On some multiscale phenomena in quantum physics, classical field theory and spacetime geometry

Daniel Alexander Weiss

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Daniel Alexander Weiss

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Daniel Alexander Weiss
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Erstgutachter: Prof. Dr. Stefan Hofmann

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Abstract:

In this thesis we deal with three instances of multiscale phenomena in, on a first glimpse, very distinct areas of theoretical physics: We consider the quantum statistical physics of Bose gases, the behaviour of classical field theories in the vicinity of cosmological singularities and the behaviour of gravity in extreme situations. In those treatments, the concept of infinite renormalization will be a leitmotif, since our approaches are all motivated by different aspects of renormalization theory. Especially we will see, how techniques and ideas which are standard in some areas of theoretical physics can lead to new perspectives and surprising results in other areas.

In particular we will do the following:

– We will analyse the thermodynamic limit of an ideal Bose gas by QFT-inspired methods that rely on infinite renormalization and techniques from ζ-regularization. Thereby we will obtain as a new insight, that the phase structure of this system is interwoven with the singularity structure of the grand canonical potential and that the thermodynamic properties in the thermodynamic limit are encoded by asymptotic expansions that resemble heat kernel expansions. This structure was not described previously and especially suggests a new approach to weakly interacting Bose gases.

– We will cure the infinities which plague a classical scalar field theory in the vicinity of the big bang singularity by distributional techniques that are inspired by the Epstein-Glaser-approach to renormalization. This shows especially that renormalization ideas do not only apply on quantum field theories but are also of use in the context of classical field theories. Moreover this treatment suggests that the benign behaviour of quantum field theories in the vicinity of cosmological singularities that was observed in the literature could be merely a property of the classical background than a quantum feature.

– Motivated by the idea that the laws of gravity could be scale-dependent, we analyse and review the behaviour of general relativity in extreme situations. We thereby obtain as an interesting result, that the geodesics in the vicinity of the FLRW singularity exhibit some kind of ultrarelativistic behaviour, which was not described before. Moreover we give convincing arguments that this is a coordinate-invariant feature. In addition we will point out various occurrences in literature, where gravity exhibits a similar ultrarelativistic behaviour in extreme situations. This motivates the claim that gravity could be described by an ultrarelativistic field theory on microscopic scales.

– To understand better which gauge symmetry groups could be relevant for ultrarelativistic theories of gravity, we will analyse the universal geometric structures and associated symmetries of microscopic tangent light cones. Thereby we will reveal surprisingly rich structures hidden in the theories of relativity and especially we will discover a microscopic symmetry group which contains infinitely many Lorentz groups and resembles structurally and conceptionally the well-known Bondi-Metzner-Sachs group. By this we have discovered a previously unknown fundamental microscopic symmetry of pseudo-Riemannian geometry.

Moreover, we will explain the relation of those investigations to the existing literature and will comment on possible drawbacks, implications, open questions and consecutive directions of research.

Zusammenfassung:

Die vorliegende Arbeit beschäftigt sich mit drei Instanzen von Vielskalen-Problemen in verschiedenen und auf den ersten Blick sehr unterschiedlichen Teilbereichen der theoretischen Physik: Wir betrachten die statistische Physik des idealen Bose-Gases, das Verhalten von klassischen Feldtheorien in der Umgebung kosmologischer Singularitäten und das Verhalten der Gravitation in extremen Situationen. In diesen Untersuchungen wird das Konzept der Renormierung ein Leitmotiv sein, da all unsere Herangehensweisen durch verschiedene Aspekte der Renormierungstheorie motiviert sind. Insbesondere werden wir sehen, wie Techniken und Ideen, welche in bestimten Bereichen der Physik zum Standardrepertoire gehören, zu neuen Perspektiven und verblüffenden Resultaten in anderen Bereichen führen können.

Wir werden uns in dieser Arbeit konkret mit dem Folgenden beschäftigen:

– Wir werden den thermodynamischen Grenzfall eines idealen Bose-Gases mit Methoden untersuchen, welche von Ideen aus der relativistischen Quantenfeldtheorie inspiriert sind. Insbesondere werden wir hierfür unendliche Renormierungen und ζ-Regularisierungsmethoden verwenden. Dabei werden wir verstehen, dass die Phasenstruktur dieses Systems eng mit den Singularitätseigenschaften des großkanonischen Potenials zusammenhängt und die thermodynamischen Eigenschaften dieses Systems durch asymptotische Entwicklungen ausgedrückt werden können, welche sogenannten Heat-Kernel-Entwicklungen ähneln. Diese Strukturen wurden zuvor nicht beschrieben und motivieren eine neue Herangehensweise an das Problem des schwach wechselwirkenden Bose-Gases.

– Wir behandeln die Unendlichkeiten, die im Kontext einer klassischen skalaren Feldtheorie in der Umgebung der Urknallsingularität auftreten können, mithilfe von distributionellen Techniken, welche durch die Epstein-Glaser-Renormierungstheorie motiviert sind. Dies zeigt, dass Renormierungstechniken nicht nur im Kontext von Quantenfeldtheorien auftreten, sondern auch im Kontext klassischer Feldtheorien nützlich sein können. Darüber hinaus legt diese Untersuchung nahe, dass das gutartige Verhalten von Quantenfeldtheorien in der Umgebung kosmologischer Singularitäten, welches in den letzten Jahren von verschiedenen Autoren beschrieben wurde, eher eine Eigenschaft der klassischen Hintergrundtheorie sein könnte.

– Motiviert durch die Idee, dass die Gesetze der Gravitation skalenabhängig sein könnten, analysieren wir das Verhalten der Allgemeinen Relativitätstheorie in extremen Situationen. Hierbei erhalten wir als ein interessantes Resultat, dass die Geodäten in der Umgebung der FLRW-Singularität ein ultrarelativistisches Verhalten aufweisen, welches zuvor noch nicht beschrieben wurde. Darüber hinaus präsentieren wir überzeugende Argumente, dass dieses Verhalten eine koordinateninvariante Eigenschaft dieser Raumzeiten sein könnte. Außerdem weisen wir auf mehrere in der Literatur beschriebene Situationen hin, in welchen die Gravitation ein ähnliches ultrarelativistisches Verhalten in extremen Situationen aufweist. Diese Beobachtungen motivieren für uns die Behauptung, dass die Gravitation auf fundamentalen Skalen durch eine ultrarelativistische Feldtheorie beschrieben werden könnte.

– Um besser zu verstehen, welche Eichsymmetriegruppen relevant für ultrarelativistische Gravitationstheorien sein könnten, analysieren wir die universellen geometrischen Strukturen und die assoziierten Symmetriegruppen mikroskopischer Tangentiallichtkegel. Hierbei werden

wir überraschend reichhaltige Strukturen aufdecken, welche auch eine mikroskopische Symmetriegruppe umfassen, die unendlich viele Lorentz-Gruppen beinhaltet und konzeptionell wie strukturell der bekannten Bondi-Metzner-Sachs Gruppe ähnelt. Somit haben wir eine bisher unbekante fundamentale mikroskopische Symmetrie innerhalb der pseudo-Riemann'schen Geometrie entdeckt.

Außerdem werden wir erklären, wie das Verhältnis dieser Untersuchungen zur bereits existierenden Literatur ist und kommentieren außerdem problematische Aspekte, mögliche Implikationen, offene Fragen und konsekutive Forschungsrichtungen.

1. Introduction

One of the most striking insights of physics is so basic and old, that its importance and fascinating nature are easily overseen: The laws of nature exhibit an extreme scale-dependence. Theories and models, which describe the universe at different scales, differ in so many aspects from each other, that it is a priori hard to understand how those descriptions should be connected to each other and how laws of nature on larger scales emerge from their microscopic siblings. This thesis deals with some instances of such *multiscale phenomena* in several and on a first glimpse very distinct areas of theoretical physics: We will consider the quantum statistical physics of Bose gases, the behaviour of field theories in the vicinity of cosmological singularities and the behaviour of gravity in extreme situations. Although those topics seem to be barely interrelated, we will see that they share some common features on conceptional and on technical levels. A leitmotif in our treatment will thereby be the concept of renormalization, which can be understood as a formalization of the scale-dependent behaviour of physical theories and will be explained thoroughly in chapter 2. Indeed, our approaches to the above mentioned scenarios are all motivated by different aspects of renormalization theory. Especially we will see thereby, how techniques which are standard in some areas of theoretical physics can lead to new perspectives or new approaches in other areas.

In this chapter we want to motivate our treatment and will present concisely the corresponding ideas and results. Therefore we will review in section 1.1 how a scale-dependent behaviour occurs in the different branches of physics that are of importance for this thesis. In section 1.2 we will then present concisely and informally the key ideas and results of this thesis and will link them to the broader context presented in section 1.1. In 1.3 we will then give an overview over the organization of this thesis while in 1.4 we will introduce the employed notation and conventions.

1.1. On scale dependence

In this thesis we define *multiscale phenoemena* as situations, which exhibit a scale dependence in their theoretical description. Hereby, a *scale* should be understood as a characteristic parameter of the system which will be, depending on the situation, for example a length scale, the number of particles or a time scale. From a modern perspective (cf. e.g. [10]) one could distinguish between two categories of such scale-dependent behaviour. One the one hand it is broadly accepted, that one needs different *fundamental laws* at different scales. For example, the fundamental laws of quantum mechanics hold on microscopic scales, while the laws of general relativity hold on cosmological scales. The laws on different scales are then sometimes related by some kind of limiting procedure: For example, the laws of general relativity reduce to Newtonian gravity in the Newtonian limit (cf. [44]), and under the classical limit $\hbar \to 0$ classically allowed trajectories become dominant in path-integral approaches to quantum mechanics (cf. [54]). On the other hand, scale-dependence of physical laws can be caused by a phenomenon, which is commonly named as *complexity*. In those cases, the fundamental laws are known, but the considered systems are so complex, that it is a very hard question to predict the collective behaviour of the system[1]. Examples for this category are given by collective phenomena in condensed matter physics, chaotic systems or systems with non-linear behaviour. In those cases, the behaviour of the system at whole can show unexpected features which are not directly obvious from the fundamental description, which is a phenomenon that is commonly denoted as *emergent behaviour* (cf. [76]). In the following we want to explain concisely, how the scale dependence arises in the different branches of physics, that are of relevance for this thesis – i.e. in statistical physics, quantum field theory and spacetime geometry.

The case of statistical mechanics: Statistical mechanics should be understood as a physical framework for the derivation of thermodynamic equilibrium properties of a macroscopic system from the microscopic properties of its constituents (cf. [101]). The multiscale aspect of this formalism is thereby usually formalized by the thermodynamic limit, which marks the transition from finite-sized systems to macroscopic systems. Under this limit, the microscopic properties of the system – as N-particle wave functions or trajectories in a $6N$-dimensional phase space – get more and more irrelevant, while thermodynamic equilibrium properties emerge as an effective, macroscopically relevant description.

An interesting aspect of statistical mechanics is the occurrence of phase transitions, which are commonly understood as abrupt changes in the macroscopic physical properties of a system under a variation of external parameters as the pressure or the temperature (cf. [64, 101]). Within the formalism of statistical mechanics, those phase transitions manifest themselves by discontinuities or singularities of thermodynamic functions and their derivatives, which predict the dependence of thermodynamic quantities on external parameters. At a first glimpse, this sounds strange since the microscopic laws of physics – as the Schrödinger equation or Hamilton's equations of motion – are usually continuous and hence one should expect, that the macroscopic behaviour of such systems should be continuous, too. But this is not the case, which is a beautiful example, how complex systems can exhibit unexpected properties on macroscopic scales. The reason for this is, that although all occuring thermodynamic functions are analytic for finite sized ensembles, their limit under the

[1] For a more complete discussion of definitions of complexity, consult [119].

thermodynamic limit has not to be analytic, since limits of analytic functions are in general not necessarily analytic (cf. [101, 81]). This *breakdown of analyticity* can be understood as a multiscale phenomenon, since on microscopic scales the analyticity properties of thermodynamic functions are of no direct relevance, while on macroscopic scales they obviously determine the qualitative thermodynamic behaviour of the system to a great extent.

Infinite renormalization in relativistic quantum field theory: While the multiscale aspects of statistical physics are intuitively plausible, since is generally known that macroscopic systems are composed out of many individual microscopic constiuents, it is less obvious how they come into the game in relativistic quantum field theory. Nevertheless, from a modern perspective (cf. [182]) the multiscale aspects of QFT are closely intertwined with a phenomenon which was once one of the biggest problems of quantum field theory, namely the occurrence of infinities associated with divergent integrals appearing in the calculation of observables. Those infinities are usually caused by the assumption, that arbitrarily high momenta are allowed in the occuring integrals. Consequently, if one introduces an additional cut-off parameter Λ which marks the maximum frequency up to which one is allowed to integrate, then all observables are finite for the sake of depending on this additional parameter. Since Λ is a momentum scale, the introduction of a cut-off obviously brings a multiscale aspect into the game, although this appears a priori neither natural nor profound.

But indeed, a lot of wisdom is hidden in this cut-off procedure. This can be understood by recalling, that it is neither known how spacetime behaves on microscopic scales – i.e. for very high momenta – nor it is understood how a putative "most fundamental field theory" could look like or if such a theory even exists. Hence one should expect, that any quantum field theoretic model should be subject to some modification for momenta larger than some momentum scale that specifies the domain of validity of this model. The interesting aspect is now, that quantum field theoretic models can be classified by their properties under variations of the cut-off Λ: For some models (so called *renormalizable* models) the Λ-dependence can be entirely absorbed by a numerical redefinition of the occurring couplings, while for other theories (so called *non-renormalizable* models) infinitely many new couplings have to be introduced to absorb the Λ-dependence completely. Pictorially spoken one could say, that renormalizable models exhibit a self-similarity on all scales while non-renormalizable models appear on different scales catastrophically different. Since the couplings should be determined by experiment anyhow, we see that in the former case the parameter Λ is of no relevance: In a colloquial language, renormalizable models have hence the property that their macroscopic behaviour is in some sense ignorant of putative microscopic modifications. Hence, for such theories, one can safely send the cut-off to infinity, as long as one absorbs the Λ-dependence of the observables simultaneously in adequate numerical redefinition of the couplings.

This procedure, which is standard in quantum field theory, goes under the name of *infinite renormalization* and due to its importance for the understanding of this thesis, we will explain it more thoroughly in chapter 2. But by the present discussion it should have become clear, that it encodes some important multi-scale aspects of quantum field theory.

Relativistic quantum field theory and distributions: A different perspective on the infinities occurring in quantum field theory comes from the Epstein-Glaser formalism (cf. e.g. [159, 148]) which identifies an incorrect treatment of distributional quantities as a cause for the infinities which occur in the standard formalism of quantum field theory. Indeed, if one looks more carefully on the integrals which occur in quantum field theoretic calculations, one observes that they are not integrals over smooth functions but should be understood in a distributional sense. A correct treatment of those distributional quantities leads then to finite quantities, but – due to a fundamental indeterminacy on the level of distributional products and extensions – also to some freedom in the choice of certain parameters. Those parameters can then be identified with the renormalized couplings in the usual picture.

On a first glimpse, the multiscale aspect has hence disappeared if one looks at QFT from this perspective. But on a second glimpse, it is still present, since distributions themselves can be understood as an instance of a multiscale phenomenon. To understand this claim, recall that distributions – which go under the name of generalized functions, too – are continuous functionals of smooth functions[2] and as such they can be always approximated by sequences of smooth functions. Take for example the famous δ-distribution, which is defined as

$$\delta[f] = \int_{\mathbb{R}} dx\, \delta(x) f(x) := f(0).$$

This distribution can be, for example, approximated by $\delta[f] = \lim_{\Lambda\to\infty} \delta_\Lambda[f]$ with the latter being defined as

$$\delta_\Lambda[f] := \int_{\mathbb{R}} dx \sqrt{\frac{\Lambda}{2\pi}} e^{-\frac{1}{2}\Lambda x^2} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy e^{-\frac{1}{2}y^2} f\left(\frac{y}{\sqrt{\Lambda}}\right),$$

where we have performed the substitution $y = \sqrt{\Lambda}x$. If we consider $x \in \mathbb{R}$ as some kind of position variable, then Λ can be understood as a momentum cut-off. What happens now, if we take Λ to be very high but still finite? Then y/Λ should be very small and we could expand $f(y/\sqrt{\Lambda})$ into a Taylor series which yields a series expansion

$$\delta_\Lambda[f] = f(0) + a_1\Lambda^{-\frac{1}{2}} + a_2\Lambda^{-1} + \mathcal{O}(\Lambda^{-3/2}) \tag{1.1}$$

with a_1, a_2 being some constants. We then see that in this case the cut-off Λ is irrelevant, since it produces error terms that vanish for $\Lambda \to \infty$. If we imagine now, that δ_Λ appears as an expression for some quantum field theoretic observable, we then understand that this observable is indeed insensitive for the "physics" at high momentum scales, since the Λ-dependent error terms in (1.1) get irrelevant for $\Lambda \to \infty$.

[2] If you are not familiar with the theory of distributions, please consult appendix A.

Hence, a correct treatment of the distributions that occur in quantum field theoretic calculations can be itself understood as an aspect of QFT's multiscale behaviour, since it corresponds to a correct manipulation of distributional limits under removal of the cut-off.

The case of spacetime geometry: If one tries to quantize general relativity by the usual strategy, one obtains that the corresponding quantum field theory is non-renormalizable (cf. e.g. [109]). Along above lines this means, that this theory is only valid up to a finite momentum cut-off of the order of the Planck scale (cf. [58]). Hence, this specific quantum field theory is not able to give reliable predictions on very small scales and one expects that a yet unknown theory of quantum gravity should provide an *UV-completion* of general relativity. Consequently, the multiscale aspects of gravity manifest themselves in the problem, that nobody knows how gravity acts on the Planck scale, although the macroscopic behaviour of gravity is well understood. Hence one expects, that the theory of gravity – and possibly also the geometric structure of spacetime – should change its appearance on smaller scales. And indeed, a lot of research has been performed in the previous decades to understand how this change could look like, which lead to a plenty of different theories and findings.

In some approaches, the structure of general relativity was explicitely modified on microscopic scales and one tried then to understand a posteriori, if the modified UV-theory is capable to imply general relativity on macroscopic scales. Examples for such strategies are given by string theory (where one assumes that particles are indeed 1-dimensional entities on Planckian scales) or Hořava–Lifshitz gravity (where an anisotropy is introduced on microscopic scales). In other approaches, the classical structure of gravity was left unchanged but the quantum theory was tried to be analysed by alternative methods. Such strategies are for example the asymptotic safety program (where general relativity is analysed by the utilization of exact renormalization group methods) or by loop quantum gravity (where the classical structure of general relativity is left unchanged but its behaviour under a non-standard quantization is analysed). And there exist also approaches, as causal dynamical triangulations or causal sets, where the spacetime is discretized by hand on microscopic scales. For more information on different approaches to quantum gravity, consult [109].

As we see, the problem of quantum gravity was tackled from many different directions, but unfortunately none of those strategies were completely satisfactory. On the other hand, all those approaches carry some information on possible properties of putative UV-completions of general relativity and one could ask if there are some unifying threads. In addition one could ask, if already classical general relativity could tell us something about its small-scale behaviour. And indeed, in a series of essays (cf. e.g. [42]) Steven Carlip pointed out, that many approaches to quantum gravity predict some kind of dimensional reduction on microscopic scales. Moreover he explained, that this behaviour is also classically present in the vicinity of the Kasner singularity. By this argumentation, a fascinating "universal" multiscale aspect seems to be hidden in general relativity, as a scale-dependent dimensionality is predicted by so many different approaches.

1.2. Informal presentation of the key ideas and results

After having explained, how scale-dependent behaviour arises in the different branches of physics that are of importance for this thesis, we now want to explain concisely what we will do in this thesis and how our approaches are motivated. Therefore we will present in this section the key ideas and findings of this thesis in an informal way and will relate them to what we have learned about scale-dependence in the last section. Please note, that we will give a more in-depth introduction to those topics at the beginnings of the respective chapters. In addition, the results of this thesis are presented from a more formal perspective in chapter 7.1.

Bose-Einstein condensation by QFT-inspired methods: If one analyses the phase structure of an ideal Bose gas, one obtains as a result that this system exhibits two phases in the thermodynamic limit. One of those phases is then distinguished by the property that the quantum mechanical ground state is macroscopically occupied, which goes under the name of Bose-Einstein condensation. Although the ideal Bose gas is a well studied system, whose behaviour was already analysed by Bose and Einstein (cf. [33, 66]) almost 100 years ago and whose treatment can be found in any textbook on statistical physics, the generalization of this analysis to more realistic, interacting scenarios remained a major obstacle for many years. This situation changed, when Lieb and Seiringer proved the occurrence of Bose-Einstein condensation for dilute Bose gases ([124, 123]). But nevertheless, the understanding of interacting Bose-gases is still limited and this topic constitutes a vivid area of research (cf. [170]).

The complicated situation in the interacting case also stimulated a lot of research in the non-interacting case. A hope thereby could be, that new insights into the behaviour of ideal Bose gases could also lead to a new perspective on the much more difficult interacting case and our treatment of chapter 3 should be understood from this perspective, too: The aim of the analysis of chapter 3 is, to reexpress the thermodynamic limit of an ideal Bose gas in a way which suggests a certain stability of the results under inclusion of interactions.

Our treatment is motivated by so called heat kernel techniques and especially by the famous Minakshisundaram–Pleijel theorem (cf. [27, 45]), which states that the trace of the Heat kernel on general compact n-dimensional Riemannian manifolds exhibits an universal asymptotic behaviour

$$\operatorname{tr}\left(e^{-t\Delta}\right) \sim \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{m=0}^{\infty} a_m t^{m-\frac{n}{2}} \tag{1.2}$$

as $t \to 0^+$ for Δ being the Laplacian, independent of the concrete geometry of the manifold under consideration. In this formula, all geometric information is encoded in the constants a_m, which are consequently geometric invariants of the considered manifold (cf. [45]). Now, the partition function of a grand canonical ensemble can be written as

$$Z(\beta, \mu) = \mathrm{tr}\left(e^{-\beta(H-\mu N)}\right),$$

which resembles obviously in some sense the trace of the heat kernel. Consequently one could wonder, if the grand canonical partition sum exhibits a similar "universal" asymptotic behaviour. In the case of the $\beta \to 0$ limit this was shown for ideal Boses gases by Klaus Kirsten in [110, 112, 111], but we are interested if this behaviour is also present in the thermodynamic limit. In analogy to the geometric case, where the qualitative form of (1.2) remains unchanged under smooth deformations of the geometry, there is then a legitimate hope that an asymptotic expansion of the partition sum could exhibit a similar stabilty under an inclusion of weak interactions. If in addition the occurence of Bose-Einstein condensation could be traced back to qualitative properties of this asymptotic expansion, one could be optimistic that this strategy could tell us something about Bose-Einstein condensation in the case of weak interactions.

The analysis of chapter 4 should be understood as a first step of this research program. We will show there by the use of QFT-inspired ζ-regularization methods, that the grand canonical potential, which is proportional to the natural logarithm of the partition function, has an asymptotic expansion under the thermodynamic limit which resembles the form of above mentioned heat kernel expansions. Moreover we will see there that the form of those expansion differs drastically between the two phases: In the non-condensation phase it exhibits – similar as the heat kernel expansion – a singularity of finite order, while in the condensation phase it exhibits a singularity of infinite order. The value of the thermodynamic observables in the thermodynamic limit is then encoded in the coefficients of those expansions, but due to the singular behaviour of the expansion in the condensation-phase, the observables diverge in this case. But we will see then, that those divergencies can be cured by an infinite renormalization of the chemical potential, which goes in full analogy to the renormalization procedure in quantum field theory and causes the system to exhibit condensation.

Renormalizing the initial singularity in classical field theory: Now to something completely different. It is commonly accepted, that cosmological singularities constitute a serious conceptional problem of general relativity since they mark portions of spacetime, where the basic geometric entities of spacetime break down (cf. [91]). Consequently, those objects are one of the major motivations for the need of quantum gravity, since it is expected that they should not be present in a more complete theory of gravity.

Hence, cosmological singularities are an important testing ground for theories of quantum quantrum gravity and a lot of research was published concerning their properties in various different approaches to quantum gravity. But one could look at them also from a different perspective: Instead of getting rid of them by modifying general relativity, one could ask if they can be probed by any realistic experiment at all. Since all matter theories are known to be quantum field theories one could ask, if either the quantum non-commutativity or the field-character of matter could wash out the singular character of cosmological singularities. This perspective was for the first time adopted by Horowitz and Marolf in [100] where they have shown that certain timelike curvature singularities appear non-singular when probed by quantum mechanical test particles. Those results were then extended by Schneider and Hofmann (cf. [96, 97]), who showed that the Schrödinger wave functional of a scalar quantum field behaves benignly in the vicinity of the Schwarzschild and the Kasner singularity, and by Ashtekar and Schneider (cf. [15]), who showed that the FLRW singularity is tame when probed by quantum field theoretic operator valued distributions.

Nevertheless, those investigations which analysed the impact of cosmological singularities had one caveat: It is not clear, if the tameness of the quantum fields is really caused by the quantum aspects of the theory or already a property of the classical theory which was used for quantization. Our analysis of chapter 4 should be understood as a first step to answer this question by analysing the properties of a classical scalar field in the vicinity of the big bang singularity. The main motivation for our treatment comes thereby from the aforementioned publication [15] by Ashtekar and Schneider. There they had shown, that the big bang singularity appears tame when probed by operator valued distributions in the context of scalar quantum field theory, where the tameness is associated with the distributional properties of the field operators. But distributions are in no sense tied to quantum field theory and one could also wonder, if distributional quantities in the context of classical field theory could show a similar behaviour.

We will therefore analyse the existence and the properties of distributional solutions to the conformally coupled Klein-Gordon equation in the vicinity of the singularity of an radiation dominated FLRW spacetime. We will thereby obtain as a result, that distributional solutions exist and that they are related to the usual smooth solutions by a procedure which resembles the above mentioned Epstein-Glaser approach to infinite renormalization. Moreover we show, that those solutions define distributional states on the algebra of multilocal classical Wick polynomials which have the property that all higher order observables stay finite on the singularity. Consequently the renormalized, distributional classical field theory shows a higher regularity than its non-renormalized, smooth cousin. Those results hence show on the one hand, that renormalization procedures are not only tied to quantum field theory but can also be of use in the context of classical field theories. On the other hand they make it plausible, that the tame behaviour of quantum fields in the context of [15] is a property which is also shared by the classical background theory.

Unfortunately our analysis has also some undesirable properties. Especially we will see that it relies on the neglection of back-reaction effects which is problematic in this context. Moreover we will understand, that the renormalization procedure brings a high amount of indeterminacy into the game, whose role has to be investigated by future research.

Ultrarelativistic behaviour of gravity in extreme situations: Another perspective on cosmological singularities is given by their geodesic geometry. Especially we are interested in the properties of freely falling classical test particles in singular spacetimes. Thereby we will encounter an interesting behaviour in the vicinity of the big bang singularity in section 5, which was to our best knowledge not described before: Time- and spacelike geodesics behave increasingly lightlike as they approach the singularity. Especially, the short-time asymptotics for time- and spacelike geodesics in the vicinity of the big bang singularity is dominated by the corresponding asymptotics of null geodesics, which is a sign for an ultra-relativistic behaviour on those short time-scales.

Motivated by this finding we will ask the question, if this behaviour is special to the concrete situation or a more general feature of gravity in extreme situations. And indeed, by reviewing the literature we will point out, that there are various situations in which gravity seems to exhibit an ultrarelativistic behaviour which accompanies the aforementioned dimensional reduction. Moreover we will see, that causal relations can be understood as an effective description of lightlike relations since two spacelike events are causally related if and only if they are related by a – possibly very large – chain of lightlike relations. Motivated by those thoughts we will then rise the question, if it could be that gravity is described by some sort of ultrarelativistic theory on microscopic scales.

The microscopic ultrarelativistic symmetries of spacetime: Motivated by the question, if gravity could be described by some sort of ultrarelativistic theory on microscopic scales we rise the consecutive question, by which microscopic gauge symmetry groups such a theory could be ruled. To answer this question at least partially, we will investigate in chapter 6 the symmetries of microscopic tangent light cones. The motivation for this investigation is twofold: On the one hand, a pseudo-Riemannian spacetime together with the additional constraint that all particles have to move on lightlike paths constitutes a very simple model for an ultrarelativistic spacetime. On the other hand, the behaviour of geodesics in the vicinity of FLRW spacetimes motivates the idea, that tangent spaces could degenerate to tangent light cones on very small scales.

But before we are able to investigate the symmetries of microscopic tangent light cones, we first have to understand their geometric structure. Thereby we will obtain as a result, that the metric tensor, although it is degenerate thereon, restricts to a kind of distance metric on infinitesimal tangent light cones. This metric, together with other more elementary geometric structures, constitutes then a set of universal geometric structures which are present on any microscopic tangent light cone, independent of the macroscopic behaviour of the gravitational field. Moreover, those structures can be understood as an instance of a weak ultra-relativistic Carroll structure (cf. [61]) and resemble the universal structures at null-infinity that appear within the *Bondi-Metzner-Sachs*-analysis (cf. [13]).

We will then analyse the symmetries of those ultrarelativistic geometric structures, where we have to distinguish between two cases: The case of isometry and the case of conformal invariance. We then obtain as a result, that the conformal symmetry group of the aforementioned geometric structures has a rich mathematical structure since it contains infinitely many Lorentz subgroups which are parametrized by so called crossed homomorphisms. The isometry group appears then as one of those Lorentz subgroups and is a non-trivial representation of the original Lorentz group. Although it is induced by the original linearly represented Lorentz group on the full tangent space, its restricted action to the light cone has a non-trivial appearance: It acts in terms of conformal transformations on the space of null-directions and simultaneously rescales the length of null-vectors.

An important question is then of course, how the occurrence of infinitely many Lorentz subgroups should be interpreted. Fortunately, there exists a quite picturesque interpretation. To understand this, we first have to recall that it is often said, that there exists no *meaningful* length-notion for null vectors, since the metric tensor is degenerate on the light cone. In this context *meaningful* is usually understood as synonymous to *Lorentz invariant*. But we will see, that there exist *meaningful* length notions for null vectors, if one assumes them to be just Lorentz *covariant*. The conformal automorphism group then comprises all possible transformation laws for such covariant length notions and each Lorentz subgroup corresponds to a Lorentz covariant length notion for null vectors.

We have already mentioned above, that the universal geometric structures which are present on infinitesimal tangent light cones resemble those at null-infinity in the context of the Bondi-Metzner-Sachs (BMS) analysis. We will see then, that this analogy can be taken further since the mathematical structure of the conformal symmetry group resembles the structure of the original BMS group to a great extent. Moreover, both groups have a similar interpretation: While the original BMS-group is tied to geometric structures on a macroscopic null surface associated with asymptotic flatness, our symmetry groups are affiliated to microscopic null surfaces associated with the equivalence principle. Moreover, the conformal symmetry group as well as the isometry group are both eligible as gauge groups for the bundle of null vectors, which qualifies them as microscopic symmetry groups of a putative ultrarelativistic gravitational theory.

Especially we have shown thereby, that BMS-like groups constitute not only macroscopic, asymptotic symmetry groups in cosmology but describe also a microscopic and apparently unknown microscopic symmetry of pseudo-Riemannian geometry.

1.3. Outline of the thesis

We now want to give a concise overview over the outline of this thesis:

– In chapter 2 we will review the principle of (infinite) renormalization from different perspectives. Therefore we will explain in section 2.1 how this formalism makes its appearance in the standard formalism of quantum field theory. In section 2.2 we will explain, that infinite renormalization is not only of importance in quantum field theory, but can also appear in other branches of physics as quantum mechanics, electrostatics or even classical mechanics. In section 2.3 it is then reviewed, how infinite renormalization can be formalized from a distributional perspective in the context of quantum field theory while in section 2.4 we will explain, why infinite renormalization can be understood as a multi-scale phenomenon.

– In chapter 3 we will investigate the thermodynamic limit of an ideal Bose gas by QFT-inspired methods and especially we will show thereby that this situation is an instance of infinite renormalization within the framework of quantum statistical mechanics. Therefore we will give in section 3.1 a more in-depth introduction to the motivation, ideas and concepts of our investigation. In section 3.2 we will then review the basics on the quantum statistics of the harmonically trapped ideal Bose gas, which will be the starting point of our analysis. In section 3.3 then the asymptotic expansions of the grand potential are calculated for the different phases by utilization of ζ-regularization methods. In section 3.4 we will then analyse the thermodynamic limit of this system by utilization of those techniques. Thereby we will see, that an infinite renormalization of the chemical potential is needed in the condensation phase. In section 3.5 we will conclude this chapter by giving a concise summary and a short outlook on further research.

– In chapter 4 we will explain, how a classical field theory can be renormalized in the vicinity of the initial singularity by utilization of distribution theory. Therefore we will review in 4.1 the basics regarding cosmological singularities and completeness concepts. Moreover we will present the motivation and the ideas behind our treatment more thoroughly. In section 4.2 we will then review the basics of a classical massless scalar field in a radiation dominated big bang spacetime. In section 4.3 we will then reformulate this field theory in terms of an algebraic language. In section 4.4 then the renormalization procedure is performed on the level of classical solutions, classical n-point functions and classical states. Finally we conclude this chapter in section 4.5 by summarizing it, giving an outlook to more general situations and pointing out the drawbacks of our investigation.

– In chapter 5 we will present some evidence, that gravity behaves ultrarelativistically in extreme situations and formulate the claim, that it could be described by an ultrarelativistic field theory on fundamental scales. Therefore we will investigate in section 5.1 the qualitative behaviour of geodesics in FLRW background and will show thereby, that they exhibit an ultrarelativistic behaviour in the vicinity of the singularity. In chapter 5.2 we will present some evidence from the literature, that gravity exhibits a dimensional reduction and an ultrarelativistic behaviour in extreme situations. In chapter 5.3 we will then conclude this section and formulate the above mentioned claim.

– In chapter 6 we will investigate the symmetry groups of microscopic tangent light cones. In section 6.1 we will give an in-depth introduction to our treatment and will explain its features and its relation to the literature. In section 6.2 we will then introduce the light cone bundle as the basic geometric entity of our study and will investigate its universal geometric structures. In section 6.3 then the automorphism groups of a single microscopic tangent light cone are derived while in section 6.4 the mathematical structure of the conformal automorphism group is analysed. In section 6.5 on the other hand it is explained, how the isometry subgroup arises as a special Lorentz subgroup of the conformal automorphism group. In section 6.6 we then explain, how the occurence of infinitely many Lorentz subgroups can be interpreted. In section 6.7 we will then show, that the identified automorphism groups are eligible as gauge groups for the light cone bundle, while in section 6.8 we will explain their relation to the original BMS analysis. In section 6.9 we will then discuss possible implications of the present investigation and in section 6.10 we will present a short summary.

– In section 7 we will conclude this thesis by giving a concise summary and by commenting on promising directions of further research.

1.4. Notation and conventions

Finally we want to present concisely the notation and conventions for those chapters where it is needed.

Conventions for chapter 2: In chapter 2 we will use the metric signature $(+1, -1, -1, -1)$, in contrary to all other parts of the thesis. The reason for this is, that this metric signature is better suited to those quantum field theoretic computations which are performed in section 2.1.

Conventions for Chapter 4 and Appendix D: We use in those chapters the following conventions, which equal the conventions used in [44]. The metric signature is given by $(-1, +1, +1, +1)$ and the Levi-Civita connection is for vector fields $X = X^\mu \partial_\mu, Y = Y^\mu \partial_\mu \in$

$\mathfrak{X}(\mathscr{M})$ given by

$$D_X(Y) = X^\mu D_\mu \left(Y^\nu \partial_\nu\right) = X^\mu \left(\partial_\mu Y^\sigma + Y^\nu \Gamma^\sigma_{\ \mu\nu}\right) \partial_\nu =: X^\mu \left(D_\mu Y^\sigma\right) \partial_\sigma$$

where we have set $D_\mu := D_{\partial_\mu}$ and defined the Christoffel symbols

$$\Gamma^\sigma_{\ \mu\nu} \partial_\sigma := D_\mu \partial_\nu.$$

Moreover, the Christoffel symbols are given by the Koszul formula

$$\Gamma^\sigma_{\ \mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

and the Riemann curvature tensor is for vector fields $X, Y, Z \in \mathfrak{X}(\mathscr{M})$ defined as

$$R(X,Y)(Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.$$

Its coordinate expression is given by

$$R^\rho_{\ \sigma\mu\nu} \partial_\rho := R(\partial_\sigma, \partial_\mu) \partial_\nu$$

and can be calculated in terms of Christoffel symbols as:

$$R^\rho_{\ \sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\ \nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\ \mu\lambda} \Gamma^\lambda_{\ \nu\sigma} - \Gamma^\rho_{\ \nu\lambda} \Gamma^\lambda_{\ \mu\sigma}$$

The Ricci tensor is given by $R_{\mu\nu} := R^\lambda_{\ \mu\lambda\nu}$ and the Ricci scalar is given by $R = R^\mu_{\ \mu}$.

Conventions for chapter 6: Throughout this chapter, $\mathscr{M}$ denotes a time- and space-orientable spacetime with pseudo-Riemannian metric g of signature $(-1,+1,+1,+1)$. η will denote the Minkowski metric $\eta = \mathrm{diag}(-1,+1,+1,+1)$. We will denote the tangent bundle of $\mathscr{M}$ by $T\mathscr{M}$ and the tangent space at $p \in \mathscr{M}$ by $T_p\mathscr{M}$. Moreover, we will write TU for the restriction of $T\mathscr{M}$ to an open set $U \subset \mathscr{M}$. The space of vector fields over an open set $U \subset \mathscr{M}$ will be written as $\mathfrak{X}(U)$ and $X \in \mathfrak{X}(\mathscr{M})$ denotes the global timelike vector field that describes the time orientation of $\mathscr{M}$. We assume in addition, that X is normed, i.e. $g(X,X) = -1$, and denote the restriction of X to $T_p\mathscr{M}$ by X_p. We denote the proper orthochronous Lorentz group by $\mathrm{SO}^+(1,3)$ and the group of all complex 2×2 matrices with unit determinant by $\mathrm{SL}(2,\mathbb{C})$. Unit elements of matrix groups will be denoted by $\mathbf{1}$. Due to orientability of $\mathscr{M}$, the structure group of $T\mathscr{M}$ is reduced to $\mathrm{SO}^+(1,3)$, i.e. there exist bundle atlases for $T\mathscr{M}$ whose transition functions lie in $\mathrm{SO}^+(1,3)$. Let $\mathcal{C}$ be an open cover. We will then denote in the sequel by $\mathcal{A} = \{(U,\psi) | U \in \mathcal{C}\}$ an atlas consisting out of local trivializations

$$\psi : TU \to U \times \mathbb{R}^4$$

that are induced by vielbein frames. I.e. for each $(U, \psi) \in \mathcal{A}$ there is an associated vielbein frame $(U, (E_\mu)_{\mu=0,\dots,3})$ with $E_\mu \in \mathfrak{X}(U)$ satisfying

$$g(E_\mu, E_\nu) = \eta_{\mu\nu}$$
$$g(E_0, X) = -1$$

such that for any $v = v^\mu E_\mu \in T_p\mathscr{M}$ with $p \in U$

$$\psi(v^\mu E_\mu) = (p, (v^\mu))$$

holds. We will denote in the sequel vielbein frames just by (U, E_μ) or (E_μ). We will write the restriction of g to $T_p\mathscr{M}$ as g_p. Let $p \in U$. The restriction of (U, ψ) to $T_p\mathscr{M}$ with $p \in U$ will be written as

$$\psi_p : T_p\mathscr{M} \to \mathbb{R}^4, v^\mu E_\mu \mapsto (v^\mu),$$

where (E_μ) is the vielbein associated to (U, ψ). The restriction of E_μ to $p \in U$ will just be denoted by E_μ. The euclidean norm on $\mathbb{R}^3$ will be denoted by $|\cdot|$. Moreover we define the 2-sphere $S^2 \subset \mathbb{R}^3$ as

$$S^2 := \left\{ \hat{e} = (\hat{e}^1, \hat{e}^2, \hat{e}^3) \in \mathbb{R}^3 \middle| |\hat{e}| = 1 \right\}$$

and denote the Riemann sphere by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. Coordinate expressions in $\mathbb{R}^4$ will always be written as (v^μ) and vectors in $\mathbb{R}^3$ by $\vec{v}$ or (v^i). Unit vectors in $\mathbb{R}^3$ will be denoted by $\hat{v}$. For a 4-vector (v^0, v^1, v^2, v^3) we define $\vec{v} := (v^1, v^2, v^3) = (v^i)$. All structures affiliated with the Riemann sphere will be introduced in the main text when they are needed and are additionally reviewed concisely in appendix C.1. Finally, we define $\mathbb{R}^+ := (0, \infty)$ and denote the set of positive valued smooth functions on $\mathbb{C}_\infty$ by $C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$. Smooth functions associated with other domains are denoted analogously by $C^\infty(\cdot, \cdot)$. Products of smooth functions and numbers will be denoted by $\cdot$.

2. On renormalization

Renormalization is one of the key techniques in quantum field theory and statistical physics. But despite of its importance, its interpretation and validity can be hard to grasp. Since a good understanding of this concept will be of great importance in this thesis, we want to review it concisely from different perspectives: In section 2.1 we will review how infinite renormalization makes its appearance in the standard formalism of relativistic quantum field theory. Therefore we will consider a simple example and explain the philosophy of renormalization within this framework. In section 2.2 we will then see, that infinite renormalization does not just appear in relativistic quantum field theory, but in many branches of physics. In 2.3 we will then take a more mathematical perspective and will explain thereby how renormalization in quantum field theory can be interpreted in the context of distributions. Finally, we will then explain in section 2.4 how infinite renormalization complies with the concept of multiscale phenomena.

2.1. Perturbative renormalization in quantum field theory

When a physics student encounters the concept of infinite renormalization in quantum field theory for the first time, this can be a disappointing or shocking experience. The calculation of higher order observables in quantum field theories as QED or QCD includes divergent integrals, which negates on a first glimpse the predictive power of such theories. The standard textbook treatment of this problem is then a rather dubious procedure in which the occuring infinities are absorbed in redefinitions of coupling constants, which finally leads to finite observables (see e.g. [144]). The validity, consistency and interpretation of this procedure is often left unclear, and the student is left with the choice of either digging deeper or accepting this unsatisfactory situation by getting used to it. It is important to point out, that even the grandfathers of quantum field theory had big problems with accepting this procedure. Richard Feynman wrote for example in [74]:

> *It is what I would call a dippy process. Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent.*

And Dirac said in [53]:

> *This is just not sensible mathematics.*

On the other hand, this renormalization procedure is obviously a great success, since it leads to correct predictions for measurable quantities. In this section we want to review this procedure and will therefore present a simple toy example: The ϕ^4-model up to first order in perturbation theory. The calculations of this section follow [128] and we choose – only in this section – the signature $(+,-,-,-)$ as a convention for the Minkowski metric.

The ϕ^4-model: The ϕ^4-model is one of the simplest interacting quantum field theoretic models and its action on flat spacetime is given by

$$S[\phi] = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right). \tag{2.1}$$

The corresponding Hamiltonian is then $H = H_0 + \lambda H_I$ with

$$H_0 = \int_{\mathbb{R}^3} d^3x \frac{1}{2} \left[\Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right]$$

being the free Hamiltonian and

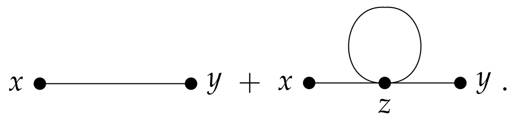
$$H_I = \int_{\mathbb{R}^3} d^3x \frac{1}{4!} \phi^4$$

being an interaction term, where Π denotes the momentum conjugate to ϕ.

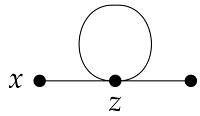
The divergency structure at first order in perturbation theory: As a warm-up, we first consider the expression for the 2-point function in momentum space up to first order in perturbation theory. I.e. we want to expand

$$\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle = \left\langle 0 \middle| T \left\{ \phi(x) \phi(y) \exp \left[-i \frac{\lambda}{4!} \int d^4z \phi^4(z) \right] \right\} \middle| 0 \right\rangle$$

up to first order in λ, where $|\Omega\rangle$ denotes the ground state of H and $|0\rangle$ denotes the ground-state of H_0. By the usual machinery (cf. e.g. [144, 128]), this is given by the following sum of Feynman graphs:



We now consider the second graph, whose expression in position space is given by the integral



$$= \frac{(-i\lambda)}{2} \int_{\mathbb{R}^4} d^4z D_F(x-z) D_F(0) D_F(z-y)$$

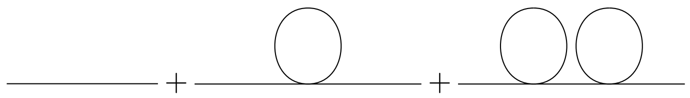
and which is called a *tadpole*-graph. The associated expression in momentum space is then, after Wick rotation and after amputation of external legs, given by

$$-i\lambda D_F(0) = \frac{(-i\lambda)}{2} \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2}$$

which diverges. By the introduction of an euclidean momentum cutoff $\Lambda > 0$ this can be reexpressed as

$$-i\lambda D_F^\Lambda(0) = \frac{(-i\lambda)}{32\pi^2} \left(\Lambda^2 + m^2 \log\left(\frac{m^2}{\Lambda^2+m^2} \right) \right)$$

and hence we see, that the 2-point function at 1-loop level exhibits a logarithmic and a quadratic divergence as $\Lambda \to \infty$. To make the connection of this *tadpole*-graph to observable quantities clearer, we now resum the 2-point function at 1-loop order. That means, that we consider the "dressed" propagator

$D^R(p) =$  $+ \cdots$

$$= \frac{i}{p^2 - m_0^2 - \lambda D_F^\Lambda(0)}. \tag{2.2}$$

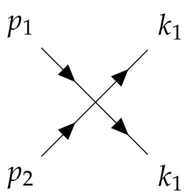
By comparing (2.2) with the free Feynman propagator, that is associated with the non-interacting Hamiltonian H_0 and is given by

$$\frac{i}{p^2 - m_0^2},$$

we observe, that the resummation of the tadpole graph has the effect to change the mass from m_0^2 to an effective mass of $m_0^2 + D_F^\Lambda(0)$. Since it seems reasonable that the mass of a particle is a measurable quantity, we can define the mass as "predicted" by the ϕ^4-model to be

$$\begin{aligned} M(\Lambda) :&= m_0^2 + \lambda D_F^\Lambda(0) \\ &= m_0^2 + \frac{\lambda}{32\pi^2} \left(\Lambda^2 + m^2 \log\left(\frac{m^2}{\Lambda^2+m^2} \right) \right), \end{aligned} \tag{2.3}$$

which obviously diverges under removal of the cutoff. On the other hand, the $2 \to 2$ scattering amplitude is non-divergent at $\mathcal{O}(\lambda)$, since it contains no loop. It is just given by

$\mathcal{M}^\Lambda_{2\to 2} =$ 

$= -i\lambda$

and is hence especially Λ-independent.

The idea behind infinite renormalization: If we regard the mass of a particle – as modeled by the pole of the resummed propagator – as an observable, the result of the last section is indeed unsatisfactory: It seems, that a naive calculation predicts an infinite value for the mass, which puts the validity of the ϕ^4-model into question. But this problem can be resolved, if one understands the action (2.1) not as the valid microscopic description of a physical system, but merely as a tool for the construction of a reasonable theory by taking an appropriate limit. Thereby the first step will be, to consider the action (2.1) only as valid, as long as a finite cutoff $\Lambda > 0$ is present. In this case all observables are finite and depend explicitely on Λ. Since one is ultimately interested in the limit $\Lambda \to 0$, one could then use this family of physical systems, parametrized by finite cutoffs $\Lambda > 0$, to approximate a limiting system with vanishing cutoff. The results of the last paragraph imply then, that the naive limit, where all physical parameters are assumed to be fixed as Λ varies, does not yield a meaningful limiting theory. But one could consider more complicated limiting procedures by demanding, that the parameters of the theory, i.e. λ and m in our case, should depend explicitely on the cut-off Λ, too. Maybe, as $\Lambda \to 0$, one could then approximate non-trivial limiting theories, which have the property, that all observables stay finite as $\Lambda \to 0$, while the initial parameters λ, m degenerate as $\Lambda \to \infty$. From this point of view, one can understand the procedure of infinite renormalization as a tool for the investigation of the boundary of the family of theories parametrized by the cutoff Λ and the occuring couplings.

The remaining question is then, how the form of those Λ-dependent couplings should be guessed. To answer this question we will consider for a moment a more general situation in which the action S depends on n couplings $g_1, ..., g_n$ and where a general regulator $\Lambda > 0$ is present, that is not necessarily a cutoff but could be an inverse lattice spacing, for example. Then recall, that the ultimate goal of any physical theory is to make numerical quantitative predictions for all possible experiments, as long as the free parameters of the theory were already fixed by measurement. The predictive power of the theory is thereby encoded in the property, that a finite amount of initial measurements should be enough to fix all parameters of the theory which determine then all other experimental outcomes. In the present situations we have n couplings and hence it is reasonable, that n observables should be enough to determine their values. Hence assume, that we have n observables $O_1, ..., O_n$ which were measured by some experimentalist to have a numerical value $v_1, ..., v_n \in \mathbb{R}$. Within the framework of our theory, those observables then can be calculated in terms of the action S and thus depend explicitely on the couplings $g_1, ..., g_n$ and the regulator Λ. Hence, they are given by functions $O_i(g_1, ..., g_n; \Lambda)$. One could then define the Λ-dependent couplings implicitly as those functions $g_i(\Lambda)$ which satisfy

$$O_i(g_1(\Lambda), ..., g_n(\Lambda), \Lambda) = v_i \tag{2.4}$$

for all $i = 1, ..., n$. Then, the observables obviously reproduce the measured values v_i in the limit $\Lambda \to \infty$. Such conditions, which fix the Λ-dependence of the *bare* couplings g_i in terms of measurable quantities v_i are called *renormalization conditions*. Please note, that one could also use more general renormalization conditions of the form

$$\lim_{\Lambda \to \infty} O_i(g_1(\Lambda), ..., g_n(\Lambda), \Lambda) = v_i, \tag{2.5}$$

which just fix the asymptotic behaviour of the Λ-dependent couplings $g_i(\Lambda)$ as $\Lambda \to \infty$. The *renormalizability* of the theory is then encoded in the statement, that one just needs a finite amount of renormalization conditions to render the limits of all reasonable observables finite as $\Lambda \to \infty$. This encodes also the predictive powe of the theory: Just a finite amount of measurements is needed to fix the functional or the asymptotic behaviour of the Λ-dependent couplings in terms of *renormalized* couplings v_i and to render all observables finite as $\Lambda \to \infty$. In addition, the observables do not depend on the microscopic couplings g_i anymore, but on the *renormalized* couplings v_i.

The renormalization procedure: As explained in the last paragraph, it is the objective of the renormalization conditions to connect the initial parameters of the theory with measurable, experimentally accessible quantities, such that the latter fix the asymptotic behaviour of the former under the limit $\Lambda \to 0$. Hence we have to decide which observables in the context of the ϕ^4-theory should be used in those conditions. Although it would be desirable to choose observables which correspond directly to measurable quantities, it is often more practicable from a theoretical perspective, to choose to choose some sort of "intermediate parameters" (cp. [137]) as renormalized couplings, which are easily calculable in the theoretical model and experimentally accessible, although not directly measurable. This also shows that renormalized parameters are no true constants of nature, but only valid in the context of the specific models (cp. [137]).

$\mathcal{M}_{2\to 2}$ as a scattering amplitude is directly measurable and it is hence instructive to use it in the renormalization conditions. On the other hand, the tadpole graph $D_F(0)$ does not even depend on external momenta and hence it seems unreasonable that it can be directly measured. But the effective mass (2.3) should be experimentally accessible and hence we want to utilize it in the renormalization conditions, too. Let us assume hence that our experimentalist has measured some quantities from which we can extract numerical values of the scattering amplitude $\mathcal{M}_{2\to 2}$ and the effective mass M. Consequently we have have

$$\lim_{\Lambda \to \infty} M(\Lambda) \stackrel{!}{=} m_R^2 \qquad \text{and} \qquad \lim_{\Lambda \to \infty} i\mathcal{M}_{2\to 2}^{\Lambda} \stackrel{!}{=} \lambda_R$$

as renormalization conditions, where m_R and λ_R are numerical values extracted from experiments which are called the renormalized mass and the renormalized coupling, respectively.

More explicitly, those renormalization conditions are given by

$$\lim_{\Lambda\to\infty}\left[m_0(\Lambda)^2+\frac{1}{32\pi^2}\left(\Lambda^2+m_0(\Lambda)^2\log\left(\frac{m_0(\Lambda)^2}{\Lambda^2+m_0(\Lambda)^2}\right)\right)\right]\stackrel{!}{=}m_R^2 \tag{2.6}$$

and

$$\lim_{\Lambda\to\infty}\lambda_0(\Lambda)\stackrel{!}{=}\lambda_R,$$

which fixes the asymptotic behaviour of $m_0(\Lambda)$ and $\lambda_0(\Lambda)$ as $\Lambda\to\infty$ (cf. [128]). In this case, the coupling λ is not renormalized at first perturbative order, i.e. $\lambda_0(\Lambda)=\lambda_R+\mathcal{O}(\Lambda)$. But the renormalization condition (2.6) means, that we have to choose the Λ-dependent coupling $m_0(\Lambda)$ in such a way, that it cancels[1] the divergency in the effective mass $M(\Lambda)$. Under this renormalization, the limit $\Lambda\to\infty$ is then well-defined and any observable is calculable up to first order in perturbation theory and depends on λ_R and m_R.

On renormalizability and non-renormalizability: Perturbative *renormalizability* then means in essence, that the procedure of the last paragraphs can be performed, at least in principle, for all possible Feynman diagrams up to any order in perturbation theory, under the requirement, that finitely many bare couplings in the initial action are sufficient for the absorption of all occurring divergences. This means especially, as said before, that renormalizable theories have predictive power and are in principle valid up to arbitrary high energy scales: Finitely many measurements are sufficient to determine the free parameters (i.e. the renormalized couplings or the cut-off dependent bare couplings) of the theory, which fixes then the experimental outcomes of all other measurements.

On the other hand, *non-renormalizable* theories, for which above procedure fails, should be regarded as either just valid up to some energy scale or as theories without predictive power: For such theories, infinitely many bare couplings have to be introduced in order to absorb the occurring infinities under removal of the cut-off. Hence, infinitely many measurements are needed to fix the free parameters of the theory and consequently it looses its predective power. Nevertheless, non-renormalizable theories are perfectly fine as low energy effective theories. We will comment on this again in section 2.4.

2.2. Some non-standard examples for infinite renormalization

After having explained thoroughly the philosophy and the formal machinery behind infinite renormalization we now want to make contact to some simple examples to illustrate our gained knowledge. People often think, that infinite renormalization is a technique which

[1] Actually, in the context of perturbative renormalization, this cancellation has to happen in a way that is compatible with the perturbative expansion in λ, which gives rise to some subtleties. Nevertheless, we won't bother with this, since we do not deal with perturbative renormalization in this thesis. More information on this can be found in [51].

just occurs in quantum field theory. But this is not true: There are many examples from many different branches of physics which fit into the template presented in the last section. In this section we will present a simple example from quantum mechanics more thoroughly and will comment at the end of the section on other examples from other branches of physics.

A non-standard example from quantum mechanics: We present a simple example of a 2-state quantum system, which is due to [137]. We therefore consider the 2-dimensional Hilbert space $\mathbb{C}^2$ and define the Hamiltonian

$$H_\Lambda = H_0 + g_0 V_\Lambda, \tag{2.7}$$

where g describes a coupling, $\Lambda > 0$ is a regulator and H_0 and V_Λ are given by the matrices

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}, \qquad V_\Lambda = \begin{pmatrix} -1 & \Lambda \\ \Lambda & 0 \end{pmatrix}.$$

Here the unperturbed Hamiltonian H_0 describes a situation of a 2-state system with ground state energy 0, whose excited state has excitation energy ω. V_Λ can be interpreted as an external potential. Now imagine, that some experimentalist has prepared a 2-state system with excitation energy ω subject to some external potential in his lab and we are ultimately interested in the question if the Hamiltonian (2.7) describes this system in the limiting case $\Lambda \to \infty$. We therefore calculate now the energy eigenvalues of H_Λ, which are understood as the only two observables of this system and are given by

$$E^\pm_\Lambda = \frac{1}{2}\left(\omega - g_0 \pm \sqrt{(\omega + g_0)^2 + 4g_0^2\Lambda^2}\right).$$

Obviously, those observables diverge in the limit $\Lambda \to \infty$. Now imagine, that the experimentalist has measured the ground state energy E^- to have the value E^-_m. I.e. we have the renormalization condition

$$\lim_{\Lambda\to\infty} E^-_\Lambda = E^-_m.$$

Choosing the Λ-dependent bare coupling $g_0(\Lambda)$ to be

$$g_0(\Lambda) = g_R \Lambda^{-1} \tag{2.8}$$

we then obtain

$$\lim_{\Lambda\to\infty} E^-_\Lambda = \frac{1}{2}\left(\omega - \sqrt{\omega^2 + 4g_R^2}\right) \stackrel{!}{=} E^-_m,$$

which can be solved for g_R. Then the renormalized model gives the excited energy E^+ as a prediction:

$$\lim_{\Lambda\to\infty} E^+_\Lambda = \frac{1}{2}\left(\omega + \sqrt{\omega^2 + 4g_R^2}\right) \tag{2.9}$$

Hence the system is renormalizable: One measurement is needed to fix the renormalized parameter g_R – i.e. the precise form of the Λ-dependent coupling (2.8) – which determines then the numerical value of all other observables – i.e. in this case just the excitation energy (2.9). In this example we can also perform the limit on the level of the Hamiltonian, since

$$H_\Lambda = H_0 + g_0(\Lambda) V_\Lambda \to H_\infty$$

as $\Lambda \to \infty$, where H_∞ is defined as

$$H_\infty = \begin{pmatrix} 0 & g_R \\ g_R & \omega \end{pmatrix}.$$

By comparison with section 2.1 we then understand, that this situation complies with the template presented there.

Other examples: Another well-studied quantum mechanical system which constitutes an example for the occurrence of an infinite renormalization is the two-dimensional Schrödinger equation subject to an attractive delta potential. This enlightening example was studied many times (e.g. [168, 24, 105, 82, 131, 102, 129, 7]) and is one of the standard examples for the usage of renormalization techniques in quantum mechanics[2]. In addition, there exist also examples for infinite renormalization in other branches of physics as electrostatics ([47]) or classical mechanics ([73]).

[2] A good reference for this is given by [102], although there is a typo in the definition of the Hamiltonian: The potential term in this reference is given by $\delta(x)^2$ which makes no sense since the delta distributions cannot be squared. The potential term should hence be replaced by $\delta(x)$.

2.3. A distributional point of view

In this section we want to give an alternative perspective on infinite renormalization in the context of quantum field theory by utilization of the theory of distributions[3]. Thereby we will understand that the infinities which plague quantum field theory in its standard formulation are caused by an inadequate treatment of distributional objects.

A simple example: As a prelude we will consider the function

$$f(x) = \theta(x)x^{-1}$$

where θ is the Heaviside step function. This example is covered in [150] and we will review their treatment here. Unfortunately, f does neither define a tempered distribution in $\mathcal{S}'(\mathbb{R})$ nor a distribution in $\mathcal{D}'(\mathbb{R})$ since

$$\int_0^1 f(x)dx$$

diverges. But it defines a distribution in $\mathcal{S}'(\mathbb{R} \setminus \{0\})$ and we could ask the question, if there exist extensions of it in $\mathcal{S}'(\mathbb{R})$, i.e. distributions $\chi \in \mathcal{S}'(\mathbb{R})$ which satisfy

$$\chi(h) = \int_{\mathbb{R}} dx f(x)h(x)$$

for all $h \in \mathcal{S}(\mathbb{R} \setminus \{0\})$. And indeed, it is easy to show, that, for any $M > 0$,

$$\chi_M(h) := \int_0^M \frac{f(x) - f(0)}{x} + \int_M^\infty \frac{f(x)}{x} \tag{2.10}$$

defines a distribution in $\mathcal{S}'(\mathbb{R})$ which extends the distribution in $\mathcal{S}'(\mathbb{R} \setminus \{0\})$ defined by f. Since the distribution χ_M is a distributional extension of the original distribution, $\chi_M - \chi_N$ should vanish on $\mathbb{R} \setminus \{0\}$ for all $M, N > 0$ and hence the distribution $\chi_M - \chi_N$ should have distributional support $\{0\}$, which is indeed the case as one easily calculates

$$\chi_M(h) - \chi_N(h) = -\ln\left(\frac{M}{N}\right)\delta(x).$$

By this we see, that there is a suitable freedom in the process of extending a given distribution: Any distribution

$$T = \sum_{\alpha=0}^{m} c_\alpha \partial^\alpha \delta \tag{2.11}$$

could be added to χ_M and the resulting distribution $\chi_M + T$ would still be an extension to the distribution defined by f. Hence it seems that there is an infinite parameter freedom in the choice of an extension. But indeed, this freedom can be reduced to a 1-parameter

[3]The basics of distribution theory are recapitulated in appendix A.

freedom by the observation that f really defines a distribution on

$$\{f \in \mathcal{S} | f(0) = 0\}.$$

The extensions in $\mathcal{S}'(\mathbb{R})$ of this distribution are then just given by the family $(\chi_M)_{M>0}$. To understand this, let $f \in \{f \in \mathcal{S} | f(0) = 0\}$ with $f'(0) \neq 0$. It then follows, that

$$\delta'(f) \neq 0$$

and hence we cannot add a distribution of the form (2.11) with $m \geq 1$ to χ_M to obtain another distributional extension. Consequently, there is a 1-parameter family of extensions in $\mathcal{S}'(\mathbb{R})$ of the distribution defined by f, which are called *renormalizations* of this distribution. We will understand soon, that the renormalization procedure in quantum field theory can be understood along similar lines: Divergences occur, since distributions are evaluated on functions for which they are ill-defined. The renormalization procedure corresponds then to an analysis of adequate distributional extensions and renormalized couplings correspond then to the residual freedom in the choice of distributional extensions analogous to the 1-parameter freedom parametrized by $M > 0$ in the above example.

Extensions of distributions: In the last paragraph we have understood how an elementary distribution can be extended from $\mathcal{S}'(\mathbb{R} \setminus \{0\})$ to $\mathcal{S}'(\mathbb{R})$. In this paragraph we now want to review concisely the general theory of distributional extensions.

Therefore let $U \subset V \subset \mathbb{R}^n$ denote two open sets and let $\chi \in \mathcal{D}(U)$ be a distribution on U. Then a distribution $\tilde{\chi} \in \mathcal{D}(V)$ is called an *extension* of χ, if

$$\tilde{\chi}(h) = \chi(h)$$

holds for all $h \in \mathcal{D}(U)$ (cf. appendix A). Of special importance in quantum field theory are thereby so called *point extensions*, which are distributions in $\mathcal{D}(\mathbb{R}^n)$ which extend distributions in $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$. As the most general distribution in $\mathcal{D}(\mathbb{R}^n)$ with distributional support $\{0\}$ can be written as

$$T = \sum_{|\alpha| \leq m} c^\alpha \partial_\alpha \delta \tag{2.12}$$

with α being a n-dimensional multi-index (cp. Thm. 2.3.4 of [98]), one expects – in analogy to the simple example of the last paragraph – that two point extensions of a given distribution should differ by a distribution of the form (2.12). Hence it seems that there is an infinite parameter freedom in the choice of point extensions. But this freedom can be further restricted by considering the so called *scaling degree*. Therefore define first a scaling map on test functions by

$$\mathbb{R}^+ \times \mathcal{D}(\mathbb{R}^n), (\lambda, h) \mapsto \lambda.h := \lambda^{-n} h(\lambda^{-1} \ \cdot \ )$$

whose induced action on distributions reads as:

$$\mathbb{R}^+ \times \mathcal{D}'(\mathbb{R}^n), (\lambda, \chi) \mapsto \lambda.\chi := \chi[\lambda. \ \cdot \ ].$$

For a distribution $\chi_f(h) := \int_{\mathbb{R}^n} d^n x f(x) h(x)$ induced by a smooth function $f \in C^\infty(\mathbb{R}^n)$ this action is for example given by:

$$\left(\lambda.\chi_f\right)(h) = \int_{\mathbb{R}^n} d^n x f(\lambda x) h(x).$$

We then define the *scaling degree* of a distribution χ as

$$\mathrm{sd}\left(\chi\right) := \inf \left\{ \omega \in \mathbb{R} \middle| \lim_{\lambda \to 0^+} \lambda^\omega \cdot (\lambda.\chi) = 0 \right\}.$$

The relevance of the scaling degree can be understood if one considers a distribution in $\mathcal{D}(\mathbb{R})$ which is induced by the function $|x|^{-n}$. It is then easy to show, that the scaling degree of $|x|^{-n}$ is given by n and hence the scaling degree measures in some sense how fast a distribution in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ "diverges" in the vicinity of 0. Hence one could expect, that the scaling degree controls the extendability of distributions which is indeed the case as stated by the following theorem:

Theorem 1 (cf. [98, 75, 4])
Let $\chi \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. Then:

1. *If $\mathrm{sd}(\chi) < n$, then $\mathrm{sd}(\chi)$ has a unique extension $\hat{\chi} \in \mathcal{D}'(\mathbb{R}^n)$ with same scaling degree.*

2. *If $n \leq \mathrm{sd}(\chi) < \infty$, then there exist distributional extensions $\hat{\chi} \in \mathcal{D}'(\mathbb{R}^n)$ with the same scaling degree. Moreover, two such extensions differ by a distribution of the form*

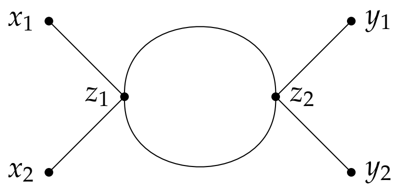
$$T = \sum_{|\alpha| \leq \mathrm{sd}(\chi) - n} c_\alpha \partial^\alpha \delta. \tag{2.13}$$

which is supported at the origin.

By this we have reviewed the essentials regarding the theory of distributional extensions, which are necessary to understand its role in quantum field theory. More information on this important topic can be found in [98].

A distributional interpretation of the divergences in QFT: We now want to understand, how the divergences occuring in relativistic QFT can be interpreted and cured in the context of distribution theory. Thereby we will focus in this section just on the concepts and ideas and will adopt a qualitative reasoning. Our main source for this is [75], while more technical and complete treatments are for example given by [159, 148].

We therefore consider the diagram



whose expression in position space is proportional to

$$\propto \int_{\mathbb{R}^4} d^4z_1 \int_{\mathbb{R}^4} d^4z_2 D_F(x_1 - z_1) D_F(x_2 - z_1) \left[D_F(z_1 - z_2)\right]^2 D_F(z_2 - y_1) D_F(z_2 - y_2)$$

and which we will rewrite as

$$\int_{\mathbb{R}^4} d^4z \int_{\mathbb{R}^4} d^4u D_F(x_1 - z) D_F(x_2 - z) \left[D_F(u)\right]^2 D_F(z_1 - u - y_1) D_F(z_1 - u - y_2).$$

This expression is not well-defined from a distributional perspective, since it contains the square of the Feynman propagator. To see that this is really a problem, recall that we have the asymptotics (cf. [75])

$$D_F(u) \sim \frac{1}{u^2} \tag{2.14}$$

as $u \to 0$ and hence the integral

$$\int_{\mathbb{R}^4} d^4x D_F(u)^2 h(u) \tag{2.15}$$

diverges for any test-function $h \in \mathcal{D}(\mathbb{R}^4)$ with $0 \in \operatorname{supp}(h)$. This tells us, that $D_F(u)^2$ does not define a distribution in $\mathcal{D}'(\mathbb{R}^4)$. But nevertheless $D_F(u)^2$ defines still a distribution in $\mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$, since the integral (2.15) converges for any function $h \in \mathcal{D}(\mathbb{R}^4)$ with $0 \notin \operatorname{supp}(h)$. This motivates a redefinition of distributional products in QFT: Instead of taking the naive ill-defined product, we take the product of two distributions on the maximal domain where it is well-defined and analyze then its distributional extensions. This will be done by application of theorem 1 and the occuring renormalization constants which parametrize the non-uniqueness of the distributional extension correspond then to the renormalized couplings of the theory. To make this explicit, we first recall (cf. [75]) that the scaling degree of D_F^2 is explicitely given by

$$\operatorname{sd}(D_F^2) = 4$$

which is not surprising due to the asymptotic behaviour (2.14). Hence, by theorem 1 and especially by equation (2.13) it then follows, that distributional point extensions of D_F^2 exist and that two such extensions differ by a distribution of the form $c\delta$. By this argumentation, the expression for the above graph gets finite and a 1-parameter renormalization freedom appears. The parameter which parametrizes this renormalization freedom corresponds then to a renormalized coupling and has to be fixed by experiment.

2.4. Renormalization and physics at different scales

In the past sections of this chapter we have looked from different perspectives on the concept of infinite renormalization. In this section we want to explain, why infinite renormalization can be understood as an instance of a multiscale phenomenon.

Infinite renormalization in QFT as a multiscale problem: When we have reviewed the formalism of infinite renormalization as it appears in relativistic quantum field theory in section 2.1, we have sketched a more mathematically footed intuition behind this procedure by saying that infinite renormalization corresponds to non-trivial approximations of "real" systems by model systems. In this paragraph we want to give a more physical perspective by which we will understand, why infinite renormalization can be understood as an instance of a multiscale phenomenon.

As a starting point we will take the observation, that currently there exists no single *theory of everything*: Any known physical theory, no matter if classical or quantum, has its specific domain of validity and in particular some energy or momentum scale Λ up to which it can be trusted. Especially, the modern viewpoint on QFT is (cf. e.g. [182]) that any quantum field theory should be understood as a low energy effective theory, valid up to a momentum scale Λ. By this, the cut-off Λ as introduced in section (2.1), has a physical meaning: It should be understood as the energy scale up to which one can trust the physical theory under consideration and above which the theory should be matched to a more complete theory.

But how can the concepts of renormalizability and non-renormalizability then be understood in this context? Therefore observe that in the context of a renormalizable theory, the renormalization condition 2.4 implies that the energy scale Λ can be entirely absorbed into a redefinition of the microscopic couplings. This means, that a variation of the energy scale Λ would not affect the form of the action, for example by addition of extra terms and by the introduction of new couplings, but would just redefine the values of the existing couplings. In this sense, for a renormalizable theory, the value of the energy scale Λ up to which the theory can be trusted is irrelevant for the theory itself, since it enters the theory just in terms of numerical redefinition of the couplings, which should be determined by measurement anyhow. By this argumentation, the cut-off can also be removed: Since it is irrelevant for the theory, we can safely send it to ∞ without modifying the experimental predictions of the theory. Nevertheless, this does not mean, that the theory has to be indeed valid up to arbitrary high momentum scales. For example, QED is renormalizable but nevertheless it gets embedded into the electroweak theory at higher energy scales. Renormalizability hence does not forbid the modification of the theory at high energy scales, but just says that the theory could be *in principle* valid up to arbitrary high energy scales. Non-renormalizability on the other hand means in this context, that the cut-off Λ should enter the theory as an extra parameter which determines the length scale up to which the theory should be trusted and above which new physics is expected to occur, which yields then a renormalizable *UV-completion* of the theory. In this sense non-renormalizable theories should be regarded as low-energy effective theories and are perfectly valid as such. If we calculate for example (cf. [182]) the four-Fermion amplitude in the non-renormalizable Fermi theory of the weak interaction, we obtain that the cross section has the behaviour

$$\mathcal{M} \sim G + G^2\Lambda^2$$

for G being the Fermi coupling. By this we see, that the "error term" $G^2\Lambda^2$ is small, if $G^2\Lambda^2 \ll 1$ but gets relevant at scales where Λ reaches $G^{-\frac{1}{2}}$. By this argumentation one should understand, why the procedure of infinite renormalization as depicted in section 2.1 could be understood as a multiscale problem since it indeed mediates between different scales and formalizes the statement, to which extent quantum field theories can be understood as effective low energy theories.

On renormalization groups: Another instance of the multiscale aspect hidden in renormalization is formalized by so called renormalization groups, which formalize the scale-dependence of quantum field theories in a more quantitative way. In the context of perturbative quantum field theory, a starting point for the introduction of renormalization groups is the aforementioned observation that a variation of the cut-off corresponds to a numerical redefinition of the couplings. Renormalization groups formalize then, how those couplings vary as the cut-off or the momentum scale varies. This change of parameters along a redefinition of the theory scale is then called a renormalization group flow. By this it becomes apparent, that a quantum field theory can change to some extent along the renormalization group flow: Some couplings – and the corresponding terms in the action – could become more and more relevant while other become less relevant as the momentum scale increases or decreases. Consequently, the renormalization group flow formalizes the scale-dependence of the theory. Since renormalization groups are not used in this thesis, we refer the interested reader to the literature (cf. [144]). Nevertheless, a qualitative understanding of this concept will be helpful to understand the motivation behind chapters 5 and 6.

3. Bose-Einstein condensation by QFT-inspired methods

In the last chapter we have reviewed the technique of infinite renormalization from different perspectives. In particular, we have seen in section 2.2 that the concept of infinite renormalization does not only make its appearance in relativistic quantum field theory, but is also of relevance in other branches of physics as quantum mechanics or electrostatics. In this section we will present another application of the technique of infinite renormalization, namely in the context of quantum statistical physics: We will reformulate the thermodynamic limit of an ideal Bose gas as an infinite renormalization procedure. Thereby some interesting aspects of this physical system will be revealed, which remain hidden in the conventional treatment and which could possibly serve as a starting point for a generalization to weakly interacting gases.

More concretely, we will analyse the thermodynamic limit of this system by the use of asymptotic expansions of the grand canonical potential, which are derived by QFT-inspired ζ-regularization techniques. Herewith we will then show, that qualitative aspects of those expansions are directly interwoven with the phase structure of the system: In the non-condensation phase the expansion has a form that resembles usual heat kernel expansions. In contrast, the expansion exhibits a singularity of infinite order above a critical density and a renormalization of the chemical potential is needed to ensure well-defined thermodynamic observables in this case. Moreover, this renormalization procedure causes the system to form a Bose-Einstein condensate. Characteristic quantities of the system, like the critical density or the internal energy, are in both cases entirely encoded in the coefficients of the asymptotic expansion.

The outline of this chapter, which is based on the author's publication [175], is as follows: In section 3.1 we will present the key ideas and the motivation of our approach. In section 3.2 we will then review the basics of the confined ideal Bose gas. In section 3.3 we will calculate the asymptotic expansions of the grand potential for different ranges of the chemical potential μ by utilization of ζ-regularization methods. In section 3.4 we will analyse the thermodynamic limit by utilization of those expansions. Thereby we will understand, that an infinite renormalization of the chemical potential is needed in the condensation phase. In section 3.4 we will then conclude this chapter by giving a short summary as well as an outlook to further research.

3.1. Introduction

It is not often the case that the frontier of contemporary research is present in our daily life. Whenever one cooks water, enjoys snow or uses a magnet, one could ask, how the rich qualitative collective properties of complex systems emerge from the relatively simple properties of its constituents and especially how different macroscopic phases of matter are connected to its microscopic description. But due to the vast complexity of macroscopic systems, the precise answer to those question is unknown in most cases. A good illustration for the large gap between the understanding of microscopic and the determination of macroscopic phenomena is the phase structure of Bose gases, since, despite of their elementary microscopic description, a rigorous understanding of their phase structure is still lacking in the most realistic scenarios.

On Bose-Einstein condensation: Bose gases usually exhibit two phases. One of them is distinguished by a macroscopic occupation of the ground state, what goes under the name of Bose-Einstein condensation (BEC). The corresponding phase transition occurs at low temperatures or high densities and has been experimentally realized in different physical systems (cp. [9, 48]), which is intriguing and fascinating by its own, since it constitutes a macroscopic quantum phenomenon where the quantum concept of indistinguishability becomes apparent in the macroscopic world. The occurence of BEC was predicted by Bose ([33]) and Einstein ([66, 67, 68]) almost 100 years ago. They analyzed a non-interacting case and argued that the system exhibits a macroscopically occupied ground state below a critical temperature. Nevertheless, a rigorous demonstration of the occurrence of a phase transition in a realistic, interacting scenario was lacking for over 70 years. This changed drastically in 2002, when Lieb and Seiringer proved the occurence of Bose-Einstein condensation in the thermodynamic limit of a dilute Bose gas ([124, 123]), which marked a huge progress in the understanding of the phase structure of continuous Bose systems. However, a rigorous understanding in other realistic regimes or for general interactions has still not been achieved. For a good review on this issues see also [170].

The difficult situation in the interacting case also continuously stimulated research in the much tamer non-interacting case. The hope could be that a new perspective on the non-interacting case also leads to valuable insights concerning the interacting case. Besides the textbook treatment, which usually utilizes integral approximation techniques (cf. [34]), notable other approaches are the loop gas technique (cf. [136, 23]) as well as a recent method which uses insights from algebraic quantum field theory (cf. [39]). Another method for the investigation of the phase structure of ideal bose gases is the method of asymptotic expansions developed in [110, 112, 111]. In those articles ζ-regularization techniques are used for the investigation of the small-β limit of trapped Bose gases. In the present document we will complement those results by an analysis of the thermodynamic or open-trap limit using similar techniques.

Motivation for our approach: Our motivation for the choice of this technique relies on the fact that in other situations the form of such expansions has been proven to be very robust with respect to smooth perturbations of the system. If one considers for example heat kernel expansions on manifolds, the qualitative form of the expansion is insensitive to the geometry of the manifold and its coefficients are calculable entirely in terms of geometric invariants, which are both non-trivial statements (cf. [169]). Hence, a hope could be that an asymptotic expansion of a characteristic thermodynamic quantity like the grand potential exhibits a similar robustness under perturbations of the system by smooth, repulsive 2-body interactions and that the occurrence of condensation could be traced back to simple qualitative properties of this expansion. Therefore, a first step is the investigation of the non-interacting case, which is achieved in this chapter. For the derivation of the asymptotics of the grand potential under the open-trap limit we utilize ζ-regularization techniques, which are mainly used in finite-temperature relativistic quantum field theory (cf. e.g. [70, 40, 71]) and are rarely applied to problems in non-relativistic quantum statistical mechanics. In particular, we will use the Mellin-Barnes integral representation and the spectral ζ-function of the 1-particle Hamiltonian to extract information on the behavior of the grand canonical partition function under the thermodynamic limit. At this stage we would like to phrase the point, that the utility of spectral ζ-functions in the current situation relies on their capability to translate qualitative properties of the eigenvalue distribution of an operator into precise analytic properties.

Informal presentation of the main ideas: We now want to present the key ideas of this chapter in a non-technical manner. The starting point for our investigation is a grand canonical treatment of the ideal Bose gas in v dimensions confined by an harmonic trap with oscillator constant κ. The limit $\kappa \to 0$ corresponds then to the open trap limit and $\Lambda = \kappa^{-1}$ should be understood as an IR-cutoff. Moreover, this limit should be understood as a thermodynamic limit in this situation, since the expectation value for the particle number diverges as $\kappa \to \infty$. If one considers then the grand canonical potential $\Omega^{(v)}(\kappa; \beta, \mu)$, where β denotes the inverse temperature and μ the chemical potential, one observes that this object diverges in the limit $\kappa \to 0$

$$\lim_{\kappa \to \infty} \Omega^{(v)}(\kappa; \beta, \mu) = \infty.$$

Moreover, the usual hermodynamic observables, calculated in terms of derivatives of the grand potential, diverge, too. This is not surprising, since extensive quantities should diverge in the thermodynamic limit and hence one shas to consider densities as proper observables in the current context. To densitize a given thermodynamic observable – as e.g. the average particle number $\langle N \rangle$ – one has to multiply it with κ^v. This follows, since κ^{-1} has dimension of length and hence κ^{-v} can be understood as the characteristic volume of the harmonic trap[1]. For example, the average density in the thermodynamic limit is given by $\lim_{\kappa \to 0} \kappa^v \langle N \rangle$.

[1]For this viewpoint, see also [136, 23].

We then observe that for $\mu \leq 0$ no pathologies appear: By utilization of ζ-regularization methods one derives

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + \mathcal{O}(\kappa). \tag{3.1}$$

as an asymptotic expansion for the grand potential. This expansion then entirely describes the behaviour of the system under the thermodynamic limit in terms of finitely many constants $\{a_{-k}^{(v)}(\beta,\mu) | k = 0, ..., v\}$. This follows, since the terms proportional to κ^{-k} can be understood as dominant contributions under the limit $\kappa \to 0$, while the terms in $\mathcal{O}(\kappa)$ are negligible in this limit. Consequently, those constants should determine the values observables in the thermodynamic limit, too. And indeed, one calculates for example

$$\kappa^v \langle N \rangle = \frac{\kappa^v}{\beta} \frac{\partial}{\partial \mu} \Omega^{(v)}(\kappa;\beta,\mu) = \frac{1}{\beta} \frac{\partial a_v^{(v)}(\beta,\mu)}{\partial \mu}$$

as an expression for the average particle density.

If one tries then to analyse the case of positive chemical potential by the same techniques, one obtains that the asymptotic expansion of the grand potential has suddenly infinitely many singular terms and exhibits additionally some kind of logarithmic singularity. I.e., as we will see later, it is of the form (cf. equation (3.24))

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{\infty} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + f(\kappa) + \mathcal{O}(\kappa), \tag{3.2}$$

where $f(\kappa)$ is the logarithmic singularity, which is not written out at the present stage to increase readability. A priori this is unacceptable, since it implies also divergent quantities for densitized thermodynamic observables. For example, one calculates for the average particle density in this case:

$$\kappa^v \langle N \rangle = \frac{\kappa^v}{\beta} \frac{\partial}{\partial \mu} \Omega^{(v)}(\kappa;\beta,\mu) \to \infty (\kappa \to 0).$$

The crucial observation is then, that this situation is very similar to the situation in relativistic quantum field theory as depicted in section 2.1, if one regards $\Lambda = \kappa^{-1}$ as a (IR)-regulator. This motivates then the idea, that an infinite renormalization of the chemical potential along the lines of section 2.1 should cure this divergent behaviour and we will see in section 3.4 that this is indeed the case: If one chooses

$$\mu_{\bar{\rho}}(\kappa) = E_0^{(v)}(\kappa) - \left[\bar{\rho} - \rho_c^{(v)}(\beta)\right]^{-1} \kappa^v + \mathcal{O}(\kappa^{v+1})$$

as a κ-dependent chemical potential $\mu(\kappa)$, then all singularities of order higher than κ^{-v} in (3.2) are cancelled. In this expression, $\bar{\rho}$ corresponds to the experimentally accessible, macroscopic average particle density and should be considered as the *renormalized* chemical potential. Moreover $\rho_c^{(v)}(\beta)$ denotes the critical density in this case. Consequently, all densitized thermodynamic observables are then rendered finite. If one analyses then finally the ground state occupation, one observes, that this renormalization procedure causes the system to exhibit Bose-Einstein condensation.

A mathematical perspective: From a mathematical perspective, the asymptotic expansion (3.1) can be understood as a statement about eigenvalue asymptotics: The grand canonical potential is initially given by a sum

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{n_1,\dots,n_v=0}^{\infty}\sum_{N=1}^{\infty}\frac{z^N}{N}\exp\left[-\beta E^{(N)}_{n_1,\dots,n_v}(\kappa)\right], \tag{3.3}$$

where $\{E^{(v)}_{N;n_1,\dots,n_v}(\kappa)|N,n_i \in \mathbb{N}_0\}$ is the set of eigenvalues of the full ∞-particle Hamiltonian. The divergence of the grand potential under the thermodynamic limit then is caused by the fact, that the energy gap between two energy eigenvalues goes to zero as $\kappa \to 0$. The asymptotic expansion (3.1) then makes it precise, how exactly the asymptotic behaviour of the eigenvalues translates into an asymptotic behaviour of the sum (3.3). Thereby, the transition from the sum (3.3) – which is actually a power series in z – to the asymptotic expansion (3.1) – which is a Laurent series in κ – is then basically an analytically non-trivial reordering of the sum (3.3) along powers of κ. This reordering is accomplished by ζ-regularization methods (see section 3.3) and constitutes, from the perspective of analytic number theory, basically an inverse Mellin transform which relates a Dirichlet series to a power series. From a physical perspective, this transition then corresponds to the extraction of those features of the system, which become relevant in the thermodynamic limit and especially translates the *microscopic* properties of the eigenvalue asymptotics into the *macroscopically relevant* properties of the grand potential with the latter being entirely encoded in the coefficients of the asymptotic expansion. This is a wonderful example how the beauty of mathematics enters physics: The macroscopic features of the model, which are encoded in the collective behaviour of the summands of (3.1), can be extracted by a standard technique from analytic number theory and are then concisely encoded in the coefficients of an asymptotic expansion.

3.2. The confined ideal Bose gas

After having presented the main ideas and the motivation behind our treatment, we will now introduce the technical foundations for our analysis. Especially, we will now review the necessary prerequisites regarding the quantum statistics of the isotropic harmonic oscillator potential. We therefore consider an ideal Bose gas in $v \geq 1$ dimensions confined to an harmonic oscillator trap. The 1-particle Hamiltonian is given by

$$T_\kappa := -\Delta + \kappa^2|\vec{x}|^2$$

where $\kappa > 0$ is the considered oscillator constant and can be understood as an infrared cutoff modeling the finite size of the trap. The eigenvalues of this operator are then given by (cf. [167])

$$E^{(v)}_{n_1,\ldots,n_v}(\kappa) = 2\kappa \left[\sum_{i=1}^{v} n_i + \frac{v}{2} \right]$$

for $n_1, \ldots, n_v \in \mathbb{N}$. The full many-body Hamiltonian is then given by the standard second quantization (cf. [11, 34])

$$H_\kappa := d\Gamma(T_\kappa)$$

on the bosonic Fock space. The grand canonical potential of the harmonically trapped Bose gas is then given by (cf. [34])

$$\Omega^{(v)}(\kappa; \beta, \mu) = \ln \left[\operatorname{tr} \left(e^{-\beta(H_\kappa - \mu N)} \right) \right], \tag{3.4}$$

where N is the bosonic number operator, β is the inverse temperature and μ is the chemical potential. The expression (3.4) is well-defined for the parameter ranges $\beta > 0$, $\kappa > 0$ and $-\infty < \mu < E^{(v)}_0(\kappa)$, as it can be shown by application of Prop. 5.2.27 of [34]. Here $E^{(v)}_0(\kappa) = \kappa v$ denotes the lowest energy eigenvalue of the 1-body Hamiltonian T_κ. As the starting point of our investigation we will then use a sum representation of (3.4) which is obtained by expanding the logarithm and utilizing the trace formula (cf. Thm. 5.11 of [11])

$$\operatorname{tr}_{\mathcal{F}(\mathcal{H})} \left(\Gamma(R) \right) = \prod_{\lambda \in \sigma(R)} \frac{1}{1 - \lambda}$$

for second quantized operators on Fock space:

$$\Omega^{(v)}(\kappa; \beta, \mu) = \sum_{n_1,\ldots,n_v=0}^{\infty} \sum_{N=1}^{\infty} \frac{z^N}{N} \exp\left[-\beta N E^{(v)}_{n_1,\ldots,n_v}(\kappa) \right] \tag{3.5}$$

Here we have defined the rapidity $z := e^{\beta\mu}$, that will be used in the sequel as an equivalent replacement for the chemical potential μ. Please note, that (3.5) constitutes a power series in z.

3.3. Asymptotic expansions by ζ-regularization methods

We now want to expand the infinite sum (3.5) in an asymptotic expansion in the trap parameter κ. This will be realized by utilization of ζ-regularization methods. The starting point is the Mellin-Barnes integral representation (cf. [111])

$$e^{-a} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} a^{-s} \Gamma(s) ds \tag{3.6}$$

which is valid for $|\arg(a)| < \frac{\pi}{2} - \delta$ with $\delta \in (0, \frac{\pi}{2}]$ and $\sigma > 0$. But before we apply the integral formula (3.6) on the sum representation (3.5), we want to investigate a simpler situation in order to make the procedure clear (cf. section 6 of [111]).

A simple example: Consider the more elementary sum

$$S(\kappa) = \sum_{l=1}^{\infty} e^{-\kappa l} \tag{3.7}$$

which could be understood as the partition sum of a simple quantum mechanical system with energy eigenvalues $E_n(\kappa) = \kappa n$. By application of (3.6) we then can write (3.7) as:

$$S(\kappa) = \sum_{l=1}^{\infty} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (\kappa l)^{-s} \, \Gamma(s) ds \tag{3.8}$$

Now recall that the Riemann ζ-function is represented for $\mathrm{Re}(s) > 1$ by the convergent sum (cf. [55])

$$\zeta_R(s) = \sum_{l=1}^{\infty} l^{-s}.$$

Hence, by demanding $\sigma > 1$, we are allowed to interchange the sum and the integral in (3.8) and obtain:

$$S(\kappa) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \kappa^{-s} \Gamma(s) \zeta_R(s) ds \tag{3.9}$$

As explained by [111], the strategy for finding the small-κ behavior is then to shift the integration contour to the left. By the residue theorem, crossing the singularities of the integrand gives then polynomial contributions in κ^{-1}. In the case of (3.9), the rightmost pole of the integrand is given by the pole of $\zeta_R(\cdot)$ at $s = 1$ and the other poles can be found at $s = -2n$ with $n \in \mathbb{N}$. We therefore shift the integral contour to $\tilde{\sigma} \in (-1, 0)$ and obtain

$$S(\kappa) = \kappa^{-1} - \frac{1}{2} + S_{res}(\kappa), \tag{3.10}$$

where we used $\mathrm{Res}_{s=1}(\zeta_R(s)) = \mathrm{Res}_{s=0}(\Gamma(s)) = 1$ and where the residual term $S_{res}(\kappa)$ is given by:

$$S_{res}(\kappa) = \frac{1}{2\pi i} \int_{\gamma(\tilde{\sigma})} \kappa^{-s} \Gamma(s) \zeta_R(s) ds$$

Here $\gamma(\tilde{\sigma})$ denotes a path in the complex plane which goes from $\sigma - i\infty$ to $\sigma + i\infty$, but intersects the real line at $\tilde{\sigma} \in (-1, 0)$. This term gives contributions in $\mathcal{O}(\kappa)$ and is hence of no relevance for us, since we are only interested in the small-κ behavior. Thus, we neglect the concrete form of this contribution and obtain the following asymptotic expansion:

$$S(\kappa) = \kappa^{-1} - \frac{1}{2} + \mathcal{O}(\kappa)$$

We now use the same strategy to derive expansions of the form (3.10) for the grand potential (3.5). Thereby we will see that we will need two different strategies for the cases $\mu \leq 0$ and

$\mu > 0$.

The case of negative chemical potential: We will now derive the small-κ asymptotics of the grand potential (3.5) in the case $\mu < 0$ or equivalently in the case $|z| < 1$. Therefore, we apply the Mellin-Barnes integral representation (3.6) on the exponential in (3.5) and obtain:

$$\Omega^{(v)}(\kappa;\beta,\mu) = \frac{1}{2\pi i}\sum_{n_1,...,n_v=0}^{\infty}\sum_{N=1}^{\infty}\frac{z^N}{N}\int_{\sigma-i\infty}^{\sigma+i\infty}\beta^{-s}N^{-s}E^{(v)}_{n_1,...,n_v}(\kappa)^{-s}\Gamma(s)ds \tag{3.11}$$

In contrast to the situation depicted before, the Riemann ζ-function is not sufficient for the analysis of this expression. Instead we need the Barnes ζ-function (see section 2.2 of [111]), which is a multidimensional generalization of the Riemann ζ-function and whose convergent sum representation is for $v \in \mathbb{N}$, $s > v$, $c > 0$ and $r > 0$ given by

$$\zeta_B^{(v)}(s,c|r) = \sum_{n_1,...,n_v\in\mathbb{N}_0}\left[c + r\left(n_1 + ... + n_v\right)\right]^{-s}$$

for $c \neq 0$ while, if $c = 0$, the sum ranges just over $(n_1,...,n_v) \neq (0,...,0)$. In addition, we need the polylogarithm, which is for $|z| < 1$ and any complex order $r \in \mathbb{C}$ given by the absolute convergent sum ([55])

$$\mathrm{Li}_r(z) = \sum_{N=1}^{\infty}\frac{z^N}{N^r}.$$

If we demand $\sigma > v$, we are then allowed to interchange the sums and the integral in (3.11) and obtain the following expression for the grand potential:

$$\Omega^{(v)}(\kappa;\beta,\mu) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\kappa^{-s}\beta^{-s}\mathrm{Li}_{s+1}(z)\zeta_B^{(v)}(s,v|2)\Gamma(s)ds. \tag{3.12}$$

Now, the location and residues of the poles are known for all functions appearing in the integrand. In particular, $\mathrm{Li}_{s+1}(z)$ has no poles for $|z| < 1$ (cf. [55]) and $\Gamma(s)$ has, as before, simple poles at $-\mathbb{N}_0$. The Barnes ζ-function has poles at $z = 1,...,v$ (cf. [111]) with residues

$$\mathrm{Res}_{s=k}\left(\zeta_B^{(v)}(s,c|r)\right) = \frac{(-1)^{v+k}}{(k-1)!(v-k)!}r^{-v}B^{(v)}_{v-k}(c,r).$$

Further, its value at zero[2] is given by (cp. [111])

$$\zeta_B^{(v)}(0,c|r) = \frac{(-1)^v}{v}r^{-v}B_v^{(v)}(c,r).$$

Here $B_m^{(v)}(c,r)$ denotes generalized Bernoulli polynomials defined as

$$\frac{e^{-xt}}{(1-e^{-rt})^v} = \left(\frac{-1}{r}\right)^v\sum_{n=0}^{\infty}\frac{(-t)^{n-v}}{n!}B_n^{(v)}(x,r).$$

[2] We will see in section 3.4, that only this value of the Barnes ζ-function is of relevance for us.

By shifting the contour to the left we can write in analogy to (3.10) equation (3.12) as

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + S_{res}(\kappa;\beta,\mu), \tag{3.13}$$

where the coefficients $a_k^{(v)}(\beta,\mu)$ are given by

$$a_{-k}^{(v)}(\beta,\mu) = \begin{cases} \beta^{-k}\mathrm{Li}_{k+1}(z)\Gamma(k)\mathrm{Res}_{s=k}\left(\zeta_B^{(v)}(s,v|2)\right) & k \in \{1,...,v\} \\ \mathrm{Li}_1(z)\zeta_B^{(v)}(0,v|2) & k=0 \end{cases} \tag{3.14}$$

and the residual term $S_{res}(\kappa;\beta,\mu)$ is given by

$$S_{res}(\kappa) = \frac{1}{2\pi i}\int_{\gamma(\tilde{\sigma})} \kappa^{-s}\beta^{-s}\mathrm{Li}_{s+1}(z)\zeta_B^{(v)}(s,v|2)\Gamma(s)ds. \tag{3.15}$$

Here $\gamma(\tilde{\sigma})$ denotes a contour that intersects the real axis at $\tilde{\sigma} \in (-1,0)$ and goes from $\sigma - i\infty$ to $\sigma + i\infty$.

Vanishing chemical potential: The case $\mu = 0$ (or equivalently $z = 1$) goes in complete analogy to the case $\mu < 0$. The only difference is that the sum over N in (3.11) reduces now to a Riemann ζ-function. By this we obtain the following integral representation of the grand potential for $\mu = 0$:

$$\Omega^{(v)}(\kappa;\beta,\mu=0) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \kappa^{-s}\beta^{-s}\zeta_R(s+1)\zeta_B^{(v)}(s,v|2)\Gamma(s)ds$$

Here, as before, $\sigma > v$ is required. And also as before, one then shifts the contour to the left and obtains the following small-κ asymptotics:

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + S_{res}(\kappa;\beta,\mu)$$

The coefficients $a_{-k}^{(v)}(\beta,\mu)$ are here given by

$$a_{-k}^{(v)}(\beta,\mu) = \begin{cases} \beta^{-k}\zeta_R(k+1)\Gamma(k)\mathrm{Res}_{s=k}\left(\zeta_B^{(v)}(s,v|2)\right) & k \in \{1,...,v\} \\ \mathrm{Res}_{s=0}\left(\zeta_R(s+1)\Gamma(s)\right)\zeta_B^{(v)}(0,v|2) & k=0 \end{cases} \tag{3.16}$$

and the residual term $S_{res}(\kappa;\beta,\mu)$ is given by

$$S_{res}(\kappa;\beta,\mu) = \frac{1}{2\pi i}\int_{\tilde{\sigma}-i\infty}^{\tilde{\sigma}+i\infty} \kappa^{-s}\beta^{-s}\zeta_R(s+1)\zeta_B^{(v)}(s,v|2)\Gamma(s)ds, \tag{3.17}$$

where $\gamma(\tilde{\sigma})$ denotes again a contour that intersects the real axis at $\tilde{\sigma} \in (-1,0)$ and goes from $\sigma - i\infty$ to $\sigma + i\infty$.

Positive chemical potential: If the chemical potential is postive, or equivalently if $|z| > 1$, the previous strategy does not work. The reason for this is twofold. On the one hand, the allowed parameter range for the chemical potential in (3.4) is $\mu \in (-\infty, E_0^{(v)}(\kappa))$ and hence one has to choose a κ-dependent μ, i.e. a map

$$\mu : \kappa \in (0, \infty) \mapsto \mu(\kappa) \in (0, E_0^{(v)}(\kappa)),$$

in this case. On the other hand, the sum representation of the polylogarithm $\sum_{N=1}^{\infty} z^N N^{-s-1}$ is in general not convergent for $|z| > 1$ and hence one is not allowed to interchange the sums and the integral in (3.11).

To circumvent this problem, we will derive a different sum representation of the grand potential by expanding the exponential that contains the chemical potential. Afterwards, we will then apply the same strategy as before on the remaining spectral functions. This gives a small-κ asymptotics where the coefficients are given as power series in the κ-dependent chemical potential. The motivation for this strategy relies on the observation, that for a positive, κ-dependent chemical potential the thermodynamic limit corresponds to a weak coupling regime, since $\mu(\kappa) \to 0$ as $\kappa \to 0$. When an adequate representation of the κ-dependent chemical potential as a series in κ is given – which will be derived in section 3.4 – this should be reinserted into the expression for the grand potential, yielding again a small-κ asymptotics.

But before turning to this program, we have to perform some preliminary steps. First, the occuring spectral functions will be much more convenient, if we perform a redefinition of the chemical potential and the energy eigenvalues by subtracting the zero point energy. We hence define

$$\tilde{\mu}(\kappa) := \mu(\kappa) - E_0^{(v)}(\kappa) \qquad \text{and} \qquad \tilde{E}^{(v)}_{n_1,\dots,n_v}(\kappa) := E^{(v)}_{n_1,\dots,n_v}(\kappa) - E_0^{(v)}(\kappa).$$

In addition, we set $\tilde{z}(\kappa) = e^{\beta\tilde{\mu}(\kappa)}$. Please observe, that this implies $\tilde{\mu}(\kappa) < 0$ and $\tilde{z}(\kappa) < 1$. Since we are interested in the phenomenon of Bose-Einstein condensation, we further split up the grand potential by separating the ground state contribution $\Omega_0^{(v)}$ from the contribution of the excited states $\Omega_1^{(v)}$:

$$\Omega^{(v)}(\kappa; \beta, \mu(\kappa)) = \Omega_0^{(v)}(\kappa; \beta, \mu(\kappa)) + \Omega_1^{(v)}(\kappa; \beta, \mu(\kappa))$$

$\Omega_0^{(v)}$ and $\Omega_1^{(v)}$ are then explicitely given by

$$\Omega_0^{(v)}(\kappa; \beta, \mu(\kappa)) = \sum_{N=1}^{\infty} N^{-1} \exp\left(\beta N \tilde{\mu}(\kappa)\right)$$

and

$$\Omega_1^{(v)}(\kappa;\beta,\mu(\kappa)) = \sum_{N=1}^{\infty} \sum_{(n_1,\dots,n_v)\neq(0,\dots,0)} N^{-1} \exp\left(-\beta N(\tilde{E}^{(v)}_{n_1,\dots,n_v}(\kappa) - \tilde{\mu}(\kappa))\right). \tag{3.18}$$

Note that in this case the ground state contribution can also written as

$$\Omega_0^{(v)}(\kappa;\beta,\mu(\kappa)) = -\ln\left(1 - \exp\left(\beta\tilde{\mu}(\kappa)\right)\right),$$

since $\tilde{z}(\kappa) < 1$ holds. For the treatment of $\Omega_1^{(v)}$ we then expand

$$\exp\left(\beta N\tilde{\mu}(\kappa)\right) = \sum_{m=0}^{\infty} \frac{1}{m!}\beta^m N^m \tilde{\mu}(\kappa)^m$$

and insert this into (3.18), which gives

$$\Omega_1^{(v)}(\kappa;\beta,\mu) = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} \sum_{(n_1,\dots,n_v)\neq(0,\dots,0)} \frac{1}{m!}\beta^m N^{m-1}\tilde{\mu}(\kappa)^m \exp\left(-\beta N\tilde{E}^{(v)}_{n_1,\dots,n_v}(\kappa)\right). \tag{3.19}$$

We then apply, as before, the Mellin-Barnes integral (3.6) on this expression which gives

$$\Omega_1^{(v)}(\kappa;\beta,\mu) = \sum_{m=0}^{\infty} \frac{\tilde{\mu}(\kappa)^m}{m!} \int_{\sigma_m - i\infty}^{\sigma_m + i\infty} \beta^{m-s}\kappa^{-s}\zeta_R(s+1-m)\zeta_B^{(v)}(s,0|2)\Gamma(s)ds,$$

where $\sigma_m > \max\{v, m\}$ is required, since otherwise we were not allowed to interchange the sums over N and $(n_1, ..., n_v)$ in (3.19) with the occurring integrals.

We then apply the same strategy as before. By shifting the integral contour to $\tilde{\sigma} \in (-1, 0)$ one obtains polynomial contributions in κ^{-1}. As before, the remaining integrals are then collected in $S_{res}(\kappa;\beta,\mu(\kappa))$. By ordering the resulting expression along powers of κ we obtain then the small-κ asymptotics

$$\Omega_1^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{\infty} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu(\kappa)) + S_{res}(\kappa;\beta,\mu(\kappa)),$$

where the coefficients $a_k^{(v)}(\beta,\mu(\kappa))$ are given as power series in $\tilde{\mu}(\kappa)$. Explicitely we obtain for those coefficients:

$$\begin{aligned} a_0^{(v)}(\beta,\mu(\kappa)) = &-\ln(\beta) + \sum_{m=1}^{\infty} \frac{\tilde{\mu}(\kappa)^m}{m!}\beta^m \zeta_R(1-m)\zeta_B^{(v)}(0,0|2) \\ &+ \mathrm{Res}_{s=0}\left(\zeta_R(s+1)\Gamma(s)\right)\zeta_B^{(v)}(0,0|2) \end{aligned} \tag{3.20}$$

and

$$a_{-k}^{(v)}(\beta,\mu(\kappa)) = \sum_{m=0,m\neq k}^{\infty} \frac{\tilde{\mu}(\kappa)^m}{m!}\beta^{m-k}\zeta_R(k+1-m)\mathrm{Res}_{s=k}(\zeta_B^{(v)}(s,0|2))\Gamma(k)$$

$$+\frac{\tilde{\mu}(\kappa)^k}{k!}\Gamma(k)\mathrm{Res}_{s=k}\left(\zeta_R(s+1-k)\zeta_B^{(v)}(s,0|2)\right)$$

for $k = 1, ..., v$ and

$$a_{-k}^{(v)}(\beta,\mu(\kappa)) = \frac{\tilde{\mu}(\kappa)^k}{k!}\Gamma(k)\zeta_B^{(v)}(k,0|2) \tag{3.21}$$

for $k \geq v+1$. All together we have then obtained the small-κ asymptotics

$$\begin{aligned}\Omega^{(v)}(\kappa;\beta,\mu(\kappa)) &= \Omega_0^{(v)}(\kappa;\beta,\mu(\kappa)) + \Omega_1^{(v)}(\kappa;\beta,\mu(\kappa))\\ \Omega_0^{(v)}(\kappa;\beta,\mu(\kappa)) &= -\ln\left(1-\exp\left(\beta\tilde{\mu}(\kappa)\right)\right)\\ \Omega_1^{(v)}(\kappa;\beta,\mu) &= \sum_{k=0}^{\infty}\kappa^{-k}a_{-k}^{(v)}(\beta,\mu(\kappa)) + S_{res}(\kappa;\beta,\mu(\kappa))\end{aligned} \tag{3.22}$$

for the case of a positive, κ-dependent chemical potentials, where the coefficients $a_k^{(v)}(\beta,\mu(\kappa))$ are given by (3.20) – (3.21) and the residual term is given by:

$$S_{res}(\kappa;\beta,\mu(\kappa)) = \sum_{m=0}^{\infty}\frac{\tilde{\mu}(\kappa)^m}{m!}\int_{\gamma(\tilde{\sigma})}\beta^{m-s}\kappa^{-s}\zeta_R(s+1-m)\zeta_B^{(v)}(s,0|2)\Gamma(s)ds \tag{3.23}$$

Here $\gamma(\tilde{\sigma})$ denotes, as before, a contour that intersects the real axis at $\tilde{\sigma} \in (-1,0)$ and goes from $\sigma - i\infty$ to $\sigma + i\infty$.

At the first glimpse, the apparent singularity of infinite order in the asymptotic expansion (3.22) seems to be unphysical, since it suggests that also all observables should exhibit a singularity of this type. The resolution of this problem relies on a good choice of the κ-dependent chemical potential $\mu(\kappa) \in (0, E_0^{(v)}(\kappa))$. This will be discussed in section 3.4 and onwards.

Summary: We want to give a short, qualitative summary of the asymptotic expansions derived in this section. In the cases $\mu < 0$ and $\mu = 0$, the asymptotic expansion has the form

$$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v}\kappa^{-k}a_{-k}^{(v)}(\beta,\mu) + S_{res}(\kappa;\beta,\mu),$$

while in the case $\mu > 0$ one has to consider a κ-dependent chemical potential $\kappa \in (0,\infty) \mapsto \mu(\kappa) \in (0, E_0^{(v)}(\kappa))$ and the asymptotic expansion exhibits a singularity of infinite order:

$$\Omega^{(v)}(\kappa;\beta,\mu(\kappa)) = -\ln\left(1-\tilde{z}(\kappa)\right) + \sum_{k=0}^{\infty}\kappa^{-k}a_{-k}^{(v)}(\beta,\mu(\kappa)) + S_{res}(\kappa;\beta,\mu(\kappa)) \tag{3.24}$$

The coefficients are explicitely given by (3.14) for the case $\mu < 0$, by (3.16) for the case $\mu = 0$ and by (3.20) – (3.21) for the case of positive, κ-dependent μ. The residual terms are explicitely given by (3.15), (3.17) and (3.23).

3.4. Thermodynamic limit and renormalization of the chemical potential

We will now utilize the small-κ asymptotics of the grand potential as derived in the last section for the analysis of the phase structure of an ideal Bose gas in the thermodynamic limit $\kappa \to 0$. As said before, the declaration of the open trap limit $\kappa \to 0$ as the thermodynamic limit is adequate, since the average particle number $\langle N \rangle$ diverges as $\kappa \to 0$. We now will first calculate the thermodynamic quantities in the non-condensation phase, before we will treat the condensation phase, where a renormalization of the chemical potential is needed.

Thermodynamic quantities in the non-condensation phase: We will now investigate the behaviour of some thermodynamic quantities under the limit $\kappa \to 0$ for fixed, negative chemical potential $\mu < 0$ and arbitrary inverse temperature $\beta > 0$. The average particle number and the average energy are given by their standard expressions

$$N^{(v)}(\kappa;\beta,\mu) := \langle N \rangle_{\kappa,\beta,\mu} = \beta^{-1}\frac{\partial}{\partial\mu}\Omega^{(v)}(\kappa;\beta,\mu),$$
$$E^{(v)}(\kappa;\beta,\mu) := \langle H_\kappa \rangle_{\kappa,\beta,\mu} = -\frac{\partial}{\partial\beta}\Omega^{(v)}(\kappa;\beta,\mu).$$

By applying those relations on the small-κ asymptotics (3.13) we obtain small-κ asymptotics for those quantities:

$$N^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} n_{-k}^{(v)}(\beta,\mu)\kappa^{-k} + \mathcal{O}(\kappa), \tag{3.25}$$
$$E^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} e_{-k}^{(v)}(\beta,\mu)\kappa^{-k} + \mathcal{O}(\kappa). \tag{3.26}$$

Here the coefficients $n_k^{(v)}(\beta,\mu)$ and $e_k^{(v)}(\beta,\mu)$ are explicitly given by

$$n_{-k}^{(v)}(\beta,\mu) := \left(\frac{1}{\beta}\frac{\partial a_{-k}^{(v)}(\beta,\mu)}{\partial\mu}\right), \tag{3.27}$$
$$e_{-k}^{(v)}(\beta,\mu) := \left(-\frac{\partial a_{-k}^{(v)}(\beta,\mu)}{\partial\beta}\right). \tag{3.28}$$

By utilizing the identity (cf. [178])

$$\frac{d}{dx}\mathrm{Li}_n(x) = \frac{1}{x}\mathrm{Li}_{n-1}(x)$$

and applying it on the expressions for the coefficients $a_{-k}^{(v)}(\beta,\mu)$ given in (3.14) we see, that the coefficients $n_{-k}^{(v)}(\beta,\mu)$ and $e_{-k}^{(v)}(\beta,\mu)$ of above expansions are all well-defined and non-zero for $k \in \{0,...,v\}$. Especially we obtain that the average particle number (3.25) and the average energy (3.26) exhibit a singularity of order κ^{-v}. This is not surprising, since in the thermodynamic limit the particle number (and hence also other extensive quantities) should diverge. The meaningful quantities in this regime are hence governed by densities. For this we consider the inverse trap parameter κ^{-1} as the characteristic length scale of the problem. This makes it plausible to think of κ^{-v} as the characteristic volume of the harmonic trap (for this viewpoint, see also [136, 23]). This motivates us to define the particle- and the energy density as[3]:

$$\begin{aligned} \rho^{(v)}(\kappa;\beta,\mu) &:= \kappa^v N^{(v)}(\kappa;\beta,\mu) \\ \rho_E^{(v)}(\kappa;\beta,\mu) &:= \kappa^v E^{(v)}(\kappa;\beta,\mu) \end{aligned}$$

We see then, that under the thermodynamic limit $\kappa \to 0$ the expressions for the average density and the average energy are directly given by the coefficients $n_{-v}^{(v)}(\beta,\mu)$ and $e_{-v}^{(v)}(\beta,\mu)$ in (3.27) and (3.28). If one inserts the expressions for the coefficients $a_{-v}^{(v)}$ one then obtains:

$$\begin{aligned} \rho^{(v)}(\beta,\mu) &:= \lim_{\kappa\to\infty} \rho^{(v)}(\kappa;\beta,\mu) = 2^{-v}\beta^{-v}\mathrm{Li}_v(z). \\ \rho_E^{(v)}(\beta,\mu) &:= \lim_{\kappa\to\infty} \rho_E^{(v)}(\kappa;\beta,\mu) = v2^{-v}\beta^{-v-1}\mathrm{Li}_{v+1}(z). \end{aligned}$$

Finally we want to show, that no condensation occurs for $\mu \leq 0$. Therefore consider the average ground-state occupation density given by (cp. [112]):

$$\rho_0^{(v)}(\kappa;\beta,\mu) = \frac{\kappa^v}{\beta}\frac{\partial}{\partial\mu}\Omega_0^{(v)}(\kappa;\beta,\mu) = \kappa^v \left(1 - \exp[-\beta(E_0^{(v)}(\kappa) - \mu)]\right)$$

Hence, $\rho_0^{(v)}(\kappa;\beta,\mu) \to 0$ as $\kappa \to 0$ for $\mu \leq 0$.

The critical density: In the last section we have analysed the case of $\mu \leq 0$. Since the function

$$\mu \in (-\infty, 0] \mapsto \rho^{(v)}(\beta,\mu)$$

is strictly monotonically increasing we can easily calculate the maximum density $\rho_c^{(v)}(\beta)$ that can be attained in this phase:

$$\rho_c^{(v)}(\beta) := \sup_{\mu<0} \rho^{(v)}(\beta,\mu) = \lim_{\mu\to 0} \rho^{(v)}(\beta,\mu).$$

This maximum density will be called the critical density. By recalling $\mathrm{Li}_v(1) = \zeta_R(v)$ (cf. [178]) we then obtain:

$$\rho_c^{(v)}(\beta) = 2^{-v}\beta^{-v}\zeta_R(v).$$

[3] A more physical approach would have been to consider the quotient $\tilde{\rho}_E^{(v)}(\kappa;\beta,\mu) = \langle N\rangle^{-1}\langle E\rangle$. But since $\langle N\rangle$ diverges as κ^{-v}, this would have been qualitatively the same and would just correspond to a change of the renormalization prescription.

This is finite for $v \geq 2$, infinite for $v = 1$ and is equivalent to the results of [23] if one takes their different conventions into account. Hence one has to choose an adequate positive, κ-dependent chemical potential to obtain higher densities. This causes the system to exhibit condensation as we will see in a moment.

Renormalization of the chemical potential: We consider from now on the case $v \geq 2$. We have seen, that the critical density $\rho_c^{(v)}(\beta)$ is finite in this case and hence one has to use a κ-dependent, positive chemical potential to obtain higher densities $\bar{\rho} > \rho_c^{(v)}(\beta)$. If one recalls the form of the small-κ asymptotics (3.24) it is a priori not clear, that such κ-dependent chemical potentials exist which imply meaningful results for thermodynamic observables in the limit $\kappa \to 0$. Nevertheless, as discussed before, this situation corresponds to a weak coupling regime where the κ-dependent chemical potential satisfies $\mu(\kappa) \in (0, E_0^{(v)}(\kappa))$ with the zero-point energy behaving as $E_0^{(v)}(\kappa) \to 0$ as $\kappa \to 0$. This suggests, that taking the thermodynamic limit corresponds to the consideration of arbitrary small neighborhoods around $\mu = 0$ and hence the asymptotic expansions obtained for $\kappa \to 0$ should not differ so drastically in the two cases $\mu \leq 0$ and $\mu > 0$. We now want to analyse, if there exist such κ-dependent chemical potentials which imply a finite density $\bar{\rho}$ in the limit $\kappa \to 0$ and which regularize the small-κ asymptotics, such that it exhibits only a singularity of finite order. More precisely stated, the question is hence if there exists for any $\bar{\rho} > \rho_c^{(v)}(\beta)$ a κ-dependent chemical potential

$$\mu_{\bar{\rho}} : \kappa \in (-\infty, 0) \mapsto \mu_{\bar{\rho}}(\kappa) \in (0, E_0^{(v)}(\kappa))$$

which satisfies the following conditions, which will be called renormalization conditions in the sequel and resemble the form of the renormalization condition (2.5) from section 2.1:

1. $\lim_{\kappa \to 0} \mu_{\bar{\rho}}(\kappa) = 0$.
2. $\lim_{\kappa \to 0} \rho^{(v)}\left(\beta, \mu_{\bar{\rho}}(\kappa)\right) = \bar{\rho}$.
3. For all $k > v$ it holds, that the coefficient $a_{-k}^{(v)}(\beta, \mu_{\bar{\rho}}(\kappa))$ in (3.24) lies in $\mathcal{O}(\kappa^{k+1})$.

Before we show that there exist such $\mu_{\bar{\rho}}(\kappa)$, we would like to draw a more explicit analogy to the renormalization procedure in quantum field theory as presented in section 2.1. Therefore recall first from there, how observables in quantum field theory are calculated: A theory is specified by a Lagrangian, which contains several microscopic ("bare") parameters, as the mass or the coupling. If one tries naivly to calculate observables by the evaluation of Feynman diagrams, one obtains divergent integrals. To cure this problems one then follows a two-step strategy. First, an UV-regulator, say a cut-off frequency Λ, is introduced to render the observables finite. In a second step, which is called the renormalization, the microscopic parameters are chosen to be regulator-dependent in a way specified by certain renormalization conditions, which ensure, that all occurring divergences in the observables are cancelled. After this procedure, the theory is reparametrized: The parameters of the theory are not given by the microscopic parameters anymore, but by renormalized, physical parameters determined by the remaining degrees of freedom of the renormalization prescription.

Apparently, our situation is quite similar. Therefore one has to view the trap-parameter κ as an IR-regulator and the chemical potential μ as a microscopic ("bare") parameter of the system. If one tries then to calculate observables in the condensation phase ($\mu > 0$) in a naive way, one obtains that all observables diverge as the IR-regulator is removed (i.e. as $\kappa \to 0$). The reason for this is that the grand potential exhibits a singularity of infinite order, as it is apparent by the asymptotic expansion (3.22). Our strategy is then to choose a regulator-dependent (i.e. κ-dependent) chemical potential $\mu(\kappa)$ which cancels all occurring divergences in the observables. As before, the precise form of this regulator-dependence is determined by certain renormalization conditions, namely in our case by above conditions (i) - (iii). After this renormalization procedure, the observables do not depend on a microscopic parameter μ anymore – especially we see, that a fixed, positive μ is not a meaningful parameter in the condensation phase – but on the macroscopic parameter $\bar{\rho}$ which can be understood as a renormalized coupling. In our case, this parameter is defined as the outcome of a density measurement and should be called the renormalized chemical potential. Hence we see that a choice of a κ-dependent μ, as determined by above conditions (i) - (iii), resembles the procedure of infinite renormalization as presented in section 2.1.

Now we are ready to perform this procedure. Therefore we will first show in the next paragraph, that κ-dependent chemical potentials that satisfy the renormalization conditions (i) and (iii) indeed exist. Consequently, they cure the singularity of infinite order in the asymptotic expansion of the grand potential. Afterwards we will show, that those κ-dependent μ satisfy the second renormalization condition, calculate additional thermodynamic observables and show that the system exhibits condensation.

Renormalized asymptotic expansions in the condensation phase: The first renormalization condition is trivially satisfied, since $\mu_{\bar{\rho}}(\kappa) \in (0, E_0^{(v)}(\kappa))$ while the last renormalization condition ensures, that the small-κ asymptotics exhibits only finitely many singularities. We show in this paragraph, that there exists a renormalized chemical potential $\mu_{\bar{\rho}}(\kappa)$, which satisfies the first and the third renormalization condition. The second renormalization condition will then be analysed afterwards.

We therefore guess the form of the κ-dependent chemical potential as:

$$\mu_{\bar{\rho}}(\kappa) = E_0^{(v)}(\kappa) - \left[\bar{\rho} - \rho_c^{(v)}(\beta)\right]^{-1} \kappa^v + \mathcal{O}(\kappa^{v+1}). \tag{3.29}$$

Since $E_0^{(v)}(\kappa) \to 0$ as $\kappa \to 0$, the κ-dependent chemical potential (3.29) trivially satisfies the first renormalization condition. We have further, by recalling (3.21), that $a_{-k}(\beta, \mu(\kappa))$ is for $k > v$ given by

$$a_{-k}(\beta, \mu_{\bar{\rho}}(\kappa)) = k^{-1} \left(\mu_{\bar{\rho}}(\kappa) - E_0^{(v)}(\kappa)\right)^k \zeta_B^{(v)}(k, 0|2)$$

and hence lies in $\mathcal{O}(\kappa^{vk})$. Consequently, the first and the third renormalization condition are satisfied. By this we obtain that the form of the asymptotic expansion of the grand potential attains the following form in the condensation phase, if a κ-dependent chemical potential (3.29) is inserted:

$$\Omega^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \Omega_0^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) + \Omega_1^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)), \tag{3.30}$$

$$\Omega_0^{(v)}(\kappa;\beta,\mu_{\rho}(\kappa)) = -\ln(1-\tilde{z}_{\bar{\rho}}(\kappa)),$$

$$\Omega_1^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta) + \mathcal{O}(\kappa). \tag{3.31}$$

Here, the coefficients $a_{-k}^{(v)}(\beta)$ are given by

$$a_{-k}^{(v)}(\beta) = \beta^{-k}\zeta_R(k+1)\Gamma(k)\mathrm{Res}_{s=k}\left(\zeta_B(s,0|2)\right) \tag{3.32}$$

for $k = 1, ..., v$ and

$$a_0^{(v)}(\beta) = -\ln(\beta) \tag{3.33}$$

for $k = 0$, where we have dropped the contributions in $\mathcal{O}(\kappa)$. We hence see, that the small-κ asymptotics of $\Omega_1^{(v)}$ almost completely resembles the form of the small-κ asymptotics in the case $\mu = 0$, if one inserts a κ-dependent chemical potential of the form (3.29).

Thermodynamic quantities in the condensation phase: We now perform an analogous analysis as in section 3.4. Thereby we will show, that the renormalized chemical potential (3.29) implies finite expressions for the considered thermodynamic quantities and hence especially satisfies the second renormalization condition from section 3.4. As in section 3.4, the average particle density and the average energy density are given by:

$$\rho^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \kappa^v\beta^{-1}\frac{\partial}{\partial\mu}\Omega^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa))$$

$$\rho_E^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = -\kappa^v\frac{\partial}{\partial\beta}\Omega^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa))$$

If one inserts the small-κ asymptotics (3.30) – (3.31) together with the expressions for the coefficients (3.32) and (3.33) in those expressions, one obtains the following expression for the densities in the thermodynamic limit:

$$\rho^{(v)}(\beta,\bar{\rho}) := \lim_{\kappa\to 0}\rho^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \bar{\rho}$$

$$\rho_E^{(v)}(\beta,\bar{\rho}) := \lim_{\kappa\to 0}\rho_E^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \rho_E^{(v)}(\beta,0)$$

Finally we want to show that the system exhibits condensation. As in section 3.4, the expression for the average ground-state occupation density is given by

$$\rho_0^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa)) = \frac{\kappa^v}{\beta}\frac{\partial}{\partial\mu}\Omega_0^{(v)}(\kappa;\beta,\mu_{\bar{\rho}}(\kappa))$$

and one obtains hence by a direct calculation, that the system exhibits condensation:

$$\lim_{\kappa \to 0} \rho_0^{(v)}(\kappa; \beta, \mu) = \bar{\rho} - \rho_c^{(v)}(\beta) \tag{3.34}$$

3.5. Conclusion

In the last sections we have understood, how the thermodynamic limit of a ideal Bose gas can be understood as a renormalization problem. In this section we now want to conclude this chapter by giving a summary and giving an outlook to possible directions of further research.

Summary: The most important observation of this section is, that there exist asymptotic expansions of the grand potential $\Omega^{(v)}(\kappa; \beta, \mu)$ as $\kappa \to 0$. Those expansions hence encode precisely the aspects of the system, which get relevant in the thermodynamic limit $\kappa \to 0$. Hence, values of densitized thermodynamic observables are entirely determined by the coefficients of this expansion in the thermodynamic limit. Although asymptotic expansions were also used in [110, 112, 111] for the analysis of the ideal Bose gas in the $\beta \to 0$ limit, the present investigation represents the first attempt to apply this technique on the analysis of a thermodynamic limit. Moreover we have gained as a result, that those expansions differ drastically between the condensation and the non-condensation phase: As said before, in the former case it exhibits a singularity of finite order, while in the latter case the occurring singularity is of infinite order. This structure was not described before, which is surprising since the thermodynamic limit of an ideal Bose gas is a very well-studied system. Finally we have shown that the thermodynamic limit corresponds in the condensation phase to a renormalization problem, since an infinite renormalization of the chemical is needed to render the densitized thermodynamic observables finite in this phase.

Universality, general traps and weakly interacting gases: A starting point for a possible generalization of the present analysis to more general traps or to weakly interacting systems is motivated by the observation, that all qualitative predictions of our analysis – as the existence of the thermodynamic limit, the occurence of condensation, etc. – rely on qualitative properties of the asymptotic expansions which seem quite universal. For example all results regarding condensation should generalize directly to all grand potentials whose asymptotic behaviour is given for $\mu \leq 0$ by

$$\Omega^{(v)}(\kappa; \beta, \mu) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta, \mu) + \mathcal{O}(\kappa)$$

and for $\mu > 0$ by

$$\Omega^{(v)}(\kappa; \beta, \mu(\kappa)) = -\ln\left(1 - \tilde{z}(\kappa)\right) + \sum_{k=0}^{\infty} \kappa^{-k} a_{-k}^{(v)}(\beta, \mu(\kappa)) + \mathcal{O}(\kappa)$$

with coefficients satisfying very general constraints, as for example differentiability and boundedness in μ. This makes it plausible, that small smooth distortions of the eigenvalue spectrum should not alter the qualitative predictions of the present analysis. A first step of an analysis of this intuition could be then inspired by [171], where only general regularitiy properties of eigenvalue sequences where used for the analysis of qualitative properties of spectral zeta functions. Analogously, one could analyse which qualitative properties eigenvalue sequences of Hamiltonians should satisfy, such that the above form of the asymptotic expansion of the grand canonical potential remain unaltered. Afterwards one should then check to which extent more general traps or the incorporation of 2-body interactions preserve or modify those properties. Moreover, motivated by the formal analogy between infinite renormalization and the thermodynamic limit in the present situation, one could ask, if renormalization group methods could be applied to this problem. From a Wilsonian perspective (cf. section 2.4 one could then consider the IR-regulator κ as a realistic entity, since any system is indeed finite. But those questions lie beyond the scope of the present analysis and should be a subject of future research.

4. Renormalizing the initial singularity in classical field theory

After having explained how renormalization can be used to eradicate infinities in physical models and having illustrated this concept by a few examples and a QFT-inspired treatment of Bose-Einstein condensation, we could now ask if there are other infinities in physics which are needed to be cured. One of the most prominent infinities in physics is the kind of infinity which arises at cosmological singularities. One could hence ask the question: Is it somehow possible to ”renormalize a cosmological singularity”? This question will be analysed in this section and we will show that it is indeed possible to define renormalized states for a classical field theory in a big bang background which have the property that a large family of observables stays finite. This renormalization procedure is thereby based on point extensions of distributions, which gives an interesting application of the techniques presented in section 2.3.

The outline of this section is as follows: In section 4.1 we will review the basics regarding cosmological singularities. Moreover, we will present the most prominent completeness concepts and will explain how the analysis of this chapter fits into the existing literature. In section 4.2 we will then review the standard formalism for the massless classical scalar field in a big bang background, whereby we will have a special emphasis on its behaviour in the vicinity of the initial singularity. In section 4.3 we will then reformulate the earlier presented classical field theory in an algebraic language, which is the foundation for the renormalization procedure which we will perform section 4.4. In section 4.5 we will then conclude this chapter by giving a summary and by commenting on drawbacks of the present analysis as well as on possible generalizations and the comparison with the quantum case.

4.1. Cosmological singularities and completeness concepts

Cosmological singularities are one of the most prominent predictions of Einstein’s theory of general relativity and as such they seem to mark a severe problem of the theory. In broad terms, a singularity is a portion of a spacetime where the most fundamental object of general relativity, the metric tensor, breaks down and hence the notion of spacetime itself gets problematic. Since singular solutions to Einstein’s field equation exist, it hence appears, that general relativity predicts its own breakdown.

On cosmological singularities: Soon after Einstein's theory of general relativity was published in its final form (cf. e.g. [69]) and Schwarzschild found the famous solution which carries his name ([160]), cosmological singularities attracted the attention of mathematicians and physicists. For example, Hilbert discussed the singular behaviour of the Schwarzschild solution in [94] and also formulated there a first attempt for the definition of a cosmological singularity (cf. [94]):

> *[Es erweisen sich bei der Schwarzschild-Lösung] $r = 0$ und [...] auch $r = \alpha$ als solche Stellen, an denen die Maßbestimmung nicht regulär ist. Dabei nenne ich eine Maßbestimmung oder ein Gravitationsfeld $g_{\mu\nu}$ an einer Stelle regulär, wenn es möglich ist, durch umkehrbar eindeutige Transormation ein solches Koordinatensystem einzuführen, dass für dieses die entsprechenden Funktionen $g'_{\mu\nu}$ an jener Stelle regulär d. h. in ihr und in ihrer Umgebung stetig und beliebig oft differenzierbar sind und eine yon Null verschiedene Determinante haben.*

Nevertheless, it took many years till it was really understood how cosmological singularities can be properly formalized and till it was shown that they are an inevitable consequence of the gravitational field equations and not only an artefact caused by oversimplifying assumptions as a high degree of symmetry. The difficulties connected with the analysis of cosmological singularities are mostly intertwined with the problem of coordinate-invariance: Even if the metric appears singular – e.g. along the lines of the definition provided by Hilbert from above – in one given set of coordinates, it is hard to show that it appears singular in all possible coordinate patches. For example (cf. [63]) it took over 15 years, till it was shown by Georges Lemaître, that the $r = 2M$ singularity of the Schwarzschild spacetime was merely a coordinate artefact.

One of the key insights for a proper treatment of cosmological singularities was their definition in terms of geodesic incompleteness. Thereby spacetimes are classified in terms of the behaviour of geodesic curves: A spacetime should be called *geodesically incomplete*, if there exist geodesics which are inextendible and have finite affine length (cf. [174]). Depending on the nature of the geodesics one can distinguish hence between *timelike, spacelike or null geodesic incompleteness*. Since any spacetime can be made incomplete by removing any subset from the initial spacetime, one should apply those criteria only to *inextendible spacetimes*, i.e. spacetimes which cannot be isometrically embedded into another spacetime. Moreover, since no known objects travels on spacelike curves one usually only considers null and timelike geodesic completeness. Along this line of argumentation, an inextendible spacetime which is null or timelike geodesically incomplete should be called *singular* (cf. [174]) and the endpoints of the corresponding inextendible geodesics can be considered as elements of the singularity, i.e. as *singular points*. Those insights can then be used as a starting point for a further classification of singular spacetimes by analysing the behaviour of spacetime curvature along such an inextendible geodesic of finite length (cf. [174, 91]):

1. The spacetime has a *(scalar polynomial) curvature singularity*, if scalars constructed as polynomials of $R_{\alpha\beta\gamma\delta}$, $g_{\mu\nu}$ and covariant derivatives thereof are unbounded along an incomplete geodesic.

2. The spacetime has a *parallelly propagated curvature singularity* if no such scalar is unbounded along an incomplete geodesic, but if components of $R_{\alpha\beta\gamma\delta}$ in a parallely propagated vielbein frame (or covariant derivatives thereof) are unbounded along an incomplete geodesic.

3. The spacetime has a *non-curvature singularity* if it is singular but neither (1.) nor (2.) holds.

The *Friedmann-Lemaître-Robinson-Walker (FLRW)* spacetime, which will be mainly considered in the next two chapters of this thesis, is an example for the occurence of a scalar polynomial curvature singularity since the Riemann scalar R blows up at its singularity. On the other hand, plane gravitational waves (cf. [117]) exhibit singularities which are of the second type. Finally, the Taub-NUT spacetime is an example for a singular spacetime which falls into the third category (cf. [91]).

By utilization of this definition, it then finally was shown[1] by Penrose and Hawking (cf. [141, 90]) that singularities are an inevitable consequence of Einstein's theory of gravity and that they can occur in realistic scenarios. The details of this proof and the different variants of the Penrose-Hawking singularity theorem are interesting by their own, but since those are not needed in this thesis, we encourage the reader to consult [91, 181] for further information.

Other completeness concepts: In the last paragraph we have explained, how cosmological singularities are defined in terms of geodesic incompleteness. One interesting aspect of this definition is, that it can be easily generalized to other classes of curves whereby it gets evident that the definition of cosmological singularities depends on the considered class of curves. For example, there exist spacetimes which are geodesically complete but which are incomplete if one considers timelike curves of bounded acceleration (cf. [91]). Colloquially spoken, such spacetimes currespond to situations where freely falling objects don't get affected by a singularity, while a rocket that accelerates in a benign and realistic way can ultimately hit a singularity after a finite amount of eigentime.

This argumentation shows that the concept of (in)completeness is not solely a property of the spacetime under consideration, but a property which is also interconnected with the behaviour of the considered class of experiments which probe the spacetime. For example, a singular spacetime whose singularity cannot be probed by any realistic experiment shouldn't be problematic. Since geodesic (in)completeness – as any completeness criterion

[1] It is important to note that the famous Penrose-Hawking singularity theorems had many precursors which are nowadays less known but equally important from a historical perspective. More information on this can be found in [63].

which is based on curves – is based on classical point particles, one could ask if this point of view is not merely an oversimplification. This motivates then the question if cosmological singularities could change their face when probed by more realistic experiments. An interesting analogy for this idea is given by the Hydrogen atom. This system is modeled by a negative point charge in a background potential given by $V(r) \propto r^{-1}$ which obviously exhibits a singularity at $r = 0$. It is then a well known fact, that the classical description of this system fails: An electron, modeled as a classical point charge orbiting in a perfect circle, would hit the singularity after a very short amount time due to its radiated energy as predicted by classical electrodynamics (see also the discussion in [121]). One could say, that the Coulomb potential is incomplete when probed by charged classical point particles. On the other hand the system is quantum mechanically stable, i.e. it is complete when probed by a charged quantum mechanical point particle (cf. [122]).

One could hence ask, if a similar argumentation could apply on cosmological singularities if probed by quantum objects. This idea was for the first time investigated by Horowitz and Marolf in [100], were timelike curvature singularities in static spacetimes where analysed by quantum mechanical probes. Thereby it was shown that under this circumstances the quantum dynamics of a point particle is well-defined and free from ambiguities. Nevertheless, their argumentation has only a very limited range of validity, since it applies only on static spacetimes with timelike singularities and cannot be generalized to more realistic singularities as the Schwarzschild or the FLRW singularity. Another direction of research was performed by Hofmann and Schneider in [96, 97], where the quantum field theoretic completeness properties of Schwarzschild and Kasner spacetimes were analysed by utilization of the Schrödinger formalism. Thereby it was shown that the wave functional of a scalar quantum field exhibits vanishing support towards the singularity and has a bounded norm in ints vicinity. This formalism was then later also applied on null singularities (cf. [155]) and lead also to a more mathematical perspective on quantum completeness by the investigation of smeared field operators and energy momentum tensors (cf. [15, 16, 17]).

Completeness and classical field theory? The thoughts of the last section motivate alternative completeness concepts which depend on the nature of the probe that is used to analyse a given singular spacetime. An interesting question is hereby if it is possible to formulate a completeness criterion in the context of classical field theory. This question is of particular interest, since any quantum field theory needs a classical background theory which is used as a starting point for the quantization. This rises then the question if the quantum completeness of singular spacetimes as depicted in the last paragraph is really a property of the quantum theory or just a property of the classical background around which the theory was quantized. Although the asymptotic behaviour of classical solutions to the wave equation in singular backgrounds was already analysed in literature (see e.g. [152, 153]), this was not done from a perspective which is valuable in the context of completeness concepts and which allows for a comparison to the situation in quantum field theory. The reason for this is, that divergent solutions of classical field equations do not imply a priori any result on the level of experimental accessible quantities, since the field itself should be merely considered as a bookkeeping device. Moreover, since the formalism of quantum field theory is based on operator algebras it is not clear from the outset, how the behaviour of classical solutions is connected to regularity properties of the associated quantum field theory.

Renormalizing classical field theory in the vicinity of the initial singularity: As a first step towards a more complete investigation of the completeness properties of singular spacetimes in classical field theory and their connection to the regularity properties of the associated quantum field theory we perform in this chapter an algebraic analysis of a classical scalar field theory in a singular big bang background. Especially we will formulate the classical field theory at the level of observable algebras and will understand how the occurring classical vacuum states could be renormalized. Our treatment is thereby highly inspired by the publication [15] which deals with a consistency analysis of a scalar quantum field theory in radiation dominated and dust filled FLRW universes. In the publication [15] it is especially shown, that operator valued distributions, 2-point functions and the energy momentum tensor of a such a theory remain well-defined throughout the initial singularity. Thereby, in the case of the radiation dominated universe, also a renormalization procedure is *implicitely* performed that is based on the extensions of occuring distributions on the singular hypersurface. Our treatment should be understood as a complementary analysis: In [15] this kind of renormalizability was considered as a quantum field theoretic property, tied to the distributional nature of the occuring operator valued distributions. Moreover, the renormalization procedure in [15] was performed rather implicitly and was not understood as a renormalization. In the present analysis on the other hand, we show that this behaviour is not solely a feature of quantum field theories but can be traced back to a modification of the associated classical field theory. To ensure comparability with [15], we hence focus on radiation dominated spacetimes and compare our results finally with the results of [15].

Outline of this chapter: We want to describe concisely the organisation of this chapter. In section 4.2 we will review the classical theory of a massless scalar field in a general big bang spacetime. In section 4.3 we will then reformulate this theory from an algebraic perspective. Finally we will then perform the renormalization procedure in section 4.4, by which we will analyse the distributional extendability to the singular hypersurface of the occuring n-point functions and classical solutions. In section 4.5 we will then conclude this section by summarizing its results and by comparing them to the results of [15]. Moreover we will discuss there the case of more general models as well as the drawbacks of the present analysis.

4.2. The massless scalar field in a big bang background

In this section we want to review the well-known theory of a conformally coupled classical scalar field on a spatially flat FLRW spacetime. The treatment of this section folllows the line of argumentation of many textbooks (compare e.g. [28, 135]). Thereby we will allow for general scale factors in this and the next section. From section 4.4 on we will then focus on the case of radiation dominated FLRW spacetimes.

Preliminaries: We consider a FLRW spacetime $\mathscr{M} \cong \mathbb{R}^4$ with pseudo-Riemannian metric

$$ds^2 = a(\eta)^2(-d\eta^2 + dx^2 + dy^2 + dz^2)$$

and smooth scale factor $a(\eta)$ satisfying $a(\eta) \neq 0$ for $\eta \neq 0$ and $a(0) = 0$ (cf. e.g. [44]). The coordinates η, x, y, z are assumed to lie in $\mathbb{R}$ and the hypersurface

$$\mathscr{X} := \{(0, x, y, z) \subset \mathscr{M} | x, y, z \in \mathbb{R}\}$$

is called the *initial singularity* of $\mathscr{M}$, since the metric is degenerate thereon. The submanifolds $\mathscr{M}_\pm := \{(\eta, x, y, z) \subset \mathscr{M} | \pm\eta > 0\}$ correspond to the big crunch and big bang epochs, respectively. We will define further the bulk spacetime

$$\mathscr{M}^\circ := \mathscr{M} \setminus \mathscr{X}$$

and will call $\mathscr{M}$ the extension of $\mathscr{M}^\circ$.

We want to probe this spacetime with a conformally coupled, real scalar field, while we will neglect backreaction effects. This simple system is chosen, since we want to focus merely on conceptional ideas than on technicalities. Nevertheless we will comment on more general models in section 4.5 and will comment there also on the question, if the neglection of backreaction is a reasonable assumption or not. The considered action is given by (cf. [28])

$$S[\phi] = \frac{1}{2} \int_{\mathscr{M}} d\mathrm{Vol}(x) \left[-g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \xi R_g \phi^2 \right],$$

where R_g is the Ricci scalar associated with g. We consider the case of conformal coupling i.e. the case of $\xi = \frac{1}{6}$. In this case the equation of motion is given by (cf. appendix B.2):

$$\frac{1}{a(\eta)^3} [\partial_\eta^2 - \Delta] a(\eta) \phi = 0 \tag{4.1}$$

To avoid IR-problems, we are – as it is common in literature – solely interested in solutions to (4.1) which have the property, that they are compactly supported on any Cauchy surface. Moreover we will define the differential operator appearing in (4.1) for later use:

$$P := \frac{1}{a(\eta)^3} [\partial_\eta^2 - \Delta] a(\eta)$$

Solving the equation of motion: The equation of motion (4.1) is obviously solved by any $\phi \in C^\infty(\mathscr{M}^\circ)$ which can be written as

$$\phi = a(\eta)^{-1}\phi_M$$

with $\phi_M \in C^\infty(\mathscr{M})$ being a solution of the flat space Klein-Gordon equation

$$\left[\partial_\eta^2 - \Delta\right]\phi_M = 0. \tag{4.2}$$

The property, that ϕ should be compactly supported on any Cauchy surface in $\mathscr{M}$ translates then to the property that ϕ_M should be compactly supported on any Cauchy surface in Minkowski spacetime. Especially we have then that ϕ_M is a smooth function in $C^\infty(\mathscr{M})$ and in addition bounded and compactly supported on the singular hypersurface $\mathscr{X}$, since this is just a regular Cauchy surface in Minkowski spacetime. A Fourier decomposition of 4.1 leads then to the statement, that any solution ϕ of 4.1 can be written as a mode expansion

$$\phi(\eta,\vec{x}) = \int_{\mathbb{R}^3} d^3k\left(z(\vec{k})X(\eta,\vec{k})e^{i\vec{k}\vec{x}} + z^*(\vec{k})X^*(\eta,\vec{k})e^{-i\vec{k}\vec{x}}\right) \tag{4.3}$$

with $X(\eta,\vec{k})$ being a solution of the mode equation

$$\left[\partial_\eta^2 + \vec{k}^2\right]a(\eta)X(\eta,\vec{k}) = 0,$$

which can be explicitly checked by inserting (4.3) into (4.1). Under the mode expansion (4.3), regularity requirements on ϕ are then translated to regularity requirements on $z(\vec{k})$ (cf. e.g. [85]). The mode equation (4.3) is then solved by any function $X(\eta,\vec{k})$ of the form

$$X(\eta,\vec{k}) = \frac{B_1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}}\frac{e^{i|\vec{k}|\eta}}{a(\eta)} + \frac{B_2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}}\frac{e^{-i|\vec{k}|\eta}}{a(\eta)}$$

with $B_1, B_2 \in \mathbb{C}$. The mode functions $X(\eta,\vec{k})$ span hence the solution space of the differential equation (4.1). The solution space of (4.1) can moreover be endowed with a symplectic form explicitely given by (cf. [108, 28])

$$(\psi_1,\psi_2) := -i\int_{\mathbb{R}^3} d^3x a(\eta)^2\left(\psi_1(\eta_0,\vec{x})\,\partial_\eta\big|_{\eta=\eta_0}\psi_2(\eta,\vec{x}) - \psi_2(\eta_0,\vec{x})\,\partial_\eta\big|_{\eta=\eta_0}\psi_1(\eta,\vec{x})\right)$$

which is defined for any $\eta_0 \in \mathbb{R}\setminus\{0\}$ and does not depend on the specific choice of η_0 (cf. [28]). This antisymmetric, bilinear form can then be used to normalize the mode functions $X(\eta,\vec{k})$. By setting $\psi_{\vec{k}}(\eta,\vec{x}) := X(\eta,\vec{k})e^{i\vec{k}\vec{x}}$ we obtain (see appendix B.2)

$$\left(\psi_{\vec{p}},\psi^*_{\vec{q}}\right) = \delta(\vec{p}-\vec{q})$$

for $|B_1|^2 - |B_2|^2 = -1$. We then choose the solution associated to $B_1 = 0$ and $B_2 = 1$ since this gives the mode functions which are conformally related to the usual choice in Minkowski spacetime. Hence our choice of mode functions (which equals the choice in [15]) is given by:

$$X(\eta, \vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2|\vec{k}|}} \frac{e^{-i|\vec{k}|\eta}}{a(\eta)} \tag{4.4}$$

From a more intrinsic perspective one could say, that ∂_η is a conformal Killing vector field compatible with the natural orientation on $\mathscr{M}$, which determines hence a natural set of positive frequency solutions (cf. [44]) given by (4.4).

Behaviour towards the singularity: A prototypical solution ϕ to the wave equation (4.1) has the asymptotic behaviour

$$\phi(\eta, \vec{x}) \sim a(\eta)^{-1}$$

towards the singularity, i.e. as $\eta \to 0$. Hence, for a scale factor which behaves asymptotically as $a(\eta) \sim \eta^c$, the solutions diverge towards the singularity. This is a hint, that the field theory behaves badly in the vicinity of the singularity. Nevertheless, the field ϕ should just be considered as a bookkeeping device and it is a priori not clear if a divergence of ϕ manifests itself in terms of a malign behaviour on the level of observables. For example, if we take as an observable a smeared field

$$F[\phi] := \int_{\mathscr{M}} d\mathrm{Vol}_g(x) f(x) \phi(x)$$

we see, that $F[\phi]$ is finite even if the interior of the support of f intersects the singularity, i.e. if $\mathrm{Int}\,(\mathrm{supp}(f)) \cap \mathscr{X} \neq \emptyset$ holds. This is due to the damping induced by the volume element

$$d\mathrm{Vol}_g(x) = d^4x\, a(\eta)^4.$$

To quantify the behaviour of the scalar field in the vicinity of the singularity, we hence would like to analyse it on a sufficiently large observable algebra. Especially we are interested in the question, which observables are infinite if they probe solutions to the equations of motion, and if one can cure this problem by some sort of renormalization procedure. But therefore, as a preliminary step, we have to embed the theory of the classical scalar field as presented in this section into a larger, algebraic context.

4.3. An algebraic perspective

For our purpose it will be convenient to use an algebraic framework, since then the considered completeness problem and the occurring renormalization procedures can be formulated very naturally. Moreover, in this formulation it will be easy to compare our results with the results from [15]. The idea behind the algebraic formalism as presented in this section is, to consider observables as elements of a suitable algebra of real valued functionals on the configuration space. Elements of the configuration space correspond then to pure states in classical field theory and evaluations of the functionals on elements of the configuration space correspond then to specific outcomes of measurements. This will be made precise in this section and we will demonstrate then at the end of this section, to which extent the theory of a conformally coupled scalar field in a big bang spacetime is classically incomplete. Please note that the formalism presented in this section is standard in the literature associated with algebraic quantum field theory. Our main sources are given by ([125, 107, 78, 85, 86]). Moreover, the adaption of this formalism to FLRW spacetimes was already performed thoroughly in [85, 86], whose results are used in this section, too.

The off-shell configuration space and observables: We define the *space of off-shell field configurations* as

$$\mathcal{C} = \{\phi : \mathscr{M}^\circ \to \mathbb{R} | \phi \in C^\infty(\mathscr{M}^\circ)\}.$$

This means that we assume field states to be smooth on $\mathscr{M}^\circ$ but allow for singular behaviour on the singularity $\mathscr{X}$. The space $\mathcal{C}$ should be understood as the space of all possible (off- and on-shell) field configurations and especially we see, that solutions to the equation of motion (4.1) lie in $\mathcal{C}$. The space $\mathcal{C}$ can be endowed with a suitable topology and a natural candidate for this is given by the Fréchet-topology induced by the family of semi-norms

$$p_{\alpha,K} : \mathcal{C} \to (0,\infty), f \mapsto \sup_{x\in K} |\partial^\alpha f|$$

for α being a multi-index and $K \subset \mathscr{M}^\circ$ being a compact set (cf. [179]). An *observable* is then a continuous, real valued functional

$$F : \mathcal{C} \to \mathbb{R},$$

which is often assumed to be *smooth* in addition. Thereby, to simplify matters, we call a functional $F : \mathcal{C} \to \mathbb{R}$ smooth, if its n-th directional derivative

$$C^\infty(\mathscr{M})^n \to \mathbb{R}, (f_1, ..., f_n) \mapsto \frac{d^n}{d\lambda_1 ... d\lambda_n} F\left(\phi + \sum_{i=1}^{n} \lambda_i f_i\right)$$

exists as a compactly supported, symmetric distribution on $\mathscr{M}^n$ for all $n \in \mathbb{N}$ and all $\phi \in \mathcal{C}$ (cf. [125, 107, 78]). Anyhow, we will focus in our discussion on observables of the form

$$F : \mathcal{C} \to \mathbb{R}, \phi \mapsto \int_{\mathscr{M}^m} d\mathrm{Vol}_g(x_1)...\mathrm{Vol}_g(x_m) f(x_1, ...x_n) \phi(x_1)^{n_1} \cdots \phi(x_m)^{n_m} \tag{4.5}$$

defined for suitable compactly supported functions $f \in C_c^\infty(\mathscr{M}^{\circ n})$, for which smoothness follows directly. We will call observables of this form *multilocal Wick monomials on* $\mathscr{M}^\circ$ and will denote the algebra spanned by all such observables as the *algebra of multilocal Wick* *observables on $\mathscr{M}^\circ$* denoted by $\mathfrak{W}[\mathscr{M}^\circ]$. Of course, one could try to topologize this algebra and analyze its closure. But this is not necessary for the present discussion and we just consider it as a (unbounded) *-algebra. Also we would like to remark that one usually focuses on a larger algebra, in which the smearing functions f are replaced by distributions satisfying certain regularity requirements (cf. e.g. [78, 5]). Anyhow, for the present analysis this is not necessary.

The solution space: We define the solution space $\mathcal{V}$ as the space of solutions to the wave equation (4.1) which satisfy the condition, that they have compact support on spacelike hypersurfaces (cf. [108]), i.e.:

$$\mathcal{V} := \{\phi \in \mathcal{C} | P\phi = 0 \text{ and for any Cauchy surface } \Sigma \subset \mathscr{M}^\circ : \operatorname{supp}(\phi) \cap \Sigma \text{ is compact } \}$$

By the results of the last section there is a 1-to-1 correspondence between the solution space of the Klein-Gordon equation on Minkowski spacetime and $\mathcal{V}$. To formalize this correspondence let $\mathcal{V}_M$ denote the space of solutions to the flat space Klein-Gordon equation $\partial_\alpha \partial^\alpha \phi = 0$ which have compact support on spacelike hypersurfaces. We then have:

$$\mathcal{V} \cong \left\{ \phi = a(\eta)^{-1} \phi_M \middle| \phi_M \in \mathcal{V}_M \right\}.$$

The space of solutions $\mathcal{V}$ will play an important role in the sequel, since we interpret it as the space of pure classical vacuum states (cf. [85]).

States and n-point functions: In algebraic approaches to quantum field theory and quantum statistical physics, states are defined as positive and normalized linear functionals on *-algebras (cf. e.g. [78]). In our situation (cf. [125]) we define a state on a observable algebra $\mathfrak{A}$ as a linear functional

$$\omega : \mathfrak{A} \to \mathbb{C}$$

which is normalized if $\mathfrak{A}$ is unital, i.e. $\omega(\mathbb{1}) = 1$, and positive, i.e. $\forall A \in \mathcal{A} : \omega(A^* A) \geq 0$. Moreover, a state ω is called pure if it is not possible to write it as a convex combination of other states. An important class of pure states in our situation is given by elements of $\mathcal{C}$ by associating the evaluation functional

$$\mathrm{ev}_\phi : \mathfrak{A} \to \mathbb{R}, F \mapsto \mathrm{ev}_\phi(F) := F(\phi)$$

to any $\phi \in \mathcal{C}$ (cf. [125]). Of special importance are then evaluation functionals of above form, for which ϕ lies in the solution space $\mathcal{V}$, since those will be considered as the pure classical vacuum states of interest. For states on the Wick algebra (4.5), we can then associate to a state ω its n-point functions. Those are distributions $\omega^{(n_1,\ldots,n_m;m)}(x_1, \ldots x_m)$ which satisfy

$$\omega[F] = \int_{\mathscr{M}^m} d\mathrm{Vol}_g(x_1) \ldots \mathrm{Vol}_g(x_m) f(x_1, \ldots x_n) \omega^{(n_1,\ldots,n_m;m)}(x_1, \ldots x_m)$$

for any F of the form (4.5). Obviously, in the case of this algebra, a state is entirely described by its n-point functions.

Behaviour at the singularity: We now want to investigate, in which sense the algebraic classical field theory defined so far exhibits a lower regularity at the singular hypersurface $\mathscr{X}$. Therefore set now $a(\eta) = \eta^c$ and let ϕ be a solution to the equations of motion, which means, that ϕ can be written as

$$\phi(x) = a(\eta)^{-1}\phi_M(x)$$

with $\phi_M(x)$ being a solution to the massless flat space wave equation (4.2). Hence $\phi(x) \sim \eta^{-c}$ as $\eta \to 0$. We then have consequently, that the higher order observable

$$\phi_f^n[\phi] = \int_{\mathscr{M}} d\mathrm{Vol}_g(x) f(x)\phi^n(x) \tag{4.6}$$

evaluated on any classical vacuum state $\phi \in \mathcal{V}$ diverges for $f \in \mathcal{D}(\mathscr{M})$ and $(4-n)c \leq -1$. This shows, that the considered classical field theory in an FLRW background is less regular, than its cousin on flat spacetime: In the latter case, the observables (4.6) evaluated on the corresponding classical vacuum is finite for any $n \in \mathbb{N}$. Also, the considered classical field theory is hence less regular on the singularity $\mathscr{X}$ than on the bulk $\mathscr{M}^\circ$.

Statement of the renormalization problem: The lower regularity of the classical field theory in the vicinity of the singularity as presented in the last paragraph can be understood in terms of the statement that the classical vacuum solutions ϕ are well-defined on $\mathscr{M}^\circ$ but are not defined on $\mathscr{M}$. Moreover, the associated vacuum states are well-defined states on $\mathfrak{W}[\mathscr{M}^\circ]$ but not well-defined on the algebra $\mathfrak{W}[\mathscr{M}]$. Here we have defined the algebra $\mathfrak{W}[\mathscr{M}]$ in almost the same way as the algebra $\mathfrak{W}[\mathscr{M}^\circ]$, with the only difference that the smearing functions are now allowed to lie in $C_c^\infty(\mathscr{M}^n)$. The question, if the infinities which arise at the singularity can be renormalized, is hence associated with an extension problem. This extension problem can be summarized in terms of the following two questions:

1. Is there for any $\phi \in \mathcal{V}$ an associated distributional extension $\hat{\phi} \in \mathcal{D}'(\mathscr{M})$?

2. Is there an extension of the vacuum states $\omega = \mathrm{ev}_\phi$ from $\mathfrak{W}[\mathscr{M}^\circ]$ to $\mathfrak{W}[\mathscr{M}]$ for $\phi \in \mathcal{V}$?

Those questions will be answered in the next section.

4.4. Renormalized states for the classical field theory

In the last section we have presented an algebraic formalism for classical field theory and explained within this framework, to which extent the theory of a classical conformally coupled, massless scalar field in a FLRW background is less regular than its cousin on Minkowski spacetime. The question is now, if the divergences which characterize this less regular behaviour, can be somehow "renormalized" along the lines of the end of the last section. This will be analysed in this section. To simplify the comparison of our results with [15], we will thereby concentrate on the case of the radiation dominated universe. I.e., we will focus in this section solely on the scale factor $a(\eta) = \eta$.

Distributional reinterpretation of the equations of motion: In this section, we want to analyse the equation of motion (4.1) in a distributional sense. I.e. we want to analyse, if the partial differential equation $P\phi = 0$ with P being defined as

$$P := \frac{1}{a(\eta)^3}\left[\partial_\eta^2 - \Delta\right] a(\eta)$$

has distributional solutions in $\mathcal{D}'(\mathscr{M})$. We therefore recall (cf. [72]), that a distribution $\phi \in \mathcal{D}'(\mathscr{M})$ is called a distributional (or weak) solution to $P\phi = 0$, if

$$\int_{\mathscr{M}} d\mathrm{Vol}_g(x)\phi(x)Pb(x) = 0$$

holds, since P is self-adjoint[2]. We then see, that $\phi \in \mathcal{D}'(\mathscr{M})$ solves (4.7) if it can be written as

$$\phi(x) = t(\eta)\phi_M(x)$$

with $\phi_M \in C^\infty(\mathscr{M})$ being a smooth solution to the flat space Klein-Gordon equation (4.2) and $t \in \mathcal{D}'(\mathbb{R})$ being a distributional solution to

$$a(\eta)t(\eta) = 1. \tag{4.8}$$

The latter means, that t is a distributional inverse to $a(\eta) = \eta$, i.e. that t satisfies

$$t[a(\cdot)b(\cdot)] = \int_{\mathbb{R}} d\eta b(\eta)$$

for any test function $b \in \mathcal{D}(\mathbb{R})$. Since η^{-1} is the smooth inverse to $a(\eta) = \eta$ one could expect, that t should be given by a distributional extension of η^{-1} from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$. Since, as discussed before, extensions of distributions are in general not unique, one should expext that there are several ways to extend η^{-1} from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$. The most obvious one would be, to generalize the procedure applied on $\theta(x)x^{-1}$ (cf. eq. (2.10)) in section 2.3. Nevertheless, we will use a different strategy for the sake of comparability with [15] and define the distribution

$$t : b \in \mathcal{D}(\mathbb{R}) \mapsto t(b) := -\int_{\mathbb{R}} d\eta \ln(|\eta|)\, b'(\eta). \tag{4.9}$$

[2] A concise calculation under utilization of $\mathrm{Vol}_g(x) = a(\eta)^4 d^4x$ shows, that for $f, g \in \mathcal{D}(\mathscr{M})$ indeed

$$\int_{\mathscr{M}} d\mathrm{Vol}_g(x)(Pf)(x)g(x) = \int_{\mathscr{M}} d\mathrm{Vol}_g(x)f(x)(Pg)(x) \tag{4.7}$$

holds.

This is motivated by the fact that the derivative of $\ln(|\eta|)$ is given by η^{-1}. It is then easy to show (cf. theorem 3), that t indeed defines a distribution in $\mathcal{D}'(\mathbb{R})$ and is a distributional extension of η^{-1}. We now show, that this t solves (4.8) in a distributional sense. Therefore let $b \in \mathcal{D}(\mathbb{R})$. We then have:

$$-\int_{\mathbb{R}} d\eta \ln(|\eta|) \frac{d}{d\eta} (\eta b(\eta)) = \int_{\mathbb{R}} d\eta\ \eta^{-1} \eta b(\eta) = \int_{\mathbb{R}} d\eta\ b(\eta).$$

From theorem 1 one now knows, that there is a certain freedom in extending η^{-1} and hence one expects, that there should be also a freedom in finding distributional solutions to (4.8). Especially one expects, that two solutions to (4.8) should differ by a linear combination of derivatives of delta distributions as in (2.13). Fortunately, this freedom can be limited to a 1-parameter freedom. To see this, observe first that

$$\int_{\mathbb{R}} d\eta\ \eta b(\eta) \delta(\eta) = 0$$

holds, but for $n > 1$

$$\int_{\mathbb{R}} d\eta\ \eta b(\eta) \delta^{(n)}(\eta) \neq 0$$

holds. For example, we have

$$\int_{\mathbb{R}} d\eta\ \eta b(\eta) \delta'(\eta) = -\int_{\mathbb{R}} d\eta\ \left(b(\eta) + \eta b'(\eta) \right) \delta(\eta) = -b(0)$$

which is non-zero in general. Hence it follows, that the distribution

$$t_K : b \mapsto t(b) + K\delta[b],$$

with t being defined as in (4.9), is for any $K \in \mathbb{R}$ a distributional solution to (4.8).

Since ϕ_M as a smooth solution to the flat space Klein-Gordon equation is smooth, we can multiply it unambigously with t_K and obtain, that

$$\phi(x) = t_K(\eta) \cdot \phi_M(x)$$

solves $P\phi = 0$ in a distributional sense. Moreover, it is easy to show, that ϕ defined as such is a distributional extension of the associated smooth solution from $\mathcal{D}'(\mathscr{M}^\circ)$ to $\mathcal{D}'(\mathscr{M})$. As it was the situation in section 2.3, the 1-parameter freedom in K corresponds to an additive renormalization freedom. Since we are in this section merely interested in the existence of renormalized states and especially in the comparison with [15], we will set $K = 0$ in the sequel but will comment on the role of the renormalization constants again in section 4.5.

Space of distributional solutions: Motivated by the outcomes of the last paragraph we define a distributional space of solutions as

$$\hat{\mathcal{V}} := \{ t(\eta) \cdot \phi_M(x) | \phi_M \in \mathcal{V}_M \}, \tag{4.10}$$

where t is defined as in (4.9). By this we have a 1-to-1 correspondence between distributional solutions $\hat{\phi} \in \hat{\mathcal{V}}$ and smooth solutions $\phi \in \mathcal{V}$. Of course it would be desirable, to obtain a more complete picture by defining a distributional space of solutions with some kind of compact spacelike support in full analogy to the original definition to $\mathcal{V}$. But since we are in the present situation merely interested in performing a proof of concept and comparing the classical theory with the situation in [15], we use the instructive definition (4.10)[3].

Renormalized n-point functions: The n-point functions $\omega_{\phi_M}^{(m_1,\ldots,m_n;n)}(x_1,\ldots,x_n)$ which describe a classical vacuum state ev_ϕ with $\phi = a(\eta)^{-1}\phi_M$ are explicitely given by

$$\omega_{\phi}^{(m_1,\ldots,m_n;n)}(x_1,\ldots,x_n) = \frac{\phi_M(x_1)^{m_1}\cdots\phi_M(x_n)^{m_n}}{a(\eta_1)^{m_1}\cdots a(\eta_n)^{m_n}}$$

for ϕ_M being a solution to the flat space Klein-Gordon equation. We will now focus for a while on the case $n = 2$ and $(m_1, m_2) = (m, 1)$ to keep things simple. But the general case follows analogously.

The 2-point function $\omega^{(m,1;2)}(x_1, x_2)$ is explicitely given by

$$\omega_{\phi}^{(m,1;2)}(x_1,x_2) = \frac{\phi_M(x_1)^m\phi_M(x_2)}{a(x_1)^m a(x_2)}.$$

and obviously we have

$$\int_{\mathscr{M}^2} d\mathrm{Vol}_g(x_1)d\mathrm{Vol}_g(x_2)f(x_1,x_2)\omega_{\phi_M}^{(m,1;2)}(x_1,x_2) < \infty \tag{4.11}$$

for all $m \in \mathbb{N}$ for $f \in \mathcal{D}(\mathscr{M}^{o2})$. We will then see in a moment, that the renormalization of states corresponds to an extension of the n-point functions towards the singularity, i.e. to the situation where (4.11) holds also for test-functions $f \in \mathcal{D}(\mathscr{M}^2)$. It is then clear, that for this a generalization of the distributional extension (4.9) to higher powers η^{-n} is needed.

We can define this extension in a straight forward way as (cf. [15])

$$t^{(n)} : b \in \mathcal{D}(\mathbb{R} \mapsto t(b) := -\frac{1}{(n-1)!}\int_{\mathbb{R}} d\eta\ \ln(|\eta|)b^{(n)}(\eta), \tag{4.12}$$

which is motivated by the fact that

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{d^n\ln(|\eta|)}{d\eta^n} = \eta^{-n}$$

holds. It is then easy to show (cf. theorem 3), that $t^{(n)}$ defines a distribution in $\mathcal{D}'(\mathbb{R})$ and is a distributional extension of η^{-n}. Since ϕ_M is smooth, we can unambigously (cf. appendix A.2 and A.4) define the product $t^{(n)} \cdot \phi_M^n$ and the tensor product distribution $\left(t^{(m)} \cdot \phi_M^m\right) \otimes$

[3]And in general, uniqueness results are for weak solutions often harder to obtain than for smooth solutions.

$(t \cdot \phi_M)$ and by this we define a distribution

$$\hat{\omega}_{\phi}^{(m,1;2)}(x_1, x_2) = \left(t^{(m)}(\eta_1) \cdot \phi_M^m(x_1)\right) \otimes (t(\eta_2) \cdot \phi_M(x_2)) \,.$$

One can then show easily, that $\hat{\omega}_{\phi}^{(m,1;2)}$ is a distributional extension of $\omega_{\phi}^{(m,1;2)}$. Therefore let $b_1, b_2 \in \mathcal{D}(\mathscr{M}^o)$ be two bump functions. We then have

$$\begin{aligned}
&\int_{\mathscr{M}^2} d\mathrm{Vol}_g(x_1) d\mathrm{Vol}_g(x_2) b_1(x_1) b_2(x_2) \hat{\omega}_{\phi_M}^{(m,1;2)}(x_1, x_2) \\
&= \frac{1}{(m-1)!} \int_{\mathbb{R}^4} d^4x_1 \ln(|\eta_1|) \frac{d^m}{d\eta_1^m} \left(a(\eta_1)^4 b_1(x_1) \phi_M(x_1)^m\right) \\
&\qquad \cdot \int_{\mathbb{R}^4} d^4x_2 \ln(|\eta_2|) \frac{d}{d\eta_2} \left(a(\eta_2)^4 b_2(x_2) \phi_M(x_2)\right)
\end{aligned}$$

and since $a(\eta)^4 b_i(x) \phi_M(x)$ is in $\mathcal{D}(\mathscr{M}^o)$ this can be written as

$$\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d\mathrm{Vol}(x_1) d\mathrm{Vol}(x_2) \eta_1^{-m} \eta_2^{-1} \phi_M(x_1)^m \phi_M(x_2) b_1(x_1) b_2(x_2).$$

Hence $\hat{\omega}_{\phi}^{(m,1;2)}$ is a distributional extension of $\omega_{\phi}^{(m,1;2)}$. The general case follows then analogously. By this we have obtained for any smooth solution ϕ to (4.1) a family of distributional n-point functions $\hat{\omega}_{\phi}^{(m_1,\dots,m_n;n)}$ which extend $\omega^{(m_1,\dots,m_n;n)} - \phi$ in the sense depicted above. Finally we want to note, that this distributional extension is, as before, not unique. Especially one can show that the distributional equation

$$\eta^n \chi = 1$$

is solved by any distribution of the form (cf. also theorem 3)

$$\chi = t^{(n)} + \sum_{i=0}^{n-1} K_i \delta^{(i)}.$$

Hence for higher order observables there is a high degree of indeterminacy in the present analysis. We will comment on this again in section 4.5. Nevertheless, since we are merely interested in performing a proof of concept and comparing the classical theory with the situation in [15], we set all renormalization constants to zero for a while.

Renormalized states: We now want to show, that the family of renormalized n-point functions which we have obtained in the last paragraph defines a state on the algebra $\mathfrak{W}[\mathscr{M}]$. We therefore define for any $\phi \in \mathcal{V}$ a map

$$\hat{\mathrm{ev}}_{\phi} : \mathfrak{W}[\mathscr{M}] \to \mathbb{C}$$

by setting

$$\hat{\text{ev}}_\phi(F) = \int_{\mathscr{M}^m} d\text{Vol}_g(x_1)...\text{Vol}_g(x_n) f(x_1, ...x_n) \hat{\omega}_\phi^{(m_1,...,m_n;n)}(x_1, ...x_n)$$

for an observable F given by

$$F[\phi] = \int_{\mathscr{M}^n} d\text{Vol}_g(x_1)...\text{Vol}_g(x_n) f(x_1, ...x_n) \phi(x_1)^{m_1} \cdots \phi(x_n)^{m_n}$$

with $f \in C_c^\infty(\mathscr{M}^n)$ and extending this to the whole algebra $\mathfrak{W}[\mathscr{M}]$ by linearity. By the argumentation of the last paragraph, the map $\hat{\text{ev}}_\phi(F)$ is hence well-defined defined on $\mathcal{W}[\mathscr{M}]$. Positivity follows, since $\hat{ev}_\phi$ has, as ev_ϕ, the desirable property that

$$\hat{\text{ev}}_\phi(F^* F) = \hat{\text{ev}}_\phi(F^*)\hat{\text{ev}}_\phi(F)$$

holds. To see this, take – in analogy to the example of the last paragraph – the observable F which is defined as

$$F[\phi] = \int_{\mathscr{M}^2} d\text{Vol}_g(x_1)\text{Vol}_g(x_2) f(x_1, x_2)\phi(x_1)^m \phi(x_2).$$

We then have, that

$$\begin{aligned}
&(F^* F)[\phi] = \\
&\int_{\mathscr{M}^4} d\text{Vol}_g(x_1)\text{Vol}_g(x_2) d\text{Vol}_g(x_3)\text{Vol}_g(x_4) f(x_1, x_2) f(x_3, x_4)\phi(x_1)^m \phi(x_2)\phi(x_3)^m \phi(x_4) \\
&= \left(\int_{\mathscr{M}^2} d\text{Vol}_g(x_1)\text{Vol}_g(x_2) f(x_1, x_2)\phi(x_1)^m \phi(x_2) \right)^2
\end{aligned}$$

holds and it then can be showed easily, that

$$\begin{aligned}
&\hat{\text{ev}}_\phi(F^* F) \\
&= \int_{\mathscr{M}^4} d\text{Vol}_g(x_1)\text{Vol}_g(x_2) d\text{Vol}_g(x_3)\text{Vol}_g(x_4) f(x_1, x_2) f(x_3, x_4)\omega_\phi^{(m,1,m,1;4)}(x_1, x_2, x_3, x_4) \\
&= \left(\int_{\mathscr{M}^2} d\text{Vol}_g(x_1)\text{Vol}_g(x_2) f(x_1, x_2)\omega_\phi^{(m,1;2)}(x_1, x_2) \right)^2
\end{aligned}$$

holds as well. The claim for general observables follows analogously. Moreover, since the n-point functions $\hat{\omega}_\phi^{n_1,...,n_m;m}(x_1, ..., x_m)$ are distributional extensions of the n-point functions $\omega_\phi^{n_1,...,n_m;m}(x_1, ..., x_m)$, for any $F \in \mathfrak{W}[\mathscr{M}^\circ]$

$$\hat{\text{ev}}_\phi(F) = \text{ev}_\phi(F) \tag{4.13}$$

holds. Hence one can think of $\hat{\text{ev}}_\phi(F)$ as an extension of $\text{ev}_\phi(F)$ from $\mathscr{M}^\circ$ to $\mathscr{M}$. In order to understand if the renormalized states exhibit a higher regularity than the unrenormalized states we consider now the higher order observable ϕ_f^n defined in equation (4.6) with $f \in$

$\mathcal{D}(\mathscr{M})$. We then have

$$\begin{aligned}\left|\hat{\mathrm{ev}}_\phi(\phi_f^n)\right| &= \left|\frac{1}{(n-1)!}\int_{\mathbb{R}^4} d^4x \frac{d^n}{d\eta^n}\left(\eta^4 f(x)\phi_M(x)^4\right)\ln(|\eta|)\right| \\ &\leq \left\|\frac{d^n}{d\eta^n}\left(\eta^4 f(x)\phi_M(x)^4\right)\right\|_\infty \int_{\mathrm{supp}(f)} d^4x\,|\ln(|\eta|)| \\ &< \infty\end{aligned}$$

for all $n \in \mathbb{N}$ and hence the claim is shown. By this we have obtained a renormalized, finite field theory on the singular FLRW spacetime, which extends the unrenormalized field theory in the sense, that (4.13) holds. Moreover this theory has the same regularity properties as its analogue on flat spacetime, since $\hat{\mathrm{ev}}_\phi(F)$ is finite for any observable in $F \in \mathfrak{W}[\mathscr{M}]$.

4.5. Conclusion

In this section we want to conclude the chapter. Especially we will summarize its findings, compare the results to the results of [15] and will discuss its drawbacks.

Summary: In this chapter we have shown, that there exist distributional solutions and distributional states for a conformally coupled classical scalar field on a radiation dominated FLRW spacetime. Those states, which are called renormalized states, extend the usual smooth states and have the property that all smeared higher order observables are finite, whereas this is not the case for the non-renormalized states. Moreover, the regularity properties of the renormalized theory resemble hence those of the corresponding theory on flat spacetime. By this we have especially shown, that the infinities which occur in the context of a classical field theory in the vicinity of a cosmological singularity can be renormalized by a renormalization scheme which is based on point-extensions of distributions.

More general situations: We want to explain concisely, how the results of this section could be generalized to more general situations. Therefore note, that the case of more general scale factors $a(\eta) = \eta^c$ with $c > 0$ can be analysed in full analogy since distributional inverses to η^c can be defined unambiguously (cf. [99]). Also the case of non-minimal coupling or the massive case can be analysed in a similar manner. In those cases, the solutions are of course not just given by a product of the inverse scale factor with flat space solutions. In those cases one has to analyse the scaling behaviour of the occurring differential equations by which one should be able to estimate the Steinmann scaling degree (cf. section 2.3) of their distributional solutions. By this one should then be able to analyse the distributional extensions towards the singular hypersurface.

Comparison to the quantum case: We want to compare our results concisely with the results of [15]. There it was shown among other things, that the smeared field operators, the smeared 2-point functions and the smeared energy-momentum tensor remain well-defined throughout the singular hypersurface if one considers scalar quantum fields in radiation dominated and dust filled FLRW spacetimes. Thereby, singularities of the form η^{-n} were replaced by distributional expressions given by the distribution $t^{(n)}$ defined in (4.12). This was then considered as a property of the distributional nature of quantum fields. Although the analysis of [15] is very interesting and rich of deep insights, the present analysis shows, that this property is *not* a property of quantum fields but already present at the classical level. Especially this suggests, that the tameness of the quantum field theory in the vicinity of the initial singularity as analysed in [15] could be merely inherited from the behaviour of its classical counterpart which was used for quantization. Moreover our analysis suggests, that one could define two inequivalent classical field theories in singular spacetimes, which are based on smooth and distributional solutions, respectively.

Drawbacks of the present analysis: Unfortunately, the present analysis has several serious drawbacks. For example, we have discarded the renormalization constants, since we were merely interested in a proof of concept and in the comparison with [15], where the renormalization constants were discarded, too. Nevertheless, discarding the renormalization constants is highly ambiguous, since – as it was depicted in chapter 2 – they correspond to free parameters of the theory which should be fixed by experiment. Now there are several ways how one could deal with those renormalization constants. The most obvious way is, to pose further restrictions on the form of the occuring distributional extensions. For example, one could demand, that the distributional extensions $t^{(n)}$ should satisfy certain algebraic relations – as e.g. $\eta t^{(n)} = t^{(n-1)}$ – by which they would mimick properties of η^{-n}. This fixes then the freedom in the renormalization constants to some extent. Another serious drawback of the present analysis – as well as of the analysis in [15] – is that it relies on the assumption that back-reaction effects can be neglected. This assumption is problematic, since one could expect that the energy density of the classical solutions exhibits even in the renormalized case a behaviour which violates this assumption.

How to do it better: A consecutive step of the present analysis would be a thorough analysis of the renormalized classical energy momentum tensor. Thereby one should understand, to which extent the occuring renormalizaion constants are physical and if the assumption that backreaction effects are negligible is appropriate. Another consecutive step of the present analysis would be, to analyse if the renormalized and non-renormalized classical theories define inequivalent quantum field theories with different completeness properties. Although the comparison of the present investigation with the results of [15] suggests that this is indeed the case, it is not shown explicitely that the resulting quantum field theories are mathematically inequivalent. Finally one should try to understand, if the present renormalization formalism can be also applied on the level of an realistic model for a classical field-theoretic measurement apparatus traversing the initial singularity. Moreover, it would be desirable to develop a measurement theory in the context of classical field theory and to apply this on cosmological singularities. We will comment on this again in section 7.2.

5. It from null?

In the last chapter we have analysed a singular FLRW spacetime in the context of classical field theory. Thereby we have obtained as a result, that the occuring states can be renormalized whereby the occurring infinites disappear. In this chapter, on the other hand, we will first perform in section 5.1 a geometric analysis of the same FLRW spacetimes by considering the qualitative behaviour of geodesics in the vicinity of the singularity. This will lead to an interesting observation: Qualitatively, time- and spacelike geodesics behave more and more lightlike as they approach the singularity. We will then complement this observation in section 5.2 by discussing various examples from the literature which suggest, that gravity exhibits in extreme situation a dimensional reduction and an ultrarelativistic behaviour. Motivated by those observations we will then rise in section 5.3 the question, if it could be, that lightlike directions are more fundamental than time- or spacelike directions, i.e. if the relativistic spacetime could be merely a effective description of a microscopically purely lightlike geometry. This idea will then be the starting point for the analysis of the next chapter, where the microscopic ultrarelativistic symmetries of spacetime will be analysed.

5.1. Qualitative behaviour of geodesics in FLRW backgrounds

In a colloquial language, the Riemann curvature tensor measures the deviation of a curved geometry from a flat pseudo-Riemannian model geometry (cf. [44, 161]). Hence, the divergent Riemann scalar of singular FLRW spacetimes should be understood as a symptom of the breakdown of the pseudo-Riemannian model in the vicinity of the singularity. Usually, this is interpreted as a sign that geometry itself breaks down at the singularity. But one could wonder also, if this point of view is not too pessimistic and if it could not be that the singularity marks merely a change of the geometric model. To gain a better geometric intuition for the FLRW singularity, we will analyse in this section the qualitative features of geodesic motion in its vicinity. Thereby we will see, that all geodesics behave more and more lightlike as they approach the singularity. This rises especially the question, if the pseudo-Riemannian model should be replaced by some ultrarelativistic model in the vicinity of the singularity. In section 5.2 we will then complement the results of this section by reviewing some situations in which gravity exhibits a similar behaviour.

A first glimpse at the equations of motion in FLRW backgrounds: We consider FLRW spacetimes given by the metric

$$ds^2 = -dt^2 + b(t)^2(dx^2 + dy^2 + dz^2)$$

in coordinates (t, x, y, z) with $t > 0$ and $(x, y, z) = (x^1, x^2, x^3) \in \mathbb{R}^3$. We demand, that $b(t) > 0$ holds for $t \neq 0$ and that $b(0) = 0$. The hypersurface at $t = 0$ is then called the singularity of this spacetime. An interesting aspect of such FLRW spacetimes is, that – due to their large symmetry – the motion of a test particle can be analysed without referring to the geodesic equations. Especially, the motion of a test particle is fully determined by the conserved momenta P^i associated with the Killing vectors ∂_i for $i \in \{1, 2, 3\}$ and by the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \sigma$, which fixes an affine parameter for time- and spacelike geodesics in the case of $\sigma \neq 0$ or corresponds to the constraint that the path should be null in the case $\sigma = 0$. By some effort (cf. appendix B.4) we obtain the following ordinary differential equations, which determine the motion:

$$\frac{dt}{d\lambda} = \sqrt{\frac{P^2}{b(t)^2} - \sigma} \tag{5.1}$$

$$\frac{dx^i}{dt} = \frac{P^i}{\sqrt{P^2 b(t)^2 - \sigma b(t)^4}} \tag{5.2}$$

Here we have set $P^2 := \sum_{i=1}^{3} \left(P^i\right)^2$. Already at this stage, some interesting aspects occure. Let therefore be from now on $P > 0$. Then, since $b(t) \to 0$ as $t \to 0$ we see, that in both equations the terms proportional to σ under the square-root are damped as $t \to 0$. To see this, multiply (5.1) and (5.2) with $b(t)$ and perform then an expansion in $b(t)$ around $b(t) = 0$:

$$\begin{aligned} b(t)\frac{dt}{d\lambda} &= \sqrt{P^2 - \sigma b(t)^2} = P - b^2\frac{\sigma}{2P} + \mathcal{O}(b^4) \\ b(t)\frac{dx^i}{dt} &= \frac{P^i}{\sqrt{P^2 - \sigma b(t)^2}} = \frac{P^i}{P} + b^2\frac{P^i\sigma}{2P^3} + \mathcal{O}(b^4) \end{aligned}$$

By observing, that the equations (5.1) and (5.2) read in the case of null geodesics ($\sigma = 0$) as

$$\begin{aligned} b(t)\frac{dt}{d\lambda} &= P \\ b(t)\frac{dx^i}{dt} &= \frac{P^i}{P}, \end{aligned}$$

we observe, that the equations of motion (5.1) and (5.2) are at the leading order determined by the behaviour of the null geodesics at the singularity. Finally, the case $P = 0$ is only allowed for timelike geodesics (see appendix B.4), i.e. for $\sigma < 0$, and the equations reduce in this situation to

$$\frac{dt}{d\lambda} = \sqrt{-\sigma},$$
$$\frac{dx^i}{dt} = 0.$$

Consequently the geodesic motion is in this case given by

$$t(\lambda) = \sqrt{-\sigma}(\lambda - \lambda_0) + t_0, \tag{5.3}$$
$$x^i(t) = x^i_0. \tag{5.4}$$

Solutions to the geodesic equations in FLRW backgrounds: We now want to get a little bit more concrete and present the solutions of the equations of motion in the case $b(t) = b_0 t^\gamma$ for $\gamma \in (0,1)$ with initial conditions $t(0) = 0$ and $x^i(0) = 0$. We then have that the null geodesics are given by

$$t(\lambda) = \left[(\gamma+1)\frac{P}{b_0}\lambda\right]^{\frac{1}{\gamma+1}}, \tag{5.5}$$
$$x^i(t) = \frac{P^i}{Pb_0(1-\gamma)} t^{1-\gamma}, \tag{5.6}$$

while the spacelike (i.e. $\sigma > 0$) and timelike (i.e. $\sigma < 0$) geodesics are given for $P > 0$ by the somewhat messy expressions

$$\lambda_\sigma(t) = \frac{b_0 t^{\gamma+1}}{P(1+\gamma)} \left[{}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right], \tag{5.7}$$
$$x^i_\sigma(t) = e^i\left[\frac{t^{1-\gamma}}{(1-\gamma)b_0}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right], \tag{5.8}$$

where (5.7) determines $t(\lambda)$ for non-null geodesics implicitely as the inverse of $\lambda(t)$ and equations (5.6), (5.8) determine the graph of null and non-null geodesics, respectively. For $P = 0$ the solution is given by (5.3) and (5.4). To gain a better intuitive understanding of those equations, please recall from the last paragraph that the equations of motion for non-null geodesics seem, as long as $P > 0$, to be dominated by null behaviour towards the singularity. Hence we expect, that this behaviour should also be represented by the solutions of the equations of motion. To see that this is really the case let from now on be $P > 0$ (we will comment on the case of $P = 0$ again at the end of this section). We then invert (5.5), which yields:

$$\lambda(t) = \frac{1}{P}\frac{b_0}{1+\gamma}t^{1+\gamma}.$$

Now let $\lambda_\star(t)$ and $x^i_\star(t)$ be the inverse parametrization and the graph of null geodesics emanating from the singularity, respecitvely, i.e.:

$$\lambda_\star(t) = \frac{1}{P}\frac{b_0}{1+\gamma}t^{1+\gamma}, \tag{5.9}$$

$$x^i_\star(t) = \frac{P^i}{Pb_0(1-\gamma)}t^{1-\gamma} \tag{5.10}$$

We then can express the functions $\lambda_\sigma(t)$ and $x^i_\sigma(t)$ given by (5.7) and (5.8) as

$$\lambda_\sigma(t) = \lambda_*(t)\tilde{G}^{(\gamma)}_{\sigma,P}(t), \tag{5.11}$$

$$x^i_\sigma(t) = x^i_\star(t)\tilde{H}^{(\gamma)}_{\sigma,P}(t), \tag{5.12}$$

with

$$\tilde{G}^{(\gamma)}_{\sigma,P}(t) = {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right),$$

$$\tilde{H}^{(\gamma)}_{\sigma,P}(t) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right).$$

Here ${}_2F_1(a,b,c;z)$ is the Gaussian hypergeometric function, which is explicitly defined by the series

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b_n)}{(c)_n}\frac{z^n}{n!}$$

for $|z| < 1$, where $(q)_n$ denotes the rising Pochhammer symbol

$$(q)_n := \frac{\Gamma(q+n)}{\Gamma(q)},$$

and analytic continuation elsewhere. Hence we have that, for a sufficiently small neighborhood of the initial singularity at $t = 0$, the functions (5.11) and (5.12) can be written as:

$$\lambda_\sigma(t) = \lambda_*(t)\left[1+\mathcal{O}\left(b(t)^2\right)\right],$$

$$x^i_\sigma(t) = x^i_\star(t)\left[1+\mathcal{O}\left(b(t)^2\right)\right].$$

By recalling, that $\lambda_\star(t)$ and $x^i_\star(t)$ as defined in (5.9) and (5.10) are the (inverse) parametrization and the graph of null geodesics, we see that time- and spacelike geodesics are (as long as $P \neq 0$) for small t dominated by the behaviour of null geodesics.

Qualitative behaviour towards the singularity: After all those abstract thoughts it is now the time for some pictures. Therefore we switch now to conformal coordinates where a new time coordinate η defined as

$$\eta(t) = \int_0^t dt' \frac{1}{b(t')}.$$

is introduced. This leads then to the metric

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dx^2 + dy^2 + dz^2\right). \tag{5.13}$$

We now choose $b(t) = b_0 t^\gamma$ with $b_0 = (1-\gamma)^{-\gamma}$ and $\gamma = (1+c)^{-1}c$ as a scale factor, which implies $a(\eta) = \eta^c$. Since the metric (5.13) is conformally equivalent to the Minkowski metric, null relations are the same as in Minkowski spacetime and especially the graphs of null geodesics are the same as in Minkowski space. We consider the same initial conditions as before and obtain then, that null geodesics are given by

$$\begin{aligned} \lambda_\star(\eta) &:= \frac{1}{P}\left[\frac{1-\gamma}{1+\gamma}\right]\eta^{\frac{1+\gamma}{1-\gamma}}, \\ x^i_\star(\eta) &:= e^i\eta + x^i_0, \end{aligned}$$

while the non-null geodesics are given, for $P > 0$, by

$$\begin{aligned} \lambda_\sigma(\eta) &= \lambda_*(\eta) G^{(\gamma)}_{\sigma,P}(\eta) \\ x^i_\sigma(\eta) &= x^i_\star(\eta) H^{(\gamma)}_{\sigma,P}(\eta) \end{aligned}$$

with

$$\begin{aligned} G^{(\gamma)}_{\sigma,P}(\eta) &:= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(1+\frac{1}{\gamma}\right), \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right), \\ H^{(\gamma)}_{\sigma,P}(\eta) &:= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right), \end{aligned}$$

where $\sigma > 0$ corresponds to affinely parametrized spacelike geodesics and $\sigma < 0$ corresponds to affinely parametrized timelike geodesics. It is valuable to note, that the hypergeometric function seems to assume a much simpler form if one sets $\gamma = 1/n$ for $n \in \mathbb{N}$. E.g. we have with Mathematica for $\gamma = \frac{1}{2}$

$$H^{(\gamma)}_{\sigma,P}(\eta) = \frac{P \arcsin \frac{\eta\sqrt{\sigma}}{P}}{\eta\sqrt{\sigma}} = 1 + \frac{\sigma}{6P^2}\eta^2 + \mathcal{O}(\eta^4)$$

and for $\gamma = \frac{1}{3}$:

$$H^{(\gamma)}_{\sigma,P}(\eta) = -\frac{2P^2\left(-1+\sqrt{1-\frac{\eta\sigma}{P^2}}\right)}{\eta\sigma} = 1 + \frac{\sigma}{4P^2}\eta + \frac{\sigma^2}{8P^4}\eta^2 + \frac{5\sigma^3}{64P^6}\eta^3 + \mathcal{O}(\eta^4).$$

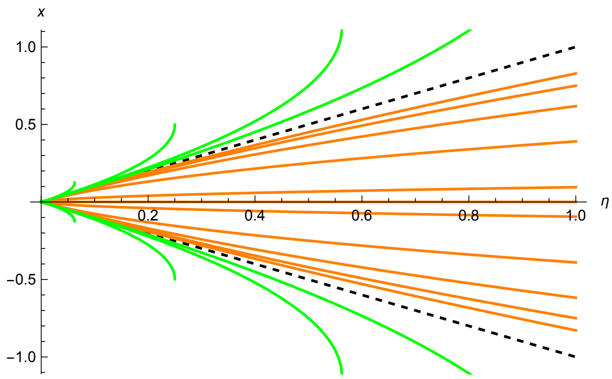


Figure 5.1.: Geodesics in the η-x^1-plane for $\gamma = 1/3$ emanating at the singularity $\eta = 0$ with initial condition $x^i = 0$ and printed in the interval $\eta \in (0, 1]$. Timelike geodesics are orange, null geodesics are dashed and spacelike geodesics are green. For the timelike geodesics the momenta are, from the center to outside, given by $P = 0, \frac{1}{20}, \frac{5}{20}, \frac{10}{20}, \frac{15}{20}, 1$. For the timelike geodesics, the momenta are, from the center to the outside, given by $P = \frac{15}{20}, \frac{10}{20}, \frac{5}{20}, \frac{1}{20}$.

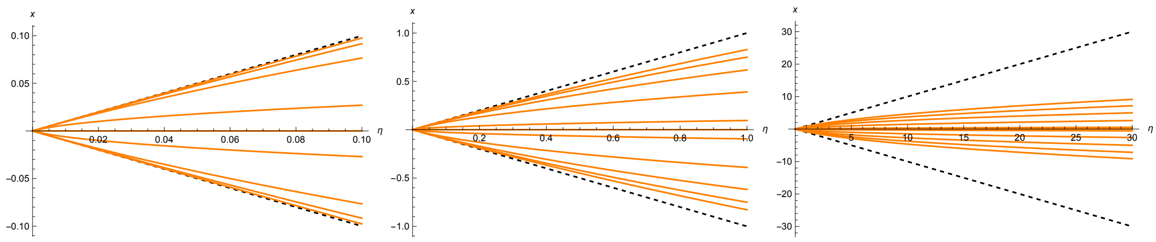


Figure 5.2.: Timelike geodesics (orange) are plotted for the same momenta, as in figure 5.1, but this time for different ranges of η: In the left picture we have $\eta \in \left(0, \frac{1}{10}\right)$, in the middle picture $\eta \in (0, 1)$ and in the right picture $\eta \in (0, 30)$. Thereby one realizes, how the timelike geodesics approach for $P \neq 0$ more and more the light cone towards the singularity.

In figure 5.1 we have now plotted the graphs of various geodesics in the η-x-plane for the case $\gamma = \frac{1}{3}$. One observes that all geodesics behave approximatively as null geodesics in the vicinity of the singularity, while for larger times their respective time- and spacelike behaviour gets more and more dominant. This is also apparent from figure 5.2 where we have plotted timelike geodesics for different regions around the singularity. Observe, that this behaviour matches the discussion of the previous paragraphs. This also suggests a way, how normal neighborhoods degenerate towards the initial singularity: If one approaches the initial singularity, normal neighborhoods of points getting more and more deformed towards the light cone. Also, as it gets apparent from the pictures, an interesting "discontinuity" occurs: Although all geodesics for $P > 0$ approach the light cone in the vicinity of the singularity, the timelike geodesic with $P = 0$ is just a straight line. If one imagines the situation, where an observer falls into a big crunch singularity along the geodesic with $P = 0$, the observer experiences a situation where any other observer moving at non-vanishing momentum $P > 0$ gets accelerated to the speed of light under approaching the singularity.

Towards a coordinate-invariant statement: In this section we have analysed the qualitative behaviour of geodesics in the vicinity of the initial singularity in a spatially flat FLRW spacetime. Thereby we obtained as a qualitative result, that the motion of non-null geodesics approaches for non-zero momenta more and more the motion of null geodesics. This observation rises then immediately the question, if this behaviour is coordinate-invariant and if it can be encoded in terms of a coordinate-invariant, quantitative statement.

Fortunately, the present analysis has several features which indeed suggest, that the behaviour found in this section is a coordinate-invariant feature of FLRW spacetimes. For example, the equations (5.7, 5.8) and (5.11, 5.12) suggest, that the behaviour of space- and timelike geodesics – and especially their deviation from null geodesics – is entirely encoded by two scalar functions

$$ {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right) \quad \text{and} \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right) $$

which are especially bounded and non-zero on the whole spacetime including the singularity, since they both approach 1 as $t \to 0$.

This statement has then also a tensorial analogue on the level of tangent vectors. Therefore let $x^\mu_{\sigma,\vec{P}}(\lambda)$ be an affinely parametrized geodesics corresponding to momentum $\vec{P} = (P^1, P^2, P^3)$ and $\sigma \neq 0$ defined. Let now $x^\mu_{\star,\vec{P}}(\lambda)$ be a null geodesic with momentum $\vec{P} \neq 0$. We then can write for the respective tangent vectors

$$ \frac{\partial x^\mu_{\sigma,\vec{P}}}{\partial\lambda} = A^\mu{}_\nu[\sigma,\vec{P}]\frac{\partial x^\nu_{\star,\vec{P}}}{\partial\lambda} \tag{5.14} $$

with

$$ \left(A^\mu{}_\nu[\sigma,\vec{P}]\right) = \begin{pmatrix} \sqrt{1-\sigma\frac{b(t)^2}{P^2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

which especially satisfies

$$ \left(A^\mu{}_\nu[\sigma,\vec{P}]\right) \to \mathbf{1}_{4\times4} \quad \text{as } t \to 0. $$

This statement has now various interesting aspects. Most importantly, the tensor field $A^\mu{}_\nu[\sigma,\vec{P}]$ approaches the identity tensor towards the singularity and makes thereby the statement precise that the velocity vectors of time- and spacelike geodesics with non-vanishing momentum approach the velocity vectors of null geodesics. Moreover, this statement is obviously covariant and is defined in terms of invariantly defined objects: The tensor A relates the velocity vector of time- and spacelike geodesics with non-vanishing momentum to their null counterparts with the same momentum. Since the quantities σ and $\vec{P}$ are invariantly defined, this statement is hence independent of the considered coordinate system, which suggests indeed that the behaviour found in this section is a coordinate-invariant feature of FLRW spacetimes. Finally observe, that the tensor $A^{\mu}{}_{\nu}[\vec{P}]$ is a tensorial quantity which is defined on the full spacetime including the singularity and is especially non-zero thereon. Since it encodes, how non-null geodesics differ from null geodesics, this corresponds to the statement, that – although the metric is degenerate on the singularity – the deviation of non-null geodesics from null geodesics is a well-defined quantity on the whole spacetime. Moreover if one inverts above equation (5.14) and considers the case $P = 0$ one obtains as a result, that timelike geodesics with $P = 0$ approach zero as compared to null geodesics.

This means: If one looks at the initial singularity from a perspective, where the inertial frame is not given by vielbein frames – i.e. tangent vectors of orthonormal time- and spacelike geodesics – but by the tangent vectors of a family of null geodesics, then the infinitesimal geometry exhibits apparently a well-defined limit at the singularity. This limit corresponds then to a situation where all time- and spacelike geodesic tangent vectors with $P \neq 0$ approach the light cone while the timelike geodesic tangent vector with $P = 0$ approaches zero (as compared to null geodesics), i.e. the apex of the light cone.

Nevertheless this observation still rises some questions. For example one should analyse precisely, to which extent it is stable with respect to coordinate transformations which degenerate on the singularity. Moreover one should make it mathematically more precise, e.g. in the context of geodesic sprays, and should analyse, how far it really corresponds to a change of the geometric model in the vicinity of the singularity. We will comment on this again in section 7.2.

5.2. On dimensional reduction and ultrarelativistic effects in gravity

In the last section we have analysed the geodesic geometry of the FLRW singularity and obtained as a qualitative result, that the geometry seems to behave ultrarelativistically as one approaches the singularity. In addition, since the 4-dimensional tangent space seems to be deformed towards the 3-dimensional light cone, one could be tempted to think that some dimensional reduction takes place in the vicinity of the singularity. In this section we will see, that this kind of behaviour is not exclusively tied to FLRW spacetimes but occurs in many situations, in which gravity is probed under extreme circumstances. In this section we will present some of those scenarios. Our main reference for this section is given by [42] while other references are stated when needed[1].

[1] We want to note, that the ultrarelativistic behaviour in those stiuation was mostly not noticed by the literature, in contrast to the dimensional reduction which is a commonly observed phenomenon.

The Kasner spacetime: The most obvious question raised by the analysis of the last section is to which extent the results presented there can be generalized to other curvature singularities. Of special interest is hereby the Kasner singularity, since, as we will see later, it has an important application within the BKL conjecture and appears also in the context of the short distance Wheeler-DeWitt equation.

The *Kasner spacetime* is a vacuum solution to Einstein's equation which is given by the metric

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$$

with real *Kasner exponents* (p_1, p_2, p_3), which have to satisfy the *Kasner conditions*

$$\sum_{i=1}^{3} p_i = 1, \tag{5.15}$$

$$\sum_{i=1}^{3} p_i^2 = 1. \tag{5.16}$$

Since (5.15) describes a plane (the so called *Kasner plane*) in $\mathbb{R}^3$ and (5.16) describes a sphere (the so called *Kasner sphere*), the Kasner conditions can be understood as the statement, that the parameters p_i have to lie on the intersection of the Kasner plane with the Kasner sphere. An important property of the Kasner exponents is, that the 3-tuple (p_1, p_2, p_3) is (up to permutations) either $(0, 0, 1)$ or contains always one negative and two positive exponents.

As it was the case for the FLRW spacetime, the vector fields ∂_i with $i \in \{1, 2, 3\}$ are Killing vector fields. The geodesics are in this case determined due to the high amount of symmetry by the equations

$$\frac{dt}{d\lambda} = \sqrt{\sum_{i=1}^{3} t^{-2p_i}(P^i)^2 - \sigma},$$

$$\frac{dx^i}{dt} = \frac{P^i t^{-2p_i}}{\sqrt{\sum_{i=1}^{3} t^{-2p_i}(P^i)^2 - \sigma}}.$$

Here $\sigma \in \mathbb{R}$ with $\sigma = 0$ corresponds to null geodesics, while $\sigma > 0$ corresponds to spacelike and $\sigma < 0$ corresponds to timelike geodesics.

On the one hand, the Kasner geodesics show a similar behaviour towards the singularity at $t = 0$ as compared to the FLRW metric. To see this, consider first the case $(p_1, p_2, p_3) =$

$(1,0,0)$. In this case the geodesic equations read

$$\begin{aligned}
t\frac{dt}{d\lambda} &= \sqrt{(P^1)^2 + \left[(P^2)^2 + (P^3)^2 - \sigma\right] t^2} \\
t\frac{dx^1}{dt} &= \frac{P^1}{\sqrt{(P^1)^2 + \left[(P^2)^2 + (P^3)^2 - \sigma\right] t^2}} \\
\frac{dx^j}{dt} &= \frac{P^j}{\sqrt{(P^1)^2 t^{-2} + (P^2)^2 + (P^3)^2 - \sigma}}
\end{aligned}$$

whose small-t behaviour is given by the following expansions:

$$\begin{aligned}
t\frac{dt}{d\lambda} &= P^1 + \frac{(P^2)^2 + (P^3)^2 - \sigma}{2P^1} t^2 + \mathcal{O}(t^4) \\
t\frac{dx^1}{dt} &= 1 - \frac{(P^2)^2 + (P^3)^2 - \sigma}{2(P^1)^2} t^2 + \mathcal{O}(t^4) && (5.17) \\
\frac{dx^j}{dt} &= \frac{P^j}{P^1} t - \frac{P^j}{2P^1} \left((P^2)^+ (P^3)^2 - \sigma\right) t^3 + \mathcal{O}(t^5) && (5.18)
\end{aligned}$$

We then see directly, that σ is again irrelevant up to leading order in t. Hence, qualitatively spoken, it seems that timelike, spacelike and null geodesics have again the same limiting behaviour towards the singularity.

But as an additional feature as compared to the FLRW case we have here a more severe dimensional reduction. To see this, observe that in above expansions (5.17 - 5.18), up to leading order in t, only P^1 is of relevance. Moreover observe, that $\dot{x}^j \sim 0$ for $j \neq 1$ while $\dot{x}^i \sim t^{-1}$ up to leading order in t, where the dot denotes the derivative with respect to t. This could be interpreted as a sign of dimensional reduction in the vicinity of the singularity (cf. [42]).

We now want to analyse, if those features are also present for more general Kasner exponents. Therefore let without loss of generality $p_1 \geq p_2 > 0 > p_3$. Set moreover $p_1 = a$, $p_2 = b$ and $p_3 = -c$ and observe, that then $a \geq b$ holds. Then the geodesics are in this case determined by:

$$\begin{aligned}
t^a\frac{dt}{d\lambda} &= \sqrt{(P^1)^2 + (P^2)^2 t^{2(a-b)} + (P^3)^2 t^{2(a+c)} - \sigma t^{2a}}, \\
t^a\frac{dx^1}{dt} &= \frac{P^i}{\sqrt{(P^1)^2 + (P^2)^2 t^{2(a-b)} + t^{2(a+c)} (P^3)^2 - \sigma t^{2a}}}, \\
t^{2b-a}\frac{dx^2}{dt} &= \frac{P^2}{\sqrt{(P^1)^2 + (P^2)^2 t^{2(a-b)} + (P^3)^2 t^{2(a+c)} - \sigma t^{2(a-b)}}}, \\
t^{2c-a}\frac{dx^3}{dt} &= \frac{P^3}{\sqrt{(P^1)^2 + (P^2)^2 t^{2(a-b)} + (P^3)^2 t^{2(a+c)} - \sigma t^{2(a+c)}}}.
\end{aligned}$$

Hereby we see again, that the terms proportional to σ are damped and the structure of the equation suggests also a dimensional reduction. Hence, qualitatively, the Kasner geodesics shows a similar behaviour in vicinity of the singularity as the FLRW geodesics, with the difference that additionally a more severe dimensional reduction occurs. More information on Kasner geodesics can be found in [89] and more information on dimensional reduction in Kasner spacetimes can be found in [42].

The BKL-conjecture and the Kasner singularity: In the last paragraph we have seen, that Kasner geodesics exhibit in some sense a ultrarelativistic behaviour in the vicinity of the singularity, since the behaviour of time- and spacelike geodesics equals the behaviour of null geodesics up to leading order in time. In this paragraph we want to present another aspect of the Kasner singularity which is also connected to ultrarelativistic behaviour.

Belinski, Khalatnikov and Lifshitz analysed in the 1960 in a series of papers (cf. e.g. [26]) generic spacelike singularities in general relativity and presented some powerful arguments supporting a claim, which goes today under the name of the *BKL-conjecture*. Roughly said, the BKL-conjecture states, that as one approaches a generic spacelike singularity, spatial points decouple and as a consequence timelike derivatives in Einstein's equations dominate over spacelike derivatives which finally leads to a reduction of Einstein's equations to a set of ordinary partial differential equations with respect to a single time variable (cf. [93, 18, 25]). This conjecture then lead also to the claim, that the behaviour of any spacetime in the vicinity of a cosmological singularity is described by a sequence of (generalized) Kasner spacetimes (cf. [92]).

This conjecture has multiple aspects, which are notable in the context of the present analysis. The most obvious aspect is, that it suggests that the Kasner singularity is in some sense the most general cosmological singularity, which motivates the conjecture that the type of ultrarelativistic behaviour that we have encountered previously is indeed a general feature of cosmological singularities. Moreover, field equations with dominant timelike derivatives are a feature of so called ultralocal field theories (cf. [114, 115]) which are also the natural field theories in ultrarelativistic contexts. Especially, field theories with Carrollian (i.e. ultrarelativistic) symmetry (cf. [19]) are often ultralocal or have an ultralocal sector (cf. [154, 20]). This suggests that the BKL-conjecture itself incorporates the statement that general relativity reduces to an ultrarelativistic field theory in the vicinity of cosmological singularities.

Short distance Wheeler-DeWitt equation, Strong gravity and stuff like that: If one tries to quantize gravity canonically, one obtains the famous Wheeler-DeWitt equation which reads as

$$\left[-16\pi\ell_P^2 G_{ijkl}\frac{\delta}{\delta q_{ij}}\frac{\delta}{\delta q_{kl}} + \frac{1}{16\pi\ell_P^2}\sqrt{\det(q)}R^{(3)}\right]\Psi[q] = 0,$$

where $q = q_{ij}$ is the spatial metric on a given Cauchy surface, $R^{(3)}$ is its Ricci scalar and

$$G_{ijkl} = \frac{1}{2}\frac{1}{\sqrt{\det(q)}}\left(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl}\right)$$

is the so called *DeWitt supermetric* (cf. [127, 42]). If one performs the strong coupling limit $\ell_P \to \infty$, which can be understood as the limiting behaviour on small scales where the Planck length gets relevant, then obviously the second term of above equation drops out. As it was pointed out by Isham in [104] (cf. also [42]), this limit is an ultralocal limit since the second term, which contains spatial derivatives in the metric, drops out. As an interesting fact, this theory of strong gravity is indeed solvable (cf. [145, 146]). In this context one then obtains as a result (cf. [77]) that the quasiclassical approximation for the evolution of wave functionals towards the initial singularity is dominated by asymptotic scatterings of Kasner wave functions, which is riminiscent of the BKL behaviour as presented in the last section. Moreover, since the Planck length is given by

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}},$$

the limit $\ell_P \to \infty$ corresponds obviously to an ultrarelativistic $c \to 0$ limit. Since the $c \to 0$ limit of the Poincaré group is the ultrarelativistic Carroll group (cf. [19]) and since classical ultrarelativistic gravity corresponds to a contraction of the Poinacré group to the Carroll group (cf. [49, 88]), this suggests a speculative connection between small-scale quantum gravity, the BKL-conjecture and an ultrarelativistic "sector" of gravity.

Effective relativity from microscopic null geometry: In the last paragraphs we have presented some evidence, that gravity exhibits an ultrarelativistic behaviour in several extreme situations. This motivates the idea, that gravity could be described on fundamental microscopic scales by some sort of ultrarelativistic field theory. Pictorially spoken this corresponds then to a situation where microscopically anything moves at the speed of light. Since our macroscopic world obviously differs from this picture, this rises the question how causal relationships should emerge from a microscopic setting where only lightlike geometry is of relevance.

To answer this question we would like to note, that it was already realised by Penrose and Kronheimer in the 60s that null relations (so called *horismus relations*) are enough to determine the causal structure of strongly causal spacetimes (cf. [118, 133]). Let therefore $(\mathscr{M}, g)$ be a time-oriented 4-dimensional pseudo-Riemannian spacetime with metric g. Let the time-orientation be modelled by a timelike vector field $X \in \mathfrak{X}(\mathscr{M})$ satisfying $g(X, X) < 0$. We then call a smooth curve

$$\gamma : (a, b) \to \mathscr{M}$$

with tangent vector field $\dot\gamma$ *timelike* if $g(\dot\gamma, \dot\gamma) < 0$, *causal* if $g(\dot\gamma, \dot\gamma) \leq 0$, *null* if $g(\dot\gamma, \dot\gamma) = 0$, *spacelike* if $g(\dot\gamma, \dot\gamma) > 0$ and *future-pointing* if $g(\dot\gamma, X) < 0$. By this one defines the following standard relations on $\mathscr{M}$ (cf. [134]):

- x *chronologically precedes* y, written $x \ll y$, if there is a future-pointing timelike curve from x to y.

- x *strictly causally precedes* y, written $x < y$, if there is a future-pointing causal curve from x to y.
- x *causally precedes* y, written $x \leq y$, if $x < y$ or $x = y$.
- x *horismos* y, written[2] $x \rightharpoonup y$, if there exists a future directed null curve from x to y or if $x = y$.

It follows then that in a large class of realistic spacetimes the causal relations are generated by null relations. Therefore we say, that x is connected to y by a *horismos chain*, written $x \preceq y$, if and only if there is a finite sequence (called the horismos chain) $(z_n)_{1 \leq n \leq N}$ such that

$$x = z_1 \rightharpoonup z_1 \rightharpoonup \ldots \rightharpoonup z_N = y \tag{5.19}$$

holds. It then can be shown (cf. [118, 133]) that for strongly causal spacetimes (and hence especially for globally hyperbolic spacetimes) $x \leq y$ holds if and only if $x \preceq y$ holds. It is important to notice, that the number N, which determines the length of the horismus chain (5.19) that connects x and y, must allowed to be arbitrarily large for this statement to hold. Pictorially spoken, one could hence understand causal relations as some sort of "thermodynamic limit" of horismos relations.

By this we see that a macroscopic causal geometry can indeed emerge as an effective description of a microscopically lightlike geometry. Another evidence which supports this picture comes from [32]. Here it was shown, that any map which maps all light cones of Minkowski space onto themselves is necessarily (up to a scale factor) an inhomogeneous Lorentz transformation. This means, that special relativity is entirely fixed by the null properties of Minkowski spacetime without the need to consider time- or spacelike relations.

5.3. Conclusion

In this chapter we have presented some examples, which provide – together with the results of the first section of this chapter – evidence to the claim, that gravity exhibits an ultrarelativistic and dimensionally reduced behaviour in extreme situations. For the sake of completeness we want to note, that the list of situations presented in this thesis is not exhaustive. For example, as we pointed out before, ultrarelativistic behaviour seems to be interconnected with the causal decoupling of nearby points. This phenomenon goes in related situations also under the name of *asymptotic silence* and arises on microscopic scales indeed in many different approaches to quantum gravity, as spin foam models ([43]), causal sets ([65]), Planckian scattering in string theory (cf. [42] and references cited therein) and loop quantum cosmology ([132]).

[2] In the literature one usually write $x \to y$ for the horismos relation. But to avoid confusion with limits, which are regularly used in this thesis, we have decided for a different notation

The results of this section then raise the question, if it is possible to describe gravity on fundamental scales by some type of ultrarelativistic field theory. The results of the last paragraph can be hereby understood as a first plausibility analysis: If spacetime is on microscopic scales dominated by some sort of ultrarelativistic behaviour, then causal and spacelike relations should be merely an effective description of null relations, what is indeed the case as we have explained there. Of course, the most important question is then if such an ultrarelativistic field theory exists which implies general relativity on macroscopic scales and what its features are. Since symmetries are usually a good guiding principle for the building of physical theories, a first preliminary step could be to analyse the possible microscopic ultrarelativistic symmetry groups, which could be eligible as gauge groups of such a theory. This will be done partially in the next chapter.

6. The microscopic ultrarelativistic geometry of spacetime

In the last chapter we have presented some evidence that gravity shows an ultrarelativistic behaviour in extreme situations. Motivated by this observation we have formulated the claim that gravity could be described by an ultrarelativistic field theory on microscopic scales. In this section we will perform a first step for the analysis of this claim: We will analyse the microscopic ultrarelativistic symmetries of a 4-dimensional pseudo-Riemannian spacetime, to understand which symmetry groups could be used for the construction of such a theory. This will be done by investigating the universal geometric structures on infinitesimal tangent light cones together with an analysis of their symmetry groups. As we will see, this will lead to surprisingly rich structures hidden in the theories of relativity. Especially, two natural microscopic symmetry groups will arise: a non-trivially represented Lorentz group and an infinite dimensional group, that resembles the famous *Bondi-Metzner-Sachs*-group. Those two groups encompass a rich mathematical structure, since the latter contains the former as a non-canonical subgroup, next to infinitely many other Lorentz subgroups. Moreover we will understand, that the non-trivially represented Lorentz group can be understood as the ultrarelativistic, *Carrollian* symmetry group of the infinitesimal light cone, while the other, BMS-like group is the corresponding conformal extension. In addition, we will compare our investigation with the classical BMS analysis and will understand thereby, that the microscopic BMS-like group is indeed conceptually and structurally a microscopic analogue of the BMS group. Finally we will show, that both symmetry groups are gauge groups for the bundle of null vectors and will discuss the intuition behind those symmetry groups. Also, our results imply, that BMS-like groups arise not only as macroscopic, asymptotic symmetry groups in cosmology, but describe also a fundamental and apparently unknown microscopic symmetry of pseudo-Riemannian geometry. This chapter is based on the author's publication [176] and an outline of this chapter can be found at the end of the introduction (i.e. section 6.1) of this section.

6.1. Introduction

In the last chapter we saw, that lightlike relations are in some sense more fundamental than causal relations, since two points are causally seperated if and only if they can be connected by a curve that is piecewise a null geodesic. This picture motivates then an elementary, though speculative, idea how a spacetime which is ultrarelativistic on microscopic scales could be described by relativity on macroscopic scales: If we assume, that microscopically only null curves are permitted and if we allow those curves additionally to "scatter" on a very small, say Planckian, scale, then effectively such a motion should be described by a timelike curve. Kinematically, this model would be characterized by the constraint that the velocity vectors of all curves had to lie on the microscopic tangent light cones while dynamical laws should then control the scattering on Planckian scales. Although it is of course not clear from the outset, that meaningful dynamical laws exist which would imply general relativity on macroscopic scales, this picture makes it clear that *any* such dynamical theory should respect the symmetries of the kinematical arena, i.e. the symmetries of the bundle of microscopic tangent light cones. From a more physical perspective this picture could also be understood as a scale-dependence of the considered inertial frames: Although macroscopically inertial frames are given by vielbein frames – since all timelike directions can be generated on a macroscopic scale by scattered null curves – microscopically only null directions are allowed and hence microscopic inertial frames should be described by frames adapted to microscopic tangent light cones. This idea then motivates the study of the universal geometric structures and the symmetries of microscopic tangent light cones, which we will perform in this section.

On ultrarelativistic structures and symmetry groups: A situation in which all motion is constrained to be lightlike can be also seen from a different perspective, namely that of ultrarelativistic geometric structures. Historically, the study of ultrarelativistic structures started with the paper [120] by Jean-Marc Lévy-Leblond and was continued in the better known paper *Possible kinematics* [19] by Henri Bacri and Lévy-Leblond, where in some sense all possible 4-dimensional kinematic symmetry groups where derived (cf. also [62]). Next to well-known examples as the Galilei, the Lorentz or the de-Sitter groups, also more exotic groups arose and especially the ultrarelativistic *Carroll group* was described.

The Carroll group can be understood as a Wigner-Inönü-Contraction (cf. [157, 103]) of the Poincaré group under the limit $c \to 0$ (cf. [120]). It hence should be understood as an *ultrarelativistic* symmetry group, describing situations where anything moves at the speed of light. It is opposed by the well-known *Galilei*-group which arises on the other hand as a contraction of the Lorentz group under the limit $c \to \infty$. The Carroll group was named by Lévy-Leblond after Lewis Carroll, the author of the book *Alice's Adventures in Wonderland*, due to its miraculous properties. E.g. Lévy-Leblond writes in [120]:

> *Le comportement d'un éventuel Univers qui serait régi par le groupe d'invariance ici n'est pas sans rappeler celui du "Pays des Merveilles". L'absence de causalité est particulièrement claire dans les aventures d'Alice ainsi que la valeur arbitraire des intervalles de temps (cf. en particulier le chapitre 7, "Un thé de fous"). C'est pourquoi il ne nous a*

> *pas paru déplacé d'associer le nom de L. Carroll à cette nouvelle limite non-relativiste du groupe de Poincaré.*

Despite of its interesting properties, it took several decades till Carrollian groups found applications in modern theoretical physics. A starting point for a renewed interest in this structure was the observation (cf. [60]), that the famous Bondi-Metzner-Sachs group, which appears in the description of gravitational waves at null infinity, is the conformal extension of the Carroll group. This finding lead then to various applications of Carrollian groups in the context of holography (cf. e.g. [57]) and is less suprising than it appears on a first glimpse: The Bondi-Metzner-Sachs group is a symmetry group associated with null infinity, which can be understood as a natural macroscopic null surface associated with asymptotically flat space times. As a null surface, null infinity can be endowed with ultrarelativistic geometric structures, whose symmetry groups should then be variants of the Carroll group.

Since microscopic tangent light cones, which are the objects of our interest, are certainly ultrarelativistic objects, too, we expect, that those geometric entities have similar properties. And indeed, we will understand lateron that their geometric properties resemble those of null-infinity to a such a great extent, that the analysis of this chapter could be understood as a microscopic analogue of the Bondi-Metzner-Sachs analysis. Therefore we will recapitulate the Bondi-Metzner-Sachs analysis concisely in the following.

The Bondi-Metzner-Sachs framework: In their seminal works from 1962 (cf. [30, 156, 31]) Bondi, Metzner, van der Burg and Sachs derived among other results, that the asymptotic symmetry group of asymptotically flat spacetimes at null infinity is not given by the Poincaré group, but by an infinite dimensional generalization, that goes today under the name of the Bondi-Metzner-Sachs (BMS) group. This discovery not only laid the foundation for a coordinate invariant analysis of gravitational waves (cf. [30, 156]), but stimulated a lot of research till today: It found application in numerical relativity (cf. [180, 29]), led more recently to the formulation of celestial holography (cf. e.g. [139]) and even triggered deep insights into the structure of gauge theories on asymptotically flat spacetimes ([165]). In a modern language (cf. [13, 14]), the original BMS-analysis can be most easily understood by performing a conformal compactification ([140]) of the considered spacetime. In this scenario, null infinity becomes a 3-dimensional null manifold which represents a boundary of the compactified bulk spacetime. As such, it inherits several universal geometric structures from the bulk, that are independent of the specific spacetime under consideration. The symmetry group of those structures is then given by the BMS-group, which describes the asymptotic symmetries that are encompassed by all asymptotically flat spacetimes (cf. [13, 14]).

The classic BMS group can hence be understood as a macroscopic symmetry group. It encodes, how spacetimes behave asymptotically at the largest scales. But one could also wonder about the microscopic asymptotics and could hence ask the opposite question: Is there a microscopic analogue of the BMS group which describes, how spacetimes behave asymptotically at microscopic scales? On the first glimpse, this question seems to be trivial. The Einstein equivalence principle (cf. e.g. [44]) tells us directly, that every spacetime behaves microscopically like flat Minkowski space. Hence, the microscopic symmetry group of general relativity should be given by the Lorentz group. This is of course true in some sense and can be formalized in terms of the vielbein formalism (cf. e.g. [44]), where the Lorentz group appears as a microscopic gauge freedom in the choice of local inertial frames. But if one compares this with the classic BMS analysis, one will realize, that this is not the correct microscopic analogue the original BMS analysis.

In the original BMS analysis, solely null infinity was analysed (cf. [13, 14]), since gravitational waves are assumed to travel along null rays. Spacelike or timelike directions were discarded. Hence, a microscopic analogue of the classical BMS-analysis should also focus just on microscopic null directions and should consider a microscopic analogue of null infinity. Or more precisely: One should identify the universal structures, that are induced on a natural microscopic null surface that is common to all spacetimes and determine their symmetry group. But which natural microscopic null surface exists in any spacetime? Here again the Einstein equivalence principle comes in, as it states, that any spacetime behaves microscopically (or better, infinitesimally) like Minkowski space. This criterion could hence be understood as a microscopic analogue of asymptotic flatness, since it says, that any spacetime behaves in the microscopic limit asymptotically like flat spacetime. And precisely, as (macroscopic) asymptotic flatness singles out null infinity as the natural macroscopic null surface for the classic BMS analysis, the Einstein equivalence principle, interpreted as a microscopic asymptotic flatness criterion, singles out infinitesimal tangent light cones as natural candidates for a microscopic BMS analysis. Consequently, one can ask in full analogy to the macroscopic case: Which universal structures on infinitesimal tangent light cones are common to all spacetimes satisfying the Einstein equivalence principle, what are the symmetries of those structures and what is the structure and the interpretation of the occuring symmetry groups? Those questions will be answered in this article and thereby we will uncover some interesting structures hidden in the theory of relativity. Especially, we will show as a main result, that a microscopic symmetry group appears, whose structure resembles the original BMS group and which is eligible as a gauge group for the bundle of null vectors for a generic spacetime.

Informal explanation of our work: After this overview of the motivation and philosophy behind our work, we would like to present now its line of argumentation in a concise way. Therefore recall first, that usually the set of null vectors is "identified" with its space of directions, which is commonly denoted as the celestial sphere. Under this identification, the Lorentz group is isomorphic to the Möbius group of conformal automorphisms of the Riemann sphere (cf. e.g. [143, 138]). Nevertheless we will see, that null vectors transform under Lorentz transformations not only by a change of their direction, but also by a rescaling of their "length". This behaviour is discarded by the common "identification" of the set of null vectors with the celestial sphere and one could ask, if there is some non-trivial information hidden in those rescalings. At a first glimpse, one might be tempted to think, that no non-trivial coordinate invariant information could be encoded in the length - and consequently also in rescalings - of null vectors, since the metric is degenerate for them. Moreover, the above mentioned isomorphism between the Lorentz and the Möbius group suggests, that there is no interesting information left, which could be encoded in non-trivial rescalings. But this intuition is only partially correct: We will see, that the metric constitutes a kind of distance function on each tangent light cone, although it is degenerate thereon. Metric degeneracy translates then to the statement, that the distance between two null vectors pointing in the same direction is zero. This distance function can then be related to a degenerate Riemannian metric on each tangent light cone and constitutes, together with other, more elementary properties inherited from the ambient tangent space, a set of universal geometric structures existent on any infinitesimal light cone. Those structures are then a microscopic analogue of the macroscopic universal structures at null infinity identified in the original BMS analysis.

One can then start to analyse those automorphisms of tangent light cones, which preserve the identified universal structures and thereby a basic question appears: How should the metric, that appears along above lines on any infinitesimal tangent light cone, be interpreted? As a fixed degenerate metric or as a representative of a conformal equivalence class of degenerate metrics? The former interpretation will lead to a group of isometries, while the latter will give rise to a conformal automorphism group. More or less surprisingly, the non-trivial rescalings of null vectors will appear again if one analyses the isometry group: Those rescalings are needed to compensate the appearing conformal factor in the metric on the light cone, which is induced by the action of Möbius transformations on the Riemann sphere.

If one analyses then the mathematical structure of the conformal automorphism group, some interesting properties appear: It can be written as a semidirect product of the Möbius group with a group of smooth, real valued functions on the Riemann sphere, which resembles the structure of the original BMS group. Moreover, it incorporates infinitely many Lorentz subgroups. The isometry group constitutes then just one of those Lorentz subgroups and can be shown to be induced by the standard representation of the Lorentz group on a single tangent space. By investigating those Lorentz subgroups more thoroughly, one realizes, that each Lorentz subgroup singles out a class of length gauges for null vectors, i.e. they set a scale for null vectors. For example, the subgroup of isometries is associated with the 3-length of null vectors in a vielbein frame. But this seems unsatisfactory: As said before, by metric degeneracy there is no preferred notion of length associated with a single null direction and by this, no structure on the light cone should single out a scale for null vectors. This suggests, that intrinsically no Lorentz subgroup of the conformal automorphism group is preferred and by this it seems, that the conformal automorphism group is intrinsically the more natural symmetry group for tangent light cones. From this point of view, we will then get also a different perspective on the conformal automorphism group: If one tries to introduce meaningful notions of length for null vectors, one necessarily has to enlarge the class of allowed coordinate systems, such that all possible length gauges are included. By doing so, also the symmetry group gets enlarged to the conformal automorphism group, which describes then rescalings of null vector lengths and associated Lorentz transformation laws in an invariant way. But by this, the original Lorentz group action on null vectors looses its distinguished role and infinitely many Lorentz transformation laws appear. This is a way, in which the microscopic BMS-like conformal automorphism group could be interpreted.

Now one could ask, what benefit one could draw from the existence of such a group. Therefore we will show finally, that the isomtery group and the conformal automorphism group are suitable as gauge groups for the bundle of future pointing null vectors. By this we have especially identified a geometric entity, that exists on the bulk of any spacetime and is associated with a BMS-like group. The benefit of this structure has of course to be proven in the future, but motivated by the various applications of the original BMS group we think, that this finding could have interesting implications: On the one hand, the existence of the BMS-like gauge group on the bundle of tangent light cones could lead to a bulk counterpart of the original BMS-anaylsis of gravitational waves and rises in addition questions regarding the fundamental symmetry group of gravity. On the other hand, given the recent discovery of connections between the BMS group, soft theorems and memory effects denoted commonly as the ”IR-triangle” [166, 165], our findings motivate the question, if there could exist an analogous ”UV-triangle”.

Finally we would like to mention, that the original BMS analysis incorporated of course not only the investigation of universal structures at null infinity and their symmetries, but also the examination of induced higher order structures, dynamical considerations and the analysis of gravitational waves. The microscopic analogues of those questions are not analysed in this article and will be, as sketched in the last paragraph, an object of future research.

Organization of the section: In section 6.2 we will introduce the light cone bundle of a generic spacetime and will identify the universal geometric structures that are induced on infinitesimal tangent light cones by the ambient spacetime geometry as motivated by Einstein’s equivalence principle. Thereby we will also explain concisely, how null vectors rescale under standard Lorentz transformations and moreover we will explain in which sense those geometric structures are ultrarelativistic. In section 6.3 we will then analyse those automorphisms of a single infinitesimal tangent light cone that preserve the universal structure either up to isometry or up to conformal equivalence. In section 6.4 we will analyse the structure of those automorphism groups thoroughly. Especially we will show, that the conformal automorphism group can be written as a right semidirect product group that contains infinitely many Lorentz subgroups. Thereby we will also explain, how those Lorentz subgroups can be parametrized. In section 6.5 we will then explain how the subgroup of isometries arises as a specific Lorentz subgroup. Moreover we will explain in section 6.6, how Lorentz subgroups correspond to length gauges for null vectors and why this motivates the claim that the conformal automorphism group could be a more natural symmetry group for tangent light cones than any Lorentz subgroup. Dually we will understand thereby, that the conformal automorphism group could be interpreted as a group which encodes all possible length gauges for null vectors as well as their Lorentz transformation properties. In section 6.7 we will then show that the conformal automorphism group and its isometry subgroup constitute gauge groups for the light cone bundle. In section 6.8 we will compare our analysis with the original BMS analysis, which will justify, why the conformal automorphism group is called a microscopic analogue of the BMS group. In section 6.9 possible implications as well as remaining open questions are discussed, while in section 6.10 a summary will be given. In appendix C we will review some basic facts on the Riemann sphere and its automorphisms. Moreover we will derive a convenient representation of coordinate systems of the light cone and will derive a transformation law (including rescalings) for null vectors under Lorentz transformations. In appendix D we will review some prerequisites from group theory.

6.2. Tangent light cones and their universal geometric structures

In this section, we will introduce the basic geometric entity of our study, namely the bundle of future pointing light cones associated with a spacetime $\mathscr{M}$, and will analyse, which universal geometric structures are present on its fibers, as induced by the microscopic geometry of the generic spacetime $\mathscr{M}$. Therefore, we will first define the notion of tangent light cones as well as the light cone bundle. In addition, we will establish a bundle atlas for the light cone bundle, that is induced by the bundle atlas $\mathcal{A}$ of $T\mathscr{M}$ (cf. section 1.4) and is hence associated with vielbein frames. Moreover, we will sketch briefly, how transition functions look like for this bundle, and thereby we will understand qualitatively how null vectors behave under Lorentz transformations. Afterwards we will then investigate, which universal geometric structures on infinitesimal tangent light cones are induced by the microscopic geometry of $\mathscr{M}$, independently of the macroscopic behaviour of the metric g.

The light cone bundle: We define the pointed future[1] tangent light cone at $p \in \mathscr{M}$ as

$$L_p^+\mathscr{M} := \{v \in T_p\mathscr{M} | g_p(v,v) = 0 \text{ and } g(X_p, v) < 0\}$$

and by this, we define the future light cone bundle as the following sub-fiberbundle of $T\mathscr{M}$:

$$L^+\mathscr{M} := \bigsqcup_{p\in\mathscr{M}} L_p^+\mathscr{M} \subset T\mathscr{M}$$

[1]The analysis presented in this article goes completely analogous for past light cones and the associated past light cone bundle. Since $\mathscr{M}$ is time orientable, we focus hence without loss of generality solely on future light cones.

For any open set $U \subset \mathcal{M}$ we will denote the restriction of $L^+\mathscr{M}$ to U by L^+U. We now introduce a class of suitable coordinate systems for $L^+\mathscr{M}$ which are induced by the bundle charts $(U, \psi) \in \mathcal{A}$ of $T\mathscr{M}$. We will then see especially, that the typical fiber of $L^+\mathscr{M}$ is diffeomorphic to $\mathbb{C}_\infty \times \mathbb{R}^+$, i.e. for any $p \in \mathscr{M}$

$$L_p^+\mathscr{M} \cong \mathbb{C}_\infty \times \mathbb{R}^+$$

holds, where $\mathbb{C}_\infty$ is the Riemann sphere (cf. 1.4). Here, $\mathbb{C}_\infty$ will be interpreted as a space of null directions and $\mathbb{R}^+$ as a "length" for null vectors. Therefore let $(U, \psi) \in \mathcal{A}$ be a bundle trivialization of $T\mathscr{M}$ and let (E_μ) be the associated vielbein. We then have for $v \in L_p^+\mathscr{M}$, that its coordinate representation $\psi_p(v) = (v^\mu)$ satisfies $v^0 = |\vec{v}|$, where we set $\vec{v} := (v^1, v^2, v^3)$. By this we can write $(v^\mu) = |\vec{v}| \cdot (1, \hat{v})$, where we have defined $\hat{v} := |\vec{v}|^{-1}\vec{v}$. Since $v^0 = |\vec{v}| > 0$ and $\hat{v} \in S^2$, we get therewith an identification

$$\tilde{\psi}_p^+ : L_p^+\mathscr{M} \to S^2 \times \mathbb{R}^+, v^\mu E_\mu \mapsto (\hat{v}, |\vec{v}|),$$

of the tangent lightcone at p with $S^2 \times \mathbb{R}^+$. This gives then rise to a smooth bundle trivialization

$$\tilde{\psi}^+ : L^+U \to U \times (S^2 \times \mathbb{R}^+), v^\mu E_\mu \in T_p\mathscr{M} \mapsto (p, (\hat{v}, |\vec{v}|)) \,.$$

We now utilize the stereographic projection defined as[2]

$$\rho : S^2 \to \mathbb{C}_\infty, \hat{v} \mapsto \rho(\hat{v}) := \frac{\hat{v}^1 + i\hat{v}^2}{1 - \hat{v}^3},$$

which constitutes a diffeomorphism. By this we get an identification

$$\psi_p^+ : L_p^+\mathscr{M} \to \mathbb{C}_\infty \times \mathbb{R}^+, v \mapsto \left(z_p^\psi(v), \lambda_p^\psi(v)\right) \tag{6.1}$$

explicitely given by:

$$\begin{aligned} z_p^\psi(v^\mu E_\mu) &:= \rho(\hat{v}) = \frac{v^1 + iv^2}{v^0 - v^3} \\ \lambda_p^\psi(v^\mu E_\mu) &:= |\vec{v}| = v^0. \end{aligned} \tag{6.2}$$

This induces a smooth bundle trivialization

$$\psi^+ : L^+U \to U \times (\mathbb{C}_\infty \times \mathbb{R}^+), v \in T_p\mathscr{M} \mapsto \left(p, (z_p^\psi(v), \lambda_p^\psi(v))\right) . \tag{6.3}$$

By this we have constructed a bundle atlas

$$\mathcal{B} := \{(U, \psi^+) | (U, \psi) \in \mathcal{A}\}$$

[2]We adopt the convention from [143], which differs from the convention that is usually used in literature.

for the fiber bundle $L^+\mathscr{M}$, where for any $(U, \psi) \in \mathcal{A}$, the trivialization ψ^+ is given by the associated map (6.3). For $p \in U$, the restriction of any trivialization $(U, \psi^+) \in \mathcal{B}$ to $L_p^+\mathscr{M}$ will be denoted by ψ_p^+ as given by (6.1) and we will denote the set of all such restrictions by $\mathcal{B}_p$. Since Lorentz transformations preserve light cones, it is easy to show, that the transition functions of $\mathcal{B}$ are indeed smooth and preserve the fibers of $L^+\mathscr{M}$. Unfortunately, at the present stage we are not able to derive the precise form of the transition functions of $\mathcal{B}$ in a convenient representation. Nevertheless, we want to sketch already briefly the result, since it is illustrative for the understanding of the paper, although not absolutely necessary. The full discussion will then follow in sections 6.3 and 6.7 as well as in appendix C.3. Hence, an impatient reader can also jump directly to the next paragraph. Now let $p \in U$, set $v = v^\mu E_\mu \in L_p^+\mathscr{M}$ and $w = \Lambda^\mu{}_\nu v^\nu E_\mu$. Define in addition

$$(z, \lambda) := \psi_p^+(v), \tag{6.4}$$

$$(z', \lambda') := \psi_p^+(w). \tag{6.5}$$

One can then show by utilization of the standard isomorphism between the Lorentz group $\mathrm{SO}^+(1,3)$ and the automorphism group of $\mathbb{C}_\infty$ (cf. appendix C.3 or [138, 143]), that there is a unique automorphism (i.e. a Möbius transformation, cf. appendix C.1) $Z_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of $\mathbb{C}_\infty$ associated to Λ s.th.

$$z' = Z_\Lambda(z) \tag{6.6}$$

holds. By utilization of (6.2) we have naivly

$$\lambda' = \Lambda^0{}_\mu v^\mu,$$

which is not a very helpful representation, since it does not depend explicitely on z and λ. But by some more advanced techniques (cf. appendix C.3) one can indeed show, that there is for each Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ an associated function $f_\Lambda \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ such that

$$\lambda' = f_\Lambda(z)\lambda \tag{6.7}$$

holds. In this form, this is to the best of our knowledge an original result of this article. The concrete form of the function f_Λ is not of importance now and will be understood in sections 6.3 and 6.5. At the present stage, just a qualitative understanding of equations (6.6) and (6.7) is sufficient: They state, that a Lorentz transformation acts on null vectors by a conformal transformation (6.6) on their space of directions $\mathbb{C}_\infty$, together with a non-trivial, direction dependent rescaling (6.7) of their "length". The latter can be understood by recalling, that

$$\lambda = |\vec{v}| = v^0$$
$$\lambda' = |\vec{w}| = w^0$$

hold by (6.4 - 6.5) . We will understand later on, which non-trivial information is encoded in those rescalings.

Universal structures on tangent light cones: We now want to analyse the geometric structures that are, independently of the macroscopic behaviour of the gravitational field, induced on tangent light cones $L_p^+\mathscr{M}$ by the geometry of their ambient tangent spaces $T_p\mathscr{M}$. Those universal structures are linked to Einstein's equivalence principle, that is commonly stated as (cf. [44]):

> *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.*

This statement should be understood as an *asymptotic* statement: In the infinitesimal limit (and hence, strictly speaking, not locally[3]), any spacetime behaves asymptotically like flat spacetime. Geometrically, this is formalized by the linear structure of any tangent space $T_p\mathscr{M}$ and by the property, that the metric g reduces to an inner product g_p on $T_p\mathscr{M}$ which can be brought to Minkowski form η by choice of a vielbein frame. Hence the question of this section will be: What universal structures are induced on $L_p^+\mathscr{M}$ by the linear structure of $T_p\mathscr{M}$ and by the inner product g_p? And what are the coordinate expressions of those structures in the coordinate systems $\mathcal{B}_p$?

For the analysis of those questions, choose a $p \in \mathscr{M}$ and let $\psi_p^+ \in \mathcal{B}_p$ be a coordinate system of the form (6.1). Observe then first, that $L_p^+\mathscr{M}$ inherits, as a subspace of the tangent space at p, a topology and a smooth structure, that can be equally characterized by the observation, that (6.1) is a diffeomorphism:

(S1) $L_p^+\mathscr{M} \cong \mathbb{C}_\infty \times \mathbb{R}^+$ as a differentiable manifold.

Moreover, although $L_p^+\mathscr{M} \subset T_p\mathscr{M}$ is no linear subspace, it is a linear cone and as such it inherits a notion of multiplication with scalars from $T_p\mathscr{M}$:

(S2) $L_p^+\mathscr{M}$ is a linear cone, in the sense, that for all $\alpha > 0$ and all $v \in L_p^+\mathscr{M}$

$$\alpha \cdot v \in L_p^+\mathscr{M}$$

holds.

Please notice at this point, that any coordinate system $\psi^+ \in \mathcal{B}$ respects this cone structure,

[3] Although "local" and "infinitesimal" are often used synonymous in the context of general relativity, there is a difference between those concepts: Infinitesimal objects are associated with the infinitesimal limit, i.e. with tangent spaces, while local objects are associated with (small) open sets and local coordinate systems. Those notions are hence not equivalent: For example, infinitesimally, one can always find vielbein frames in which the metric has Minkowski form, while it is not possible to find a local coordinate system with this property for a generic curved spacetime, cf. [147]. The latter can especially be formalized in terms of Riemann normal coordinates, cf. [147, 44] and section 6.1.

since for all $\alpha > 0$ also

$$z_p^\psi(\alpha \cdot v) = z_p^\psi(v),$$
$$\lambda_p^\psi(\alpha \cdot v) = \alpha \cdot \lambda_p^\psi(v)$$

hold. In addition, there is a kind of complex structure, which is induced by the complex structure on $\mathbb{C}_\infty$ under the identification of $L_p^+\mathscr{M}$ with $\mathbb{C}_\infty \times \mathbb{R}^+$:

(S3) There is a set $\mathcal{Z}_p$ that includes exactly all surjective maps

$$z_p : L_p^+\mathscr{M} \to \mathbb{C}_\infty$$

which satisfy the requirement, that

$$z_p^\psi \circ z_p^{-1} : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is a biholomorphic automorphism of $\mathbb{C}_\infty$ and hence a Möbius transformation.

Please consult appendix C.1, if you are not familiar with the notion of Möbius transformations. Finally we want to analyse, which structure on $L_p^+\mathscr{M}$ is induced by the metric g. Prima facie, one could be tempted to think, that no interesting structure on $L_p^+\mathscr{M}$ is induced by the metric, since the latter is degenerate on thereon. But we will see, that this is not true. Therefore we will first introduce the inverse of the stereographic projection ρ, which will be denoted by $\hat{\epsilon} := \rho^{-1}$, and is explicitely given by (cf. [143] or appendix C.1)

$$\hat{\epsilon} : \mathbb{C}_\infty \to S^2, z \mapsto \hat{\epsilon}(z) := (\hat{\epsilon}^1(z), \hat{\epsilon}^2(z), \hat{\epsilon}^3(z))$$

with:

$$\hat{\epsilon}^1(z) = \frac{z+\bar{z}}{z\bar{z}+1},$$
$$\hat{\epsilon}^2(z) = \frac{1}{i}\frac{z-\bar{z}}{z\bar{z}+1},$$
$$\hat{\epsilon}^3(z) = \frac{z\bar{z}-1}{z\bar{z}+1}.$$

We will then write the inverse of ψ_p^+ for convenience as $\theta_p := (\psi_p^+)^{-1}$ and express it in terms of the inverse stereographic projection explicitely as

$$\theta_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to L_p^+\mathscr{M}, (z,\lambda) \mapsto \lambda \cdot \hat{\epsilon}^\mu(z) E_\mu,$$

where we set $\hat{\epsilon}^0 := 1$. By this we can then define a map

$$h_p : (\mathbb{C}_\infty \times \mathbb{R}^+) \times (\mathbb{C}_\infty \times \mathbb{R}^+) \to [0, \infty) \tag{6.8}$$

as the negative of the pullback of g_p along θ_p, i.e.:

$$h_p\left((z_1,\lambda_1),(z_2,\lambda_2)\right) := -g_p(\theta_p(z_1,\lambda_1),\theta_p(z_2,\lambda_2))$$

As a somewhat surprising result we obtain then, that h_p is explicitely given by

$$h_p\left((z_1,\lambda_1),(z_2,\lambda_2)\right) = \frac{2\lambda_1\lambda_2|z_1-z_2|^2}{(|z_1|^2+1)(|z_2|^2+1)},$$

which is just

$$h_p\left((z_1,\lambda_1),(z_2,\lambda_2)\right) = \frac{\lambda_1\lambda_2}{2}d^2(z_1,z_2) \tag{6.9}$$

with d being the chordal distance (cf. (C.5) and (C.6)) on $\mathbb{C}_\infty$ given by:

$$d:\mathbb{C}_\infty\times\mathbb{C}_\infty\to[0,\infty),(z_1,z_2)\mapsto\frac{2|z_1-z_2|}{\sqrt{|z_1|^2+1}\sqrt{|z_2|^2+1}}. \tag{6.10}$$

Hence, although the metric is degenerate on $L_p^+\mathscr{M}$, it carries non-trivial information, since it describes a kind of distance function[4] (6.9) thereon, which is related to the chordal distance on $\mathbb{C}_\infty$. We now can ask, if the distance function (6.9) is related to some kind of Riemannian metric on the manifold $\mathbb{C}_\infty\times\mathbb{R}^+$. And indeed, one can show easily, that $\sqrt{h_p}$ is the distance function induced by a degenerate Riemannian metric q_p on $L_p^+\mathscr{M}$, which is given in any coordinate system $\psi_p^+\in\mathcal{B}_p$ by:

$$ds^2 = 2\lambda^2\frac{dzd\bar{z}}{(1+z\bar{z})^2} \tag{6.11}$$

I.e. we have:

(S4) There exists a degenerate metric q_p on $L_p^+\mathscr{M}$ whose coordinate expression in any coordinate system $\psi_p^+\in\mathcal{B}_p$ is given by

$$ds^2 = 2\lambda^2\frac{dzd\bar{z}}{(1+z\bar{z})^2}.$$

Please note, that q_p is really a degenerate Riemannian metric on the manifold $L_p^+\mathscr{M}$, as (6.11) is a degenerate Riemannian metric on the manifold $\mathbb{C}_\infty\times\mathbb{R}^+$. This is in contrast to the metric g, which is a pseudo-Riemannian metric on $\mathscr{M}$ and gives as such rise to an inner product g_p on $T_p^+\mathscr{M}$. As such, g_p induces then the distance function (6.9) on $L_p^+\mathscr{M}\subset T_p\mathscr{M}$, whose infinitesimalization is then given by (6.11), which describes q_p in the coordinates ψ_p^+.

Hence we have all together the following universal structures on $L_p^+\mathscr{M}$:

(S1) $L_p^+\mathscr{M}\cong\mathbb{C}_\infty\times\mathbb{R}^+$ as a differentiable manifold.

[4]Please note, that neither h_p nor $\sqrt{h_p}$ constitute (pseudo-)metrics, since they do not obey the triangle equality. Nevertheless, they can be interpreted as distance functions in the present situation, since they are induced by a degenerate Riemannian metric on $L_p^+\mathscr{M}$ and are related to the chordal distance on $\mathbb{C}_\infty$. Therefore we call them distance functions in the sequel.

(S2) $L_p^+\mathscr{M}$ is a linear cone, in the sense that for all $\alpha > 0$ and all $v \in L_p^+\mathscr{M}$

$$\alpha \cdot v \in L_p^+\mathscr{M}$$

holds.

(S3) There is a set $\mathcal{Z}_p$ which includes exactly all surjective maps

$$z_p : L_p^+\mathscr{M} \to \mathbb{C}_\infty$$

that satisfy the requirement, that

$$z_p^\psi \circ z_p^{-1} : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is biholomorphic for any coordinate system $\psi_p^+ = (z_p^\psi, \lambda_p^\psi) \in \mathcal{B}_p$ and hence a Möbius transformation.

(S4) There exists a degenerate metric q_p on $L_p^+\mathscr{M}$, whose expression in any local coordinate system $\psi_p^\psi \in \mathcal{B}_p$ is given by the metric $\tilde{q}_p$ on $\mathbb{C}_\infty \times \mathbb{R}^+$ explicitely given by

$$ds^2 = 2\lambda^2 \frac{dzd\bar{z}}{(1+z\bar{z})^2}. \tag{6.12}$$

In the next section we want to analyse the set of automorphisms of $L_p^+\mathscr{M}$, which preserve those universal geometric structures.

A Carrollian perspective on the universal structures: We now want to explain, why the universal structures discovered so far can be understood as features which encode the ultrarelativistic geometry of microscopic tangent light cones. Therefore recall from [61] that a weak $d+1$ dimensional Carroll manifold is given by a triple (C, g, ξ) where C is a smooth $d+1$ dimensional manifold and g is a twice-symmetric positive tensor field thereon whose Kernel is generated by a nowhere vanishing vector field ξ. We then see, that each tangent light cone $L_p^+\mathscr{M}$ can be understood as a weak Carroll manifold whose degenerate metric is given by q_p and where the vector field ξ is determined by the cone structure (S1).

6.3. Automorphism groups of tangent light cones

In this section we will determine the automorphisms of $L_p^+\mathscr{M}$ which preserve the universal structures (S1) - (S4) in a certain sense. As explained in the introduction, the crucial question is how to interpret the structure (S4): Should it be understood as a single degenerate metric or as a representative of a conformal equivalence class on $L_p^+\mathscr{M} \cong \mathbb{C}_\infty \times \mathbb{R}^+$? The former will lead to a group of isometries, while the latter will give a conformal automorphism group. There are arguments for both positions and we will comment on this more extensively in section 6.5 . But in this section, we won't bother with this question and just determine the respective automorphism groups for both interpretations. Therefore we will explain some generalities regarding automorphism groups of $L_p^+\mathscr{M}$ and will especially analyse, how the structures (S1) - (S3) already fix the form of suitable automorphisms to a great extent, independent of the interpretation of (S4). Moreover, we will review in this section some basics regarding Möbius transformations and metrics on the Riemann sphere, which are needed in the sequel. Finally we will then determine the group of isometries and the conformal automorphism group. Please note in addition, that due to the similarity of the universal geometric structures on $L_p^+\mathscr{M}$ with the structure of a weak Carroll manifold, the isometry group should be understood as a Carrollian symmetry group while the conformal automorphism group can be understood as its conformal extension.

Generalities: An automorphism group of $L_p^+\mathscr{M}$ is a set of all maps

$$\Pi_p : L_p^+\mathscr{M} \to L_p^+\mathscr{M}$$

which preserve the universal structures (S1) - (S4) in a specific sense, together with the composition $\circ$ as a group operation. By choosing an arbitrary but fixed coordinate system $\psi_p^+ \in \mathcal{B}_p$, one notices, that a map Π_p preserves the structures (S1) - (S4) in a specified sense if and only if the associated coordinate representation of Π_p given by

$$\psi_p^+ \circ \Pi_p \circ \theta_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+$$

preserves the induced universal structures on $\mathbb{C}_\infty \times \mathbb{R}^+$ in the analogous specified sense. Here we set again for convenience $\theta_p := (\psi_p^+)^{-1}$. Additionally, the group multiplication law $\circ$ is preserved under conjugation with ψ_p^+ and hence any automorphism group of $L_p^+\mathscr{M}$ is naturally isomorphic to the corresponding automorphism group of the induced structures on $\mathbb{C}_\infty \times \mathbb{R}^+$. Hence, we can define the automorphism group $L_p^+\mathscr{M}$ equally as a group of automorphisms of $\mathbb{C}_\infty \times \mathbb{R}^+$ together with $\circ$ as a group operation and this is what we will do in the sequel, since it is more convenient in the present situation.

As said before, there are two different ways, how one could interpret the structure (S4): Either as a degenerate metric on $L_p^+\mathscr{M}$ or as a representative of a conformal equivalence class thereon. Each of those two interpretations yields then its respective automorphism group. In the former case, the automorphisms constitute isometries of $L_p^+\mathscr{M}$, while in the latter case they correspond to conformal automorphisms of $L_p^+\mathscr{M}$. Hence, we will obtain two distinct automorphism groups: The group of isometries Iso_p^+ of $L_p^+\mathscr{M}$ and the group of conformal automorphisms Con_p^+ of $L_p^+\mathscr{M}$. Nevertheless, both automorphism groups should preserve the universal structures (S1) - (S3) in the same sense, since those structures are independent of the interpretation of (S4). Therefore, we will discuss now first the requirements on automorphisms as induced by (S1)- (S3). For this, we write first a generic map

$$\Phi_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+ \tag{6.13}$$

in full generality as

$$\Phi_p : (z, \lambda) \mapsto (Z(z, \lambda), Y(z, \lambda)).$$

We then have first and foremost the following two requirements for Φ_p, which are induced by (S1) and (S2) respectively:

(R1) Φ_p is a diffeomorphism.

(R2) Φ_p is homogeneous in the sense, that for all $\alpha > 0$ and all $(z, \lambda) \in \mathbb{C}_\infty \times \mathbb{R}^+$

$$Z(z, \alpha \cdot \lambda) = Z(z, \lambda) \tag{6.14}$$
$$Y(z, \alpha \cdot \lambda) = \alpha \cdot Y(z, \lambda) \tag{6.15}$$

should hold.

Those both requirements fix the form of Φ_p already to a great extent: Equation (6.14) and requirement (R1) are together equivalent to the statement, that Z is a smooth automorphism of $\mathbb{C}_\infty$ which does not depend on the λ-coordinate. Moreover, equation (6.15) and (R2) hold together if and only if there exists a smooth function $Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ such that

$$Y(z, \lambda) = Y(z) \cdot \lambda$$

holds for all $(z, \lambda) \in \mathbb{C}_\infty \times \mathbb{R}^+$. Consequently, we can write any map (6.13) satisfying the requirements (R1) - (R2) as

$$\Phi_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z(z), Y(z) \cdot \lambda)$$

with $Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ and Z being a diffeomorphism $Z : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Now, the complex structure (S3) is preserved if and only if Z is a biholomorphic conformal automorphism of $\mathbb{C}_\infty$, i.e. a Möbius transformation. This gives the following requirement:

(R3) Z is a biholomorphic automorphism of the Riemann sphere $\mathbb{C}_\infty$, i.e. a Möbius transformation.

All together, any automorphism of $\mathbb{C}^\infty \times \mathbb{R}^+$ which preserves the structures (S1) - (S3) and satisfies consequently the requirements (R1) - (R3) is given by a map

$$\Phi_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z(z), Y(z) \cdot \lambda) \tag{6.16}$$

with Z being a Möbius transformation and $Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$. Moreover we see, that two automorphisms are equal if and only if their associated Möbius transformations and smooth functions Y are equal, respectively.

In the sequel we will need some basic facts regarding Möbius transformations. Hence, we will recapitulate those facts concisely now, while a more complete account of the corresponding theory is presented in appendix C.1. Therefore recall first, that any Möbius transformation Z can be written as

$$Z : \mathbb{C}_\infty, \to \mathbb{C}_\infty, z \mapsto \frac{az+b}{cz+d}$$

for complex numbers $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc = 1$. By this, any matrix $A \in \mathrm{SL}(2, \mathbb{C})$ given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.17}$$

defines an associated Möbius transformation Z^A specified by

$$Z^A(z) := \frac{az+b}{cz+d}. \tag{6.18}$$

Moreover, two matrices $A, B \in \mathrm{SL}(2, \mathbb{C})$ define the same Möbius transformation if and only if $A = -B$. This defines then an isomorphism between $\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/\{\pm\mathbf{1}\}$ and the Möbius group, that is explicitely given by

$$[A] \in \mathrm{PSL}(2, \mathbb{C}) \mapsto Z^A$$

with Z^A as defined in (6.18). We will use in the sequel the notions Möbius group and $\mathrm{PSL}(2, \mathbb{C})$ interchangeably. In addition, we will denote equivalence classes $[A] \in \mathrm{PSL}(2, \mathbb{C})$ just in terms of one of their representatives, i.e. by slight abuse of notation $[A] = A$. Now recall further, that $\mathbb{C}_\infty$ is a Riemann surface and is hence especially endowed with a natural conformal structure (cf. [106]), that can be described in terms of a conformal equivalence class of Riemannian metrics. One representative of this conformal structure is given by the Riemannian metric

$$ds^2 = \frac{4}{(1+z\bar{z})^2} dz d\bar{z}$$

on $\mathbb{C}_\infty$, which is also the Riemannian metric associated with the chordal distance (6.10). The pullback of this metric along a Möbius transformation Z^A as specified by (6.18) is then explicitely given by

$$ds^2 = K^A(z)^2 \frac{4}{(1+z\bar{z})^2} dz d\bar{z}, \tag{6.19}$$

where the conformal factor $K^A(z)$ associated with a matrix $A \in \mathrm{PSL}(2, \mathbb{C})$ as specified by

(6.17) is explicitely given by

$$K^A(z) = \frac{(1+z\bar{z})}{(az+b)(\bar{a}\bar{z}+\bar{b})+(cz+d)(\bar{c}\bar{z}+\bar{d})}, \tag{6.20}$$

as one can easily calculate. Finally recall, that the Möbius group and the proper orthochronous Lorentz group are isomorphic, i.e. $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}^+(1,3)$, as said before and as described in appendix C.3 (cf. also [143, 138]). The corresponding isomorphism will be written as

$$\Lambda \in \mathrm{SO}^+(1,3) \mapsto A_\Lambda \in \mathrm{PSL}(2,\mathbb{C}).$$

The isometry group: We now want to find all maps

$$\Phi_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z,\lambda) \mapsto (Z(z), Y(z)\cdot\lambda)$$

of the form (6.16), such that the following requirement holds:

(R4a) Φ_p is an isometry, i.e.

$$\Phi_p^* \tilde{q}_p = \tilde{q}_p$$

is satisfied, where $\tilde{q}_p$ is the metric (6.12) on $\mathbb{C}_\infty \times \mathbb{R}^+$ that is induced by the metric q_p on $L_p^+\mathscr{M}$, cf. (S4).

Recall from (S4), that $\tilde{q}_p$ is given by:

$$ds^2 = 2\lambda^2 \frac{dzd\bar{z}}{(1+z\bar{z})^2}$$

Then its pullback $\Phi_p^*\tilde{q}_p$ can be easily calculated by the utilization of (6.19) and is explicitely given by

$$ds^2 = 2Y(z)^2\lambda^2 K^A(z)^2 \frac{dzd\bar{z}}{(1+z\bar{z})^2}, \tag{6.21}$$

where K^A is the conformal factor (6.39). Hence we obtain directly, that Φ_p is an isometry if and only if

$$Y(z) \overset{!}{=} K^A(z)^{-1}$$

holds. We now define

$$f^A := (K^A)^{-1} \tag{6.22}$$

and by this

$$Y(z) \overset{!}{=} f^A(z)$$

should hold for Φ_p being an isometry. By this we have, that the isometry group of the infinitesimal light cone consists out of all maps Φ_p^A of the form

$$\Phi_p^A : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z_A(z), f^A(z)\lambda),$$

where A is any matrix $A \in \mathrm{PSL}(2, \mathbb{C})$, together with the composition as the group multiplication law. This group will be denoted in the sequel as:

$$\mathrm{Iso}_p^+ = \left(\{\Phi_p^A | A \in \mathrm{PSL}(2, \mathbb{C})\}, \circ \right).$$

We see already, that it is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$ as a set, and hence also isomorphic to the Lorentz group $\mathrm{SO}^+(1, 3)$ as a set. We will understand in section 6.5, that they are also isomorphic as a group, and that Iso_p^+ is induced by local Lorentz transformations acting on $T_p\mathscr{M}$. By this, it is also not a coincidence, that the function (6.22) and the function f_Λ which determines the rescalings (6.7) of null vectors under Lorentz transformations are both denoted by the same letter. Indeed, they are the same mathematical object, as we will understand then, too.

The conformal automorphism group: We now want to find all maps

$$\Psi_p : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z(z), Y(z) \cdot \lambda)$$

of the form (6.16), such that the following requirement holds:

(R4b) Ψ_p is a conformal automorphism, i.e. there exists a function $\Omega_p \in C^\infty(\mathbb{C}_\infty \times \mathbb{R}^+, \mathbb{R}^+)$ s.th.

$$\Psi_p^* \tilde{q}_{p,(z,\lambda)} = \Omega_p(z, \lambda)^2 \tilde{q}_{p,(z,\lambda)} \tag{6.23}$$

holds for all $(z, \lambda) \in \mathbb{C}_\infty \times \mathbb{R}^+$. Here $\tilde{q}_p$ denotes, as before, the metric (6.12) on $\mathbb{C}_\infty \times \mathbb{R}^+$ and $\tilde{q}_{p,(z,\lambda)}$ denotes its pointwise evaluation on the tangent space $T_{(z,\lambda)}(\mathbb{C}_\infty \times \mathbb{R}^+)$ of $\mathbb{C}_\infty \times \mathbb{R}^+$ at (z, λ).

Observe, that, due to the form (6.16) of Ψ_p, it follows directly, that the conformal factor must be of the form $\Omega_p(z)$ with $\Omega_p \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$. We have namely as in (6.21)

$$\Psi_p^* \tilde{q}_{p,(z,\lambda)} = Y(z)^2 K^A(z)^2 \tilde{q}_{p,(z,\lambda)},$$

which says, that the conformal factor Ω_p as defined in (6.23) is explicitely given by:

$$\Omega_p(z) = Y(z) K^A(z)$$

By this, the group of conformal automorphisms of the infinitesimal light cone consists out of all maps

$$\Psi_p^{(A,Y)} : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z_A(z), Y(z) \cdot \lambda),$$

with $A \in \mathrm{PSL}(2,\mathbb{C})$ and $Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$, together with the composition as the group multiplication law. This group will be denoted in the sequel as:

$$\mathrm{Con}_p^+ := \left(\{\Psi_p^{(A,Y)} | A \in \mathrm{PSL}(2,\mathbb{C}) \text{ and } Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)\}, \circ \right).$$

We see already, that it is isomorphic to $\mathrm{PSL}(2,\mathbb{C}) \times C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ as a set. Its non-trivial group structure will be understood in section 6.4.

We want to remark finally, that the condition (R4b) can be rephrased in a different but equivalent manner. Therefore observe first, that the metric $\tilde{q}_p$ from (6.12) is compatible with the linear cone structure (S2) of $L_p^+\mathscr{M}$ in the sense, that

$$\tilde{q}_{p,(z,\alpha\lambda)} = \alpha^2 \tilde{q}_{q,(z,\lambda)}$$

holds for all $\alpha > 0$. This behaviour is induced by the compatibility of the distance function h_p (6.8) with the cone structure. If one would like to define a conformal equivalence class of metrics on $L_p^+\mathscr{M}$, all metrics in this class should have this property, because otherwise they would not be compatible with (S2). Hence, two metrics $q_p^{(1)}$ and $q_p^{(2)}$ on $L_p^+\mathscr{M}$ should be called conformally equivalent, if there exists a positive valued, smooth function $\Omega_p \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ independent of λ, such that their coordinate expressions $\tilde{q}_p^{(1)}, \tilde{q}_p^{(2)}$ satisfy:

$$\tilde{q}^{(1)}_{p,(z,\lambda)} = \Omega_p(z)^2 \tilde{q}^{(2)}_{p,(z,\lambda)}$$

The conformal structure on $L_p^+\mathscr{M}$ is then the set of all metrics that are in this sense conformally equivalent to the metric (6.12). One can then see easily, that the requirement (R4b) is, due to the requirement (R2), equivalent to the statement, that any conformal automorphism of $L_p^+\mathscr{M}$ should preserve this conformal structure under pullback.

Summary: In this section we have found two natural automorphism groups for $\mathrm{L}_p^+\mathscr{M}$. The first is the group of isometries Iso_p^+, whose underlying set is given by

$$\mathrm{Iso}_p^+ = \{\Phi_p^A | A \in \mathrm{PSL}(2,\mathbb{C})\}$$

with

$$\Phi_p^A : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z,\lambda) \mapsto (Z_A(z), f^A(z)\lambda),$$

together with the composition $\circ$ as the group operation. Here $f^A(z)$ is defined by (6.22). The other is the group of conformal automorphisms Con_p^+, whose underlying set is given by

$$\mathrm{Con}_p^+ = \{\Psi_p^{(A,Y)} | A \in \mathrm{PSL}(2,\mathbb{C}), Y \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)\}$$

with

$$\Psi_p^{(A,Y)} : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z, \lambda) \mapsto (Z_A(z), Y(z) \cdot \lambda), \tag{6.24}$$

together with the composition $\circ$ as the group operation. In the next section we will understand their mathematical structure and their interrelation.

6.4. Structure of the conformal automorphism group

In the last section we have determined two different automorphism groups of $L_p^+\mathscr{M}$: A isometry group Iso_p^+ and a conformal automorphism group Con_p^+. In this section, we want to understand their structure of the conformal automorphism group better. Thereby we will understand, that it constitutes a semidirect product of the Möbius group with the group of positive valued smooth functions on the Riemann sphere. Afterwards we will then show, that the conformal automorphism group contains infinitely many Lorentz subgroups, and explain, how those subgroups can be parametrized in terms of so called crossed homomorphisms. In the next sections we will need some prerequisites from group theory as adapted to our situation. Especially the notions of right semidirect products and crossed homomorphisms are needed. To increase readability, we have outsourced their discussion to appendix D. Please note, that we will denote from now on the element in $\mathrm{PSL}(2,\mathbb{C})$ associated with a Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ by $A_\Lambda \in \mathrm{PSL}(2,\mathbb{C})$ and the corresponding Möbius transformation by $Z_\Lambda := Z^{A_\Lambda}$.

Structure of the conformal automorphism group: Let G be the (right) semidirect product group (for a definition of this notion see appendix D.1)

$$G := \mathrm{PSL}(2,\mathbb{C}) \ltimes_\kappa C^\infty(\mathbb{C}_\infty, \mathbb{R}^+),$$

where the group antihomomorphism

$$\kappa : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{Aut}(C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)), A \mapsto \kappa_A \tag{6.25}$$

is defined as:

$$\kappa_A : C^\infty(\mathbb{C}_\infty, \mathbb{R}^+) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), Y \mapsto Y \circ Z^A \tag{6.26}$$

This means in particular, that $G \cong \mathrm{PSL}(2,\mathbb{C}) \times C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ as a set and that the product of $(A_1, Y_1), (A_2, Y_2) \in G$ is defined as:

$$(A_1, Y_1)(A_2, Y_2) = (A_1 A_2, Y_1 \circ Z^{A_2} \cdot Y_2). \tag{6.27}$$

We will show in this section, that the group Con_p^+ of conformal automorphisms is isomorphic to G and moreover, that the action of Con_p^+ on $\mathbb{C}_\infty \times \mathbb{R}^+$ can be described in terms of a faithful left group action $\star : G \curvearrowright \mathbb{C}_\infty \times \mathbb{R}^+$. The latter means, that G is indeed a transformation group (cf. [87]) acting on $\mathbb{R}^+ \times \mathbb{C}_\infty$.

Therefore observe first, that obviously Con_p^+ and G are isomorphic as sets in terms of the bijection

$$(A, Y) \in G \mapsto \Psi^{(A,Y)},$$

with $\Psi^{(A,Y)} \in \mathrm{Con}_p^+$ being defined as in (6.24). Let now $(A_1, Y_1), (A_2, Y_2) \in G$. Then their product in G is given by (6.27), while we have on the other hand:

$$\Psi^{(A_1,Y_1)} \circ \Psi^{(A_2,Y_2)} = \Psi_p^{(A_1 A_2, Y_1 \circ Z^{A_2} \cdot Y_2)}.$$

Hence the map

$$G \to \mathrm{Con}_p^+, g \mapsto \Psi_p^g$$

is indeed a group isomorphism. Define now a faithful left group action of G on $\mathbb{C}_\infty \times \mathbb{R}^+$

$$\star : G \curvearrowright \mathbb{C}_\infty \times \mathbb{R}^+, (g, (z, \lambda)) \mapsto g \star (z, \lambda) \tag{6.28}$$

as:

$$(A, Y) \star (z, \lambda) = (Z^A(z), Y(z)\lambda).$$

Let now $(z, \lambda) \in \mathbb{C}_\infty \times \mathbb{R}^+$. Then one can show easily:

$$\forall g \in G : \Psi_p^g(z, \lambda) = g \star (z, \lambda).$$

Hence we have shown, that Con_p^+ is isomorphic to G and acts on $\mathbb{C}_\infty \times \mathbb{R}^+$ in terms of the faithful group action $\star$. Therefore we will use the notions G and Con_p^+ synonymously, while we prefer the former in situations, where we focus on the mathematical structure of the conformal automorphism group, and use the latter, when we refer to its interpretation as a group of automorphisms of $L_p^+\mathscr{M}$.

Lorentz subgroups in terms of crossed homomorphisms: Let now

$$c : \mathrm{SO}^+(1,3) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), \Lambda \mapsto c_\Lambda$$

be a crossed homomorphism (see appendix D.2), i.e. c satisfies

$$c_{\Lambda_1 \Lambda_2} = c_{\Lambda_2} \cdot c_{\Lambda_1} \circ Z_{\Lambda_2}.$$

By isomorphy of the Lorentz group $\mathrm{SO}^+(1,3)$ and the Möbius group $\mathrm{PSL}(2,\mathbb{C})$, this induces a Lorentz subgroup $i_c(\mathrm{SO}^+(1,3)) \subset G$ as specified by the associated embedding

$$i_c : \mathrm{SO}^+(1,3) \hookrightarrow G, \Lambda \mapsto (A_\Lambda, c_\Lambda), \tag{6.29}$$

and moreover, any Lorentz subgroup of G is of this form. This is a general property of semidirect product groups, as explained in appendix D.2. We now want to give two examples for classes of crossed homomorphisms in the present situation. For both classes, there exist elementary examples, that we have encountered already without noticing it or that occur later on. A complete classification of crossed homomorphisms is not important for the present discussion and will be a question of further research, cf. section 6.10. For the first class, let $L \in C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$ be a smooth function. It is then easy to see, that the associated map

$$c^{(L)} : \mathrm{SO}^+(1,3) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), \Lambda \mapsto \frac{L \circ Z_\Lambda}{L}$$

constitutes a crossed homomorphism and hence, the associated embedding

$$\mathrm{SO}^+(1,3) \hookrightarrow G, \Lambda \mapsto \left(A_\Lambda, \frac{L \circ Z_\Lambda}{L}\right)$$

defines a Lorentz subgroup of G. For the second class of crossed homomorphisms, we have to introduce so called projective coordinates for $\mathbb{C}_\infty$ (cf. appendix C.1 or [143]). Those are obtained by labeling points $z \in \mathbb{C}_\infty$ not by a single complex number, but by a pair of complex numbers $(\xi, \eta) \in \mathbb{C}^2$ that are allowed to take any value other than $(0,0)$ and specify a point $z \in \mathbb{C}_\infty$ in terms of the quotient

$$z = \xi/\eta.$$

They are often more convenient, since any point on $\mathbb{C}_\infty$ can be labeled in terms of projective coordinates by two finite complex numbers, e.g. $\infty = \xi/0$. Also, the action of Möbius transformations on projective coordinates can be conveniently expressed (by slight abuse of notation) as:

$$Z^A(\xi/\eta) = A \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Please note, that two tuples (ξ, η) and $(\alpha\xi, \alpha\eta)$ specify the same $z \in \mathbb{C}_\infty$ for any $\alpha \in \mathbb{C} \setminus \{0\}$. Therefore, they are called projective coordinates.

A different kind of crossed homomorphisms can then be defined, if Q is a homogeneous, positive valued polynomial of degree d in $\mathbb{C} \times \mathbb{C}$, i.e. a map

$$Q : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+, (\xi, \eta) \mapsto Q(\xi, \eta)$$

which satisfies $Q(\alpha\xi, \alpha\eta) = \alpha^d Q(\xi, \eta)$ for all $\alpha \in \mathbb{C} \setminus \{0\}$. It is then easy to show, that the associated map (where $\mathbb{C}_\infty$ is now coordinatized in projective coordinates)

$$c^{(Q)} : \mathrm{SO}^+(1,3) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), \Lambda \mapsto c^{(Q)}_\Lambda := \frac{Q(A_\Lambda\ \cdot\ )}{Q(\cdot)},$$

which should be understood as

$$c_\Lambda^{(Q)}(z = \xi/\eta) = \frac{Q(A_\Lambda(\xi,\eta)^T)}{Q\,(\xi,\eta)}$$

gives a well defined crossed homomorphism. Hence, the associated embedding

$$\mathrm{SO}^+(1,3) \hookrightarrow G, \Lambda \mapsto \left(A_\Lambda, \frac{Q(A_\Lambda\ \cdot\ )}{Q(\cdot)} \right)$$

defines then a Lorentz subgroup of G.

6.5. Characterization of the subgroup of isometries

In the last paragraph we saw, that G comprises infinitely many Lorentz subgroups. In this section we will show, that the group of isometries Iso_p^+ is one of those Lorentz subgroups, and that it is induced by the usual representation of the Lorentz group on $T_p\mathscr{M}$. Especially, we will first present the crossed homomorphism to which the inclusion $\mathrm{Iso}_p^+ \subset \mathrm{Con}_p^+$ is associated and then we will explain, how the group Iso_p^+ is induced by the action of the Lorentz group on $T_p\mathscr{M}$ associated with a local vielbein.

Intrinsic characterization Recall, that we have defined in (6.22) a map f^A associated to an $A \in \mathrm{PSL}(2,\mathbb{C})$ as the inverse of the conformal factor K^A from (6.20). By the use of projective coordinates, this map can be conveniently written as

$$f^A : \mathbb{C}_\infty \to \mathbb{C}_\infty, z = \xi/\eta \mapsto \frac{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} A^* A \begin{pmatrix} \xi \\ \eta \end{pmatrix}}{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}}.$$

This representation makes it then easy to show, that f defines indeed a crossed homomorphism

$$f : \mathrm{PSL}(2,\mathbb{C}) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), A \mapsto f^A,$$

by checking either

$$f^{A_1 A_2} = f^{A_1} \circ Z^{A_2} \cdot f^{A_2}$$

explicitely, or by realizing, that this map is, under the isomorphism $\mathrm{SO}^+(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$, an example of the second class of crossed homomorphisms as presented in section 6.4. The latter holds especially, because the map

$$Q : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+, (\xi,\eta) \mapsto \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi\bar{\xi} + \eta\bar{\eta}$$

is a positive valued homogeneous polynomial of order 2 on $\mathbb{C} \times \mathbb{C}$. By the isomorphism $\mathrm{SO}^+(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$, this defines then a crossed homomorphism

$$f : \mathrm{SO}^+(1,3) \to C^\infty(\mathbb{C}_\infty, \mathbb{R}^+), \Lambda \mapsto f_\Lambda := f^{A_\Lambda}. \tag{6.30}$$

Recall now from section 6.3, that the isometry group Iso_p^+ is explicitely given by

$$\mathrm{Iso}_p^+ = \left(\{\Phi_p^A | A \in \mathrm{PSL}(2,\mathbb{C})\}, \circ \right)$$

with

$$\Phi_p^A : \mathbb{C}_\infty \times \mathbb{R}^+ \to \mathbb{C}_\infty \times \mathbb{R}^+, (z,\lambda) \mapsto (Z_A(z), f^A(z)\lambda).$$

Hence,

$$\Phi_p^A = \Psi_p^{(A,f^A)}$$

holds, where $\Psi_p^{(A,f^A)} \in \mathrm{Con}_p^+$ denotes the conformal automorphism associated with $(A, f^A) \in G$, as given by (6.24). Hence,

$$\mathrm{Iso}_p^+ \cong i_f(\mathrm{SO}^+(1,3)) \subset G$$

is satisfied, where i_f is the embedding (6.29) associated with the crossed homomorphism f, i.e.:

$$i_f : \mathrm{SO}^+(1,3) \hookrightarrow G, \Lambda \mapsto (A_\Lambda, f_\Lambda). \tag{6.31}$$

Consequently, Iso_p^+ is isomorphic to the Lorentz subgroup of G that is specified by the crossed homomorphism (6.30).

Extrinsic characterization: We now want to show, how the group Iso_p^+ arises equally in terms of the representation of the Lorentz group $\mathrm{SO}^+(1,3)$ on $T_p\mathscr{M}$ associated with a local vielbein. Let therefore (U,ψ) be a local trivialization with $p \in U$ and (E_μ) be an associated vielbein. We then obtain an associated action of $\mathrm{SO}^+(1,3)$ on $T_p\mathscr{M}$ defined as

$$\Lambda v := \left(\Lambda^\mu{}_\nu v^\nu\right) E_\mu$$

for any vector $v = v^\mu E_\mu \in T_p\mathscr{M}$. Let now

$$\psi_p^+ : L_p^+\mathscr{M} \to \mathbb{C}_\infty \times \mathbb{R}^+, v \mapsto \left(z_p^\psi(v), \lambda_p^\psi(v)\right)$$

be the coordinate system in $\mathcal{B}_p$ that is induced by (U,ψ). We then have by the results of appendix C.3 and especially by (C.16-C.17), that for any null vector $v \in L_p^+\mathscr{M}$

$$\psi_p^+(\Lambda v) = (A_\Lambda, f_\Lambda) \star \psi_p^+(v)$$

holds, with $(A_\Lambda, f_\Lambda) \in G$, f_Λ being defined by (6.30) and $\star$ being the group action (6.28). By the results of section 6.4 and 6.5, this can then equally be written as

$$\psi_p^+(\Lambda v) = \Phi^{A_\Lambda} \circ \psi_p^+(v).$$

Moreover we have for $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,3)$:

$$\psi_p^+(\Lambda_1 \Lambda_2 v) = ((\Lambda_1, f_{\Lambda_1}) \cdot (\Lambda_2, f_{\Lambda_2})) \star \psi_p^+(v).$$

And by this we see, that the action of $\mathrm{SO}^+(1,3)$ on $T_p\mathscr{M}$ induces the group Iso_p^+. This can be equally understood by observing, that for all $\Lambda \in \mathrm{SO}^+(1,3)$ and all $v, w \in T_p\mathscr{M}$

$$g(\Lambda v, \Lambda w) = g(v, w)$$

must hold. Hence, the restriction of any Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ to $L_p^+\mathscr{M}$ constitutes an isometriy for the induced metric q on $L_p^+\mathscr{M}$ from (S4).

6.6. On length gauges and Lorentz subgroups

In this section we want to gain a better intuition for the physical interpretation of the group $\mathrm{Con}_p^+ \cong G$ and its Lorentz subgroups. Therefore we now want to make the structure and the interpretation of Con_p^+ more lucid by explaining, how any Lorentz subgroup of Con_p^+ seems to define a Lorentz-covariant notion of scale for null vectors. By this we will argue, that Con_p^+ could be a more natural automorphism group for $L_p^+\mathscr{M}$ than any of its Lorentz subgroups.

To do so, we will now adopt a "passive" point of view and try to understand, which coordinate systems for $L_p^+\mathscr{M}$ are induced, if one composes conformal automorphisms $\Psi \in \mathrm{Con}_p^+$ with a coordinate system $\psi_p^+ \in \mathcal{B}_p$. But first, we take one step back and ask in full generality, if and how one could define meaningful notions of length for null vectors in $L_p^+\mathscr{M}$.

On the one hand, one could be tempted to think, that no such notion exists, since the inner product g_p is degenerate on $L_p^+\mathscr{M}$ and hence

$$\sqrt{g_p(v,v)} = 0$$

holds. On the other hand, $L_p^+\mathscr{M}$ is a linear cone and thus it is still possible to define meaningful notions of length as homogeneous, positive definite maps

$$\lambda_p : L_p^+\mathscr{M} \to \mathbb{R}^+, \tag{6.32}$$

which mimick hence the properites of vector space norms in the present situation. We will call in the sequel such a map (6.32) a length gauge for $L_p^+\mathscr{M}$.

Now consider first a map $\psi_p^+ \in \mathcal{B}_p$ with associated vielbein (E_μ). This map can be written as (cf. 6.1)

$$\psi_p^+ : L_p^+\mathscr{M} \to \mathbb{C}_\infty \times \mathbb{R}^+, v \mapsto \left(z_p^\psi(v), \lambda_p^\psi(v)\right) \tag{6.33}$$

with

$$\lambda_p^\psi(v^\mu E_\mu) = v^0 = |\vec{v}| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}. \tag{6.34}$$

Here, λ_p^ψ as given by (6.34) is obviously a length gauge for $L_p\mathscr{M}$ and hence, any map $\psi_p^+ \in \mathcal{B}_p$ determines a length gauge for $L_p^+\mathscr{M}$ as the 3-length (or equivalently as the 0-component) of a null vector in the associated vielbein frame. Hence, the class of coordinates $\mathcal{B}_p$ (or equivalently the bundle atlas $\mathcal{B}$) singles out a class of length gauges for $L_p^+\mathscr{M}$, namely exactly those, that are associated to vielbein frames, as specified by (6.34).

At this stage, at least the author feels a bit uncomfortable: The 3-length (or equivalently the zero component v^0) associated to a vielbein frame is not a Lorentz invariant quantity. Why should it be hence a preferred notion of length for null vectors? For example, Penrose writes in [143]:

> *The extent of a null vector cannot be characterized in an invariant way by a number, nor can null vectors of different directions be compared with respect to extent. The ratio of the extents of null vectors of the same direction is meaningful, being just the ratio of the vectors.*

Here, Penrose calls it "the extent of a null vector", what we call the 3-length as given by the length gauge (6.34). Hence, the length gauges (6.34) as singled out by the class of coordinates $\mathcal{B}_p$ don't seem to be meaningful quantities. Now, there are two strategies, how one could deal with this insight. Either one could discard length gauges completely, what leads to the usual "identification" of a future pointing light cone with the celestial sphere, together with the corresponding well known theory (cf. [143]). But instead of doing so, one could adopt a different strategy: Instead of discarding length gauges completely, one could consider contrarily the set of *all* possible length gauges as an invariant geometric structure associated with $L_p^+\mathscr{M}$ and analyse its properties. And we will see now, that this is exactly what we did in this paper. Especially we will understand, that the group G describes all possible length gauges together with their transformation properties under Lorentz transformations.

Therefore consider again a coordinate system $\psi_p^+ \in \mathcal{B}_p$ as given by (6.33). Now let $(\mathbf{1}, Y) \in G$ and consider the map $\psi_p^Y := \Psi^{(\mathbf{1},Y)} \circ \psi_p^+ = (\mathbf{1}, Y) \star \psi_p^+$. This map will be explicitely written as

$$\psi_p^Y : L_p^+\mathscr{M} \to \mathbb{C}_\infty \times \mathbb{R}^+, v \mapsto \left(z_p^Y(v), \lambda_p^Y(v)\right)$$

and by the very definition of the group action $\star$ (or equivalently by the action of conformal automorphisms as presented in section 6.3) we have then:

$$\begin{aligned} z^{Y}_{p}(v) &= z^{\psi}_{p}(v), \\ \lambda^{Y}_{p}(v) &= Y(z^{\psi}_{p}(v))\lambda^{\psi}(v). \end{aligned} \tag{6.35}$$

And hence, since $Y \in C^{\infty}(\mathbb{C}_{\infty}, \mathbb{R}^{+})$, any length gauge is induced, if we act with group elements $g \in G$ on coordinate systems in $\mathcal{B}_p$. I.e. any length gauge (6.32) can be written as (6.35) for a suitable $Y \in C^{\infty}(\mathbb{C}_{\infty}, \mathbb{R}^{+})$. Consequently, group elements $g \in G$ of the form $g = (\mathbf{1}, Y)$ could be understood as pure length gauge transformations. In addition one can now enlarge the class of coordinates $\mathcal{B}_p$ to a larger class $\mathcal{R}_p$ which incorporates all those possible length gauges, i.e.:

$$\mathcal{R}_p = \{(\mathbf{1}, Y) \star \psi^{+}_{p} | Y \in C^{\infty}(\mathbb{C}_{\infty}, \mathbb{R}^{+}) \text{ and } \psi^{+}_{p} \in \mathcal{B}_p\}. \tag{6.36}$$

Now, by the results of section 6.5, we have, that for any two $\psi^{+}_{1,p}, \psi^{+}_{2,p} \in \mathcal{B}_p$ there exists exactly one $g \in \mathrm{Iso}^{+}_{p} \subset \mathrm{Con}^{+}_{p}$, s.th. $\psi^{+}_{1,p} = g \star \psi^{+}_{2,p}$ holds. Hence, we have

$$\mathcal{R}_p = G \star \psi^{+}_{p}$$

for any $\psi^{+}_{p} \in \mathcal{B}_p$. And moreover, it follows then, that G acts simply transitive on $\mathcal{R}_p$ and hence parametrizes all possible coordinate systems for $L^{+}_{p}\mathscr{M}$ that incorporate all admissible length gauges thereon.

But how do the various Lorentz subgroups enter the game now? This is maybe the most subtle and interesting aspect of the present discussion. Therefore consider first again maps $\psi^{+}_{p} \in \mathcal{B}_p$ as given by (6.33). As explained above, the associated length gauges λ^{ψ}_{p} given by (6.34) are 3-lengths induced by vielbein frames. And as such, there is an associated transformation law under Lorentz transformations, explicitely described by

$$\psi^{+}_{p}(\Lambda v) = (A_{\Lambda}, f_{\Lambda}) \star \psi^{+}_{p}(v)$$

as presented in section 6.5. But if one enlarges the class of coordinates, such that all length gauges are allowed, i.e. if one makes a transition from $\mathcal{B}_p$ to $\mathcal{R}_p$ as defined in (6.36), then there exists no single, preferred Lorentz transformation law anymore. Instead, there are infinitely many possible Lorentz transformation laws as described by the various Lorentz subgroups of G. Moreover, if we consider again the group antihomomorphism κ from (6.25), we see, that it is not possible to absorb any crossed homomorphism other than the trivial one in the group composition law. I.e. we can't define

$$\kappa_A : C^{\infty}(\mathbb{C}_{\infty}, \mathbb{R}^{+}) \to C^{\infty}(\mathbb{C}_{\infty}, \mathbb{R}^{+}), Y \mapsto c_A \cdot Y \circ Z^{A}$$

for $c_A \neq 1$, since otherwise $\kappa_A(1) = c_A \neq 1$ and hence, G would not constitute a group.

This means, that from a group theoretic perspective, no non-trivial Lorentz subgroup of G could be preferred in terms of an alternative multiplication law for G. Especially we see, that the crossed homomorphism $A \mapsto f_A$ seems completely arbitrary from this perspective. Conversely, any Lorentz subgroup of G singles out a subclass of length gauges and a corresponding Lorentz transformation law, which is again unsatisfactory in the light of the discussion above surrounding the quote of Penrose. Hence it seems, that no Lorentz subgroup of G and especially not its subgroup of isometries is intrinsically preferred. By this argumentation, G could be considered as a more natural automorphism group for $L^+_p\mathscr{M}$ than any of its Lorentz subgroups.

6.7. The automorphism groups as gauge groups for the light cone bundle

In this section we want to sketch concisely, how the microscopic BMS-like group G, as well as its subgroup of isometries, constitute gauge groups for the bundle of future pointing null vectors. Therefore observe first, that the results of section 6.2 generalize in a straightforward way to the full light cone bundle. I.e., the bundle $L^+\mathscr{M}$ is equipped with the following universal structures:

(F1) $F = \mathbb{C}_\infty \times \mathbb{R}^+$ is the typical fiber of the fiber bundle $L^+\mathscr{M}$, i.e. $L^+_p\mathscr{M} \cong F$.

(F2) Each fiber $L^+_p\mathscr{M}$ is a linear cone.

(F3) There is a family $(U_i, z_i)_{i\in I}$ of surjective maps

$$z_i : L^+U_i \to U_i \times \mathbb{C}_\infty$$

whose transition functions are well defined and valued in the Möbius group. Here $(U_i)_{i\in I}$ is an open cover for $\mathscr{M}$.

(F4) There exists a degenerate metric q_p on any fiber $L^+_p\mathscr{M}$. Moreover, the map

$$p \in \mathscr{M} \mapsto q_p$$

is smooth.

This allows then, as common in fiber bundle theory (cf. [162, 22, 87]), the definition of adapted bundle atlases that preserve those structures in a certain sense. The matching conditions of overlapping charts are then described in terms of transition functions that are valued in a structure group and, as before, the question thereby will be, how (F4) should be interpreted: As a fixed degenerate Riemannian metric on $L^+_p\mathscr{M}$ or as a a representative of a conformal structure thereon? The former will yield a $\mathrm{SO}^+(1,3)$-structure for the lightcone bundle, while the latter gives rise to a G-structure.

Consider first the case, where q_p is interpreted as a fixed degenerate Riemannian metric on any generic fiber $L_p^+\mathscr{M}$. Then the bundle atlas $\mathcal{B}$ as constructed in section 6.2 is made up of smooth local trivializations (U, φ) given by maps

$$\varphi : L^+U \to U \times F$$

which preserve the cone structure and have the property, that the metric q_p is in each chart represented by

$$ds^2 = 2\lambda^2 \frac{dzd\bar{z}}{(1+z\bar{z})^2} \tag{6.37}$$

and whose transition functions lie, by direct generalization of the results of sections 6.3 and 6.5, in $\mathrm{Iso}_p^+ \cong \mathrm{SO}^+(1,3)$. More explicitely, the latter means that for two local trivializations $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{B}$ with $U_1 \cap U_2 \neq \emptyset$ the transition function

$$\varphi_2 \circ \varphi_1^{-1} : (U_2 \cap U_1) \times F \to (U_2 \cap U_1) \times F$$

can be written as

$$\varphi_2 \circ \varphi_1^{-1}(p, (z, \lambda)) = (p, i_f(\Lambda_p) \star (z, \lambda)),$$

where

$$\Lambda : U_1 \cap U_2 \to \mathrm{SO}^+(1,3), p \mapsto \Lambda_p$$

is a local Lorentz transformation, i_f is the embedding

$$i_f : \mathrm{SO}^+(1,3) \hookrightarrow G$$

defined in (6.31) and $\star$ is the group action (6.28). In this sense, $\mathcal{B}$ constitutes a $\mathrm{SO}^+(1,3)$-structure for $L^+\mathscr{M}$, where the Lorentz group is non-trivially represented in terms of the crossed homomorphism f.

We now interpret q_p as a representative of a conformal equivalence class of metrics on a generic fiber $L_p^+\mathscr{M}$. Then the structures (F1) - (F4) induce an adapted bundle atlas, whose smooth trivializations (U, φ) given by

$$\varphi : L^+U \to U \times F$$

have the property, that the metric q_p is in each chart (U, φ) represented by

$$ds^2 = 2\lambda^2 \Omega_p^\varphi(z)^2 \frac{dzd\bar{z}}{(1+z\bar{z})^2} \tag{6.38}$$

for an associated smooth conformal factor

$$\Omega^\varphi : U \times \mathbb{C}_\infty \to \mathbb{R}^+, (p, z) \mapsto \Omega_p^\varphi(z), \tag{6.39}$$

and whose transition functions lie hence in G. The latter means, that for two such local trivializations $(U_1, \varphi_1), (U_2, \varphi_2)$ with $U_1 \cap U_2 \neq \emptyset$ the transition function

$$\varphi_2 \circ \varphi_1^{-1} : (U_2 \cap U_1) \times F \to (U_2 \cap U_1) \times F$$

can be written as

$$\varphi_2 \circ \varphi_1^{-1}(p, (z, \lambda)) = (p, g_p \star (z, \lambda))$$

for

$$g : U_1 \cap U_2 \to G, p \mapsto g_p$$

being a local smooth[5] G-gauge transformation. In this sense, those local trivializations form a G-structure for $L^+\mathscr{M}$, which we call $\mathcal{R}$. Please note, that such an G-structure $\mathcal{R}$ indeed exists and can be constructed explicitely in terms of the atlas $\mathcal{B}$, by applying smooth G-valued maps locally on coordinate systems in $\mathcal{B}$. I.e. we define

$$\mathcal{R} = \left\{ (U, g \star \psi^+) \middle| (U, \psi^+) \in \mathcal{B} \text{ and } g \in C^\infty(U, G) \right\}$$

where $g \star \psi^+$ is defined as:

$$g \star \psi^+ : L^+U \to U \times F, v \in L^+_p\mathscr{M} \mapsto \Big(p, g_p \star \psi^+_p(v) \Big)$$

By this, it follows directly, that the restrictions of maps in $\mathcal{R}$ to $T_p\mathscr{M}$ are given by $\mathcal{R}_p$ as constructed in section 6.6.

In this sense, we have identified two extremal natural gauge groups for the bundle $L^+\mathscr{M}$: The Lorentz group $\mathrm{SO}^+(1,3)$ forms a kind of minimal gauge group, while the microscopic BMS group G constitutes some sort of maximal gauge group. By the argumentation of section 6.6, the former can be understood as a gauge group, which carries already information on the embedding $L^+\mathscr{M} \hookrightarrow T\mathscr{M}$, while the group G seems to be preferred by the intrinsic geometry of $L^+\mathscr{M}$. Nevertheless, there should be some hard criterion, by which one can answer the question, which of those gauge groups is the "correct one". We will comment on this again in section 6.9. Please note in addition, that, by the results of this section, we have now all necessary ingredients for a $\mathrm{SO}^+(1,3)$- and a G-gauge theory on $L^+\mathscr{M}$. I.e. one could define associated principal fiber bundles and analyse connections thereon.

Finally we would like to note, that there are also groups H which satisfy

$$\mathrm{SO}^+(1,3) \subsetneq H \subsetneq G$$

and could be also understood as possible gauge groups for $L^+\mathscr{M}$. For example, one could restrict the class of the allowed conformal factors $\Omega_p^\varphi(z)$ from equation (6.38). Also, any

[5]Of course, one has to specify, which notion of smoothness is meant exactly, since G is infinite dimensional. But since we just want to sketch the ideas in the present discussion, we won't bother with this question here.

of the Lorentz subgroups of G should induce a corresponding bundle atlas. A more rigorous mathematical investigation of those structures as well as their geometric interpretations would be desirable, and we will comment on this question again in section 6.10.

6.8. Relation to the BMS analysis

In this paragraph, we want to compare the original BMS analyis at null infinity with our microscopic analogue. Especially, we will first compare our methodology with the original BMS methodology. Afterwards we will compare the structure of G with the structure of the original BMS group. The similarities between those situations will then justify, why we call G a microsocpic analogue of the BMS group, as will be summarized at the end of this section.

Methodology We want to review concisely the original BMS analysis of asymptotically flat spacetimes in a modern language. Our main sources for this are [79, 13, 14]. To understand the original BMS analysis of asymptotic symmetries, it is instructive (cf. [13, 14]), to perform a conformal compactification (cf. [140]) of the asymptotically flat spacetime under consideration. Then null infinity $\mathcal{I}$ becomes a 3-dimensional submanifold of Einstein's static universe, which can be regarded as a part of the topological boundary of the considered "physical" spacetime and as such, it inherits several universal geometric structures from the physical spacetime. Those structures (cf. [13]) consist out of a degenerate metric q_{ab} with signature $(0, +, +)$ and a complete vector field n on $\mathcal{I}$, which is defined in terms of the conformal factor Ω that was used for the conformal compactification:

$$n^\mu = \nabla^\mu \Omega$$

Here, ∇ denotes the Levi-Civita connection on the compactified spacetime. In addition, there is a remaining freedom for admissible conformal transformations of the physical spacetime given by smooth redefinitions of the conformal factor Ω of the form

$$\Omega' = \omega\Omega.$$

Here, ω is a smooth function on the compactified spacetime (including its boundary) that is nowhere vanishing on $\mathcal{I}$. One can restrict this "gauge freedom" further by allowing solely conformal factors, that satisfy $\nabla_\mu n^\mu = 0$. One can then show (cf. [13, 14]) that smooth factors ω preserve this condition if and only if their Lie derivative along n vanishes, i.e. $\mathcal{L}_n\omega = 0$, under restriction to $\mathcal{I}$. Under such "gauge transformations", the universal structure $(q_{\mu\nu}, n^\mu)$ transforms as follows:

$$q'_{\mu\nu} = \omega^2 q_{\mu\nu} \qquad\qquad n'^\mu = \omega^{-1} n^\mu \tag{6.40}$$

One then calls two tuples $(q_{\mu\nu}, n^\mu)$ and $(q'_{\mu\nu}, n'^\mu)$ equivalent, if they are related to each other by a conformal transformation of the type (6.40). The BMS group can then be understood as the group of all transformations on I, which preserve this universal structure, i.e. map tuples $(q_{\mu\nu}, n^\mu)$ to equivalent tuples.

The relation to our analysis becomes apparent, if one realizes, that all objects appearing in the original BMS analysis have direct microscopic analogues in our analysis. Consider first asymptotic flatness. In the case of the original BMS framework, the property of asymptotic flatness was exactly described in terms of decay conditions for the metric along null rays towards infinity (cf. [126]). As explained in sections 6.1 and 6.2, our analogue to asymptotic flatness is given by Einstein's equivalence principle. As sketched in footnote 3, Einstein's equivalence principle demands flatness in an infinitesimal limit, which can be quantified in terms of Riemann normal coordinates: Around any point $p \in \mathscr{M}$ there exists coordinate patch (x^μ) around p (i.e. $x^\mu(p) = 0$) in which the metric assumes the form (cf. [35, 164])

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(0) x^\alpha x^\beta + \mathcal{O}(|x|^3),$$

where $R_{\mu\alpha\nu\beta}$ is a coordinate expression for the Riemann tensor. In this formulation, Einstein's equivalence principle is quantified in terms of a microscopic asymptotic decay condition and hence resembles the role of asymptotic flatness in the BMS framework. Consider now null infinity: It is a natural macroscopic null surface associated with any asymptotically flat spacetime and can be represented as the union of a pointed past and a pointed future light cone in Einstein's static universe. Moreover, it is diffeomorphic to $\mathbb{C}_\infty \times \mathbb{R}$. Analogously, the past and future tangent light cones $L_p^\pm \mathscr{M}$ are natural microscopic null surfaces associated with any spacetime satisfying Einstein's equivalence principle and are diffeomorphic to $\mathbb{C}_\infty \times \mathbb{R}^+$. In the BMS analysis, the geometry of the bulk spacetime induces a degenerate metric q and a complete null vector field n^μ on null infinity. In our situation, the geometry of $T_p\mathscr{M}$ together with the psuedo-Riemannian metric g induces also a degenerate metric as given by (S4) on $L_p^+\mathscr{M}$. The analogue to the complete null vector field n^μ is then given by the linear cone structure of $L_p^+\mathscr{M}$. The microscopic analogue to the gauge condition $\mathcal{L}_n\omega$ in the original BMS-analysis is the requirement, that conformal transformations should preserve the compatibility of the induced metric with the linear cone structure, as depicted at the end of section 6.3. The group Con_p^+ is in our situation then the group which preserves the linear cone structure and the conformal structure on $L_p^+\mathscr{M}$ up to conformal equivalence, analogously as the BMS group preserves the universal structure in the sense depicted above.

Group structure In this section we will compare the structure of G with the structure of the original BMS group. Thereby we will see, that some subtile differences appear, although the structure of G and the structure of the original BMS group are still very similar. Our main source regarding the structure of the original BMS group will be the modern review [8]. Classic sources on this topic are [156, 41, 130]. Null infinity can be coordinatized in

Bondi coordinates by $(z,u) \in \mathbb{C}_\infty \times \mathbb{R}$ (cf. [138]) and in [8] a general BMS transformation thereon is given by (cf. Formulas 5.15a and 5.15b of [8])

$$z \to z' = \frac{az+b}{cz+d} \tag{6.41}$$
$$u \to u' = K(z,\bar{z})\,[u+\alpha(z,\bar{z})] \tag{6.42}$$

where K is the conformal factor given by

$$K(z,\bar{z}) = \frac{1+z\bar{z}}{(az+b)(\bar{a}\bar{z}+\bar{b})+(cz+d)(\bar{c}\bar{z}+\bar{d})}, \tag{6.43}$$

and $\alpha \in \mathbb{C}^\infty(\mathbb{C}_\infty,\mathbb{R}^+)$ is a supertranslation. Observe now, that the formulas (6.41 - 6.42) are very similar to the coordinate expression of a transformation of $L_p^+\mathscr{M}$ that corresponds to an iterative application of a generic element $\Phi_p^A \in \mathrm{Iso}_p^+$ and an element $\Psi_p^{(\mathbf{1},Y)} \in \mathrm{Con}_p^+$, since

$$\Psi_p^{(\mathbf{1},Y)} \circ \Phi_p^A(z,\lambda) = \Big(Z^A(z), Y(z)f^A(z)\lambda\Big).$$

holds, what means, that (z,λ) transforms under $\Psi_p^{(\mathbf{1},Y)} \circ \Phi_p^A \in \mathrm{Con}_p^+$ as

$$z \to z' = \frac{az+b}{cz+d} \tag{6.44}$$
$$\lambda \to \lambda' = K(z,\bar{z})^{-1}Y(z)\lambda \tag{6.45}$$

with the conformal factor K given by (6.43). The important differences between (6.44 - 6.45) and (6.41 - 6.42) are, that our analogues of supertranslations do not act by addition, but by multiplication, and that our analogue of the conformal factor, given by the crossed homomorphism f_A, is exactly the inverse of K. The former can be easily understood, since automorphisms of the form $\Psi^{(\mathbf{1},Y)}$ are multiplicative "superrescalings" of null vectors. The occurence of the inverse conformal factor in (6.45) can be understood in the present situation, if one derives the Lorentz-Möbius correspondence not as we will do it in appendix C.3, but in terms of Bondi coordinates. Then one obtains, that the advanced coordinate $u = t + r$ rescales under Lorentz transformations exactly under the inverse prefactor as the radial coordinate r. This is for example presented in sections 4.2.1 and 4.2.2 of [138].

In all articles, which we have mentioned above, the structure of the original BMS group is described as a semidirect product of the Lorentz group $\mathrm{SO}^+(1,3)$ with the group of supertranslations $C^\infty(\mathbb{C}_\infty,\mathbb{R})$. In this sense our group G resembles the structure of the original BMS group. Especially in [8], also the original BMS group is constructed as a right semidirect product, which resembles our construction of G as performed in section 6.4. But if one digs deeper in the mentioned articles, an important difference will appear: In [8, 80], the group action which is utilized for the definition of the semidirect product includes the conformal factor K. In particular, for a supertranslation $\alpha \in C^\infty(\mathbb{C}_\infty,\mathbb{R})$ and a Lorentz group element $\Lambda \in \mathrm{SO}^+(1,3)$, the group action is defined as (cf. eq. 6.14a-b of [8]):

$$\sigma_\Lambda(\alpha) = K^{-1} \cdot \alpha \circ Z_\Lambda. \tag{6.46}$$

Adapted to our situation, the corresponding modification of the group action κ specified by (6.25 - 6.26) would be

$$\kappa : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{Aut}(C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)), A \mapsto \kappa_A \tag{6.47}$$

with:

$$\kappa_A : Y \mapsto f_A \cdot Y \circ Z^A \tag{6.48}$$

But this would not be appropriate in our situation. If we had adopted the group action (6.47 - 6.48), we would have had especially

$$\kappa_A(1) = f_A \neq 1.$$

Hence κ_A would not constitute an automorphism of $C^\infty(\mathbb{C}_\infty, \mathbb{R}^+)$, since it would not preserve the unit element. Hence G would in this case not constitute a proper group (cf. also the discussion at the end of section 6.6). Therefore we think, that f_A must be excluded from the definition of the semidirect product in the present scenario. In the case of the original BMS group, this ambiguity does not appear, since it is additive. Hence its unit element is given by 0 and fortunately the action (6.46) satisfies $\sigma_\Lambda(0) = 0$.

Conclusion We think, that the present discussion shows, that the original BMS group and the group G are very similar: Both appear as conformal automorphism groups of natural null surfaces endowed with similar universal geometric structures and both can be written as right semidirect products of the Lorentz group (or equivalently the Möbius group) with a group of smooth functions on the Möbius sphere.

6.9. On possible implications

As said in the introduction, the original BMS group found various applications in different branches of modern gravitational and high energy physics. Motivated by this success and by the similarity of the microscopic BMS-like group G presented in this article as compared to the original BMS group, we ask the question, if the group G together with its associated geometric structures could have similar applications. Therefore we will sketch three directions of research, that are motivated by the findings of the present article and by applications of the original BMS group. The scenarios depicted in this subsection will be investigated in a subsequent series of publications.

A bulk description for gravitational waves? The original BMS analysis of universal structures and associated symmetries at null infinity was a conerstone of the proof, that gravitational waves indeed exist in general relativity (cf. [126]). The reason for this is, that the description of gravitational radiative degrees of freedom simplifies drastically, if one formulates them in terms of induced higher order quantities on null infinity (cf. [12, 13, 14]). Given the similarity of the group G with the original BMS group and the similarity of the universal structures on $L^+_p\mathscr{M}$ with the universal structures on null infinity, one could ask, if an analysis of G-connections on the light cone bundle $L^+\mathscr{M}$ could also lead to a simplified description of radiative degrees of freedom on the bulk. I.e., in a more colloquial language, if infinitesimal light cones could serve as convenient probes for gravitational waves on the bulk. This could be indeed meaningful, since the equivalence classes of connections which appear at null infinity (cf. [12, 13, 14]) could correspond in our scenario to a single gauge equivalence class of G-connections on $L^+\mathscr{M}$ as induced by several inequivalent Levi-Civita connections on $T\mathscr{M}$. Consequently, also quantities like the Bondi news tensor or the leading order Weyl tensor (for both cf. [13]) could have analoga for G-connections. Finally, the tangent bundle of null infinity could be understood as an arena for boundary values of the bulk light cone bundle. By this, the description of radiative degrees of freedom on null infinity should correspond directly to boundary values for G-connections, which could define soliton-like vacuums solutions.

An "UV-Triangle"? Recently, a deep interconnection between the original BMS group, soft theorems in quantum gauge theory and memory effects in gravitational physics was discovered, going under the name of the IR-triangle (cf. [165]). Given the structural similarity of the group G and the BMS group, one could ask the question, if there could be a similar interrelation between G and the UV-structure of gauge theories. This seems appealing: Due to the infinite dimensionality of G, there could exist infinitely many charges associated with G. Similarly, as the charges of the original BMS group imply soft theorems in terms of their Ward identities, one could ask, if identities associated with G could constrain scattering amplitudes non-perturbatively in the deep UV. Moreover, the existence of distinct Lorentz subgroups of G could be related to a microscopic gravitational memory effect: A bypassing gravitational wave could link two Lorentz subgroups of G to each other, resembling the situation at null infinity, where gravitational radiation links distinct Minkowski vacua to each other, which are otherwise related by supertranslations (cf. [166, 13, 14]).

An extension of the fundamental gauge group of gravity? Finally we would like to propose a speculative scenario, in which G (or maybe also a subgroup of G that is strictly larger than the subgroup of isometries) could constitute a fundamental gauge group for a full theory of gravity. But before doing so, please note first, that G could indeed already arise as a gauge group for a specific, very realistic sector of general relativity: In the situation, where all test particles are assumed to be massless, the tangent bundle can be safely replaced by the light cone bundle and especially all gravitational quantities should influence such test particles only in terms of their induced quantities on $L^+\mathscr{M}$. One could then analyse, if Einstein's equation (or equivalently the Einstein-Hilbert action) can be reexpressed entirely in terms of induced quantities on $L^+\mathscr{M}$. By doing so, one should especially understand, which gauge freedom is dictated by the action principle (or equivalently by its canonical formulation) for the induced connections on $L^+\mathscr{M}$: Is the gauge freedom described by G or is it described by the subgroup of isometries? The discussion of section 6.6 makes it plausible, that the gauge group is indeed enlarged to G in this scenario, but of course, this has to be investigated. But however, a careful analysis of this situation should shed in any case some light on the the question regarding the "correct" gauge group for the light cone bundle, as raised in section 6.7. Please note also, that this scenario is also closely connected to a hypothetical bulk description of gravitational waves as sketched above: Both scenarios aim towards a simplification of general relativity by considering solely its effects on massless test particles.

After sketching this realistic scenario regarding a subsector of general relativity, we now want to propose a speculative, in which G could constitute a gauge group for a full theory of gravity. Therefore recall, that we have shown in section 6.4, that G contains infinitely many Lorentz subgroups. By the argumentation of section 6.6 one can realize in addition, that two distinct Lorentz subgroups are related by some kind of length gauge transformation. Moreover, all Lorentz subgroups are (trivially) isomorphic to each other and especially isomorphic to the subgroup of isometries. This finding resembles to some extent the situation in spontaneously broken gauge theories and hence one could ask, if a Higgs-like mechanism could break the gauge symmetry of a G-gauge theory to an arbitrary Lorentz subgroup. By this, general relativity could constitute a low-energy approximation of such a theory, where the Higgs-like field is near to its ground state. Interestingly, everything should move at the speed of light in situations, where the Higgs-like field is not in its ground state. Please note in addition, that the subgroup of isometries is, although not intrinsically preferred, induced by the standard representation of the Lorentz group on $\mathbb{R}^4$. By this, the subgroup of isometries seems natural from the perspective of any spontaneously choosen Lorentz subgroup of G, although it seems unnatural if one considers G as a fundamental gauge group.

Finally note, that those scenarios seem to be plausible from different perspectives. There are various hints (cf. chapter 5), that gravity behaves at fundamental scales in a 2-dimensional way (cf. [42]). In addition, the BKL-conjecture (cf. [26]) suggests, that gravity behaves at fundamental scales in an ultralocal way, which is also a property of ultrarelativistic field theories (cf. [50, 113]). Note, that those properties would be directly imprinted into any theory associated with $L^+\mathscr{M}$ and G: The light cone bundle is infinitesimally a 2-dimensional space and hence the dimensional reduction at microscopic scales would be manifest in a G-gauge theory on the light cone bundle. Moreover, tangent light cones are obviously ultrarelativistic objects and the group G should thus, analogously to the situation in [60, 59, 61], be related to some ultrarelativistic symmetry group. Hence, also the ultralocal behaviour at microscopic scales should be manifest. This could be summarized in a picturesque way by saying, that a

G-gauge theory on the light cone bundle should describe a situation, where fundamentally everything moves at the speed of light and where timelike causality relations are emergent by some yet to be invented mechanism. As argued above, this mechanism could be maybe a spontaneous symmetry breaking or a ”gauged holographic principle”.

6.10. Conclusion

Finally we want to conclude this chapter. Therefore we will first summarize the results of the present analysis and will comment then on further open questions other than those raised in the last section.

Summary: In this chapter we have performed a thorough analysis of the universal structures that are induced on the infinitesimal tangent light cones of a generic spacetime obeying Einstein’s equivalence principle. Thereby, we obtained as a main result, that those structures single out two natural microscopic symmetry groups, that arise as automorphism groups: A non-trivially represented Lorentz group as their isometry group and a group G as their conformal automorphism group. We investigated the mathematical structure of the group G and showed, that it can be described in terms of a right semidirect product of the Lorentz group (or equivalently the Möbius group) with a group of smooth, positive valued functions on the Riemann sphere. We further showed, that G contains infinitely many Lorentz subgroups which are parametrized in terms of crossed homomorphisms. We have demonstrated, how the isometry group arises as a non-canonical subgroup of G and argued, that no Lorentz subgroup seems to be intrinsically preferred from a geometric and a group theoretic perspective. Especially, we realized thereby, that G encodes all possible length gauge choices for null vectors and that any Lorentz subgroup corresponds to a subclass of such length gauge choices together with an an associated Lorentz transformation law. We also compared our methodology and results with the classic BMS analysis, and justified by this, that G can be called a microscopic analogue of the BMS group. Finally, we have sketched, how G and the isometry subgroup could constitute gauge groups for the bundle of null directions. By this we have identified a geometric structure which exists on the bulk of any spacetime obeying Einstein’s equivalence principle and which is associated with a BMS-like group. This implies especially, that BMS-like groups do not only describe macroscopic asymptotic symmetries in general relativity, but also constitute a fundamental and, to the best of our knowledge, unknown microscopic symmetry of Lorentzian geometry/ This symmetry encodes in an invariant way, how null vectors interfere with Lorentz transformations. In addition we would like to mention three results of this article, that lie not in the mainline of argumentation, but are still worthwhile to be mentioned explicitly:

- We gave a convenient representation for the rescalings of null vectors under Lorentz transformations, cf. (6.7) or section 6.5. This transformation law was derived by a reinterpretation of the implicit definition of the inverse stereographic projection (C.3), which is convenient for the calculation of null vector rescalings, cf. appendix C.3. Although it was of course realized all over literature, that null vectors transform under Lorentz transformation not only by a change of their direction, but also by a rescaling of their length (cf. e.g. [143, 130, 156]), the representation (6.7) as well as its derivation in appendix C.3 are to the best of our knowledge new.

- We have shown in section 6.2, that the square root of the negative standard Minkowski inner product constitutes a kind of distance function on the light cone of Minkowski vector space. Moreover we have shown, that this distance function is related to the chordal distance on the Riemann sphere. Although this result seems very elementary, it is, to the best of our knowledge, not present in the existent literature. Nevertheless, a related reasoning was performed in [142].

- The occurence of infinitely many Lorentz subgroups of the conformal automorphism group can be equally understood by the statement, that there seems to be no distinguished Lorentz transformation law for length gauges anymore, if one considers all possible length gauges for null vectors, cf. the end of section 6.6. I.e. if one enlarges the class of coordinate systems as described at the end of section 6.6, then the subgroup of isometries together with its associated Lorentz transformation law seem to loose their preferred role, and infinitely many Lorentz transformation laws emerge.

Finally, we want to remark, that it is in the view of the author interesting, that the group G carries no obvious canonical structure that singles out a non-trivial Lorentz subgroup. Usually, structure groups in fiber bundle theory are some kind of "group theoretic mirror" of the geometry under consideration, since they encode geometric properties of the fiber bundle (and its base manifold) in terms of group theoretic properties (cf. [22, 162]). In the present scenario, the metric (6.37) is a distinguished geometric object on $L_p^+\mathscr{M}$, since it has constant positive curvature. But as discussed at the end of section 6.6, the associated Lorentz subgroup of isometries seems not to be preferred from a group theoretic perspective. This could be interpreted as a hint, that the structure of G favours a conformal interpretation of the induced metrics on $L_p^+\mathscr{M}$.

G from the perspective of mathematical gauge theory? We have sketched in section 6.7, how the group G constitutes a gauge group for the light cone bundle. It could be an interesting question, if there exist manifolds, which admit a G-structure for a fiber bundle of linear tangent cones, but don't admit a pseudo-Riemannian metric. A further interesting question going in the same direction would be, how G-structures on the lightcone bundle interfere with spin structures on the manifold under consideration. Also, one could ask, as sketched concisely in section 6.7, how subgroups of G that are strictly larger than the isometry subgroup are related to geometric properties. Finally, it could be interesting to analyse, how the structures described in this article are related to global geometric and topological questions in pseudo-Riemannian geometry, by investigating the topology of the light cone bundle as well as its invariants (cf. [162]).

Towards a BMS geometry? It was shown in [60, 59], that a certain conformal Carroll group of the macroscopic Minkowski light cone is given by the BMS-group. In contrast, we have shown in this article, that the conformal automorphism group of the infinitesimal light cone is given by G. The geometric relation of the original BMS group to the group G resembles hence in some sense the relation of the Poincaré group to the Lorentz group: The BMS group acts on the macroscopic light cone, as the Poincaré group acts on the macroscopic Minkowski space, and the group G acts on infinitesimal tangent light cones, as the Lorentz group acts on infinitesimal tangent Minkowski spaces. Now recall, that pseudo-Riemannian geometry can be written as a Cartan geometry (cf. [161]) based on the Poincaré and the Lorentz group. Although the structural interrelation of the group G and the BMS group seem to forbid the formulation of a Cartan geometry based on those two groups, one could still ask the question, if above geometric picture could be interpreted in similar lines, giving rise to a kind of general BMS geometry in a, possibly extended, Cartan geometric framework.

The mathematical structure of G? As said in section 6.4, a general classification of crossed homomorphisms $c : \mathrm{SO}^+(1,3) \hookrightarrow G$ would be desirable. On the other hand, the existence of infinitely many Lorentz subgroups of G suggests, that each of those subgroups could encode in some sense the microscopic transformation properties of some geometric or physical object. A better qualitative understanding of crossed homomorphisms associated with Lorentz subgroups could hence yield also a better understanding of the interpretation of the occuring Lorentz subgroups. Moreover, it would be very important in the context of the discussion of section 6.6, to understand, if the crossed homomorphism f that defines the isometry subgroup is in some sense intrinsically preferred by the structure of G or just extrinsically induced by the linear representation of the Lorentz group on $T_p\mathscr{M}$.

7. Conclusion and outlook

In this chapter we will conclude this thesis. Especially we will concisely summarize its findings in section 7.1 and comment on promising directions of further research in section 7.2.

7.1. Summary

In this section we present concisely the results of this thesis from a formal perspective. A presentation which is more focused on qualitative and conceptual aspects can be found in section 1.2, while for a more in-depth treatment one should consult the conclusions of the respective chapters.

Bose-Einstein-condensation by asymptotic expansions and spectral ζ-functions:

The treatment of chapter 3 is based on the author's publication [175]. We have obtained there the following results:

- We have shown, that there exist asymptotic expansions for the grand potential of the harmonically trapped, non-interacting Bose-Gas under the open-trap limit $\kappa \to 0$. Here κ denotes the oscillator constant of the harmonic trap.
- Those asymptotic expansions resemble heat kernel expansions and differ drastically between the two phases:
  - In the non-condensation phase the asymptotic expansion is of the form

    $$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{v} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + \mathcal{O}(\kappa)$$

    and hence exhibits a singularity of finite order.
  - In the condensation phase the asymptotic expansion is of the form

    $$\Omega^{(v)}(\kappa;\beta,\mu) = \sum_{k=0}^{\infty} \kappa^{-k} a_{-k}^{(v)}(\beta,\mu) + f(\kappa) + \mathcal{O}(\kappa) \tag{7.1}$$

and hence exhibits a singularity of infinite order. Here $f(\kappa)$ represents a logarithmic singularity, cf. (3.2).

Those two expansions hence encode entirely the aspects of the system that get dominant in the thermodynamic limit $\kappa \to 0$.

- Under a renormalization of the chemical potential, which is given by the κ-dependent coupling
$$\mu_{\bar{\rho}}(\kappa) = E_0^{(v)}(\kappa) - \left[\bar{\rho} - \rho_c^{(v)}(\beta)\right]^{-1} \kappa^v + \mathcal{O}(\kappa^{v+1}),$$
the asymptotic expansion (7.1) attains a form which exhibits only a singularity of finite order and which marks the presence of condensation, cf. (3.30 - 3.31) and (3.34). Here $\rho_c^{(v)}(\beta)$ marks the critical density of the system.

- Moreover, characteristic quantities of the system, as thermodynamic observables or the critical density, are encoded in the coefficients of the asymptotic expansion.

Those structures were to our best knowledge not described before in the literature and mark a novel contribution to the theory of Bose gases.

Renormalization of a classical field theory in the vicinity of a cosmological singularity:

We have obtained the following results in chapter 4:

- The conformally coupled massless Klein-Gordon equation on a radiation dominated big bang spacetime has distributional solutions which are distributional extensions of the ordinary smooth solutions. In the concrete case of the conformally coupled massless scalar, those extensions are related to the Cauchy principal value distribution, which extends the inverse scale factor.

- The distributional solutions exhibit a 1-parameter renormalization freedom, since the coordinate expressions of two such solutions differ by a multiple of a delta distribution with support on the singular hypersurface (i.e. by a distribution proportional to $\delta(\eta - 0)$ in conformal coordinates for conformal time η).

- The classical n-point functions can be renormalized in a similar manner, which gives renormalized distributional n-point functions. Those n-point functions define then renormalized, distributional states on the algebra of multilocal Wick Polynomials. Both, states and n-point functions, can be understood as distributional extensions of their smooth counterparts.

- The renormalized, distributional states have a higher regularity than their non-renormalized, smooth counterparts. For example, for smooth solutions ϕ the expression
$$\int_{\mathcal{M}} d\mathrm{Vol}(x) \phi^n(x) f(x)$$

diverges in the radiation dominated universe for $n \geq 5$, while it is finite for all $n \in \mathbb{N}$ in the distributional case. Hence, the renormalized, distributional field theory exhibits a similar regularity as a classical field theory on Minkowski spacetime.

- Under comparison with [15] our analysis reveals, that the tameness of quantum field theoretic operator valued distributions at the initial singularity is a feature which is shared by the associated classical field theory and not a property tied to the quantum case.
- Nevertheless our analysis has several drawbacks:
  - It relies on the neglection of backreaction, which is a questionable requirement in the present situation.
  - It affects only high order Wick observables, whose conceptual importance is questionable.
  - The renormalization of higher order observables introduces a high degree of indeterminacy due to the occurring renormalization constants.

Anyhow, this treatment shows that renormalization procedures are not only of use in the context of quantum field theory but can also have applications within classical field theory. Moreover this treatment suggests, that quantum completeness properties could be induced by the classical background.

Ultrarelativistic behaviour of gravity in extreme situations:

In chapter 5 we have obtained the following results:

- Qualitatively, time- and spacelike geodesics behave increasingly lightlike as they approach the singularity of spatially flat FLRW spacetimes . This can be either seen at the level of the geodesic vector fields in general cases or at the level of the geodesics themselves in the case of a scale factor $a(\eta) = \eta^c$ given in conformal coordinates. Moreover we have conjectured, that this is a coordinate-invariant behaviour and have proposed an associated quantitative, putatively coordinate-invariant statement.
- We have pointed out, that in several situations present in the literature, where the behaviour of gravity in extreme situations is analysed, some kind of ultrarelativistic behaviour seems to be present. Especially we have discussed the Kasner singularity, the BKL-conjecture, the short distance Wheeler-deWitt equation and the case of strong gravity.

Although those results are only of qualitative nature, they led us to the conjecture that gravity could behave ultrarelativistically on fundamental scales.

The microscopic ultrarelativistic geometry of spacetime:

In chapter 6, which is based on the author's publication [176], we have obtained the following results:

- We have shown, that microscopic tangent light cones are endowed with universal geometric structures that are independent of the macroscopic behaviour of the gravitational field. Explicitly, those structures are given by (cf. section 6.1):
  1. The microscopic future tangent light cone $L_p^+\mathscr{M}$ is isomorphic to $\mathbb{C}_\infty \times (0,\infty)$ as a smooth manifold, where $\mathbb{C}_\infty$ denotes the Riemann sphere. Hereby the direction of a null vector is identified with $z \in \mathbb{C}_\infty$ while its euclidean length is identified with $\lambda \in (0,\infty)$.
  2. $L_p^+\mathscr{M}$ is a linear cone.
  3. $L_p^+\mathscr{M}$ is endowed with a sort of conformal structure.
  4. There exists a degenerate metric on $L_p^+\mathscr{M}$ which is induced by the metric tensor.
- Moreover, those structures can be identified as the structures of a weak Carroll manifold (for the definition of the latter cf. [61]).
- The conformal automorphism group of those structures is given by a right semi-direct product $\mathrm{PSL}(2,\mathbb{C}) \rtimes_\kappa C^\infty(\mathbb{C}_\infty,(0,\infty))$. It has infinitely many Lorentz subgroups which are parametrized by so called crossed homomorphisms (cf. section 6.4).
- The isometry group of above universal structures is a non-trivially represented Lorentz group which is induced by the original representation of the Lorentz group on $T_p^+\mathscr{M}$. Any proper orthochronous Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ acts then on an element $(z,\lambda) \in \mathbb{C}_\infty \times (0,\infty)$ of $L_p^+\mathscr{M}$ as

  $$(z,\lambda) \mapsto (Z_\Lambda(z), f_\Lambda(z)\lambda)$$

  with Z_Λ being the Möbius transformation associated with Λ and f_Λ being a non-trivial rescaling factor which constitutes a crossed homomorphism (cf. section 6.5 and appendix C.3). This shows especially, that the euclidean length of null vectors is a non-trivial, Lorentz covariant quantity, although it is not Lorentz invariant.
- We have introduced the notion of a *length gauge* for null vectors, which is a Lorentz-covariant map $\lambda_p : L_p^+\mathscr{M} \to \mathbb{R}^+$. We have then explained, that the conformal automorphism group could be understood as the group which encodes the transformation properties of all possible length gauges.
- We have shown, that the conformal automorphism group and the isometry group are both eligible as gauge groups for the bundle of null vectors.

- We have shown, that the structure and the interpretation of the conformal automorphism group resemble those of the original BMS group to a great extent.

By this analysis we have revealed some surprisingly rich and previously undiscovered structures hidden in the theories of relativity. Especially we have shown, that BMS-like groups arise not only as asymptotic symmetry groups in the context of cosmology, but describe also a fundamental and apparently unknown microscopic symmetry of pseudo-Riemannian geometry.

7.2. On promising directions of further research

In this section we want to present concisely some directions of further research that are motivated by the results of this thesis and which are promising from our point of view. Partly, those research ideas were already presented in the conclusions of the respective chapters.

Asymptotic expansions, universality and weakly interacting gases: As already explained in chapter 3, we think that the benefit of the analysis presented there is that it relates the phase structure of Bose gases to qualitative features of asymptotic expansions, which are robust under perturbations in formally similar geometric situations. The analysis of chapter 3 should be hence understood as a starting point for more general treatments. Especially we suggest the following directions of research:

- One could analyse on very general grounds, which "number-theoretic perturbations" of eigenvalue distributions and eigenvalue asymptotics would leave the qualitative structure of the asymptotic expansions invariant. Similarly one could analyse, to which extent one is allowed to deform the qualitative structure of the asymptotic expansions without modifying the resulting thermodynamic properties of the system.

- One should analyze the case of general traps by methods inspired by [171, 172]. There very general properties of eigenvalue sequences are related to properties of the respective spectral ζ-functions.

- One should try to tackle the weakly interacting case by analysing, how the eigenvalue distributions are qualitatively modified under the inclusion of weak interactions. Especially one should understand, to which extent the qualitative properties of the asymptotic expansions are universal properties of this system.

- The formal analogy between infinite renormalization in the context of quantum field theory and the treatment of the chemical potential in the analysis of chapter 3 suggests the question, if one could apply some kind of renormalization group analysis on this problem, although – contrary to the case of quantum field theory – the cut-off constitutes an IR-regulator in our case.

Classical field theory in the vicinity of cosmological singularities – consecutive questions: Although the treatment of section 4 was an interesting first step in the investigation of the properties of classical field theories in the vicinity of cosmological singularities, it had some serious drawbacks as depicted there. Nevertheless we think that the present results are interesting enough to anticipate, that also classical field theories could have interesting physical aspects in the vicinity of cosmological singularities. Especially we suggest the following directions of research:

- First and foremost one should understand the physical meaning of the occurring renormalization constants. Therefore a careful analysis of the distributional classical energy momentum tensor should be performed (cf. [21] for a similar situation).
- One should find more realistic scenarios for the investigation of the completeness properties of classical field theories in the vicinity of cosmological singularities. Especially one should answer the question, if any malignity associated with cosmological singularities could be probed by a measurement apparatus that is associated with classical field theory. Therefore one should either develop a realistic model for a field theoretic measurement apparatus or one should develop a general measurement theory for classical field theory (cf. [52] for a first step in this direction). This should then be applied on cosmological singularities.
- In chapter 4 we have shown, that distributional solutions to classical field equations show a different physical behaviour than their smooth counterparts in the vicinity of cosmological singularities. Hence one could wonder, if there are also other situations where distributional classical field theory and smooth classical field theory show a different behaviour. Hence one should understand, to which extent smooth and distributional classical field theory differ from a physical perspective.
- One should analyse a classical field theory in the vicinity of cosmological singularities by the use of Colombeau algebras (cf. [83, 84]). The benefit of Colombeau algebras in the present situation is given by the fact, that they can be used for a distributional treatment of general relativity, which is not possible with standard distributions (cf. [163]). Hence one should analyse, to which extent matter fields and gravitational fields can be investigated in a mutual framework, possibly under the consideration of backreaction effects.

Classical vs. quantum completenes – more rigorous results: One of the aspects of chapter 4 was, that our analysis suggested that the benign behaviour of quantum fields in the vicinity of cosmological singularities as found by [15] is indeed already present at the classical level. Nevertheless, our results are still not concrete enough to allow a confident answer to the question, to which extent the completeness properties of quantum fields are caused by the properties of the associated classical field theories. Especially we suggest the following directions of research:

- The results of chapter 4 suggest that distributional and smooth classical field theory are from a physical perspective inequivalent at cosmological singularities. One should understand if the corresponding quantum field theories are also mathematically inequivalent, i.e. if the associated quantum states define inequivalent representations of the quantum field theoretic observable algebra (cf. [173]). Moreover one should make it precise, how the behaviour of classical observables is connected to the behaviour of quantum observables. Maybe this could be done by a deformation quantization argument.

- The quantum completeness results of [96, 97] are based on the quantum field theoretic Schrödinger formalism, in which the role of the classical background is – at least for the author of this thesis – not very clear. Hence one should understand how the completeness aspects of classical field theory are related to quantum completeness properties of the Schrödinger functional.

- The distributional treatment of chapter 4 suggests, that one should be able to extend the propagators of the classical field theory by an analogous procedure. Thereby it is expected, since the wave front (cf. appendix A) set of the Cauchy principal value distribution and the delta distribution should violate the microlocal Hadamard condition (cf. [38, 78]), that the Hadamard condition has to be modified on the cosmological singularity. Since the Hadamard criterion is of great importance for the construction of quantum field theoretic models in curved spacetime, one should analyse if this modification of the Hadamard criterion is admissible or not, which should lead to some kind of *microlocal* completeness criterion.

Understanding the ultrarelativistic aspects of singularities: In section 5.1 we have shown, that the geodesics exhibit qualitatively some kind of ultrarelativistic behaviour in the vicinity of the initial singularity. Moreover we have explained in section 5.2 among other things, that the geodesics of the Kasner singularity show a similar behaviour. Although we have conjectured that this is a coordinate invariant statement, we did not prove this there and hence we suggest especially the following directions of research:

- It should be proven, that the aforementioned ultrarelativistic behaviour of time- and spacelike geodesics is really a coordinate-invariant property of FLRW and Kasner spacetimes. Especially this should be done by the strategy suggested around equation (5.14).

- One should understand, to which extent this ultrarelativistic behaviour corresponds to a modification of the geometric model. For example, this could be done in the context of Cartan geometry (cf. [161]). Especially one should understand, if the pseudo-Euclidean model geometry should be replaced by some putative ultrarelativistic model in the vicinity of the singularity, and if this change of the microscopic geometry is enough to explain the occurrence of a curvature singularity or even to yield a geometric completion.

- If the ultrarelativistic behaviour of geodesics in the vicinity of FLRW and Kasner singularities represents a coordinate invariant property, then one should understand to which extent this property can be generalized to general singularities. Especially one could analyse, if this property follows on general grounds in the context of singularity theorems, e.g. from the existence of closed trapped surfaces.

Can horismos replace causality? At the end of section 5.2 we have reviewed the fact, that causal relations are in some sense effective descriptions of horismos (i.e. null) relations, since two points are causally related if and only if they are related by a chain of null relations. This motivates the following directions of research:

- One should analyse to which extent horismos relations can replace causal relations in approaches to quantum gravity which rely on fundamental discretizations of spacetime, as *causal* dynamical triangulations and *causal* set theory. I.e. one could analyse under which conditions *horismos* dynamical triangulations and *horismos* sets could be defined. Moreover one could construct a version of Regge calculus where only null relations are allowed. Please note, that this was already suggested by [133] and a preliminary step was performed in [158].

- One should analyse the properties of a *constant speed random walk*, where all particles move with constant speed but are allowed to change the direction randomly. This situation would then be a toy model for some kind of Brownian motion on a spacetime where microscopically only null directions are allowed.

Applications of μBMS? In section 6 we have revealed some rich structures hidden in the theories of relativity, which especially comprised a microscopic analogue of the BMS-group. Motivated by the various applications of the original BMS group, one could ask if the microscopic BMS group has similar properties. Especially we suggest the following directions of research:

- One should understand, if the microscopic BMS group allows for a simplified Bulk description of gravitational waves on the light cone bundle. In addition one should understand, if it is interrelated to some kind of microscopic or infinitesimal memory effects on the level of tangent spaces.

- One should understand if the microscopic BMS group is associated with UV- or IR-properties of (quantum) field theories, in analogy to its macroscopic cousin. A possible starting point for this could be given by the concept of a length gauge as presented in section 6.6: Since length gauges are dual to null vectors they could be understood as very general Lorentz-covariant (re)definitions of momenta. One could then try to understand how scattering amplitudes behave under such momentum redefinitions.

- One should understand, to which extent general relativity can be entirely re-expressed in terms of objects that are related to the light cone bundle. In addition one should understand how a gauge theory associated with the μBMS group looks like.

Structural properties of μBMS: Although we have performed a thorough structural analysis of the μBMS-group, still many questions associated with this group are unanswered. Especially we suggest the following directions of research:

- One should understand, how the crossed homomorphisms which parametrize the Lorentz subgroups can be classified. One possible attempt for this would be to classify them in terms of their invariance properties under subgroups of $\mathrm{PSL}(2, \mathbb{C})$.
- One should understand – from the perspective of mathematical gauge theories – if there are manifolds which allow for a μBMS-structure but cannot be endowed with a pseudo-Riemannian metric. Generally spoken it would be also interesting to understand, if the various subgroups of the μBMS group are related to geometric properties of the manifold under consideration.
- In chapter 6 it was mentioned, that the μBMS-group and the original BMS group share a relationship which is reminiscent of the relation between the Lorentz and the Poincaré group. Since the Minkowski space can be written as the quotient of the Poincaré and the Lorentz group one could ask, if the μBMS- and the original BMS-group can be also used for the construction of some geometric model.

μBMS, singularities and other symmetries: One of the motivations for the investigation of the symmetries associated with microscopic tangent light cones was the observation made in section 5.1, that the geodesic geometry behaves increasingly ultrarelativistic in the vicinity of the initial singularity. While in the case of the FLRW spacetime this corresponds qualitatively to a degeneration of the tangent spaces towards microscopic tangent light cones, the picture is less clear in the Kasner case sketched in section 5.2. We hence propose the following direction of research:

- One should analyse the behaviour of geodesics in vicinity of the Kasner singularity. Are the geodesics constrained to some kind of dimensionally reduced light cone in the vicinity of the singularity? If so, one should investigate the microscopic symmetries of this geometric entity.
- Motivated by the observations, that all geodesic tangent vectors approach the light cone in the vicinity of the initial singularity and that the light cone is associated with a BMS-like group, one could ask if an asymptotic BMS-like symmetry group could be associated with universal geometric structures present on cosmological singularities.

A. A concise introduction to distribution theory

Since distributions will be of great importance in chapter 4 and 2, we want to give a concise introduction to their theory. Our main source will be [36], while other standard resources are comprised by [151, 149, 98, 179].

A.1. Basic notions regarding distributions

The space of distributions $\mathcal{D}'(\mathbb{R}^n)$ is defined as the dual space of the space of compactly supported smooth functions $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$, where the latter is endowed with its standard locally convex topology (cf. e.g. [179] or section V of [151]). As such, distributions can be understood as natural generalizations of linear functionals of the form

$$\chi_f : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}, h \mapsto \int_{\mathbb{R}^n} d^n x f(x) h(x)$$

with $f \in C(\mathbb{R}^n)$ being a continuous function. This is the reason, why distributions are often called generalized functions. If one wants to check explicitely, if a linear map

$$\chi : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}, h \mapsto \chi(h) \tag{A.1}$$

is a distribution, one has to show, that χ is continuous in the topology of $\mathcal{D}(\mathbb{R}^n)$, which is equivalent to the statement (cf. section V of [151]), that for any compact set $K \subset \mathbb{R}^n$ there is a constant C and an integer j s.th.

$$|\chi(h)| \leq C \sum_{|\alpha|<j} \|\partial^\alpha h\|_\infty$$

holds, where α is a multi-index (cf. [151]). We will also need the notion of compactly supported distributions. We therefore define the distributional support of a distribution $\chi \in \mathcal{D}'(\mathbb{R}^n)$ as the set of all $x \in \mathbb{R}^n$ s.th. for any open neighborhood $x \in U_x \subset \mathbb{R}^n$ the restriction of χ to U_x is non-zero. One can then show, that compactly supported distributions, i.e. distributions in $\mathcal{D}(\mathbb{R}^n)$ with compact distributional support, form the dual space of the space of all smooth functions $\mathcal{E}(\mathbb{R}^n) := C^\infty(\mathbb{R}^n)$ equipped with its standard topology.

Hence we denote the space of compactly supported distributions by $\mathcal{E}'(\mathbb{R}^n)$. Please note, that obviously $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ holds.

As the topological dual of $\mathcal{D}(\mathbb{R}^n)$, the space of distributions is equipped with a natural topology which has especially the property, that a sequence of distributions $(\chi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to a distribution $\chi \in \mathcal{D}(\mathbb{R}^n)$ if and only if for any bump function $b \in \mathcal{D}(\Omega)$

$$\lim_{n\to\infty} \chi_n(b) = \chi(b)$$

holds. In addition we have the useful property, that $\mathcal{D}(\mathbb{R}^n)$ can be continuously embedded into $\mathcal{D}'(\mathbb{R}^n)$ by

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n), b \mapsto \chi_b$$

with χ_b defined as in A.1. Moreover, in terms of this embedding, $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ is a dense subset, i.e. any distribution can be approximated by a sequence of bump functions. This means, that for any distribution $\chi \in \mathcal{D}'(\mathbb{R}^n)$ there exists a sequence of bump functions $(\chi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ such that

$$\chi(b) = \lim_{n\to\infty} \chi_n(b)$$

holds for any $b \in \mathcal{D}(\mathbb{R}^n)$. This property is extremely useful since it means that any distribution χ can be written as

$$\chi(b) = \lim_{n\to\infty} \int_{\mathbb{R}^n} d^n x f_n(x) b(x)$$

for a sequence $(f_n(x))_{n\in\mathbb{N}}$ of compactly supported smooth functions. For example, the famous delta distribution can be defined as

$$\delta(b) = \lim_{n\to\infty} \int_{\mathbb{R}^n} dx \left[\sqrt{\frac{n}{2\pi}} \exp\left(-\frac{n}{2}x^2\right)\right] b(x).$$

A.2. Some elementary operations on distributions

For later use we also want to introduce some elementary operations which can be performed on distributions, namely the restriction of a given distribution, the multiplication with a bump function and the distributional Fourier transform. More complicated operations (as products, extensions or tensor products) will then be introduced later.

Restrictions of distributions: We therefore define, for a distribution $\chi \in \mathcal{D}(\mathbb{R}^n)$ and an open set $U \subset \mathbb{R}^n$, the restriction of χ to U, denoted as $\chi|_U$, as the distribution in $\mathcal{D}'(U)$ which satisfies

$$\chi|_U (b) = \chi(i(b))$$

for all $b \in \mathcal{D}(U)$. Here $i : \mathcal{D}(U) \hookrightarrow \mathcal{D}(\mathbb{R}^n)$ is the canonical injection induced by the embedding $U \subset \mathbb{R}^n$.

Multiplication of distributions with smooth functions: Further we want to define the multiplication of a distribution with a smooth function, which can be easily defined as

$$f \cdot \chi = \chi \cdot f := \chi(f \ \cdot \ )$$

for a smooth function $f \in C^\infty(\mathbb{R}^n)$ and a distribution $\chi \in \mathcal{D}(\mathbb{R}^n)$.

Derivatives of distributions: Distributional derivatives are easily defined in a way, which is motivated by partial integration. Let therefore $\chi \in \mathcal{D}(\mathbb{R}^n)$ be a distribution and α be a multi-index. We then define:

$$\partial^\alpha \chi := (-1)^{|\alpha|} \chi\, (\partial^\alpha \ \cdot \ )$$

Extensions of distributions: Extensions of distribution are also easily defined as the operation reverse to restrictions: Let $U \subset \mathbb{V} \subset \mathbb{R}^n$ be open sets and $\chi \in \mathcal{D}(U)$. Then an $\tilde{\chi} \in \mathcal{D}(U)$ is called an extension of χ to V, if

$$\tilde{\chi}|_U = \chi$$

holds. Although this definition is very lucid, the operation of extending a given distribution is a very subtle procedure. We will give a concise treatment of this important topic in section 2.3.

The Fourier transform: Finally we want to define the distributional Fourier transform. Unfortunately, the Fourier transform cannot be defined for all distributions in $\mathcal{D}'(\mathbb{R}^n)$, but just for so called tempered distributions. Those are defined as members of the dual space $\mathcal{S}'(\mathbb{R}^n)$ of the Schwartz space of rapidly decreasing functions. The details of tempered distributions are not of importance in the present discussion and we refer to the literature (e.g. [151]) for their precise definition. For a tempered distribution $\chi \in \mathcal{S}'(\mathbb{R}^n)$ we can then define its distributional Fourier transform as

$$\mathcal{F}(\chi)(b) := \chi(\mathcal{F}(b))$$

which defines again a tempered distribution. Since $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ holds, the distributional Fourier transform is especially defined for distributions of compact support.

A.3. The singular support and the wave front set

We now want to understand better, to which extent distributions differ qualitatively from functions and especially we want to understand, how a certain distribution can be characterized by its singular behaviour. We therefore define first the so called *singular support* of a distribution $\chi \in \mathcal{D}'(\mathbb{R}^n)$, which is, in a colloquial language, defined as the set of all points in $\mathbb{R}^n$ on which χ cannot be represented in terms of a smooth function. More precisely, a point $x \in \mathbb{R}^n$ lies not in the singular support sing supp(χ) of a distribution $\chi \in \mathcal{D}'(\mathbb{R}^n)$, if and only if there is a open neighborhood $x \in U \subset \mathbb{R}^n$ s.th. the restriction of χ to U is can be written as

$$\chi|_U(b) = \int_U d^n x f(x) b(x)$$

for a smooth function b. For example, the singular support of the delta distribution is given by sing supp$(\delta) = \{0\}$. A more refined criterion for the singular structure of distributions is given by the concept of the *wave front set*, which characterizes, in a colloquial language, how a distribution differs from a smooth function in momentum space. Before defining it, we first state the following variant of the famous Payley-Schwartz theorem, which characterizes precisely, when compactly supported distributions can be represented in terms of compactly supported smooth functions. This will serve as a motivation for the definition of the wavefront set.

Theorem 2 (cf. Chapter 8.1 of [98])
A compactly supported distribution $\chi \in \mathcal{E}'(\mathbb{R}^n)$ is given by a compactly supported function $b \in \mathcal{D}(\mathbb{R}^n)$, i.e.

$$\forall f \in \mathcal{E}(\mathbb{R}^n) : \chi(f) = \int_{\mathbb{R}^n} d^n x b(x) f(x),$$

if and only if its distributional Fourier transform $\hat{\chi} \in C^\infty(\mathbb{R}^n)$ satisfies the following decay property:

(DP) For all $N \in \mathbb{N}$ there is a $C_N > 0$ s.th. $|\hat{\chi}(k)| \leq C_N (1 + |k|)^{-N}$ holds for all $k \in \mathbb{R}^n$.

The core message of this theorem is, that the difference between smooth functions and distributions is given by their UV-behaviour: A compactly supported distribution is precisely then a compactly supported smooth function, when its Fourier transform decays sufficiently fast. This motivates then a refined criterion for the characterization of the singular structure of distributions as compared to the singular support: Instead of collecting just the points $x \in \mathbb{R}^n$ at which a distribution fails to be a smooth function, one could in addition look at the directions in momentum space, at which its Fourier transform (localized at x) does not decay sufficiently fast. This is precisely the intuition behind the wavefront set, which will now be defined precisely. Therefore let $\chi \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution and $x \in \mathbb{R}^n$. Then define the singular fiber $\Sigma_x(\chi)$ of χ at x as the complement of the set of all tuples $(x, k) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ satisfying the property, that there is a bump function $b \in \mathcal{D}(\mathbb{R}^n)$ s.th. the Fourier transform of $b \cdot \chi$ decays according to the decay property (DP) of theorem 2 in an open cone around k. Then the wave front set of χ is defined as:

$$\mathrm{WF}(\chi) := \{(x, k) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} | k \in \Sigma_x(\chi)\}$$

A.4. Products of distributions:

After having understood how to characterize the singular structure of distributions, we now want to explain how they can be multiplied. We will present two types of product operations for distributions: We first introduce the tensor product and will then explain, when distributions can be multiplied, extending the usual multiplication of smooth functions.

Tensor products of distributions: Let X_1, X_2 be two open subsets of $\mathbb{R}^n$ and let $\chi_i \in \mathcal{D}'(X_i)$ be two distributions for $i = 1, 2$. Then there is a unique distribution $\chi_1 \otimes \chi_2 \in \mathcal{D}'(X_1 \times X_2)$ s.th. for all test functions $b_1 \in \mathcal{D}(X_1)$ and $b_2 \in \mathcal{D}(X_2)$

$$u_1 \otimes u_2\,(b_1 \otimes b_2) = u_1(b_1) \cdot u_2(b_2)$$

holds, where $b_1 \otimes b_2(x) := b_1(x) \cdot b_2(x)$ is the usual tensor product of smooth functions. The distribution $\chi_1 \otimes \chi_2$ is then called the *tensor product (distribution)* of χ_1 and χ_2. Moreover, the wave front set of $\chi_1 \otimes \chi_2$ satisfies:

$$WF(u \otimes v) \subset [\mathrm{WF}(u) \times \mathrm{WF}(v)] \cup [(\mathrm{supp}(u) \times \{0\}) \times \mathrm{WF}(v)] \cup [\mathrm{WF}(u) \times (\mathrm{supp}(v) \times \{0\})] \, .$$

Products of distributions: The multiplicability of distributions is described by Hörmanders theorem (cf. thm. 8.2.10 of [98]). Therefore let $X \subset \mathbb{R}^n$ be open and $\chi_1, \chi_2 \in \mathcal{D}'(X)$. Then, if there is no $(x, k) \in \mathrm{WF}(\chi_1)$ s.th. $(x, -k) \in \mathrm{WF}(\chi_2)$, we can define the product $\chi_1 \cdot \chi_2$ as the pull-back of the tensor product $\chi_1 \otimes \chi_2$ along the diagonal map

$$X \hookrightarrow X \times X, x \mapsto (x, x).$$

Moreover, the wave front set of $\chi_1 \cdot \chi_2$ is in this case given by

$$\mathrm{WF}(\chi_1 \cdot \chi_2) \subset (\mathrm{WF}(\chi_1) \cup (\mathrm{X} \times \{0\}) + (\mathrm{WF}(\chi_2) \cup (\mathrm{X} \times \{0\}) \, .$$

B. Calculations for FLRW-spacetimes

In this chapter, all calculations in the context of FLRW spacetime are collected that are needed in this thesis.

B.1. The metric, christoffel symbols and curvature tensors

We consider the spacetime $\mathscr{M} \cong \mathbb{R}^4$ with metric

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)$$

and $a : \mathbb{R} \to [0, \infty)$ satisfying $a(\eta) > 0$ for all $\eta \neq 0$.

Inverse metric: The inverse metric is given by

$$g^{\mu\nu} = a(\eta)^{-2} \eta^{\mu\nu}$$

since

$$a(\eta)^{-2} a(\eta)^2 \eta^{\mu\rho} \eta_{\rho\nu} = \delta^\mu_\nu$$

holds.

Christoffel symbols: The Christoffel symbols are given by the famous Koszul formula

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

which reads in our case as

$$\Gamma^\sigma{}_{\mu\nu} = \frac{\partial_\eta a(\eta)}{a(\eta)} \left[\delta^0{}_\mu \delta^\sigma{}_\nu + \delta^0{}_\nu \delta^\sigma{}_\mu - \eta^{\sigma 0} \eta_{\mu\nu}\right]$$

Hence 10 components of the Christoffel symbols are non-vanishing and those are explicitely given by:

$$\Gamma^0{}_{00} = \Gamma^0{}_{ii} = \Gamma^i{}_{0i} = \Gamma^i{}_{i0} = \frac{\partial_\eta a(\eta)}{a(\eta)} \tag{B.1}$$

Ricci tensor and Ricci scalar: Since we don't need the Riemann tensor, we will directly compute the Ricci tensor for FLRW spacetime. We therefore express first the Ricci tensor in terms of Christoffel symbols:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\lambda}{}_{\lambda\rho} - \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\rho\nu}.$$

Since all Christoffels other than (B.1) vanish, it can be easily shown that the Ricci tensor is diagonal, what follows also from symmetry considerations (cp. [183]). We now calculate the diagonal elements:

$$R_{00} = \partial_\lambda \Gamma^{\lambda}{}_{00} - \partial_0 \Gamma^{\lambda}{}_{\lambda 0} + \Gamma^{\rho}{}_{00}\Gamma^{\lambda}{}_{\lambda\rho} - \Gamma^{\rho}{}_{\lambda 0}\Gamma^{\lambda}{}_{\rho 0} = 3\left[\frac{\dot{a}\dot{a} - \ddot{a}a}{a^2}\right]$$
$$R_{ii} = \partial_\lambda \Gamma^{\lambda}{}_{ii} - \partial_i \Gamma^{\lambda}{}_{\lambda i} + \Gamma^{\rho}{}_{ii}\Gamma^{\lambda}{}_{\lambda\rho} - \Gamma^{\rho}{}_{\lambda i}\Gamma^{\lambda}{}_{\rho i} = \left[\frac{\dot{a}\dot{a} + \ddot{a}a}{a^2}\right]$$

With this we can now easily calculate the Ricci scalar as

$$R = g^{\mu\nu}R_{\mu\nu} = a^{-2}\left(-R_{00} + R_{11} + R_{22} + R_{33}\right) = 6\frac{\ddot{a}}{a^3}.$$

B.2. The wave equation and associated calculations

From the action

$$\mathcal{L} = \frac{1}{2}\sqrt{-\det(g)}\left[-g^{\mu\nu}\partial_\mu\partial_\nu\phi - \xi R_g \phi^2\right]$$

we can derive the equation of motion for ϕ by utilization of the Euler Lagrange equations. But before doing so, we first insert the expressions for the geometric quantities. By recalling the results from the last section and observing, that $\det(g) = -a(\eta)^8$ we obtain:

$$\mathcal{L} = -\frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 3\xi a\ddot{a}\phi^2$$

The Euler lagrange equations $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$ read then, since

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = -\partial_\mu\left(a^2\eta^{\mu\nu}\partial_\nu\phi\right) \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -6\xi a\ddot{a}\phi$$

holds as

$$-\partial_\mu\left(a^2\eta^{\mu\nu}\partial_\nu\phi\right) + 6\xi a\ddot{a}\phi = 0. \tag{B.2}$$

By utilization of

$$\partial_\mu\left(a^2\eta^{\mu\nu}\partial_\nu\phi\right) = a^2\left[-\partial_\eta^2 + \Delta\right]\phi - 2a\dot{a}\partial_\eta\phi,$$

where the Laplacian Δ is – as usual – defined as $\Delta = \sum_{i=1}^{3} \partial_i^2$ we can write the equation of motion (B.2) as:

$$a^2 \left[\partial_\eta^2 - \Delta\right] \phi + 2a\dot{a}\partial_\eta \phi + 6\xi a\ddot{a}\phi = 0 \tag{B.3}$$

We now introduce, as common in FLRW spacetimes (cp. [28, 135]), a scalar field χ defined as:

$$\chi := a\, \phi$$

We then obtain an equation of motion for χ by inserting $\phi = a^{-1}\chi$ in (B.3). By utilization of

$$\partial_\eta^2 \left(a^{-1}\chi\right) = \frac{-a\chi\ddot{a} - 2a\dot{a}\dot{\chi} + 2\chi\dot{a}^2 + a^2\ddot{\chi}}{a^3}, \tag{B.4}$$

$$\Delta \left(a^{-1}\chi\right) = a^{-1}\Delta\chi,$$

$$\partial_\eta \left(a^{-1}\chi\right) = \frac{a\dot{\chi} - \dot{a}\chi}{a^2}, \tag{B.5}$$

we can then write (B.3) as

$$a \left[\partial_\eta^2 - \Delta\right] \chi + (6\xi - 1)\, \ddot{a}\chi = 0$$

or equivalently as

$$a \left[\partial_\eta^2 - \Delta\right] a\phi + (6\xi - 1)\, a\ddot{a}\phi = 0. \tag{B.6}$$

We now introduce the d'Alambertian

$$\Box = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g} g^{\mu\nu} \partial_\nu\right),$$

which can be written in our FLRW spacetime as:

$$\begin{aligned}\Box &= \frac{1}{a^2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\frac{\dot{a}}{a^3}\partial_\eta \\ &= a^{-2}\left[-\partial_\eta^2 + \Delta\right] - 2\frac{\dot{a}}{a^3}\partial_\eta\end{aligned}$$

We will now see, that the equation of motions (B.2), (B.3) and (B.6) can be considered as equivalent to the wave equation. Therefore consider the massless wave operator

$$P = [-\Box + \xi R]$$

which can be hence written as

$$P = a^{-2}\left[-\partial_\eta^2 + \Delta\right] - 2\frac{\dot{a}}{a^3}\partial_\eta + \xi 6\frac{\ddot{a}}{a^3}.$$

By utilization of the formulas (B.4) - (B.5), we can write the wave operator equivalently as

$$P = a^{-3}\left[-\partial_\eta^2 + \Delta\right] a + (6\xi - 1)\, \frac{\ddot{a}}{a^3}.$$

Hence the wave equation $P\phi = 0$ is equivalent to the equations of motion (B.2), (B.3) and (B.6). In the case of conformal coupling $\xi = \frac{1}{6}$, the wave operator is then given by

$$P = a^{-3}\left[-\partial_\eta^2 + \Delta\right] a \tag{B.7}$$

and the wave equation is given by

$$a^{-3}\left[-\partial_\eta^2 + \Delta\right] a\phi = 0. \tag{B.8}$$

Obivously, the equation of motion (B.6) is equivalent to the wave equation $P\phi = 0$ as given by (B.8) with the wave operator P being defined as in (B.7).

Finally we want to show, that

$$\left(\psi_{\vec{p}}, \psi^*_{\vec{q}}\right) = \delta(\vec{p} - \vec{q})$$

holds for $\psi_{\vec{k}}(\eta, \vec{x}) := X(\eta, \vec{k})e^{i\vec{k}\vec{x}}$ with

$$X(\eta, \vec{k}) = \frac{B_1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}}\frac{e^{i|\vec{k}|\eta}}{a(\eta)} + \frac{B_2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}}\frac{e^{-i|\vec{k}|\eta}}{a(\eta)}$$

and $B_1, B_2 \in \mathbb{C}$ satisfying $|B_1|^2 - |B_2|^2 = -1$. As before we define

$$(\psi_1, \psi_2) := -i\int_{\mathbb{R}^3} d^3x a(\eta)^2 \left(\psi_1(\eta_0, \vec{x})\, \partial_\eta\big|_{\eta=\eta_0} \psi_2(\eta, \vec{x}) - \psi_2(\eta_0, \vec{x})\, \partial_\eta\big|_{\eta=\eta_0} \psi_1(\eta, \vec{x})\right).$$

Then observe first, that we can write

$$\begin{aligned}\left(\psi_{\vec{p}}, \psi^*_{\vec{q}}\right) &= -i\int_{\mathbb{R}^3} d^3x a(\eta)^2 e^{i(\vec{p}-\vec{q})\vec{x}} \left(X(\eta_0, \vec{p})\dot{X}^*(\eta_0, \vec{q}) - \dot{X}(\eta_0, \vec{p})X^*(\eta_0, \vec{q})\right) \\ &= -i\int_{\mathbb{R}^3} d^3x a(\eta)^2 e^{i(\vec{p}-\vec{q})\vec{x}}\, W\big|_{\eta=\eta_0} [X(\cdot, \vec{p}), X^*(\cdot, \vec{q})],\end{aligned}$$

where we have defined the Wronskian

$$W[y_1, y_2] = y_1\dot{y}_2 - \dot{y}_1 y_2.$$

It is now easy to show, that

$$W[gy_1, gy_2] = g^2 W[y_1, y_2]$$

holds and by this we have

$$\left(\psi_{\vec{p}}, \psi^*_{\vec{q}}\right) = -i\int_{\mathbb{R}^3} d^3x e^{i(\vec{p}-\vec{q})\vec{x}}\, W\big|_{\eta=\eta_0} [T(\cdot, \vec{p}), T^*(\cdot, \vec{q})]$$

for

$$T(\eta, \vec{q}) = \frac{B_1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}} e^{i|\vec{k}|\eta} + \frac{B_2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}|}} e^{-i|\vec{k}|\eta}.$$

Now observe, that by the famous identity

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3x e^{i(\vec{p}-\vec{q})\vec{x}} = \delta(\vec{p}-\vec{q})$$

we have

$$\left(\psi_{\vec{p}}, \psi^*_{\vec{q}}\right) = -i(2\pi)^3\delta(\vec{p}-\vec{q})\; W|_{\eta=\eta_0}\left[T(\cdot,\vec{p}), T^*(\cdot,\vec{q})\right].$$

It then can be easily calculated that this reduces to:

$$\left(\psi_{\vec{p}}, \psi^*_{\vec{q}}\right) = -(2\pi)^3\delta(\vec{p}-\vec{q})\left(\frac{|B_1|^2-|B_2|^2}{(2\pi)^3}\right).$$

And hence the claim is shown.

B.3. Distributional extensions

We prove the following theorem:

Theorem 3
Let $n \in \mathbb{N}$. Then:

1. *The map*

$$t^{(n)} : b \in \mathcal{D}(\mathbb{R}) \mapsto t(b) := -\frac{1}{(n-1)!}\int_{\mathbb{R}} d\eta\, \ln(|\eta|) b^{(n)}(\eta)$$

   is a distribution in $\mathcal{D}'(\mathbb{R})$.

2. *$t^{(n)}$ extends $\eta^{-n} \in \mathcal{D}'(\mathbb{R}\setminus\{0\})$ and moreover $\eta^n t^{(n)} = 1$ holds in a distributional sense.*

3. *Any $T^{(n)}$ with $\eta^n T^{(n)} = 1$ is given by*

$$T^{(n)} = t^{(n)} + \sum_{i=0}^{n-1} K_i^{(n)}\delta^{(i)}$$

Proof. 1.) Let $b \in \mathcal{D}(\mathbb{R})$. We then have

$$\left|t^{(n)}[b]\right| \leq \left(\int_{\mathrm{supp}(b)} d\eta\, \ln(|\eta|)\right)\left\|b^{(n)}\right\|_\infty$$

and hence the claim is shown, since $\ln(|\eta|) \in L^1_{\mathrm{loc}}(\mathbb{R})$.
2.) Let $b \in \mathcal{D}(\mathbb{R}\setminus\{0\})$. Then we have by integration by parts

$$t^{(n)}(b) = \int_{\mathbb{R}} d\eta\, \eta^{-n} b(\eta)$$

which is integrable since $0 \notin \mathrm{supp}(b)$. Moreover, for $b \in \mathcal{D}(\mathbb{R})$, one can show easily that $t^{(n)}[\eta^n b] = \int_{\mathbb{R}} d\eta\, b(\eta)$.
3.) We first show, that $T^{(n)}$ satisfies $\eta^n T^{(n)} = 1$. Therefore observe, that for any $0 \leq i \leq n-1$

$$\begin{aligned}\int_{\mathbb{R}} d\eta\, \eta^n \delta^{(i)}(\eta) b(\eta) &= (-1)^i \int_{\mathbb{R}} d\eta\, \delta(x) \frac{d^i}{d\eta^i} (\eta^n b(\eta)) \\ &= (-1)^i \int_{\mathbb{R}} d\eta\, \delta(x) \left(\sum_{j=0}^{i} \binom{i}{j} \frac{n!}{(n-j)!} \eta^{n-j} b^{(i-j)} \right) \\ &= 0.\end{aligned}$$

holds. And hence $\eta^n T^{(n)} = 1$ holds. Now assume, that $T^{(n)} \in \mathcal{D}'(\mathbb{R})$ satisfies $\eta^n T^{(n)} = 1$. It then follows, that $T^{(n)}$ restricts to η^{-n} on $\mathbb{R} \setminus \{0\}$. Hence it differs from $t^{(n)}$ by a linear combination of derivatives of delta distributions. But we have that $\eta^{(n)} \delta^i(\eta) \neq 0$ for $i \geq n$ and hence the claim follows. $\square$

B.4. Geodesics

We want to analyze the qualitative behaviour of geodesics in an FLRW background given by the metric

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)$$

with scale factor $a(\eta) = \eta^c$ and $\eta > 0$.

Other coordinate system: As a matter of fact, the geodesic equations are more convenient, if analysed in coordinates, in which the metric has the form

$$ds^2 = -dt^2 + b(t)^2 \left(dx^2 + dy^2 + dz^2\right) \tag{B.9}$$

with $b(t) = b_0 t^\gamma$ and $t > 0$. The both coordinate systems are related in terms of the coordinate transformation

$$\eta(t) = \int_0^t dt' \frac{1}{b(t')}, \tag{B.10}$$

since one obtains then easily:

$$\begin{aligned}ds^2 &= -dt^2 + b(t)^2 \left(dx^2 + dy^2 + dz^2\right) \\ &= -\left(\frac{dt}{d\eta}\right)^2 d\eta^2 + b(t(\eta))^2 \left(dx^2 + dy^2 + dz^2\right) \quad = -b(t(\eta))^2 \left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)\end{aligned}$$

Hence we have

$$a(\eta) = b \circ t(\eta)$$

with $t(\eta)$ defined as the inverse function of (B.10). If we set $b(t) = b_0 t^\gamma$ and $a(\eta) = \eta^c$ we obtain, that those scale factors are related as

$$\gamma = \frac{c}{1+c} \qquad \text{and} \qquad b_0 = (1-\gamma)^{-\gamma}.$$

since (B.10) reads then

$$\eta(t) = \frac{1}{b_0}\frac{1}{1-\gamma}t^{1-\gamma}.$$

Especially we have, since $c \in (0,\infty)$, that γ lies in $(0,1)$. The non-vanishing Christoffel symbols for the metric (B.9) are then given by

$$\Gamma^0_{\ ii} = b(t)\partial_t b(t) \qquad \text{and} \qquad \Gamma^i_{\ 0i} = \Gamma^i_{\ i0} = \frac{\partial_t b(t)}{b(t)},$$

which read in the case of $b(t) = b_0 t^\gamma$ as

$$\Gamma^0_{\ ii} = \gamma b_0^2 \frac{t^{2\gamma}}{t} \qquad \text{and} \qquad \Gamma^i_{\ 0i} = \gamma\frac{1}{t}.$$

The geodesic equations: The geodesic equations

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

read in those coordinates hence as

$$\frac{d^2 t}{d\lambda^2} + b(t)\dot{b}(t)\sum_{i=1}^{3}\left(\frac{dx^i}{d\lambda}\right)^2 = 0 \qquad \text{and} \qquad \frac{d^2 x^i}{d\lambda^2} + \frac{\dot{b}(t)}{b(t)}\sum_{i=1}^{3}\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} = 0.$$

Moreover we have, that the vector fields ∂_i are Killing vector fields for $i \in \{1,2,3\}$ and hence we have associated constants of motion, which are given by

$$b(t)^2\frac{dx^i}{d\lambda} = P^i. \tag{B.11}$$

Moreover we have the condition

$$-\left(\frac{dt}{d\lambda}\right)^2 + b(t)^2\sum_{i=1}^{3}\left(\frac{dx^i}{dt^2}\right)^2 = \sigma. \tag{B.12}$$

Here, the case of $\sigma < 0$ corresponds to affinely parametrized timelike geodesics, 0 corresponds to affinely parametrized null geodesics and $\sigma > 1$ corresponds to affinely parametrized spacelike geodesics.

Differential equations for the parametrization and the graph: Now observe, that, due to the high amount of symmetry, we don't need the geodesic equations to solve the equations of motion. We just need equations (B.11) and (B.12). We then insert first equation (B.11) into

(B.12), which gives:

$$b(t)^{-2}\sum_{i=1}^{3}\left(P^i\right)^2 = \sigma + \left(\frac{dt}{d\lambda}\right)^2.$$

By setting $P^2 := \sum_{i=1}^{3}\left(P^i\right)^2$ we can write this as

$$\frac{dt}{d\lambda} = \pm\sqrt{b(t)^{-2}P^2 - \sigma}, \tag{B.13}$$

which is a ordinary differential equation, that determines the parametrization of the geodesics. We then can write equation (B.11) as

$$\frac{dx^i}{d\lambda} = P^i b(t)^{-2}$$

which determines $x^i(\lambda)$ in terms of $t(\lambda)$. This gives a set of two ordinary differential equations, that determine the motion completely:

$$\frac{dt}{d\lambda} = \sqrt{\frac{P^2}{b(t)^2} - \sigma}$$
$$\frac{dx^i}{d\lambda} = \frac{P^i}{b(t)^2}$$

Please note, that the sign of the square root (B.13) was here choosen in a way which ensures, that the tangent vectors of our curve are future pointing.

But in the present situation, we are more interested in the graph $x^i(t)$ of the geodesics. We therefore rewrite the second differential equation as

$$\frac{dx^i}{dt}\frac{dt}{d\lambda} = \frac{P^i}{b(t)^2} \Leftrightarrow \frac{dx^i}{dt}\sqrt{\frac{P^2}{b(t)^2} - \sigma} = \frac{P^i}{b(t)^2}$$

and hence we have:

$$\frac{dx^i}{dt} = \frac{P^i}{\sqrt{P^2 b(t)^2 - \sigma b(t)^4}}$$

And hence our differential equations for the parametrization and the graph are

$$\frac{dt}{d\lambda} = \sqrt{\frac{P^2}{b(t)^2} - \sigma}, \tag{B.14}$$
$$\frac{dx^i}{dt} = \frac{P^i}{\sqrt{P^2 b(t)^2 - \sigma b(t)^4}}. \tag{B.15}$$

The equations of motion as integral equations: We now want to solve the equations of motion in the case $b(t) = b_0 t^\gamma$. We therefore write first (B.14) and (B.15) as integral equations.

We therefore write (B.14) as

$$\frac{1}{\sqrt{\frac{P^2}{b(t)^2} - \sigma}} \frac{dt}{d\lambda} = 1$$

which gives after integration

$$\int_{t(\lambda_0)}^{t(\lambda)} dt \frac{1}{\sqrt{\frac{P^2}{b(t)^2} - \sigma}} = (\lambda - \lambda_0).$$

This yields hence an implicit definition of the function $t(\lambda)$. Integrating (B.15) gives on the other hand:

$$x^i(t) - x^i(t_0) = \int_{t_0}^{t} dt \frac{P^i}{\sqrt{P^2 b(t)^2 - \sigma b(t)^4}}$$

Now consider the case $b(t) = b_0 t^\gamma$. Then the integral equations read in this case:

$$\int_{t(\lambda_0)}^{t(\lambda)} dt \frac{1}{\sqrt{\frac{P^2}{b_0^2 t^{2\gamma}} - \sigma}} = (\lambda - \lambda_0), \tag{B.16}$$

$$x^i(t) - x^i(t_0) = \int_{t_0}^{t} dt \frac{P^i}{\sqrt{P^2 b_0^2 t^{2\gamma} - \sigma b_0^4 t^{4\gamma}}}. \tag{B.17}$$

Null geodesics: In the case of null geodesics, the equations of motion can be easily solved. The integral equations read in this case:

$$\int_{t(\lambda_0)}^{t(\lambda)} dt t^\gamma = \frac{P}{b_0}(\lambda - \lambda_0),$$

$$x^i(t) - x^i(t_0) = \frac{P^i}{P b_0} \int_{t_0}^{t} dt t^{-\gamma}.$$

By recalling, that $\gamma \in (0, 1)$, those equations can be easily solved and we obtain:

$$t(\lambda) = \left[(\gamma + 1) \frac{P}{b_0} (\lambda - \lambda_0) + t(\lambda_0)^{\gamma+1} \right]^{\frac{1}{\gamma+1}},$$

$$x^i(t) = \frac{P^i}{P b_0 (1 - \gamma)} \left[t^{1-\gamma} - t_0^{1-\gamma} \right] + x^i(t_0).$$

Since we are interested in the geodesics which emanate from the singularity we set $\lambda_0 = 0$ and $t(\lambda_0) = t_0 = 0$ and obtain:

$$t(\lambda) = \left[(\gamma + 1) \frac{P}{b_0} \lambda \right]^{\frac{1}{\gamma+1}},$$

$$x^i(t) = \frac{P^i}{P b_0 (1 - \gamma)} t^{1-\gamma} + x^i(0).$$

Adapting the integrals for time- and spacelike geodesics: For $\sigma \neq 0$, solving the equations of motion is more complicated. For this, we need Euler's integral representation of the Gaussian Hypergeometric function, which is given by the formula (cf. [177])

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{s^{b-1}(1-s)^{c-b-1}}{(1-sz)^a} ds \tag{B.18} $$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $c \notin \{-m | m \in \mathbb{N}\}$ and $z \notin [1, \infty)$.

We now have to bring the integrals in a form, such that this formula is applicable. We first start with (B.16). We therefore set first for convenience $A^2 = P^2/b_0^2$ and write

$$ \int_{t(\lambda_0)}^{t(\lambda)} dt \frac{1}{\sqrt{\frac{P^2}{b_0^2 t^{2\gamma}} - \sigma}} = \frac{1}{A} \int_0^{t(\lambda)} dt \frac{t^\gamma}{\sqrt{1 - \sigma A^{-2} t^{2\gamma}}} - \frac{1}{A} \int_0^{t(\lambda_0)} dt \frac{t^\gamma}{\sqrt{1 - \sigma A^{-2} t^{2\gamma}}}. $$

We hence now want to calculate the integral

$$ I_1(t) := \frac{1}{A} \int_0^t dx \frac{x^\gamma}{\sqrt{1 - \sigma A^{-2} x^{2\gamma}}}. $$

We then make the substitution

$$ x(s) = t s^{\frac{1}{2\gamma}} $$

by which we can write

$$ I_1(t) = \frac{t^{\gamma+1}}{2\gamma A} \int_0^1 ds \frac{s^{\frac{1}{2\gamma} - \frac{1}{2}}}{\sqrt{1 - (\sigma t^{2\gamma} A^{-2})\, s}}, $$

where we have used

$$ \frac{dx}{ds} = \frac{t}{2\gamma} s^{\frac{1}{2\gamma} - 1}. $$

We then have by application of (B.18)

$$ {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3 + \frac{1}{\gamma}\right); z\right) = \frac{\Gamma\left(\frac{3}{2} + \frac{1}{2\gamma}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2\gamma}\right)} \int_0^1 ds \frac{s^{-\frac{1}{2} + \frac{1}{2\gamma}}}{\sqrt{1 - sz}} $$

which reduces by $\Gamma(1+x) = x\Gamma(x)$ to

$$ {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3 + \frac{1}{\gamma}\right); z\right) = \frac{\gamma+1}{2\gamma} \int_0^1 ds \frac{s^{-\frac{1}{2} + \frac{1}{2\gamma}}}{\sqrt{1 - sz}}. $$

And with this we obtain finally

$$ I_1(t) = \frac{t^{\gamma+1}}{A(1+\gamma)} {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3 + \frac{1}{\gamma}\right); \sigma A^{-2} t^{2\gamma}\right) $$

and hence all together:

$$\begin{aligned}(\lambda-\lambda_0) &= I_1(t(\lambda)) - I_1(t(\lambda_0)) \\ &= \left[\frac{b_0 t^{\gamma+1}}{P(1+\gamma)}{}_2F_1\left(\frac{1}{2},\frac{\gamma+1}{2\gamma},\frac{1}{2}\left(3+\frac{1}{\gamma}\right);\sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]_{t(\lambda_0)}^{t(\lambda)}\end{aligned}$$

We now want to obtain a solution for equation (B.17). We therefore write

$$\int_{t_0}^{t} dx \frac{P^i}{\sqrt{P^2 b_0^2 x^{2\gamma} - \sigma b_0^4 x^{4\gamma}}} = \frac{P^i}{Pb_0}\int_0^t dx \frac{x^{-\gamma}}{\sqrt{1-\sigma A^{-2}x^{2\gamma}}} - \frac{P^i}{Pb_0}\int_0^{t_0} dt \frac{x^{-\gamma}}{\sqrt{1-\sigma A^{-2}x^{2\gamma}}},$$

where we set again $A^2 = P^2/b_0^2$. We hence now calculate the integral

$$I_2(t) := \frac{P^i}{Pb_0}\int_0^t dx \frac{x^{-\gamma}}{\sqrt{1-\sigma A^{-2}x^{2\gamma}}}.$$

Again we make the substitution

$$x(s) = ts^{\frac{1}{2\gamma}}$$

which leads to:

$$I_2(t) = \frac{P^i}{Pb_0}\frac{t^{1-\gamma}}{2\gamma}\int_0^1 ds \frac{s^{\frac{1}{2\gamma}-\frac{3}{2}}}{\sqrt{1-(\sigma A^{-2}t^{2\gamma})\,s}}$$

We then have with (B.18)

$${}_2F_1\left(\frac{1}{2},\frac{1}{2}\left(\frac{1}{\gamma}-1\right),\frac{1}{2}\left(\frac{1}{\gamma}+1\right);z\right) = \frac{\Gamma\left(\frac{1}{2\gamma}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2\gamma}-\frac{1}{2}\right)}\int_0^1 ds \frac{s^{\frac{1}{2\gamma}-\frac{3}{2}}}{\sqrt{1-sz}}$$

which reduces by $\Gamma(1+x) = x\Gamma(x)$ to

$${}_2F_1\left(\frac{1}{2},\frac{1}{2}\left(\frac{1}{\gamma}-1\right),\frac{1}{2}\left(\frac{1}{\gamma}+1\right);z\right) = \frac{1-\gamma}{2\gamma}\int_0^1 ds \frac{s^{\frac{1}{2\gamma}-\frac{3}{2}}}{\sqrt{1-sz}}.$$

And hence we have

$$I_2(t) = \frac{P^i}{Pb_0}\frac{t^{1-\gamma}}{1-\gamma}{}_2F_1\left(\frac{1}{2},\frac{1}{2}\left(\frac{1}{\gamma}-1\right),\frac{1}{2}\left(\frac{1}{\gamma}+1\right);\sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)$$

which leads to:

$$x^i(t) = x^i(t_0) + \left[\frac{P^i}{Pb_0}\frac{t^{1-\gamma}}{1-\gamma}{}_2F_1\left(\frac{1}{2},\frac{1}{2}\left(\frac{1}{\gamma}-1\right),\frac{1}{2}\left(\frac{1}{\gamma}+1\right);\sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]_{t_0}^{t}$$

Now observe finally, that, since $P^2 = \sum_{i=1}^3 \left(P^i\right)^2$, we have, that the vector $\hat{e}$ defined as

$$\hat{e} = \left(e^1, e^2, e^3\right) := \left(\frac{P^1}{P},\frac{P^2}{P},\frac{P^3}{P}\right)$$

lies in S^2, i.e. $\hat{e}\cdot\hat{e}=1$. We hence can write $x^i(t)$ as:

$$x^i(t) = x^i(t_0) + e^i \left[\frac{t^{1-\gamma}}{(1-\gamma)b_0} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]_{t_0}^{t}$$

And hence we have all together the following solutions:

$$\begin{aligned}
(\lambda-\lambda_0) &= \left[\frac{b_0 t^{\gamma+1}}{P(1+\gamma)} {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]_{t(\lambda_0)}^{t(\lambda)} \\
x^i(t) &= x^i(t_0) + e^i \left[\frac{t^{1-\gamma}}{(1-\gamma)b_0} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]_{t_0}^{t}
\end{aligned}$$

Again, for $\lambda_0 = 0$ and $t_0 = 0$ those solutions read then:

$$\begin{aligned}
\lambda(t) &= \frac{b_0 t^{\gamma+1}}{P(1+\gamma)} \left[{}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right] \\
x^i(t) &= x^i(t_0) + e^i \left[\frac{t^{1-\gamma}}{(1-\gamma)b_0} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)\right]
\end{aligned}$$

Solutions in conformal coordinates: We now want to transform the solutions of the equations of motion to conformal coordinates. This is done by recalling, that the relationship between conformal time η and t is in our case given by

$$\eta(t) = \frac{1}{b_0}\frac{1}{1-\gamma}t^{1-\gamma} \qquad \Leftrightarrow \qquad t(\eta) = [b_0(1-\gamma)\eta]^{\frac{1}{1-\gamma}}$$

with $b_0 = (1-\gamma)^{-\gamma}$ and $\gamma = \frac{c}{1+c}$. We then have for the null solutions:

$$\begin{aligned}
\eta(\lambda) &= \left[\frac{1+\gamma}{1-\gamma}\right]^{\frac{1-\gamma}{1+\gamma}} [P\lambda]^{\frac{1-\gamma}{1+\gamma}}, \\
x^i(\eta_0) &= e^i\left[\eta-\eta_0\right] + x^i(\eta_0).
\end{aligned}$$

We now consider the time- and spacelike solutions. We have

$$\begin{aligned}
\lambda-\lambda_0 &= \left[\frac{1}{P}\frac{1-\gamma}{1+\gamma}\,\eta^{\frac{1+\gamma}{1-\gamma}}\,{}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right)\right]_{\eta(\lambda_0)}^{\eta(\lambda)}, \\
x^i(\eta) &= x^i(\eta_0) + e^i\left[\eta\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right)\right]_{\eta_0}^{\eta},
\end{aligned}$$

which gives for $\lambda_0 = 0$ $\eta_0 = \eta(\lambda_0) = 0$:

$$\begin{aligned}
\lambda(\eta) &= \frac{1}{P}\left[\frac{1-\gamma}{1+\gamma}\right]\eta^{\frac{1+\gamma}{1-\gamma}}\,{}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right), \\
x^i(\eta) &= x^i(\eta_0) + e^i\left[\eta\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right)\right].
\end{aligned}$$

A unifying form for the geodesics: We now continue working in conformal coordinates with $\lambda_0 = 0$ and $\eta_0 = \eta(\lambda_0) = 0$. We first write the null geodesics as:

$$\lambda_\star(\eta) := \frac{1}{P}\left[\frac{1-\gamma}{1+\gamma}\right]\eta^{\frac{1+\gamma}{1-\gamma}},$$
$$x^i_\star(\eta) := e^i\eta + x^i_0.$$

Let now $\sigma \neq 0$. We can then write the corresponding time-/spacelike geodesics as

$$\lambda_\sigma(\eta) = \lambda_*(\eta)G^{(\gamma)}_{\sigma,P}(\eta),$$
$$x^i_\sigma(\eta) - x^i_0 = \left[x^i_\star(\eta) - x^i_0\right] H^{(\gamma)}_{\sigma,P}(\eta),$$

with

$$G^{(\gamma)}_{\sigma,P}(\eta) := {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right),$$
$$H^{(\gamma)}_{\sigma,P}(\eta) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \frac{\sigma}{P^2}\eta^{\frac{2\gamma}{1-\gamma}}\right).$$

This discussion holds also for the other set of coordinates. We have:

$$\lambda_\sigma(t) = \lambda_*(t)\tilde{G}^{(\gamma)}_{\sigma,P}(t)$$
$$x^i_\sigma(t) - x^i_0 = \left[x^i_\star(t) - x^i_0\right] \tilde{H}^{(\gamma)}_{\sigma,P}(t)$$

with

$$\lambda_\star(t) = \frac{1}{P}\frac{b_0}{1+\gamma}t^{1+\gamma}$$
$$x^i_\star(t) - x^i_0 = e^i\frac{1}{b_0(1-\gamma)}t^{1-\gamma}$$

and

$$\tilde{G}^{(\gamma)}_{\sigma,P}(t) = {}_2F_1\left(\frac{1}{2}, \frac{\gamma+1}{2\gamma}, \frac{1}{2}\left(3+\frac{1}{\gamma}\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)$$
$$\tilde{H}^{(\gamma)}_{\sigma,P}(t) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{1}{\gamma}-1\right), \frac{1}{2}\left(\frac{1}{\gamma}+1\right); \sigma\frac{b_0^2}{P^2}t^{2\gamma}\right)$$

C. The Lorentz-Möbius correspondence as adapted to our situation

In this section we want to explain the correspondence between the Lorentz and the Möbius group as adapted to our situation. Most of the results are adapted straightforwardly from literature, but some results are in the present form also new. To keep this section self-contained, we first review all basic facts on the Riemann sphere that are required for the understanding of this article, although they were partially already presented in the main body. This will be done in section C.1. In section C.2 we will derive a useful interpretation for bundle trivializations in $\mathcal{B}$. In section C.3 we will then utilize this representation for the derivation of Lorentz transformation properties (6.6, 6.7) of null vectors.

C.1. The Riemann sphere and Möbius transformations

We review the basic theory of the Riemann sphere. Our main source for this section is given by [143]. The Riemann sphere is defined as the extended complex plane, i.e.

$$\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$$

and is coordinatized by complex numbers $z \in \mathbb{C}_\infty$. But for the derivation of the Lorentz-Möbius correspondence a different set of coordinates will be more convenient. Those coordinates are given by tuples $(\xi, \eta) \in \mathbb{C}^2$ which are allowed to take any value other than $(0,0)$ and will be called projective coordinates. A point $z \in \mathbb{C}_\infty$ on the Riemann sphere is then specified by the quotient

$$z = \xi/\eta.$$

Observe, that two tuples $(\xi_1, \eta_1), (\xi_2, \eta_2)$ represent the same $z \in \mathbb{C}_\infty$ if and only if there is an $\alpha \in \mathbb{C} \setminus \{0\}$ such that $(\xi_1, \eta_1) = (\alpha\xi_2, \alpha\eta_2)$ holds. This is the reason, why those coordinates are called projective coordinates (cf. [143]).

The Riemann sphere $\mathbb{C}_\infty$ is diffeomorphic to the standard 2-sphere S^2 in terms of the stereographic projection

$$\rho : S^2 \to \mathbb{C}_\infty, \hat{v} \mapsto \rho(\hat{v}) := \frac{\hat{v}^1 + i\hat{v}^2}{1 - \hat{v}^3},$$

where we wrote a generic unit vector $\hat{v} \in S^2$ as $\hat{v} = (\hat{v}^1, \hat{v}^2, \hat{v}^3)$. The inverse of the stereographic projection will be denoted by $\hat{e} := \rho^{-1}$ and can be explicitely written as

$$\hat{e} : \mathbb{C}_\infty \to S^2, z \mapsto \hat{e}(z) := \left(\hat{e}^1(z), \hat{e}^2(z), \hat{e}^3(z)\right)$$

with (cf. [143]):

$$\hat{e}^1\,(z = \xi/\eta) = \frac{z+\bar{z}}{\bar{z}+1} = \frac{\xi\bar{\eta}+\eta\bar{\xi}}{\xi\bar{\xi}+\eta\bar{\eta}} \tag{C.1}$$

$$\hat{e}^2\,(z = \xi/\eta) = \frac{1}{i}\frac{z-\bar{z}}{z\bar{z}+1} = \frac{1}{i}\frac{\xi\bar{\eta}-\eta\bar{\xi}}{\xi\bar{\xi}+\eta\bar{\eta}}$$

$$\hat{e}^3\,(z = \xi/\eta) = \frac{z\bar{z}-1}{z\bar{z}+1} = \frac{\xi\bar{\xi}-\eta\bar{\eta}}{\xi\bar{\xi}+\eta\bar{\eta}} \tag{C.2}$$

By introducing the Pauli matrices $(\sigma_\mu)_{\mu=0,\dots,3}$ defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and setting $\hat{e}^0(z) = 1$, we can write the relations (C.1) - (C.2) conveniently as

$$\hat{e}^\mu(z)\sigma_\mu = \frac{2}{\xi\bar{\xi}+\eta\bar{\eta}}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \tag{C.3}$$

for $z = \xi/\eta$ (cf. [143]). We now want to introduce Möbius transformations as biholomorphic automorphisms of $\mathbb{C}_\infty$ that are given by (cf. [143, 106, 56, 116])

$$Z : \mathbb{C}_\infty \to \mathbb{C}_\infty, z \mapsto \frac{az+b}{cz+d}$$

for complex numbers $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc = 1$. Strictly speaking, the requirement $ad - bc = 1$ is not necessary, but it is convenient, since it makes the structure of the Möbius group more lucid. Now, there exists a canonical surjective homomorphism (cf. [143]) between the group $\mathrm{SL}(2, \mathbb{C}) = \{A \in \mathbb{C}^{2\times 2} | \det(A) = 1\}$ and the Möbius group, which associates to a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \tag{C.4}$$

the Möbius transformation

$$Z^A(z) := \frac{az+b}{cz+d}.$$

One can then see easily, that two matrices $A, B \in \mathrm{SL}(2, \mathbb{C})$ define the same Möbius transformation, i.e. $Z^A = Z^B$, if and only if $A = \pm B$. Having this in mind, writing matrices A instead of Möbius transformations Z^A is often more practical and one gets by this an isomorphism between the Möbius group and the group $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C}) \setminus \{\pm 1\}$. Therefore, we denote the Möbius group often just by $\mathrm{PSL}(2,\mathbb{C})$. Moreover, we denote an equivalence class of matrices $[A]$ in $\mathrm{PSL}(2,\mathbb{C})$ just by one of their representatives, i.e. $[A] = A$ by slight abuse of notation. Observe in addition, that the action of a Möbius transformation can be easily expressed (by slight abuse of notation) in terms of projective coordinates as

$$Z^A\left(\xi/\eta\right) = A \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

which will be useful for the derivation of the Lorentz-Möbius correspondence. Finally we want to introduce a distance function on $\mathbb{C}_\infty$ given by

$$d : \mathbb{C}_\infty \times \mathbb{C}_\infty \to [0,\infty), (z_1, z_2) \mapsto \frac{2|z_1 - z_2|}{\sqrt{|z_1|^2+1}\sqrt{|z_2|^2+1}}. \tag{C.5}$$

This distance function makes $\mathbb{C}_\infty$ to a metric space and is the so called chordal distance (cf. [95]). It is induced by the corresponding euclidean distance of the associated points on S^2:

$$d(z_1, z_2) = |\hat{e}(z_1) - \hat{e}(z_2)| \tag{C.6}$$

The chordal distance is moreover related to a Riemannian metric on $\mathbb{C}_\infty$ that is expliecietly given by:

$$ds^2 = \frac{4}{(1+z\bar{z})^2} dz d\bar{z} \tag{C.7}$$

This metric is compatible with the natural conformal structure that $\mathbb{C}_\infty$ carries as a Riemann surface (cf. [106]). The pullback of the metric C.7 under a Möbius transformation Z^A specified by a matrix C.4 can be easily calculated as

$$ds^2 = K^A(z)^2 \frac{4}{(1+z\bar{z})^2} dz d\bar{z}$$

where the conformal factor K^A is given by:

$$K^A(z) = \frac{1+z\bar{z}}{(az+b)(\bar{a}\bar{z}+\bar{b}) + (cz+d)(\bar{c}\bar{z}+\bar{d})}$$

It can be written equivalently as

$$K^A(z) = \frac{1+z\bar{z}}{1+Z^A(z)\overline{Z^A(z)}} \frac{1}{(cz+d)(\bar{c}\bar{z}+\bar{d})}$$

or by the utilization of projective coordinates as:

$$K^A(z) = \frac{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}}{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} A^* A \begin{pmatrix} \xi \\ \eta \end{pmatrix}}$$

C.2. Spin representation of light cone bundle trivializations

We now want to utilize the results of the last section, to derive a convenient representation for local trivializations $(U, \psi^+) \in \mathcal{B}$ (for the definition of $\mathcal{B}$ consult section 1.4). This representation is an original construction of this article and is especially a reinterpretation of the relation (C.3). It will then allow a very fast computation of null vector rescalings under Lorentz transformation and will hence be useful in the next subsection, where we finally present the Lorentz-Möbius correspondence as adapted to our situation. Therefore let $(U, \psi) \in \mathcal{A}$ be a local trivialization of $T\mathscr{M}$ associated to a vielbein (E_μ) and let

$$\psi^+ : L^+U \to U \times (\mathbb{C}_\infty \times \mathbb{R}^+), v \in T_p\mathscr{M} \mapsto \left(p, (z_p^\psi(v), \lambda_p^\psi(v)\right) \tag{C.8}$$

be the associated bundle trivialization of $L^+\mathscr{M}$ as defined in (6.3). I.e. we have explicitely:

$$z_p^\psi(v^\mu E_\mu) = \rho(\hat{v}) = \frac{v^1 + iv^2}{v^0 - v^3},$$
$$\lambda_p^\psi(v^\mu E_\mu) = |\vec{v}|.$$

Let $p \in U$. As before, we denote the restriction of ψ^+ to $L_p^+\mathscr{M}$ by

$$\psi_p^+ : L_p^+\mathscr{M} \to L_p^+\mathscr{M}, v \mapsto (z_p^\psi(v), \lambda_p^\psi(v)). \tag{C.9}$$

We now want to generalize the relation (C.3), such that we can use it as an implicit definition of such bundle trivializations. Therefore observe first, that by (C.3) any null vector $v \in L_p^+\mathscr{M}$ given in the vielbein frame (E_μ) by $v = v^\mu E_\mu$ can be written as

$$v^\mu \sigma_\mu = v^0 \frac{2}{\xi\bar{\xi} + \eta\bar{\eta}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \tag{C.10}$$

for a unique tuple $(z = \xi/\eta, v^0) \in \mathbb{C}_\infty \times \mathbb{R}^+$. This follows just by multiplication of (C.3) with v^0 and by $|\hat{e}(z)| = 1$. Note, that the mentioned projective freedom in the coordinates (ξ, η) does not spoil the uniquenes of the representation (C.10), due to the occurence of the factor $(\xi\bar{\xi} + \eta\bar{\eta})^{-1}$ on the right hand side of (C.10). Equation (C.10) gives hence a correspondence between $L_p^+\mathscr{M}$ and $\mathbb{C}_\infty \times \mathbb{R}^+$ associated to a vielbein frame (E_μ). In this formulation, one realizes, that (C.10) is an implicit definition of the map ψ_p^+ from (C.9). I.e. one could have defined the diffeomorphism (C.9), and hence also the bundle trivialization (C.8), equivalently by demanding, that

$$v^\mu \sigma_\mu = \lambda_p^\psi(v) \frac{2}{\xi_p(v)\bar{\xi}_p(v) + \eta_p(v)\bar{\eta}_p(v)} \begin{pmatrix} \xi_p(v) \\ \eta_p(v) \end{pmatrix} \begin{pmatrix} \bar{\xi}_p(v) & \bar{\eta}_p(v) \end{pmatrix} \tag{C.11}$$

should hold for all $v \in L_p^+\mathscr{M}$ with $v = v^\mu E_\mu$. Here, we have set $z_p^\psi(v) =: \xi_p(v)/\eta_p(v)$. This relation (C.11) will be called the spin representation of the local bundle trivialization (C.9) and will be very useful in the next subsection.

C.3. The Lorentz-Möbius correspondence and associated null vector rescalings

We now present the well known correspondence between the Lorentz group $\mathrm{SO}^+(1,3)$ and the Möbius group $\mathrm{PSL}(2,\mathbb{C})$ as adapted to our situation. The ideas of this section come from [143] but are of course adapted to our situation. However, the utilization of the spin represention (C.3) and (C.11) for the derivation of null vector rescalings under Lorentz transformations is, to the best of our knowledge, an original development of this article. Therefore we fix again a $p \in \mathscr{M}$ and a vielbein frame (E_μ) associated to a local trivialization $(U,\psi) \in \mathcal{A}$ with $p \in U$. Let now $v \in L_p^+\mathscr{M}$ be a null vector with $v = v^\mu E_\mu$. Then, as explained in the last subsection, the induced coordinate system

$$\psi_p^+ : L_p^+\mathscr{M} \to \mathbb{C}_\infty \times \mathbb{R}^+, v \mapsto \left(z_p^\psi(v), \lambda_p^\psi(v)\right)$$

given by (C.9) is implicitely defined by the relation (C.11). Now forget first about the right hand side of (C.11). The well known correspondence between the Lorentz- and the Möbius group states then (cf. [143, 138]), that there is 1-to-1-correspondence between Lorentz transformations $\Lambda \in \mathrm{SO}^+(1,3)$ and matrices $A_\Lambda \in \mathrm{PSL}(2,\mathbb{C})$ such that

$$\left(\Lambda^\mu{}_\nu v^\nu \sigma_\mu\right) = A_\Lambda \left(v^\mu \sigma_\mu\right) A_\Lambda^* \tag{C.12}$$

holds for any $(v^\mu) \in \mathbb{R}^4$. The map $\Lambda \mapsto A_\Lambda$ constitutes then the isomorphism $\mathrm{SO}^+(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$ (cf. [138, 143]). By considering now again the right hand side of (C.11), we are then able to derive, how an active local Lorentz transformation induces a transformation on $\mathbb{C}_\infty \times \mathbb{R}^+$. Therefore let $\Lambda \in \mathrm{SO}^+(1,3)$ be a Lorentz transformation and set $w = w^\mu E_\mu \in L_p^+\mathscr{M}$ with:

$$w^\mu = \Lambda^\mu{}_\nu v^\nu$$

Set further $z_p^\psi(v) = \xi/\eta$. We then can write by (C.11) and (C.12):

$$\left(w^\mu \sigma_\mu\right) = \lambda_p^\psi(v) \frac{2}{\xi\bar{\xi} + \eta\bar{\eta}} A_\Lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} A_\Lambda^* \tag{C.13}$$

Set now

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} := A_\Lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Then, (C.13) can be equally written as:

$$w^\mu \sigma_\mu = \left(\lambda_p^\psi(v) \frac{\xi'\bar{\xi}' + \eta'\bar{\eta}'}{\xi\bar{\xi} + \eta\bar{\eta}} \right) \frac{2}{\xi'\bar{\xi}' + \eta'\bar{\eta}'} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} \begin{pmatrix} \bar{\xi}' & \bar{\eta}' \end{pmatrix}$$

This gives then, by defining the Möbius transformation $Z_\Lambda := Z^{A_\Lambda}$:

$$z_p^\psi(w) = Z_\Lambda(z_p^\psi(v)) \tag{C.14}$$

$$\lambda_p^\psi(w) = \lambda_p^\psi(v) \frac{\xi'\bar{\xi}' + \eta'\bar{\eta}'}{\xi\bar{\xi} + \eta\bar{\eta}} \tag{C.15}$$

Define then for each $A \in \mathrm{PSL}(2, \mathbb{C})$ the map

$$f^A(z) := \frac{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} A^* A \begin{pmatrix} \xi \\ \eta \end{pmatrix}}{\begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}},$$

which is just the inverse of the conformal factor (6.39), i.e. $f^A(z) = K^A(z)^{-1}$. We then can write (C.14-C.15) equally as

$$z_p^\psi(w) = Z_\Lambda(z_p^\psi(v)) \tag{C.16}$$

$$\lambda_p^\psi(w) = \lambda_p^\psi(v) f_\Lambda(z_p^\psi(v)) \tag{C.17}$$

where we have defined

$$f_\Lambda(Z) := f^{A_\Lambda}(z).$$

By this we have derived transformation formulas for coordinate systems $\psi^+ \in \mathcal{B}$ under local Lorentz transformations. Now recall, that $\lambda_p^\psi(w) = w^0 = |\vec{w}|$ and $\lambda_p^\psi(v) = v^0 = |\vec{v}|$. By this we then can summarize this finding in a more colloquial language: Any active Lorentz transformation

$$v^\mu E_\mu \mapsto (\Lambda^\mu{}_\nu w^\nu) E_\mu$$

corresponds to a unique Möbius transformation

$$z \mapsto Z_\Lambda(z)$$

on the Riemann sphere, together with a direction dependent rescaling

$$v^0 \mapsto w^0 = v^0 f_\Lambda(\rho(\hat{v})).$$

As it stands, it seems a little bit surprising, that the function f^A which determines the rescaling is just the inverse of the conformal factor (6.39), i.e. $f_A(z) = K_A^{-1}(z)$. But the reason for this was explained in section 6.5.

D. On right semidirect product groups

We introduce some concepts from group theory, that are needed for our investigation. Especially we will introduce the notion of right semidirect product groups in section D.1 and will explain in section D.2, how certain subgroups of right semidirect product groups can be parametrized in terms of so called crossed homomorphisms.

D.1. Right semidirect product groups

We first introduce the appropriate notion of a semidirect product of groups. Let P, Q be two groups and let

$$\tilde{\kappa} : P \to \mathrm{Aut}(Q), p \mapsto \tilde{\kappa}_p$$

be a group antihomomorphism. The latter is equivalent to the statement, that $\tilde{\kappa}$ defines a right group action of P on Q that acts on Q in terms of automorphisms. We then define the right semidirect product of P and Q with respect to $\tilde{\kappa}$, denoted by $P \ltimes_{\tilde{\kappa}} Q$, as follows (cf. [46, 6, 2]):

1. The underlying set is given by the cartesian product $P \times Q$.
2. The group operation is given by:

$$(p_1, q_1)(p_2, q_2) = (p_1 p_2, \tilde{\kappa}_{p_2}(q_1) q_2)$$

3. The identity element is given by (e_P, e_Q), where e_P and e_Q are the respective identity elements of P and Q.
4. The inverse element of (p, q) is given by $(p^{-1}, \tilde{\kappa}_{p^{-1}}(q^{-1}))$.

Left and right semidirect products are equivalent, in the same way as left and right actions are equivalent (cf. [6]). Nevertheless, a right semidirect product will be more instructive in the present situation. Please note in addition, that it is of greatest importance, that $\tilde{\kappa}$ acts on Q in terms of automorphisms and especially, that $\tilde{\kappa}_p$ preserves the unit element of Q for any $p \in P$, since otherwise $P \ltimes_{\tilde{\kappa}} Q$ would not even constitute a group. For later use, we also want to define the canonical projection

$$\pi_{\tilde{\kappa}} : P \ltimes_{\tilde{\kappa}} Q \to P, (p, q) \mapsto p.$$

D.2. Crossed homomorphisms

Let P denote an arbitrary group, let Q denote an abelian group and let $P \ltimes_{\tilde{\kappa}} Q$ denote their right semidirect product with respect to an antihomomorphism $\tilde{\kappa} : P \to Q$. Then one can show (cf. [3, 1] and p. 88 of [37]), that crossed homomorphisms[1], defined as maps

$$c : P \to Q, p \mapsto c_p$$

which satisfy

$$\forall p_1, p_2 \in P : c_{p_1 p_2} = \tilde{\kappa}_{p_2}(c_{p_1}) c_{p_2},$$

parametrize homomorphic sections of the canonical projection

$$\pi_{\tilde{\kappa}} : P \ltimes_{\tilde{\kappa}} Q \to P.$$

This means, they parametrize injective group homomorphisms

$$i_c : P \hookrightarrow P \ltimes_{\tilde{\kappa}} Q$$

which satisfy $\pi_{\tilde{\kappa}} \circ i = \mathrm{id}$ (i.e. homomorphic embeddings of the group P into the semidirect product $P \ltimes_{\tilde{\kappa}} Q$) by sending:

$$i_c : P \hookrightarrow P \ltimes_{\tilde{\kappa}} Q, p \mapsto (p, c_p)$$

By this, there is a 1-to-1-correspondence between subgroups $P \subset P \ltimes_{\tilde{\kappa}} Q$ and crossed homomorphisms c (cf. [3, 37]).

[1] Our definition of a crossed homomorphism is adapted to the occuring right semidirect product. Actually, it should be called a crossed antihomomorphism.

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