Macroeconomic Consequences of Demographic Change:

Capital Expenditures, Demand, and Factor Prices

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Für meine Familie.

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Preface

"*In the long run we are all dead*"[1] J.M. Keynes famously wrote, and even though he was and is right, in the meantime, we (hopefully) all age. As fortunate as it is for an individual to age and live a long and happy life, once aging affects an entire population, this has extensive consequences for the economy as well as society as a whole. Studying the potential ramifications of population aging and demographic change for the economy has a long tradition in economics. Hansen (1939) was one of the first to propose that demographic change, which entails a decrease in the population growth rate, could lead to low demand and potentially to a *secular stagnation*. Given that demographic change and population aging are already affecting many (high-income) countries in the world, they will constitute a major challenge in the 21st century; see, for example, Cooley and Henriksen (2018), Kotschy and Sunde (2018), Eggertsson *et al.* (2019a), Jones (2022), and Maestas *et al.* (2023).

The economic consequences of demographic change and population aging are vast, and there exists a multitude of literatures that study the different channels through which demographic change and population aging affect macroeconomic aggregates. Moreover, the terms demographic change and population aging encompass multiple aspects that all affect the economy, e.g., an increase in life expectancy, a change in fertility, a change in the demographic structure, etc. These factors all affect the economy through different channels. To gain a better understanding of how demographic change and population aging impact the economy, I focus on one important channel that constitutes a central part of demographic change and population aging: the reduction in (the growth rate of) the labor force that is

[1] Keynes, 1923, p. 80.

mainly due to falling birth rates (United Nations (2022)).

Therefore, in this dissertation, I aim to contribute to the literature by analyzing the effect demographic change and population aging have on the economy through reducing the overall labor supply, i.e., I model demographic change as a form of adverse supply shock that reduces the overall labor force or the growth rate of the labor force and investigate its implications for the aggregate economy. I will use the two terms, i.e., demographic change and population aging, synonymously throughout this dissertation.

This dissertation is, for the most part, theoretical. Using rich yet still tractable economic models, I analyze the channels and mechanisms through which demographic change can have an effect on macroeconomic aggregates. Moreover, this approach also allows me to study in detail the conditions and assumptions under which certain channels are present, and if there are multiple channels of opposite signs, clarify the conditions under which one channel dominates the other. Having a good understanding of the underlying economic mechanisms is important not only to understand patterns observed in the data but also when designing policy interventions. Therefore, shedding additional light on these underlying channels and mechanisms, as well as the necessary assumptions, is the main objective of this dissertation.

This dissertation consists of four self-contained chapters that cover different aspects of how demographic change and population aging affect the economy. Chapters 1–3 study the effect of demographic change and population aging on capital expenditures, aggregate demand, wages, unemployment, interest rates, and the rate of innovation. Chapter 4 diverges to some extent from the first three chapters and investigates a topic that is closely related to demographic change and population aging; namely, the increase in health expenditures as a share of GDP.

In Chapter 1 of this dissertation, I study the effect of demographic change on the *supply side* of the economy. More specifically, I analyze how demographic change affects capital expenditures of firms and how this in turn affects wages and interest rates in the economy. To that end, I assume that firms not only face linear costs when acquiring capital, which is standard, but also convex or concave costs. The model highlights two channels through which demographic change can affect the wage rate. First, with convex acquisition costs, a fall in labor supply will lead to higher wages in equilibrium. With concave acquisition costs, the effect on wages is ambiguous and depends on parameter values. This channel operates independently of the interest rate. Second, if demographic change makes labor relative to capital more scarce, this will also result in a higher equilibrium wage rate, provided that the interest rate can fall. Therefore, this channel can only operate in an environment in which interest rates are downwardly flexible, i.e., there is no binding effective lower bound. Regarding the overall capital expenditures of firms, the model shows that for constant interest rates and convex acquisition costs, a decrease in the labor force will lead to a reduction in the optimal capital demand of firms. For concave acquisition costs, the effect of demographic change on the optimal capital expenditures of firms is ambiguous and depends on parameter values. Therefore, this chapter emphasizes that even though demographic change will make labor scarcer, this does not automatically imply that the price of labor, i.e., the wage rate, will necessarily increase. Consequently, two important aspects that need to be considered when attempting to answer the question of whether demographic change will indeed lead to higher wages are: how does demographic change affect the capital-to-labor ratio? And what kind of capital acquisition costs do firms face?

In Chapter 2, I study the effect of demographic change on the *demand side* of the economy. More specifically, I ask whether demographic change can lead to a shortfall in demand that results in a demand-induced recession. As stated before, I treat demographic change as an adverse supply shock, and in this chapter I study and characterize the conditions under which demographic change can constitute a "Keynesian supply shock", i.e., a negative supply shock that produces an income loss, which in turn leads to a fall in aggregate demand such that output in the economy falls *below* potential.

The reduction of the labor force leads to a reduction in the capital expenditures of firms, which then requires a fall in the equilibrium interest rate in order for the capital market to be in equilibrium. In case the real interest rate is unable to fall due to, for example, a binding real effective lower bound (ELB). This can give rise to an endogenous "savings glut".

Savings and investments are central to this mechanism, as investments not only lay the foundation for production in the future, where they belong to the realm of *supply*, but also belong to the domain of *demand* in the period in which they are acquired. Therefore, the presence of savings and investments entails the existence of an *intertermporal* channel through which adverse supply shocks in period $t + 1$ can negatively affect demand in period t. Normally, this channel is closed as the real interest rate for period $t + 1$ adjusts to ensure savings are equal to investments. However, if the real interest rate is constrained by, for example, a binding real ELB, this channel opens up, which can give rise to an *intertermporal* "Keynesian supply shock".

The excess savings, or "savings glut", will need to be eliminated in order for the economy to reach an equilibrium. This can either take place through an adjustment in *nominal* or *real* variables. In the event that the presence of nominal rigidities prevents the economy from reaching an equilibrium through a change in nominal variables, the equilibrium will be reached through an adjustment in real variables, which will entail involuntary unemployment as well as a fall in output per capita. Involuntary unemployment can also lead to a higher real wage rate if the capital-to-labor ratio increases. However, higher real wages are not the *cause* of involuntary unemployment; rather, they are the *consequence* thereof.

Moreover, the demand-induced recession does not only negatively affect the young households that, by accumulating excess savings, can be considered the "culprits" and whose savings need to be reduced in order for the economy to reach an equilibrium, but also the old households that do not accumulate any savings in the current period, i.e., who have a marginal propensity to consume of 1.

In addition, I show that the economy can suffer a demand-induced recession even if young households recognize that accumulating excess savings will be self-defeating, i.e., cause a demand-induced recession that will lower their income and their savings. Furthermore, I show that the government, by taxing the excess savings, can stabilize output and employment and even induce a Pareto improvement by stabilizing the income of the current old generation.

In Chapter 3, I study whether demographic change can have a *positive* or *negative* effect on the rate of innovation in an economy. To that end, I analyze how demographic change affects the rate of innovation by influencing the incentive to innovate. I build a model in which output is produced using (automating) capital and labor. I assume that there are two types of labor: low-skilled and high-skilled labor. In addition, I assume that low-skilled labor serves as a substitute for capital, whereas high-skilled labor is complementary to capital. The idea behind this set-up is that it is possible to replace certain tasks done by humans in the production process with (automating) capital, e.g., robots; however, (automating) capital cannot function completely independently in the production process. For example, it is possible to replace workers on the assembly line with robots, but in order for the robots to function properly, we require human mechanics and engineers. I model innovations such that they take the form of an increase in the level of factor augmenting technology.

Unlike the existing literature, the model allows me to study the two opposing channels that demographic change can have on the incentive to innovate in one framework. On the one hand, demographic change can lead to a negative *supply* effect, i.e., a smaller population entails fewer researchers, which decreases the rate of innovation. On the other hand, demographic change can lead to a positive *demand* effect, as demographic change increases the costs of low-skilled workers by reducing their supply, which makes it more attractive to substitute low-skilled workers with capital. Thus, giving firms an incentive to invest in a higher level of capital augmenting technology, which positively affects the rate of innovation in the economy. Which effect dominates crucially depends on how the skill share is affected by demographic change, as demographic change will most likely affect the entire population, i.e., all skill groups will shrink in absolute terms to some degree due to demographic change.

In case the skill share remains constant, the negative effect always dominates the positive effect. If the skill share increases due to demographic change and the elasticity of substitution between capital and low-skilled labor is large, i.e., above a certain threshold, the result is ambiguous and depends on parameter values. Therefore, it is then possible that the positive effect dominates the negative effect, and thus that demographic change accelerates the rate of innovation.

Moreover, the main policy implication of the model is that governments should attempt to increase the share of high-skilled workers in the population, because this will not only have a positive direct effect on the rate of innovation but also a positive indirect effect, i.e., by increasing the value of an innovation.

In addition, I use the model to study how demographic change could influence the wage rate of low-skilled and high-skilled workers differently for a constant skill share. The model predicts that high-skilled workers will always benefit from demographic change. Whereas for low-skilled labor, the sign of the effect depends on the value of the elasticity of substitution between capital and low-skilled labor. In case the elasticity of substitution is high, i.e., above a certain threshold, demographic change has a negative effect on the wage rate of low-skilled workers, i.e., even though there are fewer low-skilled workers, their wage rate declines. If the elasticity of substitution is low, i.e., below a certain threshold, low-skilled workers, similarly to high-skilled workers, benefit from demographic change. The economic intuition is that fewer high-skilled workers affect the wage rate of low-skilled workers negatively, and fewer low-skilled workers have a positive effect on their wage rate. Which effect dominates then depends on the value of the elasticity of substitution between capital and low-skilled labor.

In the model, demographic change also leads to a higher capital-to-labor ratio. Whether this increases or decreases the skill premium solely depends on the elasticity of substitution between capital and low-skilled labor. If the elasticity of substitution is greater than 1, i.e., capital and low-skilled workers are substitutes relative to capital and high-skilled workers, demographic change leads to an increase in the skill premium. This effect is independent of changes in the (relative) level of technology. If the elasticity of substitution is below 1, demographic change decreases the skill premium. As the empirical evidence suggests that high-skilled labor is relatively more complementary to capital than low-skilled labor, the model illustrates that demographic change can also be seen as a contributing factor to the rise in the skill premium that has been observed in the past.

In Chapter 4, which is joint work with Johanna Rude, we argue that the rise in healthcare expenditures as a share of GDP as well as a reallocation of production factors to the healthcare sector combined with lower productivity growth in the healthcare sector relative to the rest of the economy is not as undesirable as it might seem at first. We develop a two-sector general equilibrium model in which each sector produces a different good; we refer to these two goods as health and non-health goods. Output in each sector is produced using low-skilled and high-skilled labor. We assume that high-skilled labor is unable to switch sectors due to, for example, occupation-specific skills. Low-skilled labor, however, is fully mobile in our model. Households have standard homothetic CES preferences over the two goods. In the theoretical part, we show that an increase in the level of productivity in the non-health sector leads to an *income* and a *substitution* effect. The reallocation of the flexible production factor, as well as whether the share of healthcare expenditures as a share of GDP increases in response to productivity growth in the non-health sector, depends on which effect dominates. We show that if health and non-health goods are complements the income effect dominates the substitution effect, leading to a reallocation of production factors from the non-health sector to the health sector and an increase in the share of healthcare expenditures as a share of GDP. If they are substitutes, the substitution effect dominates the income effect, and the opposite occurs. Therefore, our model is able to rationalize the stylized facts, conditional on that the elasticity of substitution between health and non-health consumption is less than 1. Moreover, the model also suggests that the pattern observed in the data is potentially optimal from the perspective of a utility-maximizing representative agent.

In the empirical part, we proceed to estimate the elasticity of substitution between health and non-health consumption using German microdata. We find strong evidence that the elasticity of substitution is indeed below 1, which confirms the sufficient condition derived in the theoretical part. In addition, our model predicts that if the elasticity of substitution is below 1 and productivity growth in the non-health sector is stronger relative to the health sector, the skill premium in the health sector will increase relative to the non-health sector due to the additional inflow of low-skilled workers into the health sector. Using German and US data, we show that the skill premium in the health sector has increased more than the rest of the economy, which provides further support for our theoretical model.

Chapter 1

Aging is Coming: The Effect of Aging and Expectations on Capital Expenditures, Interest Rates, and Wages

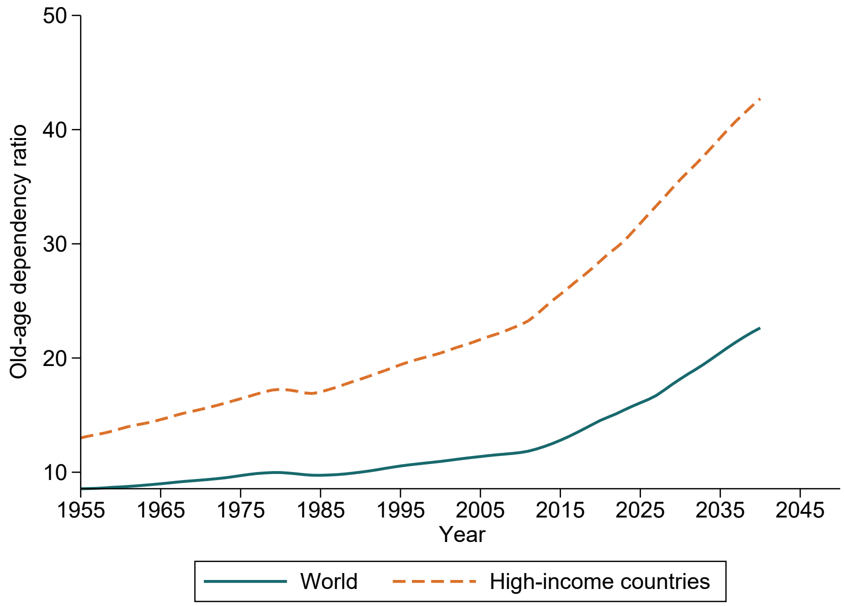
1.1 Introduction

Today, population aging and demographic change are already affecting advanced economies such as Japan and selected European countries like Germany, and projections predict that they will become even more prevalent in the foreseeable future (United Nations (2019)). This is mainly due to the fact that birth rates in many advanced countries have fallen below replacement levels, and additional migration has only partly been able to offset the resulting decrease in the labor force growth rate.[1]

Figure 1.1 displays the evolution of the old-age dependency ratio for high-income countries as well as the world. The data shows a clear upward trend that is expected to continue into

[1] United Nations (2022).

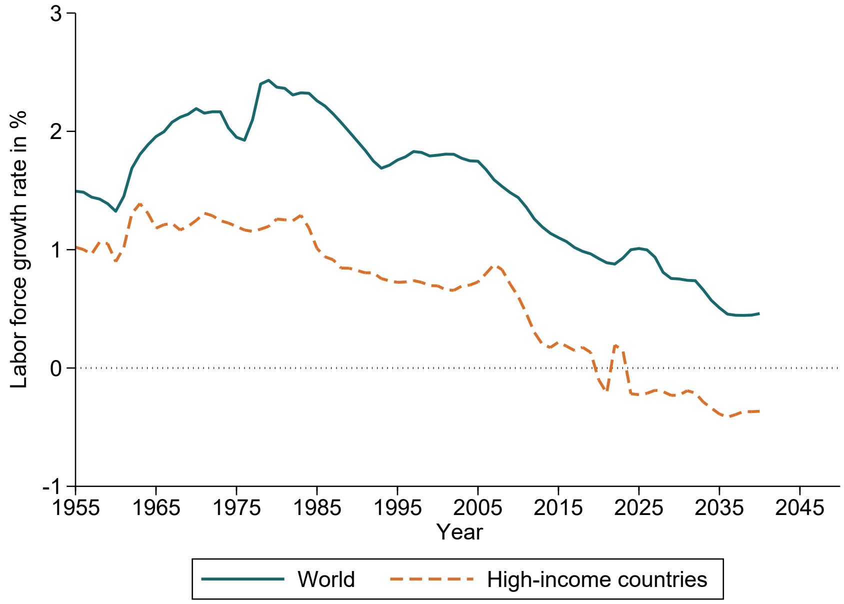
the foreseeable future.



This figure provides a graphical illustration of the trend in the old-age dependency ratio for the world and high-income countries. The old-age dependency ratio is defined here as $\frac{65+}{15-64} \cdot 100$. The data is taken from the UN, and the classification is based on the World Bank.

Figure 1.1: *Old-Age Dependency Ratio*

Figure 1.2 plots the growth rate of the (potential) workforce for high-income countries and the world in %. The data depicts a decreasing pattern. The growth rate has, on average, remained stable until the beginning of the 21st century. However, starting around 2010, the growth rate of the workforce has started to decrease. Moreover, this process is expected to continue and will imply that in high-income countries, the workforce might even start to decrease. Both figures illustrate that population aging and demographic change—I will use the two terms synonymously—are already affecting high-income countries and are expected to continue to do so in the future. This implies that the share of retired individuals in the population is growing, and at the same time, the size of the workforce stagnates or potentially even decreases. Furthermore, a similar trend, albeit less strong, can also be observed for the world as a whole.



This figure provides a graphical illustration of the trend in the labor force growth rate for the world and high-income countries. The labor force consists of people aged 15-64. The data is taken from the UN, and the classification is based on the World Bank.

Figure 1.2: *Labor Force Growth Rate*

Population aging will most likely affect societies and their economies in multiple dimensions. In this paper, I analyze how population aging can affect different macroeconomic aggregates. Namely, the capital stock per worker, interest rates, and wages, with the aim of gaining a better understanding of how demographic change could influence the investment behavior of firms as well as interest rates and the wages of workers over time.

The main finding of this paper is that demographic change decreases the investment demand of firms. Whether the capital stock per worker increases or decreases depends on the costs firms face when acquiring capital as well as on the (downward) flexibility of the interest rate. As the wage is positively linked to the capital stock per worker, this also entails that wages will *not necessarily* increase in response to population aging. The steady state wage rate will only increase if there is a *permanent* negative shock to the population growth rate. A *temporary* negative shock to the population growth rate will lead to a temporary increase in the wage rate. However, over time, the wage rate will fall back to its original (steady state) level. In contrast, if capital exhibits non-linear (level) costs, a *temporary* negative shock to the population growth rate will also influence the steady state interest rate.

I start by considering a partial equilibrium model in which capital supply and the interest rate are exogenous. I subsequently endogenize capital supply to study the joint effect population aging has on capital demand and on the interest rate.

The comparative statics for the partial equilibrium model imply that population aging leads to a lower overall capital stock, but to a larger capital stock per worker if firms face convex (level) costs when acquiring capital. This in turn leads to a higher equilibrium wage rate and hence implies a positive relationship between growth in the capital stock per worker and growth in the wage rate.

The model demonstrates how firms' expectations can influence the future wage rate through the investment decisions they make. Optimistic, and even too optimistic, expectations lead to a higher wage rate, and pessimistic expectations lead to a lower wage rate. In addition, the elastic labor supply implies that the negative effects of incorrect expectations are mitigated to some degree by allowing households to adjust their labor supply in response to the previous investment decisions of firms.

With endogenous capital supply and convex capital (level) costs and/or linear costs, a permanent decrease in the hours worked due to population aging will lead to a higher capital stock per worker, which entails a higher wage rate and a lower interest rate. The intuition is that population aging in an OLG model always reduces labor supply first, as fewer people are born, and only in the next period does it lead to a reduction in capital supply because fewer people will save for their retirement. As a lower number of workers implies reduced capital demand, this lowers the interest rate.[2] The effect on the equilibrium capital stock can either be zero or negative, depending on the value of the elasticity of intertemporal substitution of households.

A temporary shock to labor supply will leave the steady state capital stock per worker and

[2]Assuming capital supply is less than perfectly elastic in response to a change in the interest rate.

thus the wage rate unaffected, as labor only exhibits linear costs. Depending on whether capital features convex or concave (level) costs, a temporary shock to labor supply will lead to a higher or lower steady state interest rate.

In addition, I show that the results also hold for a more general CES production function, as long as I exclude the case of perfect substitutes.

With non-convex costs, e.g., fixed costs, the effect of population aging is in general ambiguous and depends on parameters. However, this also implies that population aging *can* lead to a lower capital stock per worker and hence lower wages.

Related Literature

This paper relates to several strands of the literature that study the consequences of demographic change and population aging. In the recent literature, there has been some debate on whether population aging will affect output per capita positively or negatively. Acemoglu and Restrepo (2017) highlight the potential of automation, which could be encouraged even further through population aging, as this can lead to higher wages, which in turn can make the employment of robots more attractive to firms. Other authors, however, have pointed to the possible negative ramifications of population aging, as, for example, Eggertsson *et al.* (2019a)-henceforth ELS.

Moreover, interest rates have been decreasing since the mid-1980s.[3] However, at the same time, investment has been lackluster over the past decade,[4]. This is surprising given that interest rates were low to begin with and decreased further. Here I will employ simple theoretical models to explore how population aging could help rationalize firms' reluctance to invest.

In addition, there is an ongoing debate about whether population aging will lead to permanently lower interest rates or whether the dissaving of the old will eventually reduce capital supply and lead to higher interest rates. This hypothesis has been reintroduced by Goodhart and Pradhan (2020) and has previously been known as the "asset meltdown

[3] See, for example, Eggertsson *et al.* (2019a) and Farhi and Gourio (2018).

[4] See, for example, Alexander and Eberly (2018) and Gutiérrez and Philippon (2016).

hypothesis" (Poterba (2001), Abel (2001)). Contrary to the authors mentioned before, Auclert *et al.* (2021) argue that the dissaving of the old will not be enough to offset the fall in interest rates. I will use the model outlined here to argue that in a standard OLG model with a Cobb-Douglas or CES production function and linear costs, the effect population aging has on the interest rate will be determined by the equilibrium capital-to-labor ratio, i.e., the higher the capital-to-labor ratio, the lower the interest rate.

The paper most closely related to mine is that by ELS. However, my paper differs from ELS in a number of ways. In my paper, I can explicitly solve for the capital demand of firms and also look at the effects of technological progress. Moreover, I further include elastic labor supply and analyze how population aging affects labor supply as well as the wage rate, both directly and indirectly through the capital demand of firms. In addition, I also study how expectations of firms influence their investment behavior and how this in turn affects the wage rate and labor supply.

For additional papers that study the effect of population aging on macroeconomic outcomes. See, for example, Krueger and Ludwig (2007) for an early contribution. For more recent studies. See, for example, Carvalho *et al.* (2016), Kotschy and Sunde (2018), Cooley and Henriksen (2018), Aksoy *et al.* (2019), Auclert *et al.* (2021), Gagnon *et al.* (2021), Papetti (2021), Acemoglu and Restrepo (2022), and Maestas *et al.* (2023).

Moreover, the article is also related to a broader literature on the secular stagnation hypothesis going back to Hansen (1939). See Eggertsson *et al.* (2019b) for a recent formal and quantitative analysis of the secular stagnation hypothesis, as well as Summers and Rachel (2019) and Michau (2018). Farhi and Gourio (2018) and Caballero *et al.* (2017) discuss additional factors that could potentially explain the phenomena associated with the secular stagnation hypothesis.

My work is also related to the literature that looks at the effect expectations can have on macroeconomic outcomes. For early contributions to this literature, see Kiyotaki (1988) and Benhabib and Farmer (1994). For more recent contributions, see, for example, Farmer (2012) and Benigno and Fornaro (2018).

The paper proceeds as follows. Section 1.2 presents some motivating correlations. Section 1.3 introduces the main model, and Section 1.6 discusses the effect of expectations on the equilibrium wage rate in more detail. Section 1.7 examines how investment is affected by population aging if firms face fixed capital acquisition costs. In Section 1.8, capital supply is endogenized to analyze general equilibrium effects. Section 1.9 extends the model structure to incorporate a more general CES production function, and Section 1.10 concludes.

1.2 Motivation

To investigate the relation between demographic change and the capital stock per worker, I estimate the following empirical model[5]

$$\ln(k_{i,t}) = \alpha n_{i,t-1} + \gamma \ln(y_{i,t-1}) + \delta h_{i,t-1} + \zeta_i + \eta_t + \varepsilon_{i,t}, \tag{1.1}$$

$k_{i,t}$ is the capital stock per worker in country i in year t at current PPPs. $n_{i,t-1}$ is the population growth from period $t-2$ to period $t-1$ in country i, i.e., a measure for demographic change, and thus α captures the effect of demographic change on the capital stock per worker. I control for real income per capita ($y_{i,t-1}$) and for human capital ($h_{i,t-1}$). To address the problem of reverse causality, I use lagged values of the independent variables. Moreover, to account for time-invariant cross-country heterogeneity and global trends, I include a full set of country (ζ_i) and time (η_t) fixed effects. The error term $\varepsilon_{i,t}$ captures all additional omitted variables. As GDP per capita and the capital stock per worker are highly correlated, I only include GDP per capita as an independent variable, but not the capital stock per worker from the previous period.

The empirical analysis uses 5-year and 10-year panel data from 1969-2019. The capital stock, the number of people engaged, the population size, the real income per capita, and the human capital index are all taken from the Penn World Table version 10.0 (Feenstra *et al.*

[5]In the theoretical part, I assume that capital fully depreciates after one period, and thus investments in the current period are equal to the capital stock of the next period. Moreover, wages and interest rates depend on the capital stock per worker. Therefore, I use the capital stock per worker and not capital formation as the dependent variable.

(2015)). The panels are created by taking 5- and 10-year averages to reduce the problem of measurement errors and business cycle effects.

Table 1.1 presents the results for estimating equation (1.1).[6] Column (1) reports the results for the full sample, column (2) for the OECD countries, and column (3) for the non-OECD countries for a 5-year panel. Columns (4) to (6) show the results for a 10-year panel. The effect of a decrease in the population growth rate on the capital stock per worker is positive, except for column (5), and statistically significant in the 5-year panel of the full sample. Therefore, as the population growth rate falls due to demographic change, this is correlated with a higher capital stock per worker.

[6]The following countries are used in the empirical analysis: DZA, ARG, AUS, AUT, BGD, BRB, BEL, BOL, BWA, BRA, BFA, CMR, CAN, CHL, CHN, HKG, COL, COG, CRI, CYP, CIV, COD, DNK, DOM, ECU, EGY, ETH, FIN, FRA, GAB, DEU, GHA, GRC, GTM, HTI, ISL, IND, IDN, IRN, IRL, ISR, ITA, JAM, JPN, JOR, KEN, LUX, MDG, MWI, MYS, MLI, MLT, MUS, MEX, MAR, MOZ, MMR, NAM, NLD, NZL, NER, NGA, NOR, PAK, PAN, PRY, PER, PHL, PRT, KOR, ROU, RWA, SEN, SGP, ZAF, ESP, LKA, SWE, CHE, SYR, THA, TTO, TUN, TUR, TZA, UGA, GBR, USA, URY, VEN, ZMB, ZWE.

Table 1.1: *Demographic Change and Capital Stock per Worker*

Dependent variable	Log Capital stock per worker					
	(1)	(2)	(3)	(4)	(5)	(6)
	Full sample	OECD	Non-OECD	Full sample	OECD	Non-OECD
Population growth rate	-7.803***	-4.411	-5.923*	-7.538	1.205	-5.350
	(2.875)	(5.410)	(3.250)	(4.866)	(6.514)	(5.560)
Log Income p.c.	0.723***	0.837***	0.795***	0.559***	0.809***	0.628***
	(0.0827)	(0.110)	(0.0915)	(0.0988)	(0.120)	(0.111)
Human capital	-0.225	0.0832	-0.435**	-0.0587	0.0611	-0.228
	(0.195)	(0.220)	(0.212)	(0.265)	(0.280)	(0.291)
Observations	920	270	650	460	135	325
R-squared	0.763	0.906	0.763	0.729	0.906	0.726
Number of countries	92	27	65	92	27	65
Country FE	YES	YES	YES	YES	YES	YES
Time FE	YES	YES	YES	YES	YES	YES
Panel	5-years	5-years	5-years	10-years	10-years	10-years

Notes: Results of fixed effects regressions. The dependent variable is the log capital stock at current PPPs (in mil. 2017USD) divided by the number of persons engaged (in millions). All regressions include country–specific fixed and time effects. Standard errors are clustered at the country level. *, **, *** denote significance at 10%, 5%, and 1% levels.

To examine the relationship between population aging and the return to capital, I estimate the following model

$$R_{i,t} = \alpha n_{i,t-1} + \gamma \ln(y_{i,t-1}) + \delta h_{i,t-1} + \zeta_i + \eta_t + \varepsilon_{i,t}, \tag{1.2}$$

$R_{i,t}$ denotes the real internal rate of return in country i in period t, which I use as a proxy for the return to capital, $n_{i,t-1}$ is the population growth rate from period $t-2$ to period $t-1$ in country i, $\ln(y_{i,t-1})$ denotes log real income per capita, $h_{i,t-1}$ is the human capital index, ζ_i, and η_t are the country and time fixed effects. To address the potential problem of reverse causality, I used lagged values of the independent variables.

Table 1.2 presents the results for estimating equation (1.2).[7] Columns (1) to (3) depict the results for the 5-year panel for the full sample as well as when I split the sample into OECD and non-OECD countries. Columns (4) to (6) show the results for a 10-year panel.

A higher population growth rate has a positive effect on the internal rate of return; however, the effect is only statistically significant for the full sample and the non-OECD sample for the 5-year panel.

Table 1.2: *Demographic Change and the Internal Rate of Return*

Dependent variable	Internal rate of return					
	(1)	(2)	(3)	(4)	(5)	(6)
	Full sample	OECD	Non-OECD	Full sample	OECD	Non-OECD
Population growth rate	5.409*	3.574	7.552**	3.586	7.830	5.298
	(3.015)	(6.362)	(3.359)	(4.508)	(7.162)	(5.196)
Log Income p.c.	0.0392	0.0245	0.0265	-0.101	-0.0230	-0.120
	(0.0686)	(0.0849)	(0.0782)	(0.0833)	(0.0979)	(0.0931)
Human capital	-0.0245	-1.135**	0.200	0.154	-0.929*	0.411
	(0.268)	(0.494)	(0.368)	(0.295)	(0.526)	(0.398)
Observations	780	270	510	390	135	255
R-squared	0.127	0.260	0.163	0.128	0.229	0.171
Number of countries	78	27	51	78	27	51
Country FE	YES	YES	YES	YES	YES	YES
Time FE	YES	YES	YES	YES	YES	YES
Panel	5-years	5-years	5-years	10-years	10-years	10-years

Notes: Results of fixed effects regressions. The dependent variable is the real internal rate of return. All regressions include country–specific fixed and time effects. Standard errors are clustered at the country level. *, **, *** denote significance at 10%, 5%, and 1% levels.

In the following, I will analyze different models that differ in the form of capital costs to explore under which assumptions demographic change, i.e., a decrease in the population

[7] The following countries are used in the empirical analysis: ARG, AUS, AUT, BRB, BEL, BOL, BWA, BRA, BFA, CMR, CAN, CHL, CHN, HKG, COL, CRI, CYP, CIV, DNK, DOM, ECU, EGY, FIN, FRA, GAB, DEU, GRC, GTM, ISL, IND, IDN, IRN, IRL, ISR, ITA, JAM, JPN, JOR, KEN, LUX, MYS, MLT, MUS, MEX, MAR, MOZ, NAM, NLD, NZL, NER, NGA, NOR, PAN, PRY, PER, PHL, PRT, KOR, ROU, RWA, SEN, SGP, ZAF, ESP, LKA, SWE, CHE, THA, TTO, TUN, TUR, TZA, GBR, USA, URY, VEN, ZMB, ZWE.

growth rate, leads to a higher capital stock per worker and to lower interest rates.

1.3 Model

To investigate how population aging affects capital demand and wages, I consider the following environment.[8] Time is discrete, and I consider first a small open economy, which takes the world interest rate as given. The firms in this economy have access to the international capital market, where they can buy capital at the world interest rate, which is taken as exogenous. I assume free entry, which will imply that in equilibrium, firms make zero profits. There is only one final good that can be used for investment or consumption. To illustrate the main mechanisms, I assume that the economy only runs for two periods; however, the extension to multiple periods would be straightforward under the assumption that capital fully depreciates after being used for production purposes.[9] In the first period, firms decide how much capital to acquire, for which they pay the world interest rate r_t. The gross interest rate is denoted by $R_t = 1 + r_t$. In addition, firms face non-linear (level) costs when acquiring capital, similar to the investment literature; see, for example, Summers *et al.* (1981) or Cooper and Haltiwanger (2006). In the second period, the firms combine the capital they build up with labor, which is elastically supplied by households, to produce the final good. Production takes place using a Cobb-Douglas production function with constant returns to scale.[10] In addition, I assume that capital expenditures or investments—I use the two terms synonymously—constitute a sunk cost, i.e., the firms cannot sell the capital in the second period that they installed in the first period.

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[8]Throughout the rest of the paper, whenever I refer to population aging, it implies that the old-age dependency ratio increases due to a decrease in the number of people of working-age. The model used in this paper features a two-period OLG structure, which implies that life expectancy cannot increase. Therefore, I remain mostly agnostic about the effects of an increase in longevity on capital demand and wages.

[9]This implies that investments in the current period are equivalent to the capital stock of the next period by construction.

[10]Without non-linear capital costs, the solution to the optimization problem of the firm is not well-defined, as I assume constant returns to scale. The non-linear costs ensure that a well-defined solution exists, which allows to solve for the overall capital demand in the economy.

Before continuing, I will briefly discuss the importance of the timing assumption. The assumption that capital and labor are chosen at different points in time is solely necessary for expectations to have an effect (see Section 1.6). The more important assumption, which is standard in the discrete-time OLG models, is that capital needs to be built up first, i.e., the transformation of labor into capital through the process of saving and investing requires time, in this set-up, one period. Therefore, whether capital demand and savings are matched at the end of the first period or at the beginning of the second period does not alter the results here.[11]

1.3.1 Firms

Firms have perfect foresight. The optimization problem of the representative firm ranges over two periods.[12]

In period t the firms acquire the capital with which they will produce in period $t+1$.[13] In period $t+1$ firms combine their capital stock with labor chosen in the current period to produce the final good.

The overall profits of firms are given as follows

$$\pi_{t+1} = -R_t K_t - C(K_t) + \beta \left(K_t^{\alpha} (A_{t+1} L_{t+1})^{1-\alpha} - w_{t+1} L_{t+1} \right), \tag{1.3}$$

$$\text{where} \quad C(0) = 0, \quad \frac{\partial C(K_t)}{\partial K_t} > 0, \quad \frac{\partial^2 C(K_t)}{\partial K_t^2} \neq 0.$$

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[11]The main reason being that the assumption of perfect competition implies that firms have no market power and thus output produced by the firms solely depends on the prices of the input factors, i.e., capital and labor. Results can differ if one assumes firms have market power, because in that case their production decisions can depend on the demand they face, and lower investments can be seen as a reduction in demand. In this case, the timing assumption could lead to different results.

[12]Alternatively, I could also assume that firms exist for multiple periods but that capital depreciates fully after one period.

[13]Throughout, the time subscript will refer to the period in which the decision regarding the variable is made, e.g., capital is chosen in period t and hence the subscript is t.

R_t is the world gross interest rate, and $C(K_t)$ denotes the non-linear (level) costs of acquiring capital, for example, credit market frictions.[14] I assume the degree of homogeneity of $C(K_t)$ is larger than 1, i.e., the non-linear (level) costs are convex, β is the discount factor of the firms, and w_{t+1} is the wage rate in the second period. A_{t+1} is a labor enhancing technology parameter.[15]

The optimization problem of the firms in the first period, i.e., period t, looks as follows

$$\max_{K_t} -R_t K_t - C(K_t) + \beta \left(K_t^{\alpha} (A_{t+1} L_{t+1})^{1-\alpha} - w_{t+1} L_{t+1} \right), \tag{1.4}$$

and the optimization problem of the firms in the second period, i.e., period $t+1$ is given as follows

$$\max_{L_{t+1}} K_t^{\alpha} (A_{t+1} L_{t+1})^{1-\alpha} - w_{t+1} L_{t+1}. \tag{1.5}$$

The model is solved by backward induction. Thus, I first solve for the optimal labor demand taking the capital stock as given, and in the second step, I solve for the optimal capital demand[16] in the first period.

The optimal labor demand in the second period is given as

$$L_{t+1} = \left(\frac{(1-\alpha) A_{t+1}^{1-\alpha} K_t^{\alpha}}{w_{t+1}} \right)^{\frac{1}{\alpha}}. \tag{1.6}$$

Assuming free entry into the market, additional firms will enter the market until profits are zero. As I assume firms also face non-linear (level) costs when acquiring capital, the cost curve of the firms with regard to capital is U-shaped. Hence, new firms enter the market until profits are zero, which is the case when the marginal product of capital is equal to the

[14]As capital fully depreciates after one period, $C(K_t)$ can equivalently be seen as *level* or as *adjustment* costs for capital.

[15]Assuming $C(K_t)$ is concave would *not necessarily* change the results, however, the result would no longer be unambiguous. See Section A.1.3 in the Appendix.

[16]The terms capital demand and investments will be used interchangeably in this section as I either assume the economy runs only for two periods or that capital depreciates fully after one period.

average cost of capital.[17]

$$\underbrace{\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)}_{\text{Marginal Product}}=\underbrace{\frac{C(K_t)+R_tK_t}{K_t}}_{\text{Average Costs}}. \tag{1.7}$$

$$F\equiv\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)-R_t-\frac{C(K_t)}{K_t}=0. \tag{1.8}$$

Equation (1.8) determines the optimal capital demand of the firms. Given the assumption that $C(K_t)$ is convex and because the capital stock cannot be negative, it follows that $\frac{\partial F}{\partial K_t}<0$.[18]

I can insert equations (1.6) and (1.8) into equation (1.3) to verify that firms indeed make zero profits

$$\begin{aligned}\pi_{t+1}=&\beta\left(K_t^{\alpha}(A_{t+1}L_{t+1})^{1-\alpha}-(1-\alpha)K_t^{\alpha}(A_{t+1}L_{t+1})^{1-\alpha}\right)\\&-\beta\left(\alpha K_t^{\alpha}(A_{t+1}L_{t+1})^{1-\alpha}\right)=0.\end{aligned} \tag{1.9}$$

1.3.2 Households

Labor is elastically supplied in period $t+1$ by $N_{t+1}=(1+n_{t+1})N_t$ households, where n_{t+1} is the population growth rate. Households are endowed with one unit of time, which they allocate between leisure and work. Hence, households face an opportunity cost when supplying labor. The households face the following utility maximization problem

$$\begin{aligned}&\max_{c_{t+1},\ell_{t+1}}\frac{1}{1-\theta}c_{t+1}^{1-\theta}+\frac{1}{1-\theta}(1-\ell_{t+1})^{1-\theta}\\&\text{s.t. } c_{t+1}=w_{t+1}\ell_{t+1},\end{aligned} \tag{1.10}$$

where $\ell_{t+1}\in[0,1)$ denotes the time a household supplies as labor, and correspondingly, $1-\ell_{t+1}$ denotes the leisure time of the household. $\frac{1}{\theta}$ is the elasticity of substitution between

[17] See, for example, Hellwig and Irmen (2001). Labor only exhibits linear costs, and thus, in equilibrium under free entry, the marginal costs of labor are equal to the marginal product of labor.

[18] As $C(K_t)-C'(K_t)K_t<0$, this is due to Euler's homogeneous function theorem. It implies that $C'(K_t)K_t=\lambda C(K_t)$ where λ denotes the degree of homogeneity of $C(K_t)$, which I assumed to be larger than 1.

consumption c_{t+1} and leisure $(1-\ell_{t+1})$, where $\frac{1}{\theta} \in (1, \infty)$, i.e., consumption and leisure are substitutes. The households maximize equation (1.10) subject to the constraint that yields the optimal labor supply

$$\ell_{t+1} = \frac{1}{1 + w_{t+1}^{\frac{\theta-1}{\theta}}}. \tag{1.11}$$

Assuming that the labor market clears, I can equalize labor supply and labor demand, i.e., $L_{t+1} = \ell_{t+1}(1+n_{t+1})N_t$, and solve for the market clearing equilibrium wage rate w_{t+1}. The equilibrium wage rate is implicitly determined by the following function

$$G \equiv \left(w_{t+1}^{-\frac{1}{\alpha}} + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}} \right) \left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}} K_t - \ell_{t+1}(w_{t+1})(1+n_{t+1})N_t = 0. \tag{1.12}$$

1.4 Equilibrium

In equilibrium, the labor market clears

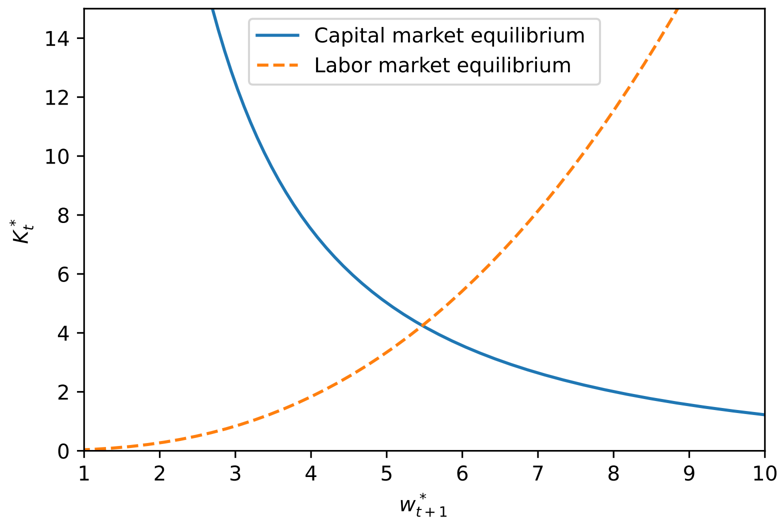
$$L_{t+1} = \ell_{t+1}(1+n_{t+1})N_t = \ell_{t+1}N_{t+1}. \tag{1.13}$$

The equilibrium can then be characterized by a system of two equations and two endogenous variables. Namely, the optimal capital demand of the firms K_t and the equilibrium wage rate w_{t+1}, which are given by the solution to the following system of equations

$$F \equiv \beta \left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right) - R_t - \frac{C(K_t)}{K_t} = 0, \tag{1.14}$$

$$G \equiv \left(w_{t+1}^{-\frac{1}{\alpha}} + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}} \right) \left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}} K_t - \ell_{t+1}(w_{t+1})(1+n_{t+1})N_t = 0. \tag{1.15}$$

Figure 1.3 illustrates graphically the system of two equations that determine the equilibrium capital demand K_t and the equilibrium wage rate w_{t+1}. The equilibrium values are given by the intersection of the two curves. As before, we have a downward sloping capital demand curve. The labor market equilibrium curve is the inverse function of the equilibrium wage rate, i.e., it is upward sloping because a larger capital stock leads to a higher equilibrium wage rate.[19]



The figure is created using the following parameter values: $N_{t+1} = 20$, $\alpha = 0.4$, $\theta = 0.5$, $A_{t+1} = 100$, $R_t = 1.05$, $\beta = 0.95$, and $C(K_t) = K_t^2$.

Figure 1.3: *Equilibrium*

1.5 Comparative Statics

To determine the effect of a change in the exogenous variables on the endogenous variables, I totally differentiate the system of equations given by (1.14) and (1.15). This yields the following comparative statics.[20]

$$\frac{\partial K_t}{\partial R_t} < 0, \quad \frac{\partial K_t}{\partial A_{t+1}} > 0, \quad \frac{\partial K_t}{\partial n_{t+1}} > 0,$$
$$\frac{\partial w_{t+1}}{\partial R_t} < 0, \quad \frac{\partial w_{t+1}}{\partial A_{t+1}} > 0, \quad \frac{\partial w_{t+1}}{\partial n_{t+1}} < 0.$$

Proof. See Appendix A.1.2. □

[19]For a formal discussion of uniqueness and existence, see Section A.1.1 in the Appendix.

[20]This rests on the assumption that either capital or labor are now chosen in the same period or that the change in population size is known by the firms one period ahead and is thus incorporated into their capital demand decisions, i.e., firms have perfect foresight.

Therefore, an increase in the exogenous interest rate lowers the demand for capital, leading to a lower capital stock, which translates into a lower equilibrium wage rate. A rise in the level of technology A_{t+1} has a positive direct and a negative indirect effect on capital demand K_t. Through the positive direct effect, it increases the marginal product of capital by raising the level of productivity, i.e., better technologies allow for the production of more of the final good with the same input factors. Through the negative indirect effect, the increase in A_{t+1} also leads to a rise in the equilibrium wage rate by increasing the marginal product of labor. However, the positive direct effect always dominates, such that technological progress always leads to additional demand for capital and higher wages.

A growing workforce leads to an increase in the demand for capital, as firms need to equip the additional workers with capital. Even though a rise in n_{t+1} leads to a larger capital stock, the equilibrium wage rate nonetheless decreases. The reason is that firms face both linear and convex (level) costs when acquiring capital. Hence, when the workforce grows, firms will install more capital, but because of the convex (level) costs, firms will acquire less capital than would be necessary to keep the capital stock per worker constant, and therefore the capital stock per worker decreases. This implies a lower marginal product of labor, which translates into a lower equilibrium wage rate. Conversely, this also means that when the workforce shrinks due to, for example, population aging, the convex (level) costs entail that the capital stock per worker increases and hence also the equilibrium wage rate.[21] Hence, without convex (level) costs, the assumption of an infinitely elastic capital supply implies that the firms will increase or reduce the total capital stock such that they keep capital per worker and thus the marginal product of labor constant.[22] Therefore, the model with convex (level) costs is able to replicate the empirical finding that population aging is correlated with a higher capital stock per worker.

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[21]See Section A.1.2 in the Appendix.

[22]With only linear costs, combining the first-order conditions yields $w_{t+1} = (1-\alpha)\left(\frac{\alpha}{R_t}\right)^{\frac{\alpha}{1-\alpha}}$, as R_t is exogenous w_{t+1} is independent of n_{t+1}. The capital stock per worker is given as $k_t = \left(\frac{\alpha}{R_t}\right)^{\frac{1}{1-\alpha}} A_{t+1}$ and thus only depends on R_t and is independent of n_{t+1}.

Furthermore, using equations (1.11) and (1.12); I can determine the effects of a change in the different exogenous variables on the optimal labor supply ℓ_{t+1} of households. This leads to the following comparative statics

$$\frac{d\ell_{t+1}}{dw_{t+1}} > 0, \quad \frac{d\ell_{t+1}}{dK_t} > 0, \quad \frac{d\ell_{t+1}}{dA_{t+1}} > 0, \quad \frac{d\ell_{t+1}}{dn_{t+1}} < 0.$$

Proof. The results follow from applying the implicit function theorem to equation (1.11) and (1.12). □

The results show that a rise in the built-up capital stock increases labor supply because capital positively affects the marginal product of labor and thus the wage rate, which in turn makes it more attractive for households to supply labor. The same logic applies to an increase in the level of technology A_{t+1}. An increase in n_{t+1}, i.e., population growth, leads to a lower wage rate and thus makes it less attractive for households to supply labor. Conversely, a decrease in the workforce, due, for example, to population aging, leads households to supply more labor.

To sum up, the model predicts that a decrease in labor supply leads to a reduction of the overall capital stock in the economy because it lowers firms' demand for capital. At the same time, the capital stock per worker increases even when the interest rate remains constant due to the assumption of convex capital acquisition costs.

1.6 Effect of Expectations

So far, I have assumed that firms possess perfect foresight. In the following, I assume that firms face risk when making their investment decision, i.e., they do not know with certainty how high the labor supply will be in the next period. For simplicity, I assume that there are only two possible outcomes for the future, i.e., labor supply can be either high or low, and firms know the probability of either state.

Therefore, I can distinguish two cases: one where the firms were too optimistic regarding labor supply in the next period and one where they were too pessimistic.

More formally, firms face risk regarding overall labor supply in the next period and hence set their capital demand according to equation (1.8) except that without perfect foresight, the wage rate is cast in expectations

$$\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{\mathbb{E}_t[w_{t+1}]}\right)^{\frac{1-\alpha}{\alpha}}\right)=\frac{C(K_t)+R_tK_t}{K_t}. \tag{1.16}$$

The expected wage rate is determined by the following equation, which is based on equation (1.12)

$$\left(\mathbb{E}_t\left[w_{t+1}^{-\frac{1}{\alpha}}\right]+\mathbb{E}_t\left[w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}\right]\right)\left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}}K_t-\mathbb{E}_t[L_{t+1}]=0. \tag{1.17}$$

Underestimating labor supply implies $\mathbb{E}_t[L_{t+1}] < L_{t+1}$ which in turn means the expected wage rate is larger than the one realized in period $t+1$, i.e., $\mathbb{E}_t[w_{t+1}] > w_{t+1}$. Given the capital stock from the first period, the firm will hire labor in the second period until the marginal product of labor is equal to the marginal cost of labor. However, as $\mathbb{E}_t[w_{t+1}] > w_{t+1}$, and because additional capital cannot be installed in period $t+1$, this implies that

$$\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)>\frac{C(K_t)+R_tK_t}{K_t}, \tag{1.18}$$

and thus, the firms will make positive profits because they installed less capital than would have been (socially) optimal.

In case the firms were too optimistic regarding the next period's labor supply, i.e., $\mathbb{E}_t[L_{t+1}] > L_{t+1}$, and thus $\mathbb{E}_t[w_{t+1}] < w_{t+1}$. Hence, firms will have installed too much capital, and hence

$$\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)<\frac{C(K_t)+R_tK_t}{K_t}. \tag{1.19}$$

Therefore, there are now too few workers to utilize the entire capital stock, and thus the costs of capital are higher than the rewards. As the firms have already installed the capital and it is not possible for them to sell it, they will make losses in that period. Furthermore, firms will still hire all available workers because the marginal product of capital increases with the number of workers. A reduction in labor demand would increase the marginal product of labor and thus lead to a rise in the wage rate, hence offsetting the savings made by reducing the number of employees. Furthermore, it would also lower the marginal product of capital. Consequently, in this framework, it is never optimal for the firms to reduce their labor demand if they have been too optimistic and installed too much capital. Moreover, I also analyze the effect of uncertainty regarding the level of productivity in the next period, i.e., A_{t+1}. This implies that capital demand is now given as follows

$$\beta\left(\alpha\mathbb{E}_t\left[A_{t+1}^{\frac{1-\alpha}{\alpha}}\right]\left(\frac{1-\alpha}{\mathbb{E}_t[w_{t+1}]}\right)^{\frac{1-\alpha}{\alpha}}\right)=\frac{C(K_t)+R_tK_t}{K_t}, \tag{1.20}$$

with the expected wage rate given by the following equation

$$\left(\mathbb{E}_t\left[w_{t+1}^{-\frac{1}{\alpha}}\right]+\mathbb{E}_t\left[w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}\right]\right)\left((1-\alpha)\mathbb{E}_t\left[A_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}K_t-L_{t+1}=0. \tag{1.21}$$

A higher expected level of technology has two effects on capital demand. On the one hand, it increases the marginal product of capital, which in turn increases the demand for capital. On the other hand, it increases the equilibrium wage rate, which lowers the demand for capital. However, in the previous section I have shown that the first effect always dominates, and hence, if firms expect technological progress to be stronger, i.e., a higher $\mathbb{E}_t[A_{t+1}]$, this implies a higher capital demand in period t.

Therefore, if firms expect a lower level of technology in the next period, they install less capital than would be (socially) optimal, i.e., $\mathbb{E}_t[A_{t+1}] < A_{t+1}$, and thus

$$\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)>\frac{C(K_t)+R_tK_t}{K_t}, \tag{1.22}$$

as a lower level of technology implies that capital will be less productive.

Conversely, too optimistic expectations lead to the installation of too much capital, i.e., $\mathbb{E}_t[A_{t+1}] > A_{t+1}$, and hence

$$\beta\left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right)<\frac{C(K_t)+R_tK_t}{K_t}. \tag{1.23}$$

Regarding the wage rate, these results imply that overly optimistic expectations lead to a higher wage rate due to the additional capital being installed. Too pessimistic expectations, conversely, mean that the wage rate will be lower because the firms will have installed too little capital.

To analyze how these results affect the profits of the firms, notice that as firms choose their labor demand once uncertainty has been resolved, their labor demand is always optimal given the current capital stock. This implies that $w_{t+1}L_{t+1} = (1-\alpha)K_t^{\alpha}(A_{t+1}L_{t+1})^{1-\alpha}$ always holds. Inserting this expression in the equation for profits yields

$$\pi_{t+1} = \beta\alpha K_t^{\alpha}(A_{t+1}L_{t+1})^{1-\alpha} - R_t K_t - C(K_t) \gtreqless 0. \tag{1.24}$$

From equations (1.17), (1.18) and (1.19) it follows that profits will be zero if expectations are correct, positive if firms are too pessimistic, and negative if firms have too optimistic expectations. Assuming perfect competition, an equilibrium would require that expected profits are equal to zero, i.e., $\mathbb{E}_t[\pi_{t+1}] = 0$, similar to a model that features risky R&D.

In the following, I will briefly sketch how this mechanism would work in a more structured version of the model. For simplicity, I assume that the number of households supplying labor in the next period can only take on two different values. In the high state, more households supply labor, and in the low state, fewer households supply labor; thus, $L_{h,t+1} > L_{l,t+1}$.

Given the economy is in state i it stays in state i with probability p_{ii} and switches to state j with probability p_{ij} with $(i, j = h, l; i \neq j)$, where h denotes the high state and l the low state. Therefore, I have the following transition matrix

$$P = \begin{pmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{pmatrix}.$$

Let $K_{h,t}^*$ denote the optimal capital demand in the high state under perfect foresight and $K_{l,t}^*$ the optimal capital demand in the low state under perfect foresight, i.e., they denote the respective capital stock firms would choose if they knew with certainty which of the two states would materialize in the next period. Given that the economy starts in state i, the expected wage rate is determined by the following equation

$$\left(\mathbb{E}_t\left[w_{t+1}^{-\frac{1}{\alpha}}\right]+\mathbb{E}_t\left[w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}\right]\right)\left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}}K_t \\ -\underbrace{\left(L_{i,t+1}p_{ii}+L_{j,t+1}p_{ij}\right)}_{\mathbb{E}_t[L_{t+1}]}=0. \tag{1.25}$$

As $p_{ii}\in(0,1)$ and $p_{ii}+p_{ij}=1$ it is clear that if the economy is initially in the high state $L_{l,t+1}<L_{h,t+1}p_{hh}+L_{l,t+1}p_{hl}<L_{h,t+1}$, i.e., the expected labor supply will be between the labor supply in the low and high state, respectively. Thus, $w_{l,t+1}<\mathbb{E}_t[w_{h,t+1}]<w_{h,t+1}$, where $\mathbb{E}_t[w_{h,t+1}]$ denotes the expected wage rate if the economy finds itself in the high state, and $w_{l,t+1}$ and $w_{h,t+1}$ respectively, denote the wage rate that would materialize if the firms knew with certainty in which state they would be in the next period.

With the assumption of perfect competition, this implies that firms choose capital in period t when the economy is initially in the high state, i.e., $K_{h,t}$, such that expected profits are zero. As firms expect an equilibrium wage rate that is between the perfect foresight wage rate of the low and the high state, I know that the capital stock chosen in period t, $K_{h,t}$, will be bounded by the two perfect foresight capital stocks, i.e., $K_{l,t}^*<K_{h,t}<K_{h,t}^*$, because the wage rate is the only stochastic variable that enters the capital demand function.

From equation (1.22) and (1.23) I know that a capital stock that is higher than the one of perfect foresight leads firms to make negative profits, and a capital stock that is lower than that of perfect foresight enables firms to generate positive profits.

More concretely, this means that if the economy is in the high state the amount of capital installed will enable firms to make positive profits in every period the economy remains in the high state because of $K_{l,t}^*<K_{h,t}<K_{h,t}^*$, i.e., in every period the economy remains in the high state the marginal product is higher than the average costs of capital. Once the economy switches to the low state the firms will make negative profits as they will have installed too much capital.

When the economy is in the low state we have $L_{l,t+1}<L_{l,t+1}p_{ll}+L_{h,t+1}p_{lh}<L_{h,t+1}$ and therefore $w_{l,t+1}<\mathbb{E}_t[w_{l,t+1}]<w_{h,t+1}$ again. This implies once more that the optimal amount

of capital $K_{l,t}$ is chosen such that $K^*_{l,t} < K_{l,t} < K^*_{h,t}$. As $L_{l,t+1}p_{ll} < L_{h,t+1}p_{hh} + L_{l,t+1}p_{hl}$ I also know that $\mathbb{E}_t[w_{l,t+1}] < \mathbb{E}_t[w_{h,t+1}]$ and hence that $K_{l,t} < K_{h,t}$. These results again imply that firms will make negative profits as long as the economy remains in the low state, because $K^*_{l,t} < K_{l,t}$, i.e., due to having installed too much capital, and firms will make positive profits as soon as the state changes, as $K_{l,t} < K^*_{h,t}$. However, perfect competition will imply that expected profits will be zero.

For the case where there is uncertainty regarding the level of productivity in the next period, assuming the economy is in state i, the expected equilibrium wage rate is determined by the following equation

$$\left(\mathbb{E}_t\left[w_{t+1}^{-\frac{1}{\alpha}}\right] + \mathbb{E}_t\left[w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}\right]\right)\left((1-\alpha)\left(A_{i,t+1}^{1-\alpha}p_{ii} + A_{j,t+1}^{1-\alpha}p_{ij}\right)\right)^{\frac{1}{\alpha}} K_t - L_{t+1} = 0. \tag{1.26}$$

Corresponding to before, I get $A_{h,t+1} < A_{h,t+1}p_{hh} + A_{l,t+1}p_{hl} < A_{h,t+1}$, i.e., the expected level of technology will be between the level of technology in the low and the high state. This further implies that $K^*_{l,t} < K_{h,t} < K^*_{h,t}$ holds again. Furthermore, as before, if the economy is in the high state firms will make positive profits in every period in which the economy remains in the high state. Once the economy switches to the low state, firms will make negative profits every period until the economy switches again to the high state. However, perfect competition will again imply that the expected profits will be zero.

To sum up, this short exposition illustrates how different expectations regarding future labor supply and technological progress can affect the wage rate of the future generation. The model suggests that positive and even too optimistic expectations always favor workers as they lead to the buildup of more capital, which in turn leads to a higher wage rate due to the complementary relationship of capital and labor as well as due to the assumption of constant returns to scale.

Conversely, low or pessimistic expectations will lead to a lower equilibrium wage rate. However, as mentioned before, the model does not feature multiple steady states, and thus the economy will converge back to the original balanced growth path as long as there are no systematic distortions in the formation of firms' expectations.

In the event that there are systematic distortions in the formation of expectations, this would lead to a higher or lower long-term capital stock, depending on whether the systematic distortions lead to more optimistic or pessimistic expectations. Furthermore, this would require that firms have some form of market power and thus can always make positive profits, as too optimistic expectations lead to lower profits, which in the case of perfect competition implies negative profits.

1.6.1 Numerical Illustration

In the following, I will solve a stylized version of the model over an infinite horizon to illustrate how the economy develops over time with expectations and uncertainty. To keep matters simple, I will only analyze the firm side, i.e., only solve the optimization problem of the firm. The representative firm solves the following profit maximization problem over an infinite horizon

$$\begin{aligned} \max_{\{I_t, L_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(K_t^{\alpha} \left(A_t L_t \right)^{1-\alpha} - I_t - w_t L_t - C(I_t) \right) \\ \text{s.t. } K_{t+1} = (1-\delta) K_t + I_t, \end{aligned} \tag{1.27}$$

where $\delta < 1$ denotes the depreciation rate and $C(I_t)$ are the convex installation cost, similarly to before. Regarding the installation costs, I assume the following functional form: $C(I_t) = \gamma I_t^2$, where $\gamma > 1$ is a parameter that determines the size of the installation costs.[23]

I can set up the Lagrangian and use the first-order conditions to solve for the steady state value of capital, assuming no technological progress, no change in the workforce, and no uncertainty, which is given as follows

$$K^* = \frac{\beta \left(\alpha \left(\frac{1-\alpha}{w^*} \right)^{\frac{1-\alpha}{\alpha}} A^{\frac{1-\alpha}{\alpha}} \right) - (1 - \beta(1-\delta))}{2\gamma\delta \left(1 - \beta(1-\delta)\right)}. \tag{1.28}$$

[23] This functional form is similar to the one that is standard in the literature (see, for example, Cooper and Haltiwanger (2006)), except that normally the installation costs are relative to the existing capital stock of the firm. However, due to the constant returns to scale production technology, I require the above form in order to be able to solve for the optimal capital stock.

w^* is the equilibrium wage rate that prevails once the economy has reached its steady state.

$$\frac{\partial K^*}{\partial \beta} > 0, \quad \frac{\partial K^*}{\partial A} > 0, \quad \frac{\partial K^*}{\partial \delta} < 0, \quad \frac{\partial K^*}{\partial \gamma} < 0, \quad \frac{\partial K^*}{\partial w^*} < 0.$$

To introduce uncertainty, I assume that, similarly to before, the labor supply becomes stochastic. To keep matters simple, I introduce the random variable z_t which can take on a value greater or lower than 1. When z_t is larger than 1, labor supply is low, and thus the equilibrium wage rate increases, and vice versa for values of z_t lower than 1. This implies that the firm now faces the following maximization problem

$$\begin{aligned} \max_{\{I_t, L_t, K_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left(K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha} - I_t - z_t w_t L_t - C(I_t) \right) \\ \text{s.t. } K_{t+1} = (1-\delta) K_t + I_t. \end{aligned} \tag{1.29}$$

I assume z_t to be continuous and to follow the subsequent AR(1) process:

$$\ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \sigma^2). \tag{1.30}$$

Therefore, the current value of the shock is determined by the size of the shock in the previous period as well as a stochastic component that follows a white noise process with mean zero and constant variance σ^2.

I set up the following value function where the continuation value is cast in expectations because, at time t the firm does not yet know the value of the shock that will materialize in the next period.

$$\begin{aligned} V(K_t, z_t) = \max_{L_t, K_{t+1}} \Big(K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha} - (K_{t+1} - (1-\delta) K_t) \\ - \gamma (K_{t+1} - (1-\delta) K_t)^2 \Big) + \beta \mathbb{E}_{z_{t+1}} V(K_{t+1}, z_{t+1}). \end{aligned} \tag{1.31}$$

I can approximate the value function around the steady state values of capital and labor. I assume the following values for the exogenous parameters: $\alpha = 0.4$, $\beta = 0.9$, $\delta = 0.1$, $A_t = 2$, $\gamma = 2.5$, $w_t = 1$, $\rho = 0.78$ and $\sigma = 0.067$.

Figure 1.4 displays the simulation of the economy over 200 periods. The first panel displays the evolution of the shock z_t, the second the progression of the capital stock of the firm, and the third the change in the current labor demand of the firm. The straight horizontal lines indicate the steady state values the variables would take on if there were no uncertainty. High values of z_t, which correspond to a high wage rate, imply that the capital stock is below its steady state value. Low values of z_t in turn imply that the capital stock is above its steady state value.

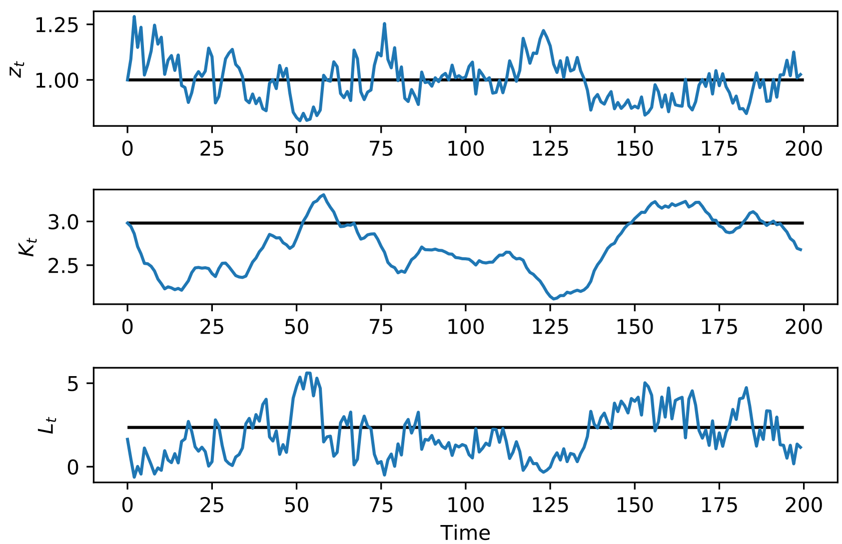

Figure 1.4: *Fluctuations Around the Steady State*

1.7 Investments with Fixed Costs

So far, I have only analyzed a model with linear and convex capital costs. In this section, I extend the analytic model to include a simple form of fixed capital cost, which corresponds to an extreme form of concave capital cost. Including fixed costs implies that models can better replicate actual investment behavior, as shown by Cooper and Haltiwanger (2006), i.e., changes in the capital stock are characterized by long periods of inactivity followed by large adjustments that take place in a short amount of time.

Assume a continuum of symmetric firms indexed on the unit interval. Firms have market power, and each produces a different variety by combining capital and labor using a Cobb-

Douglas production function. The output of all firms is then combined into a final good by a perfectly competitive final good sector using a CES aggreagtor.[24]

Assume firms rent the capital stock each period and have to pay R_t for each unit of capital, where R_t is exogenous, and in addition they are also forced to pay a fixed cost Ψ in every period in which they adjust the capital stock, i.e., when they rent more or less capital than in the previous period. Given the symmetry assumptions made above, aggregate profits in period $t+1$ can be expressed as

$$\begin{aligned} \pi_{t+1} &= \int_0^1 K_t(i)^\alpha (A_{t+1}L_{t+1}(i))^{1-\alpha} - w_{t+1}L_{t+1}(i) - R_t K_t(i) - \mathbb{1}\Psi di \\ \pi_{t+1} &= \int_0^1 \frac{1}{\varepsilon} K_t(i)^\alpha (A_{t+1}L_{t+1}(i))^{1-\alpha} - \mathbb{1}\Psi di \\ \pi_{t+1} &= \frac{1}{\varepsilon} K_t^\alpha (A_{t+1}L_{t+1})^{1-\alpha} - \mathbb{1}\Psi, \end{aligned} \tag{1.32}$$

where $\varepsilon > 1$ is the elasticity of substitution with which the different varieties are combined. $\mathbb{1}$ denotes the indicator function with

$$\mathbb{1} := \begin{cases} 1 & \text{if } K_t(i) \neq K_{t-1}(i), \\ 0 & \text{if } K_t(i) = K_{t-1}(i). \end{cases}$$

Assume, for simplicity, that labor supply is constant and that the economy only runs for two periods. Thus, the two endogenous variables will be the capital stock and the wage rate. Assume that in period t firms start with an exogenous capital stock K_0 that is already optimal given L_t.

In period $t+1$ there is a positive shock to labor supply, i.e., ΔL_{t+1}, and firms have to decide whether or not to adjust their capital stock for period $t+1$ in period t.[25]

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[24]The final good Y_t is produced according to $Y_t = \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$. $y_t(i)$ is the quantity of variety i, and the production function of producer i is $y_t(i) = K_{t-1}(i)^\alpha (A_t L_t(i))^{1-\alpha}$.

[25]Alternatively, I could assume that labor supply remains constant, but a fraction of the capital stock depreciates, and firms then have to decide whether or not to invest to keep the capital stock per worker constant.

Aggregating the first-order conditions yields

$$R_t = \frac{\varepsilon - 1}{\varepsilon} \alpha K_t^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha}, \tag{1.33}$$

$$w_{t+1} = \frac{\varepsilon - 1}{\varepsilon} (1-\alpha) K_t^{\alpha} A_{t+1}^{1-\alpha} L_{t+1}^{-\alpha}. \tag{1.34}$$

Assuming R_t and L_{t+1} are exogenous w_{t+1} and K_t will be the endogenous variables. This implies

$$\frac{\partial K_t}{\partial L_{t+1}} > 0, \quad \frac{\partial w_{t+1}}{\partial L_{t+1}} = 0.$$

Proof. See Appendix A.3. □

Notice, that the first-order conditions with and without fixed costs, i.e., only linear costs, are the same. Therefore, with only linear costs, an increase in labor supply has no effect on the wage rate if the interest rate is exogenous and capital supply is fully flexible.

This implies that if they adjust the capital stock, firms will choose the capital stock that is optimal given $L_{t+1} = L_t + \Delta L_{t+1}$ as Ψ is independent of the amount firms invest.[26]

When firms decide whether or not to adjust their capital stock, they will compare the pay-off of adjusting with the pay-off if they leave the capital stock constant. The difference in payoffs is given as

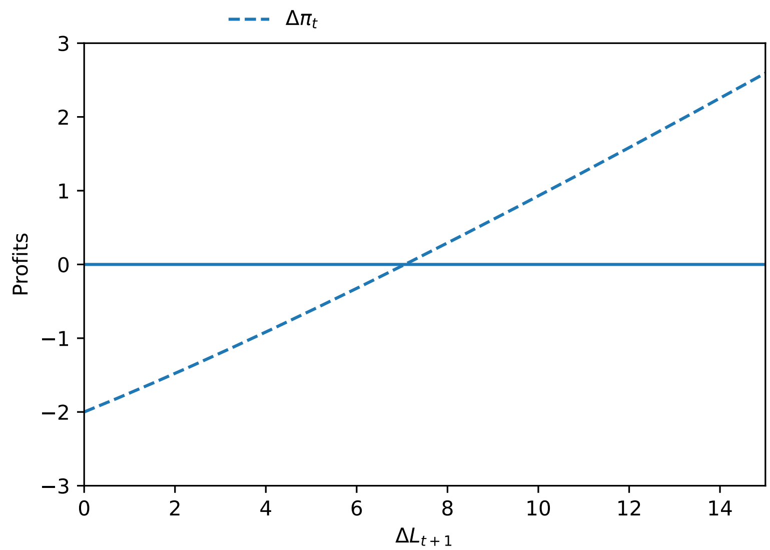
$$\Delta \pi_{t+1} = \frac{1}{\varepsilon} \left(K_t^{\alpha} A_{t+1}^{1-\alpha} L_{t+1} - K_{t-1}^{\alpha} A_{t+1}^{1-\alpha} L_{t+1} \right) - \Psi \gtreqless 0, \tag{1.35}$$

where $K_t \geq K_{t-1}$, i.e., the first term denotes output in case firms adjust the capital stock and the second term in case they keep it unchanged. Figure 1.5 illustrates this graphically.

This shows that the increase in L_{t+1} needs to be large enough such that firms find it optimal to adjust their capital stock because they have to pay the fixed cost independently of the size of the adjustment they make. Therefore, if (the growth rate of) the labor force decreases due to, for example, population aging, this can imply that an investment project is not

[26]To see this, combine the aggregated first-order conditions to get $\frac{w_{t+1}}{R_t} = \frac{1-\alpha}{\alpha} k_t$. Where $k_t = \frac{K_t}{L_t}$ is the capital stock per worker. As R_t is exogenous and w_{t+1} will not change in response to an increase in L_{t+1} if the capital stock is adjusted, this implies that in this case k_t remains constant.

undertaken if a firm is already close to its threshold. This can result in a lower capital stock per worker compared to a situation in which firms adjusted their capital stock.



The figure is created using the following parameter values: $\alpha = 0.5$, $\varepsilon = 0.5$, $\Psi = 2$, $K_0 = 10$, and $L_0 = 10$.

Figure 1.5: *Investments with Fixed Costs*

Assume firms produce output according to a more general CES production function, $F(K, L)$ which is homogeneous of degree 1 in K and L.

$$\Delta\pi_{t+1} = \frac{1}{\varepsilon}\left(F(K_t, L_{t+1}) - F(K_{t-1}, L_{t+1})\right) - \Psi \gtreqless 0, \tag{1.36}$$

where $K_t \geq K_{t-1}$. Assume that initially $\Delta\pi_{t+1} \geq 0$, i.e., the capital stock is adjusted in response to an increase in L_{t+1}. The effect of a decrease in L_{t+1} on $\Delta\pi_{t+1}$ is determined by $F_{L_{t+1}}(K_t, L_{t+1}) \gtreqless F_{L_{t+1}}(K_{t-1}, L_{t+1})$ with $K_t > K_{t-1}$.
As $F_{KL}(\cdot, \cdot) > 0$, this implies that $F_{L_{t+1}}(K_t, L_{t+1}) > F_{L_{t+1}}(K_{t-1}, L_{t+1})$ with $K_t > K_{t-1}$. And thus

$$\frac{\partial\Delta\pi_{t+1}}{\partial(-L_{t+1})} < 0.$$

Therefore, a decrease in labor supply also lowers the value of an investment project if output is produced with a two-input CES production function that features constant returns to scale.

1.8 Endogenous Capital Supply

In the following section, I endogenize capital supply in order to analyze the effect of population aging in general equilibrium. For simplicity, I assume that labor supply is now exogenous, and thus every young household in period t supplies $\bar{\ell}_t$ units of labor inelastically, where I normalize $\bar{\ell}_t$ to 1. Households now live for two periods. In the first period, they supply labor and decide how much of their income to save, and in the second period, they are retired and consume their savings. As before, I assume that at the end of the first period, the savings of the households are matched with the capital demand of the firms for the next period. In period t, there are N_t households that supply labor and N_{t-1} households that are retired. Households have CRRA preferences.

The optimization problem of a young household in period t is given as follows

$$\begin{aligned} &\max_{c_t, c_{t+1}, s_t} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma}-1}{1-\sigma} \\ &\text{s.t. } c_t = w_t - s_t \\ &c_{t+1} = s_t R_t, \end{aligned} \tag{1.37}$$

$\frac{1}{\sigma}$ is the intertemporal elasticity of substitution, with $\sigma \in (0,1)$. Hence, I assume that current and future consumption are substitutes. s_t are total savings of each household, and R_t is the gross interest rate. Optimal savings per household are given as[27]

$$s_t = \frac{\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}} w_t, \tag{1.38}$$

[27]With $\sigma < 1$ $\frac{\partial s_t}{\partial R_t} > 0$ and with $\sigma \to 1$ $\frac{\partial s_t}{\partial R_t} = 0$.

overall capital supply in the economy in period t, which will be used for production in period $t+1$, is given as: $S_t = N_t s_t$. Capital supply in period t and capital demand in period $t+1$ are now matched in period t. Hence, the firms need to choose their capital demand a period before producing with it.[28] The firm side remains as before.
Assuming the labor market and the capital market clear, I can then express the three equilibrium conditions for the economy as follows[29]

$$\text{Capital supply: } F \equiv \frac{\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}}{1+\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}} w_t N_t - K_t = 0, \tag{1.39}$$

$$\text{Capital demand: } G \equiv \beta \left[\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right] - R_t - \frac{C(K_t)}{K_t} = 0, \tag{1.40}$$

$$\text{Labor market equilbrium: } H \equiv \left(\frac{(1-\alpha) A_{t+1}^{1-\alpha} K_t^{\alpha}}{w_{t+1}} \right)^{\frac{1}{\alpha}} - (1+n_{t+1}) N_t = 0. \tag{1.41}$$

With R_t, K_t and w_{t+1} as the endogenous variables.

Note that because I make use of the equilibrium conditions, K_t now denotes the equilibrium capital stock in the economy and not the capital demand of firms.

I can derive the following comparative statics for a decrease in the population growth rate n_{t+1} and an increase in the level of technology A_{t+1}, for $\sigma \in (0,1)$.

$$\frac{\partial K_t}{\partial(-n_{t+1})} < 0, \quad \frac{\partial R_t}{\partial(-n_{t+1})} < 0, \quad \frac{\partial w_{t+1}}{\partial(-n_{t+1})} > 0,$$
$$\frac{\partial K_t}{\partial A_{t+1}} > 0, \quad \frac{\partial R_t}{\partial A_{t+1}} > 0, \quad \frac{\partial w_{t+1}}{\partial A_{t+1}} > 0.$$

[28] The reason is that capital supply is elastic, and therefore households will adapt their savings in response to the demand. If capital supply and demand were matched in the second period, the supply of savings would be fixed, i.e., completely inelastic, as households could not go back in time to consume more or less in order to adapt the savings supply. Thus, this assumption ensures that the household's intertemporal resource allocation is optimal.

[29] I could equate capital demand and supply to reduce the system to two equations; however, as I am interested in the effect of population aging on capital demand and the interest rate, I use the three equations.

Proof. See Appendix A.2.1. □

Therefore, a decrease in the number of young people, i.e., people of working-age, shifts the capital demand curve inward as before by increasing the wage rate. $\sigma \in (0, 1)$ entails that the capital supply curve is upward sloping. Combining this with the inward shift of the downward sloping capital demand curve leads to a lower equilibrium capital stock and a lower equilibrium interest rate. The equilibrium wage rate is now affected by two channels. As before, the convex (level) costs imply that a fall in n_{t+1} leads to a higher capital stock per worker, which implies an increase in the marginal product of labor and thus a higher wage rate. Moreover, the reduction in the interest rate as a result of a fall in n_{t+1} entails that the equilibrium wage rate rises, as it depends negatively on the interest rate. Hence, population aging now leads to a higher equilibrium wage rate even in the absence of convex (level) costs by increasing the capital stock per worker.[30]

Therefore, population aging implies that there is a shortfall in labor relative to capital. People exit the labor market, i.e., they retire, but their savings remain in the economy and constitute the capital stock with which the young generation produces. Population aging implies that the size of the next generation is smaller relative to the previous generation, lowering the overall amount of available labor. However, aggregate savings only decrease once the smaller generation retires, and thus the share of available overall capital relative to total labor increases in the economy in the periods in which the economy experiences population aging. Hence, in general equilibrium, this leads to a lower equilibrium interest rate and a higher capital stock per worker, which is in line with the empirical evidence presented at the beginning.

Technological progress, i.e., a rise in A_{t+1}, increases the marginal product of capital and labor, which leads to a higher demand for capital and labor that manifests itself in a higher equilibrium interest rate and wage rate. Furthermore, as capital supply is elastic, this also leads to a larger equilibrium capital stock.

[30] With only linear costs, the capital stock per worker is given as $k_t = \frac{K_t}{(1+n_{t+1})N_t} = \left(\frac{\alpha}{R_t}\right)^{\frac{1}{1-\alpha}} A_{t+1}$. As R_t is endogenous and positively depends on n_{t+1}, a fall in n_{t+1} reduces R_t, which in turn increases k_t.

In the case of log-utility, i.e., for $\sigma \to 1$, the comparative statics remain the same except for the equilibrium capital stock K_t. They are now given as

$$\frac{\partial K_t}{\partial(-n_{t+1})} = 0, \quad \frac{\partial K_t}{\partial A_{t+1}} = 0.$$

Therefore, the equilibrium capital stock K_t now remains constant. The reason being that for $\sigma \to 1$ the capital supply is completely inelastic, and thus all adjustment takes place through the interest rate.

In Section A.2.1 in the Appendix, I investigate how the above results could change if labor were to be supplied elastically. The main takeaway here is that the sufficient conditions, though not necessary conditions, for the results of the comparative statics above to remain valid are $\frac{\partial \ell_t}{\partial R_t} \geq 0$ and $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} \geq 0$.

So far, I have assumed that the interest rate can always fall in response to a decrease in n_{t+1}. However, assuming there exists a zero lower bound (ZLB), below which the interest rate cannot fall, can have interesting implications.[31] To that end, I assume that there exists an asset B_t that yields a zero return but can be used as a store of value. An example would be cash or people storing the final good and consuming it in the next period.[32]

A binding ZLB would imply that firms are not willing to invest all the capital available, and hence households would have to store parts of their savings in the form of the asset B_t. Moreover, this also implies that the interest rate is downward rigid, and thus capital deepening in response to population aging is no longer possible. Hence, population aging can only lead to higher wages if firms face convex (level) costs when acquiring capital,

[31]I abstract from inflation in this model, and thus the ZLB also applies to the real interest rate, as I assume people can always store their savings in the form of the final good similar to cash.

[32]Strictly speaking, the marginal product of capital $\alpha\beta\left(\frac{A_{t+1}N_{t+1}}{K_t}\right)^{1-\alpha}$ will only be zero if N_{t+1} is zero or K_t approaches ∞. However, for the sake of the argument, one could either assume that B_t yields a positive return $R_{f,t}$ which would then constitute the lower bound, or assume that there exists a positive probability that firms go bankrupt, which would imply that households would require a positive rate of return in order for the expected rate of return to be equal to zero. Because when a firm goes bankrupt, its capital stock will (in part) be lost. See Section A.4

similar to the partial equilibrium framework discussed in the beginning.

As I assumed perfect competition throughout the model and no rigidities, this entails that firms' production decisions are only dependent on the factor prices of the input factors and not on the price of the final good. This in turn implies that a binding ZLB will not lead to a demand shortfall. In a model where firms do have market power, population aging could lead to a fall in demand, which could then in turn affect the production and, hence, employment and investment decisions of firms. This might have more far-reaching ramifications than the model discussed here suggests.

1.8.1 Dynamics

So far, I have considered a two-period model and analyzed how a one-time shock to n_{t+1}, i.e., the population growth rate, affects the equilibrium capital stock, wages, and interest rates in the current period. To study how population aging affects the long-run outcomes in the model, I use a simplified version of the model from Section 1.8.

Assume the savings rate s_t is independent of the interest rate, i.e., $\sigma \to 1$, and the production structure is the same as in Section 1.3.1, except that the time subscript for capital now refers to the period in which it is used for production.[33] I assume that capital fully depreciates after one period. The law of motion for capital is given as

$$K_{t+1} = s_t(1-\alpha)K_t^{\alpha}(A_tL_t)^{1-\alpha} + (1-\delta)K_t, \tag{1.42}$$

as with the Cobb-Douglas production function, labor income, out of which households save, is a constant fraction of overall output.

In case the population grows at rate n_t and there is no technological progress, the steady state capital stock per worker is given as

[33] $\frac{\partial s_t}{\partial R_t} = 0 \; \forall t$ implies that non-linear capital acquisition costs do not have an impact on the law of motion of capital.

$$k^* = \left(\frac{s(1-\alpha)}{\delta + n}\right)^{\frac{1}{1-\alpha}} A, \tag{1.43}$$

which implies that $\frac{\partial k^*}{\partial(-n)} > 0$.[34] Thus, a lower population growth rate leads to a higher capital stock per worker in steady state. The intuition for this result is that the capital stock with which a generation produces is produced by the previous generation, which will be larger than the current generation if n decreases. A lower n implies that fewer workers will produce with the same capital stock, as a lower n will only lead to a fall in the overall capital stock in the next period.

On the balanced growth path (BGP), k_t will be constant, and given that $k_t = \frac{K_t}{(1+n_t)N_{t-1}} = \frac{K_t}{N_t}$, this implies that K_t needs to grow at the rate n_t in order for k^* to remain constant.

The equilibrium wage and interest rate are given as

$$R_t = \alpha\beta \left(\frac{1}{k_t}\right)^{1-\alpha} A_t^{1-\alpha} - \frac{C(K_t)}{K_t}, \tag{1.44}$$

$$w_t = (1-\alpha)k_t^{\alpha} A_t^{1-\alpha}. \tag{1.45}$$

As K_t grows at rate n_t and k_t remains constant on the BGP, the interest rate will only remain constant if n_t is zero in the long-run, as $\frac{C'(K_t)K_t - C(K_t)}{K_t^2} \lesseqgtr 0$.

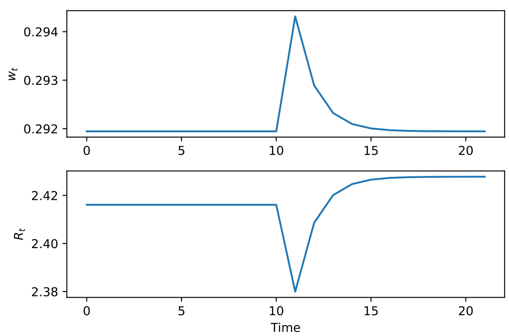
Therefore, if n_t is zero and there is a one-time decrease in the workforce due to, for example, population aging, k^* will be unaffected, and so will the steady state wage rate. In the short-run, the wage rate will still rise due to the increase in the capital stock per worker. However, the positive effect on the wage rate will fade out over time as the capital stock per worker falls back to its steady state level. The intuition for this result is that a one-time fall in N_t increases the capital stock per worker and thus the wage rate in the period in which the fall occurs. However, the reduction in N_t also implies fewer savers and thus lower

[34] Full depreciation entails $\delta = 1$ and thus $k^* = \left(\frac{s(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}} A$.

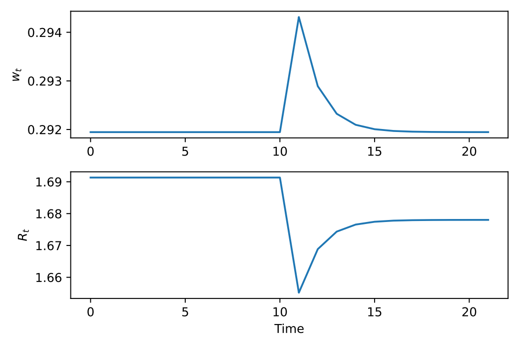
savings in the next period. This reduces the capital stock per worker relative to the period in which N_t fell, as the workforce now remains constant.

In contrast, the steady state interest rate will change if there is a one-time decrease in the labor force due to non-linear capital acquisition costs. Whether it will increase or fall depends on $\frac{C'(K_t)K_t - C(K_t)}{K_t^2} \lesseqgtr 0$, i.e., whether firms face convex or concave (level) costs.[35] A decrease in the population size will reduce the aggregate steady state capital stock, which in turn has an effect on the steady state interest rate, i.e., there is a *level* effect due to the non-linear costs (see Figure 1.6).[36]

In cases where the economy only faces linear costs in the long-run and there is a permanent fall in the population growth rate, the steady state capital stock per worker increases, which leads to higher steady state wages and lower steady state interest rates.



(a) *Convex Costs* $\left(C(K_t) = K_t^2\right)$



(b) *Concave Costs* $\left(C(K_t) = \sqrt{K_t}\right)$

The figures are created using the following parameter values: $\alpha = 0.4$, $A_t = 2$, $s_t = 0.2$, $\delta = 0.8$, $\beta = 0.9$, $L_0 = 10$, and $K_0 = K^*$. From period 10 to 11, the population falls by 2%. Before and after the population growth rate is zero.

Figure 1.6: *Dynamics*

[35]If costs are convex, the term is positive, and thus the steady state interest rate increases in response to a one-time decrease in the workforce, and vice versa if the costs are concave.

[36]A fall in N_t requires a proportional fall in K_t in order for k_t to remain constant.

1.9 CES Production Function

I modify the model from Section 1.3 such that instead of a Cobb-Douglas production function, firms use a more general CES production function, where A_{t+1} now denotes a Hicks-neutral efficiency parameter, in order to rule out factor augmenting technological progress. I employ this more general specification to test if the results derived in the previous section are robust to an elasticity of substitution that is larger or smaller than 1.
The representative firm faces the subsequent optimization problem

$$\max_{K_t, L_{t+1}} \mathbb{E}_t \left[A_{t+1} \left(\eta K_t^{\rho} + (1-\eta) L_{t+1}^{\rho} \right)^{\frac{1}{\rho}} - w_{t+1} L_{t+1} \right] - R_t K_t - C(K_t), \tag{1.46}$$

$\eta \in (0,1)$ is the share parameter, $\rho = \frac{\zeta - 1}{\zeta}$, where ζ denotes the elasticity of substitution between K_t and L_{t+1}. Hence, if $\rho > 0$, K_t and L_{t+1} are substitutes, and if $\rho < 0$ they are complements. In the following, I assume that $\{\zeta \in \mathbb{R} \mid 0 < \zeta < \infty \setminus 1\}$ thus I exclude all the limit cases, i.e., where K_t and L_{t+1} are perfect substitutes, perfect complements, or where the elasticity of substitution between them is 1.
Assuming perfect foresight and perfect competition, the profit maximization of the firms yields the following system of two equations that determine the firms' optimal capital and labor demand

$$G \equiv A_{t+1} \left(\eta^{\frac{1}{1-\rho}} + \eta^{\frac{\rho}{1-\rho}} (1-\eta) \left(\frac{L_{t+1}}{K_t} \right)^{\rho} \right)^{\frac{1-\rho}{\rho}} - R_t - \frac{C(K_t)}{K_t} = 0, \tag{1.47}$$

$$H \equiv A_{t+1} \left(\eta (1-\eta)^{\frac{\rho}{1-\rho}} \left(\frac{K_t}{L_{t+1}} \right)^{\rho} + (1-\eta)^{\frac{1}{1-\rho}} \right)^{\frac{1-\rho}{\rho}} - w_{t+1} = 0. \tag{1.48}$$

Similarly, to before, the marginal product of capital is equal to the average costs of capital, and the marginal product of labor is equal to the marginal costs of labor in equilibrium. In Section A.2.2 of the Appendix, I show that the results of the comparative statics are the same as in Section 1.8 with exogenous labor supply.
In the case that labor is supplied elastically within the model, similarly to the Cobb-Douglas case, a sufficient condition, albeit not a necessary condition, for the results of the comparative statics to remain unchanged is that $\frac{\partial \ell_t}{\partial R_t} \geq 0$ and $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} \geq 0$ hold.

Therefore, even when capital and labor are substitutes—as long as they are not perfect substitutes—a decrease in the number of workers will lower the interest rate and increase wages through the same mechanisms as in the case of a Cobb-Douglas production function. The intuition behind this result is that if labor becomes more expensive due to a decrease in the size of the workforce, firms will also reduce their demand for capital, i.e., the increase in the wage rate tightens the budget constraint of the firms, and thus there is an income effect similar to consumer theory. The corresponding substitution effect arises because the factor that has become relatively more expensive, i.e., labor, can be substituted by the factor that has become relatively cheaper, i.e., capital. However, the assumption of constant returns to scale implies that the income effect always dominates the substitution effect. With decreasing returns to scale, for example, due to a nested CES production function and a high enough elasticity of substitution, it is possible that the substitution effect dominates the income effect.[37]

1.10 Conclusion

In this article, I presented a series of analytic models that allow me to study the implications of population aging on different macroeconomic variables, namely the capital stock (per worker), interest rates, and wages.

The models illustrate how population aging or the expectation thereof leads to a reduction in capital demand, which in turn can lead to lower equilibrium interest rates if the capital stock per worker increases, as well as a lower equilibrium capital stock.[38] The effect on the capital stock and investments can be even stronger if firms face fixed costs when acquiring capital, which can lead to threshold effects. In this case, even a small decrease in the size of the workforce can lead to large shifts in the capital stock if firms are close to their investment

[37]An example of such a function would be $Y = \left(\eta x_1^{\rho_1} + (1-\eta)\left(\gamma x_2^{\rho_2} + (1-\gamma) x_3^{\rho_2}\right)^{\frac{\rho_1}{\rho_2}}\right)^{\frac{1}{\rho_1}}$. Assuming $\frac{\rho_1}{\rho_2} < 1$, then x_2 and x_3 would be combined using a decreasing returns to scale production technology. See also Section A.2.3 in the Appendix.

[38]Assuming that the intertemporal elasticity of substitution is less than 1.

threshold.

Moreover, it illustrates that the effect of population aging crucially depends on the capital costs firms face, i.e., linear, convex, or concave, as these determine whether the capital stock per worker increases in response to population aging.

The models also highlight that timing assumptions are important and that introducing risk can have profound short-run implications for wages. Therefore, the effect of population aging on wages depends strongly on the investment behavior and expectations of firms. In case firms overestimate the decrease in the future labor supply, wages will on average be lower, and in case they underestimate it, wages will on average be higher.

Hence, even though population aging is expected to make labor scarcer and thus its price, i.e., the wage rate, can be expected to increase, the models presented in this article show that this intuition can be deceptive. Wages will only increase if the complementary production factor, i.e., capital, is increased per unit of labor, i.e., if the capital-to-labor ratio increases.

In addition, the model highlights that whether wages will permanently remain higher depends on whether the negative shock to the size of the work force is permanent or transitory. Whereas the non-linear and linear (level) costs of capital entail that the long-run interest rate is not only affected by the capital-to-labor ratio but also by the aggregate capital stock.

Therefore, the models not only demonstrate how (expected) population aging can explain parts of the lackluster investment behavior observed in the past,[39] but can also illustrate under what conditions population aging will actually lead to higher wages.

[39]See, for example, Gutiérrez and Philippon (2016) andAlexander and Eberly (2018).

Chapter 2

Demand, Demography, and Disequilibrium: Can Demographic Change Constitute a Keynesian Supply Shock?

2.1 Introduction

The economic shocks associated with the COVID-19 pandemic have led to a resurgence of the concept of a "Keynesian supply shock", i.e., a negative supply shock that produces an income loss, which in turn leads to a fall in aggregate demand such that output in the economy falls *below* potential; see, for example, Cesa-Bianchi and Ferrero (2021) and Guerrieri *et al.* (2022). Whereas the negative supply shocks associated with the COVID-19 pandemic have fortunately been of a temporary nature, the world and especially high-income countries might potentially experience negative supply shocks in the future that are of a more long-lasting nature.

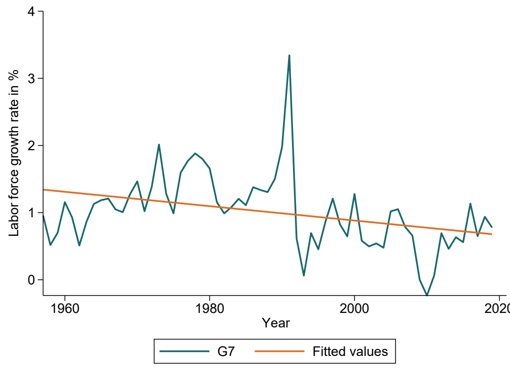
Moreover, over the past decades, population growth as well as the rate of technological progress have decreased in many advanced economies (see Figure 2.1).[1] Until recently, nominal interest rates have continuously been on the decline, which has given rise to concerns regarding a binding zero or effective lower bound and an ensuing liquidity trap. Given the recent surge in inflation central banks across the world have started to increase interest rates. Therefore, the risk that economies will face a binding effective lower bound and an ensuing liquidity trap might seem improbable. However, current projections by the International Monetary Fund predict that the natural rate of interest will remain close to or even below zero in the coming decades for many economies around the world (International Monetary Fund (2023)). Hence, the problem of a binding real effective lower bound could again become a relevant issue, assuming central banks are unwilling to commit to a higher inflation target.[2]

Demographic change can be interpreted as a negative supply shock, i.e., a decrease in aggregate labor supply. Given that demographic change will most likely prevail for multiple years, if not decades, it is natural to ask whether and under which conditions demographic change can constitute a "Keynesian supply shock"? However, so far, only a few studies have addressed this question in detail.[3] Therefore, the aim of this paper is to construct a rich yet still tractable theoretical framework that allows us to study the channels and conditions under which demographic change will compose a "Keynesian supply shock".

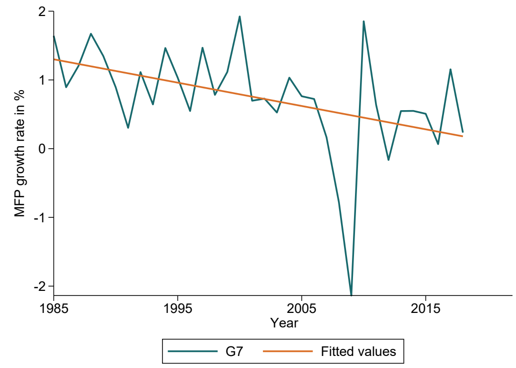
[1]See also Antolin-Diaz *et al.* (2017), Fernald *et al.* (2017), Aghion *et al.* (2019), and Eo and Morley (2020) for recent evidence that TFP growth has slowed down.

[2]See also Figure B.1.

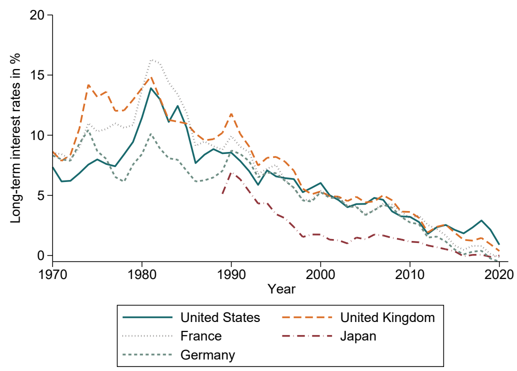
[3]See, for example, Michau (2018), Eggertsson *et al.* (2019a), and Eggertsson *et al.* (2019b).



(a) *Labor Force Growth Rate in %*



(b) *MFP Growth Rate in %*



(c) *Long Term Interest Rates in %*

The figures provides a graphical illustration of the trend in the labor force and multifactor productivity growth rate for the G7 countries, i.e., CAN, FRA, DEU, ITA, JPN, GBR, and USA. And the development of long term interest rates for selected countries. The data is taken from the OECD.

Figure 2.1: *Labor Force Growth, MFP Growth, and Interest Rates*

Examining this question in more detail is especially important, as it is closely linked to social security and the pension system, which are both especially affected by demographic change and demographic change. The combination of people having fewer children and experiencing an increase in life expectancy has led to an increase in the old-age dependency ratio[4] and it is expected that the old-age dependency ratio will increase further in the

[4]The old-age dependency ratio is defined here as $\frac{65+}{15-64} \cdot 100$.

foreseeable future.[5] Therefore, the number of people over 65 relative to the population of working-age is rising. This puts special pressure on pension systems that are based on a pay-as-you-go (PAYG) system, where pensions are directly paid for by the working population, as, for example, Germany.[6] A potential solution to this problem could be a switch to a fully funded system in which people accumulate the assets themselves that they consume when they are retired, as in the standard overlapping generations (OLG) model. This has the advantage that the working-age population no longer directly pays the pension of the retired households, as is the case in a PAYG system. There are naturally multiple aspects to consider, as both systems have their advantages and disadvantages.[7]

However, as I will detail below, switching from a PAYG to a fully funded system can have far-reaching, unintended negative consequences. Specifically, the additional savings that are accumulated in a fully funded system will lead to an endogenous savings glut and a demand-induced recession.

As alluded to in the title, the main objective of this paper is to study the conditions and channels through which demographic change, i.e., a decrease in the population growth rate as well as a change in the demographic structure, affects the economy through the *demand side*.[8] Therefore, I analyze how demographic change can lead to an aggregate demand externality. To that end, I construct a rich but still tractable OLG model with an exogenous labor supply.[9] The OLG structure entails that changes in the population growth rate not only alter labor supply but also affect the demographic structure, i.e., change the old-age dependency ratio. The model is augmented by including a financial sector that serves as a financial intermediary but also creates and extends loans to households and firms. Moreover, I modify the supply side of a standard OLG model by introducing monopolistic competition,

[5] United Nations (2022).

[6] See, for example, Fenge and Peglow (2018).

[7] See, for example, Brunner (1996), Sinn (2000) and Lindbeck and Persson (2003).

[8] See Section B.2 in the Appendix for a short discussion of the *supply* side effects of demographic change.

[9] Galí (2021) studies a New Keynesian model with overlapping generation to analyze asset price bubbles.

which allows me to study how demand directly influences the production decisions of firms. On the household side, I introduce a second asset, which yields a constant nominal return and serves two purposes. First, it introduces a nominal effective lower bound (ELB) and second, it enables me to explicitly analyze how demographic change can lead to a reduction in demand and a corresponding demand-driven recession. The main friction present in the model is the downward rigidity of the nominal wage rate and the nominal price of the final good, i.e., the price level. This entails that the *nominal* ELB becomes a *real* ELB.[10]

Therefore, this article studies how demographic change can lead to a demand-induced recession if the real interest rate is constrained. The intuition behind this result is that a decrease in the workforce reduces investment demand from firms. However, at the same time, the capital supply does not fall, as agents need to build up savings for their retirement, and thus there is a situation of low investment demand and excess savings. Under "normal" circumstances, this would simply lead to a fall in the real interest rate. However, if the real interest rate is constrained, i.e., due to a binding real ELB, there needs to be a different mechanism through which savings and investments are brought into equilibrium. As savings exceed investment demand for a given real interest rate, households are forced to hold some of their wealth in nominal assets. However, these nominal assets will not constitute demand—hence the demand-induced recession—and thus firms will be forced to reduce their labor demand, resulting in involuntary unemployment.[11] Involuntary unemployment will reduce output and thus income, which in turn will also reduce savings. This process continues until the excess savings have been eliminated and the economy has reached its new (rationing) equilibrium.[12]

[10] The main rigidity present in the model is a binding real ELB. Given that nominal interest rates have been on the rise since 2021/22, this might not seem like an important cause for concern at the moment. However, the mechanism outlined in this paper does not specifically depend on the presence of a binding real ELB. What is required is a form of rigidity that entails households finding it optimal to invest a portion of their savings in a *nominal asset* that does not constitute demand, i.e., money, and thus the economy will suffer from too little demand.

[11] As nominal wages are downward rigid, firms cannot lower their prices in order to attract more demand.

[12] Section B.8 in the Appendix analyzes how the excess savings can be absorbed through a change in *nominal* rather than *real* variables.

Hence, nominal assets, i.e., money, take the form of a mirage in this model. They technically exist as a savings device as long as agents do not attempt to utilize them for this purpose, but as soon as they attempt to store their wealth in them, they disappear. The intuition is that only money that is spent constitutes demand and thus income. Therefore, in order to save by holding money requires that agents first earn this money, which first requires that somebody spends the money. This is at the heart of the problem; saving in nominal assets implies in this model that the money is not spent, but without somebody spending the money, households never generate the income out of which they plan to save.[13]

Furthermore, the framework developed in this article illustrates how negative supply shocks that materialize in the future and affect investment demand, e.g., a decrease in the population growth rate, can, through an *intertemporal* channel, induce a negative demand shock in the current period if the real interest rate cannot adjust. Investments are by their nature forward-looking, and this entails that future events will have an effect on them. At the same time, they also constitute demand in the period in which they are built up or acquired. Normally, the real interest rate absorbs shocks that affect investment demand but not savings, i.e., the interest rate adjusts such that the capital market is in equilibrium. Therefore, as long as the real interest rate is free to adjust, the intertemporal channel is closed. However, as soon as the real interest rate is constrained, the intertemporal channel opens up, and, as a consequence, negative supply shocks can bring about a negative demand shock. Hence, an economy can suffer a higher loss in output, both directly through the negative supply shock and indirectly through the induced negative demand shock.[14] Therefore, the model illustrates how demographic change can develop into an *intertemporal* "Keynesian supply shock", i.e., the negative supply shock and the associated shortfall in demand do not materialize in the same period.[15]

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[13]This mirrors the Old-Keynesian paradox of thrift, see, for example, Eggertsson and Krugman (2012) for a more recent discussion.

[14]Thus, my model suggests that negative supply shocks are more contractionary if an economy is at or below the ELB, which is in line with the empirical evidence presented in Wieland (2019).

[15]In this model, a negative supply shock in period $t+1$ affects demand in period t, i.e., $supply(t+1) \rightarrow$

More generally, the framework developed in this article entails that there are two types of shocks that can bring about a demand-induced recession. First, shocks that push the equilibrium interest rate against the real ELB such that optimal savings exceed the maximum amount of capital that the economy can absorb in a given period, i.e., shocks that either affect optimal savings supply or optimal capital demand. Second, shocks that push the real ELB against the equilibrium interest rate, i.e., shocks that increase the real return of the nominal asset relative to the equilibrium interest rate, such as deflation.

In addition, I show that absent a commitment device or government intervention, excess savings and a corresponding demand-induced recession will be the only outcome that constitutes a Nash equilibrium. Therefore, even if agents understand that saving more than the economy can absorb leads to involuntary unemployment and reduces their income, they will nonetheless decide on this equilibrium because otherwise households would have an incentive to deviate.

Throughout the main part, I assume households are homogeneous. I relax this assumption in the Appendix and characterize the conditions under which excess savings not only lead to negative consequences for the households that are responsible for them but also for households that have not accumulated excess savings.

Contribution to the Literature

As mentioned before, one paper that is closely related to mine is Eggertsson *et al.* (2019b). In this paper, households need to borrow funds to finance their consumption when they are young, and as population growth decreases, this reduces the demand for funds, which can result in a demand shortage if the ELB is binding. The authors assume the central bank sets the inflation rate equal to target unless this would imply a negative nominal interest rate. In this case, the nominal interest rate is zero, and inflation falls below target. Thus, the model features two regimes: one where inflation is positive, which allows for full employment, and

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$demand(t)$. This differs from "standard" hysteresis or scarring effects (see, for example, Fornaro and Wolf (2023)), where $supply(t) \rightarrow demand/supply(t+1, t+2, ...)$. Therefore, in the framework developed here, it is also possible that the *expectation* of a negative supply shock in the future induces a demand shortfall in the current period similarly to Benigno and Fornaro (2018), even though the negative supply shock might never materialize.

one where inflation becomes negative, which leads to an increase in real wages as nominal wages are downward rigid, and thus involuntary unemployment and a reduction in output. Related to the previous paper, Eggertsson *et al.* (2019a) incorporates capital as well as fully rigid wages to analyze the effect of a decrease in the growth rate of population on output per capita. They find that as long as the interest rate can fall, a decrease in the number of young people leads to capital deepening, which allows output per person to potentially remain constant or even increase. However, if the interest rate cannot fall due to a binding ELB, capital deepening is no longer possible, and demographic change implies the economy enters a demand-driven recession that results in a fall in output per person.

Similarly, Michau (2018) employs a Ramsey model and imposes a ceiling on the inflation rate to introduce a real effective lower bound. Moreover, households have a preference for wealth, which can give rise to an endogenous savings glut. Moreover, wages are assumed to be downward rigid to prevent a deflationary spiral that arises if aggregate demand is lower than aggregate supply. Similarly, to Eggertsson *et al.* (2019b) the model features two steady states: the neoclassical steady state and the secular stagnation steady state. The secular stagnation steady state requires a binding downward wage rigidity as well as an excessively high interest rate. Together, they imply that aggregate demand is depressed. This paper as well as Eggertsson *et al.* (2019b) feature the *paradox of flexibility*. Thus, more flexible wages translate into lower inflation, which raises the real interest rate, further depressing demand. Moreover, Schmitt-Grohé and Uribe (2017) study how a negative confidence shock pushes inflation below target, which leads the central bank to cut the nominal interest rate to bring inflation back to target. Once the ELB becomes binding, the central bank can no longer reduce the nominal interest rate, which implies inflation remains below target. Inflation below target increases real wages because, due to downward nominal wage rigidities, nominal wages cannot fall. Thus, real wages become too high, which leads to involuntary unemployment. Hence, the mechanism that leads to involuntary unemployment is similar to the one in Eggertsson *et al.* (2019b).

In contrast to the articles discussed before, I relax the assumption that output is demand-determined by explicitly modeling how demand enters the production decisions of firms. This allows me to study how demand initially affects supply, as in the case of demand-determined output, but also how the effect of demand on supply can endogenously lead to a feedback effect, where the change in supply in turn affects demand. This entails that an initial demand shortage can lead to an endogenous redistribution of income, i.e., increasing the fraction of output received by workers or capital owners, which can worsen or alleviate the demand shortage.

Moreover, I introduce money as a potential store of value, but households do not have an inherent preference or need to hold a positive cash balance at the end of the period. Therefore, households can potentially save in two assets, i.e., invest in the capital stock of the next period or hold a positive money balance between periods. This gives rise to a no-arbitrage condition that mirrors the Fisher equation. This no-arbitrage condition constitutes a lower bound on the real interest rate. Given its property of a lower bound, this implies that the rate of inflation or deflation *can* but does *not necessarily* have an effect on the real interest rate. This then offers a potential resolution for the problem of a deflationary spiral that can emerge if aggregate demand is lower than aggregate supply (see, for example, Michau (2018). This also allows me to endogenize the demand shortfall and study how an adverse supply shock in the next period can cause a demand-induced recession in the current period.

I generalize the result by Eggertsson *et al.* (2019a) and show that any variable that affects optimal capital demand in period $t+1$ can increase (alleviate) the demand shortage in period t if the real interest rate in period $t+1$ is constrained by the ELB.

I likewise assume nominal wages are downward rigid. However, unlike the papers discussed before, once the economy reaches its ELB this does not lead to deflation or inflation below target, but instead the price level will remain constant. Nonetheless, the economy will still experience involuntary unemployment in equilibrium, which is caused by insufficient demand. Therefore, unlike Schmitt-Grohé and Uribe (2017) or Eggertsson *et al.* (2019b), the model can generate involuntary unemployment with constant nominal/real wages and thus shows that real wages are not necessarily countercyclical in secular stagnation episodes.

The model can also generate rising real wages with involuntary unemployment. This will only occur, however, if capital deepening is possible in the current period, i.e., if the interest rate can still fall in the period before the real ELB becomes binding; otherwise, the real wage will not change. Hence, in my model, rising real wages do not *cause* involuntary unemployment. Instead, higher real wages are a potential *consequence* of involuntary unemployment because each unit of employed labor will be able to produce with more capital, increasing the marginal product of labor. Therefore, in contrast to the articles discussed before, I do not require higher wages in order for involuntary unemployment to occur; instead, involuntary unemployment is directly caused by the shortage in demand.

Unlike Michau (2018) and Eggertsson *et al.* (2019b), the model does not feature the paradox of flexibility. Hence, in cases where nominal wages are partially downward flexible, i.e., they can fall by a certain amount, this will allow the economy to absorb parts of the demand shortfall through a change in *nominal* variables rather than *real* variables, which reduces the required change in *real* variables, i.e., the equilibrium involuntary unemployment rate will be lower.[16]

More broadly, this paper is related to the literature on secular stagnation that was first introduced by Hansen (1939), which has regained prominence in the aftermath of the Great Financial Crisis, see, for example, Kocherlakota (2013), Summers (2015), Gordon (2015), Eggertsson *et al.* (2016), Cervellati *et al.* (2017), Illing *et al.* (2018), Michau (2018), Eggertsson *et al.* (2019b), Eggertsson *et al.* (2019a), Summers and Rachel (2019), and Geerolf (2019). And the closely connected vast literature on the zero lower bound, liquidity trap, and aggregate demand externalities. See, for example, Krugman (1998), Christiano *et al.* (2011), Eggertsson and Krugman (2012), Farhi and Werning (2016), Korinek and Simsek (2016), Rendahl (2016), Cochrane (2017), Schmitt-Grohé and Uribe (2017), Caballero and Farhi (2018), Benigno and Fornaro (2018), Wieland (2019), Caballero and Simsek (2020), Bilbiie (2021), and Fernández-

[16] See also Kiley (2016).

Villaverde *et al.* (2023).[17] As well as the literature related to the concept of a "savings glut", first proposed by Bernanke (2005), that studies the observed decrease in the world interest rate. Recent contributions to this literature include Coeurdacier *et al.* (2015), and Del Negro *et al.* (2017, 2019). In addition, this paper relates to recent literature that investigates how a shortfall in demand can lead to output falling *below* potential. See, for example, Challe (2020), Ravn and Sterk (2021), Cesa-Bianchi and Ferrero (2021), and Guerrieri *et al.* (2022).

The remainder of the paper is structured as follows. Section 2.2.1 discusses the structure of the model. The household and production sector are introduced in Sections 2.2.2 and 2.2.3 respectively. Section 2.3 discusses the equilibrium, and Section 2.4 presents the results. In Section 2.5 I discuss whether the market outcome is efficient from a social planner's perspective. Section 2.6 studies the role of fiscal policy, and Section 2.7 concludes.

2.2 Model

2.2.1 Main Assumptions and Structure

The model consists of a two-period OLG closed economy, i.e., households live for two periods. Time is discrete: $t \in \{0, 1, ...\}$. The economy consists of households that consume, save, and work and firms that hire labor, rent capital, produce output, and set prices. There is a continuum of monopolistic competitive firms that all produce a different intermediate good using capital and labor as input factors. Capital fully depreciates after one period. A perfectly competitive final good sector combines the intermediate goods into a final good that is used for consumption and investment purposes. All agents, i.e., households and firms, face a cash-in-advance constraint, i.e., the final good can only be purchased in return for money. Therefore, there exists a perfectly competitive banking sector that creates and

[17] This is further related to the literature studying general-disequilibrium models. See, for example, Barro and Grossman (1971) as well as Michaillat and Saez (2015, 2022) for more recent studies. These papers also consider the link between aggregate demand and unemployment; however, they conceptually differ as they take a matching approach to the labor and product markets rather than a rationing or disequilibrium approach.

extends loans to firms and households at the beginning of each period.[18] This entails that agents do not need to hold a positive cash balance at the end of the period because they can always borrow the necessary funds at the beginning of the next period.

I assume that banks are constrained with regard to the maximum amount of loans that they can issue, such that the nominal price of the final good, P_t, is constant and equal to 1.[19] This assumption ensures that the economy does not exhibit excess demand. Moreover, throughout the main text, I assume that all agents correctly expect the price level to be constant and equal to 1 for all future periods.[20] As the main focus of the paper is to analyze how the economy behaves once the real ELB is binding, I abstract from a central bank in this model.[21]

I further impose the following (non-standard) assumptions that are important for the derivation of the results in this paper.

Assumption 2.1 *The final good cannot be transferred directly between periods. Households can only transfer income between periods by either investing in the capital stock (through firms) or by holding a positive amount of nominal assets, i.e., money, at the end of the period.*

This implies there exists a *nominal* effective lower bound.

Assumption 2.2 *Market exchanges cannot take place through bartering. All agents require a form of nominal goods, i.e., money, to conduct market exchanges.*

This entails that demand is always nominal.

[18]This assumption can also be used to introduce a more realistic financial sector into the model. Because rather than lending out deposits received from households that save, banks create new deposits, i.e., money, by making loans to firms and households, see, for example, Disyatat (2011), McLeay *et al.* (2014), Brunnermeier and Sannikov (2016), and Jakab and Kumhof (2020).

[19]Hence, I analyze the model at its zero-inflation rate steady state. However, one could assume that, for exogenous reasons, the inflation is positive, and if this were known by all agents, a similar result would emerge, as households would automatically demand higher nominal wages each period to keep the real wage constant. I discuss the effect of expectation regarding the price level in Section B.5.2 in the Appendix.

[20]Section B.8 in the Appendix studies how the economy behaves under perfectly flexible prices.

[21]With a biding real ELB, the central bank and thus monetary policy is constrained, and thus, by construction, the central bank is not able to stabilize the economy using conventional instruments.

Assumption 2.3 *The nominal asset, i.e., money, that is used for transaction purposes and can be used as a store of value, is in potentially infinite supply and can be produced at no costs.*

Assumption 2.4 *Households do not internalize that saving in money will reduce aggregate demand, which in turn will reduce their income.*[22]

This implies that households expect zero involuntary unemployment, as the only source that leads to unemployment in the model is insufficient demand from households.

Assumption 2.5 *The nominal wage rate is downward rigid.*

For empirical evidence regarding the downward rigidity of nominal wages, see, for example, Kahn (1997), Goette *et al.* (2007), Barattieri *et al.* (2014), Schmitt-Grohé and Uribe (2013, 2016), and Fallick *et al.* (2016).

Assumption 2.6 *The nominal price of the final good can only fall if the nominal prices of the intermediate goods fall.*

The structure of the model, i.e., the makeup of monopolistic competition, entails that the nominal price of the final good P_t is indeterminate, as the model has one more endogenous variable than equilibrium conditions.[23] The model can be closed by adding an additional equilibrium condition that relates the nominal price of the final good to nominal demand and real output. The purpose of Assumption 2.6 is to ensure that a reduction in nominal demand or nominal spending does not mechanically lead to a lower nominal price of the final good. But instead, monopolistically competitive firms need to find it optimal to lower their nominal prices in response to a fall in nominal demand, and only then can the nominal price of the final good decrease, i.e., firms will only lower their prices if they first experience a fall in their costs, ceteris paribus.[24]

[22]Alternatively, I could assume that there exists no device with which households can commit to a lower savings rate. See Section 2.5.

[23]See, for example, Hagedorn (2016) or Castillo-Martinez and Reis (2019).

[24]See also Section B.8 in the Appendix.

Assumption 2.5 allows the model to feature involuntary unemployment. The other assumptions entail that the economy can experience situations in which demand is lower than supply. Therefore, combining them allows for the possibility of a demand-induced recession.

In the following, I will briefly summarize the structure of the model. A more detailed exposition will follow in the ensuing sections.

– All agents (households and firms) face a form of cash-in-advance constraint. However, due to the banking sector, agents are not required to hold a positive amount of money between periods.

– Young households make a consumption-saving choice and, in addition, decide how to allocate their savings between the two available assets. One asset guarantees a real return and the other only a nominal return, and hence, the real return of the latter asset is affected by inflation or deflation. Therefore, optimal consumption, savings, and the allotment of savings will be given as functions of the wage rate and the respective rates of return of the two assets. Old households will simply consume all their income.

– The economy features a real ELB, and hence the real return to capital or real interest rate—I will use the two terms synonymously—cannot fall below a certain threshold in equilibrium. This entails that the capital stock used for production cannot exceed $\bar{K}_{t+1}$, i.e., the maximum capital stock the economy can sustain if the economy is below the ELB in period $t + 1$.[25]

– Firms decide on their optimal prices, labor demand, and investment for the next period. They possess perfect foresight and thus will readily take into account any shocks that materialize in the next period and that have an effect on the investment decision, e.g., shocks to technology or labor supply. Hence, optimal investments will be given as a function of the interest rate as well as the aforementioned shocks.

[25]See Section B.3 in the Appendix.

- Banks lend households and firms money at the beginning of each period so that they can finance their consumption and investment expenditures.[26] At the end of the period, the banks sell the credits extended to firms to young households in the form of bonds, such that young households will own the capital stock of the next period. Thus, they indirectly match savings and investments. All money borrowed by households and firms will constitute *ex-ante* aggregate demand. If household savings exceed $\bar{K}_{t+1}$ banks will not be able to lend out all the funds that are technically available, as households will never buy bonds with money that yield a lower real return than money.

- The final good-producing firms will take all prices as given and produce the amount of final good that is demanded, assuming demand is equal to or less than the production possibility frontier.
  The intermediate goods firms observe the demand for their product, i.e., *ex-ante* aggregate demand, and choose their optimal price as well as the optimal input quantities, taking wages and interest rates as given. The reason I use imperfect competition is that it allows me to study how demand affects the production decisions of firms.

- The endogenous variables are determined such that all markets clear either through adjustments in prices and/or rationing.

Note that this all takes place in the same period, i.e., all endogenous variables are determined simultaneously, and thus timing does not matter. Therefore, the terms *ex-ante* and *ex-post* only serve tractability purposes but do not have a time dimension.
The following diagram illustrates the economic mechanism for the case of a binding and a non-binding real ELB.

[26]The concept of banks in the model is explained in more detail in Section B.18 in the Appendix.

No binding real ELB in period $t+1$

Negative supply shock in period $t+1$ $\rightarrow$ reduces investment demand in period t $\rightarrow$ reduces the real interest rate in period $t+1$ $\rightarrow$ ensures $savings = investments$ in period t $\rightarrow$ demand and supply in period t are unaffected.

Binding real ELB in $t+1$

Negative supply shock in period $t+1$ $\rightarrow$ reduces investment demand in period t $\rightarrow$ reduces demand in period t $\rightarrow$ $supply > demand$ in period t $\rightarrow$ with nominal rigidities: demand-induced recession in period t $\rightarrow$ supply in period t falls $\rightarrow$ $supply = demand$ in period t.

2.2.2 Households

Households live for two periods. In each period, a discrete number of identical households indexed by j with $j \in \{1, 2, ..., N_t^y\}$ is born.[27] In the first period, households supply labor exogenously, which is normalized to 1, and consume and save for the second period of their lives, in which they consume all their remaining assets. At the *end* of each period, households receive a nominal income from either supplying labor or in the form of a return on savings and dividend payments.[28] Due to the cash-in-advance constraint, agents need to pay for their consumption *before* they receive their income at the end of the period. Therefore, they will borrow the funds necessary to finance their consumption in the current period from the banks at the beginning of the period. Old households will exit the economy at the end of the period and will thus borrow against their entire income. Young households, on the other hand, will need to save for when they are retired, i.e., the second period of life, and thus will only borrow against a fraction of their income. At the end of the period, households will use their income to pay back their loans, and in the case of young households, they will save the remaining fraction of their income by buying the capital stock of the next period from banks.

[27]Section B.15 in the Appendix studies the implications of household heterogeneity.

[28]Given that the nominal price level will be fixed, nominal income will be equal to real income

Therefore, at the beginning of each period, young households decide on their consumption and also implicitly determine their savings. Households can either save in real assets, i.e., by investing in the capital stock of firms, or in nominal assets, i.e., money. The *nominal* gross return to capital in period $t+1$ is denoted as R_{t+1} and the *real* gross return by r_{t+1}. The return on capital is determined in equilibrium. The nominal asset yields an exogenous and time-constant *nominal* gross return R_f, with $R_f \geq 1$, i.e., R_f constitutes the *nominal* effective lower bound (ELB) in each period.

I assume there is no risk, and thus the portfolio decision of households only depends on the respective real returns of the two assets.

In each period, the overall population consists of $N_t = N_t^y + N_t^o$ households, where N_t^y denotes the number of young households and N_t^o the number of old households.

The optimization problem in nominal terms of household j born in period t is given as[29]

$$\begin{aligned} \max_{c_{1,t}^j, c_{2,t+1}^j, s_t^j, B_t^j} \; & u(c_{1,t}^j) + \beta u(c_{2,t+1}^j) \\ \text{s.t. } P_t c_t^j &= (1-\eta_t) W_t - P_t s_t^j - B_t^j \\ P_{t+1} c_{2,t+1}^j &= s_t^j R_{t+1} + P_{t+1} d_{t+1}^j + B_t^j R_f \\ B_t^j &\geq 0. \end{aligned} \tag{2.1}$$

Where $\eta_t \in [0,1)$ determines the amount of involuntary unemployment and thus the amount of rationing of working hours each household potentially experiences. Therefore, if $\eta_t > 0$ households will not be able to supply all the hours they would like to, and thus their income will decline. However, due to Assumption 2.4 households expect η to be equal to zero.[30] W_t is the nominal wage rate, w_t is the real wage rate, s_t^j are savings invested in real assets in real terms, $P_t s_t^j$ are savings invested in real assets in nominal terms in period t, B_t^j are the savings invested in nominal assets, $b_t^j = \frac{B_t^j}{P_t}$ is the real value of savings invested in nominal assets in period t, $\mathcal{S}_t^j = s_t^j + b_t^j$ are overall savings in real terms, and d_{t+1}^j are the real dividend

[29]See Section B.4 in the Appendix for a more extensive exposition of the household problem.

[30]See also Section 2.5.

payments from firms paid to household j.[31]

Regarding preferences, I make the standard assumptions, i.e., $u(\cdot)$ is strictly increasing and strictly concave.

Therefore, households face the standard consumption-savings choice, except that they now have two assets in which they can potentially save.

At the end of the period, young households will acquire real assets by buying bonds that constitute the loans banks extended to firms, such that the firms could finance the investments in the capital stock for the next period. Therefore, banks match savings and investment, as they know how much young households intend to save; they know the income young households will receive at the end of the period and what fraction of it they borrowed for consumption purposes. In addition, banks are aware of the fact that there is a real ELB, which entails that households will only buy real assets, i.e., loans to firms, that yield a real return that is equal to or higher than the real return of the nominal asset, i.e., money.

More formally, the optimal overall savings $\mathcal{S}_t^j$ of household j are determined by a standard Euler equation

$$u'(c_{1,t}^j) = \beta r_{t+1} u'(c_{2,t+1}^j). \tag{2.2}$$

Moreover, from the first-order conditions of the household optimization problem, I can derive the necessary condition for B_t^j to be positive, i.e., for it to be optimal for young households to save in the nominal asset.

$$r_{t+1} = \frac{P_t}{P_{t+1}} R_f = \frac{1}{\Pi_{t+1}} R_f = \mathcal{R}_{t+1},$$

[31] I assume only old households receive dividend payments to keep the model as simple as possible. The qualitative implications of the model would remain unchanged if young households would also receive dividend income and dividends constitute a constant fraction of overall output, which is the case in a model of monopolistic competition à la Dixit–Stiglitz.

where $\mathcal{R}_{t+1}$ is the real return on the nominal asset and constitutes the *real* effective lower bound (ELB) in period $t+1$. Therefore, I assume that when making their investment decisions, households only compare the real return of the two assets.

r_{t+1} is the real return on capital in period $t+1$, which is endogenous. Taking into account the equilibrium on the capital market (s_t now denotes the equilibrium aggregate savings in real assets in real terms), I can express the necessary condition for $B_t^j > 0$ as an (endogenous) no-arbitrage condition[32]

$$r_{t+1}(s_t) \geq \frac{1}{\Pi_{t+1}} R_f. \tag{2.3}$$

Thus, as long as the weak inequality holds, all savings will be invested in real assets, i.e., the capital stock. However, as $\frac{\partial r_{t+1}}{\partial s_t} < 0$, higher real savings will decrease the return to capital, and thus at some point households will start saving in the nominal asset. Hence, equation (2.3) constitutes an upper bound on the amount of real savings invested in real assets.

The optimal overall savings in real terms of household j are implicitly determined by

$$u'\left((1-\eta_t)w_t - \mathcal{S}_t^j\right) = \beta r_{t+1} u'\left(r_{t+1}\mathcal{S}_t^j + d_{t+1}^j\right), \tag{2.4}$$

with $r_{t+1} = \mathcal{R}_{t+1}$ if the real ELB is binding and $r_{t+1} > \mathcal{R}_{t+1}$ otherwise. The savings rate is denoted by $\varsigma_t \in (0,1)$, with $\mathcal{S}_t^j = \varsigma_t w_t (1-\eta_t)$.

Throughout, I will assume that

$$\frac{d\varsigma_t}{dr_{t+1}} \geq 0, \quad \frac{d\varsigma_t}{dw_t} = \frac{d\varsigma_t}{d\eta_t} = 0.$$

Hence, the optimal savings rate is independent of or increasing in the interest rate and independent of income, i.e., preferences are homothetic.

[32] The condition is *endogenous* if the price level is flexible and determined within the model. This also shows how the real ELB is affected if there is inflation or deflation.

2.2.3 Production

This section introduces the production sector. Given that I assume full depreciation and either fully flexible or fully rigid wages, and thus prices, the optimization problem faced by firms is static.[33]

Let Y_t denote the maximum or potential level of output that can be produced in period t, i.e., *ex-ante* aggregate supply,

$$\begin{aligned} Y_t &= \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ Y_t &= K_t^{\phi} (A_t L_t)^{1-\phi}, \end{aligned} \tag{2.5}$$

where K_t and L_t denote the aggregate capital stock and employment, respectively. This follows from the production function of the intermediate goods firms and the fact that they are symmetric.[34] Thus, Y_t can be seen as the frictionless benchmark. This is the level of output that prevails if there is no shortfall in demand.

Recall, that households and firms face a cash-in-advance constraint and thus need to borrow funds from banks in order to pay for consumption and investments for the next period, i.e., the machines with which firms intend to produce in the next period need to be bought in the current period. Total consumption expenditures and investments will constitute *ex-ante* nominal aggregate demand, which is denoted as $\mathcal{M}_t$, where

$$\mathcal{M}_t = P_t \left(N_t^y c_{1,t}^j + N_{t-1}^y c_{2,t}^j + \int_0^1 I_t(i) di \right). \tag{2.6}$$

The representative final good producer takes all prices and the demand for the final good,

[33] The intertemporal dimension only arises through the intertemporal problem of the household side but does not alter the static optimization problem of the firms.

[34] The production function of firm i is given as $y_t(i) = K_t(i)^{\phi} (A_t L_t(i))^{1-\phi}$.

i.e., $\mathcal{M}_t$, as given. Therefore, her maximization problem in nominal terms is given as[35]

$$\begin{aligned} \max_{\mathcal{M}_t, \{y_t(i)\}_{i\in[0,1]}} & \mathcal{M}_t - \int_0^1 P_t(i) y_t(i) di \\ \text{s.t. } \mathcal{M}_t &= P_t \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ \mathcal{M}_t &\leq P_t Y_t, \end{aligned} \tag{2.7}$$

with $\varepsilon > 1$. P_t is the nominal price of the final good, i.e., in terms of money. The constraints imply that the final good firms cannot sell more than is demanded because the price of the final good is fixed. Moreover, demand cannot be larger than what the economy can produce, i.e., real demand $\frac{\mathcal{M}_t}{P_t}$ has to be equal to or smaller than the production possibility frontier of the economy Y_t.

As the price level is assumed to be equal to 1 and will not change, I simplify the notation by writing the model in real terms.

The solution to the maximization problem of the final good producers yields the demand for each variety i in nominal terms.

$$\begin{aligned} y_t(i) &= \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \frac{\mathcal{M}_t}{P_t} \\ y_t(i) &= \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \Upsilon_t, \end{aligned} \tag{2.8}$$

where $\Upsilon_t = \frac{\mathcal{M}_t}{P_t}$ denote *ex-ante* real aggregate demand.

Firm i decides how much labor and capital to employ to produce variety i taking into account the demand for variety i.

This leads to the following profit maximization problem of firm i in nominal terms

[35] As I only consider situations where $\mathcal{M}_t \leq P_t Y_t$, the complementary slackness condition will always be satisfied. Alternatively, the problem could also be formulated as an expenditure minimization problem: $\min_{\{y_t(i)\}_{i\in[0,1]}} \int_0^1 P_t(i) y_t(i) di$ s.t. $\mathcal{M}_t = P_t \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$ and defining $P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$ will yield the same demand schedule for each variety i, as the profit maximization problem.

$$\max_{P_t(i),y_t(i),K_t(i),L_t(i)} P_t(i)y_t(i) - R_tK_t(i) - W_tL_t(i)$$

$$\text{s.t. } K_t(i)^{\phi}(A_tL_t(i))^{1-\phi} - y_t(i) = 0 \tag{2.9}$$

$$\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t - y_t(i) = 0.$$

I can insert the constraint to express the profit maximization of firm i in nominal terms as

$$\max_{K_t(i),L_t(i)} P_tK_t(i)^a(A_tL_t(i))^bY_t^{\frac{1}{\varepsilon}} - R_tK_t(i) - W_tL_t(i), \tag{2.10}$$

with $a = \phi\left(\frac{\varepsilon-1}{\varepsilon}\right)$ and $b = (1-\phi)\left(\frac{\varepsilon-1}{\varepsilon}\right)$.

The first-order condition of firm i in nominal terms read

$$R_t = aP_tK_t(i)^{a-1}(A_tL_t(i))^bY_t^{\frac{1}{\varepsilon}}, \tag{2.11}$$

$$W_t = bP_tK_t(i)^aA_t^bL_t(i)^{b-1}Y_t^{\frac{1}{\varepsilon}}. \tag{2.12}$$

The optimal nominal price for good i is given as

$$P_t(i) = \left(\frac{\varepsilon}{\varepsilon-1}\right)\Psi_t, \tag{2.13}$$

where $\frac{\varepsilon}{\varepsilon-1} > 1$ is the mark-up and $\Psi_t = \left(\frac{R_t}{\phi}\right)^{\phi}\left(\frac{W_t}{A_t(1-\phi)}\right)^{1-\phi}$ denotes the nominal marginal costs of producing one additional unit of good i.[36]

Population grows at an exogenous rate n_{t+1}

$$N_{t+1}^y = (1+n_{t+1})N_t^y, \tag{2.14}$$

where N_t^y is the number of young households in period t.

[36]The real marginal costs ψ_t are given as $\psi_t = \frac{\Psi_t}{P_t} = \left(\frac{r_t}{\phi}\right)^{\phi}\left(\frac{w_t}{A_t(1-\phi)}\right)^{1-\phi}$.

The level of labor augmenting technology that is the same for all intermediate good firms grows at an exogenous rate g_{t+1}

$$A_{t+1} = (1 + g_{t+1})A_t, \tag{2.15}$$

where A_t is the level of technology in period t.

2.3 Equilibrium

The law of motion for the capital stock is given as

$$K_{t+1} = \begin{cases} s_t^j(r_{t+1}, w_t, \eta_t)N_t^y & \text{if } \bar{K}_{t+1} \geq \mathcal{S}_t^j(r_{t+1}, w_t, \eta_t)N_t^y, \\ \bar{K}_{t+1} & \text{if } \bar{K}_{t+1} < \mathcal{S}_t^j(\mathcal{R}_{t+1}, w_t, \eta_t)N_t^y. \end{cases} \tag{2.16}$$

s_t^j denotes savings invested in real assets in real terms and $\mathcal{S}_t^j(r_{t+1}, w_t, \eta_t)$ overall savings of households j in real terms, i.e., savings in real and nominal assets. If the real ELB is not binding in the next period, i.e., $\bar{K}_{t+1} \geq \mathcal{S}_t^j(r_{t+1}, w_t, \eta_t)N_t^y$, the law of motion for capital is standard. In cases where the real ELB is binding, the capital stock for the next period is bounded from above and thus independent of the amount households save.

Using the budget constraints of young and old households, *ex-ante* nominal aggregate demand in period t can be expressed as

$$\mathcal{M}_t = P_t \left(\int_0^1 \pi_t(i)di + \tilde{\pi}_t + r_t K_t + w_t L_t - N_t^y(s_t^j + b_t^j) + \int_0^1 I_t(i)di \right), \tag{2.17}$$

where $\int_0^1 \pi_t(i)di$ denotes the real profits of all intermediate goods firms and $\tilde{\pi}_t$ the profits of the final good firms, which in equilibrium will be zero. All these profits will be paid to the current old generation in the form of dividends, i.e., d_t^j.

$B_t^j = P_t b_t^j$ denote *ex-ante* nominal savings invested in the nominal asset, i.e., money. B_t^j and hence b_t^j will only be positive if the real ELB is binding. In this case, *ex-ante* savings today will be larger than investment demand for the next period. This means that banks will not lend out all available money, as firms would require a further decrease in the interest rate to increase their investments.

The model features two potential equilibria: a rationing equilibrium and a full employment equilibrium. In the following, I will focus on the rationing equilibrium, i.e., the equilibrium in which nominal wages and the nominal price of the final good are downward rigid, and relegate the discussion of the flexible price equilibrium to the Appendix.[37] I assume throughout that the interest rate in period t is perfectly (downward) flexible.[38]

As the nominal price of the final good will remain constant, and to simplify notation, I will for the remainder of this section write all variables in real terms.

Definition 2.1 *A competitive equilibrium with perfectly downward rigid nominal wages is a sequence of aggregate capital stocks, household consumption, factor prices, and unemployment rates* $\{K_t, c_{1,t}, c_{2,t}, r_t, w_t, \eta_t\}_{t=0}^{\infty}$ *that satisfy equations* (2.2), (2.4), (2.16), (2.20), (2.21), (2.28), (2.29), (2.30).

An equilibrium is characterized such that all firms maximize their profits, which gives rise to the optimal capital and labor demand. And that the markets for capital, labor, and the final good clear.[39]

Making use of the fact that all firms are symmetric, i.e., A_t is the same across all firms, and equation (2.17), the aggregate first-order conditions can be expressed as

$$r_t = aK_t^{a-1}(A_t L_t(\eta_t))^b \left(K_t^{\phi}(A_t L_t(\eta_t))^{1-\phi} - N_t^y b_t^j\right)^{\frac{1}{\varepsilon}}, \tag{2.18}$$

$$w_t = bK_t^a A_t^b \left(L_t(\eta_t)\right)^{b-1} \left(K_t^{\phi}(A_t L_t(\eta_t))^{1-\phi} - N_t^y b_t^j\right)^{\frac{1}{\varepsilon}}, \tag{2.19}$$

[37]The full employment equilibrium is discussed in Section B.8.2 in the Appendix.

[38]See Section B.12 in the Appendix for a situation in which wages and interest rates are perfectly rigid.

[39]In cases where all prices, wages, and interest rates are flexible, the nominal price of the final good P_t will also be an endogenous variable; see Sections B.8.1 and B.8.2 in the Appendix.

where K_t and L_t denote overall capital and labor demand in period t and $a = \phi\left(\frac{\varepsilon-1}{\varepsilon}\right)$ and $b = (1-\phi)\left(\frac{\varepsilon-1}{\varepsilon}\right)$.

2.3.1 Capital Market Equilibrium

The capital market equilibrium for period t determines the rate of return on capital in period t. As I assume the ELB is never binding in period t the interest rate r_t will ensure that demand is equal to supply.[40]

$$N^y_{t-1} s^j_{t-1}(r_t) = K_t(r_t, w_t), \tag{2.20}$$

2.3.2 Labor Market Equilibrium

A (rationing) equilibrium in the labor market must satisfy the following conditions,

$$\eta_t \geq 0, \quad w_t \geq \bar{w},$$

with at least one condition holding with equality. $\eta_t \in [0, 1)$ denotes the amount of involuntary unemployment, i.e., with $\eta_t = 0$ there is no involuntary unemployment, and $\bar{w}$ is the exogenous wage rate below which the equilibrium wage rate cannot fall due to the downward wage rigidity.

This gives rise to the following labor market equilibrium condition

$$N^y_t \cdot (1 - \eta_t) = L_t(r_t, w_t). \tag{2.21}$$

Therefore, if the wage rate is too high for the labor market to clear, there will be involuntary unemployment, i.e., $\eta_t > 0$. Moreover, this implies $\frac{\partial L_t(\eta_t)}{\partial \eta_t} < 0$, where $L_t(\eta_t)$ now denotes equilibrium employment.

[40]See also Section B.7.1 for a more extensive exposition.

2.3.3 Final Good Market Equilibrium

Using the capital and labor market equilibrium *ex-ante* nominal aggregate demand $\mathcal{M}_t$ can be expressed as

$$\mathcal{M}_t = P_t K_t^{\phi} (A_t L_t)^{1-\phi} - N_t^y P_t b_t^j, \tag{2.22}$$

and *ex-ante* real aggregate demand Y_t as

$$\mathrm{Y}_t = K_t^{\phi} (A_t L_t)^{1-\phi} - N_t^y b_t^j. \tag{2.23}$$

Therefore, *ex-ante* real aggregate demand can be equal to or lower than aggregate supply

$$\begin{aligned} Y_t &\geq \mathrm{Y}_t \\ K_t^{\phi} (A_t L_t)^{1-\phi} &\geq K_t^{\phi} (A_t L_t)^{1-\phi} - \mathbb{1} N_t^y b_t^j, \end{aligned} \tag{2.24}$$

with

$$N_t^y b_t^j = \mathcal{S}_t(r_{t+1}, w_t, \eta_t) - \bar{K}_{t+1},$$

where $\mathcal{S}_t(r_{t+1}, w_t, \eta_t)$ denotes aggregate savings in real terms and $\mathbb{1}$ denotes the indicator function with

$$\mathbb{1} := \begin{cases} 1 & \text{if } \bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) < 0, \\ 0 & \text{if } \bar{K}_{t+1} - \mathcal{S}_t(r_{t+1}, w_t, \eta_t) \geq 0, \end{cases}$$

and thus

$$K_t^{\phi} (A_t L_t)^{1-\phi} \geq K_t^{\phi} (A_t L_t)^{1-\phi} + \mathbb{1}(\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t)). \tag{2.25}$$

When the real ELB is binding in period $t+1$, it follows that $r_{t+1} = \mathcal{R}_{t+1}$ with $\mathcal{R}_{t+1}$ being exogenous.

In cases where $\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) \geq 0$, then b_t^j will *ex-ante* be zero, as in this situation the real interest rate will simply rise to equalize demand and supply of capital.[41]

For the economy to be in equilibrium *ex-post* it is required that[42]

$$\begin{aligned} Y_t^* &= \Upsilon_t^* \\ K_t^{\phi}(A_t L_t(\eta_t^*)^{1-\phi} &= K_t^{\phi}(A_t L_t(\eta_t^*)^{1-\phi} + \mathbb{1}(\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t^*, \eta_t^*)). \end{aligned} \tag{2.26}$$

Therefore, in order for the final good market to be in equilibrium, it must hold that

$$\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t^*, \eta_t^*) \geq 0, \tag{2.27}$$

where $\mathcal{S}_t(\mathcal{R}_{t+1}, w_t^*, \eta_t^*) = \varsigma_t w_t^* \cdot (1 - \eta_t^*) N_t^y$ denotes *ex-post* aggregate savings in real terms in period t.[43] This implies that in equilibrium, $B_t^* = 0$, i.e., the final good market only clears if the equilibrium nominal savings are equal to zero.[44]

With downward rigid wages, the equilibrium can be characterized by two or three conditions depending on whether the real ELB is binding or not and thus has either two or three endogenous variables, i.e., the interest rate, wage rate, and unemployment rate. The interest rate and wage rate will always have to be determined in equilibrium, i.e., the standard capital and labor market clearing conditions are always present. Whether the unemployment rate is determined in equilibrium depends on whether the real ELB is

[41]See also Section B.3 in the Appendix.

[42]See also Section B.7.2 in the Appendix.

[43]This implies $\frac{\partial \mathcal{S}_t(\mathcal{R}_{t+1}, w_t^*, \eta_t^*)}{\partial \eta_t^*} < 0$.

[44]I could also assume that the economy features a full employment equilibrium with a positive amount of equilibrium nominal savings ($B^* > 0$). See Section B.16 in the Appendix for details.

binding or not. In situations where the real ELB is *not* binding, the final good market will automatically clear. However, if the real ELB *is* binding, then there will be an additional equilibrium condition that ensures the final good market clears. Therefore, this condition takes the form of a complementary slackness condition. The reason this condition can be slack is that the unemployment rate η_t cannot fall below zero.
An equilibrium in the final good market requires that

$$\begin{aligned} \bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) &\geq 0 \\ \bar{K}_{t+1} - \varsigma_t w_t N_t^y \cdot (1-\eta_t) &\geq 0 \\ \bar{K}_{t+1} - \varsigma_t w_t L_t(\eta_t) &\geq 0, \end{aligned}$$

where the last line made use of the labor market clearing condition.

The equilibrium wage rate, interest rate, and unemployment rate are given by the solution to the following system of equations, where $K_t(r_t)$ and $L_t(\eta_t)$ denote the equilibrium capital stock and employment, respectively.

$$r_t - aK_t^{a-1}(A_t L_t(\eta_t))^b \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi} - (\mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} = 0, \tag{2.28}$$

$$w_t - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi} - (\mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} = 0, \tag{2.29}$$

$$\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t) \geq 0. \tag{2.30}$$

This implies that if *ex-ante* $\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t = 0) \geq 0$, the economy does not suffer from a demand shortage and can produce at full employment, and thus η_t is not an endogenous variable, i.e., the equilibrium unemployment rate cannot fall below zero, which entails that the third equation will be slack.

Assume now that the *ex-ante* equilibrium is such that $\bar{K}_{t+1} - \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t = 0) = 0$, i.e., with full employment ($1 - \eta_t^* = 1$) the economy is exactly at the kink (see Figure B.3). In this case, a fall in $\bar{K}_{t+1}$ will imply that the complementary slackness equilibrium condition becomes binding, and thus η_t becomes an endogenous variable, i.e., the unemployment rate is now positive and must be determined in equilibrium. When η_t has to be determined in equilibrium, i.e., will be larger than zero, this implies the economy has a binding real ELB, i.e., r_{t+1} is constrained by the ELB and thus no longer constitutes an endogenous variable (see also Table B.1).

2.4 Results

Lemma 2.1 *With downward rigid nominal wages, a nominal demand shortage, i.e., a fall in $\bar{K}_{t+1}$, will leave $P_t(i)$ unchanged.*

Proof. See Appendix B.6. □

Lemma 2.1 entails that the intermediate goods firms will not find it optimal to lower their nominal prices in response to a fall in nominal demand if the nominal wage rate cannot fall. Moreover, with Assumption 2.6 this implies that the nominal price of the final good P_t cannot fall, and thus the nominal ELB becomes a real ELB.

2.4.1 Intratemporal Effects

The model outlined above yields the following results when I only consider one period and view $\bar{K}_{t+1}$ as an exogenous variable. Therefore, in this section, I treat the adverse demand shock as exogenous.

Proposition 2.1 *With fully flexible wages, interest rates, and prices, a negative demand shock in period t will only have nominal effects but not real effects on the economy, i.e., classical dichotomy holds. Nominal wages and interest rates in period t will fall, leading to deflation in period t, which will be followed by inflation in period $t+1$. However, real wages, interest rates, and output in period t will remain constant.*

Proof. See Appendix B.8.2. □

The intuition is as follows: the demand shortage, which arises due to the fall in $\bar{K}_{t+1}$, needs to be endogenously eliminated, which requires a reduction in excess savings. The reduction in excess savings can either take place through an adjustment in *nominal* or *real* variables.

A change in the price level, i.e., deflation followed by inflation, decreases the real ELB in period $t+1$, which makes savings in the nominal asset less attractive and thus eliminates the excess savings. Therefore, the demand shortage can be eliminated through a change in nominal variables only.[45] Deflation can potentially trigger a deflationary spiral, which entails that the deflation is pushed to an earlier period, which raises the real effective lower bound in an earlier period, which then entails that the economy suffers from a demand deficiency in an earlier period. In Section B.5.1 I characterize the condition that ensures that the deflation and inflation necessary to eliminate the demand shortage do not lead to a deflationary spiral. Specifically, there needs to exist a period in the past in which the equilibrium real interest rate is higher than the real return on the nominal asset once deflation is taken into account. In this specific period, the deflation can then be absorbed without having any real effects on the economy.

Hence, with no nominal rigidities, the model features classical dichotomy, and thus an adverse nominal demand shock has no real effects on the economy.

In cases where nominal rigidities exist, e.g., when the nominal wage rate is downward rigid, an adverse nominal demand shock can have real effects on the economy. The following propositions summarize these effects.

Proposition 2.2 *With downward rigid wages and flexible interest rates in period t and a binding real effective lower bound in period $t+1$, a negative demand shock in period t will lead to lower interest rates, higher wages, higher unemployment, and lower output in period t. The price level in period t will remain constant.*

Proof. See Appendix B.10. □

[45] See also Section B.5 in the Appendix.

In cases where nominal wages are downward rigid, which prevents the price level from falling, excess savings need to be eliminated through a change in real variables. An increase in involuntary unemployment reduces the income of young households and thus excess savings in period t. Moreover, as I assume the interest rate in period t to be fully flexible, there will be a decrease in the interest rate. Involuntary unemployment and fully flexible interest rates in period t entail that the capital stock per worker increases. Given the production function, this leads to a higher marginal product of labor and thus a higher equilibrium wage rate, as well as a lower marginal product of capital and correspondingly a lower interest rate in period t.[46]

Proposition 2.3 *With downward rigid wages and interest rates in period t and a binding real effective lower bound in period $t+1$, a negative demand shock in period t will lead to higher unemployment, an underutilization of capital, lower output, and constant wages as well as interest rates in period t. The price level in period t will remain constant.*

Proof. See Appendix B.12. □

If the interest rate cannot fall in period t, the capital stock per worker will remain constant. As the unemployment rate increases, not all the capital that is technically available will be used in the production process.[47] This implies that the marginal product of labor as well as the wage rate will remain constant. Capital that remains idle will generate a return of zero, i.e., investors will receive exactly the same amount they invested, which, given a constant price level, is equal to the real return of saving in terms of money.

The current young generation's attempt to save more than the economy can absorb has not only negative consequences for them, i.e., involuntary unemployment and a lower income, but also adverse ramifications for the current old generation. The demand-induced recession

[46]See Section B.14 in the Appendix for an extension to a more general CES production function.

[47]If savings supply is elastic and households in period $t-1$ know that there will be a negative demand shock in period t and that the interest rate in period t is downwardly rigid, which will entail that not all of their savings will generate a (positive) return. Then households in period $t-1$ could, in response, lower their savings by enough such that there is no demand shortage in period t, i.e., $\mathcal{S}_t\left(w_t(\mathcal{S}_{t-1},\cdot),\cdot\right)$. However, I assume here that the potential decline in $\mathcal{S}_{t-1}$ in response to a fall in $\bar{K}_{t+1}$ is not strong enough to lower $\mathcal{S}_t$ such that the demand shortage in period t does not arise.

will also lower their available income by reducing the return on their savings, even though they have not actively contributed to the demand shortage, i.e., their marginal propensity to consume is 1.[48]

2.4.2 Intertemporal Effects

So far, I have treated $\bar{K}_{t+1}$ as an exogenous variable. However, once I consider two consecutive periods, $\bar{K}_{t+1}$ becomes an endogenous variable, i.e., the demand shortfall in period t will be endogenous. A binding *lower* bound on the equilibrium real interest rate implies a binding *upper* bound on the equilibrium capital stock, as the real interest rate constitutes the price of capital. This leads to the following lemma

Lemma 2.2 *A reduction in the population growth rate in period $t+1$ n_{t+1} or the growth rate of technology g_{t+1} will lower the effective upper bound on the capital stock denoted by $\bar{K}_{t+1}$.*

Proof. See Appendix B.3. □

How much capital the economy can absorb, i.e., the size of the capital stock, for a given interest rate depends positively on the population size as well as the level of technology. As labor and technology complement capital, they both increase the marginal product of capital. Conversely, in case they experience a reduction, this will decrease the marginal product of capital, and if the interest rate and hence the marginal product of capital cannot fall due to the binding real ELB, the maximum capital stock the economy can sustain, i.e., the effective upper bound, falls.

The effect of a decrease in the population growth rate or the rate of technological progress in period $t+1$ on the economy in period t is summarized in the following propositions

Proposition 2.4 *With downward rigid wages and flexible interest rates in period t and a binding real effective lower bound in period $t+1$, a decrease in the population growth rate or the rate of technological progress in period $t+1$ will lead to lower interest rates, higher wages, higher unemployment, and lower output in period t. The price level in period t will remain constant.*

[48]See Section B.15 in the Appendix for a more detailed discussion.

Proof. Follows from Proposition 2.2 and Lemma 2.2. □

Proposition 2.5 *With downward rigid wages and interest rates in period t and a binding real effective lower bound in period $t+1$, a decrease in the population growth rate or the rate of technological progress in period $t+1$ will lead to higher unemployment, an underutilization of capital, lower output, and constant wages as well as interest rates in period t. The price level in period t will remain constant.*

Proof. Follows from Proposition 2.3 and Lemma 2.2. □

Demographic change acts as a negative supply shock in period $t+1$ by reducing the number of workers in this period. This further reduces the maximum capital stock the economy can absorb in this period and, hence, also reduces the amount of savings the economy can sustain without suffering from too little demand. Moreover, if demographic change induces a fall in the real interest rate such that the real ELB becomes binding, the negative supply shock described above can turn into an *intertemporal* "Keynesian supply shock", such that output in period t falls below potential due to a demand-induced recession brought about by excess savings, i.e., an endogenous savings glut. Therefore, if demographic change occurs simultaneously with a binding real ELB, which itself can be caused by demographic change, demographic change can cause a demand-induced recession.

2.5 Allocative Efficiency

The emergence of an endogenous savings glut is central to the demand-induced recession. Young households attempting to save more than the economy can absorb leads to a disequilibrium on the final good market, which ultimately entails involuntary unemployment and a lower income for young households.

This raises two questions: first, is it possible for young households to reach an equilibrium that constitutes a Pareto improvement relative to the baseline case? And second, under which conditions can such an equilibrium be reached?

To answer the first question, recall that involuntary unemployment only arises because, for given parameter values, young households attempt to save more than the economy can absorb.

Ex-ante aggregate savings are determined by the *ex-ante* Euler equation

$$u'(\mathcal{S}_{I,t}(w_t,\eta_t=0)) = \beta r_{t+1} u'(\mathcal{S}_{I,t}(w_t,\eta_t=0)),$$

where the subscript, i.e., the Roman numeral, serves to distinguish different cases. However, as $\mathcal{S}_{I,t}(w_t,\eta_t=0) > \bar{K}_{t+1}$ this will not constitute an equilibrium. The resulting demand-induced recession will lead to a new level of aggregate savings that satisfies the *ex-post* Euler equation

$$u'(\mathcal{S}_{II,t}(w_t^*,\eta_t^*>0)) = \beta r_{t+1} u'(\mathcal{S}_{II,t}t(w_t^*,\eta_t^*>0)), \tag{2.31}$$

where $\mathcal{S}_{II,t}(w_t^*,\eta_t^*>0) = \bar{\mathcal{S}}_t = \bar{K}_{t+1}$ and thus $\mathcal{S}_{I,t}(w_t,\eta_t=0) > \mathcal{S}_{II,t}t(w_t^*,\eta_t^*>0)$. Assuming that overall savings are a fraction of overall income and that the marginal propensity to save is less than 1, the reduction in savings requires a fall in income and thus consumption.[49]

Consider a situation in which young households understand that saving more than the economy can absorb will lead to involuntary unemployment and a lower income. Thus, households *ex-ante* choose their savings $\mathcal{S}_{III,t}(w_t,\eta_t=0)$ such that $\mathcal{S}_{III,t}(w_t,\eta_t=0) = \mathcal{S}_{II,t}(w_t^*,\eta_t^*>0) = \bar{K}_{t+1}$, i.e., they save the maximum amount that the economy can absorb. This entails

$$u'(\mathcal{S}_{III,t}(w_t,\eta_t=0)) > \beta r_{t+1} u'(\mathcal{S}_{III,t}(w_t,\eta_t=0)), \tag{2.32}$$

and thus young households diverge from their *ex-ante* Euler equation.

As $\mathcal{S}_{III,t}(w_t,\eta_t=0) = \mathcal{S}_{II,t}(w_t^*,\eta_t^*>0)$, the level of consumption and utility in the second period of life, i.e., c_{t+1}, will be the same independent of whether young households reduce

[49] In cases where the savings rate is increasing with income, a fall in income due to involuntary unemployment would further reduce savings through a reduction in the savings rate. Thus, for a given drop in income, savings would fall more due to the indirect effect of income on the savings rate.

their savings themselves or whether they are reduced through involuntary unemployment.[50] This leads to the following lemma

Lemma 2.3 *A binding real ELB in period $t+1$ implies that the level of consumption that households born in period t can attain in period $t+1$ is bounded from above.*

Similarly, if I considered a benevolent social planner who, by definition, unlike young households, internalizes that aggregate savings are constrained by an effective upper bound. In cases where optimal savings of young households lead to a binding ELB, she will directly set $\mathcal{S}_t^* = \bar{K}_{t+1}$, i.e., save the maximum amount that does not cause involuntary unemployment.

Consumption in the first period of life is given as

$$c_{II,t} = \mathcal{I}_{II,t}(w_t^*, \eta_t^* > 0) - \mathcal{S}_{II,t}(w_t^*, \eta_t^* > 0), \tag{2.33}$$

$$c_{III,t} = \mathcal{I}_{III,t}(w_t, \eta_t = 0) - \mathcal{S}_{III,t}(w_t, \eta_t = 0), \tag{2.34}$$

where $\mathcal{I}_{II,t}(w_t^*, \eta_t^* > 0)$ denotes overall income in case II and $\mathcal{I}_{III,t}(w_t, \eta_t = 0)$ overall income in case III.
Whether $c_{III,t} \gtreqless c_{II,t}$ depends on whether $\mathcal{I}_{III,t}(w_t, \eta_t = 0) \gtreqless \mathcal{I}_{II,t}(w_t^*, \eta_t^* > 0)$. Lemma B.1 entails that income falls as unemployment increases and thus $\mathcal{I}_{III,t}(w_t, \eta_t = 0) > \mathcal{I}_{II,t}(w_t^*, \eta_t^* > 0)$, which implies $c_{III,t} > c_{II,t}$. Therefore, if young households voluntarily choose a lower level of savings than implied by their *ex-ante* Euler equation, they can reach a higher level of consumption and thus utility in the first period of life. Recall that the *ex-ante* Euler equation does not take into account that excess savings can lead to involuntary unemployment and thus a lower lifetime income. However, if young households internalize this, similar to a benevolent social planner, they will reduce their savings in order to avoid a demand-induced recession and increase their consumption in the first period of life.
Moreover, as an increase in η_t also lowers r_t and thus the consumption level of old house-

[50] A necessary condition for a demand-induced recession to occur in period t is that the economy is below the real ELB in period $t+1$ and hence r_{t+1} is constant.

holds in period t, a voluntary reduction of savings by young households in period t also benefits the old households in period t. These results are summarized in the following proposition

Proposition 2.6 *If the economy experiences a binding real ELB in period $t+1$, a Pareto improvement is possible if young households voluntarily choose a lower level of savings than implied by their ex-ante Euler equation.*

Hence, there exists the possibility of reaching a second equilibrium that Pareto-dominates the one that leads to a demand-induced recession and involuntary unemployment. To analyze whether this equilibrium can be reached without imposing further assumptions, consider the following:

I assume that households can strategically interact and investigate whether committing to a lower savings rate can constitute a Nash equilibrium.

As before, there is a discrete number of young households denoted by $j \in \{1, 2, ..., N_t^y\}$ and parameters are such that overall optimal savings $\mathcal{S}_t$, i.e., based on their *ex-ante* Euler equation, are larger than $\bar{\mathcal{S}}_t = \bar{K}_{t+1}$, i.e., the real ELB is binding in period $t+1$ and thus households would like to save more than the economy can absorb. Hence, should they decide to remain on the savings schedule implied by their *ex-ante* Euler equation, demand will be too low, resulting in a demand-induced recession and involuntary unemployment.

Assume initially that all households agreed to lower their savings rate to eliminate the demand shortfall. To check whether this constitutes a Nash equilibrium, i.e., no household has an incentive to unilaterally increase her savings rate, consider the following:

Consumption in period $t+1$ of household j is given as

$$c_{2,t+1}^j = r_{t+1}\varsigma_t^j \mathcal{I}_t^j, \tag{2.35}$$

ς_t^j denotes the savings rate and $\mathcal{I}_t^j$ the income of household j.

Using this, I derive the following lemma

Lemma 2.4 *In cases where the economy consists of at least two symmetric households and they initially agree on a lower savings rate than "optimal" to prevent a demand-induced recession. If then* *one agent unilaterally increases her savings rate, this will increase her lifetime utility.*

Proof. See Appendix B.13. □

Therefore, if there are at least two households, it becomes optimal to deviate from the lower savings rate because it allows the household to reach a higher level of consumption in period $t+1$. Further, as the increase in the savings rate gets her closer to her *ex-ante* Euler equation, this allows her to reach a higher level of utility. The intuition is that by unilaterally increasing the savings rate, a household gets closer to the "optimal" savings rate, which only benefits her. However, the corresponding costs, i.e., the higher unemployment rate, will have to be borne by all agents.

Hence, the only Nash equilibrium will be the one where all agents remain on their *ex-ante* Euler equation, resulting in involuntary unemployment and a lower level of consumption in period t. Consequently, the second equilibrium that Pareto-dominates the first cannot be reached absent a commitment device or government intervention, as households always have an incentive to deviate. This yields the following proposition

Proposition 2.7 *If the economy consists of at least two households and the parameters are such that optimal aggregate savings are higher than the maximum investments the economy can absorb and there exists no commitment device, the resulting Nash equilibrium will entail ex-ante savings that are too high and result in a demand-induced recession, which entails involuntary unemployment.*

2.6 Government

Consider the case with downward nominal wage rigidity and a constant price level. I assume there exists a government that levies a proportional tax τ_t on labor income and spends all the tax income in the current period.

With a binding real ELB, the economy is characterized by excess savings. In the event that the government does not levy a tax, equilibrium savings have to satisfy

$$\varsigma_t w_t (1-\eta_t) N_t^y - \bar{K}_{t+1} = 0, \tag{2.36}$$

with $\eta_t > 0$.

Assuming the government levies a positive tax rate that exactly eliminates excess savings, the tax rate has to satisfy

$$\varsigma_t w_t^{\tau}(1-\tau_t)N_t^y - \bar{K}_{t+1} = 0, \tag{2.37}$$

where $w_t^{\tau} < w_t$, as with a positive tax rate, there will be no demand-induced recession and thus no involuntary unemployment, which leaves the capital to labor ratio constant and in turn the wage rate will not increase. In addition, this also implies that output remains constant, i.e., at the level it would be if the real ELB were not binding.[51]

Therefore, young households will have the same net-of-tax income in both cases. In case the government does not levy the tax, their income will fall through involuntary unemployment, i.e., $w_t(1-\eta_t) = w_t^{\tau}(1-\tau_t)$. This is true even if government expenditures are completely wasteful. In cases where the government uses the tax funds to provide a public good that increases the utility of (young) households, taxing labor income in a situation of binding real ELB can thus lead to a Pareto improvement. The important condition that needs to be satisfied, however, is that the tax income is spent on acquiring the final good in the current period, i.e., reducing excess private savings is not enough because the government also needs to generate additional demand.

Assume the government decides to borrow the excess savings instead of taxing them. Then, as long as the real ELB is binding, there will always be excess savings that the government can borrow to roll over the debt without crowding out capital in the production sector. The important condition that this strategy is feasible is that the excess savings of the next period are equal to or larger than the excess savings of the current period plus interest, i.e., $N_{t+1}^y B_{t+1}^j \geq N_t^y B_t^j R_f$, as otherwise the government would be forced to borrow more than just the excess savings to roll-over the debt from one period to the next.

However, once the real ELB is no longer binding, continuing to roll over debt by the

[51] I assume labor supply to be exogenous, and thus the tax will not distort labor supply. With elastic labor supply, income taxation would reduce the optimal labor supply, and thus equilibrium output would be lower compared to the case with exogenous labor supply.

government will lead to crowding out as less capital will be available for production purposes. In this case, excess government debt will lead to higher interest rates as the government competes with the production sector for savings, a lower capital stock per worker compared to a situation with no excess government debt, and thus lower wages and lower output. Moreover, this would not constitute a stable equilibrium path, as the government would be forced to borrow ever larger amounts in order to pay the interest on the debt.

2.6.1 Investment Subsidy

Assume the government subsidizes each unit of capital invested in period t by the amount z_t. The subsidy is financed through a tax on labor income τ_t. Per unit of capital firms invest, they now have to pay $\mathcal{R}_{t+1} - z_t$, where $\mathcal{R}_{t+1}$ constitutes the real ELB.[52] If the economy is initially below the real ELB, this implies

$$I_t(\mathcal{R}_{t+1}) < \mathcal{S}_t(\mathcal{R}_{t+1}, w_t, \eta_t = 0).$$

The tax rate τ_t that eliminates the excess savings is then given by

$$\begin{aligned} I_t(\mathcal{R}_{t+1} - z_t) =& \mathcal{S}_t(\mathcal{R}_{t+1}, w_t(1-\tau_t), \eta_t = 0) \\ I_t\left(\mathcal{R}_{t+1} - \frac{w_t \tau_t}{I_t}\right) =& \mathcal{S}_t(\mathcal{R}_{t+1}, w_t(1-\tau_t), \eta_t = 0), \end{aligned} \tag{2.38}$$

where I have used the fact that $w_t \tau_t = z_t I_t$, i.e., the subsidy has to be fully paid for through taxation.

Assuming full depreciation, i.e., $\bar{K}_{t+1} = I_t$, I can express the first-order conditions as

$$F_{\bar{K}_{t+1}} - \left(\mathcal{R}_{t+1} - \frac{w_t \tau_t}{\bar{K}_{t+1}}\right) = 0, \tag{2.39}$$

$$F_{L_{t+1}} - w_{t+1} = 0. \tag{2.40}$$

Applying the implicit function theorem to the first-order condition with respect to capital yields $\frac{\partial \bar{K}_{t+1}}{\partial \tau_t} > 0$.

[52] Recall that if the real ELB is binding, it must hold that $r_{t+1} = \mathcal{R}_{t+1}$.

Therefore, a higher tax rate not only reduces the excess savings but also increases the maximum capital stock the economy can sustain and thus allows (young) households to accumulate and sustain higher savings.

2.7 Conclusion

This paper started by asking whether demographic change can constitute a "Keynesian supply shock." The theoretical analysis that followed documented the necessary and sufficient conditions under which the negative supply shock associated with demographic change can lead to a demand-induced recession. Specifically, if nominal wages are downward rigid, which in the model gives rise to downward rigid nominal prices, and if, in addition, the real ELB becomes binding in the economy, demographic change can indeed compose a "Keynesian supply shock", namely, an *intertemporal* "Keynesian supply shock". The main culprit is excess savings, which cannot be absorbed by the economy due to the binding real ELB. This also implies that, in theory, it would be possible to avoid a demand-induced recession and reach an equilibrium that constitutes a Pareto improvement. However, absent a commitment device, it will not be possible for the economy to evade a demand-induced recession, as such a device would be necessary in order for households to plausibly commit to lower savings. Furthermore, the government can ameliorate the situation and bring about a Pareto improvement by taxing excess savings and using the funds to finance a public good that generates an increase in utility for all households. Moreover, for the government, taxation is superior compared to borrowing the excess savings because it prevents the crowding out of private investments and the risk of excessive government debt once the real ELB is no longer binding. In addition, the demand-induced recession can have negative ramifications for households that themselves did not accumulate excess savings, such as the current old generation, by reducing their income, even though this does not curtail excess savings in the economy. Therefore, excess savings that bring about a demand-induced recession can have adverse effects for all households in the economy, independent of whether they contributed to the excess savings that bring about a demand-induced recession.

Chapter 3

Demographic Change, Wages, and Innovations

3.1 Introduction

The economic recovery after the Great Recession of 2007-2009 has been sluggish; see, for example, Summers (2015). Some researchers argue that this is, among other things, due to a slowdown in TFP growth; see, for example, Gordon (2015), Antolin-Diaz *et al.* (2017), Fernald *et al.* (2017), Aghion *et al.* (2019), and Eo and Morley (2020). Different potential explanations for why TFP growth has decreased have been put forward by researchers; some argue that it is due to ideas getting harder to find; see, for example, Bloom *et al.* (2020). Others attribute it to structural change, i.e., economic activity has been reallocated to sectors in which the potential for productivity growth is limited, such as the service sector; see, for example, Duernecker and Sanchez-Martinez (2021). In addition, another structural factor that potentially affects economic outcomes is becoming more prevalent: demographic change will be one of the important challenges of the 21st century.[1] See, for example, Kotschy and Sunde (2018), Eggertsson *et al.* (2019a), Maestas *et al.* (2023), and

[1]This paper only analyzes how demographic change affects the economy through a reduction in labor supply and a corresponding increase in the capital-to-labor ratio.

Kotschy and Bloom (2023) for empirical studies that analyze the effect of demographic change on the economy.

There are two strands of theoretical literature that studies the effect of demographic change and population aging—I will use the two terms interchangeably—on innovation and technological progress. The two strands make different predictions about whether demographic change affects output (growth) positively or negatively.

One literature encompasses endogenous and semi-endogenous growth models and argues that population size, or the growth rate of the population, positively affects the rate of technological progress. Therefore, as population growth and population size start to decrease, these models predict a negative effect on the rate of technological progress. Hence, these models stress that demographic change, through a negative *supply* channel, reduces the number of researchers in a given period, which has a detrimental effect on output (growth) in the future (see, for example, Romer (1990), Aghion and Howitt (1992), Jones (1995, 2022)).[2]

The other literature argues that demographic change, by reducing the supply of workers, increases the costs of labor and thus makes substituting labor with capital, e.g., robots or other forms of automation, more attractive. Hence, the reduction in overall labor supply can (in part) be mitigated by employing additional capital in the production process. Labor scarcity induced by demographic change and population aging has a positive effect on the incentive to innovate, as it increases the value of an innovation that allows for the substitution of labor through capital in these models. Therefore, this literature stresses that demographic change, through a positive *demand* channel, can increase the rate of innovation by incentivizing the development of new automation technologies. This, in turn, has a positive effect on output (growth) in the future (see, for example, Abeliansky and Prettner (2023), Acemoglu and Restrepo (2017, 2022)).

The aim of this paper is to combine the different channels emphasized in the two strands of the literature into one coherent framework to gain a better theoretical understanding of how

[2]In this paper, I solely focus on the effect of demographic change on output (growth) through influencing the incentive to innovate.

demographic change affects innovations and thus output (growth). I highlight two channels through which demographic change affects the rate of innovation: A *supply* channel, which influences the costs of an innovation, and a *demand* channel, which affects the value of an innovation. The demand channel consists of two effects with opposite signs. The first effect is always negative; whether the second is negative or positive depends on which channel dominates.

In addition, I use the framework to analyze how demographic change affects factor prices, i.e., the return to capital, wages, and income inequality.

I employ a nested CES framework with three production factors: capital, low- and high-skilled labor. In light of the recent discussion regarding automation and demographic change (see, for example, Abeliansky and Prettner (2023), or Acemoglu and Restrepo (2022)). it is important to study demographic change in a framework that differentiates between different skill levels, as (automation) capital likely interacts differently with high- and low-skilled workers (see, for example, Prettner and Strulik (2020)). Throughout the paper, I assume that low-skilled workers and capital are imperfect substitutes and that capital as well as low-skilled labor are complementary to high-skilled workers. In addition, I first assume that the skill share is fixed and then show how demographic change could affect the incentive to innovate through an additional channel if the skill share changes due to demographic change. However, for tractability, I do not explicitly model how the skill share is affected by demographic change, i.e., I do not model the education decision explicitly.

I then endogenize the R&D process by building a simple endogenous growth model to study how demographic change affects the incentives to innovate in terms of increasing the level of factor augmenting technology. In addition, I also analyze how population aging affects the decision of firms to enter a market in a model of expanding varieties, i.e., where technological progress is modeled as increasing the number of varieties.

The first finding of this paper is that the effects of demographic change on factor prices crucially depend on which skill group is affected and, if they are both affected by it, on how the relative size of the two groups changes. Therefore, whether demographic change leads to higher or lower wages and interest rates is at first ambiguous. Moreover, I show that with capital-skill complementarity, demographic change leads to an increase in the skill premium, absent changes in the *relative* supply of skill groups or the level of technology. Therefore, demographic change can increase income inequality by affecting the capital-to-labor ratio.

The second finding is that demographic change can affect the rate of innovation in terms of an increase in the level factor augmenting technology through three channels. First, by reducing the number of (potential) researchers, R&D becomes more expensive, which reduces the rate of innovation. Second, by reducing the number of low-skilled workers, the value of an innovation that increases the level of capital augmenting technology, i.e., automation, increases if capital and low-skilled labor are strong substitutes. This then positively affects the rate of innovation. Third, by reducing the number of high-skilled workers, the value of an innovation that increases the level of capital augmenting technology falls. Therefore, demographic change has a positive and negative effect on the demand for automation technologies. Which effect dominates depends on how the skill composition of the shrinking population changes. The positive effect is more likely to dominate if the demographic change leads to an increase in the share of high-skilled workers. Therefore, the main policy implication of this paper is that governments should attempt to increase the share of high-skilled workers in the population, as this will not only affect the rate of innovation positively through a direct effect, i.e., increasing the number of potential researchers, but also through an indirect effect, i.e., increasing the value of an innovation relative to a situation with no government intervention.

The third finding is that demographic change can also have an *intertemporal* effect. This effect acts through the R&D channel. Demographic change affects the expected return of different innovations and thus alters the incentives to conduct R&D. In general equilibrium, the production and R&D sector compete for the same production factors. Therefore, a change in the (expected) value of an innovation will either lead to an inflow or outflow of resources from the R&D sector. This in turn will influence the resources employed in the production sector and, thus, the return on all factors employed in the production sector.

This entails that the potential positive effect of demographic change on the skill premium can be weakened or strengthened through the intertemporal R&D effect.

Related Literature

This paper relates to the literature that studies factor augmenting technological change. See, for example, Acemoglu (1998, 2002, 2003, 2010). Furthermore, it builds on the literature that studies capital-skill complementarity. See, for example, Stokey (1996), Krusell *et al.* (2000), and Duffy *et al.* (2004). Moreover, it relates to the literature that analyzes how changes in the population structure affect economic growth through influencing the development and adaption of new technologies. See, for example Zeira (1998), Prettner (2013), Hashimoto and Tabata (2016), Peters and Walsh (2021), and Jones (2022). Improvements in the field of robotics have given rise to a growing literature that studies the effects of automation on economies and especially the interplay of automation and demographic change. See, for example, Abeliansky and Prettner (2023), Acemoglu and Restrepo (2018), Prettner and Strulik (2020), Acemoglu and Restrepo (2020), Acemoglu and Restrepo (2022), and Moll *et al.* (2022).

The rest of the paper proceeds as follows. In Section 3.2 I will present some motivating correlations that we observe in the data. Section 3.3, introduces the production sector and derives the first set of results. Section 3.4 outlines the R&D sector in a partial equilibrium framework to study the main mechanisms in the R&D sector. In Section 3.5 I combine the two models discussed before to show how demographic change affects the value of an innovation and how this can give rise to an *intertemporal* effect. Section 3.6 concludes.

3.2 Motivation

Demographic change entails that labor becomes scarcer, and simple economic reasoning would imply that the price of that factor, i.e., the wage rate, should increase.

To explore the relationship between demographic change and the wage rate, I follow the empirical macro growth literature (see, for example, Krueger and Lindahl (2001) for an overview) and estimate the following model

$$\ln(w_{i,t}) = \alpha d_{i,t-1} + \beta \ln(y_{i,t-1}) + \gamma h_{i,t-1} + \delta(d_{i,t-1} \cdot h_{i,t-1}) + \zeta_i + \eta_t + \varepsilon_{i,t}, \quad (3.1)$$

$w_{i,t}$ denotes the wage rate at current PPPs in country i in period t. $d_{i,t-1}$ is the old-age dependency ratio $\left(\frac{65+}{15-64} \cdot 100\right)$, i.e., population aging. I control for real income per capita ($y_{i,t-1}$) and for human capital ($h_{i,t-1}$). As demographic change can potentially affect the wage rate differently depending on the skill level, I interact the old-age dependency ratio with human capital. To address the potential problem of reverse causality, I use lagged values of the independent variables. Moreover, to account for time-invariant cross-country heterogeneity and global trends, I include a full set of country (ζ_i) and time (η_t) fixed effects. The error term $\varepsilon_{i,t}$ captures all additional omitted variables.

The empirical analysis uses 1- and 4-year panel data. The 4-year panel is created by taking 4-year averages to reduce the problem of measurement errors and business cycle effects. Real income per capita and the human capital index are taken from the Penn World Table version 10.0 (Feenstra *et al.* (2015). The data for the old-age dependency ratio is taken from the World Bank and the data for the wage rate is from the OECD.

Table 3.1 presents the results for estimating equation (3.1).[3] For the larger sample that includes a smaller observation period, i.e., columns (1) and (3), I find a negative and statistically significant effect of demographic change on log wages. This seems puzzling, as demographic change would be expected to make labor scarcer and thus, ceteris paribus, increase its price, i.e., the wage rate. The interaction term between population aging and human capital is positive and statistically significant, suggesting that the effect of demographic change on the wage rate in a country becomes less negative with a higher level of human capital. For the smaller sample that covers a longer time period, i.e., columns (2) and (4), the sign of the effects is the same, however, they are not statistically significant.

[3]Columns (1) and (3) contain AUS, AUT, BEL, CAN, CZE, DNK, EST, FIN, FRA, DEU, GRC, HUN, ISL, IRL, ISR, ITA, JPN, LTU, LUX, MEX, NLD, NZL, NOR, POL, PRT,KOR, SVK, SVN, ESP, SWE, CHE, GBR, USA. Columns (2) and (4) contain AUS, AUT, BEL, CAN, DNK, FIN, FRA, DEU, ISL, IRL, ITA, JPN, LUX, MEX, NLD, NZL, NOR, KOR, ESP, SWE, CHE, GBR, and USA.

In the theoretical part that follows, I will attempt to rationalize these findings.

Table 3.1: *Demographic Change and the Wage Rate*

Dependent variable	Log wage rate			
	(1)	(2)	(3)	(4)
	1995-2019	1991-2019	1995-2019	1991-2019
Old-age dependency ratio	-0.0578***	-0.0453	-0.0593**	-0.0448
	(0.0191)	(0.0332)	(0.0269)	(0.0352)
Log Income p.c.	0.630***	0.311***	0.618***	0.271**
	(0.0861)	(0.0661)	(0.0807)	(0.115)
Human capital	-0.318**	-0.204	-0.272	-0.201
	(0.122)	(0.219)	(0.161)	(0.230)
Old-age dependency ratio	0.0172***	0.0107	0.0172**	0.00973
· Human capital	(0.00546)	(0.00905)	(0.00757)	(0.00970)
Observations	792	644	198	161
R-squared	0.822	0.807	0.801	0.830
Number of countries	33	23	33	23
Country FE	YES	YES	YES	YES
Time FE	YES	YES	YES	YES
Panel	1-year	1-year	4-years	4-years

Notes: Results of fixed effects regressions. The dependent variable is the log wage rate at constant prices at 2021 USD PPPs. All regressions include country–specific fixed and time effects. Standard errors are clustered at the country level. *, **, *** denote significance at 10%, 5%, and 1% levels.

3.3 Production Sector

In this section, I construct a general equilibrium model with monopolistic competition and a nested CES production structure. I introduce monopolistic competition in order for firms to make positive profits, which will be necessary to cover the costs of risky R&D, that will be introduced in Section 3.4.

3.3.1 Set-up

Time is discrete and denoted by $t \in \mathbb{N}_0$. The final good is produced by a representative producer who uses a continuum of intermediate goods $i \in [0,1]$. The maximization problem is given as

$$\begin{aligned} \max_{Y_t, \{y_t(i)\}_{i\in[0,1]}} \quad & P_t Y_t - \int_0^1 P_t(i) y_t(i) di \\ \text{s.t. } Y_t = & \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \end{aligned} \tag{3.2}$$

with $\varepsilon > 1$. P_t denotes the price of the final good, which is normalized to 1. I can then rewrite the problem in real terms, i.e., by dividing by P_t, as

$$\begin{aligned} \max_{Y_t, \{y_t(i)\}_{i\in[0,1]}} \quad & Y_t - \int_0^1 p_t(i) y_t(i) di \\ \text{s.t. } Y_t = & \left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}. \end{aligned} \tag{3.3}$$

The demand for each variety i is then given as

$$y_t(i) = p_t(i)^{-\varepsilon} Y_t. \tag{3.4}$$

Assumption 3.1 *Each variety is produced using capital, low- and high-skilled labor. High-skilled labor is more complementary to capital compared to low-skilled labor, and hence low-skilled labor is a better substitute for capital than high-skilled labor. Moreover, the elasticity of substitution between high- and low-skilled labor is the same as between capital and high-skilled labor.*

Griliches (1969) was one of the first papers that presented empirical evidence for capital-skill complementarity. For more recent empirical studies that find evidence in favor of capital-skill complementarity, see Duffy *et al.* (2004) and Lewis (2011).

Firm i produces according to the following constant returns to scale production function

$$F(K_t(i), L_t(i), H_t(i)) = \left(\eta(A_{K_t}K_t(i))^{\theta} + (1-\eta)(A_{L_t}L_t(i))^{\theta}\right)^{\frac{\alpha}{\theta}} (A_{H_t}H_t(i))^{1-\alpha}, \tag{3.5}$$

with $\alpha \in (0,1)$ and $\{\theta \in \mathbb{R} \mid -\infty < \theta \leq 1 \setminus 0\}$. For $\theta \in (0,1]$ we have capital-skill complementarity, i.e., capital and high-skilled workers are complements, and capital and low-skilled labor are substitutes, i.e., the elasticity of substitution between capital and low-skilled labor is larger than 1. Here, the elasticity of substitution between capital $K_t(i)$ and low-skilled workers $L_t(i)$ ($\sigma_{K,L}$) is equal to $\frac{1}{1-\theta}$ and the elasticity of substitution between capital $K_t(i)$ and high-skilled workers $H_t(i)$ ($\sigma_{K,H}$) as well as between $L_t(i)$ and $H_t(i)$ ($\sigma_{L,H}$) is 1. $\theta > 0$ implies that $\sigma_{K,L} > \sigma_{K,H}$, i.e., capital-skill complementarity.

To keep the analysis tractable, I assume the equilibrium capital stock remains constant throughout the model.[4]

Firm i decides how much of the input factors to employ to produce variety i, taking into account the demand for variety i.

This leads to the following profit maximization problem for firm i in real terms[5]

$$\begin{aligned} \max_{p_t(i),K_t(i),L_t(i),H_t(i)} \; & p_t(i)y_t(i) - R_tK_t(i) - w_{L_t}L_t(i) - w_{H_t}H_t(i) \\ \text{s.t.} \; & \left(\eta(A_{K_t}K_t(i))^{\theta} + (1-\eta)(A_{L_t}L_t(i))^{\theta}\right)^{\frac{\alpha}{\theta}} (A_{H_t}H_t(i))^{1-\alpha} = y_t(i) \\ & p_t(i)^{-\varepsilon}Y_t = y_t(i). \end{aligned} \tag{3.6}$$

[4]Assuming the equilibrium capital stock is also affected by demographic change would lead to an additional indirect channel, i.e., $K_t(N_t, N_{t-1})$. However, the aim of this paper is to focus on the direct channels, and therefore, studying the effect of demographic change on factor prices and the incentive to innovate through the equilibrium capital stock is left for future research.

[5]Throughout, I will only consider real variables, i.e., all variables will be expressed in terms of the price of the final good P_t which is normalized to 1.

This can be expressed as

$$\max_{K_t(i),L_t(i),H_t(i)} \left(\eta(A_{K_t}K_t(i))^\theta + (1-\eta)(A_{L_t}L_t(i))^\theta\right)^{\frac{a}{\theta}} (A_{H_t}H_t(i))^b Y_t^{\frac{1}{\varepsilon}} - R_t K_t(i) \\ - w_{L_t}L_t(i) - w_{H_t}H_t(i), \tag{3.7}$$

where $a = \alpha\left(\frac{\varepsilon-1}{\varepsilon}\right)$ and $b = (1-\alpha)\left(\frac{\varepsilon-1}{\varepsilon}\right)$. This implies that the degree of homogeneity of the "effective" production function is now $\frac{\varepsilon-1}{\varepsilon} < 1$.

This yields the following first-order condition for firm i

$$\begin{aligned} R_t =& a\left(\eta(A_{K_t}K_t(i))^\theta + (1-\eta)(A_{L_t}L_t(i))^\theta\right)^{\frac{a}{\theta}-1} (A_{H_t}H_t(i))^b Y_t^{\frac{1}{\varepsilon}} \eta A_{K_t}^\theta K_t(i)^{\theta-1}, \\ w_{L_t} =& a\left(\eta(A_{K_t}K_t(i))^\theta + (1-\eta)(A_{L_t}L_t(i))^\theta\right)^{\frac{a}{\theta}-1} (A_{H_t}H_t(i))^b Y_t^{\frac{1}{\varepsilon}} (1-\eta) A_{L_t}^\theta L_t(i)^{\theta-1}, \\ w_{H_t} =& b\left(\eta(A_{K_t}K_t(i))^\theta + (1-\eta)(A_{L_t}L_t(i))^\theta\right)^{\frac{a}{\theta}} A_{H_t}^b H_t(i)^{b-1} Y_t^{\frac{1}{\varepsilon}}. \end{aligned}$$

Aggregating over all firms, applying symmetry, and using
$Y_t = \left(\eta(A_{K_t}K_t)^\theta + (1-\eta)(A_{L_t}L_t)^\theta\right)^{\frac{\alpha}{\theta}} (A_{H_t}H_t)^{1-\alpha}$, I can express the aggregated first-order conditions as

$$R_t = \mu\alpha\left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^\theta A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}}\right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}\eta, \tag{3.8}$$

$$w_{L_t} = \mu\alpha\left(\eta(A_{K_t}K_t)^\theta A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta)A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}}\right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}(1-\eta), \tag{3.9}$$

$$w_{H_t} = \mu(1-\alpha)\left(\eta(A_{K_t}K_t)^\theta + (1-\eta)(A_{L_t}L_t)^\theta\right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha} H_t^{-\alpha}, \tag{3.10}$$

where $\mu = \frac{\varepsilon-1}{\varepsilon} < 1$ denotes the inverse mark-up.

Using Euler's homogeneous function theorem, overall profits in the economy π_t can then be expressed as[6]

$$\begin{aligned}\pi_t =& \left(1-\frac{\varepsilon-1}{\varepsilon}\right)\left(\eta(A_{K_t}K_t)^{\theta}+(1-\eta)(A_{L_t}L_t)^{\theta}\right)^{\frac{\alpha}{\theta}}(A_{H_t}H_t)^{1-\alpha}\\ \pi_t =& \frac{1}{\varepsilon}\left(\eta(A_{K_t}K_t)^{\theta}+(1-\eta)(A_{L_t}L_t)^{\theta}\right)^{\frac{\alpha}{\theta}}(A_{H_t}H_t)^{1-\alpha}\\ \pi_t =& \frac{1}{\varepsilon}F(A_{K_t}K_t, A_{L_t}L_t, A_{H_t}H_t).\end{aligned} \tag{3.11}$$

Aggregate profits, pi_t, thus constitute a constant fraction of aggregate output.

3.3.2 Results

The overall population in period t consists of N_t individuals.[7] A fraction $\psi_t \in (0,1)$ is high-skilled, and a fraction $1-\psi_t$ is low-skilled. This entails

$$\begin{aligned}H_t =& \psi_t N_t,\\ L_t =& (1-\psi_t)N_t.\end{aligned} \tag{3.12}$$

Moreover, I assume the supply of capital, low- and high-skilled labor is fixed.

Using the aggregated first-order conditions and imposing that the markets for all input factors clear, I can derive the effect of a change in the stock of low- and high-skilled workers as well as the entire population on the equilibrium prices of the input factors.

The equilibrium conditions are given as follows, where R_t, w_{L_t}, and $w_{H,t}$ are the endogenous variables, and K_t, L_t, and H_t denote the equilibrium values of capital, low- and high-skilled workers, respectively.[8]

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[6] Recall, that the "effective" production function of firm i is homogenous of degree $\frac{\varepsilon-1}{\varepsilon} < 1$ in $K_t(i)$, $L_t(i)$ and $H_t(i)$. Making use of Euler's homogeneous function theorem, I can derive the profits of firm i. The aggregate profits π_t can then be derived by aggregating over all firms.

[7] As $N_t = (1+n_t)N_{t-1}$, where n_t denotes the growth rate of the population, the comparative statics of a change in N_t have the same sign as the comparative statics of a change in n_t.

[8] The results presented are generally robust to more general production functions of the form $Y_t = \left(\delta(\eta K_t^{\theta}+(1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}}+(1-\delta)H_t^{\rho}\right)^{\frac{1}{\rho}}$ or $Y_t = \left(\delta(\eta K_t^{\theta}+(1-\eta)H_t^{\theta})^{\frac{\rho}{\theta}}+(1-\delta)L_t^{\rho}\right)^{\frac{1}{\rho}}$ as in Duffy *et al.* (2004). The results might change if we assume a production function of the form $Y_t = \left(\delta(\eta H_t^{\theta}+(1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}}+(1-\delta)K_t^{\rho}\right)^{\frac{1}{\rho}}$. However, in this case, we would assume that the elasticity of substitution between high-skilled labor and capital

$$A \equiv \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}\eta - R_t = 0,$$

$$B \equiv \mu\alpha \left(\eta(A_{K_t}K_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}(1-\eta) - w_{L_t} = 0,$$

$$C \equiv \mu(1-\alpha) \left(\eta(A_{K_t}K_t)^{\theta} + (1-\eta)(A_{L_t}L_t)^{\theta} \right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha} H_t^{-\alpha} - w_{H_t} = 0.$$

Proposition 3.1 *A decrease in L_t, H_t, or N_t, with a constant ψ_t has the following effects on the equilibrium return to capital*

$$\frac{\partial R_t}{\partial(-L_t)} \begin{cases} = 0 & \text{if } \alpha = \theta, \\ > 0 & \text{if } \alpha < \theta, \\ < 0 & \text{if } \alpha > \theta, \end{cases} \qquad \frac{\partial R_t}{\partial(-H_t)} < 0, \quad \frac{\partial R_t}{\partial(-N_t)} < 0.$$

Proof. See Appendix C.1. □

If $\theta > 0$, capital and low-skilled labor are substitutes. Furthermore, if it also holds that $\theta > \alpha$, then capital and low-skilled labor are *strong* substitutes. In this case, an outflow of low-skilled labor L_t entails a rise in the return to capital. Capital becomes more useful in production due to the lower supply of low-skilled workers, and thus the equilibrium price of capital, i.e., R_t, increases. Therefore, if demographic change leads to a reduction in L_t, this can increase the equilibrium return to capital if capital and low-skilled labor are strong substitutes.

High-skilled labor is complementary to capital, and thus a reduction therein reduces the marginal product of capital, and hence R_t.

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and low-skilled labor and capital is the same, i.e., capital-skill complementarity would be ruled out. See also Section C.4.

Proposition 3.2 *The effect of a decrease in population size with a constant ψ_t has the following effect on the wages of low- and high-skilled workers*

$$\frac{\partial w_{L_t}}{\partial(-N_t)} \begin{cases} = 0 & \text{if } \alpha = \theta, \\ < 0 & \text{if } \alpha < \theta, \\ > 0 & \text{if } \alpha > \theta, \end{cases} \qquad \frac{\partial w_{H_t}}{\partial(-N_t)} > 0.$$

Proof. See Appendix C.1. □

Demographic change has two effects on the wage rate of each skill group. On the one hand, fewer low-skilled workers directly increase the wage rate of low-skilled workers. On the other hand, fewer high-skilled workers decrease the wage rate of low-skilled workers through an indirect effect, as the two types of labor are complementary, i.e., the elasticity of substitution between low- and high-skilled workers is 1. And vice versa for high-skilled workers.

For high-skilled workers, the direct effect always dominates, whereas for low-skilled workers, which effect dominates depends on the elasticity of substitution between capital and low-skilled workers, i.e., if low-skilled labor and capital are *strong* substitutes, demographic change affects the wage rate of low-skilled workers negatively.

In this framework, high-skilled workers always benefit from demographic change, whereas the effect of demographic change on the wages of low-skilled workers depends on the value of the elasticity of substitution between low-skilled labor and capital.

This is consistent with the data, as demographic change is related to a fall in the aggregate wage rate, i.e., the wage rate of low- and high-skilled workers combined. Moreover, the effect of demographic change on the wage rate, conditional on a high level of human capital, has a positive effect. This suggests that in countries with a high skill share relative to other countries, which in the theoretical model is a proxy for the level of human capital, demographic change has a more positive effect on the wage rate compared to countries with a low skill share relative to other countries.

The skill premium can be derived as follows

$$\phi_t \equiv \frac{w_{H_t}}{w_{L_t}} = \frac{(1-\alpha)\left(\eta\left(\frac{A_{K_t}}{A_{L_t}}\right)^{\theta}\left(\frac{K_t}{(1-\psi_t)N_t}\right)^{\theta} + (1-\eta)\right)}{\alpha(1-\eta)\frac{\psi_t}{1-\psi_t}}. \tag{3.13}$$

Proposition 3.3 *The effect of a decrease in population size with a constant ψ_t on the skill premium is given as*

$$\frac{\partial \phi_t}{\partial(-N_t)} = \theta \frac{(1-\alpha)\eta\left(\frac{A_{K_t}}{A_{L_t}}\right)^{\theta}\left(\frac{K_t}{(1-\psi_t)N_t}\right)^{\theta}\frac{1}{N_t}}{\alpha(1-\eta)\frac{\psi_t}{1-\psi_t}} \gtreqless 0.$$

A decrease in N_t increases the skill premium if $\theta \in (0,1]$, i.e., capital-skill complementarity. A fall in N_t increases the capital stock per worker. Capital-skill complementarity implies that high-skilled labor benefits more from the higher capital stock per worker relative to low-skilled labor, and hence the relative wage rate of high-skilled workers increases. Therefore, if the elasticity of substitution between capital and low-skilled labor is higher than the elasticity of substitution between capital and high-skilled labor, demographic change by itself leads to an increase in the skill premium, and thus to a rise in income inequality if the skill share remains constant. Hence, demographic change is an additional factor that potentially contributed to the rise in the skill premium (see Acemoglu and Autor (2011)).[9] Rewriting the skill premium in terms of capital per worker yields

$$\phi_t = \frac{(1-\alpha)\left(\eta\left(\frac{A_{K_t}}{A_{L_t}}\right)^{\theta}\left(\frac{1}{1-\psi_t}\right)^{\theta} k_t^{\theta} + (1-\eta)\right)}{\alpha(1-\eta)\frac{\psi_t}{1-\psi_t}},$$

where $k_t = \frac{K_t}{N_t}$. This implies that the sign of the effect of an increase in the capital-to-labor ratio k_t on the skill premium is the same as the effect of a decrease in N_t, i.e., $\frac{\partial \phi_t}{\partial(-N_t)} \propto \frac{\partial \phi_t}{\partial k_t}$.[10] As the production function has constant returns to scale in capital and the two types of labor

[9] The skill premium in this model looks similar to the one in the canonical model (see, for example, Acemoglu and Autor (2011)), except that in this model the skill premium changes even if the relative supply of high- and low-skilled labor $\left(\frac{\psi_t}{1-\psi_t}\right)$ and the relative level of technology remain unchanged. In addition, with $\theta \in (0,1]$ I have $\frac{\partial \phi_t}{\partial A_{K_t}} > 0$, and thus an increase in the level of capital augmenting technology increases the skill premium.

[10] $\frac{\partial \phi_t}{\partial(-N_t)} = \frac{k_t}{N_t}\frac{\partial \phi_t}{\partial k_t}$.

and costs are linear, all changes in income that are directly due to demographic change will be driven by a change in the capital-to-labor ratio.

3.4 R&D Sector

The R&D sector is modeled by combining the approaches of Aghion and Howitt (1992) and Jones (1995).[11] There is a perfectly competitive research sector that engages in risky R&D and attempts to develop a higher (exogenous) level of a factor augmenting technology. The current level of technology is denoted by A_{m_t} for $m \in \{K, L, H\}$ and if a firm is successful, this leads to a marginal increase in the level of factor augmenting technology $\gamma_m > 0$ for the production factor m in the next period. Therefore, the level of factor augmenting technology in the next period $A_{m_{t+1}}$ for production factor m is given as

$$A_{m_{t+1}} = \begin{cases} A_{m_t} + \gamma_m & \text{if successful,} \\ A_{m_t} & \text{if not successful.} \end{cases} \tag{3.14}$$

Moreover, if a R&D firm is successful, it receives a one period patent on the new technology and can sell it to intermediate goods firms. I assume that all intermediate goods firms are symmetric, and thus either all acquire the new technology or none do.[12]

The successful R&D firm will charge a price that makes the intermediate good firms indifferent between buying the new technology and keeping the old, in which case I assume they always buy the new technology. The maximum price a successful R&D firm can charge is denoted by Δ_{t+1} and is assumed to be exogenous.

I make the following two assumptions regarding the R&D process.

[11] Instead of modeling the productivity of research, I model it as the probability of successfully inventing a higher level of productivity, as this simplifies the model considerably. Alternatively, I could also employ an ideas production function, as in Jones (1995), and I would find similar results, see Section C.7 in the Appendix.

[12] I could also consider a patent for multiple periods, but this would make the model more complicated without changing the main insights.

Assumption 3.2 *It takes one period to develop a new technology through R&D. Therefore, a technology developed in period t will only be available in period $t+1$.*

Assumption 3.3 *R&D is conducted using high-skilled labor as the only input factor, and the degree of homogeneity of the "production function" for R&D is less than 1, i.e., decreasing returns to scale, and the costs are linear.*

This assumption ensures that I can solve for the optimal labor demand of the R&D sector $H_{R\&D_t}$, i.e., the optimization problem has a well-defined solution, and that the comparative statics for $H_{R\&D_t}$ are defined.[13]

The total number of high-skilled workers engaged in R&D results in a success probability per unit of time, i.e., in each time period t, $G(H_{R\&D_t}, A_{m_t})$, where $H_{R\&D_t}$ denotes the number of high-skilled workers engaged in R&D.

I assume that $G(H_{R\&D_t}, A_{m_t})$ satisfies the following properties[14]

$$G(\cdot,\cdot) \in [0,1], \quad G(0,\cdot) = 0, \quad \frac{\partial G(\cdot,\cdot)}{\partial H_{R\&D_t}} > 0, \quad \frac{\partial G(\cdot,\cdot)}{\partial A_{m_t}} \gtreqless 0, \quad \frac{\partial^2 G(\cdot,\cdot)}{\partial H^2_{R\&D_t}} < 0.$$

Therefore, if no workers conduct research, the probability of developing a new technology is zero. If more high-skilled workers are employed in R&D the probability of being successful per unit of time increases but at a decreasing rate, i.e., $\frac{\partial G(\cdot,\cdot)}{\partial H_{R\&D_t}} > 0$ and $\frac{\partial^2 G(\cdot,\cdot)}{\partial H^2_{R\&D_t}} < 0$. Depending on the sign of $\frac{\partial G(\cdot,\cdot)}{\partial A_{m_t}}$ we either have a "standing on shoulder" effect (> 0), which entails a higher value of A_{m_t} makes current researchers more productive, or a "fishing out" effect (< 0), where new ideas get continuously harder to find.

If a firm is successful, it obtains a one period patent. The value of the patent is given by Δ_{t+1}. R&D investments will only be attractive if the expected return per unit of time, i.e., $G(H_{R\&D_t}, A_{m_t})\Delta_{t+1}$, is at least as large as the costs, i.e., $w_{H_t} H_{R\&D_t}$.

Perfect competition in the R&D sector entails that the expected profits per unit of time must

[13] Alternatively, I could also assume constant returns to scale in the R&D sector but convex costs to ensure a well-defined solution.

[14] An example for the functional form is $G(\cdot,\cdot) = \delta H^{\chi-\beta}_{R\&D_t} A^{\varphi}_{m_t}$, where $\delta \in (0,1)$, $\chi, \beta \in (0,1)$ with $\chi - \beta > 0$, and $\varphi < 1$. δ is a scaling parameter. With $\varphi > 0$ the "standing on shoulders" effect dominates, and with $\varphi < 0$ the "fishing out" effect dominates.

be zero

$$\mathbb{E}_t[\pi_{R\&D_{t+1}}] = G(H_{R\&D_t}, A_{m_t})\Delta_{t+1} - w_{H_t} H_{R\&D_t} = 0.$$

R&D firms take the wage rate of skilled workers w_{H_t} as given. For simplicity, I assume there is no discounting.[15]

An equilibrium in the R&D sector requires

$$F \equiv G(H_{R\&D_t}, A_{m_t})\Delta_{t+1} - w_{H_t} H_{R\&D_t} = 0. \tag{3.15}$$

$$\frac{\partial F}{\partial \Delta_{t+1}} > 0, \quad \frac{\partial F}{\partial w_{H_t}} < 0, \quad \frac{\partial F}{\partial H_{R\&D_t}} \gtreqless 0.$$

$$\frac{\partial F}{\partial H_{R\&D_t}} = \frac{\partial G(H_{R\&D_t}, A_{m_t})\Delta_{t+1}}{\partial H_{R\&D_t}} - w_{H_t} \gtreqless 0.$$

Standard profit optimization would imply that in the optimum $\frac{\partial G(H_{R\&D_t}, A_{m_t})}{\partial H_{R\&D_t}}\Delta_{t+1} = w_{H_t}$, i.e., marginal costs are equal to marginal benefits. However, due to the assumption that the degree of homogeneity of $G(\cdot,\cdot)$ is less than 1, it follows that, in this case, firms would make positive profits due to Euler's homogeneous function theorem.[16] The assumption of perfect competition implies, however, that profits are driven down to zero. Therefore, to ensure zero profits, the average benefit will be equal to the marginal

[15]With discounting, the expected profits are given as $\frac{1}{R_{t+1}} G(H_{R\&D_t}, A_{K_t})\Delta_{t+1} - w_{H_t} H_{R\&D_t}$, where R_{t+1} denotes the gross return to capital in period $t+1$. As $\frac{\partial R_{t+1}}{\partial(-N_{t+1})} < 0$ (see proposition 3.1) demographic change can have an additional positive effect on the value of an innovation by lowering the discount factor.

[16]Setting marginal benefits equal to marginal costs entails $\frac{\partial G(\cdot,\cdot)}{\partial H_{R\&D_t}}\Delta_{t+1} = w_{H_t}$. By Euler's homogeneous function theorem, expected aggregate profits in the R&D sector are then given as $(1-\lambda)G(\cdot,\cdot)\Delta_{t+1}$, where $\lambda < 1$ denotes the degree of homogeneity of $G(\cdot,\cdot)$.

costs

$$\frac{G(H_{R\&D_t}, A_{m_t})\Delta_{t+1}}{H_{R\&D_t}} = w_{H_t}.$$

The assumption that the degree of homogeneity of $G(\cdot,\cdot)$ is less than 1 implies that $\frac{G(H_{R\&D_t},A_{m_t})\Delta_{t+1}}{H_{R\&D_t}} > \frac{\partial G(H_{R\&D_t},A_{m_t})\Delta_{t+1}}{\partial H_{R\&D_t}}$,[17] i.e., the average benefits will be higher than the marginal benefits. This implies

$$\frac{\partial F}{\partial w_{H_t}} < 0, \quad \frac{\partial F}{\partial H_{R\&D_t}} < 0,$$

$$\frac{dH_{R\&D_t}}{dw_{H_t}} = -\frac{\frac{\partial F}{\partial w_{H_t}}}{\frac{\partial F}{\partial H_{R\&D_t}}} < 0.$$

A higher wage reduces $H_{R\&D_t}$, i.e., the number of workers employed in research, and as $-\frac{\partial G(H_{R\&D_t},A_{m_t})}{\partial H_{R\&D_t}} < 0$, this also reduces the probability of a successful innovation and thus technological progress per unit of time.

The effect of an increase in the value of an innovation, Δ_{t+1}, on the optimal number of researchers is given as

$$\frac{dH_{R\&D_t}}{d\Delta_{t+1}} > 0.$$

Hence, if the value of an innovation rises, the optimal number of high-skilled workers employed in R&D increases.

[17]Using Euler's homogeneous function theorem, $\lambda f(x) = \frac{\partial f(x)}{\partial x} x \Rightarrow \lambda \frac{f(x)}{x} = \frac{\partial f(x)}{\partial x}$. With $\lambda < 1 \Rightarrow \frac{f(x)}{x} > \frac{\partial f(x)}{\partial x}$.

$$\begin{aligned}\mathbb{E}_t[A_{m_{t+1}}] &= G(H_{R\&D_t}, A_{m_t})(A_{m_t} + \gamma_m) + (1 - G(H_{R\&D_t}, A_{m_t}))A_{m_t} \\ \frac{\mathbb{E}_t[A_{m_{t+1}}]}{A_{m_t}} &= \frac{G(H_{R\&D_t}, A_{m_t})\gamma_m}{A_{m_t}} + 1 \\ \mathbb{E}_t\left[g_{A_{m_{t+1}}}\right] &= \frac{G(H_{R\&D_t}, A_{m_t})\gamma_m}{A_{m_t}}.\end{aligned} \tag{3.16}$$

The expected growth rate of the level of factor augmenting technology for factor m, $\mathbb{E}_t\left[g_{A_{m_{t+1}}}\right] = \frac{\mathbb{E}_t\left[A_{m_{t+1}}\right]}{A_{m_t}} - 1$, is increasing in the number of researchers $H_{R\&D_t}$ as well as in the size of γ_m, i.e., the magnitude of the technological progress that is taken as exogenous.[18] Therefore, as higher wages in period t reduce $H_{R\&D_t}$ this lowers the expected growth rate. Thus, if demographic change in period t leads to a higher equilibrium wage rate, this can have a negative effect on technological progress.[19]

Moreover, as $\frac{dH_{R\&D_t}}{d\Delta_{t+1}} > 0$, any variable that increases (lowers) the value of an innovation will lead to a higher (lower) research intensity in terms of workers allocated to R&D and thus a higher (lower) expected growth rate.

3.5 Full Model

3.5.1 Value of an Innovation

So far, I have assumed Δ_{t+1} to be exogenous. Instead, assume the intermediate good firms can purchase a higher level of a respective factor augmenting technology, i.e., $A_{m_{t+1}}$ for $m \in \{K, L, H\}$. As an increase in $A_{m_{t+1}}$ would always increase output and thus profits, firms would be willing to pay for the higher level of technology up to the point where the costs of acquiring it are equal to the benefits it generates. And as all firms are symmetric, they will all behave in the same way. The benefits of a marginal increase in the respective level of the

[18] $\mathbb{E}_t\left[g_{A_{m_{t+1}}}\right]$ denotes how much the level of factor augmenting technology is expected to have grown in period $t+1$ relative to period t.

[19] Using the functional form for $G(\cdot,\cdot)$ specified before, the growth rate of technology can be expressed as $\mathbb{E}_t\left[g_{A_{m_{t+1}}}\right] = \frac{\delta H_{R\&D_t}^{\chi-\beta}}{A_{m_t}^{1-\varphi}}\gamma_m$. Given that γ_m is a constant, I can also express the growth rate as $\mathbb{E}_t\left[g_{A_{m_{t+1}}}\right] = \tilde{\delta}\frac{H_{R\&D_t}^{\chi-\beta}}{A_{m_t}^{1-\varphi}}$, which is equivalent to the one derived in Jones (1995), as $\chi - \beta \in (0,1)$.

factor augmenting technology in terms of output are given as follows

$$F_{A_{K_{t+1}}} = \alpha \left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} + (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_{t+1}} H_{t+1})^{1-\alpha} \eta > 0,$$

$$F_{A_{L_{t+1}}} = \alpha \left(\eta (A_{K_{t+1}} K_{t+1})^{\theta} A_{L_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} L_{t+1}^{\frac{\theta^2}{\alpha-\theta}} + (1-\eta) A_{L_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} L_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_{t+1}} H_{t+1})^{1-\alpha} (1-\eta) > 0,$$

$$F_{A_{H_{t+1}}} = \left(\eta (A_{K_{t+1}} K_{t+1})^{\theta} + (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} \right)^{\frac{\alpha}{\theta}} (1-\alpha) A_{H_{t+1}}^{-\alpha} H_{t+1}^{1-\alpha} > 0.$$

The profits of the firms are a constant fraction of output due to the demand each intermediate good producer faces. Hence, if firms gain access to a marginal increase in the level of a factor augmenting technology, the additional profits are given as

$$\Delta_{t+1} = \frac{1}{\varepsilon} F_{A_{m_{t+1}}} (A_{K_{t+1}} K_{t+1}, A_{L_{t+1}} L_{t+1}, A_{H_{t+1}} H_{t+1}) > 0 \text{ for } m \in \{K, L, H\}. \tag{3.17}$$

Thus, the value of an innovation, Δ_{t+1}, is proportional to the "marginal product" of an increase in the level of factor augmenting technology.

Definition 3.1 *Automation capital is defined such that an increase in the capital stock and/or the level of capital augmenting technology lowers the wage rate of low-skilled workers.*

Assumption 3.4 *Parameter values are such that $\theta > \alpha$, i.e., capital and low-skilled labor are strong substitutes.*[20]

Corollary 3.1 *If Assumption 3.4 is fulfilled, K_t takes the form of automation capital in the production function.*

Proof. As $\frac{\partial w_{L_t}}{\partial A_{K_t}} < 0$ and $\frac{\partial w_{L_t}}{\partial K_t} < 0$ if $\theta > \alpha$. □

Therefore, I restrict the analysis to the case where a scarcity of low-skilled labor increases the return to capital and thus potentially leads to further automation, i.e., an endogenous increase in $A_{K_{t+1}}$.

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[20]As $\theta = \frac{\sigma_{K,L}-1}{\sigma_{K,L}}$, where $\sigma_{K,L}$ denotes the elasticity of substitution between capital and low-skilled labor, $\theta > \alpha$ implies $\sigma_{K,L} > \frac{1}{1-\alpha}$.

Assumption 3.5 *The skill share, ψ_t, is either unaffected by demographic change or, if it does respond to it, will react in such a way that a decrease in N_t will always decrease L_t and H_t.*[21]

$$\frac{\partial L_t}{\partial(-N_t)} = -\left(1-\psi_t\right) - N_t \frac{\partial \psi_t}{\partial(-N_t)} < 0,$$
$$\frac{\partial H_t}{\partial(-N_t)} = -\psi_t + N_t \frac{\partial \psi_t}{\partial(-N_t)} < 0.$$

Proposition 3.4 *Suppose Assumptions 3.4 and 3.5 hold a decrease in L_{t+1}, H_{t+1}, or N_{t+1} has the following effects on the value of an innovation that increases $A_{K_{t+1}}$, i.e., $\Delta_{t+1} = \frac{1}{\varepsilon} F_{A_{K_{t+1}}}$:*

1. *A lower value of L_{t+1} increases the value of Δ_{t+1}.*
2. *A lower value of H_{t+1} decreases the value of Δ_{t+1}.*
3. *A lower value of N_{t+1} decreases the value of Δ_{t+1} if the skill share stays constant.*
4. *A lower value of N_{t+1} has an ambiguous effect on the value of Δ_{t+1} if the skill share changes due to demographic change.*

Proof. See Appendix C.2. □

A decrease in the number of workers leads to a form of *income* and *substitution* effect. First, fewer workers implies that wages rise, which makes production more expensive and thus reduces the value of an innovation, which lowers the value of capital augmenting technology, i.e., automation, and thus the demand for it; this is the negative *income* effect. Second, the rise in wages makes capital relative to labor less expensive. This incentivizes firms to invest in technologies that raise the productivity of the relatively cheaper production factor, i.e., capital. This is the positive *substitution* effect that increases the value of an innovation, which increases the value of capital augmenting technology.

Which effect dominates depends on the elasticity of substitution between labor and capital. The elasticity of substitution between high-skilled labor and capital is 1, which implies that the income effect always dominates the substitution effect. Therefore, demographic change

[21] In Section C.6 in the Appendix, I briefly analyze how demographic change could affect ψ_t.

that reduces the number of high-skilled workers always has a detrimental effect on the value of an innovation.

The elasticity of substitution between low-skilled labor and capital is governed by the parameter θ. For $\theta > \alpha$ the positive substitution effect dominates the negative income effect, and vice versa for $\theta < \alpha$. Therefore, if low-skilled labor and capital are strong substitutes, demographic change that lowers the number of low-skilled workers increases the value of an innovation in this model.

Therefore, demographic change affects the value of an innovation through two competing channels. However, in this model, the negative demand effect, i.e., the one working through high-skilled labor, always dominates the positive demand effect if the ratio of low- to high-skilled workers remains constant, even when I consider the case where capital and low-skilled workers are perfect substitutes, i.e., $\theta = 1$.

In case demographic change also changes the skill share, then $\frac{\partial \Delta_{t+1}}{\partial N_{t+1}} \gtreqless 0$ if Assumption 3.4 holds. Moreover, if demographic change increases the share of high-skilled workers in the population, i.e., $\frac{\partial \psi_{t+1}}{\partial(-N_{t+1})} > 0$, it is more likely that $\frac{\partial \Delta_{t+1}}{\partial N_{t+1}}$ is positive. $\frac{\partial \psi_{t+1}}{\partial(-N_{t+1})} > 0$ entails that the positive demand channel is amplified as the number of low-skilled workers is reduced further and that the negative demand channel is weakened as the decrease in the number of high-skilled workers is reduced.

Regarding policy implications, the model thus suggests that policies that increase the share of high-skilled workers in the population could help alleviate the negative effect of demographic change on the rate of innovation. As this would further decrease the number of low-skilled workers and at the same time increase the number of high-skilled workers (relative to a situation with no policy intervention), both of which would affect the value of an innovation positively, which in turn would also have a positive effect on the rate of innovation.

Moreover, a larger share of high-skilled workers can increase the rate of innovation, which constitutes a positive externality for future generations. Therefore, from a social planner's perspective, it could be optimal to subsidize the costs of becoming high-skilled.

The degree of market power, i.e., ε, is exogenous, and therefore demographic change affects the value of an innovation only through a scale effect, i.e., a larger population increases the overall output and thus the value of an innovation. Therefore, demographic change can induce a negative market-size effect by reducing overall output and profits.

In the case where an innovation is modeled as increasing the number of intermediate goods and the production function of each intermediate good producer is the same as before, demographic change will always decrease the value of an innovation, i.e., an additional variety of the intermediate good; see Section C.5 in the Appendix. In such a setup, an innovation affects all production factors equally and thus leaves the level of factor augmenting technology the same. Which implies that the mechanism through which a fall in L_{t+1} increases the value of Δ_{t+1} is not present.

3.5.2 Intra and Intertemporal Effects

In this section I combine the model from Section 3.3 with the model from Section 3.4.

This gives rise to two additional equilibrium conditions, in addition to the three from Section 3.3. The first is a new labor market clearing condition for high-skilled workers. Recall that high-skilled workers can either work in the R&D sector or in the production sector; thus, in equilibrium, it has to hold that

$$H_t = H_{R\&D_t} + H_{Y_t}, \tag{3.18}$$

where H_t denotes the overall supply of high-skilled workers, $H_{R\&D_t}$ the number employed in the R&D sector and H_{Y_t} the number employed in the production sector.

I assume that high-skilled workers can freely switch between sectors, and thus the wage they receive must be the same in the R&D and the production sector. The second additional equilibrium condition is the zero-profit condition for R&D firms

$$G(H_{R\&D_t}, A_{m_t})\Delta_{t+1} - w_{H_t} H_{R\&D_t} = 0,$$

with $\Delta_{t+1} = \frac{1}{\varepsilon} F_{A_{m_{t+1}}}(K_{t+1}, L_{t+1}, H_{t+1})$ and $m \in \{K, L, H\}$. As there are now three levels of factor augmenting technology that can potentially be improved, I assume that R&D firms will only focus on one, i.e., the one that is expected to yield the highest benefits. However, it would be possible to extend this to three perfectly competitive R&D sectors, which all attempt to improve one specific factor augmenting technology, and the expected benefits would then determine how much resources would be directed to each R&D sector.

The aggregated first-order conditions from the production sector and the zero-profit condition from the R&D sector yields

$$A \equiv \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_{Y_t})^{1-\alpha}\eta - R_t = 0,$$

$$B \equiv \mu\alpha \left(\eta (A_{K_t}K_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_{Y_t})^{1-\alpha}(1-\eta) - w_{L_t} = 0,$$

$$C \equiv \mu \left(\eta(A_{K_t}K_t)^{\theta} + (1-\eta)(A_{L_t}L_t)^{\theta} \right)^{\frac{\alpha}{\theta}} (1-\alpha) A_{H_t}^{1-\alpha} H_{Y_t}^{-\alpha} - w_{H_t} = 0,$$

$$D \equiv G(H_{R\&D_t}, A_{m_t})\Delta_{t+1} - w_{H_t}H_{R\&D_t} = 0,$$

$$E \equiv H_{R\&D_t} - H_t + H_{Y_t} = 0.$$

I can rewrite this using the new labor market clearing condition for high-skilled workers. Therefore, there are now four equilibrium conditions that have to hold. The corresponding four endogenous variables are R_t, w_{L_t}, w_{H_t} and $H_{R\&D_t}$.

$$\begin{aligned} A \equiv& \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} \\ &\cdot (A_{H_t}(H_t - H_{R\&D_t}))^{1-\alpha}\eta - R_t = 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} B \equiv& \mu\alpha \left(\eta (A_{K_t}K_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} \\ &\cdot (A_{H_t}(H_t - H_{R\&D_t}))^{1-\alpha}(1-\eta) - w_{L_t} = 0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} C \equiv& \mu \left(\eta(A_{K_t}K_t)^{\theta} + (1-\eta)(A_{L_t}L_t)^{\theta} \right)^{\frac{\alpha}{\theta}} (1-\alpha) A_{H_t}^{1-\alpha} (H_t - H_{R\&D_t})^{-\alpha} \\ &- w_{H_t} = 0, \end{aligned} \tag{3.21}$$

$$D \equiv \frac{G(H_{R\&D_t}, A_{m_t})\Delta_{t+1}}{H_{R\&D_t}} - w_{H_t} = 0. \tag{3.22}$$

I can then totally differentiate the system of equilibrium conditions to find the effect of an increase in Δ_{t+1}, i.e., an increase in the value of an innovation, on the endogenous variables in period t.

Proposition 3.5 *Assuming the skill share remains constant, a fall in N_t has the following effect on the number of high-skilled workers engaged in R&D in period t*

$$\frac{\partial H_{R\&D_t}}{\partial(-N_t)} < 0.$$

Proof. See Appendix C.3. □

Thus, demographic change in period t will decrease the number of high-skilled workers employed in R&D, which will lead to a decrease in the growth rate of factor augmenting technology, i.e., through the negative *supply* channel, which increases the costs of R&D. Hence, demographic change leads to an *intratemporal* effect, where demographic change in the current period reduces the amount of resources devoted to R&D in the current period. The effect on the growth rate or the level of factor augmenting technology in the next period is then also negative, with the effect taking the form of an *intratemporal* effect, as $\frac{\partial \mathbb{E}_t\left[g_{A_{m_{t+1}}}\right]}{\partial(-N_t)} < 0$.

Proposition 3.6 *The effect of an increase in the value of an innovation, i.e., Δ_{t+1}, on the factor prices and the number of high-skilled workers in the production sector in period t is given as*

$$\frac{\partial R_t}{\partial \Delta_{t+1}} < 0, \quad \frac{\partial w_{L_t}}{\partial \Delta_{t+1}} < 0, \quad \frac{\partial w_{H_t}}{\partial \Delta_{t+1}} > 0, \quad \frac{\partial H_{R\&D_t}}{\partial \Delta_{t+1}} > 0.$$

Proof. See Appendix C.3. □

A higher value of Δ_{t+1} makes R&D in period t more attractive, which increases the equilibrium wage rate of high-skilled workers, as they are the only factor employed in R&D. This reduces the number of high-skilled workers in the production sector, i.e., H_{Y_t} falls, and as they are complements to capital and low-skilled workers, this will decrease the return to capital and the wage rate of low-skilled workers in period t.

As $\Delta_{t+1}(N_{t+1}(L_{t+1}, H_{t+1}))$ demographic change in period $t+1$ can not only have an effect on factor prices in period $t+1$ but also on factor prices in period t. The reason being that an innovation can only be employed in the period after it was developed. This links two consecutive periods and gives rise to an *intertemporal* effect, which yields the following two results

Proposition 3.7 *Suppose Assumptions 3.4 and 3.5 hold, then demographic change in period $t+1$ has the following effects on the number of high-skilled workers employed in R&D that attempts to develop a higher level of capital augmenting technology in period t:*

1. *In case the skill share remains constant:* $\frac{\partial H_{R\&D_t}}{\partial(-N_{t+1})} < 0$.

2. *In case the skill share changes due to demographic change:* $\frac{\partial H_{R\&D_t}}{\partial(-N_{t+1})} \gtreqless 0$.

Proof. Follows from Propositions 3.4 and 3.6. □

A change in N_{t+1} influences the expected return on a higher level of capital augmenting technology. As it takes one period to develop a higher level of technology, this gives rise to an *intertemporal* effect, where demographic change in the next period has an influence on the amount of resources devoted to R&D in the current period. The effect on the growth rate or the level of factor augmenting technology in the current period is then also negative, with the effect taking the form of an *intratemporal* effect, as $\frac{\partial \mathbb{E}_t\left[g_{A_{m_{t+1}}}\right]}{\partial(-N_{t+1})} < 0$.

Proposition 3.8 *Suppose Assumption 3.4 holds and parameter values are such that $\theta \in (0, 1]$ and demographic change continues for multiple periods, then the following holds:*

1. *If demographic change leaves the skill share unaffected, the skill premium will be lower relative to a situation in which demographic change only lasts for one period.*

2. *If demographic change changes the skill share, the skill premium can be lower or higher relative to a situation in which demographic change only lasts for one period.*

Proof. Follows from Propositions 3.3, 3.4, and 3.6. □

Recall that if $\theta \in (0,1]$ capital and low-skilled labor are substitutes and capital and high-skilled labor are complements, a higher capital-to-labor ratio benefits high-skilled workers relatively more. R&D introduced an additional channel that directly affects the wage of high-skilled workers and indirectly the wage of low-skilled workers. Demographic change can entail that conducting R&D becomes less attractive, which implies more high-skilled workers will work in the production sector. This, in turn, will increase the wages of low-skilled workers and thus lower the skill premium.
Therefore, for $\theta \in (0,1]$, demographic change taking place in the current period will increase wage income inequality today. In addition, demographic change that will materialize in the next period can have a mitigating influence on income inequality in the current period. Hence, if demographic change continues for multiple periods, its effect on income inequality is ambiguous due to the two opposing channels.

3.6 Conclusion

In this article, I analyze how demographic change could affect factor prices as well as the costs and value of an innovation. The theoretical analysis suggests that demographic change affects factor prices through different channels, which potentially have opposite signs. Therefore, the effect on wages and interest rates will depend on how demographic change affects the absolute as well as the relative size of the two skill groups. Furthermore, the model predicts that the skill premium will increase due to demographic change if capital and high-skilled labor are complements and, correspondingly, if capital and low-skilled labor are substitutes. In addition, if demographic change continues for multiple periods, the effect demographic change has on the skill premium can be weakened or strengthened through the indirect R&D channel. Thus, demographic change can affect income inequality directly by influencing the capital-to-labor ratio as well as indirectly through the R&D channel.
Demographic change can have a direct negative effect on the level of R&D by decreasing the supply of researchers, which increases the costs of R&D. In addition, population aging can

affect the value of an innovation, which increases the level of factor augmenting technology, positively or negatively. For demographic change to increase the value of an innovation, it requires that low-skilled labor and capital are strong substitutes, as well as that demographic change increases the share of high-skilled workers in the population. However, this will not guarantee that demographic change increases the value of an innovation. Therefore, the model highlights that the effect of demographic change on the level of R&D conducted in the economy depends crucially on how the skill composition of the shrinking population changes. This suggests that policymakers should focus on policies that aim to increase the share of high-skilled workers in the population. As a higher skill share in the population will make it more likely that the effect of demographic change on the value of an innovation and thus R&D is positive.

Chapter 4

Is Baumol's Cost Disease Really a Disease? Health Care Expenditure and Factor Reallocation[1]

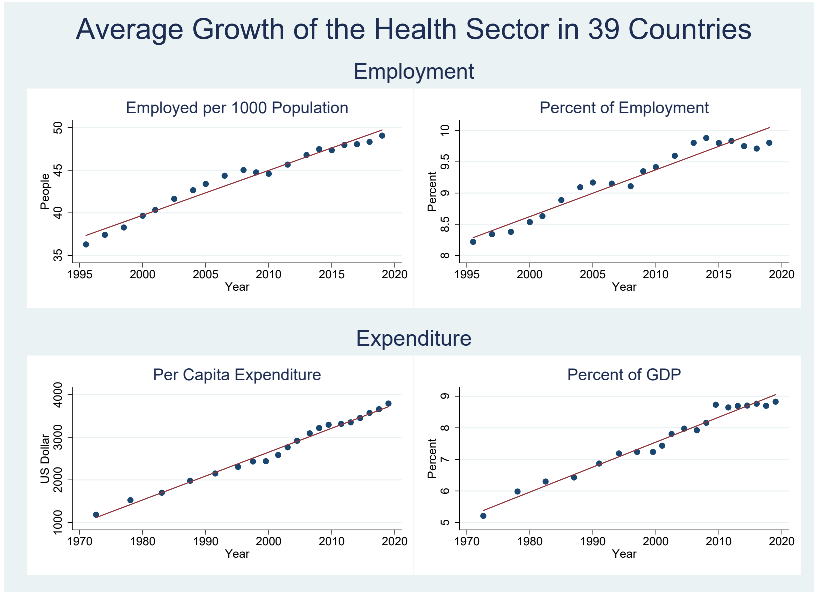
4.1 Introduction

For many decades, health expenditures as a share of GDP have been continuously on the rise in OECD countries. At the same time, employment in the health sector relative to the rest of the economy has also increased; see Figure 4.1.[2] Moreover, there is wide agreement that productivity growth in the health sector relative to the rest of the economy is lower (see Sheiner and Malinovskaya (2016) and Okunade and Osmani (2018)).

One concept that has been used in the past to study these patterns is Baumol's cost disease (Baumol (1967)): if productivity growth in one sector is higher than in the other and wages in both sectors are positively related, then this entails that the production costs and prices in the less productive sector will grow relative to the more productive sector (see

[1]This chapter is joint work with Johanna Rude.

[2]In our empirical analysis, we will focus on the case of Germany. All four trends considered in Figure 4.1 are the same for Germany. The corresponding Figure D.1 can be found in the Appendix.



This figure provides a graphical illustration of the trend in employment and expenditure in the healthcare sector. Data from the OECD on 39 countries are combined, which are listed in Appendix D.5.

Figure 4.1: *Employment and Expenditure in the Health Care Sector*

also Nordhaus (2008)). Multiple empirical studies have presented evidence that Baumol's cost disease is indeed partly responsible for the increase in healthcare expenditures as a share of GDP (see, for example, Hartwig (2008), Bates and Santerre (2013), Hartwig and Sturm (2014), and Colombier (2017)). Inspired by these findings, a large literature on how best to contain the expenditure disease in the healthcare sector emerged (for a review, see Stadhouders *et al.* (2019)).

However, an open question that remains in this context is whether the rise in health expenditures as a share of GDP and the reallocation of labor to the health sector combined with lower productivity growth in the health sector relative to the rest of the economy is necessarily inefficient or a "disease" and directly warrants government intervention. In this paper, we study this question in more detail and attempt to provide a potential answer to it. To that end, we build on Acemoglu and Guerrieri (2008) and construct a micro-founded two-sector closed economy general equilibrium model, and show under which conditions this model can rationalize the stylized facts presented before. In contrast to Baumol (1967) we explicitly model the demand side and thus the demand for the different goods. We assume preferences are homothetic and therefore rule out any effect operating through the income elasticity of demand.[3] An increase in the level of productivity in the non-health sector leads to an *income* and a *substitution* effect. The reallocation of the flexible production factor, as well as whether healthcare expenditures as a share of GDP increase in response to productivity growth in the non-health sector, depends on which effect dominates. We show that if health and non-health goods are complements the income effect dominates the substitution effect, leading to a reallocation of production factors from the non-health sector to the health sector and an increase in the share of healthcare expenditures as a share of GDP. If they are substitutes, the substitution effect dominates the income effect, and the opposite occurs. In case the elasticity of substitution is 1, the two effects exactly offset each other, and the allocation of production factors remains unchanged. Therefore, the central parameter in our framework is the elasticity of substitution between the two goods, which governs whether health and non-health goods are complements or substitutes.

This entails that our model has additional testable implications that can be examined using available data, i.e., the value of the elasticity of substitution between health and non-health consumption.[4]

Our theory does not depend on any forms of frictions or rigidities to rationalize the empirical findings and thus suggests that the patterns observed in the data are *potentially* optimal from the perspective of a utility maximizing representative household, i.e., it is optimal to spend a larger fraction of nominal income on the good that is produced in the relatively less productive sector and allocate more production factors to the relatively less productive

[3]If preferences are non-homothetic and health consumption constitutes a luxury good, an increase in the share of expenditures devoted to health consumption could be explained by higher income levels. However, studies such as Martín *et al.* (2011) and Ke *et al.* (2011) have found income elasticities with respect to health consumption of less than 1, i.e., they found evidence that health consumption is not a luxury good.

[4]Baumol (1967) predicts that wages increase in excess of productivity growth in the stagnant sector, and this is how the theory is often tested empirically (see, for example, Hartwig (2008)). Our theory can make the same prediction if we assume that there is a flexible production factor. However, in our model, this ultimately depends on the value of the elasticity of substitution between health and non-health consumption. Giving us an additional testable implication.

sector sector.[5] Therefore, our theory warrants caution when regarding the rise in health expenditures as a share of GDP combined with lower productivity growth in the health care sector relative to the rest of the economy as problematic or inefficient.

Whether the pattern in the data is indeed optimal from the perspective of the representative household depends, as hinted at before, on the value of the elasticity of substitution between health and non-health goods. More specifically, we require that the elasticity of substitution between health and non-health goods be below 1, i.e., that health and non-health goods are complements, in order for our model to rationalize the stylized facts and for the pattern observed in the data to be in line with the behavior of a utility maximizing representative household.

We, therefore, proceed to estimate the elasticity of substitution using German microdata. Our estimates suggest that the elasticity of substitution is below 1, which supports our theory. Moreover, the model makes contrasting predictions regarding the skill premium in the health and non-health sectors, depending on the value of the elasticity of substitution. More specifically, if the elasticity of substitution is below 1, an increase in the level of productivity in the non-health sector relative to the health sector leads to a higher skill premium in the health sector relative to the non-health sector. This provides us with an additional possibility to assess the validity of our model. Using German wage data, we show that the data supports the prediction our model makes if the elasticity of substitution between health and non-health goods is below 1. Subsequently, we extend our analysis to the US, where we find similar patterns.

Related Literature

This paper is related to a large body of literature on health economics. The existing literature is largely concerned with identifying the determinants of health care spending (see, for example, Erdil and Yetkiner (2009), De Meijer *et al.* (2013), Baltagi *et al.* (2017) and You and Okunade (2017)) or related productivity growth in the health care sector (see, for example,

[5]As our model does not feature any imperfections or externalities, and thus the competitive equilibrium is Pareto efficient by the 1st Welfare Theorem.

Dunn *et al.* (2022), Cutler *et al.* (2022) and Chernew and Newhouse (2011) for a review). In this strand of the literature, of which Getzen and Okunade (2017) provide a concise review, determinants of health care spending are analyzed on the macro level. This paper, in contrast, suggests a microeconomic explanation for increased health care spending.

This paper is also related to the large literature on structural change and non-balanced growth (see Herrendorf *et al.* (2014) for an overview). This literature seeks to understand structural change through mechanisms that either pertain to the supply side or the demand side. Theories concentrating on the supply side focus on factors such as differences in rates of technological progress and capital intensities (see, for example, Baumol (1967), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), and Duarte and Restuccia (2010)). In contrast, theories focusing on the demand side emphasize the role of non-homothetic preferences, i.e., the income elasticity of demand differs across income groups (see, for example, Kongsamut *et al.* (2001), Boppart (2014), Alonso-Carrera and Raurich (2015), and Comin *et al.* (2021)). In this paper, we contribute to the literature by attempting to combine the two views. To that end, we assume households consume two different goods but otherwise have standard homothetic preferences. If productivity growth in the two sectors differs, this can lead to a reallocation of production factors from one sector to the other. The direction of reallocation is solely determined by the preferences of the households, namely, by the elasticity of substitution. Thus, we highlight the importance of another elasticity, i.e., the elasticity of substitution, relative to the income elasticity of demand, in contributing to explaining structural change.

The rest of the paper is structured as follows. Section 4.2 introduces the model. In Section 4.2.4 we derive the theoretical results that serve as testable predictions. Section 4.3 empirically tests the predictions made by the model, and Section 4.4 concludes.

4.2 Model

4.2.1 Production

We consider a closed economy with no capital. Each good is produced using high- and low-skilled labor with a constant returns to scale production technology. Sector 1 produces good 1, and sector 2 produces good 2.[6]

The production function for good j with $j \in \{1,2\}$ is given as

$$Y_{j,t} = L_{j,t}^{\alpha_j}(A_{j,t}H_{j,t})^{1-\alpha_j}, \tag{4.1}$$

with $\alpha_j \in (0,1)$.

There are three groups of households: engineers and doctors, who together constitute high-skilled labor, and low-skilled workers. Engineers work in sector 1, i.e., the non-health sector, and doctors work in sector 2, i.e., the health sector. We assume that low-skilled workers can freely switch between sectors, whereas high-skilled labor cannot switch sectors.[7] Becoming a high-skilled worker requires acquiring occupation-specific skills through, for example, university studies, which takes time. In our model, we consider the short- and medium-run and therefore assume that workers cannot acquire additional occupational skills.[8]

Assuming all high-skilled labor is employed implies

$$\begin{aligned} N_t^e &= H_{1,t}, \\ N_t^d &= H_{2,t}. \end{aligned}$$

[6]Throughout the article, we use the term good, but this is only done for simplicity and does not imply that we only consider physical products.

[7]While assuming that high-skilled labor cannot switch sectors is certainly an overly restrictive and simplifying assumption, there is evidence for labor mobility to decrease with education level, which, presumably, is a proxy for skill level. Mincer and Jovanovic (1981) note that the probability of switching jobs is negatively predicted by an individual's education level. In addition, Kambourov and Manovskii (2009) find evidence for occupational-specific human capital. Neffke *et al.* (2017) find that it is mainly workers with low wages in low-skill occupations that change their employment across industry classification system.

[8]Our main results, except for Proposition 4.3, do not depend on the assumption that high-skilled labor is immobile; see Section D.3 in the Appendix.

We assume that the low-skilled labor supply is fixed and denoted by N_t^l. Moreover, we assume that unlike the other production factors, low-skilled workers are fully mobile, i.e., they can switch between sectors at no cost.
An equilibrium in the market for low-skilled labor requires

$$N_t^l = L_{1,t} + L_{2,t},$$

where $L_{1,t}$ and $L_{2,t}$, respectively, denote the number of low-skilled workers employed in either sector.
To keep the production side as simple as possible, we assume firms operate under perfect competition and thus take all prices as given and will make zero profits in equilibrium. The profit maximization problem in each sector is given as

$$\max_{L_{1,t},H_{1,t}} \pi_{1,t} = p_{1,t} L_{1,t}^{\alpha_1} (A_{1,t} H_{1,t})^{1-\alpha_1} - W_{1,t}^l L_{1,t} - W_{1,t}^h H_{1,t}, \tag{4.2}$$

$$\max_{L_{2,t},H_{2,t}} \pi_{2,t} = p_{2,t} L_{2,t}^{\alpha_2} (A_{2,t} H_{2,t})^{1-\alpha_2} - W_{2,t}^l L_{2,t} - W_{2,t}^h H_{2,t}. \tag{4.3}$$

Good 1 is used as the numeraire, and thus $p_{1,t} \equiv 1$.

Define the nominal wage of high-skilled labor, i.e., in terms of the numeraire, of each group as[9]

$$w_t^e = \frac{W_{1,t}^h}{p_{1,t}} = w_{1,t}^h,$$
$$w_t^d = \frac{W_{2,t}^h}{p_{1,t}} = w_{2,t}^h,$$

and the nominal wage of low-skilled labor in each sector as

$$w_{1,t}^l = \frac{W_{1,t}^l}{p_{1,t}},$$
$$w_{2,t}^l = \frac{W_{2,t}^l}{p_{1,t}}.$$

Using $p_t = \frac{p_{2,t}}{p_{1,t}}$, nominal wages can be written as

[9]To find the real wage rate, we need to calculate a price index, which depends on prices, the structure, as well as parameters of the utility function; see Section 4.2.2.

$$
\begin{aligned}
w_{1,t}^{h} =&(1-\alpha_1)L_{1,t}^{\alpha_1}A_{1,t}^{1-\alpha_1}H_{1,t}^{-\alpha_1},\\
w_{2,t}^{h} =&p_t(1-\alpha_2)L_{2,t}^{\alpha_2}A_{2,t}^{1-\alpha_2}H_{2,t}^{-\alpha_2},\\
w_{1,t}^{l} =&\alpha_1 L_{1,t}^{\alpha_1-1}(A_{1,t}H_{1,t})^{1-\alpha_1},\\
w_{2,t}^{l} =&p_t\alpha_2 L_{2,t}^{\alpha_2-1}(A_{2,t}H_{2,t})^{1-\alpha_2}.
\end{aligned}
$$

Aggregate nominal income of each group is given as[10]

$$
\begin{aligned}
I_t^e N_t^e =&Y_{1,t}-w_{1,t}^{l}L_{1,t},\\
I_t^d N_t^d =&p_t Y_{2,t}-w_{2,t}^{l}L_{2,t},\\
I_t^l N_t^l =&w_{1,t}^{l}L_{1,t}+w_{2,t}^{l}L_{2,t}.
\end{aligned}
$$

Aggregating over the three groups yields aggregate production

$$
I_t^e N_t^e + I_t^d N_t^d + I_t^l N_t^l = Y_{1,t} + p_t Y_{2,t}.
$$

4.2.2 Households

Preferences are homothetic, and a household of group i with $i \in \{e,d,l\}$ consumes a final good c_t^i that is produced by combing two goods, i.e., good 1 and 2, using a CES aggregator. This gives rise to the following maximization problem in nominal terms

$$
\begin{aligned}
&\max_{c_{1,t}^i, c_{2,t}^i} c_t^i(c_{1,t}^i, c_{2,t}^i) = \left(\gamma^{\frac{1}{\theta}}(c_{1,t}^i)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(c_{2,t}^i)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \quad i \in \{e,d,l\}\\
&\text{s.t. } c_{1,t}^i + p_t c_{2,t}^i = I_t^i,
\end{aligned}
\tag{4.4}
$$

with $\theta \in (0,\infty)$. θ denotes the elasticity of substitution between the two goods. For $\theta \in (0,1)$, the two goods are complements, and for $\theta \in (1,\infty)$, the two goods are substitutes.
The optimal demand for either good is given as

[10] $p_t Y_{2,t} = w_{2,t}^l L_{2,t} + w_{2,t}^h H_{2,t}$ due to perfect competition and constant returns to scale.

$$c_{1,t}^{i} = \gamma \frac{I_t^i}{\gamma + (1-\gamma)p_t^{1-\theta}}, \tag{4.5}$$

$$c_{2,t}^{i} = (1-\gamma)p_t^{-\theta} \frac{I_t^i}{\gamma + (1-\gamma)p_t^{1-\theta}}, \tag{4.6}$$

where $p_t = \frac{p_{2,t}}{p_{1,t}}$ is the relative or nominal price of good $c_{2,t}$.

The price index, i.e., the price of one unit of c_t^i, is given as

$$\mathcal{P}_t = \left(\gamma + (1-\gamma)p_t^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

We assume that preferences are the same for groups, i.e., all households are symmetric.[11]

4.2.3 Equilibrium

Market clearing requires that, for each good, demand be equal to supply

$$Y_{1,t} = \sum_i \gamma \frac{I_t^i}{\gamma + (1-\gamma)p_t^{1-\theta}} N_t^i,$$

$$Y_{2,t} = \sum_i (1-\gamma)p_t^{-\theta} \frac{I_t^i}{\gamma + (1-\gamma)p_t^{1-\theta}} N_t^i.$$

We can combine the equilibrium conditions of the two goods markets

$$\begin{aligned} \frac{Y_{2,t}}{Y_{1,t}} &= \frac{(1-\gamma)p_t^{-\theta} \sum_i I_t^i N_t^i}{\gamma \sum_i I_t^i N_t^i} \\ p_t^{\theta} \frac{Y_{2,t}}{Y_{1,t}} &= \frac{1-\gamma}{\gamma}. \end{aligned} \tag{4.7}$$

As low-skilled labor is fully mobile, we require an additional equation that determines the equilibrium division of low-skilled labor between the two sectors, i.e., we need to determine the equilibrium values of $L_{1,t}$ and $L_{2,t}$. Full mobility implies that the nominal wage rate in

[11]See Section D.4 in the Appendix for a short discussion of how heterogeneous preferences could affect the model.

both sectors needs to be equal

$$w^l_{1,t} = w^l_{2,t} = w^l_t.$$

In equilibrium, firms maximize their profits, households maximize their utility, all markets clear, and the wage rate of low-skilled labor has to be equal across both sectors.
We can characterize the equilibrium as a system of two non-linear equations

$$\begin{aligned} F \equiv & p_t^{\theta} \frac{(L_t - L_{1,t})^{\alpha_2} (A_{2,t} H_{2,t})^{1-\alpha_2}}{L_{1,t}^{\alpha_1} (A_{1,t} H_{1,t})^{1-\alpha_1}} - \frac{1-\gamma}{\gamma} = 0 \\ F \equiv & p_t^{\theta} \frac{Y_{2,t}}{Y_{1,t}} - \frac{1}{\gamma} + 1 = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} G \equiv & w^l_{1,t} - w^l_{2,t} = 0 \\ G \equiv & \alpha_1 L_{1,t}^{\alpha_1 - 1} (A_{1,t} H_{1,t})^{1-\alpha_1} - p_t \alpha_2 (L_t - L_{1,t})^{\alpha_2 - 1} (A_{2,t} H_{2,t})^{1-\alpha_2} = 0 \\ G \equiv & \alpha_1 \frac{Y_{1,t}}{L_{1,t}} - p_t \alpha_2 \frac{Y_{2,t}}{L_{2,t}} = 0, \end{aligned} \tag{4.9}$$

with p_t and $L_{1,t}$ as the endogenous variables. Where p_t is the relative price of good 2 and $L_{1,t}$ the number of low-skilled workers employed in sector 1. Equation (4.8) determines the relative price p_t such that the demand and supply for both goods are equalized. Equation (4.9) is only present if low-skilled labor is mobile.[12] It ensures that the wage in either sector is equal for low-skilled workers.

4.2.4 Results

Lemma 4.1 *An increase in $A_{1,t}$ leads ceteris paribus to an increase in the relative price of good 2, i.e., p_t.*

Proof. See Appendix D.1. □

A higher level of productivity in sector 1 relative to sector 2 entails that good 1 becomes

[12]Would we assume that low-skilled labor cannot switch between sectors, the equilibrium could be characterized by only one equation, i.e., equation (4.8).

relatively more abundant and good 2 relatively more scarce.[13] Thus, the relative price of good 2 will increase. This is in line with the empirical evidence presented in Nordhaus (2008).

Proposition 4.1 *An increase in $A_{j,t}$ has the following effect on $L_{j,t}$*

$$\frac{\partial L_{j,t}}{\partial A_{j,t}} \begin{cases} < 0 & \text{if } \theta < 1, \\ > 0 & \text{if } \theta > 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

Proof. See Appendix D.1. □

Therefore, an increase in the level of productivity in sector 1, i.e., the non-health sector, can either lead to an inflow or outflow of low-skilled labor from this sector, depending on whether the two consumption goods are complements, i.e., $\theta \in (0,1)$, or substitutes, i.e., $\theta \in (1,\infty)$. Moreover, this also implies that if $\theta \in (0,1)$ a fall in the productivity level of the health sector due to, for example, exogenous distortions or inefficiencies would lead to a reallocation of low-skilled labor to the health sector.

The economic intuition behind this result is that an increase in $A_{1,t}$ increases $w^l_{1,t}$ directly through a *scale* effect. In addition, it leads to a rise in p_t, which increases $w^l_{2,t}$, i.e., good 1 becomes more abundant and thus the inverse of its relative price increases, through an indirect *price* effect. In equilibrium, the no-arbitrage condition must be satisfied, i.e., $w^l_{1,t} = w^l_{2,t}$, thus as low-skilled labor is fully flexible, it will switch between sectors if the *scale* effect is larger or smaller than the *price* effect. For $\theta = 1$ the two effects exactly offset each other; for $\theta < 1$, i.e., complements, the *price* effect dominates the *scale* effect, which leads to an outflow of low-skilled labor from sector 1, which increases $w^l_{1,t}$ and reduces $w^l_{2,t}$. For $\theta > 1$, i.e., substitutes, the *scale* effect dominates the *price* effect, which leads to an outflow of low-skilled labor from sector 2 and a corresponding inflow into sector 1, which reduces $w^l_{1,t}$ and increases $w^l_{2,t}$.

[13] Productivity does not only encompass the level of technology but also other factors that determine how efficiently the input factors can be combined in the production process. A fall in $A_{2,t}$, i.e., the healthcare sector becoming less efficient, would yield the same qualitative results.

We can also interpret our results in terms of an income and a substitution effect. An increase in $A_{1,t}$ makes good 2 relative to good 1 more expensive, and thus consumers will consume more of the relatively cheaper good; this is the *substitution* effect. Moreover, a higher level of $A_{1,t}$ also makes the economy altogether richer.[14] This leads to an *income* effect, i.e., households will demand more of both goods. Which effect dominates depends on the elasticity of substitution between the two goods. For $\theta \in (0, 1)$, the income effect dominates the substitution effect, and to satisfy the additional demand for good 2, low-skilled labor is transferred from sector 1 to sector 2. If $\theta \in (1, \infty)$, the substitution effect dominates the income effect, leading to a reallocation of low-skilled labor to sector 1 to meet the additional demand for good 1. For $\theta = 1$, i.e., log utility, the two effects cancel each other out.

Unlike Baumol (1967), we provide a micro-founded theory and explicitly model how the flexible production factor is allocated between the two sectors. A reallocation of production factors from the sector that experiences an increase in productivity relative to the other sector might at first seem counterintuitive, as it reduces overall physical output, i.e., $Y_{1,t} + Y_{2,t}$. However, the utility of households in this economy is *not* necessarily maximized by maximizing the physical production of the two goods; that would only be the case if they are perfect substitutes, i.e., $\theta \to \infty$. Rather, households want to consume an optimal relative bundle of the two goods, which depends on preferences and relative prices as well as the elasticity of substitution.[15]

Moreover, recall that our model does not feature any form of imperfections or externalities, and thus the competitive equilibrium derived here is Pareto efficient by the 1st Welfare Theorem. Therefore, if the economy devotes more income and resources to the less productive sector, i.e., the health sector, this does not necessarily mean that the economy suffers from a form of inefficiency or "disease" that warrants government intervention. Rather, it could be the case that preferences, i.e., the elasticity of substitution, are such that the income effect

[14]As preferences are homothetic and the same for all groups, the distribution of the additional income is not relevant.

[15]Combing the first-order conditions of the representative household yields $\frac{c_{1,t}}{c_{2,t}} = \frac{\gamma}{1-\gamma}\left(\frac{p_{2,t}}{p_{1,t}}\right)^{\theta}$.

dominates the substitution effect.

Proposition 4.2 *If $\alpha_1 = \alpha_2 = \alpha$, an increase in $A_{1,t}$ always has the following effect on the share of good 1 in nominal GDP, i.e., $\xi_t = \frac{Y_{1,t}}{Y_{1,t}+p_t Y_{2,t}}$,*

$$\frac{\partial \xi_t}{\partial A_{1,t}} \begin{cases} < 0 & \text{if } \theta < 1, \\ > 0 & \text{if } \theta > 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

Proof. See Appendix D.2. □

There are three channels through which an increase in $A_{1,t}$ can influence ξ_t in this model. First, directly by increasing the output produced in sector 1. Second, by triggering a reallocation of low-skilled labor from one sector to another. Third, by influencing the relative price of good 2 and thus affecting the nominal value of output produced in sector 2. The first and third channel have opposite effects on ξ_t. The sign of the effect of the second channel on ξ_t depends on the value of θ. The result of the above proposition remains unchanged if we assume that, unlike in Baumol (1967), all production factors are immobile.

Let $\phi_{j,t}$ denote the skill premium in sector j. Thus, the skill premium in sector 1 is given as

$$\phi_{1,t} = \frac{w^h_{1,t}}{w^l_{1,t}} = \frac{1-\alpha_1}{\alpha_1}\frac{L_{1,t}}{H_{1,t}}, \tag{4.10}$$

and in sector 2 as

$$\phi_{2,t} = \frac{w^h_{2,t}}{w^l_{2,t}} = \frac{1-\alpha_2}{\alpha_2}\frac{L_{2,t}}{H_{2,t}} = \frac{1-\alpha_2}{\alpha_2}\frac{L_t - L_{1,t}}{H_{2,t}}. \tag{4.11}$$

Lemma 4.2 *An increase in $L_{j,t}$ leads ceteris paribus to a higher skill premium in sector j and a lower skill premium in sector k for $j \neq k$.*

Proof. Follows from $\frac{\partial \phi_{1,t}}{\partial L_{1,t}} > 0$ and $\frac{\partial \phi_{2,t}}{\partial L_{1,t}} < 0$. □

The elasticity of substitution between high- and low-skilled labor is 1, and the production function has positive but decreasing returns to scale with respect to high- and low-skilled labor, respectively, i.e., $Y_{L_{j,t}} > 0$ and $Y_{L_{j,t}L_{j,t}} < 0$. In addition, the cross derivatives are positive, i.e., $Y_{H_{j,t}L_{j,t}} > 0$. Thus, an inflow of low-skilled labor will decrease the wage rate of low-skilled labor and increase the wage rate of high-skilled labor, as they are complemented by the additional low-skilled workers. Hence, if low-skilled workers switch from the non-health sector to the health sector, this increases the wage rate of high-skilled workers in the health sector and decreases the wage rate of low-skilled workers in the health sector, and vice versa for the wage rate in the non-health sector.

Proposition 4.3 *An increase in $A_{j,t}$ leads ceteris paribus to a lower skill premium in sector j and a higher skill premium in sector k if $\theta \in (0, 1)$ and to a higher skill premium in sector j and a lower skill premium in sector k if $\theta \in (1, \infty)$ for $j \neq k$.*

Proof. Follows from Proposition 4.1 and Lemma 4.2. □

Therefore, a rise in the skill premium in the health sector relative to the rest of the economy can be explained by an increase in the level of technology in the non-health sector if $\theta \in (0, 1)$. The intuition for this result is that for $\theta \in (0, 1)$ an increase in $A_{1,t}$ leads to an outflow of low-skilled labor from the non-health sector and a corresponding inflow of low-skilled labor into the health sector. Thus, the ratio of low-skilled workers to high-skilled workers increases in the health sector. As this ratio governs the skill premium in our model, the change therein leads to a rise in the skill premium in the health sector relative to the rest of the economy.

As discussed in the introduction, productivity growth in the non-health sector seems to be stronger than in the health sector. Moreover, we observe a rise in health expenditures as a share of GDP and an increase in the share of workers employed in the healthcare sector. Similar to Baumol (1967) our model can potentially replicate these empirical findings. The sufficient condition for our model to do so is that the elasticity of substitution between health and non-health consumption, i.e., θ, is below 1. However, in contrast to the former, our model also provides us with additional testable implications that can be examined using available data. The first is whether the elasticity of substitution between health and non-health consumption is indeed below 1. The second is whether the skill premium in the health sector has increased relative to the rest of the economy.

4.3 Testable Model Implications

The model described in Section 4.2 can be falsified by testing its implications empirically along two lines. First, the model predicts that a reallocation of resources towards the less productive sector as documented in the introduction depends on the parameter value of θ. Specifically, the resource reallocation is expected to occur if the two consumption goods considered, in this case, health care and all other consumption, are complements. This is equivalent to $\theta < 1$, which is a necessary condition for the mechanism proposed in the model to explain the empirical facts highlighted in the introduction. Using data to test if indeed $\theta < 1$, the model can be falsified. And second, the model predicts that given $\theta < 1$ and higher mobility of unskilled than skilled labor, both the share of unskilled labor and the skill premium in the health sector increase. In the model, this is due to a shift of unskilled labor from industry to health. To assess the model's validity and assumptions, both aspects are addressed in this section.

4.3.1 Preference Estimation

In the introduction, we documented a shift of resources toward the healthcare sector. This reallocation took place despite lower productivity growth in the healthcare sector than in the rest of the economy. The model in Section 4.2 provides a micro-foundation for the mechanisms that can rationalize this finding. It predicts that a shift of resources towards the less productive sector occurs only if the goods produced in the less productive sector are complements to the goods produced in the other sector. The crucial parameter and its restriction to see such a reallocation is $\theta < 1$.

The FOCs from the household maximization can be used to derive the optimal ratio of c_1

and c_2.

$$\frac{c_2}{c_1} = \frac{1-\gamma}{\gamma}\left(\frac{p_1}{p_2}\right)^{\theta}$$
$$\ln\left(\frac{c_2}{c_1}\right) = \ln\left(\frac{1-\gamma}{\gamma}\right) + \theta \ln\left(\frac{p_1}{p_2}\right).$$

This log-linearized ratio can be used to motivate an estimation equation. Of course, other factors besides relative prices and the substitution parameter θ may influence the optimal ratio. We assume that these are captured by the error term ε. The estimation equation is given by

$$\ln\left(\frac{c_{2,t}}{c_{1,t}}\right) = \ln\left(\frac{1-\gamma}{\gamma}\right) + \theta \ln\left(\frac{p_{1,t}}{p_{2,t}}\right) + \varepsilon_t, \tag{4.12}$$

where ε_t is the error term and $\ln\left(\frac{1-\gamma}{\gamma}\right)$ is the constant.

4.3.2 Data

Equation (4.12) demonstrates how the elasticity of substitution between health care spending and all other consumption spending can be estimated. Using microdata, it can be tested if $\theta < 1$, implying that healthcare consumption is complementary to all other consumption. Specifically, in order to estimate θ in microdata, variation in both prices and quantities at the household level is necessary. These requirements are met by the German EVS data provided by the Statistisches Bundesamt. It is a triennial household-level survey, providing detailed information on household expenditures as well as socioeconomic information for roughly 40,000 representative households in each wave. In addition to reporting very granular expenditure data, the EVS also provides the user with transparently combined aggregate measures for different spending categories, one of which is health care. For the estimation, the EVS waves of 2003 and 2018 are used.

The Statistisches Bundesamt collects the EVS data with the primary purpose of constructing inflation measures from it. The official price data, also obtainable from the Statistisches

Bundesamt, is derived from the EVS data. We therefore use and combine two data sets from the same data source. This guarantees a correspondence between available price sub-indices and consumption categories in the EVS. Since price data is indispensable for the estimation proposed, this constitutes a considerable advantage of using EVS data for the estimation. For the estimation, it is indispensable to obtain price variation at the household level. The household-level price data is constructed by weighting the official prices of the sub-categories of consumption with the household-specific shares of expenditure devoted to each sub-category of consumption. Importantly, the data only covers expenditures made by the household. For healthcare expenditures, this means that only those expenditures that are not covered by health insurance are recorded in the EVS. This poses a problem for identification, which is discussed in the next section.

4.3.3 Identification

A common problem with measuring household-level healthcare expenditures is that healthcare spending is often at least partially covered by private or public health insurance. Therefore, healthcare spending by households is likely to be underestimated. In Germany, public health insurance is mandatory, and it arguably covers most, if not all, necessary treatments. If households report private healthcare spending, it is for services above and beyond the quite generous basic coverage. Formally, mandatory healthcare insurance can be modeled as the opposite of a subsistence constraint. This is in analogy to the class of Stone-Geary utility functions (going back to Geary (1950) and Stone (1954)). In that case, household preferences are given by

$$\max_{c_{1,t},c_{2,t}} c_t(c_{1,t}, c_{2,t}) = \left(\gamma^{\frac{1}{\theta}}(c_{1,t})^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(\kappa \cdot c_{2,t} + (1-\kappa)\overline{x}_2)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

$$\text{s.t. } p_{1,t}c_{1,t} + p_{2,t}c_{2,t} = I_t \text{ and } c_{2,t} \geq 0,$$

where $(1-\kappa)\overline{x}_2$ refers to healthcare spending covered by insurance, which is paid for through taxation, and I_t denotes the net-of-tax income. $\kappa \cdot c_2$ refers to healthcare spending on top of items covered by health insurance. Total healthcare consumption by the household is given by $\kappa \cdot c_{2,t} + (1-\kappa)\overline{x}_2$. In the presence of $\overline{x}_2$, the optimal value of c_2 can be zero, requiring an additional non-negativity constraint in the household maximization problem. For estimation, only households reporting positive private expenditures on healthcare are used, such that the non-negativity is met by all observations included in the estimation. c_1 refers to all other consumption. Analogous to the above, an estimation equation for θ can be derived from the FOCs

$$\kappa \cdot \frac{\kappa \cdot c_2 + (1-\kappa) \cdot \overline{x}_2}{c_1} = \frac{1-\gamma}{\gamma} \left(\frac{p_1}{p_2}\right)^{\theta}. \tag{FOCs}$$

$$\ln\left(\frac{\kappa \cdot c_2 + (1-\kappa) \cdot \overline{x}_2}{c_1}\right) = \ln\left(\frac{1-\gamma}{\gamma}\right) - \ln(\kappa) + \theta \ln\left(\frac{p_1}{p_2}\right), \tag{a}$$

$$\ln\left(\frac{c_2}{c_1}\right) = \ln\left(\frac{1-\gamma}{\gamma}\right) - \ln(\kappa) + \theta_b \ln\left(\frac{p_1}{p_2}\right). \tag{b}$$

Ideally, we would like to estimate equation (a), which theoretically is guaranteed to result in an unbiased estimate of θ. Since we do not observe $\overline{x}_2$, the only equation we can estimate is equation (b). This results in an unbiased estimate of θ if the healthcare costs covered by public health insurance are as price sensitive as private healthcare spending. Mathematically, the coefficient of interest is defined as follows

$$\theta = \frac{\kappa \cdot Cov\left(c_2, \frac{p_1}{p_2}\right) + (1-\kappa) \cdot Cov\left(\overline{x}_2, \frac{p_1}{p_2}\right)}{Var\left(\frac{p_1}{p_2}\right)}.$$

The coefficient that can be estimated given the available data is θ_b, which is defined as

$$\theta_b = \frac{Cov\left(c_2, \frac{p_1}{p_2}\right)}{Var\left(\frac{p_1}{p_2}\right)}.$$

The bias of the estimated coefficient, θ_b, relative to the true coefficient of interest, θ, can be derived mathematically. The estimated coefficient is upward biased whenever

$$\frac{Cov\left(c_2, \frac{p_1}{p_2}\right)}{Var\left(\frac{p_1}{p_2}\right)} > \frac{\kappa \cdot Cov\left(c_2, \frac{p_1}{p_2}\right) + (1-\kappa) \cdot Cov\left(\overline{x}_2, \frac{p_1}{p_2}\right)}{Var\left(\frac{p_1}{p_2}\right)}$$

$$\Leftrightarrow$$

$$Cov\left(c_2, \frac{p_1}{p_2}\right) > Cov\left(\overline{x}_2, \frac{p_1}{p_2}\right).$$

A higher covariance between private healthcare spending and relative prices than between insurance-covered healthcare spending and relative prices is a sufficient condition for the estimated value of $\hat{\theta}$ to be upward biased. The inequality of covariances is likely to hold for two reasons. One is that people are less price-conscious when seeking insurance-covered treatments than when seeking medical treatments that have to be paid for privately. Given that people don't even learn about the costs they incur when seeking treatment covered by health insurance, it seems safe to assume that that is the case.[16] Additionally, the coverage of medical treatments by public health insurance is likely to be less price sensitive than when people decide to get elective procedures for which they have to pay the costs themselves. There are binding regulations determining which medical treatments have to be covered by public health insurance. The procedure to change these regulations is lengthy and generally not initiated by price changes.[17] Therefore, the covariance between public health insurance coverage and treatment costs is likely to be lower than that between elective healthcare expenditures and treatment costs. This results in $Cov(c_2, \frac{p_1}{p_2}) > Cov(\overline{x}_2, \frac{p_1}{p_2})$.

The bias increases in the difference in price variability of $\frac{\overline{x}_2}{c_1}$ and $\frac{c_2}{c_1}$, expressed above by the respective covariances. In addition, note that the bias increases in $(1-\kappa)$, assuming that $Cov\left(\overline{x}_2, \frac{p_1}{p_2}\right) > 0$. Effectively, the bias results from an estimation that disregards an unobserved part of healthcare consumption that has a lower price sensitivity than the

[16] See https://www.krankenkassen.de/gesetzliche-krankenkassen/leistungen-gesetzliche-krankenkassen/gesetzlich-vorgeschriebene-leistungen/gesetzliche-leistungen/.

[17] See https://www.bundesgesundheitsministerium.de/service/begriffe-von-a-z/l/leistungskatalog.html and https://www.g-ba.de/ueber-den-gba/arbeitsweise/beratungsantrag/.

observed part of healthcare consumption. If the observed share of overall healthcare consumption increases, the estimation bias decreases. The upward bias can be directly derived as

$$\theta_b = \hat{\theta} = \left(\theta - (1-\kappa)\frac{Cov\left(\overline{x}_2, \frac{p_1}{p_2}\right)}{Var\left(\frac{p_1}{p_2}\right)}\right)\frac{1}{\kappa}.$$

For $\lim_{\kappa \to 1} \hat{\theta} = \theta$, whereas $\lim_{\kappa \to 0} \hat{\theta} = \infty$.

When consuming out-of-pocket healthcare, the basic healthcare needs of consumers in Germany have already been met by public health insurance. When estimating the empirical model given by (4.12) (which is equivalent to equation (b)), this is not accounted for. Thus, $\hat{\theta}$ is biased upward in the presence of relatively price-inelastic, mandatory, and sufficiently generous healthcare insurance. The bias invariably works against finding complementarity between healthcare spending and all other consumer spending.[18]

Eliminating the bias and obtaining unbiased estimates would require data on both the health insurance premium directly subtracted from income as well as a monetary estimate of the health care sought out but paid for by the insurance on an individual level. Unfortunately, this is not possible due to data availability. Based on equation (4.12), we proceed to estimate θ using the German EVS data. Keeping in mind the upward bias mandatory health insurance exudes on the estimated coefficient, this estimation can still provide us with insightful results.

4.3.4 Estimating θ

For the estimation of equation (4.12), the aggregated value for health spending relative to the rest of consumption spending is analyzed. If using aggregated values, the price for health spending is constant across all observations, as variation in the composition of health

[18]Note that the bias described here is different from a classical measurement error in the dependent variable. This would require the measurement error $\overline{x}_2$ to be independent of c_2. The generosity of public health insurance coverage, however, is very likely to be correlated with private healthcare spending, such that the problem at hand cannot adequately be described with classical measurement error.

spending across individuals cannot be used. This implies that only cross-year analysis is feasible. Results from the structural equation estimation are reported in Table 4.1.

The main regression result is reported in column (1), using the whole pooled EVS sample. In addition, columns two, three, and four report the estimated elasticity of substitution for subsamples divided along the income distribution. In theory, we would expect both preference parameters γ (estimated indirectly by the constant) and θ to be constant across all subsamples. The theory is derived with the clearly simplifying assumption of a representative agent, such that obtaining non-varying estimates of the two preference parameters in survey data is unrealistic. If, however, the estimated values of the parameters are reasonably stable across subsamples, it suggests some robustness of the results. In particular, it is of special interest to see if θ is estimated to be above or below the value of 1.

Table 4.1: *Estimating θ by Income Group*

	All	Bottom 50%	Next 40%	Top 10%
Theta	0.017	0.149	0.161	1.331
	[-0.19,0.22]	[-0.12,0.42]	[-0.17,0.49]	[0.58,2.08]
Constant	-3.973	-4.087	-3.949	-3.661
	[-3.99,-3.96]	[-4.11,-4.07]	[-3.97,-3.93]	[-3.71,-3.61]
Observations	77,501	37,089	32,219	8,193

Note: The dependent variable is the log ratio of health to all other expenditures as reported in the 2003 and 2018 waves of the EVS. Significance stars are suppressed because they are not informative in this context. The numbers in brackets report the 95%-confidence interval. The constant represents the estimate of $\ln\left(\frac{1-\gamma}{\gamma}\right)$.

From the reported confidence intervals, it is quite clear that $\hat{\theta}$ is estimated to be smaller than 1, except in the subset of the Top 10% highest income households. As detailed in the previous section, the estimated coefficients reported in Table 4.1 are biased upward because of the broad coverage public health insurance provides in Germany.

The finding of increasing estimated values of $\hat{\theta}$ along the income distribution can be rationalized with $\overline{x}_2$ being less relevant as income increases. For low levels of income, the amount of healthcare covered by insurance, $(1-\kappa)\overline{x}_2$, may be larger than optimal from the household's point of view, resulting in the non-negativity constraint of c_2 being binding, such that $c_2^* \leq 0$. However, as income increases, households may want to consume more healthcare than covered by health insurance, such that the non-negativity constraint of c_2 is no longer binding. Assuming a fixed $\overline{x}_2$ across all households, the share of healthcare costs covered by insurance decreases as income increases. This leads to an increase in the upward bias of the estimated coefficient $\hat{\theta}$, as argued above.

The mathematical explanation can be supplemented by intuitive reasoning, explaining why the upward bias is higher the higher the household income is. The majority of healthcare expenditures by low-income households, if not zero, are likely to be primarily due to co-payments on drugs, dentures, and other basic medical needs, which households have to make irrespective of their price. Households with higher incomes, in contrast, may decide to get elective medical treatments such as teeth beautification, skin care, or plastic surgery. This intuition is supported by the expenditure elasticities of healthcare ($\epsilon_{health} = 1.17$) and all other consumption ($\epsilon_{other} = 0.97$).[19] The fact that the expenditure elasticity of healthcare is larger than 1, whereas the expenditure elasticity of all other consumption is smaller than 1, signifies that healthcare is a luxury good. As a standard CES-utility function in principle cannot accommodate expenditure elasticities that are different from 1, two remarks are in order: One, the expenditure elasticities are once again estimated without taking the fixed amount of healthcare provided by insurance into account. This results in an upward bias in the estimated expenditure elasticity of health consumption. Therefore, the difference between the expenditure elasticities of total health consumption and all other consumption is likely to be smaller in reality. It highlights again the problems for empirical analysis caused by a partial observation of healthcare consumption. Two, the finding of non-unit expenditure elasticities implies once more that, in reality, preferences cannot be perfectly

[19]The expenditure elasticities are estimated by regressing the log of healthcare expenditures on the log of aggregate expenditures. Both 95%-confidence intervals of the estimated expenditure elasticities exclude the value 1 with $ci_{health} = [1.15, 1.18]$ and $ci_{other} = [0.968, 0.971]$.

described by the representative agent with a CES-utility function. Nevertheless, the model described in Section 4.2 yields valuable insights by highlighting the role played by the demand side in general and the elasticity of substitution in particular when analyzing factor reallocation across sectors.

As these estimates are based on individual consumption expenditures, noise in the data may attenuate the estimated coefficients. However, for the estimation, the consumption aggregates of the EVS were used and merged with official price data on the same consumption aggregates published by the Statistisches Bundesamt. This leaves no room for personal interpretation or discretion about the handling of the data. Therefore, any measurement error exerting attenuation bias would lie with the Statistisches Bundesamt. While classical measurement error cannot be ruled out completely, it is likely to be much smaller than the structural upward bias discussed in the previous section. If anything, we expect the estimates to be upwardly biased.

4.3.5 Alternative θ Estimation

The estimation equation is derived from the FOCs and therefore under the implicit assumption of constant income. Furthermore, there is no theoretical reason for including income as a control variable when estimating θ, as preferences are assumed to be homothetic. The variation of results across columns reported in Table 4.1 indicates, however, that the relationship between the consumption ratio and the price ratio changes with income. To investigate and control the role of income in the estimation results, we repeat the estimation, this time including income as an explanatory variable. If preferences are indeed homothetic, we would expect the corresponding coefficient $\hat{\beta}$ to equal zero.

$$\ln\left(\frac{c_{2,t}}{c_{1,t}}\right) = \ln\left(\frac{1-\gamma}{\gamma}\right) + \theta \ln\left(\frac{p_{1,t}}{p_{2,t}}\right) + \beta \ln(income) + \varepsilon_t. \tag{4.13}$$

Table 4.2: *Estimating θ with Income Effect*

	All	Bottom 50%	Next 40%	Top 10%
Theta	0.221	0.138	0.287	1.437
	[0.02,0.42]	[-0.13,0.41]	[-0.04,0.62]	[0.68,2.19]
Log(income)	0.222	0.087	0.407	0.273
	[0.21,0.24]	[0.05,0.12]	[0.33,0.49]	[0.14,0.41]
Constant	-6.018	-4.841	-7.829	-6.442
	[-6.17,-5.87]	[-5.12,-4.56]	[-8.61,-7.05]	[-7.83,-5.05]
Observations	77,473	37,061	32,219	8,193

Note: The dependent variable is the log ratio of health to all other expenditures as reported in the 2003 and 2018 waves of the EVS. Significance stars are suppressed because they are not informative in this context. The numbers in brackets report the 95%-confidence interval. The constant represents the estimate of $\ln\left(\frac{1-\gamma}{\gamma}\right)$.

The results of estimating equation (4.13), reported in Table 4.2, confirm that income plays a role in the relationship between the consumption ratio and the price ratio. The estimates for θ go up if income is included as a control variable. However, they remain smaller than 1 except in the subsample of the Top 10% of the income distribution, where it is estimated to be larger than 1 as in the baseline regression. Income, however, is positively correlated with the share of consumption made up by healthcare. This once again indicates that healthcare is a luxury good. In our model, we can rationalize a reallocation of resources towards the healthcare sector if $\theta < 1$ under homothetic preferences. The finding reported in Table 4.2 shows that some of the reallocations towards the healthcare sector may be driven by healthcare being a luxury good. Assuming non-homothetic preferences would thus facilitate modeling a reallocation. Our model, however, can explain the empirical facts with a minimum of free parameters. While non-homothetic preferences may be part of the story, our model can explain the empirical facts using homothetic preferences, which continue to be the benchmark case in economic models.

The structural estimation in (4.12) can also be separated out and reformulated, imposing equality and opposite signs for the two price coefficients. The new estimation equation is given by

$$\ln(c_{2,t}) = \ln\left(\frac{1-\gamma}{\gamma}\right) + \theta\ln(p_{1,t}) - \theta\ln(p_{2,t}) + \ln(c_{1,t}) + \varepsilon_t. \tag{4.14}$$

This does not address the problem of a structural bias in the θ estimate but allows for a more flexible and intuitive estimation. The relationship can be estimated by putting a constraint on the coefficients of $\ln(p_{2,t})$ and $\ln(p_{1,t})$ to be of the same magnitude but have different signs. Results are reported in Table 4.3.

Table 4.3: *Alternative Estimation of θ by Income Group*

	All	Bottom 50%	Next 40%	Top 10%
Other Price	0.092	0.633	1.195	1.782
	[-0.11,0.30]	[0.36,0.90]	[0.81,1.58]	[1.04,2.53]
Health Price	-0.092	-0.633	-1.195	-1.782
	[-0.30,0.11]	[-0.90,-0.36]	[-1.58,-0.81]	[-2.53,-1.04]
Other Consumption	0.895	0.701	0.307	0.294
	[0.88,0.91]	[0.67,0.73]	[0.26,0.36]	[0.21,0.38]
Constant	-3.035	-1.548	2.420	3.052
	[-3.20,-2.87]	[-1.80,-1.29]	[1.94,2.89]	[2.21,3.89]
Observations	77,501	37,089	24,264	8,193

Note: The dependent variable is log health care expenditures as reported in the 2003 and 2018 waves of the EVS. The two price coefficients are constrained to be equal but of opposite signs. Significance stars are suppressed because they are not informative in this context. The numbers in brackets report the 95%-confidence interval. The constant represents the estimate of $\ln\left(\frac{1-\gamma}{\gamma}\right)$.

In this setup, $\hat{\theta}$ is the estimated coefficient of Other Price, reported in the first row of Table 4.3. As expected from the previous regressions, it is estimated to be smaller than 1, again indicating that health consumption and other consumption are complements. This is true for the pooled sample as well as the Bottom 50% of the income distribution. As already seen in Table 4.1, the estimated $\hat{\theta}$ increases over the income distribution, which is what would be expected. While the estimated $\hat{\theta}$ is now larger than 1 for the Top 50% of the income distribution, it remains below 1 for the Bottom 50% and the pooled sample, which is reassuring.

4.3.6 Factor Reallocation

Under two assumptions, the model predicts a reallocation of unskilled labor to the less productive sector. The first assumption is that the two goods produced are complements, which is the case if $\theta < 1$, supportive evidence of which is presented in the previous section. The second assumption is that unskilled labor is more mobile than skilled labor. This assumption is based on findings in the literature investigating labor mobility. That labor mobility is negatively predicted by education is an empirical finding already shown for the US by Mincer and Jovanovic (1981). This finding is confirmed by Kambourov and Manovskii (2009), who use US data from 1968-1993 to argue that human capital is occupation-specific. Using German social security records from 1999-2008, Neffke *et al.* (2017) report that workers in high-income segments switch industries less often than those in low-income segments. Furthermore, if high-income workers do switch industries, they tend to switch to industries that are closely related to their origin industry. In summary, there is ample evidence based on data from the US and Germany that higher education results in less labor mobility in the sense of sectoral switches.

4.3.7 The Case of Germany

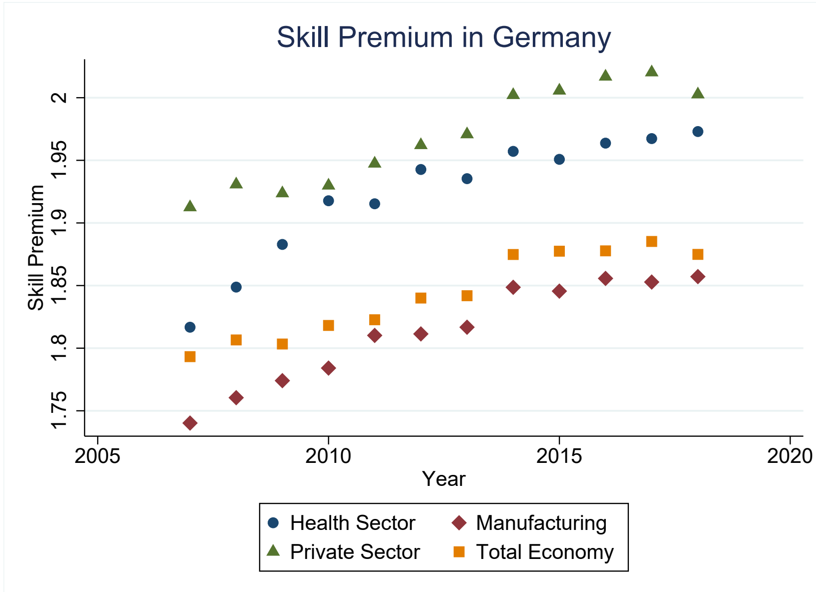
In this section, we test if there was indeed a reallocation of unskilled labor to the healthcare sector, focusing on the case of Germany. Data for this analysis is taken from the German Statistical Office.[20] Optimally, we would like to investigate data spanning the period 2003-2018, such that it is the same as the period over which the preference parameter θ is

[20] `https://www.statistischebibliothek.de/mir/receive/DESerie_mods_00000301`.

estimated. However, data is only available as far back as 2007, reducing statistical power.

In the statistic, it is distinguished between five skill levels. For the purpose of this analysis, the two top skill levels are aggregated into a high-skilled group, with the remaining three skill levels aggregated into a low-skilled group. The high-skilled group comprises workers in management positions and specialized positions who have graduated from college and/or have many years of experience and expert knowledge. This definition of high-skilled labor is in line with occupation-specific human capital accumulation, which has been found to make employment switches across sectors less likely.

There was barely a change in the share of skilled and unskilled labor in Germany from 2007 to 2018, displayed in Figure D.2. The share of high-skilled labor in the healthcare sector decreased from 40.1% in 2007 to 37.0% in 2018. This is the first indication of an increase in the amount of low-skilled labor in the healthcare sector, as predicted by the model. In the overall economy, the share of high-skilled labor increased slightly from 35.8% in 2007 to 36.0% in 2018. Given these small changes, direct analysis of employment shares by skill level is unlikely to yield meaningful results. Instead of measuring the reallocation of different kinds of labor into or out of the healthcare sector, we measure labor mobility indirectly via a skill premium. If the skill premium in one sector increases, it indicates that unskilled labor increases more than skilled labor relative to the respective demand for the different kinds of labor in that sector. One advantage of using this measure is that aggregate data is sufficient to investigate the relative mobility of labor rather than requiring individual-level data. A second advantage is that it measures supply relative to demand for the two kinds of labor, which makes the measure robust to potential structural changes and trends, such as an overall increased supply of skilled labor. The model predicts that the skill premium increased more in the healthcare sector than in the rest of the economy, which is captured by the parameter ϕ in Section 4.2.4.



This figure provides a graphical illustration of the skill premium paid in the healthcare sector, the manufacturing sector, the private sector, and the overall economy, respectively. Data is taken from the German Statistical Office for the years 2007-2018.

Figure 4.2: *Skill Premium in Germany in Different Sectors and the Overall Economy*

Figure 4.2 displays the skill premium paid to high-skilled employees in Germany in different sectors. There has been a general increase in the skill premium from 2007 to 2018 in Germany, as illustrated by the squares in Figure 4.2. While all sectors considered experienced an increase in the skill premium, the increase was fastest in the healthcare sector, as illustrated by the dots in Figure 4.2. These results are indicative of a reallocation of unskilled labor to the healthcare sector, as predicted by the model.[21]

By separately regressing the skill premium for total employment and employment in different sectors on a time variable, it can be tested if there is a statistical difference in the increase in the skill premium between the different sectors. The regression results are

[21] All the results presented in this section are calculated based on employment numbers for Germany. There are some particularities with employment in the healthcare sector in Germany, none of which pose a threat to our identification strategy. For details, see Appendix D.6.

Table 4.4: *Estimating the Time Trends of Skill Premia*

	Total	Private	Manu	Health
Year	0.00903***	0.0105***	0.0109***	0.0127***
	(10.96)	(10.24)	(12.23)	(7.61)
Constant	-16.32***	-19.15***	-20.12***	-23.66***
	(-9.84)	(-9.28)	(-11.22)	(-7.03)
R^2	0.92	0.91	0.94	0.85
Observations	12	12	12	12

Note: The dependent variable is the skill premium in the overall economy, the private sector, the manufacturing sector, and the health sector, respectively. Results are obtained using German employment Data from 2007-2018. Significance stars are defined as follows: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. t-statistics in parentheses.

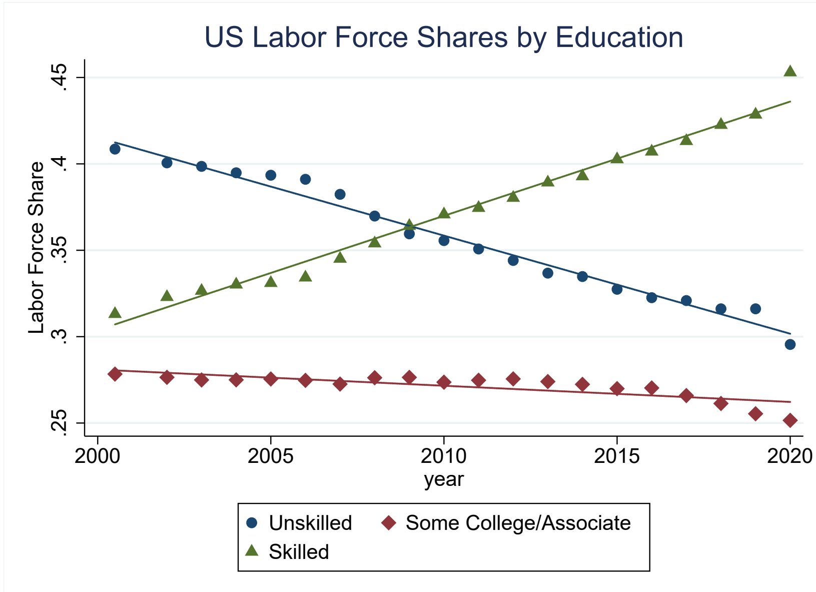
reported in Table 4.4. As foreshadowed by the graphical illustration, the time trend for the skill premium is the steepest in the healthcare sector. A Wald-test of similarity indicates that the null hypothesis of similar trends can be rejected at a $p - value = 0.06$ for total employment. The difference between the time trend in the healthcare sector and the private sector and the healthcare sector and the manufacturing sector is not statistically significant, with respective p-values of $p - value_{Private} = 0.30$ and $p - value_{Manu} = 0.37$.

The lack of statistical significance between the healthcare sector and the other two sectors may be due to the short period of available data. It may also be due to the highly regulated labor market in Germany. While providing protection for workers, it reduces the flexibility with which any sector can react to changes in labor demand. The reallocation of unskilled labor, which in principle could easily switch into the healthcare sector to help meet increased demand for healthcare, is thus inhibited by the strong German labor protection laws. This is likely to reduce the expected skill premium increase in the healthcare sector in Germany and works against finding a statistically significant difference between the healthcare sector and other sectors. Both aspects work against finding a statistically significant difference in the time trends of skill premia. The fact that we find (partially) statistically significant results despite these caveats emphasizes the relevance of the model's implications.

4.3.8 Extending the Analysis to the USA

In this section, we investigate if there has been a reallocation of unskilled labor towards the healthcare sector in the US. The purpose of this section is twofold. One, by replicating findings regarding the skill premium found for Germany using US data, the relevance and plausibility of the model is once again demonstrated. Both the healthcare system and the labor market regulations in the US are very different from those in Germany. Showing the specific pattern in the skill premium to hold in two distinct countries makes the external validity and general applicability of the model likely. Two, the US data covers a longer period, and there is a larger variation in skill shares of employment over time, rendering analyses of changes in the share of unskilled labor meaningful. First, we check if a reallocation of unskilled labor towards the healthcare sector took place. The testable implication is that the share of unskilled labor rose faster in the healthcare sector than in the rest of the economy. Second, and as discussed before, we analyze if the skill premium increased more in the health care sector than in the rest of the economy, as implied by a reallocation of unskilled labor.

Each year, the US Bureau of Labor Statistics releases wage data for different education levels in the whole US economy. According to the data, the share of unskilled labor (measured as the share of workers with a high school degree or less) decreased from 39.9% to 31.6% from 2003 to 2018, a decrease of 21%. At the same time, the share of skilled labor increased from 32.7% to 42.3%. Workers with some college experience or an associate's degree are not included in either group, as it is unclear which category they belong to. The share of that in-between-education group is rather large, on average making up 27% of the labor force. However, this share stays quite constant over time, changing from 27% to 26% between 2003 and 2018. The change over time for each skill group is depicted in Figure 4.3. Contrary to the case of Germany, there is a considerable trend in the shares of differently skilled labor in the overall labor force. Overall, unskilled labor decreased, accompanied by a simultaneous increase in skilled labor across all sectors of the US economy.



This figure provides a graphical illustration of the trend in employment shares for three different skill groups. It is based on data from the US Bureau of Labor Statistics. Workers with a high school education or less and no professional training are classified as unskilled, and workers with at least a Bachelor's degree are classified as skilled.

Figure 4.3: *Skilled and Unskilled Workers*

The statistics cited in the previous paragraph clearly show an increasing time trend in the share of skilled labor and the skill premium across all sectors. To analyze how these statistics changed over the same time within the healthcare sector, a different dataset from the US Bureau of Labor Statistics has to be employed, which reports employment and wage statistics for different occupational groups.[22] To distinguish between skilled and unskilled labor, the occupational group "Healthcare practitioners and technical occupations" is compared to the "Healthcare support occupations". The employment share of the unskilled group decreased from 34.2% to 32.3% in the healthcare sector. Thus, the share of unskilled employment decreased from 2003 to 2018 by 6% in the healthcare sector, which is much lower than the 21% recorded for the overall economy. The shares of unskilled labor in the overall economy and the healthcare sector are displayed in the first row of Table 4.5. The respective growth

[22]See `https://www.bls.gov/oes/current/oes_nat.htm`.

Table 4.5: *US Labor Force Changes 2003-2018*

	Overall Economy		Healthcare Sector	
	2003	2018	2003	2018
Unskilled Labor Force	39.9%	31.6%	34.2%	32.3%
Δ		-21%		-6%
Skill Premium	1.87	1.91	2.11	2.23
Δ		+2.1%		+5.7%

Note: Calculations based on data from the US Bureau of Labor Statistics.

rates are reported in the second row. The fact that the share of unskilled labor decreased by less in the healthcare sector than in the rest of the economy is in line with the prediction made by the model if $\theta < 1$.

Next, it is informative to compare the change in the skill premium between the overall economy and the healthcare sector. As noted before, the share of skilled labor in the overall economy increased in the period 2003-2018. At the same time, the skill premium of college graduates relative to workers with a high school diploma or less increased from 1.87 to 1.91, or by 2.1% across all sectors.[23] When adding those workers with some college or an associate's degree to the unskilled labor force, thus comparing college graduates to all other workers, the considered changes are of similar magnitude.[24] In the healthcare sector, the skill premium increased from 2.11 to 2.23, or by 5.7% in the same period.[25] So while the skill premium increased in both the overall economy and the healthcare sector, the increase was stronger in the healthcare sector. The skill premium as well as its growth rate in the overall economy and the healthcare sector are displayed in the third and fourth rows of Table 4.5, respectively. The fact that the skill premium increased more in the healthcare sector than in the overall economy is in line with the model prediction for $\theta < 1$.

Instead of looking at the share of unskilled labor and the skill premium in each sector separately, the ratio of these indicators can be constructed for each year. While this may

[23]The skill premium is measured at the median of the respective education group's wage distribution.

[24]The skill premium of college graduates relative to all other workers changed from 1.70 to 1.76, or by 3.5%. The share of unskilled labor, including all but college graduates, changed from 67.3% to 57.7% or by 14%.

[25]Again, the skill premium is measured for median wages.

Table 4.6: *Ratios of Key Indicators*

	2003	2018
Unskilled Labor Force Ratio	0.857	1.022
Skill Premium Ratio	1.128	1.168

Note: Calculations based on Table 4.5, which summarizes data from the US Bureau of Labor Statistics. The ratios implicitly account for time trends and compositional changes in the labor force.

be a less intuitive measure, it has the advantage of being unaffected by overall time trends. Specifically, the ratio is constructed as $Ratio_{measure,t} = \frac{\text{measure}_{health,t}}{\text{measure}_{overall,t}}$, with measure referring either to the share of the unskilled labor force or the skill premium. Table 4.6 displays the ratio of unskilled labor and the skill premium in the healthcare sector relative to the overall economy for 2003 and 2018. For example, the skill premium in the healthcare sector was 1.128 times larger than the skill premium in the overall economy in 2003. In 2018, it was 1.168 times larger in the healthcare sector than in the overall economy. This indicates that the skill premium increased faster in the healthcare sector than in the overall economy. The same is true for the ratio of unskilled labor force shares. By comparing the ratios across time, time trends and overall compositional changes in the labor force are implicitly accounted for.

Overall, there is supportive evidence for the testable model implication of a factor reallocation and ensuing changes in factor remuneration, summarized in Tables 4.5 and 4.6. Taking the time trends for both the unskilled labor share and skill premium into account, the development in the healthcare sector of both measures is in line with the model predictions if $\theta < 1$.

One drawback of the US data used here is that the healthcare sector is, of course, included in the data on the overall economy. The limited data availability prohibits a direct comparison between the overall economy excluding the healthcare sector and the healthcare sector. It implies that the actual difference between the two groups is larger than identified in the imperfect data, which works against finding any differences between the two groups compared here.

To facilitate the comparison of the results concerning factor reallocation in German and US data, Table 4.5 and Table 4.6 are replicated using the German data from Section 4.3.7. The tables can be found in Appendix Section D.7 as Table D.1 and Table D.2. Compared to the case of Germany, the increase in the share of unskilled labor in the healthcare sector compared to the overall economy is more pronounced in the US data. In contrast, the increase in the skill premium paid in the healthcare sector compared to the overall economy is a bit smaller in the US than in the German data. The overall pattern of a more-than-average increase in both the unskilled labor share and the skill premium in the healthcare sector is present in both German and US data. This is remarkable given the very different labor markets, especially with regard to labor protection laws and healthcare systems, in the two countries.

4.4 Conclusion

Spending on health care as a share of GDP has steadily increased for at least 50 years across 39 countries with available data. Employment in the healthcare sector has mirrored the increase in spending, documenting a reallocation of labor towards the healthcare sector. These two phenomena have been extensively studied by economists, and different explanations for the "excess growth" have been proposed and analyzed (see Getzen (2016) for a review of the literature). In the quest for explanations, the focus has been on macroeconomic variables like income per capita.

The health sector and its increasing share in GDP are often associated with Baumol's cost disease, a phrase based on Baumol (1967). We construct a micro-founded theory that can rationalize the empirical findings and provide us with additional testable implications that can be evaluated using available data.

We show that if the level of productivity increases in one sector relative to the other, this gives rise to a substitution effect and an income effect. The substitution effect entails that more resources flow into the more productive sector, whereas the income effect encompasses the opposite. Which of the two effects dominates depends on the elasticity of substitution between health and non-health consumption. If the elasticity of substitution between health and non-health goods is less than 1, i.e., $\theta \in (0, 1)$, for which we provide empirical evidence, higher productivity growth in one sector relative to the other leads to an outflow of the flexible production factor from the more productive sector. Moreover, this can potentially increase the share of the less productive sector in terms of nominal GDP. Therefore, unequal productivity growth increases the relative price of the good produced in the relatively less productive sector and leads to a reallocation of production factors from the relatively more productive sector to the relatively less productive sector. This is in line with Baumol's cost disease. However, in this case, the term "cost disease" might be misplaced because the outcome, i.e., a reallocation of production factors from the more productive sector to the less productive sector, is optimal from the perspective of a representative utility-maximizing household. Of course, this does not necessarily mean that spending an ever-larger fraction of income and production factors on healthcare is always optimal from a welfare perspective. Nonetheless, our model highlights that the intuition that reallocating production factors from the relatively more productive sector to the relatively less productive sector is inefficient or constitutes a "disease", as it will lower overall physical output, is not a priori correct and thus does not directly warrant government intervention.

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Appendix A

Appendix to Chapter 1

A.1 Partial Equilibrium

A.1.1 Equilibrium

I assume that we have $K_t, w_{t+1}, A_{t+1}, R_t, L_t \in (0, \infty)$, $n_{t+1} \in (-1, \infty)$ and $\alpha, \theta \in (0, 1)$.

$$\begin{aligned}\frac{\partial F}{\partial K_t} =& - \frac{C'(K_t)K_t - C(K_t)}{K_t^2} < 0, \\ \frac{\partial G}{\partial K_t} >& 0.\end{aligned}$$

Hence, the sign of the respective derivative is always the same for any value of w_{t+1} which implies that the solutions for $F(\cdot)$ and $G(\cdot)$ for each w_{t+1} are unique and thus the functions are well-defined. I can rewrite $F(\cdot)$ and $G(\cdot)$ such that $K_t = f(w_{t+1})$ and $K_t = g(w_{t+1})$. Assuming $C(K_t)$ takes on the following form $C(K_t) = \gamma K_t^\eta$ with $\gamma > 0$ and $\eta = 2$. This then allows me to reduce the system to one equation

$$f(w_{t+1}) - g(w_{t+1}) = 0,$$

with

$$f(w_{t+1}) = \frac{1}{\gamma} \left(\beta \left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right) - R_t \right),$$

$$g(w_{t+1}) = \left(\left(w_{t+1}^{-\frac{1}{\alpha}} + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}} \right) \left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}} \right)^{-1} \ell_{t+1}(1+n_{t+1})L_t.$$

It is straightforward to show that $\frac{\partial f(w_{t+1})}{\partial w_{t+1}} < 0$ and $\frac{\partial g(w_{t+1})}{\partial w_{t+1}} > 0$ which implies that the functions have at most one point of intersection and thus the equilibrium is unique.

I can use the intermediate value theorem to establish a condition for the existence of the equilibrium. Using the upper bound on w_{t+1} I get

$$\lim_{w_{t+1}\to\infty} f(w_{t+1}) - g(w_{t+1})$$
$$-\frac{R_t}{\gamma} - \infty < 0.$$

As

$$\lim_{w_{t+1}\to\infty} \left(\left(w_{t+1}^{-\frac{1}{\alpha}} + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}} \right) \left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}} \right)^{-1} \ell_{t+1}(1+n_{t+1})L_t$$
$$\lim_{w_{t+1}\to\infty} w_t^{\frac{1}{\alpha}} \left(\left(1 + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta}} \right) \left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}} \right)^{-1} \ell_{t+1}(1+n_{t+1})L_t$$
$$\frac{\ell_{t+1}(1+n_{t+1})L_t}{\left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}}} \cdot \lim_{w_{t+1}\to\infty} w_t^{\frac{1}{\alpha}} \cdot \lim_{w_{t+1}\to\infty} \left(\left(1 + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta}} \right) \right)^{-1}$$
$$\frac{\ell_{t+1}(1+n_{t+1})L_t}{\left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}}} \cdot \infty \cdot 1 = \infty.$$

Using the lower bound I get

$$\lim_{w_{t+1}\to 0+} f(w_{t+1}) - g(w_{t+1})$$
$$\infty - 0 > 0.$$

As

$$\lim_{w_{t+1}\to 0+} \frac{1}{\gamma} \left(\beta \left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right) - R_t \right) = \frac{1}{\gamma}(\infty - R_t) = \infty,$$
$$\frac{\ell_{t+1}(1+n_{t+1})L_t}{\left((1-\alpha) A_{t+1}^{1-\alpha} \right)^{\frac{1}{\alpha}}} \cdot \lim_{w_{t+1}\to 0+} w_t^{\frac{1}{\alpha}} \cdot \lim_{w_{t+1}\to 0+} \left(\left(1 + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta}} \right) \right)^{-1}$$

$$\frac{\ell_{t+1}(1+n_{t+1})L_t}{\left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}}} \cdot 0 \cdot 0 = 0.$$

Therefore, the conditions for the existence of an equilibrium are fulfilled.

A.1.2 Comparative Statics

To find the overall effect of the independent on the dependent variables I need to analyze the following system of two equations

$$F \equiv \beta \left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right) - R_t - \frac{C(K_t)}{K_t} = 0,$$

$$G \equiv (w_{t+1}^{-\frac{1}{\alpha}} + w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}) \left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}} K_t - \ell_{t+1}(w_{t+1})(1+n_{t+1})N_t = 0.$$

$$\frac{\partial F}{\partial K_t} = -\frac{C'(K_t)K_t - C(K_t)}{K_t^2} < 0,$$

$$\frac{\partial F}{\partial w_{t+1}} = \beta \left(\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{w_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \right) \frac{1}{w_{t+1}}(-1) < 0,$$

$$\frac{\partial F}{\partial n_{t+1}} = 0,$$

$$\frac{\partial G}{\partial K_t} = (w_{t+1}^{-\frac{1}{\alpha}} + w^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}) \left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}} > 0,$$

$$\frac{\partial G}{\partial w_{t+1}} = \left(-\frac{1}{\alpha} w_{t+1}^{-\frac{1}{\alpha}-1} + \underbrace{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}}_{<0 \text{ as } \theta \in (0,1)} w_{t+1}^{\frac{\alpha(\theta-1)-\theta}{\theta\alpha}-1} \right) \left((1-\alpha)A_{t+1}^{1-\alpha}\right)^{\frac{1}{\alpha}} K_t - \underbrace{\frac{\partial \ell_{t+1}}{\partial w_{t+1}}}_{>0}(1+n_{t+1})N_t < 0,$$

$$\frac{\partial G}{\partial n_{t+1}} = -\ell_{t+1}N_t < 0.$$

The effect of n_{t+1} on K_t and w_{t+1} is then given as

$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial n_{t+1}} & \frac{\partial F}{\partial w_{t+1}} \\ -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial w_{t+1}} \end{vmatrix}} > 0, \quad \frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & -\frac{\partial F}{\partial n_{t+1}} \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial n_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial w_{t+1}} \end{vmatrix}} < 0.$$

The effect of a change in the other exogenous variables on w_{t+1} or K_t can be found in a similar way.

To see why convex capital costs lead to a lower (higher) capital stock per worker when n_{t+1} increases (falls) with constant interest rates, note that the equilibrium wage rate by construction equates demand and supply of labor and must thus satisfy

$$\begin{aligned} w_{t+1} =& (1-\alpha)\left(\frac{K_t}{\ell_{t+1}(w_{t+1})(1+n_{t+1})N_t}\right)^{\alpha} A_{t+1}^{1-\alpha} \\ w_{t+1}\ell_{t+1}(w_{t+1})^{\alpha} =& (1-\alpha)k_t^{\alpha}A_{t+1}^{1-\alpha}, \end{aligned}$$

where $k_t = \frac{K_t}{(1+n_{t+1})N_t}$ denotes the capital stock per worker in period $t+1$. From Section 1.5, I know that the overall effect of an increase in n_{t+1} on w_{t+1} and $\ell_{t+1}(w_{t+1})$ is negative, and as α and A_{t+1} are exogenous, a fall in w_{t+1} and $\ell_{t+1}(w_{t+1})$, i.e., the LHS, thus requires a reduction in k_t, i.e., the RHS. Conversely, an increase in w_{t+1} and $\ell_{t+1}(w_{t+1})$ due to a fall in n_{t+1} requires an increase in k_t.

A.1.3 Concave Costs

Assume the non-linear capital costs are concave. More specifically, let λ denote the degree of homogeneity of $C(K_t)$, I then assume that $\lambda \in (0,1)$. Moreover, assume for simplicity that labor supply is exogenous. The equilibrium is then characterized by the following system of equations

$$\begin{aligned} F \equiv& \beta\alpha\left(\frac{(1+n_{t+1})N_t}{K_t}\right)^{1-\alpha} A_{t+1}^{1-\alpha} - R_t - \frac{C(K_t)}{K_t} = 0, \\ G \equiv& (1-\alpha)\left(\frac{K_t}{(1+n_{t+1})N_t}\right)^{\alpha} A_{t+1}^{1-\alpha} - w_{t+1} = 0. \end{aligned}$$

with $(1+n_{t+1})N_t = L_{t+1}$ and K_t and w_t as the endogenous variables.

$$\frac{\partial F}{\partial K_t} = -\beta\alpha(1-\alpha)\left(\frac{(1+n_{t+1})N_t}{K_t}\right)^{1-\alpha}\frac{1}{K_t}A_{t+1}^{1-\alpha} - \frac{C'(K_t)K_t - C(K_t)}{K_t^2} \gtreqless 0,$$

$$\frac{\partial F}{\partial w_{t+1}} = 0, \quad \frac{\partial F}{\partial n_{t+1}} > 0, \quad \frac{\partial G}{\partial K_t} > 0, \quad \frac{\partial G}{\partial w_{t+1}} < 0, \quad \frac{\partial G}{\partial n_{t+1}} < 0,$$

as $-\frac{C'(K_t)K_t - C(K_t)}{K_t^2} > 0$, by Euler's homogeneous function theorem. Therefore, the effect of n_{t+1} on K_t and w_{t+1} is ambiguous and depend on parameter values as well as the capital and labor stock.

$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{-\frac{\partial F}{\partial n_{t+1}}\frac{\partial G}{\partial w_t}}{\frac{\partial F}{\partial K_t}\frac{\partial G}{\partial w_t}} = \frac{\frac{\partial F}{\partial n_{t+1}}}{-\frac{\partial F}{\partial K_t}} \begin{cases} < 0 & \text{if} \quad \frac{\partial F}{\partial K_t} > 0 \\ > 0 & \text{if} \quad \frac{\partial F}{\partial K_t} < 0 \end{cases}$$

$$\frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{\frac{\partial F}{\partial K_t}\left(-\frac{\partial G}{\partial n_{t+1}}\right) + \frac{\partial F}{\partial n_{t+1}}\frac{\partial G}{\partial K_t}}{\frac{\partial F}{\partial K_t}\frac{\partial G}{\partial w_t}} = \frac{\frac{\partial F}{\partial K_t}\left(-\frac{\partial G}{\partial n_{t+1}}\right) + \frac{\partial F}{\partial n_{t+1}}\frac{\partial G}{\partial K_t}}{-\frac{\partial F}{\partial K_t}} \begin{cases} < 0 & \text{if} \quad \frac{\partial F}{\partial K_t} > 0 \\ \lesseqgtr 0 & \text{if} \quad \frac{\partial F}{\partial K_t} < 0 \end{cases}$$

A.2 General Equilibrium

A.2.1 Cobb-Douglas Production Function

$$\text{Capital supply: } F \equiv \frac{\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}} w_t N_t - K_t = 0,$$

$$\text{Capital demand: } G \equiv \beta\left[\alpha A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}}\right] - R_t - \frac{C(K_t)}{K_t} = 0,$$

$$\text{Labor market equilibrium: } H \equiv \left(\frac{(1-\alpha)A_{t+1}^{1-\alpha}K_t^{\alpha}}{w_{t+1}}\right)^{\frac{1}{\alpha}} - (1+n_{t+1})N_t = 0.$$

$$\mathbf{\Omega} = \begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix} = \begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}$$

$$\mathbf{\Omega} = \frac{\partial H}{\partial w_{t+1}} + \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial K_t} - \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial K_t}\frac{\partial H}{\partial w_{t+1}} < 0.$$

$$\frac{\partial F}{\partial K_t} < 0, \quad \frac{\partial F}{\partial R_t} > 0, \quad \frac{\partial F}{\partial w_{t+1}} = 0, \quad \frac{\partial F}{\partial A_{t+1}} = 0,$$

$$\frac{\partial G}{\partial K_t} < 0, \quad \frac{\partial G}{\partial R_t} = -1, \quad \frac{\partial G}{\partial w_{t+1}} < 0, \quad \frac{\partial G}{\partial A_{t+1}} > 0,$$
$$\frac{\partial H}{\partial K_t} > 0, \quad \frac{\partial H}{\partial R_t} = 0, \quad \frac{\partial H}{\partial w_{t+1}} < 0, \quad \frac{\partial H}{\partial A_{t+1}} > 0.$$

$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial n_{t+1}} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial n_{t+1}} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} 0 & \frac{\partial F}{\partial R_t} & 0 \\ 0 & -1 & \frac{\partial G}{\partial w_{t+1}} \\ 1 & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$
$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{\frac{\partial F}{\partial R_t}\frac{\partial G}{\partial w_{t+1}}}{\mathbf{\Omega}} > 0.$$

$$\frac{\partial R_t}{\partial n_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & -\frac{\partial F}{\partial n_{t+1}} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial n_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & 0 & 0 \\ \frac{\partial G}{\partial K_t} & 0 & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & 1 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$
$$\frac{\partial R_t}{\partial n_{t+1}} = \frac{\frac{\partial G}{\partial w_{t+1}}}{\mathbf{\Omega}} > 0.$$

$$\frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial n_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial n_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & -\frac{\partial H}{\partial n_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & 0 \\ \frac{\partial H}{\partial K_t} & 0 & 1 \end{vmatrix}}{\mathbf{\Omega}}$$
$$\frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{\frac{\partial F}{\partial K_t}\frac{\partial G}{\partial R_t}\left(-\frac{\partial H}{\partial n_{t+1}}\right) - \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial K_t}}{\mathbf{\Omega}} < 0.$$

$$\frac{\partial K_t}{\partial A_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial A_{t+1}} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} 0 & \frac{\partial F}{\partial R_t} & 0 \\ -\frac{\partial G}{\partial A_{t+1}} & -1 & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial A_{t+1}} & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial K_t}{\partial A_{t+1}} = \frac{\frac{\partial F}{\partial R_t}\left(\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}\right)}{\mathbf{\Omega}} > 0.$$

$$\begin{gathered}
\left|\frac{\partial G}{\partial A_{t+1}}\right| \gtreqless \left|\frac{\partial H}{\partial A_{t+1}}\right| \\
(1-\alpha)\beta A_{t+1}^{\frac{1-\alpha}{\alpha}-1}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1}{\alpha}-1} \gtreqless \frac{1-\alpha}{\alpha} A_{t+1}^{\frac{1-\alpha}{\alpha}-1}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1}{\alpha}} K_t \\
\beta\alpha\frac{w_{t+1}}{1-\alpha} \gtreqless K_t \\
\left|\frac{\partial G}{\partial w_{t+1}}\right| \gtreqless \left|\frac{\partial H}{\partial w_{t+1}}\right| \\
(1-\alpha)\beta A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1}{\alpha}-1} w_{t+1}^{-1} \gtreqless \frac{1}{\alpha} A_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\alpha}{w_{t+1}}\right)^{\frac{1}{\alpha}} w_{t+1}^{-1} K_t \\
\beta w_{t+1} \gtreqless \frac{1}{\alpha} K_t.
\end{gathered}$$

$$\begin{gathered}
\left|\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\right| \gtreqless \left|\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}\right| \\
\beta\alpha\frac{w_{t+1}}{1-\alpha}\frac{1}{\alpha}K_t \gtreqless \beta w_{t+1} K_t \\
\frac{1}{1-\alpha} > 1 \text{ as } \alpha \in (0,1) \\
\left|\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\right| > \left|\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}\right| \\
\Leftrightarrow \\
\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}} < 0.
\end{gathered}$$

Thus, the numerator is negative.

$$\frac{\partial R_t}{\partial A_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & -\frac{\partial F}{\partial A_{t+1}} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & 0 & 0 \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial R_t}{\partial A_{t+1}} = \frac{\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}}{\mathbf{\Omega}} > 0.$$

$$\frac{\partial w_{t+1}}{\partial A_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial A_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial A_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & -\frac{\partial H}{\partial A_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & -\frac{\partial G}{\partial A_{t+1}} \\ \frac{\partial H}{\partial K_t} & 0 & -\frac{\partial H}{\partial A_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial w_{t+1}}{\partial A_{t+1}} = \frac{-\frac{\partial H}{\partial A_{t+1}} - \frac{\partial F}{\partial R_t}\left(\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial K_t} + \frac{\partial G}{\partial K_t}\frac{\partial H}{\partial A_{t+1}}\right)}{\mathbf{\Omega}} > 0.$$

Log-utility, i.e., $\lim_{\sigma\to 1}$, which implies $\frac{\partial F}{\partial R_t} = 0$ and thus $\frac{\partial K_t}{\partial n_{t+1}} = 0$ as well as $\frac{\partial K_t}{\partial A_{t+1}} = 0$. The other results remain the same.

With $C(K_t) = 0$ for $K_t > 0$, I have $\frac{\partial G}{\partial K_t} = 0$. All results remain the same. An increase in n_{t+1} leads to a higher demand for capital, which, as the capital supply is upward sloping in the interest rate, leads to a higher equilibrium interest rate, which in turn implies a lower equilibrium wage rate.

With $\sigma > 1$ the optimal savings are decreasing in the interest rate, i.e., the savings supply is now also downward sloping. Hence, $\frac{\partial F}{\partial R} < 0$. This implies that the sign of $\mathbf{\Omega}$ will depend on parameter values. However, it is clear that the sign of the comparative statics $\frac{\partial K_t}{\partial n_{t+1}}$ and the sign of $\frac{\partial R_t}{\partial n_{t+1}}$ as well as the sign of $\frac{\partial K_t}{\partial A_{t+1}}$ and the sign of $\frac{\partial R_t}{\partial A_{t+1}}$ will be opposite, as the numerators are of opposite sign. Therefore, the equilibrium capital stock and the equilibrium interest rate will move in opposite direction.

Endogenous labor supply would imply that $\ell_t(w_t, R_t)$ and $\ell_{t+1}(w_{t+1}, R_{t+1})$. Hence, there are two additional channels that could affect the result. In case $\frac{\partial \ell_t}{\partial R_t} < 0$, $\frac{\partial F}{\partial R_t}$ could become negative, which could change the signs of the comparative statics. If $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} < 0$ then $\frac{\partial H}{\partial w_{t+1}}$ could become positive, which could also change the signs of the comparative statics. In case $\frac{\partial \ell_t}{\partial R_t} \geq 0$ and $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} \geq 0$ the results remain the same.

A.2.2 CES Production Function

With a CES production function, I get two optimality conditions from the firms' optimization problem and the capital supply equation derived from the households' intertemporal optimization. Assuming that the labor and capital market clear, I can use the respective equilibrium conditions, i.e., $K_t = S_t N_t$ and $L_{t+1} = (1+n_{t+1})N_t \ell_{t+1}(w_{t+1}, R_{t+1})$, to write down the three equations that determine the equilibrium.

$$F \equiv \frac{\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}}{1+\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}} w_t \ell_t - K_t = 0,$$

$$G \equiv A_{t+1}\left(\eta^{\frac{1}{1-\rho}} + \eta^{\frac{\rho}{1-\rho}}(1-\eta)\left(\frac{N_{t+1}\ell_{t+1}(w_{t+1}, R_{t+1})}{K_t}\right)^{\rho}\right)^{\frac{1-\rho}{\rho}} - R_t - \frac{C(K_t)}{K_t} = 0,$$

$$H \equiv A_{t+1}\left(\eta(1-\eta)^{\frac{\rho}{1-\rho}}\left(\frac{K_t}{N_{t+1}\ell_{t+1}(w_{t+1}, R_{t+1})}\right)^{\rho} + (1-\eta)^{\frac{1}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} - w_{t+1} = 0.$$

$$\mathbf{\Omega} = \begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix} = \begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}$$

$$\mathbf{\Omega} = \frac{\partial H}{\partial w_{t+1}} + \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial K_t} - \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial K_t}\frac{\partial H}{\partial w_{t+1}}.$$

$$\begin{aligned}
&\frac{\partial F}{\partial K_t} < 0, \quad \frac{\partial F}{\partial R_t} \gtreqless 0, \quad \frac{\partial F}{\partial w_{t+1}} = 0, \quad \frac{\partial F}{\partial A_{t+1}} = 0, \quad \frac{\partial F}{\partial n_{t+1}} = 0, \\
&\frac{\partial G}{\partial K_t} < 0, \quad \frac{\partial G}{\partial R_t} = -1, \quad \frac{\partial G}{\partial w_{t+1}} \gtreqless 0, \quad \frac{\partial G}{\partial A_{t+1}} > 0, \quad \frac{\partial G}{\partial n_{t+1}} > 0, \\
&\frac{\partial H}{\partial K_t} > 0, \quad \frac{\partial H}{\partial R_t} = 0, \quad \frac{\partial H}{\partial w_{t+1}} \gtreqless 0, \quad \frac{\partial H}{\partial A_{t+1}} > 0, \quad \frac{\partial H}{\partial n_{t+1}} < 0.
\end{aligned}$$

Throughout I assume that $\frac{\partial \ell_t}{\partial R_t} \geq 0$ and $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} \geq 0$ holds. This implies $\frac{\partial F}{\partial R_t} > 0$, $\frac{\partial G}{\partial w_{t+1}} > 0$ and $\frac{\partial H}{\partial w_{t+1}} < 0$. I will discuss the consequences of relaxing these assumptions at the end. These assumptions entail that $\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial K_t} - \frac{\partial G}{\partial K_t}\frac{\partial H}{\partial w_{t+1}} < 0$ and thus $\mathbf{\Omega} < 0$ always holds.

$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial n_{t+1}} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial n_{t+1}} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} 0 & \frac{\partial F}{\partial R_t} & 0 \\ -\frac{\partial G}{\partial n_{t+1}} & -1 & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial n_{t+1}} & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial K_t}{\partial n_{t+1}} = \frac{\frac{\partial F}{\partial R_t}\frac{\partial G}{\partial w_{t+1}}\left(-\frac{\partial H}{\partial n_{t+1}}\right) + \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial w_{t+1}}}{\mathbf{\Omega}} > 0,$$

as $\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial w_{t+1}} - \frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial n_{t+1}} < 0$.

$$\frac{\partial R_t}{\partial n_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & -\frac{\partial F}{\partial n_{t+1}} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial n_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & 0 & 0 \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial n_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial n_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial R_t}{\partial n_{t+1}} = \frac{\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial w_{t+1}} - \frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial n_{t+1}}}{\mathbf{\Omega}} > 0,$$

as $\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial w_{t+1}} - \frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial n_{t+1}} < 0$.

$$\frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial n_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial F}{\partial R_t} & -\frac{\partial G}{\partial n_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & -\frac{\partial H}{\partial n_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & -\frac{\partial G}{\partial n_{t+1}} \\ \frac{\partial H}{\partial K_t} & 0 & -\frac{\partial H}{\partial n_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial w_{t+1}}{\partial n_{t+1}} = \frac{-\frac{\partial H}{\partial n_{t+1}} - \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial K_t} + \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial K_t}\frac{\partial H}{\partial n_{t+1}}}{\mathbf{\Omega}} < 0,$$

as $-\frac{\partial G}{\partial n_{t+1}}\frac{\partial H}{\partial K_t} + \frac{\partial G}{\partial K_t}\frac{\partial H}{\partial n_{t+1}} > 0$.

$$\frac{\partial K_t}{\partial A_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial A_{t+1}} & \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial w_{t+1}} \\ -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} 0 & \frac{\partial F}{\partial R_t} & 0 \\ -\frac{\partial G}{\partial A_{t+1}} & -1 & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial A_{t+1}} & 0 & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial K_t}{\partial A_{t+1}} = \frac{\frac{\partial F}{\partial R_t}\left(\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}\right)}{\mathbf{\Omega}} > 0.$$

$$\frac{\partial R_t}{\partial A_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & -\frac{\partial F}{\partial A_{t+1}} & \frac{\partial F}{\partial w_{t+1}} \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & 0 & 0 \\ \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial A_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial A_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial R_t}{\partial A_{t+1}} = \frac{\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial w_{t+1}}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial A_{t+1}}}{\mathbf{\Omega}} > 0.$$

$$\frac{\partial w_{t+1}}{\partial A_{t+1}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial K_t} & \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial A_{t+1}} \\ \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial A_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial R_t} & -\frac{\partial H}{\partial A_{t+1}} \end{vmatrix}}{\mathbf{\Omega}} = \frac{\begin{vmatrix} -1 & \frac{\partial F}{\partial R_t} & 0 \\ \frac{\partial G}{\partial K_t} & -1 & -\frac{\partial G}{\partial A_{t+1}} \\ \frac{\partial H}{\partial K_t} & 0 & -\frac{\partial H}{\partial A_{t+1}} \end{vmatrix}}{\mathbf{\Omega}}$$

$$\frac{\partial w_{t+1}}{\partial A_{t+1}} = \frac{-\frac{\partial H}{\partial A_{t+1}} - \frac{\partial F}{\partial R_t}\left(\frac{\partial G}{\partial A_{t+1}}\frac{\partial H}{\partial K_t} + \frac{\partial G}{\partial K_t}\frac{\partial H}{\partial A_{t+1}}\right)}{\mathbf{\Omega}} > 0.$$

Similarly, to the Cobb-Douglas case, if $\frac{\partial \ell_t}{\partial R_t} < 0$, then $\frac{\partial F}{\partial R_t}$ could become negative, which could change the signs of the comparative statics. If $\frac{\partial \ell_{t+1}}{\partial w_{t+1}} < 0$ then $\frac{\partial H}{\partial w_{t+1}}$ could become positive, which could also change the signs of the comparative statics.

A.2.3 Income and Substitution Effect

Consider a general CES production function. ζ is the elasticity of substitution, where $\zeta \in (0, \infty)$. $\rho = \frac{\zeta - 1}{\zeta}$ and thus $\rho \in (-\infty, 1)$. ν denotes the returns to scale (RTS). We have decreasing returns to scale (DRTS) if $\nu < 1$, constant returns to scale (CRTS) if $\nu = 1$ and increasing returns to scale (IRTS) if $\nu > 1$.

$$\max_{\{x_i\}_{i=1}^N} A\left(\sum_{i=1}^N \eta_i x_i^\rho\right)^{\frac{\nu}{\rho}} - \sum_{i=1}^N w_i x_i.$$

Optimal demand for input factor j is given as

$$x_j = \left(\frac{\eta_j}{w_j}\right)^{\frac{1}{1-\rho}} (A\nu)^{\frac{1}{1-\nu}} \left(\sum_{i=1}^{N} \left(\frac{\eta_i}{w_i^{\rho}}\right)^{\frac{1}{1-\rho}}\right)^{\frac{\nu-\rho}{\rho(1-\nu)}}.$$

The effect of a change in the price of factor k on the demand of input j, with $k \neq j$ is given as

$$\left.\frac{\partial x_j}{\partial w_k}\right|_{k\neq j} = \frac{\nu-\rho}{\rho(1-\nu)} \left(\frac{\eta_j}{w_j}\right)^{\frac{1}{1-\rho}} (A\nu)^{\frac{1}{1-\nu}} \left(\sum_{i=1}^{N} \left(\frac{\eta_i}{w_i^{\rho}}\right)^{\frac{1}{1-\rho}}\right)^{\frac{\nu-\rho}{\rho(1-\nu)}-1} \frac{(-\rho)}{1-\rho} \left(\frac{\eta_k}{w_k^{\rho}}\right)^{\frac{1}{1-\rho}} \frac{1}{w_k} \gtreqless 0,$$

$$\left.\frac{\partial x_j}{\partial w_k}\right|_{k\neq j} = \frac{\rho-\nu}{(1-\nu)(1-\rho)} \left(\frac{\eta_j}{w_j}\right)^{\frac{1}{1-\rho}} (A\nu)^{\frac{1}{1-\nu}} \left(\sum_{i=1}^{N} \left(\frac{\eta_i}{w_i^{\rho}}\right)^{\frac{1}{1-\rho}}\right)^{\frac{\nu-\rho}{\rho(1-\nu)}-1} \left(\frac{\eta_k}{w_k^{\rho}}\right)^{\frac{1}{1-\rho}} \frac{1}{w_k} \gtreqless 0.$$

Excluding the limit cases implies that $\rho < 1$. Moreover, assume we have either CRTS or DRTS, i.e., $\nu \leq 1$.

$$\left.\frac{\partial x_j}{\partial w_k}\right|_{k\neq j} \begin{cases} > 0 & \text{if } 1 > \rho > \nu > 0, \\ \leq 0 & \text{if } 1 \geq \nu > \rho > 0. \end{cases}$$

Hence, in case of DRTS and a high enough elasticity of substitution, i.e., a large enough value for ρ, the substitution effect dominates the income effect, i.e., the relatively more expensive factor k is substituted by the relatively cheaper factor j. In case of CRTS the income effect always dominates the substitution effect. Thus, in this case an increase in the price of one factor always lowers the demand for all other factors.

A.3 Investments with Fixed Costs

Consider a standard CES production function with no convex costs and fixed factor supply, as in Section 1.7.

$$F(K_t, L_{t+1}) = A_{t+1} \left(\eta K_t^{\rho} + (1-\eta) L_{t+1}^{\rho}\right)^{\frac{1}{\rho}}.$$

The first-order conditions are given as

$$G \equiv \frac{\varepsilon - 1}{\varepsilon} F_{K_t} - R_t = 0,$$
$$H \equiv \frac{\varepsilon - 1}{\varepsilon} F_{L_{t+1}} - w_{t+1} = 0,$$

with K_t and w_{t+1} as the endogenous variables.[1]

$$\frac{\partial K_t}{\partial L_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial G}{\partial L_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial L_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{\begin{vmatrix} -\frac{\partial G}{\partial L_{t+1}} & 0 \\ -\frac{\partial H}{\partial L_{t+1}} & -1 \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial K_t} & 0 \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{F_{K_t L_{t+1}}}{-F_{K_t K_t}} > 0,$$

$$\frac{\partial w_{t+1}}{\partial L_{t+1}} = \frac{\begin{vmatrix} \frac{\partial G}{\partial K_t} & -\frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial L_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial K_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial K_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{-F_{K_t K_t} F_{L_{t+1} L_{t+1}} + F_{K_t L_{t+1}} F_{L_{t+1} K_t}}{-F_{K_t K_t}} = 0.$$

Keeping K_t fixed, a change in L_{t+1} has the following effects on R_t and w_{t+1}

$$\frac{\partial R_t}{\partial L_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial G}{\partial L_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ -\frac{\partial H}{\partial L_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{\frac{\varepsilon-1}{\varepsilon} F_{K_t L_{t+1}}}{1} > 0,$$

[1] The case of a Cobb-Douglas production function follows analogously.

$$\frac{\partial w_{t+1}}{\partial L_{t+1}} = \frac{\begin{vmatrix} \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial K_t} & -\frac{\partial H}{\partial L_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial R_t} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{-\frac{\varepsilon-1}{\varepsilon} F_{L_{t+1}K_t}}{1} < 0.$$

Hence, a fall in the equilibrium capital-to-labor ratio decreases the wage rate and increases the interest rate.

A.4 Zero Lower Bound

To analyze what happens if the interest rate cannot fall to equate capital demand and capital supply, I introduce the asset B_t which yields a constant real return R_f and is available in potentially infinite supply. The real return from investing in the production sector is the same as before, i.e., R_t. This implies the following no-arbitrage condition: $R_t \geq R_f \; \forall t$, i.e., the return of capital used in the production sector cannot be lower than the return of the risk-free asset, because otherwise households would not be willing to invest in the production sector. Therefore, R_f constitutes a lower bound, below which the equilibrium return to capital cannot fall. A negative value of R_f implies that storing wealth in B_t has a cost, e.g., a positive inflation rate in case B_t constitutes cash.

$$S_t = K_t + B_t,$$

and thus, the net return households receive on their savings is given as follows

$$R_t = \begin{cases} R_t & \text{if} \quad B_t = 0, \\ R_f & \text{otherwise.} \end{cases}$$

The capital supply with the equilibrium condition hence looks the following

$$\text{Capital supply: } F \equiv \frac{\beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}}} w_t N_t - K_t - B_t = 0.$$

The other two equations remain unchanged. The (zero) lower bound implies that R_t is fixed, and hence I use B_t to replace R_t as an endogenous variable. It is straightforward to verify that $\frac{\partial F}{\partial B_t} = -1$, $\frac{\partial G}{\partial B_t} = 0$ and $\frac{\partial H}{\partial B_t} = 0$. This implies that

$$\mathbf{\Omega} = \frac{\partial F}{\partial B_t}\frac{\partial G}{\partial w_{t+1}}\frac{\partial H}{\partial K_t} - \frac{\partial F}{\partial B_t}\frac{\partial G}{\partial K_t}\frac{\partial H}{\partial w_{t+1}} > 0,$$

as well as

$$\begin{aligned}
\frac{\partial K_t}{\partial n_{t+1}} &= \frac{\frac{\partial F}{\partial B_t}\frac{\partial G}{\partial w_{t+1}}}{\mathbf{\Omega}} > 0,\\
\frac{\partial B_t}{\partial n_{t+1}} &= \frac{\frac{\partial G}{\partial w_{t+1}}}{\mathbf{\Omega}} < 0,\\
\frac{\partial w_{t+1}}{\partial n_{t+1}} &= \frac{-\frac{\partial F}{\partial B_t}\frac{\partial G}{\partial K_t}}{\mathbf{\Omega}} < 0.
\end{aligned}$$

Population aging, i.e., a decrease in n_{t+1}, implies that K_t is reduced, and as the interest rate is fixed, B_t increases to ensure we are in equilibrium. The wage rate will again increase due to the convex costs of capital, which imply a rise in the capital stock per worker. Note that w_{t+1} is now only affected through the convex costs, as the channel working through the interest rate is now closed.

Appendix B

Appendix to Chapter 2

B.1 Figures and Tables

(a)

(b)

The figures display the ratio of gross capital formation as % of GDP relative to gross savings as % of GDP. The data is taken from the World Bank.

Figure B.1: *Investment-to-Savings Ratio*

Figure B.1 displays the ratio of gross capital formation as % of GDP relative to gross savings as % of GDP. For the OECD countries, we observe a downward trend that implies that investments relative to savings are falling, which could entail further downward pressure on real interest rates. Moreover, the ratio has fallen despite the low interest rates of the past few years.

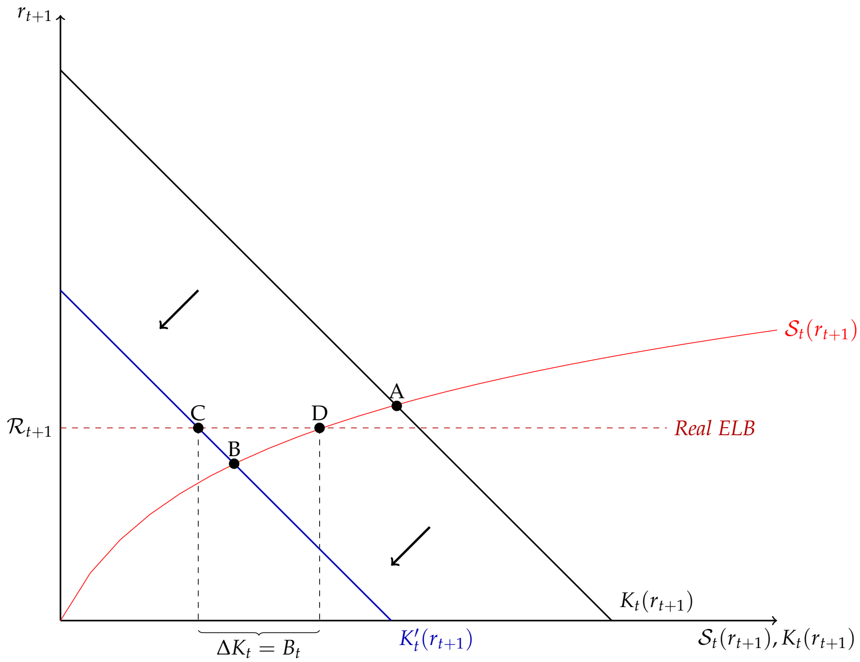


Figure B.2: *Decrease in Capital Demand*

In Figure B.2 the economy was originally at point A. capital demand, K_t, then decreases to K_t' due to an exogenous shock. Without any frictions, the new equilibrium would be at point B. However, due to the real ELB, the interest rate cannot fall below $\mathcal{R}_{t+1}$. Thus, investment is equal to point C. With the interest rate equal to $\mathcal{R}_{t+1}$ savings are equal to point D. The difference between points C and D constitutes excess savings that will not be invested by the firms, i.e., B_t.

Table B.1: *Endogenous Variables*

	Real ELB binding in period $t+1$	
	No	Yes
Endogenous variables in period t	Wage rate (w_t) Return to capital (r_t)	Wage rate (w_t) Return to capital (r_t) Unemployment rate (η_t)
Endogenous variables in period $t+1$	Wage rate (w_{t+1}) Return to capital (r_{t+1})	Wage rate (w_{t+1})

B.2 Demographic Change and Output per Capita

First, consider the standard Solow model. The production function is denoted by $F(K_t, A_t L_t)$ and has constant returns to scale in K_t and L_t. Moreover, we have $F_{K_t} > 0$ and $F_{L_t} > 0$. Let $f(k_t) := F(\frac{K_t}{A_t L_t}, 1)$ and $k_t = \frac{K_t}{A_t L_t}$ with $f'(k_t) > 0$ and $f''(k_t) < 0$.

In steady state, it holds that

$$G \equiv sf(k^*) - (n + g + \delta)k^* = 0,$$

which entails

$$\frac{\partial G}{\partial k^*} = sf'(k^*) - (n + g + \delta) \gtreqless 0.$$

Assuming $f(k_t)$ is homogeneous of degree α with $\alpha \in (0, 1)$, then by Euler's homogeneous function theorem and the definition of G it follows that

$$\begin{aligned} k^* \frac{\partial G}{\partial k^*} =& sf'(k^*)k^* - (n + g + \delta)k^* \gtreqless 0 \\ k^* \frac{\partial G}{\partial k^*} =& s\alpha f(k^*) - (n + g + \delta)k^* < 0, \end{aligned}$$

as $k^* > 0$, it follows

$$\frac{dk^*}{dn} < 0.$$

Output per capita adjusted for technology in steady state is given as[1]

$$y^* = f(k^*),$$

it follows

$$\frac{\partial y^*}{\partial(-n)} = f'(k^*)\frac{dk^*}{d(-n)} > 0.$$

Therefore, in a standard Solow model, a decrease in the population growth rate leads to a higher steady state level of output per capita. The economic intuition for this result is that a reduction in population growth increases the capital stock per worker, which in turn implies that each worker is more productive.

Second, consider a standard two-period OLG model. In this model, a decrease in the population growth rate also leads to a shift in the demographic structure, which is not the case in the Solow model. Output is produced using labor and capital and a standard neoclassical constant returns to scale production function, as in the Solow model.
Output per capita in period t is given as

$$\begin{aligned} y_t =& \frac{F(K_t, A_t(1+n_t)L_{t-1})}{(1+n_t)L_{t-1} + L_{t-1}} \\ y_t =& \frac{F(K_t, A_t L_t)}{L_t + L_{t-1}}, \end{aligned}$$

the effect of a fall in n_t and thus L_t on y_t is given as

$$\frac{\partial y_t}{\partial(-L_t)} = -\frac{\left(F_{L_t} + F_{K_t}\frac{\partial K_t}{\partial L_t}\right)(L_t + L_{t-1}) - F(K_t, A_t L_t)}{(L_t + L_{t-1})^2} \gtreqless 0.$$

[1]The level of technology is independent of population growth, and thus output per capita adjusted for technology is proportional to output per capita. Moreover, if the growth rate of technology is positive, output per capita will always grow independent of a change in the population growth rate. By adjusting for technology, I take this factor into account.

There are two competing effects at work. On the one hand, each worker becomes more productive as the marginal product of labor increases, assuming the capital stock per worker increases, which positively affects output per worker. On the other hand, overall output decreases as there are fewer workers and the ratio of non-workers to workers increases, both of which have a negative effect on output per capita.

As an example, consider an economy with a Cobb-Douglas production function:

$$\frac{\partial y_t}{\partial(-L_t)} = \frac{\alpha K_t^{\alpha}(A_t L_t)^{1-\alpha}\left(1 - \frac{\partial K_t}{\partial L_t}\frac{L_t}{K_t}\frac{L_t + L_{t-1}}{L_t}\right) - (1-\alpha)K_t^{\alpha}(A_t L_t)^{1-\alpha}\frac{L_{t-1}}{L_t}}{(L_t + L_{t-1})^2} \gtreqless 0.$$

The first term captures the increase in the marginal product of labor, which is positive if the capital stock per worker increases, i.e., the overall capital stock remains constant or does not decrease by too much. The second term is always negative and captures the fall in overall labor supply and the increase in the ratio of non-workers to workers. Note that for $\alpha = 0$, i.e., production only requires labor and has c.r.s., the first effect is shut down, and only the second effect remains.

We can also analyze the steady state of a two-period OLG model. Output per capita adjusted for technology is given as

$$\frac{Y_t}{A_t L_t\left(1 + \frac{1}{1+n_t}\right)} = \frac{f(k_t)}{1 + \frac{1}{1+n_t}} = y_t.$$

Assume the economy is in steady state and experiences a negative shock to n, i.e., the population growth rate. This entails for y^*

$$\frac{\partial y^*}{\partial(-n)} = -\frac{f'(k^*)\frac{dk^*}{dn}\left(1 + \frac{1}{1+n_t}\right) + f(k^*)\frac{1}{(1+n)^2}}{\left(1 + \frac{1}{1+n_t}\right)^2} \gtreqless 0.$$

The sign of the effect is ambigious. However, it shows that a necessary, albeit not sufficient, condition for a decrease in the population growth rate to increase output per capita is that

$\frac{dk^*}{d(-n)} > 0$, i.e., that a decrease in the population growth rate increases the steady state value of capital per efficiency unit of labor.

B.3 Effective Upper Bound

Assume a constant returns to scale aggregate CES production function

$$Y_{t+1} = F(K_{t+1}, L_{t+1}, (1 + g_{t+1})A_t).$$

With a potentially binding real effective lower bound (ELB) in the economy, i.e., the real interest rate cannot fall below $\mathcal{R}_{t+1}$, there exists a maximum amount of savings $\bar{\mathcal{S}}_t$ the economy can absorb. With full depreciation, this implies $\bar{\mathcal{S}}_t = \bar{K}_{t+1}$. Therefore, an effective lower bound on the interest rate implies there is an effective upper bound (EUB) on the capital stock the economy can absorb.

$\bar{K}_{t+1}$ hence denotes the EUB on the capital stock in period $t+1$, where $\bar{K}_{t+1}$ is implicitly defined by the following system of two equations

$$\begin{aligned}
&G \equiv \mu F_{\bar{K}_{t+1}}(\bar{K}_{t+1}, L_{t+1}, (1 + g_{t+1})A_t) - \mathcal{R}_{t+1} = 0, \\
&H \equiv L^S_{t+1}(w_{t+1}, \mathbf{\Theta}_{t+1}) - L^D_{t+1} = 0 \\
&\Leftrightarrow \\
&H \equiv L^S_{t+1}(F_{L_{t+1}}(\bar{K}_{t+1}, L_{t+1}, (1 + g_{t+1})A_t), \mathbf{\Theta}_{t+1}) - L^D_{t+1} = 0.
\end{aligned}$$

$\mu = \frac{\varepsilon-1}{\varepsilon} < 1$ denotes the inverse mark-up. In equilibrium, it must hold that $L^S_{t+1} = L^D_{t+1} = L_{t+1}$, i.e., labor supply is equal to labor demand. Here, L_{t+1} and $\bar{K}_{t+1}$ are the endogenous variables, and $\mathbf{\Theta}_{t+1}$ is a row vector that captures all exogenous parameters that affect optimal labor supply. I assume that $\frac{\partial L^S_{t+1}}{\partial w_{t+1}} \geq 0$, i.e., optimal labor supply, is never decreasing in the wage rate.

Therefore, the maximum amount of savings that can be absorbed depends on the growth rate of technology g_{t+1} where $(1 + g_{t+1})A_t = A_{t+1}$ and the parameters that affect equilibrium labor supply $\mathbf{\Theta}_{t+1}$ where $\mathbf{\Theta}_{t+1} = (\Theta_{1,t+1}, ..., \Theta_{n,t+1})$, e.g., the population growth rate n_{t+1}. Let $F_{\bar{K}_{t+1}}$ denote the first and $F_{\bar{K}_{t+1}\bar{K}_{t+1}}$ denote the second derivative of $F(K_{t+1}, L_{t+1}, (1+g_{t+1})A_t)$ with respect to $\bar{K}_{t+1}$. I make the standard assumptions that $F_{\bar{K}_{t+1}} > 0$, $F_{\bar{K}_{t+1}\bar{K}_{t+1}} < 0$, $F_{L_{t+1}} > 0$, $F_{L_{t+1}L_{t+1}} < 0$ and $F_{\bar{K}_{t+1}L_{t+1}} > 0$.

$$\begin{aligned}
\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix} &= F_{\bar{K}_{t+1}\bar{K}_{t+1}} \left(L^S_{F_{L_{t+1}}} F_{L_{t+1}L_{t+1}} - 1 \right) - F_{\bar{K}_{t+1}L_{t+1}} L^S_{F_{L_{t+1}}} F_{L_{t+1}\bar{K}_{t+1}} \\
&= -F_{\bar{K}_{t+1}\bar{K}_{t+1}} + L^S_{F_{L_{t+1}}} \underbrace{\left(F_{\bar{K}_{t+1}\bar{K}_{t+1}} F_{L_{t+1}L_{t+1}} - F_{\bar{K}_{t+1}L_{t+1}} F_{L_{t+1}\bar{K}_{t+1}} \right)}_{=0} \\
&= -F_{\bar{K}_{t+1}\bar{K}_{t+1}} > 0, \\
\begin{vmatrix} -\frac{\partial G}{\partial g_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ -\frac{\partial H}{\partial g_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix} &= -\underbrace{\frac{\partial G}{\partial g_{t+1}}}_{<0} \underbrace{\frac{\partial H}{\partial L_{t+1}}}_{<0} + \underbrace{\frac{\partial G}{\partial L_{t+1}}}_{>0} \underbrace{\frac{\partial H}{\partial g_{t+1}}}_{>0} > 0, \\
\begin{vmatrix} -\frac{\partial G}{\partial \Theta_{i,t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ -\frac{\partial H}{\partial \Theta_{i,t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix} &= -\underbrace{\frac{\partial G}{\partial \Theta_{i,t+1}}}_{=0} \underbrace{\frac{\partial H}{\partial L_{t+1}}}_{<0} + \underbrace{\frac{\partial G}{\partial L_{t+1}}}_{>0} \underbrace{\frac{\partial H}{\partial \Theta_{i,t+1}}}_{\gtreqless 0} \gtreqless 0.
\end{aligned}$$

The second-order necessary conditions for a local interior optima state that the Hessian is negative semidefinite.

This requires that $F_{\bar{K}_{t+1}\bar{K}_{t+1}} < 0$, $F_{L_{t+1}L_{t+1}} < 0$ and $F_{\bar{K}_{t+1}\bar{K}_{t+1}} F_{L_{t+1}L_{t+1}} - F_{\bar{K}_{t+1}L_{t+1}} F_{L_{t+1}\bar{K}_{t+1}} \geq 0$.

$$\frac{\partial \bar{K}_{t+1}}{\partial g_{t+1}} = \frac{\begin{vmatrix} -\frac{\partial G}{\partial g_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ -\frac{\partial H}{\partial g_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix}} > 0, \quad \frac{\partial \bar{K}_{t+1}}{\partial \Theta_{i,t+1}} = \frac{\begin{vmatrix} -\frac{\partial G}{\partial \Theta_{i,t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ -\frac{\partial H}{\partial \Theta_{i,t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix}} \gtreqless 0.$$

$$\frac{\partial \bar{K}_{t+1}}{\partial \Theta_{i,t+1}} \begin{cases} > 0 & \text{if} \quad \frac{\partial H}{\partial \Theta_{i,t+1}} > 0, \\ < 0 & \text{if} \quad \frac{\partial H}{\partial \Theta_{i,t+1}} < 0. \end{cases}$$

The overall effect of $\Theta_{i,t+1}$ on the equilibrium real interest and wage rate in period t is then given by the chain rule as

$$\begin{aligned}\frac{dr_t}{d\Theta_{i,t+1}} &= \frac{\partial r_t}{\partial \bar{K}_{t+1}} \frac{\partial \bar{K}_{t+1}}{\partial \Theta_{i,t+1}}, \\ \frac{dw_t}{d\Theta_{i,t+1}} &= \frac{\partial w_t}{\partial \bar{K}_{t+1}} \frac{\partial \bar{K}_{t+1}}{\partial \Theta_{i,t+1}}.\end{aligned}$$

$\Theta_{i,t+1}$ affects w_t and r_t only indirectly through $\bar{K}_{t+1}$. In case the real ELB is not binding, $\bar{K}_{t+1}$ is not an endogenous variable and hence has no effect on w_t or r_t, i.e., $\frac{\partial r_t}{\partial \bar{K}_{t+1}} = 0$ and $\frac{\partial w_t}{\partial \bar{K}_{t+1}} = 0$.

With $\bar{K}_{t+1}$ and w_t as the endogenous variables, we have the following system of equilibrium conditions

$$\begin{aligned}G &\equiv F_{\bar{K}_{t+1}}(\bar{K}_{t+1}, L_{t+1}(w_{t+1}, \mathbf{\Theta}_{t+1}), (1+g_{t+1})A_t) - \mathcal{R}_{t+1} = 0, \\ H &\equiv F_{L_{t+1}}(\bar{K}_{t+1}, L_{t+1}(w_{t+1}, \mathbf{\Theta}_{t+1}), (1+g_{t+1})A_t)) - w_{t+1} = 0,\end{aligned}$$

where $L_{t+1}(\cdot)$ denotes the equilibrium labor supply in the economy. Again, I assume that that $\frac{\partial L_{t+1}}{\partial w_{t+1}} \geq 0$, i.e., optimal labor supply, and thus equilibrium employment, is increasing in the wage rate.

$$\begin{aligned}\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix} &= F_{\bar{K}_{t+1}\bar{K}_{t+1}} \left(F_{L_{t+1}L_{t+1}} L_{w_{t+1}} - 1 \right) - F_{\bar{K}_{t+1}L_{t+1}} L_{w_{t+1}} F_{L_{t+1}\bar{K}_{t+1}} \\ &= -F_{\bar{K}_{t+1}\bar{K}_{t+1}} > 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial w_{t+1}}{\partial \Theta_{i,t+1}} &= \frac{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & -\frac{\partial G}{\partial \Theta_{i,t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & -\frac{\partial H}{\partial \Theta_{i,t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial w_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial w_{t+1}} \end{vmatrix}} = \frac{\left(-F_{\bar{K}_{t+1}\bar{K}_{t+1}} F_{L_{t+1}L_{t+1}} + F_{\bar{K}_{t+1}L_{t+1}} F_{L_{t+1}\bar{K}_{t+1}} \right) \frac{\partial L_{t+1}(\cdot)}{\partial \Theta_{i,t+1}}}{-F_{\bar{K}_{t+1}\bar{K}_{t+1}}} \\ &= \frac{0}{-F_{\bar{K}_{t+1}\bar{K}_{t+1}}} = 0.\end{aligned}$$

$$\frac{\partial w_{t+1}}{\partial g_{t+1}} = \frac{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & -\frac{\partial G}{\partial g_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & -\frac{\partial H}{\partial g_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial L_{t+1}} \\ \frac{\partial H}{\partial \bar{K}_{t+1}} & \frac{\partial H}{\partial L_{t+1}} \end{vmatrix}} = \frac{-F_{\bar{K}_{t+1}\bar{K}_{t+1}} F_{L_{t+1}g_{t+1}} + F_{\bar{K}_{t+1}g_{t+1}} F_{L_{t+1}\bar{K}_{t+1}}}{-F_{\bar{K}_{t+1}\bar{K}_{t+1}}} > 0.$$

Therefore, if the real ELB is binding in period $t+1$, a change in labor supply in period $t+1$ will leave the equilibrium wage rate in said period unchanged. This is due to the binding real ELB, which entails that the marginal product of capital must remain constant and so must the marginal product of labor.

In contrast, an increase in the growth rate of technology will still have a positive effect on the wage rate in period $t+1$, as this mechanism operates independently from the interest rate.

I assume full depreciation, and hence the law of motion for the capital stock is given as

$$K_{t+1} = I_t,$$

and therefore

$$\bar{K}_{t+1} = \bar{I}_t,$$

thus, the maximum investments that the economy can absorb in period t, i.e. $\bar{I}_t$, are equal to the maximum capital stock the economy can sustain in period $t+1$, i.e., $\bar{K}_{t+1}$.

Without full depreciation, the law of motion for the capital stock becomes

$$K_{t+1} = I_t + (1-\delta)K_t,$$

thus, the maximum investments that the economy can absorb in period t, i.e., $\bar{I}_t$, are given as

$$\bar{I}_t = \bar{K}_{t+1} - (1-\delta)K_t.$$

B.4 Households

As described in the main text, households live for two periods. They can save in two different assets. On the one hand, they can invest their nominal savings in the capital stock of the next period, i.e., invest in s_t. In this case, they receive the nominal gross return $R_{t+1} = P_{t+1}r_{t+1}$, where r_{t+1}, the real gross return on capital in period $t+1$, is determined endogenously through demand and supply. On the other hand, they can save in the nominal asset B_t, i.e., money, which delivers the exogenous and constant nominal gross return $R_f \geq 1$ and whose price is equal to 1 as it serves as the numeraire.

The utility maximization problem of household j born in period t is given as[2][3][4]

$$
\begin{aligned}
\max_{c^j_{1,t}, c^j_{2,t+1}, s^j_t, B^j_t} \; & u(c^j_{1,t}) + \beta u(c^j_{2,t+1}) \\
\text{s.t. } P_t c^j_t &= W_t - P_t s^j_t - B^j_t \\
P_{t+1} c^j_{2,t+1} &= s^j_t R_{t+1} + P_{t+1} d^j_{t+1} + B^j_t R_f \\
B^j_t &\geq 0 \\
&\Leftrightarrow \\
c^j_{1,t} &= w_t - s^j_t - \frac{B^j_t}{P_t} \\
c^j_{2,t+1} &= s^j_t r_{t+1} + d^j_{t+1} + \frac{B^j_t R_f}{P_{t+1}} \\
B^j_t &\geq 0.
\end{aligned}
$$

---

[2]Strictly speaking, the non-negativity constraint on B_t is not required, as I assume households cannot be indebted between periods, i.e., all debt taken on at the beginning of a period must be redeemed at the end of said period. Therefore, a negative position in B_t would be equivalent to a lower s_t, i.e., the maximum amount households can spend on consumption in period t is W_t, which due to the budget constraint implies $P_t s_t = B_t = 0$.

[3]The budget constraint for the second period contains an s_t, because the equilibrium in the capital market implies $N^y_t s^j_t = K_{t+1}$.

[4]To keep the notion as simple as possible, I have dropped the term $(1 - \eta_t)$, i.e., the potential rationing of working hours.

$$\mathcal{L} = u(c^j_{1,t}) + \beta u(c^j_{2,t+1}) + \lambda_t \left(w_t - c^j_{1,t} - s^j_t - \frac{B^j_t}{P_t} \right) + \lambda_{t+1} \left(s^j_t r_{t+1} + d^j_{t+1} + \frac{B^j_t R_f}{P_{t+1}} - c^j_{2,t+1} \right).$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c^j_{1,t}} =& u'(c^j_{1,t}) - \lambda_t = 0, \\
\frac{\partial \mathcal{L}}{\partial c^j_{2,t+1}} =& \beta u'(c^j_{2,t+1}) - \lambda_{t+1} = 0, \\
\frac{\partial \mathcal{L}}{\partial s^j_t} =& -\lambda_t + \lambda_{t+1} r_{t+1} = 0, \\
\frac{\partial \mathcal{L}}{\partial B^j_t} =& -\lambda_t \frac{1}{P_t} + \lambda_{t+1} \frac{R_f}{P_{t+1}} = 0.
\end{aligned}$$

I can derive the equations that determine the optimal demand for real savings s_t and nominal savings B_t. As all households are symmetric, I drop the index j.

$$\begin{aligned}
u'(c_t) =& \beta r_{t+1}(s_t) u'(c_{t+1}), \\
r_{t+1}(s_t) \geq& \frac{P_t}{P_{t+1}} R_f.
\end{aligned}$$

The first equation is the standard Euler equation, and the second is the (endogenous) no-arbitrage condition. If the inequality is strict, then the return from saving in real assets will always be larger than the return from saving in nominal assets, and thus households will never save in nominal assets, i.e., $B_t = 0$.[5] I denote the real return of the nominal asset, i.e., money, by $\mathcal{R}_{t+1}$.

$$\begin{aligned}
\mathcal{R}_{t+1} =& \frac{P_t}{P_{t+1}} R_f \\
\mathcal{R}_{t+1} =& \frac{1}{\Pi_{t+1}} R_f,
\end{aligned}$$

[5]Note that the second equation also corresponds to the Fisher equation, with $R_f = 1 + i$, where i corresponds to the nominal interest rate and Π_{t+1} to one plus the (expected) inflation rate.

The households face a standard consumption-saving problem, and the only difference is that they can allocate their savings to two different assets. As I assume no risk, households will allocate their savings so as to maximize real returns. The real return on asset B_t, i.e., money, represents the lower bound on the real return households will earn on their savings. However, if the equilibrium real return on the real asset is higher, then households will never find it optimal to invest in the nominal asset, i.e., B_t will be set to zero, and we have a standard textbook consumption-savings problem.[6]

B.5 Effective Lower Bound

Let $\mathcal{S}_t$ denote aggregate savings in real terms, $\omega_t \in (0, 1]$ the fraction of overall savings in real terms invested in real assets, i.e., in the capital stock of the next period, and $\varsigma_t \in (0, 1)$ the savings rate. This implies

$$\begin{aligned} P_t\mathcal{S}_t =& \varsigma_t W_t(1-\eta_t)N_t^y \\ P_t\mathcal{S}_t =& (S_t^j + B_t^j)N_t^y \\ \mathcal{S}_t =& (s_t^j + b_t^j)N_t^y. \end{aligned}$$

and thus

$$N_t^y s_t^j = \omega_t \mathcal{S}_t, \quad N_t^y b_t^j = (1-\omega_t)\mathcal{S}_t.$$

Assuming the economy is in a situation where the real ELB is binding in period $t+1$, households will choose $\omega_t < 1$[7] such that the real return on both assets is equalized[8]

$$r_{t+1}(\mathcal{S}_t\omega_t) = \mathcal{R}_{t+1},$$

[6] See also Mundell (1963) and Tobin (1965), i.e., the Mundell-Tobin effect.

[7] With no binding real ELB we always have $\omega_t = 1$.

[8] Notice that P_t has no direct effect on r_{t+1}, as savings and investments are indexed with the same price level, i.e., we either have $S_t = P_t K_{t+1}$ or $s_t = K_{t+1}$.

where $\mathcal{R}_{t+1}$ constitutes the real ELB in period $t+1$. This entails[9]

$$\frac{d\omega_t}{d\mathcal{R}_{t+1}} < 0, \quad \frac{d\omega_t}{d\mathcal{S}_t} < 0.$$

Hence, a fall in $\mathcal{R}_{t+1}$ or $\mathcal{S}_t$ will increase the share of savings invested in the real asset.

A change in $\mathcal{R}_{t+1}$ requires a change in P_t or P_{t+1}. Therefore, the effect of P_t on ω_t is given as

$$\frac{\partial\omega_t}{\partial P_t} = \frac{d\omega_t}{d\mathcal{R}_{t+1}}\frac{\partial\mathcal{R}_{t+1}}{\partial P_t} < 0.$$

From before

$$\begin{aligned}(1-\omega_t)\varsigma_t W_t =& B_t \\ (1-\omega_t)\varsigma_t w_t =& b_t,\end{aligned}$$

where b_t denotes the real value of the excess savings. Therefore, a reduction in P_t will not have a direct effect on excess savings. Only by lowering $\mathcal{R}_{t+1}$ and correspondingly increasing ω_t, does a fall in P_t eliminate the excess savings. To eliminate excess savings, we require a fall in the real ELB, which requires deflation. Alternatively, reducing income through higher unemployment will also result in lower savings and, hence, lower excess savings.

B.5.1 Deflationary Spiral

Inflation in period $t+1$ implies deflation in period t and thus increases $\mathcal{R}_t$. Therefore, it is now possible that the real ELB binds in period t. Assuming this is the case, this would require a fall in P_{t-1} to alleviate the problem of deflation in period t.[10] Hence, deflation would be pushed to an earlier period. In order for the economy to be able to absorb

[9] $\frac{\partial r_{t+1}}{\partial\omega_t} < 0$ and $\frac{\partial r_{t+1}}{\partial\mathcal{S}_t} < 0$ as the marginal product of capital is positive but decreasing. However, $\frac{\partial\mathcal{R}_{t+1}}{\partial\omega_t} = 0$ as R_f is exogenous and as I assume households do not take into account how the equilibrium price level will change in response to changes in ω_t.

[10] Assuming R_f remains constant, a fall in P_{t-1} that is equivalent to the fall in P_t will leave $\mathcal{R}_t$ unchanged and thus prevent the real ELB from binding.

the deflationary pressure, I require that there exists a period $t-k$ in which $r_{t-k} \geq \mathcal{R}_{t-k}$ continues to hold after the increase in $\mathcal{R}_{t-k}$ that is due to the fall in P_{t-k}, i.e., the equilibrium interest rate, which is determined by structural factors, such as population and technology, is higher than the new real ELB in period $t-k$, denoted by $\mathcal{R}^*_{t-k} > \mathcal{R}_{t-k}$.[11] Hence, the real equilibrium return on capital must be higher than the nominal return on money divided by one plus the rate of inflation or deflation, i.e., the real return on money. Assuming a period $t-k$ exists that satisfies the above condition, the economy can eliminate excess savings through deflation and inflation and avoid a deflationary spiral.

This implies, assuming agents have perfect foresight and fully flexible prices and wages, that the price levels are linked over time as long as the real ELB is binding in some period t and the deflationary shock has not yet been absorbed, i.e., we are before period $t-k$.

More specifically, this entails that the effect of a fall in P_t on $\mathcal{M}_{t-1}$, i.e., nominal demand in period $t-1$, is given as

$$\frac{\partial \mathcal{M}_{t-1}}{\partial(-P_t)} \begin{cases} = 0 & \text{if } r^*_t \geq \mathcal{R}^*_t, \\ < 0 & \text{if } r^*_t < \mathcal{R}^*_t, \end{cases}$$

where the * denotes the equilibrium value of the variable. The effect of a fall in P_t on $\mathcal{M}_{t-1}$ is given as

$$\frac{\partial \mathcal{M}_{t-1}}{\partial(-P_t)} = \underbrace{\frac{\partial \mathcal{M}_{t-1}}{\partial b_{t-1}}}_{<0} \underbrace{\frac{\partial b_{t-1}}{\partial \omega_{t-1}}}_{<0} \underbrace{\frac{\partial \omega_{t-1}}{\partial \mathcal{R}_t}}_{\leq 0} \underbrace{\frac{\partial \mathcal{R}_t}{\partial(-P_t)}}_{>0} \leq 0.$$

For $\left.\frac{\partial \omega_{t-1}}{\partial \mathcal{R}^*_t}\right|_{r^*_t \geq \mathcal{R}^*_t} = 0$, because the real return of the nominal asset $\mathcal{R}^*_t$ is below the real return of the real asset in equilibrium. Hence, even if $\mathcal{R}^*_t$ marginally increases, this will not induce young households to save in nominal assets, and the fraction of savings that are invested in real assets, i.e., ω_{t-1}, will not change.

For $\left.\frac{\partial \omega_{t-1}}{\partial \mathcal{R}^*_t}\right|_{r^*_t < \mathcal{R}^*_t} < 0$, as an increase in $\mathcal{R}^*_t$ will promote young households to save a larger fraction of their savings in nominal assets, captured by $(1-\omega_{t-1})$.

[11] $\mathcal{R}_{t-k} = \frac{P_{t-k-1}}{P_{t-k}}$ and thus $\frac{\partial \mathcal{R}_{t-k}}{\partial(P_{t-k})} > 0$.

B.5.2 Expectations

Assuming that the expected price level of the next period positively depends on the price level of the current period, i.e., $\frac{\partial \mathbb{E}_t[P_{t+1}]}{\partial P_t} > 0$. Thus, a fall in the current price level would induce young households to expect that the price level of the next period will fall as well, which would offset some of the effect a fall in P_t has on $\mathcal{R}_{t+1}$. A necessary and sufficient condition that a fall in P_t leads to a fall in $\mathcal{R}_{t+1}$ is

$$
\begin{aligned}
1 >& \frac{P_t}{\mathbb{E}_t[P_{t+1}]} \frac{\partial \mathbb{E}_t[P_{t+1}]}{\partial P_t} \\
1 >& \mathcal{E}(\mathbb{E}_t[P_{t+1}], P_t).
\end{aligned}
$$

Hence, the elasticity of the expected price level with respect to the current price level is less than 1.[12] Moreover, there is recent empirical evidence that households do not seem to expect deflation even in an environment that has experienced deflation before (Gorodnichenko and Sergeyev (2021)).

Moreover, as I assume P_{t+1} to be the only future variable that has an impact on current outcomes and that is not known with certainty, how exceptions regarding P_{t+1} are formed can have important ramifications. Therefore, changes in $\mathbb{E}_t[P_{t+1}]$ independent of P_t can alleviate or worsen a demand deficiency. For example, if households suddenly expect P_{t+1} to be lower than before, this makes saving in nominal assets more attractive and could thus trigger a demand shortage.

B.6 Rigid Nominal Wages

The main rigidity in the model is that nominal wages are downward rigid. However, the nominal return to capital is fully flexible, assuming it is not constrained by the nominal ELB. A negative demand shock, i.e., a fall in $\bar{K}_{t+1}$, as shown in Section B.10, leads to a lower

[12] If this condition is violated, firms decreasing their prices in period t would *increase* $\mathcal{R}_{t+1}$, because households would expect the price level to fall further. The increase in $\mathcal{R}_{t+1}$ would mean households save more in nominal assets, i.e., a higher B_t, thus reducing demand further, which would lead to a further decrease in P_t and so on. Thus, the economy would end up in a deflationary spiral.

nominal return on capital and higher nominal wages. The effect on equilibrium marginal costs is given as

$$\frac{\partial \Psi_t}{\partial \bar{K}_{t+1}} = P_t \left(\phi \left(\frac{r_t}{\phi}\right)^{\phi} \frac{1}{r_t} \frac{\partial r_t}{\partial \bar{K}_{t+1}} \left(\frac{w_t}{A_t(1-\phi)}\right)^{1-\phi} + (1-\phi)\left(\frac{r_t}{\phi}\right)^{\phi} \left(\frac{w_t}{A_t(1-\phi)}\right)^{1-\phi} \frac{1}{w_t}\frac{\partial w_t}{\partial \bar{K}_{t+1}} \right)$$
$$\frac{\partial \Psi_t}{\partial \bar{K}_{t+1}} = \Psi_t \left(\phi \frac{1}{r_t} \frac{\partial r_t}{\partial \bar{K}_{t+1}} + (1-\phi)\frac{1}{w_t}\frac{\partial w_t}{\partial \bar{K}_{t+1}} \right).$$

Recall from Section B.10 that

$$\frac{\partial r_t}{\partial \bar{K}_{t+1}} = a(1-\phi)K_t^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\frac{1}{\bar{K}_{t+1}} \cdot \left(\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t}\right)^{-1}$$
$$\frac{\partial r_t}{\partial \bar{K}_{t+1}} = \frac{1-\phi}{\bar{K}_{t+1}} r_t \cdot \left(\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t}\right)^{-1}$$
$$\frac{\partial w_t}{\partial \bar{K}_{t+1}} = -bK_t^{\phi}A_t^{1-\phi}\phi\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\frac{1}{\bar{K}_{t+1}} \cdot \left(\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t}\right)^{-1}$$
$$\frac{\partial w_t}{\partial \bar{K}_{t+1}} = -\frac{\phi}{\bar{K}_{t+1}} w_t \cdot \left(\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t}\right)^{-1}$$

$$\frac{\partial \Psi_t}{\partial \bar{K}_{t+1}} = \Psi_t \left(\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t}\right)^{-1} \left(\phi\frac{1-\phi}{\bar{K}_{t+1}} - (1-\phi)\frac{\phi}{\bar{K}_{t+1}} \right) = 0.$$

Therefore, with downward rigid nominal wages, a negative (nominal) demand shock leaves the marginal costs and thus the optimal nominal price of each intermediate good $P_t(i)$ unchanged. The zero-profit condition in the final good sector entails that the nominal price of the final good, P_t can only fall if there is a fall in $P_t(i)$. Therefore, if nominal wages are downward rigid, a negative demand shock will not lead to a fall in P_t.

B.7 Equilibrium

B.7.1 Capital Market Equilibrium

As stated in the main text, the capital market equilibrium for period t is given as

$$N_{t-1}^{y} s_{t-1}^{j}(r_t) = K_t(r_t),$$

where r_t is the endogenous variable that clears the capital market. Recall that I assume the ELB is never binding in period t and thus $K_t < \bar{K}_t$.
Therefore, the equilibrium condition for the capital market entails

$$\omega_{t-1}\mathcal{S}_{t-1} < \bar{K}_t,$$

with $\omega_{t-1} = 1$, as the ELB is not binding, and where $\mathcal{S}_{t-1}$ denotes overall savings in real terms and ω_{t-1} the share of savings invested in real assets.

The capital market equilibrium for period $t+1$ is given as

$$N_t^y s_t^j = K_{t+1},$$

assuming the ELB is binding in period $t+1$ entails $K_{t+1} = \bar{K}_{t+1}$. Hence, I can express the equilibrium condition for the capital market as

$$\omega_t \mathcal{S}_t = \bar{K}_{t+1},$$

with $\omega_t < 1$.

B.7.2 Final Good Market Equilibrium

The equilibrium on the final good market can be expressed as

$$Y_t = \underbrace{Y_t - \mathcal{S}_t(r_{t+1}, w_t, 1-\eta_t) + \bar{K}_{t+1}}_{=Y_t},$$

where $\bar{K}_{t+1} - \mathcal{S}_t(r_{t+1}, w_t, 1-\eta_t) \geq 0$ denotes the excess savings.

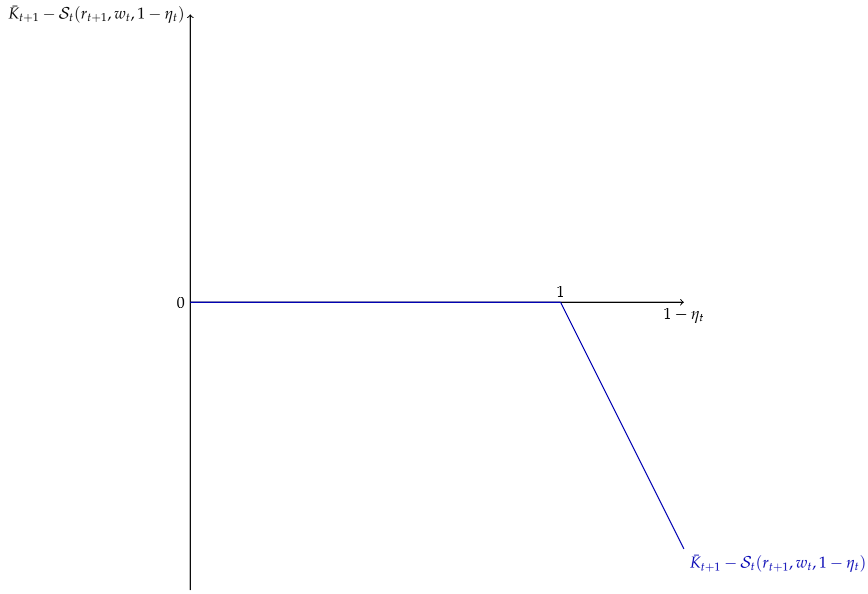


Figure B.3: *Zero Equilibrium Unemployment*

Figure B.3 illustrates the situation for an economy that is at the kink, i.e., $1 - \eta_t^* = 1$. In this case, $\bar{K}_{t+1} - \mathcal{S}_t(r_{t+1}, w_t^*, 1 - \eta_t^*) = 0$.

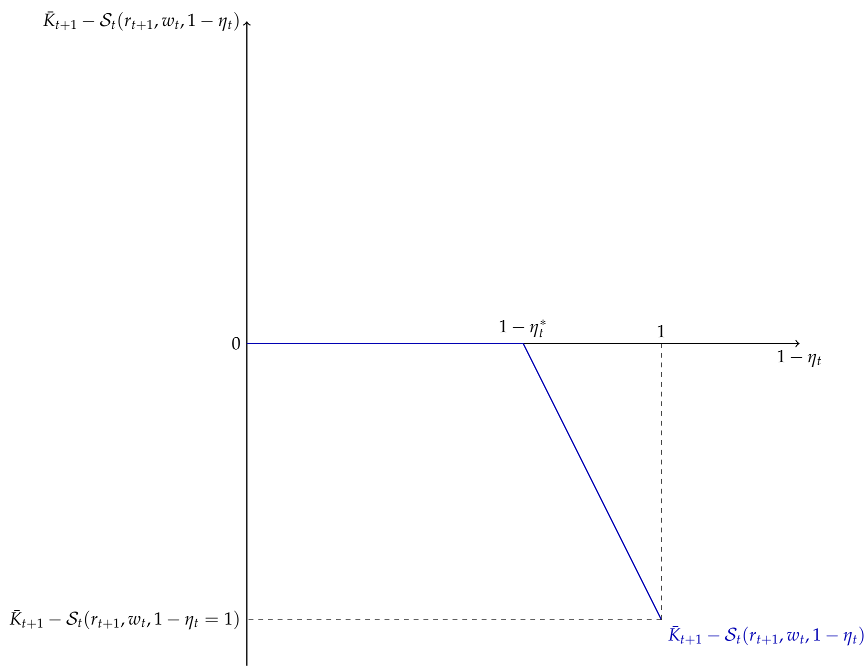


Figure B.4: *Non-Zero Equilibrium Unemployment*

Figure B.4 illustrates how a rise in the unemployment rate leads to a fall in aggregate savings, which is necessary to eliminate the demand shortage. However, once the demand shortage has been eliminated, which is the point where the complementary slackness equilibrium condition holds with equality, and the unemployment rate ceases to be an endogenous variable.

The overall income of young households is given as

$$\begin{aligned} w_t N_t^y \cdot (1-\eta_t) =& \mu(1-\phi)Y_t \\ \mathcal{I}_t^y =& \mu(1-\phi)Y_t, \end{aligned}$$

where $\mu = \frac{\varepsilon-1}{\varepsilon} < 1$ is the inverse mark-up.

Lemma B.1 *With a Cobb-Douglas production function, a higher unemployment rate will always reduce the overall income of workers and thus their savings.*

Proof.

$$\frac{\partial \mathcal{I}_t^y(\eta_t)}{\partial \eta_t} = \mu(1-\phi)\frac{\partial Y_t}{\partial L_t}\frac{\partial L_t}{\partial \eta_t} < 0.$$

□

This implies that young households can never increase their overall income by supplying fewer hours of labor.

Capital stock per hour worked

$$k_t = \frac{K_t}{N_t^y \ell_t \cdot (1-\eta_t)},$$

$$\frac{\partial k_t}{\partial \eta_t} > 0.$$

Therefore, the capital stock per worker will increase.

B.8 Flexible Price Equilibrium

B.8.1 Price Level

The optimal nominal price of intermediate good i is given as

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{R_t}{\phi}\right)^{\phi}\left(\frac{W_t}{A_t(1-\phi)}\right)^{1-\phi}$$
$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1}\Psi_t,$$

where $\frac{\varepsilon}{\varepsilon-1}$ denotes the mark-up and Ψ_t denotes the nominal marginal costs.

All firms are symmetric and will thus all set the same price.

The nominal price of the final good, which corresponds to the price index, P_t is given as

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$$
$$P_t = \frac{\varepsilon}{\varepsilon - 1}\Psi_t$$
$$1 = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{r_t}{\phi}\right)^{\phi}\left(\frac{w_t}{A_t(1-\phi)}\right)^{1-\phi}.$$

This implies that the price level is indeterminate, i.e., we require an additional equation to solve for P_t. For example, an equation that relates nominal demand and supply.

Therefore, I introduce an additional equilibrium condition that determines the equilibrium nominal price of the final good.

$$\mathcal{M}_t = P_t\left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$\mathcal{M}_t = P_t Y_t.$$

Hence, P_t is determined such that nominal demand is equal to nominal supply, i.e., that the final goods market not only clears in *real*, but also in *nominal terms*.

I assume throughout that without any form of demand shortage, i.e., the economy producing at its output potential, the maximum amount of money banks can issue each period will be such that $\{P_t\}_{t=0}^{\infty} = 1$, i.e., the nominal price of the final good is constant and equal to 1.

Moreover, as the final good firms operate under perfect competition and are thus price takers, they cannot change P_t directly. The intermediate good firms take nominal wages, nominal interest rates, and the nominal price of the final good as given but can decide on the nominal price of their output, i.e., $P_t(i)$.

In addition, I assume that all intermediate good firms can change their prices each period without incurring any costs. However, as shown before, their optimal price depends on their marginal costs; hence, they will only change their nominal prices if their marginal costs have changed.[13]

B.8.2 Flexible Nominal Wages

The maximum money balance in the economy in period t is denoted by M_t and is such that $M_t = Y_t$. where Y_t constitutes the production possibility frontier of the economy or the potential output in period t. Hence, the maximum money supply will be such that the economy will have a constant price level that is equal to 1; as long as there is no demand shortage, i.e., *ex-ante*, we have $P_t = 1$.

Money is created and issued by banks. Let $\mathcal{M}_t$ denote the amount of money issued by banks, i.e., the amount of money in circulation in a given period. The amount of money in circulation $\mathcal{M}_t$ is such that

$$\mathcal{M}_t \leq M_t.$$

Banks can only issue money if households or firms demand a loan and as long as the additional money that is created does not violate the maximum money balance. Moreover, I assume that young households only borrow the funds necessary to finance consumption for the current period, as they can directly save out of the income that they receive at the end of the period. Thus, banks can neither create money and spend it themselves, nor can they

[13]This is due to the set-up of monopolistic competition in this model. (Nominal) profits of intermediate good firms will always constitute a constant fraction of (nominal) output. Therefore, if nominal demand were to increase, firms could not absorb the entire additional demand via higher profits. Instead, they could only appropriate a fraction, and the rest would lead to higher nominal wages and higher nominal returns to capital.

issue more money than the maximum money balance in a given period.[14]

Therefore, the amount of money used for purchasing goods will be $\mathcal{M}_t$.

Assume *ex-ante* $P_t = \bar{P}_t = 1$, i.e., all agents behave as if the nominal price of the final good is fixed, and hence demand is given as $\mathcal{M}_t$.[15] The final good firms then observe nominal demand from households and firms, which is given as

$$\mathcal{M}_t = M_t - N_t^y B_t^j$$
$$\mathcal{M}_t = \bar{P}_t Y_t - (1 - \omega_t(\bar{P}_t))\, \bar{P}_t \mathcal{S}_t.$$

Where $\omega_t \in (0, 1]$ denotes the share of savings $\mathcal{S}_t$ invested in nominal assets, i.e., held in cash. Therefore, for $\omega_t < 1$, the economy experiences *ex-ante* a shortfall in nominal demand, i.e., in this case, (young) households prefer to save in nominal assets (money) as well as in real assets.

The intuition of the first line is that young households are the only agents in the economy that save, i.e., end the period with a positive amount of assets. All other agents will end the period with zero assets. $N_t^y B_t^j$ constitutes the total amount of planned nominal savings. Recall that young households make their consumption-saving decisions at the beginning of the period, and they always assume the economy will produce at its full potential. Therefore, they do not take into account that the amount of money in circulation, and hence their nominal income, is demand-determined.

Ex-ante nominal demand $\mathcal{M}_t$ is thus all money that is spent in period t to acquire the final good and thus denotes the maximum nominal sales the final good firms can make.

The rest of the production sector is as described in Section 2.2.3 of the main text.

The equilibrium conditions are given as follows, where K_t and L_t denote the equilibrium

[14]This ensures that absent any demand shortfall, the economy will have a constant price level equal to 1.

[15]Alternatively, one could also argue that a lower value of $\mathcal{M}_t$ leads to a lower level of P_t through the nominal final goods market equilibrium condition, which would then entail a positive inflation rate in the next period and thus make it unattractive for households to save in money. This way, intermediate good firms would not experience a shortfall in demand and nominal prices, i.e., intermediate goods prices, wages, and return to capital would fall due to the fall in P_t. However, this is ruled out by Assumption 2.6.

values of capital and labor.

$$
\begin{aligned}
F \equiv & K_t - \left(\frac{a}{r_t \bar{P}_t}\right) \left(\bar{P}_t A_t^b \left(\frac{b}{w_t \bar{P}_t}\right)^b \left(\frac{a}{r_t \bar{P}_t}\right)^a\right)^\varepsilon \left(\frac{\mathcal{M}_t}{\bar{P}_t}\right) = 0, \\
G \equiv & L_t - \left(\frac{b}{w_t \bar{P}_t}\right) \left(A_t^b \left(\frac{b}{w_t \bar{P}_t}\right)^b \left(\frac{a}{r_t \bar{P}_t}\right)^a\right)^\varepsilon \left(\frac{\mathcal{M}_t}{\bar{P}_t}\right) = 0,
\end{aligned}
$$

where $r_t \bar{P}_t = R_t$ is the nominal return to capital and $w_t \bar{P}_t = W_t$ is the nominal wage rate. Using[16]

$$
\begin{aligned}
\mathcal{M}_t = & M_t - \varsigma_t W_t L_t + \bar{P}_t \bar{K}_{t+1} \\
\mathcal{M}_t = & \bar{P}_t Y_t - \varsigma_t W_t L_t + \bar{P}_t \bar{K}_{t+1} \\
\mathcal{M}_t = & \bar{P}_t K_t^\phi (A_t L_t)^{1-\phi} - \varsigma_t W_t L_t + \bar{P}_t \bar{K}_{t+1},
\end{aligned}
$$

where ς_t denotes the savings rate, the equilibrium conditions can be expressed as

$$
\begin{aligned}
F \equiv & \frac{1}{L_t} - \left(\frac{a}{R_t}\right) \left(\bar{P}_t A_t^b \left(\frac{b}{W_t}\right)^b \left(\frac{a}{R_t}\right)^a\right)^\varepsilon \\
& \cdot \left(\frac{1}{\bar{P}_t} \left(\bar{P}_t K_t^{\phi-1} A_t^{1-\phi} L_t^{-\phi} - \frac{\varsigma_t W_t}{K_t} + \frac{\bar{P}_t \bar{K}_{t+1}}{K_t L_t}\right)\right) = 0, \\
G \equiv & \frac{1}{K_t} - \left(\frac{b}{W_t}\right) \left(\bar{P}_t A_t^b \left(\frac{b}{W_t}\right)^b \left(\frac{a}{R_t}\right)^a\right)^\varepsilon \\
& \cdot \left(\frac{1}{\bar{P}_t} \left(\bar{P}_t K_t^{\phi-1} A_t^{1-\phi} L_t^{-\phi} - \frac{\varsigma_t W_t}{K_t} + \frac{\bar{P}_t \bar{K}_{t+1}}{K_t L_t}\right)\right) = 0.
\end{aligned}
$$

Assuming that for the initial price level $\bar{P}_t$ we have $\omega_t < 1$ and thus $\mathcal{M}_t < M_t$, i.e., the economy suffers from a demand deficiency, I can derive the following effects on the nominal return to capital and nominal wages

$$
\frac{\partial R_t}{\partial(-\bar{K}_{t+1})} < 0, \quad \frac{\partial W_t}{\partial(-\bar{K}_{t+1})} < 0.
$$

[16] I use $\bar{P}_t \bar{K}_{t+1}$, because the capital stock for the next period must be purchased in the current period at current prices.

Proof. See Appendix B.11. □

Therefore, a demand shortfall induced by a fall in the population growth rate or a fall in the rate of technological progress will lead to a fall in the nominal return to capital as well as nominal wages.

The optimal nominal price of intermediate good i is given as

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1}\Psi_t,$$

where $\frac{\varepsilon}{\varepsilon-1} > 1$ is the mark-up and $\Psi_t = \left(\frac{R_t}{\phi}\right)^{\phi}\left(\frac{W_t}{A_t(1-\phi)}\right)^{1-\phi}$ denotes the nominal marginal costs.

A shortfall in demand, caused by a reduction in $\bar{K}_{t+1}$, has the following effect on $P_t(i)$

$$-\frac{\partial P_t(i)}{\partial \bar{K}_{t+1}} = \frac{\varepsilon}{\varepsilon - 1}\Psi_t\left(\phi\frac{1}{R_t}\frac{\partial R_t}{\partial \bar{K}_{t+1}} + (1-\phi)\frac{1}{W_t}\frac{\partial W_t}{\partial \bar{K}_{t+1}}\right) < 0.$$

Hence, a shortfall in demand will, through lowering the nominal return on capital and nominal wages, also lead to lower nominal prices for intermediate goods.

Perfect competition in the production sector for the final good entails

$$P_t\left(\int_0^1 y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 P_t(i)y_t(i)di = 0,$$

with $P_t = \bar{P}_t = 1$ a fall in $P_t(i)$ entails that the final good firms would make positive profits. This is not possible in equilibrium, and hence I require a fall in P_t to ensure that the zero-profit condition is satisfied.[17]

As full price flexibility entails full employment, $y_t(i)$ will remain constant, and hence the above equation encompasses that $\frac{P_t(i)}{P_t}$ remains constant, i.e., the real price of variety i stays constant. And thus the real return to capital as well as the real wage rate.[18]

Therefore, the *ex-ante* shortfall in nominal demand will lead *ex-post* to a lower price level,

[17]In case P_t is fully rigid, there would be no equilibrium consistent with fully flexible wages and interest rates, i.e., the demand shortage can only be eliminated if P_t falls. See Section B.11.

[18]As $\frac{P_t(i)}{P_t} = \left(\frac{r_t}{\phi}\right)^{\phi}\left(\frac{w_t}{A_t(1-\phi)}\right)^{1-\phi}$.

i.e., $P_t^* < P_t = 1$.

Assuming $P_t = P_{t+1}^* = 1$, i.e., there was a one-time demand shortage, I have $P_t^* < P_{t+1}^*$, which implies a positive rate of inflation Π_{t+1} and thus a lower value for $\mathcal{R}_{t+1}$. Which reduces the share of savings invested in nominal assets, i.e., $-\frac{\partial \omega_t}{\partial P_t} > 0$.

In equilibrium, it must hold that

$$\omega_t(P_t^*) = 1,$$

which implies the equilibrium nominal price of the final good needs to clear the market for the final good, and thus all money withdrawn at the beginning of the period must be spent, i.e., $\omega_t = 1$. Hence, *ex-post* we will have

$$\begin{aligned} \mathcal{M}_t =& P_t^* Y_t - (1 - \omega_t(P_t^*))\, P_t^* \mathcal{S}_t \\ \mathcal{M}_t =& P_t^* Y_t. \end{aligned}$$

Therefore, all money issued at the beginning of the period will be used for purchasing the final good, and the share of savings invested in nominal assets will be zero. This will allow the economy to produce at its full potential, and thus real factor returns will remain constant while nominal returns will have decreased due to the lower price level. Moreover, the amount of money in circulation in period t will be *lower* compared to a situation in which the economy had not suffered from a demand shortage in period t. The reason behind this is that the amount of money in circulation will be demand-determined and as only money in circulation will constitute nominal income, the price level will fall to ensure that nominal supply and demand are equal.

I can combine the cases of perfectly flexible and perfectly rigid nominal wages to study a situation in which nominal wages are partially flexible. Following Eggertsson *et al.* (2019b), assume households would never accept a nominal wage rate that is below $\bar{W}_t = \gamma W_{t-1} + (1-\gamma) W_t^{flex}$, where W_t^{flex} is the nominal wage rate that would prevail if all prices were fully flexible. $1 - \gamma$ thus captures by how much nominal wages in the current period can fall relative to the previous period.

Therefore, this also captures the share of the demand shortfall that can be absorbed by a fall in nominal variables rather than real variables. Hence, if nominal wages are partially flexible, a negative demand shock induces a weaker fall in output compared to a situation in which nominal wages are completely rigid.

B.9 Persistent Shocks

So far, I have mainly considered a static environment, i.e., there was only a one-time shock, i.e., the real ELB was only binding for one period. However, I can readily use the model at hand to investigate what happens if the shock persists over multiple periods and agents have perfect foresight.

As discussed in the main part, there are two possibilities to eliminate the excess savings. A higher unemployment rate or a fall in the price level followed by a corresponding increase that reduces the real return on money sufficiently, such that households no longer find it optimal to save in money. Thus, the problem can either be resolved through a change in *real* or *nominal* variables.

Resolving the issue of excess savings through higher unemployment entails that the problem of too little demand can be solved within the period, i.e., through an *intratemporal* channel. Therefore, if the shock persists for multiple periods, the economy will experience involuntary unemployment in each period in which the shock is active, and the amount of involuntary unemployment will depend on the size of the shock, i.e., the amount of excess savings.

In the event that excess savings are eliminated through a higher inflation rate, the resolution of the excess savings becomes *intertemporal* (see also Section B.5). A fall in the current price level will only induce households to reduce their investment in nominal assets, i.e., money, if the (expected) price level in the next period remains constant or decreases by less than the current price level (see also Section B.4), i.e., the (expected) inflation rate increases. If the shock persists over multiple periods, the economy requires a positive inflation rate in each period in which the real ELB is binding.

This implies that as long as the shock induces the real ELB to potentially bind, the economy requires a positive inflation rate to ensure that the real ELB does not bind. Put differently, in each period in which a shock causes *ex-ante* nominal savings to be higher than capital demand, a positive inflation rate ensures that investing in nominal assets is not optimal, i.e., inflation reduces the optimal savings by lowering the real ELB. To generate the necessary inflation in the following periods, the economy will experience a strong reduction in the price level in the last period before the shock becomes active. Over time, the economy will experience a growing price level until it reaches the original level and the shock has passed.

B.10 Proofs Rigid Nominal Wages

Consider the equilibrium conditions from the main text and assume the real ELB is binding, and thus η_t is an endogenous variable. K_t denotes the equilibrium capital stock in period t with $\frac{\partial K_t}{\partial r_t} \geq 0$. $\varsigma_t(r_{t+1})$ denotes the savings rate of the current young generation. However, if the real ELB is binding in period $t+1$, then $r_{t+1} = \mathcal{R}_{t+1}$, which is exogenous, and thus r_{t+1} is no longer an endogenous variable. Moreover, I assume that the savings rate is independent of income, i.e., $\frac{\partial \varsigma_t}{\partial w_t} = \frac{\partial \varsigma_t}{\partial \eta_t} = 0$.

$$r_t - aK_t^{a-1}(A_t L_t(\eta_t))^b \left(K_t^{\phi}(A_t L_t(\eta_t))^{1-\phi} - (\mathcal{S}_t(w_t,\eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} = 0,$$
$$w_t - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^{\phi}(A_t L_t(\eta_t))^{1-\phi} - (\mathcal{S}_t(w_t,\eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} = 0,$$
$$\bar{K}_{t+1} - \mathcal{S}_t(w_t,\eta_t) = 0,$$

with $a = \phi\left(\frac{\varepsilon-1}{\varepsilon}\right)$, $b = (1-\phi)\left(\frac{\varepsilon-1}{\varepsilon}\right)$ and where $L_t = N_t^y(1-\eta_t)$ denotes the equilibrium employment and ς_t the savings rate.

$$\bar{K}_{t+1} - \varsigma_t w_t L_t(\eta_t) = 0$$
$$\Leftrightarrow$$

$$w_t = \frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)}$$

$$\Leftrightarrow$$

$$L_t(\eta_t) = \frac{\bar{K}_{t+1}}{\varsigma_t w_t}.$$

Therefore, I can eliminate either w_t or $L_t(\eta_t)$ from the system of equations.[19] Simplifying it to a system of two equations

$$\begin{aligned}
w_t - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi} - (\varsigma_t w_t L_t(\eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} &= 0 \\
\frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)} - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi} - \left(\varsigma_t \frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)} L_t(\eta_t) - \bar{K}_{t+1}\right)\right)^{\frac{1}{\varepsilon}} &= 0 \\
\frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)} - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi}\right)^{\frac{1}{\varepsilon}} &= 0 \\
\frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)} - bK_t^\phi A_t^{1-\phi} L_t(\eta_t)^{-\phi} &= 0.
\end{aligned}$$

$$\begin{aligned}
w_t - bK_t^a A_t^b L_t(\eta_t)^{b-1} \left(K_t^\phi (A_t L_t(\eta_t))^{1-\phi} - (\varsigma_t w_t L_t(\eta_t) - \bar{K}_{t+1})\right)^{\frac{1}{\varepsilon}} &= 0 \\
w_t - bK_t^a A_t^b \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{b-1} \left(K_t^\phi (A_t \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{1-\phi} - \left(\varsigma_t w_t \frac{\bar{K}_{t+1}}{\varsigma_t w_t} - \bar{K}_{t+1}\right)\right)^{\frac{1}{\varepsilon}} &= 0 \\
w_t - bK_t^a A_t^b \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{b-1} \left(K_t^\phi (A_t \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{1-\phi}\right)^{\frac{1}{\varepsilon}} &= 0 \\
w_t - bK_t^\phi A_t^{1-\phi} \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi} &= 0.
\end{aligned}$$

Consider first the situation with r_t and η_t as the endogenous variables. The simplified equilibrium conditions are given as

$$\begin{aligned}
F \equiv & r_t - aK_t(r_t)^{\phi-1}(A_t L_t(\eta_t))^{1-\phi} = 0, \\
G \equiv & \frac{\bar{K}_{t+1}}{\varsigma_t} - bK_t(r_t)^\phi A_t^{1-\phi} L_t(\eta_t)^{1-\phi} = 0.
\end{aligned}$$

[19] In cases where labor supply is elastic and/or the savings rate depends on income, this simplification is no longer necessarily feasible. As $L_t(w_t, \eta_t) = N_t^y \ell_t(w_t)(1-\eta_t)$ and or $\varsigma_t(r_{t+1}, w_t)$.

$$
\begin{aligned}
&\frac{\partial F}{\partial r_t} > 0, \quad \frac{\partial F}{\partial \eta_t} > 0, \quad \frac{\partial F}{\partial \bar{K}_{t+1}} = 0, \\
&\frac{\partial G}{\partial r_t} < 0, \quad \frac{\partial G}{\partial \eta_t} > 0, \quad \frac{\partial G}{\partial \bar{K}_{t+1}} > 0.
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} \frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial \eta_t} \\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial \eta_t} \end{vmatrix} &= \frac{\partial F}{\partial r_t}\frac{\partial G}{\partial \eta_t} - \frac{\partial F}{\partial \eta_t}\frac{\partial G}{\partial r_t} > 0, \\
\begin{vmatrix} -\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial \eta_t} \\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial \eta_t} \end{vmatrix} &= \frac{\partial F}{\partial \eta_t}\frac{\partial G}{\partial \bar{K}_{t+1}} > 0, \\
\begin{vmatrix} \frac{\partial F}{\partial r_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}} \\ \frac{\partial G}{\partial r_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}} \end{vmatrix} &= -\frac{\partial G}{\partial \bar{K}_{t+1}} < 0.
\end{aligned}
$$

$$
\frac{\partial r_t}{\partial(-\bar{K}_{t+1})} = -\frac{\begin{vmatrix} -\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial \eta_t} \\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial \eta_t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial \eta_t} \\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial \eta_t} \end{vmatrix}} < 0, \quad \frac{\partial \eta_t}{\partial(-\bar{K}_{t+1})} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial r_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}} \\ \frac{\partial G}{\partial r_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial \eta_t} \\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial \eta_t} \end{vmatrix}} > 0.
$$

Consider the second situation with r_t and w_t as the endogenous variables. The simplified equilibrium conditions are given as

$$
\begin{aligned}
F \equiv & r_t - aK_t(r_t)^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi} = 0, \\
G \equiv & w_t - bK_t(r_t)^{\phi}A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi} = 0.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial F}{\partial r_t} > 0, \quad \frac{\partial F}{\partial w_t} > 0, \quad \frac{\partial F}{\partial \bar{K}_{t+1}} < 0, \\
&\frac{\partial G}{\partial r_t} < 0, \quad \frac{\partial G}{\partial w_t} > 0, \quad \frac{\partial G}{\partial \bar{K}_{t+1}} > 0.
\end{aligned}
$$

Proof.

$$\frac{\partial G}{\partial w_t} = 1 - \phi b K_t^{\phi} A_t^{1-\phi} \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi} \frac{1}{w_t} \gtreqless 0.$$

Recall that

$$w_t = b K_t^{\phi} A_t^{1-\phi} \left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}.$$

Inserting this in the expression for $\frac{\partial G}{\partial w_t}$ yields

$$\frac{\partial G}{\partial w_t} = 1 - \phi,$$

as $\phi < 1$

$$\frac{\partial G}{\partial w_t} > 0.$$

□

$$\begin{vmatrix} \frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial w_t} \\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial w_t} \end{vmatrix} = \frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t} - \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial r_t} > 0,$$

$$\begin{aligned} \begin{vmatrix} -\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial w_t} \\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial w_t} \end{vmatrix} = & \left(a(1-\phi)K_t^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\frac{1}{\bar{K}_{t+1}}\right) \\ & \cdot \left(1 - b\phi K_t^{\phi} A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\frac{1}{w_t}\right) \\ & + \left(a(1-\phi)K_t^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\frac{1}{w_t}\right) \\ & \cdot \left(b\phi K_t^{\phi} A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\frac{1}{\bar{K}_{t+1}}\right) \\ = & a(1-\phi)K_t^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\frac{1}{\bar{K}_{t+1}} > 0, \end{aligned}$$

$$\begin{vmatrix} \frac{\partial F}{\partial r_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}} \\ \frac{\partial G}{\partial r_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}} \end{vmatrix} = \left(1 + (1-\phi)aK_t^{\phi-1}\frac{1}{K_t}\frac{\partial K_t}{\partial r_t}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\right)$$

$$\cdot(-1)\left(\phi b K_t^{\phi} A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\frac{1}{\bar{K}_{t+1}}\right)$$
$$+\left((1-\phi)aK_t^{\phi-1}\left(A_t\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\right)^{1-\phi}\frac{1}{\bar{K}_{t+1}}\right)$$
$$\cdot\left(\phi b K_t^{\phi}\frac{1}{K_t}\frac{\partial K_t}{\partial r_t}A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\right)$$
$$=-\left(\phi b K_t^{\phi} A_t^{1-\phi}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\phi}\frac{1}{\bar{K}_{t+1}}\right)<0$$

$$\frac{\partial r_t}{\partial(-\bar{K}_{t+1})}=-\frac{\begin{vmatrix}-\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial w_t}\\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial w_t}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial w_t}\\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial w_t}\end{vmatrix}}<0,\quad \frac{\partial w_t}{\partial(-\bar{K}_{t+1})}=-\frac{\begin{vmatrix}\frac{\partial F}{\partial r_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}}\\ \frac{\partial G}{\partial r_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial w_t}\\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial w_t}\end{vmatrix}}>0.$$

Therefore, a fall in $\bar{K}_{t+1}$ has the following effects on r_t, w_t, and η_t.

$$\frac{\partial r_t}{\partial(-\bar{K}_{t+1})}<0,\quad \frac{\partial w_t}{\partial(-\bar{K}_{t+1})}>0,\quad \frac{\partial \eta_t}{\partial(-\bar{K}_{t+1})}>0.$$

Therefore, a fall in $\bar{K}_{t+1}$ has two opposing effects on the income of workers. However, due to Lemma B.1 the unemployment channel will always dominate the wage channel.

B.11 Proofs Flexible Factor Prices

Assume that nominal factor prices are fully flexible. However, as before, I assume that $P_t = \bar{P}_t = 1$, i.e., is assumed to be fixed at first. Imposing that all factor markets clear yields the following equilibrium conditions

$$F \equiv \frac{1}{L_t}-\left(\frac{a}{R_t}\right)\left(\bar{P}_t A_t^{b}\left(\frac{b}{W_t}\right)^{b}\left(\frac{a}{R_t}\right)^{a}\right)^{\varepsilon}$$
$$\cdot\left(\frac{1}{\bar{P}_t}\left(\bar{P}_t K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi}-\frac{\varsigma_t W_t}{K_t}+\frac{\bar{P}_t\bar{K}_{t+1}}{K_t L_t}\right)\right)=0$$

$$\Leftrightarrow$$

$$\begin{aligned} F \equiv &\frac{1}{L_t} - \left(\frac{a}{r_t}\right)\left(A_t^b\left(\frac{b}{w_t}\right)^b\left(\frac{a}{r_t}\right)^a\right)^\varepsilon \\ &\cdot\left(K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi} - \frac{\varsigma_t w_t}{K_t} + \frac{\bar{K}_{t+1}}{K_t L_t}\right) = 0, \\ G \equiv &\frac{1}{K_t} - \left(\frac{b}{W_t}\right)\left(\bar{P}_t A_t^b\left(\frac{b}{W_t}\right)^b\left(\frac{a}{R_t}\right)^a\right)^\varepsilon \\ &\cdot\left(\frac{1}{\bar{P}_t}\left(\bar{P}_t K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi} - \frac{\varsigma_t W_t}{K_t} + \frac{\bar{P}_t\bar{K}_{t+1}}{K_t L_t}\right)\right) = 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} G \equiv &\frac{1}{K_t} - \left(\frac{b}{w_t}\right)\left(A_t^b\left(\frac{b}{w_t}\right)^b\left(\frac{a}{r_t}\right)^a\right)^\varepsilon \\ &\cdot\left(K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi} - \frac{\varsigma_t w_t}{K_t} + \frac{\bar{K}_{t+1}}{K_t L_t}\right) = 0. \end{aligned}$$

Moreover, assume that $\frac{\partial \varsigma_t}{\partial W_t} \geq 0$, i.e., the savings rate is never decreasing in income, $\frac{\partial L_t}{\partial W_t} \geq 0$ and $\frac{\partial K_t}{\partial R_t} \geq 0$ i.e., the equilibrium factor supply is never decreasing with respect to its price.

$$\begin{aligned} \Lambda_t &\equiv \left(\frac{a}{R_t}\right)\left(A_t^b\left(\frac{b}{W_t}\right)^b\left(\frac{a}{R_t}\right)^a\right)^\varepsilon > 0, \\ \Xi_t &\equiv \left(\frac{b}{W_t}\right)\left(A_t^b\left(\frac{b}{W_t}\right)^b\left(\frac{a}{R_t}\right)^a\right)^\varepsilon > 0, \\ \Phi &\equiv \frac{1}{\bar{P}_t}\left(\bar{P}_t K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi} - \frac{\varsigma_t W_t}{K_t} + \frac{\bar{P}_t\bar{K}_{t+1}}{K_t L_t}\right) > 0, \\ K_{R_t} &\equiv \frac{\partial K_t}{\partial R_t} \geq 0, \\ L_{W_t} &\equiv \frac{\partial L_t}{\partial W_t} \geq 0, \\ \Phi_{L_{W_t}} \equiv &\frac{1}{\bar{P}_t}\left((-\phi)\bar{P}_t K_t^{\phi-1}A_t^{1-\phi}L_t^{-\phi-1}L_{W_t} - \frac{1}{K_t}\left(\varsigma_t + \frac{\partial\varsigma_t}{\partial W_t}W_t\right) - \frac{\bar{P}_t\bar{K}_{t+1}}{K_t L_t^2}L_{W_t}\right) < 0, \\ \Phi_{K_{R_t}} \equiv &\frac{1}{\bar{P}_t}\left((\phi-1)\bar{P}_t K_t^{\phi-2}A_t^{1-\phi}L_t^{-\phi}K_{R_t} + \frac{\varsigma_t W_t}{K_t^2}K_{R_t} - \frac{\bar{P}_t\bar{K}_{t+1}}{K_t^2 L_t}K_{R_t}\right) < 0. \end{aligned}$$

Proof.

$$\Phi_{K_{R_t}} = (\phi-1)\bar{P}_t K_t^{\phi-2}A_t^{1-\phi}L_t^{-\phi}K_{R_t} + \frac{\varsigma_t W_t}{K_t^2}K_{R_t} - \frac{\bar{P}_t\bar{K}_{t+1}}{K_t^2 L_t}K_{R_t} \gtreqless 0$$

$$\Phi_{K_{R_t}} = (\phi - 1)\bar{P}_t K_t^{\phi} A_t^{1-\phi} L_t^{-\phi} K_{r_t} + \varsigma_t w_t K_{r_t} - \frac{\bar{P}_t \bar{K}_{t+1}}{L_t} K_{R_t} \gtreqless 0,$$

as $\varsigma_t \leq 1$ and $W_t = \underbrace{\frac{\varepsilon - 1}{\varepsilon}}_{=\mu \in (0,1)} (1-\phi)\bar{P}_t K_t^{\phi} A_t^{1-\phi} L_t^{-\phi}$. Plugging these two values in yields

$$\Phi_{K_{R_t}} = (\mu\varsigma_t - 1)(1-\phi)\bar{P}_t K_t^{\phi} A_t^{1-\phi} L_t^{-\phi} K_{R_t} - \frac{\bar{P}_t \bar{K}_{t+1}}{L_t} K_{R_t} < 0,$$

where K_{R_t}, L_{w_t}, $\Phi_{L_{w_t}}$, and $\Phi_{K_{r_t}}$ represent the corresponding partial derivatives. □

$$\begin{aligned}
\frac{\partial F}{\partial R_t} &= \Lambda_t \left(\frac{1 + a\varepsilon}{R_t} \Phi_t - \Phi_{K_{R_t}} \right) > 0, \\
\frac{\partial F}{\partial W_t} &= -\frac{1}{L_t^2} L_{W_t} + \frac{b\varepsilon}{W_t} \Lambda_t \Phi_t - \Lambda_t \Phi_{L_{W_t}} \gtreqless 0, \\
\frac{\partial G}{\partial R_t} &= -\frac{1}{K_t^2} K_{R_t} + \frac{a\varepsilon}{R_t} \Xi_t \Phi_t - \Xi_t \Phi_{K_{R_t}} \gtreqless 0, \\
\frac{\partial G}{\partial W_t} &= \Xi_t \left(\frac{1 + b\varepsilon}{W_t} \Phi_t - \Phi_{L_{W_t}} \right) > 0, \\
\frac{\partial F}{\partial \bar{K}_{t+1}} &= -\Lambda_t \frac{1}{K_t L_t} < 0, \\
\frac{\partial G}{\partial \bar{K}_{t+1}} &= -\Xi_t \frac{1}{K_t L_t} < 0.
\end{aligned}$$

$$\begin{vmatrix} \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial W_t} \\ \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial W_t} \end{vmatrix} = \frac{\partial F}{\partial R_t}\frac{\partial G}{\partial W_t} - \frac{\partial F}{\partial W_t}\frac{\partial G}{\partial R_t}.$$

$$\begin{aligned}
\frac{\partial F}{\partial R_t}\frac{\partial G}{\partial W_t} &= \Lambda_t \left(\frac{1 + a\varepsilon}{R_t} \Phi_t - \Phi_{K_{R_t}} \right) \Xi_t \left(\frac{1 + b\varepsilon}{W_t} \Phi_t - \Phi_{L_{W_t}} \right), \\
\frac{\partial F}{\partial W_t}\frac{\partial G}{\partial R_t} &= \left(-\frac{1}{L_t^2} L_{W_t} + \Lambda_t \left(\frac{b\varepsilon}{W_t} \Phi_t - \Phi_{L_{W_t}} \right) \right) \left(-\frac{1}{K_t^2} K_{R_t} + \Xi_t \left(\frac{a\varepsilon}{R_t} \Phi_t - \Phi_{K_{R_t}} \right) \right).
\end{aligned}$$

$$\frac{\partial F}{\partial R_t}\frac{\partial G}{\partial W_t} - \frac{\partial F}{\partial W_t}\frac{\partial G}{\partial R_t} = \Lambda_t \Xi_t \Phi_t \left(\frac{1 + a\varepsilon + b\varepsilon}{R_t W_t} \Phi_t - \frac{1}{R_t} \Phi_{L_{W_t}} - \frac{1}{W_t} \Phi_{K_{R_t}} \right) + \Xi_t \frac{1}{L_t^2} L_{W_t} \left(\frac{a\varepsilon}{R_t} \Phi_t - \Phi_{K_{R_t}} \right)$$

$$+\Lambda_t \frac{1}{K_t^2} K_{R_t} \left(\frac{b\varepsilon}{W_t} \Phi_t - \Phi_{L_{W_t}} \right) - \frac{1}{K_t^2 L_t^2} L_{W_t} K_{R_t} \gtreqless 0.$$

All terms except for the last one are positive. Notice that if $K_{R_t} = 0$ or $L_{W_t} = 0$, the expression is always positive. I assume throughout that the exogenous parameters are such that $\frac{\partial F}{\partial R_t} \frac{\partial G}{\partial W_t} - \frac{\partial F}{\partial W_t} \frac{\partial G}{\partial R_t} > 0$.

$$\begin{aligned} \begin{vmatrix} -\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial W_t} \\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial W_t} \end{vmatrix} &= \left(-\frac{\partial F}{\partial \bar{K}_{t+1}} \right) \frac{\partial G}{\partial W_t} + \frac{\partial F}{\partial W_t} \frac{\partial G}{\partial \bar{K}_{t+1}} \\ &= \Lambda_t \frac{1}{K_t L_t} \Xi_t \left(\frac{1 + b\varepsilon}{W_t} \Phi_t - \Phi_{L_{W_t}} \right) + \left(-\frac{1}{L_t^2} L_{W_t} - \frac{b\varepsilon}{W_t} \Lambda_t \Phi_t - \Lambda_t \Phi_{L_{W_t}} \right) \left(-\Xi_t \frac{1}{K_t L_t} \right) \\ &= \frac{1}{K_t L_t} \Xi_t \left(\frac{1}{W_t} \Phi_t \Lambda_t + \frac{1}{L_t^2} L_{W_t} \right) > 0. \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}} \\ \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}} \end{vmatrix} &= \frac{\partial F}{\partial R_t} \left(-\frac{\partial G}{\partial \bar{K}_{t+1}} \right) + \frac{\partial F}{\partial \bar{K}_{t+1}} \frac{\partial G}{\partial R_t} \\ &= \Lambda_t \left(\frac{1 + a\varepsilon}{R_t} \Phi_t - \Phi_{K_{R_t}} \right) \Xi_t \frac{1}{K_t L_t} - \Lambda_t \frac{1}{K_t L_t} \left(-\frac{1}{K_t^2} K_{R_t} - \frac{a\varepsilon}{R_t} \Xi_t \Phi_t - \Xi_t \Phi_{K_{R_t}} \right) \\ &= \frac{1}{K_t L_t} \Lambda_t \left(\frac{1}{R_t} \Phi_t \Xi_t + \frac{1}{K_t^2} K_{R_t} \right) > 0. \end{aligned}$$

$$\frac{\partial R_t}{\partial(-\bar{K}_{t+1})} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial \bar{K}_{t+1}} & \frac{\partial F}{\partial W_t} \\ -\frac{\partial G}{\partial \bar{K}_{t+1}} & \frac{\partial G}{\partial W_t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial W_t} \\ \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial W_t} \end{vmatrix}} < 0, \quad \frac{\partial W_t}{\partial(-\bar{K}_{t+1})} = \frac{\begin{vmatrix} \frac{\partial F}{\partial R_t} & -\frac{\partial F}{\partial \bar{K}_{t+1}} \\ \frac{\partial G}{\partial R_t} & -\frac{\partial G}{\partial \bar{K}_{t+1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial R_t} & \frac{\partial F}{\partial W_t} \\ \frac{\partial G}{\partial R_t} & \frac{\partial G}{\partial W_t} \end{vmatrix}} < 0.$$

Therefore, a fall in $\bar{K}_{t+1}$ leads to a reduction in nominal returns to capital and wages, assuming the ELB is binding. Moreover, assuming $\bar{P}_t$ remains constant, a fall in $\bar{K}_{t+1}$ has the same effect on the real return to capital and real wages, i.e., r_t and w_t.

As shown in the previous section, a fall in $\bar{K}_{t+1}$ leads to a fall in R_t and W_t. In the case,

P_t is fixed, but W_t and R_t are fully flexible, and hence r_t and w_t are, i.e., the only rigidity is that the nominal price of the final good cannot change.[20] It will not be possible to reach an equilibrium in which the optimality conditions of firms and the final good market clearing condition simultaneously hold. To see this, consider the following:

Assume that the economy is *ex-ante* above the ELB. By the aggregated first-order conditions of the firms, w_t is given as

$$\begin{aligned} w_t =& bK_t^a A_t^b L_t^{b-1} \left(K_t^{\phi} (A_t L_t)^{1-\phi} - \underbrace{(\mathcal{S}_t(w_t, L_t) - \bar{K}_{t+1})}_{=0} \right)^{\frac{1}{\varepsilon}} \\ w_t =& bK_t^{\phi} A_t^{1-\phi} L_t^{-\phi}. \end{aligned}$$

Now assume there is a negative shock to $\bar{K}_{t+1}$, i.e., $\bar{K}_{t+1}$ decreases to $\bar{K}_{t+1}^{new}$ and thus $\mathcal{S}_t(w_t, L_t) - \bar{K}_{t+1}^{new} > 0$.

The *ex-post* equilibrium wage rate w_t^* is given as the solution to a fixed point

$$w_t^* =bK_t^a A_t^b L_t^{b-1} \left(K_t^{\phi} (A_t L_t)^{1-\phi} - (\mathcal{S}_t(w_t^*, L_t) - \bar{K}_{t+1}^{new}) \right)^{\frac{1}{\varepsilon}},$$

where $w_t^* < w_t$ as $\frac{\partial w_t}{\partial(-\bar{K}_{t+1})} < 0$.

Clearance of the final good market requires that

$$\mathcal{S}_t(w_t^*, L_t) - \bar{K}_{t+1}^{new} = 0,$$

which implies for the *ex-post* wage w_t^*

$$w_t^* =bK_t^{\phi} A_t^{1-\phi} L_t^{-\phi},$$

and thus $w_t^* = w_t$; however, this contradicts the statement made earlier. Therefore, the economy cannot eliminate the demand shortage if w_t and r_t are fully flexible but P_t is fixed.

[20]If factor prices are fully flexible, this implies the full employment and utilization of all production factors.

B.12 Proofs Rigid Factor Prices

Assume that the nominal interest rate as well as nominal wages are completely downward rigid; this entails, as before, that P_t is also completely downward rigid.
The capital and labor market equilibrium with completely rigid prices is given as

$$\begin{aligned} N_{t-1}^{y} s_{t-1}(1-\eta_{K_t}) =& K_t, \\ N_t^{y}(1-\eta_{L_t}) =& L_t, \end{aligned}$$

$\eta_{K_t} \in [0,1)$ denotes the degree of factor utilization of capital, i.e., for $(1-\eta_{K_t}) = 1$ all capital available is used. Similarly, $\eta_{L_t} \in [0,1)$ denotes the degree of factor utilization of labor.
This implies $\frac{\partial K_t}{\partial \eta_{K_t}}\Big|_{R_t=\bar{R}_t} < 0$ and $\frac{\partial L_t}{\partial \eta_{L_t}}\Big|_{W_t=\bar{W}_t} < 0$.
As R_t and W_t are now fixed at some exogenous level, i.e., $R_t = \bar{R}_t \Leftrightarrow r_t = \bar{r}_t$ and $W_t = \bar{W}_t \Leftrightarrow w_t = \bar{w}_t$, η_{K_t} and η_{L_t} are now the endogenous variables.

As prices are now fixed, I can interpret the problem the firms face as a cost minimization problem, i.e., firms will take wages, interest rates, prices, and the demand as given and minimize costs to produce the demanded amount.[21]
The cost minimization problem of firm i in real terms is given as

$$\begin{aligned} &\min_{K_t(i),L_t(i)} r_t K_t(i) + w_t L_t(i) \\ &\text{s.t. } K_t(i)^{\phi}(A_t L_t(i))^{1-\phi} = p_t(i)^{-\varepsilon} Y_t. \end{aligned}$$

Solving for the optimal capital and labor demand and aggregating over all firms yields

$$K_t = \left(\frac{\phi}{1-\phi}\frac{w_t}{r_t}\right)^{1-\phi} \frac{Y_t}{A_t^{1-\phi}},$$

[21] Alternatively, I could also set-up a profit maximization problem where the wage and interest rate are exogenously given.

$$L_t = \left(\frac{1-\phi}{\phi}\frac{r_t}{w_t}\right)^{\phi} \frac{Y_t}{A_t^{1-\phi}}.$$

Inserting the equilibrium conditions in the first-order conditions from above, inserting the definition of Y_t and rearranging yields

$$\begin{aligned}
F \equiv & \frac{1}{N_t^y(1-\eta_{L_t})} - \bar{\Lambda}_t \frac{\bar{\Phi}_t}{A_t^{1-\phi}} = 0, \\
G \equiv & \frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})} - \bar{\Xi}_t \frac{\bar{\Phi}_t}{A_t^{1-\phi}} = 0,
\end{aligned}$$

where

$$\begin{aligned}
\bar{\Lambda}_t \equiv & \left(\frac{\phi}{1-\phi}\frac{\bar{w}_t}{\bar{r}_t}\right)^{1-\phi} > 0, \\
\bar{\Xi}_t \equiv & \left(\frac{1-\phi}{\phi}\frac{\bar{r}_t}{\bar{w}_t}\right)^{\phi} > 0, \\
\bar{\Phi}_t \equiv & S_{t-1}(\bar{r}_t)(1-\eta_{K_t})^{\phi-1}(A_t)^{1-\phi}N_t^y(1-\eta_{L_t})^{-\phi} \\
& - \frac{S_t}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})} + \frac{\bar{K}_{t+1}}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})N_t^y(1-\eta_{L_t})} > 0.
\end{aligned}$$

$$\bar{\Phi}_{\eta_{K_t}} > 0, \quad \bar{\Phi}_{\eta_{L_t}} > 0,$$

where $\bar{\Phi}_{\eta_{K_t}}$ and $\bar{\Phi}_{\eta_{L_t}}$ represent the corresponding partial derivatives, and the first relation follows from the proof that $\Phi_{K_{r_t}} < 0$ in Section B.11.

$$\begin{aligned}
\frac{\partial F}{\partial \eta_{K_t}} =& -\bar{\Lambda}_t\bar{\Phi}_{\eta_{K_t}}\frac{1}{A_t^{1-\phi}} < 0, \quad \frac{\partial F}{\partial \eta_{L_t}} = \frac{1}{N_t^y(1-\eta_{L_t})^2} - \bar{\Lambda}_t\bar{\Phi}_{\eta_{L_t}}\frac{1}{A_t^{1-\phi}} \gtreqless 0, \\
\frac{\partial G}{\partial \eta_{L_t}} =& -\bar{\Xi}_t\bar{\Phi}_{\eta_{L_t}}\frac{1}{A_t^{1-\phi}} < 0, \quad \frac{\partial G}{\partial \eta_{K_t}} = \frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})^2} - \bar{\Xi}_t\bar{\Phi}_{\eta_{K_t}}\frac{1}{A_t^{1-\phi}} \gtreqless 0, \\
\frac{\partial F}{\partial \bar{K}_{t+1}} =& -\bar{\Lambda}_t\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})N_t^y(1-\eta_{L_t})} < 0, \\
\frac{\partial G}{\partial \bar{K}_{t+1}} =& -\bar{\Xi}_t\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})N_t^y(1-\eta_{L_t})} < 0.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial F}{\partial \eta_{K_t}}\frac{\partial G}{\partial \eta_{L_t}} - \frac{\partial F}{\partial \eta_{L_t}}\frac{\partial G}{\partial \eta_{K_t}} =& \frac{1}{N_t^y(1-\eta_{L_t})^2}\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})^2} \\
&+ \frac{1}{N_t^y(1-\eta_{L_t})^2}\bar{\Xi}_t\bar{\Phi}_{\eta_{K_t}}\frac{1}{A_t^{1-\phi}} + \frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})^2}\bar{\Lambda}_t\bar{\Phi}_{\eta_{L_t}}\frac{1}{A_t^{1-\phi}} > 0.
\end{aligned}
$$

To find $\frac{\partial \eta_{K_t}}{\partial(-\bar{K}_{t+1})}$ and $\frac{\partial \eta_{L_t}}{\partial(-\bar{K}_{t+1})}$, I totally differentiate the system of equations and apply Cramer's rule.

$$
\begin{aligned}
\left(-\frac{\partial F}{\partial \bar{K}_{t+1}}\right)\frac{\partial G}{\partial \eta_{L_t}} + \frac{\partial F}{\partial \eta_{L_t}}\frac{\partial G}{\partial \bar{K}_{t+1}} &= -\frac{1}{N_t^y(1-\eta_{L_t})^2}\bar{\Xi}_t\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})N_t^y(1-\eta_{L_t})} < 0, \\
\frac{\partial F}{\partial \eta_{K_t}}\left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right) + \frac{\partial F}{\partial \bar{K}_{t+1}}\frac{\partial G}{\partial \eta_{K_t}} &= -\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})^2}\bar{\Lambda}_t\frac{1}{S_{t-1}(\bar{r}_t)(1-\eta_{K_t})N_t^y(1-\eta_{L_t})} < 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \eta_{K_t}}{\partial(-\bar{K}_{t+1})} =& \frac{\left(-\frac{\partial F}{\partial \bar{K}_{t+1}}\right)\frac{\partial G}{\partial \eta_{L_t}} + \frac{\partial F}{\partial \eta_{L_t}}\frac{\partial G}{\partial \bar{K}_{t+1}}}{\frac{\partial F}{\partial \eta_{K_t}}\frac{\partial G}{\partial \eta_{L_t}} - \frac{\partial F}{\partial \eta_{L_t}}\frac{\partial G}{\partial \eta_{K_t}}} > 0, \\
\frac{\partial \eta_{L_t}}{\partial(-\bar{K}_{t+1})} =& \frac{\frac{\partial F}{\partial \eta_{K_t}}\left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right) + \frac{\partial F}{\partial \bar{K}_{t+1}}\frac{\partial G}{\partial \eta_{K_t}}}{\frac{\partial F}{\partial \eta_{K_t}}\frac{\partial G}{\partial \eta_{L_t}} - \frac{\partial F}{\partial \eta_{L_t}}\frac{\partial G}{\partial \eta_{K_t}}} > 0.
\end{aligned}
$$

Therefore, a fall in $\bar{K}_{t+1}$ will lead to an increase in the level of underutilization of labor and capital, which leads to a fall in output. Wages, interest rates, and the price of the final good will remain constant.

Assumption 2.1 implies that savings that are not demanded by firms have to be stored in nominal assets. If the interest rate in period t is fixed and the economy experiences a negative demand shock in period t, this entails that nominal savings in period $t-1$ increase, and thus the economy in period $t-1$ likewise experiences a negative demand shock.

B.13 Proofs Allocative Efficiency

Consumption of household j born in period t in period $t+1$ is given as

$$\begin{aligned}
c^j_{2,t+1} =& r_{t+1}\varsigma^j_t \mathcal{I}^j_t \\
c^j_{2,t+1} =& r_{t+1}\varsigma^j_t \mu(1-\phi)\frac{Y_t}{N^y_t} \\
c^j_{2,t+1} =& r_{t+1}\varsigma^j_t \mu(1-\phi)K^\phi_t A^{1-\phi}_t (1-\eta_t)^{1-\phi}(N^y_t)^{-\phi},
\end{aligned}$$

where $\mu = \frac{\varepsilon-1}{\varepsilon} < 1$ is the inverse mark-up and r_{t+1} is fixed due to the binding real ELB in period $t+1$.

Following Section B.10, the effect of an increase in ς^j_t can be calculated as follows

$$\frac{\partial \eta_t}{\partial \varsigma^j_t} = \frac{\begin{vmatrix} \frac{\partial F}{\partial r_t} & -\frac{\partial F}{\partial \varsigma^j_t} \\ \frac{\partial G}{\partial r_t} & -\frac{\partial G}{\partial \varsigma^j_t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial r_t} & \frac{\partial F}{\partial \eta_t} \\ \frac{\partial G}{\partial r_t} & \frac{\partial G}{\partial \eta_t} \end{vmatrix}} = \frac{-\frac{\partial G}{\partial \varsigma^j_t}}{\frac{\partial G}{\partial \eta_t}}.$$

With

$$G \equiv \frac{\bar{K}_{t+1}}{(1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t} - bK^\phi_t A^{1-\phi}_t \left((1-\eta_t)N^y_t\right)^{-\phi} = 0.$$

$$\begin{aligned}
\frac{\partial \eta_t}{\partial \varsigma^j_t} =& \frac{\bar{K}_{t+1}}{\left((1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t\right)^2}(1-\eta_t)\cdot\left(\frac{\bar{K}_{t+1}}{(1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t}\frac{1}{1-\eta} - b\phi K^\phi_t A^{1-\phi}_t \left((1-\eta_t)N^y_t\right)^{-\phi}\frac{1}{1-\eta}\right)^{-1} \\
\frac{\partial \eta_t}{\partial \varsigma^j_t} =& \frac{\bar{K}_{t+1}}{\left((1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t\right)^2}(1-\eta_t)\cdot\left(\frac{\bar{K}_{t+1}}{(1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t}\frac{1-\phi}{1-\eta}\right)^{-1} \\
\frac{\partial \eta_t}{\partial \varsigma^j_t} =& \frac{1}{(1-\eta_t)\sum_{j=1}^{N^y_t}\varsigma^j_t}(1-\eta_t)\cdot\left(\frac{1-\phi}{1-\eta}\right)^{-1}
\end{aligned}$$

$$\frac{\partial \eta_t}{\partial \varsigma_t^j} = \frac{1-\eta}{(1-\phi)\sum_{j=1}^{N_t^y} \varsigma_t^j} > 0.$$

The effect of an increase in the savings rate of household j on her consumption in period $t+1$ is given as

$$\begin{aligned}
\frac{\partial c_{2,t+1}^j}{\partial \varsigma_t^j} =& r_{t+1}\mathcal{I}_t^j - r_{t+1}\varsigma_t^j \mu(1-\phi)K_t^\phi A_t^{1-\phi}(1-\phi)(1-\eta_t)^{-\phi}\frac{\partial \eta_t}{\partial \varsigma_t^j}(N_t^y)^{-\phi} \\
\frac{\partial c_{2,t+1}^j}{\partial \varsigma_t^j} =& r_{t+1}\mathcal{I}_t^j - r_{t+1}\varsigma_t^j \mu(1-\phi)K_t^\phi A_t^{1-\phi}(1-\phi)(1-\eta_t)^{-\phi}\frac{1-\eta}{(1-\phi)\sum_{j=1}^{N_t^y}\varsigma_t^j}(N_t^y)^{-\phi} \\
\frac{\partial c_{2,t+1}^j}{\partial \varsigma_t^j} =& r_{t+1}\mathcal{I}_t^j\left(1-\frac{\varsigma_t^j}{\sum_{j=1}^{N_t^y}\varsigma_t^j}\right) \\
\frac{\partial c_{2,t+1}^j}{\partial \varsigma_t^j} =& r_{t+1}\mathcal{I}_t^j\left(1-\frac{1}{N_t^y}\right) \geq 0.
\end{aligned}$$

As at first all households behave symmetrically, i.e., $\varsigma_t^j = \varsigma_t^k \; \forall j \neq k$, and thus $\sum_{j=1}^{N_t^y} \varsigma_t^j = N_t^y \varsigma_t^j$. And as $N_t^y \geq 1$, i.e., there exists at least one household in the economy.

Committing to a lower savings rate implies

$$u'(c_{1,t}^j) > \beta r_{t+1} u'(c_{2,t+1}^j),$$

where as remaining on the *ex-ante* Euler equation would entail

$$u'(c_{1,t}^j) = \beta r_{t+1} u'(c_{2,t+1}^j),$$

which would also yield the highest level of utility.
Increasing the savings rate lowers $c_{1,t}^j$ and increases $c_{2,t+1}^j$ and as $u'(\cdot) > 0$, this means household j gets closer to her *ex-ante* Euler equation and thus to the highest level of utility she can reach in her lifetime given her budget constraint.

B.14 CES Production Function

With a CES production function, the share of labor and capital income in output changes with a change in the capital-to-labor ratio.

$\rho = \frac{\sigma-1}{\sigma}$, where $\sigma \in (0,\infty)$ denotes the elasticity of substitution between capital and labor and $\{\rho \in \mathbb{R} \mid -\infty < \theta \leq 1 \setminus 0\}$. As before, ε denotes the elasticity of substitution between the different intermediate goods in the production of the final good, where it is assumed that $\varepsilon \in (1,\infty)$.

The profit maximization problem of intermediate good firm i is given as

$$\max_{p_t(i),K_t(i),L_t(i)} p_t(i) \left(\phi(A_{K_t}K_t(i))^\rho + (1-\phi)(A_{L_t}L_t(i))^\rho\right)^{\frac{1}{\rho}} - r_tK_t(i) - w_tL_t(i)$$
$$\text{s.t. } \left(\phi(A_{K_t}K_t(i))^\rho + (1-\phi)(A_{L_t}L_t(i))^\rho\right)^{\frac{1}{\rho}} = p_t(i)^{-\varepsilon}Y_t.$$

$$\max_{K_t(i),L_t(i)} \left(\phi(A_{K_t}K_t(i))^\rho + (1-\phi)(A_{L_t}L_t(i))^\rho\right)^{\frac{\mu}{\rho}} Y_t^{\frac{1}{\varepsilon}} - r_tK_t(i) - w_tL_t(i),$$

where $\mu = \frac{\varepsilon-1}{\varepsilon}$ and thus $\mu \in (0,1)$. Y_t is defined analogous to the main text.

$$Y_t = \left(\phi(A_{K_t}K_t)^\rho + (1-\phi)(A_{L_t}L_t)^\rho\right)^{\frac{1}{\rho}} - \left(\mathcal{S}_t(w_t,\eta_t) - \bar{K}_{t+1}\right).$$

The aggregated first-order conditions are given as

$$r_t - \mu\left(\phi(A_{K_t}K_t)^\rho + (1-\phi)(A_{L_t}L_t)^\rho\right)^{\frac{\mu}{\rho}-1} Y_t^{\frac{1}{\varepsilon}} \phi A_{K_t}^\rho K_t^{\rho-1} = 0,$$
$$w_t - \mu\left(\phi(A_{K_t}K_t)^\rho + (1-\phi)(A_{L_t}L_t)^\rho\right)^{\frac{\mu}{\rho}-1} Y_t^{\frac{1}{\varepsilon}} (1-\phi) A_{L_t}^\rho L_t^{\rho-1} = 0.$$

Assuming the real ELB is binding in period $t+1$, I have a third equilibrium condition

$$\bar{K}_{t+1} - \varsigma_t w_t L_t(\eta_t) = 0$$

$$\Leftrightarrow$$

$$w_t = \frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)}.$$

The equilibrium can then be characterized as

$$F \equiv r_t - \mu \left(\phi(A_{K_t} K_t(r_t))^\rho + (1-\phi)(A_{L_t} L_t(\eta_t))^\rho\right)^{\frac{1-\rho}{\rho}} \phi A_{K_t}^\rho K_t(r_t)^{\rho-1} = 0,$$

$$\Leftrightarrow$$

$$F \equiv r_t - \mu \left(\phi A_{K_t}^\rho + (1-\phi)\left(\frac{A_{L_t} L_t(\eta_t)}{K_t(r_t)}\right)^\rho\right)^{\frac{1-\rho}{\rho}} \phi A_{K_t}^\rho = 0,$$

$$G \equiv \frac{\bar{K}_{t+1}}{\varsigma_t L_t(\eta_t)} - \mu \left(\phi(A_{K_t} K_t(r_t))^\rho + (1-\phi)(A_{L_t} L_t(\eta_t))^\rho\right)^{\frac{1-\rho}{\rho}} (1-\phi) A_{L_t}^\rho L_t(\eta_t)^{\rho-1} = 0,$$

where K_t and $L_t(\eta_t)$ now denote the equilibrium values of capital and labor with $\frac{\partial K_t}{\partial r_t} \geq 0$. And r_t and η_t are the endogenous variables.

$$\frac{\partial F}{\partial r_t} > 0, \quad \frac{\partial F}{\partial \eta_t} = \frac{\partial F}{\partial L_t}\frac{\partial L_t}{\partial \eta_t} > 0, \frac{\partial F}{\partial \bar{K}_{t+1}} = 0,$$
$$\frac{\partial G}{\partial r_t} < 0, \quad \frac{\partial G}{\partial \eta_t} = \frac{\partial G}{\partial L_t}\frac{\partial L_t}{\partial \eta_t} \gtreqless 0, \frac{\partial G}{\partial \bar{K}_{t+1}} > 0.$$

$$G \equiv \frac{\bar{K}_{t+1}}{\varsigma_t} - \mu \left(\phi(A_{K_t} K_t)^\rho + (1-\phi)(A_{L_t} L_t(\eta_t))^\rho\right)^{\frac{1-\rho}{\rho}} (1-\phi) A_{L_t}^\rho L_t(\eta_t)^\rho = 0$$

$$G \equiv \frac{\bar{K}_{t+1}}{\varsigma_t} - \mu \left(\phi(A_{K_t} K_t)^\rho L_t(\eta_t)^{\frac{\rho^2}{1-\rho}} + (1-\phi) A_{L_t}^\rho L_t(\eta_t)^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} (1-\phi) A_{L_t}^\rho = 0.$$

$$\frac{\partial G}{\partial L_t} = -\mu \left(\phi(A_{K_t} K_t)^\rho L_t(\eta_t)^{\frac{\rho^2}{1-\rho}} + (1-\phi) A_{L_t}^\rho L_t(\eta_t)^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}-1} (1-\phi) A_{L_t}^\rho$$
$$\cdot \left(\rho\phi(A_{K_t} K_t)^\rho L_t(\eta_t)^{\frac{\rho^2}{1-\rho}-1} + (1-\phi) A_{L_t}^\rho L_t(\eta_t)^{\frac{\rho}{1-\rho}-1}\right) \gtreqless 0.$$

A sufficient condition that $\frac{\partial G}{\partial L_t} < 0$ and thus that $\frac{\partial G}{\partial \eta_t} > 0$ is $\rho > 0$, i.e., the elasticity of substitution between capital and labor is larger than 1 and thus capital and labor are substitutes.

Assuming $\rho > 0$, it follows that

$$\frac{\partial r_t}{\partial(-\bar{K}_{t+1})} = -\frac{\frac{\partial F}{\partial \eta_t}\frac{\partial G}{\partial \bar{K}_{t+1}}}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial \eta_t} - \frac{\partial F}{\partial \eta_t}\frac{\partial G}{\partial r_t}} < 0, \quad \frac{\partial \eta_t}{\partial(-\bar{K}_{t+1})} = -\frac{\frac{\partial F}{\partial r_t}\left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right)}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial \eta_t} - \frac{\partial F}{\partial \eta_t}\frac{\partial G}{\partial r_t}} > 0.$$

Using the third equilibrium condition again

$$\bar{K}_{t+1} - \varsigma_t w_t L_t(\eta_t) = 0$$
$$\Leftrightarrow$$
$$L_t(\eta_t) = \frac{\bar{K}_{t+1}}{\varsigma_t w_t}.$$

The equilibrium can also be characterized as

$$\begin{aligned} F \equiv & r_t - \mu\left(\phi A_{K_t}^{\rho} + (1-\phi)\left(A_{L_t}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\frac{1}{K_t(r_t)}\right)^{\rho}\right)^{\frac{1-\rho}{\rho}} \phi A_{K_t}^{\rho} = 0, \\ G \equiv & w_t - \mu\left(\phi(A_{K_t}K_t(r_t))^{\rho}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\rho} + (1-\phi)A_{L_t}^{\rho}\right)^{\frac{1-\rho}{\rho}} (1-\phi)A_{L_t}^{\rho} = 0, \end{aligned}$$

with r_t and w_t as the endogenous variables.

$$G \equiv 1 - \mu\left(\phi(A_{K_t}K_t)^{\rho}\left(\frac{\bar{K}_{t+1}}{\varsigma_t}\right)^{-\rho} w_t^{-\frac{\rho^2}{1-\rho}} + (1-\phi)A_{L_t}^{\rho} w_t^{-\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} (1-\phi)A_{L_t}^{\rho} = 0.$$

$$\begin{aligned} &\frac{\partial F}{\partial r_t} > 0, \quad \frac{\partial F}{\partial w_t} > 0, \quad \frac{\partial F}{\partial \bar{K}_{t+1}} < 0, \\ &\frac{\partial G}{\partial r_t} < 0, \quad \frac{\partial G}{\partial w_t} \gtreqless 0, \quad \frac{\partial G}{\partial \bar{K}_{t+1}} > 0. \end{aligned}$$

$$\frac{\partial G}{\partial w_t} = -\mu\left(\phi(A_{K_t}K_t)^{\rho}\left(\frac{\bar{K}_{t+1}}{\varsigma_t}\right)^{-\rho} w_t^{-\frac{\rho^2}{1-\rho}} + (1-\phi)A_{L_t}^{\rho} w_t^{-\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}-1} (1-\phi)A_{L_t}^{\rho}$$

$$\cdot \left(-\rho\phi(A_{K_t}K_t)^{\rho} \left(\frac{\bar{K}_{t+1}}{\varsigma_t}\right)^{-\rho} w_t^{-\frac{\rho^2}{1-\rho}-1} - (1-\phi)A_{L_t}^{\rho} w_t^{-\frac{\rho}{1-\rho}-1} \right) \gtreqless 0.$$

A sufficient condition that $\frac{\partial G}{\partial w_t} > 0$ is $\rho > 0$, i.e., the elasticity of substitution between capital and labor is larger than 1 and thus capital and labor are substitutes.

Assuming $\rho > 0$, it follows that

$$\frac{\partial r_t}{\partial(-\bar{K}_{t+1})} = -\frac{\left(-\frac{\partial F}{\partial \bar{K}_{t+1}}\right)\frac{\partial G}{\partial w_t} + \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial \bar{K}_{t+1}}}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t} - \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial r_t}} = \frac{\frac{\partial F}{\partial \bar{K}_{t+1}}}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t} - \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial r_t}} < 0,$$

$$\frac{\partial w_t}{\partial(-\bar{K}_{t+1})} = -\frac{\frac{\partial F}{\partial r_t}\left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right) + \frac{\partial F}{\partial \bar{K}_{t+1}}\frac{\partial G}{\partial r_t}}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t} - \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial r_t}} = \frac{\frac{\partial G}{\partial \bar{K}_{t+1}}}{\frac{\partial F}{\partial r_t}\frac{\partial G}{\partial w_t} - \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial r_t}} > 0.$$

As

$$\begin{aligned}
\left(-\frac{\partial F}{\partial \bar{K}_{t+1}}\right)\frac{\partial G}{\partial w_t} + \frac{\partial F}{\partial w_t}\frac{\partial G}{\partial \bar{K}_{t+1}} =& \Lambda \frac{1}{\bar{K}_{t+1}}\left(1 - \Xi\frac{1}{w_t}\right) + \Lambda\frac{1}{w_t}\Xi\frac{1}{\bar{K}_{t+1}} \\
=& \Lambda\frac{1}{\bar{K}_{t+1}} \\
=& \left(-\frac{\partial F}{\partial \bar{K}_{t+1}}\right) > 0. \\
\frac{\partial F}{\partial r_t}\left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right) + \frac{\partial F}{\partial \bar{K}_{t+1}}\frac{\partial G}{\partial r_t} =& \left(1 + \Lambda\frac{1}{K_t}\frac{\partial K_t}{\partial r_t}\right)(-1)\left(\Xi\frac{1}{\bar{K}_{t+1}}\right) + \left(-\Lambda\frac{1}{\bar{K}_{t+1}}\right)\left(-\Xi\frac{1}{K_t}\frac{\partial K_t}{\partial r_t}\right) \\
=& -\Xi\frac{1}{\bar{K}_{t+1}} \\
=& \left(-\frac{\partial G}{\partial \bar{K}_{t+1}}\right) < 0,
\end{aligned}$$

with

$$\begin{aligned}
\Lambda \equiv& \mu(1-\rho)\left(\phi A_{K_t}^{\rho} + (1-\phi)\left(A_{L_t}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\frac{1}{K_t(r_t)}\right)^{\rho}\right)^{\frac{1-\rho}{\rho}-1}\phi A_{K_t}^{\rho}(1-\phi) \\
&\cdot \left(A_{L_t}\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)\frac{1}{K_t(r_t)}\right)^{\rho} > 0,
\end{aligned}$$

$$\Xi \equiv \mu(1-\rho)\left(\phi(A_{K_t}K_t(r_t))^\rho\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\rho}+(1-\phi)A_{L_t}^\rho\right)^{\frac{1-\rho}{\rho}-1}$$
$$\cdot(1-\phi)A_{L_t}^\rho\phi(A_{K_t}K_t(r_t))^\rho\left(\frac{\bar{K}_{t+1}}{\varsigma_t w_t}\right)^{-\rho}>0.$$

Overall labor income is given as

$$L_t(\eta_t)w_t = \mu(1-\phi)\frac{1}{\phi\left(\frac{A_{K_t}K_t}{A_{L_t}L_t(\eta_t)}\right)^\rho+(1-\phi)}\left(\phi(A_{K_t}K_t)^\rho+(1-\phi)(A_{L_t}L_t(\eta_t))^\rho\right)^{\frac{1}{\rho}}$$
$$\mathcal{I}_t^y(\eta_t) = \frac{\mu(1-\phi)}{\phi\left(\frac{A_{K_t}K_t}{A_{L_t}L_t(\eta_t)}\right)^\rho+(1-\phi)}Y_t(\eta_t).$$

$$\frac{\partial \mathcal{I}_t^y(\eta_t)}{\partial \eta_t} = \frac{\mu(1-\phi)}{\phi\left(\frac{A_{K_t}K_t}{A_{L_t}L_t}\right)^\rho+(1-\phi)}\left(\underbrace{\frac{1}{\phi\left(\frac{A_{K_t}K_t}{A_{L_t}L_t}\right)^\rho+(1-\phi)}\rho\phi\left(\frac{A_{K_t}K_t}{A_{L_t}L_t}\right)^\rho\frac{1}{L_t}\frac{\partial L_t}{\partial \eta_t}Y_t}_{\gtreqless 0}+\underbrace{\frac{\partial Y_t}{\partial L_t}\frac{\partial L_t}{\partial \eta_t}}_{<0}\right)\gtreqless 0.$$

A sufficient condition for $\frac{\partial \mathcal{I}_t^y(\eta_t)}{\partial \eta_t}<0$ is that $\rho>0$, i.e., capital and labor are substitutes. An elasticity of substitution greater than 1 implies that the share of aggregate output received by labor increases (decreases) if the amount of labor used in production rises (falls). $\rho>0$ hence ensures that the income share received by labor and thus the overall income of young workers falls as the unemployment rate increases. This is a sufficient condition for the savings glut to be endogenously eliminated.

B.15 Household Heterogeneity

So far, I have assumed that all households are symmetrical and homogeneous. However, in recent years, the macroeconomic literature has recognized the importance of household heterogeneity, especially with regard to inequality; see, for example, Kaplan *et al.* (2018). Assuming that all households are homogeneous implies that they are all responsible for the excess savings that bring about a demand-induced recession. In reality, savings are not equally distributed across all households (see, for example, Saez and Zucman (2016) and Mian *et al.* (2021)). Therefore, only some households contribute to a potential savings glut, but as I will show below, the negative ramifications will likely affect all households, even those that have no savings.

Assume that nominal wages and the price level are downward rigid and that all households have to pay mandatory contributions into a PAYG pension system. Therefore, all households will receive some income in old-age, regardless of whether they have accumulated savings or not. Moreover, assume there are two types of households: savers and hand-to-mouth households. Savers find it optimal to save beyond the mandatory amount, i.e., the amount paid into the pension system. Hand-to-mouth households do not accumulate private savings, i.e., they consume their entire income net of the pension contribution.[22]

This set-up entails that only savers are responsible for excess savings. And, thus, a demand-induced recession can only be eliminated by reducing the income of savers.

The intermediate good firms produce using capital and labor from savers and hand-to-mouth households, which are assumed to supply different types of labor.[23] Consider a situation in which savers attempt to save more than the economy can absorb. This leads to a shortfall in demand and involuntary unemployment for savers. To analyze what this entails for hand-to-mouth households, I study how the equilibrium employment of hand-to-mouth households is affected by a decrease in the labor supply of savers, i.e., due to involuntary unemployment.

Assuming factor supply is fixed, the aggregate first-order conditions yield the following system of equations, which characterize the equilibrium in the market for the two types of labor and capital

$$\mu F_{L_t^S}(L_t^S, L_t^H, K_t) - w_{L_t^S} = 0,$$
$$\mu F_{L_t^H}(L_t^S, L_t^H, K_t) - w_{L_t^H} = 0,$$

[22]This could, for example, be due to differences in time preferences or income.

[23]If they supply the same type of labor hand-to-mouth, households will automatically be affected in the same way as savers since firms will reduce aggregate labor demand due to too little demand.

$$\mu F_{K_t}(L_t^S, L_t^H, K_t) - R_t = 0.$$

$\mu = \frac{\varepsilon-1}{\varepsilon} < 1$ is the inverse mark-up. $F(L_t^S, L_t^H, K_t)$ denotes the production function of the intermediate good firms. $F(L_t^S, L_t^H, K_t)$ has constant returns to scale in L_t^S, L_t^H, K_t, is increasing, and is concave in each production factor. L_t^S and L_t^H denote the labor supply of savers and hand-to-mouth households, respectively, and K_t is capital. $w_{L_t^S}$, L_t^H, and R_t are the endogenous variables.

An increase in the unemployment rate of savers is equivalent to a decrease in L_t^S. The effect of a fall in L_t^S on L_t^H is given as

$$\frac{\partial L_t^H}{\partial(-L_t^S)} = -\frac{\begin{vmatrix} -1 & -\mu F_{L_t^S L_t^S} & 0 \\ 0 & -\mu F_{L_t^H L_t^S}) & 0 \\ 0 & -\mu F_{K_t L_t^S} & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 0 \\ 0 & \mu F_{L_t^H L_t^H} & 0 \\ 0 & 0 & -1 \end{vmatrix}} = \frac{F_{L_t^H L_t^S}}{F_{L_t^H L_t^H}},$$

as $F_{L_t^H L_t^H} < 0$, the sign of $\frac{\partial L_t^H}{\partial(-L_t^S)}$ depends on the sign of $F_{L_t^H L_t^S}$. For $F_{L_t^H L_t^S} > 0$ it follows that $\frac{\partial L_t^H}{\partial(-L_t^S)} < 0$, and thus an increase in the unemployment rate of savers will also lead to a rise in the unemployment rate of hand-to-mouth households, which reduces their income.[24] Therefore, a demand-induced recession, which is caused by the excess savings of a fraction of the population, can have negative ramifications for households that themselves have not accumulated excess savings. This is also shown in the main part of the paper. (Involuntary) unemployment reduces the return to capital in the current period and thus the income of old households, even though old households have a marginal propensity to consume of 1, and hence curtailing their income does not reduce excess savings in the economy. Thus, old households can be seen as akin to hand-to-mouth households in this section.

[24]Assuming the wage rate of hand-to-mouth households is flexible, the effect is given as $\frac{\partial w_{L_t^H}}{\partial(-L_t^S)} = \frac{\mu F_{L_t^H L_t^S}}{-1}$, which is negative if $F_{L_t^H L_t^S} > 0$.

B.16 Positive Nominal Savings in Equilibrium

Assume that, for unspecified reasons, households always find it optimal to hold a positive amount of nominal assets between periods denoted by $B^* > 0$. This entails that in period 0, the amount of funds agents can borrow from banks, $\mathcal{M}_0$, is such that

$$\mathcal{M}_0 = P_0 Y_0 - \mathcal{B}^*,$$

where $\mathcal{B}^* = N_0^y B^* > 0$ denotes aggregate equilibrium nominal savings. In the initial period, banks must not only provide sufficient funds in order for the price level to be equal to 1 given real production but also the necessary funds for households to accumulate a positive amount of nominal savings without causing a demand shortage.
In all consecutive periods, the amount agents can borrow is given by

$$\mathcal{M}_t = P_t Y_t,$$

with $\{P_t\}_{t=0}^{\infty} = 1$. Hence, after the initial period, banks are constrained such that they cannot issue more loans than are consistent with a constant price level. Therefore, for an equilibrium that features full employment and a constant price level with $B^* > 0$, I require that aggregate equilibrium nominal savings, $\mathcal{B}_t$, remain constant over time.

$$\begin{aligned}
\mathcal{B}_t &= B_t N_t^y \\
\mathcal{B}_t &= (1-\omega_t)\varsigma_t W_t N_t^y \\
\frac{\mathcal{B}_t}{P_t} &= (1-\omega_t)\varsigma_t w_t (1+n_t) N_{t-1}^y \\
\mathcal{B}_t &= (1-\omega_t)\varsigma_t w_t (1+n_t) N_{t-1}^y,
\end{aligned}$$

where $(1-\omega_t) \in [0,1]$ denotes the share of overall savings invested in nominal assets, and the last line makes use of the fact that P_t must remain constant and is equal to 1. In order for $\mathcal{B}_t$ to remain constant over time, the aggregate dissavings of the old in terms of nominal assets must be equal to the aggregate savings of the young in terms of nominal assets ,i.e.,

$$\mathcal{B}_{t-1} = \mathcal{B}_t$$

$$
\begin{aligned}
(1-\omega_{t-1})\varsigma_{t-1}w_{t-1}N^y_{t-1} =&(1-\omega_t)\varsigma_t w_t(1+n_t)N^y_{t-1}\\
1 =&\frac{1-\omega_t}{1-\omega_{t-1}}\frac{\varsigma_t}{\varsigma_{t-1}}\frac{w_t}{w_{t-1}}\frac{N^y_t}{N^y_{t-1}}.
\end{aligned}
$$

If the real wage rate and/or the population grows over time, I require either a fall in $(1-\omega_t)$ or the savings rate, ς_t, in order for $\mathcal{B}_t$ to remain constant, as otherwise $\mathcal{B}_t$ would increase over time, resulting in a demand shortage.

In case $\mathcal{B}_t$ decreases over time due to, for example, technological progress that increases real wages, this will lead to a demand shortage that can be absorbed through a change in *nominal* or *real* variables. A third option would be for the government to take on nominal debt and spend this money in the current period. This would provide the additional liquidity needed in order for households to increase the overall amount of nominal savings and, at the same time, keep the price level constant and maintain full employment in the short-run. To assess whether this would also be sustainable in the long-run, would require a more in-depth analysis.

B.17 Risk and the Effective Lower Bound

The following serves as a simple illustration of why (young) households could be willing to save in the nominal asset, i.e., money, even though the nominal asset only delivers a nominal return of zero.

Assume that investing in the capital stock for the next period is risky. Agents can either invest in the real risky asset, i.e., the capital stock of the next period, or in the riskless (safe) nominal asset. Furthermore, assume that the price level in the next period is unaffected by this risk, i.e., there is no risk regarding the rate of inflation for the next period.

Assume the real return on the next period's capital stock r_{t+1} is lognormally distributed: $\log(r_{t+1}) \sim \mathcal{N}(\mu, \sigma^2)$.

Assume the agent has already decided how much to optimally save, i.e., ς_t^*, and now must decide how to allocate her wealth between the two assets. The household has CRRA preferences, and her problem can be written as

$$\max \quad \mathbb{E}_t \left[\frac{c_{t+1}^{1-\gamma} - 1}{1-\gamma} \right]$$

$$\text{s.t. } c_{t+1} = r_{p,t+1} \varsigma_t^*,$$

$r_{p,t+1}$ denotes return on the portfolio, i.e., the combination of risky and riskless assets, and γ is the degree of relative risk aversion. Let ω_t be the portfolio share in the risky asset. Following Campbell (2017) Chapter 2.1, the optimal portfolio choice problem can be expressed as follows

$$\max_{\omega_t} \omega_t (\mathbb{E}_t[\log(r_{t+1})] - \log(R_{f,t+1})) + \frac{1}{2}\omega_t(1-\omega_t)\sigma_{t+1}^2 + \frac{1}{2}(1-\gamma)\omega_t^2 \sigma_{t+1}^2,$$

where r_{t+1} is the return on the risky asset, i.e., the capital stock, and $R_{f,t+1}$ is the return on the riskless asset, i.e., money.

This yields the solution

$$\omega_t = \frac{\mathbb{E}_t[\log(r_{t+1})] - \log(R_{f,t+1}) + \frac{\sigma^2}{2}}{\gamma\sigma^2} \approx \frac{\mathbb{E}_t[r_{t+1}] - R_{f,t+1}}{\gamma\sigma^2}.$$

I can set $\omega_t = 1$ to solve for the risk-free rate that constitutes the endogenous ELB

$$\mathbb{E}_t[r_{t+1}] - \gamma\sigma^2 = R_{f,t+1}.$$

Hence, as long as $\mathbb{E}_t[r_{t+1}] \geq R_{f,t+1} + \gamma\sigma^2$, households will invest all their savings in the real asset, i.e., the capital stock.

Moreover, even if $R_{f,t+1} = 1$ implying the nominal return of the riskless asset is zero, $\mathbb{E}_t[r_{t+1}]$ needs to be larger than 1 for households to be willing to invest all their savings in real risky assets and not hold any of the riskless (safe) assets.

The reason for this result is that agents are risk-averse. For $\gamma = 0$ agents are risk neutral, and I get the same result as in the case with no risk, i.e., $\mathbb{E}_t[r_{t+1}] = R_{f,t+1}$.

In addition, to account for the fact that the riskless asset yields a nominal return and the risky asset a real return, the above condition can be rewritten as

$$\mathbb{E}_t[r_{t+1}] - \gamma\sigma^2 \geq \frac{P_t}{P_{t+1}} R_{f,t+1}$$
$$\mathbb{E}_t[r_{t+1}] - \gamma\sigma^2 \geq \frac{1}{\Pi_{t+1}} R_{f,t+1}.$$

Hence, a higher inflation rate will decrease the value of the riskless asset and thus lower the endogenous ELB in the economy.

B.18 Financial Sector

Banks can issue money to households and firms. They operate under perfect competition and incur zero costs. Moreover, they have perfect foresight, i.e., they will always know how high the income of a borrower will be. Therefore, they will never extend a loan that is higher than the borrower's income.

In addition, banks face three constraints: First, they cannot lend money to themselves. Second, they cannot issue more money than an exogenous multiple of the production possibility frontier.[25] Third, at the end of each period, the balance sheet total has to be zero. Hence, all loans taken out at the beginning of the period must be repaid at the end of the period or sold to an agent that is not a bank.

Households and firms will both borrow from the banks in order to finance consumption in the current period or to invest in the capital stock of the next period.

Households will receive their income at the end of the current period and are thus able to repay their loans directly. Firms, however, will only generate a return on their investment in the next period and are thus unable to repay their loans directly; hence, they will have to pay interest on their loans.

Moreover, as banks cannot issue more credit than what is produced, the maximum amount of money that is available for investment will be equal to the savings of the young house-

[25]This ensures that if the economy operates at full capacity, the price level is constant. For simplicity, I will assume that the price level is equal to 1 in that case, i.e., the exogenous multiple will be 1.

holds, i.e., the share of their income that young households do not borrow at the beginning of the period.

At the end of the period, the income of young households will thus be higher than their obligations. Hence, they will be able to purchase the debt obligations that the banks issued to the firms.

Therefore, banks match investments with savings and determine the equilibrium real interest rate subject to the constraint that it has to be larger or equal to the real return on the nominal asset, as households will never exchange money for bonds that yield a lower real return than money.

So far, the financial sector's main purpose has been to help eliminate "Say's law" and thus allow demand to have an effect on production. However, given the assumption made, the financial sector itself has no effect on outcomes. Nevertheless, it is straightforward to extend the model in such a way that the decisions made in the financial sector affect outcomes in the economy.[26]

Assume that the financial sector no longer has perfect foresight and full information and hence has to decide how much to lend to households and firms. From the discussion in the main part, it is clear that in order for this decision to have an effect on outcomes, it is necessary that banks, for some reason, lend more or less money to households and firms compared to the case of perfect foresight and information. Resulting in either a demand shortage, similarly to before, except that now it is not the households who save "too much", but the banks that are too strict in their lending. In the other situation, there would be excess demand, potentially resulting in inflation.

For example, if banks are required to remain below a certain leverage ratio and for some (exogenous) reason their equity decreases, and assuming banks are not able to sufficiently replenish their equity in time, they might be forced to reduce their lending to households

[26]This relates to the vast literature on financial frictions. See Quadrini (2011) and Brunnermeier *et al.* (2012) for surveys of the mechanisms and literature.

and firms, thus inducing a demand-driven recession if wages and prices are rigid.[27]

[27] Assuming banks face the risk that some agents default on their debt, this would not necessarily lead to too little lending. Banks would charge higher interest rates, and this would make some households poorer, i.e., the ones that do not default on their debt. To see this, assume for simplicity that only households could potentially default and that banks operate under perfect competition and have zero costs. Let p_t denote the probability of default. If the banks lend out the amount $\mathcal{M}_t$ to households, the expected repayment is $(1 - p_t)\mathcal{M}_t R_t$, where R_t is the interest rate households have to pay within the period. As banks make zero profits and face no costs, the amount they lend out has to be equal to the expected repayment, i.e., $\mathcal{M}_t = (1 - p_t)\mathcal{M}_t R_t$. Hence, an increase in p_t only affects R_t but not $\mathcal{M}_t$.

Appendix C

Appendix to Chapter 3

C.1 Factor Prices

$$A \equiv \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}\eta - R_t = 0$$

$$A \equiv \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} k_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}(1-\psi_t))^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}\psi_t)^{1-\alpha}\eta - R_t = 0,$$

$$B \equiv \mu\alpha \left(\eta (A_{K_t}K_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}H_t)^{1-\alpha}(1-\eta) - w_{L_t} = 0$$

$$B \equiv \mu\alpha \left(\eta (A_{K_t}k_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} (1-\psi)^{\frac{\theta^2-\theta}{\alpha-\theta}} + (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} (1-\psi_t)^{\frac{\theta\alpha-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t}\psi_t)^{1-\alpha}(1-\eta) - w_{L_t} = 0,$$

$$C \equiv \mu(1-\alpha) \left(\eta (A_{K_t}K_t)^{\theta} + (1-\eta)(A_{L_t}L_t)^{\theta} \right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha} H_t^{-\alpha} - w_{H_t} = 0$$

$$C \equiv \mu(1-\alpha) \left(\eta (A_{K_t}k_t)^{\theta} + (1-\eta)(A_{L_t}(1-\psi_t))^{\theta} \right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha} \psi_t^{-\alpha} - w_{H_t} = 0,$$

with $k_t = \frac{K_t}{N_t}$.

$$\frac{\partial R_t}{\partial L_t} = -\frac{\frac{\partial A}{\partial L_t}}{\frac{\partial A}{\partial R_t}} \gtreqless 0.$$

$$\frac{\partial A}{\partial L_t} = (\alpha-\theta)\mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t}L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-2\theta}{\theta}} (A_{H_t}H_{Y_t})^{1-\alpha}\eta$$

$$\cdot (1-\eta) A_{L_t}^{\theta} L_t^{\theta-1} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \gtreqless 0.$$

The sign is determined by $\alpha - \theta \gtreqless 0$, i.e., depending on if L_t and K_t are *strong* substitutes or not.

$$\frac{\partial R_t}{\partial H_t} = -\frac{\frac{\partial A}{\partial H_t}}{\frac{\partial A}{\partial R_t}} > 0.$$

$$\frac{\partial R_t}{\partial N_t} = -\frac{\frac{\partial A}{\partial N_t}}{\frac{\partial A}{\partial R_t}} > 0,$$

as

$$\begin{aligned}
\frac{\partial A}{\partial N_t} =& (\alpha-\theta)\mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t} L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}-1} (A_{H_t} H_t)^{1-\alpha} \eta \\
& \cdot (1-\eta) A_{L_t}^{\theta} L_t^{\theta-1} \frac{\partial L_t}{\partial N_t} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} + \mu\alpha \left(\eta A_{K_t}^{\frac{\alpha\theta}{\alpha-\theta}} K_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}} + (1-\eta)(A_{L_t} L_t)^{\theta} A_{K_t}^{\frac{\theta^2}{\alpha-\theta}} K_t^{\frac{\theta^2-\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} \\
& \cdot (1-\alpha)(A_{H_t} H_t)^{1-\alpha} \eta \frac{1}{H_t} \frac{\partial H_t}{\partial N_t} \\
\frac{\partial A}{\partial N_t} \propto& (\alpha-\theta) \left(\eta \left(\frac{A_{K_t} K_t}{A_{L_t} L_t} \right)^{\theta} + (1-\eta) \right)^{-1} (1-\eta) \frac{1-\psi_t}{L_t} + (1-\alpha) \frac{\psi_t}{H_t} \\
\frac{\partial A}{\partial N_t} \propto& (\alpha-\theta)(1-\eta) \frac{H_t}{L_t} \frac{1-\psi_t}{\psi_t} \frac{1}{1-\alpha} + \eta \left(\frac{A_{K_t} K_t}{A_{L_t} L_t} \right)^{\theta} + (1-\eta) \\
\frac{\partial A}{\partial N_t} \propto& (1-\eta) \left(\frac{\alpha-\theta}{1-\alpha} + 1 \right) + \eta \left(\frac{A_{K_t} K_t}{A_{L_t} L_t} \right)^{\theta} \\
\frac{\partial A}{\partial N_t} \propto& (1-\eta) \left(\frac{1-\theta}{1-\alpha} \right) + \eta \left(\frac{A_{K_t} K_t}{A_{L_t}(1-\psi_t) N_t} \right)^{\theta} > 0,
\end{aligned}$$

as $\theta \in (1, -\infty)$. Moreover, as the constant of proportionality is positive this implies that $\frac{\partial A}{\partial N_t} > 0$.

$$\frac{\partial w_{L_t}}{\partial L_t} = -\frac{\frac{\partial B}{\partial L_t}}{\frac{\partial B}{\partial w_{L_t}}} < 0, \quad \frac{\partial w_{L_t}}{\partial H_t} = -\frac{\frac{\partial B}{\partial H_t}}{\frac{\partial B}{\partial w_{L_t}}} > 0, \quad \frac{\partial w_{L_t}}{\partial N_t} = -\frac{\frac{\partial B}{\partial N_t}}{\frac{\partial B}{\partial w_{L_t}}} \gtreqless 0,$$

$$\frac{\partial w_{H_t}}{\partial L_t} = -\frac{\frac{\partial C}{\partial L_t}}{\frac{\partial C}{\partial w_{H_t}}} > 0, \quad \frac{\partial w_{H_t}}{\partial H_t} = -\frac{\frac{\partial C}{\partial H_t}}{\frac{\partial C}{\partial w_{H_t}}} < 0, \quad \frac{\partial w_{H_t}}{\partial N_t} = -\frac{\frac{\partial C}{\partial N_t}}{\frac{\partial C}{\partial w_{H_t}}} < 0,$$

Effect of N_t on the wage rate of low- and high-skilled workers:

Define

$$\Lambda_t \equiv \eta (A_{K_t} K_t)^{\theta} A_{L_t}^{\frac{\theta^2}{\alpha-\theta}} L_t^{\frac{\theta^2-\theta}{\alpha-\theta}}$$
$$\Xi_t \equiv (1-\eta) A_{L_t}^{\frac{\alpha\theta}{\alpha-\theta}} L_t^{\frac{\theta\alpha-\theta}{\alpha-\theta}}$$

$$\begin{aligned}\frac{\partial B}{\partial N_t} =& \mu\alpha \left(\Lambda_t + \Xi_t\right)^{\frac{\alpha-\theta}{\theta}-1} (A_{H_t} H_t)^{1-\alpha} (1-\eta) \left((\theta-1)\Lambda_t + (\alpha-1)\Xi_t\right) \frac{1}{L_t} \frac{\partial L_t}{\partial N_t} \\ &+ \mu\alpha \left(\Lambda_t + \Xi_t\right)^{\frac{\alpha-\theta}{\theta}} (A_{H_t} H_t)^{1-\alpha} (1-\eta)(1-\alpha) \frac{1}{H_t} \frac{\partial H_t}{\partial N_t} \gtreqless 0 \\ \frac{\partial B}{\partial N_t} \propto& (\theta-\alpha)\Lambda_t \gtreqless 0,\end{aligned}$$

with $\alpha \in (0,1)$, $\theta \in (-\infty, 1]$ and a positive constant of proportionality.

$$\begin{aligned}\frac{\partial C}{\partial N_t} =& \mu(1-\alpha) \left(\eta (A_{K_t} K_t)^{\theta} + (1-\eta)(A_{L_t} L_t)^{\theta}\right)^{\frac{\alpha}{\theta}-1} A_{H_t}^{1-\alpha} H_t^{-\alpha} \alpha (1-\eta)(A_{L_t} L_t)^{\theta} \frac{1}{L_t} \frac{\partial L_t}{\partial N_t} \\ &- \mu(1-\alpha) \left(\eta (A_{K_t} K_t)^{\theta} + (1-\eta)(A_{L_t} L_t)^{\theta}\right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha} H_t^{-\alpha} \alpha \frac{1}{H_t} \frac{\partial H_t}{\partial N_t} \\ \frac{\partial C}{\partial N_t} \propto& \left(\left(\eta \frac{A_{K_t} K_t}{A_{L_t} L_t}\right)^{\theta} + (1-\eta)\right)^{-1} (1-\eta) \frac{1-\psi_t}{L_t} - \frac{\psi_t}{H_t} \\ \frac{\partial C}{\partial N_t} \propto& (1-\eta) - \left(\left(\eta \frac{A_{K_t} K_t}{A_{L_t} L_t}\right)^{\theta} + (1-\eta)\right) \\ \frac{\partial C}{\partial N_t} \propto& -\left(\eta \frac{A_{K_t} K_t}{A_{L_t} L_t}\right)^{\theta} < 0,\end{aligned}$$

as the constant of proportionality is positive, this implies that $\frac{\partial C}{\partial N_t} < 0$ and as $\frac{\partial C}{\partial w_{H_t}} < 0$, I have $\frac{\partial w_{H_t}}{\partial N_t} < 0$.

C.2 Value of an Innovation

In order to determine how the value of an innovation, i.e., Δ_{t+1}, changes if L_{t+1}, H_{t+1}, or N_{t+1} change, I use the second derivative of the production function.

$$F_{A_{K_{t+1}}L_{t+1}} = \alpha \left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} + (1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}-1} (A_{H_{t+1}}H_{t+1})^{1-\alpha}\eta$$
$$\cdot (\alpha-\theta)(1-\eta) A_{L_{t+1}}^{\theta} L_{t+1}^{\theta-1} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \gtreqless 0.$$

If Assumption 3.4 holds, the sign is negative, and thus scarcity of low-skilled labor induces capital augmenting technological change.

If $\alpha > \theta$ and $\theta > 0$, i.e., K_{t+1} and L_{t+1} are *weak* substitutes, then the sign is positive. Thus, an increase (decrease) in L_{t+1} makes acquiring a higher level of capital augmenting technology more (less) attractive.

If $\alpha < \theta$ and $\theta > 0$, i.e., K_{t+1} and L_{t+1} are *strong* substitutes, then the sign is negative. Hence, $0 < \alpha < \theta$ is a sufficient condition that $F_{A_{K_{t+1}}L_{t+1}} < 0$ and thus that a decrease (increase) in L_{t+1} makes acquiring a higher level of capital augmenting technology more (less) attractive.

If $\theta < 0$, i.e., K_{t+1} and L_{t+1} are *complements* and we have capital-skill substitutability, the sign is positive.

Therefore, depending on the value of the elasticity of substitution, demographic change increases or decreases the value of an innovation that augments capital.[1]

$$F_{A_{K_{t+1}}H_{t+1}} = \alpha(1-\alpha) \left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} + (1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} A_{H_{t+1}}^{1-\alpha} H_{t+1}^{-\alpha}\eta$$
$$\cdot \alpha \left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} + (1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}-1} (A_{H_{t+1}}H_{t+1})^{1-\alpha}\eta > 0.$$

An increase (decrease) in H_{t+1} makes acquiring a higher level of capital augmenting technology more (less) attractive.

An increase in N_{t+1} that leaves the skill share constant has the following effect on the value of an innovation

[1] The conditions derived above are equivalent to the conditions derived in Acemoglu (2010), where they determine whether a technology is strongly labor saving, i.e., if $\alpha < \theta$, or strongly labor complementary, i.e., if $\alpha > \theta$.

$$
\begin{aligned}
F_{A_{K_{t+1}}N_{t+1}} =&(\alpha-\theta)\alpha\left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}}K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}}+(1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta}A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}}K_{t+1}^{\frac{\theta^2}{\alpha-\theta}}\right)^{\frac{\alpha-\theta}{\theta}-1}(A_{H_{t+1}}H_{t+1})^{1-\alpha}\eta\\
&\cdot(1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta}\frac{1}{L_{t+1}}\frac{\partial L_{t+1}}{\partial N_{t+1}}A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}}K_{t+1}^{\frac{\theta^2}{\alpha-\theta}}\\
&+\alpha\left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}}K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}}+(1-\eta)(A_{L_{t+1}}L_{t+1})^{\theta}A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}}K_{t+1}^{\frac{\theta^2}{\alpha-\theta}}\right)^{\frac{\alpha-\theta}{\theta}}\\
&\cdot(1-\alpha)(A_{H_{t+1}}H_t)^{1-\alpha}\eta\frac{1}{H_{t+1}}\frac{\partial H_{t+1}}{\partial N_{t+1}}\\
F_{A_{K_{t+1}}N_{t+1}} \propto&(1-\eta)\left(\frac{1-\theta}{1-\alpha}\right)+\eta\left(\frac{A_{K_{t+1}}K_{t+1}}{A_{L_{t+1}}(1-\psi_{t+1})N_{t+1}}\right)^{\theta}>0,
\end{aligned}
$$

as the constant of proportionality is positive, this implies that $F_{A_{K_{t+1}}N_{t+1}}>0$. Therefore, the effect is always positive, even if capital and low-skilled labor are perfect substitutes, i.e., $\theta=1$.

Assume that an increase in N_{t+1} also changes the share of the population that is low- and high-skilled, i.e., ψ_{t+1}, but that nonetheless the stock of low- and high-skilled workers increases if N_{t+1} increases.

$$
\begin{aligned}
\frac{\partial L_{t+1}}{\partial N_{t+1}} =&(1-\psi_{t+1})-N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}>0,\\
\frac{\partial H_{t+1}}{\partial N_{t+1}} =&\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}>0.
\end{aligned}
$$

And that $\theta>\alpha$.

$$
\begin{aligned}
F_{A_{K_{t+1}}N_{t+1}} \propto&(1-\eta)\left(\frac{\alpha-\theta}{1-\alpha}\frac{\psi_{t+1}}{1-\psi_{t+1}}\frac{(1-\psi_{t+1})-N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}{\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}+1\right)\\
&+\eta\left(\frac{A_{K_{t+1}}K_{t+1}}{A_{L_{t+1}}(1-\psi_{t+1})N_{t+1}}\right)^{\theta}\gtreqless 0\\
F_{A_{K_{t+1}}N_{t+1}} \propto&(1-\eta)\left(\frac{\alpha-\theta}{1-\alpha}\frac{\psi_{t+1}}{1-\psi_{t+1}}\frac{1-\left(\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}\right)}{\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}+1\right)\\
&+\eta\left(\frac{A_{K_{t+1}}K_{t+1}}{A_{L_{t+1}}(1-\psi_{t+1})N_{t+1}}\right)^{\theta}\gtreqless 0
\end{aligned}
$$

$$F_{A_{K_{t+1}}N_{t+1}} \propto (1-\eta)\left(\underbrace{\frac{\alpha-\theta}{1-\alpha}\frac{\psi_{t+1}}{1-\psi_{t+1}}}_{<0}\left(\frac{1}{\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}-1\right)+1\right)$$
$$+\eta\left(\frac{A_{K_{t+1}}K_{t+1}}{A_{L_{t+1}}(1-\psi_{t+1})N_{t+1}}\right)^{\theta} \gtreqless 0.$$

In order for the first term to be negative, I require that $\frac{1}{\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}-1>0$. This can be rewritten as $1>\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}>0$. The second inequality holds by assumption, i.e., that the expression is larger than zero, whether the first holds depends on parameter values, but it is more likely to hold if $\frac{\partial\psi_{t+1}}{\partial N_{t+1}}<0 \Leftrightarrow \frac{\partial\psi_{t+1}}{\partial(-N_{t+1})}>0$, i.e., demographic change increases the share of high-skilled people.

$F_{A_{K_{t+1}}N_{t+1}}$ is more likely to be negative if the negative term is large, i.e., if $\frac{1}{\psi_{t+1}+N_{t+1}\frac{\partial\psi_{t+1}}{\partial N_{t+1}}}-1$ is large. The expression becomes larger if fraction becomes larger and that is the case if the denominator gets closer to zero, which happens if $\frac{\partial\psi_{t+1}}{\partial N_{t+1}}<0$.

C.3 Intra and Intertemporal Effects

$$\frac{\partial H_{R\&D_t}}{\partial N_t} = \frac{-\frac{\partial C}{\partial N_t}}{-\left(\frac{\partial D}{\partial H_{R\&D_t}}-\frac{\partial C}{\partial H_{R\&D_t}}\right)} > 0,$$
$$\frac{\partial R_t}{\partial \Delta_{t+1}} = \frac{\frac{\partial A}{\partial H_{R\&D_t}}\frac{\partial D}{\partial \Delta_{t+1}}}{-\left(\frac{\partial D}{\partial H_{R\&D_t}}-\frac{\partial C}{\partial H_{R\&D_t}}\right)} < 0,$$
$$\frac{\partial w_{L_t}}{\partial \Delta_{t+1}} = \frac{\left(-\frac{\partial D}{\partial \Delta_{t+1}}\right)\left(-\frac{\partial B}{\partial H_{R\&D_t}}\right)}{-\left(\frac{\partial D}{\partial H_{R\&D_t}}-\frac{\partial C}{\partial H_{R\&D_t}}\right)} < 0,$$
$$\frac{\partial w_{H_t}}{\partial \Delta_{t+1}} = \frac{\frac{\partial C}{\partial H_{R\&D_t}}\frac{\partial D}{\partial \Delta_{t+1}}}{-\left(\frac{\partial D}{\partial H_{R\&D_t}}-\frac{\partial C}{\partial H_{R\&D_t}}\right)} > 0,$$
$$\frac{\partial H_{R\&D_t}}{\partial \Delta_{t+1}} = \frac{\frac{\partial D}{\partial \Delta_{t+1}}}{-\left(\frac{\partial D}{\partial H_{R\&D_t}}-\frac{\partial C}{\partial H_{R\&D_t}}\right)} > 0.$$

$$\begin{vmatrix} \frac{\partial A}{\partial R_t} & \frac{\partial A}{\partial w_{L_t}} & \frac{\partial A}{\partial w_{H_t}} & \frac{\partial A}{\partial H_{R\&D_t}} \\ \frac{\partial B}{\partial R_t} & \frac{\partial B}{\partial w_{L_t}} & \frac{\partial B}{\partial w_{H_t}} & \frac{\partial B}{\partial H_{R\&D_t}} \\ \frac{\partial C}{\partial R_t} & \frac{\partial C}{\partial w_{L_t}} & \frac{\partial C}{\partial w_{H_t}} & \frac{\partial C}{\partial H_{R\&D_t}} \\ \frac{\partial D}{\partial R_t} & \frac{\partial D}{\partial w_{L_t}} & \frac{\partial D}{\partial w_{H_t}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & -1 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \\ 0 & 0 & -1 & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = (-1) \begin{vmatrix} -1 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \\ 0 & -1 & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix}$$
$$= -\left(\frac{\partial D}{\partial H_{R\&D_t}} - \frac{\partial C}{\partial H_{R\&D_t}} \right) > 0,$$

as $\frac{\partial C}{\partial H_{R\&D_t}} > 0$ and in Section 3.4 I have shown that $\frac{\partial D}{\partial H_{R\&D_t}} < 0$.

$$\begin{vmatrix} -\frac{\partial A}{\partial \Delta_{t+1}} & \frac{\partial A}{\partial w_{L_t}} & \frac{\partial A}{\partial w_{H_t}} & \frac{\partial A}{\partial H_{R\&D_t}} \\ -\frac{\partial B}{\partial \Delta_{t+1}} & \frac{\partial B}{\partial w_{L_t}} & \frac{\partial B}{\partial w_{H_t}} & \frac{\partial B}{\partial H_{R\&D_t}} \\ -\frac{\partial C}{\partial \Delta_{t+1}} & \frac{\partial C}{\partial w_{L_t}} & \frac{\partial C}{\partial w_{H_t}} & \frac{\partial C}{\partial H_{R\&D_t}} \\ -\frac{\partial D}{\partial \Delta_{t+1}} & \frac{\partial D}{\partial w_{L_t}} & \frac{\partial D}{\partial w_{H_t}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & -1 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \\ -\frac{\partial D}{\partial \Delta_{t+1}} & 0 & -1 & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = (-1) \begin{vmatrix} 0 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \\ -\frac{\partial D}{\partial \Delta\Pi_{t+1}} & -1 & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix}$$
$$= \frac{\partial A}{\partial H_{R\&D_t}} \frac{\partial D}{\partial \Delta_{t+1}} < 0.$$

$$\begin{vmatrix} \frac{\partial A}{\partial R_t} & -\frac{\partial A}{\partial \Delta_{t+1}} & \frac{\partial A}{\partial w_{H_t}} & \frac{\partial A}{\partial H_{R\&D_t}} \\ \frac{\partial B}{\partial R_t} & -\frac{\partial B}{\partial \Delta_{t+1}} & \frac{\partial B}{\partial w_{H_t}} & \frac{\partial B}{\partial H_{R\&D_t}} \\ \frac{\partial C}{\partial R_t} & -\frac{\partial C}{\partial \Delta_{t+1}} & \frac{\partial C}{\partial w_{H_t}} & \frac{\partial C}{\partial H_{R\&D_t}} \\ \frac{\partial D}{\partial R_t} & -\frac{\partial D}{\partial \Delta_{t+1}} & \frac{\partial D}{\partial w_{H_t}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & 0 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \\ 0 & -\frac{\partial D}{\partial \Delta_{t+1}} & -1 & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = \left(-\frac{\partial D}{\partial \Delta_{t+1}}\right) \begin{vmatrix} -1 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & -1 & \frac{\partial C}{\partial H_{R\&D_t}} \end{vmatrix}$$
$$= \left(-\frac{\partial D}{\partial \Delta_{t+1}}\right)\left(-\frac{\partial B}{\partial H_{R\&D_t}}\right) < 0.$$

$$\begin{vmatrix} \frac{\partial A}{\partial R_t} & \frac{\partial A}{\partial w_{L_t}} & -\frac{\partial A}{\partial \Delta_{t+1}} & \frac{\partial A}{\partial H_{R\&D_t}} \\ \frac{\partial B}{\partial R_t} & \frac{\partial B}{\partial w_{L_t}} & -\frac{\partial B}{\partial \Delta_{t+1}} & \frac{\partial B}{\partial H_{R\&D_t}} \\ \frac{\partial C}{\partial R_t} & \frac{\partial C}{\partial w_{L_t}} & -\frac{\partial C}{\partial \Delta_{t+1}} & \frac{\partial C}{\partial H_{R\&D_t}} \\ \frac{\partial D}{\partial R_t} & \frac{\partial D}{\partial w_{L_t}} & -\frac{\partial D}{\partial \Delta_{t+1}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & -1 & 0 & \frac{\partial B}{\partial H_{R\&D_t}} \\ 0 & 0 & 0 & \frac{\partial C}{\partial H_{R\&D_t}} \\ 0 & 0 & -\frac{\partial D}{\partial \Delta_{t+1}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix} = (-1) \begin{vmatrix} -1 & 0 & \frac{\partial A}{\partial H_{R\&D_t}} \\ 0 & 0 & \frac{\partial C}{\partial H_{R\&D_t}} \\ 0 & -\frac{\partial D}{\partial \Delta_{t+1}} & \frac{\partial D}{\partial H_{R\&D_t}} \end{vmatrix}$$

$$=\frac{\partial C}{\partial H_{R\&D_t}}\frac{\partial D}{\partial \Delta_{t+1}} > 0.$$

$$\begin{vmatrix} \frac{\partial A}{\partial R_t} & \frac{\partial A}{\partial w_{L_t}} & \frac{\partial A}{\partial w_{H_t}} & -\frac{\partial A}{\partial \Delta_{t+1}} \\ \frac{\partial B}{\partial R_t} & \frac{\partial B}{\partial w_{L_t}} & \frac{\partial B}{\partial w_{H_t}} & -\frac{\partial B}{\partial \Delta_{t+1}} \\ \frac{\partial C}{\partial R_t} & \frac{\partial C}{\partial w_{L_t}} & \frac{\partial C}{\partial w_{H_t}} & -\frac{\partial C}{\partial \Delta_{t+1}} \\ \frac{\partial D}{\partial R_t} & \frac{\partial D}{\partial w_{L_t}} & \frac{\partial D}{\partial w_{H_t}} & -\frac{\partial D}{\partial \Delta_{t+1}} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -\frac{\partial D}{\partial \Delta_{t+1}} \end{vmatrix} = (-1) \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -\frac{\partial D}{\partial \Delta_{t+1}} \end{vmatrix}$$

$$=\frac{\partial D}{\partial \Delta_{t+1}} > 0.$$

$$\begin{vmatrix} \frac{\partial A}{\partial R_t} & \frac{\partial A}{\partial w_{L_t}} & \frac{\partial A}{\partial w_{H_t}} & -\frac{\partial A}{\partial N_t} \\ \frac{\partial B}{\partial R_t} & \frac{\partial B}{\partial w_{L_t}} & \frac{\partial B}{\partial w_{H_t}} & -\frac{\partial B}{\partial N_t} \\ \frac{\partial C}{\partial R_t} & \frac{\partial C}{\partial w_{L_t}} & \frac{\partial C}{\partial w_{H_t}} & -\frac{\partial C}{\partial N_t} \\ \frac{\partial D}{\partial R_t} & \frac{\partial D}{\partial w_{L_t}} & \frac{\partial D}{\partial w_{H_t}} & -\frac{\partial D}{\partial N_t} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & -\frac{\partial A}{\partial N_t} \\ 0 & -1 & 0 & -\frac{\partial B}{\partial N_t} \\ 0 & 0 & -1 & -\frac{\partial C}{\partial N_t} \\ 0 & 0 & -1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 0 & -\frac{\partial B}{\partial N_t} \\ 0 & -1 & -\frac{\partial C}{\partial N_t} \\ 0 & -1 & 0 \end{vmatrix}$$

$$= -\frac{\partial C}{\partial N_t} > 0.$$

$$\begin{aligned} \frac{\partial C}{\partial N_t} =& \mu(1-\alpha)\left(\eta(A_{K_t}K_t)^\theta + (1-\eta)(A_{L_t}L_t)^\theta\right)^{\frac{\alpha}{\theta}-1} A_{H_t}^{1-\alpha}(H_t - H_{R\&D_t})^{-\alpha}\alpha(1-\eta)(A_{L_t}L_t)^\theta \frac{1}{L_t}\frac{\partial L_t}{\partial N_t} \\ &- \mu(1-\alpha)\left(\eta(A_{K_t}K_t)^\theta + (1-\eta)(A_{L_t}L_t)^\theta\right)^{\frac{\alpha}{\theta}} A_{H_t}^{1-\alpha}(H_t - H_{R\&D_t})^{-\alpha}\alpha\frac{1}{H_t}\frac{\partial H_t}{\partial N_t} \\ \frac{\partial C}{\partial N_t} \propto& (1-\eta)(A_{L_t}L_t)^\theta\left(\frac{N_t}{L_t}\frac{\partial L_t}{\partial N_t} - \frac{N_t}{H_t}\frac{\partial H_t}{\partial N_t}\right) - \eta(A_{K_t}K_t)^\theta\frac{N_t}{H_t}\frac{\partial H_t}{\partial N_t} < 0, \end{aligned}$$

an exogenous skill share ψ_t entails that $\frac{N_t}{L_t}\frac{\partial L_t}{\partial N_t} = \frac{N_t}{H_t}\frac{\partial H_t}{\partial N_t}$. And as the constant of proportionality is positive this implies that $\frac{\partial C}{\partial N_t} < 0$.

C.4 Nested CES

Consider a more general nested CES production function, as for example in Duffy *et al.* (2004).

$$F=\left(\delta(\eta K_t^{\theta}+(1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}}+(1-\delta)H_t^{\rho}\right)^{\frac{1}{\rho}},$$

with $\rho\in(-\infty,1)$ and $\theta\in(-\infty,1]$. This specification implies that the elasticity of substitution between K_t and H_t as well as between L_t and H_t is the same, i.e., determined by ρ. The elasticity of substitution between K_t and L_t is determined by θ. Capital-skill complementarity requires that $\theta>\rho$.

First-order condition with respect to capital

$$A\equiv\left(\delta(\eta K_t^{\theta}+(1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}}+(1-\delta)H_t^{\rho}\right)^{\frac{1-\rho}{\rho}}\delta(\eta K_t^{\theta}+(1-\eta)L_t^{\theta})^{\frac{\rho-\theta}{\theta}}\eta K_t^{\theta-1}-R_t=0$$

$$\Leftrightarrow$$

$$A\equiv\left(\left(\delta\left(\eta K_t^{\theta}+(1-\eta)L_t^{\theta}\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\theta}+(1-\eta)L_t^{\theta}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)K_t^{\frac{\rho(\theta-1)}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}\delta\eta$$
$$-R_t=0$$

$$\Leftrightarrow$$

$$A\equiv\left(\delta\left(\eta+(1-\eta)\left(\frac{L_t}{K_t}\right)^{\theta}\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\frac{\theta(\rho-1)}{\rho-\theta}}+(1-\eta)L_t^{\theta}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-\rho}{\rho}}\delta\eta$$
$$-R_t=0.$$

$$\frac{\partial A}{\partial L_t}=\left(\delta\left(\eta+(1-\eta)\left(\frac{L_t}{K_t}\right)^{\theta}\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\frac{\theta(\rho-1)}{\rho-\theta}}+(1-\eta)L_t^{\theta}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-2\rho}{\rho}}$$
$$\cdot\,\delta\eta\left[\delta\left(\eta+(1-\eta)\left(\frac{L_t}{K_t}\right)^{\theta}\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}-1}(1-\theta)\left(\frac{L_t}{K_t}\right)^{\theta}\frac{1}{L_t}\right.$$

$$+(1-\delta)H_t^{\rho}\left(\eta K_t^{\frac{\theta(\rho-1)}{\rho-\theta}}+(1-\eta)L_t^{\theta}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}-1}(1-\eta)(\rho-\theta)L_t^{\theta-1}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\Bigg] \gtreqless 0,$$

$$\frac{\partial A}{\partial H_t}=\left(\delta\left(\eta+(1-\eta)\left(\frac{L_t}{K_t}\right)^{\theta}\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\frac{\theta(\rho-1)}{\rho-\theta}}+(1-\eta)L_t^{\theta}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-2\rho}{\rho}}$$
$$\cdot\,\delta\eta\left((1-\delta)(1-\rho)H_t^{\rho-1}\left(\eta K_t^{\frac{\theta(\rho-1)}{\rho-\theta}}+(1-\eta)L_t^{\theta}K_t^{\frac{\theta(\theta-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)>0.$$

First-order condition with respect to low-skilled labor

$$B\equiv\left(\delta\left(\eta\left(\frac{K_t}{L_t}\right)^{\theta}+(1-\eta)\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\theta}L_t^{\frac{\theta(\theta-1)}{\rho-\theta}}+(1-\eta)L_t^{\frac{\theta(\rho-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-\rho}{\rho}}\delta(1-\eta)$$
$$-\,w_{L_t}=0.$$

$$\frac{\partial B}{\partial L_t}=\left(\delta\left(\eta\left(\frac{K_t}{L_t}\right)^{\theta}+(1-\eta)\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\theta}L_t^{\frac{\theta(\theta-1)}{\rho-\theta}}+(1-\eta)L_t^{\frac{\theta(\rho-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-2\rho}{\rho}}$$
$$\cdot\,\delta(1-\eta)\Bigg[\delta\left(\eta\left(\frac{K_t}{L_t}\right)^{\theta}+(1-\eta)\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}-1}(\theta-1)\eta\left(\frac{K_t}{L_t}\right)^{\theta}\frac{1}{L_t}$$
$$+\,(1-\delta)H_t^{\rho}\left(\eta K_t^{\theta}L_t^{\frac{\theta(\theta-1)}{\rho-\theta}}+(1-\eta)L_t^{\frac{\theta(\rho-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}-1}$$
$$\cdot\left(\eta(\theta-1)K_t^{\theta}L_t^{\frac{\theta(\theta-1)}{\rho-\theta}-1}+\eta(\rho-1)L_t^{\frac{\theta(\rho-1)}{\rho-\theta}-1}\right)\Bigg]<0,$$

$$\frac{\partial B}{\partial H_t}=\left(\delta\left(\eta\left(\frac{K_t}{L_t}\right)^{\theta}+(1-\eta)\right)^{\frac{\rho(1-\theta)}{\theta(1-\rho)}}+(1-\delta)H_t^{\rho}\left(\eta K_t^{\theta}L_t^{\frac{\theta(\theta-1)}{\rho-\theta}}+(1-\eta)L_t^{\frac{\theta(\rho-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right)^{\frac{1-2\rho}{\rho}}$$

$$\cdot\, \delta(1-\eta)\left((1-\delta)(1-\rho)H_t^{\rho-1}\left(\eta K_t^{\theta} L_t^{\frac{\theta(\theta-1)}{\rho-\theta}} + (1-\eta) L_t^{\frac{\theta(\rho-1)}{\rho-\theta}}\right)^{\frac{\rho(\rho-\theta)}{\theta(1-\rho)}}\right) > 0.$$

First-order condition with respect to high-skilled labor

$$C \equiv \left(\delta(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}} + (1-\delta)H_t^{\rho}\right)^{\frac{1-\rho}{\rho}} (1-\delta)H_t^{\rho-1} - w_{H_t} = 0$$

$$\Leftrightarrow$$

$$C \equiv \left(\delta\left(\frac{(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{1}{\theta}}}{H_t}\right)^{\rho} + (1-\delta)\right)^{\frac{1-\rho}{\rho}} (1-\delta) - w_{H_t} = 0.$$

$$\frac{\partial C}{\partial L_t} = (1-\rho)\left(\delta(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}} + (1-\delta)H_t^{\rho}\right)^{\frac{1-2\rho}{\rho}} (1-\delta)H_t^{\rho-1}\delta(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{\rho}{\theta}-1} \cdot (1-\eta)L_t^{\theta-1} > 0,$$

$$\frac{\partial C}{\partial H_t} = (1-\rho)\left(\delta\left(\frac{(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{1}{\theta}}}{H_t}\right)^{\rho} + (1-\delta)\right)^{\frac{1-2\rho}{\rho}} (1-\delta) \cdot \left(\frac{(\eta K_t^{\theta} + (1-\eta)L_t^{\theta})^{\frac{1}{\theta}}}{H_t}\right)^{\rho} \frac{1}{H_t}(-1) < 0.$$

The sign of $\frac{\partial A}{\partial L_t}$ is determined by $(1-\theta)(\cdot) + (\rho-\theta)(\cdot)$. The first term is always positive and the second can either be positive or negative. A necessary condition that $\frac{\partial A}{\partial L_t}$ has a negative sign is that $\theta > \rho$ and hence capital-skill complementarity. Notice, this is similar to simple case considered before, however, capital-skill complementarity now only constitutes a *necessary* condition and no longer a *sufficient* condition.

The simpler case considered in the main text can be derived by evaluating the limit where $\rho \to 0$.

The signs of the other derivatives are the same as in the simpler model considered in the main text.

C.5 Task-Based Framework

Consider the same set-up as in Section 3.3 with a continuum of symmetric monopolistic firms. They are indexed by i where $i \in [0, M_t]$ with $M_t \geq 1$.

Aggregate output is given as

$$Y_t = \left(\int_0^{M_t} y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$Y_t = \left(\int_0^{M_t} F(K_t(i), L_t(i), H_t(i))^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

with $\varepsilon > 1$. All firms are symmetric and thus all have the same production function $F(K_t(i), L_t(i), H_t(i))$, which has constant returns to scale in $K_t(i)$, $L_t(i)$ and $H_t(i)$. In equilibrium, all factor markets clear, and due to symmetry, all firms employ the same amount of input factors, i.e., $K_t(i) = \frac{K_t}{M_t}$, $L_t(i) = \frac{L_t}{M_t}$ and $H_t(i) = \frac{H_t}{M_t}$. Using this and the fact that $F(K_t(i), L_t(i), H_t(i))$ has c.r.s., I can rewrite aggregate output as

$$Y_t = \left(\int_0^{M_t} M_t^{\frac{\varepsilon-1}{\varepsilon}} F(K_t, L_t, H_t)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$Y_t = F(K_t, L_t, H_t) \frac{1}{M_t} \left(\int_0^{M_t} 1^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$Y_t = F(K_t, L_t, H_t) \frac{1}{M_t} M_t^{\frac{\varepsilon}{\varepsilon-1}}$$
$$Y_t = F(K_t, L_t, H_t) M_t^{\frac{1}{\varepsilon-1}}.$$

Profits in the final good sector are zero, and only the intermediate good-producing firms make positive profits. Given that they face isoelastic demand, their profits are given as a constant share of output

$$\pi_t = \frac{1}{\varepsilon} F(K_t, L_t, H_t) M_t^{\frac{1}{\varepsilon-1}}$$
$$\pi_t = \frac{1}{\varepsilon} \left(\eta (A_{K_t} K_t)^{\theta} + (1-\eta)(A_{L_t} L_t)^{\theta} \right)^{\frac{\alpha}{\theta}} (A_{H_t} H_t)^{1-\alpha} M_t^{\frac{1}{\varepsilon-1}}.$$

Overall profits in the economy (as well as output) are increasing in M_t, i.e., an increase in the number of intermediate goods has the same effect as an increase in TFP.

The profit of an individual firm i is given as

$$\pi_t(i) = \frac{1}{\varepsilon} F(K_t, L_t, H_t) M_t^{\frac{1}{\varepsilon-1}} \frac{1}{M_t}$$
$$\pi_t(i) = \frac{1}{\varepsilon} \left(\eta (A_{K_t} K_t)^\theta + (1-\eta)(A_{L_t} L_t)^\theta \right)^{\frac{\alpha}{\theta}} (A_{H_t} H_t)^{1-\alpha} M_t^{\frac{1}{\varepsilon-1}} \frac{1}{M_t}.$$

An increase in the number of firms, i.e., varieties, has two opposing effects on the profits of an individual firm

$$\frac{\partial \pi_t(i)}{\partial M_t} = \frac{1}{\varepsilon} F(K_t, L_t, H_t) \left(\underbrace{\frac{1}{\varepsilon - 1} M_t^{\frac{1}{\varepsilon-1}-1} \frac{1}{M_t}}_{>0} \underbrace{- M_t^{\frac{1}{\varepsilon-1}} \frac{1}{M_t^2}}_{<0} \right) \gtreqless 0$$
$$\frac{\partial \pi_t(i)}{\partial M_t} = F(K_t, L_t, H_t) \left(\frac{2-\varepsilon}{\varepsilon(\varepsilon-1)} M_t^{\frac{2-\varepsilon}{\varepsilon-1}-1} \right) \gtreqless 0.$$

The positive terms stem from the fact that an increase in the number of varieties increases overall output, similarly to an increase in TFP, and as profits are a share of output, this increases the profits of individual firms. The negative term is due to the fact that all available input factors need to be divided among all producing firms, and if there are more firms producing, this reduces, c.p., the amount of input factors each firm receives, which in turn reduces their overall profits.
A sufficient condition that an increase in M_t increases profits is that $1 < \varepsilon \leq 2$, i.e., each firm requires that their products are sufficiently complementary, which implies a higher level of market power.

Assume there are initially M_t firms operating in the market. The marginal firm that could enter the market would have to pay a fixed cost of $f(k)$ to start production. Once it decides to enter, it will earn the same (gross) profits as all other firms, with $M_t + dM_t$ firms in the market. The profits the marginal firm k would earn are given as

$$\pi_t(k) = \underbrace{\frac{1}{\varepsilon} F(K_t, L_t, H_t) M_t^{\frac{1}{\varepsilon-1}} \frac{1}{M_t}}_{>0} + \underbrace{\frac{\partial \pi_t(i)}{\partial M_t}}_{\gtreqless 0} - f(k)$$

$$\pi_t(k) = \frac{1}{\varepsilon} F(K_t, L_t, H_t) M_t^{\frac{2-\varepsilon}{\varepsilon-1}} \left(1 + \frac{2-\varepsilon}{\varepsilon-1}\frac{1}{M_t}\right) - f(k) \gtreqless 0.$$

The first term captures the baseline profits that are a function of the current number of operating firms and available production factors. The second term captures the effect of the additional firm entering the market on profits. As displayed before, this effect can either be positive or negative depending on the value of ε.

Whether entering the market is optimal depends on two conditions, which both constitute necessary conditions that need to be fulfilled in order for it to be optimal for an additional firm to enter the market. First, we require that the market structure, i.e., the level of market power and the number of competitors, is such that by entering the market a firm *can* make positive profits, i.e., a necessary condition that $\pi_t(k) \geq 0$ is that parameter values are such that $\left(1 + \frac{2-\varepsilon}{\varepsilon-1}\frac{1}{M_t}\right) \geq 0$. This condition is independent of the number of workers. Second, we require that the profits a firm is expected to make will be high enough to cover the entrance costs, i.e., that the firm will *make* non-negative profits when entering the market. Whether this condition is satisfied positively depends on the workforce.

$F(K_t, L_t, H_t)$ only affects $\frac{\partial \pi_t(i)}{\partial M_t}$ as a scaling parameter, and thus production, and hence the size of the workforce, only have an effect on whether the second condition is fulfilled.

The first condition, i.e., the sign of $\frac{\partial \pi_t(i)}{\partial M_t}$, is independent of $F(K_t, L_t, H_t)$, as it is completely determined by the value of ε and M_t, i.e., the market structure. In order for, for example, L_t to have an effect on whether the first condition is fulfilled, requires for example, that ε depends on the amount of output produced, i.e., $\varepsilon\left(F(K_t, L_t, H_t)\right)$.

C.6 Endogenous Education Decision

Assume there is a continuum of households. Each household has a different ability level. The ability levels are uniformly distributed over $[\underline{a}, \bar{a}]$. A higher ability level makes it cheaper for a household to become high-skilled workers. The wage households earn as high- or low-skilled workers is independent of their ability levels, i.e., there is only an *extensive* margin. Households only live for one period and first decide whether to become high-skilled or remain low-skilled, then work and earn the respective equilibrium wage rate. Capital supply is exogenous and fixed

To become high-skilled, a household faces a fixed cost $f(\cdot)$ that is decreasing in her respective ability level, i.e., households with a high ability face lower costs and households with a low ability face higher costs.

The value of being low-skilled or high-skilled, respectively, is given as

$$\begin{aligned} V_{L_t} &= w_{L_t}, \\ V_{H_t} &= w_{H_t} - f(a). \end{aligned}$$

In the following, I will only consider the marginal low-skilled household, i.e., the households with the highest ability level that remains low-skilled. Assume that if she is indifferent, she will remain low-skilled. This household's ability level is denoted by a^* and in equilibrium, it satisfies

$$w_{L_t} \geq w_{H_t} - f(a^*).$$

Define $V_t(a) = V_{L_t} - V_{H_t}$. For the marginal worker it must hold that $V_t(a^*) \geq 0$.

Assume there is an exogenous marginal decrease in the number of low-skilled workers. The marginal worker will now reevaluate whether she remains low-skilled or becomes high-skilled. She remains low-skilled if $\frac{\partial V_t(a^*)}{\partial(-L_t)} \geq 0$.

$$\frac{\partial V_t(a^*)}{\partial(-L_t)} = \underbrace{\frac{\partial w_{L_t}}{\partial(-L_t)} dL_t}_{>0} - \left(\underbrace{\frac{\partial w_{H_t}}{\partial H_t} dH_t}_{<0} + 2 \cdot \underbrace{\frac{\partial w_{H_t}}{\partial(-L_t)} dL_t}_{<0} \right) \geq 0.$$

If she remains low-skilled, her wage increases because there are fewer low-skilled workers. If she becomes a high-skilled worker, the number of high-skilled workers increases and the number of low-skilled workers decreases by two, i.e., one due to the exogenous decrease and one because the marginal worker decides to become skilled.

Assume there is an exogenous marginal decrease in the number of high-skilled workers.

$$\frac{\partial V_t(a^*)}{\partial(-H_t)} = \underbrace{\frac{\partial w_{L_t}}{\partial(-H_t)}dH_t}_{<0} - \left(\underbrace{\frac{\partial w_{H_t}}{\partial(-L_t)}dL_t}_{<0} \right) \gtreqless 0.$$

If she remains low-skilled, her wage will fall because there are fewer high-skilled workers. If instead she becomes a high-skilled worker, the number of high-skilled workers remains constant but the number of low-skilled workers decreases, which has a negative effect on the wage of high-skilled workers.

Assume there is an exogenous marginal decrease in N_t.

$$\frac{\partial V_t(a^*)}{\partial(-N_t)} = \underbrace{\frac{\partial w_{L_t}}{\partial(-N_t)}dN_t}_{\gtreqless 0} - \left(\underbrace{\frac{\partial w_{H_t}}{\partial(-N_t)}dN_t}_{>0} + \underbrace{\frac{\partial w_{H_t}}{\partial(-L_t)}dL_t}_{<0} + \underbrace{\frac{\partial w_{H_t}}{\partial H_t}dH_t}_{<0} \right) \gtreqless 0.$$

A decrease in N_t has a positive effect on w_{H_t} and an ambiguous effect on w_{L_t}. If the marginal household decides to become a high-skilled worker, there is a marginal increase in the number of high-skilled workers and a decrease in the number of low-skilled workers (in addition to the decrease that is caused by the fall in N_t). Both have a negative effect on the wages of high-skilled workers.

C.7 Ideas Production Function

The set-up of the R&D sector is as in Section 3.4 of the main text. Except that there is no risk, i.e., R&D will always yield a positive outcome, and that the size of the innovation positively depends on the number of researchers employed. Therefore, the value of an innovation Δ_{t+1} is now endogenous and depends on the number of researchers employed.

Aggregating over all R&D firms yields the level of factor augmenting technology in sector

m in period $t+1$ as

$$A_{m_{t+1}} = \delta H_{R\&D_t}^{\chi-\beta} A_{m,t}^{\varphi} + A_{m,t},$$

with $\delta > 0$ as a scaling parameter, $\chi > \beta$ and $\chi, \beta \in (0,1)$. With $\varphi > 0$ the "standing on shoulders" effect dominates, and with $\varphi < 0$ the "fishing out" effect dominates.
The value of an innovation Δ_{t+1} is given as

$$\Delta_{t+1} = \frac{1}{\varepsilon} \left(Y(A_{m_{t+1}}(H_{R\&D_t}), \cdot) - Y(A_{m_t}, \cdot) \right).$$

The maximum amount firms are willing to pay for a higher level of technology is equal to the additional profits that they make due to the higher level of technology, which is given as the difference between profits generated using the new higher level of technology and profits generated using the old level of technology.
The aggregate profits in the R&D sector are given as

$$\pi_{R\&D_{t+1}} = \Delta_{t+1}(H_{R\&D_t}) - w_{H_t} H_{R\&D_t},$$

for simplicity, I assume there is no discounting. Perfect competition implies profits will be zero in equilibrium, and thus the equilibrium amount of labor employed in the R&D sector is determined by the following equation

$$F \equiv \Delta_{t+1}(H_{R\&D_t}) - w_{H_t} H_{R\&D_t} = 0.$$

Assume for simplicity that only the level of capital augmenting technology is improved by R&D, i.e., $A_{K_{t+1}}$.

$$\begin{aligned} \frac{\partial F}{\partial H_{R\&D_t}} =& \alpha \left(\eta (A_{K_{t+1}} K_{t+1})^{\theta} + (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} \right)^{\frac{\alpha}{\theta}-1} (A_{H_{t+1}} H_{t+1})^{1-\alpha} \eta A_{K_{t+1}}^{\theta-1} K_{t+1}^{\theta} \frac{\partial A_{K_{t+1}}}{\partial H_{R\&D_t}} \\ & - w_{H_t} \gtreqless 0, \end{aligned}$$

using the equilibrium condition, this can be simplified to

$$
\begin{aligned}
\frac{\partial F}{\partial H_{R\&D_t}} =& \frac{\alpha\eta A_{K_{t+1}}^{\theta-1} K_{t+1}^{\theta} \frac{\partial A_{K_{t+1}}}{\partial H_{R\&D_t}} H_{R\&D_t}}{\eta (A_{K_{t+1}} K_{t+1})^{\theta} + (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta}} - 1 \gtreqless 0 \\
\frac{\partial F}{\partial H_{R\&D_t}} =& \alpha\delta(\chi-\beta)\eta (A_{K_{t+1}} K_{t+1})^{\theta} \left(1 - \frac{A_{K_t}}{A_{K_{t+1}}}\right) - \eta (A_{K_{t+1}} K_{t+1})^{\theta} - (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} < 0,
\end{aligned}
$$

if δ is not too large, as $\alpha, \chi - \beta, \eta \in (0,1)$ and $\frac{A_{K_t}}{A_{K_{t+1}}} < 1$.

$$
\frac{\partial F}{\partial L_{t+1}} = \frac{1}{\varepsilon} \left(\frac{\partial Y(A_{K_{t+1}}, \cdot)}{\partial L_{t+1}} - \frac{\partial Y(A_{K_t}, \cdot)}{\partial L_{t+1}} \right) \gtreqless 0,
$$

with $A_{K_{t+1}} > A_{K_t}$. Similarly to before, I can use the second derivative to examine if output increases more or less when the number of low-skilled workers changes and the level of technology differs across the two functions.

$$
\frac{dH_{R\&D_t}}{dw_{H_t}} < 0, \quad \frac{dH_{R\&D_t}}{dL_{t+1}} \gtreqless 0.
$$

C.8 Demographic Change and Capital Supply

So far, I have assumed that the capital stock remains constant. However, demographic change will likely have an effect on capital supply; the effect can be positive, i.e., if people expect to live longer, they accumulate more savings, or negative, i.e., a smaller population will generate lower aggregate savings.

A change in the capital stock has an *intertemporal* as well as an *intratemporal* effect on the incentive to innovate.

An increase in the capital stock in period $t+1$ has the following effect on the value of an innovation

$$
F_{A_{K_{t+1}} K_{t+1}} = \alpha \left(\eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}} + (1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\theta}} (A_{H_{t+1}} H_{t+1})^{1-\alpha} \eta
$$

$$\cdot \left(\alpha \eta A_{K_{t+1}}^{\frac{\theta\alpha-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\alpha\theta}{\alpha-\theta}-1} + \theta(1-\eta)(A_{L_{t+1}} L_{t+1})^{\theta} A_{K_{t+1}}^{\frac{\theta^2-\theta}{\alpha-\theta}} K_{t+1}^{\frac{\theta^2}{\alpha-\theta}-1} \right),$$

which is always positive for $\theta > 0$, i.e., when capital and low-skilled labor are substitutes relative to capital and high-skilled labor.

An increase in K_t has the following effect on $H_{R\&D_t}$

$$\frac{\partial H_{R\&D_t}}{\partial K_t} = \frac{-\frac{\partial C}{\partial K_t}}{-\left(\frac{\partial D}{\partial H_{R\&D_t}} - \frac{\partial C}{\partial H_{R\&D_t}}\right)} < 0.$$

A higher capital stock increases the productivity of high-skilled workers in the production sector and thus raises their wage rate. As they can freely switch, high-skilled workers will reallocate to the production sector, reducing the number of workers in the R&D sector.

Appendix D

Appendix to Chapter 4

D.1 Proof of Proposition 1 and 2

The effect of an increase in $A_{1,t}$ on p_t is given as

$$\frac{\partial p_t}{\partial A_{1,t}} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial A_{1,t}} & \frac{\partial F}{\partial L_{1,t}} \\ -\frac{\partial G}{\partial A_{1,t}} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}} > 0.$$

The effect on an increase in $H_{1,t}/A_{1,t}$ on $L_{1,t}$ is given as

$$\frac{\partial L_{1,t}}{\partial H_{1,t}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial p_t} & -\frac{\partial F}{\partial H_{1,t}} \\ \frac{\partial G}{\partial p_t} & -\frac{\partial G}{\partial H_{1,t}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}} = \frac{\partial L_{1,t}}{\partial H_{1,t}} = \frac{\partial L_{1,t}}{\partial A_{1,t}} \begin{cases} > 0 & \text{if } \theta > 1, \\ < 0 & \text{if } \theta < 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

The effect on an increase in $H_{2,t}/A_{2,t}$ on $L_{1,t}$ is given as

$$\frac{\partial L_{1,t}}{\partial H_{2,t}} = \frac{\begin{vmatrix} \frac{\partial F}{\partial p_t} & -\frac{\partial F}{\partial H_{2,t}} \\ \frac{\partial G}{\partial p_t} & -\frac{\partial G}{\partial H_{2,t}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}} = \frac{\partial L_{1,t}}{\partial H_{2,t}} = \frac{\partial L_{1,t}}{\partial A_{2,t}} \begin{cases} < 0 & \text{if } \theta > 1, \\ > 0 & \text{if } \theta < 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

The effect of an increase in γ, i.e., the preference for good one, on p_t and $L_{1,t}$ is given as

$$\frac{\partial p_t}{\partial \gamma} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial \gamma} & \frac{\partial F}{\partial L_{1,t}} \\ -\frac{\partial G}{\partial \gamma} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}} < 0, \quad \frac{\partial L_{1,t}}{\partial \gamma} = \frac{\begin{vmatrix} \frac{\partial F}{\partial p_t} & -\frac{\partial F}{\partial \gamma} \\ \frac{\partial G}{\partial p_t} & -\frac{\partial G}{\partial \gamma} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix}} > 0.$$

If households have a higher preference for good 1, then the relative price of good 2 will fall and more low-skilled labor will flow into sector 1.

Recall, from Section 4.2.3, that the equilibrium can be characterized as follows

$$\begin{aligned} F \equiv & p_t^{\theta} \frac{Y_{2,t}}{Y_{1,t}} - \frac{1}{\gamma} + 1 = 0, \\ G \equiv & \alpha_1 \frac{Y_{1,t}}{L_{1,t}} - p_t \alpha_2 \frac{Y_{2,t}}{L_{2,t}} = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\partial F}{\partial p_t} > 0, \quad \frac{\partial F}{\partial L_{1,t}} < 0, \quad \frac{\partial F}{\partial H_{1,t}} = \frac{\partial F}{\partial A_{1,t}} < 0, \quad \frac{\partial F}{\partial H_{2,t}} = \frac{\partial F}{\partial A_{2,t}} > 0, \quad \frac{\partial F}{\partial \gamma} > 0, \\ &\frac{\partial G}{\partial p_t} < 0, \quad \frac{\partial G}{\partial L_{1,t}} < 0, \quad \frac{\partial G}{\partial H_{1,t}} = \frac{\partial G}{\partial A_{1,t}} > 0, \quad \frac{\partial G}{\partial H_{2,t}} = \frac{\partial G}{\partial A_{2,t}} < 0, \quad \frac{\partial G}{\partial \gamma} = 0. \end{aligned}$$

$$\begin{vmatrix} \frac{\partial F}{\partial p_t} & \frac{\partial F}{\partial L_{1,t}} \\ \frac{\partial G}{\partial p_t} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix} = \frac{\partial F}{\partial p_t} \frac{\partial G}{\partial L_{1,t}} - \frac{\partial F}{\partial L_{1,t}} \frac{\partial G}{\partial p_t} < 0.$$

$$\begin{vmatrix} -\frac{\partial F}{\partial A_{1,t}} & \frac{\partial F}{\partial L_{1,t}} \\ -\frac{\partial G}{\partial A_{1,t}} & \frac{\partial G}{\partial L_{1,t}} \end{vmatrix} = \left(-\frac{\partial F}{\partial A_{1,t}}\right)\frac{\partial G}{\partial L_{1,t}} + \frac{\partial F}{\partial L_{1,t}}\frac{\partial G}{\partial A_{1,t}} < 0.$$

$$\begin{aligned}
\begin{vmatrix} \frac{\partial F}{\partial p_t} & -\frac{\partial F}{\partial H_{1,t}} \\ \frac{\partial G}{\partial p_t} & -\frac{\partial G}{\partial H_{1,t}} \end{vmatrix} =& \frac{\partial F}{\partial p_t}\left(-\frac{\partial G}{\partial H_{1,t}}\right) + \frac{\partial F}{\partial H_{1,t}}\frac{\partial G}{\partial p_t} \\
=& -\theta p_t^{\theta-1}\frac{Y_{2,t}}{Y_{1,t}}\alpha_1\frac{1}{L_{1,t}}\frac{\partial Y_{1,t}}{\partial H_{1,t}} + p_t^{\theta}\frac{Y_{2,t}}{Y_{1,t}}\frac{1}{Y_{1,t}}\frac{\partial Y_{1,t}}{\partial H_{1,t}}\alpha_2\frac{Y_{2,t}}{L_{2,t}} \gtreqless 0 \\
=& p_t\alpha_2\frac{Y_{2,t}}{L_{2,t}} - \theta\alpha_1\frac{Y_{1,t}}{L_{1,t}} \gtreqless 0 \\
=& \frac{p_t\alpha_2\frac{Y_{2,t}}{L_{2,t}}}{\alpha_1\frac{Y_{1,t}}{L_{1,t}}} - \theta \gtreqless 0 \\
=& 1-\theta \gtreqless 0.
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} \frac{\partial F}{\partial p_t} & -\frac{\partial F}{\partial H_{2,t}} \\ \frac{\partial G}{\partial p_t} & -\frac{\partial G}{\partial H_{2,t}} \end{vmatrix} =& \frac{\partial F}{\partial p_t}\left(-\frac{\partial G}{\partial H_{2,t}}\right) + \frac{\partial F}{\partial H_{2,t}}\frac{\partial G}{\partial p_t} \\
=& \theta p_t^{\theta-1}\frac{Y_{2,t}}{Y_{1,t}}p_t\alpha_2\frac{1}{L_{1,t}}\frac{\partial Y_{2,t}}{\partial H_{2,t}} - p_t^{\theta}\frac{1}{Y_{1,t}}\frac{\partial Y_{2,t}}{\partial H_{2,t}}\alpha_2\frac{Y_{2,t}}{L_{2,t}} \gtreqless 0 \\
=& \theta - 1 \gtreqless 0.
\end{aligned}$$

D.2 Proof of Proposition 3

Let ξ_t denotes the share of good 1 in nominal GDP

$$\xi_t = \frac{Y_{1,t}}{Y_{1,t} + p_t Y_{2,t}}.$$

There are three channels through which an increase in $A_{1,t}$ can influence ξ in this model. First, directly by increasing the output produced in sector 1. Second, by triggering a reallocation of low-skilled labor from one sector to the other. Third, by influencing the relative price of good 2 and thus affecting the nominal value of output produced in sector 2.

We assume first that low-skilled labor cannot switch sectors. This simplifies the analysis, as we only have one equilibrium condition in this case

$$p_t = \left(\frac{1-\gamma}{\gamma} \frac{Y_{1,t}}{Y_{2,t}} \right)^{\frac{1}{\theta}}.$$

$$\begin{aligned}
\frac{\partial p_t}{\partial A_{1,t}} &= \frac{1}{\theta} \left(\frac{1-\gamma}{\gamma} \frac{Y_{1,t}}{Y_{2,t}} \right)^{\frac{1}{\theta}} \frac{1}{Y_{1,t}} \frac{\partial Y_{1,t}}{\partial A_{1,t}} \\
\frac{\partial p_t}{\partial A_{1,t}} &= \frac{1}{\theta} p_t \frac{1}{Y_{1,t}} \frac{\partial Y_{1,t}}{\partial A_{1,t}}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \xi_t}{\partial A_{1,t}} &= \frac{\frac{\partial Y_{1,t}}{\partial A_{1,t}} p_t Y_{2,t} - Y_{1,t} \frac{\partial p_t}{\partial A_{1,t}} Y_{2,t}}{(Y_{1,t} + p_t Y_{2,t})^2} \gtreqless 0 \\
\frac{\partial \xi_t}{\partial A_{1,t}} &= \frac{\frac{p_t Y_{1,t} Y_{2,t}}{A_{1,t}} \left(\frac{\partial Y_{1,t}}{\partial A_{1,t}} \frac{A_{1,t}}{Y_{1,t}} - \frac{\partial p_t}{\partial A_{1,t}} \frac{A_{1,t}}{p_t} \right)}{(Y_{1,t} + p_t Y_{2,t})^2} \gtreqless 0 \\
\frac{\partial \xi_t}{\partial A_{1,t}} &= \frac{\frac{\partial Y_{1,t}}{\partial A_{1,t}} p_t Y_{2,t} - Y_{1,t} \frac{1}{\theta} p_t \frac{1}{Y_{1,t}} \frac{\partial Y_{1,t}}{\partial A_{1,t}} Y_{2,t}}{(Y_{1,t} + p_t Y_{2,t})^2} \gtreqless 0 \\
\frac{\partial \xi_t}{\partial A_{1,t}} &= \frac{\left(\frac{\partial Y_{1,t}}{\partial A_{1,t}} p_t Y_{2,t} \right) \left(1 - \frac{1}{\theta} \right)}{(Y_{1,t} + p_t Y_{2,t})^2} \gtreqless 0.
\end{aligned}$$

$$\frac{\partial \xi_t}{\partial A_{1,t}} \begin{cases} < 0 & \text{if } \theta < 1, \\ > 0 & \text{if } \theta > 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

If the two goods are substitutes an increase in $A_{1,t}$ leads to an increase of good 1 as a share of nominal GDP. Consequently, for $\theta < 1$, i.e., the two goods are complements, an increase in $A_{1,t}$ leads to an increase in the share of good 2 in nominal GDP.

In case low-skilled labor is fully mobile, the effect of $A_{1,t}$ on ξ_t is given as

$$\frac{\partial \xi_t}{\partial A_{1,t}} = \left(p_t \frac{\partial L_{1,t}}{\partial A_{1,t}} \left(Y_{2,t} \frac{\partial Y_{1,t}}{\partial L_{1,t}} - Y_{1,t} \frac{\partial Y_{2,t}}{\partial L_{1,t}} \right) + \frac{p_t Y_{1,t} Y_{2,t}}{A_{1,t}} \left(\frac{\partial Y_{1,t}}{\partial A_{1,t}} \frac{A_{1,t}}{Y_{1,t}} - \frac{\partial p_t}{\partial A_{1,t}} \frac{A_{1,t}}{p_t} \right) \right) \cdot \frac{1}{(Y_{1,t} + p_t Y_{2,t})^2}$$

$$\frac{\partial \xi_t}{\partial A_{1,t}} = \left(\underbrace{p_t \frac{\partial L_{1,t}}{\partial A_{1,t}}}_{\gtreqless 0} \underbrace{\left(Y_{2,t} \frac{\partial Y_{1,t}}{\partial L_{1,t}} - Y_{1,t} \frac{\partial Y_{2,t}}{\partial L_{1,t}} \right)}_{>0} + \frac{p_t Y_{1,t} Y_{2,t}}{A_{1,t}} \underbrace{\left((1-\alpha_1) - \underbrace{\frac{\partial p_t}{\partial A_{1,t}} \frac{A_{1,t}}{p_t}}_{>0} \right)}_{\gtreqless 0} \right) \cdot \frac{1}{(Y_{1,t} + p_t Y_{2,t})^2} \gtreqless 0.$$

The first term captures the effect of the reallocation of low-skilled labor that follows the increase in $A_{1,t}$. Depending on the elasticity of substitution this term can be positive or negative. The second term consists of two elements with opposite signs. The first part captures the increase in output in sector 1 due to the increase in $A_{1,t}$ and is thus positive. The second part, which is the same as in the case when low-skilled labor is immobile, captures the effect of the increase in $A_{1,t}$ on the relative price. It has a negative effect on ξ_t because an increase in $A_{1,t}$ makes good 1 relative more abundant to good 2 and this will increase the relative price of good 2, i.e., good 2 becomes more expensive and good 1 less expensive.

Assume $\alpha_1 = \alpha_2 = \alpha$, this entails that we can combine the equilibrium conditions and solve for p_t, which is given as

$$p_t = \left(\frac{1-\gamma}{\gamma} \right)^{\frac{1-\alpha}{\alpha+\theta(1-\alpha)}} \left(\frac{A_{2,t} H_{2,t}}{A_{1,t} H_{2,t}} \right)^{\frac{1-\alpha}{\alpha+\theta(1-\alpha)}}.$$

Using this, we can express $\frac{\partial p_t}{\partial A_{1,t}} \frac{A_{1,t}}{p_t}$ as

$$\frac{\partial p_t}{\partial A_{1,t}} \frac{A_{1,t}}{p_t} = \frac{1-\alpha}{\alpha + \theta(1-\alpha)}.$$

The sign of the second term of $\frac{\partial \xi_t}{\partial A_{1,t}}$ is determined by

$$\alpha + \theta(1-\alpha) - 1 \gtreqless 0,$$

which is zero for $\theta = 1$, smaller than zero for $\theta \in (0,1)$, and larger than zero for $\theta > 1$.

Proof. For $\theta = 0$, we have $\alpha - 1 < 0$, as $\alpha \in (0,1)$. For $\theta = 1$, we have $1 - 1 = 0$. As $\alpha + \theta(1-\alpha) - 1$ is strictly increasing in θ it follows that for $\theta \in (0,1)$, $\alpha + \theta(1-\alpha) - 1 < 0$ and for $\theta > 1$, $\alpha + \theta(1-\alpha) - 1 > 0$. □

Therefore, it follows that an increase in $A_{1,t}$ has the following effect on the share of sector 1 in nominal GDP

$$\frac{\partial \xi_t}{\partial A_{1,t}} \begin{cases} < 0 & \text{if } \theta < 1, \\ > 0 & \text{if } \theta > 1, \\ = 0 & \text{if } \theta = 1. \end{cases}$$

D.3 Full Factor Mobility

Consider a situation in which all production factors are fully mobile, except for the level of technology. For simplicity we assume each sector only produces with one production factor, but the production function has constant returns to scale in that factor. The production function for good j with $j \in \{1,2\}$ is given as

$$Y_{j,t} = A_{j,t} L_{j,t}.$$

The equilibrium can again be characterized by a system of two equations

$$\begin{aligned} F \equiv & p_t^{\theta} \frac{A_{2,t}(L_t - L_{1,t})}{A_{1,t} L_{1,t}} - \frac{1-\gamma}{\gamma} = 0 \\ F \equiv & p_t^{\theta} \frac{Y_{2,t}}{Y_{1,t}} - \frac{1}{\gamma} + 1 = 0, \\ G \equiv & w_{1,t}^{l} - w_{2,t}^{l} = 0 \end{aligned}$$

$$
\begin{aligned}
G \equiv & A_{1,t} - p_t A_{2,t} = 0 \\
G \equiv & \frac{Y_{1,t}}{L_{1,t}} - p_t \frac{Y_{2,t}}{L_{2,t}} = 0.
\end{aligned}
$$

This entails

$$
\begin{aligned}
&\frac{\partial F}{\partial p_t} > 0, \quad \frac{\partial F}{\partial L_{1,t}} < 0, \quad \frac{\partial F}{\partial A_{1,t}} < 0, \quad \frac{\partial F}{\partial A_{2,t}} > 0, \quad \frac{\partial F}{\partial \gamma} > 0, \\
&\frac{\partial G}{\partial p_t} < 0, \quad \frac{\partial G}{\partial L_{1,t}} = 0, \quad \frac{\partial G}{\partial A_{1,t}} > 0, \quad \frac{\partial G}{\partial A_{2,t}} < 0, \quad \frac{\partial G}{\partial \gamma} = 0.
\end{aligned}
$$

And therefore, the results of the comparative statics are the same as in Section D.1.

D.4 Heterogeneous Preferences

Assume households face the same maximization problem as before, except now preferences over the two goods are heterogeneous, i.e., γ^i can now potentially differ across groups. To simplify the analysis, we further assume that all labor is immobile, i.e., low-skilled workers cannot switch sectors. This allows us to express the equilibrium as one equation.

$$
\begin{aligned}
&\max_{c^i_{1,t}, c^i_{2,t}} c^i_t(c^i_{1,t}, c^i_{2,t}) = \left((\gamma^i)^{\frac{1}{\theta}} (c^i_{1,t})^{\frac{\theta-1}{\theta}} + (1-\gamma^i)^{\frac{1}{\theta}} (c^i_{2,t})^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \quad i \in \{e, d, l\} \\
&\text{s.t. } c^i_{1,t} + p_t c^i_{2,t} = I^i_t,
\end{aligned}
$$

with $\theta \in (0, \infty)$.

Market clearing requires

$$
\begin{aligned}
\frac{Y_{2,t}}{Y_{1,t}} =& \frac{\sum_i (1-\gamma^i) p_t^{-\theta} \frac{I^i_t}{\gamma^i + (1-\gamma^i) p_t^{1-\theta}} N^i_t}{\sum_i \gamma^i \frac{I^i_t}{\gamma^i + (1-\gamma^i) p_t^{1-\theta}} N^i_t} \\
p_t^{\theta} \frac{Y_{2,t}}{Y_{1,t}} =& \frac{\sum_i (1-\gamma^i) \frac{I^i_t}{\gamma^i + (1-\gamma^i)} N^i_t}{\sum_i \gamma^i \frac{I^i_t}{\gamma^i + (1-\gamma^i)} N^i_t}
\end{aligned}
$$

$$
\begin{aligned}
p_t^{\theta}\frac{Y_{2,t}}{Y_{1,t}} &= \frac{\sum_i (1-\gamma^i) I_t^i N_t^i}{\sum_i \gamma^i I_t^i N_t^i} \\
p_t^{\theta}\frac{Y_{2,t}}{Y_{1,t}} &= \frac{\sum_i \mathcal{I}_t^i(Y_{1,t}, Y_{2,t}, p_t)}{\sum_i \gamma^i \mathcal{I}_t^i(Y_{1,t}, Y_{2,t}, p_t)} - 1,
\end{aligned}
$$

where $\mathcal{I}_t^i(Y_{1,t}, Y_{2,t}, p_t) = I_t^i N_t^i$ denotes the aggregate income of group i.

The case of homogeneous preferences can be derived by assuming γ^i is the same for all groups.[1]

$$
F \equiv p_t^{\theta} \underbrace{\frac{Y_{2,t}}{Y_{1,t}}}_{\text{relative supply}} - \underbrace{\frac{Y_{1,t} + p_t Y_{2,t}}{\sum_i \gamma^i \mathcal{I}_t^i(Y_{1,t}, Y_{2,t}, p_t)}}_{\text{demand composition}} + 1,
$$

where we use the fact that $\sum_i \mathcal{I}_t^i(Y_{1,t}, Y_{2,t}, p_t) = Y_{1,t} + p_t Y_{2,t}$. A change in $Y_{j,t}$ with $j \in \{1, 2\}$ has an effect on the relative price p_t through the relative supply as well as by altering the demand composition. The latter channel only exists in a model with heterogeneous preferences.

D.5 List of Countries

Australia, Austria, Belgium, Canada, Chile, Colombia, Costa Rica, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Latvia, Lithuania, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Russia, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Türkiye, United Kingdom, United States.

D.6 Employment in the Healthcare Sector in Germany

This section details some of the particularities of employment in the healthcare sector in Germany. Self-employment is quite common in the healthcare sector in Germany. In

[1]This yields $p_t^{\theta}\frac{Y_{2,t}}{Y_{1,t}} = \frac{1}{\gamma} - 1$.

2012, 4.7% of all self-employed in Germany were physicians and pharmacists, making it the occupational group with the fifth most self-employed persons.[2] The income of self-employed persons in general is difficult to pin down. Nevertheless, the net income of self-employed physicians' offices in Germany in 2015 is reported to have been EUR 192,000.[3] In comparison, employed physicians earned between EUR 57,000 and EUR 125,000 in 2019.[4] Given these numbers, it seems likely that physicians earn even more than suggested in the employment data. Thus the skill premium and its increase over the year is likely underestimated in the employment data.

D.7 German Labor Force Changes 2007-2018

Table D.1 and Table D.2 display changes in the German labor force, analogously to Tables 4.5 and 4.6 in Section 4.3.8. They display the same statistics and ratios using German data. The goal is to facilitate the comparison of results found in the German and US data. Deviating from the US data, workers with medium skill levels are counted towards unskilled workers, such that the share of the unskilled labor force both in the overall economy and the healthcare sector is larger in Table D.1 than in Table 4.5. This however is irrelevant to the derived results, as the focus of the analysis is on relative, rather than absolute changes in labor force shares.

[2] See: https://www.destatis.de/DE/Methoden/WISTA-Wirtschaft-und-Statistik/2013/07/selbststaendigkei-deutschland-72013.pdf?__blob=publicationFile, p.490.

[3] See: https://www.destatis.de/DE/Themen/Branchen-Unternehmen/Dienstleistungen/Publikationen/Downloads-Dienstleistungen-Kostenstruktur/kostenstruktur-aerzte-2020161159004.pdf?__blob=publicationFile, p. 19.

[4] See: https://www.marburger-bund.de/bundesverband/tarifvertraege.

Table D.1: *German Labor Force Changes 2007-2018*

	Overall Economy		Healthcare Sector	
	2007	2018	2007	2018
Unskilled Labor Force	64.7%	64.7%	57.5%	60.3%
Δ		0%		+4.9%
Skill Premium	1.75	1.83	1.79	1.96
Δ		+4.6%		+9.1%

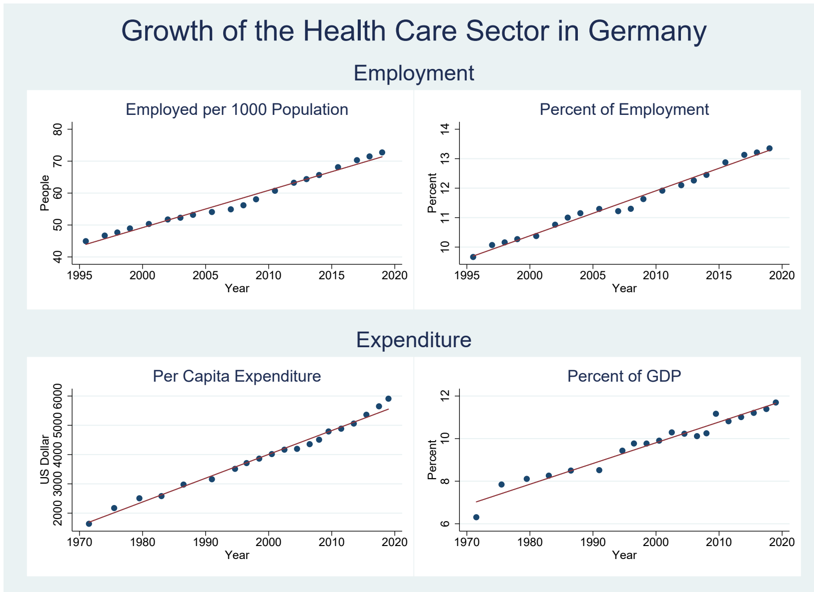
Note: Calculations based on data from the German Statistical Office.

Table D.2: *German Ratios of Key Indicators*

	2007	2018
Unskilled Labor Force Ratio	0.889	0.932
Skill Premium Ratio	1.023	1.071

Note: Calculations based on Table D.1, which summarizes data from the German Statistical Office. The ratios implicitly account for time trends and compositional changes in the labor force.

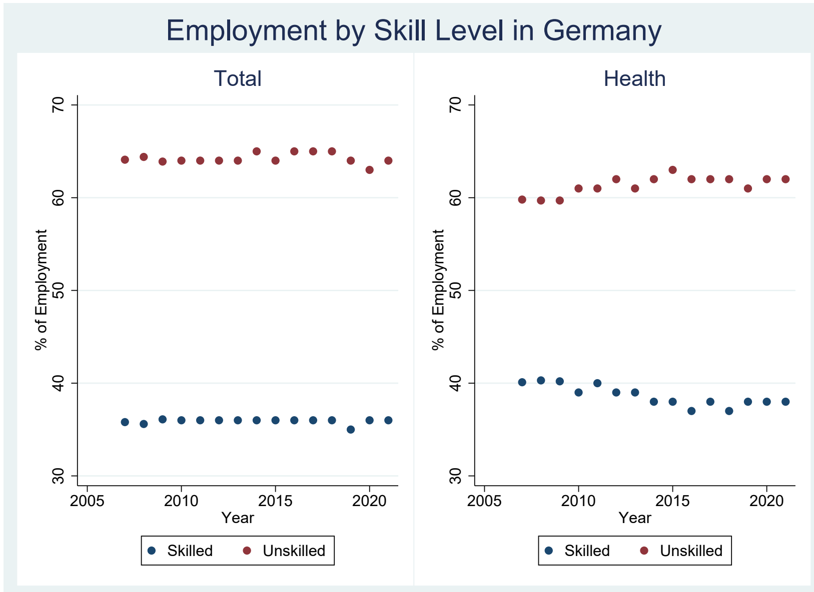
D.8 Additional Graphs



This figure provides a graphical illustration of the trend in employment and expenditure in the healthcare sector in Germany, based on data provided by the OECD

Figure D.1: *Employment and Expenditure in the Healthcare Sector*

Figure D.1 illustrates the share of employment in the healthcare sector as well as the share of overall expenditure going towards healthcare in Germany. Both measures have been continually on the rise in absolute as well as relative terms.



This figure provides a graphical illustration of the trend in the employment shares of skilled and unskilled workers in the overall economy and the health-care sector in Germany. The data used are provided by the German Statistical Office. Workers are classified as skilled if they are university graduates and unskilled otherwise.

Figure D.2: *The Share of Skilled and Unskilled Employment in Germany*

Figure D.2 illustrates the share of high- and low skilled labor for the total economy and the healthcare sector in Germany from 2007 to 2018. While there is little to no change in the total economy, there is a slight upward trend for low skilled labor in the healthcare sector. The left panel of Figure D.2 is in stark contrast to Figure 4.3 depicting the case of the US, which saw a marked increase in the share of high skilled labor. No figure equivalent to the right panel of Figure D.2 exists, due to a lack of available data.

Eidesstattliche Versicherung

Ich versichere hiermit eidesstattlich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sowie mir gegebene Anregungen sind als solche kenntlich gemacht.
Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht. Sofern ein Teil der Arbeit aus bereits veröffentlichten Papers besteht, habe ich dies ausdrücklich angegeben.

Frankfurt am Main, 02.02.2024	Lukas Weber
Datum	Unterschrift