,x_n))\right] \\ = \int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}\left[F(\eta_t)\right]\mu^{(n)}(\mathrm{d}(x_1,\ldots,x_n)) \qquad \text{(IR.1)}$$

holds for all $n\in\mathbb{N}_0$, measurable $F:\mathbf{N}_n\to[0,\infty)$, $t\geq 0$ and $\mu\in\mathbf{N}_{<\infty}$. We note that integrability is not assumed on either the left-hand or the right-hand side in (IR.1). In the case of non-integrability, we read the equation as $\infty=\infty$.

Interpreting (IR.1), the left-hand side describes n particles chosen uniformly from the evolved state of the process starting at μ. The right-hand side describes n particles chosen uniformly from the initial state μ and then evolve under the process. Using similar arguments to those presented in the proof of Proposition 2.1.2, it follows that a finite particle system that satisfies (IR.1) is conservative.

We denote by $P_t^{[0]}$ the identity operator on $\mathbb{R}$, for all $t\geq 0$. In terms of the notation J_n and the semigroups $(P_t)_{t\geq 0}$ and $(P_t^{[n]})_{t\geq 0}$, the factorial measure intertwining relation (IR.1) rewrites as

$$P_tJ_nf_n(\mu)=J_nP_t^{[n]}f_n(\mu) \qquad (2.16)$$

for $n\in\mathbb{N}$, measurable symmetric $f_n:E^n\to[0,\infty)$, $\mu\in\mathbf{N}_{<\infty}$ and $t\geq 0$.

Theorem 2.1.5. *Let $(\Omega,\mathcal{F},(\eta_t)_{t\geq 0},(\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ be a Markov family. Then, it is consistent if and only if the factorial measure intertwining relation* (IR.1) *holds true.*

The equivalence is closely related to [CGR21, Theorem 4.3] in the discrete setting. As a reminder, $\iota_n(x_1,\ldots,x_n)=\delta_{x_1}+\ldots+\delta_{x_n}$ where $x_1,\ldots,x_n\in E$.

Proof. Recall that $\mathcal{A}^0$ denotes the identity operator and $\mathcal{A}^k$ the k-fold application of $\mathcal{A}$, $k\in\mathbb{N}$. First, we claim that for $\mu\in\mathbf{N}_{<\infty}$, $\mu(E)=N$, measurable $g_n:E^n\to[0,\infty)$ and measurable $G:\mathbf{N}_{<\infty}\to[0,\infty)$ such that $G\circ\iota_n=g_n$, the equation

$$\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}\mathcal{A}^{N-n}G(\mu)=J_ng_n(\mu) \qquad (2.17)$$

holds true. Indeed, using (2.7) and (2.13):

$$
\begin{aligned}
&\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}\mathcal{A}^{N-n}G(\mu)\\
&\quad=\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}\int\cdots\int G(\mu-\delta_{y_1}-\ldots-\delta_{y_{N-n}})\\
&\qquad\qquad(\mu-\delta_{y_1}-\cdots-\delta_{y_{N-n-1}})(\mathrm{d}y_{N-n})\cdots(\mu-\delta_{y_1})(\mathrm{d}y_2)\mu(\mathrm{d}y_1)\\
&\quad=\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}\int G(\mu-\delta_{y_1}-\ldots-\delta_{y_{N-n}})\mu^{(N-n)}(\mathrm{d}(y_1,\ldots,y_{N-n}))\\
&\quad=\int G(\delta_{x_1}+\ldots+\delta_{x_n})\mu^{(n)}(\mathrm{d}(x_1,\ldots,x_n))=J_n g_n(\mu).
\end{aligned}
$$

On the one hand, we assume consistency and fix $f_n : E^n \to [0,\infty)$ measurable and symmetric. Let $F : \mathbf{N}_{<\infty} \to [0,\infty)$ be measurable such that $F\circ\iota_n = f_n$. Thus, using (2.17) both for $F\circ\iota_n = f_n$ and $P_t^{[n]}(F\circ\iota_n) = (P_tF)\circ\iota_n$, conservation of number of particles (Proposition 2.1.2) and (2.6) yields

$$
\begin{aligned}
P_tJ_nf_n(\mu)&=\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}P_t\mathcal{A}^{N-n}F(\mu)\\
&=\mathbb{1}_{\{N\geq n\}}\frac{n!}{(N-n)!}\mathcal{A}^{N-n}P_tF(\mu)=J_nP_t^{[n]}f_n(\mu).
\end{aligned}
$$

On the other hand, we assume (IR.1) and fix a measurable $F : \mathbf{N}_{<\infty} \to [0,\infty)$ and $\mu\in\mathbf{N}_{<\infty}$, $\mu(E)=N\in\mathbb{N}$. Then, by (2.17), we obtain

$$
(N-1)!\mathcal{A}F(\mu)=J_{N-1}(F\circ\iota_{N-1})(\mu). \tag{2.18}
$$

Using (2.18), conservation of number of particles together with (IR.1) yields

$$
\begin{aligned}
P_t\mathcal{A}F(\mu)&=\frac{1}{(N-1)!}J_{N-1}P_t^{[N-1]}(F\circ\iota_{N-1})(\mu)\\
&=\frac{1}{(N-1)!}J_{N-1}((P_tF)\circ\iota_{N-1})(\mu)=\mathcal{A}P_tF(\mu).
\end{aligned}
$$

□

The factorial measure intertwining relation (IR.1), stated by Theorem 2.1.5, can be reformulated in several ways. Firstly, (IR.1) is a measure-theoretic equation that states the equality between the measure $\int\mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}[\eta_t\in\cdot\,]\,\mu^{(n)}(\mathrm{d}(x_1,\ldots,x_n))$ and the push-forward measure of $\mathbb{E}_\mu\left[\int\eta_t^{(n)}(\,\cdot\,)\right]$ under the map $\iota_n(x_1,\ldots,x_n)=\delta_{x_1}+\ldots+\delta_{x_n}$. If there are transition kernels denoted by $P_t^{[n]} : E^n\times\mathcal{E}^{\otimes n}\to[0,1]$, which lead to the n-particle semigroup (also denoted by $P_t^{[n]}$ by abuse of notation), this measure-theoretic equation can be expressed as an intertwining relation in terms of kernels. Indeed, if we also consider the semigroup $(P_t)_{t\geq0}$ with its transition kernels $P_t:\mathbf{N}_{<\infty}\times\mathcal{N}\to[0,1]$ and define $\Lambda_n:\mathbf{N}_{<\infty}\times\mathcal{E}^{\otimes n}\to[0,\infty)$, where $\Lambda_n(\mu,B):=\mu^{(n)}(B)=J_n\mathbb{1}_B(\mu)$, it becomes clear that the kernel Λ_n intertwines the transition kernels $(P_t)_{t\geq 0}$ and $(P_t^{[n]})$, i.e., $P_t\Lambda_n = \Lambda_n P_t^{[n]}$ holds. This means that

$$\int P_t(\mu, \mathrm{d}\mu')\Lambda_n(\mu', B) = \int \Lambda_n(\mu, \mathrm{d}x)P_t^{[n]}(x, B)$$

holds for all $\mu \in \mathbf{N}_{<\infty}$, $t \geq 0$ and $B \in \mathcal{E}^{\otimes n}$.

Secondly, a rephrasing involves the use of the semigroup $(P_t)_{t\geq 0}$ only which highlights the *self*-intertwining aspect in the factorial measure intertwining relation (IR.1). We define the operator $\mathcal{K}$ as

$$\mathcal{K}F(\mu) := F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int F(\delta_{x_1} + \ldots + \delta_{x_n})\mu^{(n)}(\mathrm{d}(x_1, \ldots, x_n)) \tag{2.19}$$

for measurable non-negative (or bounded) $F : \mathbf{N}_{<\infty} \to \mathbb{R}$ and $\mu \in \mathbf{N}_{<\infty}$. Note that the integral vanishes for $n > \mu(E)$. The function $\mathcal{K}F$ is commonly referred to as Lenard's *K-transform* of F (see [Len75, last equation on page 243] and, e.g., [KK02, Section 3.2]). Due to linearity, it follows from (2.9) and (IR.1) that $\mathcal{K}$ intertwines $(P_t)_{t\geq 0}$ with itself, i.e.,

$$P_t\mathcal{K} = \mathcal{K}P_t. \tag{2.20}$$

For free Kawasaki dynamics, a specific case within the realm of independent particles, this result is well-known (see [KKO+23, Section 3.2]). For a more detailed examination of independent Markov processes, we refer to Section 4.2 below.

Thirdly, the left-hand side of (IR.1) corresponds to the n-th factorial moment measure of the η_t starting at the initial configuration μ. Moment measures are closely related to correlation functions which characterize uniquely a point process given a growth condition, see, e.g., [Len73] or [Len75]. We observe a corollary regarding the time evolution of correlation functions of a consistent Markov family. For a random initial condition $\xi \in \mathbf{N}_{<\infty}$ with distribution ρ, we denote

$$\alpha_n^t(B) := \int \mathbb{E}_\mu\left[\eta_t^{(n)}(B)\right]\rho(\mathrm{d}\mu), \quad B \in \mathcal{E}^{\otimes n}$$

as the n-th factorial moment measure of η_t starting at ξ.

Corollary 2.1.6. *If one of the equivalent conditions of Theorem 2.1.5 holds true, then*

$$\alpha_n^t(B) = \int P_t^{[n]}\widetilde{\mathbb{1}}_B(x)\ \alpha_n^0(\mathrm{d}x) \tag{2.21}$$

for all $n \in \mathbb{N}$, $t \geq 0$ and sets $B \in \mathcal{E}^{\otimes n}$.

Firstly, we note that the symmetrization $\widetilde{\mathbb{1}}_B$ reduces to the function $\mathbb{1}_B$ for symmetric sets B. Secondly, for deterministic initial condition ξ, the time-zero factorial moment measure is simply $\alpha_n^0 = \xi^{(n)}$.

Proof. Using $\mu^{(n)}(B) = \int \mathbb{1}_B \,\mathrm{d}\mu^{(n)} = \int \widetilde{\mathbb{1}}_B \,\mathrm{d}\mu^{(n)}$ yields

$$\alpha_n^t(B) = \int \mathbb{E}_\mu \left[\int \widetilde{\mathbb{1}}_B \,\mathrm{d}\eta_t^{(n)} \right] \rho(\mathrm{d}\mu) = \iint P_t^{[n]} \widetilde{\mathbb{1}}_B \,\mathrm{d}\mu^{(n)} \rho(\mathrm{d}\mu) = \int P_t^{[n]} \widetilde{\mathbb{1}}_B \,\mathrm{d}\alpha_n^0$$

where in the second equality we used (IR.1). □

Remark 2.1.7. Another way to rephrase (2.21) is possible, provided that there exists a σ-finite measure λ on $(E, \mathcal{E})$ and, for each $n \in \mathbb{N}$, measurable functions $u_t^{[n]} : E^n \times E^n \to [0, \infty)$ such that $u_t^{[n]}(x, y) = u_t^{[n]}(y, x)$, where $x, y \in E^n$, and

$$P_t^{[n]} f_n(x) = \int f_n(y) u_t^{[n]}(x, y) \lambda^{\otimes n}(\mathrm{d}y) \tag{2.22}$$

for all $t > 0$, $x \in E^n$ and symmetric measurable bounded functions $f_n : E^n \to [0, \infty)$. This assumption shares similarities with the notion of duality from probabilistic potential theory, see [BG68, Chapter VI]. We emphasize that (2.22) is stronger than reversibility of the measure $\lambda^{\otimes n}$. The additional condition is satisfied, for example, by independent reversible diffusions with $E = \mathbb{R}$ endowed with the Borel σ–algebra where λ is the Lebesgue measure and $u_t^{[n]}$ is the product of the densities of the transition kernels of the one particle dynamics. Corollary 2.1.6, (2.22) and the symmetry of $u_t^{[n]}$ result in

$$\alpha_n^t(B) = \int \mathbb{1}_B(x) \int_{E^n} u_t^{[n]}(y, x) \alpha_n^0(\mathrm{d}y) \lambda^{\otimes n}(\mathrm{d}x) \tag{2.23}$$

for $t > 0$. Therefore, $\alpha_n^t(B)$ is absolutely continuous with respect to $\lambda^{\otimes n}$ with Radon-Nikodym derivative $\int_{E^n} u_t^{[n]}(y, \cdot\,) \alpha_n^0(\mathrm{d}y)$.

2.2 Consistency for infinite particle systems

In [CGR21] and Section 2.1, *consistency* refers to the property that the removal of a particle uniformly at random commutes with the time evolution of the process. However, for infinitely many particles, we deliberately define consistency differently since (2.4) is not useful for extending to infinitely many particles. If we were to define consistency using the commutation property of the lowering operator $\mathcal{A}$ with the Markov semigroup P_t, $t \geq 0$, as seen in (2.6), we would encounter several problems: First, a technical subtlety arises: For infinitely many particle systems, the function $P_t F : \mathbf{N} \to \mathbb{R}$, $P_t F(\mu) = \mathbb{E}_\mu [F(\eta_t)]$ may not always be measurable for a measurable function $F : \mathbf{N} \to \mathbb{R}$, making the operator approach impractical.

However, a more significant conceptual difficulty arises. Equation (2.6) lacks a connection between the dynamics of infinitely many particles and the system with a finite number of particles. Specifically, when one particle is removed from an infinite configuration, the total number of particles in the configuration remains infinite, whereas removing a particle from a finite configuration results in a finite number of remaining particles. Nevertheless, our goal is to establish intertwining relations that connect systems with an infinite number of particles and those with a finite number. To address this issue, this section introduces a meaningful new concept of consistency that extends the notion of consistency to infinitely many particles. Furthermore, we provide a recapitulation of strongly consistent systems and demonstrate that they satisfy this new consistency condition.

A random measure $\xi \in \mathbf{N}$ is said to be *proper* if there exist random variables $Z_k \in E$, $K \in \mathbb{N}_0 \cup \{\infty\}$ such that

$$\xi = \sum_{k=1}^{K} \delta_{Z_k} \tag{2.24}$$

is satisfied. For point processes, where $\xi(E) < \infty$, the condition of being proper is automatically satisfied. For a more in-depth discussion of proper point processes, we refer to [LP17].

According to Theorem 2.1.5, our new definition below is a natural extension of the definition of consistency which coincides with the notation of consistency of finite particle systems. Let $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ be a Markov family with state space $\mathbf{N}$ such that η_t is proper for each $t \geq 0$.

Definition 2.2.1. The Markov family $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ is called *consistent* if it is conservative and satisfies the factorial measure intertwining relation (IR.1) for all $\mu \in \mathbf{N}$, $t \geq 0$, measurable $F : \mathbf{N}_{<\infty} \to [0, \infty)$ and $n \in \mathbb{N}$.

Remark 2.2.2. (i) We only define consistency for processes where each η_t, $t \geq 0$, is proper. This assumption is motivated by a technical subtlety: in general, the map $\mu \mapsto \int F(\delta_{x_1} + \ldots + \delta_{x_n})\mu^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$ for $n \geq 2$, even for measurable F, is not measurable. By requiring each η_t to be proper, we ensure the measurability of $\omega \mapsto \int F(\delta_{x_1} + \ldots + \delta_{x_n})\eta_t(\omega)^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$.

(ii) For infinitely many particles, the factorial measure intertwining relation (IR.1) does not imply that the total number of particles is a conserved quantity compared to the finite case. Thus, we include the assumption of conservation of the number of particles in the definition of consistency. A pathological counterexample is given, for example, by E containing only one single element and a Markov process that stays constant for a finite initial datum but for the initial datum ∞ it jumps immediately into the state where the number of particles follows a distribution having no finite moments.

In addition to the concept of consistency introduced in this and the preceding section, there exists a stronger notion of consistency which we refer to as *strong consistency*, called *compatibility* by Le Jan and Raimond [LR04a]. Strongly consistent families are studied in the context of stochastic flows: Le Jan and Raimond have investigated a one-to-one correspondence between strongly consistent families and stochastic flows of kernels. In this context, we begin with a family of Markov semigroups $(P_t^{[n]})_{t\geq 0}$ defined on functions on E^n, $n \in \mathbb{N}$. Strong consistency, roughly means that time evolution and removal of any *deterministic* particle commute—there is no need to choose the particle to be removed uniformly at random. The following definition is found in [LR04a, Definition 1.1].

A family of Markov semigroups $(P_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$ is called *strongly consistent* if the equation

$$P_t^{[n]} f_n(x_1, \dots, x_n) = P_t^{[l]} g_l(x_{i_1}, \dots, x_{i_l}), \quad x_1, \dots, x_n \in E \tag{2.25}$$

holds for all $l \leq n$, $i_1, \dots, i_l \in \{1, \dots, n\}$ pairwise different and $g_l : E^l \to \mathbb{R}$ bounded and measurable where $f_n : E^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto g_l(x_{i_1}, \dots, x_{i_l})$.

In the following, we demonstrate that strong consistency of the family $(P_t^{[n]})_{t\geq 0}$ implies the consistency of a Markov family that describes its unlabeled dynamics. First, note that when we have a strongly consistent family $(P_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, (2.25) implies that $(P_t^{[n]})_{t\geq 0}$ on $(E^n, \mathcal{E}^{\otimes n})$ preserves symmetry. In other words, if f_n is symmetric, then $P_t^{[n]} f_n$ is symmetric as well—we remind the reader that a function $f_n : E^n \to \mathbb{R}$ is called symmetric if $f_n(x_1, \dots, x_n) = f_n(x_{s(1)}, \dots, x_{s(n)})$ holds for all $x_1, \dots, x_n$ and all permutations s. As a result, by using the one-to-one correspondence described in (2.9), we obtain a Markov semigroup $(P_t)_{t\geq 0}$ which describes the dynamics of a Markov family $(\Omega,\ \mathcal{F},\ (\eta_t)_{t\geq 0},\ (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ of finitely many particles. Furthermore, strong consistency implies (2.10). Thus, the Markov family is consistent, i.e., it satisfies (2.4).

In the following, we demonstrate that a strongly consistent family also enables the construction of a consistent process with infinitely many particles. First, we remark that for a strongly consistent family $(P_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, the existence of a Markov family $(\Omega^{[\infty]}, \mathcal{F}^{[\infty]}, (Z_t^{[\infty]})_{t\geq 0}, (\mathbb{P}_x^{[\infty]})_{x\in E^\infty})$, $Z_t^{[\infty]} = Z_t = (Z_{k,t})_{k\in\mathbb{N}}$ describing the evolution of infinitely many particles is implied by Kolmogorov's theorem (see, e.g., [LR04a, Section 1.5.3]). $\mathbb{P}_x^{[\infty]}$ is a probability measure on a probability space $(\Omega^{[\infty]}, \mathcal{F}^{[\infty]})$ for each $x \in E^\infty$ and $Z_{k,t}$ are real-valued random variables where $k \in \mathbb{N}$, $t \geq 0$. $Z_0 = x$ holds $\mathbb{P}_x^{[\infty]}$-almost surely for all $x \in E^\infty$, the map $x \mapsto \mathbb{P}_x^{[\infty]}[Z_t \in A]$ is measurable for all measurable $A \subset E^\infty$ and $t \geq 0$ and $(Z_t)_{t\geq 0}$ satisfies the Markov property with respect to its natural filtration. This family is such that each finite subconfiguration of l-particles evolves according to $P_t^{[l]}$. More precisely, for each family $i_1, \dots, i_l \in \mathbb{N}, l \in \mathbb{N}$ of pairwise different indices, $t \geq 0$ and $x = (x_k)_{k\in\mathbb{N}} \in E^\infty$, the distribution of $(Z_{i_1,t}, \dots, Z_{i_l,t})$ under $\mathbb{P}_x^{[\infty]}$ is equal to $P_t^{[l]}((x_{i_1}, \dots, x_{i_l}), \cdot)$.

Remark 2.2.3. Using standard measure-theoretic arguments, this property extends to infinitely many particles as well: for every injection $s : \mathbb{N} \to \mathbb{N}$, the distribution of $(Z_{s(k),t})_{k\in\mathbb{N}}$ under $\mathbb{P}_x^{[\infty]}$ is equal to the distribution of $(Z_{k,t})_{k\in\mathbb{N}}$ under $\mathbb{P}^{[\infty]}_{(x_{s(k)})_{k\in\mathbb{N}}}$.

Proposition 2.2.4. *There exists a Markov family* $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ *with state space* $\mathbf{N}$ *such that* η_t *is proper for all* $t \geq 0$ *and*

(i) *for each* $\mu = \sum_{k=1}^n \delta_{x_k}$, $x_k \in E$, $n \in \mathbb{N} \cup \{\infty\}$ *the distribution of* $(\eta_t)_{t\geq 0}$ *under* $\mathbb{P}_\mu$ *is equal to the distribution of* $(\sum_{k=1}^n \delta_{Z_{k,t}})_{t\geq 0}$ *where each* $Z_{k,t}$ *is starting at* x_k*;*

(ii) *the Markov family is consistent.*

In Proposition 2.2.4, when $n = 0$, the summation is considered to be zero. This implies that the empty configuration remains empty at all times. The proposition demonstrates that it is possible to construct a consistent, unlabeled process from any strongly consistent family. In Remark 4.3.2 below, we observe that strong consistency is, in fact, a stronger property than consistency. More specifically, we present an example of a consistent Markov family that describes the evolution of unlabeled particles such that there is no labeling that satisfies the strong consistency property.

In the following proof, we use a construction of the unlabeled dynamics that combines two principles. Firstly, we combine the dynamics of different numbers of particles, choosing which dynamics to follow based on the initial configuration's particle number. Secondly, we map the labeled notation $x = (x_k)_{k=1}^n \in E^n$ to $\iota_n(x) = \sum_{k=1}^n \delta_{x_k}$. These arguments are standard: the first one is straight-forward and the second can be interpreted as a Markov mapping theorem. However, since this construction is essential for the intertwining relations, we present it in detail for the reader's benefit. The proof of consistency follows by linearity and by the definition of the factorial measure.

Remark 2.2.5. For $n = l$, the strong consistency property (2.25) corresponds to the definition of an exchangeable particle system, see, e.g., [Daw93, Section 2.3]. We refer to [Daw93, Proposition 2.3.3] for an analogy to Proposition 2.2.4 going from the labeled to an unlabeled notation.

Proof of Proposition 2.2.4. For each $n \in \mathbb{N} \cup \{\infty\}$, let $(\Omega^{[n]}, \mathcal{F}^{[n]}, (Z_t^{[n]})_{t\geq 0}, (\mathbb{P}_x^{[n]})_{x\in E^n})$ be the Markov family that describes the dynamics of n particles. The family consists of a measurable space $(\Omega^{[n]}, \mathcal{F}^{[n]})$, measurable maps $Z_t^{[n]} : \Omega^{[n]} \to E^n$, $t \geq 0$, and probability measures $\mathbb{P}_x^{[n]}$ on $(\Omega^{[n]}, \mathcal{F}^{[n]})$ with corresponding expected values denoted by $\mathbb{E}_x^{[n]}$. The Markov property is satisfied with respect to the natural filtration $\mathcal{F}_t^{[n]} := \sigma(Z_s^{[n]} : 0 \leq s \leq t)$.

- We define $\Omega := \{0\} \cup \bigcup_{n\in\mathbb{N}\cup\{\infty\}} \{n\} \times \Omega_n$ and equip it with the σ-algebra $\mathcal{F}$ generated by the sets $\{0\}$ and $\{n\} \times A_n$, $A_n \in \mathcal{F}^{[n]}$, $n \in \mathbb{N}_0 \cup \{\infty\}$.
- Put $\eta_t(n, \omega^{[n]}) := \iota_n(Z_t^{[n]}(\omega^{[n]}))$ for $n \in \mathbb{N} \cup \{\infty\}$, $\omega^{[n]} \in \Omega^{[n]}$ and $\eta_t(0) := 0$.
- For every $\mu \in \mathbf{N}$, we choose a fixed $z(\mu) \in E^n$ such that $\iota_n(z(\mu)) = \mu$ where $n = \mu(E)$. We then define $\mathbb{P}_\mu$ to be the push-forward measure of $\mathbb{P}_{z(\mu)}^{[n]}$ under the map $\omega^{[n]} \mapsto (n, \omega^{[n]})$.

The distribution $\mathbb{P}_\mu$ relies on how the components of $z(\mu)$ are permuted, but according to the strong consistency property (2.25) and Remark 2.2.3, the distribution of $(\eta_t)_{t\geq 0}$ under $\mathbb{P}_\mu$ remains unchanged regardless of the choice of permutation. In particular, this distribution is equal to the distribution of $(\iota_n(Z_t^{[n]}))_{t\geq 0}$ under $\mathbb{P}_x$, for all $x \in E^n$ that satisfy $\iota_n(x) = \mu$.

We now show that for $\mathcal{F}_t = \sigma(\eta_s : 0 \le s \le t)$ the Markov property

$$\mathbb{P}_\mu[\eta_{t+s} \in A \mid \mathcal{F}_s] = \mathbb{P}_{\eta_s}[\eta_t \in A], \quad \mathbb{P}_\mu\text{-almost surely for all } \mu \in \mathbf{N}, A \in \mathcal{N}, t, s \ge 0$$

holds by applying the Markov property of the process $(Z_t^{[n]})_{t\ge 0}$. Indeed, fix $A \in \mathcal{N}$, $s, t \ge 0$, $\mu = \sum_{k=1}^n \delta_{x_k} \in \mathbf{N}$, $x = (x_k)_{k=1}^n$, $n \in \mathbb{N} \cup \{\infty\}$ and $B \in \mathcal{F}_s$. Then, using the definition of $\mathcal{F}_s$, we find a measurable set $C \subset \{h : h : [0,t] \to \mathbf{N}\}$ such that $\mathbb{1}_B = \mathbb{1}_C((\eta_u)_{0\le u\le s})$. Therefore,

$$\begin{aligned}
\mathbb{E}_\mu[\mathbb{1}_A(\eta_{t+s})\mathbb{1}_B] &= \mathbb{E}_x^{[n]}\left[\mathbb{1}_A(\iota_n(Z_{t+s}^{[n]}))\mathbb{1}_C\left(\left(\iota_n(Z_u^{[n]})\right)_{0\le u\le s}\right)\right] \\
&= \mathbb{E}_x^{[n]}\left[\mathbb{E}_{Z_s^{[n]}}^{[n]}\left[\mathbb{1}_A(\iota_n(Z_t^{[n]}))\right]\mathbb{1}_C\left(\left(\iota_n(Z_u^{[n]})\right)_{0\le u\le s}\right)\right] \\
&= \mathbb{E}_\mu\left[\mathbb{E}_{\eta_s}\left[\mathbb{1}_A(\eta_t)\right]\mathbb{1}_B\right]
\end{aligned}$$

holds since $\mathbb{1}_C\left(\left(\iota_n(Z_u^{[n]})\right)_{0\le u\le s}\right)$ is $\mathcal{F}_s^{[n]}$-measurable. The remaining properties can be easily obtained.

By definition, $(\eta_t)_{t\ge 0}$ is conservative, so only (IR.1) requires a proof. Let $n \in \mathbb{N} \cup \{\infty\}$ be fixed and let $\mu = \sum_{k=1}^n \delta_{x_k} \in \mathbf{N}$ where $x = (x_k)_{k=1}^n$. By applying (2.12) twice and using strong consistency and linearity, we obtain consistency:

$$\begin{aligned}
\mathbb{E}_\mu\left[\int F(\delta_{y_1} + \ldots + \delta_{y_l})\eta_t^{(l)}(\mathrm{d}(y_1,\ldots,y_l))\right] &= \sum_{1\le i_1,\ldots,i_l\le n}^{\neq} \mathbb{E}_x^{[n]}\left[F(\delta_{Z_{i_1,t}^{[n]}} + \ldots, \delta_{Z_{i_l,t}^{[n]}})\right] \\
= \sum_{1\le i_1,\ldots,i_l\le n}^{\neq} \mathbb{E}_{\delta_{x_{i_1}}+\ldots+\delta_{x_{i_l}}}[F(\eta_t)] &= \int \mathbb{E}_{\delta_{y_1}+\ldots+\delta_{y_l}}[F(\eta_t)]\,\mu^{(l)}(\mathrm{d}(y_1,\ldots,y_l)). \square
\end{aligned}$$

3 Infinite-dimensional orthogonal polynomials

In this chapter, we generalize the orthogonal self-duality relation, see (1.9), to the class of consistent particle systems in uncountable spaces, e.g., $\mathbb{R}^d$. This generalization results in intertwining relations involving so-called *infinite-dimensional orthogonal polynomials* which are well-studied objects in the field of infinite-dimensional analysis.

This chapter is structured as follows: In Section 3.1, we recall the concept of infinite-dimensional orthogonal polynomials and present our primary contribution, a sufficient condition for orthogonal intertwining relations, see Theorem 3.1.6. Subsequently, in Section 3.1.3 we prove a useful product formula for these polynomials.

Next, we delve into two typical reversible measures for consistent systems, namely, the distribution of the Poisson process (Section 3.2) and the Pascal process (Section 3.3). These sections do not introduce additional intertwining relations but instead focus on a more in-depth examination of the polynomials themselves. Both polynomials fall into the Meixner-type category, see [Mei34], [BK19]. Infinite-dimensional orthogonal polynomials of Meixner's type have been extensively studied in the context of non-Gaussian white noise [Ber96], [Ber02]. Their connections with quantum probability and representations of *-Lie algebras and current algebras are investigated in [AFS02], [AB09].

Armed with the knowledge of the orthogonal polynomials associated with Poisson and Pascal processes, we dedicate Section 3.4 to particle systems with an infinite number of particles. This section presents a refined result that only requires reversible measures for the n-particle dynamics. For examples, we refer to Chapter 4.

3.1 The general case

After a review of the literature, in which we revisit infinite-dimensional polynomials (Section 3.1.1) and their orthogonalization (Section 3.1.2), we present our main theorem: Given the existence of a reversible measure for a consistent particle system, we obtain an orthogonal self-intertwining relation. The orthogonality is with respect to the reversible measure. The proof is concise and does not rely on explicit formulas of orthogonal polynomials but solely on their orthogonality.

This section closely follows the exposition in [FJRW24, Section 2.3] with the following difference: In this thesis, we relax the assumption that the reversible measure is the distribution of a finite point process, more precisely, we cover the infinite particle case as well to provide a deeper understanding of the benefits of Section 3.4 below. All proofs are entirely analogous and no technical issues arise, except for the proof of Lemma 3.1.7

below where a straightforward approximation argument is now employed.

3.1.1 Infinite-dimensional polynomials

Let $(E, \mathcal{E})$ be a Borel space. We fix a system $\mathcal{E}_{\mathrm{b}} \subset \mathcal{E}$ with the following properties:

- $A_1 \subset A_2$ with $A_1 \in \mathcal{E}$, $A_2 \in \mathcal{E}_{\mathrm{b}}$ implies $A_1 \in \mathcal{E}_{\mathrm{b}}$;
- the space E is the union of an increasing sequence of sets $E_l \in \mathcal{E}_{\mathrm{b}}$, $l \in \mathbb{N}$;
- for each $A \in \mathcal{E}_{\mathrm{b}}$ there is an $n \in \mathbb{N}$ such that $A \subset E_n$.

These conditions imply that $\mathcal{E}_{\mathrm{b}}$ is closed under forming finite unions. For example, $\mathcal{E}_{\mathrm{b}}$ could be the bounded Borel sets in a metric space. Elements of $\mathcal{E}_{\mathrm{b}}$ are called *bounded sets* in the general case as well. A measure μ on $(E, \mathcal{E})$ is called *locally finite* if $\mu(B) < \infty$ for all $B \in \mathcal{E}_{\mathrm{b}}$. We denote the set of locally finite counting measures by $\mathbf{N}_{\mathrm{lf}}$. Note that $\mathbf{N}_{\mathrm{lf}}$ is a measurable subset of $\mathbf{N}$. We equip $\mathbf{N}_{\mathrm{lf}}$ with the trace σ-algebra $\mathcal{N}_{\mathrm{lf}}$.

We consider polynomials with bounded coefficients u_k that have bounded support, ensuring their square-integrability in Section 3.1.2 below. The set of polynomials of degree at most $n \in \mathbb{N}_0$ is defined by, see, e.g., [Lyt03a, Section 5],

$$\mathcal{P}_n := \left\{ \mathbf{N}_{\mathrm{lf}} \ni \mu \mapsto u_0 + \sum_{k=1}^{n} \int u_k \, \mathrm{d}\mu^{\otimes k} : u_k \in \mathcal{C}_k \right\}. \tag{3.1}$$

Here $\mathcal{C}_k$ denotes, for $k \geq 1$, the space of measurable bounded functions $u_k : E^k \to \mathbb{R}$ such that there exists $B \in \mathcal{E}_{\mathrm{b}}$ with $\{x \in E^k : u_k(x) \neq 0\} \subset B^k$. Moreover, we put $\mathcal{C}_0 := \mathbb{R}$. Drawing inspiration from the notation of univariate polynomials, we refer to the maps $\mu \mapsto \int u_k \, \mathrm{d}\mu^{\otimes k}$ as *monomials*.

Remark 3.1.1. The designation of polynomials is justified. First, we observe that for sets $B_1, \ldots, B_N \in \mathcal{E}_{\mathrm{b}}$ and any polynomial $p(s_1, \ldots, s_N)$ in N variables of degree n, the map $\mu \mapsto p(\mu(B_1), \ldots, \mu(B_N))$ belongs to $\mathcal{P}_n$.

More generally, when we restrict ourselves to finite configurations and fix u_k, $k \in \mathbb{N}$, we can rewrite the monomial $\mathbf{N}_{<\infty} \ni \mu \mapsto \int u_n \, \mathrm{d}\mu^{\otimes n}$ as an evaluation $\mu \mapsto h(\mu, \ldots, \mu)$. Here, h is a multilinear form defined by $h(\nu_1, \ldots, \nu_k) := \int u_k \, \mathrm{d}(\nu_1 \otimes \cdots \otimes \nu_k)$ where $\nu_1, \ldots, \nu_k$ are elements of a Banach space containing at least the linear hull of $\mathbf{N}_{<\infty}$. Thus, the terminology *polynomials* aligns with the theory of polynomials on Banach spaces, as described in, for example, [HJ14, Definition 12].

The following proposition shows that the generalized falling factorial polynomials $J_n f_n(\mu) = \int f_n \, \mathrm{d}\mu^{(n)}$ discussed in the previous chapter, see (2.14), are indeed infinite-dimensional polynomials and, more specifically, span the space of polynomials.

Proposition 3.1.2. *For all $n \in \mathbb{N}_0$, the identity*

$$\mathcal{P}_n = \left\{ \mathbf{N}_{\mathrm{lf}} \ni \mu \mapsto \sum_{k=0}^{n} J_k f_k(\mu) : f_k \in \mathcal{C}_k, k \in \{0, \ldots, n\} \right\} \tag{3.2}$$

holds true.

Equation (3.2) follows from explicit formulas linking factorial measures $\mu^{(n)}$ and product measures $\mu^{\otimes n}$. These relations are similar to relations between moments and factorial moments of integer-valued random variables involving Stirling numbers, see [DVJ03, Chapter 5]. Since this proposition is crucial for proving the main result and its proof is relatively concise, we provide a brief explanation. It directly follows from the fact that every monomial $\mu \mapsto \int f_n \, \mathrm{d}\mu^{\otimes n}$ can be expressed as a linear combination of generalized falling factorial polynomials of degree $k \leq n$, and vice versa, see [FKLO21, Equations (3.1)-(3.3)].

Proof. Let $\mu = \delta_{x_1} + \cdots + \delta_{x_N} \in \mathbf{N}_{\mathrm{lf}}$ and $f_n \in \mathcal{C}_n$. Then:

$$\int f_n \, \mathrm{d}\mu^{\otimes n} = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} f_n(x_{i_1}, \ldots, x_{i_n}).$$

Each multi-index $(i_1, \ldots, i_n)$ on the right-hand side corresponds to a set partition σ of $\{1, \ldots, n\}$ where k and l belong to the same block if and only if $i_k = i_l$. Let Σ_n denote the set of partitions of the numbers $\{1, \ldots, n\}$. For $\sigma \in \Sigma_n$, let $|\sigma|$ be the number of blocks in the set partition. Furthermore, let $(f_n)_\sigma : E^{|\sigma|} \to \mathbb{R}$ be the function obtained from f_n by identifying, in the order of occurrence, those arguments that belong to the same block of σ. As an example:

$$(f_4)_{\{\{1,3\},\{2\},\{4\}\}}(x_1, x_2, x_3) = f_4(x_1, x_2, x_1, x_3).$$

By grouping multi-indices $(i_1, \ldots, i_n)$ that lead to the same partition σ, we obtain

$$\int f_n \, \mathrm{d}\mu^{\otimes n} = \sum_{\sigma \in \Sigma_n} \int (f_n)_\sigma \, \mathrm{d}\mu^{(|\sigma|)} \tag{3.3}$$

(compare [DVJ03, Exercise 5.4.5]). We conclude that $\int f_n \, \mathrm{d}\mu^{\otimes n}$ is a linear combination of generalized falling factorial polynomials of degrees $|\sigma| \leq n$.

Conversely, by expanding (2.13), we obtain:

$$\int f_n \, \mathrm{d}\mu^{(n)} = \sum_{\sigma \in \Sigma_n} (-1)^{n-|\sigma|} \int (f_n)_\sigma \, \mathrm{d}\mu^{\otimes |\sigma|}. \tag{3.4}$$

Hence the generalized falling factorial polynomial of degree n on the left-hand side is a linear combination of monomials $\mu \mapsto \int g_k \, \mathrm{d}\mu^{\otimes k}$ of degree $k \leq n$. □

3.1.2 Orthogonalization

Let ρ be a probability measure on $(\mathbf{N}, \mathcal{N})$. We use the abbreviation $L^2(\rho)$ to denote the space $L^2(\mathbf{N}, \mathcal{N}, \rho)$ with norm $\|\cdot\|_{L^2(\rho)}$. Throughout the rest of the section we assume that all moments of the number of particles in bounded sets are finite, i.e.,

$$\int \mu(B)^n \rho(\mathrm{d}\mu) < \infty \tag{3.5}$$

for all $n \in \mathbb{N}$ and $B \in \mathcal{E}_\mathrm{b}$. This condition implies that $\rho(\mathbf{N}_\mathrm{lf}) = 1$ and that every map $\mu \mapsto \int f_n \, \mathrm{d}\mu^{\otimes n}$, where $f_n \in \mathcal{C}_n$, is in $L^2(\rho)$. Hence, $\mathcal{P}_n$ is a subspace of $L^2(\rho)$. In general, it is not closed; we denote its closure in $L^2(\rho)$ as $\overline{\mathcal{P}_n}$. The linear space $\mathcal{P}_n$ and its closure share the same orthogonal complement $\mathcal{P}_n^\perp = \overline{\mathcal{P}_n}^\perp$. If the total number of particles $\mu(E)$ has finite moments with respect to ρ, then the structure of bounded sets is not needed; we can simply put $\mathcal{E}_\mathrm{b} := \mathcal{E}$.

Orthogonal polynomials in a single real variable can be constructed using an orthogonalization procedure. This definition extends to the infinite-dimensional setting, as described in [Lyt03a, Section 5] and the references therein.

Definition 3.1.3. For $n \in \mathbb{N}$ and a measurable bounded function $f_n \in \mathcal{C}_n$, we define the *infinite-dimensional orthogonal polynomial* $I_n f_n \in L^2(\rho)$ as follows:

$$I_n f_n := \text{orthogonal projection of } \left(\mu \mapsto \int f_n \, \mathrm{d}\mu^{\otimes n}\right) \text{ onto } \mathcal{P}_{n-1}^\perp. \tag{3.6}$$

If $n = 0$, we define $I_0 f_0$ as the constant function equal to $f_0 \in \mathbb{R}$.

Equivalently,

$$I_n f_n(\mu) = \int f_n \, \mathrm{d}\mu^{\otimes n} - Q(\mu)$$

with $Q \in \overline{\mathcal{P}_{n-1}}$ the orthogonal projection of $\mu \mapsto \int f_n \, \mathrm{d}\mu^{\otimes n}$ onto $\overline{\mathcal{P}_{n-1}}$. Notice that $I_n f_n(\mu)$ is only defined up to ρ-null sets. A direct consequence of the definition is the orthogonality relation

$$\int (I_n f_n)(I_m g_m) \, \mathrm{d}\rho = 0 \tag{3.7}$$

for $f_n \in \mathcal{C}_n$, $g_m \in \mathcal{C}_m$ and $n, m \in \mathbb{N}_0$, $n \neq m$. This justifies the term *orthogonal polynomials*.

Infinite-dimensional orthogonal polynomials naturally appear in the study of non-Gaussian white noise [Ber96], [Ber02]. They are used to prove chaos decompositions. The relation between polynomial chaos and chaos decompositions in terms of multiple stochastic integrals with respect to *power jump martingales* (see, e.g., [NS00]) is investigated in detail [Lyt03a]. Chaos decompositions play a role in the study of Lévy white noise and stochastic differential equations driven by Lévy white noise [DØP04], [LP06], [Mey08]. Furthermore, chaos decompositions extend to other settings, such as chaos decompositions of Rademacher [NPR10] or of Dirichlet functionals [Pec08]. Chaos decompositions find applications in numerical methods, particularly in the field of uncertainty quantification, see, e.g., [CLM09].

Moreover, $f_n \mapsto I_n f_n$ extends to an operator preserving the inner product on the space of symmetric functions that are square integrable with respect to some measure λ_n (see, e.g., [Lyt03a, Corollary 5.2] for further details)—we remind the reader that a function $f_n : E^n \to \mathbb{R}$ is called symmetric if $f_n(x_1, \dots, x_n) = f_n(x_{s(1)}, \dots, x_{s(n)})$ for all

$x_1, \ldots, x_n$ and all permutations s. When ρ is the distribution of the Poisson process with intensity measure λ, the measure λ_n takes the form of the product $\lambda_n = \lambda^{\otimes n}$. The infinite-dimensional orthogonal polynomial is given by a multiple stochastic integral with respect to the compensated Poisson measure (see [Sur83]), which we recall in Section 3.2 below, hence the notation $I_n f_n(\mu)$ similarly to [Las16, Equation (25)]. In general, the measure λ_n is more complicated. In the literature (see, e.g., [Lyt03a, Section 5]), the infinite-dimensional orthogonal polynomial $I_n f_n(\mu)$ is often denoted by $:\int f_n \,\mathrm{d}\mu^{\otimes n}:$, commonly known as *Wick dots*, see also [Kac07] or [FK20].

The following observation is crucial in the proof of Theorem 3.1.6 below: It makes no difference whether one projects, in the definition of the infinite-dimensional polynomial (3.6), the monomial $\int f_n \,\mathrm{d}\mu^{\otimes n}$ or the generalized falling factorial polynomial $\mu \mapsto J_n f_n(\mu) = \int f_n \,\mathrm{d}\mu^{(n)}$ onto $\mathcal{P}_{n-1}^{\perp}$.

Proposition 3.1.4. *For all $f_n \in \mathcal{C}_n$, where $n \in \mathbb{N}$, the equation*

$$I_n f_n = \textit{orthogonal projection of } J_n f_n \textit{ onto } \mathcal{P}_{n-1}^{\perp} \tag{3.8}$$

holds true ρ-almost surely.

Proof. We observe that (3.4) implies

$$\int f_n \,\mathrm{d}\mu^{(n)} = \int f_n \,\mathrm{d}\mu^{\otimes n} + P(\mu) \tag{3.9}$$

for some $P \in \mathcal{P}_{n-1}$. It follows that the orthogonal projections of $\mu \mapsto J_n f_n(\mu)$ and $\mu \mapsto \int f_n \,\mathrm{d}\mu^{\otimes n}$ onto $\mathcal{P}_{n-1}^{\perp}$ are the same. □

Now, we present the main theorem of this section which is the counterpart of Theorem 2.1.5 with the self-intertwiner being the infinite-dimensional orthogonal polynomial introduced above. For a Markov family $(\Omega,\ \mathcal{F},\ (\eta_t)_{t\geq 0},\ (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ that describes the evolution of finitely many particles, we recall that a probability measure ρ on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ is called *reversible* if

$$\int \mathbb{E}_\mu\left[F(\eta_t)\right] G(\mu)\rho(\mathrm{d}\mu) = \int \mathbb{E}_\mu[g(\eta_t)]F(\mu)\rho(\mathrm{d}\mu) \tag{3.10}$$

holds for all measurable non-negative (or bounded) $F, G : \mathbf{N}_{<\infty} \to \mathbb{R}$ and $t \geq 0$. Similarly, for a Markov family $(\Omega,\ \mathcal{F},\ (\eta_t)_{t\geq 0},\ (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$, where η_t is proper for all $t \geq 0$, that describes the evolution of possibly infinitely many particles, we say that a probability measure ρ on $(\mathbf{N}, \mathcal{N})$ is *reversible* if (3.10) holds for all measurable non-negative (or bounded) $F, G : \mathbf{N} \to \mathbb{R}$.

Remark 3.1.5. In the case of an infinite number of particles, we note that the map $\mathbf{N}_{\mathrm{lf}} \ni \mu \mapsto \mathbb{E}_\mu\left[F(\eta_t)\right]$ is measurable for measurable non-negative (or bounded) $F : \mathbf{N} \to \mathbb{R}$. This can be proved using arguments similar to those found in, for example, [LP17, Corollary 6.5]. Therefore, together with the fact that $\rho(\mathbf{N}_{\mathrm{lf}}) = 1$, (3.10) is well-defined.

For both the finite and infinite particle case, reversibility implies that the operator $P_tF(\mu) = \mathbb{E}_\mu\left[F(\eta_t)\right]$ is a well-defined self-adjoint operator on $L^2(\rho)$ for all $t \geq 0$. We remind the reader of the n-particle semigroup $P_t^{[n]}$ as defined in (2.9). We say that a conservative Markov family satisfies the *orthogonal polynomial intertwining relation* if

$$P_tI_nf_n(\mu) = I_nP_t^{[n]}f_n(\mu) \tag{IR.2}$$

holds true for ρ-almost all $\mu \in \mathbf{N}$, symmetric $f_n \in \mathcal{C}_n$, $n \in \mathbb{N}_0$ and $t \geq 0$.

Equation (IR.2) can be read in the following way: The evolution of the infinite-dimensional orthogonal polynomial of degree n with coefficient f_n under the unlabeled dynamics $(\eta_t)_{t\geq 0}$ is determined by the n-particle evolution and is equal to the orthogonal polynomial with coefficient $P_t^{[n]}f_n$. In Section 4.1 below, we show that the orthogonal polynomial intertwining relation (IR.2) is indeed a generalization of the orthogonal polynomial duality of the SIP, the SEP or the IRW, as expressed in (1.9). We recall the fact that a consistent Markov family is conservative, see Proposition 2.1.2 or Definition 2.2.1.

Theorem 3.1.6. *Let ρ be a probability measure satisfying* (3.5)*, i.e., all moments of the number of particles in bounded sets are finite. Then, for a consistent Markov family with reversible measure ρ, the orthogonal polynomial intertwining relation* (IR.2) *holds.*

The proof is of functional-analytic nature. It is based on the fact that P_t leaves the space of polynomials of degree at most n invariant and, thanks to reversibility, commutes with the projections onto $\mathcal{P}_n$. Apart from the minimal assumption (3.5) on the moments of the reversible measure, it does not require any additional properties of the measure or further characteristics of the stochastic process. Therefore, it offers an advantage over the methods used to obtain orthogonal dualities in the past, such as three-term recurrence relations [FG19] or Lie algebra representation theory [Gro19].

Lemma 3.1.7. *For each $k \in \mathbb{N}$ and $t \geq 0$, P_t leaves the closure of the space of polynomials of degree at most k invariant, i.e.,*

$$P_t\overline{\mathcal{P}_k} \subset \overline{\mathcal{P}_k}. \tag{3.11}$$

Proof. Using the fact that P_t is a bounded operator, together with (3.2), it is sufficient to prove $P_tJ_kf_k$ is contained in $\overline{\mathcal{P}_k}$ for an arbitrary symmetric $f_k \in \mathcal{C}_k$. Note that $P_tJ_kf_k$ is equal to $J_kP_t^{[k]}f_k$ according to the factorial measure intertwining relation (IR.1). If E is a bounded set, i.e., $E \in \mathcal{E}_\mathrm{b}$, then it immediately follows that $P_t^{[k]}f_k \in \mathcal{C}_k$ and thus, $J_kP_t^{[k]}f_k \in \mathcal{P}_k$.

In general, while $P_t^{[k]}f_k$ is indeed measurable and bounded, it does not necessarily have bounded support. Thus, an additional approximation argument is required. The approximation argument is standard and does not require any special ideas. However, we present it because this lemma is the main ingredient for the main result in this section.

Let $E_l \in \mathcal{E}_\mathrm{b}$, $l \in \mathbb{N}$ be an increasing sequence with union E. To obtain that $J_kP_t^{[k]}f_k$ belongs to $\overline{\mathcal{P}_k}$, it suffices to prove that $J_k(\mathbb{1}_{E_l^k}P_t^{[k]}f_k) \in \mathcal{P}_k$ convergences to $J_kP_t^{[k]}f_k$

in $L^2(\rho)$ as $l \to \infty$. Since $J_k|f_k|$ is in $L^2(\rho)$, it follows that $J_k P_t^{[k]}|f_k| = P_t J_k|f_k|$ is also in $L^2(\rho)$. In particular, $J_k P_t^{[k]}|f_k| < \infty$ ρ-almost surely. Since $\mathbb{1}_{E_l^k} P_t^{[k]} f_k$ converges pointwise to $P_t^{[k]} f_k$ and $\left|\mathbb{1}_{E_l^k} P_t^{[k]} f_k\right| \leq P_t^{[k]}|f_k| \in L^1(\mu^{(k)})$ for ρ-almost all μ, we can apply Lebesgue's dominated convergence theorem:

$$J_k(\mathbb{1}_{E_l^k} P_t^{[k]} f_k)(\mu) \to J_k P_t^{[k]} f_k(\mu)$$

as $l \to \infty$ for ρ-almost all μ. Again, using $\left|J_k(\mathbb{1}_{E_l^k} P_t^{[k]} f_k)\right| \leq J_k P_t^{[k]}|f_k| \in L^2(\rho)$, Lebesgue's dominated convergence theorem implies

$$J_k(\mathbb{1}_{E_l^k} P_t^{[k]} f_k) \to J_k P_t^{[k]} f_k$$

as $l \to \infty$ in $L^2(\rho)$. □

It is a general fact that a bounded self-adjoint operator that leaves a closed vector space invariant commutes with the orthogonal projection onto that space. In our case, for each n,

$$P_t \Pi_n = \Pi_n P_t \tag{3.12}$$

where Π_n denotes the orthogonal projection onto $\overline{\mathcal{P}_n}$ in $L^2(\rho)$.

Let us check this fact for our concrete operators and spaces. For $F \in \overline{\mathcal{P}_n}^{\perp}$, since P_t is a self-adjoint operator on $L^2(\rho)$ and (3.11), we find that for all $G \in \overline{\mathcal{P}_n}$, the equation $\int (P_t F) G \, \mathrm{d}\rho = \int (P_t G) F \, \mathrm{d}\rho = 0$ holds, implying that $P_t F \in \overline{\mathcal{P}_n}^{\perp}$. Therefore,

$$P_t \overline{\mathcal{P}_n}^{\perp} \subset \overline{\mathcal{P}_n}^{\perp}. \tag{3.13}$$

Thus, for all $F \in L^2(\rho)$, using (3.11), (3.13) and the fact that $F - \Pi_n F \in \overline{\mathcal{P}_n}^{\perp}$, we conclude that

$$\Pi_n P_t F = \Pi_n P_t \Pi_n F + \Pi_n P_t (F - \Pi_n F) = P_t \Pi_n F.$$

Proof of Theorem 3.1.6. Let id be the identity operator on $L^2(\rho)$. We obtain the orthogonal polynomial intertwining relation (IR.2) by

$$\begin{aligned} P_t I_n f_n &= P_t(\mathrm{id} - \Pi_{n-1}) J_n f_n \\ &= (\mathrm{id} - \Pi_{n-1}) P_t J_n f_n = (\mathrm{id} - \Pi_{n-1}) J_n P_t^{[n]} f_n = I_n P_t^{[n]} f_n. \end{aligned}$$

Here, we used Proposition 3.1.4 in the first and the fourth equality, (3.12) in the second equality and the factorial measure intertwining relation (IR.1) in the third equality. □

3.1.3 A product formula

In this section, we prove a proposition that holds under the additional assumption of complete independence. A point process ξ is said to be *completely independent* (or *completely orthogonal*) if the counting variables $\xi(A_1), \dots, \xi(A_m)$, associated with pairwise disjoint regions $A_1, \dots, A_m \in \mathcal{E}$, $m \in \mathbb{N}$, are independent. For a more detailed discussion and a characterization of completely independent random measures, we refer to [LP17, Section 6.4] and [Kin67].

Complete independence implies a factorization property of infinite-dimensional orthogonal polynomials with disjointly supported coefficients. For special cases, where measures ρ lead to orthogonal polynomials of Meixner's type, a similar factorization property can be found, for example, in [Lyt03b, Lemma 3.1]. Our proposition is new and holds true for all distributions of completely independent point processes.

Given $N \in \mathbb{N}$ and $d_1, \dots, d_N \in \mathbb{N}$ along with functions $v_i : E^{d_i} \to \mathbb{R}$ for $i \in \{1, \dots, N\}$, we define the function $v_1 \otimes \ldots \otimes v_N : E^{d_1 + \ldots + d_N} \to \mathbb{R}$ as follows: it maps $(z_1, \dots, z_N)$, where $z_i \in E^{d_i}$, to the product $v_{d_1}(z_1) \cdots v_{d_N}(z_N)$.

Proposition 3.1.8. *Suppose that ρ is the distribution of a completely independent point process. Let $N \in \mathbb{N}$, $A_1, \dots, A_N \in \mathcal{E}_\mathrm{b}$ be pairwise disjoint and $d_1, \dots, d_N \in \mathbb{N}$. Furthermore, let $v_i : E^{d_i} \to \mathbb{R}$, $i = 1, \dots, N$ be measurable bounded functions that vanish on $E^{d_i} \setminus A_i^{d_i}$. Set $n := d_1 + \cdots + d_N$. Then,*

$$I_n(v_1 \otimes \ldots \otimes v_N)(\mu) = I_{d_1} v_1(\mu) \cdots I_{d_N} v_N(\mu) \tag{3.14}$$

for ρ-almost all $\mu \in \mathbf{N}_{<\infty}$.

The product formula is interesting since it is the continuum counterpart to the product form of the discrete duality presented in (1.9). Furthermore, it becomes relevant when we prove a product formula for infinitely-dimensional Meixner polynomials, see Proposition 3.3.11 below.

Remark 3.1.9. A particularly relevant case occurs when v_i is the indicator function of $A_i^{d_i}$. In this case, Proposition 3.1.8 states that the orthogonalized version of $\mu \mapsto \prod_{i=1}^n \mu(A_i)^{d_i}$ is equal to the product of the orthogonalized versions of $\mu \mapsto \mu(A_i)^{d_i}$. When ρ is the distribution of the Poisson or Pascal process (see Sections 3.2 and 3.3 below), the orthogonalized version of $\mu(A_i)^{d_i}$ is in fact a univariate orthogonal polynomial in the variable $\mu(A_i) \in \mathbb{N}_0$ and we obtain a product of univariate orthogonal polynomials, see (3.22) and (3.38) below. However, in general, the orthogonalized version of $\mu \mapsto \mu(A_i)^{d_i}$ may not be a univariate polynomial. Indeed, when considering disjoint sets $A_1, A_2 \in \mathcal{E}_\mathrm{b}$, we can use (3.14) to obtain

$$\begin{aligned} I_n \mathbb{1}_{(A_1 \cup A_2)^n}(\mu) &= \sum_{k=0}^n \binom{n}{k} I_n \mathbb{1}_{A_1^k \times A_2^{n-k}}(\mu) \\ &= \sum_{k=0}^n \binom{n}{k} I_k \mathbb{1}_{A_1^k}(\mu) I_{n-k} \mathbb{1}_{A_2^{n-k}}(\mu). \end{aligned} \tag{3.15}$$

Therefore, if $I_n \mathbb{1}_{(A_1 \cup A_2)^n}(\mu)$, $\mathbb{1}_{A_1^k}(\mu)$, and $I_{n-k}\mathbb{1}_{A_2^{n-k}}(\mu)$ were all univariate polynomials in the variables $\mu(A_1 \cup A_2)$, $\mu(A_1)$ and $\mu(A_2)$, then (3.15) could be rewritten into a convolution formula in terms of univariate orthogonal polynomials that was studied in [Eag64, Theorem 3.1] concerning a generalization of *Runge's identity*. Al-Salam and Carlitz proved in [AC76] that this formula holds if and only if the orthogonal polynomials are of Meixner-type (see [Mei34], [BK19]) or if they are the orthogonal q-polynomials (see [AC65]).

To exploit the complete independence, it is helpful to verify that if $v : E^d \to \mathbb{R}$ has its support in A^d, $A \in \mathcal{E}_{\mathrm{b}}$, then $\mu \mapsto I_d v(\mu)$ depends solely on what occurs within A. We show a bit more. Let $\mathcal{P}_n(A) \subset \mathcal{P}_n$ be the space of linear combinations of maps $\mu \mapsto \int u_k \, \mathrm{d}\mu^{\otimes k}$, $k \leq n$ where $u_k : E \to \mathbb{R}$ is bounded, measurable and vanishes on $E^k \setminus A^k$. Notice that every function $F \in \mathcal{P}_n(A)$ only depends on the restriction μ_A defined as $\mu_A(B) := \mu(A \cap B)$.

Lemma 3.1.10. *Let $d \in \mathbb{N}$, $A \in \mathcal{E}_{\mathrm{b}}$ and $v : E^d \to \mathbb{R}$ be a measurable bounded function that vanishes on $E^d \setminus A^d$. Then, there exists a map $Q \in \overline{\mathcal{P}_{d-1}(A)}$ such that $I_d v(\mu) = \int v \, \mathrm{d}\mu^{\otimes d} - Q(\mu)$ for ρ-almost all $\mu \in \mathbf{N}$.*

Proof. Let Q be the orthogonal projection of $\mu \mapsto \int v \, \mathrm{d}\mu^{\otimes d}$ onto $\overline{\mathcal{P}_{d-1}(A)}$. Then, $Q \in \overline{\mathcal{P}_{d-1}(A)}$ and the difference $F(\mu) := \int v \, \mathrm{d}\mu^{\otimes d} - Q(\mu)$ is orthogonal to $\overline{\mathcal{P}_{d-1}(A)}$. We exploit the complete independence to show that F is actually orthogonal to the bigger space $\overline{\mathcal{P}_{d-1}}$ which implies $I_d v(\mu) = F(\mu)$ for ρ-almost all μ.

Using the definition of $\mathcal{P}_{d-1}$, it suffices to prove

$$\int F(\mu) \int f_k \, \mathrm{d}\mu^{\otimes k} \rho(\mathrm{d}\mu) = 0 \tag{3.16}$$

for each $k \in \{1, \ldots, d-1\}$ and measurable bounded $f_k : E^k \to \mathbb{R}$ with the property that $\{x \in E^k : f_k(x) \neq 0\} \subset B^k$ for some $B \in \mathcal{E}_{\mathrm{b}}$.

First, let $f_k = \mathbb{1}_{C_1 \times C_2}$ with $C_i \in \mathcal{E}^{\otimes s_i}$, where $s_1, s_2 \in \mathbb{N}_0$ and $C_1 \subset (B \cap A)^{s_1}$, $C_2 \subset (B \setminus A)^{s_2}$, then $\int f_n \, \mathrm{d}\mu^{\otimes n} = \mu^{\otimes s_1}(C_1)\mu^{\otimes s_2}(C_2)$ and by the complete independence (notice $F(\mu) = F(\mu_A)$)

$$\int F(\mu) \int f_k \, \mathrm{d}\mu^{\otimes k} \rho(\mathrm{d}\mu) = \int F(\mu)\mu^{\otimes s_1}(C_1)\rho(\mathrm{d}\mu) \int \mu^{\otimes s_2}(C_2)\rho(\mathrm{d}\mu).$$

The first integral on the right-hand side vanishes because of $C_1 \subset A^{s_1}$, $s_1 \leq d-1$ and $F \in \overline{\mathcal{P}_{d-1}(A)}^{\perp}$. Therefore, (3.16) holds.

More generally, let $f_k = \mathbb{1}_C$ where C is a Cartesian product of measurable bounded sets. C is the disjoint union of Cartesian products $C_1 \times \cdots \times C_k$ where every C_i is either contained in A or in $E \backslash A$. Taking linear combinations and exploiting that $\int f_k \mathrm{d}\mu^{\otimes k}$ does not change if we permute variables in f_k, we obtain (3.16). Then, by standard measure-theoretic arguments, (3.16) extends to all $f_k \in \mathcal{C}_k$, $1 \leq k \leq d-1$. The map F is also orthogonal to all constant functions because every constant function is in $\overline{\mathcal{P}_{d-1}(A)}$. □

When evaluating the product of two infinite-dimensional orthogonal polynomials, it is important to know that the product of two polynomials is again a polynomial.

Lemma 3.1.11. *Let $A, B \in \mathcal{E}_\mathrm{b}$ be disjoint. Then, $FG \in \overline{\mathcal{P}_{m+n}(A \cup B)}$ for all $F \in \overline{\mathcal{P}_m(A)}$ and $G \in \overline{\mathcal{P}_n(B)}$, $m, n \in \mathbb{N}_0$*

Proof. Let $(F_k)_{k\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{P}_m(A)$ and $\mathcal{P}_n(B)$, respectively, with $\|F - F_k\|_{L^2(\rho)} \to 0$ and $\|G - G_k\|_{L^2(\rho)} \to 0$. We have $F_k(\mu) = F_k(\mu_A)$ for all k and μ. Hence, $F(\mu) = F(\mu_A)$ for ρ-almost all μ. Similarly G_k and G depend on μ_B only. The triangle inequality and the complete independence yield

$$\begin{aligned}\|FG - F_kG_k\|_{L^2(\rho)} &\le \|(F - F_k)G\|_{L^2(\rho)} + \|F_k(G - G_k)\|_{L^2(\rho)} \\ &= \|F - F_k\|_{L^2(\rho)} \|G\|_{L^2(\rho)} + \|F_k\|_{L^2(\rho)} \|G - G_k\|_{L^2(\rho)} \to 0.\end{aligned}$$

As each product F_kG_k is in $\mathcal{P}_{m+n}(A\cup B)$, the limit FG is in the closure $\overline{\mathcal{P}_{m+n}(A \cup B)}$. □

Proof of Proposition 3.1.8. It is enough to treat the case $N = 2$; the general case follows by an induction over N. Let $A_1, A_2 \in \mathcal{E}_\mathrm{b}$ be disjoint. Let $d_1, d_2 \in \mathbb{N}$ and $v_1 : E^{d_1} \to \mathbb{R}$, $v_2 : E^{d_2} \to \mathbb{R}$ be measurable functions that vanish outside $A_1^{d_1}$ and $A_2^{d_2}$ respectively. By Lemma 3.1.10, there exist maps $Q_1 \in \overline{\mathcal{P}_{d_1-1}(A_1)}$, $Q_2 \in \overline{\mathcal{P}_{d_2-1}(A_2)}$ such that

$$I_{d_1}v_1(\mu) = \int v_1 \,\mathrm{d}\mu^{\otimes d_1} - Q_1(\mu), \quad I_{d_2}v_2(\mu) = \int v_2 \,\mathrm{d}\mu^{\otimes d_2} - Q_2(\mu)$$

for ρ-almost all μ. Therefore, by Lemma 3.1.11, we have

$$I_{d_1}v_1(\mu)I_{d_2}v_2(\mu) = \int v_1 \,\mathrm{d}\mu^{\otimes d_1} \int v_2 \,\mathrm{d}\mu^{\otimes d_2} - Q(\mu)$$

with $Q \in \overline{\mathcal{P}_{d_1+d_2-1}}$.

Let $s_1, s_2, s_3 \in \mathbb{N}_0$, $B \in \mathcal{E}_\mathrm{b}$ and $C_1 \in \mathcal{E}^{\otimes s_i}$, $i = 1, 2, 3$ with $s_1 + s_2 + s_3 \le d_1 + d_2 - 1$ and $C_1 \subset (B \cap A_1)^{s_1}$, $C_2 \subset (B \cap A_2)^{s_2}$, $C_3 \subset (B \setminus (A_1 \cup A_2))^{s_3}$. Then, by the complete independence (notice $I_{d_1}v_1(\mu) = I_{d_1}v_1(\mu_{A_1})$ and $I_{d_2}v_2(\mu) = I_{d_2}v_1(\mu_{A_2})$ by Lemma 3.1.10),

$$\begin{aligned}&\int I_{d_1}v_1(\mu)I_{d_2}v_2(\mu)\, \mu^{\otimes(s_1+s_2+s_3)}(C_1 \times C_2 \times C_3)\rho(\mathrm{d}\mu) \\ &\quad = \int I_{d_1}v_1(\mu)\mu^{\otimes s_i}(C_1)\rho(\mathrm{d}\mu) \int I_{d_2}v_2(\mu)\mu^{\otimes s_i}(C_2)\rho(\mathrm{d}\mu) \int \mu^{\otimes s_3}(C_3)\rho(\mathrm{d}\mu).\end{aligned}$$

We must have $s_1 \le d_1 - 1$ or $s_2 \le d_2 - 1$; therefore at least one of the first two integrals on the right-hand side vanishes and the product $I_{d_1}v_1(\mu)I_{d_2}v_2(\mu)$ is orthogonal to $\mu^{\otimes n}(C_1\times C_2\times C_3)$. We conclude with an argument similar to the proof of Lemma 3.1.10 that $I_{d_1}v_1(\mu)I_{d_2}v_2(\mu)$ is in fact orthogonal to $\overline{\mathcal{P}_{d_1+d_2-1}}$. It follows that the product is equal to $I_{d_1+d_2}(v_1 \otimes v_2)(\mu)$ for ρ-almost all μ. □

3.2 The Poisson case

This section explores the orthogonal polynomial intertwining relation (IR.2) when the reversible measure is the distribution of the Poisson process. In Section 3.2.1, which does not present any new results, we provide a concise review of multiple Wiener-Itô integrals and their connection to infinite-dimensional orthogonal polynomials. The insights enable us to reformulate the intertwining relation into a self-intertwining relation for P_t eliminating the need for n-particle semigroups, see Section 3.2.2.

Intertwining relations expressed in terms of multiple Wiener-Itô integrals have already been studied for independent particle systems (see, e.g., [Sur83], [KLR08] and [KKO$^+$23]), a topic further explored in Section 4.2 below. In particular, we delve into independent random walks in Section 4.1 below. Moreover, our objective is to investigate intertwining relations for particle systems with interactions, such as correlated Brownian motions discussed in Section 4.4 below.

3.2.1 Multiple Wiener-Itô integrals

As a reminder, the symmetrization $\widetilde{f_n}$ of a function f_n is defined as in (2.11). Let λ be a locally finite measure on $(E,\mathcal{E})$. The condition that λ is locally finite is only used for sake of simplicity in the exposition and, in principle, can also be replaced by the condition that λ is σ-finite. The *multiple Wiener-Itô integral of degree* n with respect to the Poisson process with intensity measure λ (see [LP17, Section 12.2] or [Las16, Equation (25)]) is defined by

$$I_n f_n(\mu) := \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \iint \widetilde{f_n}(x_1,\dots,x_n)\lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n))\mu^{(k)}(\mathrm{d}(x_1,\dots,x_k)) \tag{3.17}$$

for $\mu \in \mathbf{N}_{\mathrm{lf}}$ and $f_n \in \mathcal{C}_n$. When integrating with respect to $\mu^{(0)}$ or $\lambda^{\otimes 0}$, we treat the corresponding integral as if it was absent. Specifically, we obtain $I_0 c(\mu) = c$ for $c \in \mathcal{C}_0 = \mathbb{R}$.

$I_n f_n$ are multiple stochastic integrals with respect to the compensated Poisson measure $\mu - \lambda$ which go back to [Ogu72]. On a more detailed study of the stochastic analysis for Poisson processes, we refer to [Mey95], [Lyt03a] and [Las16].

Remark 3.2.1. The I_n defined in (3.17) can be viewed as a modified version of the generalized falling factorial polynomials J_n defined in (2.14). In fact, $I_n f_n(\mu) = J_n f_n(\mu)$ holds for $\mu \in \mathbf{N}_{\mathrm{lf}}$ and $f_n \in \mathcal{C}_n$ if $\lambda = 0$.

Remark 3.2.2. If E is bounded, i.e., $E \in \mathcal{E}_{\mathrm{b}}$ then I_n can be understood as an integral operator with an underlying signed kernel. To be more precise, for each $n \in \mathbb{N}$ there exists a signed kernel $\Lambda_n : \mathbf{N}_{<\infty} \times \mathcal{E}^{\otimes n} \to \mathbb{R}$ such that $I_n f_n(\mu) = \int f_n(x)\Lambda_n(\mu, \mathrm{d}x)$ for all $\mu \in \mathbf{N}_{<\infty}$ and measurable bounded $f_n : E^n \to \mathbb{R}$.

The orthogonality relation

$$\int (I_n f_n)(I_m g_m)\,\mathrm{d}\pi_\lambda = \mathbb{1}_{\{n=m\}} n! \int \widetilde{f_n}\widetilde{g_m}\,\mathrm{d}\lambda^{\otimes n} \tag{3.18}$$

holds for $f_n \in \mathcal{C}_n$, $g_m \in \mathcal{C}_m$ where $n, m \in \mathbb{N}_0$, see, e.g., [Las16, Equation (28)], or also [Sur84]. Thereby, π_λ denotes the distribution of the Poisson process with intensity measure λ.

Equation (3.18) implies that $I_n f_n$ coincides with the infinite-dimensional orthogonal polynomial defined in (3.6) for the distribution of the Poisson process. Indeed, let I_n be defined as (3.17). Then, using (3.9) it follows that for each $f_l \in \mathcal{C}_l$, $l \in \mathbb{N}$ there is a polynomial $Q \in \mathcal{P}_{l-1}$ such that

$$\int f_l\,\mathrm{d}\mu^{\otimes l} = I_l f_l(\mu) + Q_{l-1}(\mu). \tag{3.19}$$

Thus, iterating yields that each $F \in \mathcal{P}_{n-1}$ can be decomposed into $F = \sum_{l=0}^{n-1} I_l u_l$ for some $u_l \in \mathcal{C}_l$, $l \in \{0, \dots, n-1\}$. Therefore, using (3.18), we have $\int (I_n f_n) F\,\mathrm{d}\pi_\lambda = 0$. In other words, $I_n f_n \in \mathcal{P}_{n-1}^{\perp}$. Consequently, when combined with (3.19), this implies that I_n satisfies (3.6).

Moreover, (3.18) implies that I_n can be uniquely extended to a bounded linear operator mapping square-integrable symmetric functions, denoted by $L^2_{\mathrm{sym}}(\lambda^{\otimes n})$, to $L^2(\pi_\lambda)$—we remind the reader that a function $f_n : E^n \to \mathbb{R}$ is called symmetric if $f_n(x_1, \dots, x_n) = f_n(x_{s(1)}, \dots, x_{s(n)})$ for all $x_1, \dots, x_n$ and all permutations s. We put $L^2_{\mathrm{sym}}(\lambda^{\otimes 0}) := \mathbb{R}$. Additionally, for each $F \in L^2(\pi_\lambda)$, there exists a unique sequence $f = (f_n)_{n \in \mathbb{N}_0}$ that belongs to the *Fock space*, see, e.g., [Las16, Theorem 2],

$$\mathfrak{F} := \bigoplus_{n=0}^{\infty} \frac{1}{n!} L^2_{\mathrm{sym}}(\lambda^{\otimes n}) \tag{3.20}$$

equipped with the inner product $\langle f, g \rangle := f_0 g_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_n g_n\,\mathrm{d}\lambda^{\otimes n}$. This sequence is such that $F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n f_n$, referred to as the *chaos decomposition*. In other words, the operator

$$\mathfrak{U} : \mathfrak{F} \to L^2(\pi_\lambda), \quad (f_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} I_n f_n \tag{3.21}$$

is unitary.

Furthermore, it is worth noting that $I_n f_n$ has a close relationship with *Charlier polynomials*, defined in (1.5), which are part of the self-duality function (1.9) of independent random walks (see [RS18, Section 4.1.1], [FG19, Section 3.3]). More precisely, they generalize the Charlier polynomial in the following sense (see, e.g., [LP11, Equation (3.3)]):

$$I_n(\mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}})(\mu) = \prod_{k=1}^{N} \mathscr{C}_{d_k}(\mu(B_k); \lambda(B_k)) \tag{3.22}$$

for π_λ-almost all $\mu \in \mathbf{N}_{<\infty}$, $d_1 + \ldots + d_N = n$ and pairwise disjoint $B_1, \ldots, B_N \in \mathcal{E}$ with $0 < \lambda(B_1), \ldots, \lambda(B_N) < \infty$. The orthogonality relation (3.18) generalizes the univariate orthogonality relation (1.6).

The infinite-dimensional orthogonal polynomials can be characterized by the *generating functional*, see, e.g., [KKO+23, Section 4.1]. More precisely,

$$\sum_{n=0}^{\infty} \frac{1}{n!} I_n u^{\otimes n}(\mu) = \exp\left(- \int u \, \mathrm{d}\lambda + \int \log(1+u) \, \mathrm{d}\mu \right) \tag{3.23}$$

holds for π_λ-almost all μ and for all non-negative $u \in \mathcal{C}_1$. Here, $u^{\otimes n}$ is defined as the n-fold tensor product $(u \otimes \cdots \otimes u)(x_1, \ldots, x_n) = u(x_1) \cdots u(x_n)$, $x_1, \ldots, x_n \in E$.

3.2.2 Self-intertwining relations

In this section, we observe that, in the Poisson case, we can reformulate the orthogonal polynomial intertwining relation (IR.2) in terms of I_n, P_t and $P_t^{[n]}$ into a self-intertwining relation where a unitary operator becomes a self-intertwiner for $(P_t)_{t \geq 0}$. Let $(\Omega, \mathcal{F}, (\eta_t)_{t \geq 0}, (\mathbb{P}_\mu)_{\mu \in \mathbf{N}_{<\infty}})$ (or $(\Omega, \mathcal{F}, (\eta_t)_{t \geq 0}, (\mathbb{P}_\mu)_{\mu \in \mathbf{N}})$, respectively) be a consistent Markov family such that η_t, $t \geq 0$ is proper for all $t \geq 0$ and the distribution of the Poisson process π_λ with finite (or locally finite, respectively) λ is reversible. Therefore, by Theorem 3.1.6, the orthogonal polynomial intertwining relation (IR.2) holds, in other words, the multiple Wiener-Itô integral I_n given by (3.17) intertwines P_t and $P_t^{[n]}$.

On the one hand, using the unitary operator defined in (3.21), we define a semigroup of self-adjoint operators

$$P_t^{\mathfrak{F}} = \mathfrak{U}^{-1} P_t \mathfrak{U} : \mathfrak{F} \to \mathfrak{F}, \quad t \geq 0 \tag{3.24}$$

acting on the Fock space $\mathfrak{F}$. By (IR.2), it follows for $f = (f_n)_{n \in \mathbb{N}_0} \in \mathfrak{F}$, with $f_n \in \mathcal{C}_n$,

$$(P_t^{\mathfrak{F}} f)_n = P_t^{[n]} f_n, \quad n \in \mathbb{N}_0; \tag{3.25}$$

and by approximation also for all $f \in \mathfrak{F}$. This proves, in particular, that $P_t^{[n]}$ is a self-adjoint contraction on $L^2_{\mathrm{sym}}(\lambda^{\otimes n})$. This means that the push-forward measure of $\lambda^{\otimes n}$ under the map $\iota_n : (x_1, \ldots, x_n) \mapsto \delta_{x_1} + \ldots + \delta_{x_n}$ is reversible for $(\eta_t)_{t \geq 0}$. While this is clear for $\lambda(E) < \infty$ since π_λ is reversible, it becomes a remarkable fact when $\lambda(E) = \infty$.

On the other hand, if λ is a finite measure, then the operator

$$\mathfrak{V} : L^2(\pi_\lambda) \to \mathfrak{F}, \quad F \mapsto \mathrm{e}^{\frac{1}{2}\lambda(E)} \, (f_n)_{n \in \mathbb{N}_0}, \text{ where } f_n := F \circ \iota_n,$$

is unitary. In particular, we conclude by (3.25) that $\mathfrak{V}$ intertwines P_t and $P_t^{\mathfrak{F}}$ which means $\mathfrak{V} P_t = P_t^{\mathfrak{F}} \mathfrak{V}$. In summary, along with (3.24), we deduce that the composition $\mathfrak{U}\mathfrak{V}$ is a self-intertwiner for P_t, i.e.,

$$\mathfrak{U}\mathfrak{V} P_t = P_t \mathfrak{U}\mathfrak{V} \tag{3.26}$$

for all $t \geq 0$.

3.3 The Pascal case

The Pascal process, also known as the *negative binomial process*, introduced in [BR91] is a well-studied point process, see [ST98], [Ser90, Section 2.7], [Ber02] or [KP09, Proposition 1.1] for further details. We briefly recall the definition of a Pascal process. Let $p \in (0, 1)$ and let α be a locally finite measure on $(E, \mathcal{E})$. The condition that λ is locally finite is only used for sake of simplicity in the exposition and, in principle, can also be replaced by the condition that λ is σ-finite. A Pascal process with parameters p and α is a random measure $\xi \in \mathbf{N}$ satisfying the following properties.

(i) The random variables $\xi(A_1), \dots, \xi(A_N)$ are independent if the sets $A_1, \dots, A_N \in \mathcal{E}$ are pairwise disjoint.

(ii) For each $A \in \mathcal{E}$ such that $0 < \alpha(A) < \infty$, the random variable $\xi(A)$ follows the negative binomial distribution with parameters p and $\alpha(A)$, i.e.,

$$\mathbb{P}[\xi(A) = k] = \alpha(A)(\alpha(A) + 1) \cdots (\alpha(A) + k - 1)\frac{p^k}{k!}(1 - p)^{\alpha(A)}, \quad k \in \mathbb{N}_0.$$

For $k = 0$, the equation reads as $\mathbb{P}(\xi(A) = 0) = (1 - p)^{\alpha(A)}$. If $\alpha(A) = 0$, then $\xi(A) = 0$ almost surely. If $\alpha(A) = \infty$, then $\xi(A) = \infty$ almost surely.

We denote the distribution of the Pascal process by $\rho_{p,\alpha}$.

Note that the Pascal process has the structure of a measure-valued Lévy process since $\xi(A_1), \dots, \xi(A_n)$ are independent for pairwise disjoint $A_1, \dots, A_n \in \mathcal{E}$ and the distribution of $\xi(A)$ is infinite divisible and only depends on $\alpha(A)$, $A \in \mathcal{E}$. For more details, see [Kin67], [Kin93, Chapter 8], [Kal17, Section 3.3]. We remark that the Pascal process is of Meixner's type (see [Lyt03b]).

The distribution of the Pascal process is a direct generalization of the product measure of negative binomial distributions, see (1.4), which is reversible for the SIP, see Section 1.1. Indeed, if E is a countable set and we identify α with the family $\alpha_i = \alpha(\{i\})$, $i \in E$ and similarly ξ with $\xi_i = \xi(\{\xi\})$, $i \in E$, then the distribution of ξ is given by $\otimes_{i \in E}\text{NegativeBinomial}(\alpha_i, p)$.

The Pascal process is a compound Poisson process. More precisely, if we consider a Poisson process η on $E \times \mathbb{N}$ with intensity measure $\alpha \otimes \sum_{k=1}^{\infty} \frac{p^k}{k}\delta_k$, then $\xi(A) := \int_{A \times \mathbb{N}} y\eta(\mathrm{d}(x, y))$ defines a Pascal process. In particular, the Pascal process can be constructed as a proper point process, meaning it takes the form (2.24).

Remark 3.3.1. In particular, if α is finite and non-zero, we can sample a Pascal process using the following procedure:

- For each $k \in \mathbb{N}$, let Y_k follow the logarithmic distribution with parameter p, i.e., $\mathbb{P}[Y_k = n] = \frac{1}{-\log(1-p)}\frac{p^n}{n!}$, $n \in \mathbb{N}$.
- For each $k \in \mathbb{N}$, let X_k follow the distribution $\frac{\alpha}{\alpha(E)}$.
- Let N follow the Poisson distribution with parameter $-\alpha(E)\log(1 - p) > 0$.

All variables are assumed to be independent. Then, $\sum_{k=1}^{N} Y_k \delta_{X_k}$ is a Pascal process with parameters p and α.

For sampling a Pascal processes with non-finite measure α, a superposition principle can be used, similar to that of the Poisson process. Specifically, if ξ_k, $k \in \mathbb{N}$ are independent Pascal processes with parameters p and α_k, then $\sum_{k=1}^{\infty} \xi_k$ is a Pascal process with parameters p and $\sum_{k=1}^{\infty} \alpha_k$.

It can readily be checked that the Laplace functional of a Pascal process is given by

$$\mathbb{E}\left[\mathrm{e}^{-\int u \, \mathrm{d}\xi}\right] = \exp\left(-\int \log\left(\frac{1 - p\mathrm{e}^{-u(x)}}{1-p}\right) \alpha(\mathrm{d}x)\right) \tag{3.27}$$

for measurable $u : E \to [0, \infty)$.

3.3.1 The Papangelou kernel

If X is a negative binomial variable with parameters $p \in (0, 1)$ and $a > 0$, then

$$(n+1)\mathbb{P}[X = n+1] = (1-p)^a \frac{a(a+1)\cdots(a+n)}{n!} p^{n+1} = p(a+n)\mathbb{P}[X = n]$$

for all $n \in \mathbb{N}_0$, accordingly

$$\mathbb{E}[Xf(X)] = \mathbb{E}[p(a+X)f(X+1)]$$

for all $f : \mathbb{N}_0 \to [0, \infty)$. The following proposition gives the analogous property for Pascal processes. For a general discussion on *Papangelou kernels*, we refer to [MWM79] or [Raf09, Section 1.2.2].

Proposition 3.3.2. *A Papangelou kernel of the Pascal process with parameters p and α is given by $p\kappa$ where*

$$\kappa : \mathbf{N} \times \mathcal{E} \to [0, \infty) \cup \{\infty\}, \quad \kappa(\mu, A) := (\mu + \alpha)(A), \quad \mu \in \mathbf{N}, A \in \mathcal{E}, \tag{3.28}$$

i.e.,

$$\iint F(\mu, x)\mu(\mathrm{d}x)\rho_{p,\alpha}(\mathrm{d}\mu) = p \iint F(\mu + \delta_x, x)(\mu + \alpha)(\mathrm{d}x)\rho_{p,\alpha}(\mathrm{d}\mu) \tag{3.29}$$

holds for each measurable function $F : \mathbf{N} \times E \to [0, \infty)$.

Proof of Proposition 3.3.2. Let $\beta := \alpha \otimes \sum_{n=1}^{\infty} \frac{p^n}{n} \delta_n$ and let η be a Poisson process with intensity measure β. Put for a counting measure μ on $E \times \mathbb{N}$ the measure $\xi_\mu \in \mathbf{N}$ defined by $\xi_\mu(A) := \int_{A \times \mathbb{N}} n\mu(\mathrm{d}(x, n))$, $A \in \mathcal{E}$. Then, ξ_η is a Pascal process. Mecke's formula (see, e.g., [LP17, Theorem 4.1]) yields

$$\begin{aligned}
\mathbb{E}\left[\int \varphi(\xi_\eta, x)\xi_\eta(\mathrm{d}x)\right] &= \mathbb{E}\left[\int n\varphi(\xi_\eta, x)\eta(\mathrm{d}(x, n))\right] = \mathbb{E}\left[\int n\varphi(\xi_{\eta + \delta_{(x,n)}}, x)\beta(\mathrm{d}(x, n))\right] \\
&= \mathbb{E}\left[\int n\varphi(\xi_\eta + n\delta_x, x)\beta(\mathrm{d}(x, n))\right] \\
&= \sum_{n=1}^{\infty} p^n \mathbb{E}\left[\int \varphi(\xi_\eta + n\delta_x, x)\alpha(\mathrm{d}x)\right]
\end{aligned} \tag{3.30}$$

for $\varphi : \mathbf{N} \times E \to [0,\infty)$ measurable. Using (3.30) for $\varphi(\mu, x) = F(\mu + \delta_x, x)$, we obtain

$$p\mathbb{E}\left[\int F(\xi_\eta + \delta_x, x)\xi_\eta(\mathrm{d}x)\right] = p\sum_{n=1}^{\infty} p^n \mathbb{E}\left[\int F(\xi_\eta + (n+1)\delta_x, x)\alpha(\mathrm{d}x)\right]$$
$$= \sum_{n=2}^{\infty} p^n \mathbb{E}\left[\int F(\xi_\eta + n\delta_x, x)\alpha(\mathrm{d}x)\right].$$

Therefore, using (3.30) again for $\varphi = F$, we have

$$\mathbb{E}\left[\int F(\xi_\eta, x)\xi_\eta(\mathrm{d}x)\right] = \sum_{n=1}^{\infty} p^n \mathbb{E}\left[\int F(\xi_\eta + n\delta_x, x)\alpha(\mathrm{d}x)\right]$$
$$= p\mathbb{E}\left[\int F(\xi_\eta + \delta_x, x)\alpha(\mathrm{d}x)\right] + p\mathbb{E}\left[\int F(\xi_\eta + \delta_x, x)\xi_\eta(\mathrm{d}x)\right]$$
$$= p\mathbb{E}\left[\int F(\xi_\eta + \delta_x, x)(\xi_\eta + \alpha)(\mathrm{d}x)\right]. \qquad \square$$

3.3.2 The measures λ_n

In this section, we define and examine measures λ_n on $(E^n, \mathcal{E}^{\otimes n})$ that play a role analogous to the product measures $\lambda^{\otimes n}$ in the Poisson case (Section 3.2). We define λ_n as follows:

$$\lambda_n(\mathrm{d}(x_1, \ldots, x_n)) := (\alpha + \delta_{x_1} + \cdots + \delta_{x_{n-1}})(\mathrm{d}x_n) \cdots (\alpha + \delta_{x_1} + \delta_{x_2})(\mathrm{d}x_3)(\alpha + \delta_{x_1})(\mathrm{d}x_2)\alpha(\mathrm{d}x_1). \quad (3.31)$$

Thus, λ_n is formed by adding points one by one: At each step, a new point either joins a pile of existing points or is placed at a new location. This relation on the one hand connects to the very definition of the dynamics of the generalized inclusion process, see Section 4.3 below, and on the other hand is connected to the *Chinese restaurant* process used in sequential constructions for random partitions [Pit06, Chapter 3]. Note that if α is finite, then normalizing λ_n by its total mass leads to the *Blackwell-MacQueen urn scheme* [BM73] which is closely related to the *Dirichlet process* [Fer73].

For example, for $n = 1, 2$, we have $\lambda_1 = \alpha$ and

$$\lambda_2(B) = \iint \mathbb{1}_B(x, y)\alpha(\mathrm{d}x)\alpha(\mathrm{d}y) + \int \mathbb{1}_B(x, x)\alpha(\mathrm{d}x)$$

for all $B \in \mathcal{E}^{\otimes 2}$.

Remark 3.3.3. The measures λ_n can be expressed by using partitions. We denote the set of partitions σ of the set $\{1, \ldots, n\}$ by Σ_n. Let $|\sigma|$ be the number of blocks of the partition σ. For a function $f_n : E^n \to \mathbb{R}$ denote by $(f_n)_\sigma : E^{|\sigma|} \to \mathbb{R}$ the function gained by identifying the variables belonging to the same $A \in \sigma$ in the order of occurrence, as already done in the proof of Proposition 3.1.2. Let α_σ be the measure defined by $\int f_n \,\mathrm{d}\alpha_\sigma = \int (f_n)_\sigma \,\mathrm{d}\alpha^{\otimes|\sigma|}$. Then, $\lambda_n = \sum_{\sigma \in \Sigma_n} \left(\prod_{A \in \sigma} (|A| - 1)!\right) \alpha_\sigma$.

The measures λ_n can be seen as a generalization of the univariate *rising factorial* $(a)^{(n)} = a(a+1)\cdots(a+n-1)$, $(a)^{(0)} = 1$ for $n \in \mathbb{N}$ and $a \in \mathbb{R}$, also known as *Pochhammer symbol*. More precisely, consider N pairwise disjoint sets $A_1, \ldots, A_N \in \mathcal{E}$ and $d_1, \ldots, d_N \in \mathbb{N}$. Then, we have

$$\lambda_{d_1+\ldots+d_N}(A_1^{d_1} \times \cdots \times A_N^{d_N}) = \alpha(A_1)^{(d_1)} \cdots \alpha(A_N)^{(d_N)}. \tag{3.32}$$

As soon as $\alpha(A_k) = 0$ for at least one k, the right-hand side becomes 0. If all $\alpha(A_k)$ are positive and $\alpha(A_k)$ is infinity for at least one k, then the right-hand side is interpreted as being infinity.

The measures λ_n are closely related to the Pascal process. Firstly, for finite α, when we condition a Pascal process on the n-particle sector, the particles follow the distribution of the normalized λ_n, as shown in Lemma 3.3.4 below. Secondly, the n-th factorial moment measures of a Pascal process coincide with λ_n up to some multiplicative constants, as presented in Proposition 3.3.5 below. Thirdly, they appear in the orthogonality relation of the orthogonal polynomials and thus in the Fock space decomposition, as discussed in Section 3.3.3 below.

Lemma 3.3.4. *If $\alpha(E) < \infty$, then $\rho_{p,\alpha}(\mathbf{N}_{<\infty}) = 1$. Moreover,*

$$\int F \,\mathrm{d}\rho_{p,\alpha} = (1-p)^{\alpha(E)}\Big(F(0) + \sum_{n=1}^{\infty} \frac{p^n}{n!} \int F(\delta_{x_1} + \ldots + \delta_{x_n})\lambda_n(\mathrm{d}(x_1, \ldots, x_n))\Big) \tag{3.33}$$

holds for all measurable $F : \mathbf{N}_{<\infty} \to [0, \infty)$.

Lemma 3.3.4 provides another method for sampling a Pascal process in addition to Remark 3.3.1. First, let N follow the negative binomial distribution with parameters p and $\alpha(E)$. Then, $\delta_{X_1} + \ldots + \delta_{X_N}$ is a Pascal process provided that $(X_1, \ldots, X_N) \in E^N$ follows the probability distribution $\frac{1}{\alpha(E)^{(N)}}\lambda_N$. To sample $X_1, \ldots, X_n$, random partitions can be used, as described in Remark 3.3.3.

Proof. Using multiple times the Papangelou kernel, see Proposition 3.3.2, and the definition of λ_n, see (3.31), together with $\rho_{p,\alpha}(\{0\}) = (1-p)^{\alpha(E)}$, we obtain

$$\begin{aligned}
&\int F(\mu)\mathbb{1}_{\{\mu(E)=n\}}\rho_{p,\alpha}(\mathrm{d}\mu) = \frac{1}{n!}\iint \mathbb{1}_{\{\mu(E)=n\}}F(\mu)\mu^{(n)}(\mathrm{d}(x_1, \ldots, x_n))\rho_{p,\alpha}(\mathrm{d}\mu) \\
&= \frac{p^n}{n!}\iint \cdots \int \mathbb{1}_{\{\mu(E)=0\}}F(\mu + \delta_{x_1} + \ldots + \delta_{x_n}) \\
&\qquad (\mu + \alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}x_2) \cdots (\mu + \alpha + \delta_{x_1})(\mathrm{d}x_2)(\mu + \alpha)(\mathrm{d}x_1)\rho_{p,\alpha}(\mathrm{d}\mu) \\
&= \rho_{p,\alpha}(\{0\})\frac{p^n}{n!}\int \cdots \int F(\delta_{x_1} + \ldots + \delta_{x_n}) \\
&\qquad (\alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}x_2) \cdots (\alpha + \delta_{x_1})(\mathrm{d}x_2)\alpha(\mathrm{d}x_1) \\
&= (1-p)^{\alpha(E)}\frac{p^n}{n!}\int F(\delta_{x_1} + \ldots + \delta_{x_n})\lambda_n(\mathrm{d}(x_1, \ldots, x_n))
\end{aligned}$$

for each $n \in \mathbb{N}$. Summing over n provides (3.33). □

Proposition 3.3.5. *The n-th factorial moment measure of a Pascal process ξ with parameters p and α is given by $\left(\frac{p}{1-p}\right)^n \lambda_n$, i.e., $\mathbb{E}\left[\xi^{(n)}(A)\right] = \left(\frac{p}{1-p}\right)^n \lambda_n(A)$ for $A \in \mathcal{E}^{\otimes n}$ and $n \in \mathbb{N}$.*

Proof. Let ξ be a Pascal process with parameters p and α and fix $A = A_1^{d_1} \times \cdots \times A_N^{d_N}$ where $A_1, \ldots, A_N \subset E$ are pairwise disjoint measurable sets and $d_1 + \ldots + d_N = n$. By combining (2.15) and (3.32) and using the fact that the k-th factorial moment of the negative binomial distribution with parameters p and a is $\left(\frac{p}{1-p}\right)(a)^{(k)}$ we obtain $\mathbb{E}\left[\int \mathbb{1}_A \, \mathrm{d}\mu^{(n)}\right] = \left(\frac{p}{1-p}\right)^n \lambda_n(A)$. This equation can be extended to all measurable subsets of E^n using standard measure-theoretic arguments. □

Remark 3.3.6. The distribution of the Pascal process is uniquely determined by its factorial moment measures. Indeed, the unique solvability of the moment problem follows by the criterion presented in [LP17, Proposition 4.12]. Fix a measurable bounded set $B \subset E$. Since $\frac{1}{2^n n!}\alpha(B)^{(n)} \to 0$ as $n \to \infty$, there exists a $D \geq 1$ such that $\frac{1}{2^n n!}\alpha(B)^{(n)} \leq D$ for all $n \in \mathbb{N}$. Therefore, we estimate the factorial moment measure as follows:

$$\left(\frac{p}{1-p}\right)^n \lambda_n(B^n) = \left(\frac{p}{1-p}\right)^n \alpha(B)^{(n)} \leq \left(2D\frac{p}{1-p}\right)^n n!.$$

3.3.3 Fock space decomposition

For each locally finite measure α and all $p \in (0,1)$, the distribution of the Pascal process $\rho_{p,\alpha}$ satisfies the moment condition (3.5), i.e., all moments of the number of particles in bounded sets are finite.

Definition 3.3.7. For each $f_n \in \mathcal{C}_n$, we denote by $\mathcal{M}_n^{p,\alpha} f_n$ the infinite-dimensional orthogonal polynomial of degree n with coefficient f_n, as defined in (3.6), with respect to the measure $\rho_{p,\alpha}$. We call $\mathcal{M}_n^{p,\alpha} f_n$ the *infinite-dimensional Meixner polynomial.*

The study of infinite-dimensional Meixner polynomials is not new, see, e.g., [Lyt03b] and the references therein. The following proposition is similar to [Lyt03b, Corollary 5.2] or [BLR15, Theorem 1.4] and generalizes the univariate orthogonality relation (1.8). We provide a self-contained proof that does not use the machinery of Jacobi fields and distribution theory. As a reminder, the symmetrization $\widetilde{f_n}$ of a function f_n is defined as in (2.11). We define $\int c \, \mathrm{d}\lambda_0 := c$ for any $c \in \mathbb{R}$.

Proposition 3.3.8. *The orthogonality relation*

$$\int (\mathcal{M}_n^{p,\alpha} f_n)(\mathcal{M}_m^{p,\alpha} g_m) \, \mathrm{d}\rho_{p,\alpha} = \mathbb{1}_{\{n=m\}} \frac{p^n n!}{(1-p)^{2n}} \int \widetilde{f_n}\widetilde{g_m} \, \mathrm{d}\lambda_n \tag{3.34}$$

holds true for $f_n \in \mathcal{C}_n$ and $g_m \in \mathcal{C}_m$, $n, m \in \mathbb{N}_0$.

Hence, the linear operator $\mathcal{M}_n^{p,\alpha}$ extends continuously to symmetric square-integrable functions which are denoted by $L^2_{\mathrm{sym}}(\lambda_n)$—we remind the reader that a function $f_n : E^n \to \mathbb{R}$ is called symmetric if $f_n(x_1,\dots,x_n) = f_n(x_{s(1)},\dots,x_{s(n)})$ for all $x_1,\dots,x_n$ and all permutations s. We define the operator

$$\mathfrak{U} : \mathfrak{F} \to L^2(\rho_{p,\alpha}), \quad f = (f_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha} f_n \tag{3.35}$$

where $(f_n)_{n\in\mathbb{N}_0}$ is contained in a space called the *extended anyon Fock space* or the *interacting Fock space*, as explored in [Lyt03a], [Lyt03b], [BLR15]. This space $\mathfrak{F} = \bigoplus_{n=0}^{\infty} \frac{p^n}{n!} L^2_{\mathrm{sym}}(\lambda_n)$ is defined as the space of sequences $(f_n)_{n\in\mathbb{N}_0}$ that satisfy the condition $\sum_{n=0}^{\infty} \frac{p^n}{n!} \|f_n\|^2_{L^2_{\mathrm{sym}}(\lambda_n)} < \infty$ where $L^2_{\mathrm{sym}}(\lambda_0)$ is understood as $\mathbb{R}$. The space is equipped with the inner product $\langle f, g\rangle = f_0 g_0 + \sum_{n=1}^{\infty} \frac{p^n}{n!} \int f_n g_n \,\mathrm{d}\lambda_n$ for $f = (f_n)_{n\in\mathbb{N}_0}, g = (g_n)_{n\in\mathbb{N}_0} \in \mathfrak{F}$.

Proposition 3.3.9. *The operator* $\mathfrak{U}$ *is unitary.*

The operator $\mathfrak{U}$ allows us to decompose any function $F \in L^2(\rho_{p,\alpha})$ into a series that converges in $L^2(\rho_{p,\alpha})$, given by $F = \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha} f_n$. The unitary operator $\mathfrak{U}$, in a slightly different setting, is the one employed in the Nualart-Schoutens chaos decomposition for the Pascal process, see, e.g., [NS00], [Lyt03a, Corollary 5.3] or [BLR15, Theorem 1.4]. To prove that $\mathfrak{U}$ is unitary, it can readily checked that it preserves the inner product by using (3.34). At the end of Section 3.3.4 below, we provide a brief argument for the bijectivity which is a self-contained proof and does not rely on the notion of distributions.

The generating function of monic univariate Meixner polynomials is given by

$$e_t(x,a) := \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} t^n \mathscr{M}_n(x;a;p) = \left(\frac{1}{1+pt}\right)^a \left(\frac{1+t}{1+pt}\right)^x \tag{3.36}$$

for all $t, a > 0$, $x \in \mathbb{N}_0$ and a fixed $p \in (0,1)$, see, e.g., [KLS10, Equation (9.10.11)]. The following proposition generalizes (3.36) and can be found in [BLR15, Theorem 1.3]. For a proof we also refer to [Lyt03b, Proposition 3.1]. We remind the reader of the notation $u^{\otimes n}(x_1,\dots,x_n) = u(x_1)\cdots u(x_n)$, $x_1,\dots,x_n \in E$.

Proposition 3.3.10. *The generating functional of infinite-dimensional Meixner polynomials is given by*

$$\sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha} u^{\otimes n}(\mu) = \exp\left(-\int \log(1+pu)\,\mathrm{d}\alpha + \int \log\left(\frac{1+u}{1+pu}\right)\mathrm{d}\mu\right) \tag{3.37}$$

for $\rho_{p,\alpha}$*-almost all* $\mu \in \mathbf{N}$ *and non-negative* $u \in \mathcal{C}_1$.

To prove the orthogonality relation stated in Proposition 3.3.8, we employ a measure-theoretical argument to extend the orthogonality of univariate Meixner polynomials to the infinite-dimensional case. For this purpose, we use the following proposition which is a variant of [Lyt03b, Lemma 3.1] and is analogous to relation (3.22) for the Poisson process.

Proposition 3.3.11. *Let $d_1, \dots, d_N \in \mathbb{N}$, $N \in \mathbb{N}$ and $B_1, \dots, B_N \in \mathcal{E}_\mathrm{b}$ be pairwise disjoint such that $\alpha(B_k) > 0$ for all $k \in \{1, \dots, N\}$. Then, $\mathcal{M}_n^{p,\alpha}$ with $n = d_1 + \ldots + d_N$ is related to univariate Meixner polynomials via*

$$\mathcal{M}_n^{p,\alpha}(\mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}})(\mu) = \prod_{k=1}^{N} \mathscr{M}_{d_k}(\mu(B_k); \alpha(B_k); p) \tag{3.38}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$.

Note that the generating function, which is given by (3.36), satisfies $e_t(x+y, a+b) = e_t(x,a)e_t(y,b)$ for each $t > 0$, $x, y \in \mathbb{N}_0$, $a, b > 0$. As a consequence, we get the convolution property (see, e.g., [AC76])

$$\mathscr{M}_n(x+y; a+b; p) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{M}_k(x; a; p) \mathscr{M}_{n-k}(y; b; p) \tag{3.39}$$

for all $n \in \mathbb{N}_0$.

Proof of Proposition 3.3.11. By the factorization property from Proposition 3.1.8, it is enough to show

$$\mathcal{M}_d^{p,\alpha} \mathbb{1}_{A^d}(\mu) = \mathscr{M}_d(\mu(A); \alpha(A); p) \tag{3.40}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$, all $d \in \mathbb{N}$ and $A \in \mathcal{E}_\mathrm{b}$ with $\alpha(A) > 0$. As we have chosen our univariate Meixner polynomials $\mathscr{M}_d$ to have leading coefficient one, we know that $\mathscr{M}_d(\mu(A); \alpha(A); p)$ is equal to $\mu(A)^d$ plus some polynomial in $\mu(A)$ of degree $\leq d - 1$. Therefore, (3.40) follows once we know that the map $\mu \mapsto \mathscr{M}_d(\mu(A); \alpha(A); p)$ is orthogonal to the space $\mathcal{P}_{d-1}$.

We check first that $\mu \mapsto \mathscr{M}_d(\mu(A); \alpha(A); p)$ is orthogonal in $L^2(\rho_{p,\alpha})$ to all maps $\mu \mapsto \mu^{\otimes m}(C)$, for every $m \leq d-1$ and $C \in \mathcal{E}^{\otimes m}$ with $C \subset A^m$.

When $C = A^m$, we are looking at two univariate polynomials in the variable $x = \mu(A)$ and the orthogonality relation follows from the orthogonality of the univariate Meixner polynomials $x \mapsto \mathscr{M}_d(x; \alpha(A); p)$ to the monomial $x \mapsto x^m$. The orthogonality to constant functions ($m = 0$) follows from univariate orthogonality as well. Next, consider the case $C = C_1^{d_1} \times \cdots \times C_N^{d_N}$ with $N \in \mathbb{N}$, $d_1, \dots, d_n \in \mathbb{N}$, $d_1 + \cdots + d_N \leq d-1$ and pairwise disjoint measurable sets $C_i \subset A$. Suppose first that $C_1 \cup \cdots \cup C_N = A$. We use the convolution property (3.39) and the complete independence of the Pascal process to find

$$\begin{aligned} &\int \mathscr{M}_d(\mu(A); \alpha(A); p) \mu^{\otimes m}(C) \rho_{p,\alpha}(\mathrm{d}\mu) \\ &\quad = \sum_{k_1 + \cdots + k_N = m} \binom{m}{k_1, \dots, k_N} \prod_{i=1}^{N} \int \mathscr{M}_{k_i}(\mu(C_i); \alpha(C_i); p) \mu^{\otimes d_i}(C_i) \rho_{p,\alpha}(\mathrm{d}\mu). \end{aligned} \tag{3.41}$$

In each summand, we must have $d_i < k_i$ for at least one $i \in \{1, \dots, N\}$ and therefore by the orthogonality of univariate Meixner polynomials, at least one of the integrals on the right-hand side above vanishes. As a consequence,

$$\int \mathscr{M}_d(\mu(A); \alpha(A); p)\mu^{\otimes m}(C)\rho_{p,\alpha}(\mathrm{d}\mu) = 0. \tag{3.42}$$

This holds true as well when each C_i is contained in A and $C_{N+1} := A \setminus (C_1 \cup \dots \cup C_N)$ is non-empty. In that case, we use a similar decomposition but now the sum on the right-hand side of (3.41) is over $(k_1, \dots, k_{N+1})$ and the product has an additional factor $\int \mathscr{M}_{k_{N+1}}(\mu(C_{N+1}); \alpha(C_{N+1}); p)\rho_{p,\alpha}(\mathrm{d}\mu)$.

Every Cartesian product $C = D_1 \times \dots \times D_m$ contained in A^m is a disjoint union of finitely many Cartesian products where any two factors are either disjoint or equal. Therefore, by linearity, the orthogonality relation (3.42) extends to all such sets. The functional monotone class theorem (see, e.g., [Bog07, Theorem 2.12.9]) yields the orthogonality of the infinite-dimensional Meixner polynomial to all maps of the form $\mu \mapsto \int f_m \,\mathrm{d}\mu^{\otimes m}$ with measurable bounded $f_m : E^m \to \mathbb{R}$ supported in A^m.

In the notation of Lemma 3.1.10, we checked the orthogonality of $\mathscr{M}_d(\mu(A); \alpha(A); p)$ to $\mathcal{P}_{d-1}(A)$. Using complete independence and arguments similar to those in the proof of Lemma 3.1.10, we conclude that the Meixner polynomial is in fact orthogonal to $\mathcal{P}_{d-1}$. □

Proof of Proposition 3.3.8. The orthogonality of $\mathcal{M}_n^{p,\alpha} f_n$ and $\mathcal{M}_m^{p,\alpha} g_m$ for $m \neq n$ is an immediate consequence of the definition of infinite-dimensional orthogonal polynomials, it does not use any properties of the Pascal process, see (3.7). Thus, we need only treat the case $m = n$.

Using linearity and the monotone class theorem as in the proof of Proposition 3.3.11, one finds that it suffices to show the orthogonality relation for functions $\widetilde{f_n}$, $\widetilde{g_n}$ that are symmetrized versions of indicator functions $f_n, g_n : E^n \to \mathbb{R}$ of the form

$$f_n = \mathbb{1}_{B_1^{d_1} \times \dots \times B_N^{d_N}}, \quad g_n = \mathbb{1}_{B_1^{d_1'} \times \dots \times B_N^{d_N'}}$$

with $B_1, \dots, B_N \in \mathcal{E}_\mathrm{b}$ pairwise disjoint and $\sum_{i=1}^N d_i = \sum_{i=1}^N d_i' = n$. Notice that $\mathcal{M}_n^{p,\alpha} \widetilde{f_n} = \mathcal{M}_n^{p,\alpha} f_n$ and $\mathcal{M}_n^{p,\alpha} \widetilde{g_n} = \mathcal{M}_n^{p,\alpha} g_n$ but in general $\int \widetilde{f_n} \widetilde{g_n} \,\mathrm{d}\lambda_n \neq \int f_n g_n \,\mathrm{d}\lambda_n$.

We remind the reader that $(a)^{(0)} = 1$ and $(a)^{(k)} = a(a+1)\cdots(a+k-1)$ denotes the rising factorial. Proposition 3.3.11, the complete independence and the orthogonality relation (1.8) for univariate Meixner polynomials yield

$$\int (\mathcal{M}_n^{p,\alpha} \widetilde{f_n})(\mathcal{M}_n^{p,\alpha} \widetilde{g_n})\rho_{p,\alpha} = \prod_{i=1}^N \mathbb{1}_{\{d_i = d_i'\}} \frac{d_i! p^{d_i}}{(1-p)^{2d_i}} (\alpha(B_i))^{(d_i)}. \tag{3.43}$$

If $d_i \neq d_i'$ for at least one i, then the right-hand side is zero, moreover $\widetilde{f_n g_n}$ vanishes identically. Hence, in that case

$$\int (\mathcal{M}_n^{p,\alpha} \widetilde{f_n})(\mathcal{M}_n^{p,\alpha} \widetilde{g_n}) \,\mathrm{d}\rho_{p,\alpha} = 0 = \int \widetilde{f_n g_n} \,\mathrm{d}\lambda_n$$

and the required equation holds true.

If $d_i = d_i'$ for all i, then $f_n = g_n$ on E^n. By (3.32), we have

$$\int f_n^2 \,\mathrm{d}\lambda_n = \lambda_n(B_1^{d_1} \times \cdots \times B_n^{d_n}) = \prod_{i=1}^{N} (\alpha(B_i))^{(d_i)}.$$

Hence, (3.43) implies

$$\int \left(\mathcal{M}_n^{p,\alpha} \widetilde{f_n}\right)^2 \,\mathrm{d}\rho_{p,\alpha} = \frac{p^n}{(1-p)^{2n}} \left(\prod_{i=1}^{N} d_i!\right) \int f_n^2 \,\mathrm{d}\lambda_n. \tag{3.44}$$

Next we check that the product of factorials on the right-hand side turns into $n!$ if f_n is replaced by $\widetilde{f_n}$. For $\sigma \in \mathfrak{S}_n$ and $x = (x_1, \ldots, x_n) \in E^n$, let $x_\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then, using that λ_n is invariant under permutation of the coordinates, we have

$$\int \widetilde{f_n}^2 \,\mathrm{d}\lambda_n = \frac{1}{(n!)^2} \sum_{\sigma,\tau \in \mathfrak{S}_n} \int f_n(x_\sigma) f_n(x_\tau) \lambda_n(\mathrm{d}x) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \int f_n(x_\pi) f_n(x) \lambda_n(\mathrm{d}x).$$

Since the sets B_i are pairwise disjoint, the product $f_n(x_\pi) f_n(x)$ vanishes unless π leaves the sets $\{1, \ldots, d_1\}$, $\{d_1 + 1, \ldots, d_1 + d_2 - 1\}$ etc. invariant, and in the latter case $f_n(x_\pi) f_n(x) = f_n(x)^2$. The number of relevant permutations is equal to $d_1! \cdots d_N!$. As a consequence,

$$\int \widetilde{f_n}^2 \,\mathrm{d}\lambda_n = \frac{1}{n!} \left(\prod_{i=1}^{N} d_i!\right) \int f_n^2 \,\mathrm{d}\lambda_n.$$

By (3.44), we get

$$\int \left(\mathcal{M}_n^{p,\alpha} \widetilde{f_n}\right)^2 \,\mathrm{d}\rho_{p,\alpha} = \frac{p^n n!}{(1-p)^{2n}} \int \widetilde{f_n}^2 \,\mathrm{d}\lambda_n$$

which is the required equation. □

3.3.4 Infinite-dimensional Meixner polynomials: an explicit formula

This section closely follows the exposition in [Wag24, Section 5.3] with the following difference: In this thesis, we relax the assumption that $E = \mathbb{R}$, more precisely, we allow arbitrary Borel spaces $(E, \mathcal{E})$ as well. All proofs remain the same and no technical issues arise.

The kernel $\kappa_{n,n-1} : E^{n-1} \times \mathcal{E} \to [0, \infty) \cup \{\infty\}$ is defined by

$$\kappa_{n,n-1}((x_1, \ldots, x_{n-1}), \cdot\,) := \alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}},$$

while $\kappa_{n,k} : E^k \times \mathcal{E}^{\otimes(n-k)} \to [0, \infty) \cup \{\infty\}$ is defined by $\kappa_{n,k} := \kappa_{k+1,k} \otimes \kappa_{k+2,k+1} \otimes \cdots \otimes \kappa_{n,n-1}$ for all $n > k > 0$. In other words,

$$\begin{aligned}\kappa_{n,k}((x_1, \ldots, x_k), \mathrm{d}(x_{k+1}, \ldots, x_n)) = (\alpha + \delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}x_n) \\ \cdots (\alpha + \delta_{x_1} + \ldots + \delta_{x_{k+1}})(\mathrm{d}x_{k+2})(\alpha + \delta_{x_1} + \ldots + \delta_{x_k})(\mathrm{d}x_{k+1}).\end{aligned} \tag{3.45}$$

The kernels $\kappa_{n,k}$ are closely related to the *Blackwell-MacQueen prediction rule* (see [BM73]) and satisfy

$$\lambda_n = \lambda_k \otimes \kappa_{n,k}. \tag{3.46}$$

The following proposition provides a novel explicit formula for infinite-dimensional Meixner polynomials. This formula plays a crucial role in proving intertwining relations for infinitely many particles, see Theorem 3.4.1 below, where we exploit its structure as a sum of integrals involving factorial measures and the kernels $\kappa_{n,k}$. Notably, our formula differs from the one presented in [Lyt03a, Equation (6.4)], which only provides a recursion. As a reminder, the symmetrization $\widetilde{f_n}$ of a function f_n is defined as in (2.11).

Proposition 3.3.12. *For each $n \in \mathbb{N}$ the infinite-dimensional Meixner polynomial of degree n is given by*

$$\mathcal{M}_n^{p,\alpha} f_n(\mu) = \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{1}{p}\right)^{k-n} \iint \widetilde{f_n}(x_1, \dots, x_n) \\ \kappa_{n,k}((x_1, \dots, x_k), \mathrm{d}(x_{k+1}, \dots, x_n)) \mu^{(k)}(\mathrm{d}(x_1, \dots, x_k)) \tag{3.47}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$ and $f_n \in \mathcal{C}_n$.

We interpret the case $n = k$ in (3.47) as though the inner integral was not present. If $k = 0$, $\kappa_{n,0}$ reduces to the measure λ_n on $(E^n, \mathcal{E}^{\otimes n})$, and we interpret the outer integral with respect to $\mu^{(0)}$ as though it was not present.

Thus, the first infinite-dimensional Meixner polynomials are given by

$$\begin{aligned}
\mathcal{M}_0^{p,\alpha} f_0(\mu) &= f_0 \\
\mathcal{M}_0^{p,\alpha} f_1(\mu) &= \int f_1 \, \mathrm{d}\mu - \frac{p}{1-p} \int f_1 \, \mathrm{d}\alpha \\
\mathcal{M}_0^{p,\alpha} f_2(\mu) &= \int f_2 \, \mathrm{d}\mu^{(2)} - \frac{p}{1-p} \int f_2 \, \mathrm{d}(\mu \otimes \alpha) - \frac{p}{1-p} \int f_2 \, \mathrm{d}(\alpha \otimes \mu) \\
&\quad - \frac{2p}{1-p} \int f_2(x,x) \mu(\mathrm{d}x) + \frac{p^2}{(1-p)^2} \int f_2 \, \mathrm{d}\alpha^{\otimes 2} + \frac{p^2}{(1-p)^2} \int f_2(x,x) \, \mathrm{d}\alpha(\mathrm{d}x)
\end{aligned}$$

for $f_0 \in \mathcal{C}_0 = \mathbb{R}$, $f_1 \in \mathcal{C}_1$ and $f_2 \in \mathcal{C}_2$ for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$.

The explicit formula for infinite-dimensional Meixner polynomials, as given by (3.47) once again demonstrates that these polynomials represent a natural generalization of the monic univariate Meixner polynomials defined in (1.7). When going to the infinite-dimensional version of these polynomials, the rising factorial $(a+k)^{(n-k)} = (a+k)(a+k+1)\cdots(a+n-1)$ turns into an integration with respect to the kernel $\kappa_{n,k}$, while the falling factorial $(x)_k = x(x-1)\cdots(x-k+1)$ occurs as an integration with respect to the factorial measure $\mu^{(k)}$.

Our strategy for the proof is as follows: On the one hand, we already know from Proposition 3.3.11 that the infinite-dimensional orthogonal polynomial for specific f_n

reduce to a product of univariate Meixner polynomials. On the other hand, we prove that the right-hand side of (3.47) has the same product structure for this f_n, as stated in Proposition 3.3.15 below. For this proposition we need two preliminary lemmas. The equation then is obtained through standard measure-theoretical arguments.

Fix a partition of measurable sets of $B_1, \ldots, B_N$ of E, $n < m$, $z_1, \ldots, z_n \in E$ and put $c_k := (\delta_{z_1} + \ldots + \delta_{z_n})(B_k)$ for $k \in \{1, \ldots, N\}$.

Lemma 3.3.13. *Let* $i_{n+1}, \ldots, i_m \in \{1, \ldots, N\}$ *and put* $e_k := \sum_{l=n+1}^{m} \mathbb{1}_{\{i_l = k\}}$ *for* $k \in \{1, \ldots, N\}$*. Then,*

$$\int \mathbb{1}_{B_{i_{n+1}} \times \ldots \times B_{i_m}}(y_{n+1}, \ldots, y_m)\, \kappa_{m,n}((z_1, \ldots, z_n), \mathrm{d}(y_{n+1}, \ldots, y_m)) = \prod_{k=1}^{N} (\alpha(B_k) + c_k)^{(e_k)}$$

holds true.

Thereby, we put $\infty^{(k)} := \infty$ for $k \geq 1$ and $\infty^{(0)} := 0$.

Proof. We prove the equation by induction over m. For $m = n + 1$ the statement is a direct consequence of the definition of $\kappa_{n+1,n}$. Assume that the statement is true for some fixed $m > n$. Let

$$\sum_{l=n+1}^{m+1} \mathbb{1}_{\{i_l = k\}} = e_k + \mathbb{1}_{\{k = i_{m+1}\}}$$

for an arbitrary $i_{m+1} \in \{1, \ldots, N\}$. Then, by (3.45),

$$\begin{aligned}
&\int \mathbb{1}_{B_{i_{n+1}} \times \ldots B_{i_{m+1}}}(y_{n+1}, \ldots, y_{m+1}) \kappa_{m+1,n}((z_1, \ldots, z_n), \mathrm{d}(y_{n+1}, \ldots, y_{m+1})) \\
&\quad = \int \mathbb{1}_{B_{i_{n+1}} \times \ldots \times B_{i_m}}(y_{n+1}, \ldots, y_m) \kappa_{m+1,m}((z_1, \ldots, z_n, y_{n+1}, \ldots, y_m), B_{i_{m+1}}) \\
&\qquad\qquad \kappa_{m,n}((z_1, \ldots, z_n), \mathrm{d}(y_{n+1}, \ldots, y_m)) \\
&\quad = (\alpha(B_{i_{m+1}}) + c_{i_{m+1}} + e_{i_{m+1}}) \prod_{k=1}^{N} (\alpha(B_k) + c_k)^{(e_k)} \\
&\quad = \prod_{k=1}^{N} (\alpha(B_k) + c_k)^{(e_k + \mathbb{1}_{\{k = i_{m+1}\}})}.
\end{aligned}$$

□

Lemma 3.3.14. *Let* $d_1, \ldots, d_N \in \mathbb{N}_0$ *with* $d_1 + \ldots + d_N = m$*. Then,*

$$\int \widetilde{\mathbb{1}}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}(z_1, \ldots, z_n, y_{n+1}, \ldots, y_m)\, \kappa_{m,n}((z_1, \ldots, z_n), \mathrm{d}(y_{n+1}, \ldots, y_m)) = \frac{1}{(m)_n} \prod_{k=1}^{N} (d_k)_{c_k} (\alpha(B_k) + c_k)^{(d_k - c_k)} \tag{3.48}$$

holds where $\widetilde{\mathbb{1}}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}$ *denotes the symmetrization of* $\mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}$*.*

If there exists a k with $d_k < c_k$, then $(d_k)_{c_k}$ becomes zero, leading to the right-hand side of (3.48) being zero. Moreover, if there exists a k for which $d_k > c_k$ and $\alpha(B_k) = \infty$, the right-hand side of (3.48) is equal to infinity.

Proof. We decompose the integral into

$$\begin{aligned}
&\int \widetilde{\mathbb{1}}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(z_1,\ldots,z_n,y_{n+1},\ldots,y_m)\kappa_{m,n}((z_1,\ldots,z_n),\mathrm{d}(y_{n+1},\ldots,y_m)) \\
&= \sum_{i_{n+1},\ldots,i_m\in\{1,\ldots,N\}} \int \mathbb{1}_{B_{i_{n+1}}\times\ldots\times B_{i_m}}(y_{n+1},\ldots,y_m) \\
&\qquad \widetilde{\mathbb{1}}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(z_1,\ldots,z_n,y_{n+1},\ldots,y_m)\kappa_{m,n}((z_1,\ldots,z_n),\mathrm{d}(y_{n+1},\ldots,y_m)). \qquad (3.49)
\end{aligned}$$

If $x_1,\ldots,x_m \in E$ are given such that $(\delta_{x_1}+\ldots+\delta_{x_m})(B_k) = d_k$ for all $k \in \{1,\ldots,N\}$, then the symmetrization $\widetilde{\mathbb{1}}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(x_1,\ldots,x_m)$ is equal to $\frac{d_1!\cdots d_N!}{m!}$. Otherwise, it is equal to zero. Thus, the statement $\widetilde{\mathbb{1}}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(z_1,\ldots,z_n,y_{n+1},\ldots,y_m) > 0$ is equivalent to $\sum_{l=n+1}^m \mathbb{1}_{\{i_l=k\}} + c_k = d_k$. By applying Lemma 3.3.13 to (3.49), we obtain:

$$\begin{aligned}
&\frac{d_1!\cdots d_N!}{m!} \sum_{\substack{i_{n+1},\ldots,i_m\in\{1,\ldots,N\} \\ \sum_{l=n+1}^m \mathbb{1}_{\{i_l=k\}}+c_k=d_k}} \int \mathbb{1}_{B_{i_{n+1}}\times\ldots\times B_{i_m}}(y_{n+1},\ldots,y_m) \\
&\qquad\qquad \kappa_{m,n}((z_1,\ldots,z_n),\mathrm{d}(y_{n+1},\ldots,y_m)) \\
&= \frac{d_1!\cdots d_N!}{m!} \sum_{\substack{i_{n+1},\ldots,i_m\in\{1,\ldots,N\} \\ \sum_{l=n+1}^m \mathbb{1}_{\{i_l=k\}}+c_k=d_k}} \prod_{k=1}^N (\alpha(B_k)+c_k)^{(d_k-c_k)}.
\end{aligned}$$

Finally, using the identity

$$\sum_{\substack{i_{n+1},\ldots,i_m\in\{1,\ldots,N\} \\ \sum_{l=n+1}^m \mathbb{1}_{\{i_l=k\}}+c_k=d_k}} 1 = (m-n)! \prod_{k=1}^N \mathbb{1}_{\{d_k\ge c_k\}} \frac{1}{(d_k-c_k)!}$$

we conclude the proof. □

We define $\hat{\mathcal{M}}_n^{p,\alpha} f_n$ as the expression on the right-hand side of (3.47). Recall that the monic univariate Meixner polynomials are given by (1.7).

Lemma 3.3.15. *Let $d_1,\ldots,d_N \in \mathbb{N}_0$ with $d_1+\ldots+d_N = m$ and $\mu \in \mathbf{N}$. Assume $\mu(B_k) < \infty$ and $0 < \alpha(B_k) < \infty$ for all k with $d_k > 0$. Then, the equation*

$$\hat{\mathcal{M}}_m^{p,\alpha} \mathbb{1}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(\mu) = \prod_{k=1}^N \mathscr{M}_{d_k}^{p,\alpha(A_k)}(\mu(B_k))$$

is satisfied.

Proof. Let $b_k := \mu(B_k) \in \mathbb{N}_0$. By applying Lemma 3.3.13 and the definition of the factorial measure $\mu^{(n)}$, we obtain

$$\iint \widetilde{\mathbb{1}}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(z_1,\dots,z_n,y_{n+1},\dots,y_m) \\ \kappa_{m,n}((z_1,\dots,z_n),\mathrm{d}(y_{n+1},\dots,y_m))\mu^{(n)}(\mathrm{d}(z_1,\dots,z_n)) \\ = \frac{n!}{(m)_n}\sum_{\substack{c_1,\dots,c_N\in\mathbb{N}_0\\ c_1+\ldots+c_N=n}}\prod_{k=1}^{N}\frac{(b_k)_{c_k}}{c_k!}(d_k)_{c_k}\,(\alpha(B_k)+c_k)^{(d_k-c_k)}.$$

It follows that

$$\hat{\mathcal{M}}_m^{p,\alpha}\mathbb{1}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}(\mu) = \prod_{k=1}^{N}\sum_{c_k=d_k}^{\infty}\left(1-\frac{1}{p}\right)^{c_k-d_k}\frac{(b_k)_{c_k}}{c_k!}(d_k)_{c_k}\,(\alpha(B_k)+c_k)^{(d_k-c_k)} \\ = \prod_{k=1}^{N}\mathscr{M}_{d_k}^{p,\alpha(B_k)}(\mu(B_k)).$$

□

Proof of Proposition 3.3.12. Fix $m\in\mathbb{N}$ and a bounded set $B\subset E$. We consider the function

$$f_m = \mathbb{1}_{B_1^{d_1}\times\cdots\times B_N^{d_N}} \tag{3.50}$$

where $B_1,\dots,B_N$ are pairwise disjoint and measurable subsets of B, $d_1+\ldots+d_N=m$ and $d_1,\dots,d_N\in\mathbb{N}$. We claim that (3.47) holds true for the function f_m.

Indeed, if $\alpha(B_k)>0$ for all k, then applying Proposition 3.3.11 and Lemma 3.3.15 yields

$$\mathcal{M}_m^{p,\alpha}f_m(\mu) = \prod_{k=1}^{N}\mathscr{M}_{d_k}^{p,\alpha(A_k)}(\mu(B_k)) = \hat{\mathcal{M}}_m^{p,\alpha}f_m(\mu)$$

for $\rho_{p,\alpha}$-almost all $\mu\in\mathbf{N}$.

On the other hand, if there exists a $k\in\{1,\dots,N\}$ such that $\alpha(B_k)=0$, then $f_m=\mathbb{1}_{B_1^{d_1}\times\cdots\times B_N^{d_N}}=0$ λ_m-almost everywhere. Therefore, using the definition of $\hat{\mathcal{M}}_m^{p,\alpha}$, Proposition 3.3.5 and (3.46), we obtain

$$\int\left|\hat{\mathcal{M}}_m^{p,\alpha}f_m\right|\mathrm{d}\rho_{p,\alpha} \le \sum_{k=0}^{m}\binom{m}{k}\left(\frac{p}{1-p}\right)^m\int\widetilde{f_m}\,\mathrm{d}\lambda_m = 0$$

which implies $\hat{\mathcal{M}}_m^{p,\alpha}f_m=0$ $\rho_{p,\alpha}$-almost surely. Furthermore, $\mathcal{M}_m^{p,\alpha}$ is a well-defined linear operator from $L^2(\lambda_n)$ to $L^2(\rho_{p,\alpha})$, as shown by the orthogonality relation (3.34) for $\mathcal{M}_m^{p,\alpha}$. Thus, $\hat{\mathcal{M}}_m^{p,\alpha}f_m = 0 = \mathcal{M}_m^{p,\alpha}f_m$ $\rho_{p,\alpha}$-almost surely.

Using the functional monotone class theorem (see, e.g., [Bog07, Theorem 2.12.9]) we obtain that (3.34) holds true for $f_m \in \mathcal{C}_m$ as well. More precisely, let

$$\mathcal{H} = \left\{ g_m : B \to \mathbb{R} \text{ measurable bounded} : \hat{\mathcal{M}}_m^{p,\alpha} \hat{g}_m = \mathcal{M}_m^{p,\alpha} \hat{g}_m \right\}$$

where $\hat{g}_m$ is defined to be equal to g_m on B and equal to zero on $E \setminus B$. Then, $\mathcal{H}$ contains the constant functions and is closed with respect the formation of uniform and monotone limits. Furthermore, the set

$$\mathcal{H}_0 = \{ \mathbb{1}_{C_1 \times \cdots \times C_m} : C_1, \ldots, C_m \subset B \text{ measurable} \}$$

is closed under forming products and is contained in $\mathcal{H}$ since $\widetilde{\mathbb{1}}_{C_1 \times \cdots \times C_m}$ can be written as a linear combination of symmetrizations of functions of the type (3.50). Therefore, by the functional monotone class theorem, $\mathcal{H}$ equals the set of all measurable bounded functions on B. $\square$

Proof of Proposition 3.3.9. Since $\mathfrak{U}$ preserves the inner product, a consequence of (3.34), only the fact that $\mathfrak{U}$ is bijective requires a proof. First, we observe that the space of polynomials $\mathcal{P} := \cup_{n=0}^{\infty} \mathcal{P}_n$ is dense in $L^2(\rho_{p,\alpha})$. This can be proved by standard arguments: The linear hull of exponentials of the form $\mu \mapsto e_u(\mu) := \mathrm{e}^{-\int u \, \mathrm{d}\mu}$, where $u \in \mathcal{C}_1$ is non-negative, is dense in $L^2(\rho_{p,\alpha})$ thanks to the functional monotone class theorem (see, e.g., [Bog07, Theorem 2.12.9]). Using the series expansion of the exponential function, we can approximate e_u by polynomials. The fact that this approximation also converges in $L^2(\rho_{p,\alpha})$ follows by a growth condition on the moments of the Pascal process.

Thus, each $F \in L^2(\rho_{p,\alpha})$ can be approximated by a sequence $P^{(N)} \in \mathcal{P}$, $N \in \mathbb{N}$ as $N \to \infty$. Using the explicit formula for infinite-dimensional Meixner polynomials, see Proposition 3.3.12, and arguments from (3.19), we obtain $P^{(N)} = \sum_{k=0}^{n^{(N)}} \frac{(1-p)^k}{k!} \mathcal{M}_k^{p,\alpha} f_k^{(N)}$ for some symmetric $f_k^{(N)} \in \mathcal{C}_k$, $n^{(N)} \in \mathbb{N}$. In other words, there are $f^{(N)} \in \mathfrak{F}$ such that $P^{(N)} = \mathfrak{U} f^{(N)}$. Using the fact that $\mathfrak{U}$ is norm preserving, we obtain that $f^{(N)}$ is a Cauchy sequence and $f := \lim_{N \to \infty} f^{(N)}$ satisfies $\mathfrak{U} f = F$. $\square$

3.3.5 Self-intertwining relations

This section is entirely analogous to Section 3.2.2 in the Poisson case. It describes how the orthogonal polynomial intertwining relation (IR.2), which states that $\mathcal{M}_n^{p,\alpha}$ intertwines the semigroups P_t and $(P_t^{[n]})$, can be reformulated into a self-intertwining relation involving only a unitary operator and the semigroup $(P_t)_{t \geq 0}$.

Let $(\Omega, \mathcal{F}, (\eta_t)_{t \geq 0}, (\mathbb{P}_\mu)_{\mu \in \mathbf{N}_{<\infty}})$ (or $(\Omega, \mathcal{F}, (\eta_t)_{t \geq 0}, (\mathbb{P}_\mu)_{\mu \in \mathbf{N}})$, respectively) be a Markov family and assume that $\rho_{p,\alpha}$ with finite (or locally finite, respectively) α is reversible. By (IR.2), the operator $P_t^{\mathfrak{F}} = \mathfrak{U}^{-1} P_t \mathfrak{U} : \mathfrak{F} \to \mathfrak{F}$, where $\mathfrak{U}$ is defined in (3.35), satisfies $(P_t^{\mathfrak{F}} f)_n = P_t^{[n]} f_n$ for all $f \in \mathfrak{F}$ and $n \in \mathbb{N}_0$. In particular, $P_t^{[n]}$ is a self-adjoint operator on $L^2_{\mathrm{sym}}(\lambda_n)$ for each $n \in \mathbb{N}$, $t \geq 0$.

As a reminder, $\iota_n(x_1,\dots,x_n) = \delta_{x_1} + \dots + \delta_{x_n}$ where $x_1,\dots,x_n \in E$. If α is finite, then, as a consequence of Lemma 3.3.4, the operator

$$\mathfrak{V} : L^2(\rho_{p,\alpha}) \to \mathfrak{F}, \quad F \mapsto (1-p)^{\frac{1}{2}\alpha(E)}\,(f_n)_{n\in\mathbb{N}_0}, \text{ where } f_n := F \circ \iota_n, \tag{3.51}$$

is unitary. It satisfies the intertwining relation $\mathfrak{V}P_t = P_t^{\mathfrak{F}}\mathfrak{V}$ which leads to the self-intertwining relation

$$\mathfrak{U}\mathfrak{V}P_t = P_t\mathfrak{U}\mathfrak{V}, \tag{3.52}$$

or, for the semigroup $(P_t^{\mathfrak{F}})_{t\geq 0}$, to the self-intertwining relation

$$\mathfrak{V}\mathfrak{U}P_t^{\mathfrak{F}} = P_t^{\mathfrak{F}}\mathfrak{V}\mathfrak{U}. \tag{3.53}$$

3.4 Intertwining for infinite particle systems

In this section, we focus on the orthogonal polynomial intertwining relation (IR.2) in the context of infinitely many particles in an uncountable space. First, it should be noted that the results presented in Section 3.1.2, in particular the one in Theorem 3.1.6, can apply to systems with an infinite number of particles, subject to the limitation that a reversible measure ρ is needed. Obtaining a reversible measure for the infinite dynamics can often be difficult. In this section, we present a novel approach that overcomes this obstacle. We show that it is not necessary to know that ρ is reversible if we have knowledge of reversible measures for the n-particle dynamics instead. Along with consistency, which we defined for infinite particle systems in Section 2.2, we find that the infinite-dimensional orthogonal polynomial of degree n intertwines the dynamics of infinitely many particles and the dynamics of n particles where $n < \infty$. Our proof relies on explicit formulas for the infinite-dimensional orthogonal polynomials. Therefore, we restrict our considerations to the infinite-dimensional polynomials with respect to the distribution of the Poisson and the Pascal process.

We illustrate our procedure by presenting examples of strongly consistent systems, specifically focusing on correlated and sticky Brownian motions in Section 4.4 and Section 4.5 below. Orthogonal intertwiners for infinitely many particles have practical applications. In particular, we obtain new reversible measures for the dynamics of infinitely many particle systems.

This section closely follows the exposition in [Wag24, Section 4.2] with the following difference: In this thesis, we relax the assumption that $E = \mathbb{R}$, more precisely, we allow arbitrary Borel spaces $(E, \mathcal{E})$ as well. All proofs remain the same and no technical issues arise.

Let $(\Omega,\, \mathcal{F},\, (\eta_t)_{t\geq 0},\, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ be a Markov family with state space $\mathbf{N}$ such that η_t is proper for each $t \geq 0$. Recall the map $\iota_n : E^n \to \mathbf{N}$, $(x_1,\dots,x_n) \mapsto \delta_{x_1} + \dots + \delta_{x_n}$ and note that π_λ denotes the distribution of the Poisson process with intensity measure λ while $\rho_{p,\alpha}$ denotes the distribution of the Pascal process with parameters p and α, see Section 3.3. We recall the fact that a consistent Markov family is conservative, see

Definition 2.2.1. Furthermore, we remember the reader that the orthogonal polynomial intertwining relation (IR.2) states $P_t I_n f_n(\mu) = I_n P_t^{[n]} f_n(\mu)$ where $P_t F(\mu) = \mathbb{E}_\mu [F(\eta_t)]$, $P_t^{[n]}$ denotes the n-particle semigroup, see (3.11), and I_n is the infinite-dimensional orthogonal polynomial, see (3.6).

Theorem 3.4.1. *Assume that $(\eta_t)_{t\geq 0}$ is consistent.*

(i) *Let λ be a locally finite measure on $(E, \mathcal{E})$. Suppose that the push-forward measure of $\lambda^{\otimes n}$ under the map ι_n is reversible for $(\eta_t)_{t\geq 0}$ for each $n \in \mathbb{N}$. Then,* (IR.2) *holds for π_λ-almost all μ, for all $t \geq 0$, $f_n \in L^2_{\mathrm{sym}}(\lambda^{\otimes n})$ and $n \in \mathbb{N}_0$ with I_n the infinite-dimensional orthogonal polynomial with respect to π_λ.*

(ii) *Let α be a locally finite measure on $(E, \mathcal{E})$ and $p \in (0,1)$. Suppose that the push-forward measure of λ_n, defined in* (3.31)*, under the map ι_n is reversible for $(\eta_t)_{t\geq 0}$ for each $n \in \mathbb{N}$. Then,* (IR.2) *holds for $\rho_{p,\alpha}$-almost all μ, for all $t \geq 0$, $f_n \in L^2_{\mathrm{sym}}(\lambda_n)$ and $n \in \mathbb{N}_0$ with $I_n = \mathcal{M}_n^{p,\alpha}$ the infinite-dimensional Meixner polynomial.*

We remind the reader that $f_n : E^n \to \mathbb{R}$ is called symmetric if $f_n(x_1, \ldots, x_n) = f_n(x_{s(1)}, \ldots, x_{s(n)})$ for all $x_1, \ldots, x_n$ and all permutations s. In terms of $(P_t^{[n]})_{t\geq 0}$, the condition that the push-forward measure of $\lambda^{\otimes n}$ under the map ι_n is reversible is equivalent to the condition $\int (P_t^{[n]} f_n) g_n \, \mathrm{d}\lambda^{\otimes n} = \int (P_t^{[n]} g_n) f_n \, \mathrm{d}\lambda^{\otimes n}$ for all $t \geq 0$ and measurable symmetric $f_n, g_n : E^n \to [0, \infty)$. The situation is analogous for λ_n.

It is important to note that we only benefit from Theorem 3.4.1 if λ (or α) is a non-finite measure. In this case, we have $\pi_\lambda(\mathbf{N}_{<\infty}) = 0$ (or $\rho_{p,\alpha}(\mathbf{N}_{<\infty}) = 0$) and consequently, (IR.2) holds for infinite configurations. Therefore, the infinite-dimensional polynomial intertwines the dynamics of infinitely many particles with their dynamics for $n < \infty$ particles.

If λ (or α) is finite, the assumption in Theorem 3.4.1 (i) (or (ii)) is equivalent to the reversibility of π_λ (or $\rho_{p,\alpha}$) and thus the orthogonal polynomial intertwining relation (IR.2) is already ensured by Theorem 3.1.6. Nevertheless, in this case, the proof of Theorem 3.4.1 provides an alternative route.

First, we verify that under the assumptions of Theorem 3.4.1, $P_t, t \geq 0$ are well-defined bounded operators on $L^2(\pi_\lambda)$ (or $L^2(\rho_{p,\alpha})$). This is accomplished by demonstrating that π_λ (or $\rho_{p,\alpha}$) serves as an invariant measure, i.e.,

$$\int \mathbb{E}_\mu [F(\eta_t)] \, \pi_\lambda(\mathrm{d}\mu) = \int F(\mu) \pi_\lambda(\mathrm{d}\mu) \tag{3.54}$$

for all measurable $F : \mathbf{N} \to [0, \infty)$ and $t \geq 0$. Note that (3.54) is well-defined, supported by arguments analogous to those outlined in Remark 3.1.5. However, there is a stronger statement: Under the assumptions of Theorem 3.4.1, we obtain that π_λ (or $\rho_{p,\alpha}$) is even reversible, see Corollary 3.4.3 below.

Proposition 3.4.2. *Assume that $(\eta_t)_{t\geq 0}$ is consistent.*

(i) *If the push-forward measure of $\lambda^{\otimes n}$ under the map ι_n is invariant for each $n \in \mathbb{N}$, then π_λ is invariant.*

(ii) *If the push-forward measure of λ_n under the map ι_n is invariant for each $n \in \mathbb{N}$, then $\rho_{p,\alpha}$ is invariant.*

If $(\eta_t)_{t\geq 0}$ describes the evolution of independent particles (see Section 4.2 below), Proposition 3.4.2 (i) is a version of Doob's theorem (cf. [DP91, Theorem 2.9.5]) or of the displacement theorem (cf. [Kin93, page 61]). Proposition 3.4.2 is a straightforward consequence of the consistency property.

Proof. We first prove implication (i). To show (3.54), we use the fact that the moment problem for the Poisson process is uniquely solvable as shown, e.g., in [LP17, Proposition 4.12]. Consequently, it suffices to check that the factorial moment measures of a Poisson process ξ, given by $\lambda^{\otimes n}$, and η_t starting at ξ coincide. Using invariance of the push-forward measure of $\lambda^{\otimes n}$ under ι_n along with consistency, i.e., the factorial measure intertwining relation (IR.1), we obtain

$$\int f_n \, \mathrm{d}\lambda^{\otimes n} = \int P_t^{[n]} f_n \, \mathrm{d}\lambda^{\otimes n} = \iint P_t^{[n]} f_n \, \mathrm{d}\mu^{(n)} \pi_\lambda(\mathrm{d}\mu) = \int \mathbb{E}_\mu \left[\int f_n \, \mathrm{d}\eta_t^{(n)} \right] \pi_\lambda(\mathrm{d}\mu)$$

for measurable $f_n : E^n \to [0, \infty)$ and $t \geq 0$. Thus, π_λ is indeed invariant for $(\eta_t)_{t\geq 0}$.

The implication (ii) follows similarly: We have knowledge of the factorial moment measures, see Proposition 3.3.5, and we know that the moment problem is uniquely solvable, as stated in Remark 3.3.6. □

We emphasize that the argument for proving invariance is a general principle: Let the push-forward of the n-th factorial moment measure of an (infinite) point process ζ under ι_n be invariant for the dynamics of a consistent particle system for all $n \in \mathbb{N}$. Assume that the factorial moment measures uniquely characterize the distribution of ζ. Then, the distribution of ζ is invariant for the (infinite) dynamics.

Subsequently, by Proposition 3.4.2 and standard arguments involving Jensen's inequality, the semigroup $P_t F(\mu) = \mathbb{E}_\mu [F(\eta_t)]$ is well-defined for $L^2(\pi_\lambda)$-equivalence classes and satisfies

$$\|P_t F\|_{L^2(\pi_\lambda)} \leq \|F\|_{L^2(\pi_\lambda)}, \tag{3.55}$$

i.e., P_t is a bounded operator on $L^2(\pi_\lambda)$. For the Pascal case, the fact that P_t is well-defined and bounded on $L^2(\rho_{p,\alpha})$ follows analogously.

A reformulation of Theorem 3.4.1 is that a unitary transformation of the Markov semigroup of the unlabeled infinite dynamics leads to the family of the n-particle Markov semigroups, similarly as done Section 3.2.2 and Section 3.3.5. More precisely, for the Poisson case, we consider the operator $P_t^{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{F}$ that maps $(f_n)_{n\in\mathbb{N}_0}$ to $(P_t^{[n]} f_n)_{n\in\mathbb{N}_0}$ where $P_t^{[0]} f_0 := f_0$, $f_0 \in \mathbb{R}$. This operator acts on the Fock space, defined in (3.20). According to the assumptions made in Theorem 3.4.1 (i), we know that $P_t^{\mathfrak{F}}$ is a well-defined bounded self-adjoint operator for every $t > 0$. Hence, $\mathfrak{U} P_t^{\mathfrak{F}} \mathfrak{U}^{-1}$ is also self-adjoint where the operator $\mathfrak{U} : \mathfrak{F} \to L^2(\pi_\lambda)$ is defined in (3.21). Theorem 3.4.1 implies that

$$P_t F(\mu) = \mathfrak{U} P_t^{\mathfrak{F}} \mathfrak{U}^{-1} F(\mu) \tag{3.56}$$

holds for π_λ-almost all $\mu \in \mathbf{N}$ and for $F = I_n f_n$ where $f_n \in L^2_{\mathrm{sym}}(\lambda^{\otimes n})$. By an approximation argument that uses the fact that P_t is a contraction, see (3.55), (3.56) follows for all $F \in L^2(\pi_\lambda)$. In particular, $(P_t)_{t\geq 0}$ is a self-adjoint operator. The Pascal case follows analogously. That results in the following corollary.

Corollary 3.4.3. *Under the assumptions of Theorem 3.4.1, if we assume the condition stated in* (i)*, then π_λ is reversible for $(\eta_t)_{t\geq 0}$, whereas if we assume the condition stated in* (ii)*, then $\rho_{p,\alpha}$ is reversible for $(\eta_t)_{t\geq 0}$.*

To begin, we present a proof for part (i) of Theorem 3.4.1. The crucial steps in the following proof are as follows: first, we establish the intertwining relation for symmetric functions in the smaller space $\mathcal{C}_n$ using the explicit formulas for the orthogonal polynomials (3.17) and (3.47). Next, we extend this relation to all functions in $L^2(\lambda^{\otimes n})$ using an approximation argument.

Proof of Theorem 3.4.1 (i). We claim the following equation: for all $t \geq 0$, $l \in \mathbb{N}_0$ and measurable $F : \mathbf{N}_{l+1} \to [0,\infty)$,

$$\int \mathbb{E}_{\delta_{z_1}+\ldots+\delta_{z_l}+\delta_y}\left[F(\eta_t)\right]\lambda(\mathrm{d}y) = \int \mathbb{E}_{\delta_{z_1}+\ldots+\delta_{z_l}}\left[F(\eta_t+\delta_y)\right]\lambda(\mathrm{d}y) \tag{3.57}$$

holds for $z = (z_1,\ldots,z_l) \in E^l$ $\lambda^{\otimes l}$-almost everywhere. The case where $l = 0$ reads as follows: $\int \mathbb{E}_{\delta_y}\left[F(\eta_t)\right]\lambda(\mathrm{d}y) = \int F(\delta_y)\lambda(\mathrm{d}y)$.

To prove (3.57), we multiply both the right-hand side and the left-hand side of the equation by an arbitrary measurable function $\varphi : E^l \to [0,\infty)$ and integrate with respect to $\lambda^{\otimes l}$. Since both the right-hand side and the left-hand side of (3.57) are symmetric in $(z_1,\ldots,z_l)$, it is sufficient to integrate with symmetric functions φ. We remind the reader that $\varphi \otimes_{\mathrm{s}} \mathbb{1}_E$ denotes the symmetrization, see (2.11), of the function $\varphi \otimes \mathbb{1}_E :$ $(x_1,\ldots,x_l,y) \mapsto \varphi(x_1,\ldots,x_l)$. Let $t \geq 0$, $l \in \mathbb{N}$. Using reversibility, we obtain

$$\begin{aligned}
&\int \varphi(x_1,\ldots,x_l)\int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_l}+\delta_y}\left[F(\eta_t)\right]\lambda(\mathrm{d}y)\lambda^{\otimes l}(\mathrm{d}(x_1,\ldots,x_l)) \qquad (3.58)\\
&\quad= \int \varphi \otimes_{\mathrm{s}} \mathbb{1}_E(x_1,\ldots,x_l,y)\mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_l}+\delta_y}\left[F(\eta_t)\right]\lambda^{\otimes(l+1)}(\mathrm{d}(x_1,\ldots,x_l,y))\\
&\quad= \int P_t^{[l+1]}\left(\varphi \otimes_{\mathrm{s}} \mathbb{1}_E\right)(x_1,\ldots,x_l,y)F(\delta_{x_1}+\ldots+\delta_{x_l}+\delta_y)\lambda^{\otimes(l+1)}(\mathrm{d}(x_1,\ldots,x_l,y)).
\end{aligned}$$

Applying (2.10) and using reversibility once again, (3.58) can be turned into

$$\begin{aligned}
&\int \left((P_t^{[l]}\varphi) \otimes_{\mathrm{s}} \mathbb{1}_E\right)(x_1,\ldots,x_l,y)F(\delta_{x_1}+\ldots+\delta_{x_l}+\delta_y)\lambda^{\otimes(l+1)}(\mathrm{d}(x_1,\ldots,x_l,y))\\
&\qquad= \int P_t^{[l]}\varphi(x_1,\ldots,x_l)\int F(\delta_{x_1}+\ldots+\delta_{x_l}+\delta_y)\lambda(\mathrm{d}y)\lambda^{\otimes l}(\mathrm{d}(x_1,\ldots,x_l))\\
&\qquad= \int \varphi(x_1,\ldots,x_l)\int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_l}}\left[F(\eta_t+\delta_y)\right]\lambda(\mathrm{d}y)\lambda^{\otimes l}(\mathrm{d}(x_1,\ldots,x_l))
\end{aligned}$$

which implies (3.57).

Equation (3.57) enables us to prove the intertwining relation (IR.2). Let $f_n \in \mathcal{C}_n$ be symmetric and select $F : \mathbf{N}_n \to \mathbb{R}$ such that $f_n = F \circ \iota_n$. Consequently, by using (3.17) and consistency, we arrive at the following

$$\begin{aligned}
&\mathbb{E}_\mu\left[I_n f_n(\eta_t)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mathbb{E}_\mu\left[\iint f_n(x_1,\dots,x_n)\lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n))\eta_t^{(k)}(\mathrm{d}(x_1,\dots,x_k))\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \iint \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_k}}\left[F(\eta_t + \delta_{x_{k+1}} + \ldots + \delta_{x_n})\right] \\
&\qquad\qquad \lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n))\mu^{(k)}(\mathrm{d}(x_1,\dots,x_k))
\end{aligned}$$

for all $\mu \in \mathbf{N}_{\mathrm{lf}}$. Using (3.57) repeatedly $n-k$ times, we obtain

$$\begin{aligned}
\int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_k}}&\left[F(\eta_t + \delta_{x_{k+1}} + \ldots + \delta_{x_n})\right] \lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n)) \\
&= \int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}\left[F(\eta_t)\right] \lambda^{\otimes(n-k)}(\mathrm{d}(y_{x+1},\dots,x_n)) \\
&= \int P_t^{[n]} f_n(x_1,\dots,x_n)\lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n))
\end{aligned}$$

for $\lambda^{\otimes k}$-almost all $(y_1,\dots,y_k)$. Since $\lambda^{\otimes k}$ is the k-th factorial moment measure of the Poisson process with intensity measure λ, integrating with respect to the k-th factorial measure of μ is well-defined, resulting in

$$\begin{aligned}
&\mathbb{E}_\mu\left[I_n f_n(\eta_t)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \iint P_t^{[n]} f_n(x_1,\dots,x_n)\lambda^{\otimes(n-k)}(\mathrm{d}(x_{k+1},\dots,x_n))\mu^{\otimes k}(\mathrm{d}(x_1,\dots,x_k)) \\
&= I_n P_t^{[n]} f_n(\mu)
\end{aligned}$$

for π_λ-almost all $\mu \in \mathbf{N}$.

The intertwining relation (IR.2) extends to $f_n \in L^2_{\mathrm{sym}}(\lambda^{\otimes n})$ which follows by an approximation argument using (3.55) together with the fact that the subspace consisting of the symmetric functions contained in $\mathcal{C}_n$ is dense in $L^2_{\mathrm{sym}}(\lambda^{\otimes n})$. □

Proof of Theorem 3.4.1 (ii). The proof is analogous to the one for the Poisson case with minor adjustments. Notably, (3.57) turns into

$$\begin{aligned}
\int \mathbb{E}_{\delta_{z_1}+\ldots+\delta_{z_l}+\delta_y}&\left[F(\eta_t)\right](\delta_{z_1}+\ldots+\delta_{z_l}+\alpha)(\mathrm{d}y) \\
&= \mathbb{E}_{\delta_{z_1}+\ldots+\delta_{z_l}}\left[\int F(\eta_t+\delta_y)\,(\eta_t+\alpha)(\mathrm{d}y)\right] \qquad (3.59)
\end{aligned}$$

for $t \geq 0$, $l \in \mathbb{N}_0$, measurable $F : \mathbf{N}_{l+1} \to [0,\infty)$ and $z = (z_1,\dots,z_l) \in E^l$ λ_l-almost everywhere. □

4 Examples

In this section, we provide examples of consistent particle systems. The main motivation of this thesis is to generalize the self-duality functions (1.3) and (1.9) of systems of particles hopping in countable sets to uncountable sets. Therefore, we demonstrate in Section 4.1 that the self-intertwining relations (IR.1) and (IR.2) are the correct ones to recover the known self-duality functions.

Subsequently, we delve into generalizations of these models in uncountable spaces. First, we discuss systems of reversible independent Markov processes in uncountable spaces which represent a generalization of independent random walks. Moreover, we introduce a new process that generalizes the symmetric inclusion process in a natural way and prove its consistency and reversibility. This allows us to apply Theorem 2.1.5 and Theorem 3.1.6 to independent Markov processes and the generalized SIP to obtain the factorial measure intertwining relation (IR.1) and the orthogonal polynomial intertwining relation (IR.2).

We note that a direct generalization of the exclusion process analogous to the generalized SIP, would not be meaningful in general. This is because the probability of jumping on already occupied points is zero whenever the jump kernel of the single particle is not atomic. Therefore, mimicking an exclusion rule to the one in the discrete setting cannot be modeled in the continuum. Hence, we do not delve further into this process. To generalize the duality of the Brownian energy process and the symmetric inclusion process, we dedicate a separate chapter, see Chapter 5 below.

Next, we consider strongly consistent models, namely sticky and correlated Brownian motions. We demonstrate that our framework is applicable to models beyond the scope of generalizing duality functions for discrete particle systems. This leads to new intertwining relations for these models, especially highlighting the methods we developed in Section 3.4 for infinitely many particles.

4.1 Reversible interacting particle systems in a finite set

In this section, we recover the well-known self-duality functions of reversible systems of particles hopping in finite sets from the intertwining relations (IR.1) and (IR.2). Let E be a finite set and identify each $\xi \in \mathbf{N}_{<\infty}$ with $(\xi_j)_{j\in E} := (\xi(\{j\}))_{j\in E} \in \mathbb{N}_0^E$. Let $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ be a Markov family. We assume that there is a probability measure ρ on $\mathbf{N}_{<\infty} = \mathbb{N}_0^E$ that is reversible. If $\rho(\{\mu\}) > 0$ for all $\mu \in \mathbf{N}_{<\infty}$, then $\mathcal{D}^{\text{cheap}}(\xi, \eta) := \frac{\mathbb{1}_{\{\eta=\xi\}}}{\rho(\{\xi\})}$ with $\eta, \xi \in \mathbf{N}_{<\infty}$ is a self-duality function, in other words, $\mathcal{D}^{\text{cheap}}$ satisfies (1.1) where both processes are $(\eta_t)_{t\geq 0}$. This is the so-called cheap self-duality function (see [CGR21, Equation (4.2)]); see also (1.15) in the introduction.

In particular, for each $n \in \mathbb{N}$,

$$\mathcal{D}_n^{\text{cheap}}(\xi, x) := \mathcal{D}^{\text{cheap}}(\xi, \delta_{x_1} + \ldots + \delta_{x_n}), \quad \xi \in \mathbf{N}_{<\infty},\ x = (x_1, \ldots, x_n) \in E^n$$

is a duality functions for $(P_t)_{t \geq 0}$ and the n-particle semigroup $(P_t^{[n]})_{t\geq 0}$. In other words,

$$P_t \mathcal{D}_n^{\text{cheap}}(\,\cdot\,, x)(\xi) = P_t^{[n]} \mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)(x)$$

for each $\xi \in \mathbb{N}_0^E$ and $x \in E^n$. If we define $\mathcal{D}_0^{\text{cheap}}(\xi,) := \mathcal{D}^{\text{cheap}}(\xi, 0)$, we obtain $P_t \mathcal{D}_0^{\text{cheap}}(\,\cdot\,,)(\xi) = P_t^{[0]} \mathcal{D}_0^{\text{cheap}}(\xi,)$. It is a well-known fact that applying an intertwiner to a duality function, such as $\mathcal{D}_n^{\text{cheap}}(\xi, x)$, results in another duality function (see, e.g., [GKRV09, Remark 2.7] or [CFG+19, Theorem 2.5]); see also (1.16) in the introduction.

Recall that J_n represents the generalized falling factorial polynomial, defined in (2.14) and that I_n stands for the infinite-dimensional orthogonal polynomial with respect to ρ as defined in (3.6). We remark that the term *infinite-dimensional* is not truly justified given that the set E is finite.

Proposition 4.1.1. *The following two statements hold true.*

(i) *When we apply J_n to $\mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)$, we obtain*

$$\mathcal{D}_n^{\text{cl}}(\xi, \eta) := \frac{1}{n!} J_n \mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)(\eta) = \mathbb{1}_{\{\xi(E)=n\}} \frac{1}{\rho(\{\xi\})} \prod_{j \in E} \frac{1}{\xi_j!} (\eta_j)_{\xi_j}$$

for all $n \in \mathbb{N}_0$ and $\xi, \eta \in \mathbf{N}_{<\infty}$.

(ii) *Assume further that $\rho = \bigotimes_{j \in E} \rho_j$ where ρ_j are probability measures on $\mathbb{N}_0$ with finite moments. For each $j \in E$, consider the sequence of univariate monic orthogonal polynomials denoted by $(\mathscr{P}_n^j)_{n \in \mathbb{N}_0}$ with respect to ρ_j. Then, when we apply I_n to $\mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)$, we obtain*

$$\mathcal{D}_n^{\text{ort}}(\xi, \eta) := \frac{1}{n!} I_n \mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)(\eta) = \mathbb{1}_{\{\xi(E)=n\}} \prod_{j \in E} \frac{1}{\rho_j(\{\xi_j\}) \xi_j!} \mathscr{P}_{\xi_j}^j(\eta_j)$$

for all $n \in \mathbb{N}_0$ and $\xi, \eta \in \mathbf{N}_{<\infty}$.

Proof. Without loss of generality, let $E = \{1, \ldots, N\}$ and fix $\xi \in \mathbb{N}_0^N$, $n \in \mathbb{N}$. Note that

$$\mathbb{1}_{\{\xi = \delta_{x_1} + \ldots + \delta_{x_n}\}} = \mathbb{1}_{\{\xi(E)=n\}} \frac{n!}{\xi_1! \cdots \xi_N!} \widetilde{\mathbb{1}}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}}(x_1, \ldots, x_n) \tag{4.1}$$

for $x_1, \ldots, x_n \in E$ where $\widetilde{\mathbb{1}}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}}$ denotes the symmetrization, see (2.11), of $\mathbb{1}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}}$. Hence, using (2.15), we obtain

$$\begin{aligned} \frac{1}{n!} J_n \mathcal{D}_n^{\text{cheap}}(\xi, \,\cdot\,)(\eta) &= \frac{\mathbb{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\}) \xi_1! \cdots \xi_N!} \int \mathbb{1}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}} \, \mathrm{d}\eta^{(n)} \\ &= \mathbb{1}_{\{\xi(E)=n\}} \prod_{j=1}^N \frac{1}{\rho_j(\{\xi_j\}) \xi_j!} (\eta_j)_{\xi_j} \end{aligned}$$

for each $\xi \in \mathbb{N}_0^N$.

For the proof of (ii), let $\mathbf{P}_n := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp}$. Note that the function $\mathscr{P}^1_{d_1} \otimes \cdots \otimes \mathscr{P}^N_{d_N}$ is defined as $\eta \mapsto \mathscr{P}^1_{\xi_1}(\eta_1) \cdots \mathscr{P}^N_{\xi_N}(\eta_1)$ in the notation introduced above Proposition 3.1.8. By the orthogonal decomposition

$$\mathbf{P}_n = \bigoplus_{d_1+\ldots+d_N=n} \operatorname{span}\{\mathscr{P}^1_{d_1} \otimes \cdots \otimes \mathscr{P}^N_{d_N}\},$$

we obtain that the projection of $\mathbb{N}_0^N \ni \eta \mapsto \int \mathbb{1}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}} \, \mathrm{d}\eta^{\otimes n} = \eta_1^{\xi_1} \cdots \eta_N^{\xi_N}$ onto $\mathbf{P}_n$ is equal to $\eta \mapsto \mathscr{P}^1_{\xi_1}(\eta_1) \cdots \mathscr{P}^N_{\xi_N}(\eta_1)$. Therefore, using (4.1),

$$\begin{aligned}\frac{1}{n!} I_n \mathcal{D}_n^{\mathrm{cheap}}(\xi, \cdot)(\eta) &= \frac{\mathbb{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\})\xi_1! \cdots \xi_N!} I_n(\mathbb{1}_{\{1\}^{\xi_1} \times \cdots \times \{N\}^{\xi_N}})(\eta) \\ &= \mathbb{1}_{\{\xi(E)=n\}} \prod_{j=1}^{N} \frac{1}{\rho_j(\{\xi_j\})\xi_j!} \mathscr{P}^j_{\xi_j}(\eta_j)\end{aligned}$$

for each $\eta \in \mathbb{N}_0^N$. □

As a consequence, Theorem 2.1.5 ensures that $\mathcal{D}_n^{\mathrm{cl}}$ is a duality function for $(P_t)_{t \geq 0}$ and $(P_t^{[n]})_{n \in \mathbb{N}_0}$ for each $n \in \mathbb{N}_0$. Moreover, when we sum $\mathcal{D}_n^{\mathrm{cl}}$ over all $n \in \mathbb{N}_0$, we obtain the self-duality function

$$\mathcal{D}^{\mathrm{cl}}(\xi, \eta) := \prod_{j \in E} \frac{1}{\rho_j(\{\xi_j\})\xi_j!} (\eta_j)_{\xi_j}, \quad \xi, \eta \in \mathbf{N}_{<\infty}. \tag{4.2}$$

For instance, if ρ is given by a product of Poisson (or negative binomial) distributions, the self-duality function (4.2) is equal to the self-duality function provided by (1.3) up to a multiplicative constant depending on the total number of particles which is a conserved quantity.

Similarly, when we sum $\mathcal{D}_n^{\mathrm{ort}}$ over all $n \in \mathbb{N}_0$, we obtain the self-duality function

$$\mathcal{D}^{\mathrm{ort}}(\xi, \eta) = \prod_{j \in E} \frac{1}{\rho_j(\{\xi_j\})\xi_j!} \mathscr{P}^j_{\xi_j}(\eta_j). \tag{4.3}$$

Again, (4.3) recovers the orthogonal duality function (1.9) up to a multiplicative constant.

Another consequence of Proposition 4.1.1 is that the K-transform, defined in (2.19), satisfies $KF(\mu) = \int F(\xi)\mathcal{D}^{\mathrm{cl}}(\xi, \mu)\rho(\mathrm{d}\xi)$. Similarly, the operators $\mathfrak{U}\mathfrak{V}$ from Section 3.2.2 and Section 3.3.5 are integral operators (up to multiplicative constants) in terms of $\mathcal{D}^{\mathrm{ort}}$ and integration with respect to reversible measures. That shows that our intertwiners are indeed the appropriate generalizations of the duality functions within the context of (1.30).

4.2 Independent particle systems

Every system of independent Markov processes, such as free Kawasaki dynamics (see, e.g., [FKO09, Section 4.2.2]) or independent Brownian motions, is consistent. For independent particles, our theorems allow us to recover known results on intertwining relations in terms of Lenard's K-transform [KKO+23] and multiple stochastic integrals [Sur84]. Our contribution is, as explained in Section 4.1, the identification that these intertwining relations are continuum counterparts to the well-known duality functions expressed in terms of falling factorials and orthogonal polynomials for independent random walks in countable spaces.

Let $p_t : E \times \mathcal{E} \to [0,1]$, $t \geq 0$ be Markov transition kernels. The transition kernel for n independent labeled particles, with one-particle evolution governed by $(p_t)_{t\geq 0}$, is given by $p_t^{\otimes n} : E^n \times \mathcal{E}^{\otimes n} \to [0,1]$ where $p_t^{\otimes n}$ is the product measure $p_t^{\otimes n}((x_1,\dots,x_n),\,\cdot\,) = \bigotimes_{i=1}^n p_t(x_i,\,\cdot\,)$ for $x_1,\dots,x_n \in E$. We abuse the notation and refer to the corresponding Markov semigroups as p_t and $p_t^{\otimes n}$ as well.

The family $(p_t^{\otimes n})_{t\geq 0}$, $n \in \mathbb{N}$ is strongly consistent, i.e., it satisfies (2.25). Therefore, by Proposition 2.2.4, there exists a consistent Markov family $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}})$ describing the dynamics of possibly infinitely many independent particles. In particular, the factorial measure intertwining relation (IR.1) holds which can be rewritten as

$$\mathbb{E}_\mu\left[\int f_n(x_1,\dots,x_n)\eta_t^{(n)}(\mathrm{d}(x_1,\dots,x_n))\right]$$
$$= \iint \cdots \int f_n(y_1,\dots,y_n)p_t(x_1,\mathrm{d}y_1)\cdots p_t(x_n,\mathrm{d}y_n)\mu^{(n)}(\mathrm{d}(x_1,\dots,x_n))$$

for $n \in \mathbb{N}$, measurable symmetric $f_n : E^n \to [0,\infty)$, $\mu \in \mathbf{N}$ and $t \geq 0$. Moreover, Lenard's K-transform intertwines P_t with itself, as noted in (2.20), where P_t denotes the Markov semigroup corresponding to $(\eta_t)_{t\geq 0}$. In the case of free Kawasaki dynamics, we retrieve a relation from [KKO+23, Section 3.2]. In the context of independent Markov jump processes, the K-transform was also examined in [BK15].

We remark that, for an infinite number of particles, we have the state space $\mathbf{N}$ in a very general sense and we do not make any claims regarding phenomena like a possible collapse, where we might end up with an infinite number of particles within a bounded set, even though the process is starting at a locally finite configuration of particles. This issue and other properties, such as path regularity, are explored, e.g., in [KLR08], [FKO09].

Remark 4.2.1. Recall that J_n represents the generalized falling factorial polynomial, defined in (2.14), and $\varphi^{\otimes n}(x_1,\dots,x_n) = \varphi(x_1)\cdots\varphi(x_n)$, $x_1,\dots,x_n \in E$. The intertwining relation in terms of Lenard's K-transform allows us to recover the intertwining relation [KKO+23, Equation (15)] in terms of the *Bogoliubov exponential*

$$e_B(\varphi,\mu) := \prod_{k=1}^N (1+\varphi(x_k)) = \sum_{n=0}^\infty \frac{1}{n!} J_n\varphi^{\otimes n}(\mu)$$

where $\mu = \sum_{k=1}^{N} \delta_{x_k} \in \mathbf{N}_{<\infty}$ and $\varphi : E \to [0,\infty)$ is measurable. Here, the second equation follows by a combinatorial argument. Then, using $P_t J_n \varphi^{\otimes n} = J_n(p_t\varphi)^{\otimes n}$ yields the intertwining relation $P_t e_B(\varphi, \cdot)(\mu) = e_B(p_t\varphi, \mu)$.

If we find a locally finite reversible measure λ on $(E, \mathcal{E})$ for the one-particle dynamics $(p_t)_{t\geq 0}$, it follows that $\lambda^{\otimes n}$ is reversible for the n-particle semigroup $(P_t^{\otimes n})_{t\geq 0}$, a straightforward consequence. As a result, Theorem 3.4.1 applies and we deduce that the intertwining relation (IR.2) in terms of multiple Wiener-Itô integrals holds true. Furthermore, by Corollary 3.4.3, we conclude that π_λ, the distribution of the Poisson process with intensity measure λ, is reversible for $(\eta_t)_{t\geq 0}$.

When expressing (IR.2) in terms of unitary transformations, as seen in (3.56), we obtain

$$\mathfrak{U} P_t \mathfrak{U}^{-1} = P_t^{\mathfrak{F}}, \text{ where } (P_t^{\mathfrak{F}} f)_n = p_t^{\otimes n} f_n, \quad f = (f_n)_{n\in\mathbb{N}_0} \in \mathfrak{F}. \tag{4.4}$$

In the context quantum field theory, the operation that transforms the products $p_t^{\otimes n}$ to P_t is known as the *second quantization*, see, e.g., [BR13, page 8]. Equation (4.4) can be found in Surgailis' article [Sur84, Equation (5.1)] (see also [KLR04] or [KLR08]). Surgailis provided a necessary and sufficient condition on p_t when P_t defined by (4.4) becomes a Markov operator, see [Sur84, Theorem 5.1]. In the context of free Kawasaki dynamics, (4.4) was also examined in [KKO$^+$23, Section 4].

If we assume that λ is invariant for $(p_t)_{t\geq 0}$ instead of assuming reversibility, then $\lambda^{\otimes n}$ is an invariant measure for $(p_t^{\otimes n})_{t\geq 0}$. Using Proposition 3.4.2, we conclude that π_λ is an invariant measure for $(\eta_t)_{t\geq 0}$. This property is a version of Doob's theorem (cf. [DP91, Theorem 2.9.5]) or of the displacement theorem (cf. [Kin93, page 61]). Upon closer examination of the proof of Theorem 3.4.1, we find that the condition of reversibility can be relaxed and it is sufficient to require invariance only to deduce the orthogonal polynomial intertwining relation (IR.2) and the unitary transformation (4.4). Indeed, Theorem 3.4.1 does not exactly rely on reversibility but solely on the condition (3.57). This condition follows in the case of independent particles already due to invariance.

Moreover, it can be checked that (3.57) holds pointwise, rather than almost everywhere. Thus, we obtain (IR.2) pointwise as well. However, it is necessary to reduce the set of configurations. By being more rigorous, we obtain the following statement. Assume that a locally finite λ is invariant for $(p_t)_{t\geq 0}$. Let $t \geq 0$ be fixed, let $\mu \in \mathbf{N}_{\mathrm{lf}}$ such that $\mu p_t \in \mathbf{N}_{\mathrm{lf}}$, where $\mu p_t(B) := \int p_t \mathbb{1}_B \,\mathrm{d}\mu$, $B \in \mathcal{E}$, and let $f_n \in \mathcal{C}_n$. Then, (IR.2) holds true. The condition $\mu p_t \in \mathbf{N}_{\mathrm{lf}}$ was already explored in [KLR08, Equation (2.4)].

4.3 Generalized inclusion process

The inclusion process in countable sets first appeared in the homogeneous case as a dual process to a model for representing energy and momentum transport, see [GKR07, Equation (3.2)], see also [GRV10]. This process also appears with a different interpretation in the field of mathematical population genetics. In fact, in [CGGR15, Section 5], it is proved that its generator coincides with the generator of a variant of the Moran model.

As an example of an interacting system of particles jumping in a general Borel space $(E,\mathcal{E})$, we introduce a new process that serves as a natural generalization of the symmetric inclusion process. We are only considering the case with a finite number of particles. The extension to scenarios involving an infinite number of particles is not part of this thesis and we leave it for future research, see Remark 4.3.4 below.

Let α be a finite measure on $(E,\mathcal{E})$ and $c : E\times E \to [0,\infty)$ be a bounded, measurable and symmetric function. The *generalized symmetric inclusion process* (gSIP) is a continuous-time jump process on $\mathbf{N}_{<\infty}$ with jump kernel

$$Q(\mu,B) = \iint \mathbb{1}_B(\mu-\delta_x+\delta_y)c(x,y)(\alpha+\mu)(\mathrm{d}y)\mu(\mathrm{d}x), \quad \mu\in\mathbf{N}_{<\infty}, B\in\mathcal{N}_{<\infty}. \tag{4.5}$$

It can be viewed, when $E=\mathbb{R}^d$ endowed with the Borel σ–algebra, as a particular case of a Kawasaki dynamics (see, e.g., [KLR07]). Bypassing the precise description of the domain, the formal generator of the process is given by

$$\mathcal{L}F(\mu) = \iint (F(\mu-\delta_x+\delta_y)-F(\mu))c(x,y)(\alpha+\mu)(\mathrm{d}y)\mu(\mathrm{d}x). \tag{4.6}$$

Notice that $Q(\mu,E)<\infty$ for $\mu\in\mathbf{N}_{<\infty}$. Accordingly, the process $(\eta_t)_{t\geq 0}$ can be constructed with the usual jump-hold construction and the semigroup $(P_t)_{t\geq 0}$ is the minimal solution of the backward Kolmogorov equation, see [Fel71, Section X.10].

The process is non-explosive since the number of particles is conserved and

$$\sup_{\mu\in\mathbf{N}_n} Q(\mu,E)<\infty \tag{4.7}$$

for every particle number $n\in\mathbb{N}_0$. Therefore, the minimal solution $(P_t)_{t\geq 0}$ is a Markov semigroup ($P_t(\mu,E)=1$ rather than ≤ 1) and it is in fact the unique solution of the backward Kolmogorov equation.

The dynamics can be described informally as follows. Starting from an initial configuration μ with $n=\mu(E)$ particles $x_1,\dots,x_n$, set

$$q_{i,0} := \int c(x_i,y)\alpha(\mathrm{d}y), \quad q_{i,j} := c(x_i,x_j), \quad z_i := \sum_{j=0}^{n} q_{i,j}, \quad z := \sum_{i=1}^{n} z_i$$

and do the following:

(i) Wait for an exponential time with parameter $Q(\mu,E)=z$.

(ii) When time is up, choose one out of the n points $x_1,\dots,x_n$; where the point x_i is chosen with probability $\frac{z_i}{z}$. Move the chosen point $x=x_i$ to a new location y:
  - With probability $\frac{q_{i,j}}{z_i}$, the new location y is equal to $y=x_j$.
  - With probability $\frac{q_{i,0}}{z_i}$, the new location y is chosen according to the probability measure $\frac{1}{q_{i,0}}c(x_i,y)\alpha(\mathrm{d}y)$.

Then, repeat.

We notice that, for example,

$$\begin{aligned}\mathcal{L}^{[n]} f_n(x_1,\dots,x_n) \\ &= \sum_{i=1}^{n} \int c(x_i,y)(f_n(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) - f_n(x_1,\dots,x_n))\alpha(\mathrm{d}y) \\ &\quad + \sum_{i,j=1}^{n} \int c(x_i,x_j)(f_n(x_1,\dots,x_{i-1},x_j,x_{i+1},\dots,x_n) - f_n(x_1,\dots,x_n))\end{aligned} \tag{4.8}$$

is for each $n \in \mathbb{N}$ a formal generator for the evolution of n labeled particles, i.e., a formal generator of the n-particle semigroup $P_t^{[n]}$ that corresponds to the gSIP.

The gSIP has a connection to the well-known SIP of particles hopping in a finite set: Let $(\eta_t)_{t\geq 0}$ be an instance of the gSIP, let $A_1,\dots,A_N \in \mathcal{E}$, $N \in \mathbb{N}$ be a partition of E and let c be constant on $A_i \times A_j$ with $c(x,y) = c_{i,j}$ for all $x \in A_i$ and $y \in A_j$, for each i,j. Then, the process $(\eta_t(A_1),\dots,\eta_t(A_N))$ starting at $\mu \in \mathbf{N}_{<\infty}$ behaves like the SIP in the finite set $\{1,\dots,N\}$ with formal generator (1.2) with initial configuration $(\mu(A_1),\dots,\mu(A_N))$, conductances $c_{i,j}$ and $\alpha_i = \alpha(A_i)$.

Just like the SIP (refer to [CGGR15, Section 5]), the gSIP has a connection to the field of population genetics. More precisely, the gSIP is closely related to the measure-valued Moran model. The *measure-valued Moran model* (see, for example, [Daw93, Section 2.5] or [Eth00, Section 5.4] and the references therein) is an extension of the classical Moran model introduced by Moran in [Mor58] (see also [KM62]). This process describes the evolution of a population where each individual is associated with a particular type. To be more precise, a configuration $(x_1,\dots,x_n)$ represents the types $x_1,\dots,x_n \in E$ of individuals in a population of size n. In this context, E is referred to as the *type space*. The evolution is a combination of mutation and selection, followed by reproduction. More precisely, on the one hand, each individual independently follows a Markov process generated by an operator known as the *mutation operator* denoted by A. On the other hand, an individual in the population may die and be replaced by an offspring of another individual.

The gSIP exhibits a similar dynamic. Specifically, when $c(x,y)$ equals the one function, the process generated by (4.8) coincides with the n-particle Moran model (see [Daw93, Equation (2.5.2)]) with mutation operator

$$A\varphi(x) = \int (\varphi(y) - \varphi(x))\alpha(\mathrm{d}x). \tag{4.9}$$

When we pass from $x_1,\dots,x_n$ into the unlabeled configuration $\mu = \delta_{x_1} + \ldots + \delta_{x_n}$, we obtain a corresponding analogy to the *measure-valued Moran process* (see, for example, [Daw93, Section 2.6]). It is worth noting that, in general, there may be multiple labeled n-particle dynamics leading to the same unlabeled dynamics. More precisely, besides the n-particle dynamics with formal generator (4.8), there is also a *lookdown* construction introduced in [DK96] that leads to the (unlabeled) gSIP.

In population genetics, the population is often modeled not with $\mu \in \mathbf{N}_{<\infty}$ but rather with the empirical measure $\frac{\mu}{\mu(E)}$. This is because one is interested in the distribution of $\frac{\mu}{\mu(E)}$ as $\mu(E) \to \infty$ rather than in the types of single individuals. This leads to Fleming-Viot limits (see, e.g., [Shi90] and the references therein) to measure-valued Fleming-Viot processes (see [FV79], [EK93]). For more details, see Chapter 5 below.

Proposition 4.3.1. *The generalized symmetric inclusion process is a consistent Markov process.*

Thus, using Theorem 2.1.5, the factorial measure intertwining relation (IR.1) holds.

Proof. First, we observe that it is enough to check the commutation property (2.6) for the formal generator (4.6) instead of the semigroup, more precisely, to check

$$\mathcal{A}\mathcal{L}F(\mu) = \mathcal{L}\mathcal{A}F(\mu), \quad \mu \in \mathbf{N}_{<\infty} \tag{4.10}$$

for all measurable bounded functions $F : \mathbf{N}_{<\infty} \to \mathbb{R}$ that vanish outside $\mathbf{N}_n$ for some $n \in \mathbb{N}$. Indeed, since (4.7) holds, $\mathcal{L}_{|\mathbf{N}_n}$ is a bounded linear operator on the space of real-valued measurable bounded functions on $\mathbf{N}_n$ equipped with the sup-norm. Thus, $\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{L}_{|\mathbf{N}_n}^k$ converges in operator norm and is equal to $P_{t|\mathbf{N}_n}$ where $(P_t)_{t\geq 0}$ denotes the Markov semigroup of the gSIP. Thanks to this property, the commutation property (4.10) can be transferred to the Markov semigroup.

To prove (4.10), we fix F and μ. We decompose the formal generator into $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ where

$$\mathcal{L}_1 F(\mu) := \iint (F(\mu - \delta_x + \delta_y) - F(\mu))\, c(x,y)\alpha(\mathrm{d}y)\mu(\mathrm{d}x)$$

and

$$\mathcal{L}_2 F(\mu) := \iint (F(\mu - \delta_x + \delta_y) - F(\mu))\, c(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x).$$

Notice that $\mathcal{L}_1$ is the formal generator of a system of independent Markov jump processes with jump kernel given by $c(x,y)\alpha(\mathrm{d}y)$. The consistency of a system of independent Markov processes was previously discussed in Section 4.2. Thus, it remains to show that

$$\mathcal{A}\mathcal{L}_2 F(\mu) = \mathcal{L}_2 \mathcal{A}F(\mu). \tag{4.11}$$

First, we compute

$$\begin{aligned}\mathcal{L}_2\mathcal{A}F(\mu) = &\iiint F(\mu - \delta_x + \delta_y - \delta_z)\mu(\mathrm{d}z)c(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x)\\ &- \iint F(\mu - 2\delta_x + \delta_y)c(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x) + \iint F(\mu - \delta_x)c(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x)\\ &- \iiint F(\mu - \delta_z)\mu(\mathrm{d}z)c(x,y)\mu(\mathrm{d}y)\mu(\mathrm{d}x).\end{aligned}$$

Second,

$$
\begin{aligned}
&\mathcal{A}\mathcal{L}_2 F(\mu) \\
&= \iiint \left(F(\mu - \delta_z - \delta_x + \delta_y) - F(\mu - \delta_z)\right) c(x,y)(\mu - \delta_z)(\mathrm{d}y)(\mu - \delta_z)(\mathrm{d}x)\mu(\mathrm{d}z) \\
&= \mathcal{L}_2\mathcal{A}F(\mu) - \iint \left(F(\mu - \delta_x) - F(\mu - \delta_z)\right) c(x,z)\mu(\mathrm{d}x)\mu(\mathrm{d}z) \\
&\qquad\qquad + \int \left(F(\mu - \delta_z) - F(\mu - \delta_z)\right) c(z,z)\mu(\mathrm{d}z).
\end{aligned}
$$

Because the last two integrals above are both 0, we obtain (4.11) and the proof is concluded. □

Remark 4.3.2. The generalized symmetric inclusion process enables us to show the existence of a consistent process, such that there exits no strongly consistent labeling. Indeed, we consider the process with formal generator (4.6), c being the constant function equal to one and $\alpha = 0$. By Proposition 4.3.1, it is consistent.

To demonstrate the non-existence of a strongly consistent labeling, we assume the existence of transition kernels $P_t^{[n]} : E^n \times \mathcal{E}^{\otimes n} \to [0,1]$ leading to a strongly consistent Markov semigroup $P_t^{[n]}$ (denoted in the same way) satisfying (2.9). We have $\mathcal{L}F(\delta_z) = 0$ for all $z \in E$ and arbitrary F, i.e., if the process starts with exactly one particle, it is deterministic and remains constant over time. Therefore, $P_t^{[1]}(z, \cdot\,) = \delta_z$ for all $t \geq 0$. Using strong consistency (2.25), we obtain

$$
P_t^{[n]}((x_1, \dots, x_n), E^{i-1} \times A_i \times E^{n-i}) = P_t^{[1]}(x_i, A_i) = \delta_{x_i}(A_i)
$$

for $x_1, \dots, x_n$, $A_i \in \mathcal{E}$, $i \in \{1, \dots, n\}$. Therefore, by using the fact that, if all margins of a random vector are almost surely constant, then the vector itself is almost surely constant, we obtain $P_t^{[n]}((x_1, \dots, x_n), \cdot\,) = \delta_{(x_1,\dots,x_n)}$. This implies that the n-particle dynamics is deterministic and remains constant all time for all initial configurations which contradicts the fact that $\mathcal{L}F(\mu) \neq 0$ in general.

The following proposition shows reversibility of the gSIP. Thus, together with consistency (Proposition 4.3.1), the orthogonal polynomial intertwining relation (IR.2) holds. Recall that the distribution of the Pascal process with parameters p and α is denoted by $\rho_{p,\alpha}$, see Section 3.3.

Proposition 4.3.3. *For every $p \in (0,1)$, the measure $\rho_{p,\alpha}$ is reversible for the gSIP.*

The reversibility of $\rho_{p,\alpha}$ does not depend on the function $c(x,y)$ in the dynamics. Moreover, we note that we have a family of reversible measures, indexed by $p \in (0,1)$. By using Lemma 3.3.4 together with the conservation of the number of particles, we conclude that the only statement of Proposition 4.3.3 is that the push-forward measure of λ_n under the map $\iota_n : (x_1, \dots, x_n) \mapsto \delta_{x_1} + \ldots + \delta_{x_n}$ is reversible for all $n \in \mathbb{N}$. The Pascal process is a sum of these measures with individual scaling depending on p.

Proof. It is enough to check the detailed balance condition

$$(\rho_{p,\alpha} \otimes Q)(A \times B) = (\rho_{p,\alpha} \otimes Q)(B \times A), \quad A, B \in \mathcal{N}_{<\infty} \tag{4.12}$$

where

$$(\rho_{p,\alpha} \otimes Q)(A \times B) = \iiint \mathbb{1}_A(\mu)\mathbb{1}_B(\mu - \delta_x + \delta_y)c(x,y)\mu(\mathrm{d}x)(\alpha + \mu)(\mathrm{d}y)\rho_{p,\alpha}(\mathrm{d}\mu)$$

and Q is the jump kernel from (4.5). Indeed, using the Papangelou kernel for the Pascal process (see Proposition 3.3.2), we get

$$\begin{aligned}(\rho_{p,\alpha} \otimes Q)(A \times B) &= \frac{1}{p} \iiint \mathbb{1}_A(\mu - \delta_y)\mathbb{1}_B(\mu - \delta_x)c(x,y)(\mu - \delta_y)(\mathrm{d}x)\mu(\mathrm{d}y)\rho_{p,\alpha}(\mathrm{d}\mu) \\ &= \frac{1}{p} \iint \mathbb{1}_A(\mu - \delta_y)\mathbb{1}_B(\mu - \delta_x)c(x,y)\mu^{(2)}(\mathrm{d}(x,y))\rho_{p,\alpha}(\mathrm{d}\mu).\end{aligned}$$

The equation above is symmetric in A and B since c is symmetric and the factorial measure is invariant under swapping the variables x and y. □

We remark that there exists also an alternative route for proving the reversibility that does not rely on the knowledge of the Papangelou kernel of a Pascal process, see [FJRW24, Proof of Theorem 4.2 (ii)]. The idea of the proof is that for particularly simple choices of $c(x,y)$ and A, B, the relation (4.12) boils down to the well-known detailed balance relation for a discrete inclusion process which can be extended by a measure-theoretic argument. Nevertheless, in this thesis, we opt for the proof stated above using Papangelou kernels which is notably shorter and simpler than the one presented in [FJRW24] where the Papangelou kernel of the Pascal process was not introduced for the sake of brevity.

Remark 4.3.4. In this chapter, we exclusively focus on the generalized symmetric inclusion process for a finite number of particles and do not provide an answer to whether there exists an analogous dynamics for an infinite number of particles with a non-finite measure α. For a construction of infinite particle systems in countable sets, we refer to [DP91, Section 2.2.4] and especially for the case of the SIP to [KR16]. Even though we do not further explore the following approaches, we briefly mention three ideas that could prove useful for a corresponding construction.

(i) The construction of dynamics for infinitely many particles is often accomplished by determining the evolution of correlation functions first and then proving the existence and uniqueness of the corresponding dynamics, see, e.g., [KKZ06] or [BKKK13]. It is natural to apply this principle to the gSIP.

Using standard theory for Markov jump process, one can define the dynamics of the gSIP with infinite α but for finitely many particles with corresponding n-particle transition kernels denoted by $P_t^{[n]} : E^n \times \mathcal{E}^{\otimes n} \to [0,1]$. Given an initial configuration $\mu \in \mathbf{N}$ and $t \geq 0$, in view of Corollary 2.1.6, a natural candidate for the n-point correlation function is given by $\alpha_n^t(B) := \int P_t^{[n]}(x,B)\mu^{(n)}(\mathrm{d}x)$ (not to be confused with the measure α used to construct the gSIP). It remains to verify the existence and uniqueness of a point process with given correlation functions. For this purpose, the criteria of Lenard, see [Len73], [Len75] offer a suitable approach. Specifically, for uniqueness, a growth condition on the correlation functions is sufficient, while for existence there is a criterion, called *Lenard positivity*, to check. More precisely, for a suitable set of functions $G : \mathbf{N} \to \mathbb{R}$, where $G \geq 0$, the condition

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int K^{-1}G(\delta_{y_1} + \ldots + \delta_{y_n})\alpha_n^t(\mathrm{d}(y_1,\ldots,y_n))$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \iint K^{-1}G(\delta_{y_1} + \ldots + \delta_{y_n})$$
$$P_t^{[n]}((x_1,\ldots,x_n),\mathrm{d}(y_1,\ldots,y_n))\mu^{(n)}(\mathrm{d}(x_1,\ldots,x_n)) \geq 0 \quad (4.13)$$

has to be checked. Note that the K-transform, given by (2.19), becomes bijective with inverse K^{-1} by a suitable choice of domain and range, see [KK02, Proposition 3.1].

While it may seem natural in (4.13) to use consistency and conclude, using the factorial measure intertwining relation (IR.1), as follows

$$\sum_{n=0}^{\infty} \frac{1}{n!} \iint K^{-1}G(\delta_{y_1} + \ldots + \delta_{y_n})$$
$$P_t^{[n]}((x_1,\ldots,x_n),\mathrm{d}(y_1,\ldots,y_n))\mu^{(n)}(\mathrm{d}(x_1,\ldots,x_n))$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \iint K^{-1}G(\delta_{y_1} + \ldots + \delta_{y_n})\nu^{(n)}(\mathrm{d}(y_1,\ldots,y_n))P_t(\mu,\mathrm{d}\nu)$$
$$= \iint KK^{-1}G(\nu)P_t(\mu,\mathrm{d}\nu) = \int G(\nu)P_t(\mu,\mathrm{d}\nu) \geq 0,$$

this approach does not seem to work directly, as the transition kernels $P_t(\mu,\mathrm{d}\nu)$ of the gSIP are the very ones whose existence is unknown.

(ii) The unitary operator $\mathfrak{U} : \mathfrak{F} \to L^2(\rho_{p,\alpha})$ defined in (3.35) may prove useful for the construction of the dynamics of infinitely many particles, similarly as described in [Sur84, Theorem 5.1] and [KLR08] for independent particles. Indeed, consider the operator $P_t^{\mathfrak{F}}$ on $\mathfrak{F}$ that is defined by $f = (f_n)_{n\in\mathbb{N}_0} \mapsto (P_t^{[n]} f_n)_{n\in\mathbb{N}_0}$. Consequently, in the spirit of Section 3.3.5 or (3.56),
$$P_t := \mathfrak{U}P_t^{\mathfrak{F}}\mathfrak{U}^{-1} : L^2(\rho_{p,\alpha}) \to L^2(\rho_{p,\alpha})$$
is the natural candidate for the Markov semigroup describing the gSIP dynamics for infinitely many particles, provided that the Pascal distribution $\rho_{p,\alpha}$ is reversible.

(iii) One could introduce a labeled dynamics on E^∞, analogous to the lookdown construction of the Moran model (see [DK96]) and subsequently map it to the space $\mathbf{N}$ using the function $\iota_\infty : x = (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k=1}^{\infty} \delta_{x_k}$.

4.4 Correlated Brownian motions

As we have seen in Section 2.2, all strongly consistent systems are consistent and hence, our approach to intertwining relations is applicable. In this section, we focus on correlated Brownian motions, a simple example within the established setting of [LR04a]. Notably, this example stands out for its simplicity, as the interaction of the Brownian motions does not depend on local interactions when the particles meet, unlike other models such as the Harris flow [Har84] or those presented in [LR14].

We define a family of real-valued stochastic processes $(X_{k,t})_{t\geq 0}$, $k \in \mathbb{N}$ to be a family of *correlated Brownian motions with pairwise correlation* $0 \leq a \leq 1$ starting at a (deterministic) sequence $x = (x_k)_{k\in\mathbb{N}}$ of real numbers if

- the family $(X_{k,t})_{k\in\mathbb{N},t\geq 0}$ is a Gaussian process;
- for all $t \geq 0$ and $k \in \mathbb{N}$: $\mathbb{E}[X_{k,t}] = x_k$;
- for all $t, s \geq 0$ and $k \in \mathbb{N}$: $\operatorname{Cov}[X_{k,t}, X_{k,s}] = \min\{t, s\}$;
- for all $t, s \geq 0$ and $k, l \in \mathbb{N}$ with $k \neq l$: $\operatorname{Cov}[X_{k,t}, X_{l,s}] = a \min\{t, s\}$.

If $a = 0$, the $(X_{1,t})_{t\geq 0}, (X_{2,t})_{t\geq 0}, \dots$ are independent Brownian motions. On the other hand, if $a = 1$, they are modifications of each other up to an additive constant: for any $k \in \mathbb{N}$ and $t \geq 0$, we have $X_{k,t} = X_{k,1} - x_1 + x_k$ almost surely.

An explicit construction can be done as follows: Let $(B_t)_{t\geq 0}$, $(B_{1,t})_{t\geq 0}, (B_{2,t})_{t\geq 0}, \dots$ be independent Brownian motions all starting at zero. From there, define

$$X_{k,t} := \sqrt{1-a}B_{k,t} + \sqrt{a}B_t + x_k, \quad k \in \mathbb{N}, t \geq 0. \tag{4.14}$$

A family of correlated Brownian motions satisfies the Markov property. We define the *n-particle semigroup*

$$P_t^{[n]} f_n(x) := \mathbb{E}\left[f_n(\sqrt{1-a}B_{1,t} + \sqrt{a}B_t + x_1, \dots, \sqrt{1-a}B_{n,t} + \sqrt{a}B_t + x_n)\right] \tag{4.15}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ where $f_n : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$ is bounded and measurable.

It can readily be checked that $P_t^{[n]}$ is strongly consistent. The correspondence between strongly consistent families and stochastic flows, as stated in [LR04a, Theorem 2.1], in our example of correlated Brownian motions is indicated in the following: Let Z_t be a random variable that follows the standard normal distribution and let K_t be a random probability kernel such that $K_t(v, \cdot)$ is equal to the normal distribution with expected value $\sqrt{at}Z_t + v$ and variance $(1-a)t$ for $v \in \mathbb{R}$ and $t \geq 0$. By using (4.15), we obtain

$$P_t^{[n]} f_n(x_1, \dots, x_n) = \mathbb{E}\left[\int \cdots \int f_n(y_1, \dots, y_n) K_t(x_1, \mathrm{d}y_1) \cdots K_t(x_1, \mathrm{d}y_1)\right] \tag{4.16}$$

for all $t \geq 0$ and $x_1, \dots, x_n \in \mathbb{R}$.

Thanks to strong consistency, Proposition 2.2.4 applies and ensures the existence of a Markov family that describes the evolution of an unlabeled system of possibly infinitely many correlated Brownian motions. The dynamics is consistent, meaning that the factorial measure intertwining relation (IR.1) is fulfilled.

In the following, we discuss reversibility. For this purpose, let λ be the Lebesgue measure on $\mathbb{R}$. We recall that $\mathcal{C}_n$ denotes the space of measurable bounded $u_n : \mathbb{R}^n \to \mathbb{R}$ such that there exists a bounded $B \subset \mathbb{R}$ such that $\{x \in \mathbb{R}^n : u_n(x) \neq 0\} \subset B^n$.

Proposition 4.4.1. *For each $n \in \mathbb{N}$, the measure $\lambda^{\otimes n}$ is reversible for n correlated Brownian motions with pairwise correlation a, i.e., $\int (P_t^{[n]} f_n) g_n \,\mathrm{d}\lambda^{\otimes n} = \int (P_t^{[n]} g_n) f_n \,\mathrm{d}\lambda^{\otimes n}$ for $t \geq 0$, $f_n, g_n \in \mathcal{C}_n$.*

Proof. To show that $\lambda^{\otimes n}$ is reversible for the n-particle dynamics, we use the following straightforward computation. Let $n \in \mathbb{N}$ and $t > 0$ be fixed. We define $Y := (Y_k)_{k=1}^n$ where $Y_k := \sqrt{1-a} B_{k,t} + \sqrt{a} B_t$ for $k \in \{1, \ldots, n\}$. By using the construction of correlated Brownian motions given in (4.14), we obtain $X_{k,t} = Y_k + x_k$. Then, using Fubini's theorem, substitution and the fact that $-Y$ is equal in distribution to Y, we obtain

$$\begin{aligned}\int \mathbb{E}\left[f_n(Y+x)\right] g_n(x) \lambda^{\otimes n}(\mathrm{d}x) &= \mathbb{E}\left[\int_{\mathbb{R}^n} f_n(-Y+x) g_n(x) \lambda^{\otimes n}(\mathrm{d}x)\right] \\ &= \mathbb{E}\left[\int f_n(x) g_n(x+Y) \lambda^{\otimes n}(\mathrm{d}x)\right] = \int f_n(x) \mathbb{E}\left[g_n(x+Y)\right] \lambda^{\otimes n}(\mathrm{d}x)\end{aligned}$$

for all f_n, $g_n \in \mathcal{C}_n$. □

Therefore, the assumptions of Theorem 3.4.1 are satisfied and thus, the orthogonal polynomial intertwining relation (IR.2) holds true for $f_n \in L^2_{\mathrm{sym}}(\lambda_n)$, i.e., the multiple Wiener-Itô integral of degree n intertwines the dynamics of infinitely many correlated Brownian motions with correlation a and the n-particle evolution. Note that (IR.2) holds for non-symmetric $f_n \in L^2(\lambda^{\otimes n})$ as well. Indeed, since $P_t^{[n]} f_n$, $f_n \in L^2(\lambda^{\otimes n})$ is also defined for non-symmetric functions, we can apply Theorem 3.4.1 to the symmetrization $\widetilde{f_n}$, defined in (2.11), and then use

$$P_t^{[n]} \widetilde{f_n} = \widetilde{P_t^{[n]} f_n}$$

which follows from the definition of strong consistency (2.25). The conclusion follows by $I_n \widetilde{f_n} = I_n f_n$.

Moreover, Corollary 3.4.3 applies and we obtain the following result. Recall that π_λ denotes the distribution of the Poisson process with intensity measure λ, here the Lebesgue measure.

Corollary 4.4.2. *The measure π_λ is reversible for infinitely many unlabeled correlated Brownian motions with correlation a.*

4.5 Sticky Brownian motions

The focus of this section is on sticky Brownian motions. This model leads us to intertwining relations in terms of the infinite-dimensional Meixner polynomials. As an application of the intertwining relations, we obtain a new result for a system of infinitely many sticky Brownian motions: the distribution of the Pascal process is reversible.

Feller considered in his boundary classification (see [Fel52]) a single reflected Brownian motion that is sticky at the origin. However, it should be noted that this condition is more accurately described as a "slowly" reflecting boundary, rather than "true" stickiness. Specifically, the amount of time that the Brownian motion spends at zero has a strictly positive Lebesgue measure with positive probability but contains no interval. In contrast, "true" stickiness is characterized by a process getting "stuck" at zero over a random interval of time, as shown, e.g., in [KKR07] for an interest rate process.

Remark 4.5.1. Boundary conditions of a Brownian motion can be characterized by conditions on the domain of the generator. For instance, the slowly reflecting property of Feller corresponds to the condition $f'(0) = cf''(0)$, for some $c > 0$, see, e.g., [Lig10, Example 3.59]. In addition to this boundary condition, there are several others (which also can be combined with each other), see [KT81, Chapter 8] or [Pes15] and the references therein. The *Dirichlet condition*, $f(0) = 0$, corresponds to a Brownian motion being killed upon reaching zero, i.e., transitioning to a coffin state, whereas the *Neumann condition*, $f'(0) = 0$, leads to a Bessel process, namely a Brownian motions that is instantaneously reflected upon reaching zero. The case $f''(0) = 0$ corresponds to zero being infinitely sticky, meaning that the Brownian motion is absorbed upon reaching zero. The *Robin condition*, $f(0) = cf'(0)$, corresponds to an elastic boundary condition at zero, whereas the case $f(0) = -cf''(0)$ corresponds to being absorbed upon reaching zero and then killed after some independent exponentially distributed time has passed.

Based on the slowly reflecting property of Feller, a pair $(X_{1,t}, X_{2,t})_{t\geq 0}$ of Brownian motions with sticky interaction can be described, namely that $(X_{1,t})_{t\geq 0}$ and $(X_{2,t})_{t\geq 0}$ are both Brownian motions and $(|X_{1,t} - X_{2,t}|)_{t\geq 0}$ is a single reflected Brownian motion that is sticky at zero. They behave independently when they are apart and interact when they meet. The pair $(X_{1,t}, X_{2,t})_{t\geq 0}$ can be characterized by a martingale problem, as shown in [HW09a].

Howitt and Warren (see [HW09a]) generalized this concept to a family of n diffusions in $\mathbb{R}$ which is commonly known as the *Howitt-Warren martingale problem* or as system of *sticky Brownian motions*. For a full construction, we refer to [HW09a]. An alternative formulation is available in [SSS14]. This yields a family of n independent Brownian motions that move separately when they are far apart and coalesce when individual processes meet. A new non-negative quantity $\theta(i : j)$ is introduced which can be interpreted as the rate at which a group of $i+j$ particles splits into a group of i particles and a group of j particles. Furthermore, the concept of strong consistency, see (2.25), corresponds to the condition $\theta(i + 1 : j) + \theta(i : j + 1) = \theta(i : j)$ which has been characterized in [SSS14, Lemma A.4] by the existence of a finite measure ν on the interval $[0, 1]$, called *the characteristic measure* which satisfies

$$\theta(i:j) = \int x^{i-1}(1-x)^{j-1}\nu(\mathrm{d}x), \quad i,j \in \mathbb{N}.$$

In this case, Proposition 2.2.4 provides a Markov family describing the infinite dynamics that is consistent, i.e., it the factorial measure intertwining relation (IR.1) is satisfied.

The parameter $\theta := 2\theta(1:1) = 2\nu([0,1]) > 0$ is referred to as the *stickiness parameter*. An increase in the value of θ leads to a higher rate of particle separation, resulting in a reduction in the stickiness of the Howitt-Warren flow.

Various choices of the characteristic measure give rise to the following examples.

- When $\nu = 0$, we obtain a situation known as coalescing Brownian motions or the *Arratia flow*, see, e.g., [Arr79] or [BGS15]. In this case, all $\theta(i:j)$ are equal to zero.
- For $\nu = \frac{\theta}{2}(\delta_0+\delta_1)$, where $\theta > 0$, we have the *erosion flow*, see, e.g., [HW09b]. In this scenario, the splitting rates $\theta(i:j)$ are zero unless i or j equals 1. Consequently, only a single particle can split away at any given time.
- Choosing $\nu = \frac{\theta}{2}\lambda_{[0,1]}$, where $\lambda_{[0,1]}$ denotes the Lebesgue measure on $[0,1]$, leads to *uniform sticky Brownian motions with stickiness* $\theta > 0$. These processes are studied, e.g., in [BR20] in the context of large deviation analysis.

Sticky Brownian motions can be understood as the diffusive limit arising from a system of random walks in random environment, see [SSS17]. For instance, the uniform sticky Brownian motions are connected to the behavior of random walks in beta-distributed random environment, see, e.g., [BC17].

In the subsequent discussion, we focus on uniform sticky Brownian motions and examine intertwining relations in terms of orthogonal polynomials. In this model, the splitting rates are given by $\theta(i:j) = \frac{\theta}{2}\frac{(i-1)!(j-1)!}{(i+j-2)!}$ and thus, the multiparticle interactions are completely determined by the two-particle interactions. The authors of [BW23] deduce the Kolmogorov backwards equation and demonstrate that, for this particular interaction, it can be solved exactly using the Bethe ansatz. We remark that having knowledge of the transition probabilities enables us to rephrase the factorial measure intertwining relation (IR.1) into (2.23). In other words, we obtain a closed formula for the correlation functions of a system of uniform sticky Brownian motions, as noted in Remark 2.1.7. Moreover, the results in [BW23] allow us to obtain the reversible measure for the n-particle dynamics.

We provide the definition of uniform sticky Brownian motions from [SSS14, Definition 2.2] for the sake of completeness.

Definition 4.5.2. Let $n \in \mathbb{N}$. We say that $(X_t)_{t\geq 0} = (X_1,\dots,X_n) = (X_{1,t},\dots,X_{n,t})_{t\geq 0}$ are *n-particle uniform sticky Brownian motions with stickiness* $\theta > 0$ if the following conditions are satisfied.

(i) $(X_t)_{t\geq 0}$ is a continuous, square-integrable semimartingale.

(ii) The covariation of X_k and X_l is given by

$$[X_k, X_l]_t = \int_0^t \mathbb{1}_{\{X_{k,s}=X_{l,s}\}}\,\mathrm{d}s, \quad t \geq 0$$

for $k, l \in \{1, \ldots, n\}$.

(iii) For each $\Delta \subset \{1, \ldots, n\}$, $f_\Delta(X_t) - \theta \int_0^t \beta_+(g_\Delta(X_s))\,\mathrm{d}s$, $t \geq 0$ is a martingale with respect to the natural filtration of $(X_t)_{t\geq 0}$ where

$$f_\Delta(x) := \max_{k\in\Delta} x_k, \quad g_\Delta(x) := |\{k \in \Delta : x_k = f_\Delta(x)\}|, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

$\beta^+(1) := 0$ and $\beta^+(m) := 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{m-1}$, $m \geq 2$.

The existence of the n-particle uniform sticky Brownian motions starting at an arbitrary initial value $x \in \mathbb{R}^n$ is proved in [HW09a, Theorem 2.1] together with the fact that their distribution is unique. A family of sticky Brownian motions satisfies the Markov property.

Put $\alpha = \theta\lambda$ where λ denotes the Lebesgue measure on $\mathbb{R}$. We remind the reader of the measures λ_n defined in (3.31).

Proposition 4.5.3. *For unlabeled uniform sticky Brownian motions $(\eta_t)_{t\geq 0}$ with stickiness θ, the push-forward measure of λ_n under the map $(x_1, \ldots, x_n) \mapsto \delta_{x_1} + \ldots + \delta_{x_n}$ is reversible.*

Proposition 4.5.3 allows us to apply Theorem 3.4.1. Thus, the sticky Brownian motions satisfy the orthogonal polynomial intertwining relation (IR.2). We express this statement in the following theorem. We recall that $\rho_{p,\alpha}$ denotes the distribution of the Pascal process with parameters p and $\alpha = \theta\lambda$, see Section 3.3.

Theorem 4.5.4. *The infinite-dimensional Meixner polynomial $\mathcal{M}_n^{p,\alpha}$ of degree n intertwines the system $(\eta_t)_{t\geq 0}$ of infinitely many uniform sticky Brownian motions with stickiness θ and their n-particle evolution $(P_t^{[n]})_{t\geq 0}$. In other words,*

$$\mathbb{E}_\mu \left[\mathcal{M}_n^{p,\alpha} f_n(\eta_t)\right] = \mathcal{M}_n^{p,\alpha} P_t^{[n]} f_n(\mu) \tag{4.17}$$

holds for $\rho_{p,\alpha}$ almost all $\mu \in \mathbf{N}$, $t \geq 0$, $n \in \mathbb{N}_0$, $f_n \in L^2(\lambda_n)$ and $p \in (0,1)$.

Moreover, Corollary 3.4.3 yields a novel result providing a family of reversible measures for a system of infinitely many sticky Brownian motions.

Corollary 4.5.5. *Let $\theta > 0$. Then, for each $p \in (0,1)$, the distribution of the Pascal process with parameters p and $\alpha = \theta\lambda$ is a reversible measure for a system of infinitely many unlabeled uniform sticky Brownian motions with stickiness θ.*

To prove Proposition 4.5.3, we use [BW23, Theorem 4.17] which provides a reversible measure for n ordered uniform sticky Brownian motions. Starting from this reversible measure, we obtain that its symmetrization is equal to the measure λ_n up to a constant.

Let $n \in \mathbb{N}$ be fixed. We use the notation Σ_n for the set of *partitions* of the set $\{1, \dots, n\}$ and we define Π_n to be the set of *ordered partitions* of $\{1, \dots, n\}$, i.e.,

$$\Pi_n := \{(a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N}, k \in \mathbb{N}, a_1 + \dots + a_k = n\}.$$

For each $\pi = (a_1, \dots, a_k) \in \Pi_n$ and each function $f_n : \mathbb{R}^n \to \mathbb{R}$, we define the function $(f_n)_\pi : \mathbb{R}^k \to \mathbb{R}$ by

$$(x_1, \dots, x_k) \mapsto f_n(\underbrace{x_1, \dots, x_1}_{a_1 \text{ times}}, \dots, \underbrace{x_k, \dots, x_k}_{a_k \text{ times}}).$$

We define the measure $\lambda_\pi^{\geq}$ on $\mathbb{R}^n$ by

$$\int f_n \, \mathrm{d}\lambda_\pi^{\geq} := \int \mathbb{1}_{\{x_1 \geq \dots \geq x_k\}} (f_n)_\pi (x_1, \dots x_k) \lambda^{\otimes k}(\mathrm{d}(x_1, \dots, x_k))$$

and put

$$m_\theta^{(n)} := \sum_{\pi \in \Pi_n} \theta^{|\pi| - n} \left(\prod_{A \in \pi} \frac{1}{|A|} \right) \lambda_\pi^{\geq}.$$

We consider the map $\varphi : \mathbb{R}^n \to \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq \dots \geq x_n\}$ that orders the components of a vector in descending order. Let $(\Omega^{[n]}, \mathcal{F}^{[n]}, (Z_t^{[n]})_{t \geq 0}, (\mathbb{P}_x^{[n]})_{x \in \mathbb{R}^n})$ denote a Markov family associated with the n-particle dynamics as described in the proof of Proposition 2.2.4, abbreviate $Z_t = Z_t^{[n]}$, $\mathbb{E}_x = \mathbb{E}_x^{[n]}$ and put $Y_t := \varphi(Z_t)$.

We define the *symmetrization* of the measure $m_\theta^{(n)}$ by $\widetilde{m}_\theta^{(n)} := \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} (T_s)_\# \, m_\theta^{(n)}$ where $(T_s)_\# \, m_\theta^{(n)}$ denotes the push-forward of $m_\theta^{(n)}$ under the map $T_s : (x_1, \dots, x_n) \mapsto (x_{s(1)}, \dots, x_{s(n)})$. $\mathfrak{S}_n$ denotes the set of permutations of $\{1, \dots, n\}$.

Proof of Proposition 4.5.3. To prove that the push-forward measure of the symmetrization $\widetilde{m}_\theta^{(n)}$ under the map $(x_1, \dots, x_n) \mapsto \delta_{x_1} + \dots + \delta_{x_n}$ is reversible for unlabeled sticky Brownian motions $(\eta_t)_{t \geq 0}$, it suffices to verify

$$\int (P_t^{[n]} f_n) g_n \, \mathrm{d}\lambda_n = \int (P_t^{[n]} g_n) f_n \, \mathrm{d}\lambda_n$$

for symmetric $f_n, g_n \in \mathcal{C}_n$. Indeed, using $f_n \circ \varphi = f_n$ and the fact that the push-forward of the measure $\widetilde{m}_\theta^{(n)}$ under the map φ is equal to $m_\theta^{(n)}$ we obtain

$$\begin{aligned}
\int f_n(x) \mathbb{E}_x \left[g_n(Z_t) \right] \widetilde{m}_\theta^{(n)}(\mathrm{d}x) &= \int f_n(\varphi(x)) \mathbb{E}_x \left[g_n(Z_t) \right] \widetilde{m}_\theta^{(n)}(\mathrm{d}x) \\
&= \int f_n(\varphi(x)) \mathbb{E}_{\varphi(x)} \left[g_n(Y_t) \right] \widetilde{m}_\theta^{(n)}(\mathrm{d}x) \\
&= \int f_n(y) \mathbb{E}_y \left[g_n(Y_t) \right] m_\theta^{(n)}(\mathrm{d}y). \qquad (4.18)
\end{aligned}$$

Thereby, in the second equation, we use the strong consistency property (2.25). To be more precise, let $x \in \mathbb{R}^n$ be fixed and let $s \in \mathfrak{S}_n$ be such that $T_s(x) = \varphi(x)$. By applying (2.25) and using the fact that $g_n \circ T_s = g_n$, we obtain

$$\mathbb{E}_x\left[g_n(Z_t)\right] = \mathbb{E}_x\left[g_n(T_s(Z_t))\right] = \mathbb{E}_{T_s(x)}\left[g_n(Z_t)\right] = \mathbb{E}_{\varphi(x)}\left[g_n(Y_t)\right].$$

Brockington and Warren demonstrated in [BW23, Theorem 4.17] that $m_\theta^{(n)}$ is reversible for n ordered uniform sticky Brownian motions with stickiness θ. As a result, (4.18) is symmetric in f_n and g_n.

Hence, to complete the proof, it only remains to show that the symmetrization of $m_\theta^{(n)}$ coincides with λ_n up to a constant. Specifically, we claim that $\widetilde{m}_\theta^{(n)} = \frac{1}{\theta^n n!}\lambda_n$. To recall, we construct λ_n using the measure $\alpha := \theta\lambda$ through (3.31). Note that both $\widetilde{m}_\theta^{(n)}$ and λ_n are invariant under permutation of variables. Hence, it suffices to prove that $\int f_n \,\mathrm{d}\widetilde{m}_\theta^{(n)} = \int f_n \,\mathrm{d}\lambda_n$ holds for all symmetric functions $f_n \in \mathcal{C}_n$.

By using the fact that the Lebesgue measure is diffuse, meaning that $\lambda(\{x\}) = 0$ for all $x \in \mathbb{R}$, we obtain for all $\pi = (a_1, \ldots, a_k) \in \Pi_n$

$$\begin{aligned}
\int (f_n)_\pi \,\mathrm{d}\lambda^{\otimes k} &= \sum_{s \in \mathfrak{S}_k} \int \mathbb{1}_{\left\{x_{s(1)} \geq \ldots \geq x_{s(k)}\right\}} (f_n)_\pi(x_1, \ldots, x_k) \lambda^{\otimes k}(\mathrm{d}(x_1, \ldots, x_k)) \\
&= \sum_{s \in \mathfrak{S}_k} \int \mathbb{1}_{\{x_1 \geq \ldots \geq x_k\}} (f_n)_\pi(x_{s^{-1}(1)}, \ldots, x_{s^{-1}(k)}) \lambda^{\otimes k}(\mathrm{d}(x_1, \ldots, x_k)) \\
&= \sum_{s \in \mathfrak{S}_k} \int f_n \,\mathrm{d}\lambda^{\geq}_{\left(a_{s(1)}, \ldots, s_{s(k)}\right)},
\end{aligned}$$

as $(f_n)_\pi$ is symmetric.

We observe that every permutation $(a_{s(1)}, \ldots, a_{s(k)})$ of an ordered partition is itself an ordered partition. Using this fact along with Remark 3.3.3, we get

$$\begin{aligned}
\int f_n \,\mathrm{d}\widetilde{m}_\theta^{(n)} = \int f_n \,\mathrm{d}m_\theta^{(n)} &= \sum_{\pi = (a_1, \ldots, a_k) \in \Pi_n} \theta^{k-n} \frac{1}{k!} \frac{1}{a_1 \cdots a_k} \int (f_n)_\pi \,\mathrm{d}\lambda^{\otimes k} \\
&= \frac{1}{n!} \sum_{\sigma = \{B_1, \ldots, B_k\} \in \Sigma_n} \theta^{k-n} (|B_1| - 1)! \cdots (|B_k| - 1)! \int (f_n)_\sigma \lambda^{\otimes k} \\
&= \frac{1}{\theta^n n!} \int f_n \,\mathrm{d}\lambda_n.
\end{aligned}$$

We can switch from the summation over ordered partitions Π_n to the summation over partitions Σ_n since

$$\begin{aligned}
&|\{B_1, \ldots, B_k\} \in \Sigma_n : \{|B_1|, \ldots, |B_k|\} = \{l_1, \ldots, l_k\}| \\
&\qquad = \frac{n!}{k!} \frac{1}{l_1! \cdots l_k!} |(a_1, \ldots, a_k) \in \Pi_n : \{a_1, \ldots, a_k\} = \{l_1, \ldots, l_k\}|
\end{aligned}$$

holds for all $l_1, \ldots, l_k \in \mathbb{N}$, $k \in \mathbb{N}$ with $l_1 + \ldots + l_k = n$. □

5 The inclusion process and the Brownian energy process in the continuum

We dedicate this chapter to the generalization of the duality functions (1.11) and (1.12), which connect the symmetric inclusion process and the Brownian energy process, to uncountable spaces. This is achieved through two intertwining relations. The first involves moment measures, while the second relates to infinite-dimensional Laguerre polynomials.

Firstly, in Section 5.1, we do not present new results. Instead, we revisit a well-known intertwining relation in the field of population genetics that connects the *Moran model* and the *Fleming-Viot process*, see, e.g., [Daw93, Corollary 2.8.2]. Our focus is on demonstrating that this relation provides a suitable framework for the generalization and recovery of the duality function (1.11). Specifically, we formulate an intertwining relation in terms of moment measures connecting the generalized symmetric inclusion process, which is closely related to the measure-valued Moran model as we have already observed in Section 4.3, and the measure-valued Fleming-Viot process, which turns out to be an infinite-dimensional analogue of the BEP. We present our approach in an abstract manner to emphasize that our methodology is a general principle within the context of consistent particle systems.

Secondly, our main contribution is presented in Theorem 5.2.2 in Section 5.2: We apply the machinery of orthogonalization of intertwining relations developed in Chapter 3 to the intertwining relation in terms of moment measures. This leads to an intertwining relation in terms of infinite-dimensional Laguerre polynomials which are the orthogonal polynomials with respect to the distribution of the Gamma process. More precisely, we show that Laguerre polynomials intertwine any consistent particle system, that admits the distribution of the Pascal process as a reversible measure, and its many-particle limit. This result holds true, for instance, in the case of the generalized symmetric inclusion process and the infinite-dimensional analogue of the Brownian energy process. Consequently, we recover the duality function (1.12) of the SIP and the BEP.

In Chapter 6 below, we delve into the algebraic approach to intertwining relations. Particularly, in Section 6.2.3, we explore intertwining relations in terms of infinite-dimensional Laguerre polynomials in the algebraic context.

5.1 Moment intertwining

To formulate the first intertwining relation, we begin by introducing some notation and offering a reminder to the reader about measure-valued Markov processes, see, e.g., [Daw93, Section 2.1]. We denote the set of finite measures ν on $(E,\mathcal{E})$ by $\mathbf{M}_{<\infty}$ and equip it with the σ-algebra $\mathcal{M}_{<\infty}$ generated by the evaluations $A\mapsto\nu(A)$, $A\in\mathcal{E}$. For each $s\in[0,\infty)$, we denote by $\mathbf{M}_s$ the set of measures $\nu\in\mathbf{M}_{<\infty}$ with total mass s, i.e., $\nu(E)=s$. Let $(\hat{\Omega},\hat{\mathcal{F}},(\zeta_t)_{t\geq 0},(\hat{\mathbb{P}}_\nu)_{\nu\in\mathbf{M}_{<\infty}})$ be a family that consists of a measurable space $(\hat{\Omega},\hat{\mathcal{F}})$, measurable maps $\zeta_t:\hat{\Omega}\to\mathbf{M}_{<\infty}$ and probability measures $\hat{\mathbb{P}}_\nu$, $\nu\in\mathbf{M}_{<\infty}$ on $(\hat{\Omega},\hat{\mathcal{F}})$. This family is called a Markov family with state space $\mathbf{M}_{<\infty}$ if it satisfies the following conditions: $\hat{\mathbb{P}}_\nu\left[\zeta_0=\nu\right]=1$ holds for each $\nu\in\mathbf{M}_{<\infty}$; for each $B\in\mathcal{M}_{<\infty}$ and $t\geq 0$, the map $\nu\to\hat{\mathbb{P}}_\nu\left[\zeta_t\in B\right]$ is measurable; and the Markov property is assumed to be satisfied with respect to the natural filtration. We denote the expectation with respect to $\hat{\mathbb{P}}_\nu$ by $\hat{\mathbb{E}}_\nu$.

Let $(\Omega,\mathcal{F},(\eta_t)_{t\geq 0},(\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ be another Markov family with state space $\mathbf{N}_{<\infty}$. We say that the *moment intertwining relation* is satisfied if

$$\hat{\mathbb{E}}_\nu\left[\int F(\delta_{x_1}+\ldots+\delta_{x_n})\,\mathrm{d}\zeta_t^{\otimes n}(\mathrm{d}(x_1,\ldots,x_n))\right]$$
$$=\int\mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}\left[F(\eta_t)\right]\nu^{\otimes n}(\mathrm{d}(x_1,\ldots,x_n))\tag{IR.3}$$

holds for all $n\in\mathbb{N}$, $\nu\in\mathbf{M}_{<\infty}$, measurable non-negative (or bounded) $F:\mathbf{N}_n\to\mathbb{R}$ and $t\geq 0$. Relation (IR.3) states that the evolution of the n-th moment measure of $(\zeta_t)_{t\geq 0}$ can be expressed through the n-particle evolution of the particle system. This relation has already been studied in the field of population genetics, see, e.g., [DH82] or [Daw93, Corollary 2.8.2] and in the context of measure-valued flows, see [Xia09, Equation (1.1)].

If $(\eta_t)_{t\geq 0}$ is conservative, we can express the intertwining relation (IR.3) in terms of operators: Consider the n-particle semigroup $P_t^{[n]}$ of $(\eta_t)_{t\geq 0}$, as defined in (2.9), and let $(S_t)_{t\geq 0}$ be the Markov semigroup of $(\zeta_t)_{t\geq 0}$. With these notations, (IR.3) can be reformulated into $S_tM_nf_n(\nu)=M_nP_t^{[n]}f_n(\nu)$ for $\nu\in\mathbf{M}_{<\infty}$ and measurable non-negative (or bounded) functions $f_n:E^n\to\mathbb{R}$ where

$$M_nf_n(\nu):=\int f_n\,\mathrm{d}\nu^{\otimes n}\tag{5.1}$$

and $M_0f_0(\nu):=f_0$. We recall the fact that a consistent Markov family is conservative, see Proposition 2.1.2.

Proposition 5.1.1. *If the Markov process $(\eta_t)_{t\geq 0}$ is consistent, then there exists a Markov family $(\hat{\Omega},\hat{\mathcal{F}},(\zeta_t)_{t\geq 0},(\hat{\mathbb{P}}_\nu)_{\nu\in\mathbf{M}_{<\infty}})$ such that* (IR.3) *is satisfied. Moreover, the finite-dimensional distributions of $(\zeta_t)_{t\geq 0}$ are determined uniquely.*

Proposition 5.1.1 has been examined in the context of exchangeable particle systems, see [Xia09, Theorem 2.1], and is a direct consequence of de Finetti's theorem, see, e.g., [Daw93, Section 11.2]. Nevertheless, we provide a concise and self-contained proof in our context of consistent particle systems.

Proof. It suffices to construct transition kernels $S_t:\mathbf{M}_{<\infty}\times\mathcal{M}_{<\infty}\to[0,1]$ and to show their uniqueness. Fix $t\geq 0$ and $\nu\in\mathbf{M}_1$. For each $n\in\mathbb{N}$, we define the measure γ_n on

$(E^n, \mathcal{E}^{\otimes n})$ by

$$\gamma_n(B_n) := \int P_t^{[n]} \widetilde{\mathbb{1}}_{B_n} \, \mathrm{d}\nu^{\otimes n}, \quad B_n \in \mathcal{E}^{\otimes n}.$$

In other words, γ_n is equal to the distribution of the n-particle dynamics at time t where the particles in the initial configuration are sampled independently according to the probability distribution ν. Using (2.10), it can be observed that $\gamma_n, n \in \mathbb{N}$ satisfy Kolmogorov's consistency condition

$$\begin{aligned} \gamma_n(B_{n-1} \times E) &= \int P_t^{[n]} \left(\widetilde{\mathbb{1}}_{B_{n-1}} \otimes_{\mathrm{s}} \mathbb{1}_E \right) \mathrm{d}\nu^{\otimes n} \\ &= \int \left(P_t^{[n-1]} \widetilde{\mathbb{1}}_{B_{n-1}} \right) \mathrm{d}\nu^{\otimes (n-1)} = \gamma_{n-1}(B_{n-1}) \end{aligned}$$

for $B_{n-1} \in \mathcal{E}^{\otimes(n-1)}$. Thus, Kolmogorov's extension theorem provides a sequence of random variables $X_k \in E$, $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the joint distribution of $(X_1, \dots, X_n)$ is γ_n. It can be readily verified that the sequence $(X_k)_{k\in\mathbb{N}}$ is exchangeable, i.e., the joint distributions of $(X_{s(k)})_{k\in\mathbb{N}}$ and $(X_k)_{k\in\mathbb{N}}$ are equal for each bijection $s : \mathbb{N} \to \mathbb{N}$. As a result, de Finetti's theorem guarantees existence and uniqueness of a probability measure $S_t(\nu, \cdot\,)$ on $\mathbf{M}_1$ satisfying

$$\iint f_n \, \mathrm{d}\nu'^{\otimes n} S_t(\nu, \mathrm{d}\nu') = \int f_n \, \mathrm{d}\gamma_n = \int P_t^{[n]} f_n \, \mathrm{d}\nu^{\otimes n}$$

for measurable symmetric $f_n : E^n \to [0, \infty)$, i.e., (IR.3) holds. If $\nu \in \mathbf{M}_s$ for some $s > 0$, we define $S_t(\nu, B) = \int \mathbb{1}_B(s\nu') S_t(s^{-1}\nu, \mathrm{d}\nu')$ for $B \in \mathcal{M}_{<\infty}$. If $s = 0$, we set $S_t(0, \cdot\,) := \delta_0$ where 0 denotes the zero measure. It can readily be checked that this is the unique choice such that (IR.3) is satisfied.

It remains to show both measurability of the map $\nu \mapsto S_t(\nu, B)$ for all $B \in \mathcal{M}_{<\infty}$, $t \geq 0$ and the semigroup property $S_{t+h}F = S_t S_h F$. Let $u : E \to [0, \infty)$ be measurable and bounded. Recall that $u^{\otimes n}(x_1, \dots, x_n)$ is defined as $u(x_1) \cdots u(x_n)$, $x_1, \dots, x_n \in E$. By Lebesgue's dominated convergence theorem and (IR.3), we obtain

$$\nu \mapsto \int \mathrm{e}^{-\int u \, \mathrm{d}\nu'} S_t(\nu, \mathrm{d}\nu') = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int P_t^{[n]} u^{\otimes n} \, \mathrm{d}\nu^{\otimes n}$$

which is measurable as a limit of measurable functions. The claim then follows by using the functional monotone class theorem (see, e.g., [Bog07, Theorem 2.12.9]).

The semigroup property follows directly for $F(\nu) = \int f_n \, \mathrm{d}\nu^{\otimes n}$ and can be extended to all measurable non-negative (or bounded) functions F again by the functional monotone class theorem. □

Remark 5.1.2. The measure-valued process $(\zeta_t)_{t\geq 0}$ provided by Proposition 5.1.1 is a many-particle limit of the particle system $(\eta_t)_{t\geq 0}$. That is a conclusion drawn in de Finetti's theorem, see, e.g., part (b) of Theorem 11.2.1 in [Daw93]. Furthermore, this concept has also been studied in [DK96] and the references therein.

In the following discussion, we aim to provide a brief intuition behind this many-particle limit, employing the intertwining relation (IR.3). Notably, another crucial ingredient is the factorial measure intertwining relation (IR.1) for the particle system. We assume that E is a compact metric space endowed with the Borel σ–algebra and suppose that $(P_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$ are Feller semigroups. Consider configurations $\mu_k \in \mathbf{N}_{<\infty}$, $k \in \mathbb{N}$ that convergence after being scaled by constants $\epsilon_k > 0$ with $\epsilon_k \to 0$ to some $\nu \in \mathbf{M}_{<\infty}$, i.e., $\epsilon_k \mu_k \to \nu$ weakly as $k \to \infty$.

Then, for $p(\nu') := \int u_n \,\mathrm{d}\nu'^{\otimes n}$, $\nu' \in \mathbf{M}_{<\infty}$, where $u_n : E^n \to \mathbb{R}$ is symmetric and continuous, we obtain

$$\lim_{k\to\infty} \mathbb{E}_{\mu_k}\left[p(\epsilon_k \eta_t)\right] = \lim_{k\to\infty} \epsilon_k^n \,\mathbb{E}_{\mu_k}\left[\int u_n \,\mathrm{d}\eta_t^{\otimes n}\right] = \lim_{k\to\infty} \epsilon_k^n \,\mathbb{E}_{\mu_k}\left[\int u_n \,\mathrm{d}\eta_t^{(n)}\right].$$

We used the fact that $\eta_t^{\otimes n}$ is equal to $\eta_t^{(n)}$ up to lower degree product measures of η_t, see (3.3), and for $l < n$ and measurable bounded $v_l : E^l \to \mathbb{R}$ we have

$$\left|\epsilon_k^n \,\mathbb{E}_{\mu_k}\left[\int v_l \,\mathrm{d}\eta_t^{\otimes l}\right]\right| \leq \|v_l\|_\infty \,\epsilon_k^n \mu_k(E)^l = \|v_l\|_\infty \,\epsilon_k^{n-l}(\epsilon_k \mu_k(E))^l \to 0. \tag{5.2}$$

Consequently, applying (IR.1) and (IR.3) and employing the arguments of (5.2) again, we arrive at:

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}_{\mu_k}\left[p(\epsilon_k \eta_t)\right] &= \lim_{k\to\infty} \epsilon_k^n \int P_t^{[n]} u_n \,\mathrm{d}\mu_k^{(n)} \\ &= \lim_{k\to\infty} \epsilon_k^n \int P_t^{[n]} u_n \,\mathrm{d}\mu_k^{\otimes n} = \int P_t^{[n]} u_n \,\mathrm{d}\nu^{\otimes n} = \hat{\mathbb{E}}_\nu\left[p(\zeta_t)\right].\end{aligned}$$

By linearity, we conclude that

$$\mathbb{E}_{\mu_k}\left[p(\epsilon_k \eta_t)\right] \to \hat{\mathbb{E}}_\nu\left[p(\zeta_t)\right] \tag{5.3}$$

as $k \to \infty$ for all polynomials p with continuous coefficients. This property can be interpreted as follows: The scaled particle system $\epsilon_k \eta_t$, where η_t is starting at μ_k, converges to ζ_t starting at ν if $k \to \infty$. Consequently, this is a generalization of the scaling limits from the SIP to the BEP, as discussed in [CGGR13, Proposition 2.5]. A more refined analysis of the convergence, such as the convergence of finite-dimensional distributions can be found, e.g, in [Daw93, Theorem 2.7.1].

Next, we examine the intertwining relation (IR.3) for the generalized symmetric inclusion process. For a function $F : \mathbf{M}_{<\infty} \to \mathbb{R}$, if the limit exits,

$$\frac{\delta F(\nu)}{\delta \nu(x)} = \lim_{h\to 0} h^{-1}(F(\nu + \delta_x) - F(\nu))$$

denotes the derivative of F at ν in direction δ_x, $x \in E$. For instance, when $f_n : E^n \to \mathbb{R}$ is bounded and measurable, $\frac{\delta M_n f_n(\nu)}{\delta \nu(x)} = n M_{n-1} g_{n-1}(\nu)$ with $g_{n-1}(y_1, \dots, y_{n-1}) = \frac{1}{n}\sum_{k=1}^n f_n(y_1, \dots, y_{k-1}, x, y_k, \dots, y_{n-1})$. Let α be a finite measure on $(E, \mathcal{E})$, $c : E \times E \to$

$[0,\infty)$ be a bounded, measurable and symmetric function and $\hat{\mathcal{L}}$ be the formal generator given by

$$\hat{\mathcal{L}}F(\nu) = \frac{1}{2}\iint c(x,y)\left(\frac{\delta^2 F(\nu)}{\delta\nu(x)^2} - 2\frac{\delta^2 F(\nu)}{\delta\nu(x)\delta\nu(y)} + \frac{\delta^2 F(\nu)}{\delta\nu(y)^2}\right)\nu(\mathrm{d}x)\nu(\mathrm{d}y) + \iint c(x,y)\left(\frac{\delta F(\nu)}{\delta\nu(y)} - \frac{\delta F(\nu)}{\delta\nu(x)}\right)\nu(\mathrm{d}x)\alpha(\mathrm{d}y), \quad \nu \in \mathbf{M}_{<\infty}. \tag{5.4}$$

It can be easily verified that the intertwining relation (IR.3) holds true at the level of generators, i.e., $\hat{\mathcal{L}}M_n f_n = M_n\mathcal{L}^{[n]}f_n$ holds for all measurable symmetric bounded $f_n : E^n \to \mathbb{R}$ where $\mathcal{L}^{[n]}$ is a formal generator of the n-particle dynamics of the gSIP, for example, the formal generator (4.8). Thus, intuitively, the Markov family $(\hat{\Omega}, \hat{\mathcal{F}}, (\zeta_t)_{t\geq 0}, (\hat{\mathbb{P}}_\nu)_{\nu\in\mathbf{M}_{<\infty}})$ provided by Proposition 5.1.1 has the formal generator $\hat{\mathcal{L}}$.

The process $(\zeta_t)_{t\geq 0}$ is closely related to the *measure-valued Fleming-Viot* process which was introduced in [FV79] and studied, e.g., in [Shi90], [EG93a]. To be more precise, when $c(x,y)$ is constant and takes the value 1, then the formal generator (5.4) coincides with the generator [Daw93, Equation (2.6.4)]. In the latter equation, the mutation operator is chosen to be $\frac{1}{2}A$, where A is defined by (4.9), and γ is set to 4. Moreover, the intertwining relation (IR.3) is further explored in [Daw93, Corollary 2.8.2].

Furthermore, the process with formal generator (5.4) is a continuum version of the Brownian energy process. Indeed, if we consider a finite set E and identify each measure $\nu \in \mathbf{M}_{<\infty}$ with $\nu = (\nu_k)_{k\in E} \in [0,\infty)^E$, $\nu_k = \nu(\{k\})$ and the measure α with $\alpha = (\alpha_k)_{k\in E}$, $\alpha_k = \alpha(\{k\})$, then the formal generator $\hat{\mathcal{L}}$ from (5.4) coincides with the formal generator (1.10) of the BEP. The following proposition shows that the intertwining relation (IR.3) recovers the duality function (1.11) of the SIP and the BEP.

Proposition 5.1.3. *Let E be a finite set, $(\eta_t)_{t\geq 0}$ be conservative and $(\zeta_t)_{t\geq 0}$ be such that* (IR.3) *is satisfied. Let ρ be a probability measure on $\mathbf{N}_{<\infty} = \mathbb{N}_0^E$ that is reversible for $(\eta_t)_{t\geq 0}$ and fulfills $\rho(\{\mu\}) > 0$ for all $\mu \in \mathbb{N}_0^E$. Then,*

$$\mathcal{D}(\mu,\nu) = \frac{1}{\rho(\{\mu\})}\prod_{k\in E}\frac{\nu_k^{\mu_k}}{\mu_k!} \tag{5.5}$$

satisfies $\mathbb{E}_\mu\left[\mathcal{D}(\eta_t,\nu)\right] = \hat{\mathbb{E}}_\nu\left[\mathcal{D}(\mu,\zeta_t)\right]$ for all $\mu \in \mathbb{N}_0^E$ and $\nu \in [0,\infty)^E$, i.e., it is a duality function.

In the following, we assume, without loss of generality, that $E = \{1,\ldots,N\}$ where $N \in \mathbb{N}$. We remind the reader that $(a)^{(0)} = 1$, $(a)^{(l)} = a(a+1)\cdots(a+l-1)$ denotes the rising factorial. For the SIP, the product of negative binomial distributions with parameters $p \in (0,1)$ and α_k, i.e., $\rho(\{\mu\}) = (1-p)^{\alpha_1+\ldots+\alpha_N}p^{\mu_1+\ldots+\mu_N}\prod_{k=1}^N \frac{(\alpha_k)^{(\mu_k)}}{\mu_k!}$, is reversible. Therefore, Proposition 5.1.3 provides the duality function $(\mu,\nu) \mapsto \prod_{k=1}^n \frac{\nu_k^{\mu_k}}{(\alpha_k)^{(\mu_k)}}$, up to a multiplicative constant. This constant is irrelevant since it only depends on the total number of particles $\mu_1+\ldots+\mu_N$ that is a conserved quantity. In other words, we recover the duality function (1.11) of the SIP and the BEP.

Proof. Let $(P_t)_{t\geq 0}$ be the Markov semigroup of $(\eta_t)_{t\geq 0}$. We note that (IR.3) can also be expressed in terms of P_t and S_t. Specifically, we define

$$TF(\nu) := \sum_{k=0}^{\infty} \frac{1}{k!} \int F(\delta_{x_1} + \ldots + \delta_{x_k}) \nu^{\otimes k}(\mathrm{d}(x_1, \ldots, x_k)) \in [0, \infty) \cup \{\infty\}$$

for measurable functions $F : \mathbf{N}_{<\infty} \to [0, \infty)$. Then, (IR.3) rewrites as $TP_tF = S_tTF$.

We fix $\mu = (\mu_1, \ldots, \mu_N) \in \mathbf{N}_{<\infty} = \mathbb{N}_0^N$. Using (4.1), we observe

$$\begin{aligned} T\mathbb{1}_{\{\mu\}}(\nu) &= \sum_{k=0}^{\infty} \frac{1}{k!} \int \mathbb{1}_{\{\mu\}}(x_1, \ldots, x_k) \nu^{\otimes k}(x_1, \ldots, x_k) \\ &= \frac{1}{\mu_1! \cdots \mu_N!} \int \mathbb{1}_{\{1\}^{\mu_1} \times \cdots \times \{N\}^{\mu_N}} \, \mathrm{d}\nu^{\otimes \mu(E)} = \frac{\nu_1^{\mu_1} \cdots \nu_N^{\mu_N}}{\mu_1! \cdots \mu_N!} \\ &= \rho(\{\mu\}) \mathcal{D}(\mu, \nu) = \int \mathbb{1}_{\{\mu\}}(\mu') \mathcal{D}(\mu', \nu) \rho(\mathrm{d}\mu') \end{aligned}$$

for each $\nu = (\nu_1, \ldots, \nu_N) \in \mathbf{M}_{<\infty} = [0, \infty)^N$. Therefore, we have for all measurable functions $F : \mathbf{N}_{<\infty} \to [0, \infty)$

$$TF(\nu) = \int F(\mu') \mathcal{D}(\mu', \nu) \rho(\mathrm{d}\mu').$$

Consequently, we deduce by reversibility that

$$\begin{aligned} \rho(\{\mu\}) P_t \mathcal{D}(\,\cdot\,, \nu)(\mu) &= \int \mathbb{1}_{\{\mu\}}(\mu') P_t \mathcal{D}(\,\cdot\,, \nu)(\mu') \rho(\mathrm{d}\mu') \\ &= \int P_t \mathbb{1}_{\{\mu\}}(\mu') \mathcal{D}(\mu', \nu) \rho(\mathrm{d}\mu') \\ &= TP_t \mathbb{1}_{\{\mu\}}(\nu) = S_t T \mathbb{1}_{\{\mu\}}(\nu) = \rho(\{\mu\}) S_t \mathcal{D}(\mu, \,\cdot\,)(\nu). \end{aligned}$$ □

Remark 5.1.4. Beyond the generalized symmetric inclusion process, Proposition 5.1.1 is valid for all consistent particle systems. Therefore, it can be applied to all examples from Chapter 4, particularly for strongly consistent systems. Strongly consistent particle systems are characterized by stochastic flows, as shown in [LR04a, Theorem 2.1]. In particular, if a family $(P_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$ of n-particle semigroups is strongly consistent, see (2.25), then there exists a family of random probability kernels $K_t : E \times \mathcal{E} \to [0, 1]$, $t \geq 0$, such that

$$P_t^{[n]} f_n(x) = \mathbb{E}\left[\int f_n(y) K_t^{\otimes n}(x, \mathrm{d}y) \right]$$

holds for all $x \in E^n$ and all measurable non-negative (or bounded) $f_n : E^n \to \mathbb{R}$. Here, the set of probability kernels is equipped with a suitable σ-algebra, on which we do not delve into further detail. It can easily be checked that the unique S_t satisfying the intertwining relation (IR.3) is given by $S_t G(\nu) := \mathbb{E}\left[G(\nu K_t)\right]$ where $(\nu K_t)(B) := \int K_t(x,B)\nu(\mathrm{d}x)$, $B \in \mathcal{E}$, see [Xia09, Page 2]. Indeed, we have

$$\begin{aligned}\iint f_n \,\mathrm{d}\nu'^{\otimes n} S_t(\nu', \mathrm{d}\nu) &= \mathbb{E}\left[\int f_n \,\mathrm{d}(\nu K_t)^{\otimes n}\right] \\ &= \mathbb{E}\left[\iint f_n(y) K_t^{\otimes n}(x, \mathrm{d}y)\,\mathrm{d}\nu^{\otimes n}(x)\right] = \int P_t^{[n]} f_n \,\mathrm{d}\nu^{\otimes n}.\end{aligned}$$

In particular, for independent particles, where each particle evolves independently following the transition kernels $p_t : E \times \mathcal{E} \to [0,1]$, S_t is given by

$$S_t G(\nu) = G(\nu p_t).$$

In other words, S_t is the Markov semigroup of the deterministic process $\zeta_t = \nu p_t$ starting at $\zeta_0 = \nu$. That generalizes and recovers the duality of IRW in a countable space with a deterministic system of coupled differential equations [GKRV09, Section 3.5].

As another example, we already described the random kernels for correlated Brownian motions in (4.16) and, thus, we find that $S_t(\nu, \cdot\,)$ is the distribution of a random shift combined with the evolution of ν under the heat-semigroup, more precisely,

$$S_t G(\nu) = \int G\left(\delta_z * \mathcal{N}(0, (1-a)t) * \nu\right)\, \mathcal{N}(0, ta)(\mathrm{d}z)$$

where the normal distribution is denoted by $\mathcal{N}$.

5.2 Infinite-dimensional Laguerre polynomials

To generalize the duality function (1.12), we briefly revisit infinite-dimensional Laguerre polynomials. These polynomials have been extensively studied in the context of Gamma white noise analysis [KdSU98], [KL00], [Lyt03a], [Lyt03b], [GS11]. More precisely, the infinite-dimensional Laguerre polynomials are the orthogonal polynomials with respect to the distribution of the Gamma process. We recapitulate the definition of the Gamma process (see [Kin93, Section 9.2] or [LP17, Example 15.6]). We fix a measure α on $(E, \mathcal{E})$ and assume, for the sake of simplifying the notation, that α is finite. A random measure ξ is called *Gamma process with shape α and rate $\beta > 0$* if the following conditions hold:

(i) $\xi(A_1), \cdots, \xi(A_N)$ are independent for pairwise disjoint $A_1, \ldots, A_N \in \mathcal{E}$.

(ii) For $A \in \mathcal{E}$ with $\alpha(A) > 0$, $\xi(A)$ follows the Gamma distribution with shape $\alpha(A)$ and rate β, i.e.,

$$\mathbb{P}[\xi(A) \in B] = \frac{1}{\Gamma(\alpha(A))} \int_B x^{\alpha(A)-1} \mathrm{e}^{-\beta x} \beta^{\alpha(A)} \,\mathrm{d}x, \quad B \subset [0, \infty) \text{ measurable.}$$

If $\alpha(A) = 0$, then $\xi(A)$ is almost surely equal to zero.

We remind the reader of the measures λ_n defined in (3.31). The n-th moment measure of ξ is given by $\beta^n\lambda_n$, see [Pit06, Exercise 2.2.6] or also [BM73], i.e.,

$$\mathbb{E}\left[\xi^{\otimes n}(C)\right] = \beta^n\lambda_n(C), \quad C \in \mathcal{E}^{\otimes n}, n \in \mathbb{N}.$$

In the following, we assume, without loss of generality, that $\beta = 1$ since changing β reduces only in a scaling of ξ. We denote the distribution of the Gamma process with shape α and rate 1 by Γ_α. The space of polynomials $\mathcal{P}_n$ of degree at most n is defined similarly to (3.1), namely,

$$\mathcal{P}_n := \Bigg\{ \mathbf{M}_{<\infty} \ni \nu \mapsto u_0 + \sum_{k=1}^{n} \int u_k \, \mathrm{d}\nu^{\otimes k} : \\ u_0 \in \mathbb{R}, u_k : E^k \to \mathbb{R} \text{ measurable and bounded for } k \in \{1, \dots, n\} \Bigg\}. \tag{5.6}$$

Analogously to infinite-dimensional orthogonal polynomials in the setup of point processes, see (3.6), we define the *infinite-dimensional Laguerre polynomial*

$$L_n^\alpha f_n := \text{orthogonal projection of } \left(\nu \mapsto \int f_n \, \mathrm{d}\nu^{\otimes n}\right) \text{ onto } \mathcal{P}_{n-1}^\perp \tag{5.7}$$

for measurable bounded $f_n : E^n \to \mathbb{R}$ and $n \in \mathbb{N}$. Here, the orthogonal projection and the orthogonal complement are considered in $L^2(\Gamma_\alpha) := L^2(\mathbf{M}_{<\infty}, \mathcal{M}_{<\infty}, \Gamma_\alpha)$. Moreover, we put $L_n^\alpha f_0 = f_0$ for $f_0 \in \mathbb{R}$. We recall that the monic univariate Laguerre polynomials are denoted by $\mathscr{L}_n^{(a-1)}$, see (1.13).

Proposition 5.2.1. *The infinite-dimensional Laguerre polynomials satisfy the following properties:*

(i) *Let $d_1, \dots, d_N \in \mathbb{N}$ and $A_1, \dots, A_N \in \mathcal{E}$ be pairwise disjoint with $\alpha(A_k) > 0$ for all $k \in \{1, \dots, N\}$. Then,*

$$L^\alpha_{d_1+\dots+d_N}(\mathbb{1}_{A_1^{d_1}\times\dots\times A_N^{d_N}})(\nu) \\ = \mathscr{L}_{d_1}^{(\alpha(A_1)-1)}(\nu(A_1)) \cdots \mathscr{L}_{d_N}^{(\alpha(A_N)-1)}(\nu(A_N)) \tag{5.8}$$

for Γ_α-almost all $\nu \in \mathbf{M}_{<\infty}$.

(ii) *For measurable bounded $f_n : E^n \to \mathbb{R}$,*

$$L_n^\alpha f_n(\nu) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \iint \widetilde{f_n}(x, y) \kappa_{n,k}(x, \mathrm{d}y) \nu^{\otimes k}(\mathrm{d}x). \tag{5.9}$$

(iii) *For measurable bounded $f_n : E^n \to \mathbb{R}$ and $g_m : E^m \to \mathbb{R}$, the orthogonality relation*

$$\int (L_n^\alpha f_n)(L_m^\alpha g_m) \, \mathrm{d}\Gamma_\alpha = \mathbb{1}_{\{n=m\}} \, n! \int \widetilde{f_n}\widetilde{g_m} \, \mathrm{d}\lambda_n \tag{5.10}$$

holds true. Thus, L_n^α *extends uniquely to a bounded operator* $L^2(\lambda_n) \to L^2(\Gamma_\alpha)$ *satisfying* (5.10)*. Moreover, the operator*

$$\mathfrak{U} : \mathfrak{F} := \bigoplus_{n=0}^{\infty} \frac{1}{n!} L^2_{\mathrm{sym}}(\lambda_n) \to L^2(\Gamma_\alpha), \quad f = (f_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} L_n^\alpha f_n \tag{5.11}$$

is unitary.

(iv) *The generating functional is given by*

$$\sum_{n=0}^{\infty} \frac{1}{n!} L_n^\alpha u^{\otimes n}(\nu) = \exp\left(-\int \log(1+u)\,\mathrm{d}\alpha + \int \frac{u}{1+u}\,\mathrm{d}\nu \right) \tag{5.12}$$

for measurable bounded $u : E \to [0,\infty)$*.*

The theory of chaos decompositions for Gamma processes is not new, see [KdSU98, Section 4.2]. More precisely, (5.8) is a variant of [Lyt03b, Lemma 3.1], while (5.10), (5.11) and (5.12) correspond to [KdSU98, Proposition 4.6, Equation (4.9) and Equation (4.2)]. To the best of our knowledge, the explicit formula (5.9) is new. Furthermore, the proof of Proposition 5.2.1 proceeds entirely analogously to the proofs of Proposition 3.3.8, Proposition 3.3.9, Proposition 3.3.11 and Proposition 3.3.12 concerning the infinite-dimensional Meixner polynomials. For this reason, we omit the proof of Proposition 5.2.1 here.

The following theorem provides a sufficient criterion for an intertwining relation in terms of infinite-dimensional Laguerre polynomials and is our main contribution in this chapter. Let $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ be a consistent Markov family such that the intertwining relation (IR.3) holds with another Markov family $(\hat{\Omega}, \hat{\mathcal{F}}, (\zeta_t)_{t\geq 0}, (\hat{\mathbb{P}}_\nu)_{\nu\in\mathbf{M}_{<\infty}})$. We recall that $P_t^{[n]}$ denotes the n-particle semigroup of $(\eta_t)_{t\geq 0}$, see (2.9). Moreover, we recall that $\rho_{p,\alpha}$ denotes the distribution of the Pascal process with parameters p and α, see Section 3.3.

Theorem 5.2.2. *The measure* Γ_α *is reversible for* $(\zeta_t)_{t\geq 0}$ *if and only if* $\rho_{p,\alpha}$ *is reversible for* $(\eta_t)_{t\geq 0}$ *for one (and consequently all)* $p \in (0,1)$*. In this case, the intertwining relation*

$$\hat{\mathbb{E}}_\nu \left[L_n^\alpha f_n(\zeta_t) \right] = L_n^\alpha P_t^{[n]} f_n(\nu) \tag{IR.4}$$

holds true for Γ_α*-almost all* $\nu \in \mathbf{M}_{<\infty}$*,* $f_n \in L^2_{\mathrm{sym}}(\lambda_n)$*,* $n \in \mathbb{N}_0$ *and* $t \geq 0$*.*

Analogously to Section 3.2.2 or Section 3.3.5, we deduce that the unitary operator $\mathfrak{U}$ defined in (5.11) intertwines S_t, the Markov semigroup of $(\zeta_t)_{t\geq 0}$, and $P_t^{\mathfrak{F}}$, i.e., $S_t\mathfrak{U} = \mathfrak{U}P_t^{\mathfrak{F}}$ where $P_t^{\mathfrak{F}}$ is defined as

$$P_t^{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{F}, \quad (f_n)_{n\in\mathbb{N}_0} \mapsto (P_t^{[n]} f_n)_{n\in\mathbb{N}_0}.$$

The theorem applies to the generalized symmetric inclusion process together with the continuum version of the Brownian energy process with formal generator (5.4). Thus, the intertwining relation (IR.4) holds and, moreover, the measure Γ_α is reversible for $(\zeta_t)_{t\geq 0}$. The latter fact is not new. Indeed, this fact is equivalent to the reversibility of the distribution of the Dirichlet process, see Remark 5.2.3 below; this reversibility is known for the Fleming-Viot process, see [Shi90] or [EK92]. Nevertheless, Theorem 5.2.2 offers an alternative route to prove reversibility by exploiting intertwining relations.

Beyond the gSIP, Theorem 5.2.2 applies to sticky Brownian motions. To ensure the existence of a finite α such that the equivalent conditions in Theorem 5.2.2 hold true, we can consider sticky Brownian motions on the circle (see [LR04b]). The corresponding measure-valued process $(\zeta_t)_{t\geq 0}$ has already been investigated in [LL04, Proposition 7], particularly regarding the reversibility of the distribution of the Dirichlet process.

Proof. On the one hand, we assume that Γ_α is reversible. We observe that (IR.3) implies $S_t\mathcal{P}_n \subset \mathcal{P}_n$. Analogously to the proof of Theorem 3.1.6, we deduce the intertwining relation (IR.4): Since Γ_α is reversible, we obtain analogously to (3.12) that S_t commutes with the orthogonal projection onto the space $\overline{\mathcal{P}_n}$ denoted by Π_n. As a reminder, $M_n f_n(\nu) = \int f_n \, \mathrm{d}\nu^{\otimes n}$, $\nu \in \mathbf{M}_{<\infty}$ as defined in (5.1). Thus,

$$\begin{aligned}\hat{\mathbb{E}}_\nu \left[L_n^\alpha f_n(\zeta_t)\right] &= S_t L_n^\alpha f_n(\nu) = S_t(\mathrm{id} - \Pi_{n-1})M_n f_n = (\mathrm{id} - \Pi_{n-1})S_t M_n f_n \\ &= (\mathrm{id} - \Pi_{n-1})M_n P_t^{[n]} f_n = L_n^\alpha P_t^{[n]} f_n\end{aligned}$$

results in (IR.4) for measurable symmetric bounded $f_n : E^n \to \mathbb{R}$. Moreover, we obtain from the orthogonality relation (5.10)

$$\int (P_t^{[n]} f_n) g_n \, \mathrm{d}\lambda_n = \frac{1}{n!}\int (L_n^\alpha P_t^{[n]} f_n) L_n^\alpha g_n \, \mathrm{d}\Gamma_\alpha = \frac{1}{n!}\int (S_t L_n^\alpha f_n) L_n^\alpha g_n \, \mathrm{d}\Gamma_\alpha \tag{5.13}$$

for $f_n, g_n : E^n \to \mathbb{R}$. The right-hand side of Equation (5.13) is symmetric in f_n and g_n which implies the reversibility of the push-forward measure of λ_n under the map $\iota_n : (x_1, \ldots, x_n) \mapsto \delta_{x_1} + \ldots + \delta_{x_n}$. Therefore, $\rho_{p,\alpha}$ is reversible, as it is the sum of these push-forward measures up to scaling, as shown in Lemma 3.3.4. Moreover, $P_t^{[n]}$ is a well-defined bounded operator on $L^2_{\mathrm{sym}}(\lambda_n)$. Therefore, through an approximation argument, (IR.4) holds for all $f_n \in L^2_{\mathrm{sym}}(\lambda_n)$ as well.

On the other hand, let the push-forward of λ_n under ι_n be reversible. By using the arguments of the proof of Theorem 3.4.1 together with the explicit formula (5.9), we obtain (IR.4). Analogously to Corollary 3.4.3, it follows that Γ_α is reversible for $(\zeta_t)_{t\geq 0}$. □

Remark 5.2.3. We observe that within each component $\mathbf{M}_s$, the dynamics of $(\zeta_t)_{t\geq 0}$ are identical up to normalization. In other words,

$$\hat{\mathbb{E}}_\nu \left[G(s\zeta_t)\right] = \hat{\mathbb{E}}_{s\nu} \left[G(\zeta_t)\right] \tag{5.14}$$

holds for all measurable $G : \mathbf{M}_{<\infty} \to [0, \infty)$, $s \geq 0$, $t \geq 0$ and $\nu \in \mathbf{M}_{<\infty}$. Consequently, the finite dimensional distributions of $(\zeta_t)_{t\geq 0}$ are fully characterized by the initial values contained $\mathbf{M}_1$.

If Γ_α is reversible for the measure-valued process $(\zeta_t)_{t\geq 0}$, it directly follows from (5.14) that the distribution of the Dirichlet process is reversible as well. The *Dirichlet process*, introduced in [Fer73], is a random probability measure $\hat{\xi}$ whose distribution is equal to that of a normalized Gamma process, i.e., the distributions of $\hat{\xi}$ and $\frac{\xi}{\xi(E)}$ are equal when ξ is a Gamma process. The orthogonal polynomials associated with the distribution of the Dirichlet process are given by infinite-dimensional *Jacobi polynomials.* These polynomials are studied, also in connection with Poisson-Dirichlet distributions, in [Eth92], [Pec08], [GS10], [GS13]. In our setup, if we define the infinite-dimensional Jacobi polynomials $\mathcal{J}_n^\alpha$ in a manner analogous to the infinite-dimensional Laguerre polynomials, then we obtain the intertwining relation

$$\hat{\mathbb{E}}_\nu\left[\mathcal{J}_n^\alpha f_n(\zeta_t)\right] = \mathcal{J}_n^\alpha P_t^{[n]} f_n(\nu).$$

This relation can be derived using arguments analogous to those presented in Theorem 5.2.2.

The following corollary shows that the duality relation between the SIP and the BEP, as expressed in (1.12), arises from the intertwining relation (IR.4). Recall that, for a finite set E, we identify finite measures ν on $(E,\mathcal{E})$ with $\nu = (\nu_k)_{k\in E} \in [0,\infty)^E$, $\nu_k = \nu(\{k\})$. We remind the reader that $(a)^{(0)} = 1$, $(a)^{(l)} = a(a+1)\cdots(a+l-1)$ denotes the rising factorial.

Corollary 5.2.4. *Let E be a finite set. Let Γ_α be reversible for $(\zeta_t)_{t\geq 0}$, or equivalently, let $\rho_{p,\alpha}$ be reversible for $(\eta_t)_{t\geq 0}$. Then,*

$$\mathcal{D}(\mu,\nu) = \prod_{k\in E} \frac{1}{(\alpha_k)^{(\mu_k)}} \mathscr{L}_{\mu_k}^{(\alpha_k-1)}(\nu_k) \tag{5.15}$$

satisfies $\mathbb{E}_\mu\left[\mathcal{D}(\eta_t,\nu)\right] = \hat{\mathbb{E}}_\nu\left[\mathcal{D}(\mu,\zeta_t)\right]$ for all $\mu \in \mathbb{N}_0^E$ and $\nu \in [0,\infty)^E$, i.e., it is a duality function.

As a by-product of the following proof, we obtain that $\mathfrak{U}$ defined in (5.11) is an integral operator in the case of a finite set E, more precisely,

$$\mathfrak{U}f(\nu) = \sum_{n=0}^\infty \frac{1}{n!} \int f_n(x_1,\ldots,x_n)\mathcal{D}(\delta_{x_1}+\ldots+\delta_{x_n},\nu)\lambda_n(\mathrm{d}(x_1,\ldots,x_n))$$

for all $f \in \mathfrak{F}$. Thus, the intertwiner $\mathfrak{U}$ has the form given in (1.30).

Proof. We assume, without loss of generality, that E is given by $\{1,\ldots,N\}$ where $N \in \mathbb{N}$. We fix $\mu \in \mathbb{N}_0^E$ and define $f = (f_n)_{n\in\mathbb{N}_0} \in \mathfrak{F}$ by $f_n(x_1,\ldots,x_n) := \mathbb{1}_{\{\mu\}}(\delta_{x_1}+\ldots+\delta_{x_n})$, $x_1,\ldots,x_n \in E$. Using (4.1) and (5.8), we obtain

$$\mathfrak{U}f(\nu) = \frac{1}{\mu_1!\cdots\mu_N!} L_{\mu(E)}^\alpha \mathbb{1}_{\{1\}^{\mu_1}\times\cdots\times\{N\}^{\mu_N}}(\nu) = \prod_{k=1}^n \frac{1}{\mu_k!}\mathscr{L}_{\mu_k}^{(\alpha_k-1)}(\nu_k).$$

Using similar arguments as in the proof of Proposition 5.1.3, we conclude that

$$
\begin{aligned}
(\mu, \nu) \mapsto & \frac{1}{\rho_{p,\alpha}(\{\mu\})} \prod_{k=1}^{N} \frac{1}{\mu_k!} \mathscr{L}_{\mu_k}^{(\alpha_k-1)}(\nu_k) \\
& = \frac{1}{(1-p)^{\alpha_1+\ldots+\alpha_N} p^{\mu_1+\ldots+\mu_N}} \prod_{k=1}^{N} \frac{1}{(\alpha_k)^{(\mu_k)}} \mathscr{L}_{\mu_k}^{(\alpha_k-1)}(\nu_k)
\end{aligned}
$$

is a duality function. It is equal to (5.15) up to a multiplicative constant depending only on the total number of particles. □

6 An algebraic approach

In this chapter, our aim is to formulate an algebraic approach to intertwining relations for particle systems in an uncountable space, focusing on the $su(1,1)$ algebra. This approach generalizes the well-known algebraic approach to duality for particle systems in countable spaces which is reviewed in Section 1.1. To achieve this, we introduce raising, lowering and neutral operators indexed by test functions rather than lattice sites as typically done in the discrete setting. Families of operators indexed by functions are standard, for example, in canonical commutation relations for bosons in quantum many-body mechanics or quantum field theory [RS75, Chapter X.7]; they also appear in connection with current algebras and quantum probability [Ara70], [AB07], [AB09]. The operators under consideration are closely related to operators studied for infinite-dimensional orthogonal polynomials [Lyt03b] and also relate to representations of the algebra of the square of white noise, the $sl(2,\mathbb{R})$ current algebra, and the finite difference algebra [Bou91], [Ś00], [AFS02]. We present three representations of the $su(1,1)$ current algebra and explore their connections.

The first representation is employed to define a family of unitary operators $U_{\xi,\phi}$ that is parametrized by $\xi \in \mathbb{C}$ and $\theta \in \mathbb{R}$ and acts on $L^2(\rho_{p,\alpha};\mathbb{C})$. Theorem 6.5.1 states that each $U_{\xi,\phi}$ is a self-intertwiner of a consistent Markov process with Markov semigroup $(P_t)_{t\geq 0}$ that admits the law of the Pascal process as a reversible measure, meaning $U_{\xi,\phi}P_t = P_t U_{\xi,\phi}$. Consequently, we obtain a generalization of [CFG$^+$19, Theorem 3.1 1. (i)] to uncountable spaces. For a specific choice of the parameters ξ and θ, the unitary operator $U_{\xi,\theta}$ maps functions supported on n-particle configurations to infinite-dimensional Meixner polynomials of degree n up to proportionality constants, as stated in Theorem 6.4.2. This result generalizes [CFG$^+$19, Theorem 3.1 1. (ii)]. As a by-product to Theorem 6.5.1 and Theorem 6.4.2, we obtain a new algebraic proof of the self-intertwining relation provided by Theorem 3.1.6. In Section 6.5.2, our results are complemented by insights on how to express infinitesimal generators using our raising, lowering and neutral operators.

Furthermore, we demonstrate that the operator switching between the first and the second representation has a close connection to the intertwining relation (IR.2) in terms of infinite-dimensional Meixner polynomials. On the other hand, we show that the intertwining relation (IR.4) in terms of the infinite-dimensional Laguerre polynomials corresponds to the change between the first and the third representation. Moreover, we delve deeper into the algebraic structure of the $su(1,1)$ current algebra and present a Baker-Campbell-Hausdorff formula.

We emphasize that our methods also apply to consistent Markov processes that have a Poisson law as a reversible measure instead of a Pascal law. The relevant algebra is the Heisenberg algebra, the raising, lowering and neutral operators are replaced with creation, annihilation and number operators, and Charlier polynomials take the place of Meixner polynomials. The outcome is a generalization of [CFG$^+$19, Theorem 3.1 3]. Creation and annihilation operators in the continuum are well-known in the context of many-body quantum mechanics [RS75]; this is why we focus our presentation on $su(1,1)$. In contrast, our method does not apply to the $SU(2)$ symmetry and Krawtchouk polynomials relevant for exclusion processes [CFG$^+$19]. We need a reference measure on configurations with respect to which raising and lowering operators are dual, and that reference measure should be infinitely divisible. Poisson and negative binomial laws are infinitely divisible and thus have natural Lévy processes or fields as continuum counterparts. Bernoulli and binomial laws are not infinitely divisible and it is not clear what the associated continuum random field should be.

6.1 Fock representations

Various concepts related to current algebra and Fock representation are discussed in [RS75, Theorem X.43], [Kac90, Chapter 7], [Fuc95, (3.1.15)], [Mey95, starting from page 59], [AB07], [AB09, Section 5 and Section 8] and [BR13, Sections 5.2.1 and 5.2.2]. In the following definition, we fix our notion of a Fock representation of the $su(1,1)$ current algebra. Let $(E,\mathcal{E})$ be a Borel space and let $\mathcal{C}$ be an algebra of functions $E\to\mathbb{C}$ equipped with pointwise multiplication that is closed under complex conjugation. Let H be a Hilbert space (over $\mathbb{C}$) with inner product $\langle\cdot,\cdot\rangle$, antilinear in the second argument, and norm $\|\cdot\|$.

Definition 6.1.1. A family of operators $\mathrm{k}^+(\varphi)$, $\mathrm{k}^0(\varphi)$, $\mathrm{k}^-(\varphi)$, $\varphi\in\mathcal{C}$ is a *Fock representation* on H of the $su(1,1)$ current algebra with *vacuum* $\psi\in H$ if the following conditions hold:

(i) The operators $\mathrm{k}^+(\varphi)$, $\mathrm{k}^0(\varphi)$, $\mathrm{k}^-(\varphi): D\to D$ are linear for each $\varphi\in\mathcal{C}$. Thereby, D is equal to the linear hull of vectors of the form $\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi$, where $\varphi_1,\cdots,\varphi_n\in\mathcal{C}$, $n\in\mathbb{N}$, together with ψ.

(ii) The operators $\mathrm{k}^+(\varphi)$ and $\mathrm{k}^0(\varphi)$ are linear in φ, while $\mathrm{k}^-(\varphi)$ is antilinear in φ.

(iii) The vacuum ψ is cyclic for $\mathrm{k}^+(\varphi),\varphi\in\mathcal{C}$, i.e., D is dense in H.

(iv) The vacuum satisfies $\|\psi\|=1$ and has the property $\mathrm{k}^-(\varphi)\psi=0$ for all $\varphi\in\mathcal{C}$.

(v) For all $f,g\in D$ and for all $\varphi\in\mathcal{C}$, the adjoint relations hold:

$$\left\langle f,\mathrm{k}^0(\varphi)g\right\rangle=\left\langle \mathrm{k}^0(\varphi)f,g\right\rangle,\quad \left\langle f,\mathrm{k}^+(\varphi)g\right\rangle=\left\langle \mathrm{k}^-(\varphi)f,g\right\rangle. \tag{6.1}$$

(vi) For all $\varphi,\theta\in\mathcal{C}$,

$$\begin{gathered}[\mathrm{k}^0(\varphi),\mathrm{k}^+(\theta)]=\mathrm{k}^+(\varphi\theta),\quad [\mathrm{k}^0(\varphi),\mathrm{k}^-(\theta)]=-\mathrm{k}^-(\overline{\varphi}\theta),\\ [\mathrm{k}^-(\varphi),\mathrm{k}^+(\theta)]=2\mathrm{k}^0(\overline{\varphi}\theta)\end{gathered} \tag{6.2}$$

and $[\mathrm{k}^\#(\varphi),\mathrm{k}^\#(\theta)]=0$ for $\#\in\{+,0,-\}$ where $[A,B]:=AB-BA$.

By the definition of a Fock representation, it follows that all operators are closable. As we see in the following, the operators $\mathrm{k}^+$and k^- can be understood as raising and lowering operators. In contrast to the boson creation and annihilation operators, which satisfy the so-called *canonical commutation relations* (CCRs) and the *canonical anti-commutation relations* (CARs) (see, e.g., [RS75, Section X.7] and [BR13, Sections 5.2.1 and 5.2.2]), it is not assumed that $\mathcal{C}$ is a Hilbert space: The commutation relations in (6.2) above require that $\overline{\varphi}\theta$ belong to $\mathcal{C}$ for all $\varphi, \theta \in \mathcal{C}$. The operator k^0 behaves similarly to the number operator (see, e.g., [BR13, Section 5.2.1]), as the following proposition shows.

Proposition 6.1.2. *If $\mathbb{1}_E \in \mathcal{C}$, then $\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi$ is an eigenvector of $\mathrm{k}^0(\mathbb{1}_E)$ with eigenvalue $n + \langle \mathrm{k}^0(\mathbb{1}_E)\psi, \psi\rangle$ for all $n \in \mathbb{N}_0$, $\varphi_1, \ldots, \varphi_n \in \mathcal{C}$. In particular, $\mathrm{k}^0(\mathbb{1}_E)$ has discrete spectrum.*

Proof. Using (6.2) multiple times, we obtain

$$\mathrm{k}^0(\mathbb{1}_E)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n) = n\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n) + \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\mathrm{k}^0(\mathbb{1}_E).$$

Hence, the proof is concluded if we show that $\mathrm{k}^0(\mathbb{1}_E)\psi = \langle \mathrm{k}^0(\mathbb{1}_E)\psi, \psi\rangle \psi$. To do this, we make use of the fact that linear combinations of vectors in the form of $v = \mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_m)\psi$ with $\theta_1, \ldots, \theta_m \in \mathcal{C}$ and $m \in \mathbb{N}_0$ are dense in H and prove that $\langle \mathrm{k}^0(\mathbb{1}_E)\psi, v\rangle = \langle \mathrm{k}^0(\mathbb{1}_E)\psi, \psi\rangle \langle \psi, v\rangle$ holds for each v of such form. The equality for $m = 0$, i.e., $v = \psi$ is straightforward. On the one hand, we use the shortcut $x := \mathrm{k}^+(\theta_2)\cdots\mathrm{k}^+(\theta_m)\psi$ along with the adjoint relations (6.1) and the commutation relations (6.2) to obtain

$$\begin{aligned}\langle \mathrm{k}^0(\mathbb{1}_E)\psi, \mathrm{k}^+(\theta_1)x\rangle &= \langle \psi, \mathrm{k}^0(\mathbb{1}_E)\mathrm{k}^+(\theta_1)x\rangle = \langle \psi, \mathrm{k}^+(\theta_1)\mathrm{k}^0(\mathbb{1}_E)x\rangle + \langle \psi, \mathrm{k}^+(\theta_1)x\rangle \\ &= \langle \mathrm{k}^-(\theta_1)\psi, \mathrm{k}^0(\mathbb{1}_E)x\rangle + \langle \mathrm{k}^-(\theta_1)\psi, x\rangle = 0.\end{aligned}$$

On the other hand, we also have $\langle \psi, \mathrm{k}^+(\theta_1)x\rangle = \langle \mathrm{k}^-(\theta_1)\psi, x\rangle = 0$. □

Next, we present an orthogonality relation that can be derived purely through algebraic methods. We remind the reader of the measures λ_n defined in (3.31). In the following, the symmetrization, see (2.11), of the function $\theta_1 \otimes \cdots \otimes \theta_m : (x_1, \ldots, x_m) \mapsto \theta_1(x_1)\cdots\theta_m(x_m)$ is denoted by $\theta_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \theta_m$.

Proposition 6.1.3. *Assume the existence of a σ-finite measure α on $(E, \mathcal{E})$ such that $\mathcal{C} \subset L^1(E, \mathcal{E}, \alpha; \mathbb{C})$ and*

$$\mathrm{k}^0(\varphi)\psi = \frac{1}{2}\int \varphi \,\mathrm{d}\alpha\, \psi \tag{6.3}$$

holds for all $\varphi \in \mathcal{C}$. Then,

$$\begin{aligned}&\langle \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi, \mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_m)\psi\rangle \\ &\qquad = \mathbb{1}_{\{n=m\}}\, n! \int (\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n)\overline{(\theta_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \theta_m)}\,\mathrm{d}\lambda_n\end{aligned} \tag{6.4}$$

holds for all $\varphi_1, \ldots, \varphi_n, \theta_1, \ldots, \theta_m \in \mathcal{C}$, $n, m \in \mathbb{N}_0$.

The measure α is the infinite-dimensional counterpart to the *Bargmann index* going back to [Bar47, Section 9], see also [Bar61]. Indeed, in the univariate case, i.e., E contains only one element, the Bargmann index k of a representation $\mathrm{k}^+, \mathrm{k}^0, \mathrm{k}^-$ of the $su(1,1)$ algebra can be extracted as the eigenvalue of k^0 corresponding to the eigenvector ψ. In this case, our notation α relates to k as $k = \frac{\alpha}{2}$. When considering the general case of E, i.e., moving from $su(1,1)$ to its current algebra, α turns into a measure and relates to the representation through the relation (6.3).

Let $\mathrm{k}^+(\varphi)$, $\mathrm{k}^0(\varphi)$, $\mathrm{k}^-(\varphi)$, $\varphi \in \mathcal{C}$ be a Fock representation on a Hilbert space H with vacuum $\psi \in H$. Let $\mathrm{K}^+(\varphi)$, $\mathrm{K}^0(\varphi)$, $\mathrm{K}^-(\varphi)$, $\varphi \in \mathcal{C}$ be another Fock representation on a Hilbert space $\mathscr{H}$ with vacuum $\Psi \in H$ (with the same algebra $\mathcal{C}$). We say that they are unitarily equivalent, if there exists a unitary operator $\mathcal{U} : H \to \mathscr{H}$ switching between representations, i.e., $\mathcal{U}\psi = \Psi$ and $\mathcal{U}\mathrm{k}^\#(\varphi) = \mathrm{K}^\#(\varphi)\mathcal{U}$ on D for all $\# \in \{+, 0, -\}$ and $\varphi \in \mathcal{C}$.

Corollary 6.1.4. *Assume that two Fock representations with the same algebra $\mathcal{C}$ satisfy* (6.3) *with the same α. Then, they are unitarily equivalent. Moreover, the unitary operator that switches between these representations is unique.*

For the proof of Proposition 6.1.3 and Corollary 6.1.4 we show the following lemma.

Lemma 6.1.5. *The following equation holds for all $\theta, \varphi_1, \ldots, \varphi_n \in \mathcal{C}$:*

$$\begin{aligned} &\mathrm{k}^-(\theta)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n) \\ &= 2\sum_{k=1}^{n}\sum_{l=k+1}^{n} \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_{l-1})\mathrm{k}^+(\varphi_k\overline{\theta}\varphi_l)\mathrm{k}^+(\varphi_{l+1})\cdots\mathrm{k}^+(\varphi_n) \\ &\qquad + 2\sum_{k=1}^{n}\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_n)\mathrm{k}^0(\varphi_k\overline{\theta}) \\ &\qquad + \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\mathrm{k}^-(\theta). \end{aligned} \tag{6.5}$$

Proof. First, by using the commutation relations (6.2), we obtain

$$\begin{aligned} \mathrm{k}^-(\theta)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n) &= \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\mathrm{k}^-(\theta) \\ &\quad + 2\sum_{k=1}^{n}\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^0(\varphi_k\overline{\theta})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_n). \end{aligned} \tag{6.6}$$

Similarly, we have

$$\begin{aligned} \mathrm{k}^0(\theta)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n) &= \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\mathrm{k}^0(\theta) \\ &\quad + \sum_{k=1}^{n}\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_k\theta)\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_n). \end{aligned} \tag{6.7}$$

Combining (6.6) and (6.7) results in (6.5). □

Proof of Proposition 6.1.3. Applying (6.5) to ψ, we obtain from (6.3)

$$\begin{aligned}
&\mathrm{k}^-(\theta)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi \\
&= 2\sum_{k=1}^{n}\sum_{l=k+1}^{n}\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_{l-1})\mathrm{k}^+(\varphi_k\overline{\theta}\varphi_l)\mathrm{k}^+(\varphi_{l+1})\cdots\mathrm{k}^+(\varphi_n)\psi \\
&\qquad + \sum_{k=1}^{n}\left(\int \varphi_k\overline{\theta}\,\mathrm{d}\alpha\right)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_n)\psi. \qquad (6.8)
\end{aligned}$$

In other words, when applying $\mathrm{k}^-(\theta)$ to a vector $\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi$, we get a linear combination of vectors resulting from only $n-1$-times applications of k^+ to the vacuum. Consequently, by induction, we obtain that for $\varphi_1,\dots,\varphi_n$, $\theta_1,\dots,\theta_m$, $n>m$, there exists a family of $f_i^{(k)}\in\mathcal{C}$, $i\in\{1,\dots,n-m\}$, $k=1,\dots,N$ such that

$$\mathrm{k}^-(\theta_1)\cdots\mathrm{k}^-(\theta_m)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi = \sum_{k=1}^{N}\mathrm{k}^+(f_1^{(k)})\cdots\mathrm{k}^+(f_{n-m}^{(k)})\psi.$$

In particular, we have

$$\begin{aligned}
\left\langle\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi,\mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_m)\psi\right\rangle &= \left\langle\mathrm{k}^-(\theta_1)\cdots\mathrm{k}^-(\theta_m)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi,\psi\right\rangle \\
&= \sum_{k=1}^{N}\left\langle\mathrm{k}^+(f_1^{(k)})\cdots\mathrm{k}^+(f_{n-m}^{(k)})\psi,\psi\right\rangle \\
&= \sum_{k=1}^{N}\left\langle\psi,\mathrm{k}^-(f_1^{(k)})\cdots\mathrm{k}^-(f_{n-m}^{(k)})\psi\right\rangle = 0
\end{aligned}$$

by using the adjoint relation (6.1). Thus, we obtain (6.4) for $n\neq m$.

We prove the case $n=m$ inductively. The case $n=m=1$ follows by using the commutation relation (6.2) together with (6.3) and $\alpha=\lambda_1$:

$$\begin{aligned}
\left\langle\mathrm{k}^+(\varphi_1)\psi,\mathrm{k}^+(\theta_1)\psi\right\rangle &= \left\langle\mathrm{k}^-(\theta_1)\mathrm{k}^+(\varphi_1)\psi,\psi\right\rangle - \left\langle\mathrm{k}^+(\varphi_1)\mathrm{k}^-(\theta_1)\psi,\psi\right\rangle \\
&= 2\left\langle\mathrm{k}^0(\varphi_1\overline{\theta_1})\psi,\psi\right\rangle = \int\varphi_1\overline{\theta_1}\,\mathrm{d}\lambda_1.
\end{aligned}$$

Let (6.4) be true for $m-1=n-1$. Then, using (6.8),

$$\begin{aligned}
&\left\langle\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi,\mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_n)\psi\right\rangle \\
&\qquad = \left\langle\mathrm{k}^-(\theta_1)\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi,\mathrm{k}^+(\theta_2)\cdots\mathrm{k}^+(\theta_n)\psi\right\rangle
\end{aligned}$$

splits into two summands. On the one hand, the first summand is given by

$$\begin{aligned}
&\left(\int\varphi_k\overline{\theta_1}\,\mathrm{d}\alpha\right)\sum_{k=1}^{n}\left\langle\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_{k-1})\mathrm{k}^+(\varphi_{k+1})\cdots\mathrm{k}^+(\varphi_n)\psi,\mathrm{k}^+(\theta_2)\cdots\mathrm{k}^+(\theta_n)\psi\right\rangle \\
&\qquad = (n-1)!\sum_{k=1}^{n}\left(\int\varphi_k\overline{\theta_1}\,\mathrm{d}\alpha\right)\int(\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_{k-1}\otimes_{\mathrm{s}}\varphi_{k+1}\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_n) \\
&\qquad\qquad \left(\overline{\theta_2\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\theta_n}\right)\,\mathrm{d}\lambda_{n-1}.
\end{aligned}$$

Since λ_{n-1} is invariant under the permutation of the variables and

$$\sum_{k=1}^{n} \varphi_k(y)\left(\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_{k-1} \otimes_{\mathrm{s}} \varphi_{k+1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n\right)(x_1, \ldots, x_{n-1}) \\ = n\left(\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n\right)(y, x_1, \ldots, x_{n-1}) \tag{6.9}$$

holds for $x_1, \ldots, x_{n-1}, y \in E$, the first summand can be reformulated into

$$(n-1)! \sum_{k=1}^{n} \left(\int \varphi_k \overline{\theta_1} \, \mathrm{d}\alpha\right) \\ \int \left(\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_{k-1} \otimes_{\mathrm{s}} \varphi_{k+1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n\right) \left(\overline{\theta_2 \otimes \cdots \otimes \theta_n}\right) \, \mathrm{d}\lambda_{n-1} \\ = n! \int \left(\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n\right) \left(\overline{\theta_1 \otimes \cdots \otimes \theta_n}\right) \, \mathrm{d}(\alpha \otimes \lambda_{n-1}). \tag{6.10}$$

On the other hand, the second summand

$$2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} \langle \mathrm{k}^+(\varphi_1) \cdots \mathrm{k}^+(\varphi_{k-1}) \mathrm{k}^+(\varphi_{k+1}) \cdots \mathrm{k}^+(\varphi_{l-1}) \mathrm{k}^+(\varphi_k \overline{\theta_1} \varphi_l) \\ \mathrm{k}^+(\varphi_{l+1}) \cdots \mathrm{k}^+(\varphi_n)\psi, \mathrm{k}^+(\theta_2) \cdots \mathrm{k}^+(\theta_n)\psi\rangle \\ = 2(n-1)! \sum_{k=1}^{n} \sum_{l=k+1}^{n} \int \Big(\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_{k-1} \otimes_{\mathrm{s}} \varphi_{k+1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_{l-1} \\ \otimes_{\mathrm{s}} (\varphi_k \overline{\theta_1} \varphi_l) \otimes_{\mathrm{s}} \varphi_{l+1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n\Big) \left(\overline{\theta_2 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \theta_n}\right) \, \mathrm{d}\lambda_{n-1}$$

is equal to $\int \widetilde{R}\left(\overline{\theta_2 \otimes \cdots \otimes \theta_n}\right) \, \mathrm{d}\lambda_{n-1}$ since symmetrization is a linear operation. Here, $\widetilde{R}$ is the symmetrization, see (2.11), of $R : E^{n-1} \to \mathbb{C}$ given by

$$R(x_1, \ldots, x_{n-1}) \\ = (n-1)! \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} \varphi_1(x_1) \cdots \varphi_{k-1}(x_{k-1}) \varphi_k(x_l) \varphi_{k+1}(x_k) \cdots \varphi_n(x_{n-1}) \overline{\theta_1}(x_l) \\ = (n-1)! \sum_{k=1}^{n} (\varphi_1 \otimes \cdots \otimes \varphi_{k-1} \otimes \varphi_{k+1} \otimes \cdots \otimes \varphi_n)(x_1, \ldots, x_{n-1}) \\ \int \varphi_k(y) \overline{\theta_1}(y) (\delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}y).$$

Thus, using (6.9) once more, we have

$$\widetilde{R}(x_1, \ldots, x_{n-1}) \\ = n! \int (\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n)(y, x_1, \ldots, x_n) \overline{\theta_1}(y) (\delta_{x_1} + \ldots + \delta_{x_{n-1}})(\mathrm{d}y). \tag{6.11}$$

Combining (6.10) and (6.11) and using the definition of λ_n, see (3.31), we obtain

$$\begin{aligned}
&\left\langle \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi, \mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_n)\psi\right\rangle \\
&\quad = n!\iint \left(\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_n\right)(x_1,\dots,x_{n-1},y)\left(\overline{\theta_1\otimes\cdots\otimes\theta_n}\right)(x_1,\dots,x_{n-1},y) \\
&\qquad\qquad (\alpha+\delta_{x_1}+\ldots+\delta_{x_{n-1}})(\mathrm{d}y)\lambda_{n-1}(\mathrm{d}(x_1,\dots,x_{n-1})) \\
&\quad = n!\int \left(\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_n\right)\left(\overline{\theta_1\otimes\cdots\otimes\theta_n}\right)\,\mathrm{d}\lambda_n \\
&\quad = n!\int (\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_n)(\overline{\theta_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\theta_n})\,\mathrm{d}\lambda_n. \qquad\square
\end{aligned}$$

Proof of Corollary 6.1.4. We define a linear operator $\mathcal{U}$ as follows:

$$\mathcal{U}: D\to\mathscr{H},\quad \mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi\mapsto \mathrm{K}^+(\varphi_1)\cdots\mathrm{K}^+(\varphi_n)\Psi. \tag{6.12}$$

First, we need to verify that the assignment (6.12) is well-defined. It suffices to prove that, for

$$v = c\psi+\sum_{k=1}^{n}\sum_{i=1}^{j_k}\mathrm{k}^+(\varphi_1^{(i,k)})\cdots\mathrm{k}^+(\varphi_k^{(i,k)})\psi\in H, \tag{6.13}$$

$$V = c\psi+\sum_{k=1}^{n}\sum_{i=1}^{j_k}\mathrm{K}^+(\varphi_1^{(i,k)})\cdots\mathrm{K}^+(\varphi_k^{(i,k)})\Psi\in \mathscr{H} \tag{6.14}$$

with $c\in\mathbb{C}$, $n\in\mathbb{N}$, $j_1,\dots,j_n\in\mathbb{N}$, $\varphi_1^{(i,k)},\dots,\varphi_k^{(i,k)}\in\mathcal{C}$, where $i\in\{1,\dots,j_k\}, k\in\{1,\dots,n\}$, the following holds: if $v=0$, then $V=0$. So, let $v=0$. Proposition 6.1.3 implies, firstly, $\langle V,\Psi\rangle_{\mathscr{H}}=c=\langle v,\psi\rangle_H=0$, secondly,

$$\begin{aligned}
\left\langle V, \mathrm{K}^+(\theta_1)\cdots\mathrm{K}^+(\theta_m)\Psi\right\rangle_{\mathscr{H}} &= \int\sum_{i=1}^{j_m}(\varphi_1^{(i,m)}\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_m^{(i,m)})(\theta_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\theta_m)\,\mathrm{d}\lambda_m \\
&= \left\langle v, \mathrm{k}^+(\theta_1)\cdots\mathrm{k}^+(\theta_m)\psi\right\rangle_H = 0
\end{aligned}$$

for $\theta_1,\dots,\theta_m\in\mathcal{C}$ if $1\le m\le n$ and thirdly, $\langle V,\mathrm{K}^+(\theta_1)\cdots\mathrm{K}^+(\theta_m)\Psi\rangle_{\mathscr{H}}=0$ if $m>n$. Thus, we obtain $V=0$.

By Proposition 6.1.3, $\mathcal{U}$ preserves the inner product. Therefore, it is a bounded operator that can be uniquely extended to an operator $H\to\mathscr{H}$ preserving the inner product as well. $\mathcal{U}$ is invertible: The inverse operator is given by the unique continuous and linear extension of the map $\mathrm{K}^+(\varphi_1)\cdots\mathrm{K}^+(\varphi_n)\Psi\to\mathrm{k}^+(\varphi_1)\cdots\mathrm{k}^+(\varphi_n)\psi$.

By the definition of of the operator $\mathcal{U}$, the relation $\mathcal{U}\mathrm{k}^\#(\varphi)=\mathrm{K}^\#(\varphi)\mathcal{U}$ on D is straightforward when # is +. When $\#\in\{0,-\}$, this relation follows by (6.5) and (6.7), along with (6.3), by expressing both $\mathrm{k}^0(\varphi)f$ and $\mathrm{k}^-(\varphi)f$ for $f\in D$ in terms of operators $\mathrm{k}^+(\,\cdot\,)$ only.

□

6.2 Three representations

In this section, we present three examples of Fock representations of the $su(1,1)$ current algebra.

(i) The Hilbert space of the first representation is the extended Fock space. The operators in this representation are continuum counterparts to the objects in the univariate case (1.17), (1.18), (1.19). Moreover, this representation is analogous to the representation of the canonical commutation relations in the Fock space for bosons in quantum many-body theory, see, e.g., [RS75, Section X.7]).

(ii) The second representation is defined on $L^2(\rho_{p,\alpha};\mathbb{C})$ where $\rho_{p,\alpha}$ is the distribution of the Pascal process. The operators closely relate to infinite-dimensional Meixner polynomials. They serve as continuum counterparts to the representation (1.22), (1.23), (1.24).

(iii) The third representation acts on $L^2(\Gamma_\alpha;\mathbb{C})$ where Γ_α is the distribution of the Gamma process. We show that the operators have close connection to infinite-dimensional Laguerre polynomials. They are the continuum counterparts to the representation (1.25), (1.26), (1.27).

Additionally, we show that the operator that switches between the representations (i) and (ii) has a close connection to the intertwining relation (IR.2) in terms of infinite-dimensional Meixner polynomials. Furthermore, we reveal the connection between the intertwining relation (IR.4) and the operator that switches between (i) and (iii).

6.2.1 On the extended Fock space

In the following, we fix a σ-finite measure α on a Borel space $(E,\mathcal{E})$ and fix $p \in (0,1]$. We define the algebra $\mathcal{C}$ as the set of measurable bounded functions $\varphi : E \to \mathbb{C}$ such that $\alpha(\{x \in E : \varphi(x) \neq 0\}) < \infty$. For each $n \in \mathbb{N}$, we denote the set of functions $f_n : E^n \to \mathbb{C}$ that are symmetric and square-integrable with respect to λ_n by $L^2_{\mathrm{sym}}(\lambda_n;\mathbb{C})$. The Hilbert space employed in the first representation is the extended Fock space $\mathfrak{F}_\mathbb{C} := \bigoplus_{n=0}^\infty \frac{p^n}{n!} L^2_{\mathrm{sym}}(\lambda_n;\mathbb{C})$ over the field $\mathbb{C}$. This definition is analogous to the real-valued case detailed in Section 3.3.3, i.e., $\mathfrak{F}_\mathbb{C}$ consists of sequences $(f_n)_{n\in\mathbb{N}_0}$ with $f_0 \in \mathbb{C}$ and measurable symmetric $f_n : E^n \to \mathbb{C}$ satisfying $|f_0|^2 + \sum_{n=1}^\infty \frac{p^n}{n!} \int |f_n|^2 \,\mathrm{d}\lambda_n < \infty$. The space $\mathfrak{F}_\mathbb{C}$ is equipped with the inner product $\langle f, g\rangle = f_0\overline{g_0} + \sum_{n=1}^\infty \frac{p^n}{n!} \int f_n \overline{g_n} \,\mathrm{d}\lambda_n$ where $f = (f_n)_{n\in\mathbb{N}_0}$, $g = (g_n)_{n\in\mathbb{N}_0} \in \mathfrak{F}_\mathbb{C}$. The vector $\psi := (1,0,0,\ldots) \in \mathfrak{F}_\mathbb{C}$ turns out to be the vacuum of the first representation.

For $\varphi \in \mathcal{C}$ and every sequence $f = (f_n)_{n\in\mathbb{N}_0}$ consisting of $f_0 \in \mathbb{C}$ and $f_n : E^n \to \mathbb{C}$, we define the sequence $k^+(\varphi)f = ((k^+(\varphi)f)_n)_{n\in\mathbb{N}_0}$ by $(k^+(\varphi)f)_0 = 0$ and

$$
\begin{aligned}
(k^+(\varphi)f)_n(x_1,\ldots,x_n) &= \frac{n}{\sqrt{p}}(f_{n-1} \otimes_{\mathrm{s}} \varphi)(x_1,\ldots,x_n) \qquad (6.15)\\
&= \frac{1}{\sqrt{p}} \sum_{l=1}^n \varphi(x_l) f_{n-1}(x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_n)
\end{aligned}
$$

for $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$. Here, the function $f_{n-1} \otimes_{\mathrm{s}} \varphi$ denotes the symmetrization of $f_{n-1} \otimes \varphi : (x_1, \dots, x_{n-1}, y) \mapsto f_{n-1}(x_1, \dots, x_{n-1})\varphi(y)$. Let $\mathcal{D}$ be defined as the space consisting of linear combinations of

$$k^+(\varphi_1) \cdots k^+(\varphi_n)\psi, \quad \varphi_1, \dots, \varphi_n \in \mathcal{C}, \quad n \in \mathbb{N}$$

and the vacuum ψ. For each $\varphi \in \mathcal{C}$, $f \in \mathcal{D}$ and $n \in \mathbb{N}$, we define

$$(k^0(\varphi)f)_n(x_1, \dots, x_n) = f_n(x_1, \dots, x_n) \left(\sum_{l=1}^{n} \varphi(x_l) + \frac{1}{2} \int \varphi \, \mathrm{d}\alpha \right) \tag{6.16}$$

and

$$\begin{aligned} &(k^-(\varphi)f)_n(x_1, \dots, x_n) \\ &\quad = \sqrt{p} \sum_{l=1}^{n} \overline{\varphi(x_l)} f_{n+1}(x_1, \dots, x_n, x_l) + \sqrt{p} \int \overline{\varphi(y)} f_{n+1}(x_1, \dots, x_n, y) \alpha(\mathrm{d}y). \end{aligned} \tag{6.17}$$

In the case where $n = 0$, we apply the convention that summing from $l = 1$ to n equals 0. The proof that both $k^-(\varphi)f$ and $k^0(\varphi)f$ are well-defined is presented in the proof of Theorem 6.2.1 below.

Theorem 6.2.1. *The family $k^+(\varphi), k^0(\varphi), k^-(\varphi)$, $\varphi \in \mathcal{C}$ is a Fock representation on $\mathfrak{F}_{\mathbb{C}}$ of the $su(1,1)$ current algebra with vacuum ψ.*

The operators $k^+(\varphi)$, $k^0(\varphi)$ and $k^-(\varphi)$ are not new. In fact, they coincide up to constants to the operators studied in [Lyt03a, Proposition 6.1] in the context of *Jacobi fields* of polynomials of Meixner's type (see also [BLM03], [Lyt03b] and [Lyt15]).

Remark 6.2.2. An alternative formulation of this Fock representation can be obtained using point process notation. We can identify each $f = (f_n)_{n \in \mathbb{N}_0}$ in the extended Fock space $\mathcal{F}_{\mathbb{C}}$ with an $F \in L^2(w_{p,\alpha}; \mathbb{C}) := L^2(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty}, w_{p,\alpha}; \mathbb{C})$ that satisfies $F(0) = f_0$ and $F(\delta_{x_1} + \ldots + \delta_{x_n}) = f_n(x_1, \dots, x_n)$ for each $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$. Here, $w_{p,\alpha}$ is a measure on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ such that, for measurable $G : \mathbf{N}_{<\infty} \to [0, \infty)$,

$$\int G \, \mathrm{d}w_{p,\alpha} = G(0) + \sum_{n=1}^{\infty} \frac{p^n}{n!} \int G(\delta_{x_1} + \ldots + \delta_{x_n}) \lambda_n(\mathrm{d}(x_1, \dots, x_n)). \tag{6.18}$$

If α is a finite measure, we have $(1-p)^{\alpha(E)} w_{p,\alpha} = \rho_{p,\alpha}$ for all $p \in (0,1)$, see Lemma 3.3.4. By abusing notation, we also denote $L^2(w_{p,\alpha}; \mathbb{C})$ as $\mathfrak{F}_{\mathbb{C}}$. In this alternative formulation, the operators $k^+(\varphi), k^-(\varphi), k^0(\varphi)$ defined in (6.15), (6.16) and (6.17) become

$$k^+(\varphi)F(\mu) = \frac{1}{\sqrt{p}} \int \varphi(x) F(\mu - \delta_x) \mu(\mathrm{d}x), \tag{6.19}$$

$$k^0(\varphi)F(\mu) = F(\mu) \int \varphi(x) \left(\mu + \frac{1}{2}\alpha \right) (\mathrm{d}x), \tag{6.20}$$

$$k^-(\varphi)F(\mu) = \sqrt{p} \int \overline{\varphi(x)} F(\mu + \delta_x) \, (\alpha + \mu) \, (\mathrm{d}x) \tag{6.21}$$

for $F \in \mathcal{D}$ and $\mu \in \mathbf{N}_{<\infty}$ and $\varphi \in \mathcal{C}$. In particular, $k^+(\mathbb{1}_E)$ coincides with $\mathcal{A}$ defined in (2.7) up to the constant $\frac{1}{\sqrt{p}}$. Moreover, the vacuum ψ is given by $\mathbb{1}_{\{0\}}$.

The proof boils down to check the properties listed in Definition 6.1.1 for the operators $k^{\#}(\varphi)$, $\# \in \{+, 0, -\}$. Since some of these properties are trivially satisfied we focus on the ones strongly relying on the definitions of the operators we introduced.

Proof of Theorem 6.2.1. On the one hand, by the definition of $\mathcal{D}$, it follows directly that if $f \in \mathcal{D}$, then $k^+(\varphi)f \in \mathcal{D}$. On the other hand, the relations (6.5) and (6.7) can be verified through a direct computation using only the definitions of $k^+(\varphi)$, $k^0(\varphi)$, $k^-(\varphi)$ which implies $k^0(\varphi)f, k^-(\varphi)f \in \mathcal{D}$ for each $f \in \mathcal{D}$.

Since $k^+(\varphi_1) \cdots k^+(\varphi_n)\psi$ is equal to the symmetrized tensor product $\varphi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \varphi_n$ on the $L^2_{\mathrm{sym}}(\lambda_n; \mathbb{C})$-component up to a multiplicative constant, the fact that $\mathcal{D}$ is dense in $\mathfrak{F}_{\mathbb{C}}$ follows through standard arguments using the functional monotone class theorem (see, e.g., [Bog07, Theorem 2.12.9]).

For (6.1), notice that $k^0(\varphi)$ acts as multiplication with a real-valued function on each $L^2_{\mathrm{sym}}(\lambda_n; \mathbb{C})$-component, making it essentially self-adjoint. Using the definition of $k^+(\varphi)$ and the fact that λ_n is invariant under the permutation of the variables, we obtain

$$\begin{aligned}\big\langle f, k^+(\varphi)g \big\rangle &= f_0 \overline{(k^+(\varphi)g)_0} + \sum_{n=1}^{\infty} \frac{p^n}{n!} \int f_n \overline{(k^+(\varphi)g)_n} \, \mathrm{d}\lambda_n \\ &= \frac{1}{\sqrt{p}} \sum_{n=1}^{\infty} \frac{p^n}{(n-1)!} \int f_n \overline{(\varphi \otimes_{\mathrm{s}} g_{n-1})} \, \mathrm{d}\lambda_n \\ &= \sqrt{p} \sum_{n=1}^{\infty} \frac{p^{n-1}}{(n-1)!} \int f_n \overline{(\varphi \otimes g_{n-1})} \, \mathrm{d}\lambda_n\end{aligned}$$

for all $f, g \in \mathcal{D}$. Using (3.46) and (6.17), we have

$$\begin{aligned}\big\langle f, k^+(\varphi)g \big\rangle = \sqrt{p} \sum_{n=1}^{\infty} \frac{p^{n-1}}{(n-1)!} \iint & f_n(x_1, \ldots, x_{n-1}, y) \overline{\varphi(y) g_{n-1}(x_1, \ldots, x_{n-1})} \\ & (\delta_{x_1} + \ldots + \delta_{x_{n-1}} + \alpha)(\mathrm{d}y) \lambda_{n-1}(\mathrm{d}(x_1, \ldots, x_{n-1})) \\ = \sqrt{p} \sum_{n=1}^{\infty} \frac{p^{n-1}}{(n-1)!} \int & (k^-(\varphi)f)_{n-1} \overline{g_{n-1}} \, \mathrm{d}\lambda_{n-1}.\end{aligned}$$

The last term above coincides with $\langle k^-(\varphi)f, g \rangle$ and thus, the proof of (6.1) is concluded.

Regarding the commutation relations (6.2), we focus on the term $[k^-(\varphi_1), k^+(\varphi_2)]$ only since the other commutation relations are proved similarly. Firstly, using that f_n

is a symmetric function, we have

$$
\begin{aligned}
&(k^-(\varphi_1)k^+(\varphi_2)f)_n(x_1,\dots,x_n)\\
&\quad = \int \overline{\varphi_1(y)} \sum_{l=1}^{n} \varphi_2(x_l) f_n(x_1,\dots,x_{l-1},x_{l+1},\dots,x_n,y)\,(\delta_{x_1}+\dots+\delta_{x_n}+\alpha)(\mathrm{d}y)\\
&\qquad + f_n(x_1,\dots,x_n)\left(\sum_{l=1}^{n}\overline{\varphi_1(x_l)}\varphi_2(x_l) + \int \overline{\varphi_1}\varphi_2\,\mathrm{d}\alpha\right). \qquad (6.22)
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
&(k^+(\varphi_2)k^-(\varphi_1)f)_n(x_1,\dots,x_n)\\
&\quad = \sum_{l=1}^{n} \varphi_2(x_l) \int \overline{\varphi_1(y)} f_n(x_1,\dots,x_{l-1},x_{l+1},\dots,x_n,y)\,(\delta_{x_1}+\dots+\delta_{x_n}+\alpha)(\mathrm{d}y)\\
&\qquad - f_n(x_1,\dots,x_n)\sum_{l=1}^{n}\overline{\varphi_1(x_l)}\varphi_2(x_l). \qquad (6.23)
\end{aligned}
$$

Subtracting the above expressions leads to

$$
\begin{aligned}
([k^-(\varphi_1),k^+(\varphi)]f)_n(x_1,\dots,x_n) &= 2f_n(x_1,\dots,x_n)\left(\sum_{l=1}^{n}\overline{\varphi_1(x_l)}\varphi_2(x_l) + \frac{1}{2}\int \overline{\varphi_1}\varphi_2\,\mathrm{d}\alpha\right)\\
&= 2(k^0(\varphi)f)_n(x_1,\dots,x_n),
\end{aligned}
$$

concluding the proof. □

6.2.2 On Pascal functionals

We now construct a second Fock representation of the $su(1,1)$ current algebra generalizing (1.22), (1.23), (1.24). To do this, we begin by fixing $p \in (0,1)$ and a σ-finite measure α on $(E,\mathcal{E})$. We recall that the probability measure $\rho_{p,\alpha}$, defined on $(\mathbf{N},\mathcal{N})$, denotes the distribution of the Pascal process, see Section 3.3. Let $L^2(\rho_{p,\alpha};\mathbb{C}) := L^2(\mathbf{N},\mathcal{N},\rho_{p,\alpha};\mathbb{C})$ be the Hilbert space of functions $F:\mathbf{N}\to\mathbb{C}$ that are square-integrable with respect to $\rho_{p,\alpha}$ equipped with the inner product $\langle F,G\rangle = \int F\overline{G}\,\mathrm{d}\rho_{p,\alpha}$. We remind the reader of the unitary operator $\mathfrak{U}:\mathfrak{F}\to L^2(\rho_{p,\alpha})$ as defined in (3.35). In the following, we shift our focus from $\mathfrak{U}$ to its complexification denoted by

$$
\mathfrak{U}_{\mathbb{C}}:\mathfrak{F}_{\mathbb{C}}\to L^2(\rho_{p,\alpha};\mathbb{C}), \qquad (6.24)
$$

i.e., $\mathfrak{U}_{\mathbb{C}}(u+\mathrm{i}v)=\mathfrak{U}u+\mathrm{i}\mathfrak{U}v$ for $u,v\in\mathfrak{F}$.

Now, we explore the Fock representation that arises from the first representation $k^+(\varphi)$, $k^0(\varphi)$, $k^-(\varphi)$ under the transformation $\mathfrak{U}_{\mathbb{C}}$. Specifically, the vacuum is given by $\mathfrak{U}_{\mathbb{C}}\psi = \mathcal{M}_0^{p,\alpha}1 = \mathbb{1}_{\mathbf{N}}$, while we define

$$
K^{\#}(\varphi) := \mathfrak{U}_{\mathbb{C}}k^{\#}\mathfrak{U}_{\mathbb{C}}^{-1}(\varphi) \qquad (6.25)
$$

for $\varphi \in \mathcal{C}$ and $\# \in \{+, 0, -\}$. These operators are defined on the domain $\mathbb{D}$ that is the image of $\mathcal{D}$ under $\mathfrak{U}_{\mathbb{C}}$. Here, $\mathcal{C}$ is the same algebra as defined in Section 6.2.1. It can be readily verified that the family $K^+(\varphi)$, $K^0(\varphi)$, $K^-(\varphi)$, $\varphi \in \mathcal{C}$ is a Fock representation on $L^2(\rho_{p,\alpha}; \mathbb{C})$ of the $su(1,1)$ current algebra with vacuum $\mathbb{1}_{\mathbf{N}}$. Relation (6.25) generalizes the univariate case [Gro19, Proposition 4.7].

Theorem 6.2.3. *The operators $K^{\#}(\varphi)$ are given by*

$$K^+(\varphi)F(\mu) = \frac{1}{1-p}\Bigg(\frac{1}{\sqrt{p}}\int \varphi(x)F(\mu-\delta_x)\mu(\mathrm{d}x) - 2\sqrt{p}F(\mu)\int \varphi(x)\left(\mu + \frac{\alpha}{2}\right)(\mathrm{d}x) + p\sqrt{p}\int \varphi(x)F(\mu+\delta_x)(\mu+\alpha)(\mathrm{d}x)\Bigg) \tag{6.26}$$

$$K^0(\varphi)F(\mu) = \frac{1}{1-p}\Bigg(-\int \varphi(x)F(\mu-\delta_x)\mu(\mathrm{d}x) + (p+1)F(\mu)\int \varphi(x)\left(\mu + \frac{\alpha}{2}\right)(\mathrm{d}x) - p\int \varphi(x)F(\mu+\delta_x)(\mu+\alpha)(\mathrm{d}x)\Bigg) \tag{6.27}$$

$$K^-(\varphi)F(\mu) = \frac{1}{1-p}\Bigg(\sqrt{p}\int \overline{\varphi(x)}F(\mu-\delta_x)\mu(\mathrm{d}x) - 2\sqrt{p}F(\mu)\int \overline{\varphi(x)}\left(\mu + \frac{\alpha}{2}\right)(\mathrm{d}x) + \sqrt{p}\int \overline{\varphi(x)}F(\mu+\delta_x)(\mu+\alpha)(\mathrm{d}x)\Bigg) \tag{6.28}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$, $F \in \mathbb{D}$ and $\varphi \in \mathcal{C}$.

In the following, we discuss properties of these operators. We prove Theorem 6.2.3 at the end of this section. Using the definitions of k^+, K^+ and $\mathfrak{U}_{\mathbb{C}}$, see (6.15), (6.25) and (6.24), we obtain

$$K^+(\varphi_1)\cdots K^+(\varphi_n)\mathbb{1}_{\mathbf{N}} = \mathfrak{U}_{\mathbb{C}}k^+(\varphi_1)\cdots k^+(\varphi_n)\psi = c^n \mathcal{M}_n^{p,\alpha}(\varphi_1 \otimes \cdots \otimes \varphi_n) \tag{6.29}$$

or, equivalently,

$$K^+(\theta)\mathcal{M}_n^{p,\alpha}(\varphi_1 \otimes \cdots \otimes \varphi_n) = c\mathcal{M}_{n+1}^{p,\alpha}(\theta \otimes \varphi_1 \otimes \cdots \otimes \varphi_n) \tag{6.30}$$

where $c = \frac{1}{\sqrt{p}} - \sqrt{p}$. Note that in the above equations, the infinite-dimensional Meixner polynomial $\mathcal{M}^{p,\alpha}f_n$ with complex-valued coefficients f_n is involved. We define this as the complexification of the polynomial with real-valued coefficients (see Definition 3.3.7) obtained by splitting the real and imaginary part of f_n and using linearity. We recall that $\mathscr{M}_n$ denotes the univariate monic Meixner polynomial of degree n, see (1.7). Proposition 3.3.11 yields

$$K^+(\mathbb{1}_{B_1})^{n_1}\cdots K^+(\mathbb{1}_{B_l})^{n_l}\mathbb{1}_{\mathbf{N}}(\mu) = \prod_{j=1}^{l} c^{n_j}\mathscr{M}_{n_j}\left(\mu(B_j); \alpha(B_j); p\right) \tag{6.31}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}$, $l \in \mathbb{N}$, pairwise disjoint $B_1, \ldots, B_l \in \mathcal{E}$ with $0 < \alpha(B_i) < \infty$ and $n_1, \ldots, n_l \in \mathbb{N}$. Note that (6.31) can be found in [BLR15, Proposition 4.6].

Remark 6.2.4. In this section, we proceed as follows: by using the unitary operator $\mathfrak{U}_{\mathbb{C}}$, the complexification of $\mathfrak{U}$ studied in Chapter 3.3.3, we can directly construct the second representation from the first one. It remains to determine $K^{\#}(\varphi)$ for $\# \in \{+, 0, -\}$ in Theorem 6.2.3.

Alternatively, there is another approach: First, we define $K^{\#}(\varphi)$ directly through the right-hand sides in Theorem 6.2.3. Subsequently, we prove that the family $K^{+}(\varphi)$, $K^{0}(\varphi)$, $K^{-}(\varphi)$, $\varphi \in \mathcal{C}$ is a Fock representation on $L^2(\rho_{p,\alpha}; \mathbb{C})$. Since Fock representations with the same reference measure α are always equivalent, as shown in Corollary 6.1.4, we retrieve the unitary operator. Notably, the fact that it is unitary is gained through algebraic methods only. In particular, we achieve an alternative—algebraic—proof of Proposition 3.3.8, namely the orthogonality relation of infinite-dimensional Meixner polynomials.

We note that, regardless of domain issues, $K^{+}(\varphi)$, $K^{0}(\varphi)$ and $K^{-}(\varphi)$ are linear combinations of the operators $k^{+}(\varphi)$, $k^{0}(\varphi)$, $k^{-}(\varphi)$ of the first Fock representation using the alternative formulation in Remark 6.2.2.

Furthermore, we note that the operators $K^{\#}(\varphi)$ have probabilistic interpretations: For real-valued $\varphi \geq 0$, $K^{-}(\varphi)$ reads as the formal generator of a birth-death process. Indeed, we have

$$\begin{aligned} K^{-}(\varphi)F(\mu) = \frac{\sqrt{p}}{1-p}\bigg(&\int \varphi(x)\left(F(\mu-\delta_x) - F(\mu)\right)\mu(\mathrm{d}x) \\ &+ \int \varphi(x)\left(F(\mu+\delta_x) - F(\mu)\right)(\mu+\alpha)(\mathrm{d}x)\bigg). \end{aligned}$$

On the other hand, if $K^{-}(\varphi)$ generates the Markov semigroup $(P_t)_{t\geq 0}$, then $K^{+}(\varphi)$ can be understood as the formal generator of the $L^2(\rho_{p,\alpha}; \mathbb{C})$-adjoint semigroup $(P_t^*)_{t\geq 0}$. This has the following heuristic interpretation: If γ_0 is a probability measure on $(\mathbf{N}, \mathcal{N})$ that is absolutely continuous with respect to $\rho_{p,\alpha}$ with density F_0, then the measure $\gamma_t := \gamma P_t$ is absolutely continuous with respect to $\rho_{p,\alpha}$ with density $P_t^* F_0$.

Moreover, $K^{0}(\varphi)$ behaves analogously to the *Ornstein-Uhlenbeck generator* in the context of the Poisson process (see, e.g., [Las16, Proposition 4]): Firstly, $K^{0}(\varphi)$ coincides for non-negative $\varphi \in \mathcal{C}$, up to negation and the addition of a constant times the identity operator, with the formal generator of a birth-death process:

$$\begin{aligned} \left(\mathcal{L}(\varphi)F\right)(\mu) :&= -\left(K^{0}(\varphi)F(\mu) - \frac{1}{2}F(\mu)\int \varphi\,\mathrm{d}\alpha\right) \\ &= \frac{1}{1-p}\bigg(\int \varphi(x)\left(F(\mu-\delta_x) - F(\mu)\right)\mu(\mathrm{d}x) \\ &\qquad + p\int \varphi(x)\left(F(\mu+\delta_x) - F(\mu)\right)(\mu+\alpha)(\mathrm{d}x)\bigg). \end{aligned} \tag{6.32}$$

Secondly, for finite α, it follows from Proposition 6.1.2 that $\mathcal{L}(\mathbb{1}_E)$ has spectrum $-\mathbb{N}_0$. Thus, together with the definition of $K^0(\varphi)$, we obtain $\mathcal{L}(\mathbb{1}_E)\mathcal{M}_n^{p,\alpha} f_n = -n\mathcal{M}_n^{p,\alpha} f_n$ for all $f_n \in L^2(\lambda_n)$, $n \in \mathbb{N}_0$. Therefore, the corresponding birth-death process is reversible and its Markov semigroup $Q_t : L^2(\rho_{p,\alpha}) \to L^2(\rho_{p,\alpha})$, $t \geq 0$ satisfies

$$Q_t \mathcal{M}_n^{p,\alpha} f_n = \mathrm{e}^{-tn} \mathcal{M}_n^{p,\alpha} f_n \tag{6.33}$$

for all $t \geq 0$, $f_n \in L^2(\lambda_n)$, $n \in \mathbb{N}_0$.

The following proposition demonstrates that, analogous to *Mehler's formula* for the Ornstein-Uhlenbeck semigroup in the Gaussian case (see [Nua06, Equation (1.67)]) or in the Poisson case (see [Las16, Equations (71) and (80)]), there exists a similar formula for the semigroup $(Q_t)_{t\geq 0}$. For each $t > 0$, define

$$q_t := \frac{(1-\mathrm{e}^{-t})p}{1-\mathrm{e}^{-t}p} \in (0,1) \tag{6.34}$$

and let γ_t be the probability distribution on $\mathbb{N}_0$ with probability mass function

$$\gamma_t(k) := \begin{cases} \mathrm{e}^{-t} q_t^{k-1}(1-q_t)^2 & \text{if } k \geq 1 \\ 1-\mathrm{e}^{-t}(1-q_t) & \text{if } k = 0. \end{cases} \tag{6.35}$$

Proposition 6.2.5. *Let α be a finite measure. Fix $t > 0$ and let ζ_t be a Pascal process with parameters q_t and α. Moreover, let $Y_{t,k}$ be γ_t distributed for all $k \in \mathbb{N}$ such that ζ_t, $Y_{t,1}, Y_{t,2}, \dots$ are independent. Then,*

$$Q_t F(\mu) = \mathbb{E}\left[F\left(\zeta_t + \sum_{k=1}^{N} Y_{t,k}\delta_{x_k}\right)\right] \tag{6.36}$$

holds for $\rho_{p,\alpha}$-almost all $\mu = \sum_{k=1}^{N} \delta_{x_k} \in \mathbf{N}_{<\infty}$ and $F \in L^2(\rho_{p,\alpha})$.

Equation (6.36) has the following probabilistic interpretation: Starting from an initial configuration $\sum_{k=1}^{N} \delta_{x_k}$, individual particles x_k may be removed with probability $\gamma_t(0)$ until time $t > 0$. Alternatively, with probability $\gamma_t(k)$, they can grow into a stack of k particles. Additionally, particles are independently added according to a Pascal process with parameters q_t and α.

We prove Proposition 6.2.5 using the generating functional of infinite-dimensional Meixner polynomials. A similar strategy can be found, for instance, in the proof in [Sur84, Theorem 5.1].

Proof. For a measurable bounded function $h : E \to [0,\infty)$, we rearrange the generating functional (3.37) of infinite-dimensional Meixner polynomials:

$$\begin{aligned} e_h(\mu) :&= \exp\left(-\int h \,\mathrm{d}\mu\right) \\ &= \exp\left(\int \log\left(\frac{1-p}{1-\mathrm{e}^{-h}p}\right)\mathrm{d}\alpha\right) \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha}\left(-\frac{1-\mathrm{e}^{-h}}{1-\mathrm{e}^{-h}p}\right)^{\otimes n}(\mu) \end{aligned}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}_{<\infty}$. Consequently, by using (6.33), we have

$$\begin{aligned} Q_t e_h(\mu) &= \exp\left(\int \log\left(\frac{1-p}{1-\mathrm{e}^{-h}p}\right) \mathrm{d}\alpha\right) \sum_{n=0}^{\infty} \mathrm{e}^{-tn} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha}\left(-\frac{1-\mathrm{e}^{-h}}{1-\mathrm{e}^{-h}p}\right)^{\otimes n}(\mu) \\ &= \exp\left(\int \log\left(\frac{1-p}{1-\mathrm{e}^{-h}p}\right) \mathrm{d}\alpha\right) \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \mathcal{M}_n^{p,\alpha}\left(-\frac{1-\mathrm{e}^{-g_t}}{1-\mathrm{e}^{-g_t}p}\right)^{\otimes n}(\mu) \\ &= \exp\left(\int \log\left(\frac{1-\mathrm{e}^{-g_t}p}{1-\mathrm{e}^{-h}p}\right) \mathrm{d}\alpha\right) \exp\left(-\int g_t \,\mathrm{d}\mu\right) \end{aligned} \tag{6.37}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}_{<\infty}$ and all $t \geq 0$. Here, g_t is chosen such that

$$\mathrm{e}^{-t} \frac{1-\mathrm{e}^{-h}}{1-\mathrm{e}^{-h}p} = \frac{1-\mathrm{e}^{-g_t}}{1-\mathrm{e}^{-g_t}p}$$

holds, i.e.,

$$g_t = \log\left(\frac{\mathrm{e}^{-t}\mathrm{e}^{-h}p - \mathrm{e}^{-t}p - \mathrm{e}^{-h}p + 1}{\mathrm{e}^{-t}\mathrm{e}^{-h} - \mathrm{e}^{-t} - \mathrm{e}^{-h}p + 1}\right). \tag{6.38}$$

By using (6.34), (6.37) and (6.38), we obtain

$$Q_t e_h(\mu) = \exp\left(-\int \log\left(\frac{1-q_t\mathrm{e}^{-h}}{1-q_t}\right) \mathrm{d}\alpha\right) \exp\left(-\int g_t \,\mathrm{d}\mu\right)$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}_{<\infty}$. The first factor is the Laplace transform of a Pascal process with parameters q_t and α, see (3.27). For the second factor, a direct computation reveals that the Laplace transform of γ_t is given by

$$\sum_{k=0}^{\infty} \mathrm{e}^{-sk}\gamma_t(k) = \frac{\mathrm{e}^{-t}\mathrm{e}^{-s} - \mathrm{e}^{-t} - \mathrm{e}^{-s}p + 1}{\mathrm{e}^{-t}\mathrm{e}^{-s}p - \mathrm{e}^{-t}p - \mathrm{e}^{-s}p + 1}, \quad s \geq 0.$$

Therefore, by (6.38), we have $\sum_{k=0}^{\infty} \mathrm{e}^{-h(x)k}\gamma_t(k) = \mathrm{e}^{-g_t(x)}$ for all $x \in E$. Consequently,

$$\exp\left(-\int g_t \,\mathrm{d}\mu\right) = \prod_{k=1}^{N} \mathrm{e}^{-g_t(x_k)} = \prod_{k=1}^{N} \mathbb{E}\left[\mathrm{e}^{-Y_{t,k}h(x_k)}\right] = \mathbb{E}\left[e_h\left(\sum_{k=1}^{N} Y_{t,k}\delta_{x_k}\right)\right].$$

Thus, (6.36) holds for all functions e_h. The proof concludes with the observation that the linear hull of the exponentials e_h is dense in $L^2(\rho_{p,\alpha})$, as demonstrated in the proof of Proposition 3.3.9. □

To prove Theorem 6.2.3, we present a preliminary result: a recursive formula for univariate Meixner polynomials. Similar formulas can be found in the literature, however in a slightly different setting and consequently with different constants, from which (6.39) can be deduced, see [LR09, Corollary 2.1] and [Gro19, Lemma 4.6]. We provide a straightforward and self-contained proof for the reader's convenience. We remark that this recursion does not coincide with the three-term recursion for Meixner polynomials (see [KLS10, Equation (9.10.3)]) and cannot be directly deduced from Rodrigues' formula (see [KLS10, Equation (9.10.10)]). The recursive formula (6.39) coincides with (6.30) in the univariate setting.

Lemma 6.2.6. *The monic Meixner polynomials* $\mathcal{M}_n(x;a;p)$ *defined in* (1.7) *satisfy the recursion*

$$\begin{aligned}\mathcal{M}_{n+1}(x;a;p) = \frac{1}{(1-p)^2}\big(&x\mathcal{M}_n(x-1;a;p)\\ &- p(a+2x)\mathcal{M}_n(x;a;p) + p^2(a+x)\mathcal{M}_n(x+1;a;p)\big) \qquad (6.39)\end{aligned}$$

for all $p \in (0,1)$, $a > 0$, $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

Proof. We fix a, x, p and write $\mathcal{M}_n(x) = \mathcal{M}_n(x;a;p)$. Let $t \geq 0$. We recall that the generating function of the monic Meixner polynomials is given by $\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{M}_n(x) = \mathrm{e}^{xA(t)+aB(t)}$ for $t \geq 0$, see (3.36), where

$$A(t) = \log(1-p+t) - \log(1-p+tp) \text{ and } B(t) = \log(1-p) - \log(1-p+tp)\,.$$

We prove (6.39) by multiplying both the right and the left-hand side by $\frac{t^n}{n!}$, summing over n and showing that the two terms obtained are equal. On the one hand,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{M}_{n+1}(x) = \frac{\partial}{\partial t}\sum_{n=1}^{\infty}\frac{t^n}{n!}\mathcal{M}_n(x) = \mathrm{e}^{xA(t)+aB(t)}\left(x\frac{\partial}{\partial t}A(t) + a\frac{\partial}{\partial t}B(t)\right).$$

On the other hand,

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{M}_n(x-1) = \mathrm{e}^{xA(t)+aB(t)}\mathrm{e}^{-A(t)} \text{ and } \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{M}_n(x+1) = \mathrm{e}^{xA(t)+aB(t)}\mathrm{e}^{A(t)}.$$

The proof is concluded by using that $A(t)$ solves the differential equation

$$x\frac{\partial}{\partial t}A(t) + a\frac{\partial}{\partial t}B(t) = \frac{1}{(1-p)^2}\left(x\mathrm{e}^{-A(t)} - p(a+2x) + p^2(a+x)\mathrm{e}^{A(t)}\right). \qquad \square$$

Proof of Theorem 6.2.3. Fix $\varphi \in \mathcal{C}$. To begin the proof of (6.26), let $\hat{K}^+(\varphi)$ be defined as its right-hand side. Furthermore, let $B_1, \dots, B_l \in \mathcal{E}$ be pairwise disjoint with $0 < a_j := \alpha(B_j) < \infty$ for all $j \in \{1, \dots, l\}$. For each $f : \mathbb{N}_0^l \to \mathbb{C}$, we define another function $F(\mu) := f(x)$ where $x := (x_1, \dots, x_l) := (\mu(B_1), \dots, \mu(B_l)) \in \mathbb{N}_0^l$. Then,

$$\hat{K}^+(\mathbb{1}_{B_j})F(\mu) = c\frac{1}{(1-p)^2}\left(x_j f(x-\delta_j) - 2p\Big(x_j + \frac{1}{2}a_j\Big)f(x) + p^2(x_j+a_j)f(x+\delta_j)\right)$$

for $j \in \{1, \dots, l\}$ where $\delta_j = (0, \dots, 0, 1, 0, \dots, 0)$ has only the j-th component equal to one. Therefore, applying $\hat{K^+}(\mathbb{1}_{B_j})$ to F yields the recursive formula (6.39) on the

j-th variable of f up to the constant c. Consequently, not only K^+ but also $\hat{K}^+$ satisfies (6.31). By standard measure-theoretic arguments, it follows that $\hat{K}^+$ satisfies (6.30) as well. Thus, for $F = K^+(\theta_1)\cdots K^+(\theta_n)\mathbb{1}_{\mathbf{N}} \in \mathbb{D}$,

$$\begin{aligned}\hat{K^+}(\varphi)F &= c^n \hat{K^+}(\varphi)\mathcal{M}_n^{p,\alpha}(\theta_1 \otimes \cdots \otimes \theta_n) \\ &= c^{n+1}\mathcal{M}_{n+1}^{p,\alpha}(\varphi \otimes \theta_1 \otimes \cdots \otimes \theta_n) = K^+(\varphi)K^+(\theta_1)\cdots K^+(\theta_n)\mathbb{1}_{\mathbf{N}} = K^+(\varphi)F\end{aligned}$$

and therefore $\hat{K^+}(\varphi) = K^+(\varphi)$ on $\mathbb{D}$.

Secondly, let $\hat{K}^-(\varphi)$ be defined as the right-hand side of (6.28). Using the Papangelou kernel for the Pascal process, see Lemma 3.3.2, we have

$$\begin{aligned}&\int G(\mu)\int \varphi(x)F(\mu - \delta_x)\mu(\mathrm{d}x)\rho_{p,\alpha}(\mathrm{d}\mu) \\ &\qquad = p\int F(\mu)\int \varphi(x)G(\mu+\delta_x)(\mu+\alpha)(\mathrm{d}x)\rho_{p,\alpha}(\mathrm{d}\mu)\end{aligned}$$

for $F, G \in \mathbb{D}$. Therefore, employing (6.26), we obtain $\langle F, K^+(\varphi)G\rangle = \langle \hat{K}^-(\varphi)F, G\rangle$. Thus, by (6.1) we conclude $K^-(\varphi) = \hat{K}^-(\varphi)$ on $\mathbb{D}$, i.e., (6.28) holds true.

A brief examination reveals that the operators $k^+(\varphi)$, $k^-(\varphi)$ and $k^0(\varphi)$, defined in (6.19), (6.20) and (6.21), are also well-defined on $\mathbb{D} \subset L^2(\rho_{p,\alpha};\mathbb{C})$ and satisfy the commutation relations (6.2) on $\mathbb{D}$. Therefore, using (6.26) and (6.28), we obtain

$$\begin{aligned}K^-(\varphi) &= \frac{1}{1-p}\left(pk^+(\overline{\varphi}) - 2\sqrt{p}k^0(\overline{\varphi}) + k^-(\varphi)\right) \\ K^+(\varphi) &= \frac{1}{1-p}\left(k^+(\varphi) - 2\sqrt{p}k^0(\varphi) + pk^-(\overline{\varphi})\right).\end{aligned}$$

Choose $\theta \in \mathcal{C}$ such that $\varphi = \varphi\overline{\theta}$. Then,

$$\begin{aligned}K^0(\varphi) &= \frac{1}{2}[K^-(\theta), K^+(\varphi)] \\ &= \frac{1}{2(1-p)^2}\Big(p[k^+(\overline{\theta}), k^+(\varphi)] + 2\sqrt{p}p[k^0(\varphi), k^+(\overline{\theta})] - p^2[k^-(\overline{\varphi}), k^+(\overline{\theta})] \\ &\qquad - 2\sqrt{p}[k^0(\overline{\theta}), k^+(\varphi)] + 4p[k^0(\overline{\theta}), k^0(\varphi)] - 2\sqrt{p}p[k^0(\overline{\theta}), k^-(\overline{\varphi})] \\ &\qquad + [k^-(\theta), k^+(\varphi)] + 2\sqrt{p}[k^0(\varphi), k^-(\theta)] + p[k^-(\theta), k^-(\overline{\varphi})]\Big) \\ &= \frac{1}{2(1-p)^2}\Big(2\sqrt{p}pk^+(\varphi\overline{\theta}) - 2p^2k^0(\varphi\overline{\theta}) - 2\sqrt{p}k^+(\varphi\overline{\theta}) + 2\sqrt{p}pk^-(\overline{\varphi}\theta) \\ &\qquad + 2k^0(\varphi\overline{\theta}) - 2\sqrt{p}k^-(\overline{\varphi}\theta)\Big) \\ &= \frac{1}{1-p}\left(-\sqrt{p}k^+(\varphi) + (p+1)k^0(\varphi) - \sqrt{p}k^-(\overline{\varphi})\right)\end{aligned}$$

which is equal to the right-hand side of (6.27). □

6.2.3 On gamma functionals

In the following, let α be a finite measure on a Borel space $(E, \mathcal{E})$. The arguments in this section work for σ-finite measures α as well; we only assume finiteness for the sake of simplicity in the exposition. We recall that the distribution of the Gamma process (see Section 5.2) with shape α and rate 1 is denoted by Γ_α. In this section, we investigate a Fock representation on $L^2(\Gamma_\alpha; \mathbb{C}) := L^2(\mathbf{M}_{<\infty}, \mathcal{M}_{<\infty}, \Gamma_\alpha; \mathbb{C})$ of the $su(1,1)$ current algebra. This representation generalizes the representation (1.25), (1.26), (1.27) (see also [Gro19, Section 4.2]) to general state spaces E.

Let $\mathfrak{F}_\mathbb{C}$ be the extended Fock space over the field $\mathbb{C}$ from Section 6.2.1 with $p = 1$. We denote by $\mathfrak{U}_\mathbb{C} : \mathfrak{F}_\mathbb{C} \to L^2(\Gamma_\alpha; \mathbb{C})$ the complexification of the unitary operator $\mathfrak{U} : \mathfrak{F} \to L^2(\Gamma_\alpha)$ defined in (5.11). Then, $\mathscr{K}^+(\varphi) := \mathfrak{U}_\mathbb{C} k^\#(\varphi)\mathfrak{U}_\mathbb{C}^{-1}$, $\varphi \in \mathcal{C}$, $\# \in \{+, 0, -\}$ is a Fock representation on $L^2(\Gamma_\alpha; \mathbb{C})$ of the $su(1,1)$ current algebra with vacuum $\mathfrak{U}_\mathbb{C}\psi = \mathbb{1}_{\mathbf{M}_{<\infty}}$. These operators are defined on the domain $\mathscr{D}$ that is the image of $\mathcal{D}$ under $\mathfrak{U}_\mathbb{C}$. Moreover, $\mathcal{C}$ is the same algebra as defined in Section 6.2.1.

The operators are given by

$$\mathscr{K}^+(\varphi)F(\nu) = \int \varphi \,\mathrm{d}(\nu - \alpha)\, F(\nu) + \int \varphi(x) \frac{\delta F(\nu)}{\delta \nu(x)} (\alpha - 2\nu)(\mathrm{d}x) + \int \varphi(x) \frac{\delta^2 F(\nu)}{\delta \nu(x)^2} \nu(\mathrm{d}x) \tag{6.40}$$

$$\mathscr{K}^0(\varphi)F(\nu) = \frac{1}{2} \int \varphi \,\mathrm{d}\alpha\, F(\nu) + \int \varphi(x) \frac{\delta F(\nu)}{\delta \nu(x)} (\nu - \alpha)(\mathrm{d}x) - \int \varphi(x) \frac{\delta^2 F(\nu)}{\delta \nu(x)^2} \nu(\mathrm{d}x) \tag{6.41}$$

$$\mathscr{K}^-(\varphi)F(\nu) = \int \overline{\varphi(x)} \frac{\delta F(\nu)}{\delta \nu(x)} \alpha(\mathrm{d}x) + \int \overline{\varphi(x)} \frac{\delta^2 F(\nu)}{\delta \nu(x)^2} \nu(\mathrm{d}x) \tag{6.42}$$

for Γ_α-almost all ν, $\varphi \in \mathcal{C}$ and $F \in \mathscr{D}$. The operators were already studied in the context of Gamma white noise analysis. More precisely, the operators $\mathscr{K}^+(\varphi)$ and $\mathscr{K}^-(\varphi)$ correspond to those presented in [KL00, Theorem 7.1] and [KL00, Theorem 7.3]. Furthermore, $\mathscr{K}^0(\varphi)$ corresponds, up to the term $\frac{1}{2}\int \varphi \,\mathrm{d}\alpha\, F(\nu)$, to the one in [KL00, Theorem 7.2].

We omit the proof of the explicit formulas (6.40), (6.41) and (6.42). Firstly, because the operators have already been studied in the literature, and secondly, because the proof is analogous to the second representation in Section 6.2.2. Indeed, like in the proof of Theorem 6.2.3, the formula (6.40) follows by using the recursion

$$\mathscr{L}_{n+1}^{(a-1)}(x) = (x - a)\mathscr{L}_n^{(a-1)}(x) + (a - 2x)\frac{\partial}{\partial x}\mathscr{L}_n^{(a-1)}(x) + x\frac{\partial^2}{\partial x^2}\mathscr{L}_n^{(a-1)}(x)$$

for the univariate monic Laguerre polynomials defined in (1.13), while (6.41) follows by the commutation relations and (6.42) follows by adjoining.

The operators $\mathscr{K}^+(\varphi)$ are closely connected to infinite-dimensional Laguerre polynomials, defined in (5.7), specifically

$$\mathscr{K}^+(\varphi_1)\cdots\mathscr{K}^+(\varphi_n)\mathbb{1}_{\mathbf{M}_{<\infty}} = L_n^\alpha(\varphi_1\otimes\cdots\otimes\varphi_n) \tag{6.43}$$

holds for $\varphi_1,\dots,\varphi_n\in\mathcal{C}$. Particularly,

$$\mathscr{K}^+(\mathbb{1}_{B_1})^{n_1}\cdots\mathscr{K}^+(\mathbb{1}_{B_l})^{n_l}\mathbb{1}_{\mathbf{M}_{<\infty}}(\nu) = \prod_{j=1}^{l}\mathscr{L}_{n_j}^{(\alpha(B_j)-1)}(\nu(B_j);\alpha(B_j)) \tag{6.44}$$

holds for Γ_α-almost all $\nu\in\mathbf{M}_{<\infty}$, $l\in\mathbb{N}$, $n_1,\dots,n_l\in\mathbb{N}$ and $B_1,\dots,B_l\in\mathcal{E}$ pairwise disjoint with $\alpha(B_i)>0$ for $i\in\{1,\dots l\}$. The relation (6.43) can be found in [KL00, second equation on page 324] while (6.44) is a variant of [Lyt03b, Lemma 3.1].

Like in the second representation, we have a probabilistic interpretation for the operators for real-valued non-negative $\varphi\in\mathcal{C}$: Both the operator $\mathscr{K}^-(\varphi)$ and the operator

$$\begin{aligned}(\hat{\mathcal{L}}(\varphi)F)(\nu) :&= -\left(\mathscr{K}^0(\varphi)F(\nu)-\frac{1}{2}F(\nu)\int\varphi\,\mathrm{d}\alpha\right)\\ &= \int\varphi(x)\frac{\delta^2F(\nu)}{\delta\nu(x)^2}\nu(\mathrm{d}x)-\int\varphi(x)\frac{\delta F(\nu)}{\delta\nu(x)}(\nu-\alpha)(\mathrm{d}x)\end{aligned} \tag{6.45}$$

can be interpreted as the formal generator of a measure-valued Markov process. In particular, since α is a finite measure, $\hat{\mathcal{L}}(\mathbb{1}_E)L_n^\alpha f_n = -nL_n^\alpha f_n$ for $f_n\in L^2(\lambda_n)$—that is, it has discrete spectrum. The corresponding Markov semigroup $(\hat{Q}_t)_{t\geq 0}$ defined on $L^2(\Gamma_\alpha)$ satisfies $\hat{Q}_tL_n^\alpha f_n = \mathrm{e}^{-tn}L_n^\alpha f_n$.

The process with formal generator (6.45) is known as *Dawson-Watanabe measure-valued continuous-state branching process with immigration*, see [EG93b, Equation (1.1)] or [SL16, Equation (38)]. See also [Eth00] for a general discussion on Dawson-Watanabe superprocesses. Additionally, if E contains only one element, (6.45) reduces to the operator $(\alpha-y)\frac{\partial}{\partial y}+y\frac{\partial^2}{\partial y^2}$ which is the formal generator of a non-negative process known as the *Cox-Ingersoll-Ross process* (CIR) in mathematical finance to model interest rate movements, see [CIR85]. The Laguerre polynomials are characterized by being eigenfunctions of this operator, see, e.g., [KLS10, Equation (9.12.5)], i.e., they satisfies the so-called *Laguerre equation.* See also [BM03] for a characterization of Markov semigroups having orthogonal polynomials as eigenfunctions.

Using the same arguments as in the second representation, see Proposition 6.2.5, it can be easily checked that a formula analogous to Mehler's formula for the Markov semigroup $(\hat{Q}_t)_{t\geq 0}$ can be obtained. In a more general setup, the transition kernels of $(\hat{Q}_t)_{t\geq 0}$ were already examined in [EG93b, Theorem 1.1] where it is proved that the kernels are equal to mixtures of distributions of Gamma distributions. In Proposition 6.2.7 below, our approach takes a distinct route: the transition kernels are a mixture of a Gamma process and the initial configuration $\nu=\sum_{k=1}^N z_k\delta_{x_k}$ where the weights z_k are randomly adjusted.

Let γ_t^{*z}, $z>0$ be the convolution semigroup defined as follows: γ_t^{*z} is the distribution of the compound Poisson random variable $\sum_{k=1}^M Y_k$ where M follows the Poisson distribution with parameter $z\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}$, every Y_k follows an exponential distribution with scale

$\frac{1}{1-\mathrm{e}^{-t}}$ and $M, Y_1, Y_2, \dots$ are assumed to be independent. In other words, the Laplace transform of γ_t^{*z} is given by

$$\int \mathrm{e}^{-sx} \gamma_t^{*z}(\mathrm{d}x) = \exp\left(-z \frac{s\mathrm{e}^{-t}}{1+(1-\mathrm{e}^{-t})s}\right), \quad s \geq 0.$$

Proposition 6.2.7. *Let α be a finite measure. Then,*

$$\hat{Q}_t F(\nu) = \mathbb{E}\left[F\left((1-\mathrm{e}^{-t})\zeta + \sum_{k=1}^{N} Y_{t,k,z_k} \delta_{x_k}\right)\right]$$

holds for Γ_α-almost all $\nu = \sum_{k=1}^N z_k \delta_{x_k} \in \mathbf{M}_{<\infty}$, $t \geq 0$ and $F \in L^2(\Gamma_\alpha)$ where Y_{t,k,z_k} follows the distribution $\gamma_t^{*z_k}$, ζ is a Gamma process with shape α and rate 1, and ζ and all Y_{t,k,z_k} are independent.

The proof is analogous to the one of Proposition 6.2.5 using the generating functional (5.12).

6.3 The Baker-Campbell-Hausdorff formula

We present a *Baker-Campbell-Hausdorff formula* which generalizes the *Weyl relation* for the free field (see, e.g., [RS75, page 231]) to our context. Although the Baker-Campbell-Hausdorff relation for the $su(1,1)$ algebra can be found for $\varphi = \xi \mathbb{1}_E$, $\xi \in \mathbb{C}$ in the literature (see, e.g., [Tru85], [CDP06, Equation (22)] or [CFG$^+$19, Theorem 3.4]), we proof this relation with a more detailed look to mathematical subtleties. The difficulty here relates to *Nelson's example* (see, e.g., [RS80, Section VIII.5]). Our strategy relies on the techniques used in [RS75, Section X.6] when proving the canonical commutation relations ([BR13]).

Let $\mathrm{k}^+(\varphi)$, $\mathrm{k}^0(\varphi)$, $\mathrm{k}^-(\varphi)$, $\varphi \in \mathcal{C}$ be a Fock representation on a Hilbert space H with vacuum $\psi \in H$. Let α be a σ-finite measure such that (6.3) is satisfied.

Theorem 6.3.1. *For each $\varphi \in \mathcal{C}$, the Baker-Campbell-Hausdorff formula*

$$\mathrm{e}^{\mathrm{k}^+(\varphi)-\mathrm{k}^-(\varphi)} f = \mathrm{e}^{\mathrm{k}^+(v)} \mathrm{e}^{\mathrm{k}^0(w)} \mathrm{e}^{-\mathrm{k}^-(v)} f, \quad f \in D \tag{6.46}$$

holds true where $v := \mathbb{1}_{\{\varphi \neq 0\}} \frac{\varphi}{|\varphi|} \tanh|\varphi| \in D$ and $w := -2 \log\cosh|\varphi| \in D$.

The individual exponential functions in (6.46) require a closer explanation: To define $\mathrm{e}^{\mathrm{i}A}$ on the left-hand side, where $A := -\mathrm{i}(\mathrm{k}^+(\varphi) - \mathrm{k}^-(\varphi))$, we prove in Lemma 6.3.2 below that the operator A is essentially self-adjoint using Nelson's analytic vector theorem (see, e.g., [RS75, Theorem X.39]). For the right-hand side of (6.46), $\mathrm{e}^{\mathrm{k}^0(w)}$ can be defined using that $\mathrm{k}^0(w)$ is essentially self-adjoint. This follows from the fact that $\mathrm{k}^0(w)$ acts as multiplication with a real-valued function on each $L^2_{\mathrm{sym}}(\lambda_n; \mathbb{C})$-component. Consequently, $\mathrm{e}^{\mathrm{k}^0(w)}$ is defined using standard techniques. $\mathrm{e}^{\mathrm{k}^+(v)}$ and $\mathrm{e}^{-\mathrm{k}^-(v)}$ represent exponential power series series that are proved to converge strongly in $\mathfrak{F}_{\mathbb{C}}$-norm when applied to $f \in D$, see Lemma 6.3.3 below.

The Baker-Campbell-Hausdorff formula enables us to prove, within the context of the first representation (see Section 6.2.1), that a specific choice of φ in (6.46) leads to infinitely-dimensional Meixner polynomials, see Theorem 6.4.2 below. The proof of the Baker–Campbell–Hausdorff formula is divided in several parts. Firstly, we address the well-definedness of the left-hand side of (6.46), a result that follows from Lemma 6.3.2 below. We observe that Corollary 6.1.4 implies that the Fock representation $\mathrm{k}^+(\varphi)$, $\mathrm{k}^0(\varphi)$, $\mathrm{k}^-(\varphi)$, $\varphi \in \mathcal{C}$ is, without loss of generality, given by the representation $k^+(\varphi)$, $k^0(\varphi)$, $k^-(\varphi)$, $\varphi \in \mathcal{C}$ from Section 6.2.1 for an arbitrary but fixed $p \in (0, 1)$.

Lemma 6.3.2. *The operator $-\mathrm{i}\,(k^+(\varphi) - k^-(\varphi)) : \mathcal{D} \to \mathcal{D}$ is essentially self-adjoint for each $\varphi \in \mathcal{C}$.*

The operator is similar to the Segal field operators for free quantum fields; we adapt the proof of essential self-adjointness from [RS75, Theorem X.41(a)].

Proof. The commutation relation provided in (6.1) implies that $-\mathrm{i}(k^+(\varphi) - k^+(\varphi))$ is a symmetric operator. Symmetric operators are always closable. By Nelson's analytic vector theorem (see, e.g., [RS75, Theorem X.39]), the result follows if we show that each $f \in \mathcal{D}$ is an *analytic vector* for $k^+(\varphi) - k^-(\varphi)$—that is, there exists $\varepsilon > 0$ such that $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \| \left(k^+(\varphi) - k^-(\varphi)\right)^n f\| < \infty$.

Fix $f = (f_l)_{l \in \mathbb{N}_0} \in \mathcal{D}$ and $\varphi \in \mathcal{C}$. By the definition of $\mathcal{D}$ and $k^+(\varphi)$, there exist $m \in \mathbb{N}$, $C > 0, D > 0$ and $B \in \mathcal{E}$ such that $\alpha(B) < \infty$, $|f_l| \leq C \mathbb{1}_{\{l \leq m\}} \mathbb{1}_{B^l}$ for each $l \in \mathbb{N}_0$ and $|\varphi| \leq D \mathbb{1}_B$. Therefore,

$$\begin{aligned} \left|(k^+(\varphi)f)_l\right| &\leq \frac{1}{\sqrt{p}}(m+1)CD\mathbb{1}_{\{l \leq m+1\}}\mathbb{1}_{B^l}, \\ \left|(k^-(\varphi)f)_l\right| &\leq \sqrt{p}CD(\alpha(B)+m-1)\mathbb{1}_{\{l \leq m-1\}}\mathbb{1}_{B^l} \end{aligned}$$

for each $l \in \mathbb{N}_0$ which implies

$$\left|(k^{\#}(\varphi)f)_l\right| \leq \frac{1}{\sqrt{p}}(\alpha(B)+m+1)CD\mathbb{1}_{\{l \leq m+1\}}\mathbb{1}_{B^l} \tag{6.47}$$

for $\# \in \{+, -\}$. We remind the reader that $(a)^{(0)} = 1$, $(a)^{(l)} = a(a+1)\cdots(a+l-1)$ denotes the rising factorial. Inequality (6.47) used multiple times yields

$$\left|(k^{\#_1}(\varphi)\cdots k^{\#_n}(\varphi)f)_l\right| \leq C\left(\frac{1}{\sqrt{p}}D\right)^n (\alpha(B)+m+1)^{(n)}\mathbb{1}_{\{l \leq m+n\}}\mathbb{1}_{B^l}$$

for $\#_1, \ldots, \#_n \in \{+, -\}$, $n \in \mathbb{N}_0$ which implies, together with $\lambda_l(B^l) = \alpha(B)^{(l)}$,

$$\begin{aligned} \left\|k^{\#_1}(\varphi)\cdots k^{\#_n}(\varphi)f\right\| &\leq C\left(\frac{1}{\sqrt{p}}D\right)^n (\alpha(B)+m+1)^{(n)}\sqrt{\sum_{l=0}^{m+n}\frac{p^l}{l!}\lambda_l(B^l)} \\ &\leq C\left(\frac{1}{\sqrt{p}}D\right)^n (\alpha(B)+m+1)^{(n)}(1-p)^{-\frac{\alpha(B)}{2}}. \end{aligned}$$

As a result,

$$\begin{aligned}\left\|\left(k^+(\varphi)-k^-(\varphi)\right)^n f\right\| &\le \sum_{\#_1,\dots,\#_n\in\{+,-\}} \left\|k^{\#_1}(\varphi)\cdots k^{\#_n}(\varphi)f\right\| \\ &\le C\left(\frac{2}{\sqrt{p}}D\right)^n(\alpha(B)+m+1)^{(n)}(1-p)^{-\frac{\alpha(B)}{2}}\end{aligned}$$

and thus

$$\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\left\|\left(k^+(\varphi)-k^-(\varphi)\right)^n f\right\| \le C(1-p)^{-\frac{\alpha(B)}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\varepsilon}{\sqrt{p}}D\right)^n(\alpha(B)+m+1)^{(n)}<\infty$$

if ε is chosen such that $\frac{2\varepsilon}{\sqrt{p}}D<1$. □

Next, we prove that the right-hand side of (6.46) is well-defined. Since $k^0(w)$ is a multiplication operator on each $L^2_{\mathrm{sym}}(\lambda_n;\mathbb{C})$-component, we have

$$(\mathrm{e}^{k^0(w)}f)_l(x_1,\dots,x_l)=\mathrm{e}^{w(x_1)+\ldots+w(x_l)+\frac{1}{2}\int w\,\mathrm{d}\alpha}f_l(x_1,\dots,x_l) \tag{6.48}$$

for λ_l-almost all $(x_1,\dots,x_l)\in E^l$ and $f\in\mathcal{D}$.

Lemma 6.3.3. *For each $f\in\mathcal{D}$:*

(i) *The series $\mathrm{e}^{-k^-(v)}f:=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-k^-(v)\right)^n f$ reduces to a finite sum. In particular, $\mathrm{e}^{-k^-(v)}f\in\mathcal{D}$.*

(ii) $\mathrm{e}^{k^0(w)}f\in\mathcal{D}$.

(iii) *The series $\mathrm{e}^{k^+(v)}f:=\sum_{n=0}^{\infty}\frac{1}{n!}k^+(v)^n f$ converges absolutely in norm.*

Proof. By the definition of $\mathcal{D}$ and linearity, the proofs for the three claims are only required for a fixed $f=k^+(\varphi_1)\cdots k^+(\varphi_m)\psi$ where $\varphi_1,\dots,\varphi_m\in\mathcal{C}$, $m\in\mathbb{N}_0$. The first claim follows from the fact that $k^-(\varphi)^n f=0$ for $n>m$.

Regarding the second claim, we note that the definition of $k^+(w)$, together with (6.48), results in

$$\mathrm{e}^{k^0(w)}k^+(\varphi_1)\cdots k^+(\varphi_m)\psi=\mathrm{e}^{\frac{1}{2}\int w\,\mathrm{d}\alpha}k^+(\varphi_1\mathrm{e}^w)\cdots k^+(\varphi_m\mathrm{e}^w)\psi\in\mathcal{D}. \tag{6.49}$$

For the third claim, let $C>0$, $0<c<1$ and $B\in\mathcal{E}$ be such that $\alpha(B)<\infty$ and $|\varphi_j|\le C\mathbb{1}_B$ for $j\in\{1,\dots,m\}$ and $|v|<c\mathbb{1}_B$. By the definition of $k^+(v)$,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\|k^+(v)^n f\right\|=\sum_{n=0}^{\infty}\frac{1}{n!}\sqrt{(n+m)!\int|g_{n+m}|^2\,\mathrm{d}\lambda_{n+m}}$$

where g_{n+m} is defined as the symmetrization of $v^{\otimes n}\otimes\varphi_1\otimes\cdots\otimes\varphi_m$. Consequently,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\|k^+(v)^n f\right\|\le C^m\sum_{n=0}^{\infty}\frac{c^n}{n!}\sqrt{(n+m)!\alpha(B)^{(n+m)}}<\infty$$

where we used that $|g_{n+m}|\le c^nC^m\mathbb{1}_{B^{n+m}}$ and $\lambda_{n+m}(B^{n+m})=\alpha(B)^{(n+m)}$. □

We proceed with the proof of the Baker–Campbell–Hausdorff formula, following the strategy of [Tru85]. However, we pay special attention to the difficulties caused by the unboundedness of operators employed in this chapter. For $t \geq 0$, we put

$$v_t := \mathbb{1}_{\{\varphi \neq 0\}} \frac{\varphi}{|\varphi|} \tanh(t|\varphi|), \quad w_t := -2 \log \cosh(t|\varphi|).$$

Furthermore, we define the linear operator $S_t : \mathcal{D} \to \mathfrak{F}_{\mathbb{C}}$ by

$$S_t f := \mathrm{e}^{k^+(v_t)} \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} f, \quad f \in \mathcal{D}.$$

By Lemma 6.3.3, S_t is well-defined for all $t \geq 0$. The key idea from [Tru85] is to differentiate $S_t f$ with respect to t and show that it satisfies a differential equation.

Remark 6.3.4. Alternatively, the Baker-Campbell-Hausdorff formula (6.46) can be deduced from the Baker-Campbell-Hausdorff formula in the univariate setting. Indeed, fix $\xi \in \mathbb{C}$ and $A \in \mathcal{E}$ with $\alpha(A) < \infty$. Since $k^+(\mathbb{1}_A), k^0(\mathbb{1}_A), k^-(\mathbb{1}_A)$ is a representation of the $su(1,1)$ algebra, we can apply the Baker-Campbell-Hausdorff formula (see, e.g., [CDP06, Equation (22)]) for $\xi \mathbb{1}_A \in \mathcal{C}$ implying (6.46). The commutation relations imply that both the left-hand side and the right-hand side of (6.46) factorize if φ is a linear combination of indicator functions over disjoint sets. Therefore, an approximation argument yields the result for all $\varphi \in \mathcal{C}$. Nevertheless, we opt for a direct proof without using the Baker-Campbell-Hausdorff formula in the univariate setting. This choice allows for a deeper insight into the commutation relations of the $su(1,1)$ current algebra.

When we differentiate $t \mapsto S_t f$, we get the following product formula in a natural way. The proof of the lemma is straightforward but requires lengthy estimates. As this lemma is an essential part of the proof of the Baker-Campbell-Hausdorff formula, we provide at least a portion of these estimates. We abbreviate the derivative with respect to t as $\partial_t = \frac{\partial}{\partial t}$. Note that $\partial_t v_t, \partial_t w_t \in \mathcal{C}$.

Lemma 6.3.5. *The function $t \mapsto S_t f$ is norm-differentiable on $(0, \infty)$ for each $f \in \mathcal{D}$ and*

$$\begin{aligned} \partial_t S_t f = \Big(&\mathrm{e}^{k^+(v_t)} k^+(\partial_t v_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} + \mathrm{e}^{k^+(v_t)} k^0(\partial_t w_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} \\ &+ \mathrm{e}^{k^+(v_t)} \mathrm{e}^{k^0(w_t)} k^-(-\partial_t v_t) \mathrm{e}^{-k^-(v_t)} \Big) f. \end{aligned} \tag{6.50}$$

Proof. Using the triangular inequality, we obtain

$$\begin{aligned} \Big\| \Big(&\frac{S_{t+s} - S_t}{s} - \mathrm{e}^{k^+(v_t)} k^+(\partial_t v_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} - \mathrm{e}^{k^+(v_t)} k^0(\partial_t w_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} \\ &- \mathrm{e}^{k^+(v_t)} \mathrm{e}^{k^0(w_t)} k^-(-\partial_t v_t) \mathrm{e}^{-k^-(v_t)} \Big) f \Big\| \\ \leq &\left\| \left(\frac{\mathrm{e}^{k^+(v_{t+s})} - \mathrm{e}^{k^+(v_t)}}{s} - \mathrm{e}^{k^+(v_t)} k^+(\partial_t v_t) \right) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} f \right\| \\ &+ \left\| \mathrm{e}^{k^+(v_{t+s})} \left(\frac{\mathrm{e}^{k^0(w_{t+s})} - \mathrm{e}^{k^0(w_t)}}{s} - k^0(\partial_t w_t) \mathrm{e}^{k^0(w_t)} \right) \mathrm{e}^{-k^-(v_t)} f \right\| \end{aligned}$$

$$+\left\|\mathrm{e}^{k^+(v_{t+s})}\mathrm{e}^{k^0(w_{t+s})}\left(\frac{\mathrm{e}^{-k^-(v_{t+s})}-\mathrm{e}^{-k^-(v_t)}}{s}-k^-(-\partial_t v_t)\mathrm{e}^{-k^-(v_t)}\right)f\right\|.$$

We only prove that the first term tends to 0 as $s \to 0$; the convergence for the remaining two terms follows similarly. Since $\mathrm{e}^{k^0(w_t)}\mathrm{e}^{-k^-(v_t)}f \in \mathcal{D}$ by Lemma 6.3.3, it suffices to show that

$$\begin{aligned}&\left\|\left(\frac{\mathrm{e}^{k^+(v_{t+s})}-\mathrm{e}^{k^+(v_t)}}{s}-\mathrm{e}^{k^+(v_t)}k^+(\partial_t v_t)\right)k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right\|\\ &\le \sum_{n=1}^{\infty}\frac{1}{n!}\left\|\left(\frac{k^+(v_{t+s})^n-k^+(v_t)^n}{s}-nk^+(v_t)^{n-1}k^+(\partial_t v_t)\right)k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right\| \end{aligned} \tag{6.51}$$

tends to 0 as $s \to 0$ for each $\varphi_1, \ldots, \varphi_m \in \mathcal{C}$. Note that, for each $l \in \mathbb{N}_0$ and $x \in E^l$,

$$\left(k^+(v_t)^n k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right)_l(x) = \mathbb{1}_{\{l=n+m\}}\frac{(n+m)!}{\sqrt{p}^{n+m}}\left(v_t^{\otimes n}\otimes_{\mathrm{s}}\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_m\right)(x)$$

is differentiable with respect to t with derivative

$$\begin{aligned}\mathbb{1}_{\{l=n+m\}}&\frac{(n+m)!}{\sqrt{p}^{n+m}}n\left((\partial_t v_t)\otimes_{\mathrm{s}} v_t^{\otimes(n-1)}\otimes_{\mathrm{s}}\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_m\right)(x)\\ &=\left(nk^+(v_t)^{n-1}k^+(\partial_t v_t)k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right)_l(x)\end{aligned}$$

for each $l \in \mathbb{N}_0$ and $x \in E^l$. Here, $v_t^{\otimes n}\otimes_{\mathrm{s}}\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_m$ denotes the symmetrization of $v_t^{\otimes n}\otimes\varphi_1\otimes\cdots\otimes\varphi_m$; the function $(\partial_t v_t)\otimes_{\mathrm{s}} v_t^{\otimes(n-1)}\otimes_{\mathrm{s}}\varphi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\varphi_m$ is defined analogously. Let $C > 0$ and $B \in \mathcal{E}$ with $\alpha(B) < \infty$ be such that $|\varphi_1|, \ldots, |\varphi_m|, |\varphi| \le C\mathbb{1}_B$. Using the mean value theorem and $|v_t| \le \mathbb{1}_B$, we obtain

$$\begin{aligned}&\left|\left(\left(\frac{k^+(v_{t+s})^n-k^+(v_t)^n}{s}-nk^+(v_t)^{n-1}k^+(\partial_t v_t)\right)k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right)_l\right|\\ &\qquad\le \mathbb{1}_{\{l=n+m\}}\frac{2(n+m)!}{\sqrt{p}^{n+m}}n\left(\tanh\left((t+1)C\right)\right)^{n-1}C^m\mathbb{1}_{B^{n+m}}\end{aligned}$$

which implies that each summand of (6.51) converges to 0 as $s \to 0$ by using Lebesgue's dominated convergence theorem. Finally, since $\tanh\left((t+1)C\right) < 1$,

$$\begin{aligned}\frac{1}{n!}&\left\|\left(\frac{k^+(v_{t+s})^n-k^+(v_t)^n}{s}-nk^+(v_t)^{n-1}k^+(\partial_t v_t)\right)k^+(\varphi_1)\cdots k^+(\varphi_m)\psi\right\|\\ &\le \frac{1}{n!}2\sqrt{(n+m)!}n\left(\tanh\left((t+1)C\right)\right)^{n-1}C^m\sqrt{(\alpha(B))^{(n+m)}}\end{aligned}$$

is summable over n. Thus, (6.51) converges to 0. □

As a next step, we work on the right-hand side of (6.50). Specifically, we shift the terms $k^0(\partial_t w_t)$ and $k^-(-\partial_t v_t)$ to the left, taking into account the commutation relations. This procedure results in a differential equation for S_t. We denote the closure of $k^+(\varphi)-k^-(\varphi)$ by $(k^+(\varphi)-k^-(\varphi))^{\mathrm{cl}}$.

Lemma 6.3.6. *Let $f \in \mathcal{D}$. Then, $S_t f$ is in the domain $(k^+(\varphi) - k^-(\varphi))^{\mathrm{cl}}$ and*

$$\partial_t S_t f = \left(k^+(\varphi) - k^-(\varphi)\right)^{\mathrm{cl}} S_t f.$$

Proof. Let $g = k^+(\varphi_1) \cdots k^+(\varphi_m)\psi$ where $\varphi_1, \ldots, \varphi_m \in \mathcal{C}$ and $m \in \mathbb{N}_0$. Firstly, as a consequence of (6.7), we deduce

$$k^+(v_t)^n k^0(\theta) = k^0(\theta) k^+(v_t)^n - n k^+(\theta v_t) k^+(v_t)^{n-1} \tag{6.52}$$

for $n \in \mathbb{N}$ and $\theta \in \mathcal{C}$, implying

$$\mathrm{e}^{k^+(v_t)} k^0(\partial_t w_t) g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k^0(\partial_t w_t) - k^+((\partial_t w_t) v_t)\right) k^+(v_t)^n g. \tag{6.53}$$

Secondly, we use (6.5), (6.49) and the fact that w_t is real-valued to obtain

$$\begin{aligned}
&\mathrm{e}^{k^0(w_t)} k^-(\partial_t v_t) g \\
&= 2\mathrm{e}^{\frac{1}{2}\int w_t \,\mathrm{d}\alpha} \sum_{j=1}^{m} \sum_{l=j+1}^{m} k^+(\varphi_1 \mathrm{e}^{w_t}) \cdots k^+(\varphi_{j-1}\mathrm{e}^{w_t}) k^+(\varphi_{j+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_{l-1}\mathrm{e}^{w_t}) \\
&\qquad\qquad k^+(\varphi_j \overline{\partial_t v_t} \varphi_l \mathrm{e}^{w_t}) k^+(\varphi_{l+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_m \mathrm{e}^{w_t})\psi \\
&\quad + \mathrm{e}^{\frac{1}{2}\int w_t \,\mathrm{d}\alpha} \sum_{j=1}^{m} \int \varphi_j \overline{\partial_t v_t} \,\mathrm{d}\alpha \, k^+(\varphi_1 \mathrm{e}^{w_t}) \cdots k^+(\varphi_{j-1}\mathrm{e}^{w_t}) k^+(\varphi_{j+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_m \mathrm{e}^{w_t})\psi \\
&= 2\mathrm{e}^{\frac{1}{2}\int w_t \,\mathrm{d}\alpha} \sum_{j=1}^{m} \sum_{l=j+1}^{m} k^+(\varphi_1 \mathrm{e}^{w_t}) \cdots k^+(\varphi_{j-1}\mathrm{e}^{w_t}) k^+(\varphi_{j+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_{l-1}\mathrm{e}^{w_t}) \\
&\qquad\qquad k^+(\varphi_j \mathrm{e}^{w_t} \overline{(\partial_t v_t)\mathrm{e}^{-w_t}} \varphi_l \mathrm{e}^{w_t}) k^+(\varphi_{l+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_m \mathrm{e}^{w_t})\psi \\
&\quad + \mathrm{e}^{\frac{1}{2}\int w_t \,\mathrm{d}\alpha} \sum_{j=1}^{m} \int \varphi_j \mathrm{e}^{w_t} \overline{(\partial_t v_t)\mathrm{e}^{-w_t}} \,\mathrm{d}\alpha \\
&\qquad\qquad k^+(\varphi_1 \mathrm{e}^{w_t}) \cdots k^+(\varphi_{j-1}\mathrm{e}^{w_t}) k^+(\varphi_{j+1}\mathrm{e}^{w_t}) \cdots k^+(\varphi_m \mathrm{e}^{w_t})\psi \\
&= k^-((\partial_t v_t)\mathrm{e}^{-w_t}) \mathrm{e}^{k^0(w_t)} g.
\end{aligned} \tag{6.54}$$

Thirdly, as a consequence of the commutation relations (6.2), we find by induction that

$$\begin{aligned}
&k^+(v_t)^n k^-(\theta) \\
&\quad = k^-(\theta) k^+(v_t)^n - 2n k^+(v_t)^{n-1} k^0(v_t \overline{\theta}) - n(n-1) k^+(v_t^2 \overline{\theta}) k^+(v_t)^{n-2}
\end{aligned} \tag{6.55}$$

holds for all $\theta \in \mathcal{C}$. Combining (6.52) and (6.55) yields

$$k^+(v_t)^n k^-(\theta) = k^-(\theta) k^+(v_t)^n - 2n k^0(v_t \overline{\theta}) k^+(v_t)^{n-1} + n(n-1) k^+(v_t^2 \overline{\theta}) k^+(v_t)^{n-2}.$$

Consequently,

$$\mathrm{e}^{k^+(v_t)} k^-(\theta) g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k^-(\theta) - 2k^0(v_t \overline{\theta}) + k^+(v_t^2 \overline{\theta})\right) k^+(v_t)^n g. \tag{6.56}$$

The proof is completed by combining (6.53), (6.54), (6.56) and Lemma 6.3.5:

$$\begin{aligned}\partial_t S_t f &= \Big(\mathrm{e}^{k^+(v_t)} k^+(\partial_t v_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} + \mathrm{e}^{k^+(v_t)} k^0(\partial_t w_t) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} \\ &\qquad + \mathrm{e}^{k^+(v_t)} k^- \left(-(\partial_t v_t)\mathrm{e}^{-w_t}\right) \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)}\Big) f \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(k^+ \left(\partial_t v_t - (\partial_t w_t) v_t - v_t^2 \overline{(\partial_t v_t)} \mathrm{e}^{-w_t}\right) + k^0 \left(\partial_t w_t + 2 v_t \overline{(\partial_t v_t)} \mathrm{e}^{-w_t}\right) \\ &\qquad - k^- \left((\partial_t v_t)\mathrm{e}^{-w_t}\right)\Big) k^+(v_t)^n \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} f \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(k^+(\varphi) - k^-(\varphi)\right) k^+(v_t)^n \mathrm{e}^{k^0(w_t)} \mathrm{e}^{-k^-(v_t)} f.\end{aligned}$$

In the last equality, we used that v_t and w_t solve the differential equations

$$\begin{aligned}\partial_t v_t - (\partial_t w_t) v_t - v_t^2 \overline{(\partial_t v_t)} \mathrm{e}^{-w_t} &= \varphi, \\ \partial_t w_t + 2 v_t \overline{(\partial_t v_t)} \mathrm{e}^{-w_t} &= 0, \\ (\partial_t v_t) \mathrm{e}^{-w_t} &= \varphi.\end{aligned}$$

□

Now we have all the ingredients to prove the Baker-Campbell-Hausdorff formula.

Proof of Theorem 6.3.1. As stated in Lemma 6.3.2, the closure A^{cl} of the operator $A := -\mathrm{i}\,(k^+(\varphi) - k^-(\varphi))$ is self-adjoint. By Stone's theorem, for every $f \in \mathcal{D}$, the function $g_t := \mathrm{e}^{\mathrm{i}tA^{\mathrm{cl}}} f$ is norm-differentiable with derivative $\partial_t g_t = \mathrm{i}A^{\mathrm{cl}} g_t$. By Lemma 6.3.6, the map $t \mapsto S_t f$ satisfies the same differential equation. Moreover, $S_0 f = g_0 = f$; therefore, $g_t = S_t f$ for all $t \geq 0$. In particular, $t = 1$ corresponds to the Baker-Campbell-Hausdorff formula. □

6.4 A family of unitary operators

In [CFG+19], the connection between orthogonal duality functions and unitary symmetries of underlying Lie-algebras is shown. This chapter is dedicated to a generalization of [CFG+19, Theorem 3.1 1. (ii)] to uncountable spaces E. We address the other part of the theorem, [CFG+19, Theorem 3.1 1. (i)], in Section 6.5.1 below.From now on, we assume that α is a finite measure. Let $p \in (0,1)$ and let $k^+(\varphi), k^-(\varphi), k^0(\varphi)$, $\varphi \in \mathcal{C}$ be the Fock representation on $\mathfrak{F}_{\mathbb{C}}$ introduced in Section 6.2.1. We study the family of the unitary operators

$$U_{\xi,\phi} := \mathrm{e}^{\xi k^+(\mathbb{1}_E) - \bar{\xi} k^-(\mathbb{1}_E)} \mathrm{e}^{2\mathrm{i}\phi k^0(\mathbb{1}_E)} \tag{6.57}$$

parametrized by $\xi \in \mathbb{C}$, $\phi \in \mathbb{R}$.

Remark 6.4.1. The family (6.57) has a close connection to the unitary representation of the Lie group $SU(1,1)$ given by [CDP06, Equation (21)]. However, to deduce that $U_{\xi,\phi}$, $\xi \in \mathbb{C}$, $\phi \in [0, 2\pi)$ is a representation of the Lie group $SU(1,1)$, we need to assume $\frac{\alpha(E)}{2} \in \mathbb{N}$, or equivalently, the spectrum of $k^0(\mathbb{1}_E)$ is a subset of $\mathbb{N}$, see [BNS65, Equation (II.13)].

We remind the reader of the unitary operator $\mathfrak{U}_{\mathbb{C}} : \mathfrak{F}_{\mathbb{C}} \to L^2(\rho_{p,\alpha};\mathbb{C})$, defined in (6.24), and consider the complexification $\mathfrak{V}_{\mathbb{C}}$ of the operator $\mathfrak{V}$, defined in (3.51), which is given by $\mathfrak{V}_{\mathbb{C}} : L^2(\rho_{p,\alpha};\mathbb{C}) \to \mathfrak{F}_{\mathbb{C}}$ where $F \mapsto (1-p)^{\frac{1}{2}\alpha(E)}\,(f_n)_{n\in\mathbb{N}_0}$, $f_n := F \circ \iota_n$ and $\iota_n(x_1,\dots,x_n) = \delta_{x_1} + \ldots + \delta_{x_n}$. The following theorem shows that, for a specific choice of ξ and ϕ, the operator $U_{\xi,\phi}$ is equal to the composition $\mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}}$.

Theorem 6.4.2. *If* $\tanh\xi = \sqrt{p}$ *and* $\phi = 0$*, then*

$$U_{\xi,\phi} = \mathrm{e}^{\xi k^+(\mathbb{1}_E) - \xi k^-(\mathbb{1}_E)} = \mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}}. \tag{6.58}$$

The fact that this theorem is a generalization of [CFG+19, Theorem 3.1 1. (ii)] becomes clear when we reformulate (6.58) using infinite-dimensional Meixner polynomials instead of the operator $\mathfrak{U}_{\mathbb{C}}$. Indeed, let $f_k \in L^2_{\mathrm{sym}}(\lambda_k)$ and define $f = (f_n)_{n\in\mathbb{N}_0}$ where $f_n := \mathbb{1}_{\{k=n\}} f_k$. Then, we have

$$(U_{\xi,\phi}f)_n(x_1,\dots,x_n) = (1-p)^{\frac{1}{2}\alpha(E)}\frac{(1-p)^k}{k!}\mathcal{M}_k^{p,\alpha} f_k(\delta_{x_1} + \ldots + \delta_{x_n})$$

for λ_n-almost all $(x_1,\dots,x_n) \in E^n$ and $n \in \mathbb{N}_0$.

We prove Theorem 6.4.2 using the Baker-Campbell-Hausdorff formula, see Theorem 6.3.1. Note that it is possible to prove the theorem without that formula by using generating functions, see [FJW23, Section 4], following the approach in [CFG+19]. Nevertheless, in this thesis, we opt for the proof stated below which is notably shorter and simpler than the one presented in [FJRW24] provided that the Baker-Campbell-Hausdorff formula is already known. The latter formula was omitted in [FJW23] for the sake of conciseness.

Proof. Since $\mathcal{D}$ is dense in $\mathfrak{F}_{\mathbb{C}}$, it is enough to verify (6.58) on $\mathcal{D}$, so let $f \in \mathcal{D}$ be fixed. By the Baker-Campbell-Hausdorff formula, the equation $U_{\xi,\phi}f = \mathrm{e}^{\sqrt{p}k^+(\mathbb{1}_E)}g$ holds where $g := \mathrm{e}^{\log(1-p)k^0(\mathbb{1}_E)}\mathrm{e}^{-\sqrt{p}k^-(\mathbb{1}_E)}f$. By Lemma 6.3.3, we have $g \in \mathcal{D}$. Let $G : \mathbf{N}_{<\infty} \to \mathbb{C}$ be the function satisfying $g_n(x_1,\dots,x_n) = G(\delta_{x_1} + \ldots + \delta_{x_n})$ for all $x_1,\dots,x_n \in E$ and $n \in \mathbb{N}_0$. By (2.17) and Remark 6.2.2, we obtain

$$\begin{aligned}\left(\mathrm{e}^{\sqrt{p}k^+(\mathbb{1}_E)}g\right)_N(z_1,\dots,z_N) &= \sum_{n=0}^{N}\frac{1}{n!}\frac{n!}{(N-n)!}((\sqrt{p}k^+(\mathbb{1}_E))^{N-n}g)_N(x_1,\dots,x_N)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int g_n\,\mathrm{d}\mu^{(n)}\end{aligned}$$

where $\mu := \delta_{z_1} + \ldots + \delta_{z_N}$. This can be rewritten as

$$\mathfrak{V}_{\mathbb{C}}^{-1}\mathrm{e}^{\sqrt{p}k^+(\mathbb{1}_E)}g(\mu) = (1-p)^{-\frac{\alpha(E)}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\int g_n\,\mathrm{d}\mu^{(n)} \tag{6.59}$$

for $\rho_{p,\alpha}$-almost all $\mu \in \mathbf{N}_{<\infty}$.

On the other hand, using the definitions of k^0 and k^-, see (6.16) and (6.17), and the kernel $\kappa_{n+k,n}$ defined in (3.45), yields

$$\begin{aligned} g_n(x_1,\dots,x_n) &= (1-p)^{\frac{1}{2}\alpha(E)+n} \sum_{k=0}^{\infty} \frac{(-\sqrt{p})^k}{k!} ((k^-(\mathbb{1}_E)^k f)_n(x_1,\dots,x_n) \\ &= (1-p)^{\frac{1}{2}\alpha(E)+n} \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \int f_{n+k}(x_1,\dots,x_{n+k}) \\ &\qquad \kappa_{n+k,n}((x_1,\dots,x_n), \mathrm{d}(x_{n+1},\dots,x_{n+k})). \end{aligned} \tag{6.60}$$

Combining (6.59) and (6.60), substituting $m = n + k$ and using the explicit formula, see Theorem 3.3.4, yields

$$\begin{aligned} \mathfrak{V}_{\mathbb{C}}^{-1} \mathrm{e}^{\sqrt{p}k^+(\mathbb{1}_E)} g(\mu) &= \sum_{m=0}^{\infty} \frac{(1-p)^m}{m!} \sum_{n=0}^{m} \binom{m}{n} \left(1 - \frac{1}{p}\right)^{n-m} \\ &\iint f_m(x_1,\dots,x_m) \kappa_{m,n}((x_1,\dots,x_n), \mathrm{d}(x_{n+1},\dots,x_{n+k})) \mu^{(n)}(\mathrm{d}(x_1,\dots,x_n)) \\ &= \sum_{m=0}^{\infty} \frac{(1-p)^m}{m!} \mathcal{M}_m^{p,\alpha} f_m(\mu) = \mathfrak{U}_{\mathbb{C}} f. \end{aligned}$$

□

6.5 Consistent particle systems

Loosely speaking, two crucial steps in the algebraic approach to duality are as follows:

(i) Exploit the commutation relations in a Lie algebra to discover symmetries of the generator, i.e., operators that commute with the generator. Then, use these symmetries to construct self-duality functions or self-intertwining relations. For instance, in [CFG+19] unitary symmetries were employed to produce orthogonal polynomials (see, e.g., [KLS10]) as duality functions (see also [FRS22]).

(ii) Recognize the generator of the particle system under investigation as an element of the universal enveloping algebra of the Lie algebra in a given representation. Subsequently, build duality relations using operators that intertwine between different representations (see, e.g., [CGRS16a]).

For an overview of this approach, we refer to [SSV20]. So far, this methodology has been applied exclusively in the discrete setting. Our interest lies in developing the algebraic approach to self-duality (or, more generally, to self-intertwining relations) within the framework of particle systems evolving in uncountable spaces, such as $\mathbb{R}^d$. Therefore, we delve into (i) in Section 6.5.1. Additionally, we provide an intuition for (ii) in Section 6.5.2.

6.5.1 Symmetries

We fix $p \in (0,1)$ and a finite measure α on a Borel space $(E, \mathcal{E})$. To simplify notation in this chapter, we identify $\mathfrak{F}_\mathbb{C}$ with $L^2(w_{p,\alpha};\mathbb{C})$. Furthermore, we identify the operators $k^+(\varphi), k^0(\varphi), k^-(\varphi)$ as operators on $\mathcal{D} \subset L^2(w_{p,\alpha};\mathbb{C})$, as described in Remark 6.2.2.

Let $(\Omega, \mathcal{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathbf{N}_{<\infty}})$ be a Markov family with state space $\mathbf{N}_{<\infty}$. Note that $k^+(\mathbb{1}_E)$, $k^0(\mathbb{1}_E)$, $k^-(\mathbb{1}_E)$ in (6.19), (6.20), (6.21) are also well-defined for measurable functions $F : \mathbf{N}_{<\infty} \to [0,\infty)$. Then, consistency defined in (2.6) rewrites as

$$P_t^{\mathfrak{F}} k^+(\mathbb{1}_E)F = k^+(\mathbb{1}_E)P_t^{\mathfrak{F}} F \tag{6.61}$$

where $(P_t^{\mathfrak{F}})_{t\geq 0}$ denotes the corresponding Markov semigroup. Note that we use the notation $(P_t^{\mathfrak{F}})_{t\geq 0}$ instead of $(P_t)_{t\geq 0}$ here, as the operators $k^+(\varphi), k^0(\varphi), k^-(\varphi)$ act on the Fock space $\mathfrak{F}_\mathbb{C} = L^2(w_{p,\alpha};\mathbb{C})$ and thus $P_t^{\mathfrak{F}}$ also acts on the Fock space. If $(\eta_t)_{t\geq 0}$ preserves the number of particles, we obtain

$$P_t^{\mathfrak{F}} k^0(\mathbb{1}_E)F = k^0(\mathbb{1}_E)P_t^{\mathfrak{F}} F. \tag{6.62}$$

If $(\eta_t)_{t\geq 0}$ is consistent and the measure $w_{p,\alpha}$ (or equivalently, the distribution of the Pascal process $\rho_{p,\alpha} = (1-p)^{\alpha(E)} w_{p,\alpha}$, see Lemma 3.3.4) is reversible for $(\eta_t)_{t\geq 0}$, then

$$P_t^{\mathfrak{F}} k^-(\mathbb{1}_E)F(\mu) = k^-(\mathbb{1}_E)P_t^{\mathfrak{F}} F(\mu) \tag{6.63}$$

for $w_{p,\alpha}$-almost all $\mu \in \mathbf{N}_{<\infty}$. Indeed, the arguments of the proof of Theorem 6.2.1 show that the adjoint relation (6.1) holds for non-negative measurable functions as well. Therefore, reversibility and (6.61) yields

$$\begin{aligned}\int \left(k^-(\mathbb{1}_E)P_t^{\mathfrak{F}} F\right) \varphi \, \mathrm{d}w_{p,\alpha} &= \int F \left(P_t^{\mathfrak{F}} k^+(\mathbb{1}_E)\varphi\right) \mathrm{d}w_{p,\alpha} \\ &= \int F \left(k^+(\mathbb{1}_E)P_t^{\mathfrak{F}}\varphi\right) \mathrm{d}w_{p,\alpha} = \int \left(P_t^{\mathfrak{F}} k^-(\mathbb{1}_E)F\right) \varphi \, \mathrm{d}w_{p,\alpha}\end{aligned}$$

for all $\varphi : \mathbf{N}_{<\infty} \to [0,\infty)$ measurable. We note that we have already encountered (6.63) in different notation, namely in (3.59).

The upcoming theorem generalizes [CFG$^+$19, Theorem 3.1 1. (i)] and applies, for instance, to the generalized symmetric inclusion process introduced in Section 4.3. Remember the operator family $U_{\xi,\phi} = \mathrm{e}^{\xi k^+(\mathbb{1}_E) - \bar{\xi} k^-(\mathbb{1}_E)} \mathrm{e}^{2\mathrm{i}\phi k^0(\mathbb{1}_E)}$, defined in (6.57).

Theorem 6.5.1. *Let $p \in (0,1)$ and let α be a finite measure. Suppose $(\eta_t)_{t\geq 0}$ is a consistent Markov process that admits the distribution of the Pascal process $\rho_{p,\alpha}$ as a reversible measure. Then, $P_t^{\mathfrak{F}}$ commutes with all unitary operators $U_{\xi,\phi}$, i.e., $U_{\xi,\phi}P_t^{\mathfrak{F}} = P_t^{\mathfrak{F}} U_{\xi,\phi}$ for all $\xi \in \mathbb{C}$, $\phi \in \mathbb{R}$ and $t \geq 0$.*

The operator $\exp(2\mathrm{i}\phi k^0(\mathbb{1}_E))$ acts as multiplication with $\exp(2\mathrm{i}\phi(\frac{1}{2}\alpha(E) + \mu(E)))$. It commutes with $P_t^{\mathfrak{F}}$ since consistency implies that the total number of particles is preserved, see Proposition 2.1.2. As for $\exp(\xi k^+(\mathbb{1}_E) - \bar{\xi} k^-(\mathbb{1}_E))$, the short informal explanation is simple: $P_t^{\mathfrak{F}}$ commutes with $k^+(\mathbb{1}_E)$, $k^-(\mathbb{1}_E)$ and, thus, also with the difference

$\xi k^+(\mathbb{1}_E)-\overline{\xi}k^-(\mathbb{1}_E)$ and the exponential $\exp(\xi k^+(\mathbb{1}_E)-\overline{\xi}k^-(\mathbb{1}_E))$. The precise reasoning is slightly longer because the operators are unbounded, necessitating careful consideration of domains.

By Theorem 6.4.2, choosing $\tanh\xi = \sqrt{p}$, $\phi = 0$ results in $\mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}} = U_{\xi,\phi}$. Consequently, Theorem 6.5.1 shows that $\mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}}$ commutes with $P_t^{\mathfrak{F}}$, recovering the intertwining relation (3.53). Therefore, we obtain an alternative—algebraic—proof for Theorem 3.1.6.

Corollary 6.5.2. *Under that assumptions of Theorem 6.5.1, the orthogonal polynomial intertwining relation* (IR.2) *holds true.*

We denote by $(P_t)_{t\geq 0}$ the Markov semigroup on $L^2(\rho_{p,\alpha};\mathbb{C})$ of $(\eta_t)_{t\geq 0}$.

Proof. Using $\mathfrak{V}_{\mathbb{C}}P_t\mathfrak{U}_{\mathbb{C}} = P_t^{\mathfrak{F}}\mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}} = \mathfrak{V}_{\mathbb{C}}\mathfrak{U}_{\mathbb{C}}P_t^{\mathfrak{F}}$ yields $P_t\mathfrak{U}_{\mathbb{C}} = \mathfrak{U}_{\mathbb{C}}P_t^{\mathfrak{F}}$. The restriction of this equation to $L^2_{\mathrm{sym}}(\lambda_n)$ corresponds to (IR.2). □

Remark 6.5.3. We note that the K-transform $\mathcal{K}$, see (2.19), can also be expressed in terms of the operator $k^+(\mathbb{1}_E)$. In fact, using (6.59), we obtain

$$\mathcal{K}F(\mu) = \mathrm{e}^{\sqrt{p}k^+(\mathbb{1}_E)}F(\mu) \tag{6.64}$$

for measurable $F : \mathbf{N}_{<\infty} \to [0,\infty)$ and $\mu \in \mathbf{N}_{<\infty}$. Here, the right-hand side reads as the finite sum $\sum_{n=0}^{\infty}\frac{\sqrt{p}^n}{n!}k^+(\mathbb{1}_E)^nF(\mu)$ where $k^+(\mathbb{1}_E)^nF(\mu)$ is zero for $n > \mu(E)$. Equation (6.64) serves as the continuum counterpart to [CGR21, Lemma 4.2].

We only prove Theorem 6.5.1 for $\phi = 0$; the commutation property of the operator $\mathrm{e}^{2\mathrm{i}\phi k^0(\mathbb{1}_E)}$ for $\phi \neq 0$ in the definition of $U_{\xi,\phi}$, see (6.57), follows similarly. Fix $\xi \in \mathbb{C}$ and set $A := -\mathrm{i}(\xi k^+(\mathbb{1}_E) - \overline{\xi}k^-(\mathbb{1}_E)) : \mathcal{D} \to \mathcal{D}$. We denote its closure by A^{cl}. Before proving Theorem 6.5.1, we present the following preliminary lemma. Note that $k^+(\mathbb{1}_E)F$ and $k^-(\mathbb{1}_E)F$, defined in (6.19) and (6.21), are also well-defined for functions $F \in L^\infty(w_{p,\alpha};\mathbb{C})$. Moreover, $k^+(\mathbb{1}_E)F, k^-(\mathbb{1}_E)F \in L^2(w_{p,\alpha};\mathbb{C})$.

Lemma 6.5.4. *Each $F \in L^\infty(w_{p,\alpha})$ is contained in the domain of A^{cl}. Moreover, $A^{\mathrm{cl}}F = -\mathrm{i}(\xi k^+(\mathbb{1}_E) - \overline{\xi}k^-(\mathbb{1}_E))F$.*

Proof. Let $B := -\mathrm{i}(\xi k^+(\mathbb{1}_E) - \overline{\xi}k^-(\mathbb{1}_E)) : L^\infty(w_{p,\alpha}) \to L^2(w_{p,\alpha};\mathbb{C})$. The same strategy used in the proof of Lemma 6.3.2 yields that B is essentially self-adjoint, i.e., both B and A have unique self-adjoint extensions. Consequently, we deduce that $A^{\mathrm{cl}} = B^{\mathrm{cl}}$ and, therefore, $BF = A^{\mathrm{cl}}F$ for $F \in L^\infty(w_{p,\alpha})$. □

Proof of Theorem 6.5.1. We claim that

$$A^{\mathrm{cl}}P_t^{\mathfrak{F}}F = P_t^{\mathfrak{F}}A^{\mathrm{cl}}F \tag{6.65}$$

holds for all F in the domain of A^{cl} and $t \geq 0$. First, note that, since $w_{p,\alpha}$ is reversible, $P_t^{\mathfrak{F}}$ is a well-defined, self-adjoint and bounded operator on $L^2(w_{p,\alpha};\mathbb{C})$. Thus, the commutation relations (6.61) and (6.63), together with Lemma 6.5.4, yield the equation $A^{\mathrm{cl}}P_t^{\mathfrak{F}}G = P_t^{\mathfrak{F}}AG$ for $G \in \mathcal{D}$. Taking limits results in (6.65).

By using the fact that $P_t^{\mathfrak{F}}$ is bounded and commutes with the closure of A, [Sch12, Proposition 5.26] applies and shows that $P_t^{\mathfrak{F}}$ commutes with all spectral projections of A^{cl}. This implies by spectral calculus that $U_{\xi,\phi} = \mathrm{e}^{\mathrm{i}A}$ commutes with $P_t^{\mathfrak{F}}$. □

6.5.2 Algebraic expressions for generators

Let α be a finite measure. When E is a countable set, the generator of the SIP can be expressed in terms of the representation k^+, k^0, k^- of the $su(1,1)$ algebra acting on lattice sites, see (1.21). This connection generalizes to the formal generator of the gSIP, at least for a specific choice of c. More precisely, the formal generator relates to the Fock representation $k^+(\varphi)$, $k^0(\varphi)$, $k^-(\varphi)$, $\varphi \in \mathcal{C}$ of the $su(1,1)$ current algebra provided in Remark 6.2.2 in the following manner: Let $\varphi_1, \varphi_2 \in \mathcal{C}$ be non-negative and define

$$C(\varphi_1, \varphi_2) := \frac{1}{2} \int \varphi_1 \, \mathrm{d}\alpha \int \varphi_2 \, \mathrm{d}\alpha - \int \varphi_1 \varphi_2 \, \mathrm{d}\alpha.$$

In the notation of counting measures, (6.22) and (6.23) rewrite as

$$\begin{aligned} & k^+(\varphi_1)k^-(\varphi_2)F(\mu) \\ & \quad = \iint \varphi_1(y)\varphi_2(x)F(\mu - \delta_y + \delta_x)\mu(\mathrm{d}y)(\mu+\alpha)(\mathrm{d}x) - F(\mu) \int \varphi_1\varphi_2 \, \mathrm{d}\mu, \\ & k^-(\varphi_1)k^+(\varphi_2)F(\mu) \\ & \quad = \iint \varphi_1(x)\varphi_2(y)F(\mu - \delta_y + \delta_x)\mu(\mathrm{d}y)(\mu+\alpha)(\mathrm{d}x) + F(\mu) \int \varphi_1\varphi_2 \, \mathrm{d}(\mu+\alpha) \end{aligned}$$

for $F \in \mathcal{D}$ and $\mu \in \mathbf{N}_{<\infty}$. When combined with

$$\begin{aligned} 2k^0(\varphi_1)k^0(\varphi_2)F(\mu) = F(\mu) \iint \left(\varphi_1(x)\varphi_2(y) + \varphi_2(x)\varphi_1(y)\right)(\mu+\alpha)(\mathrm{d}x)\mu(\mathrm{d}y) \\ + \frac{1}{2}F(\mu) \int \varphi_1 \, \mathrm{d}\alpha \int \varphi_2 \, \mathrm{d}\alpha, \end{aligned}$$

we arrive at

$$\mathcal{L} = k^+(\varphi_1)k^-(\varphi_2) + k^-(\varphi_1)k^+(\varphi_2) - 2k^0(\varphi_1)k^0(\varphi_2) + C(\varphi_1,\varphi_2)\mathrm{id}. \tag{6.66}$$

Here, id denotes the identity operator and $\mathcal{L}$ is the formal generator of the gSIP, see (4.6), with conductances

$$c(x,y) = \varphi_1(x)\varphi_2(y) + \varphi_2(x)\varphi_1(y). \tag{6.67}$$

Equation (6.66) is a generalization of [CFG+19, last equation of Section 2.2.1] to uncountable E.

Remark 6.5.5. Equation (6.66) holds only when c is of the form given in (6.67). In a more general setting, say on $E = \mathbb{R}^d$ with homogeneous measures $\alpha = \alpha\lambda$, where $\alpha > 0$ and λ denotes the Lebesgue measure, one may aim to assign meaning to expressions such as

$$\mathcal{L} = \int c(x,y)\left(k_x^+k_y^- + k_y^+k_x^- - 2k_x^0k_y^0 + \frac{\alpha^2}{2}\right)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) - \alpha \int c(x,x)\lambda(\mathrm{d}x)$$

by first defining it as (6.66). Subsequently, one can approximate suitably nice functions $c(x, y)$ by linear combinations of symmetrized tensor products. We leave as an open problem to carry out this challenge, to clarify its relevance for Markov processes and to figure out its relation to Hamilton operators such as $\int \mu(k)a(k)^\dagger a(k)\,\mathrm{d}k$ in quantum field theory and quantum many-body theory (see, e.g., the proposition preceding Theorem X.45 in [RS75], or [Tal22, Chapter 3.7]).

In Section 4.3, we proved consistency of the gSIP and showed that it admits the distribution of the Pascal process as a reversible measure. Consistency and reversibility can also be proved algebraically if c takes the form specified in (6.67). More precisely, using (6.66), then consistency—the commutation property with $k^+(\mathbb{1}_E)$—follows from the commutation relations (6.2). Furthermore, reversibility is a consequence of the adjoint relations (6.1).

Furthermore, it can readily be verified that, in terms of the second Fock representation $K^+(\varphi), K^0(\varphi), K^-(\varphi)$, $\varphi \in \mathcal{C}$, as defined in Section 6.2.2, the formal generator of the gSIP for c of the form given in (6.67) can be expressed as follows:

$$\mathcal{L} = K^+(\varphi_1)K^-(\varphi_2) + K^-(\varphi_1)K^+(\varphi_2) - 2K^0(\varphi_1)K^0(\varphi_2) + C(\varphi_1, \varphi_2)\mathrm{id}.$$

This has the following conceptual background: The orthogonal polynomial intertwining relation (IR.2), which switches between the two representations, is a self-intertwining relation for the gSIP since $\mathcal{L}$ has the same algebraic structure in both representations. This property is the continuum counterpart to [Gro19, Lemma 4.2] which states that the *Casimir* remains invariant under a change of representation.

On the other hand, the formal generator $\hat{\mathcal{L}}$ of continuum version of the Brownian energy process, see (5.4), can be rewritten as

$$\hat{\mathcal{L}} = \mathscr{K}^+(\varphi_1)\mathscr{K}^-(\varphi_2) + \mathscr{K}^-(\varphi_1)\mathscr{K}^+(\varphi_2) - 2\mathscr{K}^0(\varphi_1)\mathscr{K}^0(\varphi_2) + C(\varphi_1, \varphi_2)\mathrm{id}$$

in the Fock representation $\mathscr{K}^+(\varphi), \mathscr{K}^0(\varphi), \mathscr{K}^-(\varphi)$, $\varphi \in \mathcal{C}$ defined in Section 6.2.3. Also, here we have the intuition that the intertwining relation in terms of infinite-dimensional Laguerre polynomials 5.2, which switches between the representations, is an intertwining relation for $\mathcal{L}$ and $\hat{\mathcal{L}}$ since both have the same algebraic structure.

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